MATH 2640 EXAMPLES 4 Workshop problems

B1

a) Find stat. points of $f(x,y) = x + y^2$ subject to the constraint $h = x^2 + y^2 = 1$

Lagrangian is: $L(x,y,\lambda) = x + y^2 - \lambda(x^2 + y^2 - 1)$

First-order conditions

$$\begin{cases} L_x = 1 - 2\lambda x = 0 & (i) \\ L_y = 2y - 2\lambda y = 0 & (ii) \\ -L_\lambda = x^2 + y^2 - 1 = 0 & (iii) \end{cases}$$

From (i) $\Rightarrow$ $2\lambda x = 1$, while (ii) $\Rightarrow$ $y(1-\lambda) = 0$

From the latter, either $y = 0$ or $\lambda = 1$

If $y = 0$, then from (iii) $x^2 = 1 \Rightarrow x = \pm 1$
and from (i) $\Rightarrow \lambda = \pm 1/2$.
Thus, we obtain the points $(1,0)$ with $\lambda = 1/2$
and $(-1,0)$ with $\lambda = -1/2$.

If $\lambda = 1$, then from (i) we have $x = 1/2$
and then from (iii): $y^2 = 1 - 1/4 = 3/4 \Rightarrow y = \pm\frac{1}{2}\sqrt{3}$
giving the points $\left(\frac{1}{2}, \frac{1}{2}\sqrt{3}\right)$ and $\left(\frac{1}{2}, -\frac{1}{2}\sqrt{3}\right)$
with both $\lambda = 1$.

Evaluate f at these points:

$f(1,0) = 1$, $f(-1,0) = -1$, $f\left(\frac{1}{2}, \pm\frac{1}{2}\sqrt{3}\right) = \frac{5}{4}$
suggesting that the latter points are maxima.

b) Find the points on the ellipse $x^2+xy+y^2=9$ closest to and farthest away from the origin.

Take $f(x,y)=(\text{distance})^2 = x^2+y^2$ and find its max/min subject to $h(x,y)=x^2+xy+y^2=9$.

Lagrangian: $L(x,y,\lambda) = x^2+y^2-\lambda(x^2+xy+y^2-9)$.

Stationary points, 1st order conditions:

$$\begin{cases} L_x = 2x-2\lambda x-\lambda y = 2(1-\lambda)x-\lambda y = 0 & (i) \\ L_y = 2y-\lambda x-2\lambda y = 2(1-\lambda)y-\lambda x = 0 & (ii) \\ -L_\lambda = x^2+xy+y^2-9 = 0 & (iii) \end{cases}$$

Add (i), (ii) $\Rightarrow$ $2(1-\lambda)(x+y)-\lambda(x+y) = (x+y)(2-3\lambda) = 0$

Thus, either $x+y=0$ or $\lambda = 2/3$.

If $x+y=0$, then from (iii) $x^2-x^2+x^2=9 \Rightarrow x^2=9$

$\Rightarrow$ $x=\pm 3$ yielding $(3,-3)$, $(-3,3)$ as solutions.

then from (i) $(3,-3)$ gives $\lambda=2$

$(-3,3)$ gives $\lambda=2$.

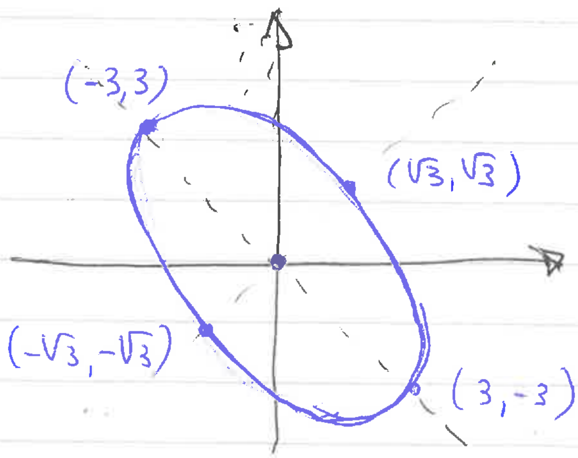
If $\lambda=2/3$, both (i), (ii) yield $x=y$, and then from (iii) we get: $3x^2=9 \Rightarrow x^2=3$, $x=\pm\sqrt{3}$

yielding points $(\sqrt{3},\sqrt{3})$ and $(-\sqrt{3},-\sqrt{3})$ both with $\lambda=2/3$.

PTO.

