ROCO218: Control Engineering
Dr Ian Howard

Tutorial 2

ROCO218 2016R Exam examples
Derive transfer function for electrical system

Q3: Canonical 2nd order form

Q4. The open-loop transfer function of a unity feedback closed-loop film transport system is given below

$$G(s) = \frac{K}{s(0.1s + 1)}$$

(a) Determine the damping rate ζ when K=10;

(5 marks)

(b) Determine the natural frequency ω_n when K=10;

(5 marks)

(c) What affects the percentage overshoot $\sigma\%$ response?

(5 marks)

(d) Explain the influence of the K on the response of the system.

(5 marks)

Q3: Canonical 2nd order form

The open loop function is

$$G(s) = \frac{K}{s(0.1s+1)}$$

Therefore the unity feedback closed loop transfer function is

$$CL(s) = \frac{G(s)}{1+G(s)} = \frac{\frac{K}{s(0.1s+1)}}{1+\frac{K}{s(0.1s+1)}} = \frac{K}{s(0.1s+1)+K}$$

$$\Rightarrow CL(s) = \frac{K}{0.1s^2+s+K} = \frac{10K}{s^2+10s+10K}$$

Substituting in K=10

$$\Rightarrow CL(s) = \frac{100}{s^2+10s+100}$$

Q3: Canonical 2nd order form

Comparing with canonical form

$$CL(s) = \frac{100}{s^2 + 10s + 100} \quad \Leftrightarrow \frac{\omega_n^2}{s^2 + 2\xi\omega + \omega_n^2}$$

$$\omega_n^2 \Leftrightarrow 100 \quad \Rightarrow \omega_n = 10$$

$$2\xi\omega_n \Leftrightarrow 10 \quad \Rightarrow \xi = 0.5$$

The percentage overshoot is given by

$$M_p = 100 \times e^{\frac{-\zeta\pi}{\sqrt{1-\xi^2}}} = 100 \times e^{\frac{-0.5\pi}{\sqrt{1-0.5^2}}} = 16.3\%$$

Matlab
100*exp(-0.5*pi/(sqrt(1-0.5^2)))

Peak time T_p of the step response

$$T_p = \frac{\pi}{w_n\sqrt{1-\xi^2}} = \frac{\pi}{10\sqrt{1-0.5^2}} = 0.3628\text{s}$$

Matlab
pi/(10 * sqrt(1-0.5^2))

As K increases overshoot increases but peak time is not affected

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Tutorial 2

2015 exam for ROCO316 Modern Control
Solutions to relevant questions

Q1: Continuous-time nonlinear control system

Consider the following continuous-time nonlinear control system

$$\begin{aligned} \dot{x}_1(t) &= x_1^3(t) - x_2(t) + 1 && (1) \\ \dot{x}_2(t) &= \mathrm{e}^{x_2(t)} + u(t) && (2) \\ y(t) &= -x_1(t) + u(t) && (3) \end{aligned}$$

(a) We assume a constant input $u(t) = \bar{u} = -1$, and we admit that the equilibrium state $(\bar{x}_1, \bar{x}_2)$ for this constant input is $(-1, 0)$. Determine the linearized system about this equilibrium state.

(5 marks)

(b) Write the linearized system in state-space form.

(5 marks)

(c) Determine the stability of the linearized system.

(5 marks)

(d) Determine the eigenvectors, and the solution of the linear system.

(5 marks)

Q1: Continuous-time nonlinear control system

- Solution: Substituting in value of u(t) = -1 gives the equations

$$\dot{x}_1(t) = x_1^3(t) - x_2(t) + 1$$

$$\dot{x}_2(t) = e^{x_2(t)} - 1$$

$$y(t) = -x_1(t) - 1$$

To linearize we need to calculate the Jacobian matrix and evaluate it at the given equilibrium position (-1,0).

