MATH2640 Introduction to Optimisation

Example Sheet 5

Please hand the assessed questions by by Thursday 12^{th} December 2019, 5pm

Inequality constraints, Kuhn-Tucker method, applications to economics & finance.
Based on material in Lectures 17 to 22

Assessed Questions

A1.

i) Consider maximising the function $f(x,y) = x + y$ subject to the simultaneous constraints

$$x \le \tfrac{3}{2}y + 1\ , \quad 2x + y \le 10\ , \quad \text{and} \quad x \ge \tfrac{1}{6}y + 1\ .$$

Write down the relevant Lagrangian and the corresponding first order equations, including the complementary slackness conditions, and the relevant inequalities. Analyse these to find the unique point that gives the maximiser, and the corresponding values of the Lagrange multipliers. (Which constraints are binding?) What is the maximum value of f under the constraints?

ii) Draw a graph of the region defined by the inequalities in the xy–plane. Solve the problem of part (i) graphically by drawing the level curves (i.e., level lines) of the function f and finding the maximiser in this way.

A2.

i) Consider the problem of maximising the function

$$f(x,y) = 2xy - x - 2y,$$

subject to the constraints

$$x^2 + 4y^2 \le 18, \quad x \ge 0, \quad y \ge 0.$$

(a) Solving this system of equations find the candidates (x, y, λ) for the maximiser. By evaluating f for these candidates find its maximum value. Solve the problem using Lagrange multipliers for all inequality constraints. Analyse the bordered Hessian for the appropriate cases.

(b) Write down the Kuhn-Tucker (KT) Lagrangian for this problem, and the corresponding KT equalities and inequalities. Solving this system of equations find the candidates (x, y, λ) for the maximiser. By evaluating f for these candidates find its maximum value.

ii) Consider the problem of maximising the function $f(x,y,z) = x + y + z$ subject to the constraints

$$3x + 2y + 4z \le 9\ , \quad x \ge 0, \quad y \ge 0, \quad z \ge 0\ .$$

Write down the KT Lagrangian for this problem, and the corresponding KT equalities and inequalities. Solve this problem and show that there is a unique maximiser. Sketch a graph of the situation indicating the region in the space of xyz-variables and the level surfaces (i.e., level planes) of the function f (optional).

P.T.O.

Further Questions for Workshop Practice

B1.

a) Find the solution candidates for maximising $f(x,y) = x^2 + y^2$ subject to $\dfrac{x^2}{9} + \dfrac{y^2}{25} \leq 1$ by forming a suitable Lagrangian and applying the first order conditions.

b) Consider the relevant Hessians for the solution candidates, taking into account whether the constraint is binding or no for those points. Evaluate f at all the solution candidates, and hence determine which are maximisers.

c) Sketch the level sets of f and the region defined by the constraint and so confirm your solution.

B2.

a) Write down the classical Lagrangian for the maximisation of $f(x,y) = 2y^2 - x$ subject to $x^2 + y^2 \leq 1$, $x \geq 0$ and $y \geq 0$. Write down the five equalities and six inequalities that form the first order conditions.

b) Find the solution candidates, and state which constraints are binding for each solution, and which solution gives the maximum f. Sketch the constraint region and the level sets of f and hence confirm your findings.

c) Write down the Kuhn-Tucker Lagrangian for this problem, and state the three equalities and six inequalities for the first order conditions. Solve these and show they lead to the same solutions as found in part **a)**.

B3. Use the Kuhn-Tucker method to maximise $f(x,y,z) = xyz + z$ subject to $x^2 + y^2 + z \leq 6$, $x \geq 0$, $y \geq 0$, $z \geq 0$. Show that there are two solution candidates for this problem and determine which is the maximiser. Which constraints are binding?

B4. A firm's revenue R depends on the rate of production y and the advertising expenditure a, according to $R = 3y^{1/6}a^{1/6}$. The cost of production $C = y$, so the profit is

$$\Pi = R - C - a = 3y^{1/6}a^{1/6} - y - a.$$

a) The firm wants to maximise revenue while maintaining the profit $\Pi \geq 1$. The constraints $y \geq 0$ and $a \geq 0$ are applied. Explain why the Kuhn-Tucker Lagrangian is

$$\bar{L} = 3y^{1/6}a^{1/6} - \lambda(1 + y + a - 3y^{1/6}a^{1/6})$$

and show from the profit constraint that neither a nor y can be zero. Establish that at maximum R,

$$1 - 2\lambda y^{5/6}a^{-1/6} + \lambda = 0,$$
$$1 - 2\lambda y^{-1/6}a^{5/6} + \lambda = 0,$$
$$1 + y + a - 3y^{1/6}a^{1/6} = 0.$$

Deduce that $y = a$ and $2y - 3y^{1/3} + 1 = 0$. Show that $y = 1$ and $y^{1/3} = (\sqrt{3} - 1)/2$ are the only positive roots of this equation, and find the corresponding values of λ. Why is $y^{1/3} = (\sqrt{3} - 1)/2$ not a valid solution? Deduce that the maximum revenue is $R = 3$ at $y = a = 1$, and that the profit there is the minimum acceptable.

b) A new managing director decides to change strategy from maximising revenue to maximising profit. Maximise Π without constraining R, and hence show that the profit rises to $\Pi = \sqrt{2}$ while the revenue falls to $R = 3/\sqrt{2}$.