This question paper consists of
6 printed pages, each of which
is identified by the reference MATH264001.

All calculators must carry
an approval sticker issued
by the School of Mathematics.

School of Mathematics

January 2018

MATH264001

Introduction to Optimisation

Time Allowed: 2 hours

You must attempt to answer 4 questions.
If you answer more than 4 questions, only your best 4 answers will be counted towards your final mark for this exam.
All questions carry equal marks.

1. (a) Consider the function

$$h(x,y,z) = x^2 - 2y^2 + 3z^2 \ ,$$

and its level surface S through the point $(1,1,1)$. Determine the shortest distance from S to the origin by considering the function $f(x,y,z) = x^2 + y^2 + z^2$ and writing down the Lagrangian for finding the minimum of f subject to (x,y,z) lying on S. Determine the critical points and by comparison of the values of f at these points determine the minimum distance.

Answer: The Lagrangian is

$$L(x,y,z,\lambda) = x^2 + y^2 + z^2 - \lambda(x^2 - 2y^2 + 3z^2 - 2) \ ,$$

with first order conditions:

$$L_x = 2(1-\lambda)x = 0 \ , \quad L_y = 2(1+2\lambda)y = 0 \ , \quad L_z = (2(1-3\lambda)z = 0 \ ,$$

$$\text{and} \quad L_\lambda = 2 - x^2 + 2y^2 - 3z^2 = 0 \ .$$

The stationary points are $(\pm\sqrt{2},0,0)$ with $\lambda = 1$ and $(0,0,\pm\sqrt{\frac{2}{3}})$ with $\lambda = 1/3$. Since $f(\pm\sqrt{2},0,0) = 2$ and $f(0,0,\pm\sqrt{\frac{2}{3}}) = 2/3$ we have that the latter points are nearest to the surface with distance $\sqrt{2/3}$.

(b) Let $z(x,y)$ be defined implicitly by the relation

$$yz^4 - 2xz^2 + x^2y = 1 \ .$$

Find the partial derivatives z_x and z_y in terms of x, y and z, and evaluate these derivatives at the point $x = 3$, $y = 1$, $z = 2$. If, furthermore, x and y are related by the relation $x^2 - 2y^2 = 7$, find dz/dx at the point $(3,1,2)$.

Answer: The z_x and z_y partial derivatives are

$$z_x = \frac{1}{2}\,\frac{xy - z^2}{xz - yz^3} \ , \quad z_x = \frac{1}{4}\,\frac{x^2 + z^4}{xz - yz^3} \ .$$

At $(3,1,2)$ they acquire the values: $1/4$ and $-25/8$ respectively. At the curve $x^2 - 2y^2 = 7$ through that point, we have $dy/dx = 3/2$ and $dz/dx = -71/16$.

(c) Calculate the gradient of the function h given in part (a), and hence determine the (unit) normal vector to the surface S at $(1,1,1)$. Derive the equation of the tangent plane to S at that point, and the rate of increase of h, $D_{\mathbf{u}}h(1,1,1)$, in the direction $\mathbf{u}$ parallel to the z-axis.

Answer: The gradient is $\nabla h = (2x, -4y, 6z)$ and at $(1,1,1)$ we have $\nabla h(1,1,1) = (2,-4,6)$. The tangent plane to S at that point has the equation $2x - 4y + 6z = 4$ with unit normal given by $\hat{\mathbf{n}} = (1/\sqrt{14}, -2/\sqrt{14}, 3/\sqrt{14})$. The directional derivative in the direction $\mathbf{u} = (0,0,1)$ is $D_{\mathbf{u}}h(1,1,1) = 6$.

2. (a) Consider the function

$$f(x,y) = xy - 3x^2y^2$$

with x, y constrained by the relation $h(x,y) = x^2 + y^2 = 2$. Find the partial derivatives f_x, f_y and the total derivatives df/dx and df/dy, and explain the difference between partial and total derivatives geometrically (if possible through a sketch).

