Correlated Equilibria & Learning

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Supervised by

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PhD Thesis Defense

Presentation Outline

Introduction

- Importance of the topic:
  - from a research viewpoint
  - practical point of view

Correlated Equilibria

- A **generalization** of Nash equilibria (Aumann, 74)(Aumann, 87) resulting from Bayesian rationality
- A solution concept with appealing **computational properties**
- **Regret-learning** dynamics naturally lead to correlated equilibria (Hart & Mas-Colell, 00)
- Many other theoretical and applied contributions in engineering, economics, etc.
- Extensions to extensive form (Von Stengel & Forges, 08) and infinite games (Hart & Schmeidler, 89)
- Other generalizations in (Forges, 20), (Brandenburger & Dekel, 92) and (Grant & Stauber, 22)

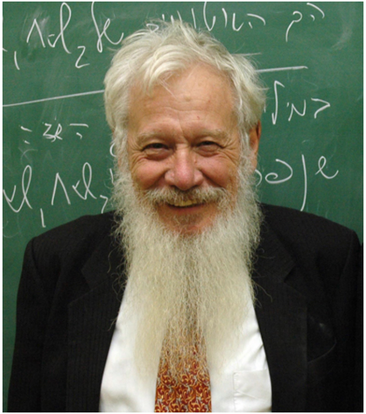


Figure 1: Robert John Aumann

Related Work

Correlated Equilibria

- Defined in (Aumann, 1974) and (Aumann, 1987)
- A proof of existence and a generalization to infinite games in (Hart & Schmeidler, 1989)
- Extensive form correlated equilibrium defined in (Von Stengel & Forges, 2008)
- Other extensions in (Forges, 2020), (Brandenburger & Dekel, 1992) and (Grant & Stauber, 2022)

Learning Correlated Equilibria

- Regret-matching (Hart & Mas-Colell, 2000) implying the convergence to the set of correlated equilibria
- Correlated Q-learning for selection of specific correlated equilibrium points (Greenwald & Hall, 2003)
- Convergence to the social welfare maximizing correlated equilibrium (Borowski et al., 2015)
- Learning algorithms for specific applications e.g., resource allocation game (Cigler & Faltings, 2011)

Constraints & Correlated Equilibria

- (Debreu, 1952) defines the concept of generalized equilibrium
- (Arrow & Debreu, 1954) shows a proof of existence
- (Rosen, 1965) shows existence and uniqueness of equilibria in concave games with coupled constraints
- (Bernasconi et al., 23) introduces the concept of constrained Phi-equilibrium

Many Other Theoretical and Applied Contributions

- XXX (Debreu, 1952)
- XXX (Arrow & Debreu, 1954)

Model & Definitions

Finite non-cooperative game $G = (\mathcal{N}, (\mathcal{A}_i)_{i\in\mathcal{N}}, (u_i)_{i\in\mathcal{N}})$

- Set of players $\mathcal{N}$
- Action set $\mathcal{A}_i$ for each $i \in \mathcal{N}$
- Utility function $u_i : \prod_{i\in\mathcal{N}} \mathcal{A}_i = \mathcal{A} \to \mathbb{R}$

Correlation device $d = (\Omega, (\mathcal{P}_i)_{i\in\mathcal{N}}, \boldsymbol{q})$

- A sample space Ω
- Partition $\mathcal{P}_i$ of Ω for each $i \in \mathcal{N}$
- Probability distribution $\boldsymbol{q}$ over Ω

Player i**'s set of strategies** is $\mathcal{S}_{i,d} = \{f_i : \Omega \to \mathcal{A}_i$ s.t. f_i is $\mathcal{P}_i$-measurable$\}$ and the **set of strategy profiles** is $\mathcal{S}_d = \mathcal{S}_{1,d} \times ... \times \mathcal{S}_{n,d}$.

Definition 1 – Correlated equilibrium (Aumann, 1974)

A correlated equilibrium of G is a pair $(d, \boldsymbol{\alpha}^*)$ where $\boldsymbol{\alpha}^* : \Omega \to \mathcal{A}$ is a correlated strategy profile such that

$$\forall i \in \mathcal{N}, \forall \alpha_i' : \Omega \to \mathcal{A}_i \quad \sum_{\omega\in\Omega} \boldsymbol{q}(\omega) u_i(\boldsymbol{\alpha}_i^*(\omega), \boldsymbol{\alpha}_{-i}^*(\omega)) \geq \sum_{\omega\in\Omega} \boldsymbol{q}(\omega) u_i(\alpha_i'(\omega), \boldsymbol{\alpha}_{-i}^*(\omega)) \tag{1}$$

Definition 2 – Correlated equilibrium distribution (Forges, 2020)

A correlated equilibrium distribution of G is a probability distribution $\boldsymbol{p} \in \Delta(\mathcal{A})$ such that,

$$\forall i \in \mathcal{N}, \forall \beta_i : \mathcal{A}_i \to \mathcal{A}_i \quad \sum_{\boldsymbol{a}\in\mathcal{A}} \boldsymbol{p}(\boldsymbol{a}) u_i(\boldsymbol{a}_i, \boldsymbol{a}_{-i}) \geq \sum_{\boldsymbol{a}\in\mathcal{A}} \boldsymbol{p}(\boldsymbol{a}) u_i(\beta_i(a_i), \boldsymbol{a}_{-i}) \tag{2}$$

