MATH 2640 EXAMPLES 1 Workshop Problems

B1

i) Function $f(x) = \frac{1}{4}x^4 - \frac{3}{2}x^2 + 2x + 1$

critical points $f'(x) = x^3 - 3x + 2 = 0$

$x = 1$ clearly a solution.

Factor: $x-1 \,/\, x^3 - 3x + 2 \,\backslash\, x^2 + x - 2$

$$\begin{array}{r} x^3 - x^2 \\ \hline x^2 - 3x \\ x^2 - x \\ \hline -2x + 2 \\ -2x + 2 \\ \hline 0 \end{array}$$

$\Rightarrow f'(x) = (x-1)(x^2+x-2) = (x-1)(x-1)(x+2)$

$= (x-1)^2(x+2) = 0 \Leftrightarrow$ $x=1$ double root
$x=-2$ other root.

characterize:

$f''(x) = 3x^2 - 3 \Rightarrow f''(1) = 0$ saddle point (point of inflection)

$f''(-2) = 9 > 0$ local min.

Values $f(1) = \frac{7}{4} = 1.75$

$f(-2) = -5$

In domain $[0,4]$

limit values

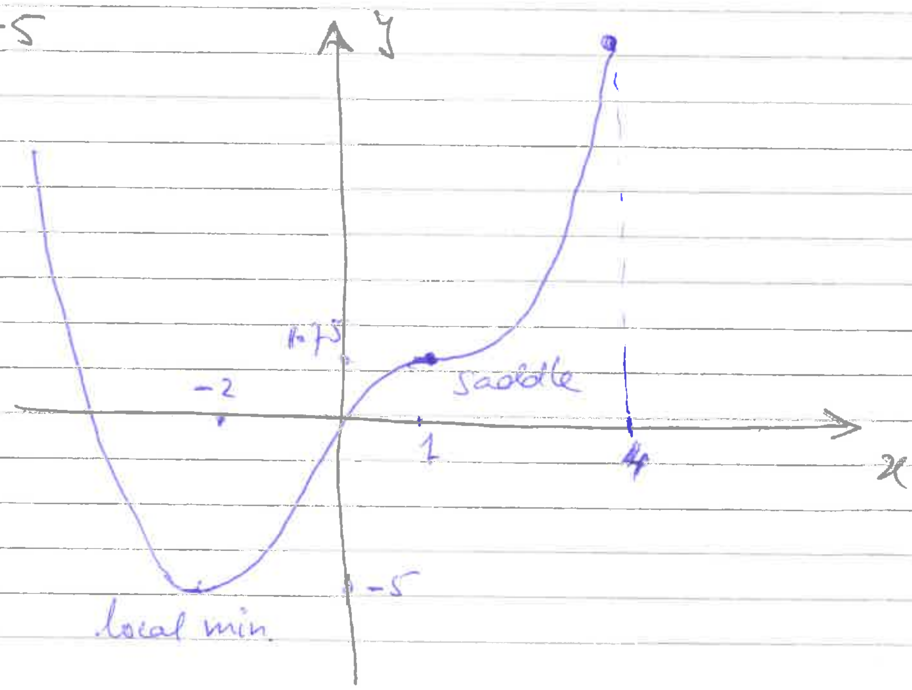
$f(0) = 1$

$f(4) = 49$

So in this domain

$(4, 49)$ would be global max.

(not critical pt!)

