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1. (a) The function $z(x, y)$ is defined implicitly by the relation

$$x^2y + xyz^2 = -5.$$

Find the partial derivatives z_x and z_y in terms of x, y and z, and evaluate them at the point $x = 1$, $y = -1$, $z = 2$. Next, let the variable y be constrained so that $y = y(x)$. Write down the general formula for $\dfrac{dz}{dx}$ in terms of z_x, z_y and $\dfrac{dy}{dx}$, and evaluate $\dfrac{dz}{dx}$ along the curve $y^3 + x^2 = 0$ in terms of x, y and z. Find also the numerical value of $\dfrac{dz}{dx}$ at the point $x = 1$, $y = -1$, $z = 2$.

Answer: Partial derivatives:

$$z_x = -\frac{2x + z^2}{2xz} , \quad z_y = -\frac{x + z^2}{2yz} ,$$

and at $(1, -1, 2)$ they yield $z_x = -3/2$, $z_y = 5/4$. Along the curve we get

$$\frac{dy}{dx} = -\frac{2x}{3y^2} , \quad \text{and} \quad \frac{dz}{dx} = -\frac{2x + z^2}{2xz} + \frac{x + z^2}{2yz}\,\frac{2x}{3y^2} .$$

At $(1, -1, 2)$ this gives $dz/dx = -7/3$.

(b) Define the gradient, ∇f, and the directional derivative $D_{\mathbf{u}}f$, in the direction of a unit vector $\mathbf{u}$, of a function $f(x, y, z)$. For the function

$$f(x, y, z) = x^2 + 2xy + yz^3,$$

find ∇f and $D_{\mathbf{u}}f$ in the direction

$$\mathbf{u} = \frac{2}{\sqrt{6}}\mathbf{i} - \frac{1}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}$$

at the point $(x, y, z) = (2, 3, 1)$. Find also the equation for the tangent plane to the surface $f(x, y, z) = 19$ at the given point.

Answer: Gradient of f: $\nabla f = (f_x, f_y, f_z) = (2x+2y, 2x+z^3, 3yz^2)$. At $(2, 3, 1)$: $\nabla f(2, 3, 1) = (10, 5, 9)$. Directional derivative along unit vector $\mathbf{u}$ is given by: $D_{\mathbf{u}}f = \mathbf{u} \cdot \nabla f$. For given vector, $D_{\mathbf{u}}f(2, 3, 1) = \sqrt{6}$. Noting that $f(2, 3, 1) = 19$ the equation of the tangent plane at this point to the level surface is given by: $10x + 5y + 9z = 44$.

(c) A company produces three products in quantities x, y and z at prices £3, £2 and £5 respectively. Write down the corresponding profit function $\Pi(x, y, z)$. The budget is in balance if

$$xy + yz = 50 \quad \text{and} \quad xz + y^2 - z^2 = 33 .$$

Currently the production is given by $x = 6$, $y = 5$ and $z = 4$. Show that at this point the budget is in balance.

Next, the company manager wants to increase the production z. Compute what effect a small increase dz will have on x and y if the budget is kept in balance (do these increase or decrease?). Furthermore, will the profit Π go up or down if z increases while keeping the budget in balance?

Answer: At $(6, 5, 4)$ both conditions given are satisfied, implying balance. The differentials give:

$$\left.\begin{array}{l} ydx + (x+z)dy = -ydz \\ zdx + 2ydy = -(x-2z)dz \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{l} 5dx + 10dy = -5dz \\ 4dx + 10dy = 2dz \end{array}\right.$$

at the given point, leading to $dx = -7dz$, $dy = 3dz$, hence x decreases and y increases as z increases. Change in the profit is $d\Pi = -10dz$, so profit goes down from the value $\Pi(6, 5, 4) = 48$.

2. (a) Use the leading principal minor test to determine whether the quadratic form

$$Q(x, y, z) = -4x^2 - 3y^2 - z^2 + 4xy + 2yz$$

is positive definite, negative definite or indefinite.

Answer: The matrix of Q is

$$\mathbf{A} = \begin{pmatrix} -4 & 2 & 0 \\ 2 & -3 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

with $LPM_3 = -4 < 0$, $LPM_2 = 8 > 0$, $LPM_1 = -4 < 0$. Alternating signs starting from negative LPM_1, hence negative definite.