Part 1 : Learning Correlated Equilibria

Online Learning with Regret

A finite game G is played repeatedly through time $t = 1, 2, 3, \ldots$

Definition 3 – Regret (Hart & Mas-Colell, 2000)

The regret player i experiences at time t for any pair of actions $j, k \in \mathcal{A}_i$ is

$$R_i^t(j,k) = max\left(0, \frac{1}{t} \sum_{\tau \leq t : a_i^\tau = j} \left[u_i\left(k, \boldsymbol{a}_{-i}^\tau\right) - u_i(\boldsymbol{a}^\tau)\right]\right) \tag{3}$$

Regret-Matching algorithm (Hart & Mas-Colell, 2000) applied to a 3-by-2 game

	D	E
A	(2, 29)	(16, 7)
B	(4, 7)	(6, 13)
C	(4, 4)	(6, 6)

Table 1: Utility matrix.

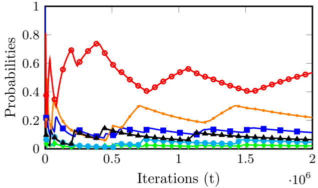


Figure 2: Evolution of the empirical probability distribution.

- The empirical distribution does not seem to converge towards a correlated equilibrium point
- We propose a new dynamic called Correlated Perturbed Regret Minimization (CPRM)

CPRM algorithm

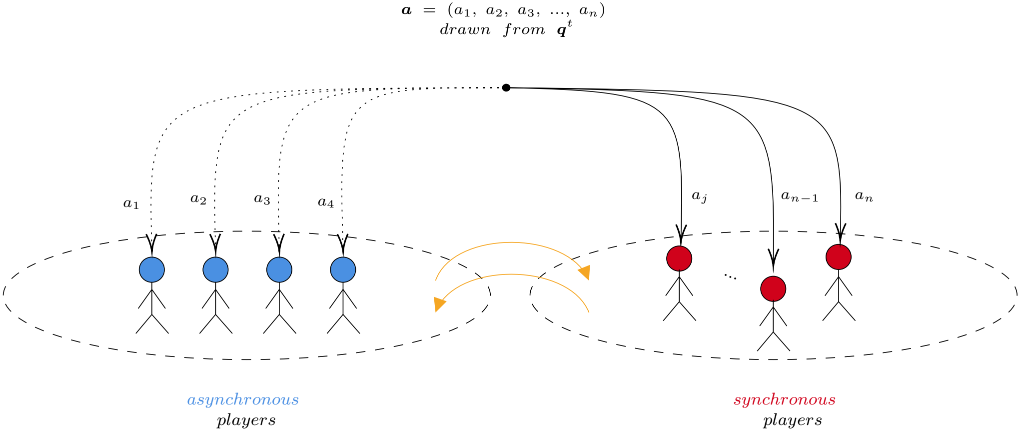
Inspired from the learning algorithm introduced in (Young, H. P., 2009) for learning pure Nash equilibria.

At time t, each player i is characterized by a mood $m_i^t \in \{syn, asyn\}$

asynchronous
player

synchronous
player

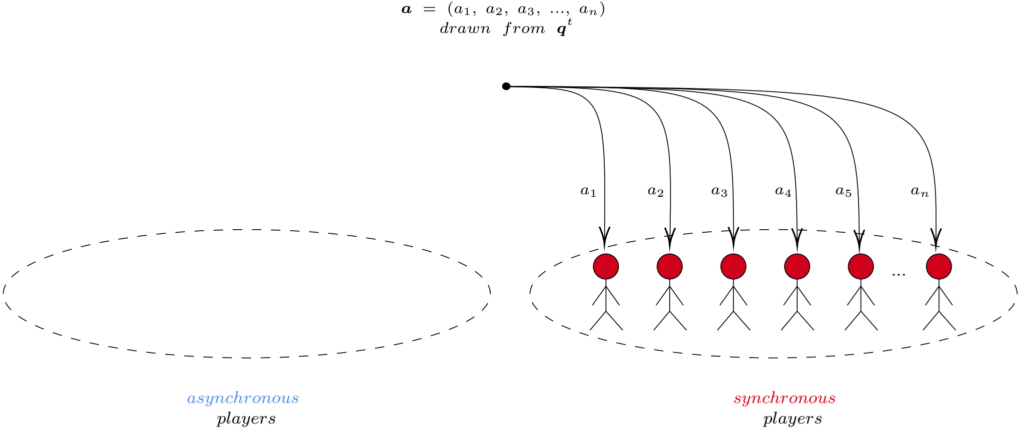
CPRM algorithm



Players implementing a regret-minimizing strategy

Players play the actions drawn from the empirical distribution

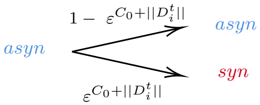
CPRM algorithm

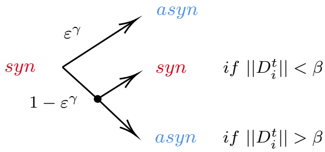


In the long run
↪ We expect players to play the action profiles drawn from the current empirical distributions to stabilize the play at an equilibrium point.

