4 tutorials uploaded to DLE

Module Information

News forum

Suggested Reading

Feedback systems: an introduction for scientists and engineers.

Aström, K. J., & Murray, R. M. (2010).

Princeton university press.

http://www.cds.caltech.edu/~murray/amwiki/index.php/Main_Page

Control Tutorials for MATLAB and Simulink

"Welcome to the Control Tutorials for MATLAB and Simulink (CTMS): They are designed to help you learn how to use MATLAB and Simulink for the analysis and design of automatic control systems. They cover the basics of MATLAB and Simulink and introduce the most common classical and modern control design techniques."

http://ctms.engin.umich.edu/CTMS

Getting started with MATLAB.

J.M. Maciejowski,

http://www-h.eng.cam.ac.uk/help/documentation/docsource/matlab1.pdf

Matlab get started guide

http://www.mathworks.co.uk/help/pdf_doc/matlab/getstart.pdf

Learn with MATLAB and Simulink Tutorials

https://uk.mathworks.com/support/learn-with-matlab-tutorials.html

Tutorials

Lectures

Lectures

Tutorials

Tutorials sheets

Laboratory Practical Assignments

Here are the practicals for the course. You will not be marked on these practical assignments, they are provided to reinforce the theory from the lectures with Matlab implementations of some important algorithms

Lab Practicals

Coursework

Coursework

Tutorials show working through typical exam questions

I will go through tutorials in extra sessions

Session tomorrow
Room: BGB 005
Date: Tuesday, 20/03/2018
Time: 12:00-14:00

Today I will also go through tips and hints to help you do the coursework

Feedback time

Write down 3 things you like about the module

Write down 3 things you think could be improved

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Dr Ian Howard

Lecture 7

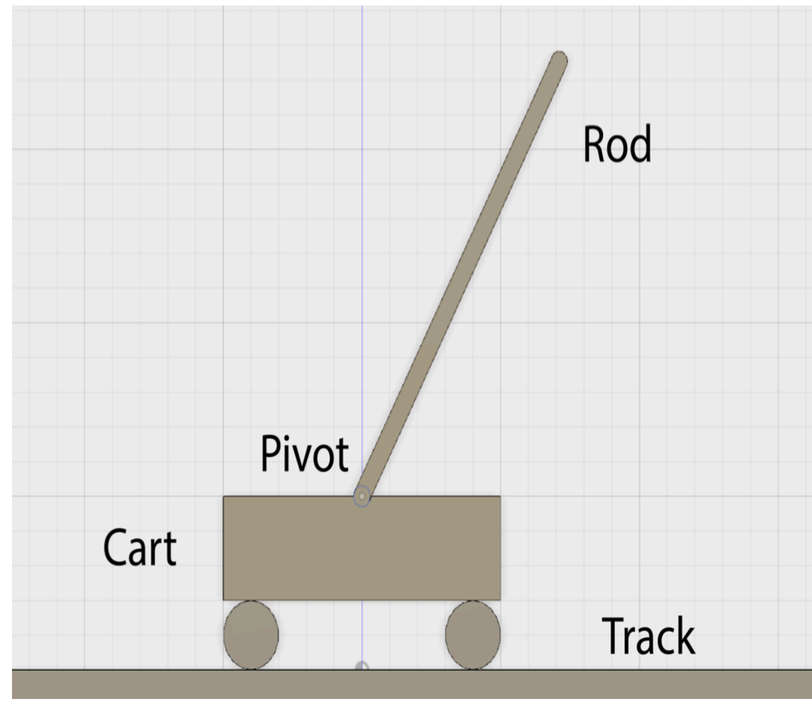
Inverted pendulum velocity control

Inverted pendulum state space velocity control

The linearized differential equation describing the inverted pendulum is given by

$$\left(I+ml^2\right)\frac{d^2\theta}{dt^2}+\mu\frac{d\theta}{dt}=mgl\theta+ml\frac{d^2x_p}{dt^2}$$

Where:
The angle to the vertical is denoted by θ
The coefficient of viciously is denoted by μ
The mass of the pendulum is denoted by m
The moment of inertial of the rod about the center of mass is denoted by I
The length to the centre of mass is denoted by *l*
The displacement of the pivot is given by x_p



Velocity controlled inverted pendulum

Re-writing the differential equation describing the inverted pendulum

$$\Rightarrow \frac{d^2\theta}{dt^2} = \frac{-\mu}{\left(I+ml^2\right)}\frac{d\theta}{dt} + \frac{mgl}{\left(I+ml^2\right)}\theta + \frac{ml}{\left(I+ml^2\right)}\frac{d^2x_p}{dt^2}$$

To control the pendulum by setting cart velocity we can write

$$\frac{d^2x_p}{dt^2} = \frac{dv_c}{dt}$$

This leads to the equation

$$\Rightarrow \frac{d^2\theta}{dt^2} = \frac{-\mu}{\left(I+ml^2\right)}\frac{d\theta}{dt} + \frac{mgl}{\left(I+ml^2\right)}\theta + \frac{ml}{\left(I+ml^2\right)}\frac{dv_c}{dt}$$

Velocity controlled inverted pendulum

Given the equation

$$\frac{d^2\theta}{dt^2} = \frac{-\mu}{\left(I + ml^2\right)}\frac{d\theta}{dt} + \frac{mgl}{\left(I + ml^2\right)}\theta + \frac{ml}{\left(I + ml^2\right)}\frac{dv_c}{dt}$$

Let the constant terms be represented by the coefficients

$$a_1 = \frac{\mu}{\left(I + ml^2\right)} \qquad b_0 = \frac{ml}{\left(I + ml^2\right)}$$

$$a_2 = \frac{-mgl}{\left(I + ml^2\right)}$$

$$\Rightarrow \frac{d^2\theta}{dt^2} = -a_1\frac{d\theta}{dt} - a_2\theta + b_0\frac{dv_c}{dt}$$

Choosing state space representations

$$x_1 = \theta$$

$$x_2 = \frac{d\theta}{dt} - b_0 v_c$$

Velocity controlled inverted pendulum

From the state space representations

$$x_1 = \theta \qquad \Rightarrow \dot{x}_1 = \frac{d\theta}{dt}$$

$$x_2 = \frac{d\theta}{dt} - b_0 v_c \qquad \Rightarrow \frac{d\theta}{dt} = x_2 + b_0 v_c \qquad \Rightarrow \dot{x}_1 = x_2 + b_0 v_c$$

$$\Rightarrow \dot{x}_2 = \frac{d^2\theta}{dt^2} - b_0 \frac{dv_c}{dt}$$

From before

$$\frac{d^2\theta}{dt^2} = -a_1 \frac{d\theta}{dt} - a_2\theta + b_0 \frac{dv_c}{dt} \qquad \Rightarrow \frac{d^2\theta}{dt^2} = -a_1\left(x_2 + b_0 v_c\right) - a_2 x_1 + b_0 \frac{dv_c}{dt}$$

Substituting into

$$\Rightarrow \dot{x}_2 = \frac{d^2\theta}{dt^2} - b_0 \frac{dv_c}{dt} \Rightarrow \dot{x}_2 = -a_1\left(x_2 + b_0 v_c\right) - a_2 x_1 + b_0 \frac{dv_c}{dt} - b_0 \frac{dv_c}{dt}$$

$$\Rightarrow \dot{x}_2 = -a_1 x_2 - a_2 x_1 - a_1 b_0 v_c$$

Velocity controlled inverted pendulum

From the state space representations

$$\dot{x}_1 = x_2 + b_0 v_c$$

$$\dot{x}_2 = -a_1 x_2 - a_2 x_1 - a_1 b_0 v_c$$

State space notation takes the form

$$\dot{X} = AX + BU$$

and

$$Y = CX + DU$$

Writing in matrix format we therefore have

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_0 \\ -a_1 b_0 \end{bmatrix} v_c$$

