7 ☐ TECHNIQUES OF INTEGRATION

7.1 Integration by Parts

1. Let $u = x$, $dv = e^{2x}\,dx \quad \Rightarrow \quad du = dx$, $v = \frac{1}{2}e^{2x}$. Then by Equation 2,

$\int xe^{2x}\,dx = \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\,dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C.$

2. Let $u = \ln x$, $dv = \sqrt{x}\,dx \quad \Rightarrow \quad du = \dfrac{1}{x}\,dx$, $v = \frac{2}{3}x^{3/2}$. Then by Equation 2,

$$\int \sqrt{x}\ln x\,dx = \tfrac{2}{3}x^{3/2}\ln x - \int \tfrac{2}{3}x^{3/2}\cdot\frac{1}{x}\,dx = \tfrac{2}{3}x^{3/2}\ln x - \int \tfrac{2}{3}x^{1/2}\,dx = \tfrac{2}{3}x^{3/2}\ln x - \tfrac{4}{9}x^{3/2} + C.$$

3. Let $u = x$, $dv = \cos 4x\,dx \quad \Rightarrow \quad du = dx$, $v = \frac{1}{4}\sin 4x$. Then by Equation 2,

$\int x\cos 4x\,dx = \frac{1}{4}x\sin 4x - \int \frac{1}{4}\sin 4x\,dx = \frac{1}{4}x\sin 4x + \frac{1}{16}\cos 4x + C.$

4. Let $u = \sin^{-1}x$, $dv = dx \quad \Rightarrow \quad du = \dfrac{1}{\sqrt{1-x^2}}\,dx$, $v = x$. Then by Equation 2,

$$\begin{aligned}\int \sin^{-1}x\,dx &= x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}\,dx = x\sin^{-1}x - \int \frac{1}{\sqrt{t}}\left(-\frac{1}{2}\,dt\right) \qquad \begin{bmatrix} t = 1-x^2, \\ dt = -2x\,dx \end{bmatrix} \\ &= x\sin^{-1}x + \tfrac{1}{2}\int t^{-1/2}\,dt = x\sin^{-1}x + \tfrac{1}{2}\cdot 2t^{1/2} + C = x\sin^{-1}x + \sqrt{1-x^2} + C\end{aligned}$$

Note: A mnemonic device which is helpful for selecting u when using integration by parts is the LIATE principle of precedence for u:

Logarithmic
Inverse trigonometric
Algebraic
Trigonometric
Exponential

If the integrand has several factors, then we try to choose among them a u which appears as high as possible on the list. For example, in $\int xe^{2x}\,dx$ the integrand is xe^{2x}, which is the product of an algebraic function (x) and an exponential function (e^{2x}). Since Algebraic appears before Exponential, we choose $u = x$. Sometimes the integration turns out to be similar regardless of the selection of u and dv, but it is advisable to refer to LIATE when in doubt.

5. Let $u = t$, $dv = e^{2t}\,dt \quad \Rightarrow \quad du = dt$, $v = \frac{1}{2}e^{2t}$. Then by Equation 2,

$\int te^{2t}\,dt = \frac{1}{2}te^{2t} - \int \frac{1}{2}e^{2t}\,dt = \frac{1}{2}te^{2t} - \frac{1}{4}e^{2t} + C.$

6. Let $u = y$, $dv = e^{-y}\,dy \quad \Rightarrow \quad du = dy$, $v = -e^{-y}$. Then by Equation 2,

$\int ye^{-y}\,dy = -ye^{-y} - \int -e^{-y}\,dy = -ye^{-y} + \int e^{-y}\,dy = -ye^{-y} - e^{-y} + C.$

7. Let $u = x$, $dv = \sin 10x\,dx \quad \Rightarrow \quad du = dx$, $v = -\frac{1}{10}\cos 10x$. Then by Equation 2,

$$\begin{aligned}\int x\sin 10x\,dx &= -\tfrac{1}{10}x\cos 10x - \int -\tfrac{1}{10}\cos 10x\,dx = -\tfrac{1}{10}x\cos 10x + \tfrac{1}{10}\int \cos 10x\,dx \\ &= -\tfrac{1}{10}x\cos 10x + \tfrac{1}{100}\sin 10x + C\end{aligned}$$

8. Let $u = \pi - x$, $dv = \cos \pi x\, dx \quad \Rightarrow \quad du = -dx$, $v = \dfrac{1}{\pi} \sin \pi x$. Then by Equation 2,

$$\int (\pi - x) \cos \pi x\, dx = \frac{1}{\pi}(\pi - x) \sin \pi x - \int -\frac{1}{\pi} \sin \pi x\, dx = \frac{1}{\pi}(\pi - x) \sin \pi x - \frac{1}{\pi^2} \cos \pi x + C.$$

9. Let $u = \ln w$, $dv = w\, dw \quad \Rightarrow \quad du = \dfrac{1}{w}\, dw$, $v = \dfrac{1}{2} w^2$. Then by Equation 2,

$$\int w \ln w\, dw = \frac{1}{2} w^2 \ln w - \int \frac{1}{2} w^2 \cdot \frac{1}{w}\, dw = \frac{1}{2} w^2 \ln w - \frac{1}{2} \int w\, dw = \frac{1}{2} w^2 \ln w - \frac{1}{4} w^2 + C.$$

10. Let $u = \ln x$, $dv = \dfrac{1}{x^2}\, dx = x^{-2}\, dx \quad \Rightarrow \quad du = \dfrac{1}{x}\, dx = x^{-1}\, dx$, $v = -x^{-1}$. Then by Equation 2,

$$\int \frac{\ln x}{x^2}\, dx = -\frac{\ln x}{x} - \int -x^{-1} \cdot x^{-1}\, dx = -\frac{\ln x}{x} + \int x^{-2}\, dx = -\frac{\ln x}{x} - x^{-1} + C = -\frac{\ln x}{x} - \frac{1}{x} + C.$$

11. First let $u = x^2 + 2x$, $dv = \cos x\, dx \quad \Rightarrow \quad du = (2x + 2)\, dx$, $v = \sin x$. Then by Equation 2,

$I = \int (x^2 + 2x) \cos x\, dx = (x^2 + 2x) \sin x - \int (2x + 2) \sin x\, dx$. Next let $U = 2x + 2$, $dV = \sin x\, dx \quad \Rightarrow$

$dU = 2\, dx$, $V = -\cos x$, so $\int (2x + 2) \sin x\, dx = -(2x + 2) \cos x - \int -2 \cos x\, dx = -(2x + 2) \cos x + 2 \sin x$.

Thus, $I = (x^2 + 2x) \sin x + (2x + 2) \cos x - 2 \sin x + C$.

12. First let $u = t^2$, $dv = \sin \beta t\, dt \quad \Rightarrow \quad du = 2t\, dt$, $v = -\frac{1}{\beta} \cos \beta t$. Then by Equation 2,

$I = \int t^2 \sin \beta t\, dt = -\frac{1}{\beta} t^2 \cos \beta t - \int -\frac{2}{\beta} t \cos \beta t\, dt$. Next let $U = t$, $dV = \cos \beta t\, dt \quad \Rightarrow \quad dU = dt$,

$V = \dfrac{1}{\beta} \sin \beta t$, so $\int t \cos \beta t\, dt = \dfrac{1}{\beta} t \sin \beta t - \displaystyle\int \frac{1}{\beta} \sin \beta t\, dt = \dfrac{1}{\beta} t \sin \beta t + \dfrac{1}{\beta^2} \cos \beta t$. Thus,

$$I = -\frac{1}{\beta} t^2 \cos \beta t + \frac{2}{\beta}\left(\frac{1}{\beta} t \sin \beta t + \frac{1}{\beta^2} \cos \beta t\right) + C = -\frac{1}{\beta} t^2 \cos \beta t + \frac{2}{\beta^2} t \sin \beta t + \frac{2}{\beta^3} \cos \beta t + C.$$

13. Let $u = \cos^{-1} x$, $dv = dx \quad \Rightarrow \quad du = \dfrac{-1}{\sqrt{1 - x^2}}\, dx$, $v = x$. Then by Equation 2,

$$\begin{aligned} \int \cos^{-1} x\, dx &= x \cos^{-1} x - \int \frac{-x}{\sqrt{1 - x^2}}\, dx = x \cos^{-1} x - \int \frac{1}{\sqrt{t}} \left(\frac{1}{2}\, dt\right) \qquad \begin{bmatrix} t = 1 - x^2, \\ dt = -2x\, dx \end{bmatrix} \\ &= x \cos^{-1} x - \tfrac{1}{2} \cdot 2t^{1/2} + C = x \cos^{-1} x - \sqrt{1 - x^2} + C \end{aligned}$$

14. Let $u = \ln \sqrt{x}$, $dv = dx \quad \Rightarrow \quad du = \dfrac{1}{\sqrt{x}} \cdot \dfrac{1}{2\sqrt{x}}\, dx = \dfrac{1}{2x}\, dx$, $v = x$. Then by Equation 2,

$$\int \ln \sqrt{x}\, dx = x \ln \sqrt{x} - \int x \cdot \frac{1}{2x}\, dx = x \ln \sqrt{x} - \int \frac{1}{2}\, dx = x \ln \sqrt{x} - \frac{1}{2} x + C.$$

Note: We could start by using $\ln \sqrt{x} = \frac{1}{2} \ln x$.

15. Let $u = \ln t$, $dv = t^4\, dt \quad \Rightarrow \quad du = \dfrac{1}{t}\, dt$, $v = \dfrac{1}{5} t^5$. Then by Equation 2,

$$\int t^4 \ln t\, dt = \frac{1}{5} t^5 \ln t - \int \frac{1}{5} t^5 \cdot \frac{1}{t}\, dt = \frac{1}{5} t^5 \ln t - \int \frac{1}{5} t^4\, dt = \frac{1}{5} t^5 \ln t - \frac{1}{25} t^5 + C.$$

16. Let $u = \tan^{-1}(2y)$, $dv = dy \quad\Rightarrow\quad du = \dfrac{2}{1+4y^2}\,dy$, $v = y$. Then by Equation 2,

$$\int \tan^{-1}(2y)\,dy = y\tan^{-1}(2y) - \int \frac{2y}{1+4y^2}\,dy = y\tan^{-1}(2y) - \int \frac{1}{t}\left(\frac{1}{4}\,dt\right) \qquad \begin{bmatrix} t = 1+4y^2, \\ dt = 8y\,dy \end{bmatrix}$$

$$= y\tan^{-1}(2y) - \tfrac{1}{4}\ln|t| + C = y\tan^{-1}(2y) - \tfrac{1}{4}\ln(1+4y^2) + C$$

17. Let $u = t$, $dv = \csc^2 t\,dt \quad\Rightarrow\quad du = dt$, $v = -\cot t$. Then by Equation 2,

$$\int t\csc^2 t\,dt = -t\cot t - \int -\cot t\,dt = -t\cot t + \int \frac{\cos t}{\sin t}\,dt = -t\cot t + \int \frac{1}{z}\,dz \qquad \begin{bmatrix} z = \sin t, \\ dz = \cos t\,dt \end{bmatrix}$$

$$= -t\cot t + \ln|z| + C = -t\cot t + \ln|\sin t| + C$$

18. Let $u = x$, $dv = \cosh ax\,dx \quad\Rightarrow\quad du = dx$, $v = \dfrac{1}{a}\sinh ax$. Then by Equation 2,

$$\int x\cosh ax\,dx = \frac{1}{a}x\sinh ax - \int \frac{1}{a}\sinh ax\,dx = \frac{1}{a}x\sinh ax - \frac{1}{a^2}\cosh ax + C.$$

19. First let $u = (\ln x)^2$, $dv = dx \quad\Rightarrow\quad du = 2\ln x \cdot \frac{1}{x}\,dx$, $v = x$. Then by Equation 2,

$I = \int (\ln x)^2\,dx = x(\ln x)^2 - 2\int x\ln x \cdot \frac{1}{x}\,dx = x(\ln x)^2 - 2\int \ln x\,dx$. Next let $U = \ln x$, $dV = dx \quad\Rightarrow$

$dU = 1/x\,dx$, $V = x$ to get $\int \ln x\,dx = x\ln x - \int x \cdot (1/x)\,dx = x\ln x - \int dx = x\ln x - x + C_1$. Thus,

$I = x(\ln x)^2 - 2(x\ln x - x + C_1) = x(\ln x)^2 - 2x\ln x + 2x + C$, where $C = -2C_1$.

20. $\displaystyle\int \frac{z}{10^z}\,dz = \int z\,10^{-z}\,dz$. Let $u = z$, $dv = 10^{-z}\,dz \quad\Rightarrow\quad du = dz$, $v = \dfrac{-10^{-z}}{\ln 10}$. Then by Equation 2,

$$\int z\,10^{-z}\,dz = \frac{-z\,10^{-z}}{\ln 10} - \int \frac{-10^{-z}}{\ln 10}\,dz = \frac{-z}{10^z\ln 10} - \frac{10^{-z}}{(\ln 10)(\ln 10)} + C = -\frac{z}{10^z\ln 10} - \frac{1}{10^z(\ln 10)^2} + C.$$

21. First let $u = e^{3x}$, $dv = \cos x\,dx \quad\Rightarrow\quad du = 3e^{3x}\,dx$, $v = \sin x$. Then

$I = \int e^{3x}\cos x\,dx = e^{3x}\sin x - 3\int e^{3x}\sin x\,dx$. Next, let $U = e^{3x}$, $dV = \sin x\,dx \quad\Rightarrow\quad dU = 3e^{3x}\,dx$, $V = -\cos x$,

so $\int e^{3x}\sin x\,dx = -e^{3x}\cos x + 3\int e^{3x}\cos x\,dx$. Substituting in the previous formula gives

$I = e^{3x}\sin x - 3(-e^{3x}\cos x + 3I) = e^{3x}\sin x + 3e^{3x}\cos x - 9I \quad\Rightarrow\quad 10I = e^{3x}\sin x + 3e^{3x}\cos x + C_1 \quad\Rightarrow$

$I = \frac{1}{10}e^{3x}\sin x + \frac{3}{10}e^{3x}\cos x + C$, where $C = \frac{1}{10}C_1$.

