MATH2640 Introduction to Optimisation

Example Sheet 3

Please hand the assessed questions by by Thursday 14th November 2019, 5 pm

Eigenvalues & normal forms, characterisation of critical points, economic optimisation.
Based on material in Lectures 9 to 13

Assessed Questions

A1. Using the results on determining the sign of quadratic forms, find and classify (where possible) the *stationary points* of the following function:

$$f(x, y, z) = x^2 - x(y + z) + \tfrac{1}{3}y^3 + y^2 + \tfrac{1}{2}z^2 - 2z.$$

A2.

i) Find the symmetric matrix $\mathbf{A}$ associated with the quadratic form

$$Q(x_1, x_2, x_3) = x_1^2 - 2x_2^2 - 7x_3^2 - 10x_1x_3 + 16x_2x_3.$$

Use the Principal Minor Test to establish whether Q is positive/negative (semi-)definite, or indefinite.

ii) Find the eigenvalues and unit eigenvectors of the matrix $\mathbf{A}$, and from the unit eigenvectors give the normal form of Q. Use this to confirm the result of part i).

A3. A company produces and sells on two products x and y for which the demand functions are

$$x = 30 - 0.5P_x\ , \quad y = 25 - P_y,$$

with P_x and P_y being their prices. The combined production cost is

$$C = x^2 + kxy + y^2 + 10$$

for some constant k. Find (a) the profit maximising level of output for each product, (b) the profit maximising price of each product, and (c) the maximum profit, for $k = 2$. (d) Determine for which values of k, by analysing the Hessian matrix, the profit is indeed maximised; are conditions on the prices met?

A4. Cobb and Douglas (1928) defined the function with their name using data from the American economy from 1899 to 1922, relating an output function $Q = Q(K, L)$ to capital K and labour L as follows

$$Q(K, L) = pL^aK^b$$

with fitting coefficients p, a and b. Using modern least-square data techniques, Felipe and Adams (2005) found that $p = 0.8353 = e^{-0.18}, a = 0.807$ and $b = 0.233$.
a) Assuming a cost function $C(K, L) = w_LL+w_KK$ and assuming a stable economy in equilibrium, and given the 1899 (equilibrium) data with $Q^* = 100, K^* = 100, L^* = 100$ determine w_k, w_L in 1899 and
b) calculate the cost (as a formula in terms of Q^*, a, b, p and for the values found).
c) In 1922, $Q^* = 240, K^* = 431$ and $L^* = 161$, determine w_K, w_L again and calculate the cost.
d) How well does the Cobb-Douglas function fit these data for Q (also give errors in percentages)?
e) Is the profit maximised at these two equilibria? (See wikipedia for the "Cobb-Douglas production function".)

P.T.O.

Further Questions for Workshop Practice

B1. Using the results on determining the sign of quadratic forms, find and classify (where possible) the *stationary points* of the following functions:

(i) $f(x, y, z) = 2x^2 + xy + 4y^2 + xz + z^2 + 2$.

(ii) $f(x, y, z) = -x^3 + 3xz + 2y - y^2 - 3z^2$.

(iii) $f(x, y) = 3x^4 + 3x^2y - y^3$.

B2.

a) Give the symmetric matrix $\boldsymbol{A}$ associated with the quadratic form

$$Q(x, y) = 10xy - x^2 - y^2 \ .$$

Find the eigenvalues λ_1, λ_2 of this matrix and hence characterize Q. (Confirm your results using the Principal Minor Test.) Find also the unit eigenvectors of the matrix and thus the explicit transformation to new variables $\widetilde{x}$, $\widetilde{y}$, expressing them in terms of x and y, in order to write Q in standard form, i.e.

$$Q(x, y) = \lambda_1 \widetilde{x}^2 + \lambda_2 \widetilde{y}^2 \ .$$

b) Give the symmetric matrix $\boldsymbol{A}$ associated with the quadratic form

$$Q(x_1, x_2, x_3) = 8x_1x_2 + 2x_1x_3 + 18x_2x_3 - 5x_1^2 - 13x_2^2 - 10x_3^2 \ .$$

