ROCO218: Control Engineering
Dr Ian Howard

Lecture 9

1st order systems

Response of 1st order system

- Consider a 1st order differential equation system

$$a\frac{dx}{dt}+bx=f(t)$$

- Taking Laplace transforms

$$L[a\frac{dx}{dt}+bx]=L[f(t)]$$

$$\Rightarrow asX(s)+bX(s)=F(s)$$

$$\Rightarrow X(s)[as+b]=F(s)$$

Response of 1st order system

- From

$$X(s)[as+b]=F(s)$$

$$\Rightarrow X(s)=\frac{F(s)}{[as+b]}$$

- We now drive the system by a unit step function starting at t=0
- Therefore f(t) is a constant =1

$$\Rightarrow F(s)=\frac{1}{s}$$

- Ignoring initial conditions the response is therefore

$$\Rightarrow X(s)=\frac{1}{s[as+b]}$$

Response of 1st order system

- From

$$X(s) = \frac{1}{s[as+b]} = \frac{1/b}{s[sa/b+1]} = \frac{k}{s[\tau s+1]}$$

Writing in the form

Where τ is the time constant

- Re-writing as partial fractions

$$\frac{k}{s[\tau s+1]} = \frac{A}{(\tau s+1)} + \frac{B}{s} = \frac{As}{(\tau s+1)s} + \frac{B(\tau s+1)}{s(\tau s+1)}$$

- Equating coefficients

$$\Rightarrow k = (A + B\tau)s + B \qquad \Rightarrow B = k$$

$$\Rightarrow 0 = A + B\tau \qquad \Rightarrow A = -B\tau \quad \Rightarrow A = -k\tau$$

Response of 1st order system

- Therefore

$$X(s)=\frac{k}{s}-\frac{k\tau}{(\tau s+1)}=k\left[\frac{1}{s}-\frac{\tau}{(\tau s+1)}\right]=k\left[\frac{1}{s}-\frac{1}{(s+1/\tau)}\right]$$

- From Laplace pairs table we see that

$$\frac{1}{s}\Leftrightarrow 1 \qquad \frac{1}{(s-a)}\Leftrightarrow e^{at}$$

- Thus the inverse Laplace transform gives the time solution

$$x(t)=k(1-e^{-t/\tau})$$

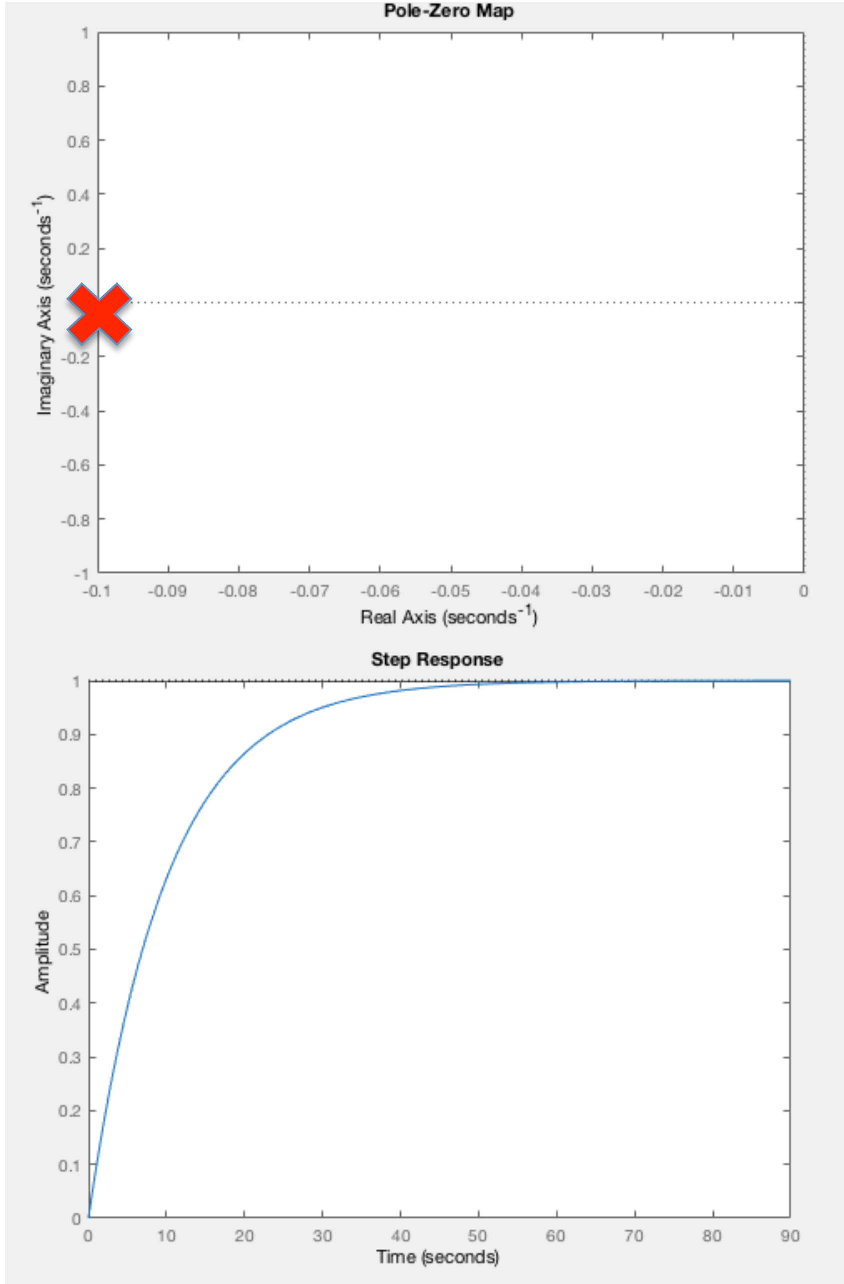
1st order system pole Matlab simulation

```
% 1st order system
k = 1;
tau = 10;

% build 1st order transfer function
s=tf('s');
sys = k/(tau * s + 1);

% plot pole
figure
hold on
pzmap(sys);

% generate step response
figure
hold on
step(sys);
```



1^{st} order system time response

We see 1^{st} order system has the time response to a step given by

$$f(t) = k(1 - e^{-t/\tau})$$

Where

- k is the system gain
- τ is the system time constant
-
- If we set time t= τ then

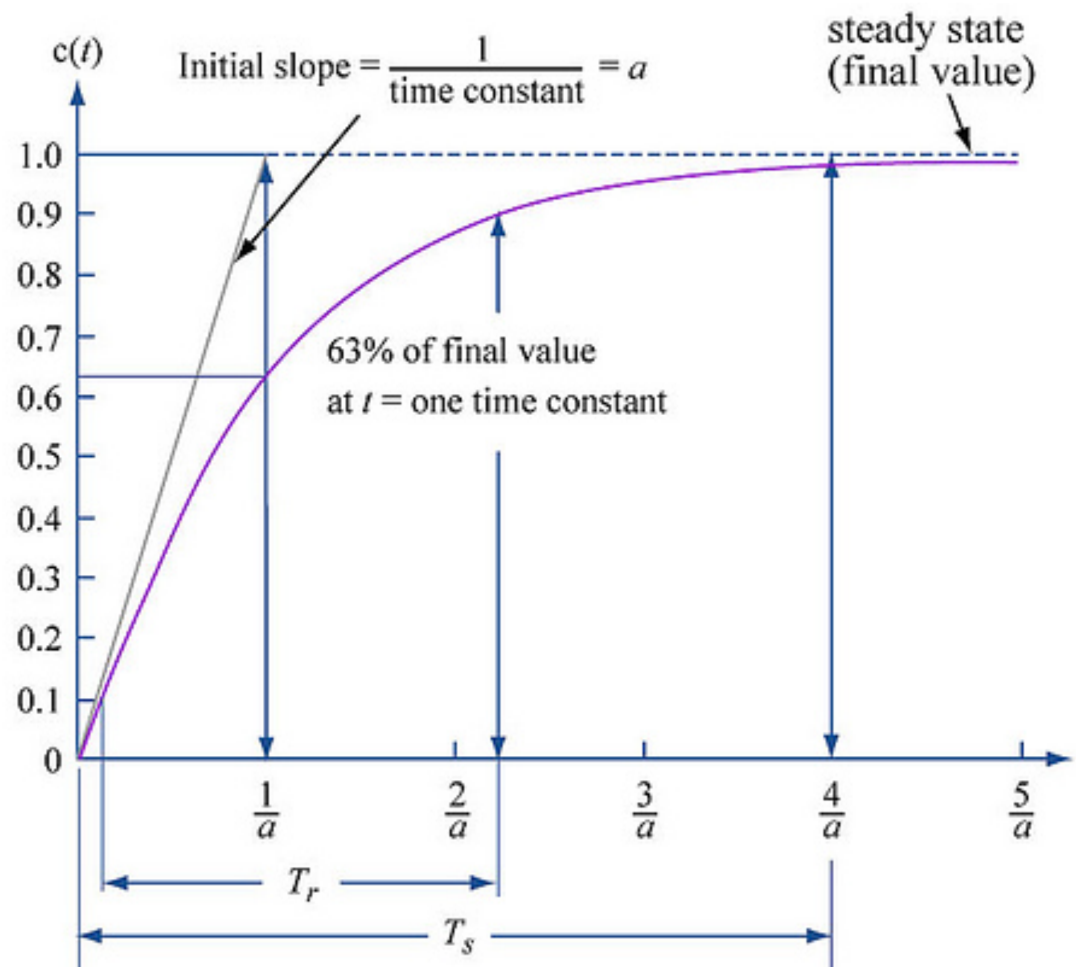
$$f(t) = k(1 - e^{-1})$$

$$f(t) = k(1 - 0.378) = 0.632$$

- Thus the time constant is the time taken to reach 63.2% of maximum change. Note also $\frac{\mathrm{d}}{\mathrm{d}x} e^{f(x)} = f'(x) e^{f(x)}$

$$\frac{df}{dt} = \frac{d}{dt}\left[k(1 - e^{-t/\tau})\right] = \frac{1}{\tau} e^{-t/\tau}$$

- Thus initial slope at t=0 is $\dfrac{1}{\tau}$



Remember: Voltages in an RC circuit

Adding up voltages

$$V_S = V_R + V_C$$

$$V_R = IR$$ Ohms law

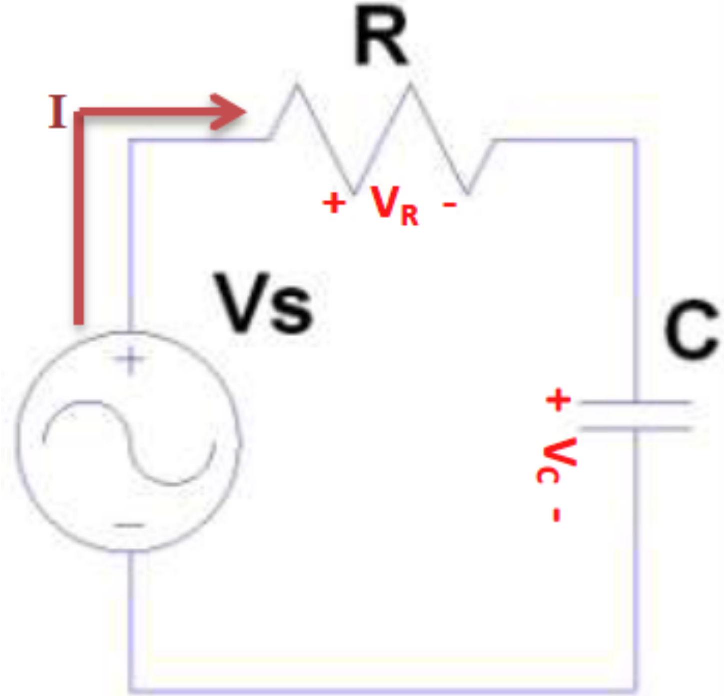
$$I = \frac{dQ}{dt} = C\frac{dV_C}{dt}$$

Current proportional to rate of change of charge

$$\Rightarrow V_R = IR = C\frac{dV_C}{dt}R$$

Substitute I from capacitor expression

$$\Rightarrow V_S = RC\frac{dV_C}{dt} + V_C$$



Rearranging gives

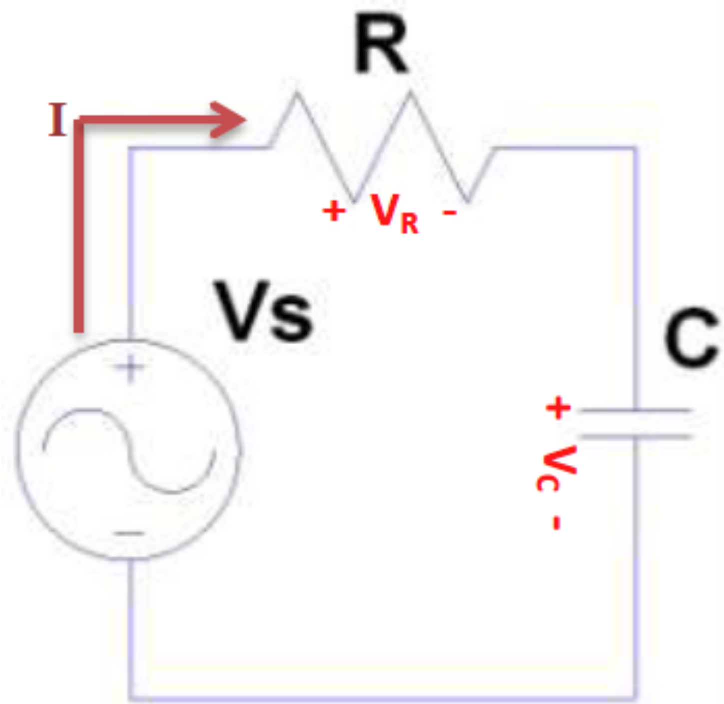
$$\frac{dV_C}{dt} = \frac{1}{RC}\left(V_S - V_C\right)$$

This is a first order linear differential equation

RC circuit time constant

From

$$\frac{dV_C}{dt} = \frac{1}{RC}\left(V_S - V_C\right)$$



Taking Laplace transformation with zero initial conditions

$$\Rightarrow sV_C(s) = \frac{1}{RC}\left(V_S(s) - V_C(s)\right)$$

Collecting V_c terms together

$$\Rightarrow sV_C(s) + \frac{1}{RC}V_C(s) = \frac{1}{RC}V_S(s)$$

$$\Rightarrow V_C(s)\left(s + \frac{1}{RC}\right) = \frac{1}{RC}V_S(s)$$

$$\Rightarrow V_C(s) = \frac{\frac{1}{RC}V_S(s)}{\left(s + \frac{1}{RC}\right)}$$

