Exam question topics overview

1. Derive a transfer function for a dynamical system e.g. mechanical system
  Be able to write down differential equations of mechanical and electrical systems and applying Laplace transforms

2. Analyze a canonical 2nd order system analysis
  Compare with canonical equations and specify canonical parameters (damping, natural frequency, overshoot etc.)

3. Derive the state space model for a linear dynamical system
  Indicate if the system is stable, controllable and observable

4. Analyze a non-linear dynamical system
  Use linearization based on the Jacobian method to derive a linear state space model in matrix form

5. Design a state feedback controller
  Specify state feedback gain using pole/eigenvalue placement by algebraic manipulation by hand.

6. Observer and integral control
  Understand these concepts and how to implement them

7. Design an optimal feedback controller
  Specify state feedback gain using Algebraic ricotta equations

ROCO218: Control Engineering
Dr Ian Howard

Tutorial 4

ROCO218 2016 Exam examples
Derive transfer function for mechanical system

Q1: Mechanical system

Q1. Calculate the transfer function from the force f(t) to the position x(t) for the mechanical system in **Figure Q1** below. Note the definition of position x(t)!

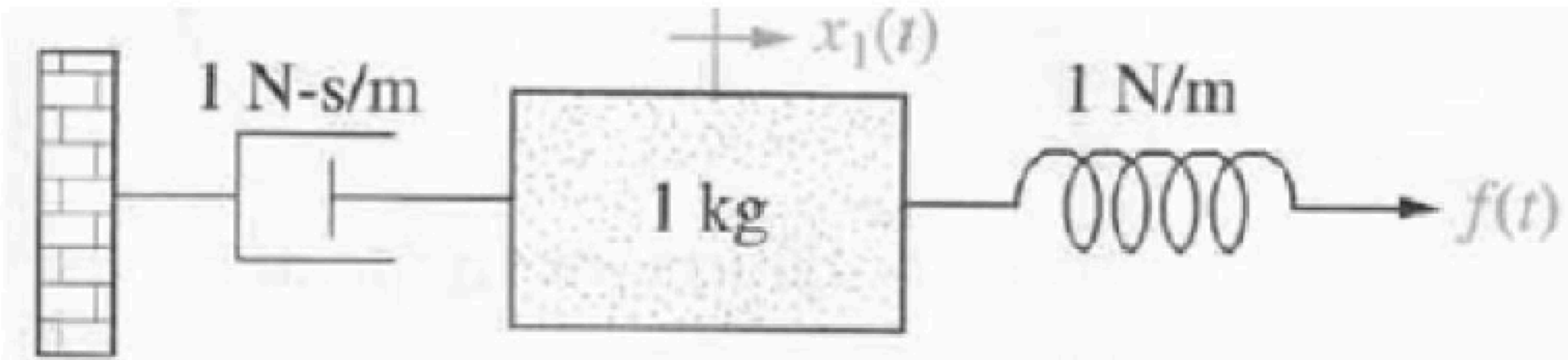
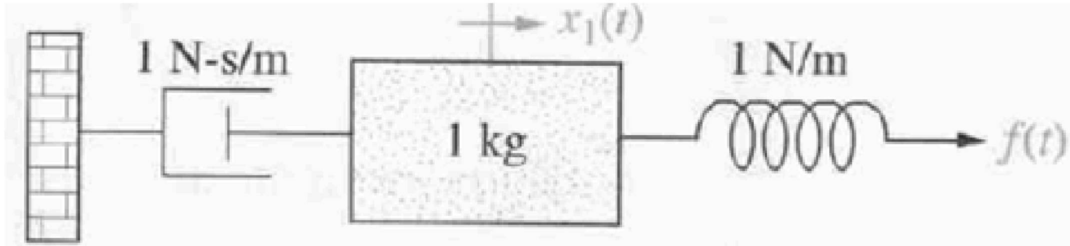


Figure Q1.

