MATH2640 Introduction to Optimisation

Example Sheet 2 Solutions to Assessed Questions

Thursday 31st October 2019 homework

Gradients & tangent plane, multivariable Taylor series, first order conditions, quadratic forms.
Based on material in Lectures 5 to 9

Assessed Questions

A1. The economy in Yorkshire is in equilibrium with the system of equations

$$f(x,y,z) = 2xz + xy + z - 2\sqrt{z} = 11 \quad \text{and} \quad g(x,y,z) = xyz = 6.$$

One solution of this set of equations is $x = 3, y = 2, z = 1$, and the economy of Yorkshire is in equilibrium at this point. Suppose the Yorkshire government discovers that the variables z can be controlled by a simple decree. If the Yorkshire government decides to raise z to 1.1 estimate the change of x and y. Why is it an estimate? **4 points**

Answer:
Verify that indeed $f(3,2,1) = 6+6+1-2 = 11$ and $g(3,2,1) = 3\times 2\times 1 = 6$.
Calculate the differentials of the functions f and g defined above:

$$df = (2z+y)dx + xdy + (2x+1-1/\sqrt{z})dz = 0 \quad \text{and} \quad dg = yzdx + xzdy + xydz = 0.$$

At $(x,y,z) = (3,2,1)$ we then find that:

$$df = 4dx + 3dy + 6dz = 0, \quad dg = 2dx + 3dy + 6dz = 0$$

Hence,

$$\begin{pmatrix} 4 & 3 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} = -6 \begin{pmatrix} 1 \\ 1 \end{pmatrix} dz. \tag{1}$$

Hence, by inverting the matrix and with a determinant $D = 6$, we find

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = -\frac{1}{6} \begin{pmatrix} 3 & -3 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 6 \\ 6 \end{pmatrix} dz = \begin{pmatrix} 0 \\ -2 \end{pmatrix} dz,$$

such that $dx = 0, dy = -2dz$, and for $dz = 1.1 - 1 = 0.1$ we obtain the increment estimates $dx = 0$ and $dy = -0.2$. These are estimates because we have chosen to linearise by evaluating df and dg at the old equilibrium $(x,y,z) = (3,2,1)$, while we should determine the new equilibrium point to be exact, which would likely require a numerical evaluation.

A2.
(i) In what direction should one move from the point (1,1,2) to increase $f(x,y,z) = e^{\frac{1}{2}xyz}$ most rapidly? Present your answer as a unit vector. **2 points**
Answer:
The gradient of f is:

$$\nabla f = \frac{1}{2}(yz, xz, xy)e^{\frac{1}{2}xyz},$$

which evaluated at the point $(1,1,2)$ gives/yields

$$\nabla f(1,1,2) = (1,1,1/2)e.$$

Note that $|\nabla f(1,1,2)| = 3e/2$. The maximum increase is in the direction of $\nabla f(1,1,2)$. Hence, $\mathbf{u} = \nabla f(1,1,2)/|\nabla f(1,1,2)| = \frac{2}{3}(1,1,1/2)$.

(ii) Find the unit normal vector to the surface $g(x,y,z) = \cos(x+y^2+z) = 0$ at the point $(2, \sqrt{\pi/2}, -2)$, and give an equation for the tangent plane to the surface at that point. **2 points**

Answer:

Gradient of g is:

$$\nabla g = (-\sin(x+y^2+z), -2y\sin(x+y^2+z), -\sin(x+y^2+z)) = -\sin(x+y^2+z)(1,2y,1).$$

At the point $(2, \sqrt{\pi/2}, -2)$, one finds that

$$\nabla g(2, \sqrt{\pi/2}, -2) = -\sin(2+\pi/2-2)(1, 2\sqrt{\pi/2}, 1) = -(1, \sqrt{2\pi}, 1),$$

with $|\nabla g| = \sqrt{2+2\pi}$ at that point such that at the normal vector at that points reads/is:

$$\mathbf{u} = \nabla g/|\nabla g| = -\frac{1}{\sqrt{2+2\pi}}(1, \sqrt{2\pi}, 1)$$

The tangent plane at that point therefore reads/is:

$$-\mathbf{u}\cdot(\mathbf{x}-\mathbf{x}_0) = (x-2) + \sqrt{2\pi}(y-\sqrt{\pi/2}) + (z+2) = 0$$

with $\mathbf{x} = (x,y,z)$ and $\mathbf{x}_0 = (2, \sqrt{\pi/2}, -2)$.

