MATH2640 EXAMPLE 2 Workshop problems

B1 From the conditions on the variables we deduce conditions on differentials dx, dy, dz:

$$xy + 2yz + z - 4\sqrt{z} = 7 \Rightarrow y\,dx + (x+2z)\,dy + \left(2y+1-\frac{2}{\sqrt{z}}\right)dz = 0$$

$$xyz = 6 \Rightarrow yz\,dx + xz\,dy + xy\,dz = 0$$

Check that $(x,y,z) = (-3,2,1)$ fulfills both conditions.

Substitute these values in the cond^s on differentials:

$$\begin{cases} 2\,dx + 5\,dy + 3\,dz = 0 \\ 2\,dx + 3\,dy + 6\,dz = 0 \end{cases}$$

We want to increase dz, and hence express dx and dy in terms of dz.

Eliminate $d\cancel{x}$: $\quad 2\,dy - 3\,dz = 0 \Rightarrow dy = \frac{3}{2}\,dz$

so $dy > 0$ if $dz > 0$, y increases.

Eliminate dy: $\quad -4\,dx - 21\,dz = 0 \Rightarrow dx = -\frac{21}{4}\,dz$

So $dx < 0$ if $dz > 0$, x decreases.

B2 i) If $f(x,y) = 4xy^2 \Rightarrow \underline{\nabla} f = (4y^2, 8xy)$

at $(2,3)$: $\underline{\nabla} f(2,3) = (36, 48)$

$\|\underline{\nabla} f(2,3)\| = \sqrt{36^2+48^2} = 12\cdot\sqrt{3^2+4^2} = 60 \Rightarrow$ unit vector

$\underline{u} = \frac{1}{60}(36,48) = \left(\frac{3}{5}, \frac{4}{5}\right)$ direction of most rapid increase.

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ii) If $g(x,y) = ye^{2x} \Rightarrow \nabla g = (2ye^{2x}, e^{2x})$

at $(0,3)$: $\underline{\nabla} g(0,3) = (6,1) \Rightarrow \|\nabla g(0,3)\| = \sqrt{37}$

unit vector $\underline{u} = \left(\frac{-6}{\sqrt{37}}, \frac{-1}{\sqrt{37}}\right)$ is the direction of most rapid <u>decrease</u> of g at $(0,3)$

iii) If $g(x,y,z) = x^2y + 2xy^2 - 3z^2 = 0$ level surface of function g. Gradient:

$$\underline{\nabla} g = (2xy + 2y^2, x^2 + 4xy, -6z)$$

at $(1,1,1)$: $\underline{\nabla} g(1,1,1) = (4,5,-6)$ perp. to tangent plane.

unit normal $\hat{\underline{n}} = \left(\frac{4}{\sqrt{77}}, \frac{5}{\sqrt{77}}, \frac{-6}{\sqrt{77}}\right)$

B3) Surface given by <u>graph</u> of function $z = f(x,y) = x^2 + y^3$

$\Rightarrow$ level surface of $g(x,y,z) = z - f(x,y) = z - x^2 - y^3 = 0$

$$\underline{\nabla} g = (-2x, -3y^2, 1)$$

At $(1,1,2)$, which is a point of surface: $g(1,1,2) = 0$

$$\underline{\nabla} g(1,1,2) = (-2,-3,1)$$

$\Rightarrow$ Equation of tangent plane:

$$(-2,-3,1) \cdot \big((x,y,z) - (1,1,2)\big) = 0 \Rightarrow \boxed{-2x - 3y + z = -3}$$

B4) Consider $z(x,y) = \sin x \sin y$ and consider the Taylor series around $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$.

$$z\left(\frac{\pi}{4}, \frac{\pi}{4}\right) = \frac{1}{2}\sqrt{2} \cdot \frac{1}{2}\sqrt{2} = \frac{1}{2}.$$

Gradient:

$$\underline{\nabla} z = (\cos x \sin y, \sin x \cos y) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

Hessian: $z_{xx} = -\sin x \sin y$, $z_{yy} = -\sin x \sin y$

$$z_{xy} = \cos x \cos y$$

$$\Rightarrow \quad H = \begin{pmatrix} z_{xx} & z_{xy} \\ z_{xy} & z_{yy} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \quad \text{at } \left(\frac{\pi}{4}, \frac{\pi}{4}\right)$$

Taylor series:

$$z\left(\frac{\pi}{4}+h, \frac{\pi}{4}+k\right) = \frac{1}{2} + \left(\frac{1}{2}, \frac{1}{2}\right) \cdot \begin{pmatrix} h \\ k \end{pmatrix} + \frac{1}{2}(h,k) H \begin{pmatrix} h \\ k \end{pmatrix} + \dots$$

$$= \frac{1}{2} + \underbrace{\frac{1}{2}(h+k)}_{\text{linear}} + \underbrace{\frac{1}{4}\left(-h^2 - k^2 + 2hk\right)}_{\text{quadratic}} + \dots$$

Note: If we expand the usual Taylor series for

$$\sin\left(\frac{\pi}{4}+h\right) = \sin\left(\frac{\pi}{4}\right) + h \cdot \cos\left(\frac{\pi}{4}\right) + \frac{1}{2}h^2\left(-\sin\frac{\pi}{4}\right) + \dots)$$

$$= \frac{1}{\sqrt{2}}\left(1 + h - \frac{1}{2}h^2 + \dots\right)$$

$$\Rightarrow \sin\left(\frac{\pi}{4}+h\right)\sin\left(\frac{\pi}{4}+k\right) = \frac{1}{\sqrt{2}}\left(1+h-\frac{1}{2}h^2+\dots\right)\frac{1}{\sqrt{2}}\left(1+k-\frac{1}{2}k^2+\dots\right)$$

$$= \frac{1}{2}\left(1 + h + k - \frac{1}{2}h^2 - \frac{1}{2}k^2 + kh + \dots\right)$$

ignoring higher-order terms $\Rightarrow$ previous Taylor series.

