MATH2640 Introduction to Optimisation

Example Sheet 5 Solutions to Assessed Questions

Thursday 12$^{\text{th}}$ December 2019 homework

Constrained optimisation, equality constraints, Lagrange multipliers, NDCQ, bordered Hessians.
Based on material in Lectures 17 to 22

Assessed Questions

A1.

i) Consider maximising the function $f(x, y) = x + y$ subject to the simultaneous constraints

$$x \leq \tfrac{3}{2}y + 1\ , \quad 2x + y \leq 10\ , \quad \text{and} \quad x \geq \tfrac{1}{6}y + 1\ .$$

Write down the relevant Lagrangian and the corresponding first order equations, including the complementary slackness conditions, and the relevant inequalities. Analyse these to find the unique point that gives the maximiser, and the corresponding values of the Lagrange multipliers (which constraints are binding?). What is the maximum value of f under the constraints?

Answer:
Lagrangian:

$$L(x, y, \lambda) = x + y - \lambda_1(x - \frac{3}{2}y - 1) - \lambda_2(2x + y - 10) - \lambda_3(\frac{1}{6}y + 1 - x). \tag{1}$$

Conditions:

$$\begin{aligned}
FOCs: \text{(i)} \quad L_x &= 1 - \lambda_1 - 2\lambda_2 + \lambda_3 = 0 && (2)\\
\text{(ii)} \quad L_y &= 1 + \frac{3}{2}\lambda_1 - \lambda_2 - \frac{1}{6}\lambda_3 = 0 && (3)\\
CSCs: \text{(iii)} \quad & \lambda_1(x - \frac{3}{2}y - 1) = 0 && (4)\\
\text{(iv)} \quad & \lambda_2(2x + y - 10) = 0 && (5)\\
\text{(v)} \quad & \lambda_3(\frac{1}{6}y + 1 - x) = 0 && (6)\\
Ineq: \quad & \lambda_1, \lambda_2, \lambda_3 \geq 0 && (7)\\
x - \frac{3}{2}y - 1 &\leq 0, 2x + y - 10 \leq 0, \frac{1}{6}y + 1 - x \leq 0. && (8)
\end{aligned}$$

From (ii)-(i) find (a) $\frac{5}{2}\lambda_1 + \lambda_2 = \frac{7}{6}\lambda_3$.
From (v) we find $\lambda_3 = 0$ or $x = 1 + y/6$.
If $\lambda_3 = 0$ then from (a) it follows $\lambda_1 = \lambda_2 = \lambda_3 = 0$ since $\lambda_i \geq 0$ but then (i), (ii) are violated.
When $x = 1 + y/6$ and $\lambda_3 \neq 0$ then from (iii) $\lambda_1(-8y/6) = 0$ so $\lambda_1 = 0$ or $y = 0$.
If $\lambda_1 = 0$ then we have the system $1 - 2\lambda_2 + \lambda_3 = 0$ and $1 - \lambda_2 - \lambda_3/6 = 0$, which system has a solution by solving for $\lambda_{2,3}$ as a system, giving : $\lambda_2 = 7/8, \lambda_3 = 3/4$.
Then (iv) and (v) are satisfied when $2x + y = 10$ and $y/6 - x = -1$ which system has as

solution $x = 2, y = 6$.
So stationary point is: $(x^*, y^*, \lambda_1^*, \lambda_2^*, \lambda_3^*) = (2, 6, 0, 7/8, 3/4)$.
If $y = 0$ then from (v) $x = 1$ and from (iv) $\lambda_2 = 0$ but then (i,ii) yield agan $\lambda_1 < 0$ so not allowed. The inequality constraint with $\lambda_1 = 0$, such that $x - 3y/2 - 1 \leq 0$, is nonbinding.

ii) Draw a graph of the region defined by the inequalities in the xy–plane. Solve the problem of part (i) graphically by drawing the level curves (i.e., level lines) of the function f and finding the maximiser in this way.