22. First let $u = e^x$, $dv = \sin \pi x\,dx \quad\Rightarrow\quad du = e^x\,dx$, $v = -\dfrac{1}{\pi}\cos \pi x$. Then

$I = \displaystyle\int e^x\sin \pi x\,dx = -\frac{1}{\pi}e^x\cos \pi x + \frac{1}{\pi}\int e^x\cos \pi x\,dx$. Next, let $U = e^x$, $dV = \cos \pi x\,dx \quad\Rightarrow$

$dU = e^x\,dx$, $V = \dfrac{1}{\pi}\sin \pi x$, so $\displaystyle\int e^x\cos \pi x\,dx = \frac{1}{\pi}e^x\sin \pi x - \frac{1}{\pi}\int e^x\sin \pi x\,dx$. Substituting in the previous formula

gives $I = -\dfrac{1}{\pi}e^x\cos \pi x + \dfrac{1}{\pi}\left(\dfrac{1}{\pi}e^x\sin \pi x - \dfrac{1}{\pi}I\right) = -\dfrac{1}{\pi}e^x\cos \pi x + \dfrac{1}{\pi^2}e^x\sin \pi x - \dfrac{1}{\pi^2}I \quad\Rightarrow$

$\left(1 + \dfrac{1}{\pi^2}\right)I = -\dfrac{1}{\pi}e^x\cos \pi x + \dfrac{1}{\pi^2}e^x\sin \pi x + C_1 \quad\Rightarrow\quad I = -\dfrac{\pi}{\pi^2+1}\,e^x\cos \pi x + \dfrac{1}{\pi^2+1}\,e^x\sin \pi x + C$,

where $C = \dfrac{\pi^2}{\pi^2+1}\,C_1$.

23. First let $u = \sin 3\theta$, $dv = e^{2\theta}\,d\theta \quad\Rightarrow\quad du = 3\cos 3\theta\,d\theta$, $v = \frac{1}{2}e^{2\theta}$. Then $I = \int e^{2\theta}\sin 3\theta\,d\theta = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{2}\int e^{2\theta}\cos 3\theta\,d\theta$. Next let $U = \cos 3\theta$, $dV = e^{2\theta}\,d\theta \quad\Rightarrow\quad dU = -3\sin 3\theta\,d\theta$, $V = \frac{1}{2}e^{2\theta}$ to get $\int e^{2\theta}\cos 3\theta\,d\theta = \frac{1}{2}e^{2\theta}\cos 3\theta + \frac{3}{2}\int e^{2\theta}\sin 3\theta\,d\theta$. Substituting in the previous formula gives $I = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{4}e^{2\theta}\cos 3\theta - \frac{9}{4}\int e^{2\theta}\sin 3\theta\,d\theta = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{4}e^{2\theta}\cos 3\theta - \frac{9}{4}I \quad\Rightarrow$ $\frac{13}{4}I = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{4}e^{2\theta}\cos 3\theta + C_1$. Hence, $I = \frac{1}{13}e^{2\theta}(2\sin 3\theta - 3\cos 3\theta) + C$, where $C = \frac{4}{13}C_1$.

24. First let $u = e^{-\theta}$, $dv = \cos 2\theta\,d\theta \quad\Rightarrow\quad du = -e^{-\theta}\,d\theta$, $v = \frac{1}{2}\sin 2\theta$. Then $I = \int e^{-\theta}\cos 2\theta\,d\theta = \frac{1}{2}e^{-\theta}\sin 2\theta - \int \frac{1}{2}\sin 2\theta\left(-e^{-\theta}\,d\theta\right) = \frac{1}{2}e^{-\theta}\sin 2\theta + \frac{1}{2}\int e^{-\theta}\sin 2\theta\,d\theta$.
Next let $U = e^{-\theta}$, $dV = \sin 2\theta\,d\theta \quad\Rightarrow\quad dU = -e^{-\theta}\,d\theta$, $V = -\frac{1}{2}\cos 2\theta$, so $\int e^{-\theta}\sin 2\theta\,d\theta = -\frac{1}{2}e^{-\theta}\cos 2\theta - \int\left(-\frac{1}{2}\right)\cos 2\theta\left(-e^{-\theta}\,d\theta\right) = -\frac{1}{2}e^{-\theta}\cos 2\theta - \frac{1}{2}\int e^{-\theta}\cos 2\theta\,d\theta$.
So $I = \frac{1}{2}e^{-\theta}\sin 2\theta + \frac{1}{2}\left[\left(-\frac{1}{2}e^{-\theta}\cos 2\theta\right) - \frac{1}{2}I\right] = \frac{1}{2}e^{-\theta}\sin 2\theta - \frac{1}{4}e^{-\theta}\cos 2\theta - \frac{1}{4}I \quad\Rightarrow$ $\frac{5}{4}I = \frac{1}{2}e^{-\theta}\sin 2\theta - \frac{1}{4}e^{-\theta}\cos 2\theta + C_1 \quad\Rightarrow\quad I = \frac{4}{5}\left(\frac{1}{2}e^{-\theta}\sin 2\theta - \frac{1}{4}e^{-\theta}\cos 2\theta + C_1\right) = \frac{2}{5}e^{-\theta}\sin 2\theta - \frac{1}{5}e^{-\theta}\cos 2\theta + C$.

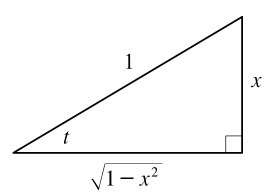
25. First let $u = z^3$, $dv = e^z dz \quad\Rightarrow\quad du = 3z^2 dz$, $v = e^z$. Then $I_1 = \int z^3 e^z dz = z^3 e^z - 3\int z^2 e^z dz$. Next let $u_1 = z^2$, $dv_1 = e^z dz \quad\Rightarrow\quad du_1 = 2z\,dz$, $v_1 = e^z$. Then $I_2 = z^2 e^z - 2\int z e^z dz$. Finally, let $u_2 = z$, $dv_2 = e^z dz \quad\Rightarrow\quad du_2 = dz$, $v_2 = e^z$. Then $\int z e^z dz = z e^z - \int e^z dz = z e^z - e^z + C_1$. Substituting in the expression for I_2, we get $I_2 = z^2 e^z - 2(z e^z - e^z + C_1) = z^2 e^z - 2z e^z + 2e^z - 2C_1$. Substituting the last expression for I_2 into I_1 gives $I_1 = z^3 e^z - 3(z^2 e^z - 2z e^z + 2e^z - 2C_1) = z^3 e^z - 3z^2 e^z + 6z e^z - 6e^z + C$, where $C = 6C_1$.

26. First let $u = (\arcsin x)^2$, $dv = dx \quad\Rightarrow\quad du = 2\arcsin x \cdot \dfrac{1}{\sqrt{1-x^2}}\,dx$, $v = x$. Then

$I = \displaystyle\int (\arcsin x)^2 dx = x(\arcsin x)^2 - 2\int \frac{x\arcsin x}{\sqrt{1-x^2}}\,dx$. To simplify the last integral, let $t = \arcsin x$ $[x = \sin t]$, so

$dt = \dfrac{1}{\sqrt{1-x^2}}\,dx$, and $\displaystyle\int \frac{x\arcsin x}{\sqrt{1-x^2}}\,dx = \int t\sin t\,dt$. To evaluate just the last integral, now let $U = t$, $dV = \sin t\,dt \quad\Rightarrow$ $dU = dt$, $V = -\cos t$. Thus,

$$\begin{aligned}\int t\sin t\,dt &= -t\cos t + \int \cos t\,dt = -t\cos t + \sin t + C\\ &= -\arcsin x \cdot \frac{\sqrt{1-x^2}}{1} + x + C_1 \quad \text{[refer to the figure]}\end{aligned}$$



Returning to I, we get $I = x(\arcsin x)^2 + 2\sqrt{1-x^2}\,\arcsin x - 2x + C$, where $C = -2C_1$.

27. First let $u = 1 + x^2$, $dv = e^{3x}\,dx \quad\Rightarrow\quad du = 2x\,dx$, $v = \frac{1}{3}e^{3x}$. Then $I = \int(1+x^2)e^{3x}\,dx = \frac{1}{3}e^{3x}(1+x^2) - \frac{2}{3}\int x e^{3x}\,dx$. Next, let $U = x$, $dV = e^{3x}\,dx \quad\Rightarrow\quad dU = dx$, $V = \frac{1}{3}e^{3x}$, so $\int x e^{3x}\,dx = \frac{1}{3}x e^{3x} - \frac{1}{3}\int e^{3x}\,dx = \frac{1}{3}x e^{3x} - \frac{1}{9}e^{3x} + C_1$. Substituting in the previous formula gives

$$\begin{aligned}I &= \tfrac{1}{3}e^{3x}(1+x^2) - \tfrac{2}{3}\left(\tfrac{1}{3}x e^{3x} - \tfrac{1}{9}e^{3x} + C_1\right) = \tfrac{1}{3}e^{3x} + \tfrac{1}{3}x^2 e^{3x} - \tfrac{2}{9}x e^{3x} + \tfrac{2}{27}e^{3x} - \tfrac{2}{3}C_1\\ &= \tfrac{11}{27}e^{3x} - \tfrac{2}{9}x e^{3x} + \tfrac{1}{3}x^2 e^{3x} + C, \text{ where } C = -\tfrac{2}{3}C_1\end{aligned}$$

28. Let $u = \theta$, $dv = \sin 3\pi\theta\, d\theta \quad\Rightarrow\quad du = d\theta$, $v = -\dfrac{1}{3\pi}\cos 3\pi\theta$. By (6),

$$\int_0^{1/2} \theta \sin 3\pi\theta\, d\theta = \left[-\frac{1}{3\pi}\theta\cos 3\pi\theta\right]_0^{1/2} + \frac{1}{3\pi}\int_0^{1/2}\cos 3\pi\theta\, d\theta = (0+0) + \frac{1}{9\pi^2}\left[\sin 3\pi\theta\right]_0^{1/2}$$
$$= \frac{1}{9\pi^2}(-1-0) = -\frac{1}{9\pi^2}$$

29. Let $u = x$, $dv = 3^x\, dx \quad\Rightarrow\quad du = dx$, $v = \dfrac{1}{\ln 3}3^x$. By (6),

$$\int_0^1 x3^x\, dx = \left[\frac{1}{\ln 3}\, x\, 3^x\right]_0^1 - \frac{1}{\ln 3}\int_0^1 3^x\, dx = \left(\frac{3}{\ln 3} - 0\right) - \frac{1}{\ln 3}\left[\frac{1}{\ln 3}\, 3^x\right]_0^1 = \frac{3}{\ln 3} - \frac{1}{(\ln 3)^2}(3-1)$$
$$= \frac{3}{\ln 3} - \frac{2}{(\ln 3)^2}$$

30. Let $u = xe^x$, $dv = \dfrac{1}{(1+x)^2}\, dx \quad\Rightarrow\quad du = (xe^x + e^x)\, dx = e^x(x+1)\, dx$, $v = -\dfrac{1}{1+x}$. By (6),

$$\int_0^1 \frac{xe^x}{(1+x)^2}\, dx = \left[-\frac{xe^x}{1+x}\right]_0^1 - \int_0^1\left(-\frac{1}{1+x}\right)e^x(1+x)\, dx = \left(-\frac{e}{2}+0\right) + \int_0^1 e^x\, dx = -\frac{1}{2}e + \left[e^x\right]_0^1$$
$$= -\tfrac{1}{2}e + e - 1 = \tfrac{1}{2}e - 1$$

31. Let $u = y$, $dv = \sinh y\, dy \quad\Rightarrow\quad du = dy$, $v = \cosh y$. By (6),

$$\int_0^2 y\sinh y\, dy = \left[y\cosh y\right]_0^2 - \int_0^2 \cosh y\, dy = 2\cosh 2 - 0 - \left[\sinh y\right]_0^2 = 2\cosh 2 - \sinh 2.$$

32. Let $u = \ln w$, $dv = w^2\, dw \quad\Rightarrow\quad du = \dfrac{1}{w}\, dw$, $v = \frac{1}{3}w^3$. By (6),

$$\int_1^2 w^2 \ln w\, dw = \left[\tfrac{1}{3}w^3\ln w\right]_1^2 - \int_1^2 \tfrac{1}{3}w^2\, dw = \tfrac{8}{3}\ln 2 - 0 - \left[\tfrac{1}{9}w^3\right]_1^2 = \tfrac{8}{3}\ln 2 - \left(\tfrac{8}{9} - \tfrac{1}{9}\right) = \tfrac{8}{3}\ln 2 - \tfrac{7}{9}.$$

33. Let $u = \ln R$, $dv = \dfrac{1}{R^2}\, dR \quad\Rightarrow\quad du = \dfrac{1}{R}\, dR$, $v = -\dfrac{1}{R}$. By (6),

$$\int_1^5 \frac{\ln R}{R^2}\, dR = \left[-\frac{1}{R}\ln R\right]_1^5 - \int_1^5 -\frac{1}{R^2}\, dR = -\tfrac{1}{5}\ln 5 - 0 - \left[\frac{1}{R}\right]_1^5 = -\tfrac{1}{5}\ln 5 - \left(\tfrac{1}{5} - 1\right) = \tfrac{4}{5} - \tfrac{1}{5}\ln 5.$$

