ROCO218: Control Engineering
Dr Ian Howard

Lecture 8

Transfer function of a SFC system

Transfer functions of SFC system

Given the state space equations

$$\dot{X} = AX + BU$$

$$Y = CX + DU$$

We showed previously that the transfer function of this state space model is given by the expression

$$G(s) = C(sI - A)^{-1} B + D$$

So assuming D is zero we have

$$G(s) = C(sI - A)^{-1} B$$

Transfer functions of SFC system

As we know, by applying full state feedback leads to the relations

$$\dot{X} = \left(A - BK\right)X$$

$$Y = \left(C - DK\right)X$$

The SFC gain K effectively changes the A matrix

Therefore we can modify the transfer function G(s) accordingly

So the transfer function for the state space system

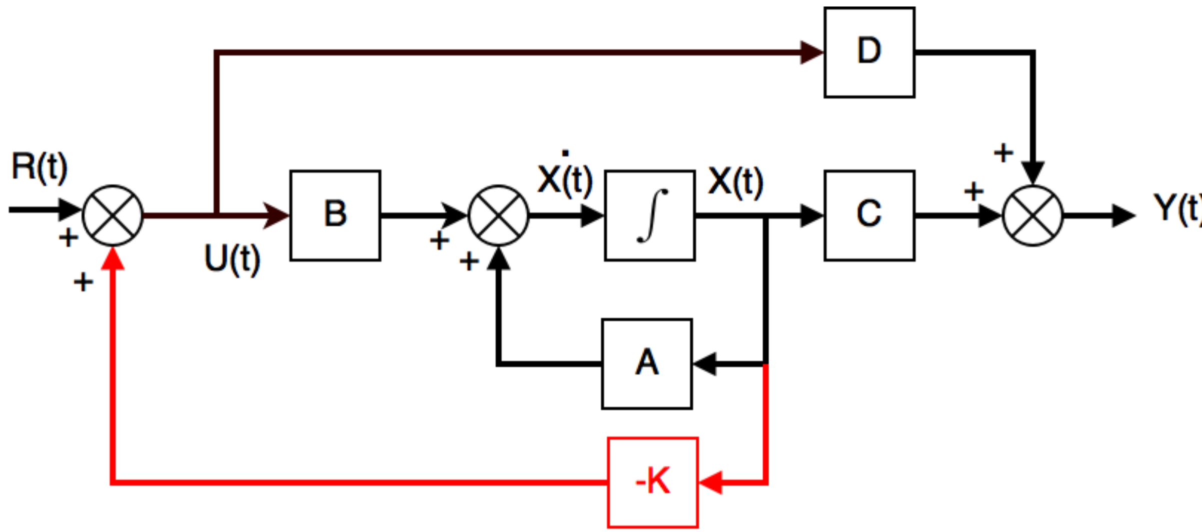
$$G(s) = C(sI - A)^{-1}B$$

Becomes

$$G_{sfc}(s) = C(sI - (A - BK))^{-1}B$$

Effect of transfer function on reference input

- The reference input R(t) will be processed by the SFC system



- To achieve good tracking of the reference input we require that

$$y(t) \approx r(t) \quad \text{as} \quad t \to \infty$$

- Driving the system with a unit step input, we require the output to be unity

$$lim_{t\to\infty}\left(y(t)\right) = 1$$

- To achieve this we need to scale the reference input by a gain $\bar{N}$

Effect of transfer function on reference input

- Using the final value theorem, we require that time output driven by step lends to unity

$$\Rightarrow lim_{s\to 0}\left(\frac{1}{s}sY(s)\right)=1$$

- So to achieve this we will scale the input so that

$$Y(s)=\bar{N}G_{sfc}(s) \qquad \Rightarrow lim_{s\to 0}\left(\bar{N}G_{sfc}(s)\right)=1$$