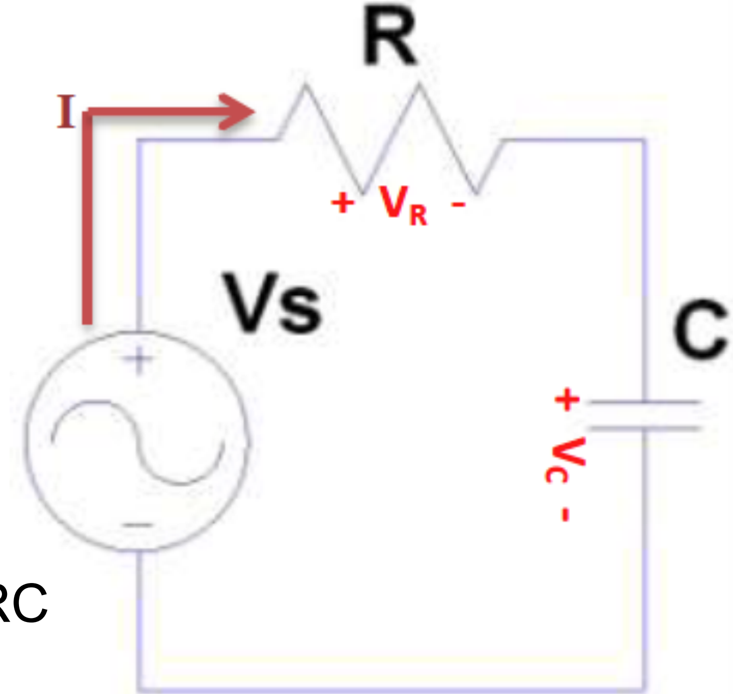
Multiply though by RC

$$\Rightarrow V_C(s) = \frac{V_S(s)}{\left(sRC + 1\right)}$$

RC circuit time constant

From

$$V_C(s) = \frac{V_S(s)}{(sRC+1)}$$



Driving using a step of voltage V at t=0 $\Rightarrow V_S(s) = \frac{V}{s}$

$$\Rightarrow V_C(s) = \frac{V}{s[\tau s+1]}$$

Where time constant τ = RC

From general case before the inverse Laplace transform gives the time solution

$$F(s) = \frac{k}{s[\tau s+1]} \Leftrightarrow f(t) = k(1-e^{-t/\tau})$$

$$\Rightarrow V_C(t) = V(1-e^{-t/\tau})$$

RC circuit current

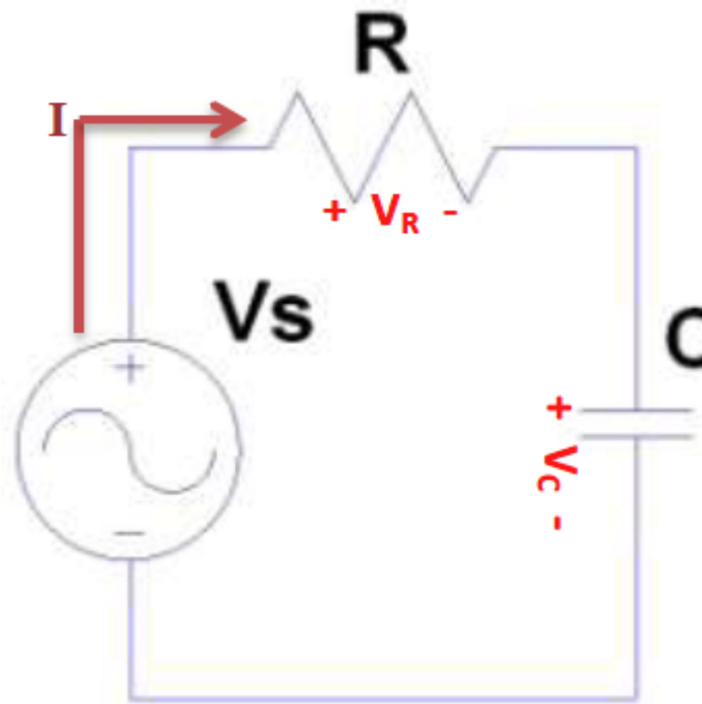
From

$$\frac{dV_C}{dt} = \frac{1}{RC}\left(V_S(t) - V_C(t)\right)$$

And

$$I(t) = C\frac{dV_C}{dt}$$

$$\Rightarrow I(t) = C\left(\frac{1}{RC}\left(V_S(t) - V_C(t)\right)\right)$$



Since

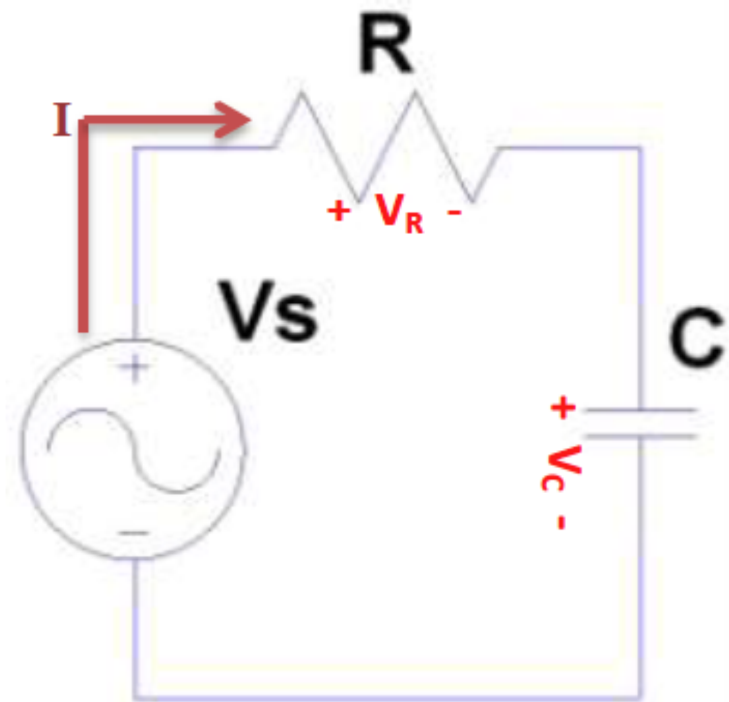
$$V_c(t) = \frac{1}{C}\int I(t)\,dt$$

$$\Rightarrow I(t) = C\left(\frac{1}{RC}\left(V_S(t) - \frac{1}{C}\int I(t)\,dt\right)\right)$$

RC circuit current

From

$$I(t) = C\left(\frac{1}{RC}\left(V_S(t) - \frac{1}{C}\int I(t)\,dt\right)\right)$$



Collecting current terms together

$$\Rightarrow I(t) + C\frac{1}{RC}\frac{1}{C}\int I(t)\,dt = C\frac{1}{RC}V_S(t)$$

$$\Rightarrow I(t) + \frac{1}{RC}\int I(t)\,dt = \frac{V_S(t)}{R}$$

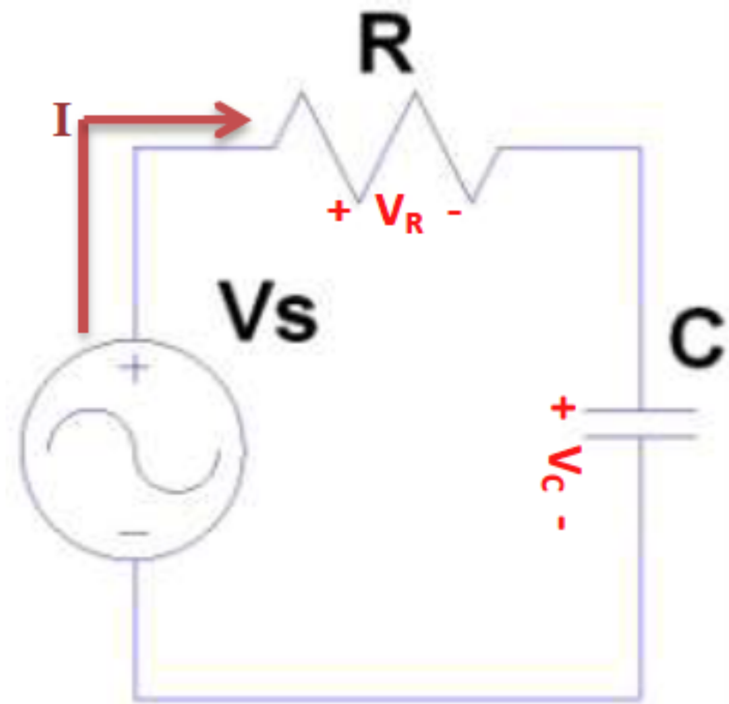
Taking Laplace transformation with zero initial conditions

$$\Rightarrow I(s) + \frac{1}{RC}\frac{I(s)}{s} = \frac{V_S(s)}{R} \quad \Rightarrow I(s)\left(1 + \frac{1}{sRC}\right) = \frac{V_S(s)}{R}$$

RC circuit current

- From

$$I(s)\left(1+\frac{1}{sRC}\right)=\frac{V_S(s)}{R}$$



$$\Rightarrow I(s)=\frac{\frac{V_S(s)}{R}}{\left(1+\frac{1}{sRC}\right)} \qquad \Rightarrow I(s)=\frac{s\frac{V_S(s)}{R}}{\left(s+\frac{1}{RC}\right)}$$

Driving using a step function if voltage V at t=0 $\Rightarrow V_S(s)=\frac{V}{s}$

$$\Rightarrow I(s)=\frac{\frac{V}{R}}{\left(s+\frac{1}{RC}\right)} = \frac{\frac{V}{R}}{\left(s+\frac{1}{\tau}\right)}$$

Where time constant τ = RC

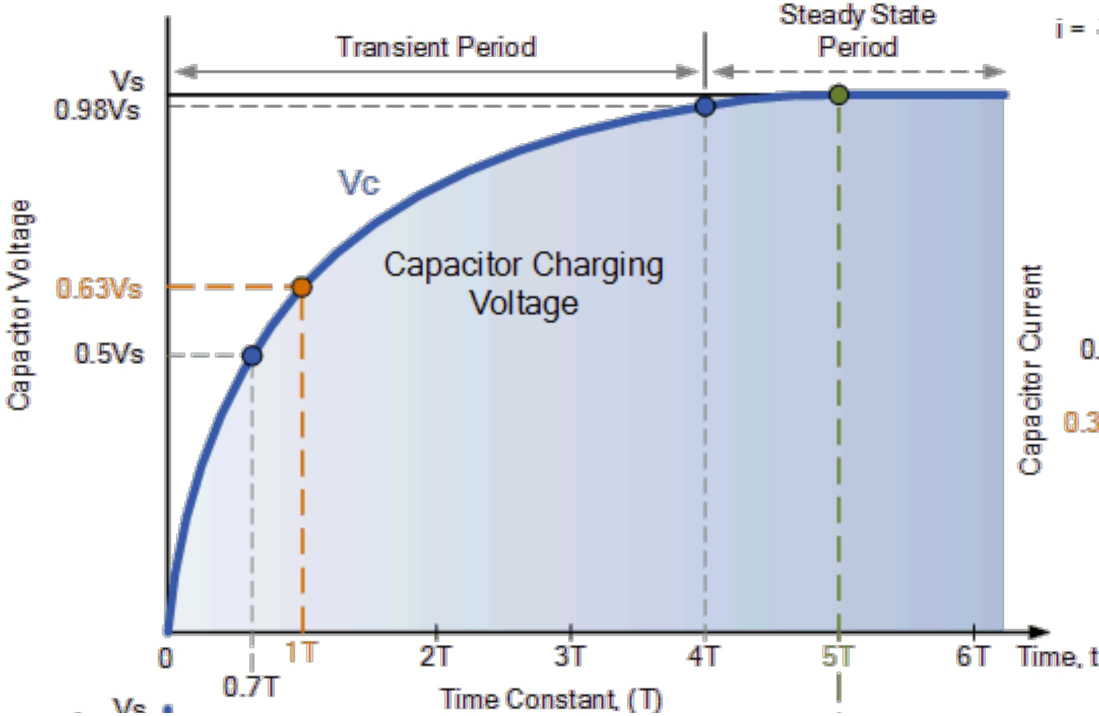
- From Laplace pairs table we see that

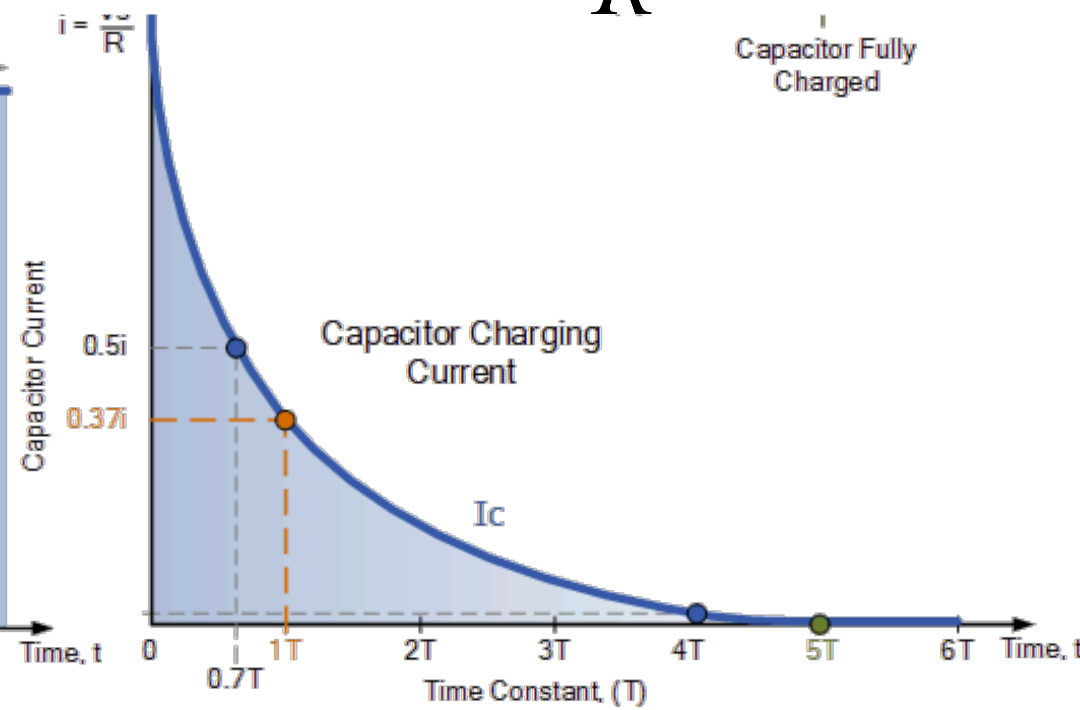
$$\frac{1}{(s-a)} \Leftrightarrow e^{at} \qquad \Rightarrow I(t)=\frac{V}{R}e^{-t/\tau}$$

