Q1)

a) $g(x,y,z) = xy^2 + x^3y - 2z^3 = 0$ 1

Note $g(1,1,1) = 0$, so $(1,1,1)$ point of the surface.

Gradient: $\nabla g = (y^2 + 3x^2y,\ 2xy + x^3,\ -6z^2)$ 1

At $(1,1,1)$: $\nabla g(1,1,1) = (4, 3, -6)$

$\Rightarrow \|\nabla g\| = \sqrt{4^2 + 3^2 + (-6)^2} = \sqrt{61}$ 1

$\Rightarrow$ unit normal vector $\underline{n} = \left(\frac{4}{\sqrt{61}}, \frac{3}{\sqrt{61}}, \frac{-6}{\sqrt{61}}\right)$

Equation for the tangent plane:

$$(4, 3, -6) \cdot (x-1, y-1, z-1) = 0$$ 2

$$\Rightarrow 4(x-1) + 3(y-1) - 6(z-1) = 0 \Rightarrow \boxed{4x + 3y - 6z = 1}$$

g decreases most rapidly in the direction opposite to $\underline{\nabla g}$, i.e. in the direction $-\underline{n} = \left(-\frac{4}{\sqrt{61}}, \frac{-3}{\sqrt{61}}, \frac{6}{\sqrt{61}}\right)$

Rate of increase of g in the direction $(1,0,0)$ is given by 1

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$$(1,0,0) \cdot \nabla g(1,1,1) = (1,0,0) \cdot (4,3,-6) = 4$$ 1

b) Let $f(x,y) = x^2y + xy^3$ subject to $y^2 + yx^2 = x^3$.

partial derivatives $f_x = 2xy + y^3$, $f_y = x^2 + 3xy^2$. 1

From $y^2 + yx^2 = x^3$ taking differentials we have

PTO

$$2y\,dy + x^2 dy + 2yx\,dx = 3x^2 dx$$ 1

$\Rightarrow \quad (2y + x^2)\,dy = (3x^2 - 2yx)\,dx$, leading to

$$\frac{dy}{dx} = \frac{3x^2 - 2yx}{2y + x^2} \quad \text{and} \quad \frac{dx}{dy} = \frac{2y + x^2}{3x^2 - 2yx}.$$ 1

Then the total derivatives,

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 2xy + y^3 + (x^2 + 3xy^2)\,\frac{3x^2 - 2yx}{2y + x^2}$$ 1

and

$$\frac{df}{dy} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial x}\frac{dx}{dy} = x^2 + 3xy^2 + (2xy + y^3)\,\frac{2y + x^2}{3x^2 - 2yx}.$$ 1

~~The total derivative $\frac{df}{dx}$ expresses the derivative of f w.r.t. x when f is considered as the function of one variable obtained by substituting y(x) into the 2-variable function, i.e. f(x, y(x)), if we could solve y from the second relation.~~ 2

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~~c) $f(x,y,z) = x^2 + xy + y^2 + 3xz + z^3$.~~
~~Find the critical points:~~

$$\cancel{\begin{cases} f_x = 2x + y + 3z = 0 & (i) \\ f_y = x + 2y = 0 & (ii) \\ f_z = 3x + 3z^2 = 0 & (iii) \end{cases}}$$ 1

PTO

~~From (ii) $\boxed{x = -2y}$, then from (i) →~~

$$\cancel{2x + y + 3z = 8y + y + 3z = 9y + 3z = 0 \Rightarrow \boxed{z = -3y}}$$

PTO.

(4)

e) $z(x,y)$ defined through $\boxed{z^2x - yz + 2xy = 4}$

$$\partial_x(z^2x - yz + 2xy) = 2zz_x x + z^2 - yz_x + 2y = 0 \qquad (*)$$

$$\Rightarrow (2zx - y)\,z_x + (z^2 + 2y) = 0 \Rightarrow \boxed{z_x = -\frac{z^2 + 2y}{2zx - y}}$$

2

$$\partial_y(z^2x - yz + 2xy) = 2zz_y x - z - yz_y + 2x = 0$$

$$\Rightarrow (2zx - y)\,z_y + (2x - z) = 0 \Rightarrow \boxed{z_y = -\frac{2x - z}{2zx - y}}$$

z_{xx} from $(*)$:

2

$$\partial_x(2zz_x x + z^2 - yz_x + 2y) =$$

$$= 2z_x^2 x + 2zz_{xx}x + 2zz_x + 2zz_x - yz_{xx} = 0$$

$$\Rightarrow (2zx - y)\,z_{xx} + 2xz_x^2 + 4zz_x = 0$$

$$\Rightarrow z_{xx} = -\frac{2x}{2zx - y}z_x^2 - \frac{4z}{2zx - y}z_x$$

$$\boxed{z_{xx} = -\frac{2x(z^2 + 2y)^2}{(2zx - y)^3} + \frac{4z(z^2 + 2y)}{(2xz - y)^2}}$$

2

If $x = y = 1 \Rightarrow z^2 - z + 2 = 4 \Rightarrow (z^2 - z - 2) = 0$

$$\Rightarrow (z - 2)(z + 1) = 0 \Rightarrow z = 2 \text{ or } z = -1$$

So $z = -1$ is the other value.

PTO

⑤

At $x=y=1$, $z=2$ we have:

$$z_x = -\frac{6}{3} = -2\ ,\quad z_y = 0\ ,\quad z_{xx} = -\frac{2}{3}z_x^2 - \frac{8}{3}z_x$$

$$= -\frac{8}{3} + \frac{16}{3} = \frac{8}{3}.$$ 2

$\frac{8}{20}$

b) products x, y, prices are $P_x = 30 - x$, $P_y = 50 - 2y$

$\Rightarrow$ revenue: $R = xP_x + yP_y = x(30-x) + y(50-2y)$ 1

Profit: $\Pi = R - C = x(30-x) + y(50-2y) - (x^2+2xy+y^2+10)$

$\Rightarrow \Pi(x,y) = 30x - x^2 + 50y - 2y^2 - x^2 - 2xy - y^2 - 10$ 1

$= 30x - 2x^2 + 50y - 3y^2 - 2xy - 10$

First-order relations

$$\begin{cases} \Pi_x = 30 - 4x - 2y = 0 & \Rightarrow\ 4x + 2y = 30 \\ \Pi_y = 50 - 6y - 2x = 0 & \Rightarrow\ 2x + 6y = 50 \end{cases}$$ 1

$\Rightarrow y = 7,\ x = 4.$ profit maximizing output.

1

Profit maximising prices are

$P_x = 30 - 4 = 26$, $P_y = 50 - 14 = 36$

Maximum profit $\Pi(4,7) = 120 - 32 + 350 - 147 - 56 - 10$

$= 225$ 1

Hessian:

~~Π_{xx} ... $-2y$~~, ~~Π_{gy}~~

$\Pi_{xx} = -4$ $\quad \Pi_{xy} = -2$, $\Pi_{yy} = -6$

PTO

Q2

(a) Consider the quadratic form

$$Q(x_1,x_2,x_3) = 3x_1^2 + \frac{3}{2}x_2^2 + \frac{11}{2}x_3^2 + 4x_1x_2 + 4x_1x_3 + x_2x_3.$$

$$= (x_1, x_2, x_3)\begin{pmatrix} 3 & 2 & 2 \\ 2 & 3/2 & 1/2 \\ 2 & 1/2 & 11/2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underline{x}^T A \underline{x}.$$

$\Rightarrow$ symmetric matrix $A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3/2 & 1/2 \\ 2 & 1/2 & 11/2 \end{pmatrix}$ 1

Eigenvalues:

$$\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 2 & 2 \\ 2 & 3/2-\lambda & 1/2 \\ 2 & 1/2 & \frac{11}{2}-\lambda \end{vmatrix} = \begin{vmatrix} 3-\lambda & 2 & 0 \\ 2 & \frac{3}{2}-\lambda & \lambda-1 \\ 2 & 1/2 & 5-\lambda \end{vmatrix}$$

$$= (3-\lambda)\left[\left(\tfrac{3}{2}-\lambda\right)(5-\lambda) - \tfrac{1}{2}(\lambda-1)\right] - 2\left[2(5-\lambda) - 2(\lambda-1)\right]$$

$$= (3-\lambda)\left(\lambda^2 - \tfrac{13}{2}\lambda + \tfrac{15}{2} - \tfrac{1}{2}\lambda + \tfrac{1}{2}\right) - 4(6-2\lambda)$$

$$= (3-\lambda)\left[\lambda^2 - 7\lambda + 8 - 8\right] = (3-\lambda)\lambda(\lambda-7).$$

