

Correlated Equilibria & Learning

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PhD Defense

Presentation Outline

1. Background & Preliminaries

- 1.1 Related work & problem definition
- 1.2 Highlight of contributions
- 1.3 Notations & model

2. Correlated Equilibria & Properties

- 2.1 Extended Game & Correlated Strategies
- 2.2 Coupled Constraints in the Extended Game

3. Learning Correlated Equilibria

- 3.1 X
- 3.2 X
- 3.3 Simulation

4. Constrained Correlated Equilibrium Strategies

- 4.1 Definition & Example
- 4.2 Alternative Characterizations
- 4.3 Definition of Constrained Correlated Equilibria
- 4.4 Properties of Constrained Correlated Equilibrium Strategies
- 4.5 Constrained Correlated Equilibrium Distributions
- 4.6 Existence of Constrained Correlated Equilibria
- 4.7 Simulation Results

5. Conclusions & Perspectives

6. X

Related Work

Correlated Equilibria

- ▶ Defined in (Aumann, 1974) and (Aumann, 1987)
- ▶ A second proof of existence and a generalization to infinite games in (Hart & Schmeidler, 1989)
- ▶ Defined for the extensive form in (Von Stengel & Forges, 2008)
- ▶ Other extensions in (Forges, 2020), (Brandenburger & Dekel, 1992) and (Grant & Stauber, 2022)

Constraints & Generalized Equilibria

- ▶ (Debreu, 1952) defines the concept of generalized equilibrium
- ▶ (Arrow & Debreu, 1954) shows a proof of existence
- ▶ (Rosen, 1965) considers coupled constraints and shows the existence and uniqueness of equilibria in concave games
- ▶ Many other theoretical and applied contributions.

⇒ A solution concept combining **correlation and constraints** has **not yet been studied** in the literature

⇒ We consider this problem for **finite non-cooperative games** and **propose a solution**

Methodology



While working on this topic, another paper dealing with the same issue came out
The author contacted us

Introduction

- ▶ Importance of the topic:
 - ▶ from a research viewpoint
 - ▶ practical point of view

State-of-the-art literature

Please tick relevant boxes

- ▶ is genuinely the work of the student
- ▶ forms a distinct contribution to knowledge of the subject
- ▶ affords evidence of originality: 1) by the discovery of new facts and/or 2) by the exercise of independent critical power
- ▶ is an integrated whole and presents a coherent argument
- ▶ gives critical assessment of the relevant literature
- ▶ describes the method of research and its findings
- ▶ includes discussion of those findings and how they advance the study
- ▶ demonstrates a deep and synoptic understanding of the field of study, objectivity and the capacity for judgment in complex situations and autonomous work in that field
- ▶ is satisfactory as regards literary presentation
- ▶ includes a full bibliography and references
- ▶ demonstrates research skills relevant to the thesis
- ▶ is of a standard to merit publication in whole, in part or in a revised form

Motivation

Currently, ¹

¹Hart, Sergiu, and Andreu Mas-Colell (2000). *"A simple adaptive procedure leading to correlated equilibrium"*, in Econometrica

Background & Context

Correlated Equilibria

- ▶ A **generalization** of Nash equilibria (Aumann, 74)(Aumann, 87) resulting from Bayesian rationality
- ▶ A solution concept with appealing **computational properties**
- ▶ **Regret-learning** dynamics naturally lead to correlated equilibria (Hart & Mas-Colell, 00)
- ▶ Many other theoretical and applied contributions in engineering, economics, etc.
- ▶ Extensions to extensive form (Von Stengel & Forges, 08) and infinite games (Hart & Schmeidler, 89)
- ▶ Other generalizations in (Forges, 20), (Brandenburger & Dekel, 92) and (Grant & Stauber, 22)