RC circuit voltage and current Time constant

$$V_C(t) = V(1 - e^{-t/\tau})$$

$$I(t) = \frac{V}{R} e^{-t/\tau}$$

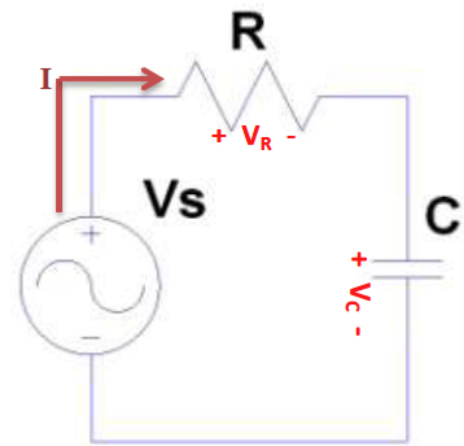




The time constant τ is related to the cutoff frequency f_c

$$f_c = \frac{1}{2\pi\tau} = \frac{1}{2\pi RC}$$

Relationship between rise time & BW for 1st order system



- For the series RC circuit, capacitor voltage is given by

$$V_C(s) = \frac{V_S(s)}{(s\tau+1)} \quad \Rightarrow \frac{V_C(s)}{V_S(s)} = \frac{1}{(s\tau+1)}$$

- Can calculate frequency response by substituting s = jω

$$\Rightarrow H(s) = \frac{1}{(j\omega\tau+1)}$$

Now write $\omega_{bw} = \frac{1}{\tau} \Rightarrow f_{bw} = \frac{1}{2\pi\tau}$

$$\Rightarrow H(s) = \frac{1}{\left(\frac{j\omega}{\omega_{bw}}+1\right)}$$

- Where f_{bw} will be the -3dB bandwidth in Hz

If we let $\omega = \omega_{bw} \Rightarrow H(s) = \frac{1}{(j+1)}$

$$\Rightarrow H(s) = \frac{(1-j)}{(1+j)(1-j)} = \frac{(1-j)}{2} \Rightarrow |H(s)| = \frac{\sqrt{(1^2+1^2)}}{2} = \frac{1}{\sqrt{2}}$$

$$dB = 20\log 10\left(\frac{1}{\sqrt{2}}\right) = \text{-}3.0103$$

- So when input frequency is f_{bw} the response drops by 3dB

Relationship between rise time & BW for 1st order system

We saw that the voltage across capacitor in response to step given by

$$V_C(s) = \frac{V}{s[\tau s+1]}$$

Where time constant τ = RC

And the inverse Laplace transform gives

$$\frac{V_C}{V}(t) = (1-e^{-t/\tau})$$

Rise time t_r is time between reaching 10% and 90% - will discuss more shortly

These times are respectively given by

$$0.1 = (1-e^{-t_{0.1}/\tau}) \Rightarrow e^{-t_{0.1}/\tau} = 0.9 \Rightarrow t_{0.1} = -\tau \ln\left(0.9\right)$$

$$0.9 = (1-e^{-t_{0.9}/\tau}) \Rightarrow e^{-t_{0.9}/\tau} = 0.1 \Rightarrow t_{0.9} = -\tau \ln\left(0.1\right)$$

$$\Rightarrow t_r = t_{0.9} - t_{0.1} \qquad \Rightarrow t_r = -\tau \ln\left(0.1\right) + \tau \ln\left(0.9\right)$$

$$\Rightarrow t_r = \tau \ln\left(\frac{0.9}{0.1}\right) = 2.1972\tau$$

Relationship between rise time & BW for 1st order system

So we see that

$$t_r = 2.1972\tau$$

From before we can also write

$$\omega_{bw} = \frac{1}{\tau}$$

$$\Rightarrow f_{bw} = \frac{1}{2\pi\tau} \quad \Rightarrow \tau = \frac{1}{2\pi f_{bw}}$$

Where f_{bw} is the -3dB bandwidth in Hz

$$\Rightarrow t_r = 2.1972\tau = \frac{2.1972}{2\pi f_{bw}} = \frac{0.3497}{f_{bw}}$$

$$\Rightarrow t_r \approx \frac{0.35}{f_{bw}}$$

This gives a rule of thumb for the relation between bandwidth and rise time for 1st order systems

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Lecture 9

2^{nd} order systems

Response of 2nd order system

Consider 2nd order response behavior described by the differential equation

$$a\frac{d^2x}{dt^2}+b\frac{dx}{dt}+cx=f(t)$$

Taking Laplace transforms

$$L[a\frac{d^2x}{dt^2}+b\frac{dx}{dt}+cx]=L[f(t)]$$

$$\Rightarrow as^2X(s)+bsX(s)+cX(s)=F(s)$$

Factorizing out X(s)

$$\Rightarrow X(s)[as^2+bs+c]=F(s)$$

2nd order system canonical form

Transfer function of the 2nd order system is

$$\frac{X(s)}{F(s)} = H(s) = \frac{1}{[as^2 + bs + c]}$$

Writing in canonical form

$$H(s) = \frac{w_n^2}{[s^2 + 2\xi w_n s + w_n^2]}$$

Poles occurs when denominator is zero

$$s^2 + 2\xi w_n s + w_n^2 = 0$$

Writing in canonical form is informative because we can then directly identify:

- W_n which is the natural frequency of the system
- ξ which is the damping ratio of the system

Remember: Solving quadratic equations

The quadratic equation

$$ax^2 + bx + c = 0$$

Has the solution

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

if $b^2 \geq 4ac$ The solution will be a real number

if $b^2 < 4ac$ The solution will be an imaginary number

2^{nd} order system canonical form

Solving for denominator poles

$$s^2 + 2\xi w_n s + w_n^2 = 0$$

$$\Rightarrow s = \frac{-2\xi w_n \pm \sqrt{(2\xi w_n)^2 - 4w_n^2}}{2} = \frac{-2\xi w_n \pm \sqrt{4w_n^2\xi^2 - 4w_n^2}}{2}$$

$$\Rightarrow s = \frac{-2\xi w_n \pm 2w_n\sqrt{\xi^2 - 1}}{2}$$

$$\Rightarrow s = -\xi w_n \pm w_n\sqrt{\xi^2 - 1}$$

for $\xi^2 < 1$

- Square root term negative so get complex result

$$\Rightarrow s = -\xi w_n \pm jw_n\sqrt{1 - \xi^2}$$

2^{nd} order system canonical form

The time response of a 2^{nd} system depends on the value of damping factor ξ and natural frequency W_n

For $\xi > 1$ Real solution and system is over-damped

$$\Rightarrow s = -\xi w_n \pm w_n\sqrt{\xi^2 - 1}$$

For $\xi = 1$ Real solution and system is critically damped

$$\Rightarrow s = -\xi w_n$$

The solution is stable and has –ve real poles
Exhibits exponential decay

For $0 < \xi < 1$ System is under-damped

$$\Rightarrow s = -\xi w_n \pm j w_n\sqrt{1 - \xi^2}$$

The solution is stable and has complex poles and exhibits oscillatory decay

For $\xi = 0$ System is un-damped

$$\Rightarrow s = \pm j w_n$$

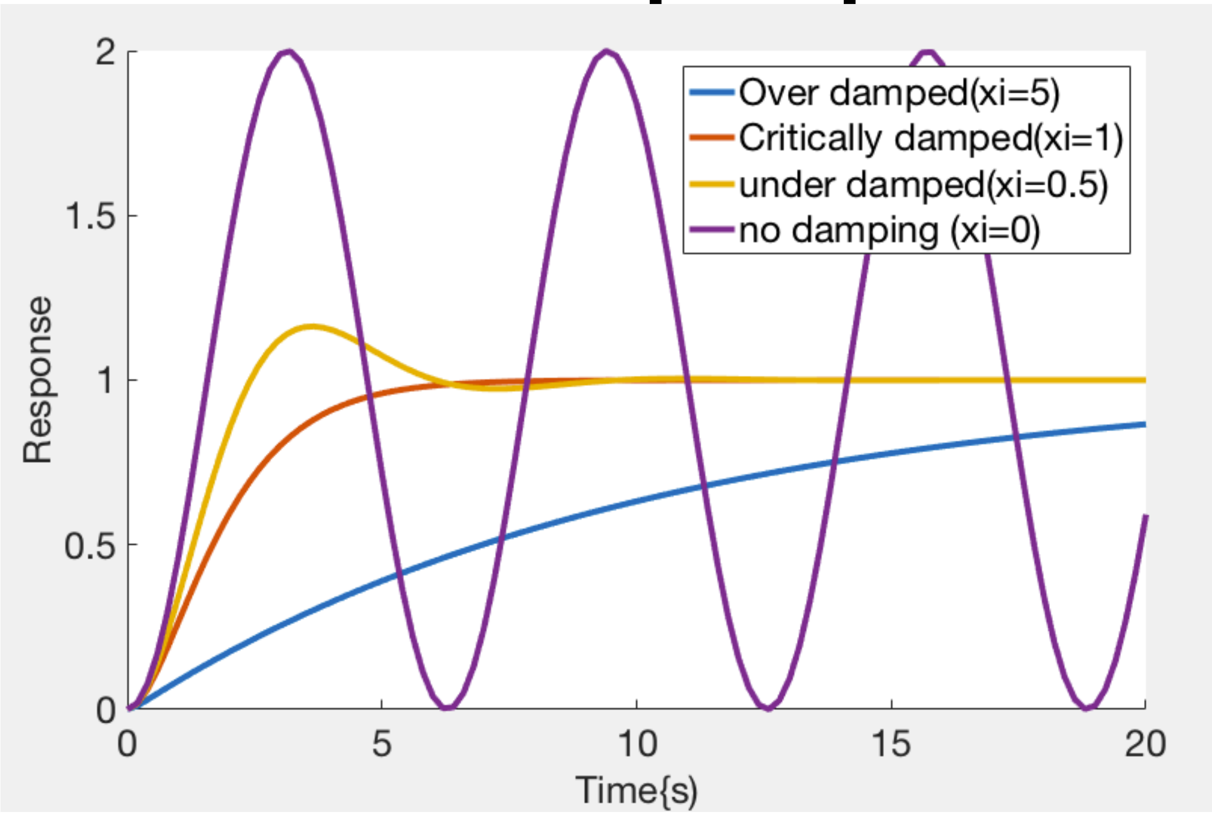
Continuous oscillation
So marginally stable

For $\xi < 0$ System is unstable!

$$\Rightarrow s = \xi w_n \pm w_n\sqrt{\xi^2 - 1}$$

System exhibits unlimited growth

2nd order step response Matlab simulation




```
% build 2nd order transfer function
wn = 1;
s=tf('s');
% over damped 2nd order system
xi=5;
sysOD = wn^2/( s^2 + 2 * xi * wn * s + wn^2);
% critically damped 2nd order system
xi=1;
sysCD = wn^2/( s^2 + 2 * xi * wn * s + wn^2);
% under damped 2nd order system
xi=0.5;
sysUD = wn^2/( s^2 + 2 * xi * wn * s + wn^2);
% no damping 2nd order system
xi=0;
sysND = wn^2/( s^2 + 2 * xi * wn * s + wn^2);
```

```
% generate step response
TDur=20;
figure
hold on
[Y,T] = step(sysOD, TDur)
h = plot(T,Y);
set(h,'LineWidth', 4);
[Y,T] = step(sysCD, TDur);
h = plot(T,Y);
set(h,'LineWidth', 4);
[Y,T] = step(sysUD, TDur);
h = plot(T,Y);
set(h,'LineWidth', 4);
[Y,T] = step(sysND, TDur);
h = plot(T,Y);
set(h,'LineWidth', 4);
h=legend('Over damped(xi=5)', 'Critically damped(xi=1)', ...
    'under damped(xi=0.5)', 'no damping (xi=0)');
set(h, 'FontSize', 25);
set(gca, 'FontSize', 25);
h = xlabel('Time{s)');
set(h, 'FontSize', 25);
h = ylabel('Response');
set(h, 'FontSize', 25);
```

When $\zeta < 1$ response under damped and sinusoidal
As ζ increases response becomes progressively less oscillatory
When $\zeta=1$ response *critically damped*
When $\zeta>1$ response *over damped* for.
Control systems normally designed with a damping factor $\zeta<1$

2nd order system analytical step response

Starting with the transfer function

$$H(s) = \frac{w_n^2}{[s^2 + 2\xi w_n s + w_n^2]}$$

Assuming zero initial conditions the step response given by

$$R(s) = \frac{1}{s}\frac{w_n^2}{[s^2 + 2\xi w_n s + w_n^2]}$$

Partial fraction expansion and inverse Laplace yields

$$y(t) = 1 - e^{-\xi w_n t} \cdot \frac{\sin\left[\left(w_n\sqrt{1-\xi^2}\right)t + a\cos(\xi)\right]}{\sqrt{1-\xi^2}}$$

2^{nd} order system analytical step response

- Note following points

Damped oscillatory frequency $= w_n\sqrt{\xi^2 - 1}$

$$y(t) = 1 - e^{-\xi w_n t} \cdot \frac{\sin\left[\left(w_n\sqrt{1-\xi^2}\right)t + a\cos(\xi)\right]}{\sqrt{1-\xi^2}}$$

Time constant $= \xi w_n$

Natural frequency $= w_n$

Damping $= \xi$

2nd order system analytical step response

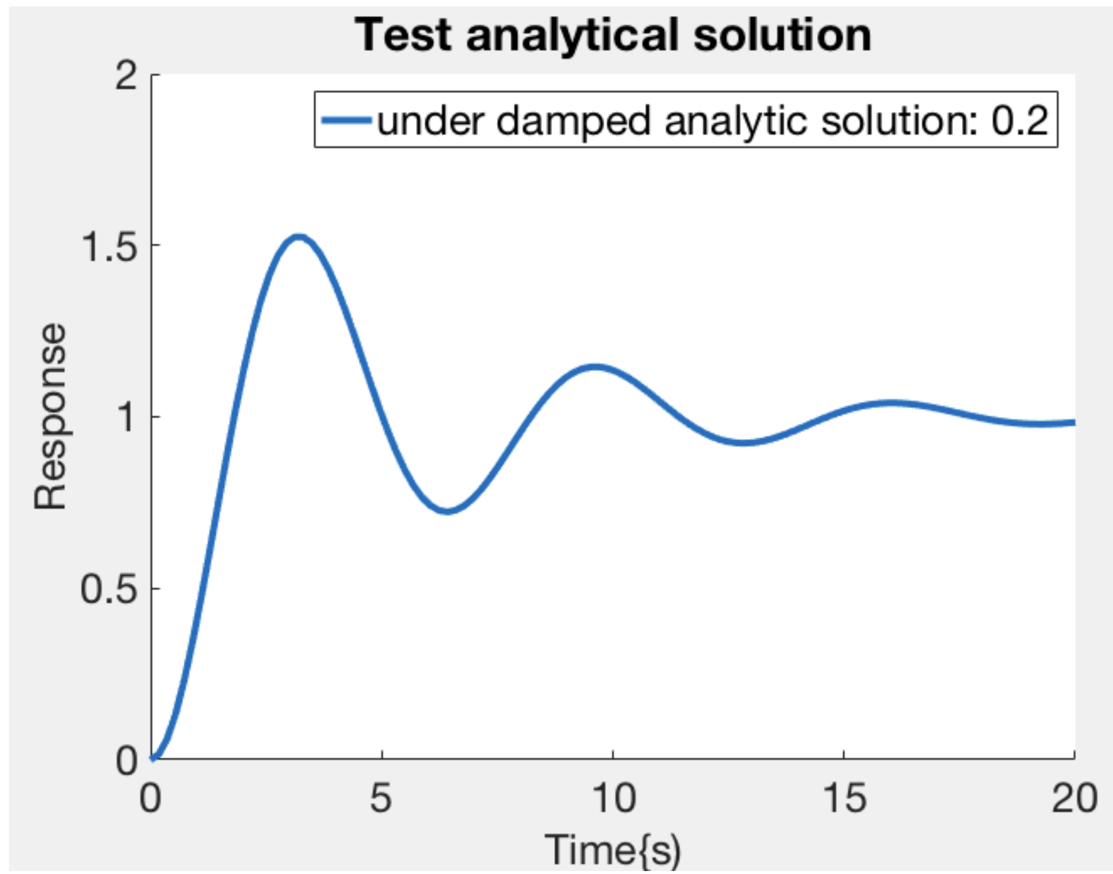
```
function y = Get2ndOrderReponse(wn, xi, T)
% compute second order response

et = exp(-xi*wn*T);
dt = sqrt(1-xi^2);
st = sin(wn*dt*T + acos(xi));
y = ones(size(T)) - et .* (st / dt);
end
```

$$(t) = 1 - e^{-\xi w_n t} \cdot \frac{\sin\left[\left(w_n\sqrt{1-\xi^2}\right)t + a\cos(\xi)\right]}{\sqrt{1-\xi^2}}$$

```
% build 2nd order transfer function
% for underdamped case
s=tf('s');
wn=1;
xi=0.2;
TDur=20;
Ta = 0:0.1:TDur;

% direct analytical simulation
Rt = Get2ndOrderResponse(wn, xi, Ta);
figure
hold on
h = plot(Ta,Rt);
set(h,'LineWidth', 4);
h=legend('under damped analytic solution: 0.2');
set(h, 'FontSize', 25);
set(gca, 'FontSize', 25);
h = xlabel('Time{s)');
set(h, 'FontSize', 25);
h = ylabel('Response');
set(h, 'FontSize', 25);
title('Test analytical solution')
```