A3. The functions $f(x,y) = \cosh(2x^2+y^3)$ and $g(x,y) = \sinh(2x^2+y^3)$ are expanded as a Taylor series about the point $(x,y) = (2,-2)$.

(i) Find the gradient vector and the Hessian matrix of f at this point, and hence give the Taylor series for $f(2+h, -2+k)$ up to linear and quadratic terms in h and k. **2 points**

Answer:

Gradient:

$$\nabla f = (4x, 3y^2)\sinh(2x^2+y^3) \quad \text{at} \quad (2,-2): \nabla f = (8,12)\sinh(8-8) = (0,0).$$

Hessian:

$$\begin{aligned} H &= \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} \\ &= \begin{pmatrix} 4\sinh(2x^2+y^3) + 16x^2\cosh(2x^2+y^3) & 12xy^2\cosh(2x^2+y^3) \\ 12xy^2\cosh(2x^2+y^3) & 6y\sinh(2x^2+y^3) + 9y^4\cosh(2x^2+y^3) \end{pmatrix} \\ &\to_{(x,y)=(2,-2)} \begin{pmatrix} 64 & 96 \\ 96 & 144 \end{pmatrix}, \end{aligned} \tag{2}$$

whence

$$f(2+h,-2+k) = 1 + (h,k)\cdot(0,0) + \frac{1}{2}(h,k)\begin{pmatrix} 64 & 96 \\ 96 & 144 \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix} = 1 + 32h^2 + 72k^2 + 96hk + \dots.$$

(ii) Find the Taylor expansion of the function $g(x,y)$ at this point. **2 points**

Answer:
Similarly. Gradient:

$$\nabla g = (4x, 3y^2)\cosh(2x^2+y^3) \quad \text{at} \quad (2,-2): \nabla g = (8,12)\cosh(8-8) = (8,12).$$

Hessian:

$$\begin{aligned} H &= \begin{pmatrix} g_{xx} & g_{xy} \\ g_{xy} & g_{yy} \end{pmatrix} \\ &= \begin{pmatrix} 4\cosh(2x^2+y^3) + 16x^2\sinh(2x^2+y^3) & 12xy^2\sinh(2x^2+y^3) \\ 12xy^2\sinh(2x^2+y^3) & 6y\cosh(2x^2+y^3) + 9y^4\sinh(2x^2+y^3) \end{pmatrix} \\ &\to_{(x,y)=(2,-2)} \begin{pmatrix} 4 & 0 \\ 0 & -12 \end{pmatrix}, \end{aligned} \tag{3}$$

whence

$$g(2+h,-2+k) = 0 + (h,k)\cdot(8,12) + \frac{1}{2}(h,k)\begin{pmatrix} 4 & 0 \\ 0 & -12 \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix} = 8h + 12k + 2h^2 - 6k^2 + \ldots.$$

A4. Use the first-order conditions to find all the critical points of

(i) $f(x,y) = 3x^4 + 6x^2y - 2y^3$. **2 points**
Answer:
The gradient components are:

$$f_x = 12x^3 + 12xy \quad \text{and} \quad f_y = 6x^2 - 6y^2,$$

Critical point conditions are that $f_y = 0 = 6(x^2-y^2)$, giving two solutions $y = \pm x$;
for (a) $y = x$, the $f_x = 0$ yields/gives that $12x(x^2+x) = 0 \Longrightarrow x^2(x+1) = 0$ such that:
$x = y = 0$ or $x = y = -1$;
for (b) $y = -x$, $f_x = 0$ yields $12x(x^2-x) = 0 \Longrightarrow x^2(x-1) = 0$ such that:
$x = y = 0$ (again) and $x = 1, y = -1$. Hence, the three stationary points where $f_x = 0, f_y = 0$ are $(x,y) = (0,0)$, $(x,y) = -(1,1)$, $(x,y) = (1,-1)$.

