

# **Numerical and Algorithmic Implementation Treatise for Universal Stochastic Predictors**

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# Chapter 1

# Discretization Fundamentals and Monte Carlo Simulations

## 1.1 Pseudo-Random Number Generation

The stochastic integrator relies on an entropy source  $\xi \sim \mathcal{D}$ .

- **Gaussian:** For Brownian motion  $dW_t \approx \sqrt{\Delta t}Z$ , with  $Z \sim \mathcal{N}(0, 1)$ . Recommended generators are Mersenne Twister or PCG64 for long periods.
- **Levy/Jumps:** Use the Chambers-Mallows-Stuck method to simulate stable variables  $S(\alpha, \beta, \gamma, \delta)$ .

### 1.1.1 Chambers-Mallows-Stuck (CMS) Algorithm

To generate a standard  $\alpha$ -stable random variable  $S(\alpha, \beta = 0, \gamma = 1, \delta = 0)$  with  $\alpha \neq 1$ :

1. Sample  $U \sim \text{Uniform}(-\pi/2, \pi/2)$  and  $W \sim \text{Exponential}(1)$ .

2. Compute:

$$X = \frac{\sin(\alpha U)}{(\cos U)^{1/\alpha}} \cdot \left[ \frac{\cos((1-\alpha)U)}{W} \right]^{(1-\alpha)/\alpha}$$

3. Return  $X$ . For the general case  $Y \sim S(\alpha, \beta, \gamma, \delta)$ , apply the corresponding affine transform.

## 1.2 Stochastic Integration Schemes

### 1.2.1 Euler-Maruyama Scheme

For the stochastic ODE  $dX_t = b(X_t)dt + \sigma(X_t)dW_t$ , the first-order discretization is:

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#### Algorithm 1 Euler-Maruyama Integrator

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- 1: **Input:**  $X_0, T, N, b(\cdot), \sigma(\cdot)$
  - 2:  $\Delta t \leftarrow T/N$
  - 3:  $X \leftarrow \text{array of length } N + 1$
  - 4: **for**  $k \leftarrow 0$  **to**  $N - 1$  **do**
  - 5:      $Z \sim \mathcal{N}(0, 1)$
  - 6:      $X_{k+1} \leftarrow X_k + b(X_k)\Delta t + \sigma(X_k)\sqrt{\Delta t}Z$
  - 7: **end for**
  - 8: **Return**  $X$
-

### 1.2.2 Milstein Scheme

Improves strong convergence to order 1.0. Requires the derivative of volatility  $\sigma'(x)$ .

$$\hat{X}_{k+1} = \hat{X}_k + b_k \Delta t + \sigma_k \Delta W_k + \frac{1}{2} \sigma_k \sigma'_k ((\Delta W_k)^2 - \Delta t)$$

**Note:** If  $\sigma(x)$  is constant (additive volatility), Milstein reduces to Euler-Maruyama.

## 1.3 Jump Process Simulation (Branch C)

For  $dX_t = b(X_t)dt + \sigma(X_t)dW_t + dJ_t$ , where  $J_t$  is a compound Poisson process with intensity  $\lambda$  and jump size  $Y \sim F_Y$ :

1. Simulate the number of jumps in  $[t, t + \Delta t]$ :  $N_{\text{jump}} \sim \text{Poisson}(\lambda \Delta t)$ .
2. If  $N_{\text{jump}} > 0$ , generate sizes  $Y_1, \dots, Y_{N_{\text{jump}}}$ .
3. Update:  $X_{k+1} = X_{k+1}^{\text{diff}} + \sum Y_i$ .

## Chapter 2

# System Identification Engine (SIA) Implementation

### 2.1 Multifractal Estimation (WTMM)

The WTMM (Wavelet Transform Modulus Maxima) algorithm extracts the singularity spectrum  $D(h)$  in quasi-real time.

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#### Algorithm 2 Detailed Discrete WTMM - Maxima Tracking