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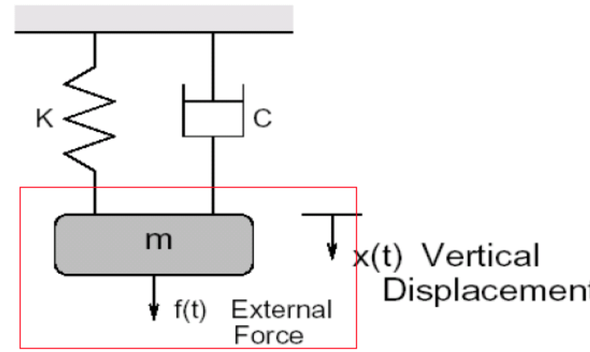
Lecture 9

Example canonical 2nd order system

Spring mass damper system: Differential eqn

From differential equation of movement

$$f(t) = m\frac{d^2x}{dt^2} + C\frac{dx}{dt} + kx$$



Let us now predict system response

Remember in the general case the Laplace transform of differentials include initial conditions

$$\mathfrak{L}\{y'\} = sY(s) - y(0)$$
$$\mathfrak{L}\{y''\} = s^2Y(s) - sy(0) - y'(0)$$

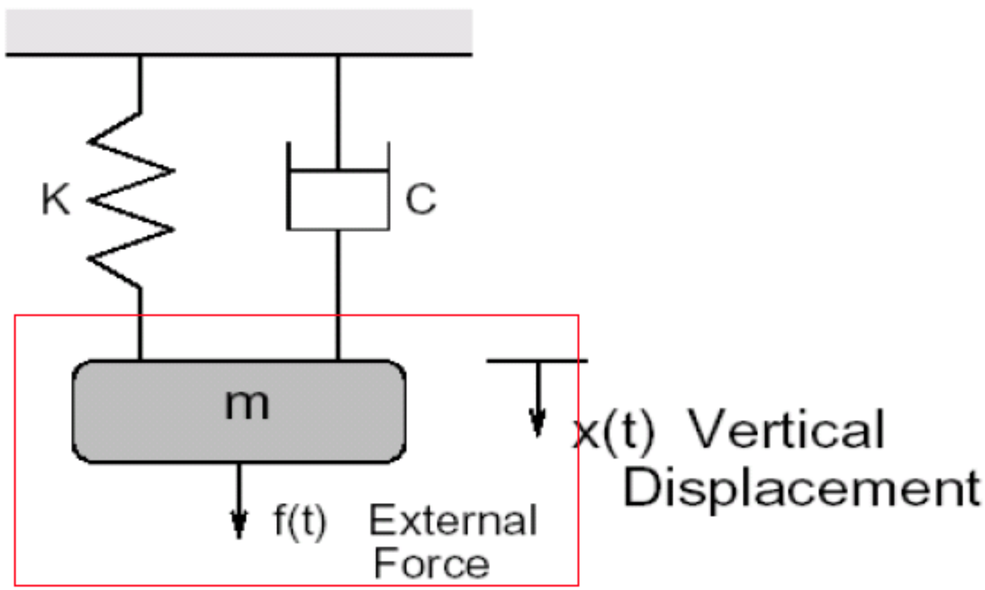
Taking Laplace transforms of mass-spring-damper system gives

$$F(s) = m\left[s^2X(s) - sx(0) - \frac{dx}{dt}(0)\right] + C\left[sX(s) - x(0)\right] + kX(s)$$

Spring mass damper system: Laplace representation

From Laplace equation

$$F(s) = m\left[s^2 X(s) - sx(0) - \frac{dx}{dt}(0)\right] + C\left[sX(s) - x(0)\right] + kX(s)$$



Consider case where

1. We pull down the mass
2. We hold still
3. Then we release at t=0

$x(0) = x_0$ Start at x_0

$\frac{dx}{dt} = 0$ Not moving

$f(t) = 0$ No external force

Spring mass damper system: Transfer function

So given

At x_0 Not moving At x_0

$$F(s) = m\left[s^2 X(s) - sx(0) - \frac{dx}{dt}(0)\right] + C\left[sX(s) - x(0)\right] + kX(s)$$

Substituting in initial conditions

$$0 = m\left[s^2 X(s) - sx_0)\right] + C\left[X(s)s - x_0\right] + kX(s)$$

Rearranging to factor out X(s)

$$\left(ms^2 + Cs + k\right)X(s) = x_0\left(ms + C\right)$$

Gives

$$X(s) = \frac{x_0\left(ms + C\right)}{\left(ms^2 + Cs + k\right)}$$

Spring mass damper system: Canonical form

To write in a canonical form

$$X(s)=\frac{x_0\left(ms+C\right)}{\left(ms^2+Cs+k\right)}$$

We need to have a unity s^2 term so

$$X(s)=\frac{x_0\left(s+\dfrac{C}{m}\right)}{\left(s^2+\dfrac{C}{m}s+\dfrac{k}{m}\right)} \Leftrightarrow \frac{x_0\left(s+2\xi\omega_n\right)}{\left(s^2+2\xi\omega_n s+\omega_n^2\right)}$$

By inspection we see that

$$2\xi\omega_n=\frac{C}{m}$$

$$\omega_n^2=\frac{k}{m}$$

So

$$\Rightarrow \omega_n=\sqrt{\frac{k}{m}}$$

The natural frequency of the system

$$\Rightarrow \xi=\frac{C}{m}\frac{1}{2}\sqrt{\frac{m}{k}}=\frac{C}{2\sqrt{mk}}$$

The damping ratio of the system

Spring mass damper system: Zero and poles

The characteristic equation

$$\frac{x_0\left(s+2\xi\omega_n\right)}{\left(s^2+2\xi\omega_n s+\omega_n^2\right)}$$

Has a zero when the numerator goes to zero
This happen when

$$s=-2\xi\omega_n$$

For $\xi^2 < 1$ there are poles when the denominator goes to zero
This happens when

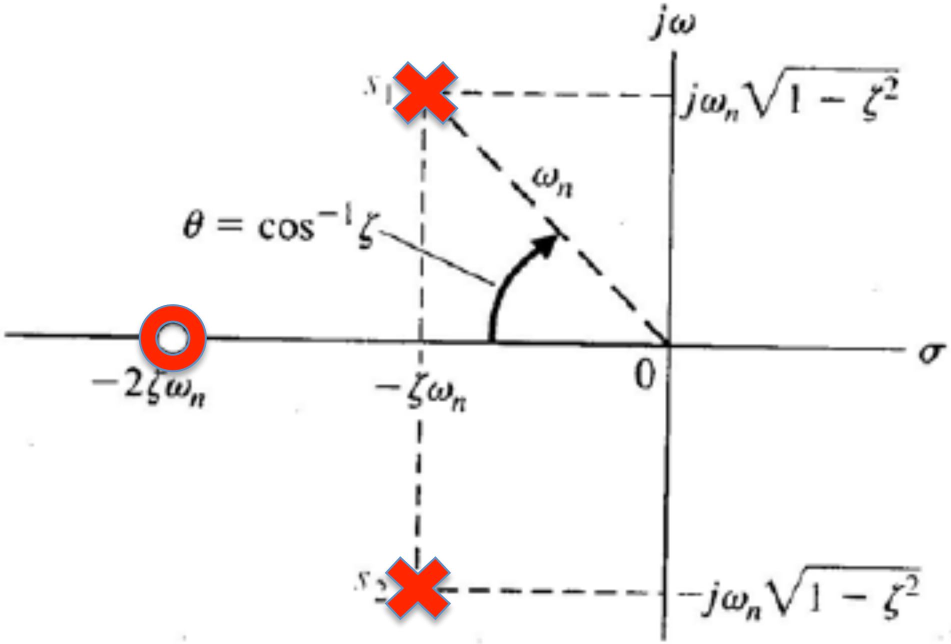
Remember for equation: $ax^2+bx+c=0.$ Solution: $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

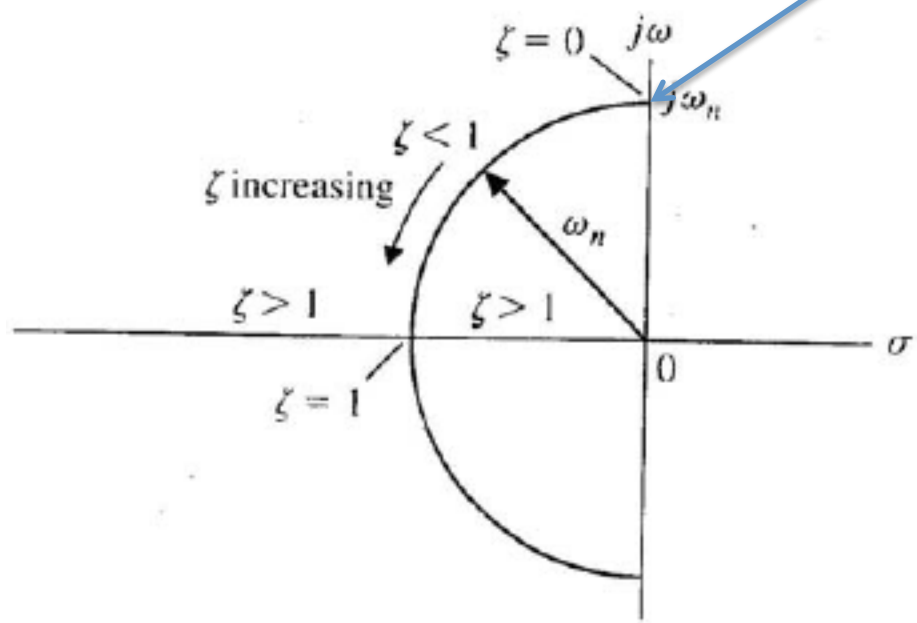
$$s=\frac{-2\xi\omega_n\pm\sqrt{4\xi^2\omega_n^2-4\omega_n^2}}{2}=-\xi\omega_n\pm\sqrt{\xi^2\omega_n^2-\omega_n^2}\quad=-\xi\omega_n\pm j\omega_n\sqrt{1-\xi^2}$$

Spring mass damper system: Zero and poles plot

Plotting the poles and zeros for ω_n=1 and 0 < ζ< 1

No damping
Only imaginary poles





Angle θ to real axis given by

$$\theta = \arccos(\xi)$$

As damping increases point pole rotates down onto real axis

Spring mass damper system Matlab plot

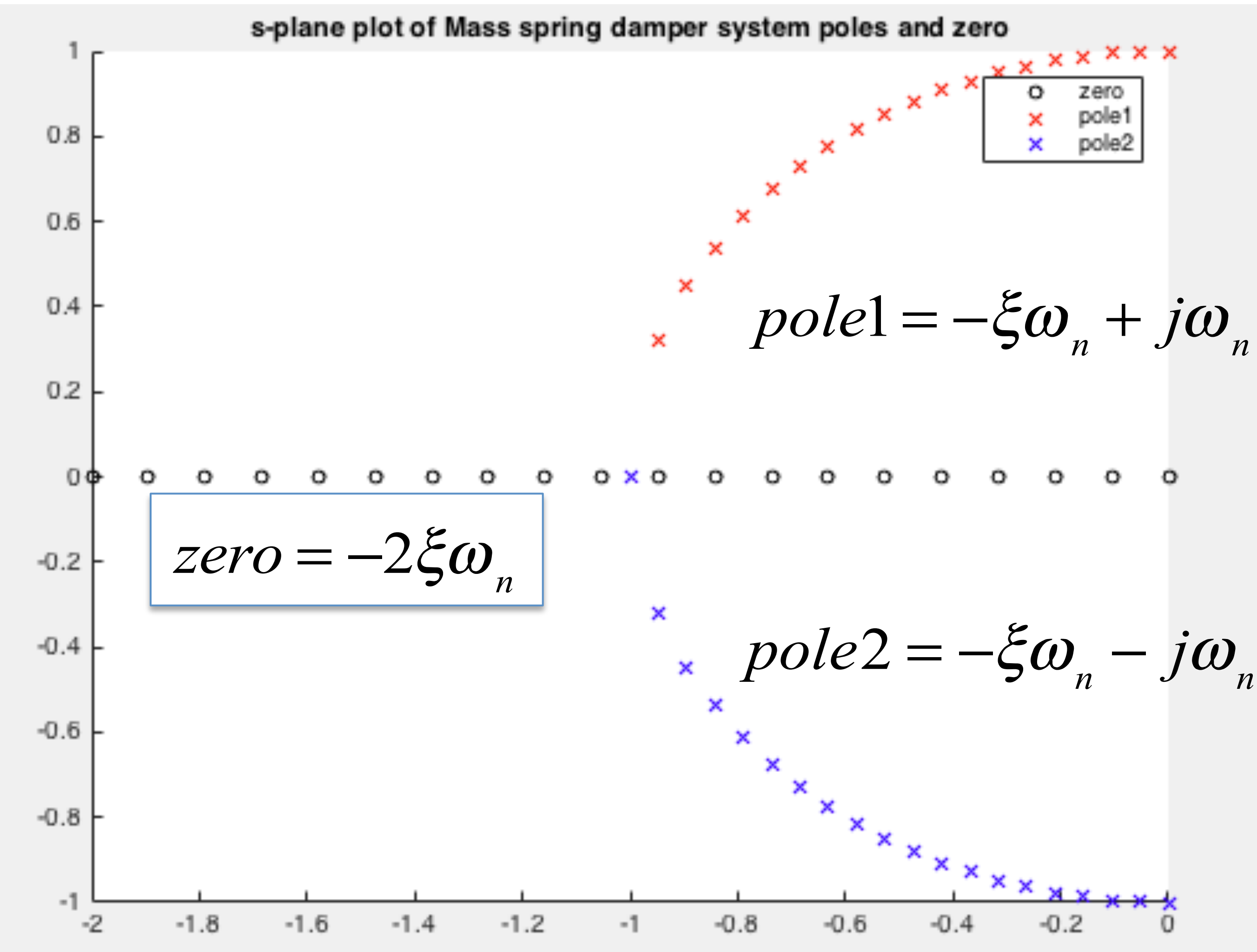
Plotting the poles and zeros for ω_n=1 and $0 < \zeta < 1$

$$zero = -2\xi\omega_n \qquad poles = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$$

```
% choose unity angular frequency
wn = 1;

% range of dampling rations
dp = linspace(0, 1, 20);

zero = complex(-2 * dp * wn);
pole1 = complex(-dp * wn + sqrt(dp .^ 2-1));
pole2 = complex(-dp * wn - sqrt(dp .^ 2-1));

% plot the plot- 2 *
figure
hold on
title('s-plane plot of Mass spring damper system poles and zero');
h = plot(zero,'ok');
set(h,'MarkerSize', 5);
h = plot(pole1,'xr');
set(h,'MarkerSize', 5);
h = plot(pole2,'xb');
set(h,'MarkerSize', 5);
legend('zero', 'pole1', 'pole2');
```