- Substituting in the transfer function for the SFC system

$$G_{sfc}(s)=C(sI-(A-BK))^{-1}B$$

$$\Rightarrow lim_{s\to 0}\left(\bar{N}C(sI-(A-BK))^{-1}B\right)=1 \quad \Rightarrow -\bar{N}C(A-BK)^{-1}B=1$$

- This leads to an expression for the scaling

$$\Rightarrow \bar{N}=-\left[C(A-BK)^{-1}B\right]^{-1}$$

- We need to scale the reference input by this value to ensure unity DC gain of the SFC system

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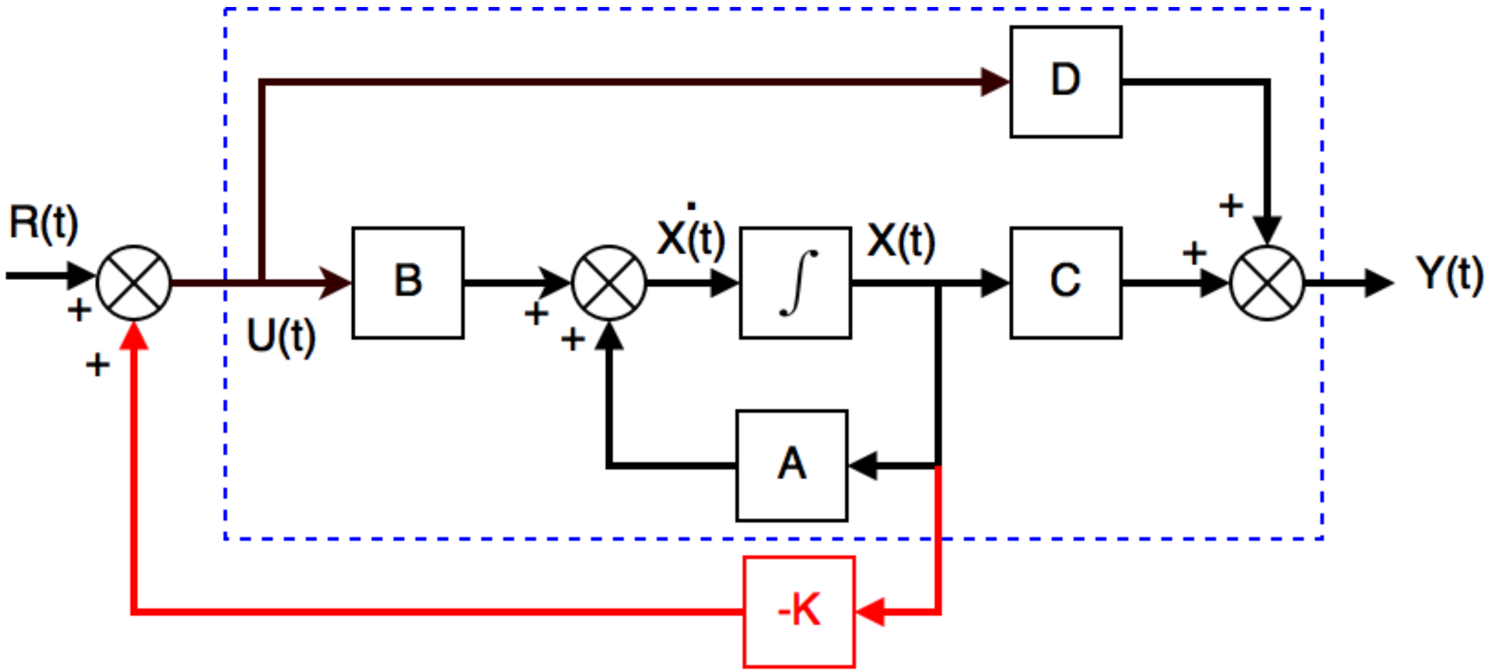
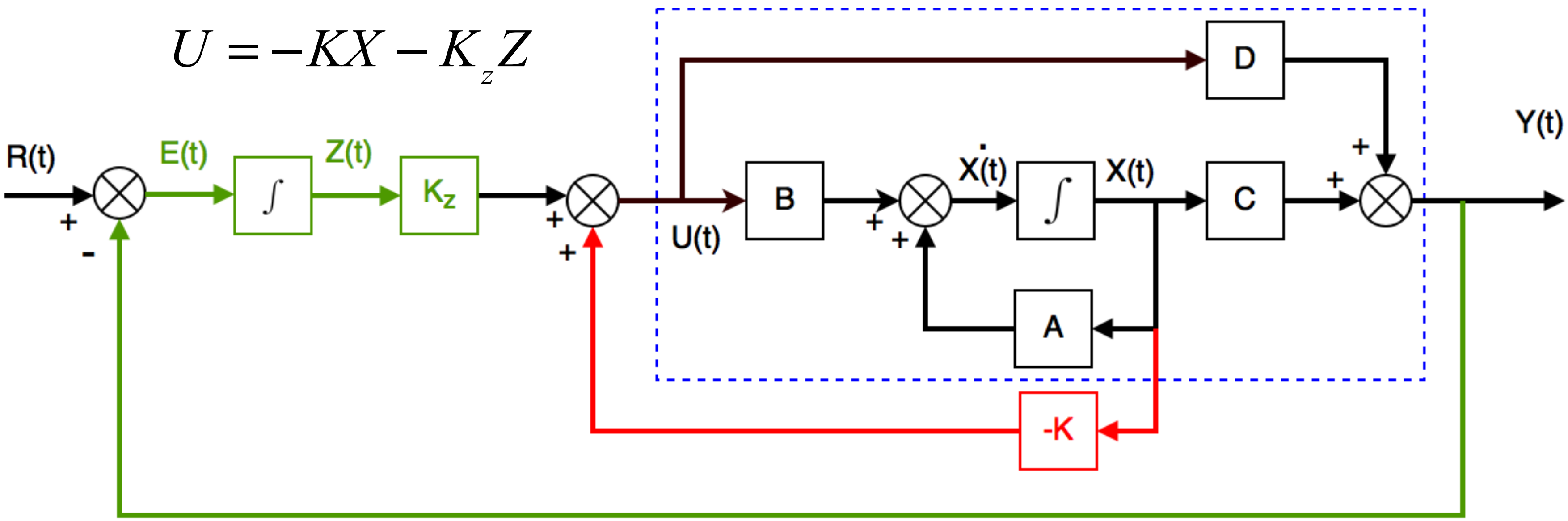
Integral action

