ROCO218: Control Engineering
Dr Ian Howard

Lecture 3

Brief introduction to Laplace transforms

Definition of Laplace transform

- The Laplace transform is a linear operator that maps a function f(t) to F(s).
- Specifically:

$$F(s) = L[f(t)] = \int_0^\infty f(t) e^{-st}\, dt$$

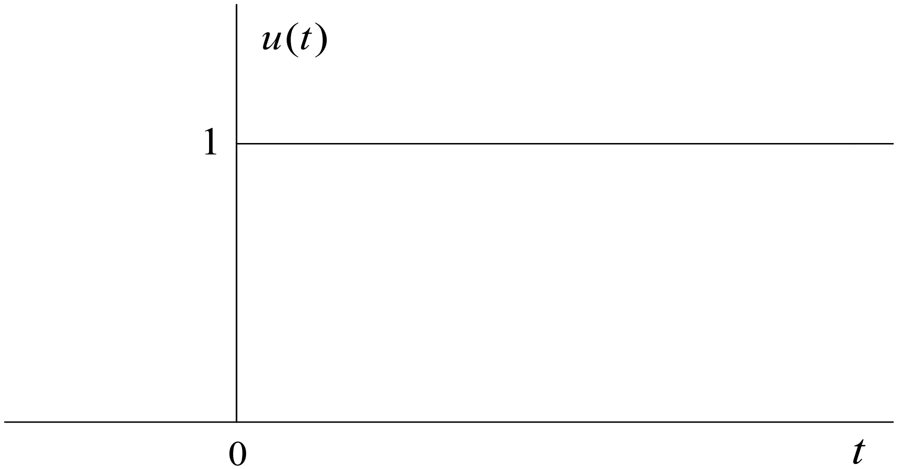
where:

$$s = \sigma + i\omega$$

- It maps from a time domain representation with real input to an angular frequency representation which is complex.

- We can use integral to calculate Laplace transforms of functions – but it can be hard work!

Laplace transform of the unit step



For t < 0 output zero
For t > zero, output constant value

$$u(t) = 1, \; t \geq 0$$

$$u(t) = 0, \; t < 0$$

For the step unit function

$$f(t) = 1$$

Since

$$\boxed{\int e^{ax} dx = \frac{1}{a} e^{ax}}$$

$$F(s) = \int_0^{\infty} (1) e^{-st} dt$$

$$F(s) = \left. \frac{e^{-st}}{-s} \right]_0^{\infty} = 0 - \left(\frac{e^{-0}}{-s} \right) = \frac{1}{s}$$

Laplace transform of exponential function

For step negative exponential function

$$f(t) = e^{-\alpha t}$$

$$\Rightarrow F(s) = \int_0^{\infty} e^{-\alpha t} e^{-st} dt = \int_0^{\infty} e^{-(\alpha+s)t} dt$$

Since

$$\boxed{\int e^{ax} dx = \frac{1}{a} e^{ax}}$$

$$= \frac{e^{-(s+\alpha)t}}{-(s+\alpha)} \Bigg]_0^{\infty} = \frac{e^{-\infty}}{-(s+\alpha)} - \frac{e^{-0}}{-(s+\alpha)}$$

$$= \frac{0}{-(s+\alpha)} - \frac{1}{-(s+\alpha)} \qquad = \frac{1}{s+\alpha}$$

Laplace transform of sine function

For sin function

$$f(t) = \sin(t)$$

$$\Rightarrow F(s) = \int_0^\infty e^{-st}\sin(t)\,dt$$

Using integration by parts

$$\boxed{\int \mathrm{udv} = \mathrm{uv} - \int \mathrm{vdu}}$$

We let

$$u = e^{-st} \Rightarrow du = -se^{-st}dt$$

$$dv = \sin(t)dt \Rightarrow v = -\cos(t)$$

$$\Rightarrow \int_0^\infty e^{-st}\sin(t)\,dt = [-e^{-st}\cos(t)]_0^\infty - s\int_0^\infty e^{-st}\cos(t)\,dt$$

$$= \left[\left(-e^{-\infty}\right) - \left(-e^{0}(1)\right)\right] - s\int_0^\infty e^{-st}\cos(t)\,dt \;\; = 1 - s\int_0^\infty e^{-st}\cos(t)\,dt$$

Laplace transform of sine function

So now we have

$$\int_0^{\infty} e^{-st}\sin(t)\,dt = 1 - s\int_0^{\infty} e^{-st}\cos(t)\,dt$$

Now dealing with right hand term

Using integration by parts again

$$\int_0^{\infty} e^{-st}\cos(t)\,dt$$

$$\boxed{\int \mathrm{udv} = \mathrm{uv} - \int \mathrm{vdu}}$$

We let

$$u = e^{-st} \Rightarrow du = -se^{-st}dt$$

$$dv = \cos(t)dt \Rightarrow v = \sin(t)$$

$$\Rightarrow \int_0^{\infty} e^{-st}\cos(t)\,dt = [e^{-st}\sin(t)]_0^{\infty} + s\int_0^{\infty} e^{-st}\sin(t)\,dt$$

$$= \left[0 - e^{-0}(0)\right] + s\int_0^{\infty} e^{-st}\sin(t)\,dt \quad = s\int_0^{\infty} e^{-st}\sin(t)\,dt$$

Laplace transform of sine function

So from

$$\int_0^\infty e^{-st}\sin(t)\,dt = 1 - s\int_0^\infty e^{-st}\cos(t)\,dt$$

Substituting in

$$\int_0^\infty e^{-st}\cos(t)\,dt \qquad = s\int_0^\infty e^{-st}\sin(t)\,dt$$

$$\Rightarrow \int_0^\infty e^{-st}\sin(t)\,dt = 1 - s^2\int_0^\infty e^{-st}\sin(t)\,dt$$

$$\Rightarrow (1+s^2)\int_0^\infty e^{-st}\sin(t)\,dt = 1$$

$$\Rightarrow \int_0^\infty e^{-st}\sin(t)\,dt = \frac{1}{(1+s^2)}$$

As we can see doing analytical solutions to get Laplace transforms of functions starts to get tricky! So its better to use tables!

Common useful Laplace transforms

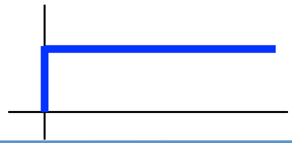
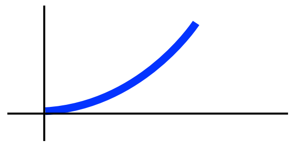
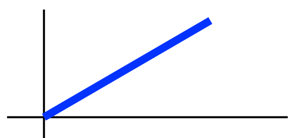
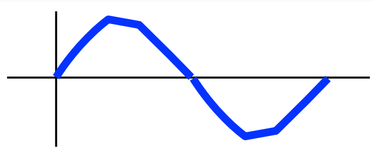
Delta function		$f(t)=\delta(t) \Leftrightarrow F(s)=1$
Unit step		$f(t)=u(t) \Leftrightarrow F(s)=\dfrac{1}{s}$
Rising exponential		$f(t)=e^{at} \Leftrightarrow F(s)=\dfrac{1}{s-a}$
Decaying exponential		$f(t)=e^{-at} \Leftrightarrow F(s)=\dfrac{1}{s+a}$
Ramp		$f(t)=t \Leftrightarrow F(s)=\dfrac{1}{s^2}$
Sine		$f(t)=\sin(at) \Leftrightarrow F(s)=\dfrac{1}{s^2+a^2}$
Cosine		$f(t)=\cos(at) \Leftrightarrow F(s)=\dfrac{s}{s^2+a^2}$

Table of Laplace transform pairs

	$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$		$f(t)=\mathfrak{L}^{-1}\{F(s)\}$	$F(s)=\mathfrak{L}\{f(t)\}$
1.	1	$\frac{1}{s}$	2.	$\mathbf{e}^{at}$	$\frac{1}{s-a}$
3.	$t^n,\quad n=1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$	4.	$t^p, p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6.	$t^{n-\frac{1}{2}},\quad n=1,2,3,\ldots$	$\frac{1\cdot 3\cdot 5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7.	$\sin(at)$	$\frac{a}{s^2+a^2}$	8.	$\cos(at)$	$\frac{s}{s^2+a^2}$
9.	$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$	10.	$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
11.	$\sin(at)-at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$	12.	$\sin(at)+at\cos(at)$	$\frac{2as^2}{(s^2+a^2)^2}$
13.	$\cos(at)-at\sin(at)$	$\frac{s(s^2-a^2)}{(s^2+a^2)^2}$	14.	$\cos(at)+at\sin(at)$	$\frac{s(s^2+3a^2)}{(s^2+a^2)^2}$
15.	$\sin(at+b)$	$\frac{s\sin(b)+a\cos(b)}{s^2+a^2}$	16.	$\cos(at+b)$	$\frac{s\cos(b)-a\sin(b)}{s^2+a^2}$

Table of Laplace transform pairs

17.	$\sinh(at)$	$\dfrac{a}{s^2-a^2}$	18.	$\cosh(at)$	$\dfrac{s}{s^2-a^2}$
19.	$\mathbf{e}^{at}\sin(bt)$	$\dfrac{b}{(s-a)^2+b^2}$	20.	$\mathbf{e}^{at}\cos(bt)$	$\dfrac{s-a}{(s-a)^2+b^2}$
21.	$\mathbf{e}^{at}\sinh(bt)$	$\dfrac{b}{(s-a)^2-b^2}$	22.	$\mathbf{e}^{at}\cosh(bt)$	$\dfrac{s-a}{(s-a)^2-b^2}$
23.	$t^n\mathbf{e}^{at},\quad n=1,2,3,\ldots$	$\dfrac{n!}{(s-a)^{n+1}}$	24.	$f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right)$
25.	$u_c(t)=u(t-c)$ Heaviside Function	$\dfrac{\mathbf{e}^{-cs}}{s}$	26.	$\delta(t-c)$ Dirac Delta Function	$\mathbf{e}^{-cs}$
27.	$u_c(t)f(t-c)$	$\mathbf{e}^{-cs}F(s)$	28.	$u_c(t)g(t)$	$\mathbf{e}^{-cs}\mathfrak{L}\{g(t+c)\}$
29.	$\mathbf{e}^{ct}f(t)$	$F(s-c)$	30.	$t^n f(t),\quad n=1,2,3,\ldots$	$(-1)^n F^{(n)}(s)$
31.	$\dfrac{1}{t}f(t)$	$\int_s^\infty F(u)\,du$	32.	$\int_0^t f(v)\,dv$	$\dfrac{F(s)}{s}$
33.	$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	34.	$f(t+T)=f(t)$	$\dfrac{\int_0^T \mathbf{e}^{-st}f(t)\,dt}{1-\mathbf{e}^{-sT}}$
35.	$f'(t)$	$sF(s)-f(0)$	36.	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
37.	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$			

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Inverse Laplace transforms

Inverse Laplace Transforms by identification

- The inverse transform is formed in order to determine the time response.

$$f(t) = L^{-1}\{F(s)\} = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} F(s)e^{st}\, ds$$

- For example, when a differential equation is solved by Laplace transforms, the solution is obtained as a function of the variable *s*.

- The simplest forms are those that can be recognized within the tables and a few of those will now be considered.