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```
1: Input: Time series  $X$ , scales  $a_i \in \{2^0, 2^{0.1}, \dots, 2^J\}$  (dense dyadic scales).
2: Step 1: CWT (FFT) and Local Maxima
3: For each scale  $a_j$ , extract the maxima set  $M_j = \{(b, |W_{a_j}(b)|)\}$ .
4: Step 2: Maxima Linking (Tracking)
5: Initialize lines  $\mathcal{L} = \{(b, |W_{a_J}(b)|)\}_{b \in M_J}$  (from coarse scale).
6: for  $j \leftarrow J - 1$  downto 1 do
7:   for each line  $L \in \mathcal{L}$  with last point  $(b_{\text{last}}, \text{mod})$  do
8:     Search  $(b_{\text{curr}}, \text{mod}_{\text{curr}}) \in M_j$  such that  $|b_{\text{curr}} - b_{\text{last}}| < C \cdot a_j$  (cone of influence).
9:     If multiple candidates, choose the one with highest modulus.
10:    Extend  $L \leftarrow L \cup \{(b_{\text{curr}}, \text{mod}_{\text{curr}})\}$ .
11:   end for
12: end for
13: Step 3: Partition Function For moments  $q \in [-5, 5]$ :
14:  $Z(q, a) = \sum_{L \in \mathcal{L}} (\sup_{(b, \text{mod}) \in L \cap \text{scale}(a)} \text{mod})^q$ 
15: Step 4: Exponents
16:  $\tau(q) \leftarrow$  slope of the linear regression  $\log Z(q, a)$  vs  $\log a$ .
17: Output: Legendre spectrum  $D(h) = \min_q (qh - \tau(q))$ .
```

---

### 2.2 Regime Change Detection (CUSUM Test)

The `RegimeChangedEvent` is emitted when the Page statistic of cumulative residuals exceeds an adaptive threshold. To improve robustness in heavy-tail regimes, we incorporate a kurtosis adjustment.

**Implementation Note 2.1 (Kurtosis Adjustment Rationale)** *The term  $(1 + \ln(\kappa_t/3))$  adjusts the threshold based on tail heaviness:*

- For Gaussian distributions:  $\kappa \approx 3 \Rightarrow \ln(\kappa/3) \approx 0$ , threshold remains  $h_t \approx k\sigma_t$

---

**Algorithm 3** Discrete CUSUM with Kurtosis Adjustment

---

```

1: Input: Standardized residuals  $e_t$ , base factor  $k$ , rolling window  $W$ .
2:  $S_0 \leftarrow 0$ ,  $G_0^+ \leftarrow 0$ ,  $G_0^- \leftarrow 0$ 
3: Initialize buffer  $\mathcal{B} \leftarrow []$  (rolling residual window)
4: for  $t \leftarrow 1$  to  $N$  do
5:   Add  $e_t$  to  $\mathcal{B}$  and keep only the last  $W$  values
6:   Compute rolling statistics:
7:      $\mu_t \leftarrow \text{mean}(\mathcal{B})$ 
8:      $\sigma_t \leftarrow \text{std}(\mathcal{B})$ 
9:      $m_4 \leftarrow \frac{1}{W} \sum_{i \in \mathcal{B}} (e_i - \mu_t)^4$                                  $\triangleright$  Fourth moment
10:     $\kappa_t \leftarrow \frac{m_4}{\sigma_t^4}$                                                $\triangleright$  Kurtosis
11:   Compute adaptive threshold:
12:      $h_t \leftarrow k \cdot \sigma_t \cdot (1 + \ln(\kappa_t/3))$                                  $\triangleright$  Log tail adjustment
13:   Update CUSUM statistic:
14:      $G_t^+ \leftarrow \max(0, G_{t-1}^+ + e_t - k)$ 
15:      $G_t^- \leftarrow \max(0, G_{t-1}^- - e_t - k)$ 
16:   if  $G_t^+ > h_t$  or  $G_t^- > h_t$  then
17:     Emit RegimeChangedEvent
18:      $G_t^+, G_t^- \leftarrow 0$                                                          $\triangleright$  Reset CUSUM
19:   end if
20: end for

```

---

- For leptokurtic distributions (heavy tails):  $\kappa > 3 \Rightarrow \ln(\kappa/3) > 0$ , the threshold increases proportionally, reducing false alarms during high-volatility non-Gaussian periods without structural change
- The logarithmic adjustment avoids explosive growth for extreme  $\kappa$

This mechanism is consistent with the Adaptive Threshold with Kurtosis Lemma in the theory document.

## 2.3 Sensitivity Computation (Malliavin/AAD)

Instead of perturbing inputs (finite differences), we compute the exact derivative of the computational graph.

### 2.3.1 Tangential Processes and Bismut-Elworthy-Li

For a general diffusion  $dX_t = b(X_t)dt + \sigma(X_t)dW_t$ , the Malliavin weight formula generalizes as:

$$\partial_{X_0} E[f(X_T)] = E \left[ f(X_T) \int_0^T (\sigma^{-1}(X_s) Y_s \nabla b(X_s))^{\top} dW_s \right]$$

where  $Y_t = \nabla_{X_0} X_t$  is the **first variation process**, satisfying the linearized ODE:

$$dY_t = \nabla b(X_t) Y_t dt + \sum_{k=1}^d \nabla \sigma_k(X_t) Y_t dW_t^k, \quad Y_0 = I$$

We must solve the coupled system  $(X_t, Y_t)$  or use automatic differentiation (forward-mode AD) to propagate the Jacobian along the trajectory.

### 2.3.2 Delta-Malliavin Monte Carlo Algorithm

To compute  $\Delta = \partial_{X_0} E[f(X_T)]$  in the simplified case:

$$\Delta \approx E \left[ f(X_T) \frac{W_T}{\sigma X_0 T} \right]$$

In computation graphs (TensorFlow/PyTorch):

1. Define the computational graph of the payoff  $L = f(X_T)$ .
2. Simulate forward paths  $X_0 \rightarrow X_1 \cdots \rightarrow X_T$ .
3. Run the backward pass to obtain  $\nabla_{X_0} L$ .
4. Average  $\nabla_{X_0} L$  over  $M$  paths.

# Chapter 3

# Numerical Solvers for Prediction Kernels

## 3.1 Branch A: Hilbert Projection and Wiener Filtering

### 3.1.1 Levinson-Durbin Recursive Algorithm

To solve the discrete Yule-Walker normal equations (the discrete equivalent of Wiener-Hopf) and obtain the optimal linear predictor of order  $p$ ,  $\hat{X}_{t+1} = \sum_{k=1}^p \phi_k X_{t-k+1}$ : **Note:** For  $O(N \log N)$  efficiency in

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#### Algorithm 4 Levinson-Durbin Recursion

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```
1: Input: Autocorrelations  $R_0, R_1, \dots, R_p$ .
2:  $E_0 \leftarrow R_0$ 
3: for  $k \leftarrow 1$  to  $p$  do
4:    $\lambda_k \leftarrow (R_k - \sum_{j=1}^{k-1} \phi_j^{(k-1)} R_{k-j})/E_{k-1}$ 
5:    $\phi_k^{(k)} \leftarrow \lambda_k$ 
6:   for  $j \leftarrow 1$  to  $k-1$  do
7:      $\phi_j^{(k)} \leftarrow \phi_j^{(k-1)} - \lambda_k \phi_{k-j}^{(k-1)}$ 
8:   end for
9:    $E_k \leftarrow E_{k-1}(1 - \lambda_k^2)$ 
10: end for
11: Output: Filter coefficients  $\phi^{(p)}$ .
```