Velocity controlled inverted pendulum

From the state space representation

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_0 \\ -a_1 b_0 \end{bmatrix} v_c$$

$$\Rightarrow \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \dfrac{mgl}{\left(I+ml^2\right)} & -\dfrac{\mu}{\left(I+ml^2\right)} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \dfrac{ml}{\left(I+ml^2\right)} \\ \dfrac{-\mu ml}{\left(I+ml^2\right)^2} \end{bmatrix} v_c$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Where output y is the pendulum angle θ

Velocity controlled inverted pendulum

From

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \dfrac{mgl}{\left(I+ml^2\right)} & -\dfrac{\mu}{\left(I+ml^2\right)} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \dfrac{ml}{\left(I+ml^2\right)} \\ \dfrac{-\mu ml}{\left(I+ml^2\right)^2} \end{bmatrix} v_c$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

We can write the state space system matrices as

$$A = \begin{bmatrix} 0 & 1 \\ \dfrac{mgl}{\left(I+ml^2\right)} & -\dfrac{\mu}{\left(I+ml^2\right)} \end{bmatrix} \qquad B = \begin{bmatrix} \dfrac{ml}{\left(I+ml^2\right)} \\ \dfrac{-\mu ml}{\left(I+ml^2\right)^2} \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Augmenting positional state

In practice we may want to control cart position as well as angle and angular velocity!
Otherwise it might never stop moving!

We can add a third state x_3 to represent cart position
Since the control signal is cart velocity, the differential of x_3 is simply given by the input velocity control signal

Therefore we can write

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -a_2 & -a_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} b_0 \\ -a_1 b_0 \\ 1 \end{bmatrix} v_c$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Augmenting positional state

Substituting in value for the coefficients leads to the expression

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \dfrac{mgl}{\left(I+ml^2\right)} & -\dfrac{\mu}{\left(I+ml^2\right)} & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \dfrac{ml}{\left(I+ml^2\right)} \\ \dfrac{-\mu ml}{\left(I+ml^2\right)^2} \\ 1 \end{bmatrix} v_c$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

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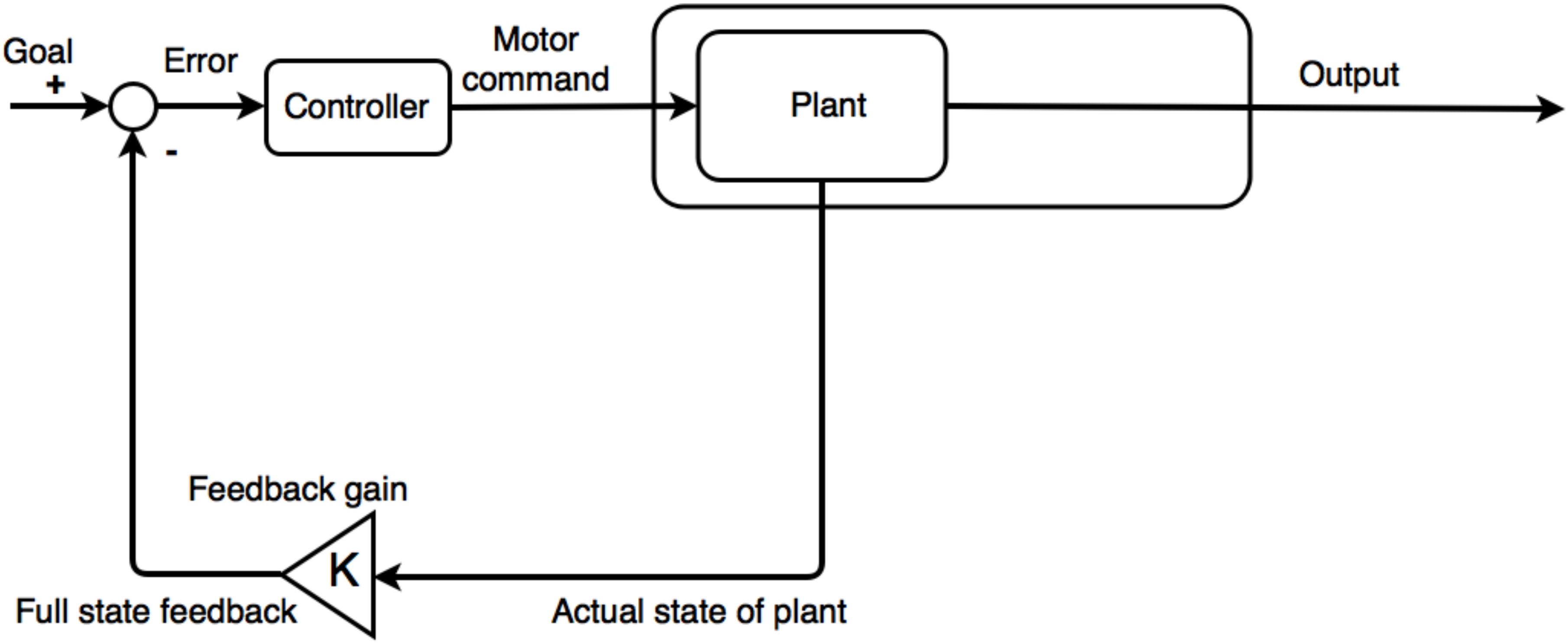
Lecture 7

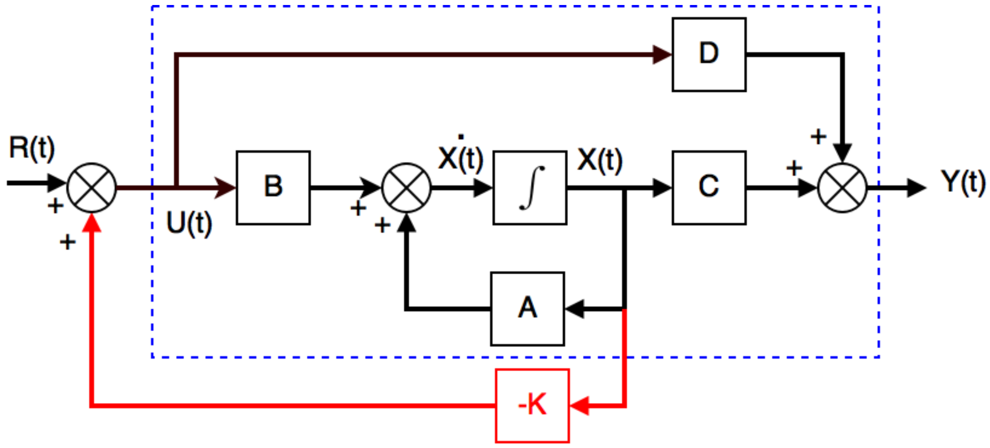
Luenberger observer

Direct state feedback control



- Full state feedback can be used to effectively control the plant
- This lets us specify behavior by using suitable feedback gain K

