MATH2640 Introduction to Optimisation

3. Quadratic forms and Eigenvalues, Economic applications (production functions): summary

3(A) Quadratic forms and eigenvalues

(i) Eigenvalues are the solutions, λ, of the matrix equation

$$A\mathbf{v} = \lambda\mathbf{v} , \quad \mathbf{v} \neq \mathbf{0} . \tag{3.1}$$

In general, an $n \times n$ matrix has n eigenvalues, but they may be complex valued, and some eigenvalues may coincide. The non-zero vectors $\mathbf{v}$ satisfying this equation are called the eigenvectors. For each eigenvalue λ there is, in fact, an infinity of corresponding eigenvectors $\mathbf{v}$, since any non-zero multiple of an eigenvector is again an eigenvector. Therefore, we sometimes want to restrict ourselves to unit (or normalised) eigenvectors: those with length $\|\mathbf{v}\| = 1$.

(ii) 2×2 case. The quadratic form associated with the symmetric matrix

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad \text{is} \quad Q(x,y) = (x,y)\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = ax^2 + 2bxy + cy^2.$$

To find the eigenvalues, we write equation (3.1) as $(A - \lambda I)\mathbf{x} = 0$, where I is the unit (identity) matrix. Then for this to have nontrivial solutions,

$$\det(A - \lambda I) = \begin{vmatrix} a-\lambda & b \\ b & c-\lambda \end{vmatrix} = 0, \quad \text{so} \quad \lambda^2 - (a+c)\lambda + ac - b^2 = 0$$

is the quadratic equation whose solutions are the two eigenvalues of the matrix A. These eigenvalues are always real for a symmetric matrix. (Prove this!)

(iii) **Normalising eigenvectors**
To find the eigenvector corresponding to a particular eigenvalue, solve $(A - \lambda I)\mathbf{x} = 0$ for that value of λ. You can only solve these equations up to an arbitrary constant k. Example:

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

has eigenvalues $\lambda = 2$ and $\lambda = 4$. To find the eigenvector corresponding to $\lambda = 2$, solve

$$\begin{matrix} x_1 + x_2 = 0 \\ x_1 + x_2 = 0 \end{matrix} \quad \text{giving} \quad (k, -k)$$

as the eigenvector. To normalise this, choose k so that $(k, -k)$ has unit length, $2k^2 = 1$, so the normalised eigenvector is $(1/\sqrt{2}, -1/\sqrt{2})$. The normalised eigenvector corresponding to $\lambda = 4$ is $(1/\sqrt{2}, 1/\sqrt{2})$.

(iv) The same methods work for finding eigenvalues of an $n \times n$ matrix: the characteristic equation

$$\det(A - \lambda I) = (-\lambda)^n + \operatorname{tr}(A)(-\lambda)^{n-1} + \cdots + \det(A) = 0 ,$$

(where $\operatorname{tr}(A)$ denotes the trace of the matrix, i.e., the sum of the diagonal entries), is now an n^{th}–order polynomial equation which, in principle, has n (possibly complex valued, possibly coinciding) roots, which are the eigenvalues $\lambda_1, \ldots, \lambda_n$. Having found the eigenvalues, the next step is to find the (unit) eigenvector $\mathbf{v}_i$ corresponding to each eigenvalue λ_i, by solving the homogeneous linear system $(A - \lambda_i I)\mathbf{v}_i = 0$. The latter can be done for instance by elementary row manipulations (e.g, see the first year Linear Algebra module) and will always have a nontrivial solution because of the condition on the λ_i.

Quadratic forms $Q(\mathbf{x}) = \mathbf{x}^T A\,\mathbf{x}$, where $\mathbf{x} = (x_1, \ldots, x_n)$, are always associated with *symmetric* matrices $A = (a_{i,j})_{i,j=1,\ldots,n}$, for which $A = A^T$ (where the symbol T denotes matrix transposition) which means $a_{ij} = a_{ji}$. Such matrices have the following remarkable properties (prove these!):

- all eigenvalues of a (real-valued) symmetric matrix A are real (i.e., are real numbers);
- eigenvectors $\mathbf{v}_i$ of a symmetric matrix for distinct eigenvalues λ_i and λ_j are orthogonal to each other, i.e., $\mathbf{v_i} \cdot \mathbf{v_j} = 0$ for $\lambda_i \neq \lambda_j$;

- a symmetric matrix A can be "diagonalised" by an orthogonal matrix O, i.e., written in the form $A = O\Lambda O^T$ where Λ is the diagonal matrix of eigenvalues.