In general the Jacobian is given by

$$J = \left(\frac{\partial f_i}{\partial x_j} \right) \Big| x = x_{equilibrium} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & & \ddots & \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

Q1: Continuous-time nonlinear control system

Here we only have a 2x2 matrix
The corresponding functions we need f_1 and f_2 are given by the two state expressions

$$f_1 = \dot{x}_1(t) = x_1^3(t) - x_2(t) + 1$$

$$f_2 = \dot{x}_2(t) = e^{x_2(t)} - 1$$

Taking partial derivatives w.r.t. to the state variables

$$\Rightarrow \frac{\partial f_1}{\partial x_1} = 3x_1^2(t) \qquad \Rightarrow \frac{\partial f_1}{\partial x_2} = -1$$

$$\Rightarrow \frac{\partial f_2}{\partial x_1} = 0 \qquad \Rightarrow \frac{\partial f_2}{\partial x_2} = e^{x_2(t)}$$

The equilibrium point is given as (-1,0) so we evaluate the Jacobian matrix at this point

$$\Rightarrow J_{|(-1,0)} = \begin{bmatrix} 3x_1^2(t) & -1 \\ 0 & e^{x_2(t)} \end{bmatrix}_{|(-1,0)} = \begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix}$$

Q1: Continuous-time nonlinear control system

(b) Write the linearized system in state-space form.

(5 marks)

From the Jacobian

$$J = \begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix}$$

we can directly write

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \dot{x}_1 = 3x_1 - x_2$$

$$\Rightarrow \dot{x}_2 = x_2$$

Q1: Continuous-time nonlinear control system

(c) Determine the stability of the linearized system.

- To determine the stability of the system we need to calculate the eigenvalues of Jacobian matrix J which is now our system matrix A

$$A = \begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix}$$

- The eigenvalues of matrix A must satisfy the equation

$$AX = \lambda X$$

Where X is the state vector $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\Rightarrow AX - \lambda X = 0$$

$$\Rightarrow (A - \lambda I)X = 0$$

Where I is the identity matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

- This equation has a non-zero solution X if and only if the determinant of the matrix (A-I)=0

$$\Rightarrow |A - \lambda I| = 0$$

Where straight bracket signifies the determinant

Q1: Continuous-time nonlinear control system

- Substituting A and I into the expression $|A-\lambda I|=0$

$$\Rightarrow \left|\begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right| = 0 \qquad \Rightarrow \left|\begin{bmatrix} 3-\lambda & -1 \\ 0 & 1-\lambda \end{bmatrix}\right| = 0$$

- For a 2x2 matrix the determinant is given by

$$\det(A) = \left|\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right| = (ad-bc)$$

$$\Rightarrow (3-\lambda)(1-\lambda)-(-1)\times 0 = 0$$

$$\Rightarrow (3-\lambda)(1-\lambda) = 0$$

$$\Rightarrow \lambda = 3, \lambda = 1$$

Note also that since the A matrix is of triangular form, its leading diagonal values indicate the eigenvalues of 3 and 1 directly. However we calculated the values for thoroughness of methodology since this will not always be the case

- Both eigenvalues are strictly positive and therefore the linearized system is unstable

Q1: Continuous-time non-linear control system

(d) Determine the eigenvectors

The eigenvalues of a matrix A satisfy the equation

$$AX = \lambda X$$

$$\Rightarrow \begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- The values of X gives the eigenvector
- When λ=1

$$\begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \begin{matrix} \Rightarrow 3x_1 - x_2 = x_1 \\ \Rightarrow x_2 = x_2 \end{matrix} \quad \Rightarrow 2x_1 = x_2 \quad \Rightarrow v_{\lambda=1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

- When λ=3

$$\begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \begin{matrix} \Rightarrow 3x_1 - x_2 = 3x_1 \\ \Rightarrow x_2 = 3x_2 \end{matrix} \quad \begin{matrix} \Rightarrow x_2 = 0 \\ \Rightarrow x_1 = x_1 \end{matrix} \quad \Rightarrow v_{\lambda=3} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

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2016R exam for ROCO319 Modern Control
Solutions to relevant questions

