Exam question topics overview

1.Derive a transfer function for a dynamical system e.g. mechanical system
Be able to write down differential equations of mechanical and electrical systems and applying Laplace transforms

2.Analyze a canonical 2nd order system analysis
Compare with canonical equations and specify canonical parameters (damping, natural frequency, overshoot etc.)

3.Derive the state space model for a linear dynamical system
Indicate if the system is stable, controllable and observable

4.Analyze a non-linear dynamical system
Use linearization based on the Jacobian method to derive a linear state space model in matrix form

5.Design a state feedback controller
Specify state feedback gain using pole/eigenvalue placement by algebraic manipulation by hand.

6.Observer and integral control
Understand these concepts and how to implement them

7.Design an optimal feedback controller
Specify state feedback gain using Algebraic ricotta equations

ROCO218: Control Engineering
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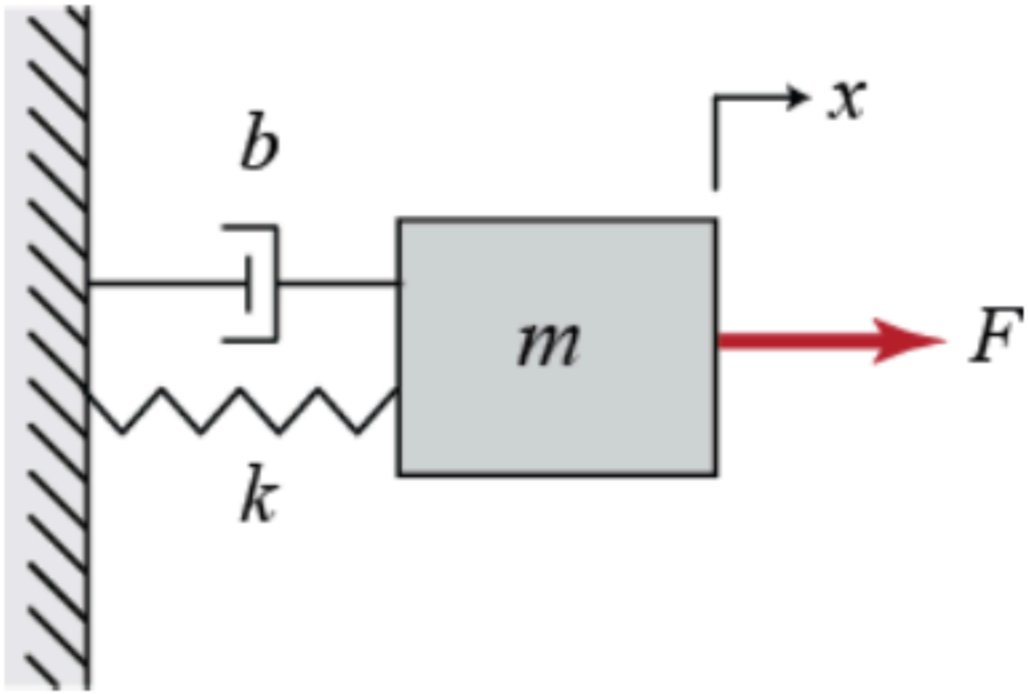
Tutorial 3

ROCO218 2017 Exam example

Q2: Mechanical system

Q2. Consider the following mechanical mass-spring damper system:



A) Using the parameter values:
m =1kg,
b=6 Nm^{-2}
K = 8 Nm^{-1}
Derive the differential equations that describe system behavior.

(5 marks)

Q2: Mechanical system

B) Assume zero initial conditions, take Laplace transformations and write down the system transfer function X(s)/F(s), where X(s) is the s-domain representation of block position and F(s) is the s-domain representation of the input force.

