MATH2640 Introduction to Optimisation

Example Sheet 1 Solutions homework

Partial differentiation, Gradient & Directional Derivative, Implicit functions, Differentials.
Based on material in Lectures 1 to 5

Solutions Assessed Questions

A 1.

(i) Find the critical points of the function $f(x) = 2x + x^3 - \frac{5}{2}x^2$ and characterize them. Sketch a graph of the function. Find also the absolute minimum in the domain $0 \leq x \leq 1$. **2 points**

Answer (i):
Critical points of $f(x) = 2x + x^3 - \frac{5}{2}x^2$:
$f'(x) = 2 + 3x^2 - 5x = (3x - 2)(x - 1) = 0$
so critical points are $x = 2/3$ and $x = 1$.
Characterise these critical points so consider $f''(x) = 6x - 5$;
hence $f''(2/3) = 6(2/3) - 5 = 4 - 5 = -1 < 0$ so $x = 2/3$ is a local maximum;
and $f''(1) = 6 - 5 = 1 > 0$ so $x = 1$ is a local minimum.
The function values at $x = 2/3$ and $x = 1$ are:
$f(2/3) = 2 \times \frac{2}{3} + (2/3)^3 - (5/2)(2/3)^2 = 4/3 + 8/27 - 10/9 = 36/27 + 8/27 - 30/9 = 14/27$ and $f(1) = 2 + 1 - 5/2 = 1/2$;
hence $(x, f(x)) = (2/3, 14/27)$ and $(1, 1/2)$ are critical points.

A sketch is given in Fig. 1.
Note that $f(0) = 0$ and $f(1) = 1/2$ such that on $0 \leq x \leq 1$ as domain:
$(x, f(x)) = (2/3, 14/27)$ is a global maximum and $(x, f(x)) = (0, 0)$ a global minimum.

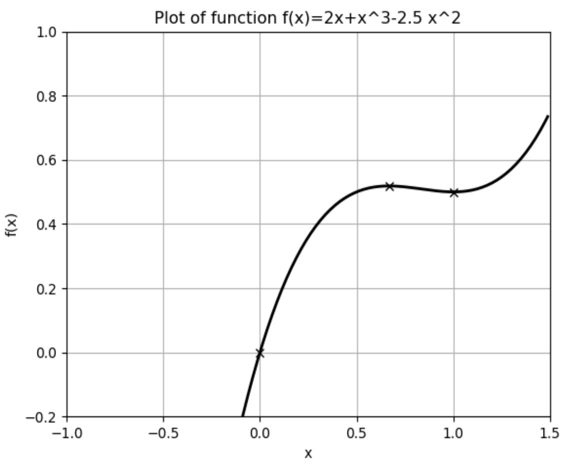

Figure 1: x versus $f(x)$.

(ii) Make a contour plot of the function $z(x,y) = x + y^2$ in the xy-plane, by drawing a collection of curves $z(x,y) = c$ for different (positive and negative) values of the constant c. **3 points**
Note extra not for marking: Try and sketch a graph of the function in the space of xyz coordinates.

Answer (ii):
Contours plot of $z(x,y) = x + y^2$. For any value of $z(x,y) = c \Longrightarrow x + y^2 = c$, which is the formula for a parabola $x = c - y^2$ with x a function of y; note that $x \leq c$ with $x = c$ for $y = 0$.
A sketch is given in Fig. 2.

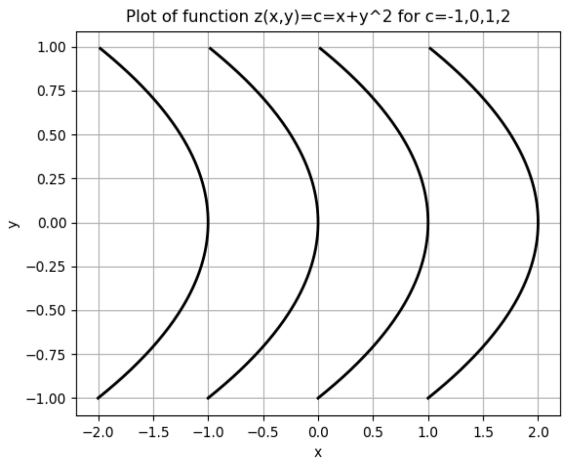

Figure 2: $z(x,y) = x + y^2 = c$ for $c = -1, 0, 1, 2$ (left to right).

From this we can deduce the sketch of the function in x, y, z–coordinates. Not that in the plane $y = 0$, we find that $z = x$.
Note: extra, not for marking: The contour curves for different values of $z = c$ sweep out a parabolic surface, which is the graph of the function $z(x,y)$, see Fig. 3.