Q1: Mechanical system



- The spring has no effect on position x(t), since the force f(t) is simply transmitted through the spring to the mass (Newton's 1st law).
- The differential equation that describes the motion of the mass therefore doesn't involve a term for the spring:

$$m\ddot{x}(t)+\mu\dot{x}(t)=f(t)$$

Taking the Laplace transform with zero initial conditions

$$ms^2X(s)+\mu sX(s)=F(s)$$

Inserting m = 1, μ = 1.

$$\Rightarrow F(s)=s^2X(s)+sX(s)=\left(s^2+s\right)X(s)$$

- Therefore transfer function from F(s) to X(s)

$$\frac{X(s)}{F(s)}=\frac{1}{\left(s^2+s\right)}$$

Q2: Example canonical 2nd order system

The open loop transfer function G(s) of a unity gain closed-loop film transport feedback system is given below:

$$G(s) = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)}$$

The unity gain closed-loop system has the error function:

$$e(t) = 1.4e^{-1.07t} - 0.4e^{-3.73t}$$

(a) Determine the damping rate ζ

(5 marks)

(b) The natural frequency ω_n

(5 marks)

(c) The open loop transfer function and the close loop transfer function.

(5 marks)

(d) The steady-state error

(5 marks)

Q2: Example canonical 2nd order system

Given the open loop function

$$G(s) = \frac{\omega_n^2}{s(s + 2\xi\omega_n)}$$

The unity feedback closed loop transfer function C(s) is therefore given by

$$C(s) = \frac{G(s)}{1 + G(s)}$$

Substituting in open loop transfer function gives

$$\Rightarrow C(s) = \frac{\dfrac{\omega_n^2}{s(s + 2\xi\omega_n)}}{1 + \dfrac{\omega_n^2}{s(s + 2\xi\omega_n)}} = \frac{\omega_n^2}{s(s + 2\xi\omega_n) + \omega_n^2} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

Q2: Example canonical 2^{nd} order system

In the time domain the error response is

$$e(t) = 1.4e^{-1.07t} - 0.4e^{-3.73t}$$

Using the Laplace transform $\quad e^{kt} \Leftrightarrow \dfrac{1}{s-k}$

e(t) in the s-domain it is give by

$$E(s) = \frac{1.4}{s+1.07} - \frac{0.4}{s+3.73}$$

Now combine both terms together over common denominator

$$E(s) = \frac{1.4(s+3.73) - 0.4(s+1.07)}{(s+1.07)(s+3.73)} = \frac{1.4s - 0.4s + 5.222 - 0.428}{s + 3.73s + 1.07s + 3.9911}$$

After rounding the values

$$E(s) \approx \frac{s+4.8}{s^2 + 4.8s + 4}$$

Q2: Example canonical 2nd order system

The error E(s) is the difference between input and output

$$\Rightarrow E(s) = R(s) - Y(s)$$

For the closed loop system the error E(s) is given by

$$E(s) = R(s) - Y(s) \qquad = R(s)[1 - T(s)]$$

Where T(s) is the closed loop transfer function.

Substituting the closed loop transfer function we derived previously

$$E(s) = R(s)\left[1 - \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}\right] = R(s)\frac{s^2 + 2\xi\omega_n s + \omega_n^2 - \omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$= R(s)\frac{s^2 + 2\xi\omega_n s}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

This is written in a canonical 2nd order form

- Where
- Damping is ξ
- Natural frequency is ω_n

Q2: Example canonical 2nd order system

Comparing the error function of our system with the desired error response for step input which sets R(s)=1/s

$$\frac{s+4.8}{s^2+4.8s+4} \Leftrightarrow \frac{1}{s}\left(\frac{s^2+2\xi\omega_n s}{s^2+2\xi\omega_n s+\omega_n^2}\right) \Leftrightarrow \frac{s+2\xi\omega_n}{s^2+2\xi\omega_n s+\omega_n^2}$$

By comparing coefficients:

- $4=\omega^2_n$. Therefore natural frequency $\omega_n = 2.0$
- $4.8=2\xi 2$ Therefore damping $\xi = 1.2$

open loop transfer function is therefore

$$G(s)=\frac{\omega_n^2}{s(s+2\xi\omega_n)} = \frac{4}{s(s+4.8)}$$

closed loop transfer function is therefore

$$C(s)=\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} = \frac{4}{s^2+4.8s+4}$$

Q2: Example canonical 2nd order system

From the given error function

$$e(t) = 1.4e^{-1.07t} - 0.4e^{-3.73t}$$

Steady state error is when t tends to infinity

$$\frac{\lim}{t \to \infty}[e(t)] = \frac{\lim}{t \to \infty}\left[1.4e^{-1.07t} + 0.4e^{-3.73t}\right] = 0$$

So Steady state error goes to to zero as t goes to infinity

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2015R exam for ROCO316 Modern Control
Solutions to relevant questions

Q1: Continuous-time nonlinear control system

Consider the following continuous-time nonlinear state-space model

$$
\begin{aligned}
\dot{x}_1 &= -x_1 + x_2 \\
\dot{x}_2 &= x_1 - x_2 - x_2^3
\end{aligned} \qquad (1)
$$

(a) Find the equilibrium position of system (1); justify your answer.

(5 marks)

(b) Derive the linearised version of system (1) about its equilibrium position; write it in matrix form (state-space form).

(5 marks)

(c) Compute the characteristic polynomial of the resulting linear system, and its associated eigenvalues. Is this linear system stable?

(5 marks)

(d) Compute the associated eigenvectors and write down the solution to the linear system.

Q1: Continuous-time nonlinear control system

Equilibria occur when the system us stationary
To find the equilibria we must therefore solve the equations

$$\dot{x}_1 = -x_1 + x_2 = 0 \qquad \Rightarrow x_1 = x_2$$

$$\dot{x}_2 = x_1 - x_2 - x_2^3 = 0$$

Substituting in $x_1 = x_2$

$$x_1 - x_1 - x_1^3 = 0 \qquad \Rightarrow x_1^3 = 0 \qquad \Rightarrow x_1 = 0 \qquad \Rightarrow x_2 = 0$$

Therefore, the only equilibrium position of system (1) is (x1, x2) = (0; 0)

To linearize the system at this point we need to evaluate the Jacobian at this point

From the state equations we can write

$$\dot{x}_1 = -x_1 + x_2 = f_1$$

$$\dot{x}_2 = x_1 - x_2 - x_2^3 = f_2$$

Q1: Continuous-time nonlinear control system

Given these two functions

$$f_1 = -x_1 + x_2 \qquad f_2 = x_1 - x_2 - x_2^3$$

Taking partial derivatives w.r.t. to state variables

$$\Rightarrow \frac{\partial f_1}{\partial x_1} = -1 \qquad \Rightarrow \frac{\partial f_1}{\partial x_2} = 1$$

$$\Rightarrow \frac{\partial f_2}{\partial x_1} = 1 \qquad \Rightarrow \frac{\partial f_2}{\partial x_2} = -1 - 3x_2^2$$

The Jacobian of the system is a 2x2 matrix since we have 2 state variables

$$J = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \dfrac{\partial f_1}{\partial x_2} & \dfrac{\partial f_2}{\partial x_2} \end{bmatrix}$$

$$\Rightarrow J = \begin{bmatrix} -1 & 1 \\ 1 & -1 - 3x_2^2 \end{bmatrix}$$

Evaluating at (0,0)

$$\Rightarrow J\big|(0,0) = \begin{bmatrix} -1 & 1 \\ 1 & -1 - 3x_2^2 \end{bmatrix}\Big|(0,0) = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

Q1: Continuous-time nonlinear control system

The Jacobian

$$J = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

Has the associated linear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The characteristic polynomial is found by finding the eigenvalues λ of the Jacobian matrix which is achieved by solving the equation

$$JX = \lambda X \qquad \Rightarrow JX - \lambda X = 0 \quad \Rightarrow (J - \lambda I)X = 0$$