Constraints in Games

- ▶ Generalized Nash equilibrium problem (Facchinei, et al., 07) (Fischer, et al., 14)
- ▶ (Debreu, 52) defines the concept of generalized equilibrium
- ▶ A proof of existence is presented in (Arrow & Debreu, 54)
- ▶ (Rosen, 65) shows existence and uniqueness of equilibria in concave games with coupled constraints
- ▶ (Bernasconi et al., 23) introduces the concept of constrained Φ -equilibrium

In this work,

- ▶ We **define** and **characterize** a new solution concept called **constrained correlated equilibrium**
- ▶ Several characterizations of equilibrium strategies and study their properties
 - ▶ sufficient **conditions** of **existence** in case of **constraints on probability distributions over action profiles**
 - ▶ characterization by **canonical correlation devices**
 - ▶ linearly constrained correlated equilibrium problem \rightarrow **MILP**

Part 1 : Learning Correlated Equilibria

Online Learning with Regret

Definition – Regret

The regret player i experiences at time t for any pair of actions $j, k \in \mathcal{A}_i$ is

$$R_i^t(j, k) = \max \left(0, \frac{1}{t} \sum_{\tau \leq t: a_i^\tau = j} \left[u_i(k, a_{-i}^\tau) - u_i(a^\tau) \right] \right) = \max(0, D_i^t(j, k)) \quad (1)$$

	D	E
A	(2, 29)	(16, 7)
B	(4, 7)	(6, 13)
C	(4, 4)	(6, 6)

Table 1: Utility matrix.

- ▶ The empirical distribution does not seem to converge towards a correlated equilibrium point
- ▶ We propose a new dynamic called Correlated Perturbed Regret Minimization (CPRM) which stabilizes the empirical distribution

¹Hart S, Mas-Colell A. (2000). "A simple adaptive procedure leading to correlated equilibrium". *Econometrica*.

Online Learning with Regret

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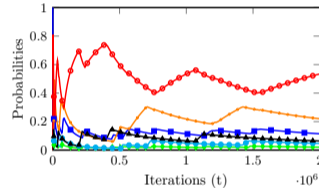


Figure 1: Evolution of the empirical probability distribution

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Evolution of the Empirical Probability Distribution

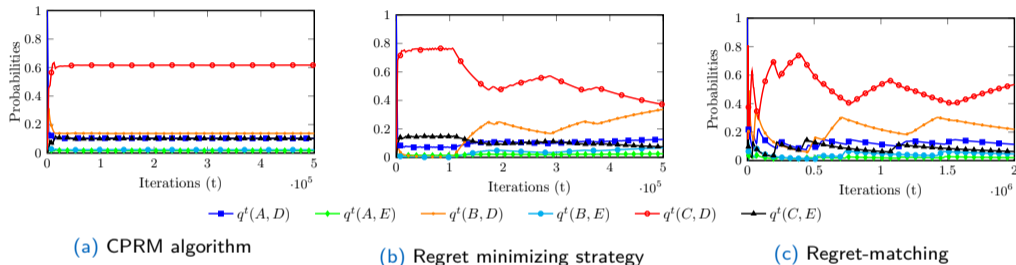
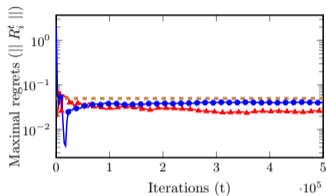
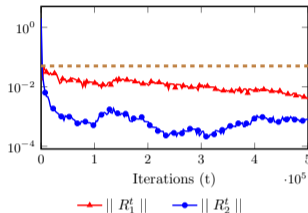


Figure 2: Evolution of the empirical distribution over action profiles.

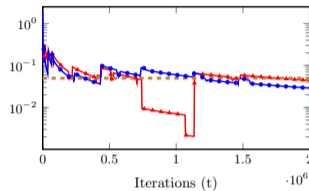
Evolution of the Regrets



(a) CPRM algorithm



(b) Regret minimizing strategy



(c) Regret-matching

Figure 3: Evolution of players' regrets for the three algorithms.

