

### 3. Quadratic forms and Eigenvalues, Economic applications (production functions): summary

#### 3(A) Quadratic forms and eigenvalues

- (i) Eigenvalues are the solutions,  $\lambda$ , of the matrix equation

$$A\mathbf{v} = \lambda\mathbf{v}, \quad \mathbf{v} \neq \mathbf{0}. \quad (3.1)$$

In general, an  $n \times n$  matrix has  $n$  eigenvalues, but they may be complex valued, and some eigenvalues may coincide. The non-zero vectors  $\mathbf{v}$  satisfying this equation are called the eigenvectors. For each eigenvalue  $\lambda$  there is, in fact, an infinity of corresponding eigenvectors  $\mathbf{v}$ , since any non-zero multiple of an eigenvector is again an eigenvector. Therefore, we sometimes want to restrict ourselves to unit (or normalised) eigenvectors: those with length  $\|\mathbf{v}\| = 1$ .

- (ii)  $2 \times 2$  case. The quadratic form associated with the symmetric matrix

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad \text{is} \quad Q(x, y) = (x, y) \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = ax^2 + 2bxy + cy^2.$$

To find the eigenvalues, we write equation (3.1) as  $(A - \lambda I)\mathbf{x} = 0$ , where  $I$  is the unit (identity) matrix. Then for this to have nontrivial solutions,

$$\det(A - \lambda I) = \begin{vmatrix} a - \lambda & b \\ b & c - \lambda \end{vmatrix} = 0, \quad \text{so} \quad \lambda^2 - (a + c)\lambda + ac - b^2 = 0$$

is the quadratic equation whose solutions are the two eigenvalues of the matrix  $A$ . These eigenvalues are always real for a symmetric matrix.

- (iii) **Normalising eigenvectors**

To find the eigenvector corresponding to a particular eigenvalue, solve  $(A - \lambda I)\mathbf{x} = 0$  for that value of  $\lambda$ . You can only solve these equations up to an arbitrary constant  $k$ . Example:

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

has eigenvalues  $\lambda = 2$  and  $\lambda = 4$ . To find eigenvector corresponding to  $\lambda = 2$ , solve

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_1 + x_2 &= 0 \end{aligned} \quad \text{giving} \quad (k, -k)$$

as the eigenvector. To normalise this, choose  $k$  so that  $(k, -k)$  has unit length,  $2k^2 = 1$ , so the normalised eigenvector is  $(1/\sqrt{2}, -1/\sqrt{2})$ . The normalised eigenvector corresponding to  $\lambda = 4$  is  $(1/\sqrt{2}, 1/\sqrt{2})$ .

- (iv) The same methods work for finding eigenvalues of an  $n \times n$  matrix: the characteristic equation

$$\det(A - \lambda I) = (-\lambda)^n + \text{tr}(A)(-\lambda)^{n-1} + \dots + \det(A) = 0,$$

(where  $\text{tr}(A)$  denotes the trace of the matrix, i.e., the sum of the diagonal entries), is now an  $n^{\text{th}}$  order polynomial equation which, in principle, has  $n$  (possibly complex valued, possibly coinciding) roots, which are the eigenvalues  $\lambda_1, \dots, \lambda_n$ . Having found the eigenvalues, the next step is to find the (unit) eigenvector  $\mathbf{v}_i$  corresponding to each eigenvalue  $\lambda_i$ , by solving the homogeneous linear system  $(A - \lambda_i I)\mathbf{v}_i = 0$ . The latter can be done for instance by elementary row manipulations (see 1st year Linear Algebra module) and will always have a nontrivial solution because of the condition on the  $\lambda_i$ .

Quadratic forms  $Q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$ , where  $\mathbf{x} = (x_1, \dots, x_n)$ , are always associated with *symmetric* matrices  $A = (a_{ij})_{i,j=1,\dots,n}$ , for which  $A = A^T$  (where the symbol  $T$  denotes matrix transposition) which means  $a_{ij} = a_{ji}$ . Such matrices have the following remarkable properties:

- all eigenvalues of a (real-valued) symmetric matrix  $A$  are real (i.e., are real numbers);
- eigenvectors  $\mathbf{v}_i$  of a symmetric matrix for distinct eigenvalues  $\lambda_i$  and  $\lambda_j$  are orthogonal to each other, i.e.,  $\mathbf{v}_i \cdot \mathbf{v}_j = 0$  for  $\lambda_i \neq \lambda_j$ ;
- a symmetric matrix  $A$  can be “diagonalised” by an orthogonal matrix  $O$ , i.e., written in the form  $A = O\Lambda O^T$  where  $\Lambda$  is the diagonal matrix of eigenvalues.