34. First let $u = t^2$, $dv = \sin 2t\, dt \quad\Rightarrow\quad du = 2t\, dt$, $v = -\frac{1}{2}\cos 2t$. By (6),

$$\int_0^{2\pi} t^2\sin 2t\, dt = \left[-\tfrac{1}{2}t^2\cos 2t\right]_0^{2\pi} + \int_0^{2\pi} t\cos 2t\, dt = -2\pi^2 + \int_0^{2\pi} t\cos 2t\, dt.$$

Next let $U = t$, $dV = \cos 2t\, dt \quad\Rightarrow\quad dU = dt$, $V = \frac{1}{2}\sin 2t$. By (6) again,

$$\int_0^{2\pi} t\cos 2t\, dt = \left[\tfrac{1}{2}t\sin 2t\right]_0^{2\pi} - \int_0^{2\pi}\tfrac{1}{2}\sin 2t\, dt = 0 - \left[-\tfrac{1}{4}\cos 2t\right]_0^{2\pi} = \tfrac{1}{4} - \tfrac{1}{4} = 0. \text{ Thus, } \int_0^{2\pi} t^2\sin 2t\, dt = -2\pi^2.$$

35. $\sin 2x = 2\sin x\,\cos x$, so $\int_0^\pi x\,\sin x\,\cos x\, dx = \frac{1}{2}\int_0^\pi x\,\sin 2x\, dx$. Let $u = x$, $dv = \sin 2x\, dx \quad\Rightarrow$ $du = dx$, $v = -\frac{1}{2}\cos 2x$. By (6),

$$\tfrac{1}{2}\int_0^\pi x\,\sin 2x\, dx = \tfrac{1}{2}\left[-\tfrac{1}{2}x\cos 2x\right]_0^\pi - \tfrac{1}{2}\int_0^\pi -\tfrac{1}{2}\cos 2x\, dx = -\tfrac{1}{4}\pi - 0 + \tfrac{1}{4}\left[\tfrac{1}{2}\sin 2x\right]_0^\pi = -\frac{\pi}{4}$$

36. Let $u = \arctan(1/x)$, $dv = dx \quad \Rightarrow \quad du = \dfrac{1}{1+(1/x)^2} \cdot \dfrac{-1}{x^2}\,dx = \dfrac{-dx}{x^2+1}$, $v = x$. By (6),

$$\int_1^{\sqrt{3}} \arctan\left(\frac{1}{x}\right) dx = \left[x \arctan\left(\frac{1}{x}\right)\right]_1^{\sqrt{3}} + \int_1^{\sqrt{3}} \frac{x\,dx}{x^2+1} = \sqrt{3}\,\frac{\pi}{6} - 1 \cdot \frac{\pi}{4} + \frac{1}{2}\left[\ln(x^2+1)\right]_1^{\sqrt{3}}$$
$$= \frac{\pi\sqrt{3}}{6} - \frac{\pi}{4} + \frac{1}{2}(\ln 4 - \ln 2) = \frac{\pi\sqrt{3}}{6} - \frac{\pi}{4} + \frac{1}{2}\ln\frac{4}{2} = \frac{\pi\sqrt{3}}{6} - \frac{\pi}{4} + \frac{1}{2}\ln 2$$

37. Let $u = M$, $dv = e^{-M}\,dM \quad \Rightarrow \quad du = dM$, $v = -e^{-M}$. By (6),

$$\int_1^5 \frac{M}{e^M}\,dM = \int_1^5 Me^{-M}\,dM = \left[-Me^{-M}\right]_1^5 - \int_1^5 -e^{-M}\,dM = -5e^{-5} + e^{-1} - \left[e^{-M}\right]_1^5$$
$$= -5e^{-5} + e^{-1} - (e^{-5} - e^{-1}) = 2e^{-1} - 6e^{-5}$$

38. Let $u = (\ln x)^2$, $dv = x^{-3}\,dx \quad \Rightarrow \quad du = \dfrac{2\ln x}{x}\,dx$, $v = -\frac{1}{2}x^{-2}$. By (6),

$I = \displaystyle\int_1^2 \frac{(\ln x)^2}{x^3}\,dx = \left[-\frac{(\ln x)^2}{2x^2}\right]_1^2 + \int_1^2 \frac{\ln x}{x^3}\,dx$. Now let $U = \ln x$, $dV = x^{-3}\,dx \quad \Rightarrow \quad dU = \dfrac{1}{x}\,dx$, $V = -\frac{1}{2}x^{-2}$.

Then

$$\int_1^2 \frac{\ln x}{x^3}\,dx = \left[-\frac{\ln x}{2x^2}\right]_1^2 + \tfrac{1}{2}\int_1^2 x^{-3}dx = -\tfrac{1}{8}\ln 2 + 0 + \tfrac{1}{2}\left[-\tfrac{1}{2x^2}\right]_1^2 = -\tfrac{1}{8}\ln 2 + \tfrac{1}{2}\left(-\tfrac{1}{8} + \tfrac{1}{2}\right) = \tfrac{3}{16} - \tfrac{1}{8}\ln 2.$$

Thus $I = \left(-\frac{1}{8}(\ln 2)^2 + 0\right) + \left(\frac{3}{16} - \frac{1}{8}\ln 2\right) = -\frac{1}{8}(\ln 2)^2 - \frac{1}{8}\ln 2 + \frac{3}{16}$.

39. Let $u = \ln(\cos x)$, $dv = \sin x\,dx \quad \Rightarrow \quad du = \dfrac{1}{\cos x}(-\sin x)\,dx$, $v = -\cos x$. By (6),

$$\textstyle\int_0^{\pi/3} \sin x\,\ln(\cos x)\,dx = \Big[-\cos x\,\ln(\cos x)\Big]_0^{\pi/3} - \int_0^{\pi/3} \sin x\,dx = -\frac{1}{2}\ln\frac{1}{2} - 0 - \Big[-\cos x\Big]_0^{\pi/3}$$
$$= -\tfrac{1}{2}\ln\tfrac{1}{2} + \left(\tfrac{1}{2} - 1\right) = \tfrac{1}{2}\ln 2 - \tfrac{1}{2}$$

40. Let $u = r^2$, $dv = \dfrac{r}{\sqrt{4+r^2}}\,dr \quad \Rightarrow \quad du = 2r\,dr$, $v = \sqrt{4+r^2}$. By (6),

$$\int_0^1 \frac{r^3}{\sqrt{4+r^2}}\,dr = \left[r^2\sqrt{4+r^2}\right]_0^1 - 2\int_0^1 r\sqrt{4+r^2}\,dr = \sqrt{5} - \tfrac{2}{3}\left[(4+r^2)^{3/2}\right]_0^1$$
$$= \sqrt{5} - \tfrac{2}{3}(5)^{3/2} + \tfrac{2}{3}(8) = \sqrt{5}\left(1 - \tfrac{10}{3}\right) + \tfrac{16}{3} = \tfrac{16}{3} - \tfrac{7}{3}\sqrt{5}$$

41. Let $u = \cos x$, $dv = \sinh x\,dx \quad \Rightarrow \quad du = -\sin x\,dx$, $v = \cosh x$. By (6),

$$I = \int_0^\pi \cos x \sinh x\,dx = \Big[\cos x \cosh x\Big]_0^\pi - \int_0^\pi -\sin x \cosh x\,dx = -\cosh\pi - 1 + \int_0^\pi \sin x \cosh x\,dx.$$

Now let $U = \sin x$, $dV = \cosh x\,dx \quad \Rightarrow \quad dU = \cos x\,dx$, $V = \sinh x$. Then

$$\int_0^\pi \sin x \cosh x\,dx = \Big[\sin x \sinh x\Big]_0^\pi - \int_0^\pi \cos x \sinh x\,dx = (0 - 0) - \int_0^\pi \cos x \sinh x\,dx = -I.$$

Substituting in the previous formula gives $I = -\cosh\pi - 1 - I \quad \Rightarrow \quad 2I = -(\cosh\pi + 1) \quad \Rightarrow \quad I = -\dfrac{\cosh\pi + 1}{2}$.

[We could also write the answer as $I = -\frac{1}{4}(2 + e^\pi + e^{-\pi})$.]

42. Let $u = \sin(t-s)$, $dv = e^s\,ds \Rightarrow du = -\cos(t-s)\,ds$, $v = e^s$. Then

$I = \int_0^t e^s \sin(t-s)\,ds = \left[e^s \sin(t-s)\right]_0^t + \int_0^t e^s \cos(t-s)\,ds = e^t \sin 0 - e^0 \sin t + I_1$. For I_1, let $U = \cos(t-s)$,

$dV = e^s\,ds \Rightarrow dU = \sin(t-s)\,ds$, $V = e^s$. So $I_1 = \left[e^s \cos(t-s)\right]_0^t - \int_0^t e^s \sin(t-s)\,ds = e^t \cos 0 - e^0 \cos t - I$.

Thus, $I = -\sin t + e^t - \cos t - I \Rightarrow 2I = e^t - \cos t - \sin t \Rightarrow I = \frac{1}{2}(e^t - \cos t - \sin t)$.

43. Let $t = \sqrt{x}$, so that $t^2 = x$ and $2t\,dt = dx$. Thus, $\int e^{\sqrt{x}}\,dx = \int e^t(2t)\,dt$. Now use parts with $u = t$, $dv = e^t\,dt$, $du = dt$, and $v = e^t$ to get $2\int te^t\,dt = 2t\,e^t - 2\int e^t\,dt = 2t\,e^t - 2e^t + C = 2\sqrt{x}\,e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$.

44. Let $t = \ln x$, so that $e^t = x$ and $e^t\,dt = dx$. Thus, $\int \cos(\ln x)\,dx = \int \cos t \cdot e^t\,dt = I$. Now use parts with $u = \cos t$, $dv = e^t\,dt$, $du = -\sin t\,dt$, and $v = e^t$ to get $\int e^t \cos t\,dt = e^t \cos t - \int -e^t \sin t\,dt = e^t \cos t + \int e^t \sin t\,dt$. Now use parts with $U = \sin t$, $dV = e^t\,dt$, $dU = \cos t\,dt$, and $V = e^t$ to get

$\int e^t \sin t\,dt = e^t \sin t - \int e^t \cos t\,dt$. Thus, $I = e^t \cos t + e^t \sin t - I \Rightarrow 2I = e^t \cos t + e^t \sin t \Rightarrow$

$I = \frac{1}{2}e^t \cos t + \frac{1}{2}e^t \sin t + C = \frac{1}{2}x\cos(\ln x) + \frac{1}{2}x\sin(\ln x) + C$.

45. Let $x = \theta^2$, so that $dx = 2\theta\,d\theta$. Thus, $\displaystyle\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3 \cos\left(\theta^2\right)\,d\theta = \int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^2 \cos\left(\theta^2\right) \cdot \tfrac{1}{2}(2\theta\,d\theta) = \tfrac{1}{2}\int_{\pi/2}^{\pi} x\cos x\,dx$. Now use parts with $u = x$, $dv = \cos x\,dx$, $du = dx$, $v = \sin x$ to get

$$\begin{aligned}\frac{1}{2}\int_{\pi/2}^{\pi} x\cos x\,dx &= \frac{1}{2}\left(\left[x\sin x\right]_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi} \sin x\,dx\right) = \frac{1}{2}\left[x\sin x + \cos x\right]_{\pi/2}^{\pi} \\ &= \tfrac{1}{2}(\pi\sin\pi + \cos\pi) - \tfrac{1}{2}\left(\tfrac{\pi}{2}\sin\tfrac{\pi}{2} + \cos\tfrac{\pi}{2}\right) = \tfrac{1}{2}(\pi\cdot 0 - 1) - \tfrac{1}{2}\left(\tfrac{\pi}{2}\cdot 1 + 0\right) = -\tfrac{1}{2} - \tfrac{\pi}{4}\end{aligned}$$

46. Let $x = \cos t$, so that $dx = -\sin t\,dt$. Thus,

$\int_0^{\pi} e^{\cos t} \sin 2t\,dt = \int_0^{\pi} e^{\cos t}(2\sin t\,\cos t)\,dt = \int_1^{-1} e^x \cdot 2x\,(-dx) = 2\int_{-1}^{1} xe^x\,dx$. Now use parts with $u = x$,

$dv = e^x\,dx$, $du = dx$, $v = e^x$ to get

$2\int_{-1}^{1} xe^x\,dx = 2\left(\left[xe^x\right]_{-1}^{1} - \int_{-1}^{1} e^x\,dx\right) = 2\left(e^1 + e^{-1} - \left[e^x\right]_{-1}^{1}\right) = 2(e + e^{-1} - [e^1 - e^{-1}]) = 2(2e^{-1}) = 4/e$.

47. Let $y = 1 + x$, so that $dy = dx$. Thus, $\int x\ln(1+x)\,dx = \int (y-1)\ln y\,dy$. Now use parts with $u = \ln y$, $dv = (y-1)\,dy$, $du = \frac{1}{y}\,dy$, $v = \frac{1}{2}y^2 - y$ to get

$$\begin{aligned}\int (y-1)\ln y\,dy &= \left(\tfrac{1}{2}y^2 - y\right)\ln y - \int\left(\tfrac{1}{2}y - 1\right)dy = \tfrac{1}{2}y(y-2)\ln y - \tfrac{1}{4}y^2 + y + C \\ &= \tfrac{1}{2}(1+x)(x-1)\ln(1+x) - \tfrac{1}{4}(1+x)^2 + 1 + x + C,\end{aligned}$$

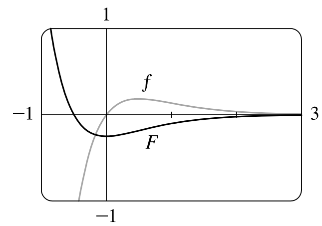
which can be written as $\frac{1}{2}(x^2-1)\ln(1+x) - \frac{1}{4}x^2 + \frac{1}{2}x + \frac{3}{4} + C$.