Example 1: Determine inverse transform

Given the Laplace transformation

$$F(s) = \frac{5}{s} + \frac{12}{s^2} + \frac{8}{s+3} = 5\left(\frac{1}{s}\right) + 12\left(\frac{1}{s^2}\right) + 8\left(\frac{1}{s+3}\right)$$

From Laplace transform pairs table we see that

$$\frac{1}{s} \Leftrightarrow 1 \qquad \frac{1}{s^2} \Leftrightarrow t \qquad \frac{1}{(s+a)} \Leftrightarrow e^{-at}$$

Thus the inverse Laplace transform gives the time solution

$$f(t) = 5 + 12t + 8e^{-3t}$$

Example 2: Determine inverse transform

Given the Laplace transformation

$$V(s) = \frac{200}{s^2 + 100} \qquad = 20\left(\frac{10}{s^2 + (10)^2}\right)$$

From Laplace pairs table we see that

$$\frac{a}{\left(s^2 + a^2\right)} \Leftrightarrow \sin\left(at\right)$$

Thus the inverse Laplace transform gives the time solution

$$v(t) = 20\sin(10t)$$

Example 3: Determine inverse transform

Given the Laplace transformation

$$V(s) = \frac{8s+4}{s^2+6s+13}$$

Look at denominator terms and try to write in standard form

$$\begin{aligned} s^2+6s+13 &= s^2+6s+(3)^2+13-(3)^2 \\ &= s^2+6s+9+4 \\ &= (s+3)^2+(2)^2 \end{aligned}$$

$$\Rightarrow V(s) = \frac{8s+4}{(s+3)^2+(2)^2}$$

Example 3: Determine inverse transform

Look at numerator terms and try to write in standard form

$$V(s) = \frac{8s+4}{(s+3)^2 + (2)^2}$$

$$8s+4 = \left[8\left(s+3\right)\right] + \left[4-24\right]$$

$$= \left[8\left(s+3\right)\right] + \left[10(2)\right]$$

$$\Rightarrow V(s) = \frac{8(s+3)}{(s+3)^2 + (2)^2} - \frac{10(2)}{(s+3)^2 + (2)^2}$$

Example 3: Determine inverse transform

Can now use tables to get inverse Laplace transform

$$V(s) = 8\left(\frac{(s+3)}{(s+3)^2+(2)^2}\right) - 10\left(\frac{(2)}{(s+3)^2+(2)^2}\right)$$

From Laplace pairs table we see that

$$\frac{s-a}{\left(s-a\right)^2+b^2} \Leftrightarrow e^{at}\cos\left(bt\right) \qquad \frac{b}{\left(s-a\right)^2+b^2} \Leftrightarrow e^{at}\sin\left(bt\right)$$

Thus the inverse Laplace transform gives the time solution

$$v(t) = 8e^{-3t}\cos 2t - 10e^{-3t}\sin 2t$$

Partial fraction expansions

- Often we will have transfer function is form that we cant directly relate to expressions in Laplace transform tables
- Then we can try to use partial fraction expansions

- Partial fractions are several fractions whose sum equals a given fraction
- Working with transforms requires breaking complex fractions into simpler fractions to allow use of tables of transforms

- Consider the transfer function:

$$\frac{(s+1)}{(s+2)(s+3)}$$

Example partial fraction expansion

Expand into a term for each factor in the denominator.

$$\frac{(s+1)}{(s+2)(s+3)} = \frac{A}{(s+2)} + \frac{B}{(s+3)}$$

Now multiply top and bottom on both RHS with terms so both denominators are the same

$$\frac{(s+1)}{(s+2)(s+3)} = \frac{A(s+3)}{(s+2)(s+3)} + \frac{B(s+2)}{(s+3)(s+2)}$$

Recombining RHS

$$\frac{(s+1)}{(s+2)(s+3)} = \frac{A(s+3)+B(s+2)}{(s+2)(s+3)}$$

Example partial fraction expansion

From $\dfrac{(s+1)}{(s+2)(s+3)} = \dfrac{A(s+3)+B(s+2)}{(s+2)(s+3)}$

Equate terms in s $\Rightarrow 1 = A + B$

Equate constant terms. $\Rightarrow 1 = 3A + 2B$

From 1st equation $\Rightarrow A = 1 - B$

Substituting A into 2nd equation

$$\Rightarrow 1 = 3(1-B) + 2B \;= 3 - 3B + 2B \;= 3 - B$$

$$\Rightarrow B = 2$$

$$\Rightarrow A = 1 - 2 \;= -1$$

So: B=2, A=-1

$$\Rightarrow \frac{(s+1)}{(s+2)(s+3)} = \frac{-1}{(s+2)} + \frac{2}{(s+3)}$$

Each term is now in a form so that inverse Laplace transforms can be applied.

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Lecture 3

Laplace procedure for Solving DEs

Laplace transform of 1st time differential

Laplace transform of 1st order differential

$$L\left[\frac{d}{dt}f(t)\right] = \int_0^\infty \frac{d}{dt}f(t)e^{-st}\,dt$$

Using integration by parts

$$\boxed{\int \mathrm{udv} = \mathrm{uv} - \int \mathrm{vdu}}$$

We let

$$u = e^{-st} \Rightarrow du = -se^{-st}$$

and

$$dv = \frac{d}{dt}f(t)dt \Rightarrow v = f(t)$$

Laplace transform of 1st time differential

From

$$u = e^{-st} \Rightarrow du = -se^{-st} \quad \text{and} \quad dv = \frac{d}{dt} f(t)dt \Rightarrow v = f(t)$$

$$\boxed{\int \mathrm{udv} = \mathrm{uv} - \int \mathrm{vdu}}$$

$$\Rightarrow \int_0^\infty \frac{d}{dt} f(t)e^{-st}\,dt \quad = [e^{-st} f(t)]_0^\infty + s\int_0^\infty f(t)e^{-st}\,dt$$

$$= [e^{-\infty} - e^{-0} f(0)] + s\int_0^\infty f(t)e^{-st}\,dt$$

Since

$$\int_0^\infty f(t)e^{-st}\,dt = F(s)$$

$$\Rightarrow L\left[\frac{d}{dt} f(t)\right] = -f(0^+) + sF(s)$$

Where f(0) is the function's initial value at t=0

Laplace transform of time integral

Laplace transform of integral

$$L\left[\int f(t)\,dt\right] \quad = \int_0^\infty g(t)e^{-st}\,dt$$

where

$$g(t) = \int_0^\infty f(t)\,dt$$

Using integration by parts

$$\boxed{\int \mathrm{udv} = \mathrm{uv} - \int \mathrm{vdu}}$$

We let

$$u = g(t) \Rightarrow du = f(t)dt$$

and

$$dv = e^{-st}dt \Rightarrow v = -\frac{1}{s}e^{-st}$$

Laplace transform of time integral

We let

$$u = g(t) \Rightarrow du = f(t)dt \quad \text{and} \quad dv = e^{-st}dt \Rightarrow v = -\frac{1}{s}e^{-st}$$

$$\boxed{\int \mathrm{udv} = \mathrm{uv} - \int \mathrm{vdu}}$$

$$\Rightarrow \int_0^\infty g(t)e^{-st}\,dt = [-\frac{1}{s}g(t)e^{-st}]_0^\infty + \frac{1}{s}\int f(t)e^{-st}dt$$

$$= \frac{1}{s}[e^{-\infty} - e^{-0}g(0)] + \frac{1}{s}\int_0^\infty f(t)e^{-st}\,dt = \frac{1}{s}[0 - g(0)] + \frac{1}{s}\int_0^\infty f(t)e^{-st}\,dt$$

Since

$$g(t) = \int_0^t f(t)dt \Rightarrow g(0) = 0 \quad \text{and} \quad \int_0^\infty f(t)e^{-st}\,dt = F(s)$$

$$\Rightarrow L\left[\int f(t)\,dt\right] = \frac{1}{s}F(s)$$

Summary of operations for solving differential equations

Laplace transform of 1st order differential

$$L\left[\frac{d}{dt}f(t)\right]=sF(s)-f(0)$$

Where f(0) is the function's initial value at t=0

Laplace transform of 2nd order differential

$$L\left[\frac{d^2}{dt^2}f(t)\right]=s^2F(s)-sf(0)-f'(0)$$

Where f(0) is the function's value at t=0 and f'(0) is the1st derivative value value at t=0

Laplace transform of integral

$$L\left[\int_0^t f(t)\,dt\right]=\frac{F(s)}{s}$$

General Laplace procedure for Solving DEs

Consider a 2nd order differential equation

$$b_2 \frac{d^2 y}{dt^2} + b_1 \frac{dy}{dt} + b_0 y = f(t)$$

Taking Laplace transforms and accounting for initial conditions

$$L\left[b_2 \frac{d^2 y}{dt^2} + b_1 \frac{dy}{dt} + b_0 y \right] = L\left[f(t) \right]$$

$$b_2 \left[s^2 Y(s) - s y(0) - y'(0) \right] + b_1 \left[s Y(s) - y(0) \right] + b_0 Y(s) = F(s)$$

General Laplace procedure for Solving DEs

From the equation

$$b_2\left[s^2Y(s)-sy(0)-y'(0)\right]+b_1\left[sY(s)-y(0)\right]+b_0Y(s)=F(s)$$

Grouping Y(s) terms

$$Y(s)\left(b_2s^2+b_1s+b_0\right)-\left(sb_2y(0)+b_2y'(0)+b_1y(0)\right)=F(s)$$

Response Y(s) then given by

$$Y(s)=\frac{\overbrace{F(s)}^{\text{Driving function F(s)}}}{\left(b_2s^2+b_1s+b_0\right)}+\frac{\overbrace{\left(sb_2y(0)+b_2y'(0)+b_1y(0)\right)}^{\text{Initial conditions terms}}}{\left(b_2s^2+b_1s+b_0\right)}$$

Example 1: Laplace solution of an ODE

ODE with zero initial conditions

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = 2 \quad y(0) = y'(0) = 0$$

Apply Laplace transform to each term

$$s^2 Y(s) + 6sY(s) + 8Y(s) = \frac{2}{s}$$

Laplace transform of step

Solve for Y(s)

$$Y(s) = \frac{2}{s(s+2)(s+4)}$$

Apply partial fraction expansion

$$\frac{2}{s(s+2)(s+4)} = \frac{A}{s} + \frac{B}{(s+2)} + \frac{C}{(s+4)}$$

Example 1: Laplace solution of an ODE

From $$\frac{2}{s(s+2)(s+4)}=\frac{A}{s}+\frac{B}{(s+2)}+\frac{C}{(s+4)}$$

Recombining RHS

$$\frac{2}{s(s+2)(s+4)}=\frac{A(s+2)(s+4)}{s(s+2)(s+4)}+\frac{Bs\,(s+4)}{(s+2)s\,(s+4)}+\frac{Cs(s+2)}{(s+4)s(s+2)}$$

$$\Rightarrow 2=A(s+2)(s+4)+Bs\,(s+4)+Cs(s+2)$$

$$\Rightarrow 2=As^2+6As+8A+Bs^2+4Bs+Cs^2+2Cs$$

$$\Rightarrow 2=(A+B+C)s^2+(6A+4B+2C)s+8A$$

Equating terms

$$\Rightarrow 2=8A$$

$$\Rightarrow 0=A+B+C$$

$$\Rightarrow 0=6A+4B+2C$$

Example 1: Laplace solution of an ODE

From the first equation

$$2 = 8A \quad \Rightarrow A = \frac{1}{4}$$

From the second equation

$$0 = A + B + C \qquad \Rightarrow -\left(\frac{1}{4} + B\right) = C$$

From the third equation

$$0 = 6A + 4B + 2C \;\Rightarrow 0 = \frac{6}{4} + 4B - 2\left(\frac{1}{4} + B\right) \;= \frac{6}{4} + 4B - \frac{2}{4} - 2B \quad = 1 + 2B$$

$$\Rightarrow -1 = 2B \Rightarrow B = -\frac{1}{2}$$

Putting this back into the second equation

$$C = -\left(\frac{1}{4} + B\right) \;= -\left(\frac{1}{4} - \frac{1}{2}\right) \quad = \frac{1}{4}$$

Example 1: Laplace solution of an ODE

This gives the final partial fraction representation

$$Y(s) = \frac{1}{4}\left(\frac{1}{s}\right) - \frac{1}{2}\left(\frac{1}{(s+2)}\right) + \frac{1}{4}\left(\frac{1}{(s+4)}\right)$$

From Laplace pairs table we see that

$$\frac{1}{s} \Leftrightarrow 1 \qquad\qquad \frac{1}{\left(s+a\right)} \Leftrightarrow e^{-at}$$

Thus the inverse Laplace transform gives the time solution

$$y(t) = \frac{1}{4} - \frac{e^{-2t}}{2} + \frac{e^{-4t}}{4}$$

Example 2: Laplace solution of an ODE

Consider 1st order differential equation with non-zero Initial conditions

$$\frac{dy}{dt} + 2y = 12 \qquad y(0) = 10$$

Taking Laplace transforms and accounting for initial conditions

$$L\left[\frac{dy}{dt}\right] + 2L\left[y\right] = L\left[12\right]$$

Laplace transform of step

$$\Rightarrow \left[sY(s) - y(0)\right] + 2\left[Y(s)\right] = \left[\frac{12}{s}\right]$$

$$\Rightarrow sY(s) - 10 + 2Y(s) = \frac{12}{s}$$

$$\Rightarrow Y(s)\left(s+2\right) = \frac{12}{s} + 10 \qquad \Rightarrow Y(s) = \frac{10}{\left(s+2\right)} + \frac{12}{s(s+2)}$$

Example 2: Laplace solution of an ODE

From

$$Y(s) = \frac{10}{\left(s+2\right)} + \frac{12}{s(s+2)}$$

Simplifying second term using partial fractions

$$\frac{12}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2} \quad \Rightarrow \frac{12}{s(s+2)} = \frac{A(s+2)}{s(s+2)} + \frac{sB}{s(s+2)}$$

$$\Rightarrow 12 = A(s+2) + sB = s(A+B) + 2A$$

Equating terms

$$\Rightarrow 12 = 2A \qquad (A+B) = 0$$

$$\Rightarrow A = 6, B = -6$$

Example 2: Laplace solution of an ODE

Therefore

$$Y(s) = \frac{10}{(s+2)} + \frac{6}{s} - \frac{6}{(s+2)} \qquad = 6\left(\frac{1}{s}\right) + 4\left(\frac{1}{(s+2)}\right)$$

From Laplace pairs table we see that

$$\frac{1}{s} \Leftrightarrow 1 \qquad\qquad \frac{1}{(s+a)} \Leftrightarrow e^{-at}$$

Thus the inverse Laplace transform gives the time solution

$$y(t) = 6 + 4e^{-2t}$$

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Lecture 3

Laplace initial and final value theorems

Initial value theorem

Initial value theorem gives the initial value of f(t) when t is zero

If the function f(t) and its first derivative are Laplace transformable and f(t) has the Laplace transform F(s) and

$$\lim_{s \to \infty} sF(s) \quad \text{exists}$$

Then

$$\lim_{s \to \infty} sF(s) = \lim_{t \to 0} f(t) = f(0)$$

Initial Value Theorem

Initial value theorem

Procedure:
1) Given a transfer function X(s)
First find the response F(s) of the transfer function X(s) by multiplying it by the Laplace transform of the requested driving function
E.g. if input unit step then

$$F(s) = \frac{X(s)}{s}$$

2.) Next multiply F(s) by s
3.) Take the limit of sF(s) as s goes to infinity
4.) The result is the value of f(t) when t =0

The utility of this theorem lies in not having to take the inverse of F(s) in order to find out the initial condition in the time domain. This is particularly useful in circuits and systems.