$\Rightarrow$ Eigenvalues $\lambda = 3$, $\lambda = 0$, $\lambda = 7$. 2

Eigenvectors:

$\boxed{\lambda = 3}$ $\quad (A-3I)\underline{v} = \begin{pmatrix} 0 & 2 & 2 \\ 2 & -3/2 & 1/2 \\ 2 & 1/2 & 5/2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Gauss elimination:

$$\begin{pmatrix} 0 & 2 & 2 \\ 2 & -3/2 & 1/2 \\ 2 & 1/2 & 5/2 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 1 \\ 2 & -3/2 & 1/2 \\ 0 & 2 & 2 \end{pmatrix} \Rightarrow \begin{array}{l} v_2 + v_3 = 0 \\ v_1 = \frac{3}{4}v_2 - \frac{1}{4}v_3 \end{array}$$

3

$\Rightarrow \underline{v} \propto \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \Rightarrow$ unit eigenvector $\underline{v} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$

PTO.

$\boxed{\lambda=0}$ $\Rightarrow$ ~~xxx~~ $A\,\underline{v} = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3/2 & 1/2 \\ 2 & 1/2 & 11/2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Gauss: $\begin{pmatrix} 3 & 2 & 2 \\ 2 & 3/2 & 1/2 \\ 2 & 1/2 & 11/2 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & -1/4 & 5/4 \\ 1 & 3/4 & 1/4 \\ 0 & -1 & 5 \end{pmatrix} \Rightarrow$ $v_2 = 5v_3$

$v_1 = -\frac{3}{4}v_2 - \frac{1}{4}v_3$

$\Rightarrow \underline{v} \propto \begin{pmatrix} -4 \\ 5 \\ 1 \end{pmatrix}$ and unit eigenvector $\underline{v} = \frac{1}{\sqrt{42}}\begin{pmatrix} -4 \\ 5 \\ 1 \end{pmatrix}$ 2

$\boxed{\lambda=7}$

$(A-7I)\,\underline{v} = \begin{pmatrix} -4 & 2 & 2 \\ 2 & -11/2 & 1/2 \\ 2 & 1/2 & -3/2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Gauss:

$\begin{pmatrix} -4 & 2 & 2 \\ 2 & -11/2 & 1/2 \\ 2 & 1/2 & -3/2 \end{pmatrix} \rightarrow \begin{pmatrix} -2 & 1 & 1 \\ 0 & -9/2 & 3/2 \\ 0 & 3/2 & -1/2 \end{pmatrix} \Rightarrow$ $3v_2 = v_3$

$2v_1 = v_2 + v_3$

$\underline{v} \propto \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \Rightarrow$ unit eigenvector $\underline{v} = \frac{1}{\sqrt{14}}\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ 2

Normal form:

$Q = \lambda_1 \tilde{x}_1^2 + \lambda_2 \tilde{x}_2^2 + \lambda_3 \tilde{x}_3^2$ 1 , with $\tilde{x}_i = \underline{v}_i \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\boxed{\Rightarrow Q(x_1,x_2,x_3) = 3\left(\frac{x_1+x_2-x_3}{\sqrt{3}}\right)^2 + 0\cdot\left(\frac{-4x_1+5x_2+x_3}{\sqrt{42}}\right)^2 + 7\left(\frac{2x_1+x_2+3x_3}{\sqrt{14}}\right)^2}$$ 2

Since one eigenvalue 0 and the other positive $\Rightarrow$ Q <u>pos. semi-def.</u> 1

$\boxed{13/20}$

(b) Impose the constraint $h(x_1,x_2,x_3) = 4x_1 - 5x_2 - x_3 = 0$

We use the Hessian to find the character of the reduced quadratic form. The bordered Hessian is:

$$H_B = \begin{pmatrix} 0 & 4 & -5 & -1 \\ 4 & 3 & 2 & 2 \\ -5 & 2 & 3/2 & 1/2 \\ -1 & 2 & 1/2 & 11/2 \end{pmatrix}$$