---

long convolutions, use FFT (convolution theorem) instead of direct time recursion.

## 3.2 Branch B: HJB Equation and Viscosity Methods

### 3.2.1 Monotone Finite Difference Schemes

The Barles-Souganidis theorem (1991) establishes necessary conditions for convergence to viscosity solutions.

**Numerical Scheme 3.1 (Generalized Upwind Scheme)** *For the equation  $H(u, u_x, u_{xx}) = 0$ , we use:*

$$D_x^+ u_i = \frac{u_{i+1} - u_i}{\Delta x}, \quad D_x^- u_i = \frac{u_i - u_{i-1}}{\Delta x}$$
$$D_{xx} u_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2}$$

The time step is updated explicitly:

$$u_i^{n+1} = u_i^n - \Delta t \cdot H_{\text{num}}(u_i^n, D_x^+ u_i^n, D_x^- u_i^n, D_{xx} u_i^n)$$

**Monotonicity Condition:** The numerical Hamiltonian  $H_{\text{num}}(u, p, q, r)$  must be non-decreasing in  $u$ ,  $p$ ,  $q$ , and  $r$  (depending on characteristic flow direction).

### 3.2.2 Deep Galerkin Method (DGM)

For high dimension ( $d > 3$ ), where grids are infeasible (curse of dimensionality).

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#### Algorithm 5 DGM Neural Network Training

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- 1: **Input:** Network  $f_\theta(t, x)$ , PDE  $\mathcal{L}u = 0$ , domain  $\Omega$ , steps  $M$ .
  - 2: **for**  $i \leftarrow 1$  **to**  $M$  **do**
  - 3:     Sample random points:  
   4:      $\{t_j, x_j\}_j \sim \text{Unif}([0, T] \times \Omega)$  (interior)  
   5:      $\{\tau_k, \xi_k\}_k \sim \text{Unif}(\{T\} \times \Omega)$  (terminal condition)  
   6:      $\{\zeta_l, \gamma_l\}_l \sim \text{Unif}([0, T] \times \partial\Omega)$  (boundary)
  - 7:     Compute loss:  
   8:      $L_1 = \frac{1}{N} \sum (\partial_t f + \mathcal{L}f(t_j, x_j))^2$   
   9:      $L_2 = \frac{1}{K} \sum (f(T, \xi_k) - g(\xi_k))^2$   
 10:      $L_3 = \frac{1}{L} \sum (f(\zeta_l, \gamma_l) - h(\gamma_l))^2$   
 11:      $L(\theta) = L_1 + L_2 + L_3$   
 12:     Update  $\theta \leftarrow \theta - \eta \nabla_\theta L(\theta)$  (Adam/SGD)
  - 13: **end for**
- 

