MATH2640 EXAMPLES 3 WORKSHOP PROBLEMS

B1)

i) $f(x,y,z) = 2x^2 + xy + 4y^2 + xz + z^2 + 2$.

First order conditions:

$$\begin{cases} f_x = 4x + y + z = 0 \\ f_y = x + 8y = 0 \quad \Rightarrow \quad y = -1/8\,x \\ f_z = x + 2z = 0 \quad \Rightarrow \quad z = -1/4\,x \end{cases}$$

Substitute the latter in first relation:

$$4x - \frac{1}{8}x - \frac{1}{4}x = 0 \Rightarrow x = 0, \quad y = 0, \quad z = 0$$

So $(0,0,0)$ only critical point.

(Could also be done by row elimination on system)

Hessian

$$H = \begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{xy} & f_{yy} & f_{yz} \\ f_{xz} & f_{yz} & f_{zz} \end{pmatrix} = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 8 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

$$\Rightarrow \det(H) = 4 \cdot 16 - 1 \cdot 2 + 1(-8) = 54 > 0$$

$$LPM_2 = \begin{vmatrix} 4 & 1 \\ 1 & 8 \end{vmatrix} = 31 > 0 \qquad LPM_1 = |4| = 4 > 0$$

$\Rightarrow$ H positive definite, hence $(0,0,0)$ local minimum

ii) $f(x,y,z) = -x^3 + 3xz + 2y - y^2 - 3z^2$

First order conditions:

$$\begin{cases} f_x = -3x^2 + 3z = 0 \\ f_y = 2 - 2y = 0 \quad \Rightarrow \quad y = 1 \\ f_z = 3x - 6z = 0 \quad \Rightarrow \quad z = \frac{1}{2}x \end{cases}$$

PTO

Substitute $z = \frac{1}{2}x$ into first eq. $\Rightarrow$ $3x^2 = \frac{3}{2}x$

$\Rightarrow x(x - \frac{1}{2}) = 0$, Thus $x = 0$ or $x = \frac{1}{2}$

$x = 0 \Rightarrow$ point $(0, 1, 0)$

$x = \frac{1}{2} \Rightarrow$ point $(\frac{1}{2}, \cancel{1}, \frac{1}{4})$.

Hessian:
$$H = \begin{pmatrix} -6x & 0 & 3 \\ 0 & -2 & 0 \\ 3 & 0 & -6 \end{pmatrix}$$

$\det(H) = -6x \cdot 12 + 3 \cdot 6 = 18(1 - 4x) = LPM_3$

$LPM_2 = \begin{vmatrix} -6x & 0 \\ 0 & -2 \end{vmatrix} = 12x$, $LPM_1 = -6x$

$(0, 1, 0) \Rightarrow LPM_3 > 0$, $LPM_2 = 0$, $LPM_1 = 0$

indefinite point

(LPM test is inconclusive)

$(\frac{1}{2}, 1, \frac{1}{4}) \Rightarrow LPM_3 < 0$, $LPM_2 > 0$, $LPM_1 < 0$

$\Rightarrow H$ neg def. $\Rightarrow$ local maximum

iii) $f(x, y) = 3x^4 + 3x^2y - y^3$

First-order conditions

$$\begin{cases} f_x = 12x^3 + 6xy = 0 \\ f_y = 3x^2 - 3y^2 = 0 \end{cases} \Rightarrow x^2 = y^2 \Rightarrow x = \pm y$$

$x = y \Rightarrow 12x^3 + 6x^2 = 0 \Rightarrow x^2(2x + 1) = 0 \Rightarrow x = 0$ or $x = -\frac{1}{2}$

$\Rightarrow$ crit points $(0, 0)$, $(-\frac{1}{2}, -\frac{1}{2})$

$x = -y \Rightarrow 12x^3 - 6x^2 = 0 \Rightarrow x^2(2x - 1) = 0 \Rightarrow x = 0$, or $x = \frac{1}{2}$

gives additional point $(\frac{1}{2}, -\frac{1}{2})$

Hessian $\begin{pmatrix} 36x^2 + 6y & 6x \\ 6x & -6y \end{pmatrix}$ At $(0, 0)$ all PMs zero, so indefinite point (test is inconclusive)

p.T.O

At both points $\left(\pm\frac{1}{2}, \mp\frac{1}{2}\right)$ we get $H = \begin{pmatrix} 6 & \pm 3 \\ \pm 3 & 3 \end{pmatrix}$

$\Rightarrow$ $LPM_2 = \det(H) = 9 > 0$, $LPM_1 = 6 > 0$ $\Rightarrow$ H pos. def.
local minimum