Sketch: write ellipse as

$$x^2+y^2+xy = \frac{3}{4}(x+y)^2 + \frac{1}{4}(x-y)^2 = 9$$



The corresponding distances are $\left(f(\pm\sqrt{3},\pm\sqrt{3})\right)^{1/2} = \sqrt{6}$ (minimum distance)

and

$$\left(f(\pm 3, \mp 3)\right)^{1/2} = \sqrt{18} = 3\sqrt{2}.$$

(maximum distance).

c) Distance of plane $4x+2y+z=5$ to the origin, which means the smallest distance from any point of the plane to the origin.

We can use $(\text{distance})^2$ for convenience to minimise, hence take $f(x,y,z) = x^2+y^2+z^2$, subject to the constraint $h(x,y,z) = 4x+2y+z=5$.

Lagrangian:

$$L(x,y,\lambda) = x^2+y^2+z^2 - \lambda(4x+2y+z-5).$$

First order conditions:

$$\begin{cases} L_x = 2x - 4\lambda = 0 & \text{(i)} \\ L_y = 2y - 2\lambda = 0 & \text{(ii)} \\ L_z = 2z - \lambda = 0 & \text{(iii)} \\ -L_\lambda = 4x+2y+z-5=0 & \text{(iv)} \end{cases}$$

From (i) – (iii) $\Rightarrow$ $x = 2\lambda$, $y=\lambda$, $z = \frac{1}{2}\lambda$.

insert this into (iv) $\Rightarrow$ $8\lambda + 2\lambda + \frac{1}{2}\lambda = 5 \Rightarrow \frac{21}{2}\lambda = 5$

$$\Rightarrow \boxed{\lambda = 10/21}$$

PTO.

Thus, by backsubstituting the value for λ:

$$x = \frac{20}{21}, \quad y = \frac{10}{21}, \quad z = \frac{5}{21} \Rightarrow \left(\frac{20}{21}, \frac{10}{21}, \frac{5}{21}\right).$$

Then the distance is

$$\left(f\left(\frac{20}{21}, \frac{10}{21}, \frac{5}{21}\right)\right)^{1/2} = \frac{1}{21}\sqrt{525} = \frac{5}{\sqrt{21}}$$

Geometrically, $(4,2,1)$ is perpendicular vector to surface

$\Rightarrow$ unit vector $\frac{1}{\sqrt{21}}(4,2,1)$,

then $\frac{1}{\sqrt{21}}(4,2,1) \cdot (x,y,z) = \frac{5}{\sqrt{21}}$ = distance.

B2)

a) Find maximum and minimum of $f = x^2 + y + z$ subject to $h_1 = x^2 + y^2 + z^2 = 50$, $h_2 = y + z = 6$.

Lagrangian is: $L(x, y, z, \lambda_1, \lambda_2) = f - \lambda_1(h_1 - 50) - \lambda_2(h_2 - 6)$

$$= x^2 + y + z - \lambda_1(x^2 + y^2 + z^2 - 50) - \lambda_2(y + z - 6)$$

First order conditions:

$$\begin{cases} L_x = 2x - 2\lambda_1 x = 0 & \text{(i)} \\ L_y = 1 - 2\lambda_1 y - \lambda_2 = 0 & \text{(ii)} \\ L_z = 1 - 2\lambda_1 z - \lambda_2 = 0 & \text{(iii)} \\ -L_{\lambda_1} = x^2 + y^2 + z^2 - 50 = 0 & \text{(iv)} \\ -L_{\lambda_2} = y + z - 6 = 0 & \text{(v)} \end{cases}$$

From (i) we have $(1 - \lambda_1)x = 0 \Rightarrow \lambda_1 = 1$ or $x = 0$

Subtracting (ii), (iii) $\Rightarrow$ $\lambda_1(y - z) = 0 \Rightarrow \lambda_1 = 0$ or $y = z$

- If $\underline{\underline{\lambda_1 = 0}}$ then $x = 0$ (since $\lambda_1 \neq 1$), the (iv), (v) determine y, z: $\begin{cases} y^2 + z^2 = 50 \\ y + z = 6 \end{cases} \Rightarrow$

$$\Rightarrow \quad y^2 + (6 - y)^2 = 2y^2 - 12y + 36 = 50$$

$$\Rightarrow \quad y^2 - 6y - 7 = 0 \Rightarrow (y - 7)(y + 1) = 0$$

So $y = 7 \Rightarrow z = -1$ or $y = -1 \Rightarrow z = 7$

This gives the points $(0, 7, -1)$, $(0, -1, 7)$ with $\lambda_1 = 0$ and $\lambda_2 = 1$ (from (ii)).