Algorithm Analysis

Mood Dynamics

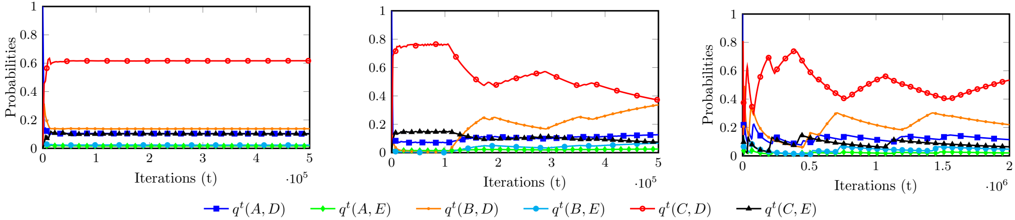




Characteristics of The Stochastic Process

- Transition probabilities depend on the regrets and a perturbation parameter $\varepsilon > 0$
- The stochastic process $\boldsymbol{X}^t = (\boldsymbol{X}_1^t, ..., \boldsymbol{X}_n^t)$ describing players' states is a perturbed non-homogeneous Markov chain
- X X X X X X X
- For a regularly perturbed homogenous Markov chain, the process spends almost all time in the stochastically stable states when $\varepsilon \to 0$

Evolution of the Empirical Probability Distribution



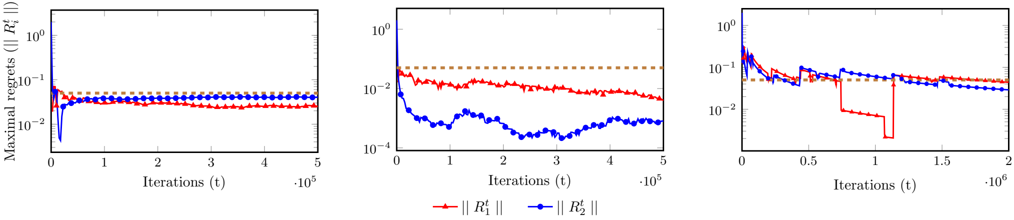
(a) CPRM algorithm

(b) Regret minimizing strategy

(c) Regret-matching

Figure 3: Evolution of the empirical distribution over action profiles.

Evolution of the Regrets



(a) CPRM algorithm.

(b) Regret minimizing strategy.

(c) Regret-matching.

Figure 4: Evolution of players' regrets for the three algorithms.

Time-varying Game

	D	E
A	(2, 29)	(16, 7)
B	(4, 7)	(6, 13)
C	(4, 4)	(6, 6)

↓

	D	E
A	(2, 29, 2)	(16, 7, 8)
B	(4, 7, 2)	(6, 13, 0)
C	(4, 4, 1)	(6, 6, 5)

X

	D	E
A	(9, 4, 0)	(4, 1, 4)
B	(8, 0, 1)	(6, 7, 2)
C	(11, 9, 3)	(2, 0, 4)

Y

↓

	D	E
A	(2, 29)	(16, 7)
B	(4, 7)	(6, 13)
C	(4, 4)	(6, 6)

↓

	D	E
A	(2, 29, 2)	(16, 7, 8)
B	(4, 7, 2)	(6, 13, 0)
C	(4, 4, 1)	(6, 6, 5)

X

	D	E
A	(9, 4, 0)	(4, 1, 4)
B	(8, 0, 1)	(6, 7, 2)
C	(11, 9, 3)	(2, 0, 4)

Y

Figure 5: Sequence of stage games in the dynamic case.

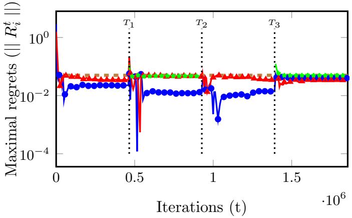


Figure 6: Evolution of maximal regrets

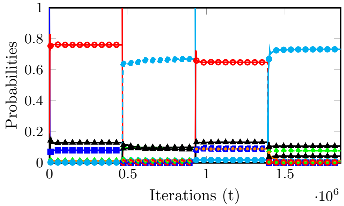


Figure 7: Evolution of the empirical distribution

An application to a congestion game

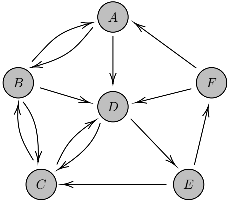


Figure 8: Network graph

Player	Source node	Destination node
1	B	F
2	B	E
3	D	B
4	F	E

Figure 9: Source-destination nodes for each player

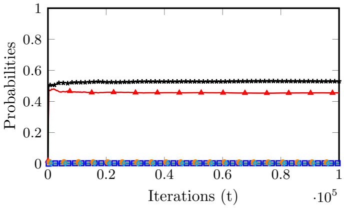


Figure 10: Evolution of the empirical distribution

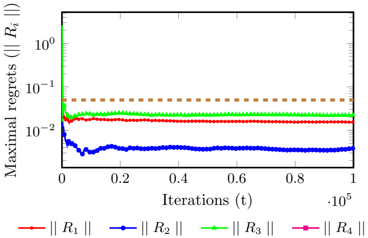


Figure 11: Evolution of maximal regrets

Is this a coincidence ?