2

where $n = 3$
$m = 1$
$\Rightarrow$ we need $n-m = 2$ LPMs
LPM_4 and LPM_3.

$$LPM_4 = \det(H_B) = \begin{vmatrix} 0 & 4 & -5 & -1 \\ 4 & 3 & 2 & 2 \\ -5 & 2 & 3/2 & 1/2 \\ -1 & 2 & 1/2 & 11/2 \end{vmatrix} \underset{\substack{C_2 \to C_2+4C_4 \\ C_3 \to C_3 - 5C_4}}{=} \begin{vmatrix} 0 & 0 & 0 & -1 \\ 4 & 11 & -8 & 2 \\ -5 & 4 & -1 & 1/2 \\ -1 & 24 & -27 & 11/2 \end{vmatrix}$$

$$= \begin{vmatrix} 4 & 11 & -8 \\ -5 & 4 & -1 \\ -1 & 24 & -27 \end{vmatrix} \underset{\substack{R_1+4R_3 \\ R_2-5R_3}}{=} \begin{vmatrix} 0 & 107 & -116 \\ 0 & -116 & 134 \\ -1 & 24 & -27 \end{vmatrix} = -107 \cdot 134 + 116^2 > 0$$

2

$$LPM_3 = \begin{vmatrix} 0 & 4 & -5 \\ 4 & 3 & 2 \\ -5 & 2 & 3/2 \end{vmatrix} = -4(6+10) - 5(8+15) < 0$$

1

So same signs as $\operatorname{Sgn}(LPM_4) = (-1)^m \Rightarrow$ pos. def

1

If $4x_1 - 5x_2 - x_3 = 0$ then from normal form we have $Q(x_1,x_2,x_3) = 3\tilde{x}_1^2 + 7\tilde{x}_2^2$ and with $\tilde{x}_1, \tilde{x}_2$ arbitrary this is positive definite.

1

$\boxed{\frac{7}{20}}$

Q3

a) Function $f(x,y,z) = \frac{1}{2}x^2 + yz + \frac{1}{3}y^3 - z^2$

subject to $h = x+y+z = 2$.

Lagrangian $L(x,y,z,\lambda) = \frac{1}{2}x^2 + yz + \frac{1}{3}y^3 - z^2 - \lambda(x+y+z-2)$ 1

First order conditions.

$$\begin{cases} L_x = x - \lambda = 0 & \text{(i)} \\ L_y = z + y^2 - \lambda = 0 & \text{(ii)} \\ L_z = y - 2z - \lambda = 0 & \text{(iii)} \\ -L_\lambda = x+y+z-2 = 0 & \text{(iv)} \end{cases}$$

1

From (i) we have $\lambda = x$. Then from (iii) and (iv)

$$\begin{cases} y - 2z = \lambda \\ y + z = 2 - \lambda \end{cases} \Rightarrow 3z = 2 - 2\lambda \Rightarrow z = \frac{2}{3}(1-\lambda)$$

and $y = \frac{1}{3}(4-\lambda)$

Plug this into (ii):

$$\frac{2}{3}(1-\lambda) + \frac{1}{9}(4-\lambda)^2 - \lambda = 0 \Rightarrow (\lambda-4)^2 + 6(1-\lambda) - 9\lambda = 0$$

$$\Rightarrow \lambda^2 - 23\lambda + 22 = 0$$ 3

$$\Rightarrow (\lambda - 22)(\lambda - 1) = 0,$$

Thus $\lambda = 22$ or $\lambda = 1$

$\lambda = 22 \Rightarrow x = 22,\ y = -6,\ z = -14 \Rightarrow (22,-6,-14)_{\lambda=22}$ 1

$\lambda = 1 \Rightarrow x = 1,\ y = 1,\ z = 0 \Rightarrow (1,1,0)_{\lambda=1}$ 1

$\boxed{\frac{7}{20}}$

P.T.O.

b) Bordered Hessian

$\nabla h = (1, 1, 1)$

and $L_{xx} = 1$, $L_{xy} = 0$, $L_{xz} = 0$

$L_{yy} = 2y$, $L_{yz} = 1$, $L_{zz} = -2$

Thus

$$H_B = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 2y & 1 \\ 1 & 0 & 1 & -2 \end{pmatrix}$$