Q3: State space analysis

Consider a simple model for cruise control as shown in Figure (2). The model is constructed as follows. The car is a rigid body of *mass* m, onto which a force (control) u applies. The car moves with a speed v and acceleration a. The resistance of the road and due to the wind is gathered in a counter-acting force proportional to the speed of the car: bv. The equation of system is given as follows:

$$m\dot{v} + bv \quad = \quad u \tag{3}$$

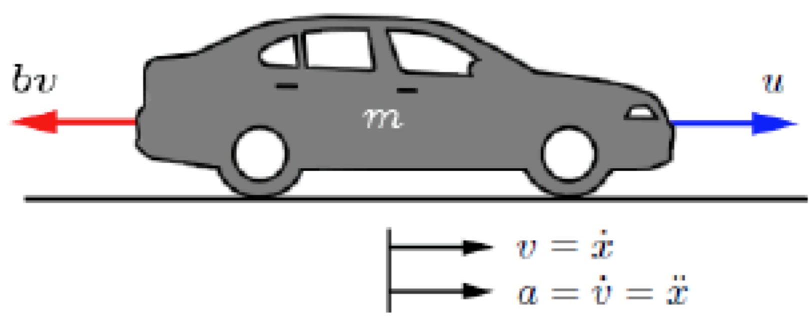

Figure 2: One-dimensional model of cruise control in a car.

(a) Assume the car weighs $m = 1000$kg and $b = 50$N sec/m. Design a state feedback $u(t) = -kv(t)$ so that the closed-loop eigenvalues are placed at -1.

(10 marks)

Q3: State space analysis

Dynamic equation is

$$m\dot{v} + bv = u$$

Which can be rewritten as

$$\dot{v} = -\frac{b}{m}v + \frac{1}{m}u$$

State space equations take the form

$$\dot{X} = AX + BU \qquad Y = CX + DU$$

Representing the single state by x, the system state space equation is

$$[\dot{x}] = -\left[\frac{b}{m}\right][x] + \left[\frac{1}{m}\right]U$$

Substituting in values

$$\Rightarrow [\dot{x}] = -\left[\frac{50}{1000}\right][x] + \left[\frac{1}{1000}\right]U$$

Therefore the system matrices are

$$\Rightarrow A = -[0.05] \qquad \Rightarrow B = [0.001]$$

Q2: State space analysis

Given system matrix A

$$A = [-0.05]$$

We went asked to, but lets calculate the open loop stability first for practice

The eigenvalues of matrix A must satisfy the equation

$$AX = \lambda X$$

Where X is the state vector

$$\Rightarrow AX - \lambda X = 0$$

$$\Rightarrow (A - \lambda I)X = 0$$

Where I is the identity matrix

This equation has a non-zero solution X if and only if the determinant of the matrix (A-I)=0

$$\Rightarrow |A - \lambda I| = 0 \qquad \Rightarrow |-0.05 - \lambda I| = 0$$

$$\Rightarrow |-0.05 - \lambda| = 0 \quad \Rightarrow (-0.05 - \lambda) = 0$$

$$\Rightarrow \lambda = -0.05$$

- For a 1x1 matrix the determinant is given by

$$\det(A) = |(a)| = a$$

The eigenvalue has a negative real value

Therefore the open loop system is stable

Q3: State space analysis

State space equations take the form

$$\dot{X} = AX + BU$$

Therefore when we apply state feedback

$$U = -KX$$

$$\dot{X} = AX - BKX \qquad \Rightarrow \dot{X} = \left(A - BK\right)X$$

Where

$$A = -[0.05]$$

$$B = [0.001]$$

The stability is now determined by location of poles which are the eigenvalue of matrix (A-BK)

The eigenvalue λ of the closed loop system are thus given by

$$\det\left(A - BK - \lambda I\right) = 0 \qquad \Rightarrow \left|-0.05 - 0.001K - \lambda\right| = 0 \quad \Rightarrow \left(-0.05 - 0.001K\right) = \lambda$$

To set the eigenvalue λ of the closed loop system to -1

$$-0.05 - 0.001K = -1 \qquad \Rightarrow -0.001K = 0.05 - 1 \qquad \Rightarrow K = \frac{0.05 - 1}{-0.001} = 950$$

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2016 exam for ROCO319 Modern Control
Solutions to relevant questions

Q1: Linearizing a non-linear system

Consider the earth and a satellite as particles as shown in Figure (1) below:

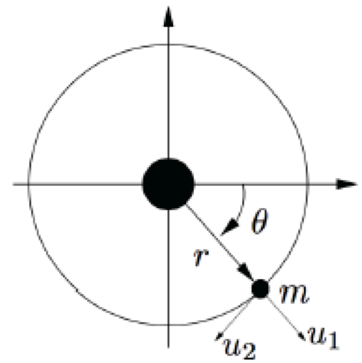

Figure 1: Satelite.

