MATH2640 EXAMPLES 5 WORKSHOP PROBLEMS

B1) a) Maximise $f = x^2+y^2$ subject to $\frac{x^2}{9}+\frac{y^2}{25} \leq 1$.

Lagrangian: $L(x,y,\lambda) = x^2+y^2-\lambda\left(\frac{x^2}{9}+\frac{y^2}{25}-1\right)$

First-order conditions:

(i) $L_x = 2x - \frac{2}{9}\lambda x = 0$

(ii) $L_y = 2y - \frac{2}{25}\lambda y = 0$

(iii) $\lambda\left(\frac{x^2}{9}+\frac{y^2}{25}-1\right) = 0$ – complementary slackness condition.

inequalities: $\lambda \geq 0$, $\frac{x^2}{9}+\frac{y^2}{25} \leq 1$.

From (i) $\Rightarrow$ $x\left(1-\frac{1}{9}\lambda\right) = 0 \Rightarrow$ $x=0$ or $\lambda = 9$

From (ii) $\Rightarrow$ $y\left(1-\frac{1}{25}\lambda\right) = 0 \Rightarrow$ $y=0$ or $\lambda = 25$

Possibilities: $(0,0)$, $(x,0)$, $(0,y)$
as we cannot have simultaneously $x\neq 0$, $y \neq 0$

$(0,0)$ then from (iii) $\underline{\underline{\lambda = 0}}$

$(x,0)$ with $x \neq 0 \Rightarrow$ $\underline{\underline{\lambda = 9}}$ then from (iii)

$$\frac{x^2}{9}+\frac{y^2}{25} = \frac{x^2}{9} = 1 \Rightarrow x = \pm 3$$

$(0,y)$ with $y \neq 0 \Rightarrow$ $\underline{\underline{\lambda = 25}}$ then from (iii)

$$\frac{x^2}{9}+\frac{y^2}{25} = \frac{y^2}{25} = 1 \Rightarrow y = \pm 5.$$

PTO.

Thus we get the five candidates

$(0,0)_{\lambda=0}$, $(\pm 3,0)_{\lambda=9}$, $(0,\pm 5)_{\lambda=25}$

At the last four points the constraint is binding.

b) When the constraint is non-binding the crit. point lies in the interior of the constraint region.

$\Rightarrow$ use the usual Hessian

$$H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

$\det(H) = 4 > 0$, so $LPM_2 > 0$, $LPM_1 = 2 > 0$

H pos. def. , we have local minimum.

When the constraint is binding we need the bordered Hessian:

$$\nabla g = \left(\frac{2x}{9}, \frac{2y}{25}\right)$$

$$L_{xx} = 2\left(1-\frac{1}{9}\lambda\right), \quad L_{xy} = 0, \quad L_{yy} = 2\left(1-\frac{1}{25}\lambda\right)$$

$$\Rightarrow H_B = \begin{pmatrix} 0 & 2x/9 & 2y/25 \\ 2x/9 & 2\left(1-\frac{1}{9}\lambda\right) & 0 \\ 2y/25 & 0 & 2\left(1-\frac{1}{25}\lambda\right) \end{pmatrix}$$

$n=2$
$m=1$

We only need $\det(H_B) = LPM_3 = -\left(\frac{2x}{9}\right)^2 \cdot 2\left(1-\frac{1}{25}\lambda\right)$

$- \left(\frac{2y}{25}\right)^2 2\left(1-\frac{1}{9}\lambda\right)$

For $(\pm 3,0)_{\lambda=9}$: $LPM_3 = -\frac{32}{25}\left(\frac{2}{3}\right)^2 < 0$

For $(0,\pm 5)_{\lambda=25}$: $LPM_3 = \frac{32}{9}\left(\frac{2}{5}\right)^2 > 0$

PTO

So for $(\pm 3, 0)$ $\operatorname{sgn}(LPM_3) = (-1)^m$ H pos. def.
local minimum.

