Introduction to optimisation

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Organisation

- See the handout online.
- Semester: 1; No. of credits: 10; Level: 2
- Prerequisites: MATH1010 or (MATH1050 & MATH1060) or (MATH1050 & MATH1331) or equivalent.
- All material will be available electronically via the MATH2640 Minerva pages.
- If you wish to have paper copies, then please email me, given our low-carbon use policy.

Introduction

- **What is optimisation**?
- Answer: Optimisation is about "the quest for the best", i.e. designing strategies for obtaining the best result. (Optimum is Latin for best.) In many cases of human endeavour we search for conditions under which we attain the optimum. So often, we are trying to find maxima (or minima).
- In economics, this can concern the highest profit, best rewards and maximum growth.
- In river or coastal flood-mitigation, this concerns mitigation or minimisation of flood damage, expressed in terms of staying below a specified water level at given locations, properties saved or economic damage saved, along a river or a coast.
- For wave-energy devices, this concerns maximising energy output given incoming waves.
- In real life, optimisation will include uncertainty, but we will not deal with the effects of uncertainty in this course.

Introduction

- In mathematics, optimisation is the subject of dealing with finding maxima or minima of functions.
- The definition of a function or functions: an expression or expressions in terms of variables $x, y, \dots$ that can be altered, often also under additional conditons.
- Then we are tasked to find maxima or minima of the expression $f(x, y)$ (or expressions $f(x, y)$ and $g(x, y)$) subject to such additional conditions.
- Here the functions f, g can can describe the quantities we are interested in, such as profit, revenue, tax, energy output, flood damage, etc.
- We will (mainly) consider applications in economics.

Breakdown of the course

Chapter 1: Several variable calculus (~ 6 lectures)

- Develop techniques of multivariate functions, partial derivatives, total derivatives, gradients, directional derivatives, implicit differentiation, chain rule (in many variables), Taylor series, Hessian and stationary points.

Chapter 2: Unconstrained optimization (~ 6 lectures)

- Quadratic forms, eigenvalues, definiteness and principal minor test, stationary points, local extrema, applications to economics (Cobb-Douglas functions).

Chapters 3 & 4: Constrained optimization (~ 10 lectures)

- A) Equality constraints (conditions involving an equality $=$–symbol)
- B) Inequality constraints (conditions involving an inequality $<$–symbol)
- Lagrange multiplier formalism to find:
  - critical points and maximisers;
  - (bordered) Hessians to classify critical points;
  - Kuhn-Tucker theory; and,
  - applications to economics, etc.

Updates and feedback

- In 2018, the course structure was updated: e.g., 5 homework assignments.
- In 2019, some notes/assignment solutions will be typed out, with accompanying Python graphs.
- Every lecture, I will ask two volunteers to give me feedback immediately after the lecture (email/in person).
- There will be online questionnaires regarding comments and feedback, halfway and at the end of the course.
- Please feel free to ask questions.

2.5 Applications to economics & Cobb-Douglas function

- Unconstrained optimisation in economics: we are interested in optimising functions describing profits based on production, costings, revenues, etc. These can involve various kinds of quantities (denoted by $x_1, x_2, \ldots, x_n$, say) describing production outputs or variables influencing production. We use symbols:
- $R =$ revenue $R(x_1, \ldots, x_n)$
- $p_i =$ price per product
- $Q_i =$ production functions $\#$ outputs
- Π –profit function
- C –cost function (cost to produce items).

Applications to economics & Cobb-Douglas function

Situation #1:

- A firm produces Q items per year, which sell at a price p per item
- The revenu is:

$$R(x_1, \ldots, x_n) = pQ(x_1, \ldots, x_n)$$

- R, Q may depend on variables x_1, x_2 influencing the production (such as # of employees, amount spent on equipment, etc.).
- Vector $(x_1, \ldots, x_n)$ is called <u>input bundle</u> vector.
- <u>Cost function</u> $C(x_1, \ldots, x_n)$ measures how much the firm spends on production, giving a negative contribution.

Applications to economics & Cobb-Douglas function

- Hence, the profit is given by:

$$\Pi(x_1, \dots, x_n) = R - C = pQ(x_1, \dots, x_n) - C(x_1, \dots, x_n)$$

- To maximise profit we want to determine variables $x_1, \dots, x_n$ such that Π has maximum value.
- Stationary points follow from first-order conditions (FOCs):

$$\frac{\partial \Pi}{\partial x_i} = 0 \Longrightarrow p\frac{\partial Q}{\partial x_i} = \frac{\partial C}{\partial x_i}, \quad \text{for} \quad i = 1, \dots, n.$$

- A stationary points Hessian needs to be ND to maximise profit.

Applications to economics & Cobb-Douglas function

- In the case we have multiple products, we can have several production functions $Q_j(x_1, \ldots, x_n)$ each with its price p_j:

$$R(x_1, \ldots, x_n) = \sum_{j=1}^{m} p_j Q_j(x_1, \ldots, x_n) = \mathbf{p} \cdot \mathbf{Q}$$

  with cost function $C(\mathbf{x}) = C(x_1, \ldots, x_n)$.