## 3.3 Branch C: Jump Integro-Differential Equation

### 3.3.1 Delta-Malliavin Algorithm on Poisson Spaces

For processes with jump component  $J_t$ , sensitivity is based on Malliavin integration by parts with probability weights:

$$\partial_{X_0} E[f(X_T)] \approx E \left[ f(X_T) \left( \frac{W_T}{\sigma T} + \sum_{i=1}^{N_T} \frac{\partial_X \Delta X_{\tau_i}}{\Delta X_{\tau_i}} \right) \right]$$

Implementation requires tracking jump times  $\tau_i$  and amplitudes  $\Delta X_{\tau_i}$  during the forward Monte Carlo step.

### 3.3.2 IMEX (Implicit-Explicit) Scheme for PIDEs

To solve the Fokker-Planck equation with integral term  $\mathcal{I}[p](x) = \int p(y) \nu(dy)$ :

$$\frac{p_i^{n+1} - p_i^n}{\Delta t} = \underbrace{\mathcal{L}_{\text{diff}} p_i^{n+1}}_{\text{Implicit}} + \underbrace{\mathcal{I}[p^n]_i}_{\text{Explicit}}$$

The diffusion part is solved by inverting a tridiagonal matrix (Thomas algorithm). The convolution integral is evaluated explicitly using FFT at each time step  $O(N \log N)$ .

## 3.4 Branch D: Signature Computation

### 3.4.1 Chen Identity and Truncation

The signature tensor  $\mathbf{S}(X)_{0,t}$  up to level  $M$  lives in  $T^{(M)}(\mathbb{R}^d)$ . **Iterative Algorithm:** Given a discretized path with increments  $\Delta X_k = X_{t_{k+1}} - X_{t_k}$ : 1. Compute the signature of the linear segment  $\mathbf{S}(\Delta X_k) = \exp(\Delta X_k)$  in the tensor algebra. - Level 1:  $\Delta X_k$  - Level 2:  $\frac{1}{2}\Delta X_k \otimes \Delta X_k$  2. Concatenate using Chen multiplicativity:

$$\mathbf{S}(X)_{0,t_{k+1}} = \mathbf{S}(X)_{0,t_k} \otimes \mathbf{S}(\Delta X_k)$$

This tensor product is implemented efficiently by exploiting the triangular structure of tensors.

### 3.4.2 Log-Signatures

To reduce the feature vector dimension, we project the signature to the free Lie algebra via the Baker-Campbell-Hausdorff (BCH) formula. Recommended libraries: `iisignature` (Python/C++) or `signatory` (PyTorch, differentiable).

## Chapter 4

# Orchestrator: Regularized Optimal Transport

### 4.1 Robustness Circuit Breaker (Pre-Orchestrator)

Before Wasserstein weighting, apply strong conditional logic based on the Robustness Postulate for Singularities.

1. **Input:** SIA vector  $V_s$  and current weights  $w_t$ .
2. If  $\alpha(t) < \alpha_{\text{threshold}}$  (critical roughness) or  $d > 1.5$ :
  - Force  $w_D \leftarrow 1.0$  (Signature).
  - Switch Wasserstein cost function to Huber metric  $\rho_\delta(x - y)$ .
3. If **RegimeChangedEvent**:
  - Reset entropy:  $w_t \leftarrow \text{Softmax}(\mathbf{0})$  (uniform).
4. **Output:** Adjusted weights to initialize Sinkhorn.

### 4.2 Sinkhorn-Knopp Algorithm (Dual Space)

The classic algorithm is numerically unstable for small  $\varepsilon$ . Implement via **LogSumExp** with dual potentials  $f = \varepsilon \log u, g = \varepsilon \log v$ .