Final value theorem

Final value theorem gives the initial value of f(t) when t goes to infinity

If the function f(t) and its first derivative are Laplace transformable and f(t) has the Laplace transform F(s) and if

$$\lim_{s \to 0} sF(s)$$

exist

Then

$$\lim_{s \to 0} sF(s) = \lim_{t \to \infty} f(t) = f(\infty)$$

Final Value Theorem

Final value theorem

Procedure:

1) Given a transfer function X(s)

First find the response F(s) of the transfer function X(s) by multiplying it by the Laplace transform of the requested driving function

E.g. if input unit step then

$$F(s) = \frac{X(s)}{s}$$

2.) The multiply F(s) by s

3.) Take the limit of sF(s) as s goes to zero

4.) The result is value of f(t) when t = infinity

Again, the utility of this theorem lies in not having to take the inverse of F(s) in order to find out the final value of f(t) in the time domain. This is particularly useful in circuits and systems.

Example: Initial- and final-value

Laplace transform of the function.

$$Y(s) = \frac{2}{s(s+2)(s+4)} \quad \Rightarrow sY(s) = \frac{2}{(s+2)(s+4)}$$

final-value theorem

$$\lim_{t\to\infty}\left[f(t)\right] = \lim_{s\to 0}[sF(s)] = \frac{2}{(0+2)(0+4)} = \frac{1}{4}$$

initial-value theorem

$$\lim_{t\to 0}\left[f(t)\right] = \lim_{s\to\infty}\left[sF(s)\right] = \frac{2}{(\infty+2)(\infty+4)} = 0$$

Interlude

10 minute break

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Lecture 3

Complex numbers

Definition of an imaginary number

The square root of a negative real number is not a real number.
Thus, we introduce imaginary numbers by letting

$$i = \sqrt{-1}$$

Therefore

$$i^2 = -1 \qquad i^3 = -i \qquad i^4 = 1$$

- A complex number has a real part and an imaginary part
- Its standard form is:

Real part

$$a + bi$$

Imaginary part

Example: z = 5+4i

Rel(z} = 5

Img{z } = 4

Complex number manipulation

Each complex number of the form

$$\boldsymbol{a + bi}$$

has a conjugate of the form

$$\boldsymbol{a - bi}$$

The product of a complex number and its conjugate is a real number

$$(\boldsymbol{a + bi})(\boldsymbol{a - bi}) = \mathbf{a^2 + b^2}$$

Two complex numbers

$$a + bi \text{ and } c + di$$

are added by adding real number parts and imaginary coefficients respectively

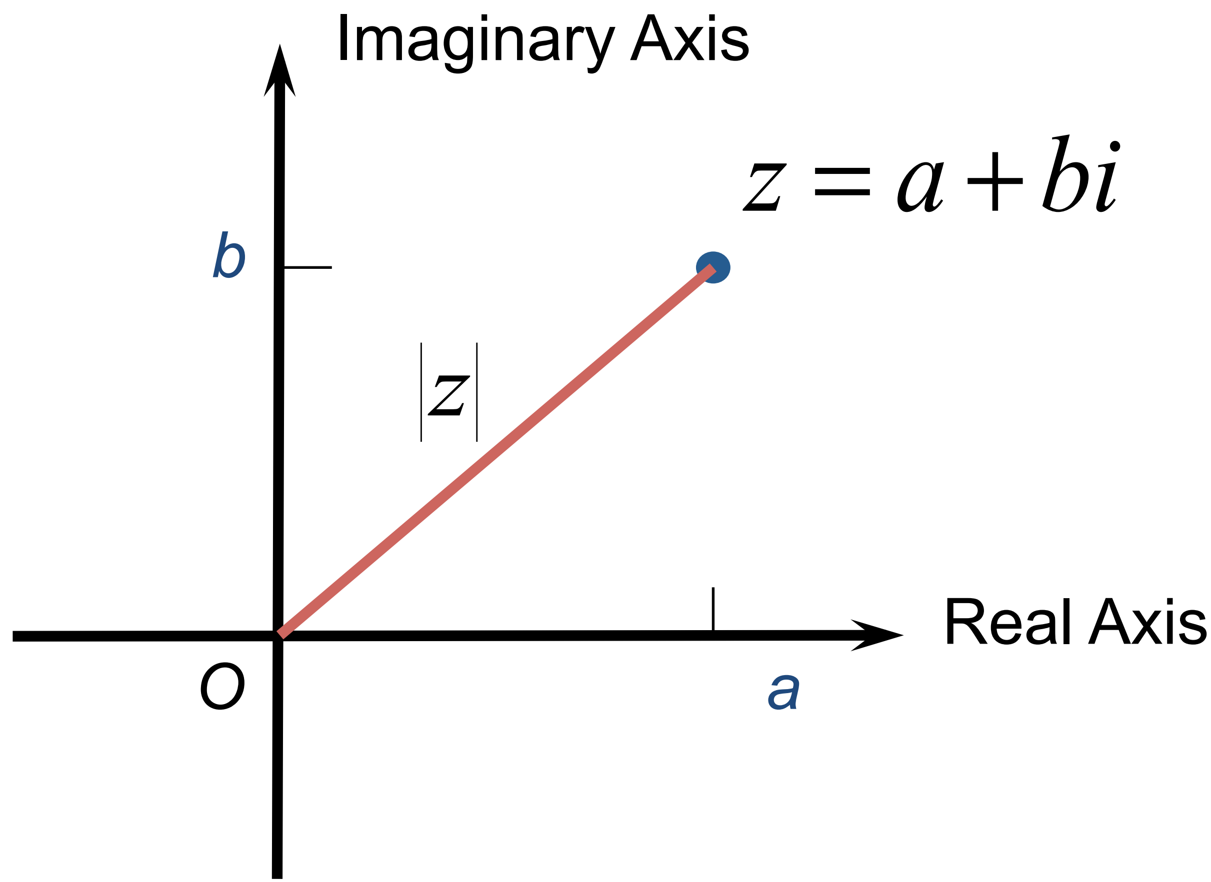
$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

are subtracted by subtracting real number parts and imaginary coefficients respectively

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

The complex plane – Argand diagrams

Just like 2 components vectors complex numbers represent both magnitude and direction



The **magnitude** or **modulus z** is defined as the distance from the origin to the point (x, y).

The modulus of z is denoted by

$$|z| = \sqrt{a^2 + b^2}$$

Converting cartesian to polar forms

Cartesian Form

$$z = x + yi = (r\cos\theta) + (r\sin\theta)i$$

Polar Form

$$z = r(\cos\theta + i\sin\theta)$$

where θ, $0 \le \theta < 2\pi$, is the **argument of z**

$$|z| = \sqrt{(r\cos\theta)^2 + (r\sin\theta)^2}$$

$$= \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} =$$

$$= \sqrt{r^2(\cos^2\theta + \sin^2\theta)}$$

$$= \sqrt{r^2} = r$$

$$\boxed{|z| = r}$$

Product and quotient theorem in polar form

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

be two complex numbers. Then

Product theorem

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

Quotient theorem

if $z_2 \neq 0$, then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

Further useful identities

Euler: $e^{ix} = \cos x + i \sin x$

$\cos x = (e^{ix} + e^{-ix})/2$

$\sin x = (e^{ix} - e^{-ix})/2i$

DeMoivre's Theorem

If $z = r(\cos\theta + i\sin\theta)$ is a complex number,

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

where $n \geq 1$ is a positive integer.

Raising a unit z to the n^{th} power is equivalent to multiplying its angle by n

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Lecture 3

Brief introduction to Fourier analysis

Frequency response methods

- The sinusoid is a unique input signal
- Resulting output signal y(t) for a linear time invariant system with sinusoidal input x(t) is sinusoidal in the steady state
- Output differs from the input waveform only in amplitude and phase angle
- Issue in frequency response methods is how to describe the amplitude and phase angle of the system

Background to Fourier analysis

- Any function that periodically repeats itself can be expressed as the sum of sines and/or cosines of different frequencies, each multiplied by a different coefficient (Fourier series).
- Even functions that are not periodic (but whose area under the curve is finite) can be expressed as the integral of sines and/or cosines multiplied by a weighting function (Fourier transform).

  - There are different ways to represent a function of signal
  - We can expand out functions
  - For example the Taylor Series

$$f(x) = \underbrace{f(a)}_{\text{constant}} + \underbrace{f'(a)(x-a)}_{\text{first-order term}} + \underbrace{\frac{f''(a)}{2!}(x-a)^2}_{\text{second-order term}} + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.$$

...

Fourier series expansion

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx),$$

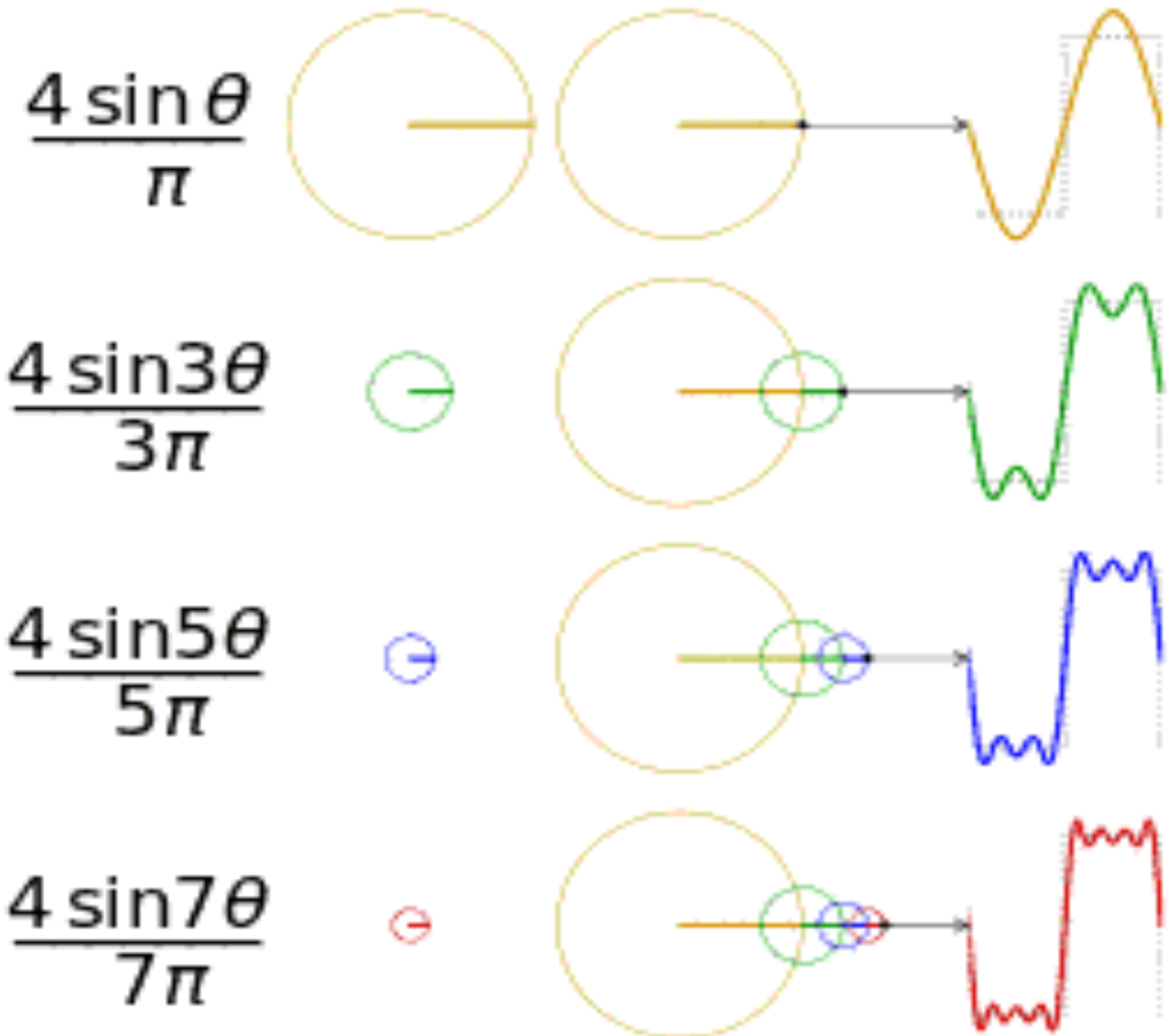
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