48. Let $y = \ln x$, so that $dy = \dfrac{1}{x}\,dx$. Thus, $\displaystyle\int \frac{\arcsin(\ln x)}{x}\,dx = \int \arcsin y\,dy$. Now use parts with $u = \arcsin y$, $dv = dy$, $du = \dfrac{1}{\sqrt{1-y^2}}\,dy$, and $v = y$ to get

$\displaystyle\int \arcsin y\,dy = y\arcsin y - \int \frac{y}{\sqrt{1-y^2}}\,dy = y\arcsin y + \sqrt{1-y^2} + C = (\ln x)\arcsin(\ln x) + \sqrt{1-(\ln x)^2} + C.$

49. Let $u = x$, $dv = e^{-2x}\,dx \quad\Rightarrow\quad du = dx$, $v = -\frac{1}{2}e^{-2x}$. Then

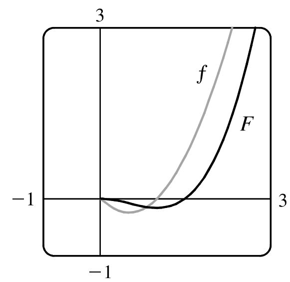
$\int xe^{-2x}\,dx = -\frac{1}{2}xe^{-2x} + \int \frac{1}{2}e^{-2x}\,dx = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C$. We see from the graph that this is reasonable, since F has a minimum where f changes from negative to positive. Also, F increases where f is positive and F decreases where f is negative.



50. Let $u = \ln x$, $dv = x^{3/2}\,dx \quad\Rightarrow\quad du = \frac{1}{x}\,dx$, $v = \frac{2}{5}x^{5/2}$. Then

$$\int x^{3/2}\ln x\,dx = \tfrac{2}{5}x^{5/2}\ln x - \tfrac{2}{5}\int x^{3/2}\,dx = \tfrac{2}{5}x^{5/2}\ln x - \left(\tfrac{2}{5}\right)^2 x^{5/2} + C$$
$$= \tfrac{2}{5}x^{5/2}\ln x - \tfrac{4}{25}x^{5/2} + C$$

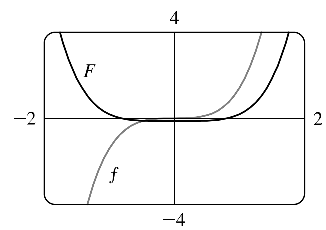
We see from the graph that this is reasonable, since F has a minimum where f changes from negative to positive.



51. Let $u = \frac{1}{2}x^2$, $dv = 2x\sqrt{1+x^2}\,dx \quad\Rightarrow\quad du = x\,dx$, $v = \frac{2}{3}(1+x^2)^{3/2}$.

Then

$$\int x^3\sqrt{1+x^2}\,dx = \tfrac{1}{2}x^2\left[\tfrac{2}{3}(1+x^2)^{3/2}\right] - \tfrac{2}{3}\int x(1+x^2)^{3/2}dx$$
$$= \tfrac{1}{3}x^2(1+x^2)^{3/2} - \tfrac{2}{3}\cdot\tfrac{2}{5}\cdot\tfrac{1}{2}(1+x^2)^{5/2} + C$$
$$= \tfrac{1}{3}x^2(1+x^2)^{3/2} - \tfrac{2}{15}(1+x^2)^{5/2} + C$$



We see from the graph that this is reasonable, since F increases where f is positive and F decreases where f is negative. Note also that f is an odd function and F is an even function.

Another method: Use substitution with $u = 1 + x^2$ to get $\frac{1}{5}(1+x^2)^{5/2} - \frac{1}{3}(1+x^2)^{3/2} + C$.

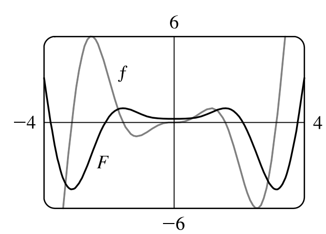
52. First let $u = x^2$, $dv = \sin 2x\,dx \quad\Rightarrow\quad du = 2x\,dx$, $v = -\frac{1}{2}\cos 2x$.

Then $I = \int x^2 \sin 2x\,dx = -\frac{1}{2}x^2\cos 2x + \int x\cos 2x\,dx$.

Next let $U = x$, $dV = \cos 2x\,dx \quad\Rightarrow\quad dU = dx$, $V = \frac{1}{2}\sin 2x$, so

$\int x\cos 2x\,dx = \frac{1}{2}x\sin 2x - \int \frac{1}{2}\sin 2x\,dx = \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C$.

Thus, $I = -\frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C$.



We see from the graph that this is reasonable, since F increases where f is positive and F decreases where f is negative. Note also that f is an odd function and F is an even function.

53. (a) Take $n = 2$ in Example 6 to get $\displaystyle\int \sin^2 x\,dx = -\frac{1}{2}\cos x\sin x + \frac{1}{2}\int 1\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$.

(b) $\int \sin^4 x\,dx = -\frac{1}{4}\cos x\sin^3 x + \frac{3}{4}\int \sin^2 x\,dx = -\frac{1}{4}\cos x\sin^3 x + \frac{3}{8}x - \frac{3}{16}\sin 2x + C$.

54. (a) Let $u = \cos^{n-1} x$, $dv = \cos x\, dx \quad\Rightarrow\quad du = -(n-1)\cos^{n-2} x\, \sin x\, dx$, $v = \sin x$ in (2):

$$\begin{aligned}\textstyle\int \cos^n x\, dx &= \cos^{n-1} x\, \sin x + (n-1) \textstyle\int \cos^{n-2} x\, \sin^2 x\, dx \\ &= \cos^{n-1} x\, \sin x + (n-1) \textstyle\int \cos^{n-2} x\,(1 - \cos^2 x) dx \\ &= \cos^{n-1} x\, \sin x + (n-1) \textstyle\int \cos^{n-2} x\, dx - (n-1) \textstyle\int \cos^n x\, dx\end{aligned}$$

Rearranging terms gives $n \int \cos^n x\, dx = \cos^{n-1} x\, \sin x + (n-1) \int \cos^{n-2} x\, dx$ or

$$\int \cos^n x\, dx = \frac{1}{n} \cos^{n-1} x\, \sin x + \frac{n-1}{n} \int \cos^{n-2} x\, dx$$

(b) Take $n = 2$ in part (a) to get $\int \cos^2 x\, dx = \frac{1}{2} \cos x\, \sin x + \frac{1}{2} \int 1\, dx = \dfrac{x}{2} + \dfrac{\sin 2x}{4} + C$.

(c) $\int \cos^4 x\, dx = \frac{1}{4} \cos^3 x\, \sin x + \frac{3}{4} \int \cos^2 x\, dx = \frac{1}{4} \cos^3 x\, \sin x + \frac{3}{8} x + \frac{3}{16} \sin 2x + C$

55. (a) From Example 6, $\displaystyle\int \sin^n x\, dx = -\frac{1}{n} \cos x\, \sin^{n-1} x + \frac{n-1}{n} \int \sin^{n-2} x\, dx$. Using (6),

$$\begin{aligned}\int_0^{\pi/2} \sin^n x\, dx &= \left[-\frac{\cos x\, \sin^{n-1} x}{n}\right]_0^{\pi/2} + \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x\, dx \\ &= (0 - 0) + \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x\, dx = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2} x\, dx\end{aligned}$$

(b) Using $n = 3$ in part (a), we have $\int_0^{\pi/2} \sin^3 x\, dx = \frac{2}{3} \int_0^{\pi/2} \sin x\, dx = \left[-\frac{2}{3} \cos x\right]_0^{\pi/2} = \frac{2}{3}$.

Using $n = 5$ in part (a), we have $\int_0^{\pi/2} \sin^5 x\, dx = \frac{4}{5} \int_0^{\pi/2} \sin^3 x\, dx = \frac{4}{5} \cdot \frac{2}{3} = \frac{8}{15}$.

(c) The formula holds for $n = 1$ (that is, $2n + 1 = 3$) by (b). Assume it holds for some $k \geq 1$. Then

$\displaystyle\int_0^{\pi/2} \sin^{2k+1} x\, dx = \frac{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2k)}{3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2k+1)}$. By Example 6,

$$\begin{aligned}\int_0^{\pi/2} \sin^{2k+3} x\, dx &= \frac{2k+2}{2k+3} \int_0^{\pi/2} \sin^{2k+1} x\, dx = \frac{2k+2}{2k+3} \cdot \frac{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2k)}{3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2k+1)} \\ &= \frac{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2k)[2\,(k+1)]}{3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2k+1)[2\,(k+1)+1]},\end{aligned}$$

so the formula holds for $n = k + 1$. By induction, the formula holds for all $n \geq 1$.

56. Using Exercise 53(a), we see that the formula holds for $n = 1$, because $\int_0^{\pi/2} \sin^2 x\, dx = \frac{1}{2} \int_0^{\pi/2} 1\, dx = \frac{1}{2}\left[\, x\, \right]_0^{\pi/2} = \frac{1}{2} \cdot \frac{\pi}{2}$.

Now assume it holds for some $k \geq 1$. Then $\displaystyle\int_0^{\pi/2} \sin^{2k} x\, dx = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2k-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2k)} \frac{\pi}{2}$. By Exercise 53(a),

$$\begin{aligned}\int_0^{\pi/2} \sin^{2(k+1)} x\, dx &= \frac{2k+1}{2k+2} \int_0^{\pi/2} \sin^{2k} x\, dx = \frac{2k+1}{2k+2} \cdot \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2k-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2k)} \frac{\pi}{2} \\ &= \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2k-1)(2k+1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2k)(2k+2)} \cdot \frac{\pi}{2},\end{aligned}$$

so the formula holds for $n = k + 1$. By induction, the formula holds for all $n \geq 1$.

57. Let $u = (\ln x)^n$, $dv = dx \quad\Rightarrow\quad du = n(\ln x)^{n-1}(dx/x)$, $v = x$. By Equation 2,

$\int (\ln x)^n\, dx = x(\ln x)^n - \int nx(\ln x)^{n-1}(dx/x) = x(\ln x)^n - n \int (\ln x)^{n-1}\, dx$.

58. Let $u = x^n$, $dv = e^x\,dx \;\Rightarrow\; du = nx^{n-1}\,dx$, $v = e^x$. By Equation 2, $\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx$.

59. $\int \tan^n x\,dx = \int \tan^{n-2} x\,\tan^2 x\,dx = \int \tan^{n-2} x\,(\sec^2 x - 1)\,dx = \int \tan^{n-2} x\,\sec^2 x\,dx - \int \tan^{n-2} x\,dx$

$= I - \int \tan^{n-2} x\,dx.$

Let $u = \tan^{n-2} x$, $dv = \sec^2 x\,dx \;\Rightarrow\; du = (n-2)\,\tan^{n-3} x\,\sec^2 x\,dx$, $v = \tan x$. Then, by Equation 2,

$$\begin{aligned} I &= \tan^{n-1} x - (n-2)\int \tan^{n-2} x\,\sec^2 x\,dx \\ 1I &= \tan^{n-1} x - (n-2)I \\ (n-1)I &= \tan^{n-1} x \\ I &= \frac{\tan^{n-1} x}{n-1} \end{aligned}$$

Returning to the original integral, $\int \tan^n x\,dx = \dfrac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx$.

60. Let $u = \sec^{n-2} x$, $dv = \sec^2 x\,dx \;\Rightarrow\; du = (n-2)\,\sec^{n-3} x\,\sec x\,\tan x\,dx$, $v = \tan x$. Then, by Equation 2,

$$\begin{aligned} \int \sec^n x\,dx &= \tan x\,\sec^{n-2} x - (n-2)\int \sec^{n-2} x\,\tan^2 x\,dx \\ &= \tan x\,\sec^{n-2} x - (n-2)\int \sec^{n-2} x\,(\sec^2 x - 1)\,dx \\ &= \tan x\,\sec^{n-2} x - (n-2)\int \sec^n x\,dx + (n-2)\int \sec^{n-2} x\,dx \end{aligned}$$

so $(n-1)\int \sec^n x\,dx = \tan x\,\sec^{n-2} x + (n-2)\int \sec^{n-2} x\,dx$. If $n - 1 \neq 0$, then

$$\int \sec^n x\,dx = \frac{\tan x\,\sec^{n-2} x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx.$$

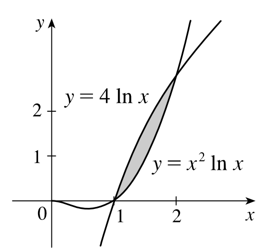
61. By repeated applications of the reduction formula in Exercise 57,

$$\begin{aligned} \int (\ln x)^3\,dx &= x\,(\ln x)^3 - 3\int (\ln x)^2\,dx = x(\ln x)^3 - 3\big[x(\ln x)^2 - 2\int (\ln x)^1\,dx\big] \\ &= x\,(\ln x)^3 - 3x(\ln x)^2 + 6\big[x(\ln x)^1 - 1\int (\ln x)^0\,dx\big] \\ &= x\,(\ln x)^3 - 3x(\ln x)^2 + 6x\ln x - 6\int 1\,dx = x\,(\ln x)^3 - 3x(\ln x)^2 + 6x\ln x - 6x + C \end{aligned}$$

