ROCO218: Control Engineering
Dr Ian Howard

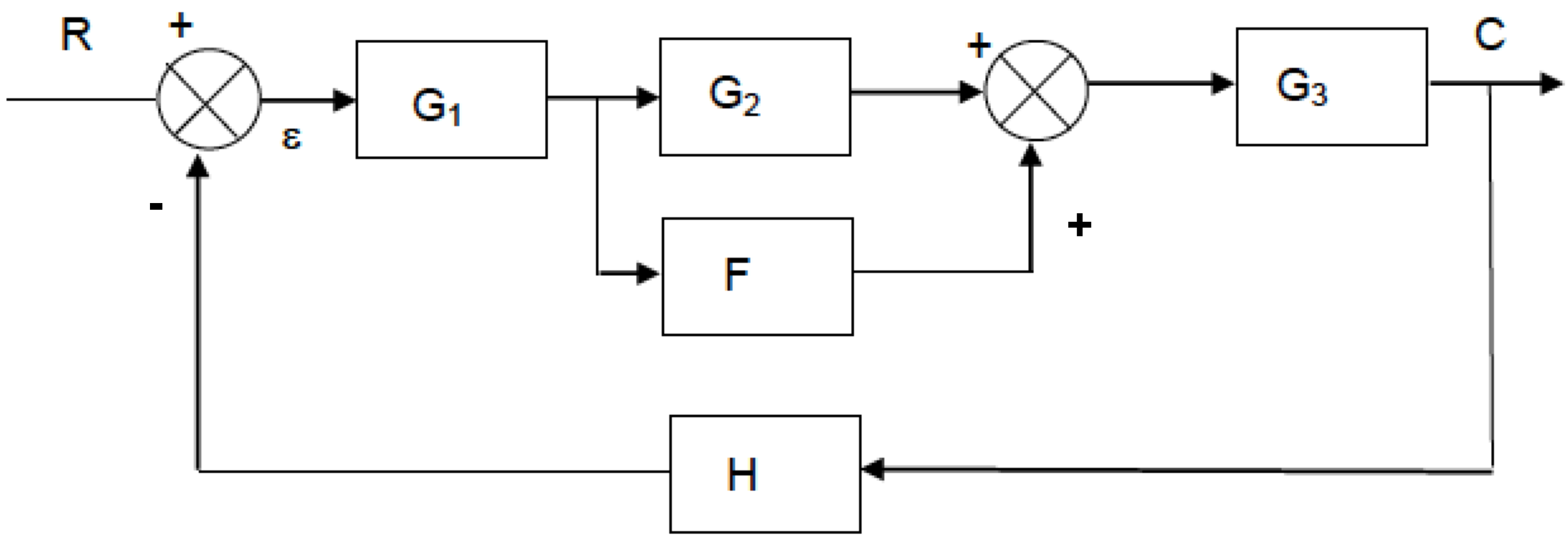
Tutorial 1

ROCO204 2012 Exam example
Transfer function of block diagram

Q1: Simplify diagram

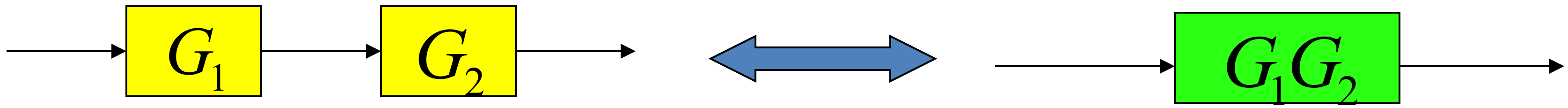
(b) Deduce the system transfer function relating C to R for the block diagram shown in **Figure Q2(b)**.

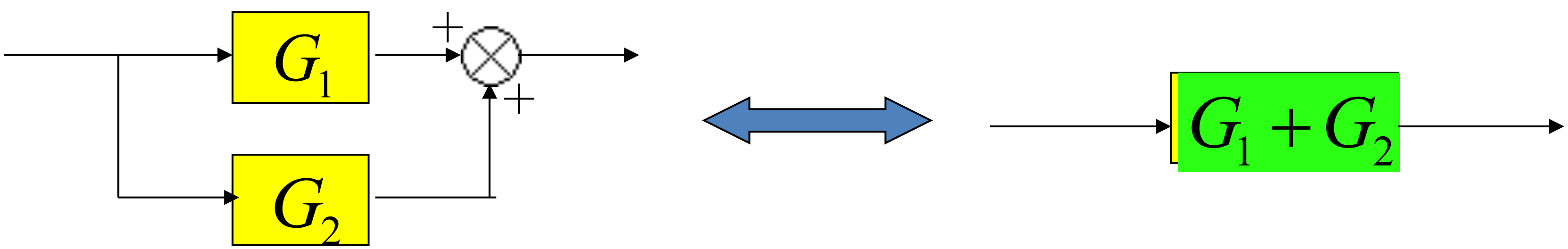
(10 marks)