Spring mass damper system Matlab plot

Plotting the poles and zeros for $\omega_n=1$ and $0 < \zeta < 1$



$$pole1 = -\xi\omega_n + j\omega_n\sqrt{1-\xi^2}$$

$$zero = -2\xi\omega_n$$

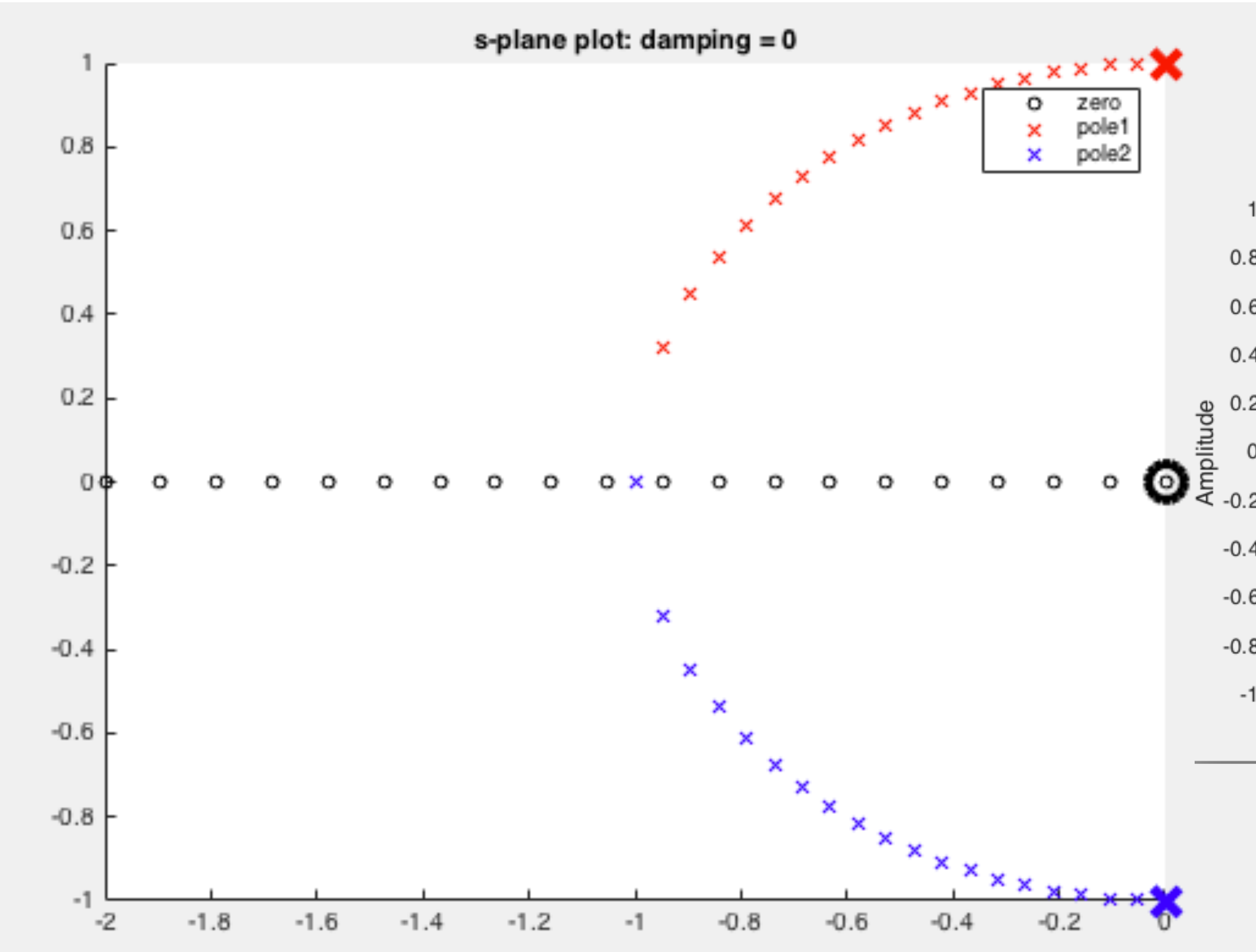
$$pole2 = -\xi\omega_n - j\omega_n\sqrt{1-\xi^2}$$

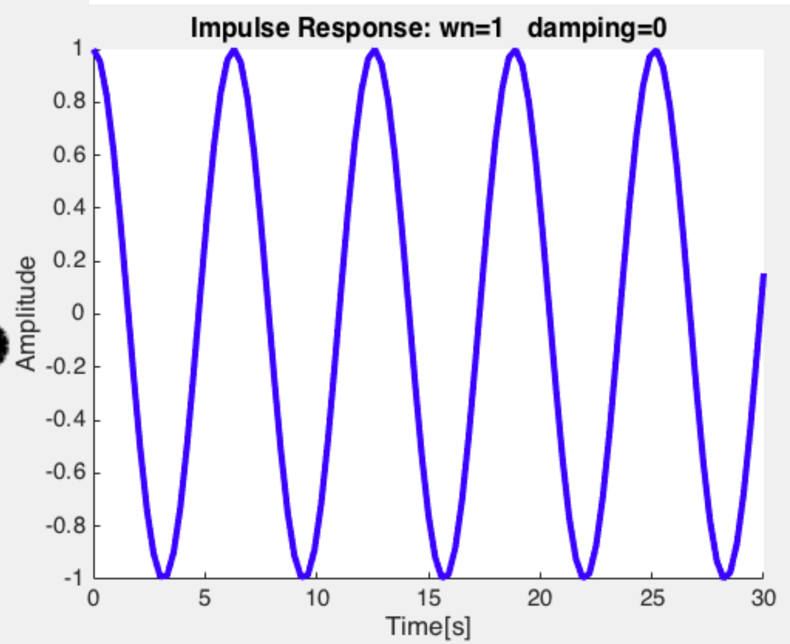
Plotting the impulse responses in Matlab

```
% for s range of damping factors
wn = 1;
tFinal = 30;
dpA =  [0 0.5 1 ];
s = tf('s');
for idx = 1:length(dpA)
    dp = dpA(idx);
    sys(idx) = (s + 2 * dp * wn)/(s^2 + s * (2 * dp * wn) + wn^2);
end

for idx = 1:length(dpA)
    figure
    hold on
    [Y T] = impulse( sys(idx), tFinal);
    %h = plot(T,Y, plotPar{idx})
    h = plot(T,Y, 'b')
    set(h,'MarkerSize', 15,'LineWidth', 4);
    xlabel('Time[s]')
    ylabel('Amplitude')
    title(sprintf('Impulse Response: wn=%g   damping=%g', wn, dpA(idx)) );
    set(gca,'fontSize', 15);
end
```

Spring mass damper system in Matlab

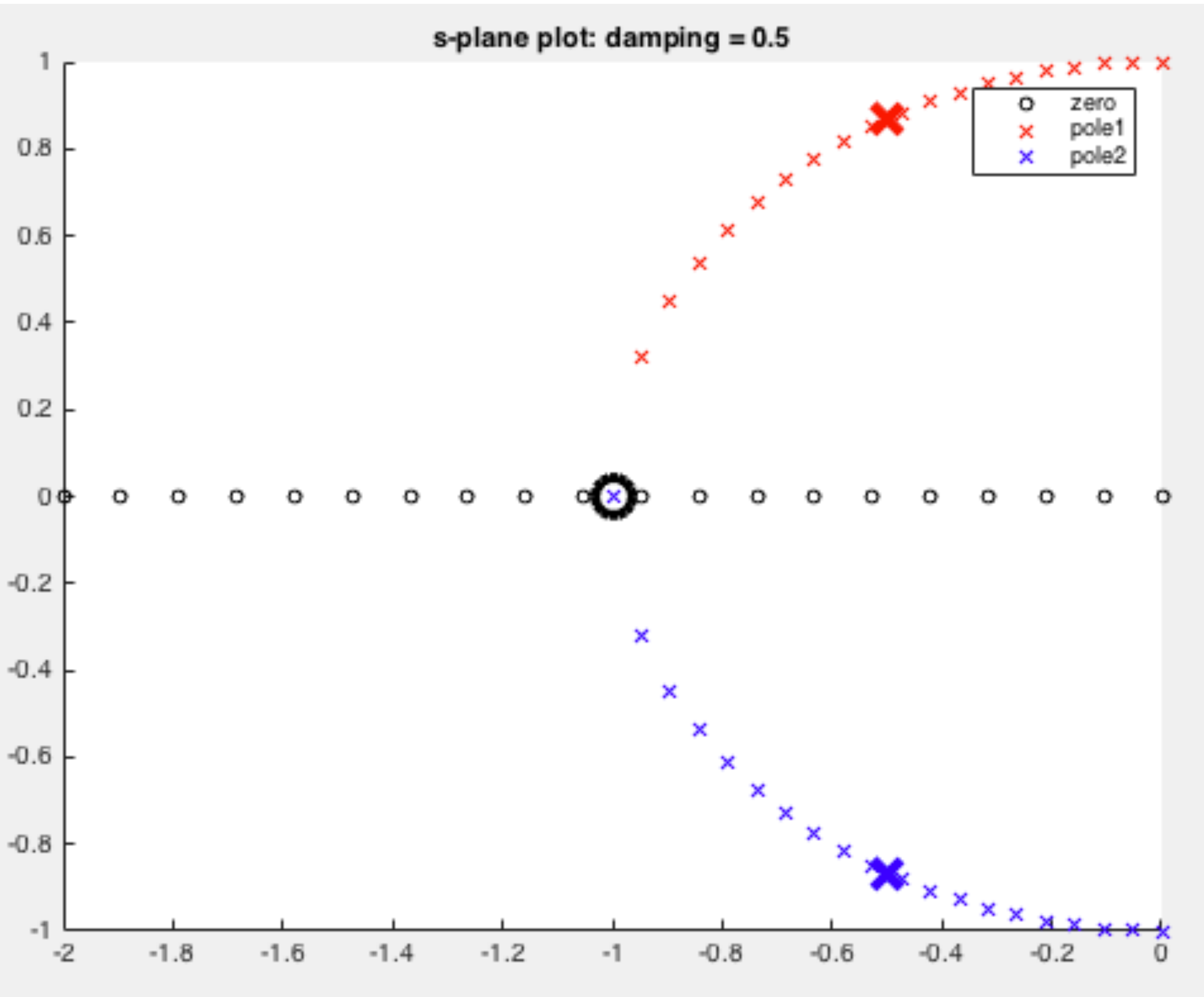
Highlighting poles and zero for damping = 0