Statistics

Part 2 : Constrained Correlated Equilibria

Constrained Correlated Equilibrium Strategies

Definition 1 – Correlated equilibrium (Forges, 2020)
A correlated equilibrium of G is a pair $(d, \boldsymbol{\alpha}^*)$ where $\boldsymbol{\alpha}^* : \Omega \to \mathcal{A}$ is a correlated strategy profile such that

$$\forall i \in \mathcal{N}, \forall \alpha'_i : \Omega \to \mathcal{A}_i \quad \sum_{\omega \in \Omega} \boldsymbol{q}(\omega) u_i(\boldsymbol{\alpha}^*_i(\omega), \boldsymbol{\alpha}^*_{-i}(\omega)) \geq \sum_{\omega \in \Omega} \boldsymbol{q}(\omega) u_i(\alpha'_i(\omega), \boldsymbol{\alpha}^*_{-i}(\omega)) \tag{4}$$

	P	A
P	8, 8	3, 10
A	10, 3	0, 0

Table 2: Utility matrix of the game of Chicken.

- $\Omega = \{H, M, L\}$
- $\mathcal{P}_1 = \{\{H\}, \{M, L\}\}$ and $\mathcal{P}_2 = \{\{H, M\}, \{L\}\}$
- $\boldsymbol{q}(H) = \boldsymbol{q}(M) = \boldsymbol{q}(L) = 1/3$

Constrained Correlated Equilibrium Strategies

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- $\boldsymbol{q}(H) = \boldsymbol{q}(M) = \boldsymbol{q}(L) = 1/3$

	$L \mapsto P$ $L^c \mapsto P$	$L \mapsto A$ $L^c \mapsto A$	$L \mapsto A$ $L^c \mapsto P$	$L \mapsto P$ $L^c \mapsto A$
$H \mapsto P$ $H^c \mapsto P$	$8, 8$	$3, 10$	$6.33, 8.67$	$4.67, 9.33$
$H \mapsto A$ $H^c \mapsto A$	$10, 3$	$0, 0$	$6.67, 2$	$3.33, 1$
$H \mapsto A$ $H^c \mapsto P$	$8.67, 6.33$	$2, 6.67$	$7, 7$	$3.67, 6$
$H \mapsto P$ $H^c \mapsto A$	$9.33, 4.67$	$1, 3.33$	$6, 3.67$	$4.33, 4.33$

Figure 12: Extension of the game of Chicken.

	$L \mapsto P$ $L^c \mapsto P$	$L \mapsto A$ $L^c \mapsto A$	$L \mapsto A$ $L^c \mapsto P$	$L \mapsto P$ $L^c \mapsto A$
$H \mapsto P$ $H^c \mapsto P$	$8, 8$	$3, 10$	$\cancel{6.33, 8.67}$	$\cancel{4.67, 9.33}$
$H \mapsto A$ $H^c \mapsto A$	$\cancel{10, 3}$	$0, 0$	$6.67, 2$	$3.33, 1$
$H \mapsto A$ $H^c \mapsto P$	$\cancel{8.67, 6.33}$	$\cancel{2, 6.67}$	$7, 7$	$3.67, 6$
$H \mapsto P$ $H^c \mapsto A$	$\cancel{9.33, 4.67}$	$1, 3.33$	$\cancel{6, 3.67}$	$4.33, 4.33$

Figure 13: Constrained extension of the game of Chicken.

Constrained Correlated Equilibrium Strategies

	$L \mapsto P$ $L^c \mapsto P$	$L \mapsto A$ $L^c \mapsto A$	$L \mapsto A$ $L^c \mapsto P$	$L \mapsto P$ $L^c \mapsto A$
$H \mapsto P$ $H^c \mapsto P$	$8, 8$	$3, 10$	$6.33, 8.67$	$4.67, 9.33$
$H \mapsto A$ $H^c \mapsto A$	$10, 3$	$0, 0$	$6.67, 2$	$3.33, 1$
$H \mapsto A$ $H^c \mapsto P$	$8.67, 6.33$	$2, 6.67$	$7, 7$	$3.67, 6$
$H \mapsto P$ $H^c \mapsto A$	$9.33, 4.67$	$1, 3.33$	$6, 3.67$	$4.33, 4.33$

Figure 14: Extension of the game of Chicken.

	$L \mapsto P$ $L^c \mapsto P$	$L \mapsto A$ $L^c \mapsto A$	$L \mapsto A$ $L^c \mapsto P$	$L \mapsto P$ $L^c \mapsto A$
$H \mapsto P$ $H^c \mapsto P$	$8, 8$	$3, 10$	$\cancel{6.33, 8.67}$	$\cancel{4.67, 9.33}$
$H \mapsto A$ $H^c \mapsto A$	$\cancel{10, 3}$	$0, 0$	$6.67, 2$	$3.33, 1$
$H \mapsto A$ $H^c \mapsto P$	$\cancel{8.67, 6.33}$	$\cancel{2, 6.67}$	$7, 7$	$3.67, 6$
$H \mapsto P$ $H^c \mapsto A$	$\cancel{9.33, 4.67}$	$1, 3.33$	$\cancel{6, 3.67}$	$4.33, 4.33$

Figure 15: Constrained extension of the game of Chicken.