Answer: The partial derivatives of f are given by

$$f_x = y - 6xy^2 \ , \quad f_y = x - 6x^2y \ ,$$

and from he constraint $h = 2$ by implicit differentiation we get $dy/dx = -x/y$, or equivalently $dx/dy = -y/x$. Thus, the total derivatives are

$$\frac{df}{dx} = y - 6xy^2 - (x - 6x^2y)\frac{x}{y} \ , \quad \frac{df}{dy} = x - 6x^2y - (y - 6xy^2)\frac{y}{x} \ .$$

(b) Write down the Lagrangian for the problem of finding the critical points of the function f of part (a), subject to the constraint $h = 2$. Write down the first order conditions, and find all 8 stationary points for the function $f(x,y)$ subject to the given constraint.
Hint: It may be useful to consider sums and differences of the first order conditions on the Lagrangian.

Answer: The Lagrangian is

$$L(x,y,\lambda) = xy - 3x^2y^2 - \lambda(x^2 + y^2 - 2) \ ,$$

with first order conditions

$$L_x = y - 6xy^2 - 2\lambda x = 0 \ , L_y = x - 6x^2y - 2\lambda y = 0 \ , \quad L_\lambda = 2 - x^2 - y^2 = 0 \ .$$

The stationary points are $(1,1)$ and $(-1,-1)$ both with $\lambda = -5/2$, the points $(1,-1)$ and $(-1,1)$, both with $\lambda = -7/2$, and the additional four points:

$$\left(\sqrt{1 + \tfrac{1}{6}\sqrt{35}}, \sqrt{1 - \tfrac{1}{6}\sqrt{35}}\right) \ , \quad \left(-\sqrt{1 + \tfrac{1}{6}\sqrt{35}}, -\sqrt{1 - \tfrac{1}{6}\sqrt{35}}\right) \ ,$$

$$\left(\sqrt{1 - \tfrac{1}{6}\sqrt{35}}, \sqrt{1 + \tfrac{1}{6}\sqrt{35}}\right) \ , \quad \left(-\sqrt{1 - \tfrac{1}{6}\sqrt{35}}, -\sqrt{1 + \tfrac{1}{6}\sqrt{35}}\right) \ ,$$

all four with $\lambda = 0$.

(c) Give the general form of the bordered Hessian H_B for the problem of part (b), (i.e., give its entries in terms of x, y and the Lagrange multiplier λ, but without explicitly calculating the Hessian at the stationary points). Assuming that $\det(H_B) \neq 0$, which leading principal minors determine the character of the critical points?

Answer: Since $\nabla h = (2x, 2y)$ and the second order derivatives

$$L_{xx} = -6y^2 - 2\lambda \ , \quad L_{yy} = -6x^2 - 2\lambda \ , m \quad L_{xy} = 1 - 12xy \ ,$$

we have the bordered Hessian

$$H_B = \begin{pmatrix} 0 & 2x & 2y \\ 2x & -6y^2 - 2\lambda & 1 - 12xy \\ 2y & 1 - 12xy & -6x^2 - 2\lambda \end{pmatrix}$$

With $n = 2$, $m = 1$ we only need $n - m = 1$ LPM, namely $LPM_3 = \det(H_B)$.

3. (a) Consider the quadratic form

$$Q(x_1, x_2, x_3) = \tfrac{1}{2}(x_1^2 + x_2^2) - 3x_1x_2 + 2(x_1x_3 + x_2x_3) = (x_1, x_2, x_3)\boldsymbol{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} .$$

Find the symmetric matrix $\boldsymbol{A}$ and apply the leading principal minor test to show that Q is indefinite.

Answer: The symmetric matrix $\boldsymbol{A}$ is given by

$$\boldsymbol{A} = \begin{pmatrix} 1/2 & -3/2 & 1 \\ -3/2 & 1/2 & 1 \\ 1 & 1 & 0 \end{pmatrix} .$$

With $LPM_3 = \det(\boldsymbol{A}) = -4 < 0$, $LPM_2 = -2 < 0$, $LPM_1 = 1/2 > 0$ the LPM test tells us we have an *indefinite* quadratic form.

(b) Compute the eigenvalues and the corresponding unit eigenvectors, and use them to construct the orthogonal matrix $\boldsymbol{O}$ diagonalising $\boldsymbol{A}$. Give the normal form of Q and construct the variables in terms of which Q can be written as a sum of squares, i.e., determine variables $\tilde{x}_i$, $i = 1, 2, 3$, such that Q adopts the form

$$Q = \lambda_1 \tilde{x}_1^2 + \lambda_2 \tilde{x}_2^2 + \lambda_3 \tilde{x}_3^2,$$

and confirm from the eigenvalues that Q is indefinite.

