MATH2640 Introduction to Optimisation

Example Sheet 3 Solutions to Assessed Questions

Thursday 14$^{\text{th}}$ November 2019 homework

Eigenvalues & normal forms, characterisation of critical points, economic optimisation.
Based on material in Lectures 9 to 13

Assessed Questions

A1. Using the results on determining the sign of quadratic forms, find and classify (where possible) the *stationary points* of the following function:

$$f(x,y,z) = x^2 - x(y+z) + \tfrac{1}{3}y^3 + y^2 + \tfrac{1}{2}z^2 - 2z.$$

4 points

Answer:
First-order conditions

$$\begin{aligned} f_x =& 2x - y - z = 0 \Longrightarrow \text{a)} \quad z = 2x - y \\ f_y =& y^2 + 2y - x = 0 \Longrightarrow \text{b)} \quad x = y^2 + 2y \\ f_z =& -x + z - 2 = 0. \quad \text{c)} \end{aligned}$$

Plug a) into c) to obtain d): $-x + 2x - y - 2 = 0 \Longrightarrow x = y + 2$. Use d) in b) to obtain: $2y + y^2 - y - 2 = 0 \Longrightarrow y^2 + y - 2 = (y+2)(y-1) = 0$.
$y = -2$ gives stationary point $(x,y,z) = (0,-2,2)$ and
$y = 1$ gives stationary point $(x,y,z) = (3,1,5)$.
Hessian:

$$\begin{aligned} f_{xx} =& 2, f_{xy} = -1, f_{xz} = -1 \\ f_{yx} =& -1, f_{yy} = 2y+2, f_{yz} = 0 \\ f_{zx} =& -1, f_{zy} = 0, f_{zz} = 1 \\ H =& \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2y+2 & 0 \\ -1 & 0 & 1 \end{pmatrix} \\ \det(H) =& 2y + 2 - 1 = 2y + 1; LPM_2 = 2(2y+2) - 1 = 4y + 3; LPM_1 = 2. \end{aligned}$$

At $(0,-2,2)$: $\det H = -3 < 0, LPM_2 = -5 < 0, LPM_1 = 2 > 0$ so ID.
At $(3,1,5)$: $\det H = 3 > 0, LPM_2 = 7 > 0, LPM_1 = 2 > 0$ so PD, minimum.

A2.

i) Find the symmetric matrix $\mathbf{A}$ associated with the quadratic form

$$Q(x_1,x_2,x_3) = x_1^2 - 2x_2^2 - 7x_3^2 - 10x_1x_3 + 16x_2x_3.$$

Use the Principal Minor Test to establish whether Q is positive/negative (semi-)definite, or indefinite.

Answer:
Write Q as:

$$Q = (x_1, x_2, x_3) \begin{pmatrix} 1 & 0 & -5 \\ 0 & -2 & 8 \\ -5 & 8 & -7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{1}$$

Principal minor test:

$$\det(A) = \begin{vmatrix} 1 & 0 & -5 \\ 0 & -2 & 8 \\ -5 & 8 & -7 \end{vmatrix} = 1(14 - 64) - 5(-10) = -50 + 50 = 0. \tag{2}$$

Since $\det(A) = 0$ all principal minors are needed.

$$PM_2 : \begin{vmatrix} 1 & 0 \\ 0 & -2 \end{vmatrix} = -2 < 0, \quad \begin{vmatrix} 1 & -5 \\ -5 & -7 \end{vmatrix} = -7 - 25 = -32 < 0,$$
$$\begin{vmatrix} -2 & 8 \\ 8 & -7 \end{vmatrix} = 14 - 64 = -50 < 0. \tag{3}$$
$$PM_1 : |1| = 1 > 0, |-2| = -2 < 0, |-7| < 0. \tag{4}$$

Hence, ID case.

ii) Find the eigenvalues and unit eigenvectors of the matrix $\mathbf{A}$, and from the unit eigenvectors give the normal form of Q. Use this to confirm the result of part i).
Answer:
Eigenvalues of A, $\det(A - \lambda I)$:

$$\begin{vmatrix} 1-\lambda & 0 & -5 \\ 0 & -2-\lambda & 8 \\ -5 & 8 & -7-\lambda \end{vmatrix} = (1-\lambda)((2+\lambda)(7+\lambda) - 64) + 5(5(2+\lambda))$$
$$= (1-\lambda)(\lambda^2 + 9\lambda - 50) + 50 + 25\lambda = -\lambda(\lambda + 14)(\lambda - 6) = 0. \tag{5}$$

So $\lambda = 0, \lambda = -14, \lambda = 6$.
$\lambda = 0$:

$$\begin{pmatrix} 1 & 0 & -5 \\ 0 & -2 & 8 \\ -5 & 8 & -7 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0 \tag{6}$$