Definition 4 – Constrained Correlated equilibrium (Boufous et al., 2024)

A correlated equilibrium of G is a pair $(d, \boldsymbol{\alpha}^*)$ where $\boldsymbol{\alpha}^* : \Omega \to \mathcal{A}$ is a correlated strategy profile such that

$$\forall i \in \mathcal{N}, \forall \alpha'_i : \Omega \to \mathcal{A}_i \quad \sum_{\omega \in \Omega} \boldsymbol{q}(\omega) u_i(\boldsymbol{\alpha}_i^*(\omega), \boldsymbol{\alpha}_{-i}^*(\omega)) \geq \sum_{\omega \in \Omega} \boldsymbol{q}(\omega) u_i(\alpha'_i(\omega), \boldsymbol{\alpha}_{-i}^*(\omega)) \tag{5}$$

Properties of Constrained Correlated Equilibrium Strategies

Consider a finite non-cooperative game $G = (\mathcal{N}, (\mathcal{A}_i)_{i\in\mathcal{N}}, (u_i)_{i\in\mathcal{N}})$, a correlation device $d = (\Omega, (\mathcal{P}_i)_{i\in\mathcal{N}}, \boldsymbol{q})$ and a constraint set $\mathcal{R}_d \subset \mathcal{S}_d$.

Proposition 1 : If $(d, \boldsymbol{\alpha}^*)$ is a correlated equilibrium and $\boldsymbol{\alpha}^* \in \mathcal{R}_d$, then $(d, \mathcal{R}_d, \boldsymbol{\alpha}^*)$ is a constrained correlated equilibrium.

Proposition 2 : If $\boldsymbol{\alpha}^* \in \mathcal{R}_d$ and for any $i \in \mathcal{N}$, for any α_i' s.t. $(\alpha_i', \boldsymbol{\alpha}_{-i}) \in \mathcal{R}_d$, for any $\omega \in \Omega$

$$\sum_{\omega' \in P_i(\omega)} \boldsymbol{q}(\omega') \left[u_i(\alpha_i^*(\omega), \boldsymbol{\alpha}_{-i}^*(\omega)) - u_i(\alpha_i'(\omega'), \boldsymbol{\alpha}_{-i}^*(\omega')) \right] \geq 0 \tag{6}$$

then $(d, \mathcal{R}_d, \boldsymbol{\alpha}^*)$ is a constrained correlated equilibrium.

Proposition 3 : The triplet $(d, \mathcal{R}_d, \boldsymbol{\alpha}^*)$ is a constrained correlated equilibrium if and only if $\boldsymbol{\alpha}^* \in \mathcal{R}_d$ and for any $i \in \mathcal{N}$, for any $\alpha_i' \in \mathcal{S}_{i,d}$,

$$\sum_{\omega \in \Omega} \boldsymbol{q}(\omega) \left[u_i(\alpha_i^*(\omega), \boldsymbol{\alpha}_{-i}^*(\omega)) - u_i(\alpha_i'(\omega), \boldsymbol{\alpha}_{-i}^*(\omega)) \right] \geq 0 \text{ or } (\alpha_i', \boldsymbol{\alpha}_{-i}^*) \notin \mathcal{R}_d \tag{7}$$

Constraints induced by a set of feasible probability distributions

- Let $\mathcal{C} \subseteq \Delta(\mathcal{A})$ be a set of probability distributions. For each correlation device d,

$$\mathcal{R}_d = \{\boldsymbol{\alpha} \in \mathcal{S}_d \mid \boldsymbol{p}_{\boldsymbol{\alpha}} \in \mathcal{C}\} \tag{8}$$

- Performance measures in many applications can be expressed in terms of probability distribution over action profiles. Examples include applications in smart grids, wireless networks, etc.

  - Constraints on the social welfare : $\sum_{i \in \mathcal{N}} \sum_{\boldsymbol{a} \in \mathcal{A}} \boldsymbol{p}(\boldsymbol{a}) u_i(\boldsymbol{a}) \geq D_1$
  - Constraints on the Nash product : $\prod_{i \in \mathcal{N}} \left(\sum_{\boldsymbol{a} \in \mathcal{A}} \boldsymbol{p}(\boldsymbol{a}) u_i(\boldsymbol{a}) \right) \geq D_2$

Theorem 1 – Characterization of the Set of Constrained Correlated Equilibrium Distributions