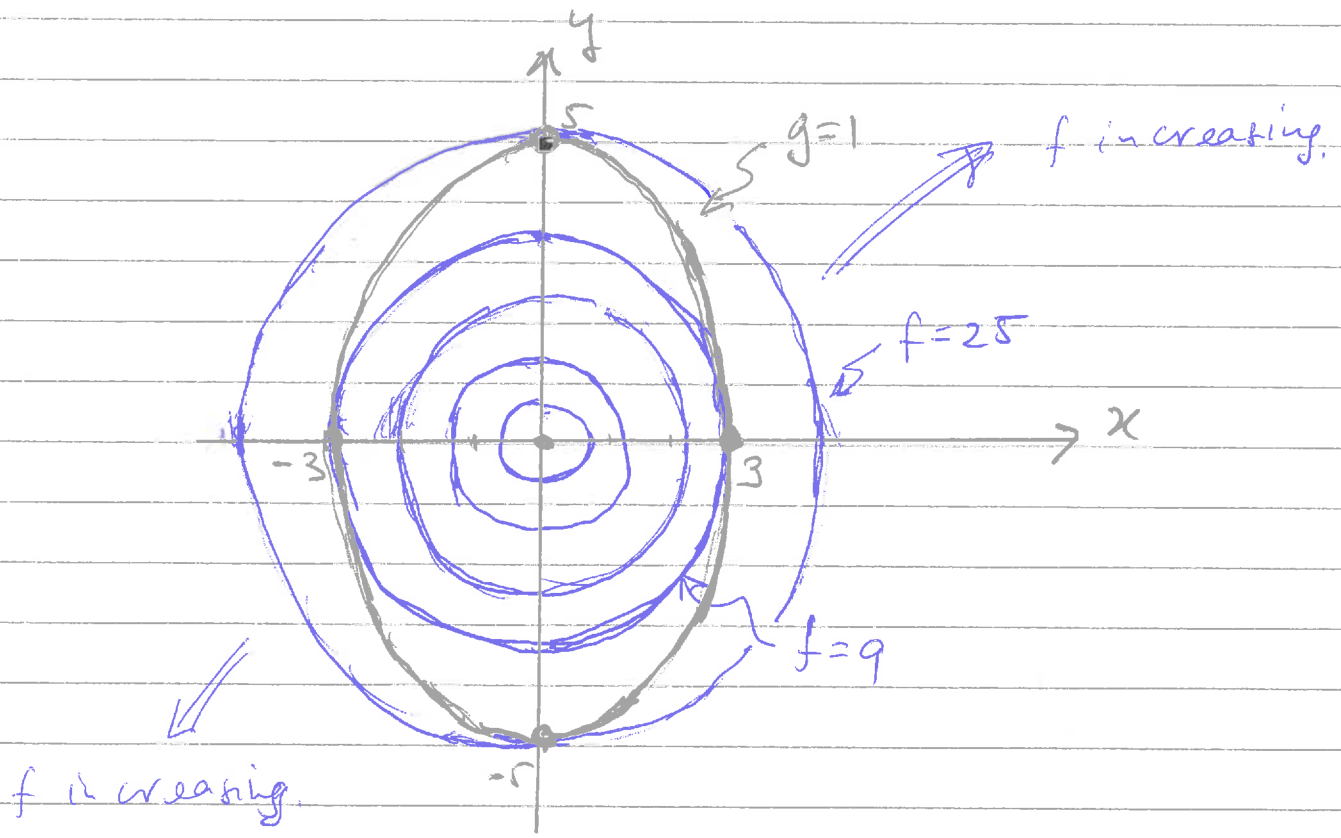
for $(0, \pm 5)$ $\operatorname{sgn}(LPM_3) = (-1)^n$ H neg. def
local maximum.

note: these are local max/min on the binding constraint set (not on entire xy-plane!)

c) The level sets $f = x^2 + y^2 = c$ constant form concentric circles in xy-plane, while the constraint set

$$\frac{x^2}{9} + \frac{y^2}{25} \leq 1$$ forms the interior

of an ellipse. Sketch:



So points $(0, \pm 5)$ are maximisers, with $f = 25$.

B2)

a) Maximise $f = 2y^2 - x$ subject to $g = x^2 + y^2 \leq 1$
and $x \geq 0$, $y \geq 0 \Rightarrow -x \leq 0, -y \leq 0$

Lagrangian:

$$L(x, y, \lambda, \mu_1, \mu_2) = 2y^2 - x - \lambda(x^2 + y^2 - 1) + \mu_1 x + \mu_2 y.$$

First order conditions:

$$L_x = -1 - 2\lambda x + \mu_1 = 0 \qquad (i)$$
$$L_y = 4y - 2\lambda y + \mu_2 = 0 \qquad (ii)$$

and the complementary slackness conditions:

$$\lambda(x^2 + y^2 - 1) = 0, \quad \mu_1 x = 0, \quad \mu_2 y = 0 \qquad (iii)$$

inequalities:

$$x^2 + y^2 \leq 1, \quad x \geq 0, \quad y \geq 0, \quad \lambda \geq 0, \mu_1 \geq 0, \mu_2 \geq 0.$$

b) From (i) $\Rightarrow$ $\mu_1 = 1 + 2\lambda x > 0$ so μ_1 nonzero.
(since λ, x positive).

