MATH2640 Introduction to Optimisation

Example Sheet 4

Please hand the assessed questions by by Thursday 28$^{\text{th}}$ November 2019, 5pm

Constrained optimisation, equality constraints, Lagrange multipliers, NDCQ, bordered Hessians. Based on material in Lectures 13 to 17

Assessed Questions

A1.

(i) Find the maximum (x^*, y^*, z^*) of the Cobb-Douglas production function

$$Q(x, y, z) = x^{1/4}y^{1/4}z^{1/4}$$

subject to the budget constraint $h(x, y, z) = ax + by + cz - d = 0$, (where a, b, c, d are positive constants), in terms of these constants. Hence, find an expression for the maximum value Q^* of the budget in terms of a, b, c, d and the corresponding value λ^* of the Lagrange multiplier. Check also that the NDCQ is satisfied.

(ii) Minimise $x^2 + \frac{1}{2}y^2 + (\frac{z}{2})^2$ subject to the constraints given by the intersection of two planes $x - y + z = 1$ and $x + y + z = -1$. Check that the NDCQ is satisfied at the stationary point. What is the "distance" from the origin to that point under the alternative distance norm $\sqrt{x^2 + \frac{1}{2}y^2 + (\frac{z}{2})^2}$? Make a sketch to illustrate the geometry of the situation (optional).

A2. Use Bordered Hessians to determine the sign properties (definiteness) of the following constrained quadratic form:

$$Q(x_1, x_2, x_3) = x_1^2 + 2x_2^2 - 2x_3^2 + 4x_1x_2 - 2x_2x_3$$

subject to the constraints $2x_1 + x_2 + x_3 = 0$ and $x_1 - x_2 - x_3 = 0$. Verify the result by eliminating two of the variables using the constraints, and determining the sign property of the reduced quadratic form.

A3.

i) Write down the Lagrangian, and hence find the two stationary points of the problem

$$f(x, y, z) = -x^2 + 2y^2 + \tfrac{4}{3}z^3 + 2yz, \quad \text{subject to} \quad h(x, y, z) = x + y - z - 1 = 0.$$

ii) Find the Bordered Hessian for this problem, and evaluate the required leading principal minors for the (two) solutions.

P.T.O.

Further Questions for Workshop Practice

B1.

a) Use the Lagrangian approach to find the maximum values of $f(x,y) = x + y^2$ subject to the constraint $h(x,y) = x^2 + y^2 = 1$.

b) Find the points on the ellipse in the xy-plane, given by $x^2 + xy + y^2 = 9$, closest to and farthest away from the origin, and hence find the maximum and minimum distances of the ellipse to the origin. Make a sketch of the situation in the xy-plane. [Hint: use $x^2 + y^2$ as the objective function.]

c) Use the Lagrangian approach to find the distance of the plane $4x + 2y + z = 5$ to the origin, and compare it to the result known from geometric considerations. [Hint: use $x^2 + y^2 + z^2$ as objective function.]

B2.

Use the method of Lagrange multipliers to solve the following constrained optimisation problems. In each case, check that the *non-degenerate constraint qualification* (NDCQ) is satisfied at the stationary points.

a) Find the maximum and minimum values of $f(x,y,z) = x^2 + y + z$ subject to the constraints $x^2 + y^2 + z^2 = 50$ and $y + z = 6$.

b) Maximise $f(x,y,z) = xz + yz$ subject to $y^2 + z^2 = 1$ and $xz = 3$.

B3. Use Bordered Hessians to determine the sign properties (definiteness) of the following constrained quadratic forms:

a) $Q(x,y) = x^2 + 2xy - y^2$ subject to $x - y = 0$,

b) $Q(x,y,z) = -6y^2 + 3z^2 + 8xy + 2yz - 2xz$ subject to $x + y - z = 0$,

c) $Q(x_1,x_2,x_3) = x_1^2 - x_2^2 - x_3^2 + 4x_1x_2 + 2x_1x_3$ subject to $x_1 + x_2 + x_3 = 0$ and $x_1 - x_2 - x_3 = 0$.

In the first two cases verify, by elimination of one of the variables, that the thus obtained reduced quadratic form has the sign behaviour as predicted by the bordered Hessian approach.

B4.

a) Write down the Lagrangian, and hence find all the stationary points of the problem of maximising the function

$$f(x,y,z) = 2x + 4y + 3z^2 \ , \quad \text{subject to} \quad h(x,y,z) = x^2 + y^2 + z^2 = 1 \ .$$

b) Find the Bordered Hessian for this problem, and evaluate the relevant leading principal minors for both solutions. Hence classify the stationary points found in part (a).