Let G be a finite non-cooperative game and $\mathcal{C}$ a set of feasible probability distributions. The distribution $\boldsymbol{p} \in \Delta(\mathcal{A})$ is a constrained correlated equilibrium distribution if and only if for any player $i \in \mathcal{N}$, for any strategy $\beta_i : \mathcal{A}_i \to \mathcal{A}_i$, if $\boldsymbol{z}_{\beta_i, \boldsymbol{p}} \in \mathcal{C}$, then

$$\sum_{\boldsymbol{a} \in \mathcal{A}} \boldsymbol{p}(\boldsymbol{a}) \left[u_i(a_i, \boldsymbol{a}_{-i}) - u_i(\beta_i(a_i), \boldsymbol{a}_{-i}) \right] \geq 0 \tag{9}$$

where $\boldsymbol{z}_{\beta_i, \boldsymbol{p}}(\boldsymbol{a}) = \Sigma_{b_i \in \mathcal{A}_i} \boldsymbol{p}(b_i, \boldsymbol{a}_{-i}) \mathbb{1}_{\beta_i(b_i) = a_i}$ for any $\boldsymbol{a} \in \mathcal{A}$ is the distribution resulting from player i's unilateral deviation β_i.

Existence of Constrained Correlated Equilibria

Consider the following two-player game:

	L	R
U	$(2,2)$	$(1,1)$
D	$(3,0)$	$(0,5)$

Figure 16: Two-player game in matrix form.

Let $\mathcal{C} \subset \Delta(\mathcal{A})$ feasible set of probability distributions such that,

$$\mathcal{C} = \{\boldsymbol{p} \in \Delta(\mathcal{A}) \mid \boldsymbol{p}(U,L) = 1 \text{ or } \boldsymbol{p}(U,R) = 1 \text{ or } \boldsymbol{p}(D,L) = 1 \text{ or } \boldsymbol{p}(D,R) = 1\} \tag{10}$$

- The set of feasible strategies is the set of pure action profiles
- The players must play a correlated strategy profile inducing a pure action profile in G.
- There does not exist a constrained correlated equilibrium for this game

Theorem 2 - Existence of Constrained Correlated Equilibria

Let G be a finite non-cooperative game and $\mathcal{C}$ a feasible set of probability distributions. If $\mathcal{C}$ is **non-empty**, **compact** and **convex**, then a constrained correlated equilibrium of G exists.

Computation of Constrained Correlated Equilibria

$$\begin{aligned} & \text{maximize } 0 \\ \text{s.t. } & \boldsymbol{p}(\boldsymbol{a}) \geq 0 \quad \forall \boldsymbol{a} \in \mathcal{A}, \quad \Sigma_{\boldsymbol{a} \in \mathcal{A}} \boldsymbol{p}(\boldsymbol{a}) = 1 && (11) \\ & \forall i \in \mathcal{N}, \forall \beta_i : \mathcal{A}_i \to \mathcal{A}_i \\ & \sum_{\boldsymbol{a} \in \mathcal{A}} \boldsymbol{p}(\boldsymbol{a}) \left[u_i(\boldsymbol{a}) - u_i(\beta_i(a_i), \boldsymbol{a}_{-i}) \right] \geq 0. && (12) \end{aligned}$$

$$\begin{aligned} & \text{maximize } 0 \\ \text{s.t. } & \boldsymbol{p}(\boldsymbol{a}) \geq 0 \quad \forall \boldsymbol{a} \in \mathcal{A}, \quad \Sigma_{\boldsymbol{a} \in \mathcal{A}} \boldsymbol{p}(\boldsymbol{a}) = 1 && (13) \\ & \forall i \in \mathcal{N}, \forall \beta_i : \mathcal{A}_i \to \mathcal{A}_i \text{ s.t. } \boldsymbol{z}_{\beta_i, \boldsymbol{p}} \in \mathcal{C} \\ & \sum_{\boldsymbol{a} \in \mathcal{A}} \boldsymbol{p}(\boldsymbol{a}) \left[u_i(\boldsymbol{a}) - u_i(\beta_i(a_i), \boldsymbol{a}_{-i}) \right] \geq 0, && (14) \\ & \boldsymbol{p} \in \mathcal{C} && (15) \end{aligned}$$

Assume linear constraints e.g., $\mathcal{C} = \{\boldsymbol{p} \in \Delta(\mathcal{A}) \mid F\boldsymbol{p} \leq 0\}$.

Constraints induced by a set of feasible probability distributions

Let $\mathcal{C} \subseteq \Delta(\mathcal{A})$ be a set of probability distributions, called feasible set of probability distributions and for each correlation device d, define the coupled constraint set $\mathcal{R}_d$ generated by $\mathcal{C}$ such that,

$$\mathcal{R}_d = \{\boldsymbol{\alpha} \in \mathcal{S}_d \mid \boldsymbol{p}_{\boldsymbol{\alpha}} \in \mathcal{C}\} \tag{16}$$

Theorem 1 : Let G be a finite non-cooperative game and $\mathcal{C}$ a set of feasible probability distributions. The distribution $\boldsymbol{p} \in \Delta(\mathcal{A})$ is a constrained correlated equilibrium distribution if and only if for any player $i \in \mathcal{N}$, for any strategy $\beta_i : \mathcal{A}_i \to \mathcal{A}_i$, if $\boldsymbol{z}_{\beta_i, \boldsymbol{p}} \in \mathcal{C}$, then

$$\sum_{\boldsymbol{a} \in \mathcal{A}} \boldsymbol{p}(\boldsymbol{a}) \left[u_i(a_i, \boldsymbol{a}_{-i}) - u_i(\beta_i(a_i), \boldsymbol{a}_{-i})\right] \geq 0 \tag{17}$$

$\rightarrow$ The set of constrained correlated equilibrium distributions is obtained using the class of canonical correlation devices.