Time-varying Game

		D	E
A	(2, 29)	(16, 7)	
B	(4, 7)	(6, 13)	
C	(4, 4)	(6, 6)	

↓

	D	E
A	(2, 29, 2)	(16, 7, 8)
B	(4, 7, 2)	(6, 13, 0)
C	(4, 4, 1)	(6, 6, 5)

X

	D	E
A	(9, 4, 0)	(4, 1, 4)
B	(8, 0, 1)	(6, 7, 2)
C	(11, 9, 3)	(2, 0, 4)

Y

↓

	D	E
A	(2, 29)	(16, 7)
B	(4, 7)	(6, 13)
C	(4, 4)	(6, 6)

↓

	D	E
A	(2, 29, 2)	(16, 7, 8)
B	(4, 7, 2)	(6, 13, 0)
C	(4, 4, 1)	(6, 6, 5)

X

	D	E
A	(9, 4, 0)	(4, 1, 4)
B	(8, 0, 1)	(6, 7, 2)
C	(11, 9, 3)	(2, 0, 4)

Y

Figure 4: Sequence of stage games in the dynamic case.

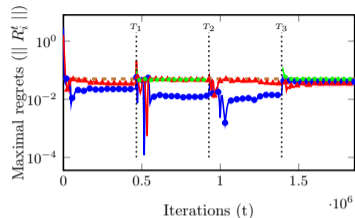


Figure 5: Evolution of maximal regrets

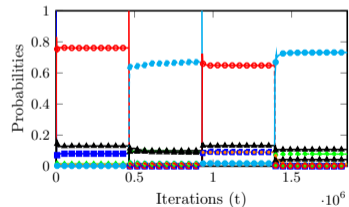


Figure 6: Evolution of the empirical distribution

Discussion

Learning is fine but...

Introduce the constraints topic with an example
for different reasons (economical, etc.)

Part 2 : Constrained Correlated Equilibria

Model & Definitions

Finite non-cooperative game $G = (\mathcal{N}, (\mathcal{A}_i)_{i \in \mathcal{N}}, (u_i)_{i \in \mathcal{N}})$

- ▶ Set of players \mathcal{N}
- ▶ Action set \mathcal{A}_i for each $i \in \mathcal{N}$
- ▶ Utility function $u_i : \prod_{i \in \mathcal{N}} \mathcal{A}_i = \mathcal{A} \rightarrow \mathbb{R}$

Correlation device $d = (\Omega, (\mathcal{P}_i)_{i \in \mathcal{N}}, \mathbf{q})$

- ▶ A sample space Ω
- ▶ Partition \mathcal{P}_i of Ω for each $i \in \mathcal{N}$
- ▶ Probability distribution \mathbf{q} over Ω

Player i 's **set of strategies** is $\mathcal{S}_{i,d} = \{f_i : \Omega \rightarrow \mathcal{A}_i \text{ s.t. } f_i \text{ is } \mathcal{P}_i\text{-measurable}\}$ and the **set of strategy profiles** is $\mathcal{S}_d = \mathcal{S}_{1,d} \times \dots \times \mathcal{S}_{n,d}$.

Definition 1 – Correlated equilibrium (Aumann, 1974)

A correlated equilibrium of G is a pair (d, α^*) where $\alpha^* : \Omega \rightarrow \mathcal{A}$ is a correlated strategy profile such that

$$\forall i \in \mathcal{N}, \forall \alpha'_i : \Omega \rightarrow \mathcal{A}_i \quad \sum_{\omega \in \Omega} \mathbf{q}(\omega) u_i(\alpha_i^*(\omega), \alpha_{-i}^*(\omega)) \geq \sum_{\omega \in \Omega} \mathbf{q}(\omega) u_i(\alpha'_i(\omega), \alpha_{-i}^*(\omega)) \quad (2)$$

Given a **constraint set** $\mathcal{R}_d \subseteq \mathcal{S}_d$, we define a constrained correlated equilibrium,