B2 a) $Q(x,y) = 10xy - x^2 - y^2 = (x,y)\, A \begin{pmatrix} x \\ y \end{pmatrix}$

Symmetric matrix $A = \begin{pmatrix} -1 & 5 \\ 5 & -1 \end{pmatrix}$

Eigenvalues

$$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 5 \\ 5 & -1-\lambda \end{vmatrix} = (1+\lambda)^2 - 25 = 0$$

$\Rightarrow$ $1 + \lambda = \pm 5$ $\Rightarrow$ $\lambda = 4, -6$ roots

$\Rightarrow$ indefinite quadratic form. In fact, $LPM_2 = -24$, $LPM_1 = -1$ $\Rightarrow$ ID.

Eigenvectors:

$\boxed{\lambda = 4}$ $\begin{pmatrix} -5 & 5 \\ 5 & -5 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \boxed{v_1 - v_2 = 0}$

$\Rightarrow$ vectors $\sim \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ unit eigenvector $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\boxed{\lambda = -6}$ $\begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \boxed{v_1 + v_2 = 0}$

$\Rightarrow$ vectors $\sim \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ unit eigenvector $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

new variables:

$$\tilde{x} = \frac{1}{\sqrt{2}} (1, 1) \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \frac{x + y}{\sqrt{2}}$$

$$\tilde{y} = \frac{1}{\sqrt{2}} (1, -1) \begin{pmatrix} x \\ y \end{pmatrix} = \frac{x - y}{\sqrt{2}}$$

PTO

$$\Rightarrow \quad Q(x,y) = 4\left(\frac{x+y}{\sqrt{2}}\right)^2 - 6\left(\frac{x-y}{\sqrt{2}}\right)^2.$$

b) Quadratic form:

$$Q(x_1,x_2,x_3) = 8x_1x_2 + 2x_1x_3 + 18x_2x_3 - 5x_1^2 - 13x_2^2 - 10x_3^2$$

$$\Rightarrow \quad A = \begin{pmatrix} -5 & 4 & 1 \\ 4 & -13 & 9 \\ 1 & 9 & -10 \end{pmatrix}$$

Eigenvalues:

$$\det(A-\lambda I) = \begin{vmatrix} -5-\lambda & 4 & 1 \\ 4 & -13-\lambda & 9 \\ 1 & 9 & -10-\lambda \end{vmatrix} =$$

$$= -(5+\lambda)\left[(13+\lambda)(10+\lambda) - 81\right] - 4\left[-4\cdot(10+\lambda) - 9\right] + \left[36 + 13 + \lambda\right]$$

$$= -\lambda^3 - 28\lambda^2 - 245\lambda - 650 + 81\lambda + 405 + 16\lambda + 160 + 36$$
$$+ 49 + \lambda$$

$$= -\lambda^3 - 28\lambda^2 - 147\lambda = -\lambda(\lambda+7)(\lambda+21)$$

$\Rightarrow$ Eigenvalues $\lambda_1 = 0$, $\lambda_2 = -7$, $\lambda_3 = -21$ negative

NSD: negative semidefinite!

Check by principal minor test:

$LPM_3 = \det A = 0 \Rightarrow$ PMs of order 2:

$$\begin{vmatrix} -5 & 4 \\ 4 & -13 \end{vmatrix} = 49 > 0, \quad \begin{vmatrix} -5 & 1 \\ 1 & -10 \end{vmatrix} = 49 > 0, \quad \begin{vmatrix} -13 & 9 \\ 9 & -10 \end{vmatrix} = 49 > 0$$

PTO

PMs of order 1:

$\underbrace{|-5|}=-5<0, \quad |-13|=-13<0, \quad |-10|=-10<0$

means: 1×1 determinant.

So odd-order PMs ≤ 0, even-order PMs $\geq 0 \Rightarrow$ NSD.

Eigenvectors

$\boxed{\lambda=0}$

$$\begin{pmatrix} -5 & 4 & 1 \\ 4 & -13 & 9 \\ 1 & 9 & -10 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

System of eqs.

$$\begin{cases} v_1 + 9v_2 - 10v_3 = 0 \\ 4v_1 - 13v_2 + 9v_3 = 0 \\ -5v_1 + 4v_2 + v_3 = 0 \end{cases}$$