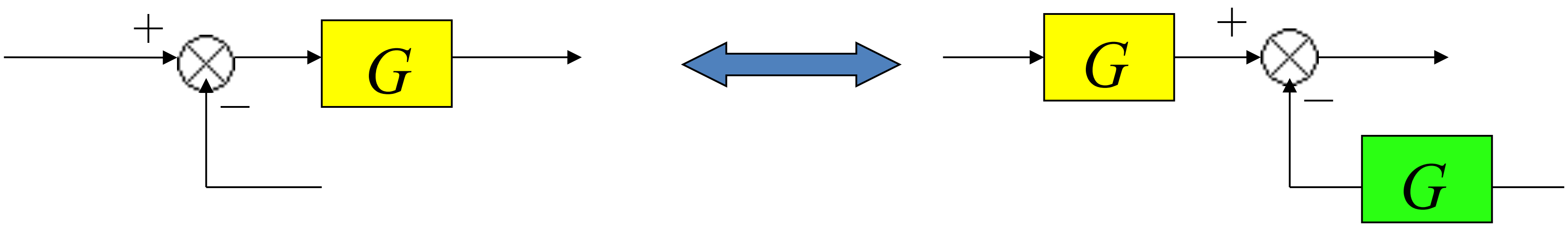
Block diagram reduction techniques

1. Combining blocks in cascade or in parallel



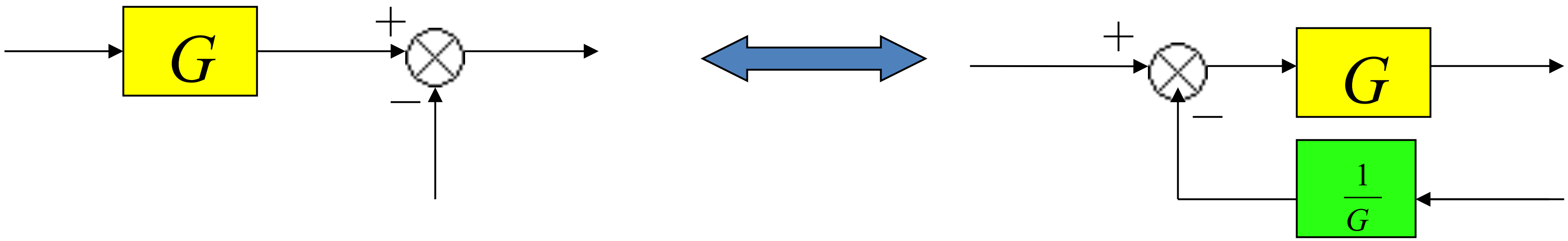


2. Moving a summing point from behind a block

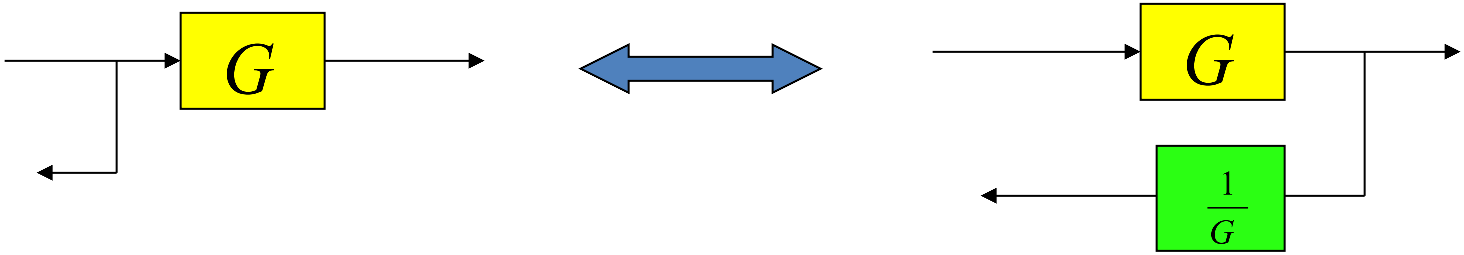


Block diagram reduction techniques

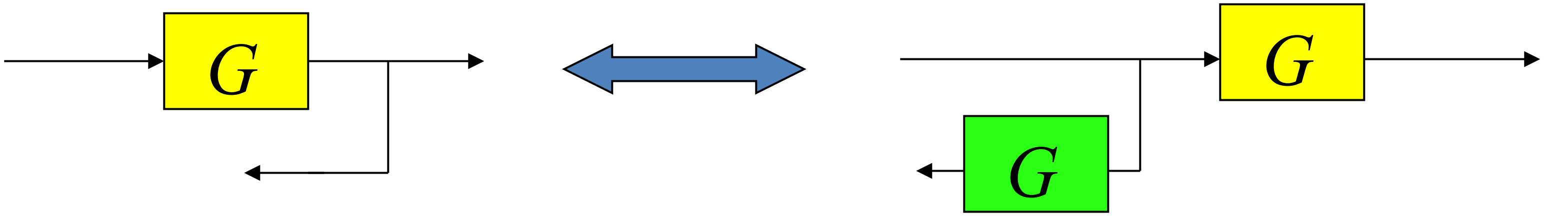
3. Moving a summing point ahead of a block