(5 marks)

C) Assuming zero initial conditions, show that in the Laplace domain the system response to a 2N step applied at t=0 is described by following partial fraction expression:

$$\frac{1}{4s}-\frac{1}{2(s+2)}+\frac{1}{4(s+4)}$$

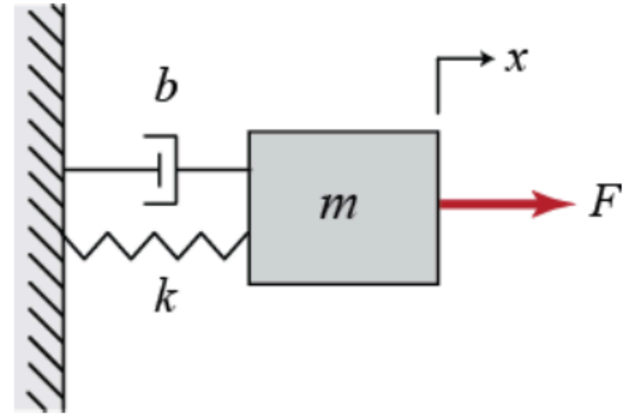
(5 marks)

D) Derive an expression for the time domain response of the system from t=0 onwards.

(5 marks)

Q2: Mechanical system

There differential equation that describes the system is



$$m\ddot{x}(t)+b\dot{x}(t)+kx=f(t)$$

Taking the Laplace transform with zero initial conditions

$$ms^2X(s)+bsX(s)+kX(s)=F(s)$$

inserting m = 1, b = 6, k=8.

$$\Rightarrow F(s)=s^2X(s)+6sX(s)+8X(s)$$

$$\Rightarrow F(s)=\left(s^2+6s+8\right)X(s)$$

- Therefore transfer function from F(s) to X(s) $\dfrac{X(s)}{F(s)}=\dfrac{1}{\left(s^2+6s+8\right)}$

Q2: Mechanical system

A step input f 2N corresponds to

$$F(s) = \frac{2}{s}$$

The system response to a step input is therefore

$$X(s) = \frac{2}{s}\frac{1}{\left(s^2 + 6s + 8\right)}$$

Factorising this denominator

$$X(s) = \frac{1}{s}\frac{1}{\left(s^2 + 6s + 8\right)} \qquad = \frac{2}{s}\frac{1}{\left(s+2\right)\left(s+4\right)}$$

Representing in partial fractions

$$\frac{2}{s}\frac{1}{\left(s+2\right)\left(s+4\right)} = \frac{A}{s} + \frac{B}{\left(s+2\right)} + \frac{C}{\left(s+4\right)}$$

Q2: Mechanical system

Setting RHS to common denominator

$$\frac{A}{s}+\frac{B}{(s+2)}+\frac{C}{(s+4)} = \frac{A(s+2)(s+4)}{s(s+2)(s+4)}+\frac{Bs(s+4)}{s(s+2)(s+4)}+\frac{Cs(s+2)}{s(s+2)(s+4)}$$

$$\Rightarrow \frac{A(s+2)(s+4)+Bs(s+4)+Cs(s+2)}{s(s+2)(s+4)} \qquad \frac{2}{s(s+2)(s+4)}$$

So matching numerator coefficients on both sizes

$$\Rightarrow A(s+2)(s+4)+Bs(s+4)+Cs(s+2)=2$$

Multiplying out

$$\Rightarrow As^2 + A6s + A8 + Bs^2 + 4Bs + Cs^2 + 2sC = 2$$

$$\Rightarrow s^2(A+B+C)+s(6A+4B+2C)+(8A)=2$$

Q2: Mechanical system

So given

$$s^2(A+B+C)+s(6A+4B+2C)+\left(8A\right)=2$$

Matching powers of s^2

$$\Rightarrow (A+B+C)=0$$

Matching powers of s

$$\Rightarrow (6A+4B+2C)=0$$

Matching constant values

$$\Rightarrow 8A=2 \qquad \Rightarrow A=\frac{1}{4}$$

Solving the remaining equations by eliminating C

$$\Rightarrow C=-(A+B) \qquad \Rightarrow 6A+4B-2(A+B)=0 \qquad \Rightarrow 4A+2B=0$$

$$\Rightarrow B=\frac{-4A}{2} \qquad \Rightarrow B=-\frac{1}{2}$$

$$\Rightarrow C=-(A+B) \quad \Rightarrow C=-\left(\frac{1}{4}-\frac{1}{2}\right) \qquad \Rightarrow C=\frac{1}{4}$$