By row elimination: $\Rightarrow \begin{cases} v_1 + 9v_2 - 10v_3 = 0 \\ -49v_2 + 49v_3 = 0 \\ 49v_2 - 49v_3 = 0 \end{cases} \Rightarrow v_2 = v_3$

then $v_1 = v_3 \Rightarrow$

$\Rightarrow$ eigenvector $\sim \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow$ unit eigenvector $\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\boxed{\lambda=-7}$

$$\begin{pmatrix} 2 & 4 & 1 \\ 4 & -6 & 9 \\ 1 & 9 & -3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} v_1 + 9v_2 - 3v_3 = 0 \\ 2v_1 + 4v_2 + v_3 = 0 \\ 4v_1 - 6v_2 + 9v_3 = 0 \end{cases} \Rightarrow \begin{cases} v_1 + 9v_2 - 3v_3 = 0 \\ -14v_2 + 7v_3 = 0 \\ -42v_2 + 21v_3 = 0 \end{cases}$$

$\Rightarrow$ $v_3 = 2v_2$, $v_1 = -3v_2$ $\Rightarrow$ eigenvector $\sim \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$

$\Rightarrow$ unit eigenvector : $\frac{1}{\sqrt{14}} \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$

$\boxed{\lambda = -21}$

$$\begin{pmatrix} 16 & 4 & 1 \\ 4 & 8 & 9 \\ 1 & 9 & 11 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \begin{cases} v_1 + 9v_2 + 11v_3 = 0 \\ 4v_1 + 8v_2 + 9v_3 = 0 \\ 16v_1 + 4v_2 + v_3 = 0 \end{cases} \Rightarrow \begin{cases} v_1 + 9v_2 + 11v_3 = 0 \\ -28v_2 - 35v_3 = 0 \\ -140v_2 - 175v_3 = 0 \end{cases}$

$\Rightarrow \begin{cases} 4v_2 + 5v_3 = 0 \\ v_1 + v_2 + v_3 = 0 \end{cases}$ $\Rightarrow$ vector $\sim \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix}$

$\Rightarrow$ unit eigenvector $\frac{1}{\sqrt{42}} \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix}$

$\Rightarrow$ orthogonal diagonalizing matrix :

$$O = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{3}{\sqrt{14}} & \frac{1}{\sqrt{42}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} & \frac{-5}{\sqrt{42}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} & \frac{4}{\sqrt{42}} \end{pmatrix}$$

new variables

check orthogonal $O \cdot O^T = I$.

$$\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{pmatrix} = O^T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{3}{\sqrt{14}} & \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} \\ \frac{1}{\sqrt{42}} & \frac{-5}{\sqrt{42}} & \frac{4}{\sqrt{42}} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

PTO

$$\Rightarrow \quad \tilde{x}_1 = \frac{x_1 + x_2 + x_3}{\sqrt{3}}, \quad \tilde{x}_2 = \frac{-3x_1 + x_2 + 2x_3}{\sqrt{14}}, \quad \tilde{x}_3 = \frac{x_1 - 5x_2 + 4x_3}{\sqrt{42}}$$

and

$$Q(x_1, x_2, x_3) = 0 \cdot \left(\frac{x_1 + x_2 + x_3}{\sqrt{3}}\right)^2 - 7\left(\frac{-3x_1 + x_2 + 2x_3}{\sqrt{14}}\right)^2 - 21\left(\frac{x_1 - 5x_2 + 4x_3}{\sqrt{42}}\right)^2$$

B3 (a) Revenue is

$$R = p_1 Q_1 + p_2 Q_2$$

express p_1, p_2 in terms of Q_1, Q_2

$$\Rightarrow \quad p_1 = 210 - Q_1, \quad p_2 = 90 - Q_2$$

$$\Rightarrow \quad R(Q_1, Q_2) = Q_1(210 - Q_1) + Q_2(90 - Q_2)$$

Cost: $\quad C(Q_1, Q_2) = 6000 + Q_1^2 + Q_1 Q_2 + Q_2^2$

$\rightarrow$ profit function:

$$\Pi(Q_1, Q_2) = R - C = Q_1(210 - Q_1) + Q_2(90 - Q_2) - 6000 - Q_1^2 - Q_1 Q_2 - Q_2^2$$

$$= 210 Q_1 - 2Q_1^2 + 90 Q_2 - 2Q_2^2 - Q_1 Q_2 - 6000$$

First-order conditions:

$$\Pi_{Q_1} = 210 - 4Q_1 - Q_2 = 0$$

$$\Pi_{Q_2} = 90 - 4Q_2 - Q_1 = 0$$

$$\Rightarrow \begin{cases} 4Q_1 + Q_2 = 210 \\ Q_1 + 4Q_2 = 90 \end{cases}$$

$$\Rightarrow 15 Q_2 = 150 \Rightarrow Q_2 = 10$$

$$15 Q_1 = 750 \Rightarrow Q_1 = 50$$

$\Rightarrow$ corresponding prices:

$$P_1^* = 210 - Q_1 = 160, \quad p_2^* = 90 - Q_2 = 80.$$

Hessian:

$$H = \begin{pmatrix} \Pi_{Q_1Q_1} & \Pi_{Q_1Q_2} \\ \Pi_{Q_1Q_2} & \Pi_{Q_2Q_2} \end{pmatrix} = \begin{pmatrix} -4 & -1 \\ -1 & -4 \end{pmatrix}$$

$\det(H) = LPM_2 = 15 > 0$; $LPM_1 = -4 < 0 \Rightarrow$ ND

critical point is local max.