4. Moving a pickoff point from behind a block

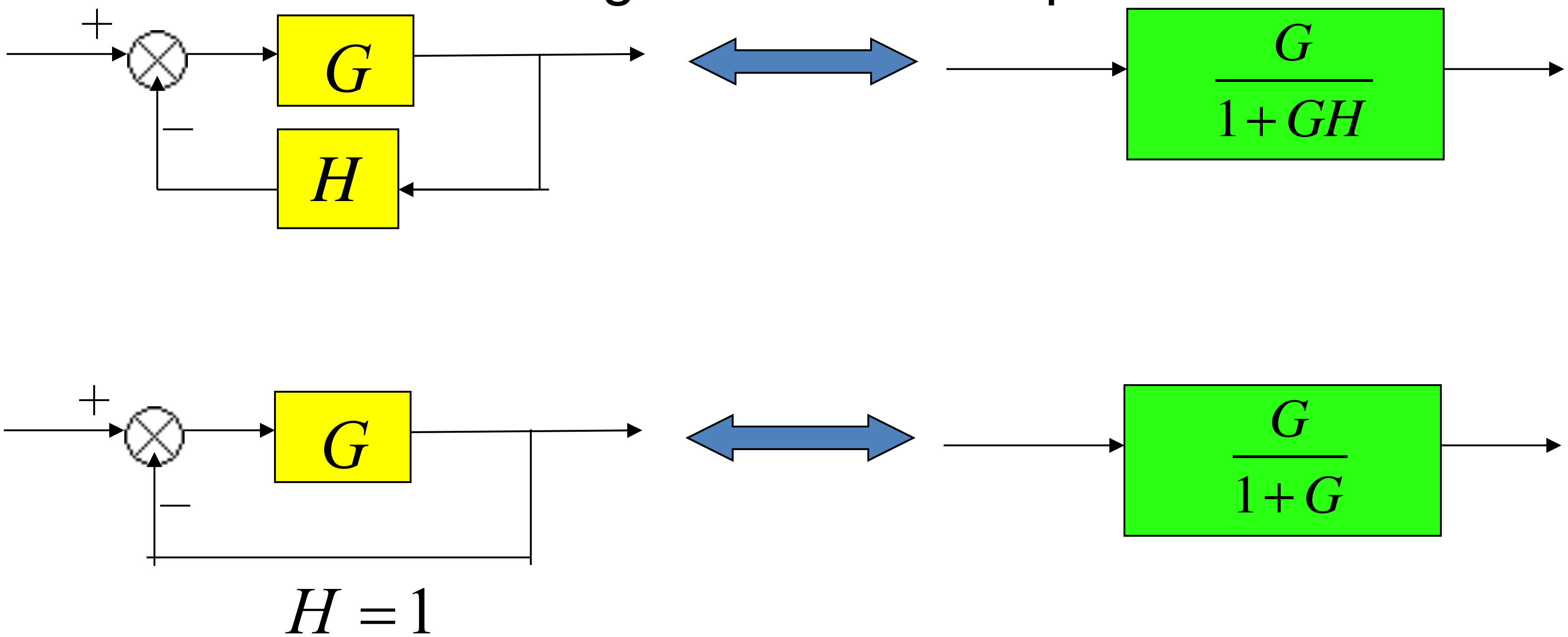


5. Moving a pickoff point from ahead of a block

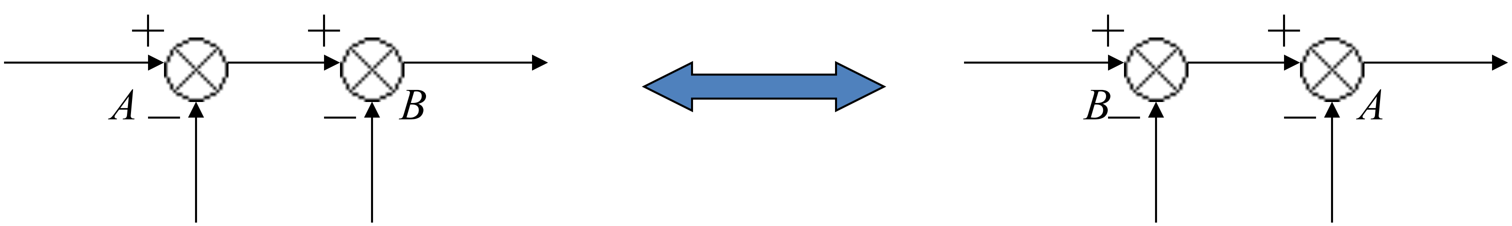