Thus the constraint $\mu_1 x = 0$ is binding $\Rightarrow$ $\boxed{x = 0}$

From (iii) we have $\mu_2 y = 0 \Rightarrow \mu_2 = 0$ or $y = 0$

Let us consider these two cases:

$\underline{\mu_2 = 0}$: then from (ii) $\Rightarrow$ $2y(2 - \lambda) = 0$
$\Rightarrow y = 0$ (which we consider next)
or $\lambda = 2$. In latter case the constraint
$x^2 + y^2 \leq 1$ is binding $\Rightarrow$ $x^2 + y^2 = y^2 = 1$
$\Rightarrow$ $y = 1$ (since y positive)

PTO.

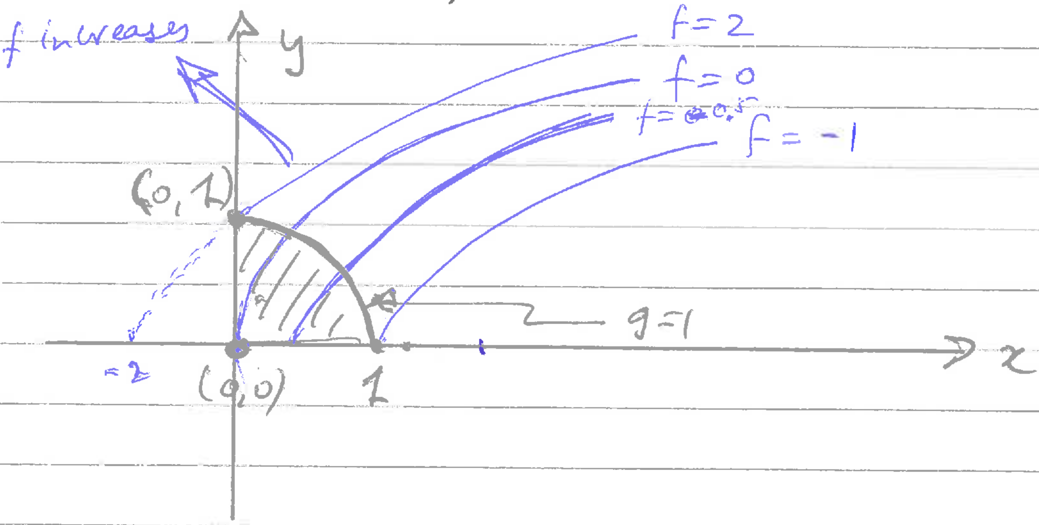
Thus, we get as solution candidate $(0,1)$ with
$\mu_1 = 1$, $\mu_2 = 0$, $\lambda = 2$.

$\underline{y=0}$: in this case we have $x=y=0$, then from (iii) $\lambda = 0$, from (ii) $\mu_2 = 0$, and $\mu_1 = 1$

$\Rightarrow$ point $(0,0)$ with $\mu_1 = 1, \mu_2 = 0, \lambda = 0$.

Evaluate f: $f(0,1) = 2$, $f(0,0) = 0$
so the point $(0,1)$ is the maximiser.

Sketch



The level sets $f = c \Rightarrow 2y^2 - x = c \Rightarrow x = 2y^2 - c$
are parabola

Since f increases in the direction indicated the point $(0,1)$ at the edge of the constraint region is the maximiser.

c) Kuhn-Tucker approach to the same problem.

The KT Lagrangian is:

$$L(x,y,\lambda) = 2y^2 - x - \lambda(x^2+y^2-1)$$

PTO

The KT relations are:

(i) $x\bar{L}_x = x(-1-2\lambda x) = 0$, $\bar{L}_x = -1-2\lambda x \leq 0$ (iv)

(ii) $y\bar{L}_y = y(4y-2\lambda y) = 0$, $\bar{L}_y = 4y-2\lambda y \leq 0$ (v)

(iii) $-\lambda \bar{L}_\lambda = \lambda(x^2+y^2-1) = 0$, $\bar{L}_\lambda = -(x^2+y^2-1) \geq 0$ (vi)

and in addition we have

$$x \geq 0, \quad y \geq 0, \quad \lambda \geq 0$$

The analysis now proceeds as follows.