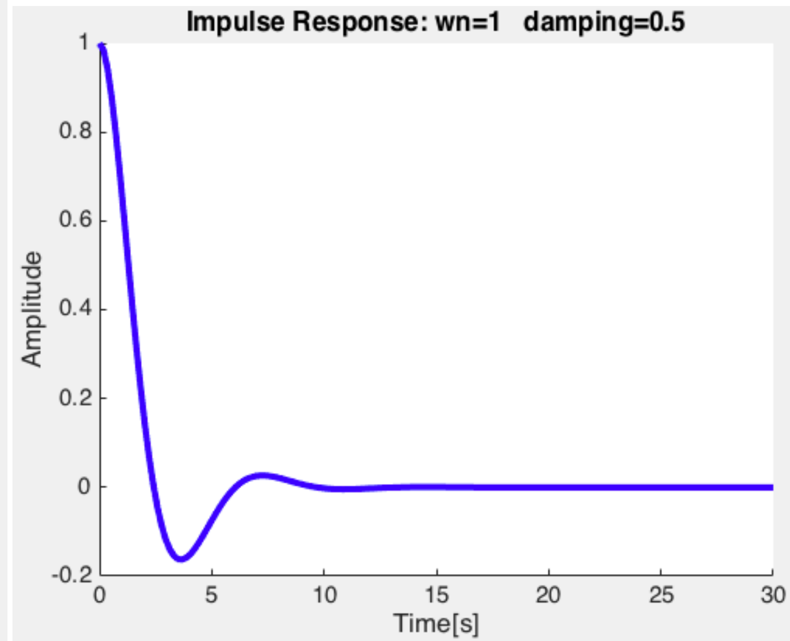




Spring mass damper system in Matlab

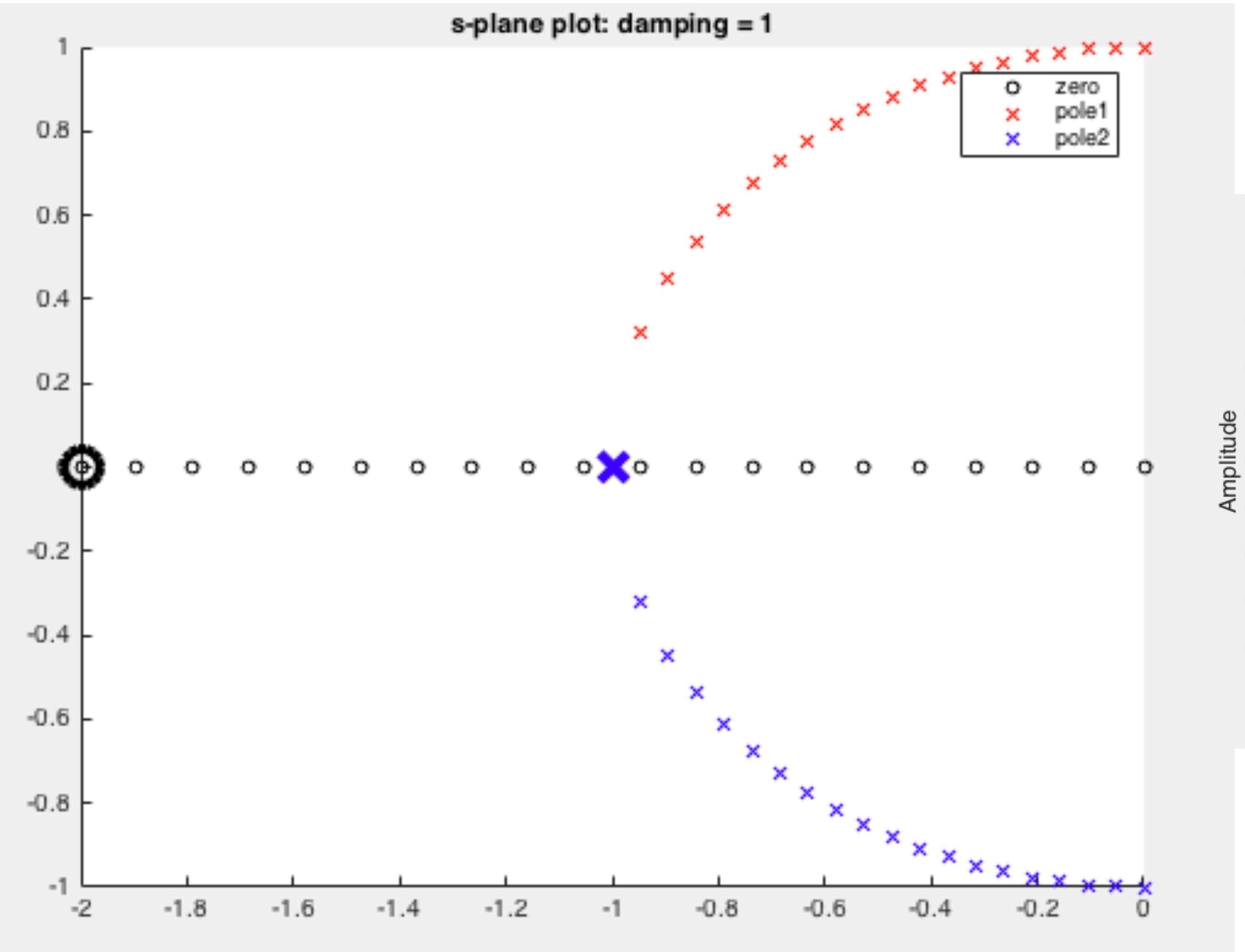
Highlighting poles and zero for damping = 0.5

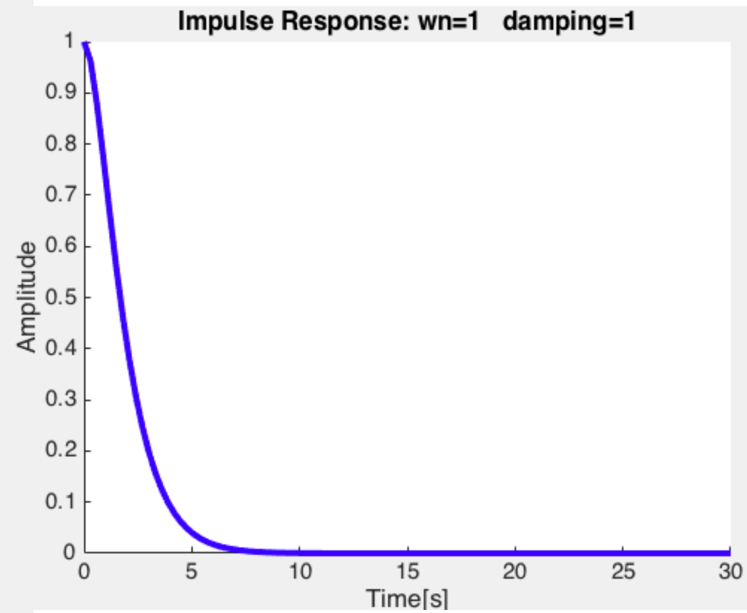




Spring mass damper system in Matlab

Highlighting poles and zero for damping = 1





Interlude

10 minute break

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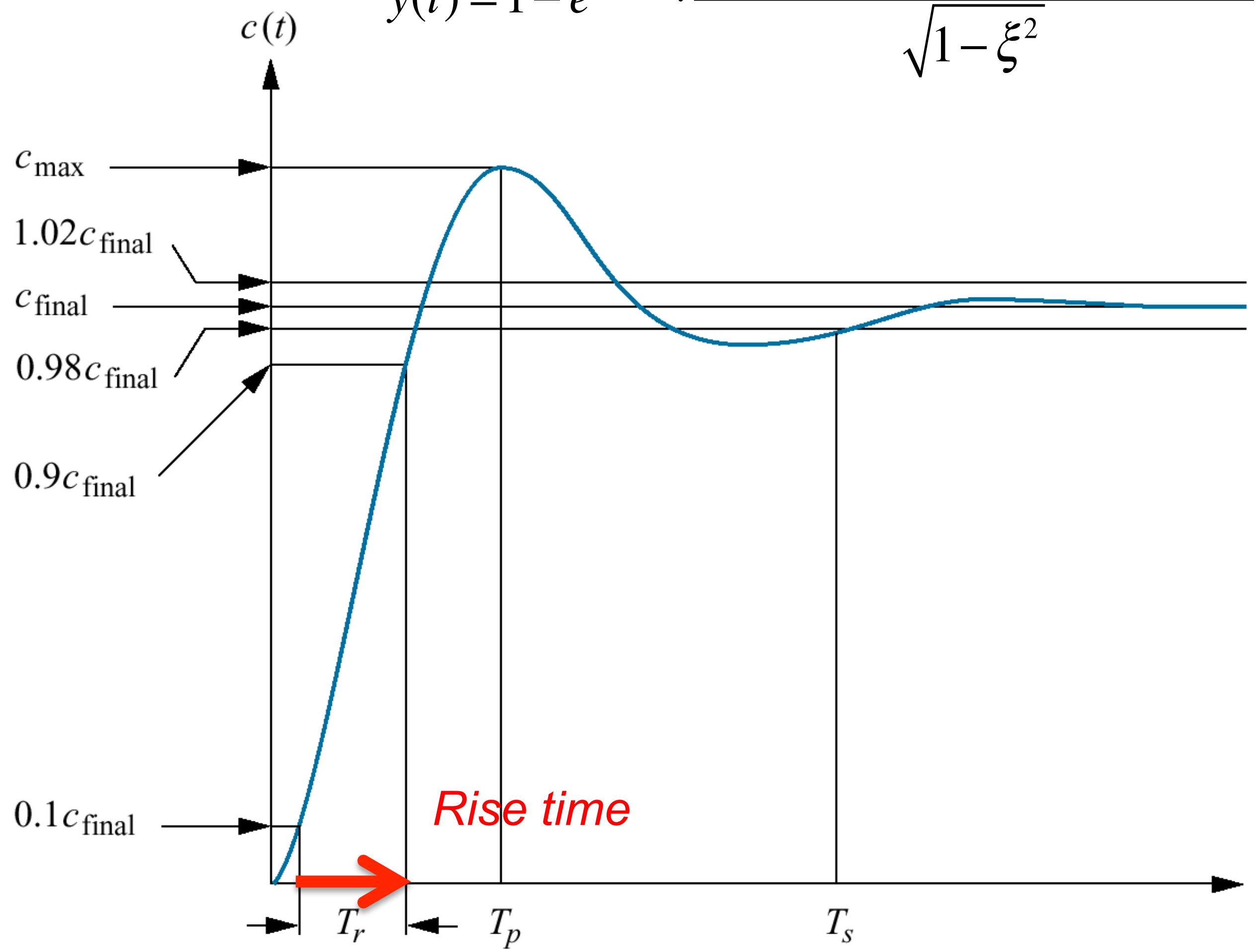
Lecture 9

2nd order performance measures

2nd order system: Rise time

Rise time T_r : It the time required for the response to rise from 10% to 90% of the final value.

$$y(t) = 1 - e^{-\xi w_n t} \cdot \frac{\sin\left[\left(w_n\sqrt{1-\xi^2}\right)t + a\cos(\xi)\right]}{\sqrt{1-\xi^2}}$$



2^{nd} order system: Peak time

Peak time T_p : It is the time required to reach the peak of the time response or peak overshoot.

T_p corresponds to half a cycle of damped oscillation w_d

Since

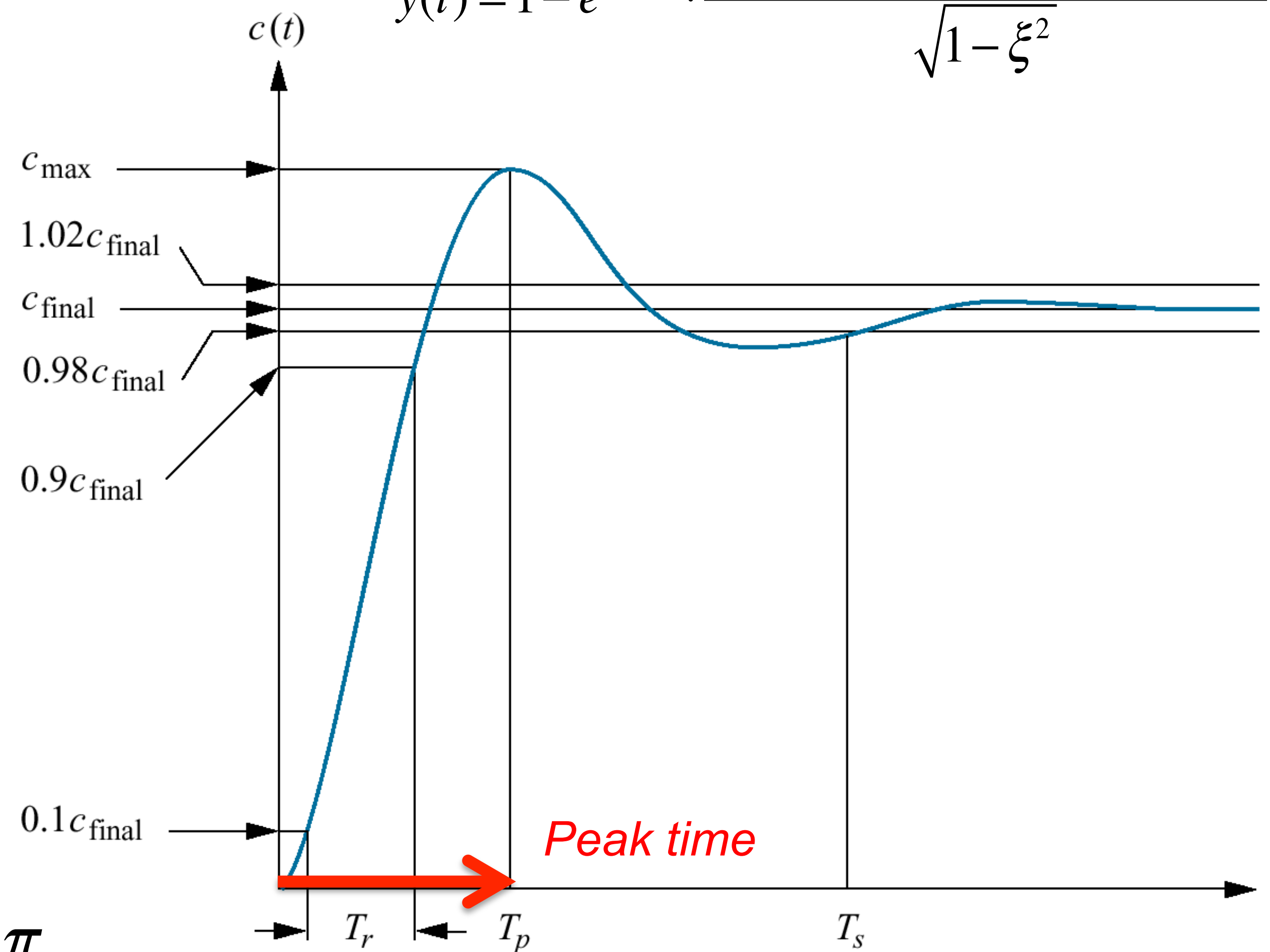
$$w_d = w_n\sqrt{1-\xi^2}$$

$$\Rightarrow f_d = \frac{w_n\sqrt{1-\xi^2}}{2\pi}$$

$$\Rightarrow t_d = \frac{2\pi}{w_n\sqrt{1-\xi^2}}$$

$$\Rightarrow T_p = \frac{1}{2}\frac{2\pi}{w_n\sqrt{1-\xi^2}} = \frac{\pi}{w_n\sqrt{1-\xi^2}}$$

$$y(t) = 1 - e^{-\xi w_n t}\cdot\frac{\sin\left[\left(w_n\sqrt{1-\xi^2}\right)t + a\cos(\xi)\right]}{\sqrt{1-\xi^2}}$$



2nd order system: Peak overshoot

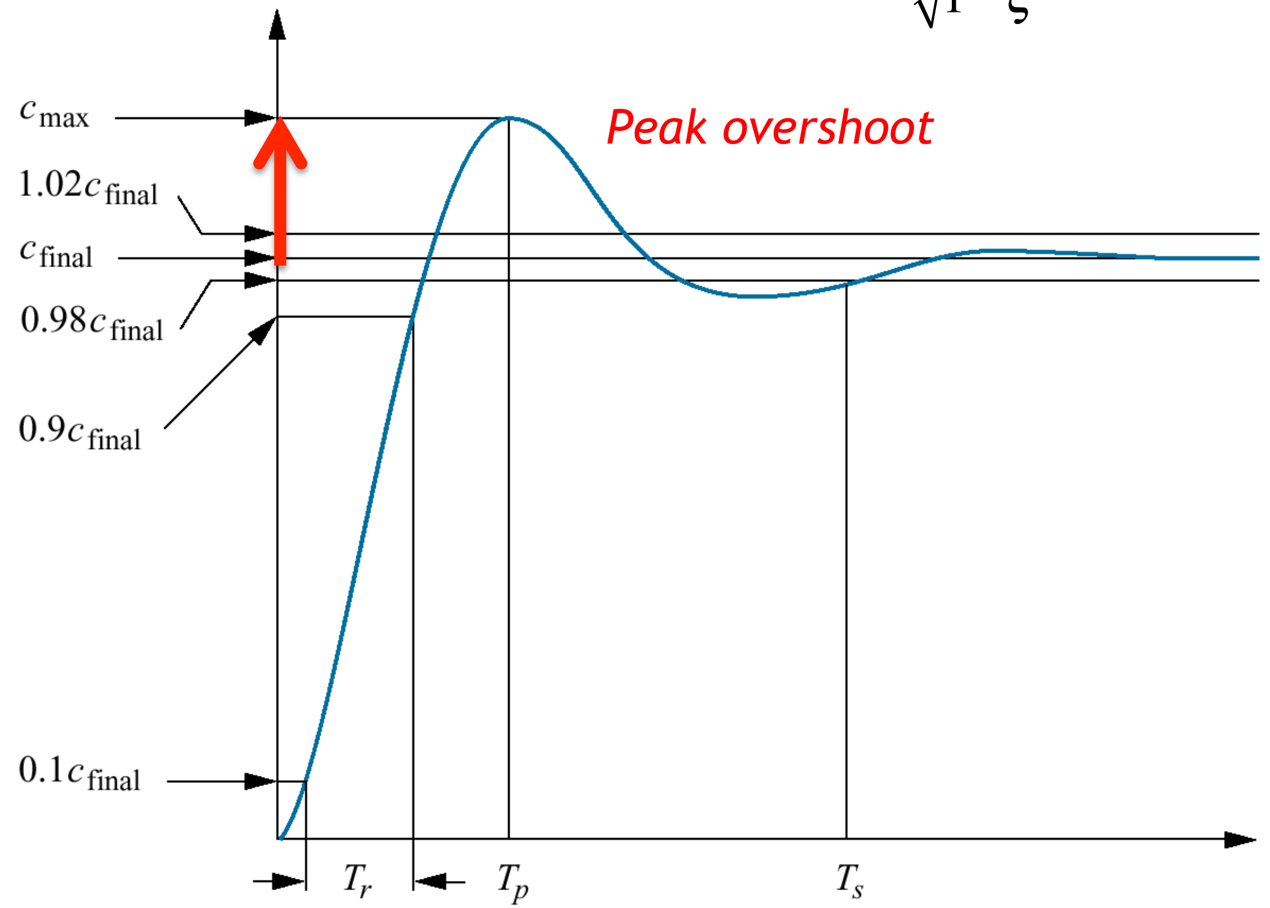
Peak overshoot M_p : It indicates the difference between the peak and the steady output.
It occurs at the peak time
t = T_p

The maximum overshoot is determined by taking the time derivative of the time response and equating to zero
Can find time to maximum overshoot like this too.

