Polynomial semantics of probabilistic circuits

Oliver Broadrick, UCLA

May, 2024

Based on joint work with Honghua Zhang and Guy Van den Broeck

Outline

- Tractable probabilistic models

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- Probabilistic circuits

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- Several circuit semantics in the literature ...

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- Probabilistic circuits
- Several circuit semantics in the literature ...
- are equivalent! (for binary random variables)
- And, don't all extend to non-binary variables

Probabilistic Models

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How we think about the world: models with uncertainty

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AI research

- Deep learning and formal methods: “neuro-symbolic AI”
- Applications: images, language, audio, medicine, science, economics, etc.

The problem

X_1	X_2	Pr
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0	1	.2
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- Expressive-efficient representation

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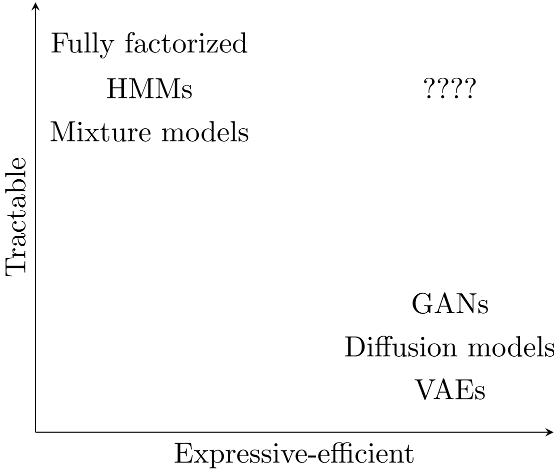

Marginal inference

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$$\begin{aligned}\Pr[X_1 = 1] &= \Pr[X_1 = 1, X_2 = 0] + \Pr[X_1 = 1, X_2 = 1] \\ &= 0.3 + 0.4 \\ &= 0.7\end{aligned}$$

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Goal: Find a model of polysize that supports marginal inference in polytime, for as large a set of probability distributions as possible.

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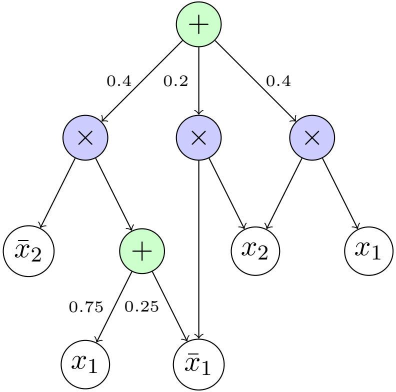

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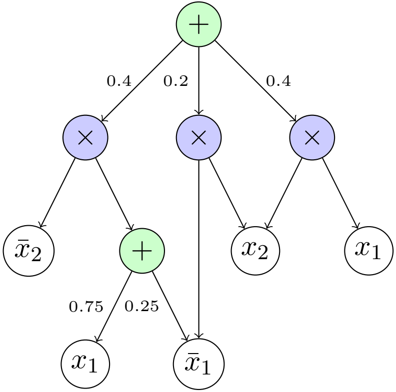

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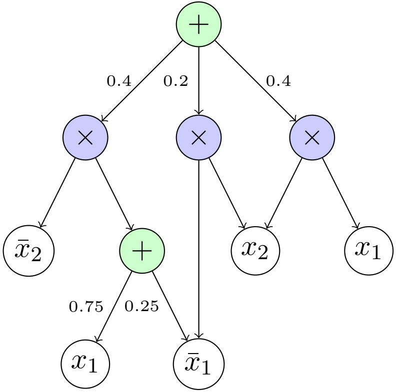

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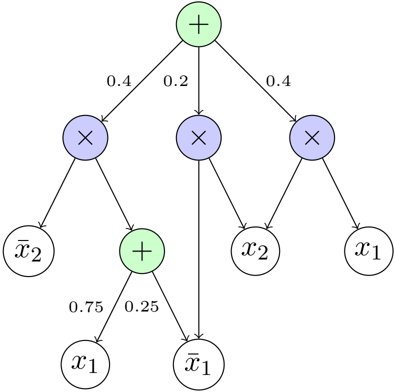

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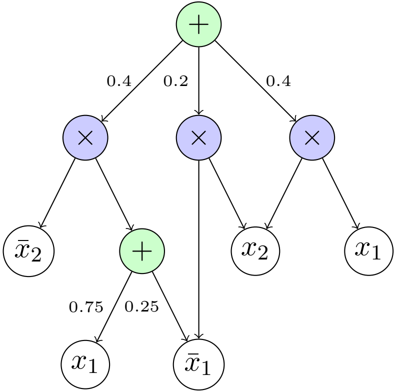

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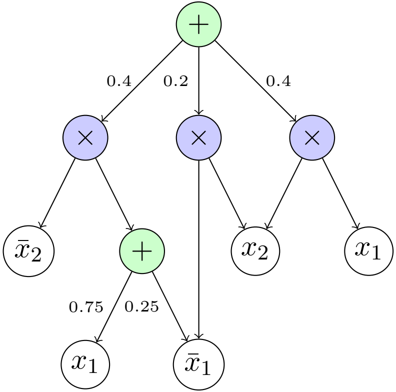