State space feedback control



- Scaling the input reference a good practice to ensure output reaches the input target
- However with pure SFC we do not make use the system output Y(t)!
- If we don’t use the output, the system will need to be calibrated to operate well
- Therefore requiring we have an exact process model is undesirable!

Integral action



- Remember one of the primary uses of feedback is to allow good performance in the presence of uncertainty!
- So we can use integral feedback to make use of the output
- Here the controller uses an integrator to achieve zero steady state error
- To achieve integral feedback we create a state within the controller that computes the integral of the error signal
- This is is then used as a feedback term as denoted by the green path on the schematic above

Integral action

- To achieve integral feedback we augment the system by adding another state Z
- The state Z is the integral of the error between the desired output R(t) and actual output Y(t)
- Thus the standard state space equation

$$\frac{d}{dt}\left[X\right]=\left[AX+BU\right]$$

augmented with the state Z becomes

$$\frac{d}{dt}\begin{bmatrix} X \\ Z \end{bmatrix}=\begin{bmatrix} AX+BU \\ y-r \end{bmatrix} \qquad =\begin{bmatrix} AX+BU \\ CX-r \end{bmatrix}$$

Using feedback control if the augmented state we see that

$$U=-KX-K_zZ$$

Remember: velocity control inverted pendulum

Given the equation

$$\frac{d^2\theta}{dt^2} = \frac{-\mu}{\left(I + ml^2\right)}\frac{d\theta}{dt} + \frac{mgl}{\left(I + ml^2\right)}\theta + \frac{ml}{\left(I + ml^2\right)}\frac{dv_c}{dt}$$