The *normalized* equations of motion can be simplified to the following two-variable model:

$$\ddot{r} = r\dot{\theta}^2 - \frac{k}{r^2} + u_1, \tag{1}$$

$$\ddot{\theta} = -2\frac{\dot{\theta}\dot{r}}{r} + \frac{1}{r}u_2, \tag{2}$$

with u_1, u_2 representing respectively the radial and tangential forces due to thrusters. The reference orbit with $u_1 = u_2 = 0$ is circular with radius, $r(t) \equiv p$, and angular velocity $\theta(t) = \omega t$. From the first equation it follows that $p^3\omega^2 = k$ (Kepler's law).

(a) Obtain the linearized equation about this orbit.

(10 marks)

(b) Write system (1)-(2) in state-space format.

(5 marks)

(c) Compute the stability of the uncontrolled system (i.e., when the control $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$).

Q1: Linearizing a non-linear system

- The differential equations that describe the system are

$$\ddot{r} = r\dot{\theta}^2 - \frac{k}{r^2} + u_1$$

$$\ddot{\theta} = -2\frac{\dot{\theta}\dot{r}}{r} + \frac{1}{r}u_2$$

  - Choosing state variables, which will be lower order than highest order terms in the equations we have

$$x_1 = \dot{r} \qquad x_2 = r \qquad x_3 = \dot{\theta} \qquad x_4 = \theta$$

$$\Rightarrow \dot{x}_2 = x_1$$

$$\ddot{r} = r\dot{\theta}^2 - \frac{k}{r^2} + u_1 \qquad \Rightarrow \dot{x}_1 = x_2 x_3^2 - \frac{k}{x_2^2} + u_1$$

$$\ddot{\theta} = -2\frac{\dot{\theta}\dot{r}}{r} + \frac{1}{r}u_2 \qquad \Rightarrow \dot{x}_3 = -2\frac{x_3 x_1}{x_2} + \frac{1}{x_2}u_2$$

Q1: Linearizing a non-linear system

So we now have 3 single order equations
So we can write the functions

$$f_1 = \dot{x}_1 = x_2 x_3^2 - \frac{k}{x_2^2} + u_1$$

$$f_2 = \dot{x}_2 = x_1$$

$$f_3 = \dot{x}_3 = -2\frac{x_3 x_1}{x_2} + \frac{1}{x_2} u_2$$

To linearize the system we need to calculate the system Jacobian

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_3} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{bmatrix}$$

- Evaluating the partial derivates w.r.t. the state variables

$$\Rightarrow \frac{\partial f_1}{\partial x_1} = 0 \qquad \Rightarrow \frac{\partial f_1}{\partial x_2} = x_3^2 + 2\frac{k}{x_2^3} \qquad \Rightarrow \frac{\partial f_1}{\partial x_3} = x_2 2 x_3$$

$$\Rightarrow \frac{\partial f_2}{\partial x_1} = 1 \qquad \Rightarrow \frac{\partial f_2}{\partial x_2} = 0 \qquad \Rightarrow \frac{\partial f_2}{\partial x_3} = 0$$

$$\Rightarrow \frac{\partial f_3}{\partial x_1} = -2\frac{x_3}{x_2} \qquad \Rightarrow \frac{\partial f_3}{\partial x_2} = 2\frac{x_1 x_3}{x_2^2} - \frac{u_2}{x_2^2} \qquad \Rightarrow \frac{\partial f_3}{\partial x_3} = -2\frac{x_1}{x_2}$$

Q1: Linearizing a non-linear system

So we now have 3 single order equations
So we can write the functions

$$f_1 = \dot{x}_1 = x_2 x_3^2 - \frac{k}{x_2^2} + u_1$$

$$f_2 = \dot{x}_2 = x_1$$

$$f_3 = \dot{x}_3 = -2\frac{x_3 x_1}{x_2} + \frac{1}{x_2} u_2$$

$$\Rightarrow J_A = \begin{bmatrix} 0 & x_3^2 + 2\frac{k}{x_2^3} & x_2 2x_3 \\ 1 & 0 & 0 \\ -2\frac{x_3}{x_2} & 2\frac{x_1 x_3}{x_2^2} - \frac{u_2}{x_2^2} & -2\frac{x_1}{x_2} \end{bmatrix}$$