In the latter case, an *orthogonal* matrix is a matrix possessing the property that $O\,O^T = O^T O = I$, and hence it is invertible with $O^{-1} = O^T$. O can be constructed by creating the matrix with as columns the *normalised* (column) eigenvectors: $O = (\mathbf{v}_1 \mathbf{v}_2 \ldots \mathbf{v}_n)$.

(v) **Normal forms**

When the normalised eigenvectors $\mathbf{v}_i$, $i = 1, \ldots, n$, corresponding to the n eigenvalues $\lambda_1, \ldots, \lambda_n$, are found, say with components $\mathbf{v_i} = (v_1^{(i)}, v_2^{(i)}, \cdots v_n^{(i)})$, then the quadratic form $Q(\mathbf{x})$ can be re-written, in terms of some new variables $\tilde{\mathbf{x}} = (\tilde{x}_1, \ldots, \tilde{x}_n)$, as a sum of squares. In fact, introducing the vector of the new variables as $\tilde{\mathbf{x}} = O^T \mathbf{x}$, we have from the diagonalisation formula (given above):

$$Q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x} = \mathbf{x}^T (O\Lambda O^T)\mathbf{x} = (\mathbf{x}^T O)\Lambda(O^T \mathbf{x}) = \tilde{\mathbf{x}}^T \Lambda \tilde{\mathbf{x}} \ ,$$

using that $\mathbf{x}^T O = (O^T \mathbf{x})^T$. Noting that $\mathbf{x}^T O = \mathbf{x}^T(\mathbf{v}_1, \ldots, \mathbf{v}_n) = (\mathbf{x}\cdot\mathbf{v}_1, \ldots, \mathbf{x}\cdot\mathbf{v}_n)$ we get the new variables in terms of dot products of the form $\tilde{x}_i = \mathbf{x}\cdot\mathbf{v}_i$, $i = 1, \ldots, n$. Thus, the quadratic form can be rewritten as a sum of squares (with coefficients being the eigenvalues) of dot products; in components:

$$Q(x_1, x_2, \ldots) = \lambda_1(x_1 v_1^{(1)} + x_2 v_2^{(1)} + \cdots)^2 + \lambda_2(x_1 v_1^{(2)} + x_2 v_2^{(2)} + \cdots)^2 + \cdots \ ,$$

which is said to be the *normal form* of the quadratic form Q. For the above example in part (iii), we get:

$$Q = 3x_1^2 + 2x_1x_2 + 3x_2^2 = 2(\frac{x_1}{\sqrt{2}} - \frac{x_2}{\sqrt{2}})^2 + 4(\frac{x_1}{\sqrt{2}} + \frac{x_2}{\sqrt{2}})^2 = 2\tilde{x}^2 + 4\tilde{x}_2^2.$$

Verify this formula by multiplying it out and comparing it to the original form of Q in this example. It is now obvious that the fact that the matrix A has positive eigenvalues implies that Q is positive definite. In general, writing a quadratic form as a sum of squares in some new variables $\tilde{x}_i$ makes the sign behaviour of these quadratic forms self-evident.

(vi) **Classification rules**

The rules for classifying quadratic forms using the eigenvalues is as follows.

a) If all eigenvalues are > 0, Q is positive definite (PD)
b) If all eigenvalues are < 0, Q is negative definite (ND)
c) If some eigenvalues are ≤ 0 and some are ≥ 0, Q is indefinite (ID).
d) If all eigenvalues are ≥ 0, Q is positive semi-definite (PSD)
e) If all eigenvalues are ≤ 0, Q is negative semi-definite (NSD)

Note that although this classification by means of eigenvalues is more transparent than the principal minor test (see Handout **2**), it requires the finding of the eigenvalues of the matrix A, which in turn requires the solving of a $\mathrm{n^{th}}$–order polynomial equation, whereas the principal minor test only requires the computation of determinants, which in general is much easier to do (and requires less computer time when done numerically).

3(B) **Unconstrained Optimisation in Economics.**

(i) **Production functions**

A firm produces Q items per year, which sell at a price p per item. The Revenue, $R = pQ$. Q depends on quantities x_1, x_2, etc. which are quantities such as the number of employees, amount spent on equipment, and so on. The vector $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ is called the input bundle vector. The Cost function measures how much a firm spends on production, and this will also be a function of x_1, x_2 and so on. The profit,

$$\Pi = R - C = pQ(x_1, x_2, \cdots, x_n) - C(x_1, x_2, \cdots, x_n).$$

To maximise the profit, we find the maximum of Π as a function of the inputs $x_1, \cdots, x_n$, leading to

$$p\frac{\partial Q}{\partial x_i} = \frac{\partial C}{\partial x_i}, \quad i = 1, \ldots, n.$$

Problems where a firm produces several different products can also lead to optimisation problems.