Definition 2 – Constrained correlated equilibrium (Boufous et al., 2024)

A constrained correlated equilibrium of G is a triplet $(d, \mathcal{R}_d, \alpha^*)$ where $\alpha^* : \Omega \rightarrow \mathcal{A}$ is a correlated strategy profile such that $\alpha^* \in \mathcal{R}_d$ and

$$\forall i \in \mathcal{N}, \forall \alpha'_i : \Omega \rightarrow \mathcal{A}_i \text{ s.t. } (\alpha'_i, \alpha_{-i}^*) \in \mathcal{R}_d \quad \sum_{\omega \in \Omega} \mathbf{q}(\omega) u_i(\alpha_i^*(\omega), \alpha_{-i}^*(\omega)) \geq \sum_{\omega \in \Omega} \mathbf{q}(\omega) u_i(\alpha'_i(\omega), \alpha_{-i}^*(\omega)) \quad (3)$$

Example : a 2-by-2 Matrix Game

Two-player game G

	P	A
P	8, 8	3, 10
A	10, 3	0, 0

Correlation device d

- ▶ $\Omega = \{H, M, L\}$
- ▶ $\mathcal{P}_1 = \{\{H\}, \{M, L\}\}$ and $\mathcal{P}_2 = \{\{H, M\}, \{L\}\}$
- ▶ $q(H) = q(M) = q(L) = 1/3$

Game G extended with correlation device d

	$L \mapsto P$ $L^c \mapsto P$	$L \mapsto A$ $L^c \mapsto A$	$L \mapsto A$ $L^c \mapsto P$	$L \mapsto P$ $L^c \mapsto A$
$H \mapsto P$ $H^c \mapsto P$	8, 8	3, 10	6.33, 8.67	4.67, 9.33
$H \mapsto A$ $H^c \mapsto A$	10, 3	0, 0	6.67, 2	3.33, 1
$H \mapsto A$ $H^c \mapsto P$	8.67, 6.33	2, 6.67	7, 7	3.67, 6
$H \mapsto P$ $H^c \mapsto A$	9.33, 4.67	1, 3.33	6, 3.67	4.33, 4.33

Constrained extended game

	$L \mapsto P$ $L^c \mapsto P$	$L \mapsto A$ $L^c \mapsto A$	$L \mapsto A$ $L^c \mapsto P$	$L \mapsto P$ $L^c \mapsto A$
$H \mapsto P$ $H^c \mapsto P$	8, 8	3, 10	6.33, 8.67	4.67, 9.33
$H \mapsto A$ $H^c \mapsto A$	10, 3	0, 0	6.67, 2	3.33, 1
$H \mapsto A$ $H^c \mapsto P$	8.67, 6.33	2, 6.67	7, 7	3.67, 6
$H \mapsto P$ $H^c \mapsto A$	9.33, 4.67	1, 3.33	6, 3.67	4.33, 4.33

Properties of Constrained Correlated Equilibrium Strategies

Consider a finite non-cooperative game $G = (\mathcal{N}, (\mathcal{A}_i)_{i \in \mathcal{N}}, (u_i)_{i \in \mathcal{N}})$, a correlation device $d = (\Omega, (\mathcal{P}_i)_{i \in \mathcal{N}}, q)$ and a constraint set $\mathcal{R}_d \subset \mathcal{S}_d$.

Proposition 1 : If (d, α^*) is a correlated equilibrium and $\alpha^* \in \mathcal{R}_d$, then $(d, \mathcal{R}_d, \alpha^*)$ is a constrained correlated equilibrium.

Proposition 2 : If $\alpha^* \in \mathcal{R}_d$ and for any $i \in \mathcal{N}$, for any α'_i s.t. $(\alpha'_i, \alpha_{-i}^*) \in \mathcal{R}_d$, for any $\omega \in \Omega$

$$\sum_{\omega' \in P_i(\omega)} q(\omega') \left[u_i(\alpha_i^*(\omega), \alpha_{-i}^*(\omega)) - u_i(\alpha'_i(\omega'), \alpha_{-i}^*(\omega')) \right] \geq 0 \quad (4)$$

then $(d, \mathcal{R}_d, \alpha^*)$ is a constrained correlated equilibrium.