Let the constant terms be represented by the coefficients

$$a_1 = \frac{\mu}{\left(I + ml^2\right)} \qquad b_0 = \frac{ml}{\left(I + ml^2\right)}$$

$$a_2 = \frac{-mgl}{\left(I + ml^2\right)}$$

$$\Rightarrow \frac{d^2\theta}{dt^2} = -a_1\frac{d\theta}{dt} - a_2\theta + b_0\frac{dv_c}{dt}$$

Choosing state space representations

$$x_1 = \theta$$

$$x_2 = \frac{d\theta}{dt} - b_0 v_c$$

Remember: Velocity control inverted pendulum

From the state space representation of the velocity controlled inverted pendulum

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_0 \\ -a_1 b_0 \end{bmatrix} v_c$$

$$\Rightarrow \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \dfrac{mgl}{\left(I+ml^2\right)} & -\dfrac{\mu}{\left(I+ml^2\right)} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \dfrac{ml}{\left(I+ml^2\right)} \\ \dfrac{-\mu ml}{\left(I+ml^2\right)^2} \end{bmatrix} v_c$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Where output y is the pendulum angle θ

Remember: *Augmented* velocity control inverted pendulum

In practice we can to control cart position as well as angle and angular velocity!
Otherwise it might never stop moving!

To do so we can add a third state x_3 to represent cart position
Since the control signal is cart velocity, the differential of x_3 is simply given by the input velocity control signal

Therefore we can write

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -a_2 & -a_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} b_0 \\ -a_1 b_0 \\ 1 \end{bmatrix} v_c$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Adding integral action to the VCIP

We now want to use the cart position to generate error integrated over time
To do so we can add a fourth state x_4 to represent integrated cart position error
To achieve this we further augment the system matrices

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -a_2 & -a_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} b_0 \\ -a_1 b_0 \\ 1 \\ 0 \end{bmatrix} v_c$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Notice we update the integral state by selecting the position state x_3 to generate the (y-r) term used for integral action
Here we assume the reference value is zero

We didn't change y to implement this

Adding integral action to the VCIP

Substituting in the coefficients leads to the equations

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \dfrac{mgl}{\left(I+ml^2\right)} & -\dfrac{\mu}{\left(I+ml^2\right)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} \dfrac{ml}{\left(I+ml^2\right)} \\ \dfrac{-\mu ml}{\left(I+ml^2\right)^2} \\ 1 \\ 0 \end{bmatrix} v_c$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Finally we can design the feedback gain vector K needed to implement full feedback state control using the Matlab place command

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Integral control exam solutions

Q4: SFC, integral control and observer

Q4. Consider the following system:

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned} \tag{3}$$

(a) Design a state feedback $u(t) = -k_1 x_1(t) - k_2 x_2(t)$ so that the closed-loop eigenvalues are placed at $\{-3, \ -2\}$. **You should use the direct method** for computing the gains k_1 and k_2.

(10 marks)

(b) Without fully solving it, write down all the equations of the integral control problem associated to the above system.

(5 marks)

(c) Without fully solving it, write down all the equations of the observer-based control problem associated to the above system.

(5 marks)

Q4: SFC, integral control and observer

- For system

$$\dot{X} = AX + BU$$
$$Y = CX + DU$$

  - Full state feedback requires that

$$U = -KX$$

- So system equations become

$$\dot{X} = AX - BKX \quad = (A - BK)X$$

$$Y = CX - DKX \quad = (C - DK)X$$

Q4: SFC, integral control and observer

- From state space system

$$\dot{X} = \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U$$

$$Y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix}$$

$$\Rightarrow B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Q4: SFC, integral control and observer

- We require that the eigenvalues λ of the controller system are at -3,-2

- This means we will want the following characteristic equation for the eigenvalues

$$\Rightarrow (\lambda+3)(\lambda+2)=0 \qquad \Rightarrow \lambda^2+5\lambda+6=0$$