Block diagram reduction techniques

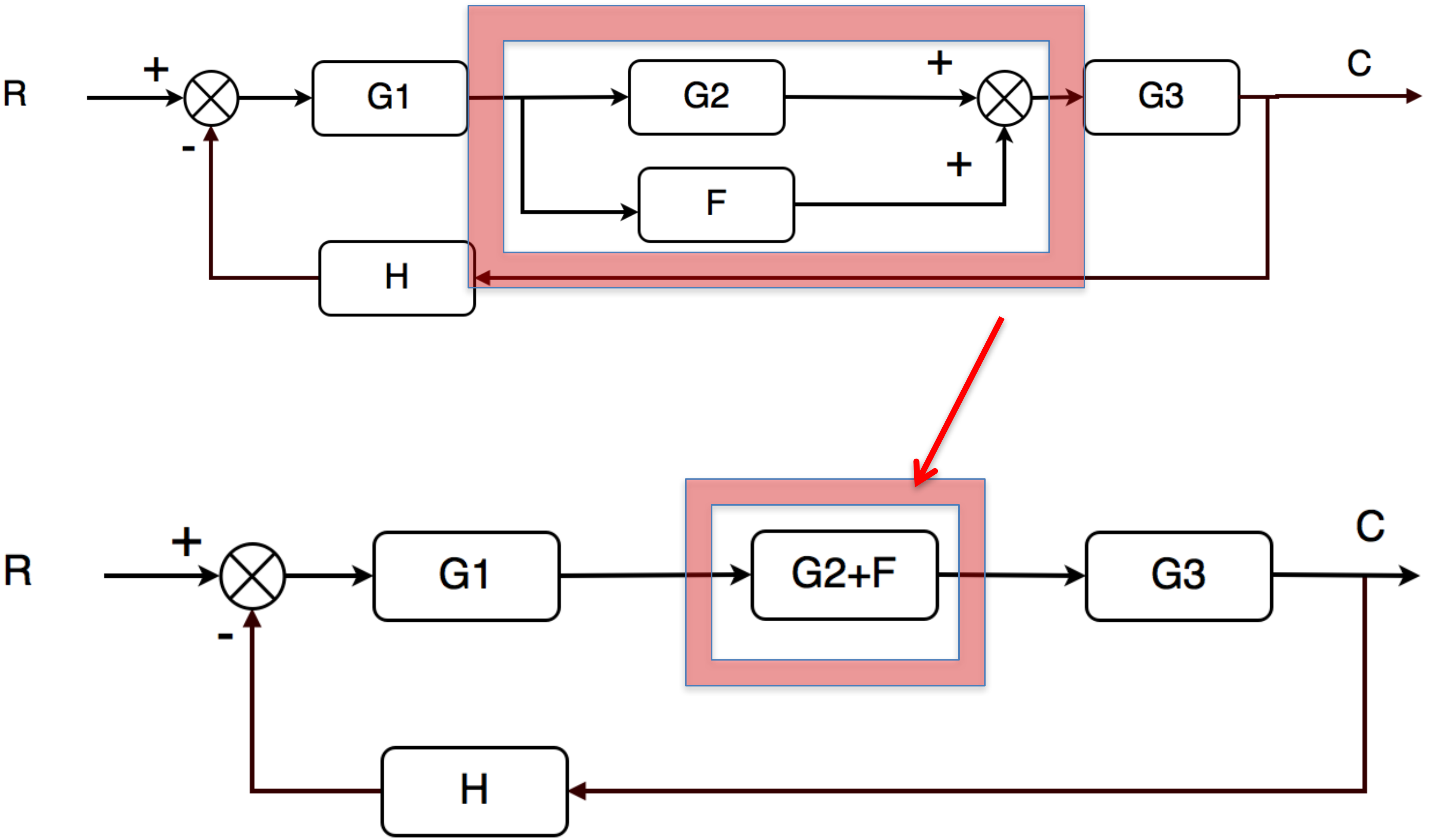
6. Eliminating a feedback loop



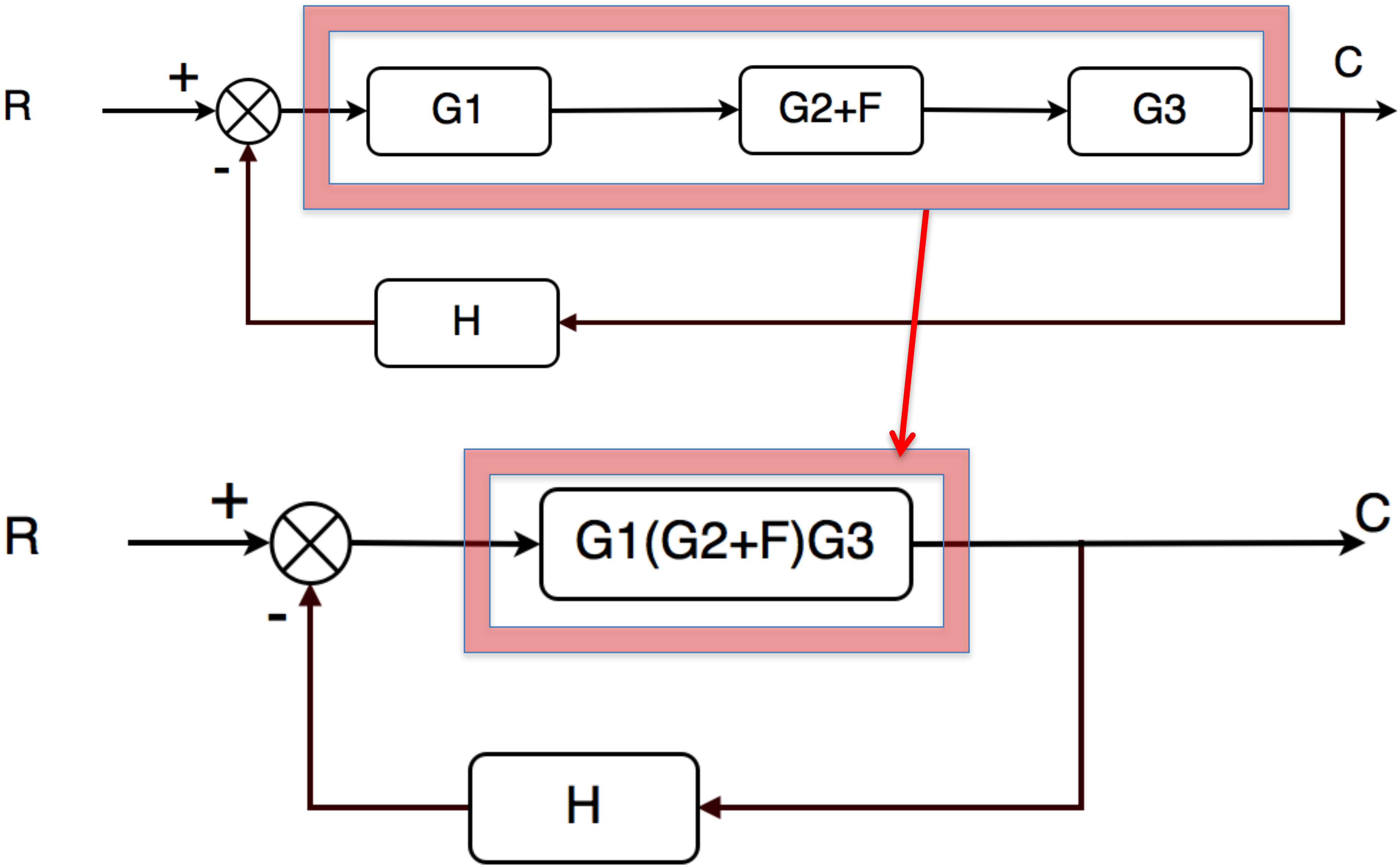
7. Swap order of two neighboring summing points