- Evaluating the partial derivates w.r.t. the state variables

$$\Rightarrow \frac{\partial f_1}{\partial x_1} = 0 \qquad \Rightarrow \frac{\partial f_1}{\partial x_2} = x_3^2 + 2\frac{k}{x_2^3} \qquad \Rightarrow \frac{\partial f_1}{\partial x_3} = x_2 2x_3$$

$$\Rightarrow \frac{\partial f_2}{\partial x_1} = 1 \qquad \Rightarrow \frac{\partial f_2}{\partial x_2} = 0 \qquad \Rightarrow \frac{\partial f_2}{\partial x_3} = 0$$

$$\Rightarrow \frac{\partial f_3}{\partial x_1} = -2\frac{x_3}{x_2} \qquad \Rightarrow \frac{\partial f_3}{\partial x_2} = 2\frac{x_1 x_3}{x_2^2} - \frac{u_2}{x_2^2} \qquad \Rightarrow \frac{\partial f_3}{\partial x_3} = -2\frac{x_1}{x_2}$$

Q1: Linearizing a non-linear system

- Again using the function definitions

$$f_1 = \dot{x}_1 = x_2 x_3^2 - \frac{k}{x_2^2} + u_1$$

$$f_2 = \dot{x}_2 = x_1$$

$$f_3 = \dot{x}_3 = -2\frac{x_3 x_1}{x_2} + \frac{1}{x_2} u_2$$

- Evaluating the partial derivates w.r.t. the control variables

$$\Rightarrow \frac{\partial f_1}{\partial u_1} = 1 \qquad \Rightarrow \frac{\partial f_1}{\partial u_2} = 0$$

$$\Rightarrow \frac{\partial f_2}{\partial u_1} = 0 \qquad \Rightarrow \frac{\partial f_2}{\partial u_2} = 0$$

$$\Rightarrow \frac{\partial f_3}{\partial u_1} = 0 \qquad \Rightarrow \frac{\partial f_3}{\partial u_2} = \frac{1}{x_2}$$

To linearize the control input we need to calculate the control Jacobian

$$J_U = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \\ \frac{\partial f_3}{\partial u_1} & \frac{\partial f_3}{\partial u_2} \end{bmatrix}$$

$$\Rightarrow J_U = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & \frac{1}{x_2} \end{bmatrix}$$

Q1: Linearizing a non-linear system

We need to evaluate the Jacobians at the equilibrium corresponding to the reference orbit using the values

$u_1=0$, $u_2= 0$

$x_1=0$, $x_2=p$, $x_3=\omega$

$k=p^3\omega^2$

$$\Rightarrow J_A = \begin{bmatrix} 0 & x_3^2 + 2\dfrac{k}{x_2^3} & x_2 2x_3 \\ 1 & 0 & 0 \\ -2\dfrac{x_3}{x_2} & 2\dfrac{x_1 x_3}{x_2^2} - \dfrac{u_2}{x_2^2} & -2\dfrac{x_1}{x_2} \end{bmatrix} = \begin{bmatrix} 0 & \omega^2 + 2\dfrac{p^3\omega^2}{p^3} & p2\omega \\ 1 & 0 & 0 \\ -2\dfrac{\omega}{p} & 0 & 0 \end{bmatrix}$$

$$J_U = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & \dfrac{1}{x_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & \dfrac{1}{p} \end{bmatrix}$$

Q1: Linearizing a non-linear system

The state space equations are therefore

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & \omega^2 + 2\dfrac{p^3\omega^2}{p^3} & p2\omega \\ 1 & 0 & 0 \\ -2\dfrac{\omega}{p} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & \dfrac{1}{p} \end{bmatrix} \begin{bmatrix} u_1 \\ u_{12} \end{bmatrix}$$