(b) Consider the quadratic form

$$Q(x_1, x_2) = 3x_1^2 - 3x_1x_2 + 3x_2^2 = \mathbf{x}^T A\mathbf{x} \ ,$$

where $\mathbf{x}^T = (x_1, x_2)$ and A a symmetric matrix. Write down the 2×2 matrix A and find the eigenvalues of this matrix, and use these to determine whether Q is positive definite, negative definite or indefinite. Find also the unit eigenvectors, and hence write Q as a sum of two squares (determine the combinations of the variables x_1 and x_2 which enter in these squares).

Answer: The matrix is

$$\mathbf{A} = \begin{pmatrix} 3 & -3/2 \\ -3/2 & 3 \end{pmatrix} ,$$

having eigenvalues $\lambda_1 = 9/2$, with unit eigenvector $\frac{1}{\sqrt{2}}(1, -1)$, and $\lambda_2 = 3/2$ with unit eigenvector $\frac{1}{\sqrt{2}}(1, 1)$. Hence, Q can be written as

$$Q(x_1, x_2) = \frac{9}{2}\left(\frac{x_1 - x_2}{\sqrt{2}}\right)^2 + \frac{3}{2}\left(\frac{x_1 + x_2}{\sqrt{2}}\right)^2 .$$

(c) Find all six stationary points of the function

$$f(x, y, z) = \frac{1}{16}x^4 + \frac{1}{3}y^3 + \frac{1}{2}z^2 + xy + xz + yz - 2y .$$

Write down the Hessian matrix for this problem, and evaluate its leading principal minors. Evaluating these at the stationary points, use the leading principal minor test to determine whether the critical points are maxima, minima or indefinite points.

Answer: The first order conditions are:

$$f_x = \frac{1}{4}x^3 + y + z = 0 \ , \quad f_y = y^2 + x + z - 2 = 0 \ , \quad f_z = x + y + z = 0 \ ,$$

leading to the stationary (critical) points $(0,2,-2)$ (indefinite point), $(0,-1,1)$ (indefinite point), $(2,2,-4)$ (local minimum), $(2,-1,-1)$ (indefinite point), $(-2,2,0)$ (local minimum) and $(-2,-1,3)$ (indefinite point). The latter characterisation is obtained from signs of the Hessian:

$$\mathbf{H} = \begin{pmatrix} \frac{3}{4}x^2 & 1 & 1 \\ 1 & 2y & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

with leading principal minors: $LPM_3 = (\frac{3}{4}x^2 - 1)(2y-1)$, $LPM_2 = \frac{3}{2}x^2 y - 1$, $LPM_1 = \frac{3}{4}x^2$, inserting the values for x, y of the critical points and considering their sign behaviour.

3. (a) Use the method of Bordered Hessians to determine the sign property (definiteness) of the following constrained quadratic form

$$Q(x_1, x_2, x_3) = x_1^2 - x_2^2 - x_3^2 + 4x_1x_2 + 2x_1x_3 \ ,$$

subject to the simultaneous constraints:

$$x_1 + x_2 + x_3 = 0 \quad \text{and} \quad x_1 - x_2 - x_3 = 0 \ .$$

Answer: The bordered Hessian is given by

$$\mathbf{H}_B = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & -1 & 2 & -1 & 0 \\ 1 & -1 & 1 & 0 & -1 \end{pmatrix} ,$$

where $n = 3$, $m = 2$. We only need $LPM_5 = -8$ with $\mathrm{sgn}(LMP_5) = (-1)^n$, and hence Q is negative definite.

(b) Write down the Lagrangian, and hence find the two stationary points of the function

$$f(x,y,z) = \frac{1}{2}x^2 + xz + \frac{1}{2}y^2 + yz + \frac{5}{24}z^3,$$

subject to the constraint

$$x + y + z = 1.$$

Answer: The Lagrangian is:

$$L(x,y,z,\lambda) = \frac{1}{2}x^2 + xz + \frac{1}{2}y^2 + yz + \frac{5}{24}z^3 - \lambda\,(x+y+z-1)\ ,$$

lambda=~~3/2~~

with the first order conditions leading to the stationary points: $(-\frac{1}{2}, -\frac{1}{2}, 2)$ with $\lambda = \frac{3}{10}$, and $(\frac{3}{10}, \frac{3}{10}, \frac{2}{5})$ with $\lambda = \frac{7}{10}$.

(c) Find the 4×4 Bordered Hessian for the problem specified in part (b), and evaluate the two relevant leading principal minors. Hence show that one of the two stationary points is a local minimum.