$n = 3$, $m = 1$

we need LPM_4, LPM_3

2

$$\det(H_B) = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 2y & 1 \\ 1 & 0 & 1 & -2 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & -1 \\ 1 & 0 & 2y & 1 \\ 1 & 0 & 1 & -2 \end{vmatrix}$$

$$= - \begin{vmatrix} 1 & -1 & -1 \\ 1 & 2y & 1 \\ 1 & 1 & -2 \end{vmatrix} = - \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2y+1 & 2 \\ 1 & 2 & -1 \end{vmatrix}$$

1

$$= -\left[-(2y+1) - 4\right] = 2y + 5 = LPM_4$$

$$LPM_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 2y \end{vmatrix} = -1 \cdot 2y + 1 \cdot (-1) = -(2y+1)$$

1

Then for $\boxed{y = -6}$ $\Rightarrow$ $LPM_4 = -7 < 0$, $LPM_3 = 11 > 0$

alternating signs and $\operatorname{sgn}(LPM_4) = (-1)^n$

$\Rightarrow$ negative definite, so $(22, -6, -14)$ maximum.

1

For $\boxed{y = 1}$ $LPM_4 = 7$ cannot be identified with either $(-1)^n$ or $(-1)^m$, so indefinite point.

1

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c) Cobb-Douglas production $Q = x^{1/3} y^{1/2}$, price p.

Then revenue function is $R(x,y) = pQ = p x^{1/3} y^{1/2}$

and with cost function $C = ax + by \Rightarrow$ profit: $\Pi(x,y) = R - C$

$$\Rightarrow \quad \Pi(x,y) = p\, x^{1/3} y^{1/2} - (ax + by)$$ 2

Subject to $C(x,y) = ax + by = C_0$ constant.

Lagrangian:

$$L(x,y,\lambda) = \Pi(x,y) - \lambda\big(C(x,y) - C_0\big)$$

$$= p x^{1/3} y^{1/2} - (1+\lambda)(ax + by) + \lambda C_0$$ 1

First-order conditions:

$$\begin{cases} \Pi_x = \frac{1}{3} p x^{-2/3} y^{1/2} - (1+\lambda) a = 0 \Rightarrow R(x,y) = 3(1+\lambda) a x & (i) \\ \Pi_y = \frac{1}{2} p x^{1/3} y^{-1/2} - (1+\lambda) b = 0 \Rightarrow R(x,y) = 2(1+\lambda) b y & (ii) \\ -L_\lambda = ax + by - C_0 = 0 & (iii) \end{cases}$$ 1

From (i), (ii) $\Rightarrow (1+\lambda) 3ax = (1+\lambda) 2by$, but $1+\lambda \neq 0$

$\Rightarrow 3ax = 2by$. Then from (iii) $ax + by = C_0$

$\Rightarrow 3ax = 2(C_0 - ax) \Rightarrow 5ax = 2C_0$, $\quad \boxed{x^* = \dfrac{2C_0}{5a}}$

and $2by = 3(C_0 - by) \Rightarrow 5by = 3C_0$, $\quad \boxed{y^* = \dfrac{3C_0}{5b}}$ 1

Furthermore,

$$3(1+\lambda) a x = p x^{1/3} y^{2/3}$$

$$\Rightarrow \quad 1 + \lambda^* = \frac{p}{3ax} x^{1/3} y^{2/3} = \frac{5p}{6C_0} \left(\frac{2C_0}{5a}\right)^{1/3} \left(\frac{3C_0}{5b}\right)^{1/2}$$ 1

Profit at critical point:

$$\Pi^* = \Pi(x^*, y^*) = p \left(\frac{2C_0}{5a}\right)^{1/3} \left(\frac{3C_0}{5b}\right)^{1/2} - C_0.$$ 1

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Q4

a) Function $f(x,y) = 3x+2y$ subject to

$y \leq x+1, \quad 2y \geq x-3, \quad x \leq 3.$

Lagrangian $\boxed{L(x,y,\lambda_1,\lambda_2,\lambda_3) = 3x+2y - \lambda_1(y-x-1) - \lambda_2(x-3-2y) - \lambda_3(x-3)}$

First-order equalities:

1

$$\begin{cases} L_x = 3+\lambda_1-\lambda_2-\lambda_3 = 0 & \text{(i)} \\ L_y = 2-\lambda_1+2\lambda_2 = 0 & \text{(ii)} \\ \lambda_1(y-x-1)=0 & \text{(iii)} \\ \lambda_2(x-3-2y)=0 & \text{(iv)} \\ \lambda_3(x-3)=0 & \text{(v)} \end{cases}$$

Complementary slackness conditions.