Where X are the corresponding eigenvectors

If the term $(J - \lambda I)$ has an inverse then

$$(J - \lambda I)X = 0 \quad \Rightarrow (J - \lambda I)^{-1}(J - \lambda I)X = 0 \quad \Rightarrow IX = 0 \quad \Rightarrow X = 0$$

This is the trivial solution!
However we get a non-trivial solution when the Jacobian term has no inverse, which occurs when

$$\det(J - \lambda I) = 0$$

Q1: Continuous-time nonlinear control system

Expanding out the expression

$$\det(J-\lambda I)=0$$

$$\Rightarrow \left\| \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\| = 0$$

$$\Rightarrow \left\| \begin{bmatrix} -1-\lambda & 1 \\ 1 & -1-\lambda \end{bmatrix} \right\| = 0$$

- For a 2x2 matrix the determinant is given by

$$\det(A) = \left| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right| = (ad-bc)$$

Characteristic equation is therefore

$$\Rightarrow (-1-\lambda)(-1-\lambda)-1=0 \qquad \Rightarrow 1+\lambda+\lambda-1+\lambda^2=0 \qquad \Rightarrow \lambda^2+2\lambda=0$$

Factorizing given

$$\Rightarrow \lambda(\lambda+2)=0 \qquad \Rightarrow \lambda = 0,-2$$

Since have one eigenvalue with a zero real value, although the other has a negative real value, the system is only marginally stable

Q1: Continuous-time nonlinear control system

The eigenvectors X and eigenvalues λ satisfy the equation

$$JX = \lambda X$$

$$\Rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

For the eigenvalue λ=-2

$$\Rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -2 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Multiplying out terms gives

$$\Rightarrow -x_1 + x_2 = -2x_1$$

$$\Rightarrow x_1 - x_2 = -2x_2$$

Both show

$$\Rightarrow x_2 = -x_1$$

$$\Rightarrow X_{\lambda=-2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

For the eigenvalue λ=0

$$\Rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Multiplying out terms gives

$$\Rightarrow -x_1 + x_2 = 0$$

$$\Rightarrow x_1 - x_2 = 0$$

Both show

$$\Rightarrow x_2 = x_1$$

$$\Rightarrow X_{\lambda=0} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Q2: State space analysis

Consider the following linear control system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2}$$

(a) Justify that the uncontrolled system (i.e. $u = 0$) is unstable.

(5 marks)

(b) Compute the controllability matrix $\mathcal{C}$ of system (3). Is this system controllable?

(5 marks)

(c) Compute the observability matrix $\mathcal{O}$ of system (3). Is this system observable?

(5 marks)

Q2: State space analysis

- State space equations take the form

$$\dot{X} = AX + BU \qquad\qquad Y = CX + DU$$

The given system state space equations are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}U \qquad\qquad y = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- Therefore the system matrices are

$$\Rightarrow A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad \Rightarrow B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \Rightarrow C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

- Therefore the since system matrix A is diagonal, by inspection we can see that the eigenvalues of the system are given the diagonal elements of the system matrix A
- Therefore $\lambda = 1,2$

- However we will proceed and adopt an analysis here that will deal with the general case

Q2: State space analysis

Given system matrix A

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

- The eigenvalues of matrix A must satisfy the equation

$AX = \lambda X$ Where X is the state vector

$$\Rightarrow AX - \lambda X = 0$$

$\Rightarrow (A - \lambda I)X = 0$ Where I is the identity matrix

- This equation has a non-zero solution X if and only if the determinant of the matrix (A-I)=0

$$\Rightarrow |A - \lambda I| = 0$$

Where straight bracket signifies the determinant

Q2: State space analysis

- Substituting A and I into the expression

$$\Rightarrow\left|\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}-\lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right|=0 \qquad \Rightarrow\left|\begin{bmatrix}1-\lambda & 0\\ 0 & 2-\lambda\end{bmatrix}\right|=0$$