Using open loop observer to estimate state



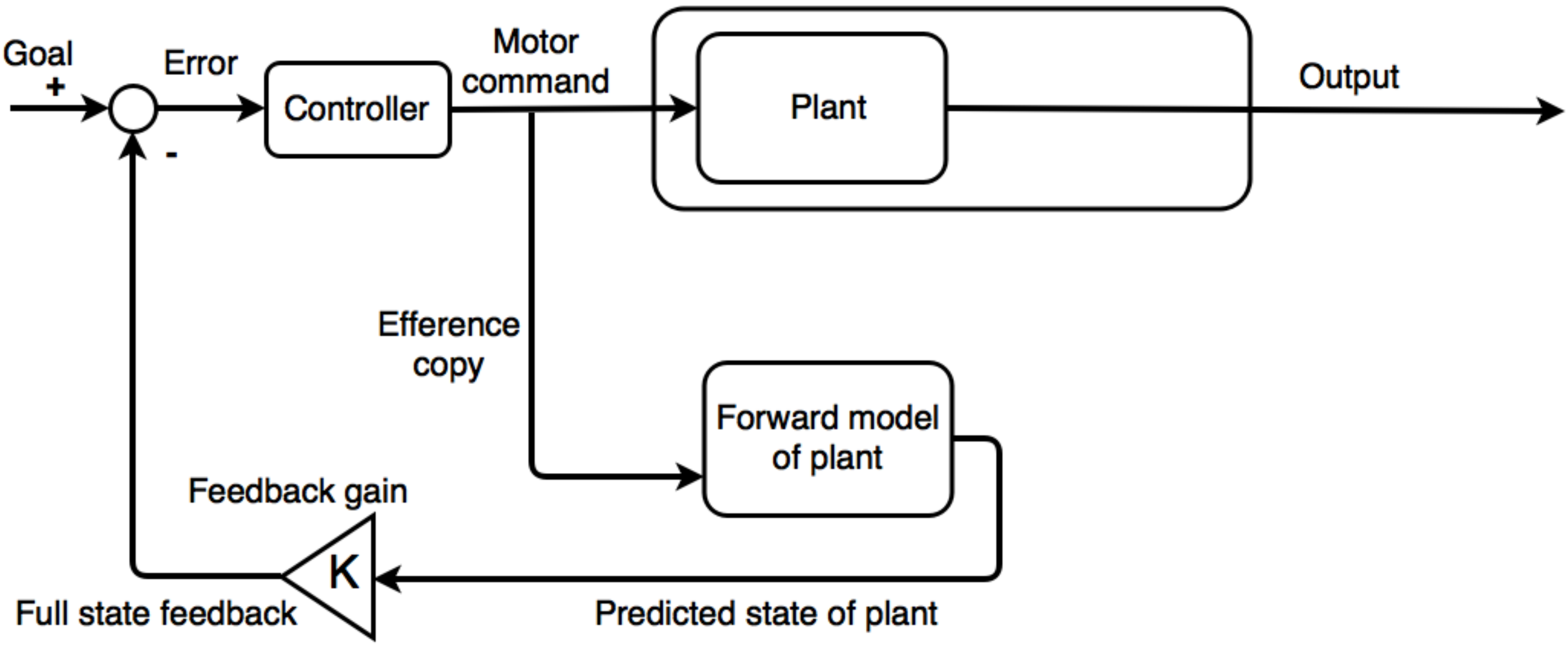
- Many procedures for control design assume the full state vector is available
- This is not always the case!

- Measurements may require costly measurement devices
- It may also be impossible to measure all state variables

- However it may be possible to construct an approximation to the full state vector using available measurements
- The method of estimation essentially provides you with a virtual sensor!

- An observer can estimate the states of the system from the output measurements.

State feedback control using an observer



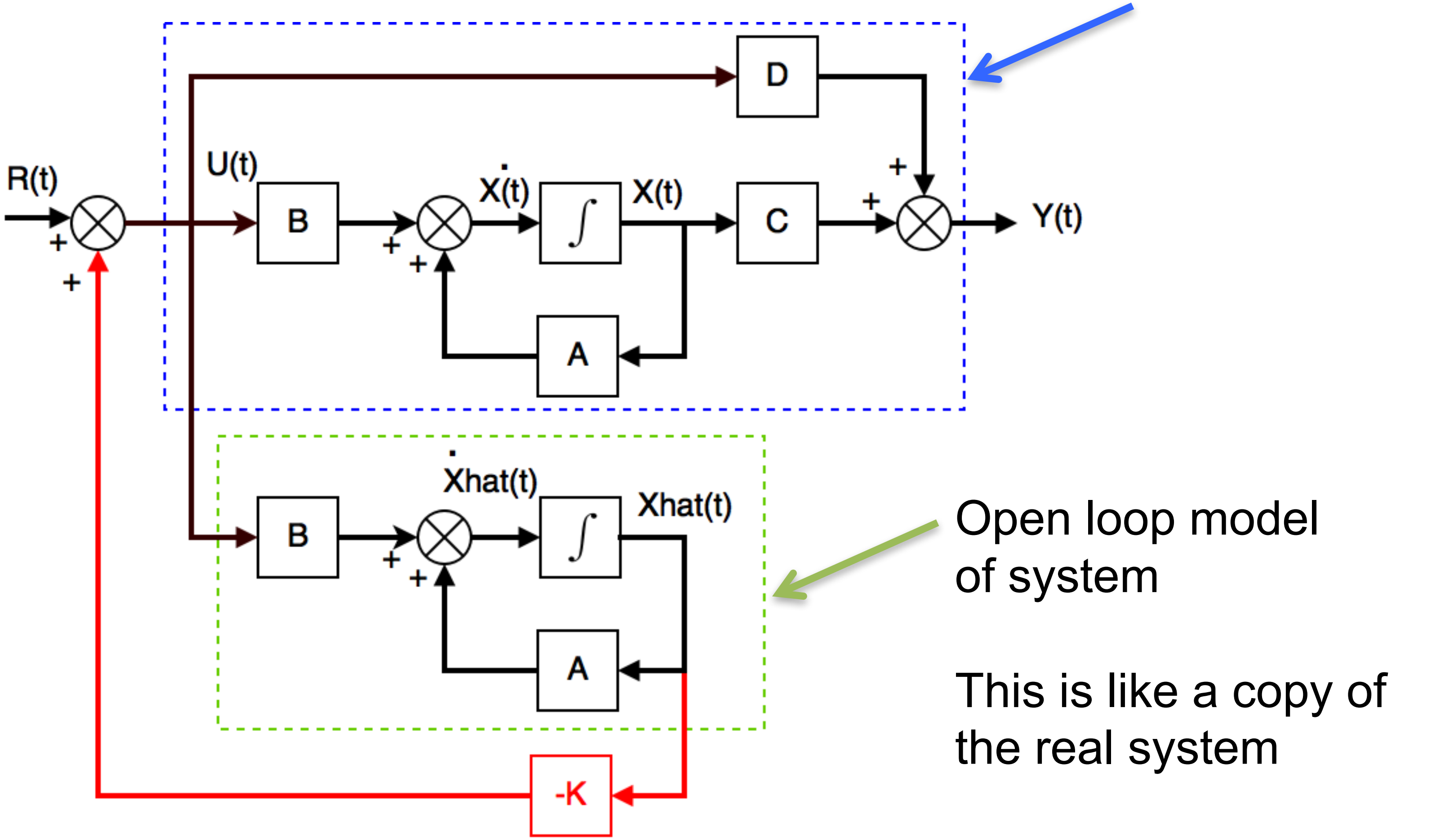
- Full state often not available
- So use an observer to estimate the full state
- The estimator here is open loop it can easily diverge from the true states
- This can happen due to disturbances or an imprecise model of the plant

Using open loop observer to estimate state

We can make an open loop estimate the state simulation using a model of the real state space system