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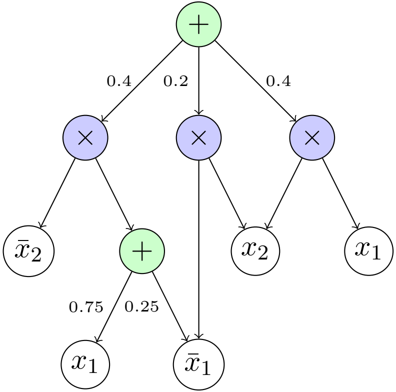

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"Network polynomial"

Circuit Semantics

Polynomial	Notation	Inference	Citation
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Fourier transform	$\hat{p}(x_1, \ldots, x_n)$	✓	Yu et al. [2023]

How do they relate?

$p(x,\bar{x})$	Network polynomial
$p(x)$	Likelihood polynomial
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Transformation from likelihood to network

Theorem 1. *Let* $\Pr$ *be a probability distribution on* n *binary random variables. Then a circuit of size* s *computing the likelihood polynomial for* $\Pr$ *can be transformed to a circuit of size* $O(sn^2)$ *computing the network polynomial for* $\Pr$.

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&= .2x_1(x_2 + \bar{x}_2) + .1x_2(x_1 + \bar{x}_1) + .1(x_1 + \bar{x}_1)(x_2 + \bar{x}_2) \\
&= .2x_1x_2 + .2x_1\bar{x}_2 + .1x_1x_2 + .1\bar{x}_1x_2 + .1x_1x_2 + .1\bar{x}_1x_2 + .1x_1\bar{x}_2 + .1\bar{x}_1\bar{x}_2
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&= .1\bar{x}_1\bar{x}_2 + .2\bar{x}_1x_2 + .3x_1\bar{x}_2 + .4x_1x_2 = p(x_1,x_2,\bar{x}_1,\bar{x}_2)
\end{aligned}
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Divisions?

Wait a second... $\left(\prod_{i=1}^{n}(x_i + \bar{x}_i)\right) p\left(\frac{x_1}{x_1 + \bar{x}_1}, \ldots, \frac{x_n}{x_n + \bar{x}_n}\right)$

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Lemma 3. *For a circuit computing f of degree d, we can obtain circuits computing $H_0[f], H_1[f], \ldots, H_d[f]$ the homogeneous parts of f, i.e. $H_i[f]$ has degree i and $f = \sum_i H_i[f]$.*

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Idea: Move divisions to the root using $a/b + c/d = (ad + bc)/bd$ and $a/b * c/d = ac/bd$.

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Theorem 2 (Strassen). *You can remove the divisions in polynomial time!*

Lemma 3. *For a circuit computing f of degree d, we can obtain circuits computing $H_0[f], H_1[f], \ldots, H_d[f]$ the homogeneous parts of f, i.e. $H_i[f]$ has degree i and $f = \sum_i H_i[f]$.*

Idea: Move divisions to the root using $a/b + c/d = (ad+bc)/bd$ and $a/b * c/d = ac/bd$.

Then for circuit a/b computing polynomial $f = a/b$ of degree d, assume $b(0) = 1$, and we have

$$H_i[f] = H_i[a(1+(1-b)+(1-b)^2+\ldots+(1-b)^d].$$

Progress update

$p(x,\bar{x})$	Network polynomial
$p(x)$	Likelihood polynomial
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Generating functions:

Generating functions: Why?

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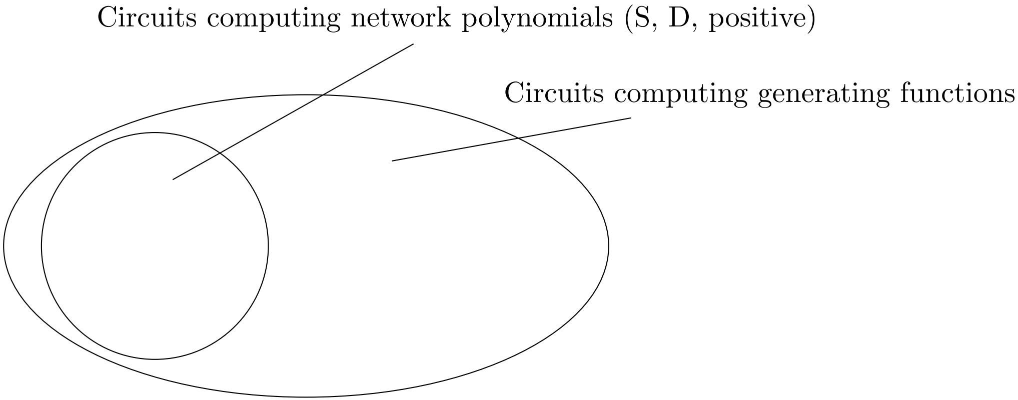

