Module Title: Introduction to Optimisation

School of Mathematics **Semester One 201819**

Calculator instructions:

- You are allowed to use a calculator which has had an approval sticker issued by the School of Mathematics.

Exam information:

- There are 3 pages to this exam.
- There will be **2 hours** to complete this exam.
- **Answer all questions.**
- All questions are worth equal marks.
- You must show all your calculations

1. (a) Find the unit normal vector to the surface $g(x,y,z) = xy^2 + x^3y - 2z^3 = 0$ at the point $(1,1,1)$, and give an equation for the tangent plane to the surface at that point. Find the direction, expressed as a unit vector, along which the function g decreases most rapidly from the point $(1,1,1)$. What is the rate of increase of g in the direction $(1,0,0)$?

(b) If $f(x,y) = x^2y + xy^3$ and $y^2 + yx^2 = x^3$, find expressions for the partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ and the total derivatives $\dfrac{df}{dx}$ and $\dfrac{df}{dy}$ in terms of x and y. (In the latter you don't need to simplify your answer.)

(c) Let the function $z(x,y)$ be defined implicitly by the relation

$$z^2x - yz + 2xy = 4.$$

Find z_x, z_y and z_{xx} in terms of x, y and z. Show that at $x = y = 1$, $z = 2$ is a value of z. Is this the only possible value of z at $x = y = 1$? Find also the numerical values of z_x, z_y and z_{xx} at $x = y = 1$, $z = 2$.

2. (a) Find the symmetric matrix associated with the quadratic form

$$Q(x_1, x_2, x_3) = 3x_1^2 + \frac{3}{2}x_2^2 + \frac{11}{2}x_3^2 + 4x_1x_2 + 4x_1x_3 + x_2x_3.$$

Find also the eigenvalues and unit eigenvectors of this matrix and hence show Q is positive semi-definite, and confirm this using the Principal Minor Test. From the unit eigenvectors give the normal form of Q.

(b) Now impose on the quadratic form of part (a) the additional constraint $4x_1 - 5x_2 - x_3 = 0$. Use the bordered Hessian to show the sign character of the reduced quadratic form. Explain how the normal form obtained in part (a) reduces under the additional constraint.

3. (a) Write down the Lagrangian, and hence find the two stationary points of the problem

$$f(x,y,z) = \frac{1}{2}x^2 + yz + \frac{1}{3}y^3 - z^2, \quad \text{subject to } h(x,y,z) = x + y + z = 2.$$

(b) Find the bordered Hessian for this problem, and evaluate the required leading principal minors for both solutions. Hence show that one solution is a local maximum and the other is indefinite.

(c) A company produces baskets at a rate $Q(x,y) = x^{1/3}y^{1/2}$ which it sells for a price p. The inputs x and y are positive quantities. The cost of production is $C = ax + by$ pounds, where a and b are positive constants. Write down the revenue and profit functions. Write down the Lagrangian for the problem of maximising the profit under the condition that the cost is kept at a constant value C_0. Show that at the critical point (x^*, y^*) of the variables and λ^* of the Lagrange multiplier we have:

$$x^* = \frac{2C_0}{5a}, \quad y^* = \frac{3C_0}{5b}, \quad 1 + \lambda^* = \frac{5p}{6C_0}\left(\frac{2C_0}{5a}\right)^{1/3}\left(\frac{3C_0}{5b}\right)^{1/2}.$$

Give a formula for the profit at the critical point.

4. Consider maximising the function $f(x, y) = 3x + 2y$ subject to the simultaneous constraints

$$y \leq x + 1\ , \quad 2y \geq x - 3\ , \quad \text{and} \quad x \leq 3\ .$$

(a) Write down the relevant Lagrangian and the corresponding first order equations, including the complementary slackness conditions, and the relevant inequalities. Analyse these to find the unique point that gives the maximiser, and the corresponding values of the Lagrange multipliers (which constraints are binding?). What is the maximum value of f under the constraints?

(b) Sketch a graph indicating the region defined by the inequalities, and explain the solution found in terms of the level curves (i.e., level lines) of f.

(c) Consider the problem of maximising the function $f(x, y, z) = x + y + z$ subject to the constraints

$$x^2 + y^2 + z^2 \leq 9\ , \quad x \geq 0, \quad y \geq 0, \quad z \geq 0\ .$$

Write down the Kuhn-Tucker (KT) Lagrangian for this problem, and the corresponding KT equalities and inequalities. Solve this problem and show that there is a unique maximiser. Give the value of the Lagrange multiplier at the maximum and the maximum value of the function.

End.