We call estimated state Xhat

$$\dot{\hat{X}} = A\hat{X} + BU$$



This is like a copy of the real system

Using open loop observer to estimate state

- Analyzing the system

$$\dot{\hat{X}} = A\hat{X} + BU$$

where Xhat is an open loop estimated state

- The state estimation error is given by the different between this estimate and the actual state

$$\tilde{X} = X - \hat{X}$$

Differentiating both sides of this equation w.r.t. time

$$\Rightarrow \dot{\tilde{X}} = \dot{X} - \dot{\hat{X}}$$

From the actual state space model we know that

$$\dot{X} = AX + BU$$

- Substituting in the state space equations

$$\Rightarrow \dot{\tilde{X}} = AX + BU - A\widehat{X} - BU$$

Using open loop observer to estimate state

- So from

$$\dot{\tilde{X}} = AX + BU - A\hat{X} - BU$$

$$\Rightarrow \dot{\tilde{X}} = AX - A\hat{X}$$

$$\Rightarrow \dot{\tilde{X}} = A\left(X - \hat{X}\right)$$

- But from before the state error is given by

$$\tilde{X} = X - \hat{X}$$

$$\Rightarrow \dot{\tilde{X}} = A\tilde{X}$$

- Therefore the state error at time t is given by the solution to this equation

$$\Rightarrow \tilde{X}(t) = e^{At}\tilde{X}(0)$$

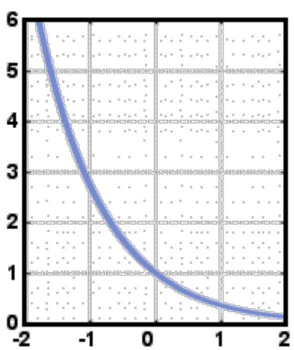
Using open loop observer to estimate state

- So from the equation of error dynamics

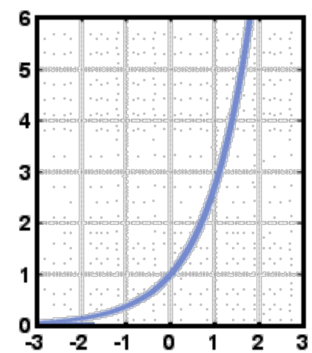
$$\tilde{X}(t) = e^{At}\tilde{X}(0)$$

- It can been seen that the error dynamics depend on eigenvalues of A

Want exponential decay of error

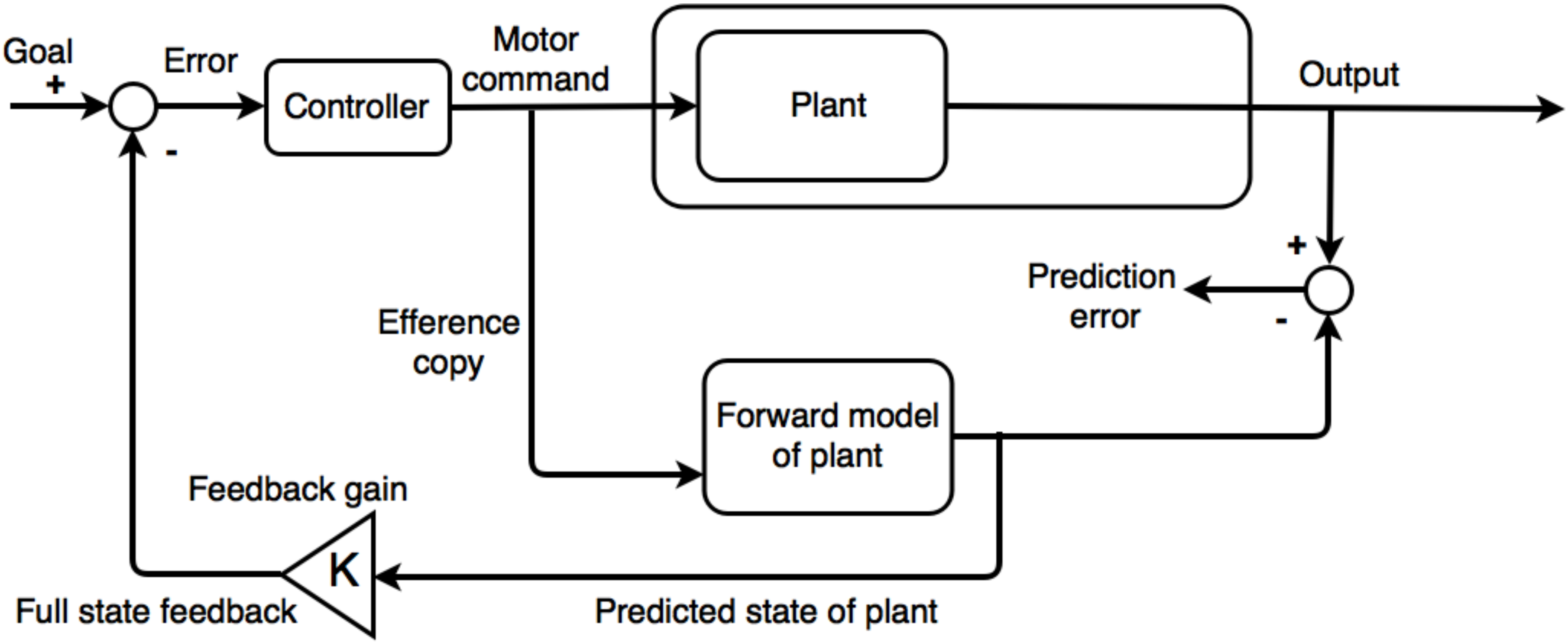


Don’t want exponential growth



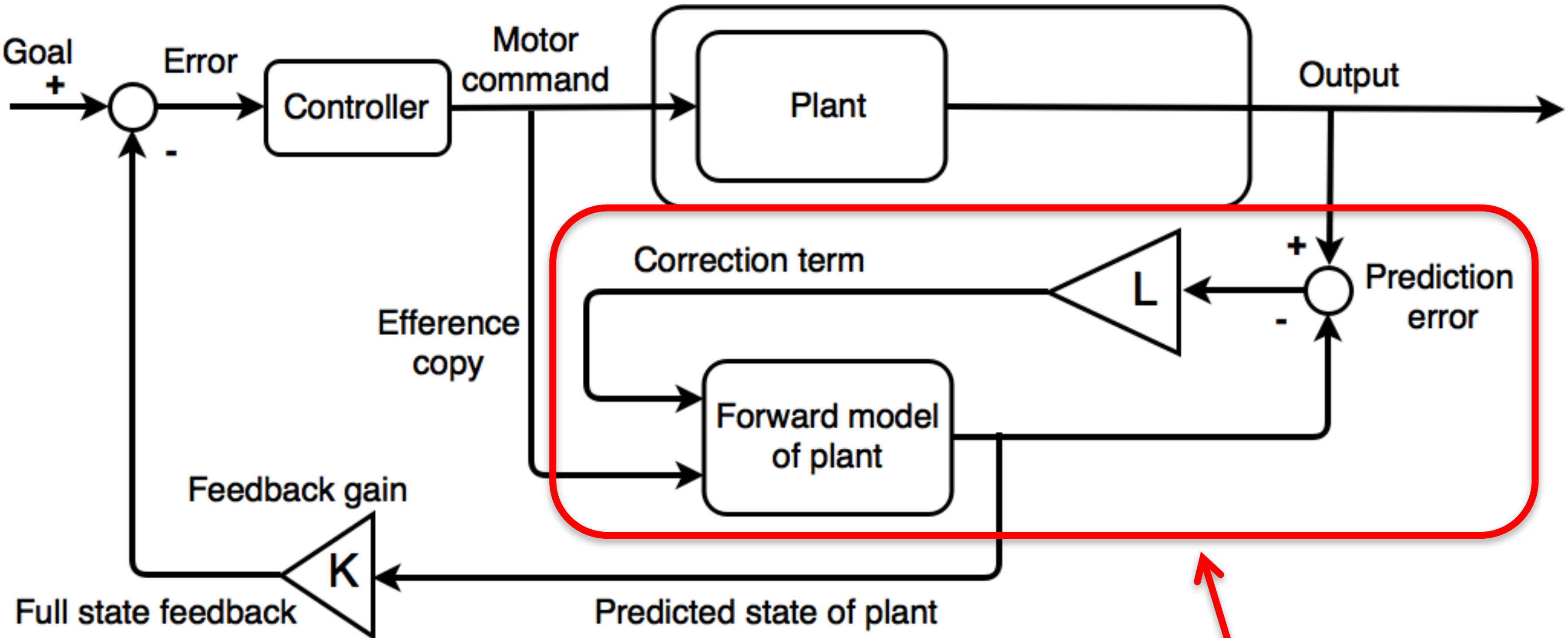
- Clearly we want error to go to zero at some (ideally short) value of time t
- If A unstable then this will not happen and the estimated state will diverge from the true state
- We note that for a given system, we cannot arbitrarily change the location of the eigenvalues of A, because we cannot change A!
- We need a formulation that lets us select the eigenvalue locations