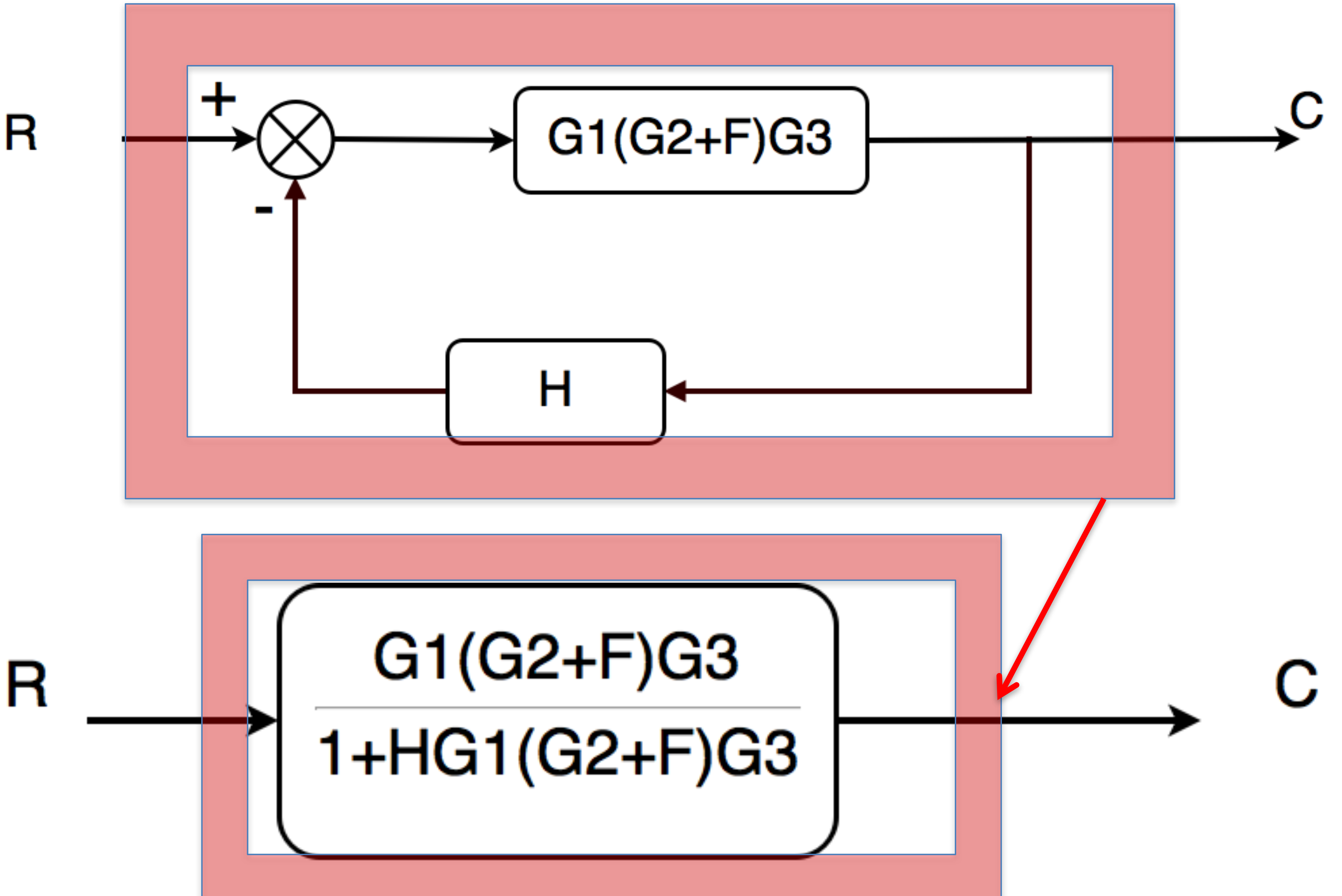
Q1: Simplify diagram



Q1: Simplify diagram



Q1: Simplify diagram



ROCO218: Control Engineering
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Tutorial 1

2016R exam for ROCO319 Modern Control
Solutions to relevant questions

Q2: State space analysis

Consider the following linear control system:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2}$$

(a) Justify that the uncontrolled system (i.e. $u = 0$) is unstable.

(5 marks)

(b) Compute the controllability matrix $\mathcal{C}$ of system (2). Is this system controllable?

(5 marks)

(c) Compute the observability matrix $\mathcal{O}$ of system (2). Is this system observable?

(5 marks)

Q2: State space analysis

- State space equations take the form

$$\dot{X} = AX + BU$$
$$Y = CX + DU$$

The given system state space equations are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} U \qquad y = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- Therefore the system matrices are

$$\Rightarrow A = \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} \qquad \Rightarrow B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \Rightarrow C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

- NB: Since system matrix A is diagonal, by inspection we can see that the eigenvalues of the system are given the diagonal elements of the system matrix A
- Therefore λ = 1,7
- However here we will proceed and adopt an analysis here that will deal with the general case

Q2: State space analysis

Given system matrix A

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix}$$

- The eigenvalues of matrix A must satisfy the equation

$$AX = \lambda X$$

Where X is the state vector

$$\Rightarrow AX - \lambda X = 0$$

Where I is the identity matrix

$$\Rightarrow (A - \lambda I)X = 0$$

- This equation has a non-zero solution X if and only if the determinant of the matrix (A-I)=0

$$\Rightarrow |A - \lambda I| = 0$$

Where straight bracket signifies the determinant

Q2: State space analysis

- Substituting A and I into the expression $\left|A-\lambda I\right|=0$

$$\Rightarrow\left\|\begin{bmatrix}1 & 0\\ 0 & 7\end{bmatrix}-\lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right\|=0 \qquad \Rightarrow\left\|\begin{bmatrix}1-\lambda & 0\\ 0 & 7-\lambda\end{bmatrix}\right\|=0$$

- For a 2x2 matrix the determinant is given by

$$\det(A)=\left\|\begin{pmatrix}a & b\\ c & d\end{pmatrix}\right\|=\left(ad-bc\right)$$

$$\Rightarrow(1-\lambda)(7-\lambda)-0\times 0=0$$

$$\Rightarrow(1-\lambda)(7-\lambda)=0$$

$$\Rightarrow\lambda=1,7$$

- Both of these eigenvalues are strictly positive
- Therefore the uncontrolled system is unstable

Q2: State space analysis

- Since our system has a 2x2 system matrix, the system controllability is given by

$$M_c = \begin{bmatrix} B & AB \end{bmatrix}$$

Calculating the AB term gives

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

$$\Rightarrow M_c = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 7 \end{bmatrix}$$

- Using Gaussian elimination to try to achieve echelon form

$$M_c \big| R_2 \to R_2 - R_1 = \begin{bmatrix} 1 & 1 \\ 1-1 & 7-1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 6 \end{bmatrix}$$

- It can be seen directly that the (reduced echelon form) matrix has rank of 2, which is full rank
- Therefore the system is controllable

Q2: State space analysis

- Since our system has a 2x2 system matrix, the system observability matrix is given by

$$M_o = \begin{bmatrix} C \\ CA \end{bmatrix}$$

Calculating the CA term gives

$$CA = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 7 \end{bmatrix}$$

$$\Rightarrow M_o = \begin{bmatrix} 1 & 1 \\ 1 & 7 \end{bmatrix}$$

- Using Gaussian elimination to try to achieve echelon form

$$M_c \Big| R_2 \rightarrow R_2 - R_1 = \begin{bmatrix} 1 & 1 \\ 1-1 & 7-1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 6 \end{bmatrix}$$

- It can be seen that the reduced echelon form matrix has rank of 2
- Therefore the system is observable

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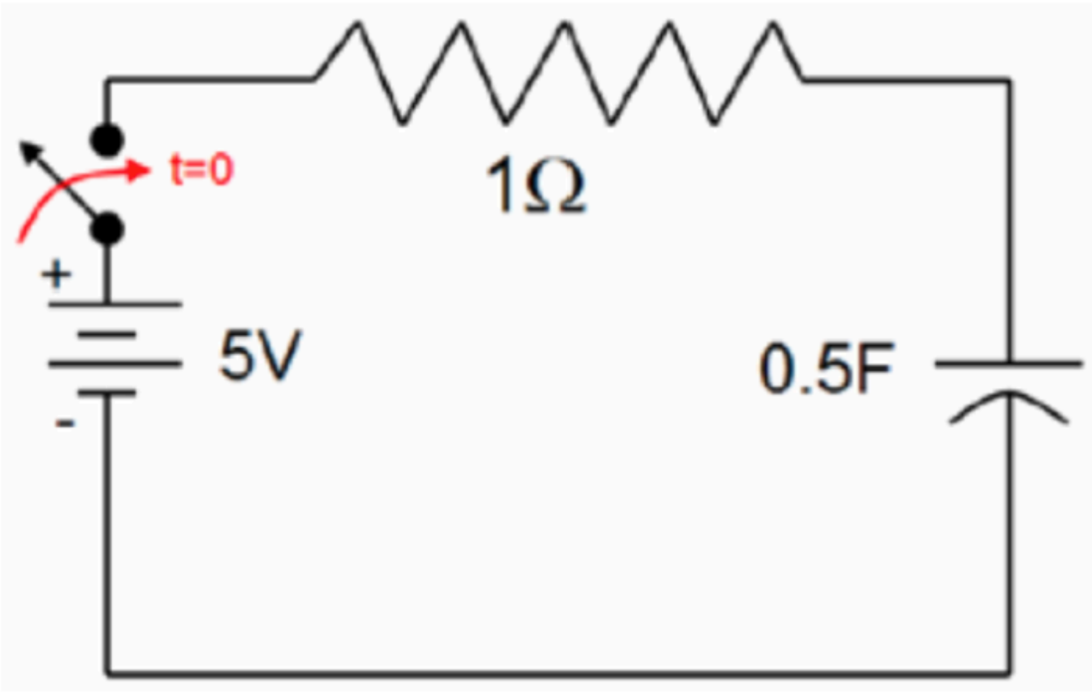
Tutorial 1