(ii) **The discriminating monopolist.**

A monopolist produces two products at a rate Q_1 and Q_2. The price obtained is not constant, but reduces if the monopolist floods the market, so in a linear model $p_1 = a_1 - b_1 Q_1$ and $p_2 = a_2 - b_2 Q_2$ are the prices for the two products. The Cost function is then taken as $C = c_1 Q_1 + c_2 Q_2$, being simply proportional to the quantity produced. Here, the a_i, b_i, c_i for $i = 1, 2$ are constants. The profit

$$\Pi = R - C = p_1 Q_1 + p_2 Q_2 - c_1 Q_1 - c_2 Q_2 = (a_1 - c_1)Q_1 + (a_2 - c_2)Q_2 - b_1 Q_1^2 - b_2 Q_2^2,$$

so there is a best choice of Q_1 and Q_2 to maximise profit, when

$$\Pi_{Q_1} = (a_1 - c_1) - 2b_1 Q_1 = 0, \quad \text{and} \quad \Pi_{Q_2} = (a_2 - c_2) - 2b_2 Q_2 = 0.$$

Inspection of the associated Hessian shows that Π has a maximum here provided b_1 and b_2 are positive. (Why?) We then also require $a_1 \geq c_1$ and $a_2 \geq c_2$, such that Q_1, Q_2 are positive at the stationary points. The monopolist then has a *profit maximising strategy* allowing discrimination between the amounts of the two products to be produced. The same approach can be used when the cost function and the pricing are more complicated functions of the Q_i.

(iii) **The Cobb-Douglas production function.**

A commonly used model in microeconomics is the Cobb-Douglas production function

$$Q(x_1, x_2) = x_1^a x_2^b, \quad \text{where} \quad a > 0, \quad b > 0.$$

The input bundle here is then just a two-dimensional vector, and we assume here the price p is independent of Q, unlike in the monopolist problem. If the Cost is linear in the input bundle, $C = w_1 x_1 + w_2 x_2$, then the Profit

$$\Pi = R - C = pQ - C = p x_1^a x_2^b - w_1 x_1 - w_2 x_2$$

and the conditions for a stationary point are

$$\Pi_{x_1} = ap x_1^{a-1} x_2^b = w_1, \quad \Pi_{x_2} = bp x_1^a x_2^{b-1} = w_2$$

giving positive critical values $x_1^* = apQ^*/w_1$ and $x_2^* = bpQ^*/w_2$, where Q^* is the production function at the critical value. (Check these calculations!)

We want to know when this stationary value is a maximum, so we must examine the Hessian

$$H = \begin{pmatrix} a(a-1)p x_1^{a-2} x_2^b & abp x_1^{a-1} x_2^{b-1} \\ abp x_1^{a-1} x_2^{b-1} & b(b-1)p x_1^a x_2^{b-2} \end{pmatrix},$$

which is negative definite provided

$$LPM_1 = a(a-1)p x_1^{a-2} x_2^b < 0 \quad \text{and} \quad LPM_2 = ab(1-a-b)p^2 x_1^{2a-2} x_2^{2b-2} > 0, \tag{1}$$

which requires

$$0 < a < 1 \quad \text{and} \quad 0 < b < 1 \quad \text{and} \quad a + b < 1.$$

If these conditions are satisfied, then the Cobb-Douglas production function gives a *profit maximising strategy*. If they are not satisfied, the profit increases indefinitely as x_1 and x_2 get very large. This is possible economic behaviour: if there is no profit maximising strategy, all it means is that the bigger the firm the more profit it makes, which is of course entirely possible.

Cobb and Douglas (2005) based their analysis and function on data from the USA economy from 1899-1922. Filipe and Adams (2005) revisited these data using with modern least-squares techniques, revealing some shortcomings in the data fitting of Cobb and Douglas from 1928, but also offer some alternatives.

References

`https://en.wikipedia.org/wiki/CobbDouglas_production_function`

Cobb, C. W., Douglas, P. H. 1928: A theory of production. *American Economic Review* **18** (Supplement): 139–165.

Filipe, J., Adams, F.G. 2005: The estimation of the Cobb-Douglas function: a retrospective view. *Eastern Economic Journal* **31**, 427–445.

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