Consider observer prediction error



- Note we can calculate prediction error as difference between predicted and actual output
- Want to use a feedback circuit so we can get error to decay faster than it would open loop
- Also want to make sure we use real output of system in case model is inaccurate

Observer with correction term



- Observer can uses prediction error to correct its state estimate
- Correction term is in another control loop that tries to drive the prediction error to zero

Use feedback controller with gain L to help reduce prediction error

Using closed loop observer to estimate state

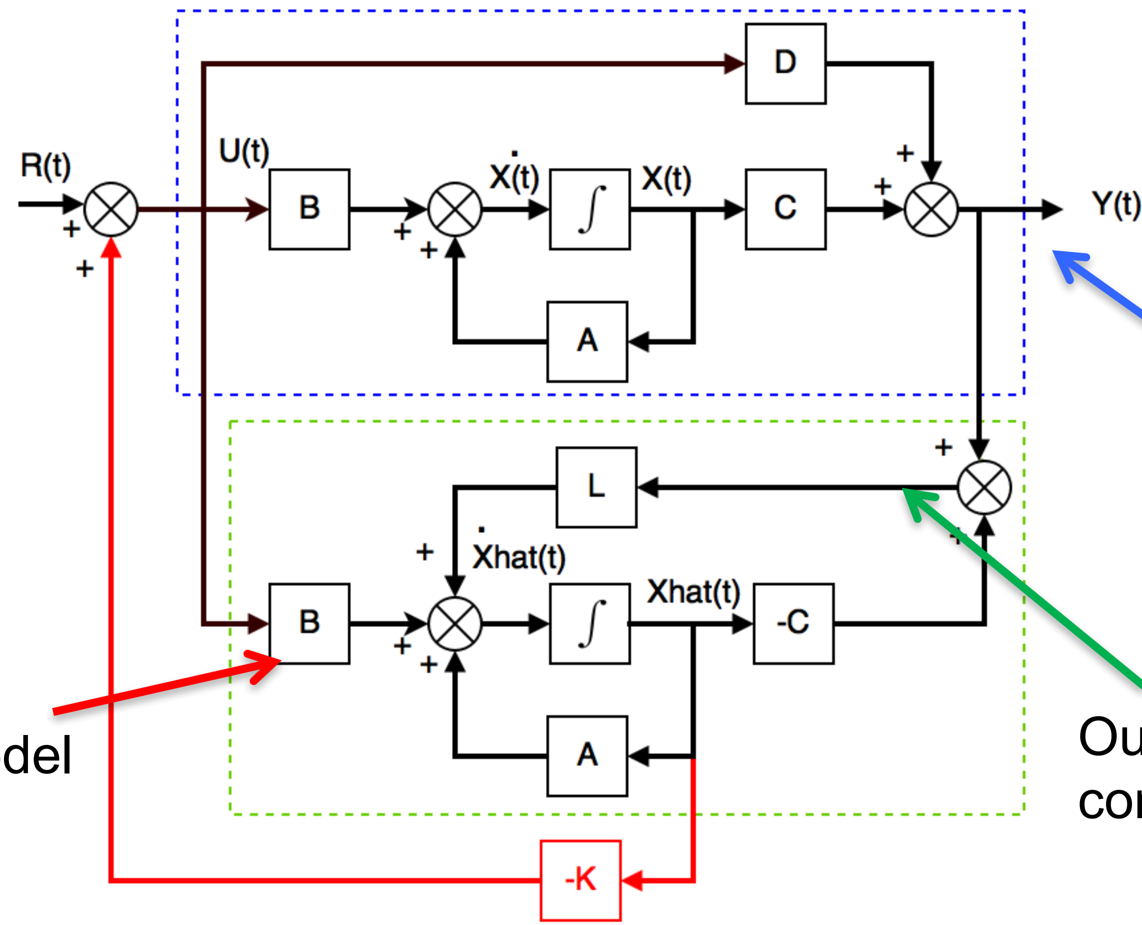
- The modified observer structure appears as follows

$$\dot{\hat{X}} = \boxed{A\hat{X} + BU} + \boxed{L(y - C\hat{X})}$$

State estimate

Open loop term

Output error correction term



State space system

Open loop model of system

Output error correction term

Using closed loop observer to estimate state

- Given the state estimate with output error correction

$$\dot{\hat{X}} = A\hat{X} + BU + L(y - C\hat{X})$$

- The state estimation error is given by

$$\tilde{X} = X - \hat{X} \qquad \Rightarrow \dot{\tilde{X}} = \dot{X} - \dot{\hat{X}}$$

- Substituting in the state space equations now gives

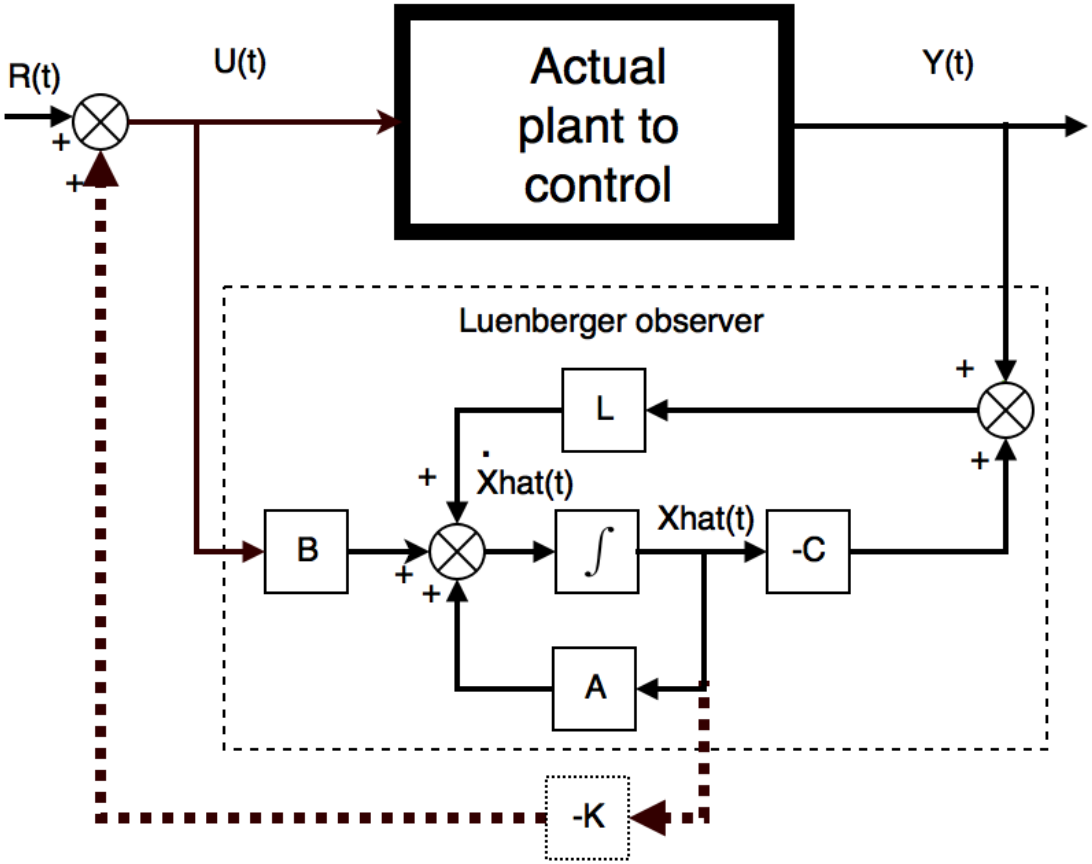
$$\Rightarrow \dot{\tilde{X}} = AX + BU - A\widehat{X} - BU - L(y - C\hat{X})$$

$$\Rightarrow \dot{\tilde{X}} = AX - A\widehat{X} - L(CX - C\hat{X}) \qquad = A\left(X - \widehat{X}\right) - LC(X - \hat{X})$$

$$\Rightarrow \dot{\tilde{X}} = A\tilde{X} - LC\tilde{X} \qquad = (A - LC)\tilde{X}$$