Q2: Mechanical system

So since

$$X(s) = \frac{A}{s} + \frac{B}{(s+2)} + \frac{C}{(s+4)}$$

Reminder: From Laplace pairs table we see that

$$\frac{1}{s} \Leftrightarrow 1$$

$$\frac{1}{(s-a)} \Leftrightarrow e^{at}$$

Substituting in the values

$$A = \frac{1}{4} \qquad B = -\frac{1}{2} \qquad C = \frac{1}{4}$$

$$\Rightarrow X(s) = \frac{1}{4s} - \frac{1}{2(s+2)} + \frac{1}{4(s+4)}$$

Taking inverse Laplace transforms we see the time response to step input is given by

$$\Rightarrow x(t) = \frac{1}{4} - \frac{1}{2}e^{-2t} + \frac{1}{4}e^{-4t}$$

Q3b: Continuous-time nonlinear control system

Q3. Consider the continuous non-linear system described by the following pair of differential equations

$$\dot{x}_1 = x_1^2 + x_2^2 + 2$$

$$\dot{x}_2 = x_1 - x_2$$

A) Determine the equilibrium points of the system

(5 marks)

B) Linearize the system around its equilibrium by evaluating the Jacobian of the system at this point. Write the linearized system in state space matrix format and specify the system matrix A where

$$\dot{X} = AX$$

(5 marks)

C) Calculate the eigenvalues of the linearized system. What can you say about its stability?

(5 marks)

D) Calculate the eigenvectors of the linearized system

(5 marks)

Q3b: Continuous-time nonlinear control system

Equilibria occur when the system is stationary
To find the equilibria we must therefore solve the equations

$$\dot{x}_1 = x_1^2 + x_2^2 + 2 = 0$$

$$\dot{x}_2 = x_1 - x_2 = 0 \qquad \Rightarrow x_1 = x_2$$

Substituting in $\quad x_1 = x_2$

$$\Rightarrow 2x_1^2 = -2 \Rightarrow x_1 = \sqrt{-x1} \qquad \Rightarrow x_1 = \pm j \qquad \Rightarrow x_2 = \pm j$$

Here we consider the +ve equilibrium position of system (1) = (j,j)

To linearize the system we need to evaluate the Jacobian at this point

From the state equations we can write

$$\dot{x}_1 = x_1^2 + x_2^2 + 2 = f_1$$

$$\dot{x}_2 = x_1 - x_2 = f_2$$

Q3b: Continuous-time nonlinear control system

Given these two functions

$$x_1^2 + x_2^2 + 2 = f_1$$

$$x_1 - x_2 = f_2$$

Taking partial derivatives w.r.t. to state variables

$$\Rightarrow \frac{\partial f_1}{\partial x_1} = 2x_1 \qquad \Rightarrow \frac{\partial f_1}{\partial x_2} = 2x_2$$

$$\Rightarrow \frac{\partial f_2}{\partial x_1} = 1 \qquad \Rightarrow \frac{\partial f_2}{\partial x_2} = -1$$

$$\Rightarrow J = \begin{bmatrix} 2x_1 & 2x_2 \\ 1 & -1 \end{bmatrix}$$

Evaluating at (j,j)

$$\Rightarrow J = \begin{bmatrix} 2j & 2j \\ 1 & -1 \end{bmatrix}$$

The Jacobian of the system is a 2x2 matrix since we have 2 state variables

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_1}{\partial x_2} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$

Q3b: Continuous-time nonlinear control system

Linearized system in state space matrix format is therefore

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2j & 2j \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

System stability is is found by examining the eigenvalues λ of the Jacobian matrix which is achieved by solving the equation

$$JX = \lambda X \qquad \Rightarrow JX - \lambda X = 0 \quad \Rightarrow (J - \lambda I)X = 0$$

Where X are the corresponding eigenvectors

If the term $(J - \lambda I)$ has an inverse then

$$(J - \lambda I)X = 0 \qquad \Rightarrow (J - \lambda I)^{-1}(J - \lambda I)X = 0 \qquad \Rightarrow IX = 0 \quad \Rightarrow X = 0$$

This is the trivial solution!