$v_1 - 5v_3 = 0 \Longrightarrow$ a) $v_1 = 5v_3$; $-2v_2 + 8v_3 = 0 \Longrightarrow$ b) $v_2 = 4v_3$. So $\mathbf{v} = (5, 4, 1)$ with length $\sqrt{5^2 + 4^2 + 1^2} = \sqrt{42}$ and $\mathbf{u}_1 = (5, 4, 1)/\sqrt{42}$.
$\lambda = -14$:

$$\begin{pmatrix} 15 & 0 & -5 \\ 0 & 12 & 8 \\ -5 & 8 & 7 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0 \tag{7}$$

$15v_1 - 5v_3 = 0 \Longrightarrow$ a) $v_1 = \frac{1}{3}v_3$; $12v_2 + 8v_3 = 0 \Longrightarrow$ b) $v_2 = -\frac{2}{3}v_3$. So $\mathbf{v} = (1, -2, 3)$ with length $\sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$ and $\mathbf{u}_2 = (1, -2, 3)/\sqrt{14}$.
$\lambda = 6$:

$$\begin{pmatrix} -5 & 0 & -5 \\ 0 & -8 & 8 \\ -5 & 8 & -13 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0. \tag{8}$$

So $\mathbf{v} = (1, -1, -1)$ with length $\sqrt{1+1+1} = \sqrt{3}$ and $\mathbf{u}_3 = (1, -1, -1)/\sqrt{3}$.
Thus,

$$\lambda = 0 : \tilde{x}_1 = (5x_1 + 4x_2 + x_3)/\sqrt{42} \tag{9}$$
$$\lambda = -14 : \tilde{x}_2 = (x_1 - 2x_2 + 3x_3)/\sqrt{14} \tag{10}$$
$$\lambda = 6 : \tilde{x}_3 = (x_1 - x_2 - x_3)/\sqrt{3} \tag{11}$$

and the canonical form is

$$\begin{aligned} Q =& 0\tilde{x_1}^2 - 14\tilde{x_2}^2 + 6\tilde{x_3}^2 \\ =& 0\left((5x_1 + 4x_2 + x_3)/\sqrt{42}\right)^2 - 14\left((x_1 - 2x_2 + 3x_3)/\sqrt{14}\right)^2 + 6\left((x_1 - x_2 - x_3)/\sqrt{3}\right)^2 \\ =& x_1^2 - 2x_2^2 - 7x_3^2 - 10x_1x_3 + 16x_2x_3 \end{aligned} \tag{12}$$

with the last expression as verification.

A3. A company produces and sells on two products x and y for which the demand functions are

$$x = 30 - 0.5P_x \ , \quad y = 25 - P_y,$$

with P_x and P_y being their prices. The combined production cost is

$$C = x^2 + kxy + y^2 + 10$$

for some constant k. Find (a) the profit maximising level of output for each product, (b) the profit maximising price of each product, and (c) the maximum profit, for a suitable k. (d) Determine for which values of k, by analysing the Hessian matrix, the profit is indeed maximised; are conditions on the prices met?

Answer:
The profit is given by the revenue, defined as prices $P_x = 60 - 2x, P_y = 25 - y$ times products (x, y), i.e. the revenue is $R = P_x x + P_y y$, minus the costs, as follows:

$$\Pi(x, y) = R(x, y) - C(x, y) = (60 - 2x)x + (25 - y)y - (x^2 + kxy + y^2 + 10). \tag{13}$$

Find the FOCs of $\Pi(x, y)$:

$$\Pi_x = 60 - 4x - 2x - ky = 60 - 6x - ky = 0 \Longrightarrow \text{a) } 6x + ky = 60 \tag{14}$$
$$\Pi_y = 25 - 2y - 2y - kx = 25 - 4y - kx = 0 \Longrightarrow \text{b) } kx + 4y = 25. \tag{15}$$

a) Combine k(a)-6(b) and 4(a)-k(b) to obtain the critical points (x^*, y^*) and profit maximising outputs:

$$x^* = \frac{240 - 25k}{24 - k^2}, \quad y^* = \frac{150 - 60k}{24 - k^2}. \tag{16}$$

The Hessian matrix reads/is:

$$H = \begin{pmatrix} \Pi_{xx} & \Pi_{xy} \\ \Pi_{xy} & \Pi_{yy} \end{pmatrix} = \begin{pmatrix} -6 & -k \\ -k & -4 \end{pmatrix}, \tag{17}$$

such that $\det(H) = 24 - k^2 > 0$; $LPM_1 = -6 < 0$; so for ND/maximum (alternating signs needed) $LPM_2 = \det(H) = 24 - k^2 > 0 \Longrightarrow k^2 < 24$.
b) Profit maximising prices are: $P_x^* = 60 - 2x^* = 60 + \frac{50k-480}{24-k^2}$, $P_y^* = 25 - y^* = 25 + \frac{60k-150}{24-k^2}$.