- So error dynamics now depend on eigenvalues of (A-LC) and not just on A
- Need to choose L so that (A-LC) has –ve eigenvalues so error decays away
- This is very similar to choosing gain to get controllable SFC system

Control of real system using SFC with observer

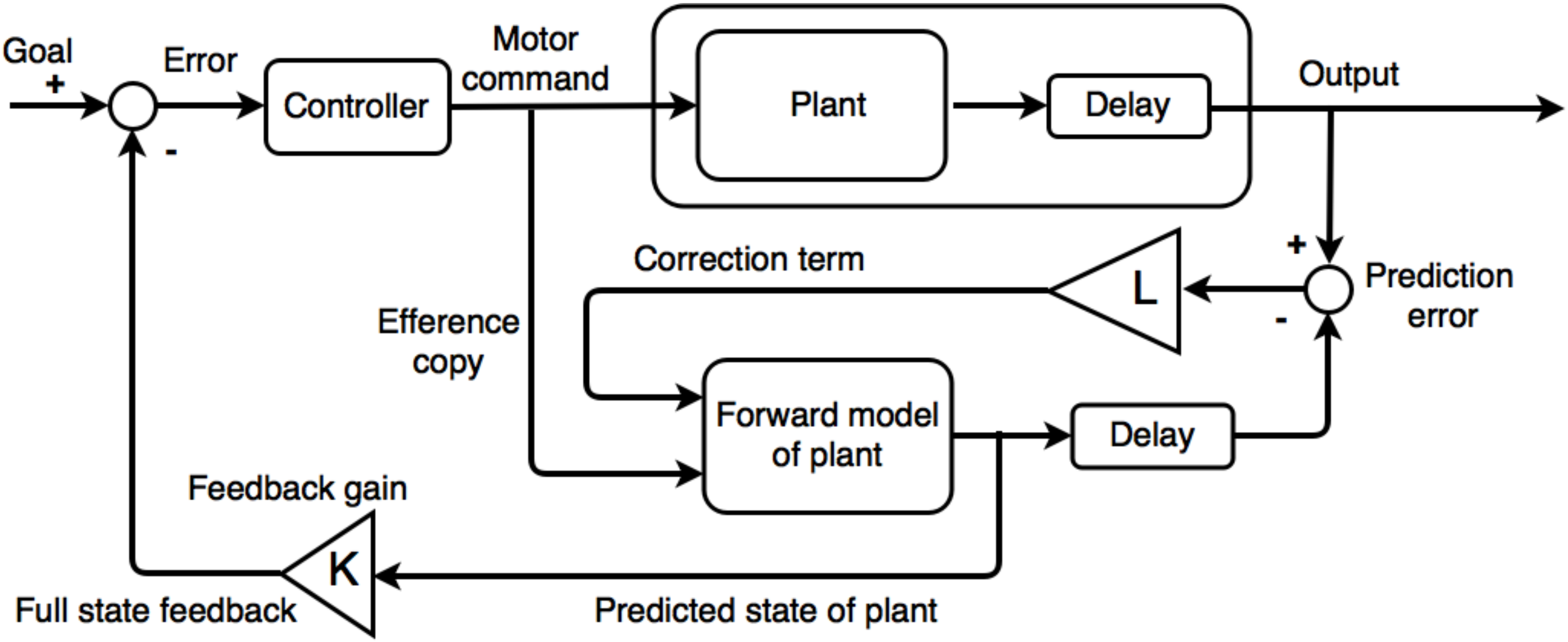


To summarize:

- We do not generally have access to the actual state of the system we want to control
- Therefore we *need* to estimate its internal states using an observer
- But we can use the real output to improve our state estimate

- The input control signal is denoted as U(t).
- The actual system output is denoted as Y(t) which is feedback to the observer to improve its state estimate Xhat(t).
- The blocks A, B, C, and D represent multiplication by the state space matrices that form the model of the actual plant.

Dealing with correction term delay



- An observer can also deal with delays
- Prediction error can be calculated after same delay as plant delay
- And then used to correct the state estimate

Ackermann observer gain calculation

- To choose Luenberger estimation gain L

$$\dot{\hat{X}} = A\hat{X} + BU + L(y - C\hat{X})$$

- Can choose appropriate observer gains L by pole placement or using an optimality criteria
- Can find gain by algebraic manipulation by hand just like when finding SFC feedback gains
- However easier to perform pole placement can be achieved using the Ackermann formula or the place command in Matlab

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% build observer just for angle and angular velocity states