Answer: The bordered Hessian is given by

$$\mathbf{H}_B = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & \frac{5}{4}z \end{pmatrix}\ ,$$

with $n=3$, $m=1$ where we need $LPM_4 = 3 - \frac{5}{2}z$, and $LPM_3 = -2$. If $z=2$ we have $LPM_4 = -2 < 0$, $LPM_3 = -2 < 0$, hence $\mathrm{sgn}(LPM_4) = (-1)^m$ and the $\mathbf{H}_B$ positive definite, hence the stat. point a local minimum. If $z = 2/5$ we have $LPM_4 = 2 > 0$, $LPM_3 = -2 < 0$, and $\mathrm{sgn}(LPM_4)$ can be identified with neither $(-1)^n$ nor $(-1)^m$, hence we have an indefinite point.

4. (a) A company's production depends on two input variables x and y according to a Cobb-Douglas production function given by

$$Q(x,y) = x^{1/2}y^{1/4}\ .$$

If the price per product is p, and the cost of production is $C(x,y) = ax + by$, where a and b are positive constants, write down the profit function $\Pi(x,y)$. Find the stationary point for the function Π and hence compute the critical values x^* and y^* and show that the profit at the stationary point is given by

$$\Pi^* = \frac{p^4}{64a^2b}\ .$$

Answer: The profit function is given by $\Pi(x,y) = px^{1/2}y^{1/4} - ax - by$. The stationary point is $(x^*, y^*) = (\frac{p^4}{32a^3b}, \frac{p^4}{64a^2b^2})$, and inserting these values we get $\Pi^* = \Pi(x^*, y^*)$ as given.

(b) Compute the Hessian matrix for the problem of part (b) and evaluate it at the stationary point (x^*, y^*). Use the leading principal minor test to show that the profit described by the model has a local maximum there.

Answer: The Hessian evaluated at the critical point is given by

$$\mathbf{H} = \frac{16a^2b}{p^4}\begin{pmatrix} -a^2 & ab \\ ab & -3b^2 \end{pmatrix}\ ,$$

and hence $LPM_2 = \det(\mathbf{H}) = 2(\frac{16a^2b}{p^4})^2a^2b^2 > 0$, $LPM_1 = -\frac{16a^2b}{p^4}a^2 < 0$, and hence $\mathbf{H}$ is negative definite, leading to a local maximum at the critical point.

(c) Suppose that the company's managing director decides to embark on a new strategy and wants to maximise the revenue $R(x, y)$ instead of the profit, while keeping the profit function $\Pi(x, y)$, found in part (a), above a threshold value c (where c is some strictly positive number $c > 0$). Taking into account that $x \geq 0$ and $y \geq 0$, show that the corresponding Kuhn-Tucker Lagrangian is given by

$$\bar{L} = px^{1/2}y^{1/4} - \lambda\left(ax + by + c - px^{1/2}y^{1/4}\right) \ .$$

Write down the three relevant equality conditions and the six relevant inequality conditions for this Lagrangian, and show that at the critical point neither x nor y can be zero. Hence show that at the solution candidates the constraint on the profit must be binding. (It is not required to actually compute the solution candidates explicitly).

Answer: The revenue is given by $R(x, y) = px^{1/2}y^{1/4}$ which is to be maximised subject to the constraint $\Pi(x, y) \geq c$, i.e., $c + ax + by - R(x, y) \leq 0$, as well as the positivity constraints $x \geq 0$, $y \geq 0$. This leads to the KT Lagrangian as given. The relevant KT conditions are:

$$x\bar{L}_x = y\bar{L}_y = \lambda\bar{L}_\lambda = 0 \ , \quad \bar{L}_x \leq 0 \ , \ \bar{L}_y \leq 0 \ , \ \bar{L}_\lambda \geq 0 \ , x \geq 0 \ , \ y \geq 0 \ , \ \lambda \geq 0 \ .$$

If either $x = 0$ or $y = 0$, the expression for $\Pi(x, y) = 0$ contradicting the requirement $\Pi(x, y) \geq c > 0$. Then KT relations:

$$x\bar{L}_x = \frac{1}{2}x^{1/2}y^{1/4}(1+\lambda) - \lambda ax = 0 \ , \quad y\bar{L}_y = \frac{1}{4}x^{1/2}y^{1/4}(1+\lambda) - \lambda by = 0 \ ,$$

lead to $\lambda \neq 0$ which in turn implies that the constraint on the profit is binding: $\Pi(x^*, y^*) = c$.