- The eigenvalues λ of the closed loop system are given by

$$\det(A-BK-\lambda I)=0$$

Substituting in values for B we have

$$BK=\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix}$$

Q4: SFC, integral control and observer

- So e expanding $\det\left(A - BK - \lambda I\right) = 0$

$$\Rightarrow 0 = \left| \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \left| \begin{bmatrix} -\lambda & 3 \\ \left(2 - k_1\right) & \left(1 - k_2 - \lambda\right) \end{bmatrix} \right|$$

This gives the characteristic equation for the system

$$\Rightarrow \left(-\lambda\right)\left(1 - k_2 - \lambda\right) - \left(2 - k_1\right)\left(3\right) = 0$$

$$\Rightarrow \lambda^2 + \left(k_2 - 1\right)\lambda + \left(3k_1 - 6\right) = 0$$

- We now need to match the coefficients in the desired eigenvalues characteristic equation using the appropriate gain vector K

We want the characteristic equation

$$\Rightarrow \lambda^2 + 5\lambda + 6 = 0 \quad \Leftrightarrow \lambda^2 + \left(k_2 - 1\right)\lambda + \left(3k_1 - 6\right) = 0$$

$$\Rightarrow k_2 - 1 = 5 \qquad \Rightarrow k_2 = 6$$

$$\Rightarrow 3k_1 - 6 = 6 \qquad \Rightarrow 3k_1 = 12 \quad \Rightarrow k_1 = 4$$

Q4: SFC, integral control and observer

- Integral action can be implemented by introducing another state z in the system such that

$$\frac{d}{dt}[z] = y - r \quad \Rightarrow \frac{d}{dt}[z] = CX - r \quad = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - r$$

- For a pure state feedback system

$$\dot{X} = AX + BU \qquad U = -KX$$

- Adding in the additional state for integral action this leads to he expression

$$\frac{d}{dt}\begin{bmatrix} X \\ z \end{bmatrix} = \begin{bmatrix} AX + BU \\ CX - r \end{bmatrix}$$

- We also amend the feedback control to include weighted contributions from the state z

$$U = -KX - K_z z$$

Q4: SFC, integral control and observer

- An observer can be used to estimate system state using efference copy of the motor command
- To make the process, robust the output from the system can also be used to correct the state estimate

- For the plant of the system

$$\dot{X} = AX + BU \qquad\qquad Y = CX + DU$$

- For state estimator

$$\dot{\hat{X}} = A\hat{X} + BU + L(Y - C\hat{X})$$

Where matrix 2x2 L represent a scaling of the error between actual and predicted output

- To ensure the estimator is stable and initial conditions decay away, it is necessary to choose L such that the expression

$$\left(A - LC\right)$$

Has eigenvalues with negative real parts

- The feedback control is given by

$$U = -K\hat{X}$$

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Optimal control

Remember: State feedback control

We showed previously state feedback control provides a means to move the eigenvalues of a system from their open loop values