% observer gain
PX=20 * [-1 -1.2];
ssm.L = place(ssm.A, ssm.C',PX);
ssm.LT = ssm.L';
```

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Lecture 7

Gaussian elimination to achieve echelon form

Gaussian elimination

- Gaussian Elimination is a method often used to solve simultaneous linear equations of the form AX=b
- Can use forward elimination is to transform a coefficient matrix into an upper triangular matrix
- After forward elimination done, back substitution can be employed to to solve equations (we wont need that here)
- It can also inform us of the rank of a matrix A
- We will need t determine the rank of a matrix is our subsequent analyses of controllability and operability of state space models

Gaussian elimination

Forward elimination

- Starting with the first row, add or subtract multiples of that row to eliminate the first coefficient from the second row and beyond.

- Continue this process with the second row to remove the second coefficient from the third row and beyond.

- Stop when an upper triangular matrix remains

$$A = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

$$\Downarrow$$

$$A = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a'_{22} & a'_{23} & b'_2 \\ 0 & 0 & a''_{33} & b''_3 \end{array}\right]$$

Matrix rank

- If a matrix can be reduced to upper triangular form and rows are nonzero then matrix is full rank

- Rank given by number of non-zero rows of triangular matrix

Rank = 3

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

Rank = 2

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & 0 \end{bmatrix}$$

Rank = 1

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Matrix rank examples

- Example1: consider the 2x2 matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$A \,\middle|\, R_2 \rightarrow R_2 - R_1 = \begin{bmatrix} 1 & 1 \\ 1-1 & -1-1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$$

Both rows contain non-zero elements
Therefore rank = 2, which is full rank for 2x2 matrix

Matrix rank examples

- Example2: consider s 2x2 matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$A \middle| R_2 \rightarrow R_2 - 2R_1 = \begin{bmatrix} 1 & 2 \\ 2-2 & 4-(2\times 2) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

Only top rows contains non-zero elements
Therefore rank = 1, which is less than full rank for 2x2 matrix
Note that here the 1st and 2nd row vectors are linearly dependent

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Lecture 7

Controllability

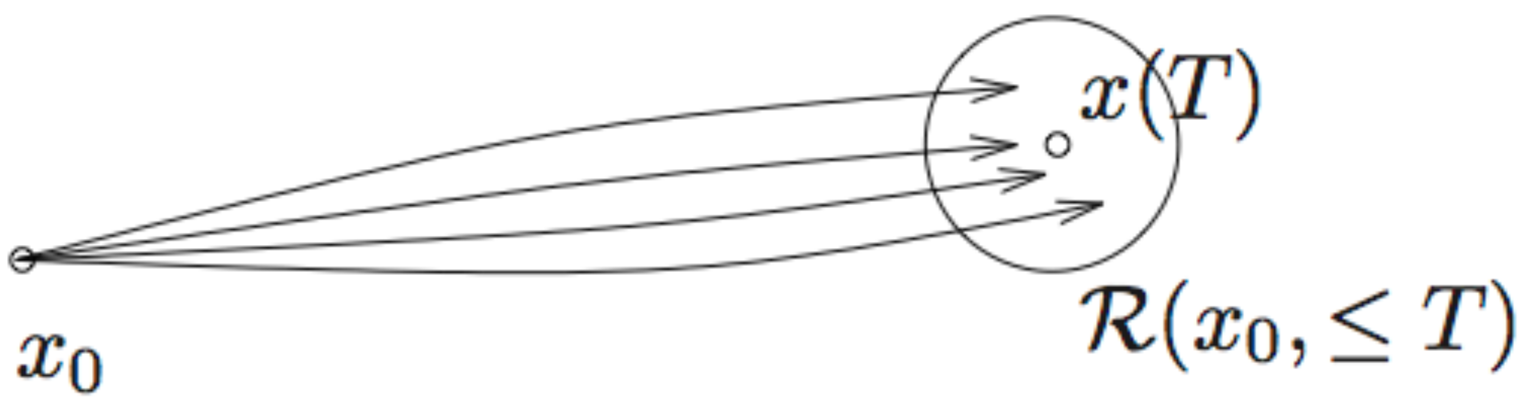
Controllability

- Given the state space system

$$\dot{X} = AX + BU$$

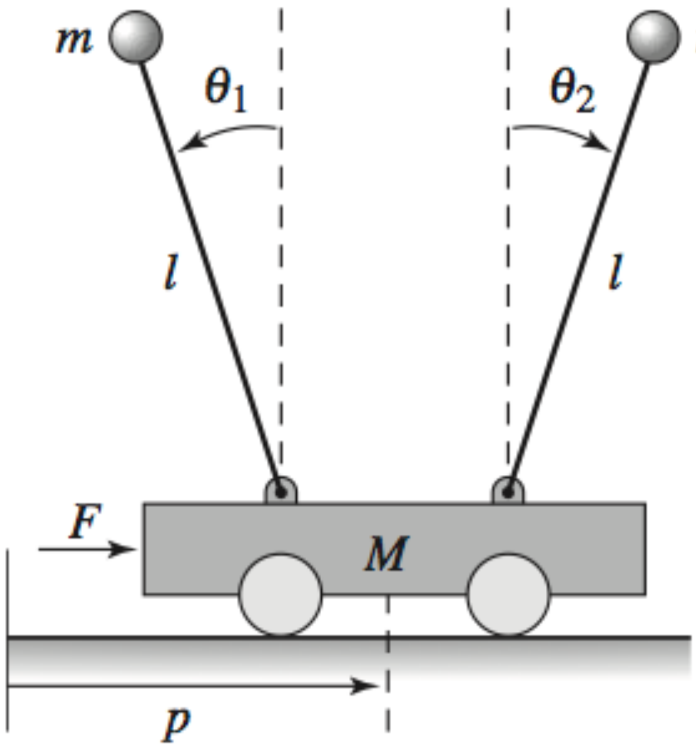
- The LTI system is controllable if, for every x*(t) and every finite T > 0, there exists an input function u(t), 0 < t ≤ T, such that the system state goes from x(0) = 0 to x(T) = x*

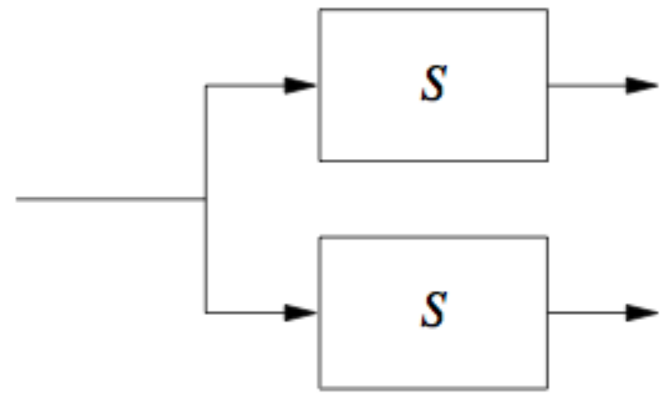
- Here we disregard the output measurements of the system and only focus on the evolution of the state



Controllability

Intuitive example of a non-controllable system





- Above system consists of two identical sub-systems with the same input.
- Same force acts on two pendula
- Not possible to arbitrarily control both pendulums at the same time.
- Cannot reach arbitrary state so the system is not controllable/reachable.

Controllability

- If system matrix A has a diagonal form the system is controllability if column vector B has no non-zero elements
- For example

$$\dot{X} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} X + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} U$$

A diagonal matrix A means at the states are uncoupled

Therefore

$$\dot{X}_1 = \lambda_1 X_1 + b_1 U$$
$$\dot{X}_2 = \lambda_2 X_2 + b_2 U$$
$$\dot{X}_3 = \lambda_3 X_3 + b_3 U$$

- In this case the control input can drive and influence all state variables
- Therefore the system is controllable

Controllability

- Another example

$$\dot{X} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} X + \begin{bmatrix} b_1 \\ b_2 \\ 0 \end{bmatrix} U$$

Therefore

$$\dot{X}_1 = \lambda_1 X_1 + b_1 U$$
$$\dot{X}_2 = \lambda_2 X_2 + b_2 U$$
$$\dot{X}_3 = \lambda_3 X_3$$

- In this case the control input cannot drive and influence all state variables
- Therefore the system is not controllable

Controllability matrix key result

- In the general case the matrix A will not be diagonal o states can be coupled together

$$\dot{X} = AX + BU$$

- In this case we cannot just use inspection and need to examine the controllability matrix to say if the system is controllable

- The controllability matrix is given by the expression

$$M_c = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$$

- The system is only fully controllable if the controllability matrix is full rank

Controllability example

- Given the state space system

$$\dot{X} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} X + \begin{bmatrix} 2 \\ 1 \end{bmatrix} U$$

$$Y = \begin{bmatrix} 3 & 0 \end{bmatrix} X$$

$$\Rightarrow A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

- Since this has a 2x2 system matrix, the system controllability is given by

$$M_c = \begin{bmatrix} B & AB \end{bmatrix}$$

- Calculating the AB term gives

$$AB = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ -1 \end{bmatrix}$$

Controllability example

- Substituting in A and AB we have

$$M_c = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 1 & -1 \end{bmatrix}$$

- Using Gaussian elimination to try to achieve echelon form

$$M_c \big| R_2 \rightarrow 2R_2 - R_1 = \begin{bmatrix} 2 & -4 \\ 2-2 & -2-(-4) \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 0 & 2 \end{bmatrix}$$

- It can be seen that the reduced echelon form matrix has full rank of 2

- Therefore the system is controllable

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Lecture 7

Observability

Observability matrix key result

- Given the state space system

$$\dot{X} = AX + BU$$
$$Y = CX + DU$$

- The LTI system is observable if the initial state x(0) can be uniquely deduced from the knowledge of the input u(t) and output y(t) for all t between 0 and any finite T > 0

- In many application, it is not practical to measure all of the states directly

- A system is observable if there is no hidden dynamics inside it

Observability matrix key result

If system matrix A has a diagonal form system controllability if output row vector C has no non-zero elements
For example

$$\dot{X} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} X + BU$$

A diagonal matrix A means at the states are uncoupled

$$Y = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} X$$

In this case we have $Y = C_1X_1 + C_2X_2 + C_3X_3$

- In this case all state variables are coupled to the output and have an effect on the output Y
- Therefore the system is observable

Observability matrix key result

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For example

$$\dot{X} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} X + BU$$

A diagonal matrix A means at the states are uncoupled

$$Y = \begin{bmatrix} c_1 & c_2 & 0 \end{bmatrix} X$$

In this case we have $Y = C_1 X_1 + C_2 X_2$

- In this case not all state variables are coupled to the output and have an effect on the output Y
- X_3 does not influence the output Y