62. By repeated applications of the reduction formula in Exercise 58,

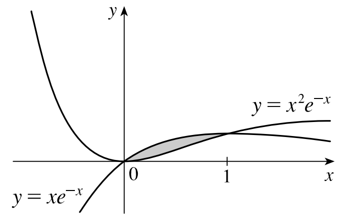
$$\begin{aligned} \int x^4 e^x\,dx &= x^4 e^x - 4\int x^3 e^x\,dx = x^4 e^x - 4\big(x^3 e^x - 3\int x^2 e^x\,dx\big) \\ &= x^4 e^x - 4x^3 e^x + 12\big(x^2 e^x - 2\int x^1 e^x\,dx\big) = x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24\big(x^1 e^x - \int x^0 e^x\,dx\big) \\ &= x^4 e^x - 4x^3 e^x + 12x^2 e^x - 24xe^x + 24e^x + C \quad \big[\text{or } e^x(x^4 - 4x^3 + 12x^2 - 24x + 24) + C\big] \end{aligned}$$

63. The curves $y = x^2\ln x$ and $y = 4\ln x$ intersect when $x^2\ln x = 4\ln x \;\Leftrightarrow$ $x^2\ln x - 4\ln x = 0 \;\Leftrightarrow\; (x^2 - 4)\ln x = 0 \;\Leftrightarrow$ $x = 1$ or 2 [since $x > 0$]. For $1 < x < 2$, $4\ln x > x^2\ln x$. Thus, area $= \int_1^2 (4\ln x - x^2\ln x)\,dx = \int_1^2 [(4 - x^2)\ln x]\,dx$. Let $u = \ln x$, $dv = (4 - x^2)\,dx \;\Rightarrow\; du = \frac{1}{x}\,dx$, $v = 4x - \frac{1}{3}x^3$. Then



$$\begin{aligned} \text{area} &= \big[(\ln x)\big(4x - \tfrac{1}{3}x^3\big)\big]_1^2 - \int_1^2 \left[\big(4x - \tfrac{1}{3}x^3\big)\frac{1}{x}\right] dx = (\ln 2)\big(\tfrac{16}{3}\big) - 0 - \int_1^2 \big(4 - \tfrac{1}{3}x^2\big)\,dx \\ &= \tfrac{16}{3}\ln 2 - \big[4x - \tfrac{1}{9}x^3\big]_1^2 = \tfrac{16}{3}\ln 2 - \big(\tfrac{64}{9} - \tfrac{35}{9}\big) = \tfrac{16}{3}\ln 2 - \tfrac{29}{9} \end{aligned}$$

64. The curves $y = x^2 e^{-x}$ and $y = xe^{-x}$ intersect when $x^2 e^{-x} = xe^{-x} \quad \Leftrightarrow$ $x^2 - x = 0 \quad \Leftrightarrow \quad x(x-1) = 0 \quad \Leftrightarrow \quad x = 0$ or 1.



For $0 < x < 1$, $xe^{-x} > x^2 e^{-x}$. Thus,

area $= \int_0^1 (xe^{-x} - x^2 e^{-x})\,dx = \int_0^1 (x - x^2)e^{-x}\,dx$. Let $u = x - x^2$,

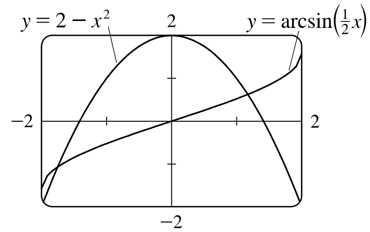
$dv = e^{-x}dx \quad \Rightarrow \quad du = (1-2x)\,dx$, $v = -e^{-x}$. Then

area $= \left[(x - x^2)(-e^{-x})\right]_0^1 - \int_0^1 [-e^{-x}(1-2x)]\,dx = 0 + \int_0^1 (1-2x)e^{-x}\,dx$.

Now let $U = 1 - 2x$, $dV = e^{-x}dx \quad \Rightarrow \quad dU = -2\,dx$, $V = -e^{-x}$. Now

area $= \left[(1-2x)(-e^{-x})\right]_0^1 - \int_0^1 2e^{-x}\,dx = e^{-1} + 1 - \left[-2e^{-x}\right]_0^1 = e^{-1} + 1 + 2(e^{-1} - 1) = 3e^{-1} - 1$.

65. The curves $y = \arcsin\left(\frac{1}{2}x\right)$ and $y = 2 - x^2$ intersect at $x = a \approx -1.75119$ and $x = b \approx 1.17210$. From the figure, the area bounded by the curves is given by



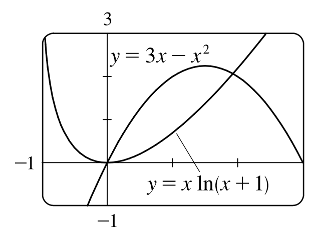
$A = \int_a^b [(2 - x^2) - \arcsin\left(\frac{1}{2}x\right)]\,dx = \left[2x - \frac{1}{3}x^3\right]_a^b - \int_a^b \arcsin\left(\frac{1}{2}x\right)dx$.

Let $u = \arcsin\left(\frac{1}{2}x\right)$, $dv = dx \quad \Rightarrow \quad du = \dfrac{1}{\sqrt{1 - \left(\frac{1}{2}x\right)^2}} \cdot \dfrac{1}{2}\,dx$, $v = x$.

Then

$$\begin{aligned} A &= \left[2x - \frac{1}{3}x^3\right]_a^b - \left\{\left[x \arcsin\left(\frac{1}{2}x\right)\right]_a^b - \int_a^b \frac{x}{2\sqrt{1 - \frac{1}{4}x^2}}\,dx\right\} \\ &= \left[2x - \tfrac{1}{3}x^3 - x\arcsin\left(\tfrac{1}{2}x\right) - 2\sqrt{1 - \tfrac{1}{4}x^2}\,\right]_a^b \approx 3.99926 \end{aligned}$$

66. The curves $y = x\ln(x+1)$ and $y = 3x - x^2$ intersect at $x = 0$ and $x = a \approx 1.92627$. From the figure, the area bounded by the curves is given by



$A = \int_0^a [(3x - x^2) - x\ln(x+1)]\,dx = \left[\frac{3}{2}x^2 - \frac{1}{3}x^3\right]_0^a - \int_0^a x\ln(x+1)\,dx$.

Let $u = \ln(x+1)$, $dv = x\,dx \quad \Rightarrow \quad du = \dfrac{1}{x+1}\,dx$, $v = \frac{1}{2}x^2$. Then

$$\begin{aligned} A &= \left[\frac{3}{2}x^2 - \frac{1}{3}x^3\right]_0^a - \left\{\left[\frac{1}{2}x^2\ln(x+1)\right]_0^a - \frac{1}{2}\int_0^a \frac{x^2}{x+1}\,dx\right\} \\ &= \left[\frac{3}{2}x^2 - \frac{1}{3}x^3\right]_0^a - \left[\frac{1}{2}x^2\ln(x+1)\right]_0^a + \frac{1}{2}\int_0^a \left(x - 1 + \frac{1}{x+1}\right)dx \\ &= \left[\tfrac{3}{2}x^2 - \tfrac{1}{3}x^3 - \tfrac{1}{2}x^2\ln(x+1) + \tfrac{1}{4}x^2 - \tfrac{1}{2}x + \tfrac{1}{2}\ln|x+1|\right]_0^a \approx 1.69260 \end{aligned}$$

67. Volume $= \int_0^1 2\pi x\cos(\pi x/2)\,dx$. Let $u = x$, $dv = \cos(\pi x/2)\,dx \quad \Rightarrow \quad du = dx$, $v = \frac{2}{\pi}\sin(\pi x/2)$.

$$V = 2\pi\left[\frac{2}{\pi}x\sin\left(\frac{\pi x}{2}\right)\right]_0^1 - 2\pi\cdot\frac{2}{\pi}\int_0^1 \sin\left(\frac{\pi x}{2}\right)dx = 2\pi\left(\frac{2}{\pi} - 0\right) - 4\left[-\frac{2}{\pi}\cos\left(\frac{\pi x}{2}\right)\right]_0^1 = 4 + \frac{8}{\pi}(0 - 1) = 4 - \frac{8}{\pi}.$$

68. Volume $= \int_0^1 2\pi x(e^x - e^{-x})\,dx = 2\pi\int_0^1 (xe^x - xe^{-x})\,dx = 2\pi\left[\int_0^1 xe^x\,dx - \int_0^1 xe^{-x}\,dx\right]$ [both integrals by parts]

$$= 2\pi\left[(xe^x - e^x) - \left(-xe^{-x} - e^{-x}\right)\right]_0^1 = 2\pi[2/e - 0] = 4\pi/e$$

69. Volume $= \int_{-1}^{0} 2\pi(1-x)e^{-x}\,dx$. Let $u = 1-x$, $dv = e^{-x}\,dx \Rightarrow du = -\,dx$, $v = -e^{-x}$.

$V = 2\pi\big[(1-x)(-e^{-x})\big]_{-1}^{0} - 2\pi\int_{-1}^{0} e^{-x}\,dx = 2\pi\big[(x-1)(e^{-x}) + e^{-x}\big]_{-1}^{0} = 2\pi\big[xe^{-x}\big]_{-1}^{0} = 2\pi(0+e) = 2\pi e.$

70. $y = e^x \Leftrightarrow x = \ln y$. Volume $= \int_1^3 2\pi y\,\ln y\,dy$. Let $u = \ln y$, $dv = y\,dy \Rightarrow du = \dfrac{1}{y}\,dy$, $v = \frac{1}{2}y^2$.

$$\begin{aligned} V &= 2\pi\big[\tfrac{1}{2}y^2\ln y\big]_1^3 - 2\pi\textstyle\int_1^3 \tfrac{1}{2}y\,dy = 2\pi\big[\tfrac{1}{2}y^2\ln y - \tfrac{1}{4}y^2\big]_1^3 \\ &= 2\pi\big[\big(\tfrac{9}{2}\ln 3 - \tfrac{9}{4}\big) - \big(0 - \tfrac{1}{4}\big)\big] = 2\pi\big(\tfrac{9}{2}\ln 3 - 2\big) = (9\ln 3 - 4)\,\pi \end{aligned}$$

71. (a) Use shells about the y-axis:

$$\begin{aligned} V &= \int_1^2 2\pi x\ln x\,dx \qquad \begin{bmatrix} u = \ln x, & dv = x\,dx \\ du = \frac{1}{x}\,dx, & v = \frac{1}{2}x^2 \end{bmatrix} \\ &= 2\pi\Big\{\big[\tfrac{1}{2}x^2\ln x\big]_1^2 - \textstyle\int_1^2 \tfrac{1}{2}x\,dx\Big\} = 2\pi\Big\{(2\ln 2 - 0) - \big[\tfrac{1}{4}x^2\big]_1^2\Big\} = 2\pi\big(2\ln 2 - \tfrac{3}{4}\big) \end{aligned}$$

(b) Use disks about the x-axis:

$$\begin{aligned} V &= \int_1^2 \pi(\ln x)^2\,dx \qquad \begin{bmatrix} u = (\ln x)^2, & dv = dx \\ du = 2\ln x\cdot\frac{1}{x}\,dx, & v = x \end{bmatrix} \\ &= \pi\Big\{\big[x(\ln x)^2\big]_1^2 - \int_1^2 2\ln x\,dx\Big\} \qquad \begin{bmatrix} u = \ln x, & dv = dx \\ du = \frac{1}{x}\,dx, & v = x \end{bmatrix} \\ &= \pi\Big\{2(\ln 2)^2 - 2\Big(\Big[x\ln x\Big]_1^2 - \textstyle\int_1^2 dx\Big)\Big\} = \pi\Big\{2(\ln 2)^2 - 4\ln 2 + 2\Big[x\Big]_1^2\Big\} \\ &= \pi[2(\ln 2)^2 - 4\ln 2 + 2] = 2\pi[(\ln 2)^2 - 2\ln 2 + 1] \end{aligned}$$

72.
$$\begin{aligned} f_{\text{ave}} &= \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{\pi/4 - 0}\int_0^{\pi/4} x\sec^2 x\,dx \qquad \begin{bmatrix} u = x, & dv = \sec^2 x\,dx \\ du = dx, & v = \tan x \end{bmatrix} \\ &= \frac{4}{\pi}\left\{\Big[x\tan x\Big]_0^{\pi/4} - \int_0^{\pi/4}\tan x\,dx\right\} = \frac{4}{\pi}\left\{\frac{\pi}{4} - \Big[\ln|\sec x|\Big]_0^{\pi/4}\right\} = \frac{4}{\pi}\left(\frac{\pi}{4} - \ln\sqrt{2}\right) \\ &= 1 - \tfrac{4}{\pi}\ln\sqrt{2} \;\text{ or }\; 1 - \tfrac{2}{\pi}\ln 2 \end{aligned}$$

73. $S(x) = \displaystyle\int_0^x \sin\big(\tfrac{1}{2}\pi t^2\big)\,dt \Rightarrow \int S(x)\,dx = \int\left[\int_0^x \sin\big(\tfrac{1}{2}\pi t^2\big)\,dt\right]dx.$

Let $u = \displaystyle\int_0^x \sin\big(\tfrac{1}{2}\pi t^2\big)\,dt = S(x)$, $dv = dx \Rightarrow du = \sin\big(\tfrac{1}{2}\pi x^2\big)\,dx$, $v = x$. Thus,

$$\begin{aligned} \int S(x)\,dx &= xS(x) - \int x\sin\big(\tfrac{1}{2}\pi x^2\big)\,dx = xS(x) - \int \sin y\,\big(\tfrac{1}{\pi}\,dy\big) \qquad \begin{bmatrix} u = \frac{1}{2}\pi x^2, \\ du = \pi x\,dx \end{bmatrix} \\ &= xS(x) + \tfrac{1}{\pi}\cos y + C = xS(x) + \tfrac{1}{\pi}\cos\big(\tfrac{1}{2}\pi x^2\big) + C \end{aligned}$$

74. (a) The rocket will have height $H = \int_0^T v(t)\,dt$ after T seconds.

$$\begin{aligned} H &= \int_0^T\left[-gt - v_e\ln\left(\frac{m-rt}{m}\right)\right]dt = -g\big[\tfrac{1}{2}t^2\big]_0^T - v_e\left[\int_0^T \ln(m-rt)\,dt - \int_0^T \ln m\,dt\right] \\ &= -\tfrac{1}{2}gT^2 + v_e(\ln m)T - v_e\textstyle\int_0^T \ln(m-rt)\,dt \end{aligned}$$

[continued]

Let $u = \ln(m - rt)$, $dv = dt \Rightarrow du = \dfrac{1}{m - rt}(-r)\,dt$, $v = t$. Then

$$\int_0^T \ln(m - rt)\,dt = \Big[t\ln(m - rt)\Big]_0^T + \int_0^T \frac{rt}{m - rt}\,dt = T\ln(m - rT) + \int_0^T \left(-1 + \frac{m}{m - rt}\right)dt$$
$$= T\ln(m - rT) + \Big[-t - \frac{m}{r}\ln(m - rt)\Big]_0^T$$
$$= T\ln(m - rT) - T - \frac{m}{r}\ln(m - rT) + \frac{m}{r}\ln m$$

So $H = -\frac{1}{2}gT^2 + v_e(\ln m)T - v_eT\ln(m - rT) + v_eT + \dfrac{m}{r}v_e\ln(m - rT) - \dfrac{m}{r}v_e\ln m$. Substituting $T = 60$, $g = 9.8$, $m = 30{,}000$, $r = 160$, and $v_e = 3000$ gives us $H \approx 14{,}844$ m.

