MATH2640 Introduction to Optimisation

Example Sheet 1

Please hand the assessed questions by Wednesday 16th October 2019, 5pm

Partial differentiation, Gradient & Directional Derivative, Implicit functions, Differentials.
Based on material in Lectures 1 to 5

Assessed Questions

A 1.

(i) Find the critical points of the function $f(x) = 2x + x^3 - \frac{5}{2}x^2$ and characterize them. Sketch a graph of the function. Find also the absolute minimum in the domain $0 \leq x \leq 1$.
(ii) Make a contour plot of the function $z(x, y) = x + y^2$ in the xy-plane, by drawing a collection of curves $z(x, y) = c$ for different (positive and negative) values of the constant c.

A 2.

(i) Find f_x, f_y, f_{xy}, f_{xx} and f_{yy} for the functions:

$$f(x, y) = \cosh(x + y) + x \ln(y) \ .$$

Verify that $f_{xy} = f_{yx}$.
(ii) Find the gradient of the function $g(x, y, z) = (2\sqrt{y} - x)z^2$ at the point $(3, 4, 1)$ and give the coordinates of the unit vector $\mathbf{u}$ in the direction of the vector $(1, 2, 3)$. Hence, calculate the directional derivative $D_{\mathbf{u}}g(3, 4, 1)$.

A 3. The function $z(x, y)$ is defined implicitly by the relation $zy^4 - xz^2 + x^2y^3 = 3$. Find

z_x, z_y and z_{yy} in terms of x, y and z. Find two possible values for z at $x = 2$, $y = 1$ (**note this correction**), and show that one value is an integer. For that value, compute the corresponding numerical values of z_x, z_y and z_{yy}.

A 4.

(i) If $f(x, y) = \exp(xy^2)$ and $x^2 + y^3 = 2xy$, find expressions for the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ and the total derivatives df/dx and df/dy in terms of x and y. (In the latter you don't need to simplify your answer.)
(ii) Let variables x, y and z be linked by the two relationships

$$f(x, y, z) = -x^2 + y^2 + z^2 - 1 = 0,$$

$$g(x, y, z) = 3x^3 + y^3 + 2z^3 - 6 = 0.$$

Derive conditions on the differentials dx, dy, and dz if the functions f and g are kept at these values. If $y = 1$, find the two points with values of y and z satisfying $f = g = 0$. For the point containing only integer values find the numerical values of dx/dz and dy/dz at that point.

P.T.O.

Further Questions for Workshop Practice

B 1.

(i) Given that $x = 1$ is a critical point of the function $f(x) = \frac{1}{4}x^4 - \frac{3}{2}x^2 + 2x + 1$, find all critical points and characterize them. Sketch a graph of the function. Find also the absolute maxmimum when the domain for x is the interval $0 \le x \le 4$.
(ii) Draw a contour plot of the function $z(x, y) = x^2 - y^2$ in the xy-plane, and sketch a graph of this function in the xyz Cartesian frame.

B 2. Find f_x, f_y, f_{xy}, f_{xx} and f_{yy} for the functions

(i) $f(x, y) = x^2y^3 + x^3y^5$; (ii) $f(x, y) = x^2 \sin^2 y - x \ln(xy)$.

B 3. Find the gradient of the following functions at the given point and calculate the directional derivative in the direction of the unit vector $\mathbf{u}$.

(i) $f(x, y) = x^2 - y/x^2$ at (1,2), $\mathbf{u} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$.

(ii) $g(x, y) = x^2 + 2xy + \frac{1}{2}y^2$ at (1,1), $\mathbf{u} = s\mathbf{i} + t\mathbf{j}$.

In case (ii), find the values of s and t that make $\mathbf{u} \cdot \nabla g$: (a) a maximum, (b) a minimum, and (c) zero. Hint: remember that $\mathbf{u}$ is a unit vector, so $s^2 + t^2 = 1$. Interpret your results geometrically.

B 4. $z(x, y)$ is defined implicitly by the relation

$$z^2x - 2yz + xy^2 = 4.$$

Find z_x, z_y and z_{xx} in terms of x, y and z. Show that at $x = y = 1$, $z = 3$ is a value of z. Is this the only possible value of z at $x = y = 1$?
Find also the numerical values of z_x, z_y and z_{xx} at $x = y = 1$, $z = 3$.

B 5. If $f(x, y) = xy^2 + x^3y$ and $y^2 = x^3 + y^3$, find expressions for the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ and the total derivatives $\frac{df}{dx}$ and $\frac{df}{dy}$ in terms of x and y.

B 6. The variables x, y and z are linked by the two relationships

$$f(x, y, z) = x + y + z - 1 = 0,$$
$$g(x, y, z) = x^2 - 2y^2 + 3z^2 - 2 = 0.$$

Show that the differentials dx, dy, and dz satisfy

$$dx + dy + dz = 0,$$
$$xdx - 2ydy + 3zdz = 0.$$

Hence find $\frac{dy}{dx}$ and $\frac{dz}{dx}$ in terms of x, y and z. If $x = 1$, find the two possible numerical values of y and z satisfying $f = g = 0$, and hence find the numerical values of $\frac{dy}{dx}$ and $\frac{dz}{dx}$ at these two points. $f = g = 0$ defines two curves lying in the three-dimensional space xyz. Find the unit vectors tangent to these curves at the two points with $x = 1$.