---

**Algorithm 6** Stabilized Sinkhorn Iterations (Log-Domain)

---

- 1: **Input:** Cost  $C$ , marginals  $a, b$  (in log:  $\alpha = \log a, \beta = \log b$ ),  $\varepsilon$ .
  - 2: Initialize duals  $f \leftarrow \mathbf{0}_N, g \leftarrow \mathbf{0}_N$
  - 3: **function** SMIN( $M, \epsilon$ )
    - 4:     **Return**  $-\epsilon \cdot \text{LogSumExp}(-M/\epsilon)$  row-wise.
    - 5: **end function**
    - 6: **while** not converged **do**
      - 7:      $f \leftarrow \text{Smin}(C - g^\top, \varepsilon) + \alpha$
      - 8:      $g \leftarrow \text{Smin}(C - f, \varepsilon) + \beta$
    - 9: **end while**
    - 10: Sinkhorn distance  $W_\varepsilon \approx \langle \exp(f/\varepsilon), (K \odot C) \exp(g/\varepsilon) \rangle$
-

### 4.3 JKO Proximal Scheme

The weight update  $w^{(k+1)} = \operatorname{argmin}_w \dots$  requires differentiation through the Sinkhorn loop. **Differentiable Implementation:** Use autodiff libraries (JAX/PyTorch) with `custom_vjp` (vector-Jacobian product) at the Sinkhorn fixed point, avoiding unrolling the loop to save memory:

$$\partial L / \partial C = P^* \quad (\text{Optimal Transport Plan})$$

This feeds the exact gradient  $\nabla_{W_2} \mathcal{F}$  to the L-BFGS optimizer. The weight update  $w^{(k)}$  is implemented as an implicit gradient step on the Wasserstein manifold:

$$w^{(k+1)} = \operatorname{Prox}_{\tau \mathcal{F}}^{W_2}(w^{(k)})$$

This is solved by nesting a Sinkhorn loop inside an L-BFGS optimizer or by projected gradient descent if entropic regularization is sufficient to smooth the energy landscape.

### 4.4 Dynamic Sinkhorn Regularization: Coupling to Local Volatility

**Motivation:** Static entropic annealing (doubling  $\varepsilon$  on failure) is robust but discrete. In highly turbulent markets, Wasserstein topology becomes rough gradually. The solution is to dynamically couple the entropic regularization parameter  $\varepsilon_t$  to local process volatility:

$$\varepsilon_t = \max(\varepsilon_{\min}, \varepsilon_0 \cdot (1 + \alpha \cdot \sigma_t))$$

where:

- $\varepsilon_0$ : nominal base regularization (typically  $10^{-2}$  or  $10^{-1}$ )
- $\varepsilon_{\min}$ : lower bound for numerical precision (e.g.,  $10^{-6}$ )
- $\sigma_t$ : local realized volatility of contemporaneous prediction error
- $\alpha > 0$ : sensitivity parameter (volatility-entropy coupling)

#### Theoretical Justification:

Under the Wasserstein flow model, the cost geometry  $C$  in the Kantorovich problem is proportional to the first variation of free energy  $\delta \mathcal{F} / \delta \rho$ . In high turbulence regimes:

1. The energy Hessian  $\nabla^2 \mathcal{F}$  has Lipschitz constants scaling with  $\|\sigma_t\|^2$  (amplified curvature).
2. The Sinkhorn operator contraction constant satisfies  $\rho_{\text{contraction}} \leq 1 - c \cdot \varepsilon$  (where  $c > 0$ ).
3. If  $\varepsilon$  is fixed and small while  $\|\sigma_t\|$  is large, convergence slows exponentially.
4. Increasing  $\varepsilon$  proportionally to  $\sigma_t$  re-accelerates convergence without losing transport precision when volatility normalizes.

#### Implementation Algorithm:

##### Numerical Example:

Suppose  $\varepsilon_0 = 0.1$ ,  $\alpha = 0.5$ ,  $\varepsilon_{\min} = 10^{-6}$ .

- **Normal regime:**  $\sigma_t = 0.02 \Rightarrow \varepsilon_t = \max(10^{-6}, 0.1 \times (1 + 0.5 \times 0.02)) = 0.101$
- **Moderate volatility:**  $\sigma_t = 0.1 \Rightarrow \varepsilon_t = 0.1 \times (1 + 0.05) = 0.105$
- **Stress:**  $\sigma_t = 0.5 \Rightarrow \varepsilon_t = 0.1 \times (1 + 0.25) = 0.125$
- **Crisis:**  $\sigma_t = 2.0 \Rightarrow \varepsilon_t = 0.1 \times (1 + 1.0) = 0.2$  (full smoothing)