In the latter, an *orthogonal* matrix is a matrix possessing the property that  $OO^T = O^T O = I$ , and hence it is invertible with  $O^{-1} = O^T$ .  $O$  can be constructed by creating the matrix with as columns the *normalised* eigenvectors:  $O = (\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n)$ .

(v) **Normal forms**

When the normalised eigenvectors  $\mathbf{v}_i$ ,  $i = 1, \dots, n$ , corresponding to the  $n$  eigenvalues  $\lambda_1, \dots, \lambda_n$ , are found, say with components  $\mathbf{v}_i = (v_1^{(i)}, v_2^{(i)}, \dots, v_n^{(i)})$ , then the quadratic form  $Q(\mathbf{x})$  can be re-written, in terms of some new variables  $\tilde{\mathbf{x}} = (\tilde{x}_1, \dots, \tilde{x}_n)$ , as a sum of squares. In fact, introducing the vector of the new variables as  $\tilde{\mathbf{x}} = O^T \mathbf{x}$ , we have from the diagonalisation formula (given above):

$$Q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x} = \mathbf{x}^T (O \Lambda O^T) \mathbf{x} = (\mathbf{x}^T O) \Lambda (O^T \mathbf{x}) = \tilde{\mathbf{x}}^T \Lambda \tilde{\mathbf{x}},$$

using that  $\mathbf{x}^T O = (O^T \mathbf{x})^T$ . Noting that  $\mathbf{x}^T O = \mathbf{x}^T (\mathbf{v}_1, \dots, \mathbf{v}_n) = (\mathbf{x} \cdot \mathbf{v}_1, \dots, \mathbf{x} \cdot \mathbf{v}_n)$  we get the new variables in terms dot products of the form  $\tilde{x}_i = \mathbf{x} \cdot \mathbf{v}_i$ ,  $i = 1, \dots, n$ . Thus, the quadratic form can be rewritten as a sum of squares (with coefficients being the eigenvalues) of dot products; in components:

$$Q(x_1, x_2, \dots) = \lambda_1(x_1 v_1^{(1)} + x_2 v_2^{(1)} + \dots)^2 + \lambda_2(x_1 v_1^{(2)} + x_2 v_2^{(2)} + \dots)^2 + \dots$$

which is said to be the *normal form* of the quadratic form  $Q$ . For the above example in part (iii), we get:

$$Q = 3x_1^2 + 2x_1x_2 + 3x_2^2 = 2\left(\frac{x_1}{\sqrt{2}} - \frac{x_2}{\sqrt{2}}\right)^2 + 4\left(\frac{x_1}{\sqrt{2}} + \frac{x_2}{\sqrt{2}}\right)^2.$$

verify this formula by multiplying it out and comparing it to the original form of  $Q$  in this example. Its now obvious that the fact that the matrix  $A$  has positive eigenvalues implies that  $Q$  is positive definite. In general, writing a quadratic form as a sum of squares in some new variables  $\tilde{x}_i$  makes the sign behaviour of these quadratic forms self-evident.

(vi) **Classification rules**

The rules for classifying quadratic forms using the eigenvalues is as follows.

- a) If all eigenvalues are  $> 0$ ,  $Q$  is positive definite (PD)
- b) If all eigenvalues are  $< 0$ ,  $Q$  is negative definite (ND)
- c) If some eigenvalues are  $\leq 0$  and some are  $\geq 0$ ,  $Q$  is indefinite (ID).
- d) If all eigenvalues are  $\geq 0$ ,  $Q$  is positive semi-definite (PSD)
- e) If all eigenvalues are  $\leq 0$ ,  $Q$  is negative semi-definite (NSD)

Note that although this classification by means of eigenvalues is more transparent than the principal minor test (see Handout 2), it requires the finding of the eigenvalues of the matrix  $A$ , which in turn requires the solving of a  $n^{\text{th}}$  order polynomial equation, whereas the principal minor test only requires the computation of determinants, which in general is much easier to do (and requires less computer time if done numerically).