However we get a non-trivial solution when the Jacobian term has no inverse, which occurs when

$$\det(J - \lambda I) = 0$$

Q3b: Continuous-time nonlinear control system

Expanding out the expression

$$\det(J-\lambda I)=0$$

$$\Rightarrow\left|\begin{bmatrix} 2j & 2j \\ 1 & -1 \end{bmatrix}-\lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right|=0 \qquad \Rightarrow\left|\begin{bmatrix} 2j-\lambda & 2j \\ 1 & -1-\lambda \end{bmatrix}\right|=0$$

- For a 2x2 matrix the determinant is given by

$$\det(A)=\left|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right|=(ad-bc)$$

Characteristic equation is therefore

$$\Rightarrow(2j-\lambda)(-1-\lambda)-2j=0 \quad \Rightarrow -2j-2j\lambda+\lambda+\lambda^2-2j=0$$

Solution is therefore

$$\Rightarrow \lambda^2+\lambda(1-2j)-4j=0$$

$$\Rightarrow \lambda=\frac{-(1-2j)\pm\sqrt{(1-2j)^2-4(-4j)}}{2}=\frac{2j-1\pm\sqrt{1-4j-4+16j}}{2}$$

$$\Rightarrow \lambda=\frac{2j-1\pm\sqrt{-3+12j}}{2}$$

Therefore–ve real eigenvalue so system stable

Q3b: Continuous-time nonlinear control system

The eigenvectors X and eigenvalues λ satisfy the equation

$$JX = \lambda X$$

$$\Rightarrow \begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Multiplying out terms gives

$$\Rightarrow 2x_1 + 2x_2 = \lambda x_1$$

$$\Rightarrow x_1 - x_2 = \lambda x_2$$

$$\Rightarrow x_1 = \lambda x_2 + x_2$$

$$\Rightarrow x_1 = x_2 (1 + \lambda)$$

$$\Rightarrow X_\lambda = \begin{bmatrix} 1+\lambda \\ 1 \end{bmatrix}$$ Where $$\lambda = \frac{2j - 1 \pm \sqrt{-3 + 12j}}{2}$$

Q4: State feedback control

A) Explain two limitations of classical (Laplace transform based) approaches to control theory and how modern control theory overcomes these issues.

(5 marks)

Now consider the following state space system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

B) We wish to implement a state feedback scheme. Write down updated state space equations for the feedback system in terms of the matrices above and the gain matrix K where

$$K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$$

(5 marks)

Q4: State feedback control

C) We wish to place the eigenvalues of the closed loop system at values at -1 and -3. Write down the characteristic equation for a system matrix that represents these eigenvalues.

(5 marks)

D) Using the direct method, calculate the values in the gain matrix K that places the eigenvectors of our state feedback system at -1 and -3.

(5 marks)

Q4: State feedback control

- State space equations take the form

$$\dot{X} = AX + BU$$

$$Y = CX + DU$$

The given system state space equations are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- Therefore the system matrices are

$$\Rightarrow A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \qquad \Rightarrow B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \Rightarrow C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Q4: State feedback control

- We now apply feedback to the state space equation

$$\dot{X} = AX + BU$$

- This is achieved by setting the input U to

$$U = -KX$$

Where the gain vector is given by $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$

- This leads to the the modified system dynamics equation

$$\Rightarrow \dot{X} = AX - BKX \qquad \Rightarrow \dot{X} = \left(A - BK\right)X$$

- So the stability of the feedback system is determined by location of its poles which are the eigenvalue of the matrix (A-BK)

- The eigenvalues λ of the closed loop system are thus given by

$$\det\left(A - BK - \lambda I\right) = 0$$

Q4: State feedback control

- Using the system matrices

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\Rightarrow BK = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix}$$

Substituting in values

$$\det\left(A - BK - \lambda I\right) = 0$$

$$\Rightarrow 0 = \left| \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \left| \begin{bmatrix} -\lambda & 2 \\ \left(1 - k_1\right) & \left(1 - k_2 - \lambda\right) \end{bmatrix} \right|$$