(b) The time taken to consume 6000 kg of fuel is $T = \dfrac{6000}{r} = \dfrac{6000}{160} = 37.5$ s. The rocket will have height $H = \int_0^{37.5} v(t)\,dt$ after 37.5 seconds. Evaluating this integral using the results of part (a) with $T = 37.5$, $g = 9.8$, $m = 30{,}000$, $r = 160$, and $v_e = 3000$ gives us $H \approx 5195$ m.

75. Since $v(t) > 0$ for all t, the desired distance is $s(t) = \int_0^t v(w)\,dw = \int_0^t w^2e^{-w}\,dw$.

First let $u = w^2$, $dv = e^{-w}\,dw \Rightarrow du = 2w\,dw$, $v = -e^{-w}$. Then $s(t) = \big[-w^2e^{-w}\big]_0^t + 2\int_0^t we^{-w}\,dw$.

Next let $U = w$, $dV = e^{-w}\,dw \Rightarrow dU = dw$, $V = -e^{-w}$. Then

$$s(t) = -t^2e^{-t} + 2\left(\big[-we^{-w}\big]_0^t + \int_0^t e^{-w}\,dw\right) = -t^2e^{-t} + 2\left(-te^{-t} + 0 + \big[-e^{-w}\big]_0^t\right)$$
$$= -t^2e^{-t} + 2(-te^{-t} - e^{-t} + 1) = -t^2e^{-t} - 2te^{-t} - 2e^{-t} + 2 = 2 - e^{-t}(t^2 + 2t + 2) \text{ meters}$$

76. Suppose $f(0) = g(0) = 0$ and let $u = f(x)$, $dv = g''(x)\,dx \Rightarrow du = f'(x)\,dx$, $v = g'(x)$.

Then $\int_0^a f(x)\,g''(x)\,dx = \Big[f(x)\,g'(x)\Big]_0^a - \int_0^a f'(x)\,g'(x)\,dx = f(a)\,g'(a) - \int_0^a f'(x)\,g'(x)\,dx$.

Now let $U = f'(x)$, $dV = g'(x)\,dx \Rightarrow dU = f''(x)\,dx$ and $V = g(x)$, so

$\int_0^a f'(x)\,g'(x)\,dx = \Big[f'(x)\,g(x)\Big]_0^a - \int_0^a f''(x)\,g(x)\,dx = f'(a)\,g(a) - \int_0^a f''(x)\,g(x)\,dx$.

Combining the two results, we get $\int_0^a f(x)\,g''(x)\,dx = f(a)\,g'(a) - f'(a)\,g(a) + \int_0^a f''(x)\,g(x)\,dx$.

77. For $I = \int_1^4 xf''(x)\,dx$, let $u = x$, $dv = f''(x)\,dx \Rightarrow du = dx$, $v = f'(x)$. Then

$I = \big[xf'(x)\big]_1^4 - \int_1^4 f'(x)\,dx = 4f'(4) - 1\cdot f'(1) - [f(4) - f(1)] = 4\cdot 3 - 1\cdot 5 - (7 - 2) = 12 - 5 - 5 = 2$.

We used the fact that f'' is continuous to guarantee that I exists.

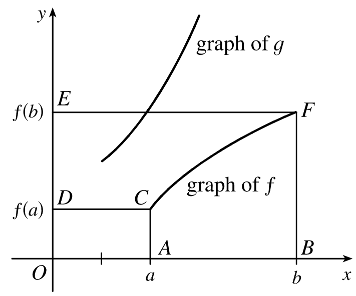
78. (a) Take $g(x) = x$ and $g'(x) = 1$ in Equation 1.

(b) By part (a), $\int_a^b f(x)\,dx = bf(b) - a\,f(a) - \int_a^b x\,f'(x)\,dx$. Now let $y = f(x)$, so that $x = g(y)$ and $dy = f'(x)\,dx$. Then $\int_a^b x\,f'(x)\,dx = \int_{f(a)}^{f(b)} g(y)\,dy$. The result follows.

(c) Part (b) says that the area of region $ABFC$ is

$$= \quad bf(b) \quad - \quad af(a) \quad - \quad \int_{f(a)}^{f(b)} g(y)\,dy$$

$$= \text{(area of rectangle } OBFE\text{)} - \text{(area of rectangle } OACD\text{)} - \text{(area of region } DCFE\text{)}$$



(d) We have $f(x) = \ln x$, so $f^{-1}(x) = e^x$, and since $g = f^{-1}$, we have $g(y) = e^y$. By part (b),

$$\int_1^e \ln x\,dx = e\ln e - 1\ln 1 - \int_{\ln 1}^{\ln e} e^y\,dy = e - \int_0^1 e^y\,dy = e - \left[e^y\right]_0^1 = e - (e-1) = 1.$$

79. (a) Assuming $f(x)$ and $g(x)$ are differentiable functions, the Quotient Rule for differentiation states $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{g(x)\,f'(x) - f(x)\,g'(x)}{[g(x)]^2}$. Writing in integral form gives $\dfrac{f(x)}{g(x)} = \displaystyle\int \frac{g(x)\,f'(x) - f(x)\,g'(x)}{[g(x)]^2}\,dx = \int \frac{1}{g(x)}\,f'(x)\,dx - \int \frac{f(x)}{[g(x)]^2}\,g'(x)\,dx$. Now let $u = f(x)$ and $v = g(x)$ so that $du = f'(x)\,dx$ and $dv = g'(x)\,dx$. Substituting into the above equation gives $\dfrac{u}{v} = \displaystyle\int \frac{1}{v}\,du - \int \frac{u}{v^2}\,dv \quad\Rightarrow$ $\displaystyle\int \frac{u}{v^2}\,dv = -\frac{u}{v} + \int \frac{1}{v}\,du.$

(b) Let $u = \ln x$, $v = x \quad\Rightarrow\quad du = \dfrac{1}{x}\,dx$. Then, using the formula from part (a), we get

$$\int \frac{\ln x}{x^2}\,dx = -\frac{\ln x}{x} + \int \frac{1}{x}\left(\frac{1}{x}\,dx\right) = -\frac{\ln x}{x} + \int \frac{1}{x^2}\,dx = -\frac{\ln x}{x} - \frac{1}{x} + C.$$

80. (a) We note that for $0 \le x \le \frac{\pi}{2}$, $0 \le \sin x \le 1$, so $\sin^{2n+2} x \le \sin^{2n+1} x \le \sin^{2n} x$. So by the second Comparison Property of the Integral, $I_{2n+2} \le I_{2n+1} \le I_{2n}$.

(b) Substituting directly into the result from Exercise 56, we get

$$\frac{I_{2n+2}}{I_{2n}} = \frac{\dfrac{1\cdot 3\cdot 5\cdot\cdots\cdot[2(n+1)-1]}{2\cdot 4\cdot 6\cdot\cdots\cdot[2(n+1)]}\dfrac{\pi}{2}}{\dfrac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}\dfrac{\pi}{2}} = \frac{2(n+1)-1}{2(n+1)} = \frac{2n+1}{2n+2}$$

(c) We divide the result from part (a) by I_{2n}. The inequalities are preserved since I_{2n} is positive: $\dfrac{I_{2n+2}}{I_{2n}} \le \dfrac{I_{2n+1}}{I_{2n}} \le \dfrac{I_{2n}}{I_{2n}}$. Now from part (b), the left term is equal to $\dfrac{2n+1}{2n+2}$, so the expression becomes $\dfrac{2n+1}{2n+2} \le \dfrac{I_{2n+1}}{I_{2n}} \le 1$. Now $\displaystyle\lim_{n\to\infty} \frac{2n+1}{2n+2} = \lim_{n\to\infty} 1 = 1$, so by the Squeeze Theorem, $\displaystyle\lim_{n\to\infty} \frac{I_{2n+1}}{I_{2n}} = 1$.

(d) We substitute the results from Exercises 55 and 56 into the result from part (c):

$$1 = \lim_{n\to\infty} \frac{I_{2n+1}}{I_{2n}} = \lim_{n\to\infty} \frac{\dfrac{2\cdot4\cdot6\cdot\dots\cdot(2n)}{3\cdot5\cdot7\cdot\dots\cdot(2n+1)}}{\dfrac{1\cdot3\cdot5\cdot\dots\cdot(2n-1)}{2\cdot4\cdot6\cdot\dots\cdot(2n)}\dfrac{\pi}{2}} = \lim_{n\to\infty}\left[\frac{2\cdot4\cdot6\cdot\dots\cdot(2n)}{3\cdot5\cdot7\cdot\dots\cdot(2n+1)}\right]\left[\frac{2\cdot4\cdot6\cdot\dots\cdot(2n)}{1\cdot3\cdot5\cdot\dots\cdot(2n-1)}\left(\frac{2}{\pi}\right)\right]$$

$$= \lim_{n\to\infty} \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\dots\cdot\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\cdot\frac{2}{\pi} \qquad \text{[rearrange terms]}$$

Multiplying both sides by $\frac{\pi}{2}$ gives us the *Wallis product*:

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\dots$$

(e) The area of the kth rectangle is k. At the $2n$th step, the area is increased from $2n-1$ to $2n$ by multiplying the width by $\dfrac{2n}{2n-1}$, and at the $(2n+1)$th step, the area is increased from $2n$ to $2n+1$ by multiplying the height by $\dfrac{2n+1}{2n}$. These two steps multiply the ratio of width to height by $\dfrac{2n}{2n-1}$ and $\dfrac{1}{(2n+1)/(2n)} = \dfrac{2n}{2n+1}$, respectively. So, by part (d), the limiting ratio is $\dfrac{2}{1}\cdot\dfrac{2}{3}\cdot\dfrac{4}{3}\cdot\dfrac{4}{5}\cdot\dfrac{6}{5}\cdot\dfrac{6}{7}\cdot\dots = \dfrac{\pi}{2}$.

81. Using the formula for volumes of rotation and the figure, we see that

Volume $= \int_0^d \pi b^2\,dy - \int_0^c \pi a^2\,dy - \int_c^d \pi[g(y)]^2\,dy = \pi b^2 d - \pi a^2 c - \int_c^d \pi[g(y)]^2\,dy$. Let $y = f(x)$,

which gives $dy = f'(x)\,dx$ and $g(y) = x$, so that $V = \pi b^2 d - \pi a^2 c - \pi\int_a^b x^2 f'(x)\,dx$.

Now integrate by parts with $u = x^2$, and $dv = f'(x)\,dx \quad\Rightarrow\quad du = 2x\,dx$, $v = f(x)$, and

$\int_a^b x^2 f'(x)\,dx = \left[x^2 f(x)\right]_a^b - \int_a^b 2x\,f(x)\,dx = b^2 f(b) - a^2 f(a) - \int_a^b 2x\,f(x)\,dx$, but $f(a) = c$ and $f(b) = d \quad\Rightarrow$

$V = \pi b^2 d - \pi a^2 c - \pi\left[b^2 d - a^2 c - \int_a^b 2x\,f(x)\,dx\right] = \int_a^b 2\pi x\,f(x)\,dx$.

7.2 Trigonometric Integrals

The symbols $\overset{s}{=}$ and $\overset{c}{=}$ indicate the use of the substitutions $\{u = \sin x, du = \cos x\,dx\}$ and $\{u = \cos x, du = -\sin x\,dx\}$, respectively.

