MATH2640 Introduction to Optimisation

Example Sheet 4 Solutions to Assessed Questions

Thursday 28^{th} November 2019 homework

Constrained optimisation, equality constraints, Lagrange multipliers, NDCQ, bordered Hessians. Based on material in Lectures 13 to 17

Assessed Questions

A1.

(i) Find the maximum (x^*, y^*, z^*) of the Cobb-Douglas production function

$$Q(x,y,z) = x^{1/4}y^{1/4}z^{1/4}$$

subject to the budget constraint $h(x,y,z) = ax + by + cz - d = 0$, (where a, b, c, d are positive constants), in terms of these constants. Hence, find an expression for the maximum value Q^* of the budget in terms of a, b, c, d and the corresponding value λ^* of the Lagrange multiplier. Check also that the NDCQ is satisfied.

Answer: The Lagrangian is:

$$L(x,y,z) = Q(x,y,z) - \lambda h(x,y,z) = x^{1/4}y^{1/4}z^{1/4} - \lambda(ax + by + cz - d). \tag{1}$$

FOCs are:

$$L_x = \frac{1}{4}x^{-3/4}y^{1/4}z^{1/4} - \lambda a = 0 \Longrightarrow \frac{1}{4}Q = \lambda ax \tag{2}$$
$$L_y = \frac{1}{4}x^{1/4}y^{-3/4}z^{1/4} - \lambda b = 0 \Longrightarrow \frac{1}{4}Q = \lambda by \tag{3}$$
$$L_z = \frac{1}{4}x^{1/4}y^{1/4}z^{-3/4} - \lambda c = 0 \Longrightarrow \frac{1}{4}Q = \lambda cz \tag{4}$$
$$-L_\lambda = ax + by + cz - d = 0. \tag{5}$$

Substituting the first three relations into the last one yields/gives $\lambda = 3Q/(4d)$. We note from these first three FOCs that $ax = by = cz$. Hence by using this (i.e. $by = ax, cz = ax$) into the last FOC we obtain $3ax = d$ such that $x^* = \frac{d}{3a}, y^* = \frac{d}{3b}, z^* = \frac{d}{3c}$, which in turn gives

$$Q^* = (x^*)^{1/4}(y^*)^{1/4}(z^*)^{1/4} = (d/3)^{3/4}\frac{1}{(abc)^{1/4}};$$

and, also $\lambda^* = 3Q^*/(4d) = \frac{1}{4}\left(\frac{3}{(abcd)}\right)^{1/4}$. Hence, the stationary point becomes

$$(x^*, y^* z^*, \lambda^*) = \left(\frac{d}{3a}, \frac{d}{3b}, \frac{d}{3c}, \frac{1}{4}\left(\frac{3}{(abcd)}\right)^{1/4}\right).$$

The NCDQ condition is $\nabla h = (a,b,c)^T \neq (0,0,0)^T$, so the NCDQ is satisfied.

(ii) Minimise $x^2 + \frac{1}{2}y^2 + (\frac{z}{2})^2$ subject to the constraints given by the intersection of two planes $x - y + z = 1$ and $x + y + z = -1$. Check that the NDCQ is satisfied at the stationary point. What is the "distance" from the origin to that point under the alternative distance norm $\sqrt{x^2 + \frac{1}{2}y^2 + (\frac{z}{2})^2}$? Make a sketch to illustrate the geometry of the situation (optional).

Answer: The relevant Lagrangian reads/is:

$$L(x, y, z) = x^2 + \tfrac{1}{2}y^2 + (\tfrac{z}{2})^2 - \lambda_1(x - y + z - 1) - \lambda_2(x + y + z + 1). \tag{6}$$