- Therefore the system is not observable

Observability matrix key result

- In the general case A will not be diagonal

$$\dot{X} = AX + BU$$
$$Y = CX + DU$$

- In this case we need to examine the observability matrix.
- The system observability matrix is given by

$$M_o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

- The system is only fully observable if the observability matrix is full rank

Observability example

- Given the state space system

$$\dot{X} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} X + \begin{bmatrix} 2 \\ 1 \end{bmatrix} U$$

$$Y = \begin{bmatrix} 3 & 0 \end{bmatrix} X$$

$$\Rightarrow A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow C = \begin{bmatrix} 3 & 0 \end{bmatrix}$$

- Since this has a 2x2 system matrix, the system observability is given by

$$M_o = \begin{bmatrix} C \\ CA \end{bmatrix}$$

- Calculating the CA term gives

$$CA = \begin{bmatrix} 3 & 0 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -6 & 0 \end{bmatrix}$$

Observability example

- Substituting in C and CA we have

$$M_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -6 & 0 \end{bmatrix}$$

  - Using Gaussian elimination to try to achieve echelon form

$$M_o \big| R_2 \to R_2 + 2R_1 = \begin{bmatrix} 3 & 0 \\ -6+2\times 3 & 0+0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$$

- It can be seen that the reduced echelon form matrix has rank of 1 which is less than full rank of 2

- This can be seen by inspection since the two rows in M_o were linearly *dependent*, differing only in scaling factor of -2

- Therefore the system is not observable

Stability, controllability and observability

- The eigenvalues of matrix A must satisfy the equation

$$AX = \lambda X$$

- And this has a non-zero solution if the determinant of the matrix (A-I)=0

$$\Rightarrow |A - \lambda I| = 0$$

- System has a 2x2 system matrix, so controllability matrix is given by

$$M_c = \begin{bmatrix} B & AB \end{bmatrix}$$

- System has a 2x2 system matrix, so observability matrix is given by

$$M_o = \begin{bmatrix} C \\ CA \end{bmatrix}$$

ROCO218: Control Engineering

Lecture 7

2015 exam for ROCO316 Modern Control
Solutions to relevant questions

Q2: Electrical circuit

Consider the electrical circuit in Figure 1 below.

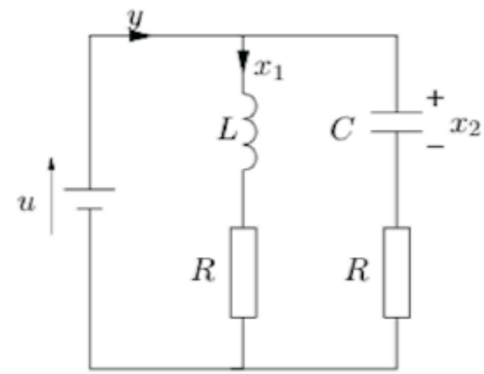

Figure 1: Electrical circuit.

We will admit that the state-space model for this circuit is given by

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -\frac{R}{L} & 0 \\ 0 & -\frac{1}{RC} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ \frac{1}{RC} \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & -\frac{1}{R} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \frac{1}{R} u \end{aligned} \tag{4}$$

We consider the following parameter values: $R = 1$ (Ohm), $L = 1$ (H), $C = 1$ (F).

(a) Compute the controllability matrix $\mathcal{C}$ associated with system (4). Is the system controllable?

(5 marks)

(b) Compute the observability matrix $\mathcal{O}$ associated with system (4). Is the system observable?

(5 marks)

Q2: Electrical circuit

- Substituting in the value of L and R into

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \dfrac{-R}{L} & 0 \\ 0 & \dfrac{-1}{RC} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \dfrac{1}{L} \\ \dfrac{1}{RC} \end{bmatrix} U \qquad \Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} U$$

$$y = \begin{bmatrix} 1 & \dfrac{-1}{R} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \frac{1}{R} U \qquad \Rightarrow y = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + U$$

- Therefore the system matrices are given by

$$\Rightarrow A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow C = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

Q2: Electrical circuit

- Since our system has a 2x2 system matrix, the system controllability is given by

$$M_c = \begin{bmatrix} B & AB \end{bmatrix}$$

Calculating the AB term gives

$$AB = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$\Rightarrow M_c = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

- Using Gaussian elimination to try to achieve echelon form

$$M_c \,|\, R_2 \to R_2 - R_1 = \begin{bmatrix} 1 & -1 \\ 1-1 & -1--1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

- It can be seen (also by inspection) that the reduced echelon form matrix only has rank of 1, therefore the system is not controllable

Q2: Electrical circuit

- Since our system has a 2x2 system matrix, the system observability matrix is given by

$$M_o = \begin{bmatrix} C \\ CA \end{bmatrix}$$

Calculating the CA term gives

$$CA = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \end{bmatrix}$$

$$\Rightarrow M_o = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

- Using Gaussian elimination to try to achieve echelon form

$$M_c \big| R_2 \rightarrow R_2 + R_1 = \begin{bmatrix} 1 & -1 \\ -1+1 & 1+-1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

- It can be seen (also by inspection) that the reduced echelon form matrix only has rank of 1, so system is not observable

Q3: State space control

Consider the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{5}$$

(a) Design a state feedback $u(t) = -k_1 x_1(t) - k_2 x_2(t)$ so that the closed-loop eigenvalues are placed at $\{-2, \ -1\}$. **You should use the direct method** for computing the gains k_1 and k_2.

(10 marks)

Q3: State space control

- From state space system

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U \qquad \Rightarrow A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \qquad \Rightarrow B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Applying state feedback control

$U = -KX$ where $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$

- To the state space equation

$$\dot{X} = AX + BU$$

- Leads to the the modified relationship

$$\dot{X} = AX - BKX \qquad \Rightarrow \dot{X} = \left(A - BK\right)X$$

- Stability determined by location of poles which are the eigenvalue of matrix (A-BK)

Q3: State space control

- We require that the eigenvalues λ of the controller system are at -2,-1
- This means we will want the following characteristic equation for the eigenvalues

$$\Rightarrow (\lambda+2)(\lambda+1)=0 \qquad \Rightarrow \lambda^2+3\lambda+2=0$$

- The eigenvalues λ of the closed loop system are given by

$$\det(A-BK-\lambda I)=0$$

Substituting in values for B we have

$$BK=\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix}$$

Q3: State space control

- Expanding $\det\left(A - BK - \lambda I\right) = 0$

$$\Rightarrow 0 = \left| \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \left| \begin{bmatrix} -\lambda & 1 \\ \left(2 - k_1\right) & \left(1 - k_2 - \lambda\right) \end{bmatrix} \right|$$

$$\Rightarrow \left(-\lambda\right)\left(1 - k_2 - \lambda\right) - \left(2 - k_1\right) = 0$$

$$\Rightarrow \lambda^2 + \left(k_2 - 1\right)\lambda + \left(k_1 - 2\right) = 0$$

- We now need to match the coefficients in the desired eigenvalues characteristic equation using the appropriate gain vector K

$$\lambda^2 + 3\lambda + 2 = 0 \qquad \Leftrightarrow \lambda^2 + \left(k_2 - 1\right)\lambda + \left(k_1 - 2\right) = 0$$

$$\Rightarrow k_2 - 1 = 3 \qquad \Rightarrow k_2 = 4$$

$$\Rightarrow k_1 - 2 = 2 \qquad \Rightarrow k_1 = 4$$