Answer:
Draw the equality parts of the three inequality constraints as lines in the x, y–plane for $y, x \geq 0$ (here). E.g. by finding intersections with axes and one or two extra points; $\nabla f = (1, 1)$ gives the direction of increase of f so draw contour lines $f + y = c$.

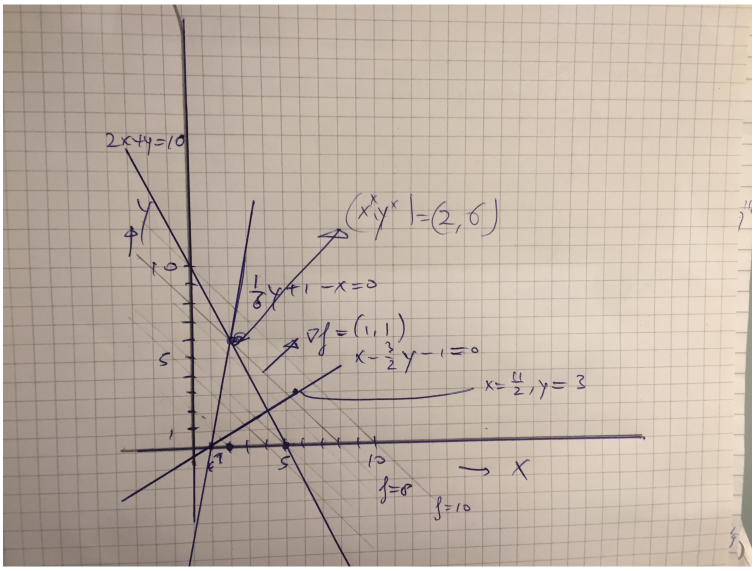

Figure 1: Sketches of case: $x = 11/2, y = 3$ lies on the line $x - 3y/2 - 1 = 0$. Maximum at $(x^*, y^*) = (2, 6)$.

A2.

i) Consider the problem of maximising the function

$$f(x, y) = 2xy - x - 2y,$$

subject to the constraints

$$x^2 + 4y^2 \leq 18, \quad x \geq 0, \quad y \geq 0.$$

(a) Solving this system of equations find the candidates (x, y, λ) for the maximiser. By evaluating f for these candidates find its maximum value. Solve the problem using Lagrange multipliers for all inequality constraints. Analyse the bordered Hessian for the appropriate cases.

Answer:
The Lagrangian is:

$$L(x, y, \lambda_1, \lambda_2, \lambda_3) = 2xy - x - 2y - \lambda_1(x^2 + 4y^2 - 18) + \lambda_2 x + \lambda_3 y. \tag{9}$$

Relations:

$$\begin{aligned} FOCS:\text{(i)}\quad L_x =& 2y-1-2\lambda_1 x+\lambda_2=0 &(10)\\ \text{(ii)}\quad L_y =& 2x-2-8\lambda_1 y+\lambda_3=0 &(11)\\ CSCs:\text{(iii)}\quad & \lambda_1(x^2+4y^2-18)=0 &(12)\\ \text{(iv)}\quad & \lambda_2 x=0 &(13)\\ \text{(v)}\quad & \lambda_3 y=0 &(14)\\ Ineqs:\quad & \lambda_1,\lambda_2,\lambda_3\geq 0 &(15)\\ x^2+4y^2-18 \leq & 0, x\geq 0, y\geq 0. &(16)\end{aligned}$$

From (iii) we find that $\lambda_1=0$ or $x^2+4y^2=18$.
If $\lambda_1=\lambda_2=\lambda_3=0$ then from (i,ii) $x=1, y=1/2$ so stationary point I: $(x^*,y^*,\lambda_1^*,\lambda_2^*,\lambda_3^*)=(1,1/2,0,0,0)$ and all constraints are nonbinding.

If $\lambda_2\neq 0, \lambda_3\neq 0$ then $x=y=0$ and (i,ii) yield $\lambda_2=1, \lambda_3=2$, so stationary point II: $(x^*,y^*,\lambda_1^*,\lambda_2^*,\lambda_3^*)=(0,0,0,1,2)$. Constraint $x^2+4y^2-18=0$ is nonbinding.