Proposition 3 : The triplet $(d, \mathcal{R}_d, \alpha^*)$ is a constrained correlated equilibrium if and only if $\alpha^* \in \mathcal{R}_d$ and for any $i \in \mathcal{N}$, for any $\alpha'_i \in \mathcal{S}_{i,d}$,

$$\sum_{\omega \in \Omega} q(\omega) \left[u_i(\alpha_i^*(\omega), \alpha_{-i}^*(\omega)) - u_i(\alpha'_i(\omega), \alpha_{-i}^*(\omega)) \right] \geq 0 \quad \text{or} \quad (\alpha'_i, \alpha_{-i}^*) \notin \mathcal{R}_d \quad (5)$$

Computation of Constrained Correlated Equilibria

maximize 0

$$\text{s.t. } p(\mathbf{a}) \geq 0 \quad \forall \mathbf{a} \in \mathcal{A}, \quad \Sigma_{\mathbf{a} \in \mathcal{A}} p(\mathbf{a}) = 1 \quad (6)$$

$$\forall i \in \mathcal{N}, \forall \beta_i : \mathcal{A}_i \rightarrow \mathcal{A}_i$$

$$\sum_{\mathbf{a} \in \mathcal{A}} p(\mathbf{a}) [u_i(\mathbf{a}) - u_i(\beta_i(a_i), \mathbf{a}_{-i})] \geq 0. \quad (7)$$

maximize 0

$$\text{s.t. } p(\mathbf{a}) \geq 0 \quad \forall \mathbf{a} \in \mathcal{A}, \quad \Sigma_{\mathbf{a} \in \mathcal{A}} p(\mathbf{a}) = 1 \quad (8)$$

$$\forall i \in \mathcal{N}, \forall \beta_i : \mathcal{A}_i \rightarrow \mathcal{A}_i \text{ s.t. } z_{\beta_i, \mathbf{p}} \in \mathcal{C}$$

$$\sum_{\mathbf{a} \in \mathcal{A}} p(\mathbf{a}) [u_i(\mathbf{a}) - u_i(\beta_i(a_i), \mathbf{a}_{-i})] \geq 0, \quad (9)$$

$$\mathbf{p} \in \mathcal{C} \quad (10)$$

Assume linear constraints e.g., $\mathcal{C} = \{\mathbf{p} \in \Delta(\mathcal{A}) \mid F\mathbf{p} \leq 0\}$.

Computation of Constrained Correlated Equilibria

Mixed-Integer Linear Program

$$\mathbf{p} \geq 0, \Sigma_{\mathbf{a} \in \mathcal{A}} \mathbf{p}(\mathbf{a}) = 1, F\mathbf{p} \leq 0 \quad (11)$$

$$\left[U_i - U_i B_{\beta_i} \right] \mathbf{p} \leq M_i \times \mathbf{y}_{\beta_i} \quad \forall i \in \mathcal{N}, \quad \forall \beta_i \quad (12)$$

$$FB_{\beta_i} \mathbf{p} \geq -K \left(1 - \mathbf{y}_{\beta_i} \right) + \delta \quad \forall i \in \mathcal{N}, \quad \forall \beta_i \quad (13)$$

$$FB_{\beta_i} \mathbf{p} \leq K \mathbf{y}_{\beta_i} \quad \forall i \in \mathcal{N}, \quad \forall \beta_i \quad (14)$$