---

**Algorithm 7** Adaptive Sinkhorn with Volatility-Based Regularization

---

- 1: **Input:** Cost  $C$ , marginals  $a, b$ , contemporaneous error  $e_t$ , EMA volatility  $\sigma_t$
  - 2: Compute scaled volatility:  $\sigma_t \leftarrow \sqrt{\text{EMA}(e_t^2, \lambda)}$
  - 3: Dynamic regularization:  $\varepsilon_t \leftarrow \max(\varepsilon_{\min}, \varepsilon_0 \cdot (1 + \alpha\sigma_t))$
  - 4: Initialize duals  $f, g \sim 0$
  - 5: **while** iteration < iter\_max **and** not converged **do**
  - 6:      $f \leftarrow \text{Smin}(C - g^\top, \varepsilon_t) + \log a$
  - 7:      $g \leftarrow \text{Smin}(C - f, \varepsilon_t) + \log b$
  - 8: **end while**
  - 9: Sinkhorn distance:  $W_{\varepsilon_t} = \langle \exp(f/\varepsilon_t), K_{\varepsilon_t} \exp(g/\varepsilon_t) \rangle$
  - 10: **Return**  $W_{\varepsilon_t}, f, g$  (duals for plan extraction)
- 

**Advantages:**

1. **Continuous transition:** No discrete jumps. Sinkhorn convergence adapts smoothly to the current regime.
2. **Reduced failures:** Avoids uniform fallback (except in extreme cases) while preserving transport precision.
3. **Self-calibration:** The parameter  $\alpha$  can be tuned via rolling validation (walk-forward) of cost vs precision.
4. **Autograd compatibility:** The dynamics  $\varepsilon_t(\sigma_t)$  is differentiable, enabling end-to-end optimization of  $\alpha$  if desired.

**Suggested Parameters:**

- $\varepsilon_0 \in [10^{-2}, 10^{-1}]$ : Depends on cost scale; typically calibrated empirically.
- $\alpha \in [0.3, 1.0]$ : Medium sensitivity. High values ( $\alpha > 1$ ) may over-smooth; low values ( $\alpha < 0.1$ ) reduce adaptation.
- Volatility estimator:  $\sigma_t = \sqrt{\text{EMA}(e_t^2, \lambda)}$  with  $\lambda \in [0.05, 0.1]$  (short memory, reactive to recent changes).

# Chapter 5

# Software Architecture and Parallelism

## 5.1 Object-Oriented Construction Patterns

The system follows SOLID principles to ensure modularity and extensibility of predictive kernels.

### 5.1.1 Suggested Class Structure

1. **AbstractStochasticProcess**: Base class defining the interface `simulate(dt, steps)`.
2. **ModelIdentifier (SIA)**: Singleton that consumes data streams and emits `RegimeChangedEvent`.  
Uses the Strategy pattern to swap estimation methods (WTMM, DFA).
3. **PredictionKernel**: Abstract interface for predictors (A, B, C, D).
  - `fit(historical_data)`: Parameter calibration.
  - `predict(horizon)`: Future trajectory generation.
  - `compute_risk()`: VaR/ES computation.
4. **Orchestrator**: Implements the Mediator pattern. Owns a `WassersteinOptimizer` and coordinates kernel weighting.

## 5.2 Heterogeneous Computing and Acceleration

### 5.2.1 GPU (CUDA/OpenCL)

Neural network training (DGM) and large Monte Carlo simulations are delegated to the GPU.

- **Kernels**: Implement random number generation (coalesced memory access) and parallel reduction for expectation computation.
- **Sinkhorn**: Matrix operations ( $K \cdot v$ ) are executed via optimized BLAS libraries (cuBLAS).

**Implementation Note 5.1 (Shared Memory Optimization for Branch D (Signatures))** *Iterative signature computation involves tensor products of the form  $\mathbf{S}_{0,t} \otimes \Delta X_k$  operating on high-dimensional tensors ( $d^M$  components for depth  $M$ ). On GPU architectures, efficiency depends critically on memory hierarchy.*

#### *CUDA Memory Management Strategy:*

1. **Shared Memory (SMEM) as explicit cache:**

- *Split the discretized path into blocks of  $B$  consecutive increments*
- *Load each block  $\{\Delta X_k, \Delta X_{k+1}, \dots, \Delta X_{k+B-1}\}$  into SMEM at kernel start*

- Compute the signature concatenation  $\bigotimes_{i=k}^{k+B-1} \mathbf{S}(\Delta X_i)$  entirely in SMEM
- Write the partial result to global memory once per block

2. **Minimize Global <-> Shared transfers:**

- Avoid redundant reads of  $\Delta X$  from global memory
- Reuse previously computed tensor components within the block
- Typical  $B \in [16, 32]$  to balance occupancy and SMEM size (48-96 KB per SM depending on architecture)