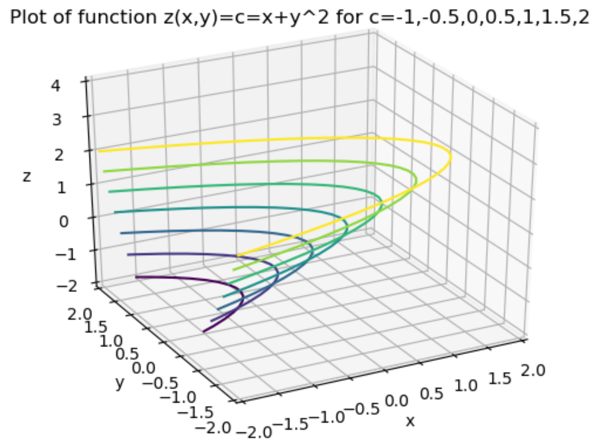

Figure 3: 3D-plot of $z(x,y) = x + y^2 = c$ for $c = -1, -0.5, 0, 0.5, 1, 1.5, 2$ (left to right).

A 2.

(i) Find f_x, f_y, f_{xy}, f_{xx} and f_{yy} for the functions:

$$f(x,y) = \cosh(x+y) + x\ln(y) \ .$$

Verify that $f_{xy} = f_{yx}$. **2 points**

Answer (i):
$f(x,y) = \cosh(x+y) + x\ln(y)$, such that $f_x = \sinh(x+y) + \ln(y)$ and
$f_y = \sinh(x+y) + x/y$
$f_{xx} = \cosh(x+y)$
$f_{yy} = \cosh(x+y) - x/y^2$
$f_{xy} = \cosh(x+y) + 1/y$
$f_{yx} = \cosh(x+y) + 1/y$, whence $f_{xy} = f_{yx}$ as expected.

(ii) Find the gradient of the function $g(x,y,z) = (2\sqrt{y} - x)z^2$ at the point $(3,4,1)$ and give the coordinates of the unit vector $\mathbf{u}$ in the direction of the vector $(1,2,3)$. Hence, calculate the directional derivative $D_{\mathbf{u}}g(3,4,1)$. **3 points**
Note extra not for marking: Also give the direction of maximum increase of the function g at that point.
Answer (ii):
Gradient of $g(x,y,z) = (2\sqrt{y} - x)z^2$.
$g_x = -z^2, g_y = z^2/\sqrt{y}, g_z = 2z(2\sqrt{y} - x)$. Evaluate $\nabla g = (g_x, g_y, g_z)$ at $(3,4,1)$ to obtain: $\nabla g|_{(3,4,1)} = (-1, 1/2, 2)$.

The unit vector in the direction of vector $(1,2,3)$ is $\mathbf{u} = (1,2,3)/(||(1,2,3)||) = (1,2,3)/\sqrt{14}$, since $||(1,2,3)|| = \sqrt{14}$.

The directional derivative therefore becomes:

$$D_{\mathbf{u}}g|_{(3,4,1)} = \frac{1}{\sqrt{14}}(1,2,3)\cdot(-1,1/2,2) = \frac{-1+1+6}{\sqrt{14}} = \frac{6}{\sqrt{14}}.$$

Note extra not for marking: The direction of maximum increase of the function g is

$$\nabla g/|\nabla g| = \frac{(-1,1/2,2)}{\sqrt{21}/2}.$$

A 3. The function $z(x,y)$ is defined implicitly by the relation $y^4 z - xz^2 + x^2y^3 = 3$. Find z_x, z_y and z_{yy} in terms of x, y and z. Find two possible values for z at $x = 2$, $y = 1$, and show that one value is an integer. For that value, compute the corresponding numerical values of z_x, z_y and z_{yy}. **5 points**

Answer:
Function $z(x,y)$ is implicitly defined by $zy^4 - xz^2 + x^2y^3 = 3$.

Implicit differentiation with respect to x yields:

$$\begin{aligned} y^4 z_x - 2xzz_x - z^2 + 2xy^3 &= 0 \\ \Longrightarrow z_x(y^4 - 2xz) &= z^2 - 2xy^3 \\ z_x &= \frac{z^2 - 2xy^3}{y^4 - 2xz}. \end{aligned}$$

Implicit differentiation with respect to y yields:

$$\begin{aligned} y^4 z_y - 2xzz_y + 4y^3z + 3x^2y^2 =& 0 \\ z_y(y^4 - 2xz) =& -4y^3z - 3x^2y^2 \\ z_y =& -\frac{(4y^3z + 3x^2y^2)}{y^4 - 2xz}. \end{aligned} \tag{1}$$