ROCO218 2016R Exam examples
Derive transfer function for electrical system

Q1: Electrical system

Q1. Calculate the transfer function from the battery voltage to the current in **Figure Q1** below.

(20 marks)



Solution: Summing voltages around circuit we have

$$v(t) = Ri(t) + \frac{1}{C}\int i(t)\,dt$$

Q1: Electrical system

Taking Laplace transforms of the voltage equation

$$v(t) = Ri(t) + \frac{1}{C}\int i(t)\,dt \qquad \Rightarrow V(s) = RI(s) + \frac{1}{sC}I(s)$$

Taking out the factor I(s)

$$\Rightarrow V(s) = I(s)\left(R + \frac{1}{sC}\right) = I(s)\frac{(sCR+1)}{sC}$$

Rearranging we get the following transfer function

$$\frac{I(s)}{V(s)} = \frac{sC}{(sCR+1)}$$

Substituting in C=0.5F, R = 1Ω

$$\Rightarrow \frac{I(s)}{V(s)} = \frac{0.5s}{(0.5s+1)}$$

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Tutorial 1

2016 exam for ROCO319 Modern Control
Solutions to relevant questions

Q3: State space control

Consider the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{3}$$

(a) Design a state feedback $u(t) = -k_1 x_1(t) - k_2 x_2(t)$ so that the closed-loop eigenvalues are placed at $\{-3,\ -2\}$. **You should use the direct method** for computing the gains k_1 and k_2.

(10 marks)

Q3: State space control

State space equations take the form

$$\dot{X} = AX + BU$$

$$Y = CX + DU$$

The given system state space equations are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Therefore the system matrices are

$$\Rightarrow A = \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix} \qquad \Rightarrow B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \Rightarrow C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Q3: State space control

We now apply feedback to the state space equation

$$\dot{X} = AX + BU$$

By setting input U to

$$U = -KX$$

where $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$

This leads to the the modified relationship

$$\Rightarrow \dot{X} = AX - BKX \qquad \Rightarrow \dot{X} = \left(A - BK\right)X$$

So the stability is now determined by location of poles which are the eigenvalue of matrix (A-BK)

The eigenvalues λ of the closed loop system are thus given by

$$\det\left(A - BK - \lambda I\right) = 0$$

Q3: State space control

Using the system matrices

$$A = \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\Rightarrow BK = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix}$$

Substituting in values

$$\det\left(A - BK - \lambda I\right) = 0$$

$$\Rightarrow 0 = \left| \begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \left| \begin{bmatrix} -\lambda_1 & 3 \\ 2 - k_1 & 1 - k_2 - \lambda \end{bmatrix} \right|$$

The characteristic equation is therefore

$$\Rightarrow \left(-\lambda\right)\left(1 - k_2 - \lambda\right) - 3\left(2 - k_1\right) = 0$$

Q2: State space analysis

Simplifying the characteristic equation

$$\Rightarrow (-\lambda)(1-k_2-\lambda)-3(2-k_1)=0$$

$$\Rightarrow \lambda^2+\lambda(k_2-1)+3k_1-6=0$$

We require that the eigenvalues λ of the controller system are at -3,-2
Therefore we want the following characteristic equation

$$\Rightarrow (\lambda+3)(\lambda+2)=0 \qquad \Rightarrow \lambda^2+5\lambda+6=0$$

We now need to match the coefficients in the desired eigenvalues characteristic equation using the appropriate gain vector K

$$\lambda^2+5\lambda+6=0 \qquad \Leftrightarrow \lambda^2+\lambda(k_2-1)+3k_1-6=0$$

$$\Rightarrow 5=k_2-1 \quad \Rightarrow k_2=6$$

$$\Rightarrow 3k_1-6=6 \quad \Rightarrow k_1=4$$

Feedback law is therefore $\qquad u(t)=-4x_1-6x_2$

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Tutorial 1

2015 exam for ROCO316 Modern Control
Solutions to relevant questions

Q1: Continuous-time nonlinear control system

Consider the following continuous-time nonlinear control system

$$
\begin{aligned}
\dot{x}_1(t) &= x_1^3(t) - x_2(t) + 1 && (1)\\
\dot{x}_2(t) &= \mathrm{e}^{x_2(t)} + u(t) && (2)\\
y(t) &= -x_1(t) + u(t) && (3)
\end{aligned}
$$

(a) We assume a constant input $u(t) = \bar{u} = -1$, and we admit that the equilibrium state $(\bar{x}_1, \bar{x}_2)$ for this constant input is $(-1, 0)$. Determine the linearized system about this equilibrium state.

(5 marks)

(b) Write the linearized system in state-space form.

(5 marks)

(c) Determine the stability of the linearized system.

(5 marks)

(d) Determine the eigenvectors, and the solution of the linear system.

(5 marks)

Q1: Continuous-time nonlinear control system

- Solution: Substituting in value of u(t) = -1 gives the equations

$$\dot{x}_1(t) = x_1^3(t) - x_2(t) + 1$$

$$\dot{x}_2(t) = e^{x_2(t)} - 1$$

$$y(t) = -x_1(t) - 1$$

To linearize we need to calculate the Jacobian matrix and evaluate it at the given equilibrium position (-1,0).