- For a 2x2 matrix the determinant is given by

$$\det(A)=\left|\begin{pmatrix}a & b\\ c & d\end{pmatrix}\right|=\left(ad-bc\right)$$

$$\Rightarrow (1-\lambda)(2-\lambda)-0\times 0=0$$

$$\Rightarrow (1-\lambda)(2-\lambda)=0$$

$$\Rightarrow \lambda=1,2$$

- Both of these eigenvalues are strictly positive and therefore the open loop system is unstable

Q2: State space analysis

- Since our system has a 2x2 system matrix, the system controllability is given by

$$M_c = \begin{bmatrix} B & AB \end{bmatrix}$$

Calculating the AB term gives

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow M_c = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

- It can be seen directly that the (reduced echelon form) matrix only has rank of 1
- Therefore the system is not controllable

Q2: State space analysis

- Since our system has a 2x2 system matrix, the system observability matrix is given by

$$M_o = \begin{bmatrix} C \\ CA \end{bmatrix}$$

Calculating the CA term gives

$$CA = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

$$\Rightarrow M_o = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

- Using Gaussian elimination to try to achieve echelon form

$$M_c \Big| R_2 \rightarrow R_2 - R_1 = \begin{bmatrix} 1 & 1 \\ 1-1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

- It can be seen that the reduced echelon form matrix has rank of 2, so the system is observable

Q3: State space control

Consider the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{3}$$

(a) Design a state feedback $u(t) = -k_1 x_1(t) - k_2 x_2(t)$ so that the closed-loop eigenvalues are placed at $\{-1, \ -1\}$. **You should use the direct method** for computing the gains k_1 and k_2.

(10 marks)

Q2: State space analysis

- State space equations take the form

$$\dot{X} = AX + BU$$

$$Y = CX + DU$$

The given system state space equations are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- Therefore the system matrices are

$$\Rightarrow A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \Rightarrow B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \Rightarrow C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Q2: State space analysis

- We now apply feedback to the state space equation

$$\dot{X} = AX + BU$$

- This is achieved by setting the input U to

$$U = -KX$$

Where the gain vector is given by $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$

- This leads to the the modified system dynamics equation

$$\Rightarrow \dot{X} = AX - BKX \qquad \Rightarrow \dot{X} = \left(A - BK\right)X$$

- So the stability of the feedback system is determined by location of its poles which are the eigenvalue of the matrix (A-BK)

- The eigenvalues λ of the closed loop system are thus given by

$$\det\left(A - BK - \lambda I\right) = 0$$

Q2: State space analysis

- Using the system matrices

$$A=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad B=\begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad C=\begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\Rightarrow BK=\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix}$$

Substituting in values

$$\det\left(A-BK-\lambda I\right)=0$$

$$\Rightarrow 0=\left\|\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}-\begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix}-\lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\| = \left\|\begin{bmatrix} -\lambda & 1 \\ \left(1-k_1\right) & \left(-k_2-\lambda\right) \end{bmatrix}\right\|$$

- The characteristic equation is therefore

$$\Rightarrow\left(-\lambda\right)\left(-k_2-\lambda\right)-\left(1-k_1\right)=0$$

Q2: State space analysis

- Simplifying the characteristic equation

$$(-\lambda)(-k_2-\lambda)-(1-k_1)=0$$

$$\Rightarrow \lambda^2+\lambda k_2+k_1-1=0$$

- We require that the eigenvalues λ of the controller system are at -1,-1
- Therefore we want the following characteristic equation

$$\Rightarrow (\lambda+1)(\lambda+1)=0 \qquad \Rightarrow \lambda^2+2\lambda+1=0$$

- We now need to match the coefficients in the desired eigenvalues characteristic equation using the appropriate gains k_1 and k_2 therefore:

$$\lambda^2+2\lambda+1=0 \qquad \Leftrightarrow \lambda^2+\lambda k_2+(k_1-1)=0$$

$$\Rightarrow k_2=2$$

$$\Rightarrow k_1-1=1 \quad \Rightarrow k_1=2$$

- Feedback law is therefore $u(t)=-2x_1-2x_2$