- Fourier series make use of the orthogonality relationships of the sine and cosine functions

Illustration of square wave approximation



Fourier Transform

- The Fourier transform is a generalization of the complex Fourier series in the limit
- Fourier analysis = frequency domain analysis
  - Low frequency: sin(nx),cos(nx) with a small n
  - High frequency: sin(nx),cos(nx) with a large n
- Note that sine and cosine waves are infinitely long – this is a shortcoming of Fourier analysis, which explains why a more advanced tool, wavelet analysis, is more appropriate for certain signals

Fourier Series in exponential form

Consider the Fourier series of the *2T* periodic function:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\frac{nx\pi}{T} + b_n \sin\frac{nx\pi}{T})$$

Using the Euler formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

It can be rewritten as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

With the decomposition coefficients calculated as:

$$c_n = \frac{1}{2T}\int_{-T}^{T} e^{-in\frac{\pi}{T}t} f(t)$$

Summary of Fourier Transform

A continuous time (CT) signal *x*(*t*) and its frequency domain, Fourier transform signal, *X*(*jw*), are related by

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt \qquad \textit{analysis}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t}\, d\omega \qquad \textit{synthesis}$$

This is denoted by:

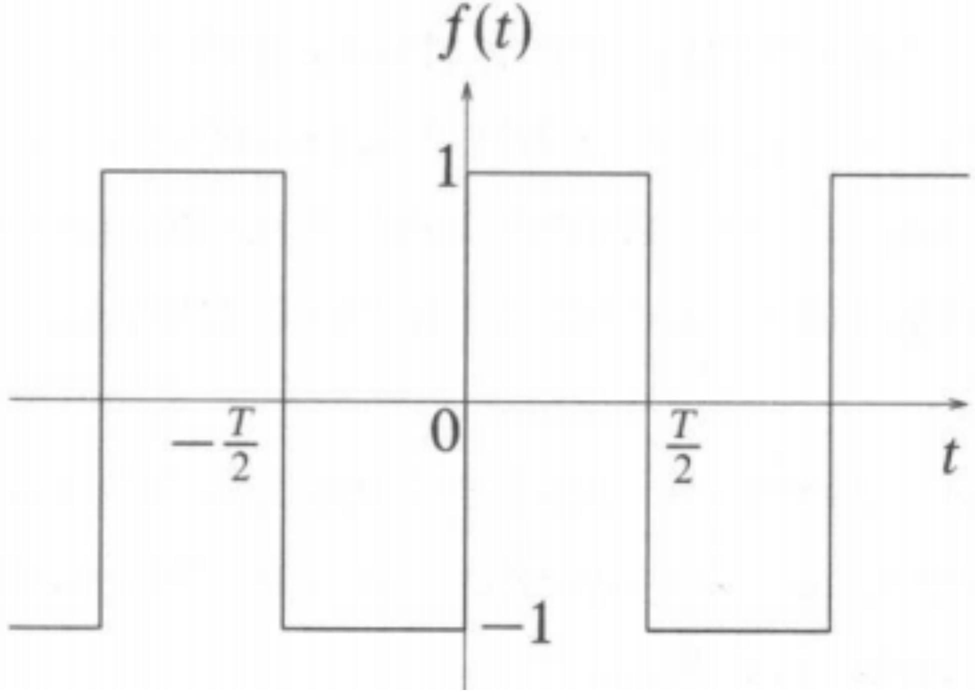
$$x(t) \overset{F}{\leftrightarrow} X(j\omega)$$

For example:

$$e^{-at}u(t) \overset{F}{\leftrightarrow} \frac{1}{a + j\omega}$$

- Where u(t) is a step signal.
- Often you have tables for common Fourier transforms
- The Fourier transform, *X*(*jw*), represents the **frequency content** of *x*(*t*).
- It exists either when *x*(*t*)->0 as |*t*|->∞ or when *x*(*t*) is periodic (it generalizes the Fourier series)

Fourier series of a square wave



The square wave may be represented by

$$f(t) = \begin{cases} -1 & \text{for } -\frac{1}{2}T \leq t < 0, \\ +1 & \text{for } 0 \leq t < \frac{1}{2}T. \end{cases}$$

Fourier series of a square wave

Note that the function is an odd function and so the series will contain only sine terms. To evaluate the coefficients in the sine series, we use Eq. (3). Hence

$$\begin{aligned} b_r &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\left(\frac{2\pi r t}{T}\right) dt \\ &= \frac{4}{T}\int_{0}^{T/2} \sin\left(\frac{2\pi r t}{T}\right) dt \\ &= \frac{2}{\pi r}\left[1-(-1)^r\right] \end{aligned}$$

Fourier series of a square wave

Thus the sine coefficients are zero if r is even and equal to $4/\pi r$ if r is odd. hence the Fourier series for the square-wavefunction may be written as

$$f(t) = \frac{4}{\pi}\left(\sin\omega t + \frac{\sin 3\omega t}{3} + \frac{\sin 5\omega t}{5} + \cdots\right)$$

where $\omega = 2\pi/T$ is called the angular frequency.

Fourier series of a square wave

The Fourier decomposition of a square wave is given by

$$f(x) = \frac{4}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi x}{L}\right).$$

- Write a Matlab GenSin function
- Takes parameters amplitude, frequency, sampleRate and duration

```
function [signal, timeValues] = GenSin(amplitude, frequency, sampleRate, duration)
% generate a sine wave
% amplitude in arbitrary units
% frequency in Hz
% sampleRate in Hz
% duration in seconds

% get data sampling points
samples = sampleRate * duration;

% get time at each sample
timeValues = duration * (1:samples)/samples;

% get sin output = sin(2pift)
signal = amplitude * sin(2 * pi * frequency * timeValues');
```

Synthesize a square wave in Matlab

- Generate four sine wave signals H1, H3, H5 and H7 representing the first 4 Fourier coefficients.
- Use sampleRate = 1000.

```
Fo = 5;               % specify fundamental frequency(Hz)
sampleRate = 1000;    % 1000 (Hz)
duration = 0.3;       % specify duration of signals (s)

% generate sine wave H1
amplitude = 4/(pi);           % 1st harmonic (fundamental)
frequency = Fo;               % 1st harmonic (fundamental)
[H1, T1] = GenSin(amplitude, frequency, sampleRate, duration);
% generate sine wave H3
amplitude = 4/(3*pi);         % 3rd harmonic
frequency = Fo*3;             % 3rd harmonic
[H3, T3] = GenSin(amplitude, frequency, sampleRate, duration);
% generate sine wave H5
amplitude = 4/(5*pi);         % 5th harmonic
frequency = Fo*5;             % 5th harmonic
[H5, T5] = GenSin(amplitude, frequency, sampleRate, duration);
% generate sine wave H7
amplitude = 4/(7*pi);         % 7th harmonic
frequency = Fo*7;             % 57th harmonic
[H7, T7] = GenSin(amplitude, frequency, sampleRate, duration);


% add all components together
SumSignal = H1+H3+H5+H7;
```

Synthesize a square wave in Matlab

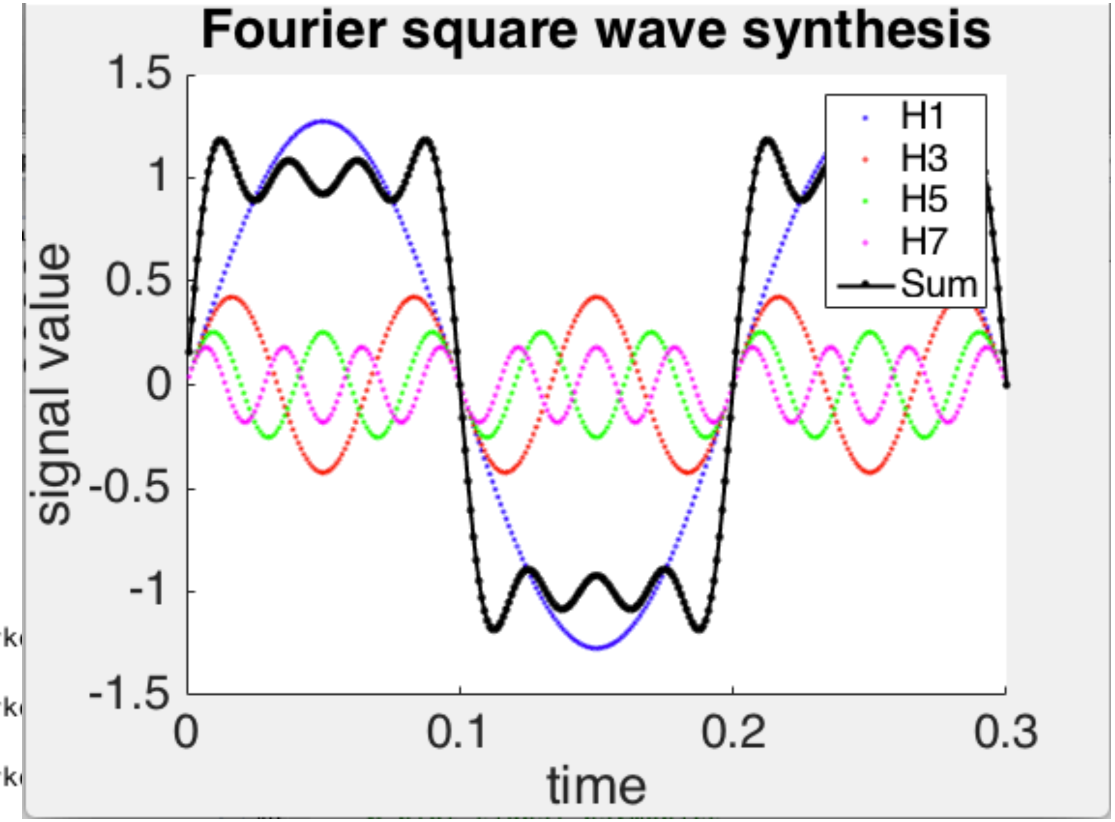
```
% generate new figure
figure
fontSize=25;
markerSizeSmall=6;
markerSizeLarge=12;
linWidthSmall=1;
linWidthLarge=2;
hold on

% write a title
h = title('Fourier square wave synthesis');
set(h, 'FontSize' , fontSize);
h = xlabel('time');
set(h, 'FontSize' , fontSize);
h = ylabel('signal value');
set(h, 'FontSize' , fontSize);

% plot signal harmonics
h = plot(T1, H1, 'b.');
set(h, 'LineWidth' , linWidthSmall, 'MarkerSize' , mark
h = plot(T3, H3, 'r.');
set(h, 'LineWidth' , linWidthSmall, 'MarkerSize' , mark
h = plot(T5, H5, 'g.');
set(h, 'LineWidth' , linWidthSmall, 'MarkerSize' , mark
h = plot(T7, H7, 'm.');
set(h, 'LineWidth' , linWidthSmall, 'MarkerSize' , markerSizeSmall);

h = plot(T7, SumSignal, 'k.-');
set(h, 'LineWidth' , linWidthLarge, 'MarkerSize' , markerSizeLarge);

% idenfity plots
h=legend('H1','H3','H5','H7','Sum' );
set(h, 'FontSize' , 20);
% make numbering on axis larger
set(gca,'fontsize',fontSize);
```