To determine stability of the uncontrolled system we need to examine the eigenvalues of the system matrix A
The eigenvalues of matrix A must satisfy the equation

$$AX = \lambda X \quad \Rightarrow (A - \lambda I)X = 0 \quad \Rightarrow |A - \lambda I| = 0$$

$$\Rightarrow \left| \begin{bmatrix} 0 & \omega^2 + 2\dfrac{p^3\omega^2}{p^3} & p2\omega \\ 1 & 0 & 0 \\ -2\dfrac{\omega}{p} & 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0 \qquad \Rightarrow \left\| \begin{matrix} -\lambda & 3\omega^2 & p2\omega \\ 1 & -\lambda & 0 \\ -2\dfrac{\omega}{p} & 0 & -\lambda \end{matrix} \right\| = 0$$

Q1: Linearizing a non-linear system

The determinant of

$$\left\|\begin{bmatrix} -\lambda & 3\omega^2 & 2p\omega \\ 1 & -\lambda & 0 \\ -2\dfrac{\omega}{p} & 0 & -\lambda \end{bmatrix}\right\| = 0$$

Has the characteristic equation

3x3 matrix determinant

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$\det(A) = a * \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b * \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c * \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Finally det(A) is:

$$\det(A) = aei - afh - bdi + bfg + cdh - ceg$$

$$(-\lambda)\begin{vmatrix} -\lambda & 0 \\ 0 & -\lambda \end{vmatrix} - (3\omega^2)\begin{vmatrix} 1 & 0 \\ -2\dfrac{\omega}{p} & -\lambda \end{vmatrix} + (2p\omega)\begin{vmatrix} 1 & -\lambda \\ -2\dfrac{\omega}{p} & 0 \end{vmatrix} = 0$$

$$\Rightarrow (-\lambda)(-\lambda)(-\lambda) - (3\omega^2)(-\lambda) + -2p\omega(-\lambda)\left(2\frac{\omega}{p}\right) = 0 \quad \Rightarrow -\lambda^3 + \lambda\left(3\omega^2 - p2\omega 2\frac{\omega}{p}\right) = 0$$

$$\Rightarrow -\lambda^3 - \lambda\omega^2 = 0 \quad \Rightarrow \lambda\left(-\omega^2 - \lambda^2\right) = 0 \quad \Rightarrow -\lambda\left(\omega^2 + \lambda^2\right) = 0$$

So solutions are

$$\Rightarrow -\lambda = 0$$

$$\Rightarrow \lambda^2 = -\omega^2 \quad \Rightarrow \lambda = \pm j\omega$$

Eigenvalues λ=0, ±jω

Q2: Controllability

Given the linear time-invariant system:

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 1 & -2 \\ 3 & 3 & 2 \\ -3 & -1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mathbf{u}$$

(a) Check the controllability using the controllability matrix.

(5 marks)

Q2: Controllability

- State space equations take the form

$$\dot{X} = AX + BU$$

The given system state space equations is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -2 \\ 3 & 3 & 2 \\ -3 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} U$$

- Therefore the system matrices are

$$\Rightarrow A = \begin{bmatrix} 1 & 1 & -2 \\ 3 & 3 & 2 \\ -3 & -1 & 0 \end{bmatrix} \qquad \Rightarrow B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

- The system matrix is 3x3
- Therefore the system will have a 3x3 system matrix

$$M_c = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$$

Q2: Controllability

Our system will have a 3x3 system matrix

Calculating the AB term gives

$$AB = \begin{bmatrix} 1 & 1 & -2 \\ 3 & 3 & 2 \\ -3 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}$$

Calculating the A^2B term gives

$$A^2B = A\left(AB\right) = \begin{bmatrix} 1 & 1 & -2 \\ 3 & 3 & 2 \\ -3 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 10 \\ 6 \\ -6 \end{bmatrix}$$

$$\Rightarrow M_c = \begin{bmatrix} 1 & 1 & 10 \\ 0 & 3 & 6 \\ 0 & -3 & -6 \end{bmatrix}$$

Q2: Controllability

- We now reduce the controllability matrix to echelon form

$$M_c = \begin{bmatrix} 1 & 1 & 10 \\ 0 & 3 & 6 \\ 0 & -3 & -6 \end{bmatrix}$$

$$M_c \big| R_3 \rightarrow R_2 + R_3 = \begin{bmatrix} 1 & 1 & 10 \\ 0 & 3 & 6 \\ 0 & -3+3 & -6+6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 10 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

- The controllability matrix thus has rank = 2, which is not full rank
- Therefore the system is not controllable