1

Furthermore the inequalities:

$y-x-1 \leq 0, \quad x-3-2y \leq 0, \quad x-3 \leq 0$

$\lambda_1 \geq 0, \quad \lambda_2 \geq 0, \quad \lambda_3 \geq 0,$ ~~ ~~

1

From (v) $\Rightarrow \lambda_3 = 0$ or $x=3$

Now if $\lambda_3=0$, then from i), ii): $\begin{cases} \lambda_1-\lambda_2+3=0 \\ -\lambda_1+2\lambda_2+2=0 \end{cases}$

1

$\Rightarrow \lambda_2+5=0, \quad \lambda_2=-5 \leq 0$ which is not allowed.

So $\lambda_3=0$ must be rejected

$\Rightarrow \boxed{x=3}$.

1

Then from iv) $\lambda_2 y = 0$ so $\lambda_2=0$ or $y=0$

If $y=0 \Rightarrow$ from (iii) $\lambda_1=0$, but then from (ii) $\lambda_2=-1 \leq 0$ not allowed

So $\boxed{\lambda_2=0} \overset{(ii)}{\Rightarrow} \lambda_1=2 \overset{(i)}{\Rightarrow} \lambda_3=5$ and

PTO.

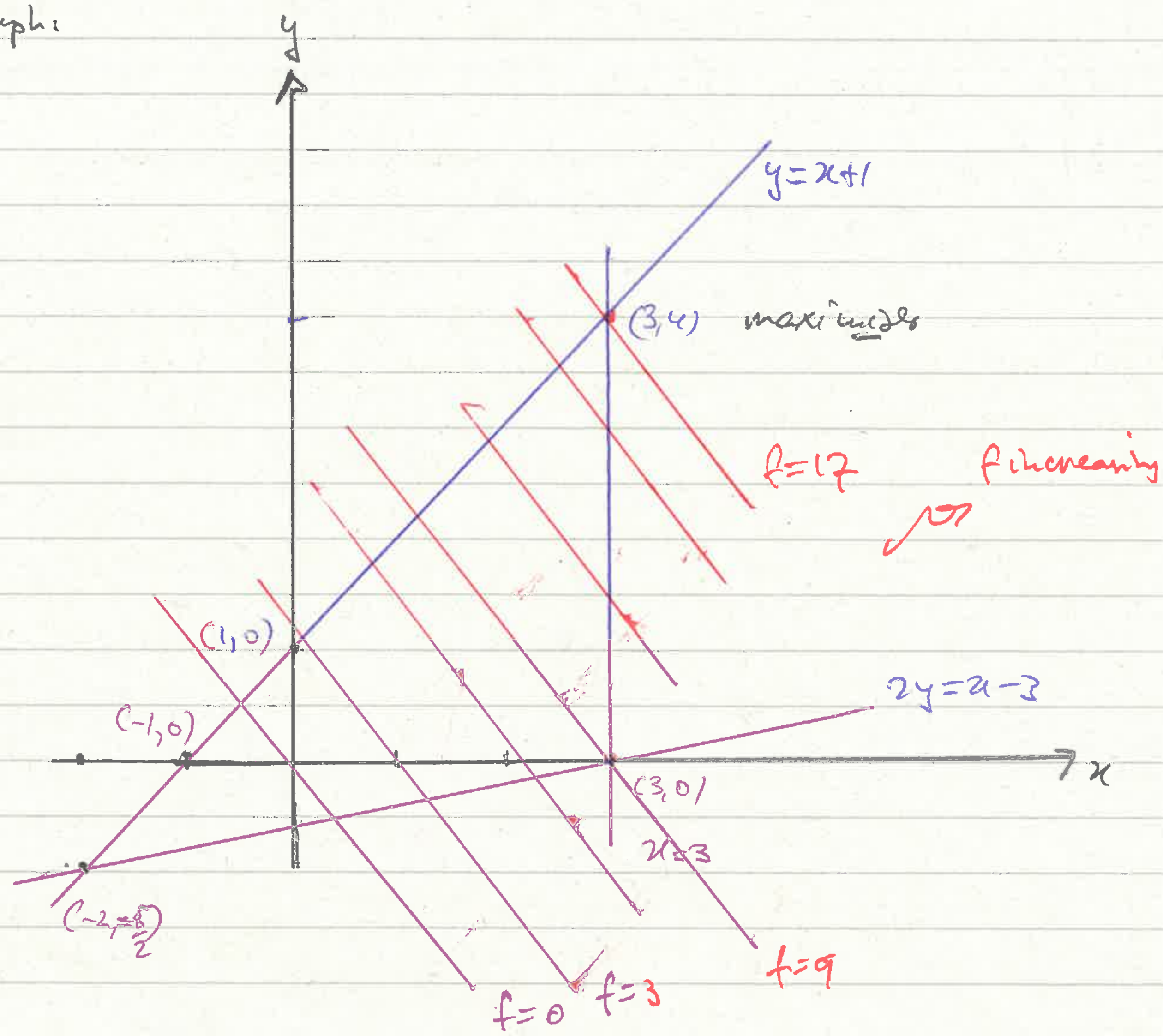
i(i) $\Rightarrow$ $y = x+1 = 4$, $\Rightarrow$ point $(3,4)$ with $\lambda_1 = 2$, $\lambda_2 = 0$, $\lambda_3 = 5$. 2