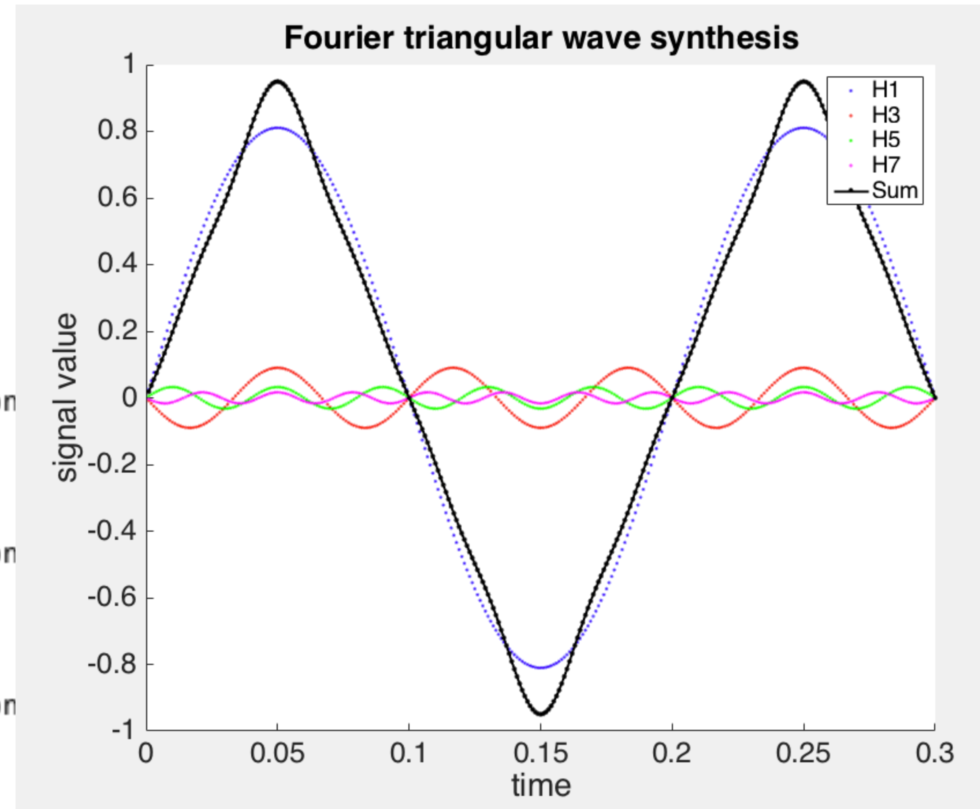


- Slowly approximates the square wave.
- We need more harmonic components to get a better approximation.

Synthesize a square wave in Matlab

The Fourier decomposition of a triangular wave is given by

$$f(x) = \frac{8}{\pi^2} \sum_{n=1,3,5,\ldots}^{\infty} \frac{(-1)^{(n-1)/2}}{n^2} \sin\left(\frac{n\pi x}{L}\right).$$




```
Fo = 5;                  % specify fundamental frequency(Hz)
sampleRate = 1000;       % 1000 (Hz)
duration = 0.3;          % specify duration of signals (s)

% generate sine wave H1
amplitude = 8/(pi^2);           % 1st harmonic (fundamental)
frequency = Fo;               % 1st harmonic (fundamental)
[H1, T1] = GenSin(amplitude, frequency, sampleRate, duration
% generate sine wave H3
amplitude = 8/(3*3*pi^2);        % 3rd harmonic
frequency = -Fo*3;            % 3rd harmonic
[H3, T3] = GenSin(amplitude, frequency, sampleRate, duration
% generate sine wave H5
amplitude = 8/(5*5*pi^2);        % 5th harmonic
frequency = Fo*5;             % 5th harmonic
[H5, T5] = GenSin(amplitude, frequency, sampleRate, duration
% generate sine wave H7
amplitude = 8/(7*7*pi^2);        % 7th harmonic
frequency = -Fo*7;            % 57th harmonic
[H7, T7] = GenSin(amplitude, frequency, sampleRate, duration);
```

Laplace versus Fourier transform

Laplace transform:

$$F(s) = L[f(t)] = \int_0^\infty f(t)e^{-st}\,dt$$

where $s = \sigma + j\omega$

Fourier transform:

$$F(j\omega) = \int_0^\infty f(t)e^{-j\omega t}\,dt$$

- Setting s=jω in F(s) yields the Fourier transform of f (t)!

- NB: Fourier Transform is often used for signals that exist for t<0)
- The frequency response in the entire right-half s- plane with s=-j∞ to s=+j∞ is the critical section of the Nyquist contour

Frequency response plots

Can substitute s = jω to get frequency response from transfer function

$$G(j\omega) = G(s)\Big|_{s=j\omega}$$

$$= |G(j\omega)|\, e^{j\phi(\omega)}$$ Exponential representation

$$= |G(j\omega)|\, \angle\phi(\omega)$$ Angle and modulus representation

$$= R(\omega) + jX(\omega)$$ Complex representation

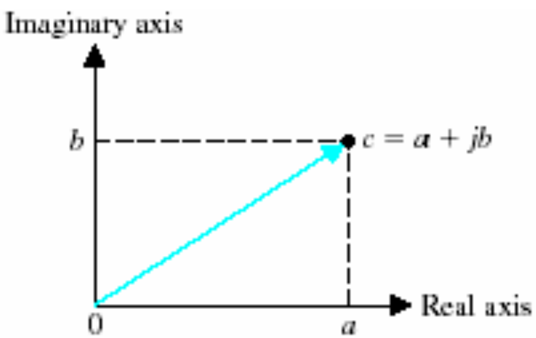


FIGURE G.1 Rectangular form of a complex number.

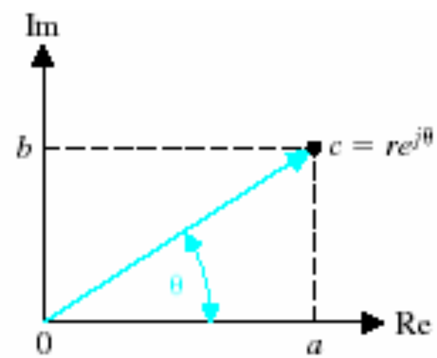
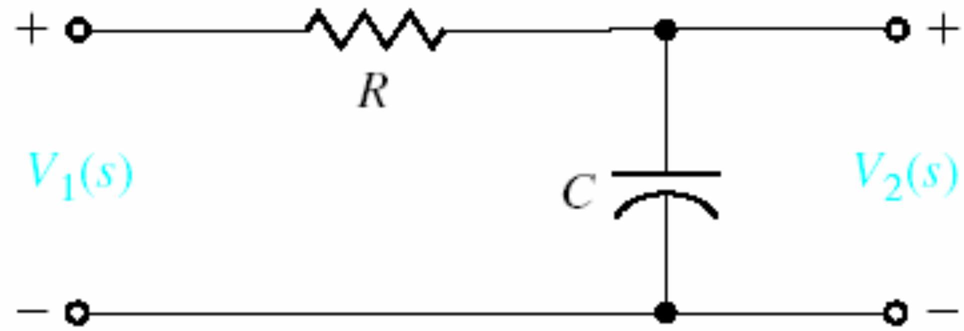


FIGURE G.2 Exponential form of a complex number.

RC network: 1st order frequency response

Consider voltages in RC network



$$v_{in} = v_R + v_C$$

$$v_R = iR$$

$$v_c = \frac{Q}{C} \quad = \frac{1}{C}\int i\,dt$$

$$v_{in} = iR + \frac{1}{C}\int i\,dt$$

Talking Laplace transforms

$$\Rightarrow V_c(s) = \frac{1}{sC} I(s) \qquad \Rightarrow V_{in}(s) = I(s)R + \frac{1}{sC} I(s)$$

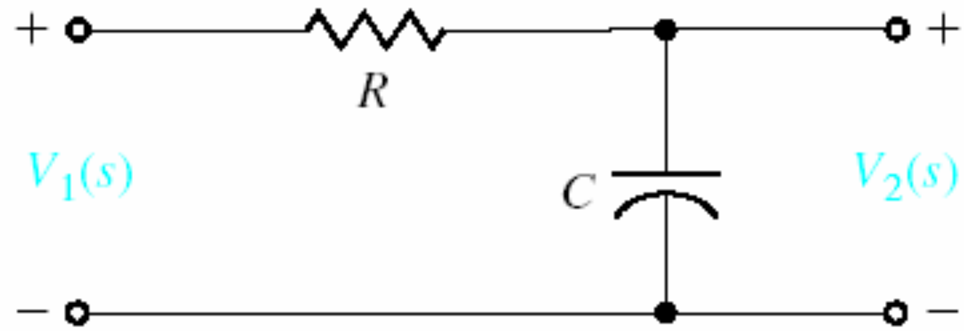
Transfer function for RC network

$$\frac{V_C(s)}{V_{in}(s)} = \frac{\frac{1}{sC} I(s)}{I(s)R + \frac{1}{sC} I(s)} \quad = \frac{1}{sRC+1} = G(s)$$

RC network: 1st order frequency response

From transfer function

$$G(s) = \frac{1}{sRC + 1}$$



Frequency response is

$$G(j\omega) = \frac{1}{j\omega RC + 1}$$

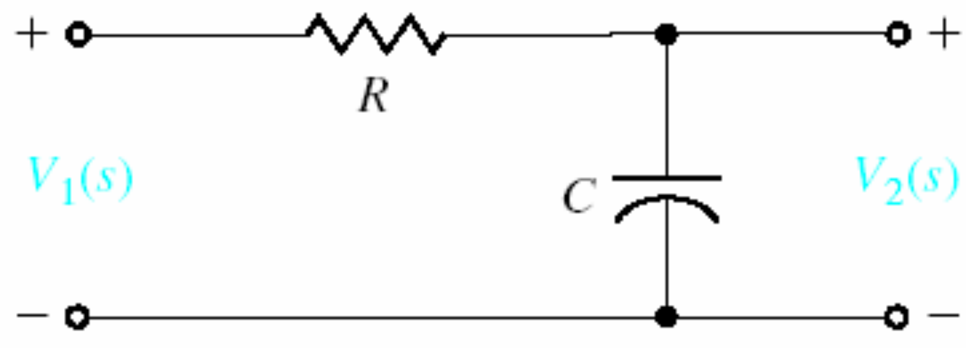
Writing $\omega_1 = \dfrac{1}{RC}$ gives

$$G(j\omega) = \frac{1}{j\omega/\omega_1 + 1}$$

NB: RC is time constant equivalent freq in Hz $\quad f_1 = \dfrac{1}{2\pi RC}$

RC network: 1st order frequency response

$$G(j\omega)=\frac{1}{j\omega/\omega_1+1}$$



Splitting into real and imaginary components

$$G(j\omega)=\frac{1-j\omega/\omega_1}{(1+j\omega/\omega_1)(1-j\omega/\omega_1)}=\frac{1-j\omega/\omega_1}{1+(\omega/\omega_1)^2}$$

$$=\left[\frac{1}{1+(\omega/\omega_1)^2}\right]-j\left[\frac{\omega/\omega_1}{1+(\omega/\omega_1)^2}\right]$$