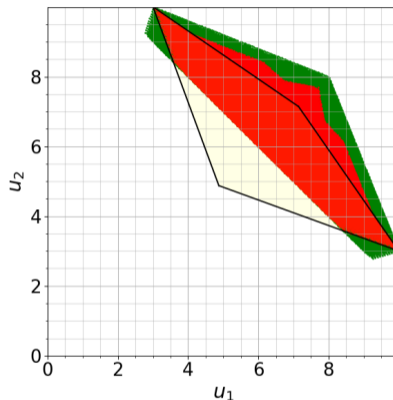
$$\mathbf{y}_{\beta_i} \in \{0, 1\} \quad \forall i \in \mathcal{N}, \quad \forall \beta_i \quad (15)$$

Simulation Example

	P	A
P	8, 8	3, 10
A	10, 3	0, 0

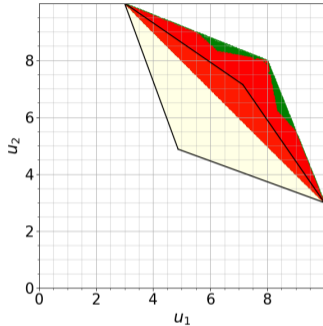
Game of Chicken

- ▶ Feasible utilities in green
- ▶ CE utilities in yellow
- ▶ CCE utilities in red

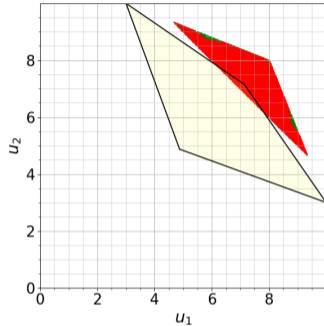


Utilities - Constraint: Social Welfare ≥ 12

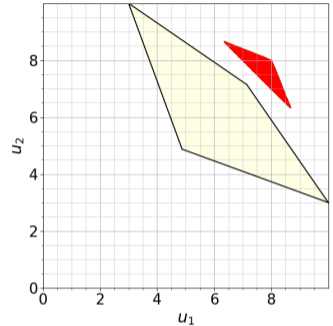
Simulation Results



Utilities - Constraint: Social Welfare
 ≥ 13



Utilities - Constraint: Social Welfare
 ≥ 14



Utilities - Constraint: Social Welfare
 ≥ 15

- ▶ The set may not be convex
- ▶ There are constrained correlated equilibria outside the set of correlated equilibrium distributions

Discussion & Highlight of Contributions

- ▶ X
- ▶ X

1. Arbitrary constraints and arbitrary correlation device

1.1 Characterizations of Equilibrium Strategies

- ▶ Definition derived from the generalized Nash equilibrium and correlated equilibrium
- ▶ Alternative characterizations of constrained correlated strategies

1.2 Properties and relationship to (unconstrained) Correlated Equilibria

- ▶ A feasible correlated equilibrium is a constrained correlated equilibrium
- ▶ The set of constrained correlated equilibrium distributions may not be convex
- ▶ There exist constrained correlated equilibrium distributions outside the set of correlated equilibrium distributions

2. Constraints on probability distributions

2.1 Characterization of constrained correlated equilibrium distributions

- ▶ Closed-form expression of the set of constrained correlated equilibrium distribution

2.2 Sufficient Conditions of Existence

- ▶ There does not always exist a constrained correlated equilibrium in finite games
- ▶ The existence of a constrained correlated equilibrium

2.3 Constrained correlated equilibrium distributions of the mixed extension of a game

- ▶ The set of constrained correlated equilibrium distribution of ΔG is included in the set of constrained correlated equilibrium of G

Open Perspectives

▶ Research questions

- ▶ subjective correlated equilibrium
- ▶ existence conditions
- ▶ extensions to infinite games
- ▶ connection to Bayesian rationality
- ▶ existence problem should be studied for weaker or alternative assumptions

▶ Unexplored areas

- ▶ Learning with constraints
- ▶ X

▶ Potential applications in various fields

- ▶ Constrained correlated equilibria
 - ▶ A
 - ▶ B
- ▶ Learning correlated equilibria
 - ▶ C
 - ▶ D

Thank you!



Questions?

For more information:

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