B5 i) Find critical points of $\boxed{f(x,y) = x^3 + y^3 - 9xy}$

First-order conditions:

$$\begin{cases} f_x = 3x^2 - 9y = 0 & \Rightarrow \quad x^2 = 3y \\ f_y = 3y^2 - 9x = 0 & \Rightarrow \quad y^2 = 3x \end{cases}$$

Substitute y from first relation into the second:

$$\left(\tfrac{1}{3}x^2\right)^2 = 3x \Rightarrow x^4 = 27x \Rightarrow x(x^3 - 27) = 0$$

$\Rightarrow$ solutions $x = 0$ or $x = 3$

- $x = 0 \Rightarrow y = 0 \Rightarrow$ critical point $(0,0)$
- $x = 3 \Rightarrow y = 3 \Rightarrow$ critical point $(3,3)$

ii) Critical points of $g(x,y) = x^4 + 2x^2y - 6y^3$

First-order conditions:

$$\begin{cases} g_x = 4x^3 + 4xy = 0 & \Rightarrow \quad x(x^2 + y) = 0 \\ g_y = 2x^2 - 18y^2 = 0 & \Rightarrow \quad x^2 = 9y^2 \end{cases}$$

From the ~~second~~ first relation: $x = 0$ or $y = -x^2$

- $x = 0 \Rightarrow y = 0$ (from 2nd relation) $\Rightarrow \boxed{(0,0) \text{ crit. pt.}}$
- $y = -x^2 \Rightarrow x^2 = 9x^4 \Rightarrow x^2(9x^2 - 1) = 0$
  $x = 0$ (already found) or $x = \pm\frac{1}{3}$

$x = \pm\frac{1}{3} \Rightarrow y = -x^2 = -\frac{1}{9} \Rightarrow \boxed{\text{crit. pts } \left(\tfrac{1}{3}, -\tfrac{1}{9}\right), \left(-\tfrac{1}{3}, -\tfrac{1}{9}\right)}$

(iii) Critical points of $h(x,y,z) = x^3 + 4xy + y^2 + 2yz - z^2$

First-order conditions:

$$\begin{cases} h_x = 3x^2 + 4y = 0 \\ h_y = 4x + 2y + 2z = 0 \quad \Rightarrow \quad 2x + y + z = 0 \\ h_z = 2y - 2z = 0 \quad \Rightarrow \quad y = z \end{cases}$$

From 2^{nd} and 3^{d} relation: $\quad 2x + 2y = 0 \Rightarrow y = -x$.

Substitute this into the 1^{st} relation: $3x^2 - 4x = 0$

$\Rightarrow \quad x = 0 \quad$ or $\quad x = \frac{4}{3}$

- $x = 0 \Rightarrow y = 0, \; z = 0 \Rightarrow$ critical pt. $(0,0,0)$
- $x = \frac{4}{3} \Rightarrow y = -\frac{4}{3}, \; z = -\frac{4}{3} \Rightarrow$ critical pt. $\left(\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3}\right)$

B6) Consider sign properties of quadratic forms.

i) $Q(x,y,z) = 2(xy - xz + yz) = (x,y,z) \underbrace{\begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}}_{=A} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\det(A) = \begin{vmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{vmatrix} = -1 \cdot 1 - 1 \cdot 1 = -2 < 0$$

Use leading principal minor test.

$$LPM_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 < 0, \quad LPM_1 = 0$$

Q indefinite.

P.T.O.

ii) $Q(x_1,x_2,x_3) = 2x_1^2+5x_2^2+x_3^2-4x_1x_2-2x_1x_3$

$$= (x_1,x_2,x_3)\begin{pmatrix} 2 & -2 & -1 \\ -2 & 5 & 0 \\ -1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

$$\Rightarrow \det(A) = \begin{vmatrix} 2 & -2 & -1 \\ -2 & 5 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 2\cdot 5 + 2(-2) - 1\cdot 5 = 1 > 0$$

Leading principal minor test applies.

$$LPM_2 = \begin{vmatrix} 2 & -2 \\ -2 & 5 \end{vmatrix} = 6 > 0, \quad LPM_1 = 2 > 0$$

All LPM's positive $\Rightarrow$ Q positive definite

iii) $Q(x_1,x_2,x_3) = -x_1^2-2x_2^2-4x_3^2+2x_1x_2+4x_2x_3$

$$= (x_1,x_2,x_3)\begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 2 \\ 0 & 2 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\Rightarrow \det(A) = \begin{vmatrix} -1 & 1 & 0 \\ 1 & -2 & 2 \\ 0 & 2 & -4 \end{vmatrix} = -1\cdot 4 - 1(-4) = 0$$

Here we need all principal minors:

order 2: $\begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix} = 1 > 0$, $\begin{vmatrix} -1 & 0 \\ 0 & -4 \end{vmatrix} = 4 > 0$, $\begin{vmatrix} -2 & 2 \\ 2 & -4 \end{vmatrix} = 4 > 0$

order 1: $|-1| = -1$, $|-2| = -2$, $|-4| = -4$

(Here $|\cdot|$ denotes 1×1 determinants!) $\Rightarrow$ odd PMs negative even PMs positive $\Rightarrow$ Q negative semidefinite