- We then get as FOCs:

$$\begin{aligned} \Pi(\mathbf{x}) &= \sum_{j=1}^{m} p_j Q_j - C = \mathbf{p} \cdot \mathbf{Q}(\mathbf{x} - C(\mathbf{x}) \\ &\implies \frac{\partial \Pi}{\partial x_i} = \sum_{j=1}^{m} p_j \frac{\partial Q_j}{\partial x_i} - \frac{\partial C}{\partial x_i} = 0 \\ &\implies \mathbf{p} \cdot \frac{\partial \mathbf{Q}}{\partial x_i} - \frac{\partial C}{\partial x_i} = 0. \end{aligned} \tag{1}$$

Applications to economics & Cobb-Douglas function

Situation $\#2$ (discriminating monopolist):

- In some cases the price per product p_i is not constant but can be influenced by the demand.
- E.g. if a monopolist floods the market the price may be negatively influence by the production, so we get for example $p_i = a_i - b_i Q_i$ with constants a_i, b_i.
- Or we get a situation where the demand, given by the demand function $F_i(p_i)$ is influenced by the price, which then influences the production.
- Thus $Q_i = F_i(p_i)$ –output of product x_i is influenced by the price p_i.
- Usually we can then invert, with inverse function $G_i(Q_i)$, determining the price:

$$Q_i = F_i(p_i) \Longleftrightarrow p_i = G_i(Q_i)$$

  and we can try and determine optimal output. (i.e. find maximum of profit as function of the productions Q_i).

Cobb-Douglas production function

- The Cobb-Douglas production function is used in many economic models and involves two (or more) variables.
- For instance in models where production Q of a product depends on: $x_1 = K =$ Capital input and $x_2 = L =$ Labour input, so $Q = Q(K, L) = Q(x_1, x_2)$.
- The form of the Cobb-Douglas (CD, 1928) function in two variables is:

$$Q(x_1, x_2) = x_1^a x_2^b = K^a L^b, \quad \text{with constants} \quad a, b > 0.$$

- In more variables, we would have:

$$Q(x_1, x_2, \ldots, x_n) = x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}, \quad \text{with} \quad a_1, a_2, \ldots, a_n > 0.$$

Cobb-Douglas production function

- Consider the two-variables case. Assume the price of the product is p and the cost function

$$C(x_1, x_2) = w_1 x_1 + w_2 x_2 \qquad (\text{constants } w_1, w_2 > 0).$$

- Hence revenue $R(x_1, x_2) = pQ(x_1, x_2) = px_1^a x_2^b$ and the profit function Π becomes

$$\Pi(x_1, x_2) = R(x_1, x_2) - C(x_1, x_2) = px_1^a x_2^b - w_1 x_1 - w_2 x_2.$$

- Let us find the critical points of Π, to investigate the strategy to maximise profit. FOCs:

$$\begin{aligned}\frac{\partial \Pi}{\partial x_1} =& apx_1^{a-1} x_2^b - w_1 = 0 \Longrightarrow w_1 = apQ/x_1\\ \frac{\partial \Pi}{\partial x_2} =& bpx_2^a x_2^{b-1} - w_2 = 0 \Longrightarrow w_2 = bpQ/x_2.\end{aligned}$$

- Denote the critical point by $(\cdot)^*$: $x_1^* = apQ^*/w_1, x_2^* = bpQ^*/w_2$.

Cobb-Douglas production function

- Furthermore, we have that $Q^* = (x_1^*)^a (x_2^*)^b$ and hence we can find (x_1^*, x_2^*) by reinserting

$$Q^* = \left(\frac{apQ^*}{w_1}\right)^a \left(\frac{bpQ^*}{w_2}\right)^b \Longrightarrow 1 = \left(\frac{ap}{w_1}\right)^a \left(\frac{bp}{w_2}\right)^b (Q^*)^{a+b-1}.$$

- The critical production Q^* subsequently follows and then reinsert back into the formulas for $x_1^* = apQ^*/w_1, x_2^* = bpQ^*/w_2$ to obtain the critical values. But: $a + b < 1$!
- We are, however, interested in the conditions on a, b such that this critical point is a maximum.
- Hence, we need to examine the Hessian at the critical point (Why?).

Cobb-Douglas production function

- Second-order conditions require the Hessian:

$$H = \begin{pmatrix} \Pi_{x_1x_1} & \Pi_{x_1x_2} \\ \Pi_{x_1x_2} & \Pi_{x_2x_2} \end{pmatrix} = \begin{pmatrix} a(a-1)px_1^{a-2}x_2^b & abpx_1^{a-1}x_2^{b-1} \\ abpx_1^{a-1}x_2^{b-1} & b(b-1)px_1^a x_2^{b-2} \end{pmatrix}.$$

- Conditions for maximum (ND ...):

$$\begin{aligned} LPM_1 =& a(a-1)px_1^{a-2}x_2^b < 0 \\ LPM_2 =& ab(a-1)(b-1)p^2x_2^{2a-2}x_2^{2b-2} - a^2b^2p^2x_1^{2a-2}x_2^{2b-2} > 0 \\ \implies LPM_2 =& abp^2x_1^{2a-2}x_2^{2b-2}\left((a-1)(b-1) - ab\right) \\ =& abp^2x_1^{2a-2}x_2^{2b-2}(1-a-b) > 0. \end{aligned}$$

- Since all variables are positive, we have a maximum when (draw a graph of $a(a-1)$ versus a)

$$a(a-1) < 0, \qquad a+b < 1.$$

- So, we have $0 < a < 1$ and $0 < b < 1$ such that $a+b < 1$.

Cobb-Douglas production function

- If these conditions are not satisfied, then profit increases as x_1, x_2 get large.
- Felipe and Adams (2005) reanalysed the data from Cobb and Douglas (1928) and found a surprising result (links via wiki on CD).
- See exercises! Caution: bespoke production functions required based on appropriate/approximate fitting to data.