PTO

- If $\lambda_1 \neq 0 \Rightarrow y = z$, implying from (v) $y = z = 3$.
  then from (iv) $x^2 = 50 - y^2 - z^2 = 32 \Rightarrow x = \pm 4\sqrt{2}$
  giving the points $(4\sqrt{2}, 3, 3)$, $(-4\sqrt{2}, 3, 3)$
  Then $\lambda_1 = 1$, $\lambda_2 = 1 - 2y = -5$ (from (ii)).

Thus, we get stationary points:

$(0, 7, -1)$, $(0, -1, 7)$ with $\lambda_1 = 0$ $\lambda_2 = 1$

$(4\sqrt{2}, 3, 3)$, $(-4\sqrt{2}, 3, 3)$ with $\lambda_1 = 1$, $\lambda_2 = -5$.

Check NDCQ:

$$\underline{\nabla} h_1 = (2x, 2y, 2z), \quad \underline{\nabla} h_2 = (0, 1, 1)$$

It is easy to see that for all four points we have that $\underline{\nabla} h_1$, $\underline{\nabla} h_2$ independent and nonzero
$\Rightarrow$ NDCQ satisfied.

b) Maximise $f = xz + yz$ subject to
$h_1 = y^2 + z^2 = 1$, $h_2 = xz = 3$.

Lagrangian:

$$L(x, y, z, \lambda_1, \lambda_2) = xz + yz - \lambda_1 (y^2 + z^2 - 1) - \lambda_2 (xz - 3)$$

1st order conditions:

$$\begin{cases} L_x = z - \lambda_2 z = 0 & \text{(i)} \\ L_y = z - 2\lambda_1 y = 0 & \text{(ii)} \\ L_z = x + y - 2\lambda_1 z - \lambda_2 x = 0 & \text{(iii)} \\ -L_{\lambda_1} = y^2 + z^2 - 1 = 0 & \text{(iv)} \\ -L_{\lambda_2} = xz - 3 = 0 & \text{(v)} \end{cases}$$

PTO

From (i) $\Rightarrow$ $(1-\lambda_2)z=0$ implying $\lambda_2=1$ or $z=0$

However, from (v) it follows that $z\neq 0$, hence $\boxed{\lambda_2=1}$

Then (ii), (iii) give: $\begin{cases} z-2\lambda_1 y=0 \\ y-2\lambda_1 z=0 \end{cases} \Rightarrow 2\lambda_1=\frac{z}{y}=\frac{y}{z}$

(since $y\neq 0$ as can be easily seen) $\Rightarrow$ $z^2=y^2$

So $z=\pm y$, then (iv) implies $2y^2=1 \Rightarrow y=\pm\frac{1}{\sqrt{2}}$

Thus, there are two cases:

$\lambda_1=\frac{1}{2}$, $z=y=\pm\frac{1}{\sqrt{2}}$ $\Rightarrow$ points $\left(3\sqrt{2},\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$

and $\left(-3\sqrt{2},-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$

$\lambda_1=-\frac{1}{2}$, $z=-y=\pm\frac{1}{\sqrt{2}}$

gives points $\left(3\sqrt{2},-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$

and $\left(-3\sqrt{2},\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$

Evaluate f: $f\left(\pm 3\sqrt{2},\pm\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}}\right)=\pm\frac{1}{\sqrt{2}}\left(\pm 3\sqrt{2}\pm\frac{1}{\sqrt{2}}\right)$

$=3+\frac{1}{2}=7/2$ $\underline{\underline{\text{max}}}$

$f\left(\pm 3\sqrt{2},\mp\frac{1}{\sqrt{2}},\pm\ \right)=\pm\frac{1}{\sqrt{2}}\left(\pm 3\sqrt{2}\mp\frac{1}{\sqrt{2}}\right)$

$=3-\frac{1}{2}=5/2$. $\underline{\underline{\text{min}}}$.

<u>NDCQ</u>: $\nabla h_1=(0,2y,2z)$, $\nabla h_2=(z,0,x)$

which are clearly nonzero and independent for all four points $\Rightarrow$ NDCQ satisfied.