- In an open loop state space system

$$\dot{X} = AX + BU$$

$$Y = CX + DU$$

- Stability determined by location of poles which are the eigenvalue of matrix A

- Substituting in state feedback as input

$$U = -KX$$

- Gives rise to the closed loop system

$$\dot{X} = \left(A - BK\right)X$$

$$Y = \left(C - DK\right)X$$

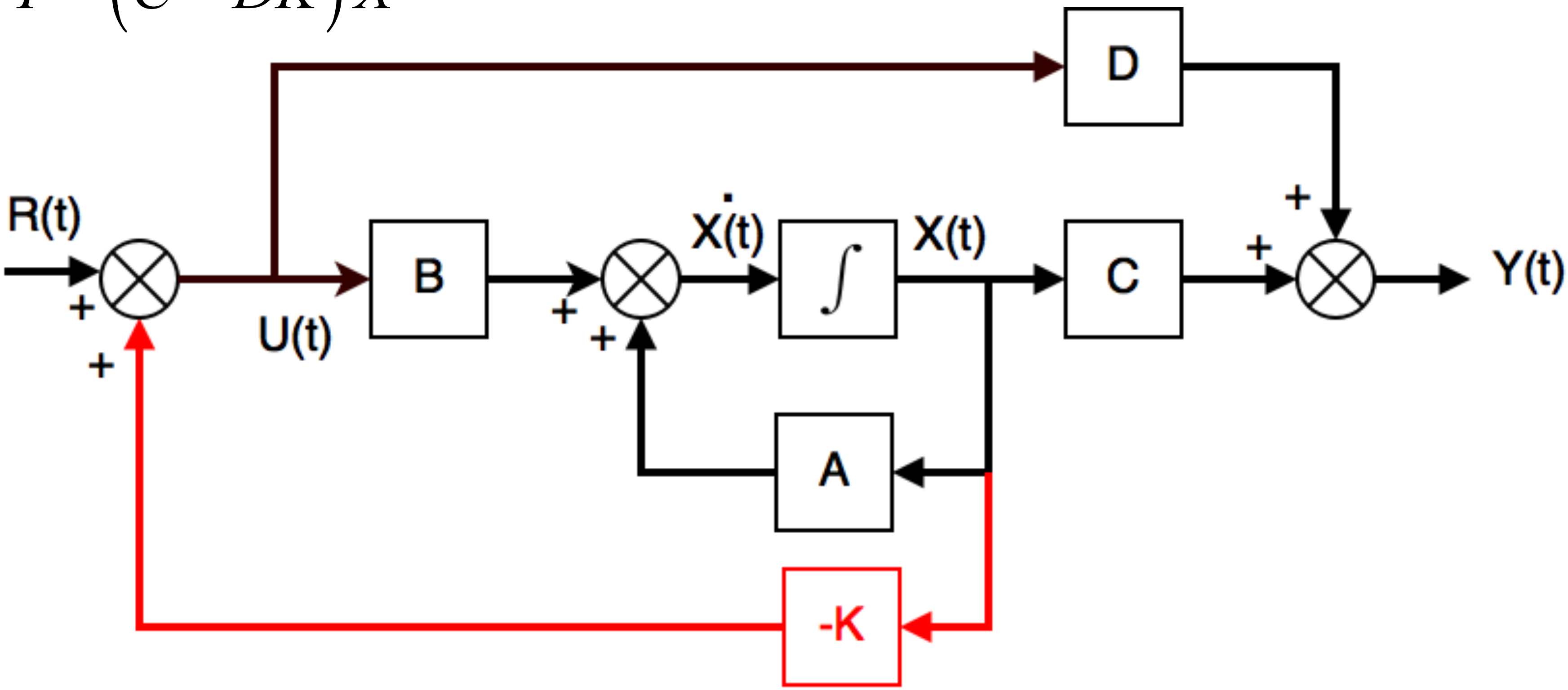
- Stability determined by location of poles which are the eigenvalue of matrix (A-BK)

Remember: State feedback control

$$\dot{X} = \left(A - BK\right)X$$

$$Y = \left(C - DK\right)X$$

The corresponding signal flow graph is shown below



- When a system is fully controllable its eigenvectors poles can be placed arbitrarily by choosing the value of K
- But how to place poles is often not obvious!
- Therefore a more principled methods to choose K is often useful

Optimal control

- Instead of direct pole placement, optimal control makes use of a performance criterion
- To do so we define a 'cost function' which lets us minimize some quantity of the control process
- For example imaging we a driving a car:

- We could minimize the time required to go from A to B

$$J = \int_{0}^{finalTime} (1)dt$$ = Final Time

- We could minimize the fuel used to go from A to B

$$J = \int_{0}^{finaLlocation} (fuelflow)dR$$ = Fuel Used

- We could minimize the financial cost to go from A to B

$$J = \int_{0}^{finaLTime} (\cos tPerHour)dt$$ = £ Cost of trip

Optimal control

- Often a quadratic performance measure is adopted
- To make analysis easier
- leads to well behaved solutions

$$J = \int_0^\infty \left(x^T Q x + u^T R u \right) dt$$

- The term $x^T Q x$ Where typically $Q = C^T C$
- Is the weighted square distance of state from its origin
- Affects convergence rate: rise time and settling time
- Large Q means get good tracking as expense of large input
- The term $u^T R u$
- Is the weighted square of control input activity
- Penalizes large and aggressive inputs
- Large R means little input needed at expense of good tracking
- We need to optimize the performance index for the given state space system