Modulus of a complex number is given by

$$\boxed{|z|=\sqrt{a^2+b^2}}$$

$$\Rightarrow|G(jw)|^2=\frac{1+(\omega/\omega_1)^2}{\left[1+(\omega/\omega_1)^2\right]\left[1+(\omega/\omega_1)^2\right]}=\frac{1}{\left[1+(\omega/\omega_1)^2\right]}$$

$$\Rightarrow|G(jw)|=\frac{1}{\left[1+(\omega/\omega_1)^2\right]^{1/2}}$$

As $w\to\infty\Rightarrow|G(jw)|=\dfrac{1}{(\omega/\omega_1)}$

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Lecture 3

Bode plots

dB Scale

- The decibel is a logarithmic unit used to express the ratio of one value of a physical property to another value
- To express ratios of amplitudes of signals, the decibel (dB) quantity is given by the expression

$$dB = 20\log_{10}\left(\frac{signalAmp_{mes}}{signalAmp_{ref}}\right)$$

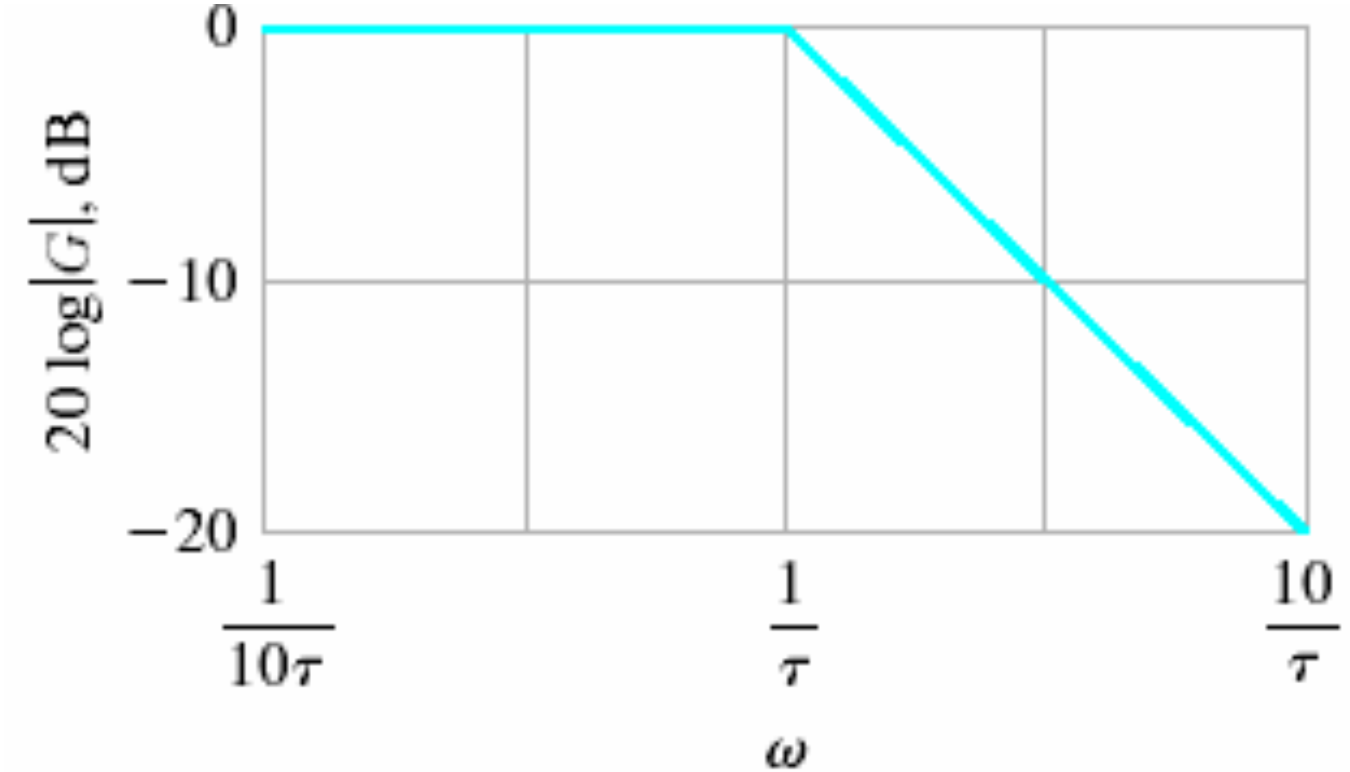
- $SignalAmp_{mes}$ is the measured signal amplitude value
- $SignalAmp_{ref}$ is a reference amplitude value (which can be 1)

- NB: to compute power ratios, a scaling factor of 10 is used instead of a scaling factor of 20

$$dB = 10\log_{10}\left(\frac{signalPow_{mes}}{signalPow_{ref}}\right)$$

- $SignalPow_{mes}$ is the measured signal power value
- $SignalPow_{ref}$ is a reference power value (which can be 1)

Bode Diagram



- PA Bode plot shows the response in dB again log frequency
- Advantage of logarithmic plot is that multiplicative factors are converted into additive ones
- We can then decompose a high order transfer function into a product of simple standard components to sketch the broad features of the Bode diagram

1st order roll-off

- A frequency increases by an octave when its value doubles

$$\Rightarrow \omega_{high} = 2\omega_{low}$$

- Consider the effect on a system with a 1st order transfer function

$$H(s) = \frac{1}{(s+a)}$$

- As frequency becomes large, the s term will dominate as the 'a' term will become insignificant. and the system behaves like an integrator

$$\Rightarrow H(s) \approx \frac{1}{(s)}$$

- So amplitude of the transfer function becomes

$$\Rightarrow |H(s)| \approx \frac{1}{\omega}$$

1^{st} order roll-off

- Therefore the dB ratio of responses when frequency increases by an octave is given by

$$dB = 20\log_{10}\left(\frac{response_{2\omega}}{response_{\omega}}\right)$$

where

$$\left(\frac{response_{2\omega}}{response_{\omega}}\right) = \frac{1/2\omega}{1/\omega} \qquad = \frac{1}{2}$$

$$\Rightarrow dB = 20\log_{10}\left(\frac{1}{2}\right) \qquad \approx -6dB$$

Thus for a 1st order system, the response falls off 6dB per octave

1st order roll-off

- For a 1st order HPF characteristic

$$\left(\frac{response_{2\omega}}{response_{\omega}}\right) = \frac{2\omega}{\omega} = 2$$

- A 1st HPF characteristic would thus give +6dB/Octave increase

- NB: change in 1st order LP system response per decade is would be 20dB

$$\left(\frac{response_{2\omega}}{response_{\omega}}\right) = \frac{1/10\omega}{1/\omega}$$

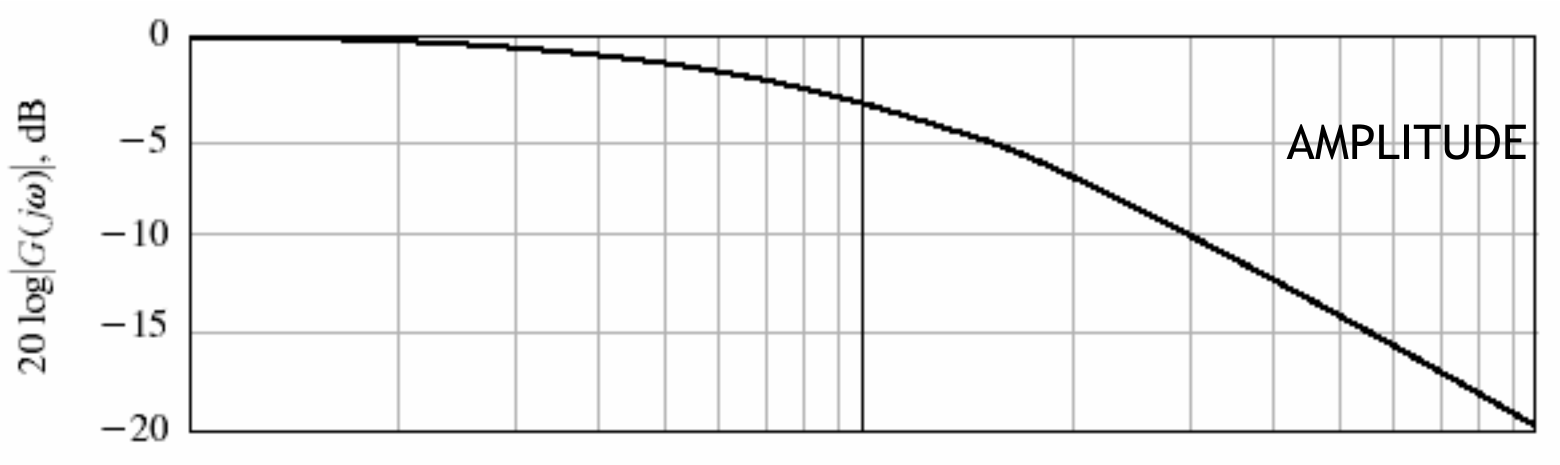
$$\Rightarrow dB = 20\log_{10}\left(\frac{1}{10}\right) \quad = -20dB$$

1st and 2nd order roll-off

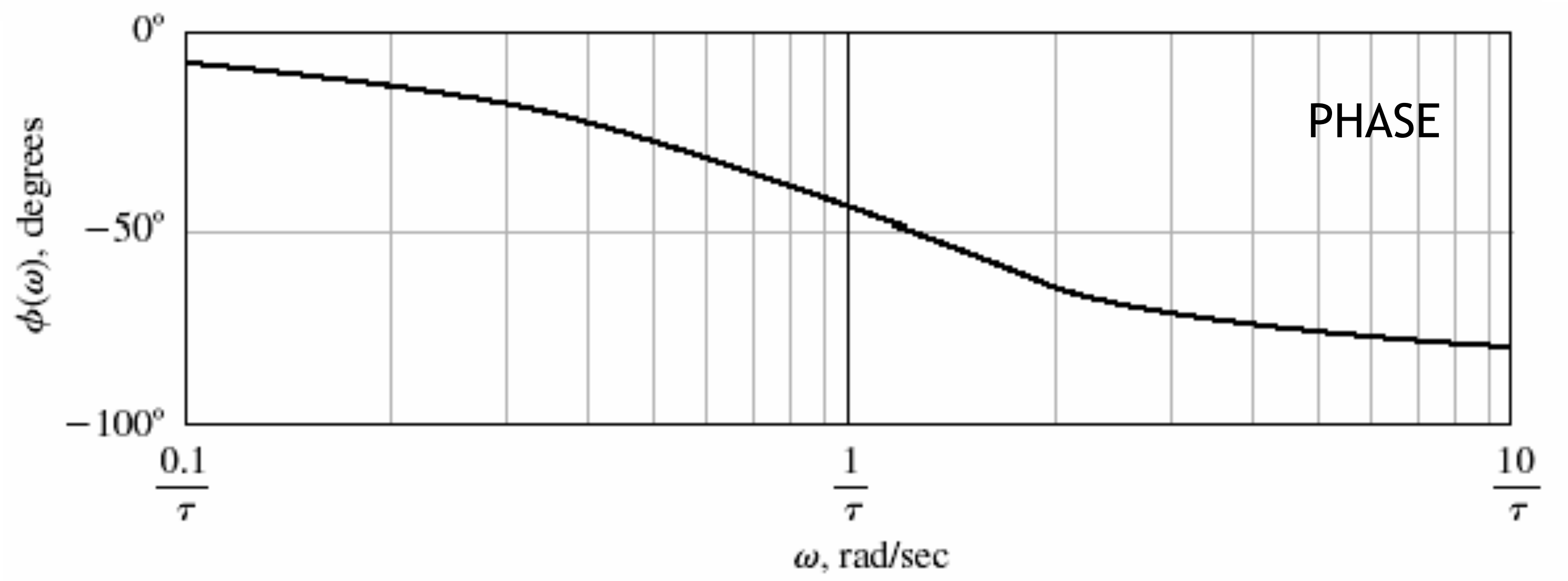
Form of transfer function		Asymptotic roll-off per octave
1st order low pass characteristic	$H_{lpf}(s) = \dfrac{1}{(s+a)}$	$=-6dB$
1st order high pass characteristic	$H_{hpf}(s) = (s+a)$	$=6dB$
2nd order low pass characteristic	$H_{lpf}(s) = \dfrac{1}{s^2+bs+c}$	$=-12dB$
2nd order high pass characteristic	$H_{hpf}(s) = s^2+bs+c$	$=12dB$