Define the mixed extension ΔG,

Theorem 2 : Let G be a finite non-cooperative game, $\mathcal{C}$ a feasible set of probability distributions, d a correlation device and $\boldsymbol{\gamma}^* \in \tilde{\mathcal{S}}_d$ a correlated strategy profile. If $(d, \tilde{\mathcal{R}}_d, \boldsymbol{\gamma}^*)$ is a constrained correlated equilibrium of ΔG, then it exists a constrained correlated equilibrium $(d', \mathcal{R}_{d'}, \boldsymbol{\alpha}^*)$ of G such that $\boldsymbol{p}_{\boldsymbol{\gamma}^*} = \boldsymbol{p}_{\boldsymbol{\alpha}^*}$.

$\rightarrow$ The set of constrained correlated equilibrium distributions of the game ΔG is included in the set of constrained correlated equilibrium distributions of G.

Computation of Constrained Correlated Equilibria

$$
\begin{aligned}
&\text{maximize } 0\\
\text{s.t. } & \boldsymbol{p}(\boldsymbol{a}) \geq 0 \quad \forall \boldsymbol{a} \in \mathcal{A}, \quad \Sigma_{\boldsymbol{a}\in\mathcal{A}} \boldsymbol{p}(\boldsymbol{a}) = 1 && (18)\\
& \forall i \in \mathcal{N}, \forall \beta_i : \mathcal{A}_i \to \mathcal{A}_i\\
& \sum_{\boldsymbol{a}\in\mathcal{A}} \boldsymbol{p}(\boldsymbol{a}) \left[u_i(\boldsymbol{a}) - u_i(\beta_i(a_i), \boldsymbol{a}_{-i})\right] \geq 0. && (19)
\end{aligned}
$$

$$
\begin{aligned}
&\text{maximize } 0\\
\text{s.t. } & \boldsymbol{p}(\boldsymbol{a}) \geq 0 \quad \forall \boldsymbol{a} \in \mathcal{A}, \quad \Sigma_{\boldsymbol{a}\in\mathcal{A}} \boldsymbol{p}(\boldsymbol{a}) = 1 && (20)\\
& \forall i \in \mathcal{N}, \forall \beta_i : \mathcal{A}_i \to \mathcal{A}_i \text{ s.t. } \boldsymbol{z}_{\beta_i,\boldsymbol{p}} \in \mathcal{C}\\
& \sum_{\boldsymbol{a}\in\mathcal{A}} \boldsymbol{p}(\boldsymbol{a}) \left[u_i(\boldsymbol{a}) - u_i(\beta_i(a_i), \boldsymbol{a}_{-i})\right] \geq 0, && (21)\\
& \boldsymbol{p} \in \mathcal{C} && (22)
\end{aligned}
$$

Assume linear constraints e.g., $\mathcal{C} = \{\boldsymbol{p} \in \Delta(\mathcal{A}) \mid F\boldsymbol{p} \leq 0\}$.

Computation of Constrained Correlated Equilibria

Mixed-Integer Linear Program

$$\boldsymbol{p} \geq 0, \Sigma_{\boldsymbol{a} \in \mathcal{A}} \boldsymbol{p}(\boldsymbol{a}) = 1, F\boldsymbol{p} \leq 0 \tag{23}$$

$$\left[U_i - U_i B_{\beta_i}\right] \boldsymbol{p} \leq M_i \times \boldsymbol{y}_{\beta_i} \quad \forall i \in \mathcal{N}, \quad \forall \beta_i \tag{24}$$

$$F B_{\beta_i} \boldsymbol{p} \geq -K \left(1 - \boldsymbol{y}_{\beta_i}\right) + \delta \quad \forall i \in \mathcal{N}, \quad \forall \beta_i \tag{25}$$

$$F B_{\beta_i} \boldsymbol{p} \leq K y_{\beta_i} \quad \forall i \in \mathcal{N}, \quad \forall \beta_i \tag{26}$$

$$y_{\beta_i} \in \{0, 1\} \quad \forall i \in \mathcal{N}, \quad \forall \beta_i \tag{27}$$

Simulation Example

	P	A
P	8, 8	3, 10
A	10, 3	0, 0

Table 4: Game of Chicken

- Feasible utilities in green
- CE utilities in yellow
- **CCE** utilities in red