Optimal control

- From the expression for cost

$$J = \int_0^\infty \left(x^T Q x + u^T R u \right) dt$$

- We want to find the corresponding gain K that minimizes J

$$\underbrace{\min}_{K} J = \int_0^\infty \left(x^T Q x + u^T R u \right) dt \qquad \begin{cases} \dot{x} = Ax + Bu \\ u = -Kx \end{cases}$$

- The solution by dynamic programming
- This leads to the expression for the optimal gain

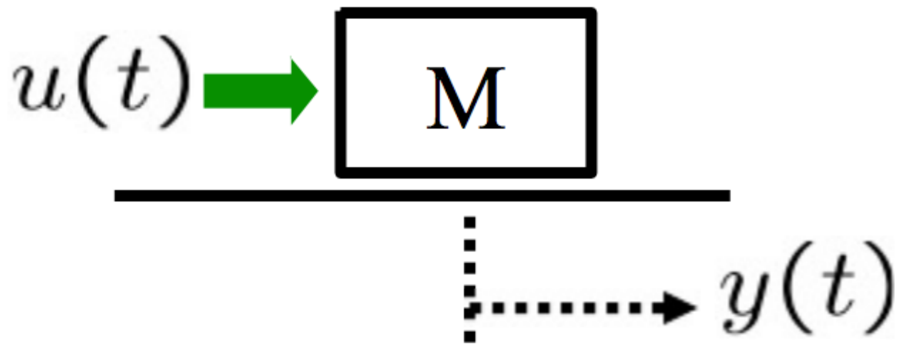
$$K = R^{-1} B^T P$$

- where R is invertible and P is symmetric and satisfies the algebraic Riccati equation

$$A^T P + PA - PBR^{-1} B^T P + Q = 0$$

Example: Force optimal control of mass position

Consider control of a 1Kg mass.
Since f = ma



$$\Rightarrow u = \frac{d^2 y}{dt^2}$$

Defining the two state variables

$$x_1 = y \qquad x_2 = \frac{dy}{dt}$$

$$\Rightarrow \dot{x}_1 = x_2 \qquad \Rightarrow \dot{x}_2 = u$$

Writing in state space notation we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Therefore

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Example: Force optimal control of mass position

We want to find optimal control that minimizes

$$J = \int_0^\infty \left(y^2(t) + \rho u^2(t) \right) dt$$

This can be formulated as an LQR problem with

$$Q = C^T C = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$R = \rho \quad \Rightarrow R^{-1} = \frac{1}{\rho}$$

To solve the minimization problem, we first solve the algebraic Riccati equation:

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

Since we have a 2x2 A matrix, we solve for the matrix P which is symmetric Matrix with the following entries:

$$P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$$

Example: Force optimal control of mass position

- Substituting in the system matrices algebraic Riccati equation becomes

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^T \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdots$$

$$- \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \left(\frac{1}{\rho} \right) \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

- Evaluating terms

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ p_1 & p_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & p_1 \\ 0 & p_2 \end{bmatrix}$$

Example: Force optimal control of mass position

- Evaluating terms

$$\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \left(\frac{1}{\rho} \right) \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$$

$$= \begin{bmatrix} p_2 \\ p_3 \end{bmatrix} \left(\frac{1}{\rho} \right) \begin{bmatrix} p_2 & p_3 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{p_2 p_2}{\rho} & \dfrac{p_2 p_3}{\rho} \\ \dfrac{p_2 p_3}{\rho} & \dfrac{p_3 p_3}{\rho} \end{bmatrix}$$