So constraints $y \leq x+1$ and $x \leq 3$ are binding.

Value at maximiser: $f(3,4) = \underline{\underline{17}}$ 1

$\boxed{\frac{8}{20}}$

ii) Graph:



$\boxed{\frac{4}{20}}$

c) maximize the function $f(x,y,z) = x+y+z$

subject to $x^2+y^2+z^2 \le 9$, $x \ge 0, y \ge 0, z \ge 0$

Kuhn-Tucker Lagrangian:

$$\bar{L}(x,y,z,\lambda) = x+y+z-\lambda(x^2+y^2+z^2-9)$$ 1

KT equalities: inequalities

$$\begin{cases} x\bar{L}_x = x(1-2\lambda x) = 0 & (i) & 1-2\lambda x \le 0 & (v) \\ y\bar{L}_y = y(1-2\lambda y) = 0 & (ii) & 1-2\lambda y \le 0 & (vi) \\ z\bar{L}_z = z(1-2\lambda z) = 0 & (iii) & 1-2\lambda z \le 0 & (vii) \\ \lambda\bar{L}_\lambda = -\lambda(x^2+y^2+z^2-9) = 0 & (iv) & x^2+y^2+z^2 \le 9 & (viii) \end{cases}$$

and: $\lambda \ge 0$, $x \ge 0$, $y \ge 0$, $z \ge 0$ (ix). 3

From (v), (vi) and (vii) it follows that neither x, y, z, λ can vanish, otherwise we get $1 \le 0$

So $\lambda \ne 0 \Rightarrow$ from (iv) $\boxed{x^2+y^2+z^2 = 9}$ (x)

and from (i)–(iii) $\Rightarrow$ $2\lambda x = 2\lambda y = 2\lambda z = 1$

$\Rightarrow$ $x = y = z = \frac{1}{2\lambda}$, then from (x) $\frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} = 9$

$\Rightarrow$ $\lambda^2 = 1/12 \Rightarrow \lambda = 1/\sqrt{12}$ 3

and $x = y = z \Rightarrow$ from (x) $x = y = z = \sqrt{3}$

Thus, we get the point $(\sqrt{3}, \sqrt{3}, \sqrt{3})$ with $\lambda = \sqrt{12}$ 1

The maximum of the function is

$$f(\sqrt{3}, \sqrt{3}, \sqrt{3}) = \sqrt{3}+\sqrt{3}+\sqrt{3} = \underline{\underline{3\sqrt{3}}}$$

$\boxed{\frac{8}{20}}$