If $\lambda_1\neq 0$ then take $\lambda_2=\lambda_3=0$ and solve system $2y-2\lambda_1 x=1$ and $2x-8\lambda_1 y=2$; solving which allows us to find $x=2y$ (take ratio of rewritten equations); when $x=2y$ is substituted in $x^2+4y^2=18$, it yield $y=3/2$ and then $x=2y=3$ and (using one of equations) $\lambda_1=(2y-1)/(2x)=1/3$. Hence, stationary point III: $(x^*,y^*,\lambda_1^*,\lambda_2^*,\lambda_3^*)=(3,3/2,1/3,0,0)$ with binding first constraint.

For point III, we have $m=1, 2m+1=3, n=2$ so we need to evaluate

$$LPM_s=\det(H_B)=\begin{vmatrix} 0 & -2x & -8y\\ -2x & -2\lambda & 2\\ -8y & 2 & -8\lambda\end{vmatrix}_{(3,3/2,1/3)}=\begin{vmatrix} 0 & -6 & -12\\ -6 & -2/3 & 2\\ -12 & 2 & -8/3\end{vmatrix}=+\text{Big}>0. \tag{17}$$

Hence, $sign(LPM_3)=(-1)^{n=2}=1$ so rule (Bn) holds and the associated local quadratic form at teh stationary point is ND so we have a maximum with value $f(3,3/2)=3$ at the stationary point.
Note that at stationary point II: $f(0,0)=0$ and at stationary point I: $f(1,1/2)=-1$.

(b) Write down the Kuhn-Tucker (KT) Lagrangian for this problem, and the corresponding KT equalities and inequalities. Solving this system of equations find the candidates (x,y,λ) for the maximiser. By evaluating f for these candidates find its maximum value.

Answer:
KT Lagrangian reads:

$$\bar{L}(x,y,\lambda)=2xy-x-2y-\lambda(x^2+4y^2-18). \tag{18}$$

KT relations are:

$$\begin{aligned} FOCs:\text{(i)}\quad x\bar{L}_x =& x(2y-1-2x\lambda)=0 &(19)\\ \text{(ii)}\quad y\bar{L}_y =& y(2x-2-8y\lambda)=0 &(20)\\ CSCs:\text{(iii)}\quad \lambda\bar{L}_\lambda =& \lambda(x^2+4y^2-18)=0 &(21)\\ Ineq:\quad x^2-4y^2 \leq & 18, x\geq 0, y\geq 0, \lambda\geq 0 &(22)\\ L_x =& 2y-1-2x\lambda\leq 0, L_y=2x-2-8y\lambda\leq 0. &(23)\end{aligned}$$

From (iii): $\lambda = 0$ or $x^2 + 4y^2 = 18$.
If $\lambda = 0$ then (i,ii) give $x = 1$ and $y = 1/2$ so stationary point I: $(x^*, y^*, \lambda^*) = (1, 1/2, 0)$.

If $x = y = 0$ then (iii) gives $\lambda = 0$ so stationary point II: $(x^*, y^*, \lambda^*) = (0, 0, 0)$.

Note that $x \neq 0, y = 0$ or $x = 0, y \neq 0$ are not allowed since $x \geq 0, y \geq 0, \lambda \geq 0$.

If $\lambda \neq 0$ and $x \neq 0, y \neq 0$ then solve the syetem $2y - 1 = 2x\lambda$ and $2x - 2 = 8y\lambda$, giving $x = 2y$ by reorganising. Via (iii) this gives $y = 3/2$ and then $x = 2y = 3$ and (via one of relations) $\lambda = (2y-1)/(2x) = 1/3$. So stationary point I: $(x^*, y^*, \lambda^*) = (3, 3/2, 1/3)$ with $f(3, 3/2) = 3$; note that $f(1, 1/2) = -1$.

ii) Consider the problem of maximising the function $f(x, y, z) = x + y + z$ subject to the constraints

$$3x + 2y + 4z \leq 9\ , \quad x \geq 0, \quad y \geq 0, \quad z \geq 0\ .$$

Write down the KT Lagrangian for this problem, and the corresponding KT equalities and inequalities. Solve this problem and show that there is a unique maximiser. Sketch a graph of the situation indicating the region in the space of xyz-variables and the level surfaces (i.e., level planes) of the function f (optional).