1. $\int \sin^3 x\cos^2 x\,dx = \int \sin^2 x\cos^2 x\sin x\,dx = \int (1-\cos^2 x)\cos^2 x\sin x\,dx \overset{c}{=} \int (1-u^2)u^2\,(-du)$

$= \int (u^4 - u^2)\,du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C = \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C$

2. $\int \cos^6 y\sin^3 y\,dy = \int \cos^6 y\sin^2 y\sin y\,dy = \int \cos^6 y(1-\cos^2 y)\sin y\,dy \overset{c}{=} \int u^6(1-u^2)\,(-du)$

$= \int (u^8 - u^6)\,du = \frac{1}{9}u^9 - \frac{1}{7}u^7 + C = \frac{1}{9}\cos^9 y - \frac{1}{7}\cos^7 y + C$

3. $\displaystyle\int_0^{\pi/2} \cos^9 x\sin^5 x\,dx = \int_0^{\pi/2} \cos^9 x\,(\sin^2 x)^2\sin x\,dx = \int_0^{\pi/2} \cos^9 x\,(1-\cos^2 x)^2\sin x\,dx$

$\displaystyle\overset{c}{=} \int_1^0 u^9(1-u^2)^2\,(-du) = \int_0^1 u^9(1-2u^2+u^4)\,du = \int_0^1 (u^9 - 2u^{11} + u^{13})\,du$

$= \left[\frac{1}{10}u^{10} - \frac{1}{6}u^{12} + \frac{1}{14}u^{14}\right]_0^1 = \left(\frac{1}{10} - \frac{1}{6} + \frac{1}{14}\right) - 0 = \frac{1}{210}$

4. $\displaystyle\int_0^{\pi/4} \sin^5 x\,dx = \int_0^{\pi/4} (\sin^2 x)^2 \sin x\,dx = \int_0^{\pi/4} (1-\cos^2 x)^2 \sin x\,dx \overset{c}{=} \int_1^{1/\sqrt{2}} (1-u^2)^2\,(-du)$

$\displaystyle = \int_{1/\sqrt{2}}^1 (1-2u^2+u^4)\,du = \left[u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right]_{1/\sqrt{2}}^1 = \left(1-\frac{2}{3}+\frac{1}{5}\right) - \left(\frac{1}{\sqrt{2}} - \frac{2}{3\sqrt{8}} + \frac{1}{5\sqrt{32}}\right)$

$\displaystyle = \frac{8}{15} - \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{6} + \frac{\sqrt{2}}{40}\right) = \frac{8}{15} - \frac{43\sqrt{2}}{120}$

5. $\int \sin^5(2t)\cos^2(2t)\,dt = \int \sin^4(2t)\,\cos^2(2t)\,\sin(2t)\,dt = \int [1-\cos^2(2t)]^2 \cos^2(2t)\,\sin(2t)\,dt$

$= \int (1-u^2)^2\,u^2\,\left(-\frac{1}{2}\,du\right) \qquad [u=\cos(2t),\ du=-2\sin(2t)\,dt]$

$= -\frac{1}{2}\int (u^4-2u^2+1)u^2\,du = -\frac{1}{2}\int (u^6-2u^4+u^2)\,du$

$= -\frac{1}{2}\left(\frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3\right) + C = -\frac{1}{14}\cos^7(2t) + \frac{1}{5}\cos^5(2t) - \frac{1}{6}\cos^3(2t) + C$

6. $\displaystyle\int \cos^3\left(\frac{t}{2}\right)\sin^2\left(\frac{t}{2}\right)dt = \int \cos^2\left(\frac{t}{2}\right)\sin^2\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)dt = \int \left(1-\sin^2\left(\frac{t}{2}\right)\right)\sin^2\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)dt$

$\displaystyle = \int (1-u^2)u^2(2\,du) \qquad \left[u=\sin\left(\frac{t}{2}\right),\ du=\frac{1}{2}\cos\left(\frac{t}{2}\right)dt\right]$

$\displaystyle = 2\int (u^2-u^4)\,du = 2\left(\frac{1}{3}u^3 - \frac{1}{5}u^5\right) + C = \frac{2}{3}\sin^3\left(\frac{t}{2}\right) - \frac{2}{5}\sin^5\left(\frac{t}{2}\right) + C$

7. $\int_0^{\pi/2} \cos^2\theta\,d\theta = \int_0^{\pi/2} \frac{1}{2}(1+\cos 2\theta)\,d\theta \qquad$ [half-angle identity]

$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \frac{1}{2}\left[\left(\frac{\pi}{2}+0\right) - (0+0)\right] = \frac{\pi}{4}$

8. $\displaystyle\int_0^{\pi/4} \sin^2(2\theta)\,d\theta = \int_0^{\pi/4} \frac{1}{2}(1-\cos 4\theta)\,d\theta \qquad$ [half-angle identity]

$\displaystyle = \frac{1}{2}\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/4} = \frac{1}{2}\left[\left(\frac{\pi}{4}-0\right) - 0\right] = \frac{\pi}{8}$

9. $\int_0^{\pi} \cos^4(2t)\,dt = \int_0^{\pi} [\cos^2(2t)]^2\,dt = \int_0^{\pi} \left[\frac{1}{2}(1+\cos(2\cdot 2t))\right]^2 dt \qquad$ [half-angle identity]

$= \frac{1}{4}\int_0^{\pi} [1+2\cos 4t+\cos^2(4t)]\,dt = \frac{1}{4}\int_0^{\pi} [1+2\cos 4t+\frac{1}{2}(1+\cos 8t)]\,dt$

$= \frac{1}{4}\int_0^{\pi} \left(\frac{3}{2}+2\cos 4t+\frac{1}{2}\cos 8t\right)dt = \frac{1}{4}\left[\frac{3}{2}t + \frac{1}{2}\sin 4t + \frac{1}{16}\sin 8t\right]_0^{\pi} = \frac{1}{4}\left[\left(\frac{3}{2}\pi+0+0\right) - 0\right] = \frac{3}{8}\pi$

10. $\int_0^{\pi} \sin^2 t\,\cos^4 t\,dt = \frac{1}{4}\int_0^{\pi} (4\sin^2 t\,\cos^2 t)\cos^2 t\,dt = \frac{1}{4}\int_0^{\pi} (2\sin t\,\cos t)^2\,\frac{1}{2}(1+\cos 2t)\,dt$

$= \frac{1}{8}\int_0^{\pi} (\sin 2t)^2(1+\cos 2t)\,dt = \frac{1}{8}\int_0^{\pi} (\sin^2 2t + \sin^2 2t\,\cos 2t)\,dt$

$= \frac{1}{8}\int_0^{\pi} \sin^2 2t\,dt + \frac{1}{8}\int_0^{\pi} \sin^2 2t\,\cos 2t\,dt = \frac{1}{8}\int_0^{\pi} \frac{1}{2}(1-\cos 4t)\,dt + \frac{1}{8}\left[\frac{1}{3}\cdot\frac{1}{2}\sin^3 2t\right]_0^{\pi}$

$= \frac{1}{16}\left[t - \frac{1}{4}\sin 4t\right]_0^{\pi} + \frac{1}{8}(0-0) = \frac{1}{16}\left[(\pi-0)-0\right] = \frac{\pi}{16}$

11. $\int_0^{\pi/2} \sin^2 x\,\cos^2 x\,dx = \int_0^{\pi/2} \frac{1}{4}(4\sin^2 x\,\cos^2 x)\,dx = \int_0^{\pi/2} \frac{1}{4}(2\sin x\,\cos x)^2 dx = \frac{1}{4}\int_0^{\pi/2} \sin^2 2x\,dx$

$= \frac{1}{4}\int_0^{\pi/2} \frac{1}{2}(1-\cos 4x)\,dx = \frac{1}{8}\int_0^{\pi/2} (1-\cos 4x)\,dx = \frac{1}{8}\left[x - \frac{1}{4}\sin 4x\right]_0^{\pi/2} = \frac{1}{8}\left(\frac{\pi}{2}\right) = \frac{\pi}{16}$

12. $\int_0^{\pi/2} (2-\sin\theta)^2\,d\theta = \int_0^{\pi/2} (4-4\sin\theta+\sin^2\theta)\,d\theta = \int_0^{\pi/2} \left[4-4\sin\theta+\frac{1}{2}(1-\cos 2\theta)\right]d\theta$

$= \int_0^{\pi/2} \left(\frac{9}{2} - 4\sin\theta - \frac{1}{2}\cos 2\theta\right)d\theta = \left[\frac{9}{2}\theta + 4\cos\theta - \frac{1}{4}\sin 2\theta\right]_0^{\pi/2}$

$= \left(\frac{9\pi}{4}+0-0\right) - (0+4-0) = \frac{9}{4}\pi - 4$

13. $\int \sqrt{\cos\theta}\,\sin^3\theta\,d\theta = \int \sqrt{\cos\theta}\,\sin^2\theta\,\sin\theta\,d\theta = \int (\cos\theta)^{1/2}(1-\cos^2\theta)\sin\theta\,d\theta$

$\stackrel{c}{=} \int u^{1/2}(1-u^2)\,(-du) = \int (u^{5/2}-u^{1/2})\,du$

$= \frac{2}{7}u^{7/2} - \frac{2}{3}u^{3/2} + C = \frac{2}{7}(\cos\theta)^{7/2} - \frac{2}{3}(\cos\theta)^{3/2} + C$

14. $\displaystyle\int \left(1+\sqrt[3]{\sin t}\right)\cos^3 t\,dt = \int \left(1+(\sin t)^{1/3}\right)\cos^2 t\cos t\,dt = \int \left(1+(\sin t)^{1/3}\right)(1-\sin^2 t)\cos t\,dt$

$\stackrel{s}{=} \int (1+u^{1/3})(1-u^2)\,du = \int (1-u^2+u^{1/3}-u^{7/3})\,du$

$= u - \frac{1}{3}u^3 + \frac{3}{4}u^{4/3} - \frac{3}{10}u^{10/3} + C$

$= \sin t - \frac{1}{3}\sin^3 t + \frac{3}{4}\sqrt[3]{\sin^4 t} - \frac{3}{10}\sqrt[3]{\sin^{10} t} + C$

15. $\displaystyle\int \sin x\,\sec^5 x\,dx = \int \frac{\sin x}{\cos^5 x}\,dx \stackrel{c}{=} \int \frac{1}{u^5}\,(-du) = \frac{1}{4u^4} + C = \frac{1}{4\cos^4 x} + C = \tfrac{1}{4}\sec^4 x + C$

16. $\displaystyle\int \csc^5\theta\,\cos^3\theta\,d\theta = \int \frac{\cos^2\theta}{\sin^5\theta}\cos\theta\,d\theta = \int \frac{1-\sin^2\theta}{\sin^5\theta}\cos\theta\,d\theta \stackrel{s}{=} \int \frac{1-u^2}{u^5}\,du = \int (u^{-5}-u^{-3})\,du$

$\displaystyle = -\frac{1}{4}u^{-4} + \frac{1}{2}u^{-2} + C = -\frac{1}{4\sin^4\theta} + \frac{1}{2\sin^2\theta} + C$

Alternate solution:

$\displaystyle\int \csc^5\theta\cos^3\theta\,d\theta = \int \left(\frac{\cos^3\theta}{\sin^3\theta}\right)\left(\frac{1}{\sin^2\theta}\right)d\theta = \int \cot^3\theta\,\csc^2\theta\,d\theta$

$= \int u^3\,(-du) \qquad [u=\cot\theta,\ du=-\csc^2\theta\,d\theta]$

$= -\int u^3\,du = -\frac{1}{4}u^4 + C = -\frac{1}{4}\cot^4\theta + C$

17. $\displaystyle\int \cot x\,\cos^2 x\,dx = \int \frac{\cos x}{\sin x}\,(1-\sin^2 x)\,dx$

$\displaystyle \stackrel{s}{=} \int \frac{1-u^2}{u}\,du = \int \left(\frac{1}{u}-u\right)du = \ln|u| - \tfrac{1}{2}u^2 + C = \ln|\sin x| - \tfrac{1}{2}\sin^2 x + C$

18. $\displaystyle\int \tan^2 x\,\cos^3 x\,dx = \int \frac{\sin^2 x}{\cos^2 x}\cos^3 x\,dx = \int \sin^2 x\,\cos x\,dx \stackrel{s}{=} \int u^2\,du = \tfrac{1}{3}u^3 + C = \tfrac{1}{3}\sin^3 x + C$

19. $\int \sin^2 x\,\sin 2x\,dx = \int \sin^2 x\,(2\sin x\,\cos x)\,dx \stackrel{s}{=} \int 2u^3\,du = \frac{1}{2}u^4 + C = \frac{1}{2}\sin^4 x + C$

20. $\int \sin x\,\cos\left(\frac{1}{2}x\right)dx = \int \sin\left(2\cdot\frac{1}{2}x\right)\cos\left(\frac{1}{2}x\right)dx = \int 2\sin\left(\frac{1}{2}x\right)\cos^2\left(\frac{1}{2}x\right)dx$

$= \int 2u^2\,(-2\,du) \qquad \left[u=\cos\left(\frac{1}{2}x\right),\ du=-\frac{1}{2}\sin\left(\frac{1}{2}x\right)dx\right]$

$= -\frac{4}{3}u^3 + C = -\frac{4}{3}\cos^3\left(\frac{1}{2}x\right) + C$

21. $\int \tan x\,\sec^3 x\,dx = \int \tan x\,\sec x\,\sec^2 x\,dx = \int u^2\,du \qquad [u=\sec x,\ du=\sec x\,\tan x\,dx]$

$= \frac{1}{3}u^3 + C = \frac{1}{3}\sec^3 x + C$

22. $\int \tan^2\theta\,\sec^4\theta\,d\theta = \int \tan^2\theta\,\sec^2\theta\,\sec^2\theta\,d\theta = \int \tan^2\theta\,(\tan^2\theta+1)\,\sec^2\theta\,d\theta$

$= \int u^2(u^2+1)\,du \qquad [u=\tan\theta,\ du=\sec^2\theta\,d\theta]$

$= \int (u^4+u^2)\,du = \frac{1}{5}u^5 + \frac{1}{3}u^3 + C = \frac{1}{5}\tan^5\theta + \frac{1}{3}\tan^3\theta + C$

23. $\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx = \tan x - x + C$

24. $\int(\tan^2 x + \tan^4 x)\,dx = \int \tan^2 x\,(1+\tan^2 x)\,dx = \int \tan^2 x\,\sec^2 x\,dx = \int u^2\,du \qquad [u = \tan x,\, du = \sec^2 x\,dx]$