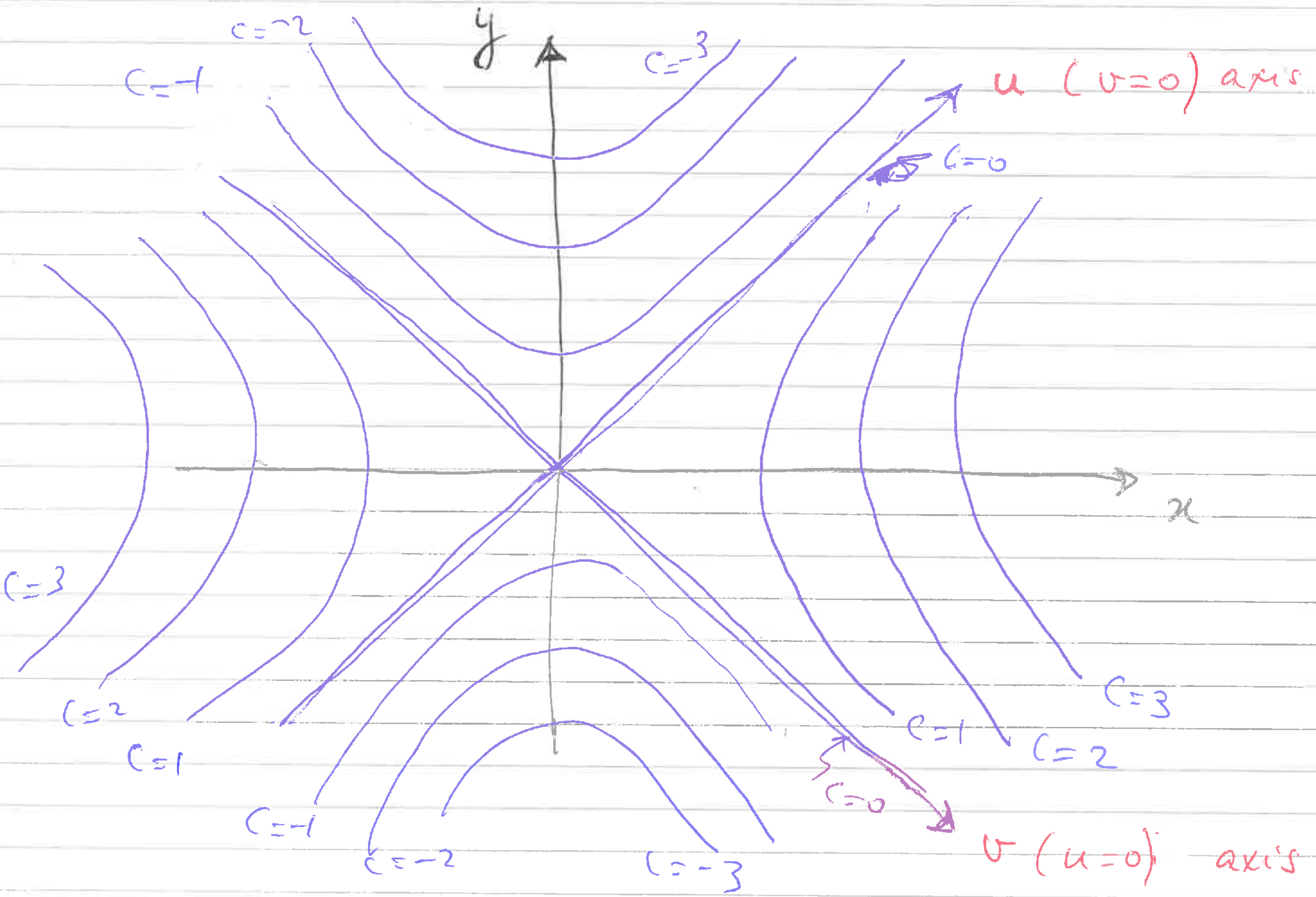


ii) Consider $z = x^2 - y^2$

Contour plot: set $z(x,y) = x^2 - y^2 = c$

If $c = 0 \Rightarrow x^2 = y^2 \Rightarrow y = \pm x$

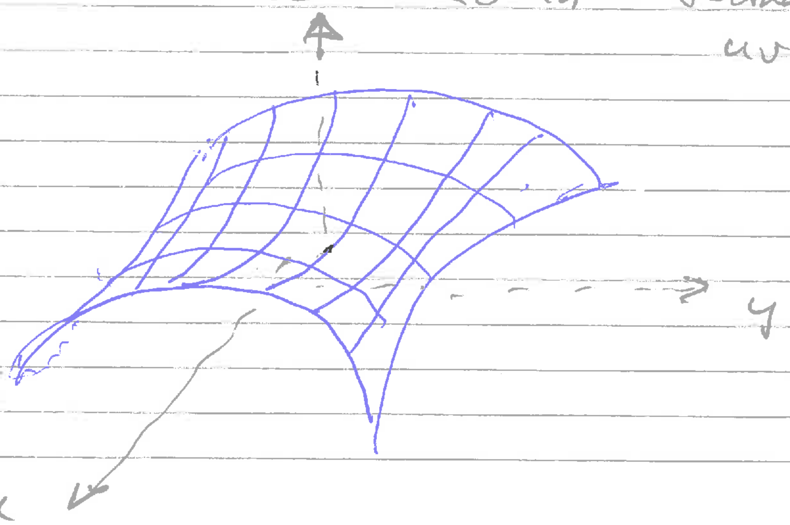
If $c > 0 \Rightarrow$ write $\left.\begin{array}{l} x + y = u \\ x - y = v \end{array}\right\} \quad uv = c > 0$



$uv = c \Rightarrow$ hyperbola with $c > 0$ in first/third uv-quadrant

and $c < 0$ in second/fourth uv-quadrant

graph:



surface in shape of a saddle!