3. **Coalesced access pattern:**

- Organize tensors with stride that enables warp-coalesced access
- For rank- $M$  tensors, flatten indices in a consistent row-major or column-major order

**Example Gain:** For  $d = 3$ ,  $M = 4$ ,  $B = 32$  on a V100 GPU:

- Without SMEM optimization: 15 GB/s effective bandwidth (global memory latency bound)
- With SMEM blocks: 120 GB/s (leveraging > 10 TB/s internal SMEM bandwidth)
- Speedup: 8x in signature concatenation kernel

### 5.2.2 FPGA (Field-Programmable Gate Array)

For ultra-low-latency applications (HFT), Branch D (Signatures) is synthesized in reconfigurable hardware.

- **Pipeline:** Iterative signature computation  $S_{0,t} \otimes \Delta X$  is implemented as a systolic pipeline.
- **Fixed-Point Arithmetic:** Fixed-point arithmetic maximizes throughput after analyzing tensor dynamic ranges.

# Chapter 6

# Numerical Stability Considerations

## 6.1 CFL Condition (Courant-Friedrichs-Lowy)

For explicit finite difference schemes in the HJB equation (Branch B), the time step must satisfy:

$$\Delta t \leq \frac{(\Delta x)^2}{2 \max \sigma^2}$$

If volatility is high, the time step becomes prohibitively small. In that case, switch to an **Implicit** or **Semi-Lagrangian** scheme.

## 6.2 Log-Signature Stability

Log-signature computation involves the Baker-Campbell-Hausdorff series, which converges only if increments are small.

---

### Algorithm 8 Adaptive Step Control for Signatures

---

```
1: Input: Path  $X$ , tolerance  $\epsilon$ .
2: function COMPUTESIG( $X$ )
3:   if  $\|\Delta X\| > \epsilon$  then
4:      $X_{\text{mid}} \leftarrow \text{Interpolate}(X)$  (midpoint)
5:      $S_1 \leftarrow \text{ComputeSig}(X_{\text{left}})$ 
6:      $S_2 \leftarrow \text{ComputeSig}(X_{\text{right}})$ 
7:     Return  $S_1 \otimes S_2$ 
8:   else
9:     Return  $\exp(\Delta X)$ 
10:  end if
11: end function
```

---

# Chapter 7

# Governance of Heuristic Metaparameters

Stochastic systems implemented on finite hardware require regularization and truncation parameters that do not exist in continuous probability theory. This chapter defines the **Control Taxonomy** to ensure numerical instantiation remains stable, reactive, and causal.

## 7.1 Taxonomy and Analytical Bounds (Safe Harbors)

The following mathematical limits are mandatory to avoid numerical collapse (NaNs), gradient explosions, or causal violations.

### 7.1.1 Discretization and Truncation Parameters

Define the resolution of the simulated world.

- **Time Step ( $\Delta t$ ):** Not free. Must satisfy the generalized CFL condition for stochastic PIDEs.

$$\Delta t \leq \frac{C_{\text{safe}} \cdot (\Delta x)^2}{2 \cdot \sup |\sigma(x)|^2 + \sup |b(x)| \cdot \Delta x}$$

Where  $C_{\text{safe}} \approx 0.9$ . This is a mixed advective-diffusive CFL condition because the dynamics have both drift (advection) and volatility (diffusion) terms. Violating this limit induces spurious oscillations in the DGM/IMEX solver.

- **Signature Depth ( $M$ ):** Truncation of the tensor algebra  $T((\mathbb{R}^d))$  defines topological memory.
  - **Safe range:**  $M \in [3, 5]$ .
  - **Justification:**  $M < 3$  loses non-commutativity (event ordering).  $M > 5$  invokes the curse of dimensionality (feature growth as  $d^M$ ), saturating RAM without marginal predictive gain.

### 7.1.2 Regularization and Stability Parameters

Control solution smoothness in ill-posed problems.

- **Sinkhorn Entropy ( $\varepsilon$ ):** Turns hard Wasserstein transport into a smooth convex problem.
  - **Initialization:**  $\varepsilon \approx 10^{-2}$ .
  - **Lower bound:**  $\varepsilon \geq 10^{-4}$  (for float32). Smaller values cause numerical underflow in  $K = e^{-C/\varepsilon}$ .