Example: Force optimal control of mass position

- So therefore

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^T \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdots$$

$$- \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \left(\frac{1}{\rho}\right) \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 0 & 0 \\ p_1 & p_2 \end{bmatrix} + \begin{bmatrix} 0 & p_1 \\ 0 & p_2 \end{bmatrix} - \begin{bmatrix} \dfrac{p_2 p_2}{\rho} & \dfrac{p_2 p_3}{\rho} \\ \dfrac{p_2 p_3}{\rho} & \dfrac{p_3 p_3}{\rho} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 1 - \dfrac{p_2 p_2}{\rho} & p_1 - \dfrac{p_2 p_3}{\rho} \\ p_1 - \dfrac{p_2 p_3}{\rho} & 2 p_2 - \dfrac{p_3 p_3}{\rho} \end{bmatrix} = 0$$

Thus the elements of the matrix must be zero

$$\Rightarrow 1 - \frac{p_2 p_2}{\rho} = 0$$

$$\Rightarrow p_1 - \frac{p_2 p_3}{\rho} = 0$$

$$\Rightarrow 2 p_2 - \frac{p_3 p_3}{\rho} = 0$$

Example: Force optimal control of mass position

- Solving the equations

$$1-\frac{p_2 p_2}{\rho}=0 \quad \Rightarrow (p_2)^2=\rho \quad \boxed{\Rightarrow p_2=\pm\sqrt{\rho}}$$

$$2p_2-\frac{p_3 p_3}{\rho}=0 \quad \Rightarrow 2p_2=\frac{p_3 p_3}{\rho} \quad \Rightarrow 2\sqrt{\rho}=\frac{(p_3)^2}{\rho}$$

$$\Rightarrow 2\rho\sqrt{\rho}=(p_3)^2 \quad \boxed{\Rightarrow p_3=\pm\sqrt{2\rho\sqrt{\rho}}}$$

$$p_1-\frac{p_2 p_3}{\rho}=0 \quad \Rightarrow p_1=\frac{p_2 p_3}{\rho} \quad \Rightarrow p_1=\frac{\sqrt{\rho}\sqrt{2\rho\sqrt{\rho}}}{\rho} \quad \boxed{\Rightarrow p_1=\pm\sqrt{2\sqrt{\rho}}}$$

- will have P positive definite (P > 0), if and only if p1 > 0 and det(P) > 0

$$\Rightarrow P=\begin{bmatrix}\sqrt{2\sqrt{\rho}} & \sqrt{\rho} \\ \sqrt{\rho} & \sqrt{2\rho\sqrt{\rho}}\end{bmatrix}$$

Example: Force optimal control of mass position

- the expression for the optimal gain

$$K = R^{-1}B^{T}P$$

$$\Rightarrow K = \left(\frac{1}{\rho}\right)\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{2\sqrt{\rho}} & \sqrt{\rho} \\ \sqrt{\rho} & \sqrt{2\rho\sqrt{\rho}} \end{bmatrix}$$

$$\Rightarrow K = \left(\frac{1}{\rho}\right)\begin{bmatrix} \sqrt{\rho} & \sqrt{2\rho\sqrt{\rho}} \end{bmatrix}$$

Optimal control gain design using Matlab

- In Matlab the optimal gain can be found using the lqr command
- The Matlab document page is as follows:

```
[K,S,e] = LQR(A,B,Q,R,N)
```

For a continuous time system, the state-feedback law $u = -Kx$ minimizes the quadratic cost function

$$J(u) = \int_0^\infty (x^T Q x + u^T R u + 2x^T N u)dt$$

subject to the system dynamics

$$\dot{x} = Ax + Bu.$$

In addition to the state-feedback gain K, `lqr` returns the solution S of the associated Riccati equation

$$A^T S + SA - (SB + N)R^{-1}(B^T S + N^T) + Q = 0$$

and the closed-loop eigenvalues `e = eig(A-B*K)`. K is derived from S using

$$K = R^{-1}(B^T S + N^T)$$