Answer: The eigenvalues are given by $\lambda_1 = 2$, $\lambda_2 = -2$ and $\lambda_3 = 1$. The corresponding unit eigenvectors are

$$\boldsymbol{v}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} , \quad \boldsymbol{v}_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} , \quad \boldsymbol{v}_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} ,$$

respectively, and hence the diagonalizing orthogonal matrix is given by

$$\boldsymbol{O} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & -1/\sqrt{3} & 2/\sqrt{6} \end{pmatrix} .$$

The normal form for Q is

$$Q(x_1, x_2, x_3) = 2\left(\frac{x_1 - x_2}{\sqrt{2}}\right)^2 - 2\left(\frac{x_1 + x_2 - x_3}{\sqrt{3}}\right)^2 + \left(\frac{x_1 + x_2 + 2x_3}{\sqrt{6}}\right)^2 ,$$

which is manifestly indefinite.

(c) Consider next the quadratic form of part (a), but subject to the constraint

$$x_1 + x_2 - x_3 = 0 \ .$$

Write down the relevant bordered Hessian, and determine the sign character of the constrained quadratic form using the leading principal minor test for bordered Hessians.

Answer: If we impose the constraint $x_1 + x_2 - x_3$ we observe that the second term in the normal form of part (b) disappears, and the constrained quadratic form reduces to $Q = 2\widetilde{x}_1^2 + \widetilde{x}_3^2$ on the constrained variables. Thus, the corresponding reduced quadratic form is positive definite. This is confirmed by considering the bordered Hessian

$$H_B = \begin{pmatrix} 0 & 1 & 1 & -1 \\ 1 & 1/2 & -3/2 & 1 \\ 1 & -3/2 & 1/2 & 1 \\ -1 & 1 & 1 & 0 \end{pmatrix} ,$$

which has $LPM_4 = \det(H_B) = -6 < 0$, $LPM_3 = -4 < 0$, so that successive LPMs have the same sign and we identify $\mathrm{sgn}(LPM_4) = (-1)^m$ (with $n = 3$, $m = 1$), hence the constrained quadratic form is PD.

4. (a) Consider the problem of maximising a function $f(x, y, z)$ subject to an inequality constraint $g(x, y, z) \leq b$, with g some function and b a constant value, and subject to the positivity conditions $x \geq 0$, $y \geq 0$, $z \geq 0$. Write down the general form of the Kuhn-Tucker (KT) Lagrangian and all four relevant equality and eight inequality conditions on the variables and Lagrangians.

Answer: The general form of the KT Lagrangian in this situation is

$$\bar{L}(x, y, z, \lambda) = f(x, y, z) - \lambda(g(x, y, z) - b) \ .$$

The relevant equalities are:

$$x\frac{\partial \bar{L}}{\partial x} = 0 \ ,$$

(b) Consider the function

$$f(x, y, z) = xyz + z$$

subject to the conditions

$$x^2 + y^2 + z^2 \leq 25 \ , \quad x \geq 0 \ , \quad y \geq 0 \ , \quad z \geq 0 \ .$$

Use the KT method to find all possible maximiser candidates, and by evaluating the value of f at those points determine the maximum. State which constraints are binding at the maximum.

5. (a) A company produces umbrellas at a rate $Q = x^{1/2}y^{1/2}$ in terms of inputs x and y which are positive quantities. Each umbrella sells at a rate of $£12$, while the production cost is $C(x, y) = x^{3/2} + 12y$ pounds. Write down the revenue and profit functions and compute the critical values of x and y and the stationary values of the profit. By computing the Hessian and using the second order conditions, verify that this stationary value is a maximum.

(b) A company produces a product which sells for $£1$. The production rate Q and cost function are given by

$$Q = 20 - \tfrac{1}{3}x^3 - 2y^2 - z^2 \ , \quad C = 6x + 3y + 5z \ ,$$

where x, y and z are input variables which are subject to the constraint $x+y+z = 3$. Write down the profit function and the Lagrangian for the problem of maximising the profit subject to the constraint. Find the values of x, y and z at the stationary point where all input variables are positive.

(c) For the problem in part (b) find the 4×4 bordered Hessian matrix and evaluate the two relevant leading principal minors. Hence, show that the stationary point with x, y and z all positive is a local maximum and find the profit at this point.

End.