- In general, asymptotic roll-off is 6dB per octave per order

Bode Diagram



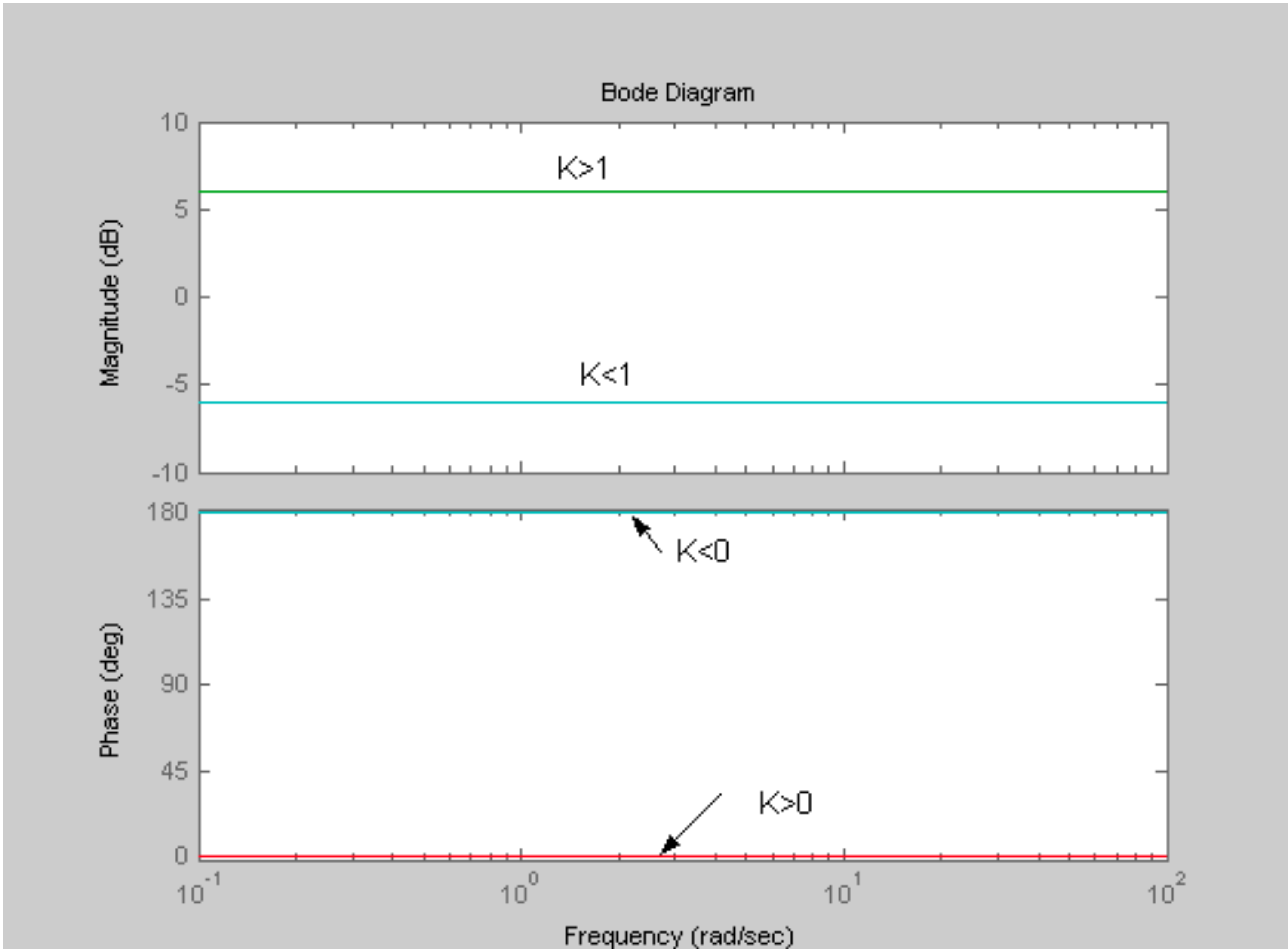
(a)



Factors in transfer function: Constant

$$K$$

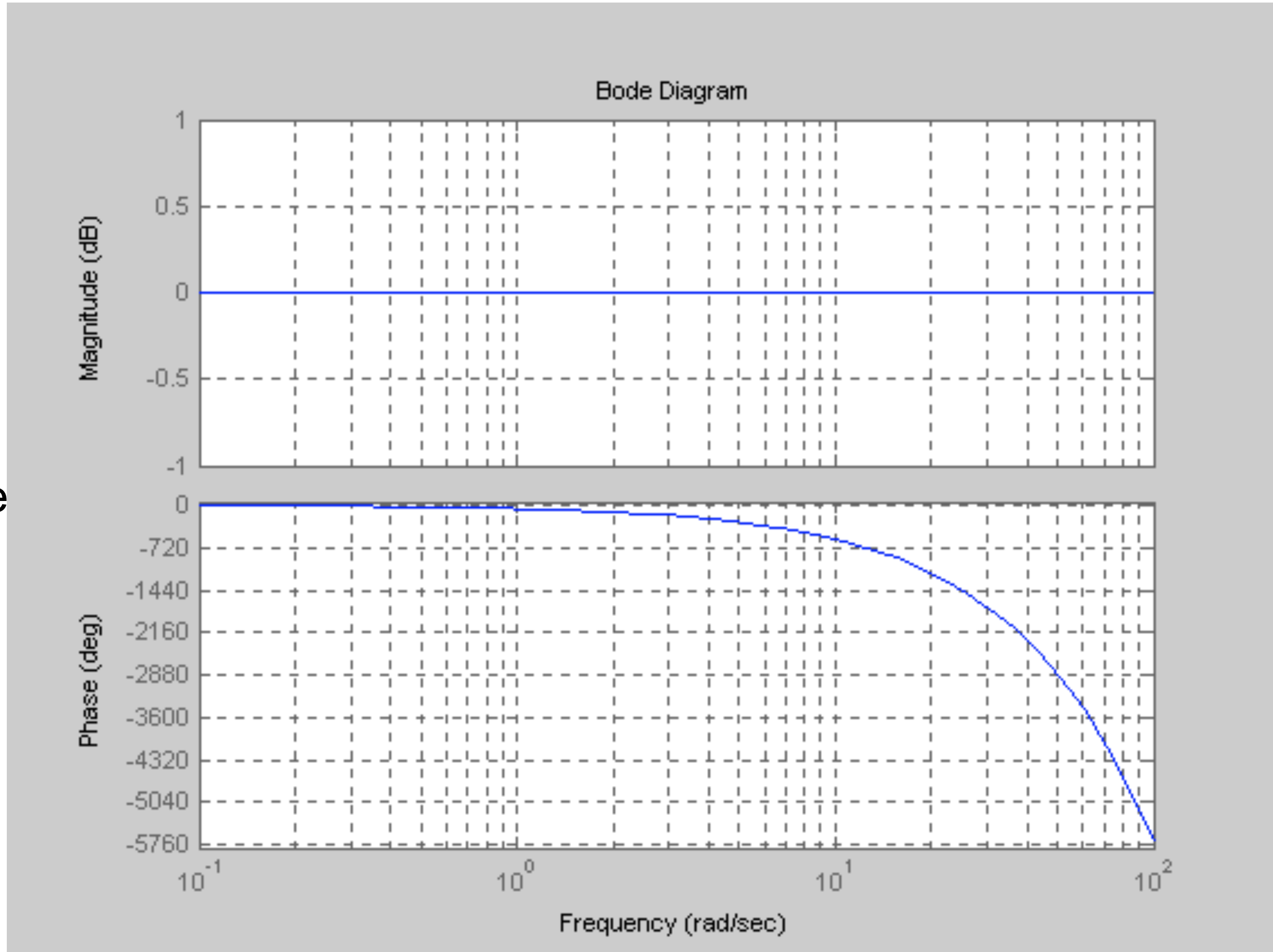
Flat amplitude
and phase responses
A direct connection



Factors in transfer function: Transport delay

$$e^{-j\omega L}$$

Flat amplitude response
Frequency dependent
phase responses



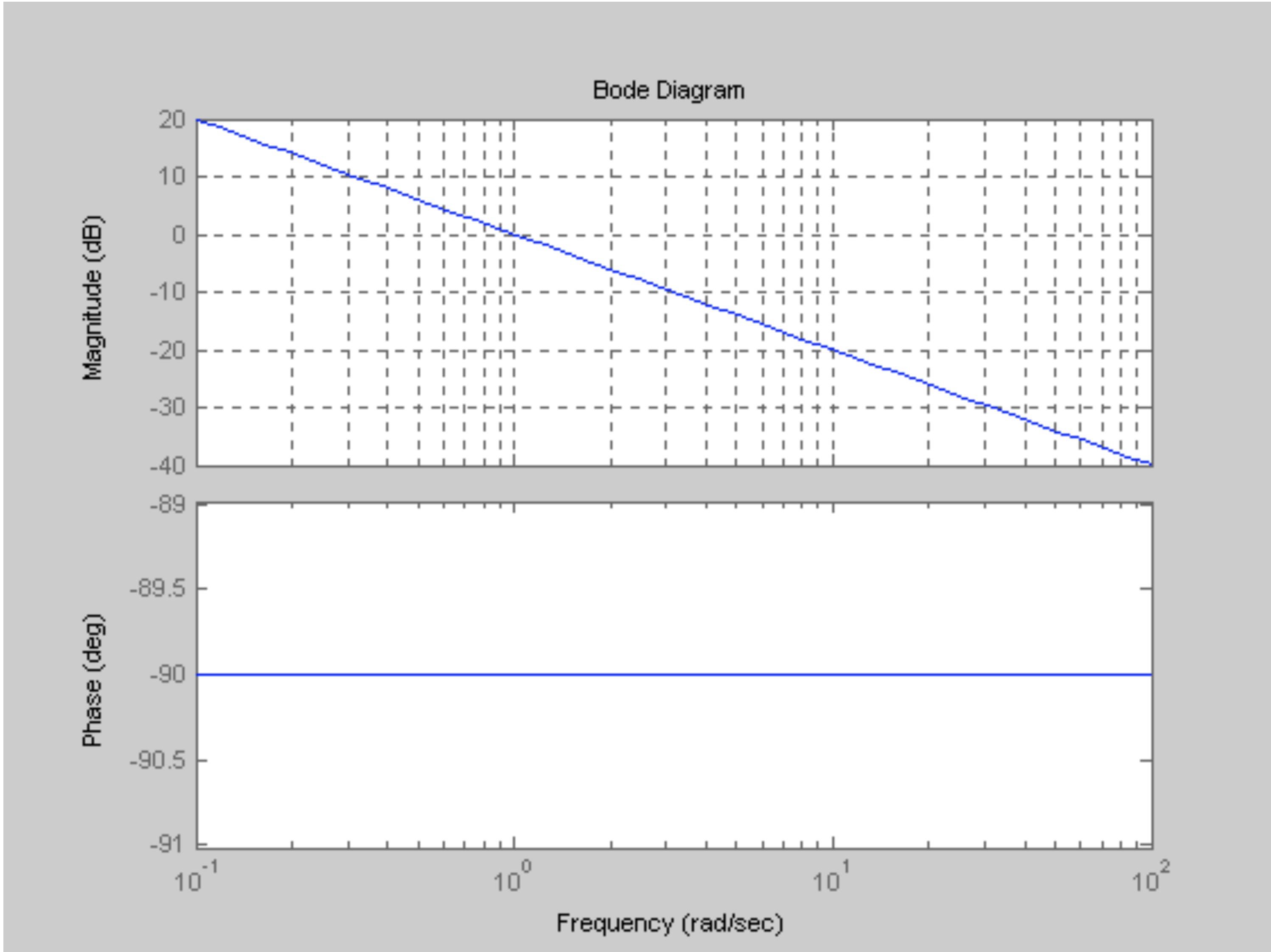
Factors in transfer function: Integrators