- The characteristic equation is therefore

$$\Rightarrow \left(-\lambda\right)\left(1 - k_2 - \lambda\right) - 2\left(1 - k_1\right) = 0$$

Q4: State feedback control

- Simplifying the characteristic equation

$$\Rightarrow (-\lambda)(1-k_2-\lambda)-2(1-k_1)=0$$

$$\Rightarrow -\lambda+\lambda k_2+\lambda^2-2+2k_1=0 \qquad \Rightarrow \lambda^2+\lambda(k_2-1)+2k_1-2=0$$

- We require that the eigenvalues λ of the controller system are at -1,-3
- Therefore we want the following characteristic equation

$$\Rightarrow (\lambda+1)(\lambda+3)=0 \qquad \Rightarrow \lambda^2+4\lambda+3=0$$

- We now need to match the coefficients in the desired eigenvalues characteristic equation using the appropriate gains k_1 and k_2 therefore:

$$\lambda^2+4\lambda+3=0 \qquad \Leftrightarrow \lambda^2+\lambda(k_2-1)+2k_1-2=0$$

$$\Rightarrow k_2-1=4 \qquad \Rightarrow k_2=5$$

$$\Rightarrow 2k_1-2=3 \Rightarrow k_1=2.5$$

- Feedback law is therefore $u(t)=-2.5x_1-5x_2$

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Tutorial 3

ROCO316 2015 Exam solutions

Q5: Optimal control

Q5. Consider the following continuous-time infinite horizon LQR problem:

$$\min_u \int_0^\infty \left[y(t)^2 + u(t)^2\right] dt, \tag{6}$$

subject to

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{x}(t) \end{aligned} \tag{7}$$

(a) Obtain the Algebraic Riccati Equation (ARE) associated with the infinite horizon LQR problem above.

(5 marks)

(b) Find the positive definite solution to the obtained ARE.

(5 marks)

(c) Determine the optimal control law.

(5 marks)

(d) Verify that the closed-loop system is asymptotically stable.

(5 marks)

Q5: Optimal control

- From state space system

$$\dot{X} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} X + \begin{bmatrix} 1 \\ -1 \end{bmatrix} U \qquad Y = \begin{bmatrix} 0 & 1 \end{bmatrix} X$$

$$\Rightarrow A = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} \qquad \Rightarrow B = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \Rightarrow C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

- Computing a value of Q to penalize state error

$$Q = C^T C \qquad = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \qquad = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

- Setting a value of R to penalize control

$$R = 1$$

Q5: Optimal control

- The algebraic Riccati equation is given by

$$A^T P + PA + Q - PBR^{-1}B^T P = 0$$

  - Matrix P is unknown and symmetric so we write

$$P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$$

- Thus the algebraic Riccati equation becomes

$$\Rightarrow \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}^T \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \cdots$$

$$- \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} 1 \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = 0$$

Q5: Optimal control

- So from

$$\begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}^T \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \cdots$$

$$- \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} 1 \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

- Evaluating terms

$$\Rightarrow \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = \begin{bmatrix} -p_1 + p_2 & -p_2 + p_3 \\ -p_2 & -p_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -p_1 + p_2 & -p_2 \\ -p_2 + p_3 & -p_3 \end{bmatrix}$$

Q5: Optimal control

$$\Rightarrow \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} 1 \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} =$$

$$\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$$

$$= \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} p_1 - p_2 & p_2 - p_3 \\ -p_1 + p_3 & -p_2 + p_3 \end{bmatrix} =$$

$$\begin{bmatrix} p_1\left(p_1 - p_2\right) + p_2\left(-p_1 + p_2\right) & p_1\left(p_2 - p_3\right) + p_2\left(-p_2 + p_3\right) \\ p_2\left(p_1 - p_2\right) + p_3\left(-p_1 + p_2\right) & p_2\left(p_2 - p_3\right) + p_3\left(-p_2 + p_3\right) \end{bmatrix}$$