Its FOCs are

$$\begin{aligned}
L_x &= 2x - \lambda_1 - \lambda_2 = 0 \Longrightarrow 2x = \lambda_1 + \lambda_2 && (7)\\
L_y &= y + \lambda_1 - \lambda_2 = 0 \Longrightarrow y = -\lambda_1 + \lambda_2 && (8)\\
L_z &= \frac{1}{2}z - \lambda_1 - \lambda_2 = 0 \Longrightarrow z = 2(\lambda_1 + \lambda_2) && (9)\\
-L_{\lambda_1} &= x - y + z - 1 = 0 \Longrightarrow 2x - 2y + 2z = 2 && (10)\\
-L_{\lambda_2} &= x + y + z + 1 = 0 \Longrightarrow 2x + 2y + 2z = -2. && (11)
\end{aligned}$$

Substitution of the first three FOCs into the last two FOCs gives the system:

$$\begin{aligned}
\lambda_1 + \lambda_2 + 2\lambda_1 - 2\lambda_2 + 4\lambda_1 + 4\lambda_2 = 7\lambda_1 + 3\lambda_2 &= 2 && (12)\\
\lambda_1 + \lambda_2 - 2\lambda_1 + 2\lambda_2 + 4\lambda_1 + 4\lambda_2 = 3\lambda_1 + 7\lambda_2 &= -2; && (13)
\end{aligned}$$

Multiplication of each equation by 3 and -7 respectively 7 and -3 gives $\lambda_1 = 1/2, \lambda_2 = -1/2$ such that we obtain the stationary points, i.e. by using the first there FOCs again to obtain:

$$(x^*, y^*, z^*, \lambda_1^*, \lambda_2^*) = (0, -1, 0, 1/2, -1/2).$$

The distance of this stationary point from the origin *under this alternative norm* is:

$$||\sqrt{0^2 + 1/2 + 0^2}|| = 1/\sqrt(2)!$$

A2. Use Bordered Hessians to determine the sign properties (definiteness) of the following constrained quadratic form:

$$Q(x_1, x_2, x_3) = x_1^2 + 2x_2^2 - 2x_3^2 + 4x_1x_2 - 2x_2x_3$$

subject to the constraints $2x_1 + x_2 + x_3 = 0$ and $x_1 - x_2 - x_3 = 0$. Verify the result by eliminating two of the variables using the constraints, and determining the sign property of the reduced quadratic form.

Answer: The associated bordered Hessian is:

$$H_B = \begin{pmatrix} 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 2 & 1 & 1 & 2 & 0 \\ 1 & -1 & 2 & 2 & -1 \\ 1 & -1 & 0 & -1 & -2 \end{pmatrix} \tag{14}$$

with $m = 2, n = 3$; hence $2m + 1 = 5$ so we only need to check LPM_5.

$$LPM_5 = \det(H_B) = \begin{vmatrix} 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 2 & 1 & 1 & 2 & 0 \\ 1 & -1 & 2 & 2 & -1 \\ 1 & -1 & 0 & -1 & -2 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 2 & 1 & 1 & 2 & 0 \\ 1 & -1 & 2 & 2 & -1 \\ 1 & -1 & 0 & -1 & -2 \end{vmatrix}$$
$$=3 \begin{vmatrix} 0 & 0 & -1 & -1 \\ 2 & 1 & 2 & 0 \\ 1 & -1 & 2 & -1 \\ 1 & -1 & -1 & -2 \end{vmatrix} = 3 \begin{vmatrix} 0 & 0 & -1 & -1 \\ 2 & 1 & 2 & 0 \\ 1 & -1 & 2 & -1 \\ 1 & -1 & -1 & -2 \end{vmatrix}$$
$$= \begin{vmatrix} 0 & 0 & -1 & -1 \\ 3 & 1 & 2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & -2 \end{vmatrix} = -9 \begin{vmatrix} 0 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & -2 \end{vmatrix} = -9(-2) = 18 > 0. \tag{15}$$

Hence, so $sign(LMP_5) = (-1)^m = 1 > 0$; case PD and the stationary point is a minimum. Given that $x_1 = 0, x_3 = -x_2$ is the solution to the two constrained equations, the quadratic form reduces to $Q(x_1, x_2, x_3)_{(h_1=0,h_2=0)} = \tilde{Q}(x_2) = 2x_2^2 - 2x_2^2 + 2x_2^2 = 2x_2^2 > 0$, which is definitely PD as function of x_2. Hence, our conclusions are consistent.