- **Impact:**  $\varepsilon \rightarrow \infty$  yields uniform mixture (max uncertainty).  $\varepsilon \rightarrow 0$  yields unstable winner-takes-all.
- **JKO Proximal Step** ( $\tau$ ): Controls the rate of change of weight distribution  $\rho$  on the Wasserstein manifold.

$$\rho_{k+1} = \text{Prox}_{\tau E}^W(\rho_k)$$

High  $\tau$  allows fast but noisy adaptation. Low  $\tau$  induces excessive inertia. Recommend  $\tau$  adaptive and inversely proportional to prediction error volatility.

### 7.1.3 Decision Thresholds (Hard Boundaries)

Convert continuous probabilities into discrete actions (e.g., Circuit Breaker activation).

- **CUSUM Threshold** ( $h_t$ ): Must not be a magic constant. Define dynamically with kurtosis adjustment:

$$h_t = k \cdot \sigma_{\text{resid}} \cdot (1 + \ln(\kappa_t/3))$$

where:

- $\sigma_{\text{resid}}$  is the rolling standard deviation of prediction residuals
- $k \in [3, 5]$  is the base sensitivity factor (three-sigma rule)
- $\kappa_t$  is kurtosis (fourth standardized moment) computed over a rolling window
- $\ln(\kappa_t/3)$  adjusts the threshold in heavy-tail regimes, reducing false positives during non-Gaussian high volatility

This adaptive threshold matches the Adaptive Threshold with Kurtosis Lemma in the theory document.

- **Singularity Tolerance** ( $H_{\min}$ ): Holder exponent threshold to activate emergency mode (Signatures). Typically  $H_{\min} \in [0.4, 0.5]$  to detect violent mean-reversion or market crash regimes.

## 7.2 Causal Cross-Validation (Walk-Forward Validation)

Static validation methods (traditional K-Fold) are prohibited as they violate the arrow of time and leak future information (look-ahead bias). The only acceptable validation scheme is rolling walk-forward with a sliding window to avoid dilution of recent regimes.

## 7.3 Derivative-Free Meta-Optimization (Bayesian Optimization)

Many hyperparameters are discrete (tree depth  $M$ , decision thresholds) or the error surface is noisy and non-convex, making gradient descent inapplicable.

We prescribe the use of **Gaussian Processes (GP)** for efficient search of the next optimal candidate  $\theta_{\text{next}}$ :

$$\theta_{\text{next}} = \arg \max_{\theta \in \Theta} \text{Expected Improvement}(\theta | \mathcal{D}_{\text{obs}})$$

The objective function is the negative return of Walk-Forward Validation ( $-\mathcal{E}$ ). After  $N$  iterations, the estimated global optimum  $\theta^*$  is the candidate that empirically minimized the error  $\mathcal{E}$ .

1. **Prior:** Define safe ranges (Section 7.1) for each hyperparameter.
2. **Surrogate Model:** Train a GP on observed pairs  $(\theta_i, \text{Performance}_i)$ .
3. **Acquisition Function:** Select  $\theta_{\text{next}}$  that maximizes the probability of improving the current best (balancing exploration vs exploitation).

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**Algorithm 9** Strict Walk-Forward Validation Protocol (Rolling Window)

---

```
1: Input: Data stream  $\mathcal{D} = \{x_1, \dots, x_T\}$ , initial window  $L_{\text{train}}$ , horizon  $H$ , maximum memory  $W_{\max}$ .
2: Output: Aggregated generalization error  $\mathcal{E}$ .
3:  $t \leftarrow L_{\text{train}}$ 
4:  $\text{errors} \leftarrow []$ 
5: while  $t + H \leq T$  do
6:    $start\_idx \leftarrow \max(1, t - W_{\max})$ 
7:    $\mathcal{D}_{\text{train}} \leftarrow \{x_{start\_idx}, \dots, x_t\}$  ▷ Rolling window
8:    $\mathcal{D}_{\text{test}} \leftarrow \{x_{t+1}, \dots, x_{t+H}\}$  ▷ Immediate unknown future
9:   Training: Optimize meta-predictor ( $\theta$ ) on  $\mathcal{D}_{\text{train}}$ 
10:  Inference:  $\hat{y} \leftarrow \text{Predict}(\mathcal{D}_{\text{test}}, \theta)$ 
11:  Evaluate:  $e_t \leftarrow \text{Metric}(\hat{y}, \mathcal{D}_{\text{test}})$ 
12:   $\text{errors.append}(e_t)$ 
13:   $t \leftarrow t + H$  ▷ Advance time step by step
14: end while
15: return Mean(errors)
```

---

4. **Costly Evaluation:** Run the full walk-forward protocol only for  $\theta_{next}$ .

This approach drastically reduces computational cost compared to grid search or random search in high-dimensional spaces, converging to the predictor's optimal "personality" in a few iterations.