In general the Jacobian is given by

$$J = \left(\frac{\partial f_i}{\partial x_j} \right) \Bigg| x = x_{equilibrium} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & & \ddots & \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

Q1: Continuous-time nonlinear control system

Here we only have a 2x2 matrix
The corresponding functions we need f_1 and f_2 are given by the two state expressions

$$f_1 = \dot{x}_1(t) = x_1^3(t) - x_2(t) + 1$$

$$f_2 = \dot{x}_2(t) = e^{x_2(t)} - 1$$

Taking partial derivatives w.r.t. to the state variables

$$\Rightarrow \frac{\partial f_1}{\partial x_1} = 3x_1^2(t) \qquad \Rightarrow \frac{\partial f_1}{\partial x_2} = -1$$

$$\Rightarrow \frac{\partial f_2}{\partial x_1} = 0 \qquad \Rightarrow \frac{\partial f_2}{\partial x_2} = e^{x_2(t)}$$

The equilibrium point is given as (-1,0) so we evaluate the Jacobian matrix at this point

$$\Rightarrow J_{|(-1,0)} = \begin{bmatrix} 3x_1^2(t) & -1 \\ 0 & e^{x_2(t)} \end{bmatrix}_{|(-1,0)} = \begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix}$$

Q1: Continuous-time nonlinear control system

(b) Write the linearized system in state-space form.

(5 marks)

From the Jacobian

$$J = \begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix}$$

we can directly write

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \dot{x}_1 = 3x_1 - x_2$$
$$\Rightarrow \dot{x}_2 = x_2$$

Q1: Continuous-time nonlinear control system

(c) Determine the stability of the linearized system.

- To determine the stability of the system we need to calculate the eigenvalues of Jacobian matrix J which is now our system matrix A

$$A = \begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix}$$

- The eigenvalues of matrix A must satisfy the equation

$$AX = \lambda X$$

Where X is the state vector $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\Rightarrow AX - \lambda X = 0$$

$$\Rightarrow (A - \lambda I)X = 0$$

Where I is the identity matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

- This equation has a non-zero solution X if and only if the determinant of the matrix (A-I)=0

$$\Rightarrow |A - \lambda I| = 0$$

Where straight bracket signifies the determinant

Q1: Continuous-time nonlinear control system

- Substituting A and I into the expression $|A-\lambda I| = 0$

$$\Rightarrow \left| \begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0 \qquad \Rightarrow \left| \begin{bmatrix} 3-\lambda & -1 \\ 0 & 1-\lambda \end{bmatrix} \right| = 0$$

- For a 2x2 matrix the determinant is given by

$$\det(A) = \left| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right| = (ad - bc)$$

$$\Rightarrow (3-\lambda)(1-\lambda) - (-1)\times 0 = 0$$

$$\Rightarrow (3-\lambda)(1-\lambda) = 0$$

$$\Rightarrow \lambda = 3, \lambda = 1$$

Note also that since the A matrix is of triangular form, its leading diagonal values indicate the eigenvalues of 3 and 1 directly. However we calculated the values for thoroughness of methodology since this will not always be the case

- Both eigenvalues are strictly positive and therefore the linearized system is unstable

Q1: Continuous-time non-linear control system

(d) Determine the eigenvectors

The eigenvalues of a matrix A satisfy the equation

$AX = \lambda X$

$$\Rightarrow \begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- The values of X gives the eigenvector
- When λ=1

$$\begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \begin{matrix} \Rightarrow 3x_1 - x_2 = x_1 \\ \Rightarrow x_2 = x_2 \end{matrix} \quad \Rightarrow 2x_1 = x_2 \quad \Rightarrow v_{\lambda=1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

- When λ=3

$$\begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \begin{matrix} \Rightarrow 3x_1 - x_2 = 3x_1 \\ \Rightarrow x_2 = 3x_2 \end{matrix} \quad \begin{matrix} \Rightarrow x_2 = 0 \\ \Rightarrow x_1 = x_1 \end{matrix} \quad \Rightarrow v_{\lambda=3} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

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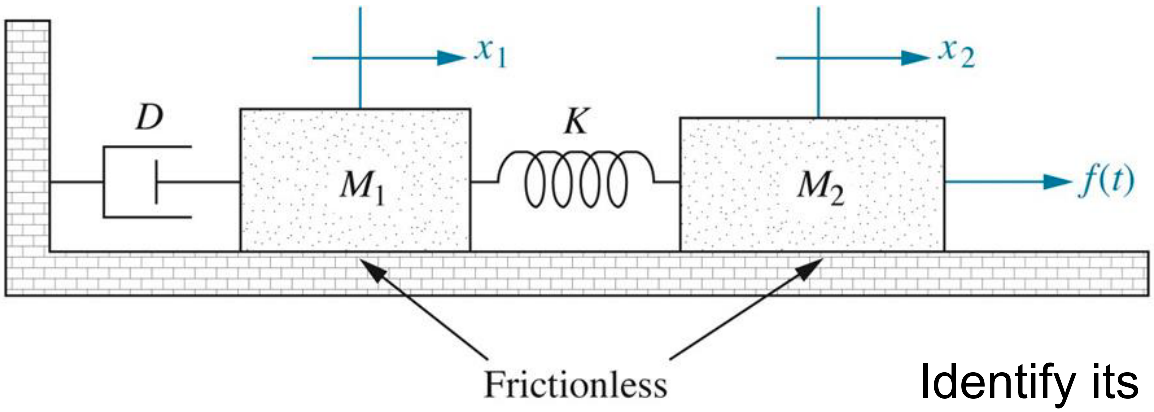
Tutorial 1

State space model of spring coupled masses

Modelling spring coupled masses

Derive the state space model for the following translational mechanical system whose input is $f(t)$ and whose output is x_2:



Identify its states: highest state term must be less than highest differential equation term