A3.

i) Write down the Lagrangian, and hence find the two stationary points of the problem

$$f(x, y, z) = -x^2 + 2y^2 + \tfrac{4}{3}z^3 + 2yz, \quad \text{subject to} \quad h(x, y, z) = x + y - z - 1 = 0.$$

Answer: The corresponding Lagrangian is:

$$L(x, y, z) = -\,x^2 + 2y^2 + \tfrac{4}{3}z^3 + 2yz - \lambda(x + y - z - 1). \tag{16}$$

FOCs:

$$L_x = -\,2x - \lambda = 0 \Longrightarrow \text{(a) } \lambda = -2x \tag{17}$$
$$L_y = 4y + 2z - \lambda = 0 \tag{18}$$
$$L_z = 4z^2 + 2y + \lambda = 0 \tag{19}$$
$$-L_\lambda = x + y - z - 1 = 0. \tag{20}$$

Put (a) into second FOC to obtain $4y + 2z + 2x = 0$ or (b) $x + 2y + z = 0$.
Put (a) into third FOC to obtain (c) $4z^2 + 2y - 2x = 0$.
Solve (b) and the fourth FOC:

$$\text{(b)}: \; x + 2y + z = 0 \tag{21}$$
$$\text{(d)}: \; x + y - z - 1 = 0, \tag{22}$$

giving (multiply (b) by one minus (d) to eliminate x) $y = -1 - 2z$ and (multiply (b) by one minus two times (d) to eliminate y) $x = 3z+2$. Using these expressions $y = -1-2z, x = 3z+2$ into (c) yields/gives:

$$4z^2 - 2 - 4z - 6z - 4 = 2(2z^2 - 5z - 3) = 0 \Longrightarrow (2z + 1)(z - 3) = 0, \tag{23}$$

such that we find the pair of critical points, after using $\lambda = -2x$ again, $(x^*, y^*, z^*, \lambda^*) = (1/2, 0, -1/2, -1)$ and $(x^*, y^*, z^*, \lambda^*) = (11, -7, 3, -22)$.

ii) Find the Bordered Hessian for this problem, and evaluate the required leading principal minors for the (two) solutions.

Answer: The bordered Hessian for this Lagrangian follows from the FOCs by further differentiations:

$$H_B = \begin{pmatrix} 0 & 1 & 1 & -1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & 4 & 2 \\ -1 & 0 & 2 & 8z \end{pmatrix} \tag{24}$$

with $m = 1, n = 3$ such that $2m+1 = 3$ and we need to investigate $LPM_3, LPM_4 = \det(H_B)$.

$$LPM_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & 4 \end{vmatrix} = 2 - 4 = -2 < 0 \tag{25}$$

$$LPM_4 = \det(H_B) = \begin{vmatrix} 0 & 1 & 1 & -1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & 4 & 2 \\ -1 & 0 & 2 & 8z \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 & -1 \\ 1 & -2 & 0 & 0 \\ 0 & 2 & 4 & 2 \\ 0 & -2 & 2 & 8z \end{vmatrix} \tag{26}$$

$$= -1 \begin{vmatrix} 1 & 1 & -1 \\ 2 & 4 & 2 \\ -2 & 2 & 8z \end{vmatrix} = -1(32z - 4 - 4 - 8 - 4 - 16z) = 20 - 16z. \tag{27}$$

Hence for $z = -1/2$ we have $LPM_4 = 28 > 0, LPM_3 < 0$, different signs so not PD; also $sign(\det(LPM_4)) > 0 \neq (-1)^n = -1$ so also not ND; hence ID.
For $z = 3$ we have $LPM_3 < 0, LPM_4 = -28 < 0$, so now $sign(LPM_4) = (-1)^m = -1$ and all sucessive LPM's (namely LPM_3) have the same sign (negative); hence PD, and the stationary point is therefore a minimum.