### 3(B) Unconstrained Optimisation in Economics.

(i) **Production functions**

A firm produces  $Q$  items per year, which sell at a price  $p$  per item. The Revenue,  $R = pQ$ .  $Q$  depends on quantities  $x_1, x_2$ , etc. which are quantities such as the number of employees, amount spent on equipment, and so on. The vector  $(x_1, x_2, \dots, x_n)$  is called the input bundle vector. The Cost function measures how much a firm spends on production, and this will also be a function of  $x_1, x_2$  and so on. The profit,

$$\Pi = R - C = pQ(x_1, x_2, \dots, x_n) - C(x_1, x_2, \dots, x_n).$$

To maximise the profit, we find the maximum of  $\Pi$  as a function of the inputs  $x_1, \dots, x_n$ , leading to

$$p \frac{\partial Q}{\partial x_i} = \frac{\partial C}{\partial x_i}, \quad i = 1, n.$$

Problems where a firm produces several different products can also lead to optimisation problems.

(ii) **The discriminating monopolist.**

A monopolist produces two products at a rate  $Q_1$  and  $Q_2$ . The price obtained is not constant, but reduces if the monopolist floods the market, so  $p_1 = a_1 - b_1 Q_1$ , and  $p_2 = a_2 - b_2 Q_2$ , are the prices for the two products. The Cost function is taken as  $C = c_1 Q_1 + c_2 Q_2$ , being simply proportional to the quantity produced. The profit

$$\Pi = R - C = p_1 Q_1 + p_2 Q_2 - c_1 Q_1 - c_2 Q_2 = (a_1 - c_1) Q_1 + (a_2 - c_2) Q_2 - b_1 Q_1^2 - b_2 Q_2^2$$

so there is a best choice of  $Q_1$  and  $Q_2$  to maximise profit, when

$$\Pi_{Q_1} = (a_1 - c_1) - 2b_1 Q_1 = 0, \quad \text{and} \quad \Pi_{Q_2} = (a_2 - c_2) - 2b_2 Q_2 = 0.$$

Inspection of the Hessian shows that  $\Pi$  has a maximum here provided  $b_1$  and  $b_2$  are positive. We then also require  $a_1 \geq c_1$  and  $a_2 \geq c_2$ . The monopolist then has a *profit maximising strategy* allowing discrimination between the amounts of the two products to be produced.

The same approach can be used when the cost function or the pricing is a more complicated function of the  $Q_i$ .

(iii) **The Cobb-Douglas production function.**

A commonly used model in microeconomics is the Cobb-Douglas production function

$$Q(x_1, x_2) = x_1^a x_2^b, \quad \text{where } a > 0, \quad b > 0.$$

The input bundle here is then just a two-dimensional vector, and we assume here the price  $p$  is independent of  $Q$ , unlike in the monopolist problem. If the Cost is linear in the input bundle,  $C = w_1 x_1 + w_2 x_2$ , then the Profit

$$\Pi = p x_1^a x_2^b - w_1 x_1 - w_2 x_2$$

and the conditions for a stationary point are

$$a p x_1^{a-1} x_2^b = w_1, \quad b p x_1^a x_2^{b-1} = w_2$$

giving positive critical values  $x_1^* = a p Q^* / w_1$  and  $x_2^* = b p Q^* / w_2$ , where  $Q^*$  is the production function at the critical value.

We want to know when this stationary value is a maximum, so we must examine the Hessian

$$H = \begin{pmatrix} a(a-1)p x_1^{a-2} x_2^b & a b p x_1^{a-1} x_2^{b-1} \\ a b p x_1^{a-1} x_2^{b-1} & b(b-1)p x_1^a x_2^{b-2} \end{pmatrix}$$

which is negative definite provided  $LPM_1 = a(a-1)p x_1^{a-2} x_2^b < 0$  and  $LPM_2 = ab(1-a-b)p^2 x_1^{2a-2} x_2^{2b-2} > 0$ , which requires

$$0 < a < 1 \quad \text{and} \quad 0 < b < 1 \quad \text{and} \quad a + b < 1.$$

If these conditions are satisfied, then the Cobb-Douglas production function gives a *profit maximising strategy*. If they are not satisfied, the profit increases indefinitely as  $x_1$  and  $x_2$  get very large. This is possible economic behaviour: if there is no profit maximising strategy, all it means is that the bigger the firm the more profit it makes, which is of course entirely possible.