Answer:
KT Lagrangian is:

$$\bar{L}(x, y, \lambda) = x + y + z - \lambda(3x + 2y + 4z - 9). \tag{24}$$

KT relations are:

$$\begin{aligned}
FOCs: \text{(i)} \quad x\bar{L}_x =& x(1 - 3\lambda) = 0 && (25)\\
\text{(ii)} \quad y\bar{L}_y =& y(1 - 2\lambda) = 0 && (26)\\
\text{(iii)} \quad z\bar{L}_z =& z(1 - 4\lambda) = 0 && (27)\\
CSCs: \text{(iv)} \quad \lambda\bar{L}_\lambda =& \lambda(3x + 2y + 4z - 9) = 0 && (28)\\
Ineq: \quad 3x + 2y + 4z \leq& 9, x \geq 0, y \geq 0, z \geq 0, \lambda \geq 0 && (29)\\
L_x =& 1 - 3\lambda \leq 0, L_y = 1 - 2\lambda \leq 0, L_z = 1 - 4\lambda \leq 0. && (30)
\end{aligned}$$

From (iv) one has $\lambda = 0$ or $3x + 2y + 4z = 9$.

If $\lambda = 0$ then $x = y = z = 0$ then the constraint is nonbinding and $f(0, 0, 0) = 0$.

If $\lambda \neq 0$ then:
(a) $x \neq 0, y = z = 0$; $\lambda = 1/3, x = 3$. Binding constraint.
(b) $y \neq 0, x = z = 0$; $\lambda = 1/2, y = 9/2$. Binding constraint.
(c) $z \neq 0, y = x = 0$; $\lambda = 1/4, z = 9/4$. Binding constraint.
But since $\lambda \geq 1/3, \lambda \geq 1/2, \lambda \geq 1/4$ only case (b) matters. So stationary point and maximum is $(x^*, y^*, z^*, \lambda^*) = (0, 9/2, 0, 1/2)$ with $f(0, 9/2, 0) = 9/2$.

Note that $\nabla f = (1, 1, 1)$ and constraint $3x + 2y + 4z = 9$ is a plane intersecting the axes at $(3, 0, 0), (0, 9/2, 0), (0, 0, 9/4)$ and the maximum is reached when the plane $f = c = 9/2$ hits the intersection point furthest away.

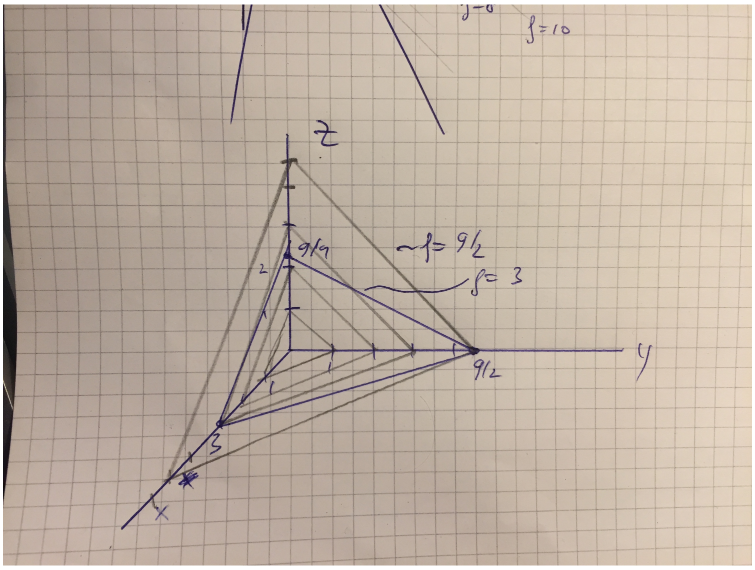

Figure 2: Sketches of case.