MATH2640 Introduction to Optimisation

Example Sheet 2

Please hand the assessed questions by Thursday 31^{st} October 2019, 5 pm

Gradients & tangent plane, multivariable Taylor series, first order conditions, quadratic forms. Based on material in Lectures 5 to 9

Assessed Questions

A1. The economy in Yorkshire is in equilibrium with the system of equations

$$2xz + xy + z - 2\sqrt{z} = 11 \quad \text{and} \quad xyz = 6.$$

One solution of this set of equations is $x = 3, y = 2, z = 1$, and the economy of Yorkshire is in equilibrium at this point. Suppose the Yorkshire government discovers that the variables z can be controlled by a simple decree. If the Yorkshire government decides to raise z to 1.1 estimate the change of x and y. Why is it an estimate?

A2.
(i) In what direction should one move from the point (1,1,2) to increase $f(x, y, z) = e^{\frac{1}{2}xyz}$ most rapidly? Present your answer as a unit vector.
(ii) Find the unit normal vector to the surface $g(x, y, z) = \cos(x + y^2 + z) = 0$ at the point $(2, \sqrt{\pi/2}, -2)$, and give an equation for the tangent plane to the surface at that point.

A3. The functions $f(x, y) = \cosh(2x^2 + y^3)$ and $g(x, y) = \sinh(2x^2 + y^3)$ are expanded as a Taylor series about the point $(x, y) = (2, -2)$.
(i) Find the gradient vector and the Hessian matrix of f at this point, and hence give the Taylor series for $f(2 + h, -2 + k)$ up to linear and quadratic terms in h and k.
(ii) Find the Taylor expansion of the function $g(x, y)$ at this point.

A4. Use the first-order conditions to find all the critical points of

(i) $f(x, y) = 3x^4 + 6x^2y - 2y^3$,

(ii) $g(x, y, z) = -6x^2 + 3xy + 3y^2 + 9yz + z^3$.

A5. Using the results about *leading principal minors* and/or *principal minors*, determine the sign properties (definite, semidefinite, indefinite) of the following quadratic forms Q in three variables.

(i) $Q(x, y, z) = -2x^2 - 5y^2 - 9z^2 + 2xy + 6xz + 6yz$.

(ii) $Q(x_1, x_2, x_3) = 2x_1^2 + 3x_2^2 + 7x_3^2 + 2x_1x_2 + 2x_1x_3 + 8x_2x_3$.

P.T.O.

Further Questions for Workshop Practice

B1. A simple economy has three variables, x, y and z. The economy is in equilibrium when

$$xy + 2yz + z - 4\sqrt{z} = 7, \quad \text{and} \quad xyz = 6.$$

Currently the economy is in equilibrium at $x = 3$, $y = 2$ and $z = 1$. The prime minister wants to increase z slightly. Assuming the economy stays in equilibrium, find whether each of x and y will increase or decrease.

B2.
(i) In what direction should one move from the point (2,3) to increase $4xy^2$ most rapidly? Present your answer as a unit vector.
(ii) In what direction should one move from the point (0,3) to decrease ye^{2x} most rapidly? Present your answer as a unit vector.
(iii) Find the unit normal vector to the surface $g(x, y, z) = x^2y + 2xy^2 - 3z^2 = 0$ at the point $(1, 1, 1)$.

B3. Find the equation of the tangent plane to the surface $z = f(x, y) = x^2 + y^3$ at the point $(1, 1, 2)$ in the form $ax + by + cz = d$.

B4. The function $z = \sin x \sin y$ is expanded as a Taylor series about the point $x = \frac{\pi}{4}$, $y = \frac{\pi}{4}$. Find the gradient vector and the Hessian matrix at this point.
Find the terms in $\sin(\frac{\pi}{4} + h)\sin(\frac{\pi}{4} + k)$ that are

(i) Linear in h and k

(ii) Quadratic in h and k.

B5. Use the first order conditions to find all the critical points of

(i) $x^3 + y^3 - 9xy$,

(ii) $x^4 + 2x^2y - 6y^3$,

(iii) $x^3 + 4xy + y^2 + 2yz - z^2$.

B6. Using the results about *leading principal minors* and/or *principal minors*, determine the sign properties (definite, semidefinite, indefinite) of the following quadratic forms Q in three variables.

(i) $Q(x, y, z) = 2(xy - xz + yz)$.

(ii) $Q(x_1, x_2, x_3) = 2x_1^2 + 5x_2^2 + x_3^2 - 4x_1x_2 - 2x_1x_3$.

(iii) $Q(x_1, x_2, x_3) = -x_1^2 - 2x_2^2 - 4x_3^2 + 2x_1x_2 + 4x_2x_3$.