B3) a) Consider $Q(x,y) = x^2 + 2xy - y^2$ subject to $h = x - y = 0 \Rightarrow \nabla h = (1,-1)$

Bordered Hessian:
$$H_B = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & -1 \end{pmatrix}$$

where $n=2$, $m=1 \Rightarrow$ we need $n-m=1$ LPM, LPM_3

$$LPM_3 = \det H_B = \begin{vmatrix} 0 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & -1 \end{vmatrix} = -1 \cdot (0) - 1 \cdot 2 = -2 < 0$$

Thus $\operatorname{sgn}(LPM_3) = (-1)^m \Rightarrow$ positive definite

Indeed, if we set $y = x$ (from the constraint) Q reduces to $Q(x) = 2x^2$ which is pos. def. as a single-variable function.

b) Consider $Q(x,y,z) = -6y^2 + 3z^2 + 8xy + 2yz - 2xz$ subject to $x + y - z = 0$

$\underline{\nabla} h = (1, 1, -1)$

Bordered Hessian:

$$H_B = \begin{pmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & 4 & -1 \\ 1 & 4 & -6 & 1 \\ -1 & -1 & 1 & 3 \end{pmatrix} \qquad \begin{matrix} n=3 \\ m=1 \end{matrix}$$

So we need $n-m=2$ LPMs, LPM_4 and LPM_3.

PTO

$$LPM_4 = \det(H_B) = \begin{vmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & 4 & -1 \\ 1 & 4 & -6 & 1 \\ -1 & -1 & 1 & 3 \end{vmatrix} \underset{\substack{C_2 \to C_2 + C_4 \\ C_3 \to C_3 + C_4}}{=} \begin{vmatrix} 0 & 0 & 0 & -1 \\ 1 & -1 & 3 & -1 \\ 1 & 5 & -5 & 1 \\ -1 & 2 & 4 & 3 \end{vmatrix}$$

$$= 1 \cdot \begin{vmatrix} 1 & -1 & 3 \\ 1 & 5 & -5 \\ -1 & 2 & 4 \end{vmatrix} \underset{\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 + R_1}}{=} \begin{vmatrix} 1 & -1 & 3 \\ 0 & 6 & -8 \\ 0 & 1 & 7 \end{vmatrix} = \begin{vmatrix} 6 & -8 \\ 1 & 7 \end{vmatrix}$$

$$= 42 + 8 = 50 > 0$$

$$LPM_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 4 \\ 1 & 4 & -6 \end{vmatrix} = -1 \cdot (-10) + 1 \cdot 4 = 14 > 0$$

Since $\operatorname{sgn}(LPM_4) \neq (-1)^n$ or $(-1)^m$ we have an indefinite quadratic form.

Indeed, if we solve the constraint $\Rightarrow$ $z = x + y$ and substitute, the reduced quadratic form becomes:

$$Q(x,y) = -6y^2 + 3(x+y)^2 + 8xy + 2(y-x)(x+y)$$

$$= -y^2 + x^2 + 14xy = (x, y) \begin{pmatrix} 1 & 7 \\ 7 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Applying the LPM test on the latter:

$LPM_2 = \det(A) = -1 - 49 = -50 < 0$

$LPM_1 = 1 > 0$ also implies the reduced quadratic form is indefinite

c) $Q(x_1, x_2, x_3) = x_1^2 - x_2^2 - x_3^2 + 4x_1x_2 + 2x_1x_3$, subject to

$h_1 = x_1 + x_2 + x_3 = 0$, $h_2 = x_1 - x_2 - x_3 = 0$

$\Rightarrow \nabla h_1 = (1,1,1)$, $\nabla h_2 = (1,-1,-1)$

and we get the bordered Hessian:

$$H_B = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & -1 & 2 & -1 & 0 \\ 1 & -1 & 1 & 0 & -1 \end{pmatrix}$$

$n = 3$

$m = 2$

$\Rightarrow$ we need $n - m = 1$: LPM_5

$$LPM_5 = \begin{vmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & -1 & 2 & -1 & 0 \\ 1 & -1 & 1 & 0 & -1 \end{vmatrix} \underset{R_1 \to R_1 + R_2}{=} \begin{vmatrix} 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & -1 & 2 & -1 & 0 \\ 1 & -1 & 1 & 0 & -1 \end{vmatrix}$$

$$= 2 \begin{vmatrix} 0 & 0 & -1 & -1 \\ 1 & 1 & 2 & 1 \\ 1 & -1 & -1 & 0 \\ 1 & -1 & 0 & -1 \end{vmatrix} \underset{C_1 \to C_1 + C_2}{=} 2 \begin{vmatrix} 0 & 0 & -1 & -1 \\ 2 & 1 & 2 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{vmatrix} =$$

$$= -4 \begin{vmatrix} 0 & -1 & -1 \\ -1 & -1 & 0 \\ -1 & 0 & -1 \end{vmatrix} = -4 \cdot \left(1 \cdot 1 - 1(-1) \right) = -8 < 0$$

Hence we can identify $\operatorname{sgn}(LPM_5) = (-1)^n$ and hence the quadratic form is <u>negative definite</u>.