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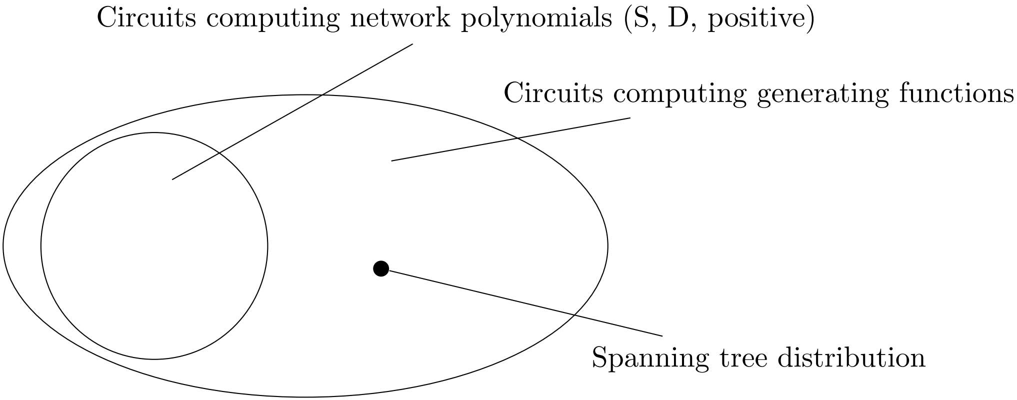

Generating functions

$$g(x) = .1 + .2x_2 + .3x_1 + .4x_1x_2$$

Can we do marginal inference?

X_1	X_2	Pr
0	0	.1
0	1	.2
1	0	.3
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Generating functions

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Can we do marginal inference?
For $X_i = 1$, set $x_i = t$

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$p(x_1, x_2, 1, 1) = .1 + .2x_2 + .3x_1 + .4x_1x_2 = g(x_1, x_2)$

Transformation from generating to network

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Theorem 4. *Let* Pr *be a probability distribution on* n *binary random variables. Then a circuit of size* s *computing the probability generating function for* Pr *can be transformed to a circuit of size* $O(sn^2)$ *computing the network polynomial for* Pr.

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Strassen to the rescue!

Progress update

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Proposition 1. *For binary random variables, probability generating functions $g(x)$ and Fourier polynomials $\hat{p}(x)$ are the same function(!), on respective domains $\{-1, 1\}^n$ and $\{0, 1\}^n$, up to the bijection $\phi : \{0, 1\} \to \{-1, 1\}$ given by $\phi(b) = (-1)^b$ applied bitwise.*

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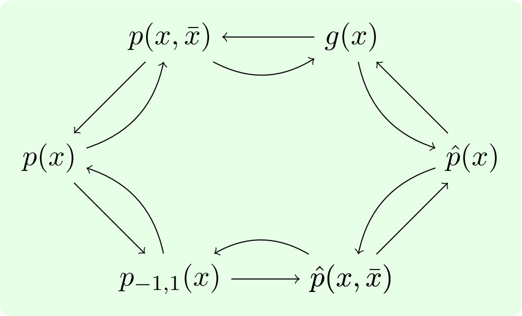

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- several distinct circuit-based models are equally succinct

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What have we done?

- several distinct circuit-based models are equally succinct
- distinct inference algorithms in a common framework

Non-binary distributions?

Non-binary distributions?

Let $\Pr : K^n \to \mathbb{R}$ be a probability mass function with $K = \{0, 1, 2, \ldots, k-1\}$.
Then the probability generating polynomial of Pr is

$$g(x) = \sum_{(d_1, d_2, \ldots, d_n) \in K^n} \Pr(d_1, \ldots, d_n) x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}. \tag{1}$$

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Theorem 5. *For $k \geq 4$, computing likelihoods on a circuit for $g(x)$ is #P-hard.*

Proof idea: Reduce from $0, 1$-permanent.

Conclusion

What we did:

- Several distinct circuit-based models are equally succinct
- Distinct inference algorithms in a common framework
- Inference is hard in circuits computing generating functions for $k \geq 4$ categories

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What's next?

- Are there more succinct tractable representations? e.g., do we need multilinearity?
- Can we characterize *all* tractable marginal inference?
- How can theoretically more expressive models be learned/constructed in practice?

Adnan Darwiche. A differential approach to inference in bayesian networks. *J. ACM*, 50(3):280–305, may 2003. ISSN 0004-5411. doi: 10.1145/765568.765570. URL `https://doi.org/10.1145/765568.765570`.

Dan Roth and Rajhans Samdani. Learning multi-linear representations of distributions for efficient inference. *Machine Learning*, 76(2):195–209, 2009. doi: 10.1007/s10994-009-5130-x. URL `https://doi.org/10.1007/s10994-009-5130-x`.

Zhongjie Yu, Martin Trapp, and Kristian Kersting. Characteristic circuit. In *Proceedings of the 37th Conference on Neural Information Processing Systems (NeurIPS)*, 2023.

Honghua Zhang, Brendan Juba, and Guy Van den Broeck. Probabilistic generating circuits. In *International Conference on Machine Learning*, pages 12447–12457. PMLR, 2021.