From (i), since λ, x positive $\Rightarrow \underbrace{(1+2\lambda x)}_{\neq 0} x = 0$

$\Rightarrow \boxed{x=0}$

From (ii), we have $y^2(2-\lambda) = 0 \Rightarrow y=0$ or $\lambda=2$

If $\underline{\lambda=2}$, then from (iii) $x^2+y^2 = y^2 = 1 \Rightarrow y=1$

$\Rightarrow$ point $(0,1)$ with $\lambda = 2$.

If $\underline{y=0}$: we get point $(0,0)$ and from (iii) we have $\lambda=0$.

The inequalities (v)–(vi) are satisfied for both points

Once again $f(0,0) = 0$, $f(0,1) = 2$

so $(0,1)$ maximizer.

Thus, the KT approach gives the same result.

B3) Maximise $f = xyz + z$ subject to $x^2+y^2+z \leq 6$ and $x \geq 0$, $y \geq 0$, $z \geq 0$, using the KT approach.

KT Lagrangian:

$$\bar{L}(x,y,z,\lambda) = xyz + z - \lambda(x^2+y^2+z-6)$$

KT relations:

(i) $x\bar{L}_x = x(yz - 2\lambda x) = 0$, $\bar{L}_x = yz - 2\lambda x \leq 0$ (v)

(ii) $y\bar{L}_y = y(xz - 2\lambda y) = 0$, $\bar{L}_y = xz - 2\lambda y \leq 0$ (vi)

(iii) $z\bar{L}_z = z(xy + 1 - \lambda) = 0$, $\bar{L}_z = xy + 1 - \lambda \leq 0$ (vii)

(iv) $-\bar{L}_\lambda = \lambda(x^2+y^2+z-6) = 0$, $\bar{L}_\lambda = -(x^2+y^2+z-6) \geq 0$ (viii)

together with $x \geq 0$, $y \geq 0$, $z \geq 0$, $\lambda \geq 0$.

From (vii) $\Rightarrow$ $\lambda \geq 1 + xy \geq 1 > 0$ so $\lambda \neq 0$

$\Rightarrow$ constraint $x^2+y^2+z = 6$ is binding.

(i),(ii) $\Rightarrow$ $\lambda x^2 = \lambda y^2 \Rightarrow x^2 = y^2$, so $\boxed{x = y}$
(using that x, y both positive)

Then from (i) $\Rightarrow$ $x^2(z - 2\lambda) = 0 \Rightarrow x = 0$ or $z = 2\lambda$

If $\underline{x = 0}$ $\Rightarrow$ $y = 0$, from binding constraint (iv)
we have $\underline{z = 6}$

$\Rightarrow$ point $(0,0,6)$ with $\lambda = 1$ (from (iii))

Value of f: $f(0,0,6) = 6$. PTO

If $z = 2\lambda$, then from (iii) with $y = x$ we get

$$2\lambda(x^2+1-\lambda) = 0 \Rightarrow x^2+1 = \lambda \neq 0$$

while from (iv) $2x^2 + 2\lambda - 6 = 0$

$$\Rightarrow 2x^2 + 2(x^2+1) = 6 \Rightarrow 4x^2 = 4 \Rightarrow x = 1$$

while $\lambda = 2$.

Thus we get the point $(1,1,4)$ with $\lambda = 2$

and value $f(1,1,4) = 8$.

So latter point is the maximiser, and the constraint is binding for this point.

B4) The revenue is given by $R = 3y^{1/6}a^{1/6}$ as a function of y and a. The profit is given by

$$\Pi = R - C - a = 3y^{1/6}a^{1/6} - y - a.$$

a) We want first to maximise the revenue $R(y,a)$ subject to the constraint on the profit $\Pi \geq 1$

$$\Rightarrow \quad -\Pi \leq -1$$

KT Lagrangian $\bar{L} = R - \lambda(-\Pi + 1) =$

$$= 3y^{1/6}a^{1/6} - \lambda(1 + y + a - 3y^{1/6}a^{1/6})$$

$$= 3(1+\lambda)y^{1/6}a^{1/6} - \lambda(1+y+a)$$

KT relations:

$$y\bar{L}_y = y\left(\tfrac{1}{2}(1+\lambda)y^{-5/6}a^{1/6} - \lambda\right) = 0 \qquad \text{(i)}$$

$$a\bar{L}_a = a\left(\tfrac{1}{2}(1+\lambda)y^{1/6}a^{-5/6} - \lambda\right) = 0 \qquad \text{(ii)}$$

$$\lambda\bar{L}_\lambda = \lambda\left(3y^{1/6}a^{1/6} - 1 - y - a\right) = 0 \qquad \text{(iii)}$$

with inequalities:

$$\bar{L}_y = \tfrac{1}{2}(1+\lambda)y^{-5/6}a^{1/6} - \lambda \leq 0$$

$$\bar{L}_a = \tfrac{1}{2}(1+\lambda)y^{1/6}a^{-5/6} - \lambda \leq 0$$

$$\bar{L}_\lambda = 3y^{1/6}a^{1/6} - 1 - y - a \geq 0$$

and $a \geq 0$, $y \geq 0$, $\lambda \geq 0$.

PTO.

From (i), (ii) we have

$$\tfrac{1}{2}(1+\lambda)\, y^{1/6} a^{1/6} = \lambda y = \lambda a \qquad (*)$$

while $\Pi \geq 1$ implies $y \neq 0$ and $a \neq 0$. In fact,

if $y = 0 \Rightarrow \Pi = -a \leq 0$
$a = 0 \Rightarrow \Pi = -y \leq 0$ while $\Pi \geq 1$ by constraint.

The the above relation tells us that $\lambda \neq 0$ otherwise $(*)$ would lead to a contradiction.

Thus, the constraint is binding $\Rightarrow \Pi = 1$.
At the same time, from $(*)$ we have $\boxed{y = a}$

Then, the constraint $\Pi = 1$ leads to:

$3y^{1/3} - 1 - 2y = 0$, The latter factorizes:

$$3y^{1/3} - 1 - 2y = (-y^{1/3} + 1)(2y^{2/3} + 2y^{1/3} - 1) = 0$$

$$\Rightarrow y^{1/3} = 1 \quad \text{or} \quad y^{1/3} = \frac{-2 \pm \sqrt{4 + 4\cdot 2}}{4} = \frac{-1 \pm \sqrt{3}}{2}$$

Thus, the positive roots for y are:

$y^{1/3} = 1 \Rightarrow y = 1$ ~~and~~ or $y^{1/3} = \dfrac{-1+\sqrt{3}}{2}$.

$\boxed{y = 1} \Rightarrow \lambda = 1$ (from (i) or (ii)) and $\underline{a = 1}$.
then the revenu $R(1,1) = 3$

$\boxed{y^{1/3} = \dfrac{-1+\sqrt{3}}{2}}$ However this gives, from $(*)$,
$1 + \lambda = 2\lambda y^{2/3} = \lambda\cdot(2 - \sqrt{3}) \Rightarrow \lambda < 0$
so this is not admissable!

PTO.

Thus, the ~~ma~~maximiser is : $\boxed{y = a = 1, \ \lambda = 1, \ R = 3}$

profit $\Pi = 1$; constraint is binding.

b) Change of strategy. New manager wants to maximise profit without any constraints.

$\Rightarrow$ maximise $\Pi = 3 y^{1/6} a^{1/6} - y - a.$

First order conditions:

$$\Pi_a = \frac{1}{2} y^{1/6} a^{-5/6} - 1 = 0 \qquad (i)$$

$$\Pi_y = \frac{1}{2} y^{-5/6} a^{1/6} - 1 = 0 \qquad (ii)$$

(i),(ii) $\Rightarrow$ $R = 6a = 6y \Rightarrow \boxed{a = y}$

Furthermore from (i) $\Rightarrow$ $\frac{1}{2} y^{-2/3} = 1 \Rightarrow y = \frac{1}{2\sqrt{2}} = a$

The profit is then computed as:

$$\Pi = 3y^{1/3} - 2y = \frac{3}{\sqrt{2}} - \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \underline{\sqrt{2}}$$

revenue

$R = 3y^{1/3} = \frac{3}{\sqrt{2}}$ (while $R = 3$ under the previous strategy!)

The general theory of Cobb-Douglas functions tells us that the critical point $\left(\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$ is a maximum.