Q5: Optimal control

$$
\Rightarrow \begin{bmatrix} -p_1 + p_2 & -p_2 + p_3 \\ -p_2 & -p_3 \end{bmatrix} + \begin{bmatrix} -p_1 + p_2 & -p_2 \\ -p_2 + p_3 & -p_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \cdots\cdot
$$

$$
- \begin{bmatrix} p_1\left(p_1 - p_2\right) + p_2\left(-p_1 + p_2\right) & p_1\left(p_2 - p_3\right) + p_2\left(-p_2 + p_3\right) \\ p_2\left(p_1 - p_2\right) + p_3\left(-p_1 + p_2\right) & p_2\left(p_2 - p_3\right) + p_3\left(-p_2 + p_3\right) \end{bmatrix}
$$

$$
= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
$$

Q5: Optimal control

- Equating terms leads to 3 simultaneous equations

$$2p_2 - 2p_1 - p_1(p_1 - p_2) + p_2(p_1 - p_2) = 0$$

$$p_3 - 2p_2 - p_2(p_1 - p_2) + p_3(p_1 - p_2) = 0$$

$$p_3(p_2 - p_3) - p_2(p_2 - p_3) - 2p_3 + 1 = 0$$

From (8), we obtain either $p_1 = p_2$ or $p_1 - p_2 = -2$. If we assume $p_1 - p_2 = -2$, then by (9) we get that $p_3 = 0$, which does not lead to a positive definite P. Therefore, we must have $p_1 = p_2$. Then, equation (9) implies that : $p_3 = 2p_2$. And equation (10) leads to the following quadratic equation for p_2 : $-4p_2 - p_2^2 + 1 = 0$, whose solutions are : $-\frac{1}{2}(4 \pm \sqrt{16+4}) = -2 \pm \sqrt{5}$. However, p_2 has to be strictly positive. Hence, we

- Want positive definite matrix P
- Eqn 1 $\Rightarrow p_1 = p_2$
- Eqn 2 $\Rightarrow 2\,p_2 = p_3$
- Eqn 3 $\Rightarrow p_2 = -2 + \sqrt{5}$

$$\Rightarrow P = \left(-2+\sqrt{5}\right)\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

Q5: Optimal control

- Control law given by

$$u(t) = -KX \qquad = -R^{-1}B^T PX$$

$$u(t) = -inv(1)\begin{bmatrix} 1 & -1 \end{bmatrix}\left(-2+\sqrt{5}\right)\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}X$$

$$\Rightarrow u(t) = \left(-2+\sqrt{5}\right)\begin{bmatrix} 0 & 1 \end{bmatrix}X \qquad = \begin{bmatrix} 0 & \left(-2+\sqrt{5}\right) \end{bmatrix}X$$

- The matrix of the resulting closed-loop system is given by

$$A_{cl} = A - BK \qquad = A - BR^{-1}B^T P$$

$$A_{cl} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}\begin{bmatrix} 0 & \left(-2+\sqrt{5}\right) \end{bmatrix} = \begin{bmatrix} -1 & \left(-2+\sqrt{5}\right) \\ 1 & \left(1-\sqrt{5}\right) \end{bmatrix}$$

Q5: Optimal control

- Eigenvalues λ of A_{cl} given by

$$\det\left|\begin{bmatrix} -1 & \left(-2+\sqrt{5}\right) \\ 1 & \left(1-\sqrt{5}\right) \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right| = \det\left|\begin{bmatrix} \left(-1-\lambda\right) & \left(-2+\sqrt{5}\right) \\ 1 & \left(1-\sqrt{5}-\lambda\right) \end{bmatrix}\right| = 0$$

$$\Rightarrow \left(-1-\lambda\right)\left(1-\sqrt{5}-\lambda\right)-\left(-2+\sqrt{5}\right)=0$$

$$\Rightarrow \lambda^2+\sqrt{5}\lambda+1=0$$

$$\Rightarrow \lambda = \frac{-\sqrt{5}\pm\sqrt{5-4}}{2} \qquad \Rightarrow \lambda = \frac{-\sqrt{5}\pm 1}{2}$$

- Real part of eigenvalues always negative therefore system stable