Values of profit at critical point:

$\Pi^* = \Pi(Q_1^*, Q_2^*) = \Pi(50, 10) = -300 < 0$

so company makes a loss.

(b) Revenue: $R = p_1 Q_1 + p_2 Q_2 = p_1(40 - p_1) + p_2(60 - 2p_2)$

Cost: $C = K + 20(40 - p_1) + 10(60 - 2p_2)$

$= K + 1400 - 20p_1 - 20p_2$

$\Rightarrow$ Profit $\Pi(p_1, p_2) = R - C = p_1(40 - p_1) + p_2(60 - 2p_2)$

$- K - 1400 + 20p_1 + 20p_2$

$= 60p_1 - p_1^2 + 80p_2 - 2p_2^2 - K - 1400$

First-order conditions:

$$\left.\begin{aligned} \Pi_{p_1} &= 60 - 2p_1 = 0 \\ \Pi_{p_2} &= 80 - 4p_2 = 0 \end{aligned}\right\} \Rightarrow p_1 = 30, \; p_2 = 20$$

PTO

Hessian

$$H = \begin{pmatrix} \Pi_{p_1p_1} & \Pi_{p_1p_2} \\ \Pi_{p_1p_2} & \Pi_{p_2p_2} \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -4 \end{pmatrix}$$

$LPM_2 = 8 > 0$, $LPM_1 < 0 \Rightarrow$ N.D., (p_1^*, p_2^*) local max.

Profit at crit. point $\Pi(p_1^*, p_2^*) = 300 - K$

So company breaks even if $\underline{K = 300}$.

B4] $Q(x,y) = x^{1/2}y^{1/4}$, price $p \Rightarrow$ revenue $R = p\,x^{1/2}y^{1/4}$

Cost $C(x,y) = ax + by$

a) Profit: $\Pi(x,y) = R - C = p\,x^{1/2}y^{1/4} - ax - by$.

First order conditions:

$$\begin{cases} \Pi_x = \frac{1}{2}p\,x^{-1/2}y^{1/4} - a & \Rightarrow \quad pQ = 2ax \\ \Pi_y = \frac{1}{4}p\,x^{1/2}y^{-3/4} - b & \Rightarrow \quad pQ = 4by \end{cases}$$

$$\Rightarrow \boxed{2ax = 4by}$$

Furthermore $\frac{1}{8}p^2x^{1/2}y^{-3/4} \cdot x^{-1/2}y^{1/4} = ab$

$$\Rightarrow p^2y^{-1/2} = 8ab \Rightarrow y = \left(\frac{p^2}{8ab}\right)^2 = y_*$$

and

$$x = \frac{2b}{a}y = \frac{2b}{a}\left(\frac{p^2}{8ab}\right)^2 = x_*$$

$$\Rightarrow x_* = \frac{p^4}{32a^3b}, \quad y_* = \frac{p^4}{64a^2b^2}$$

Profit at the critical point:

$$\Pi_* = \Pi(x_*, y_*) = p\left(\frac{p^4}{32a^3b}\right)^{1/2}\left(\frac{p^4}{64a^2b^2}\right)^{1/4} - a\frac{p^4}{32a^3b} - b\frac{p^4}{64a^2b^2}$$

$$= \frac{p^4}{16a^2b}\left(1 - \frac{1}{2} - \frac{1}{4}\right) = \frac{p^4}{64a^2b}$$

(b) Hessian:

$$H = \begin{pmatrix} \Pi_{xx} & \Pi_{xy} \\ \Pi_{xy} & \Pi_{yy} \end{pmatrix} = \begin{pmatrix} -\frac{1}{4}px^{-3/2}y^{1/4} & \frac{1}{8}px^{-1/2}y^{-3/4} \\ \frac{1}{8}px^{-1/2}y^{-3/4} & -\frac{3}{16}px^{1/2}y^{-7/4} \end{pmatrix}$$

$$= \frac{1}{4}px^{-3/2}y^{-7/4}\begin{pmatrix} -y^2 & \frac{1}{2}xy \\ \frac{1}{2}xy & -\frac{3}{4}x^2 \end{pmatrix}$$

$$\Rightarrow \quad \det(H) = \left(\frac{1}{4}px^{-3/2}y^{-7/4}\right)^2\left(\frac{3}{4}x^2y^2 - \frac{1}{4}x^2y^2\right) > 0$$

$$LPM_1 = -\frac{1}{4}px^{-3/2}y^{-7/4}y^2 < 0$$ so ND indeed max.