c) Maximising profit for $k = 2$ with $x^* = (240 - 50)/(24 - 4) = 9.5; y^* = (150 - 120)/(24 - 4) = 1.5; P_x = 60 - 19 = 41, P_y = 25 - 1.5 = 23.5$:

$$\begin{aligned}\Pi(x^* = 9.5, y^* = 1.5) =&(41 \times 9.5) \times 9.5 + (23.5 \times 1.5) \times 1.5 - (19.5^2 + 2 \times 9.5 \times 1.5 + 1.5^2 + 10)\\ =&293.75.\end{aligned}$$

d) In addition to $k < \sqrt{24}$, also $x^* > 0 \Longrightarrow k < 240/25; y^* > 0 \Longrightarrow k < 150/60 = 2.5$, so the condition $k < 2.5$ is most stringent. In addition prices need to be positive, i.e. $x < 30, y < 25$, which is the case at equilibrium.

A4. Cobb and Douglas (1928) defined the function with their name using data from the American economy from 1899 to 1922, relating an output function $Q = Q(K, L)$ to capital K and labour L as follows

$$Q(K, L) = pL^aK^b$$

with fitting coefficients p, a and b. Using modern least-square data techniques, Felipe and Adams (2005) found that $p = 0.8353, a = 0.807$ and $b = 0.233$.
a) Assuming a cost function $C(K, L) = w_LL+w_KK$ and assuming a stable economy in equilibrium, and given the 1899 (equilibrium) data with $Q^* = 100, K^* = 100, L^* = 100$ determine w_k, w_L in 1899 and
b) calculate the cost (as a formula in terms of Q^*, a, b, p and for the values found).
c) In 1922, the data are $Q^* = 240, K^* = 431$ and $L^* = 161$; determine w_K, w_L again and calculate the cost.
d) How well does the Cobb-Douglas function fit these data for Q (also give errors in percentages)?
e) Is the profit maximised at these two equilibria?

See wikipedia for the "Cobb-Douglas function" with links to Felipe and Adams (2005) and Cobb and Douglas (1928).

Answer:
a) The profit function is output minus costs:

$$\Pi(K, L) =Q(K, L) - C(K, L) = pL^aK^b - w_LL - w_KK. \tag{18}$$

FOC's are:

$$\Pi_K =bpK^{b-1}L^a - w_K = 0 \Longrightarrow w_K = \frac{bQ^*}{K^*} = b = 0.233. \tag{19}$$

$$\Pi_L =apK^bL^{a-1} - w_L = 0 \Longrightarrow w_L = \frac{aQ^*}{L^*} = a = 0.807. \tag{20}$$

b) Costs at equilibrium: $C(K^*, L^*) = w_KK^* + w_LL^* = Q^*(a + b) = 104$.
c) At 1922-equilibrium $w_K = \frac{bQ^*}{K^*} = 0.1297, w_L = \frac{aQ^*}{L^*} = 1.203$. Costs at 1922-equilibrium: $C(K^*, L^*) = w_KK^* + w_LL^* = Q^*(a + b) = 249.60$.
d) For the first equilibrium in 1899: $p(L^*)^a(K^*)^b = 100.43$ so that is a 0.43% error and for the second equilibrium in 1922: $p(L^*)^a(K^*)^b = 207.29$ so that is a $|207.29/240 - 1| = 13.63\%$ error.
e) Profit is maximal at the critical point when in both cases $0 < a < 1, 0 < b < 1$ such that $a+b < 1$ as derived as maximising conditions in class. The Hessian for general a, b reads/is:

$$H = \begin{pmatrix} b(b-1)pK^{b-2}L^a & abpK^{b-1}L^{a-1} \\ abpK^{b-1}L^{a-1} & b(b-1)pK^bL^{a-2} \end{pmatrix}. \tag{21}$$

Conditions for a maximum/profit are: $LPM_1 = b(b - 1)pK^{b-1}L^a < 0 \Longrightarrow 0 < a < 1$ and $LPM_2 = abp^2K^{2b-2}L^{2a-2}(1 - a - b) > 0 \Longrightarrow a + b < 1$, which when combined yields $0 < a < 1, 0 < b < 1$

such that $a + b \leq 1$. While $0 < a, b < 1$, here we have $a + b = 0.807 + 0.233 = 1.04$ such that the maximising conditions are *not met for these data.* Note that Cobb and Douglas (1928) imposed $a + b = 1$ and took $a = 3/4, b = 1/4$ and $p = 1.01$ as fit. *Comment:* Felipe and Adams (2005) discuss how the general approach can be salvaged by using more appropriate output functions and data fitting.