$$\frac{1}{(j\omega)^{y}}$$

Reduced amplitude response with frequency

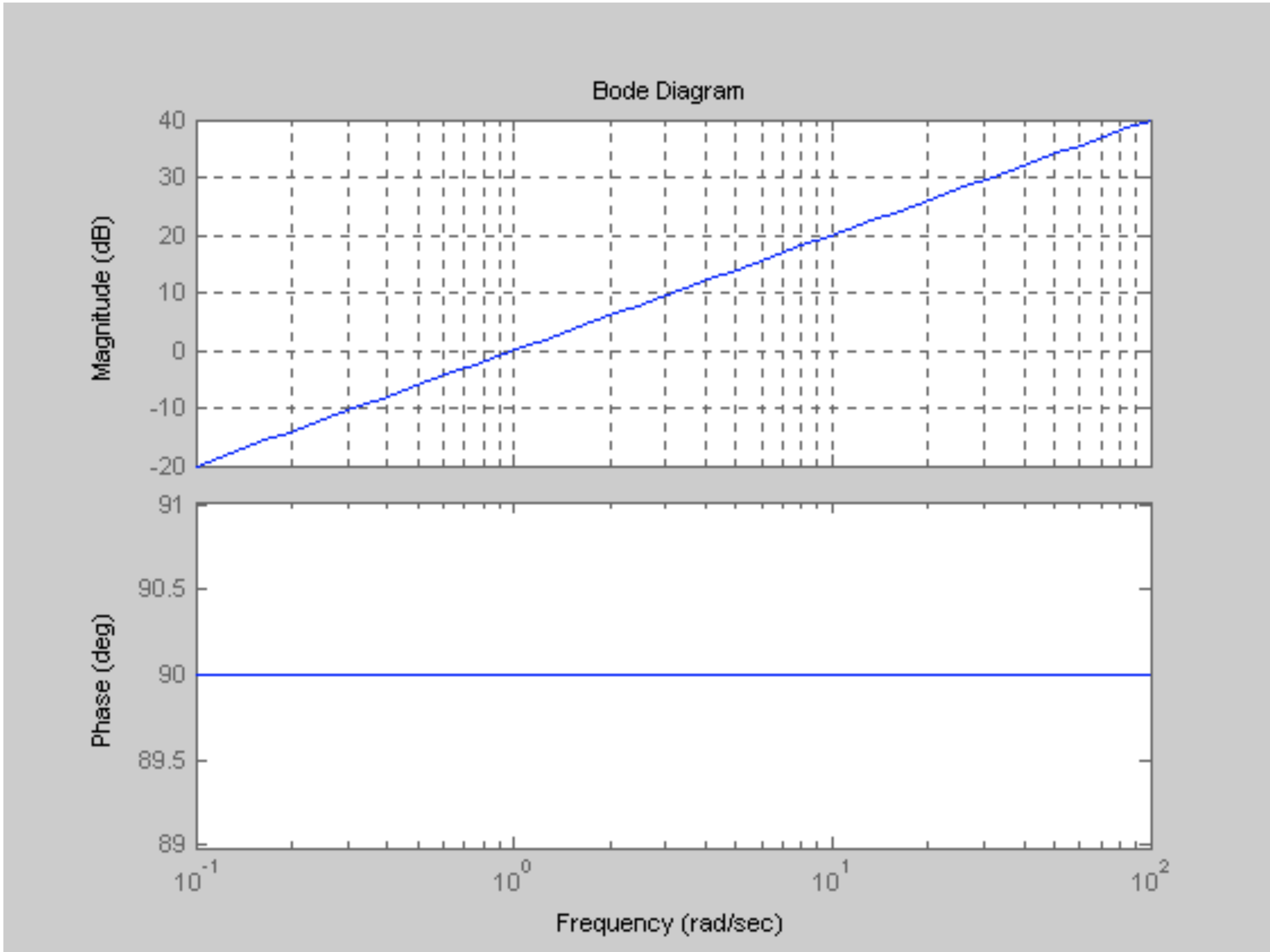
Integrators have a -90°phase responses

Output lags input



Factors in transfer function: Differentiators

$$(j\omega)^r$$

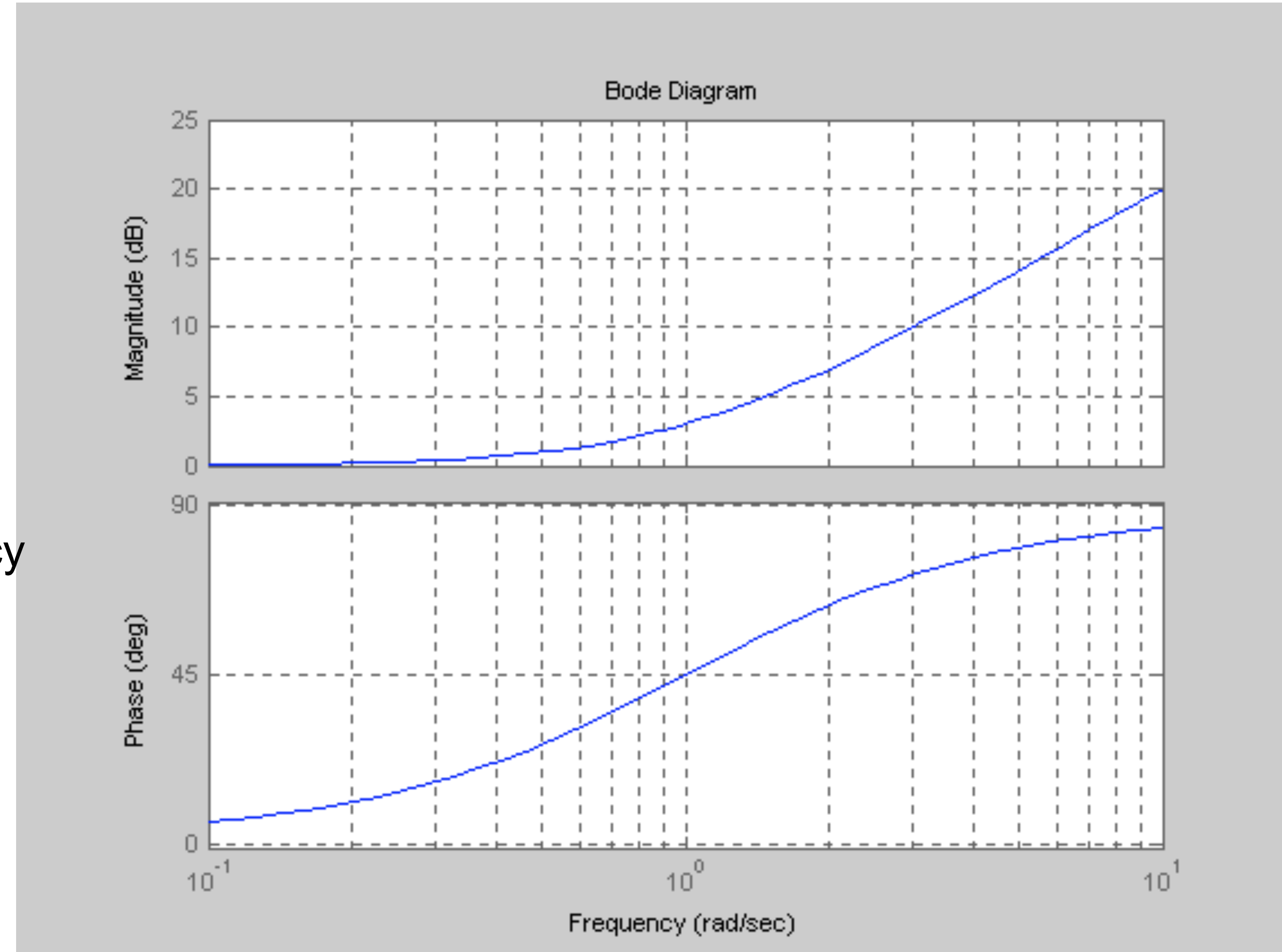


Increasing amplitude response with frequency
Differentiators have +90°phase responses
Output leads input

Factors in transfer function: First order lead terms (real zeros)

$$(1 + j\omega T_{zi})^{r}$$

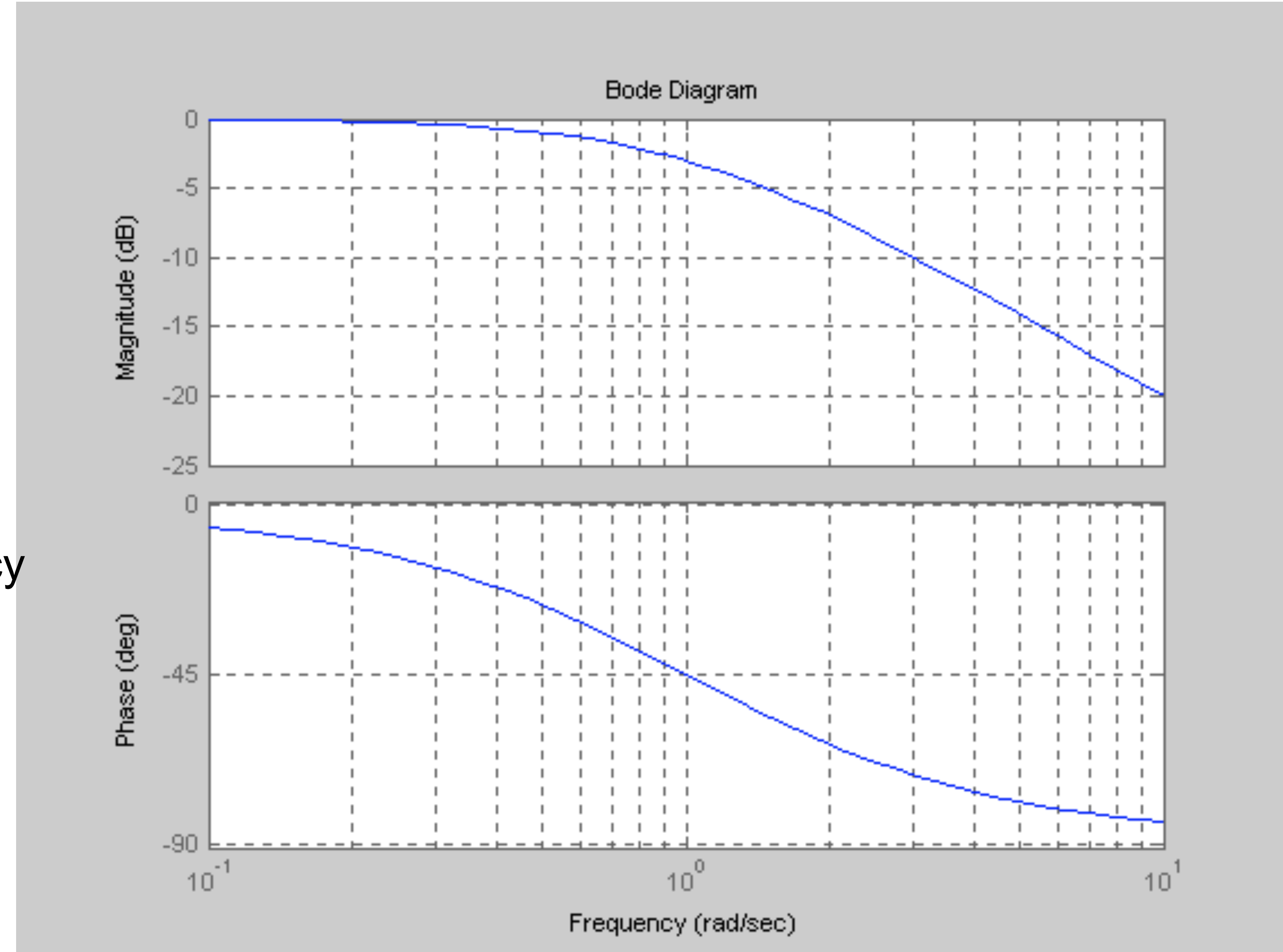
Increasing amplitude response with frequency
Frequency dependent Phase
- High pass filter



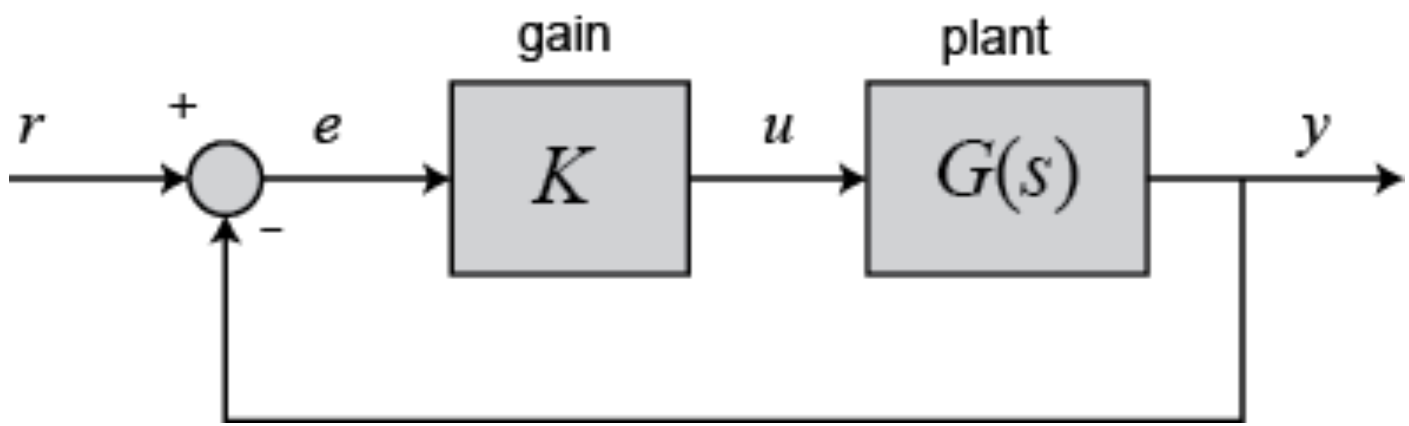
Factors in transfer function: First order lag terms (real poles)

$$\frac{1}{(1+j\omega T_{pj})^{r}}$$

Decreasing amplitude response with frequency
Frequency dependent phase
= Low pass filter



Gain and phase margins



K a variable (constant) gain
G(s) is the plant under consideration

- The gain margin is defined as the change in open-loop gain required to make the system unstable
- The phase margin is defined as the change in open-loop phase shift required to make a closed-loop system unstable
- Feedback system goes unstable when we get positive feedback!

Gain and Phase Margins on the Bode Diagram