(ii) $g(x,y,z) = -6x^2 + 3xy + 3y^2 + 9yz + z^3$. **2 points**
Answer:
The gradient compoments are:

$$g_x = -12x + 3y, \quad g_y = 3x + 6y + 9z, \quad g_z = 9y + 3z^2.$$

Critical point conditions are that:
(a) $g_x = -12x + 3y = 0 \Longrightarrow y = 4x$;
(b) $g_y = 0 \Longrightarrow x + 2y + 3z = 0$;
(c) $g_z = 0 \Longrightarrow 3y + z^2 = 0$.
Combining (a) in (b) gives (d): $x + 8x + 3z = 0$ or $3x + z = 0$ or $z = -3x$;
Putting $z = -3x$ and $y = 4x$ into (c) gives: $12x + 9x^2 = 0$ or $3x(4+3x)$, such that $x = 0, y = 0, z = 0$ or $x = -4/3, y = -16/3, z = 4$. Hence, stationary points are $(0,0,0)$ (easily verified directly that $\nabla g = 0$ at that point) and $(-4/3, -16/3, 4)$ (also true by direct verification that $\nabla g = 0$ at that point).

A5. Using the results about *leading principal minors* and/or *principal minors*, determine the sign properties (definite, semidefinite, indefinite) of the following quadratic forms Q in three variables.

(i) $Q(x, y, z) = -2x^2 - 5y^2 - 9z^2 + 2xy + 6xz + 6yz$. **2 points**
Answer:
Hence,

$$Q = (x, y, z) \begin{pmatrix} -2 & 1 & 3 \\ 1 & -5 & 3 \\ 3 & 3 & -9 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \equiv \mathbf{x}^T A \mathbf{x}, \tag{4}$$

in which the determinant of A is:

$$\det A = \begin{vmatrix} -2 & 1 & 3 \\ 1 & -5 & 3 \\ 3 & 3 & -9 \end{vmatrix} = -90 + 9 + 9 + 45 + 18 + 9 = 0.$$

Since $\det A = 0$ we need to the the principal minor test; PMs of order 2 are as follows

$$\begin{vmatrix} -2 & 1 \\ 1 & -5 \end{vmatrix} = 9, \quad \begin{vmatrix} -2 & 3 \\ 3 & -9 \end{vmatrix} = 9, \quad \begin{vmatrix} -5 & 3 \\ 3 & -9 \end{vmatrix} = 36,$$

so all positive. PMs or order one are:

$$|-2| = -2, \quad |-5|, = -5, \quad |-9| = -9,$$

so all negative. Since all even PMs are positive and all odd PMs are ≤ 0, then A is negative semi-definite (NSD).

(ii) $Q(x_1, x_2, x_3) = 2x_1^2 + 3x_2^2 + 7x_3^2 + 2x_1x_2 + 2x_1x_3 + 8x_2x_3$. **2 points**
Answer:
Now

$$Q = (x, y, z) \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 4 \\ 1 & 4 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \equiv \mathbf{x}^T A \mathbf{x}, \tag{5}$$

in which the determinant of A is:

$$\det A = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 3 & 4 \\ 1 & 4 & 7 \end{vmatrix} = 42 + 4 + 4 - 3 - 32 - 7 = 8.$$

Hence, the leading PM-test applies:

$$LPM_3 = \det A = 8 > 0, \quad LPM_2 = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 6 - 1 = 5 > 0 \quad LPM_1 = 2 > 0;$$

these are all positive so Q is positive definite.