B2)

i) $f(x,y) = x^2y^3 + x^3y^5$

$f_x = 2xy^3 + 3x^2y^5$, $f_y = 3x^2y^2 + 5x^3y^4$

$f_{xx} = 2y^3 + 6xy^5$, $f_{yy} = 6x^2y + 20x^3y^3$

$(f_x)_y = 6xy^2 + 15x^2y^4$

$(f_y)_x = 6xy^2 + 15x^2y^4$

$\Rightarrow f_{xy} = f_{yx}$.

ii)

$f(x,y) = x^2 \sin^2(y) - x\ln(xy)$

$f_x = 2x\sin^2(y) - \ln(xy) - 1$

$f_y = 2x^2 \sin(y)\cos(y) - \frac{x}{y}$

$f_{xx} = 2\sin^2(y) - \frac{1}{x}$

$f_{yy} = 2x^2\cos^2(y) - 2x^2\sin^2(y) + \frac{x}{y^2}$

$$\left.\begin{aligned} (f_x)_y &= 4x\sin(y)\cos(y) - \frac{1}{y} \\ (f_y)_x &= 4x\sin(y)\cos(y) - \frac{1}{y} \end{aligned}\right\} \Rightarrow f_{xy} = f_{yx}.$$

B3)

i) $f = x^2 - y/x^2$ at $(1,2)$, $\underline{u} = \left(\frac{3}{5}, -\frac{4}{5}\right)$

$\underline{\nabla} f = \left(2x + 2y/x^3 , -1/x^2\right) \Rightarrow \underline{\nabla} f(1,2) = (6,-1)$

directional derivative:

$D_{\underline{u}} f(1,2) = \underline{u} \cdot \underline{\nabla} f(1,2) = \frac{3}{5}\cdot 6 - \frac{4}{5}\cdot(-1) = \frac{22}{5}$

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ii) $g = x^2 + 2xy + \frac{1}{2}y^2$ at $(1,1)$

$\underline{u} = (s,t)$ is unit vector if $s^2+t^2=1$

$\underline{\nabla} g = (2x+2y, 2x+y) \Rightarrow \underline{\nabla} g(1,1) = (4,3)$

$D_{\underline{u}} g(1,1) = (s,t)\cdot(4,3) = 4s+3t$

is max. if $\underline{u} \parallel \underline{\nabla} g(1,1)$ (parallel to gradient)

min if $\underline{u}$ anti-parallel to gradient

$\Rightarrow (s,t) = \alpha(4,3)$ with $\alpha > 0$ max $D_{\underline{u}} g$
$\alpha < 0$ min. $D_{\underline{u}} g$

Since (s,t) unit vector $\Rightarrow$ $s^2+t^2 = (4\alpha)^2 + (3\alpha)^2$
$= 25\alpha^2 = 1$

$\Rightarrow \alpha^2 = \frac{1}{25}$, $\alpha = \pm\frac{1}{5}$

Thus, $\underline{u} = \frac{1}{5}(4,3)$ for $D_{\underline{u}} g(1,1)$ to be max.
$\underline{u} = -\frac{1}{5}(4,3)$ for $D_{\underline{u}}(1,1)$ to be min.

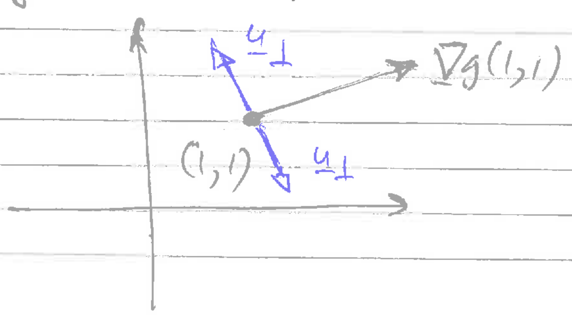
In that case: $D_{\underline{u}} g(1,1) = \pm 5$. (+ max, − min).

$D_{\underline{u}} g = 0$ if $\underline{u} \perp \underline{\nabla} g$ (perpendicular to gradient)

then: $4s+3t=0 \Rightarrow (s,t) = \beta(-3,4)$ and $\beta = \pm\frac{1}{5}$
by some computation.