For z_{yy} start from (1):

$$\begin{aligned} (y^4 - 2xz)z_{yy} + 8y^3z_y - 2xz_y^2 + 12y^2z + 6x^2y = 0 \\ z_{yy} =& \frac{-8y^3z_y + 2xz_y^2 - 12y^2z - 6x^2y}{y^4 - 2xz} \\ =& \frac{8y^3(4y^3z + 3x^2y^2)}{(y^4 - 2xz)^2} + \frac{2x(4y^3z + 3x^2y^2)^2}{(y^4 - 2xz)^3} - 6\frac{2y^2z + x^2y}{y^4 - 2xz}. \end{aligned} \tag{2}$$

At $x = 2, y = 1$, $z - 2z^2 + 4 = 3 \Longrightarrow (2z + 1)(z - 1) = 0$, so $z = -1/2$ or $z = 1$.
When $x = 2, y = 1, z = 1$, we find that

$$\begin{aligned} z_x =& \frac{z^2 - 2xy^3}{y^4 - 2xz} = \frac{1 - 4}{1 - 4} = 1. \\ z_y =& -\frac{4y^3z + 3x^2y^2}{y^4 - 2xz} = -\frac{4 + 12}{1 - 4} = 16/3 \\ z_{yy} =& \frac{-8(16/3) + 4(16/3)^2 - 12 - 24}{1 - 4} = -316/27, \end{aligned} \tag{3}$$

in the latter case, e.g., using (2) and the results $z_x = 1$ and $z_y = 16/3$ may be easiest.

A 4.

(i) If $f(x, y) = \exp(xy^2)$ and $x^2 + y^3 = 2xy$, find expressions for the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ and the total derivatives df/dx and df/dy in terms of x and y. (In the latter you don't need to simplify your answer.) **2 points**

Answer (i):
Consider $f(x, y) = \exp(xy^2)$ on the curve $x^2 + y^3 = 2xy$.
Now $f_x = y^2 \exp(xy^2)$ and $f_y = 2xy \exp(xy^2)$.
Total derivatives are: $\frac{df}{dx} = f_x + f_y \frac{dy}{dx}$ and $\frac{df}{dy} = f_y + f_x \frac{dx}{dy}$. For $\frac{dy}{dx}$ and $\frac{dx}{dy}$ use implicit differentiation of the curve:

$$\begin{aligned} 2xdx + 3y^2dy =& 2ydx + 2xdy \\ \Longrightarrow 2(x - y)dx =& (2x - 3y^2)dy \\ \Longrightarrow \frac{dy}{dx} =& \frac{2(x - y)}{2x - 3y^2}, \frac{dx}{dy} = \frac{2x - 3y^2}{2(x - y)}. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{df}{dx} =& y^2 \exp(xy^2) + 2xy \exp(xy^2) \frac{2(x - y)}{2x - 3y^2} \\ \frac{df}{dy} =& 2xy \exp(xy^2) + y^2 \exp(xy^2) \frac{2x - 3y^2}{2(x - y)}. \end{aligned}$$

(ii) Let variables x, y and z be linked by the two relationships

$$f(x,y,z) = -x^2 + y^2 + z^2 - 1 = 0,$$

$$g(x,y,z) = 3x^3 + y^3 + 2z^3 - 6 = 0.$$

Derive conditions on the differentials dx, dy, and dz if the functions f and g are kept at these values. If $y = 1$, find the two points with values of y and z satisfying $f = g = 0$. For the point containing only integer values find the numerical values of dx/dz and dy/dz at that point. **3 points**

Answer (ii):
Simultaneous conditions are given by the functions f and g as

$$\begin{aligned} f(x,y,z) &= -x^2 + y^2 + z^2 - 1 = 0, \\ g(x,y,z) &= 3x^3 + y^3 + 2z^3 - 6 = 0 \end{aligned}$$

such that

$$\begin{aligned} df &= -2x dx + 2y dy + 2z dz = 0, \\ dg &= 9x^2 dx + 3y^2 dy + 6z^2 dz = 0 \end{aligned}$$

At $y = 1$ we find $f = 1 + z^2 - x^2 - 1 = 0$ such that $z^2 = x^2$ giving $z = \pm x$.
Also for $y = 1$ we find $g = 2z^3 + 3x^3 - 5 = 0$.
For $z = x$ this gives $5x^3 = 5$ such that $x = 1, y = 1, z = 1$.
For $z = -x$ this gives $x^3 = 5$ such that $x = 5^{1/3}, y = 1, z = -5^{1/3}$.
For the point with the integer values, i.e., $(1,1,1)$,

$$\begin{aligned} -2dx + 2dy &= -2dz, \\ 3dx + dy &= -2dz. \end{aligned}$$

Giving

$$\begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} dz \tag{4}$$

Hence,

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 & 1 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} dz = -\frac{1}{4} \begin{pmatrix} 1 \\ 5 \end{pmatrix} dz,$$

such that $dx/dz = -1/4, dy/dz = -5/4$.