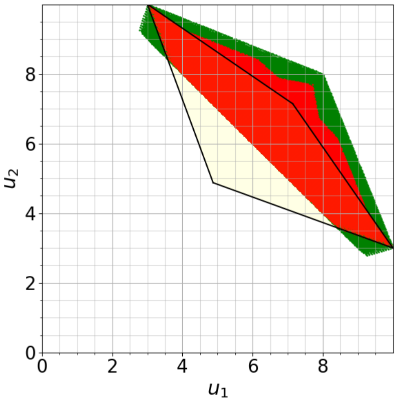
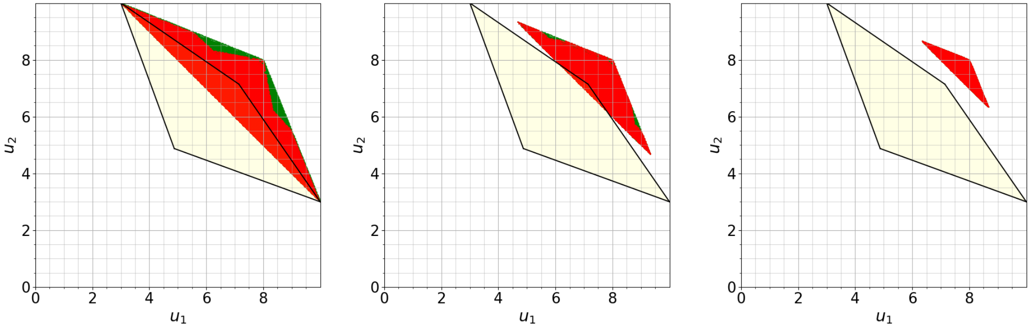


Figure 17: Utilities - Constraint: Social Welfare ≥ 12

Simulation Results



Utilities - Constraint: Social Welfare ≥ 13

Utilities - Constraint: Social Welfare ≥ 14

Utilities - Constraint: Social Welfare ≥ 15

- The set may not be convex
- There are constrained correlated equilibria outside the set of correlated equilibrium distributions

Conclusion & Highlight of Contributions

1. **Arbitrary constraints and arbitrary correlation device**

   1.1 **Characterizations of Equilibrium Strategies**
   - Definition through generalized Nash equilibrium and correlated equilibrium with alternative characterizations

   1.2 **Properties and relationship to (unconstrained) Correlated Equilibria**
   - A feasible correlated equilibrium is a constrained correlated equilibrium
   - The set of constrained correlated equilibrium distributions may not be convex
   - There exist constrained correlated equilibrium distributions outside the set of correlated equilibrium distributions

2. X

Conclusion & Highlight of Contributions

1. **Arbitrary constraints and arbitrary correlation device**

   1.1 **Characterizations of Equilibrium Strategies**
   - Definition through generalized Nash equilibrium and correlated equilibrium with alternative characterizations

   1.2 **Properties and relationship to (unconstrained) Correlated Equilibria**
   - A feasible correlated equilibrium is a constrained correlated equilibrium
   - The set of constrained correlated equilibrium distributions may not be convex
   - There exist constrained correlated equilibrium distributions outside the set of correlated equilibrium distributions

2. **Constraints on probability distributions**

   2.1 **Characterization of constrained correlated equilibrium distributions**
   - Closed-form expression of the set of constrained correlated equilibrium distribution

   2.2 **Conditions of Existence**
   - There does not always exist a constrained correlated equilibrium in finite games

   2.3 **Constrained correlated equilibrium distributions of the mixed extension of a game**
   - Every constrained correlated equilibrium distribution of ΔG is a constrained correlated equilibrium distribution of G

   2.4 **Computation of linearly constrained correlated equilibria**

Open Perspectives

- **Research questions**
  - subjective correlated equilibrium
  - existence conditions
  - extensions to infinite games
  - connection to Bayesian rationality
  - existence problem should be studied for weaker or alternative assumptions
- **Unexplored areas**
  - Learning with constraints
  - X
- **Potential applications in various fields**
  - Constrained correlated equilibria
    - A
    - B
  - Learning correlated equilibria
    - C
    - D

Bibliography

R. J. Aumann. *Subjectivity and correlation in randomized strategies*. Journal of Mathematical Economics, 1974.

R. J. Aumann. *Correlated equilibrium as an expression of bayesian rationality*. Econometrica, 1987.

S. Hart & D. Schmeidler. *Existence of correlated equilibria*. Mathematics of Operations Research, 1989.

B. von Stengel & F. Forges. *Extensive-form correlated equilibrium: Definition and computational complexity*. Mathematics of Operations Research, 2008.

F. Forges. *Correlated equilibria and communication in games*. Complex Social and Behavioral Systems: Game Theory and Agent-Based Models, 2020.

A. Brandenburger, E. Dekel, and John Geanakoplos. *Correlated equilibrium with generalized information structures*. Games and Economic Behavior, 1992.

S. Grant and R. Stauber. *Delegation and ambiguity in correlated equilibrium*. Games and Economic Behavior, 2022.

G. Debreu. *A social equilibrium existence theorem*. Proceedings of the National Academy of Sciences, 1952.

K. J. Arrow and G Debreu. *Existence of an equilibrium for a competitive economy*. Econometrica: Journal of the Econometric Society, 1954.

J. B. Rosen. *Existence and Uniqueness of Equilibrium Points for Concave N-Person Games*. Econometrica, 1965.

Young, H. P. (2009). *Learning by trial and error*. Games and economic behavior.

Bibliography

Thank you!

Questions?

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