$\Rightarrow D_{\underline{u}} g = 0$ if

$\underline{u} = \frac{1}{5}(\mp 3, \pm 4)$



B4 $z(x,y)$ defined by $\boxed{z^2x - 2yz + xy^2 = 4}$

- Derivative w.r.t. x, keeping y constant:

$$2zz_x x + z^2 - 2yz_x + y^2 = 0$$

$$\Rightarrow \quad 2(zx - y)\, z_x = -(z^2 + y^2) \Rightarrow \boxed{z_x = \frac{1}{2}\,\frac{z^2+y^2}{y - xz}}$$

- Derivative w.r.t. y, keeping x constant:

$$2zz_y x - 2z - 2yz_y + 2xy = 0$$

$$\Rightarrow \quad 2(zx - y)\, z_y = 2(z - xy) \Rightarrow \boxed{z_y = \frac{z - xy}{zx - y}}$$

Second order derivative:

$$\frac{\partial}{\partial x}\left(2zz_x x + z^2 - 2yz_x + y^2\right) =$$

$$= 2xz_x^2 + 2zxz_{xx} + 2zz_x + 2zz_x - 2yz_{xx} = 0$$

$$\Rightarrow \quad (zx - y)\, z_{xx} + xz_x^2 + 2zz_x = 0$$

$$\Rightarrow \quad z_{xx} = -\frac{xz_x^2 + 2zz_x}{zx - y}$$

Substitute z_x obtained earlier: $\boxed{z_{xx} = -\frac{1}{4}\frac{x(z^2+y^2)^2}{(zx-y)^3} + z\frac{z^2+y^2}{(zx-y)^2}}$

Verify $(1,1,3)$ satisfies the condition.

If $x=y=1 \Rightarrow \quad z^2 - 2z + 1 = 4 \Rightarrow (z-3)(z+1) = 0 \Rightarrow z = 3$ or $z = -1$

So $z = -1$ another possible value for $x = y = 1$.

At $(1,1,3)$:

$$z_x = -\frac{5}{2}, \quad z_y = 1, \quad z_{xx} = \frac{35}{8}.$$

B5 $f(x,y) = xy^2 + x^3y$ with y satisfying $\boxed{y^2 = x^3 + y^3}$

$$\frac{\partial f}{\partial x} = f_x = y^2 + 3x^2y \quad , \quad \frac{\partial f}{\partial y} = f_y = 2xy + x^3$$

From relation for y by implicit differentiation:

$$2y\,dy = 3x^2dx + 3y^2dy \Rightarrow (2y - 3y^2)dy = 3x^2dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{3x^2}{2y-3y^2} \quad \text{or} \quad \frac{dx}{dy} = \frac{2y-3y^2}{3x^2}.$$

By chain rule: total derivative:

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = y^2 + 3x^2y + (2xy + x^3)\frac{3x^2}{2y-3y^2}$$

$$\frac{df}{dy} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial x}\frac{dx}{dy} = 2xy + x^3 + (y^2 + 3x^2y)\frac{2y-3y^2}{3x^2}.$$

B6 Relations:

$$f = x + y + z - 1 = 0$$

$$g = x^2 - 2y^2 + 3z^2 - 2 = 0$$

Differentials

$$df = dx + dy + dz = 0$$

$$dg = 2x\,dx - 4y\,dy + 6z\,dz = 0$$

$$\Rightarrow \begin{cases} dx + dy + dz = 0 \\ x\,dx - 2y\,dy + 3z\,dz = 0 \end{cases}$$

Eliminate dy $\Rightarrow$ $(2y + x)dx + (2y + 3z)dz = 0$

$$\Rightarrow \boxed{\frac{dz}{dx} = -\frac{2y+x}{2y+3z}}$$

PTO

Eliminate $dz \Rightarrow (3z-x)\,dx + (3z+2y)\,dy = 0$

$$\Rightarrow \boxed{\frac{dy}{dx} = -\frac{3z-x}{3z+2y}}$$

If $x=1$ we have $\begin{cases} y+z=0 & \Rightarrow z=-y \\ -2y^2+3z^2=1 \end{cases}$

then $y^2=1 \Rightarrow y=\pm 1$

Thus we get two solutions: $x=1 \;\; y=1 \;\; z=-1$

$x=1 \;\; y=-1 \;\; z=1$

$\Rightarrow$ two points $(1,1,-1)$ and $(1,-1,1)$.

Gradients at these points are proportional to

$$(dx, dy, dz) = dx\left(1, \frac{dy}{dx}, \frac{dz}{dx}\right)$$

- at $(1,1,-1)$: $\frac{dy}{dx} = -4$, $\frac{dz}{dx} = 3$

$\Rightarrow$ tangent vector $\sim (1,-4,3) \Rightarrow \|(1,-4,3)\| = \sqrt{26}$

corresponding unit vector $\frac{1}{\sqrt{26}}(1,-4,3)$

- at $(1,-1,1)$: $\frac{dy}{dx} = -2$, $\frac{dz}{dx} = 1$

$\Rightarrow$ tangent vector $\sim (1,-2,1) \Rightarrow \|(1,-2,1)\| = \sqrt{6}$

corresponding unit vector: $\frac{1}{\sqrt{6}}(1,-2,1)$.