Equations of motion are

$$M_1 \frac{d^2 x_1}{dt^2} + D \frac{dx_1}{dt} = K(x_2 - x_1)$$

$$M_2 \frac{d^2 x_2}{dt^2} + K(x_2 - x_1) = f(t)$$

$$x_1$$

$$v_1 = \dot{x}_1$$

$$x_2$$

$$v_2 = \dot{x}_2$$

Modelling spring coupled masses

From 1st equation of motion

$$M_1 \frac{d^2 x_1}{dt^2} + D\frac{dx_1}{dt} = K(x_2 - x_1)$$

$$\Rightarrow \frac{d^2 x_1}{dt^2} = -\frac{D}{M_1}\frac{dx_1}{dt} + \frac{K}{M_1}(x_2 - x_1)$$

$$\Rightarrow \dot{v}_1 = -\frac{D}{M_1}v_1 + \frac{K}{M_1}(x_2 - x_1)$$

From definition of state variables

$$\dot{x}_1 = v_1$$

$$\dot{x}_2 = v_2$$

Reordering

$$\Rightarrow \dot{v}_1 = -\frac{K}{M_1}x_1 - \frac{D}{M_1}v_1 + \frac{K}{M_1}x_2$$

Modelling spring coupled masses

From 2nd equation of motion

$$M_2 \frac{d^2 x_2}{dt^2} + K(x_2 - x_1) = f(t)$$

$$\Rightarrow \frac{d^2 x_2}{dt^2} = \frac{-K}{M_2}(x_2 - x_1) + \frac{f(t)}{M_2}$$

$$\Rightarrow \dot{v}_2 = \frac{-K}{M_2}(x_2 - x_1) + \frac{f(t)}{M_2}$$

Reordering

$$\Rightarrow \dot{v}_2 = \frac{K}{M_2} x_1 - \frac{K}{M_2} x_2 + \frac{f(t)}{M_2}$$

From definition of state variables

$$\dot{x}_1 = v_1$$

$$\dot{x}_2 = v_2$$

Modelling spring coupled masses

Thus we have

$$\dot{x}_1 = v_1$$

$$\dot{v}_1 = -\frac{K}{M_1}x_1 - \frac{D}{M_1}v_1 + \frac{K}{M_1}x_2$$

$$\dot{x}_2 = v_2$$

$$\dot{v}_2 = \frac{K}{M_2}x_1 - \frac{K}{M_2}x_2 + \frac{f(t)}{M_2}$$

Want matrix format is

$$\boxed{\dot{X} = AX + BU}$$

$$\boxed{Y = CX + DU}$$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{v}_1 \\ \dot{x}_2 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{K}{M_1} & -\frac{D}{M_1} & \frac{K}{M_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{K}{M_2} & 0 & -\frac{K}{M_2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{M_2} \end{bmatrix} f(t)$$

Modelling spring coupled masses

Since want output to be x_2

Want matrix format is

$$\boxed{\dot{X} = AX + BU}$$

$$\boxed{Y = CX + DU}$$

$$\Rightarrow y = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix}$$

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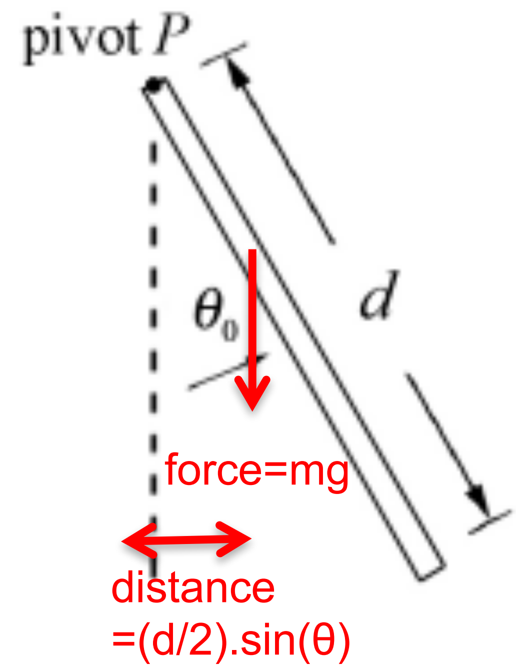
Tutorial 1

Analysis of rod pendulum

Fixed rod pendulum

Gravitation force acting on center of mass leads to torque

$$T_g = -mg\left(d/2\right)\sin(\theta)$$



Resistance torque to movement by moment of inertia of system

$$T_I = I_r \frac{d^2\theta}{dt^2} \Rightarrow I_r \frac{d^2\theta}{dt^2} = -mg\left(d/2\right)\sin(\theta)$$

For rod $\quad I_r = \dfrac{md^2}{3}$ For rod around one end

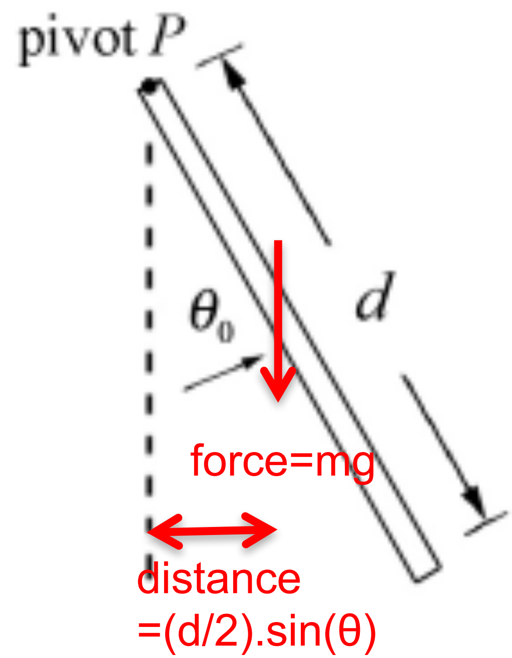
$$\Rightarrow \frac{md}{3}\frac{d^2\theta}{dt^2} = -mg\left(d/2\right)\sin(\theta)$$

$$\Rightarrow \frac{d^2\theta}{dt^2} = -\frac{mg\left(d/2\right)}{\left(md^2/3\right)}\sin(\theta)$$

For small angles $\quad \sin(\theta) \approx \theta$

$$\Rightarrow \frac{d^2\theta}{dt^2} = -\frac{3g\theta}{2d}$$

Fixed rod pendulum



Linearized equation of motion for rod pendulum

$$\frac{d^2\theta}{dt^2} = -\frac{3g\theta}{2d}$$

Taking Laplace transforms with zero initial condition

$$\Rightarrow s^2\Phi(s) = -\frac{3g}{2d}\Phi(s)$$

$$\Rightarrow s = \pm i\sqrt{\frac{3g}{2d}} \qquad \Rightarrow \omega = \pm\sqrt{\frac{3g}{2d}}$$

$$\Rightarrow f = \frac{1}{2\pi}\sqrt{\frac{3g}{2d}}$$