B4)

a) Function $f = 2x + 4y + 3z^2$ subject to constraint $h = x^2 + y^2 + z^2 = 1$.

Lagrangian: $L(x, y, z, \lambda) = 2x + 4y + 3z^2 - \lambda(x^2 + y^2 + z^2 - 1)$

First-order conditions:

$$\begin{cases} L_x = 2 - 2\lambda x = 0 & \text{(i)} \\ L_y = 4 - 2\lambda y = 0 & \text{(ii)} \\ L_z = 6z - 2\lambda z = 0 & \text{(iii)} \\ -L_\lambda = x^2 + y^2 + z^2 - 1 = 0 & \text{(iv)} \end{cases}$$

From (i), (ii) : $\lambda x = 1$, $\lambda y = 2$

and from (iii) we have $z(3 - \lambda) = 0$

$\Rightarrow$ $z = 0$ or $\lambda = 3$

If $\boxed{z = 0}$ then with $\lambda x = 1$, $\lambda y = 2$ we have

from (iv) $\lambda^2 = (\lambda x)^2 + (\lambda y)^2 = 1 + 4 = 5$

$\Rightarrow$ $\lambda = \pm\sqrt{5}$ $\Rightarrow$ $x = \pm\frac{1}{\sqrt{5}}$, $y = \pm\frac{2}{\sqrt{5}}$

Thus, we get two critical points:

$\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right)$ with $\lambda = \sqrt{5}$

$\left(-\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}, 0\right)$ with $\lambda = -\sqrt{5}$

If $\boxed{\lambda = 3}$ then from (i), (ii) we have $x = \frac{1}{3}$, $y = \frac{2}{3}$

and then from (iv) we get $z^2 = 1 - \frac{1}{9} - \frac{4}{9} = \frac{4}{9}$ $\Rightarrow$ $z = \pm 2/3$

PTO

Thus, we get the points $\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$ and $\left(\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right)$ both with $\lambda = 3$

b) From the constraint we have $\nabla h = (2x, 2y, 2z)$.

Furthermore, $L_{xx} = L_{yy} = -2\lambda$, $L_{zz} = 6 - 2\lambda$

$$L_{xy} = L_{xz} = L_{yz} = 0.$$

Thus, the general form of the bordered Hessian is

$$H_B = \begin{pmatrix} 0 & 2x & 2y & 2z \\ 2x & -2\lambda & 0 & 0 \\ 2y & 0 & -2\lambda & 0 \\ 2z & 0 & 0 & 6-2\lambda \end{pmatrix} = 2 \begin{pmatrix} 0 & x & y & z \\ x & -\lambda & 0 & 0 \\ y & 0 & -\lambda & 0 \\ z & 0 & 0 & 3-\lambda \end{pmatrix}$$

Since $n = 3$, $m = 1$ we need $n - m = 2$ LPMs: LPM_4, LPM_3.

$$LPM_4 = \det(H_B) = 16 \begin{vmatrix} 0 & x & y & z \\ x & -\lambda & 0 & 0 \\ y & 0 & -\lambda & 0 \\ z & 0 & 0 & 3-\lambda \end{vmatrix} = 16 \left\{ -x \begin{vmatrix} x & y & z \\ 0 & -\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} - \lambda \begin{vmatrix} 0 & y & z \\ y & -\lambda & 0 \\ z & 0 & 3-\lambda \end{vmatrix} \right\} =$$

$$= 16\left[-\lambda^2 (x^2 + y^2 + z^2) + 3\lambda (x^2 + y^2) \right]$$

$$LPM_3 = 8 \begin{vmatrix} 0 & x & y \\ x & -\lambda & 0 \\ y & 0 & -\lambda \end{vmatrix} = 8\lambda (x^2 + y^2).$$

- At $\left(\pm\frac{1}{\sqrt{5}}, \pm\frac{2}{\sqrt{5}}, 0\right)$ with $\lambda = \pm\sqrt{5}$ we have

  $LPM_4 = 16(\pm 3\sqrt{5} - 5)$, $LPM_3 = \pm 8\sqrt{5}$

  $+$ sign $LPM_4 > 0$, $LPM_3 > 0$ indefinite point

  $-$ sign $LPM_4 < 0$, $LPM_3 < 0$ pos. definite $\Rightarrow$ local min.

- At $\left(\frac{1}{3}, \frac{2}{3}, \pm\frac{2}{3}\right)$ with $\lambda = 3$, $LPM_4 = -64 < 0$, $LPM_3 = 40/3 > 0$

  Hence neg. definite $\Rightarrow$ local max