B 5] Production rate $Q(x,y,z) = x^{1/4} y^{1/4} z^{1/4}$, $p=4$.

$\Rightarrow$ revenue $R = 4x^{1/4}y^{1/4}z^{1/4}$

Cost: $C = x + \frac{1}{2}y + \frac{1}{3}z$

$\Rightarrow$ Profit $\Pi(x,y,z) = R - C = 4x^{1/4}y^{1/4}z^{1/4} - x - \frac{1}{2}y - \frac{1}{3}z.$

First-order conditions:

$$\begin{cases} \Pi_x = x^{-3/4}y^{1/4}z^{1/4} - 1 = 0 & \Rightarrow \quad Q = x \\ \Pi_y = x^{1/4}y^{-3/4}z^{1/4} - \frac{1}{2} = 0 & \Rightarrow \quad Q = \frac{1}{2}y \\ \Pi_z = x^{1/4}y^{1/4}z^{-3/4} - \frac{1}{3} = 0 & \Rightarrow \quad Q = \frac{1}{3}z \end{cases}$$

$\Rightarrow$ $y = 2Q$, $z = 3Q$, $x = Q$

$\Rightarrow$ $xyz = 6Q^3 \Rightarrow Q = (xyz)^{1/4} = (6Q^3)^{1/4}$

$\Rightarrow$ $\boxed{Q = 6} \Rightarrow x = 6, \; y = 12, \; z = 18.$
at. stat. point.

Hessian:

$$\Pi_{xx} = -\frac{3}{4}x^{-7/4}y^{1/4}z^{1/4} = -\frac{3}{4}\frac{Q}{x^2}$$

$$\Pi_{xy} = \frac{1}{4}x^{-3/4}y^{-3/4}z^{1/4} = \frac{1}{4}\frac{Q}{xy}$$

$$\Pi_{xz} = \frac{1}{4}x^{-3/4}y^{1/4}z^{-3/4} = \frac{1}{4}\frac{Q}{xz}$$

$$\Pi_{yy} = -\frac{3}{4}x^{1/4}y^{-7/4}z^{1/4} = -\frac{3}{4}\frac{Q}{y^2}$$

$$\Pi_{yz} = \frac{1}{4}x^{1/4}y^{-3/4}z^{-3/4} \qquad \Pi_{zz} = -\frac{3}{4}x^{1/4}y^{1/4}z^{-7/4}$$

$$\Rightarrow \quad H = \begin{pmatrix} -\frac{3}{4}\frac{a}{x^2} & \frac{1}{4}\frac{a}{xy} & \frac{1}{4}\frac{a}{xz} \\ \frac{1}{4}\frac{a}{xy} & -\frac{3}{4}\frac{a}{y^2} & \frac{1}{4}\frac{a}{yz} \\ \frac{1}{4}\frac{a}{xz} & \frac{1}{4}\frac{a}{yz} & -\frac{3}{4}\frac{a}{z^2} \end{pmatrix}$$

$$\det(H) = -\frac{3}{4}\frac{a}{x^2}\left(\frac{9}{16}\frac{a^2}{y^2z^2} - \frac{1}{16}\frac{a^2}{y^2z^2}\right)$$

$$-\frac{1}{4}\frac{a}{xy}\left(-\frac{3}{16}\frac{a^2}{xyz^2} - \frac{1}{16}\frac{a^2}{xyz^2}\right)$$

$$+\frac{1}{4}\frac{a}{xz}\left(\frac{1}{16}\frac{a^2}{xy^2z} + \frac{3}{16}\frac{a^2}{y^2xz}\right)$$

$$= \frac{a^3}{64x^2y^2z^2}\left(-3\cdot 8 + 1\cdot 4 + 1\cdot 4\right) < 0$$

$$LPM_2 = \frac{9}{16}\frac{a^2}{x^2y^2} - \frac{1}{16}\frac{a^2}{x^2y^2} = \frac{a^2}{2x^2y^2} > 0$$

$$LPM_1 = -\frac{3}{4}\frac{a}{x^2} < 0$$

so alternating signs with $LPM_1 < 0$

$\Rightarrow$ H is ND $\Rightarrow$ <u>max</u>.