Result is

$$M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\xi^2}}}$$

$$y(t) = 1 - e^{-\xi w_n t} \cdot \frac{\sin\left[\left(w_n\sqrt{1-\xi^2}\right)t + a\cos(\xi)\right]}{\sqrt{1-\xi^2}}$$



2^{nd} order system: Settling time

Settling time T_s : Time required for the response to reach and stay within a specified tolerance band (usually 2%) of its final value

$$\Rightarrow e^{-\xi w_n T_s} = 0.02$$

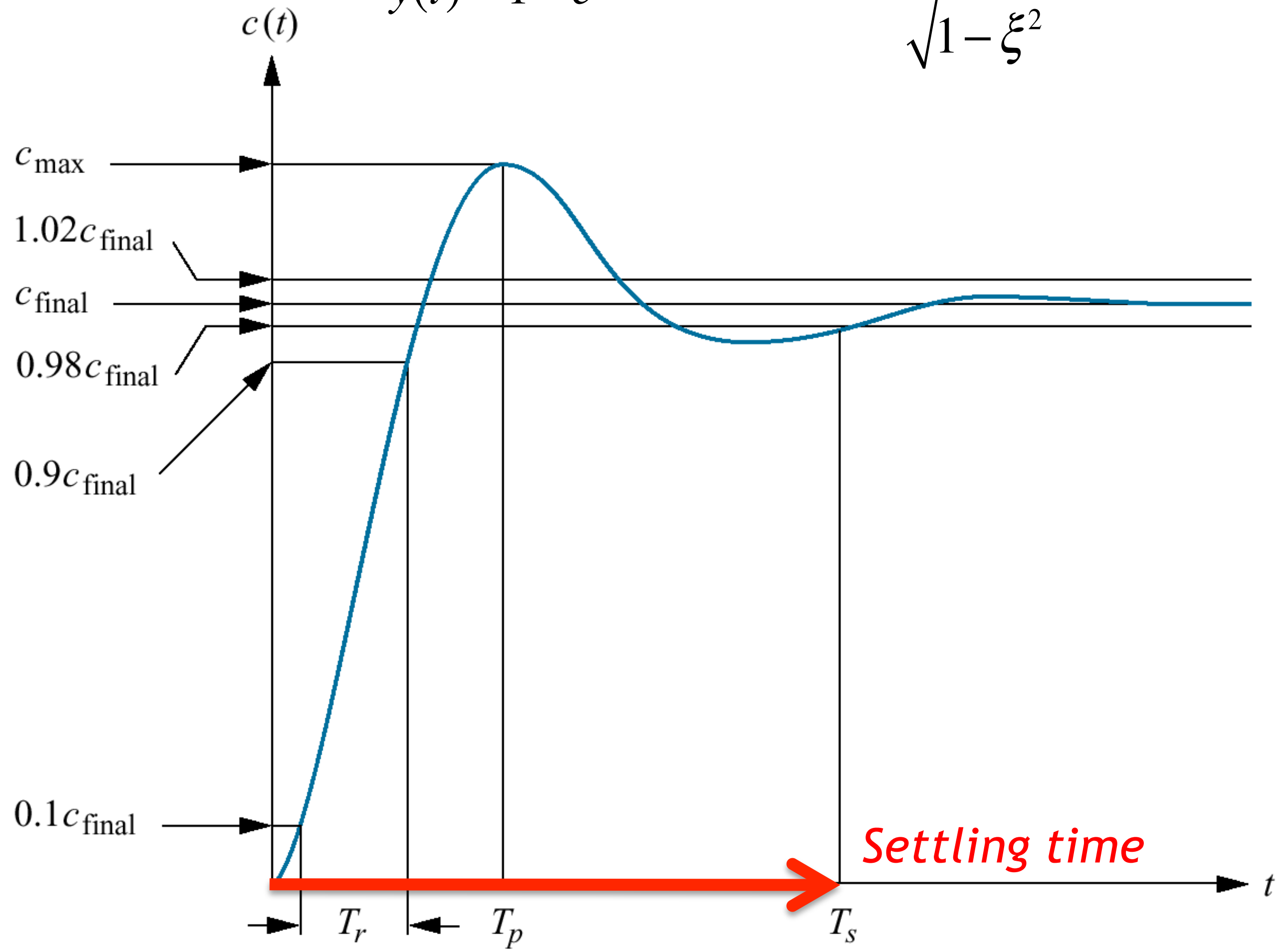
$$\Rightarrow -\xi w_n T_s = \ln(0.02)$$

$$\Rightarrow \xi w_n T_s = 3.912$$

$$\Rightarrow T_s = \frac{3.912}{\xi w_n}$$

So $\Rightarrow T_s \approx \dfrac{4}{\xi w_n}$

$$y(t) = 1 - e^{-\xi w_n t} \cdot \frac{\sin\left[\left(w_n\sqrt{1-\xi^2}\right)t + a\cos(\xi)\right]}{\sqrt{1-\xi^2}}$$



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Lecture 9

ROCO218 2016R Exam examples
Derive transfer function for electrical system

Q3: Canonical 2nd order form

Q4. The open-loop transfer function of a unity feedback closed-loop film transport system is given below

$$G(s) = \frac{K}{s(0.1s+1)}$$

(a) Determine the damping rate ζ when K=10;

(5 marks)

(b) Determine the natural frequency ω_n when K=10;

(5 marks)

(c) What affects the percentage overshoot $\sigma\%$ response?

(5 marks)

(d) Explain the influence of the K on the response of the system.

(5 marks)

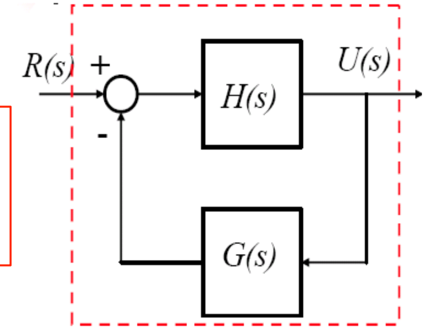
Q3: Canonical 2nd order form

The open loop function is

$$G(s) = \frac{K}{s(0.1s+1)}$$

Remember

$$\frac{U(s)}{R(s)} = \frac{H(s)}{(1+H(s)G(s))}$$



Therefore the unity feedback closed loop transfer function is

$$CL(s) = \frac{G(s)}{1+G(s)} = \frac{\dfrac{K}{s(0.1s+1)}}{1+\dfrac{K}{s(0.1s+1)}} = \frac{K}{s(0.1s+1)+K}$$

$$\Rightarrow CL(s) = \frac{K}{0.1s^2+s+K} = \frac{10K}{s^2+10s+10K}$$

Substituting in K=10

$$\Rightarrow CL(s) = \frac{100}{s^2+10s+100}$$

Q3: Canonical 2nd order form

Comparing with canonical form

$$CL(s) = \frac{100}{s^2 + 10s + 100} \Leftrightarrow \frac{\omega_n^2}{s^2 + 2\xi\omega + \omega_n^2}$$

$$\omega_n^2 \Leftrightarrow 100 \quad \Rightarrow \omega_n = 10$$

$$2\xi\omega_n \Leftrightarrow 10 \quad \Rightarrow \xi = 0.5$$

The percentage overshoot is given by

$$M_p = 100 \times e^{\frac{-\zeta\pi}{\sqrt{1-\xi^2}}} = 100 \times e^{\frac{-0.5\pi}{\sqrt{1-0.5^2}}} = 16.3\%$$

Matlab
100*exp(-0.5*pi/(sqrt(1-0.5^2)))

Peak time T_p of the step response

$$T_p = \frac{\pi}{w_n\sqrt{1-\xi^2}} = \frac{\pi}{10\sqrt{1-0.5^2}} = 0.3628\text{s}$$

Matlab
pi/(10 * sqrt(1-0.5^2))

As K increases overshoot increases but peak time is not affected

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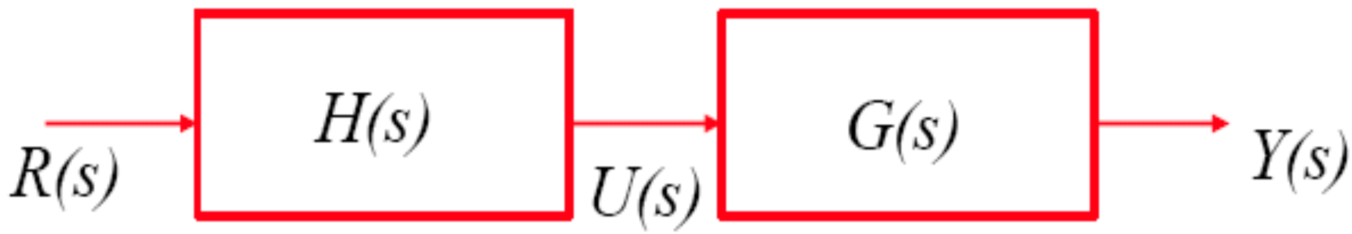
Lecture 9

Inversion

Control and inversion

In open loop control, the controller makes use of a model of the process without knowing how things are actually going

Consider the open loop system



Perfect servo control is achieved if Y(s)=R(s)

$$\Rightarrow R(s)H(s)G(s) = Y(s) \qquad = R(s)$$

$$\Rightarrow H(s)G(s) = 1$$

$$\Rightarrow H(s) = \frac{1}{G(s)}$$

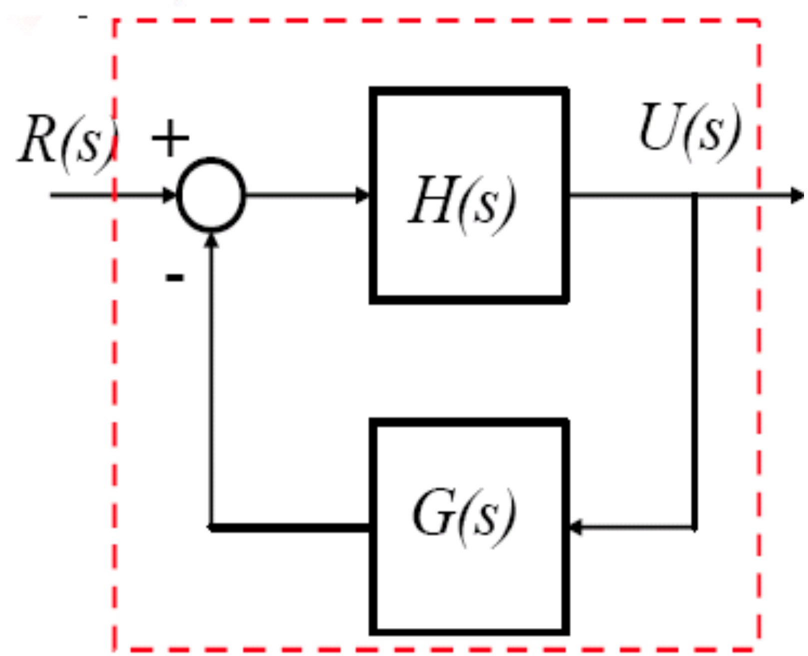
Thus controller need to have inverse transfer function of the plant

Control and inversion

- The concept of plant inversion is very general in control
- Perfect control requires open-loop inversion of the plant
- Various ways to approximate it
- Only possible if the model is perfect and there are no disturbances since open loop systems are also poor at rejecting disturbances
- True inversion cannot be applied in practice
- Relies on very accurate model
- Also requires the plant and its inverse to be stable

Remember: Advantages of feedback

In closed-loop, the feedback depends on what actually happens, since it is the output of the plant



This will bring two benefits:

- Reduced sensitivity to modeling errors
- Reduced sensitivity to disturbances

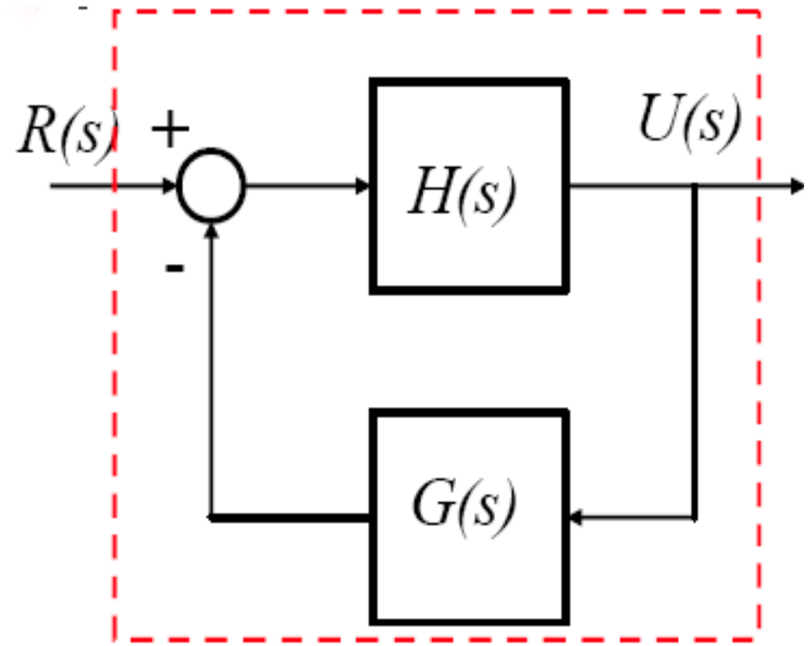
Control and inversion

- Consider feedback system

$$\frac{U(s)}{R(s)} = \frac{H(s)}{\left(1+H(s)G(s)\right)}$$

Note that for $H(s) >> 1$

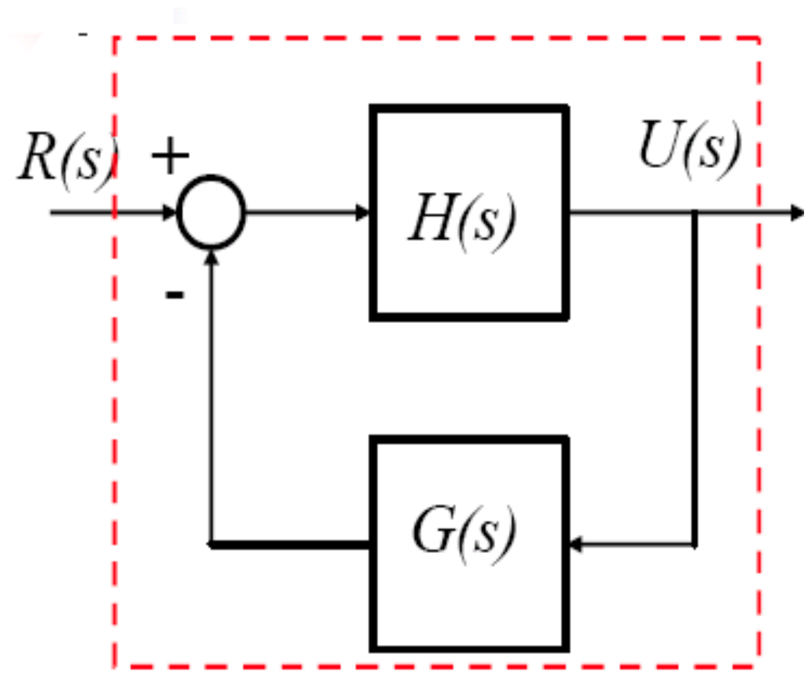
$$\frac{U(s)}{R(s)} \approx \frac{H(s)}{H(s)G(s)} \approx \frac{1}{G(s)}$$



- High gain feedback implicitly generates the inverse of G(s)
- It does so without us having to actually carry the inversion!

Trade-off

- Although it seems that all is needed is high gain feedback, there is a cost attached to the use of high-gain feedback



When choosing the feedback gain, one mus the trade off between those various issues

- This is the essence of control design
- It will result in very large control actions
- It increases the risk of instability
- It increase the sensitivity to measurement noise

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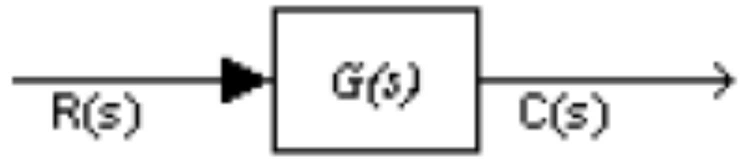
Lecture 9

Steady state error

Steady state error: open loop

The steady state error (SSE) is the steady state value of the difference between the desired and the actual output of the control system

For the open loop system



$$E(s) = R(s) - C(s) \qquad = R(s)[1 - G(s)]$$

Using the final value theorem of Laplace transforms

$$SSE = \lim_{s \to 0} sE(s) \qquad = \lim_{s \to 0} sR(s)\left(1 - G(s)\right)$$

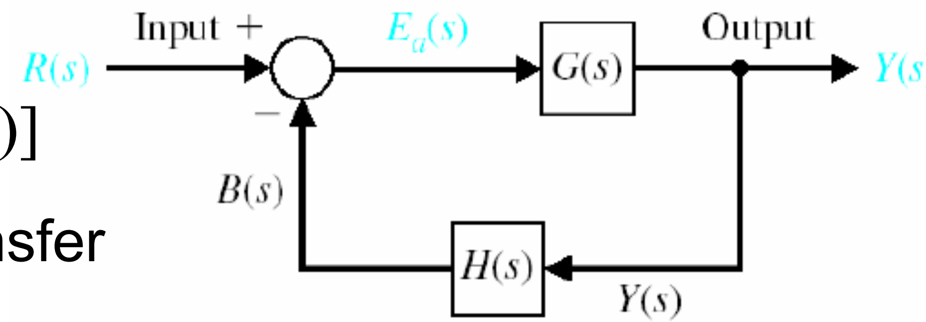
If the input R(s) is a step then

$$SSE = \lim_{s \to 0} s\left(1 - G(s)\right)\frac{1}{s} \qquad = 1 - G(0)$$

Where G(0) is the DC gain of the system

Steady state error: closed loop

For the closed loop system



$$E(s) = R(s) - Y(s) \quad = R(s)[1 - T(s)]$$

Where T(s) is the closed loop transfer function

$$T(s) = \frac{G(s)}{\left(1 + H(s)G(s)\right)}$$

$$\Rightarrow E(s) = R(s)\left[1 - \frac{G(s)}{1 + H(s)G(s)}\right] \quad = R(s)\left[\frac{1 + H(s)G(s) - G(s)}{1 + H(s)G(s)}\right]$$

If H(s) = 1

$$E(s) = R(s)\left[\frac{1 + G(s) - G(s)}{1 + G(s)}\right] = R(s)\frac{1}{1 + G(s)}$$

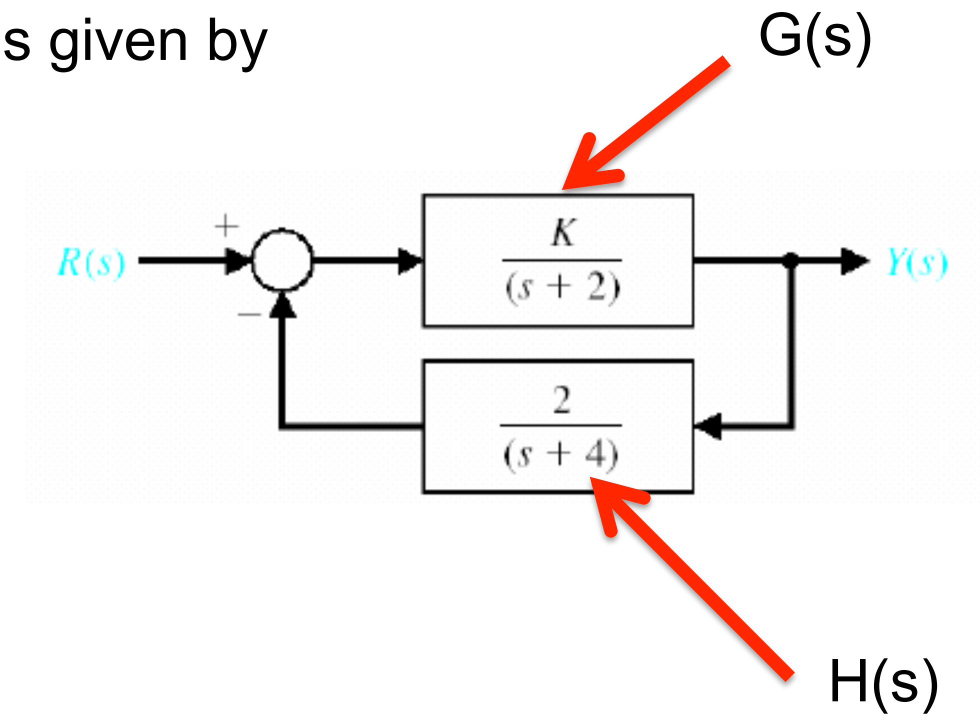
$$SSE = \lim_{s \to 0} sE(s) \quad = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)}$$

If the input R(s) is a step then

$$= \frac{1}{1 + G(0)}$$

Steady state error example 1

Closed loop transfer function T(s) is given by



$$T(s) = \frac{G(s)}{1 + H(s)G(s)}$$

$$\Rightarrow T(s) = \frac{\dfrac{K}{(s+2)}}{1 + \dfrac{K}{(s+2)}\dfrac{2}{(s+4)}}$$

$$\Rightarrow T(s) = \frac{K(s+4)}{(s+4)(s+2) + 2K}$$

Steady state error example 1

So given $T(s) = \dfrac{K(s+4)}{(s+4)(s+2)+2K}$ and $E(s) = R(s)[1-T(s)]$

$$SSE = \lim_{s\to 0} sE(s) \qquad = \lim_{s\to 0} sR(s)\left[1-\frac{K(s+4)}{(s+4)(s+2)+2K}\right]$$

If the input is a step then R(s)= 1/s so

$$SSE = \lim_{s\to 0}\left[1-\frac{K(s+4)}{(s+4)(s+2)+2K}\right]$$

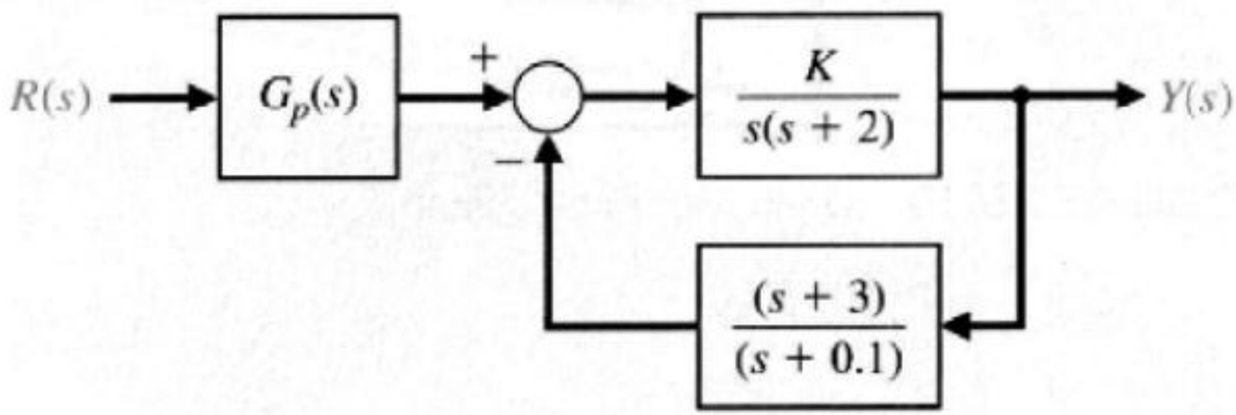
$$\Rightarrow SSE = \left[1-\frac{K(0+4)}{(0+4)(0+2)+2K}\right] \quad = 1-\frac{4K}{8+2K}$$

If SSE = 0 then

$$\Rightarrow 0 = 1-\frac{4K}{8+2K} \quad \Rightarrow 1 = \frac{4K}{8+2K} \quad \Rightarrow 8+2K = 4K \quad \Rightarrow 8 = 2K$$

Therefore SSE=0 iff K=4

Steady state error example 2

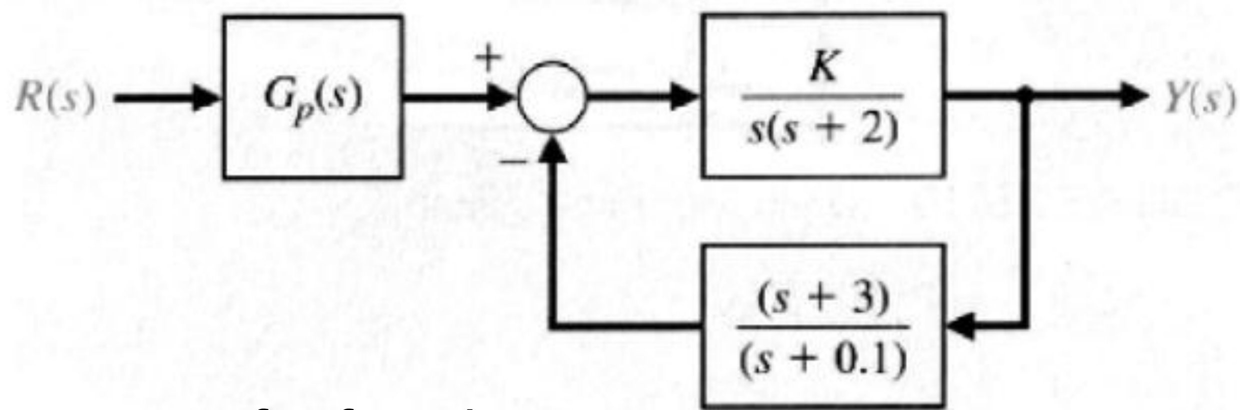


A feedback system with an input amplifier $G_p(s)$ is shown above

a) Determine the steady state error for a unit step input when $K = 0.4$ and $G_p(s) = 1$.

b) Select an appropriate constant value for $G_p(s)$ so that the steady state error is zero for a unit step input and $K = 0.4$.

Steady state error example 2



Writing down transfer function

T(s) = Y(s)/R(s)

$$T(s) = G_p(s)\left(\frac{\dfrac{K}{s(s+2)}}{1+\dfrac{K}{s(s+2)}\dfrac{(s+3)}{(s+0.1)}}\right) = G_p(s)\left(\frac{K(s+0.1)}{s(s+2)(s+0.1)+K(s+3)}\right)$$

$$= G_p(s)\left(\frac{0.4(s+0.1)}{s(s+2)(s+0.1)+0.4(s+3)}\right)$$

Steady state error example 2

Determine the steady state error for a unit step input when K = 0.4 and $G_p(s) = 1$

$$T(s) = \left(\frac{0.4(s+0.1)}{s(s+2)(s+0.1)+0.4(s+3)} \right)$$

We know error E(s) is given by

$$E(s) = R(s)[1-T(s)]$$

So steady state error is

$$ESS = \lim_{s\to 0} sR(s)\left(1-T(s)\right)$$

So steady state error for a unit step is

$$ESS = \lim_{s\to 0}\left(1-T(s)\right) \qquad = 1-T(0) \quad = 1-\left(\frac{0.4(0+0.1)}{0(2)(0+0.1)+0.4(0+3)} \right)$$

$$= 1-\frac{0.04}{1.2} \qquad = 0.9667$$

Steady state error example 2

Select an appropriate constant value for $G_p(s)$ so that the steady state error is zero for a unit step input and K = 0.4.

So steady state error for a unit step is now

$$ESS = \lim_{s \to 0}\left(1 - G_p T(s)\right) = 1 - G_p T(0) = 1 - G_p \frac{0.04}{1.2}$$

We require

$$1 - G_p \frac{0.04}{1.2} = 0$$

Therefore

$$\Rightarrow G_p \frac{0.04}{1.2} = 1 \quad \Rightarrow G_p = \frac{1.2}{0.04}$$

So error condition fulfilled if $G_p = 30$