Find the eigenvalues λ_1, λ_2, λ_3 of this matrix and hence characterize Q. (Confirm your results using the Principal Minor Test.) Find also the unit eigenvectors of $\boldsymbol{A}$, and hence an orthogonal matrix $\boldsymbol{O}$ that diagonalizes $\boldsymbol{A}$. Use this to give the explicit transformation to new variables $\widetilde{x}_1$, $\widetilde{x}_2$, $\widetilde{x}_3$, expressing them in terms of x_1, x_2, x_3, such that $Q(x_1, x_2, x_3)$ can be written in the form

$$Q(x, y) = \lambda_1 \widetilde{x}_1^2 + \lambda_2 \widetilde{x}_2^2 + \lambda_3 \widetilde{x}_3^2 \ .$$

B3.

a) A railway company sells tickets to first class and economy class customers at prices p_1 and p_2 respectively. They can sell $Q_1 = 210 - p_1$ first class seats, and $Q_2 = 90 - p_2$ economy class seats. The cost to the company is

$$C = 6000 + Q_1^2 + Q_1Q_2 + Q_2^2 \ .$$

Write down the profit $\Pi(Q_1, Q_2)$ as a function of Q_1, Q_2, and determine the prices p_1^* and p_2^* that give a stationary value for the profit. By computing the Hessian and applying the leading principal minor test, establish that the stationary values give a maximum. Can the company make a profit, or must it cut costs to avoid running at a loss?

b) A theatre company sells tickets to first class and second class customers at prices p_1 and p_2 respectively. They can sell $Q_1 = 40 - p_1$ first class seats, and $Q_2 = 60 - 2p_2$ second class seats. The cost to the company is

$$C = K + 20Q_1 + 10Q_2,$$

where K is a constant. Write down the profit in terms of p_1 and p_2. Find the prices p_1 and p_2 that give a stationary value for the profit and show it is a maximum. Determine the value of K at which the company just breaks even.

P.T.O.

B4. A Cobb-Douglas production function is given by

$$Q(x,y) = x^{\frac{1}{2}} y^{\frac{1}{4}},$$

where x and y are the input variables. The price of each product is p, and the cost of production is $C(x,y) = ax + by$.

a) Write down the profit, Π. Find the stationary point for the profit Π and show that the critical values for x and y are given by

$$x_* = \frac{p^4}{32a^3b}\ , \quad y_* = \frac{p^4}{64a^2b^2}\ ,$$

and that the profit at the stationary point is given by

$$\Pi_* = \frac{p^4}{64a^2b}\ .$$

b) Find the Hessian matrix at the stationary point, and verify that it is indeed a maximum.

B5. A company produces saucepans at a rate $Q = x^{1/4}y^{1/4}z^{1/4}$ which it sells for £4 per item. The inputs x, y and z are positive quantities. The cost of production is $C = x + \frac{1}{2}y + \frac{1}{3}z$ pounds. Write down the profit, Π, in terms of x, y and z.

a) Express the values of x, y, z at the stationary point in terms of Q, and hence find the value of x, y and z at which the profit takes a stationary value. Find also Q, C and Π at this stationary point.

b) Show that the Hessian matrix for this problem can be written as

$$H = \begin{pmatrix} \frac{-3Q}{4x^2} & \frac{Q}{4xy} & \frac{Q}{4xz} \\ \frac{Q}{4xy} & \frac{-3Q}{4y^2} & \frac{Q}{4yz} \\ \frac{Q}{4xz} & \frac{Q}{4yz} & \frac{-3Q}{4z^2} \end{pmatrix},$$

and hence find the leading principal minors LPM_1, LPM_2 and LPM_3. Deduce that the profit is a maximum at the stationary point.