5. (a) Write down the Lagrangian for a general function $f(x, y, z)$ to be maximised subject to an inequality constraint $g(x, y, z) \leq b$ where b is a constant, and state the first order conditions, as well as the relevant inequalities and the complementary slackness condition.

Answer: The Lagrangian is

$$L(x, y, z, \lambda) = f(x, y, z) - \lambda\left(g(x, y, z) - b\right) \ .$$

The first order conditions are

$$L_x = 0 \ , \quad L_y = 0 \ , \quad L_z = 0 \ , \quad \lambda(g(x, y, z) - b) = 0 \ ,$$

where the latter is the complementary slackness condition. The inequalities are $\lambda \geq 0$ and $g(x, y, z \leq b$.

(b) Use the method of part (a) to find all seven solution candidates for maximising

$$f(x, y, z) = 9x^2 + y^2 + 4z^2$$

subject to

$$g(x, y, z) = x^2 + y^2 + z^2 \leq 1$$

by forming, according to part (a), the relevant Lagrangian and applying the first order conditions and the complementary slackness condition. For which solution candidates is the constraint binding?

Answer: The Lagrangian is given by

$$L(x,y,z,\lambda) = 9x^2 + y^2 + 4z^2 - \lambda(x^2 + y^2 + z^2 - 1) \ ,$$

yielding the first order conditions:

$$L_x = 2(9-\lambda)x = 0 \ , \quad L_y = 2(1-\lambda)y = 0 \ , \quad L_z = 2(4-\lambda)z = 0 \ ,$$

together with the complementary slackness condition $\lambda(x^2+y^2+z^2-1) = 0$. The inequalities are:

$$\lambda \geq 0 \ , \quad x^2 + y^2 + z^2 \leq 1 \ .$$

The solution candidates are the six points: $(\pm 1, 0, 0)$ with $\lambda = 9$, $(0, \pm 1, 0)$ with $\lambda = 1$, and $(0, 0, \pm 1)$ with $\lambda = 4$, and the point $(0,0,0)$ with $\lambda = 0$. In all cases, except for the latter point, the constraint is binding.

(c) For the solution candidates found in part (a) compute the Hessian (for those where the constraint is non-binding) and the bordered Hessian (for those where the constraint is binding) and use these to determine their character (local maximum/minimum or indefinite). Evaluate also the function value f at all the solution candidates, and hence determine which two points are the maximisers.

Answer: In case of $(0,0,0)$ the constraint is non-binding, hence we use the usual Hessian for the function f evaluated at that point :

$$\mathbf{H} = \begin{pmatrix} 18 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{pmatrix} \ ,$$

and this is clearly positive definite, hence we have a local minimum. In the other cases we need the bordered Hessian:

$$\mathbf{H}_B = \begin{pmatrix} 0 & 2x & 2y & 2z \\ 2x & 18-2\lambda & 0 & 0 \\ 2y & 0 & 2-2\lambda & 0 \\ 2z & 0 & 0 & 8-2\lambda \end{pmatrix} \ ,$$

where $n = 3$, $m = 1$. We need LPM_4 and LPM_3 for all the solution candidates:

$$LPM_4 = -4\left[x^2(2-2\lambda)(8-2\lambda) + y^2(18-2\lambda)(8-2\lambda) + z^2(18-2\lambda)(2-2\lambda)\right] \ ,$$

$$LPM_3 = -4\left[x^2(2-2\lambda) + y^2(18-2\lambda)\right] \ .$$

For $(\pm 1, 0, 0)$ $LPM_4 < 0$, and $LPM_3 > 0$, hence $\mathbf{H}_B$ is negative definite, and hence these points are local maxima. For $(0, \pm 1, 0,)$ $LPM_4 < 0$, and $LPM_3 < 0$, hence $\mathbf{H}_B$ is positive definite, and hence these points are local minima. For $(0, 0, \pm 1)$ $LPM_4 > 0$, and hence $\operatorname{sgn}(LPM_4)$ cannot be identified, and hence these points are indefinite points. The values of f are given by $f(\pm 1, 0, 0) = 9$, $f(0, \pm 1, 0) = 1$, $f(0, 0, \pm 1) = 4$ and $f(0,0,0) = 0$.

End.