$= \frac{1}{3}u^3 + C = \frac{1}{3}\tan^3 x + C$

25. Let $u = \tan x$. Then $du = \sec^2 x\,dx$, so

$$\begin{aligned}\int \tan^4 x \sec^6 x\,dx &= \int \tan^4 x \sec^4 x\,(\sec^2 x\,dx) = \int \tan^4 x(1+\tan^2 x)^2\,(\sec^2 x\,dx)\\ &= \int u^4(1+u^2)^2\,du = \int(u^8+2u^6+u^4)\,du\\ &= \tfrac{1}{9}u^9 + \tfrac{2}{7}u^7 + \tfrac{1}{5}u^5 + C = \tfrac{1}{9}\tan^9 x + \tfrac{2}{7}\tan^7 x + \tfrac{1}{5}\tan^5 x + C\end{aligned}$$

26. $\int_0^{\pi/4} \sec^6\theta\,\tan^6\theta\,d\theta = \int_0^{\pi/4} \tan^6\theta\,\sec^4\theta\,\sec^2\theta\,d\theta = \int_0^{\pi/4} \tan^6\theta(1+\tan^2\theta)^2\sec^2\theta\,d\theta$

$= \int_0^1 u^6(1+u^2)^2\,du \qquad \left[\begin{array}{c} u = \tan\theta, \\ du = \sec^2\theta\,d\theta \end{array}\right]$

$= \int_0^1 u^6(u^4+2u^2+1)\,du = \int_0^1(u^{10}+2u^8+u^6)\,du$

$= \left[\frac{1}{11}u^{11} + \frac{2}{9}u^9 + \frac{1}{7}u^7\right]_0^1 = \frac{1}{11} + \frac{2}{9} + \frac{1}{7} = \frac{63+154+99}{693} = \frac{316}{693}$

27. $\int \tan^3 x\,\sec x\,dx = \int \tan^2 x\,\sec x\,\tan x\,dx = \int(\sec^2 x - 1)\,\sec x\,\tan x\,dx$

$= \int(u^2-1)\,du \quad [u = \sec x, du = \sec x\,\tan x\,dx] \quad = \frac{1}{3}u^3 - u + C = \frac{1}{3}\sec^3 x - \sec x + C$

28. Let $u = \sec x$, so $du = \sec x \tan x\,dx$. Thus,

$$\begin{aligned}\int \tan^5 x \sec^3 x\,dx &= \int \tan^4 x \sec^2 x\,(\sec x \tan x)\,dx = \int(\sec^2 x - 1)^2 \sec^2 x\,(\sec x \tan x\,dx)\\ &= \int(u^2-1)^2u^2\,du = \int(u^6-2u^4+u^2)\,du\\ &= \tfrac{1}{7}u^7 - \tfrac{2}{5}u^5 + \tfrac{1}{3}u^3 + C = \tfrac{1}{7}\sec^7 x - \tfrac{2}{5}\sec^5 x + \tfrac{1}{3}\sec^3 x + C\end{aligned}$$

29. $\int \tan^3 x\,\sec^6 x\,dx = \int \tan^3 x\,\sec^4 x\,\sec^2 x\,dx = \int \tan^3 x\,(1+\tan^2 x)^2\sec^2 x\,dx$

$= \int u^3(1+u^2)^2\,du \qquad \left[\begin{array}{c} u = \tan x, \\ du = \sec^2 x\,dx \end{array}\right]$

$= \int u^3(u^4+2u^2+1)\,du = \int(u^7+2u^5+u^3)\,du$

$= \frac{1}{8}u^8 + \frac{1}{3}u^6 + \frac{1}{4}u^4 + C = \frac{1}{8}\tan^8 x + \frac{1}{3}\tan^6 x + \frac{1}{4}\tan^4 x + C$

30. $\int_0^{\pi/4}\tan^4 t\,dt = \int_0^{\pi/4}\tan^2 t\,(\sec^2 t - 1)\,dt = \int_0^{\pi/4}\tan^2 t\,\sec^2 t\,dt - \int_0^{\pi/4}\tan^2 t\,dt$

$= \int_0^1 u^2\,du\ \ [u = \tan t] \ \ - \int_0^{\pi/4}(\sec^2 t - 1)\,dt = \left[\frac{1}{3}u^3\right]_0^1 - \left[\tan t - t\right]_0^{\pi/4}$

$= \frac{1}{3} - \left[\left(1 - \frac{\pi}{4}\right) - 0\right] = \frac{\pi}{4} - \frac{2}{3}$

31. $\int \tan^5 x\,dx = \int(\sec^2 x - 1)^2\,\tan x\,dx = \int \sec^4 x\,\tan x\,dx - 2\int \sec^2 x\,\tan x\,dx + \int \tan x\,dx$

$= \int \sec^3 x\,\sec x\,\tan x\,dx - 2\int \tan x\,\sec^2 x\,dx + \int \tan x\,dx$

$= \frac{1}{4}\sec^4 x - \tan^2 x + \ln|\sec x| + C \quad [\text{or } \frac{1}{4}\sec^4 x - \sec^2 x + \ln|\sec x| + C\,]$

32. $\int \tan^2 x\,\sec x\,dx = \int(\sec^2 x - 1)\,\sec x\,dx = \int \sec^3 x\,dx - \int \sec x\,dx$

$= \frac{1}{2}(\sec x\,\tan x + \ln|\sec x + \tan x|) - \ln|\sec x + \tan x| + C \qquad$ [by Example 8 and (1)]

$= \frac{1}{2}(\sec x\,\tan x - \ln|\sec x + \tan x|) + C$

33. $\displaystyle\int \frac{1-\tan^2 x}{\sec^2 x}\,dx = \int \left(\cos^2 x - \sin^2 x\right) dx = \int \cos 2x\,dx = \tfrac{1}{2}\sin 2x + C$

34. $\displaystyle\int \frac{\tan x\sec^2 x}{\cos x}\,dx = \int \sec^2 x \tan x \sec x\,dx = \int u^2\,du \qquad [u = \sec x,\ du = \sec x\ \tan x\,dx]$

$\displaystyle= \tfrac{1}{3}u^3 + C = \tfrac{1}{3}\sec^3 x + C$

35. $\displaystyle\int_0^{\pi/4} \frac{\sin^3 x}{\cos x}\,dx = \int_0^{\pi/4} \frac{\sin^2 x}{\cos x}\sin x\,dx = \int_0^{\pi/4} \frac{1-\cos^2 x}{\cos x}\sin x\,dx \overset{c}{=} \int_1^{1/\sqrt{2}} \frac{1-u^2}{u}(-du)$

$\displaystyle= \int_{1/\sqrt{2}}^{1}\left(\frac{1}{u} - u\right) du = \left[\ln|u| - \frac{1}{2}u^2\right]_{1/\sqrt{2}}^{1} = \left(\ln 1 - \frac{1}{2}\right) - \left(\ln\frac{1}{\sqrt{2}} - \frac{1}{4}\right) = -\frac{1}{4} - \ln\frac{\sqrt{2}}{2}$

36. $\displaystyle\int \frac{\sin\theta + \tan\theta}{\cos^3\theta}\,d\theta = \int \frac{\sin\theta}{\cos^3\theta}\,d\theta + \int \frac{\tan\theta}{\cos^3\theta}\,d\theta = \int \frac{\sin\theta}{\cos^3\theta}\,d\theta + \int \sec^2\theta\tan\theta\sec\theta\,d\theta$

$\displaystyle= -\int \frac{1}{u^3}\,du + \int v^2\,dv \qquad \left[\begin{array}{l} u = \cos\theta,\ du = -\sin\theta\,d\theta \\ v = \sec\theta,\ dv = \sec\theta\tan\theta\,d\theta \end{array}\right]$

$\displaystyle= \frac{1}{2u^2} + \frac{1}{3}v^3 + C = \frac{1}{2\cos^2\theta} + \frac{1}{3}\sec^3\theta + C = \frac{1}{2}\sec^2\theta + \frac{1}{3}\sec^3\theta + C$

37. $\int_{\pi/6}^{\pi/2} \cot^2 x\,dx = \int_{\pi/6}^{\pi/2}(\csc^2 x - 1)\,dx = \left[-\cot x - x\right]_{\pi/6}^{\pi/2} = \left(0 - \frac{\pi}{2}\right) - \left(-\sqrt{3} - \frac{\pi}{6}\right) = \sqrt{3} - \frac{\pi}{3}$

38. $\displaystyle\int_{\pi/4}^{\pi/2} \cot^3 x\,dx = \int_{\pi/4}^{\pi/2} \cot x\,(\csc^2 x - 1)\,dx = \int_{\pi/4}^{\pi/2} \cot x\ \csc^2 x\,dx - \int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin x}\,dx$

$\displaystyle= \left[-\tfrac{1}{2}\cot^2 x - \ln|\sin x|\right]_{\pi/4}^{\pi/2} = (0 - \ln 1) - \left[-\tfrac{1}{2} - \ln\tfrac{1}{\sqrt{2}}\right] = \tfrac{1}{2} + \ln\tfrac{1}{\sqrt{2}} = \tfrac{1}{2}(1 - \ln 2)$

39. $\int_{\pi/4}^{\pi/2} \cot^5\phi\ \csc^3\phi\,d\phi = \int_{\pi/4}^{\pi/2} \cot^4\phi\ \csc^2\phi\ \csc\phi\ \cot\phi\,d\phi = \int_{\pi/4}^{\pi/2}(\csc^2\phi - 1)^2\ \csc^2\phi\ \csc\phi\ \cot\phi\,d\phi$

$\displaystyle= \int_{\sqrt{2}}^{1}(u^2 - 1)^2 u^2\,(-du) \qquad [u = \csc\phi,\ du = -\csc\phi\ \cot\phi\,d\phi]$

$\displaystyle= \int_1^{\sqrt{2}}(u^6 - 2u^4 + u^2)\,du = \left[\tfrac{1}{7}u^7 - \tfrac{2}{5}u^5 + \tfrac{1}{3}u^3\right]_1^{\sqrt{2}} = \left(\tfrac{8}{7}\sqrt{2} - \tfrac{8}{5}\sqrt{2} + \tfrac{2}{3}\sqrt{2}\right) - \left(\tfrac{1}{7} - \tfrac{2}{5} + \tfrac{1}{3}\right)$

$\displaystyle= \frac{120 - 168 + 70}{105}\sqrt{2} - \frac{15 - 42 + 35}{105} = \frac{22}{105}\sqrt{2} - \frac{8}{105}$

40. $\int_{\pi/4}^{\pi/2} \csc^4\theta\ \cot^4\theta\,d\theta = \int_{\pi/4}^{\pi/2} \cot^4\theta\ \csc^2\theta\ \csc^2\theta\,d\theta = \int_{\pi/4}^{\pi/2} \cot^4\theta\,(\cot^2\theta + 1)\ \csc^2\theta\,d\theta$

$= \int_1^0 u^4(u^2 + 1)\,(-du) \qquad \left[\begin{array}{c} u = \cot\theta, \\ du = -\csc^2\theta\,d\theta \end{array}\right]$

$= \int_0^1(u^6 + u^4)\,du$

$= \left[\frac{1}{7}u^7 + \frac{1}{5}u^5\right]_0^1 = \frac{1}{7} + \frac{1}{5} = \frac{12}{35}$

41. $\displaystyle I = \int \csc x\,dx = \int \frac{\csc x\,(\csc x - \cot x)}{\csc x - \cot x}\,dx = \int \frac{-\csc x\ \cot x + \csc^2 x}{\csc x - \cot x}\,dx$. Let $u = \csc x - \cot x \ \Rightarrow$

$du = (-\csc x\ \cot x + \csc^2 x)\,dx$. Then $I = \int du/u = \ln|u| = \ln|\csc x - \cot x| + C$.

42. Let $u = \csc x$, $dv = \csc^2 x\,dx$. Then $du = -\csc x \cot x\,dx$, $v = -\cot x \quad \Rightarrow$

$$\begin{aligned}\int \csc^3 x\,dx &= -\csc x \cot x - \int \csc x \cot^2 x\,dx = -\csc x \cot x - \int \csc x\,(\csc^2 x - 1)\,dx \\ &= -\csc x \cot x + \int \csc x\,dx - \int \csc^3 x\,dx\end{aligned}$$

Solving for $\int \csc^3 x\,dx$ and using Exercise 41, we get

$\int \csc^3 x\,dx = -\frac{1}{2} \csc x \cot x + \frac{1}{2} \int \csc x\,dx = -\frac{1}{2} \csc x \cot x + \frac{1}{2} \ln|\csc x - \cot x| + C$. Thus,

$$\begin{aligned}\int_{\pi/6}^{\pi/3} \csc^3 x\,dx &= \left[-\tfrac{1}{2}\csc x \cot x + \tfrac{1}{2}\ln|\csc x - \cot x|\right]_{\pi/6}^{\pi/3} \\ &= -\tfrac{1}{2} \cdot \tfrac{2}{\sqrt{3}} \cdot \tfrac{1}{\sqrt{3}} + \tfrac{1}{2}\ln\left|\tfrac{2}{\sqrt{3}} - \tfrac{1}{\sqrt{3}}\right| + \tfrac{1}{2} \cdot 2 \cdot \sqrt{3} - \tfrac{1}{2}\ln\left|2 - \sqrt{3}\right| \\ &= -\tfrac{1}{3} + \sqrt{3} + \tfrac{1}{2}\ln\tfrac{1}{\sqrt{3}} - \tfrac{1}{2}\ln\left(2 - \sqrt{3}\right) \approx 1.7825\end{aligned}$$

43. $\int \sin 8x \cos 5x\,dx \overset{2a}{=} \int \frac{1}{2}[\sin(8x - 5x) + \sin(8x + 5x)]\,dx = \frac{1}{2}\int(\sin 3x + \sin 13x)\,dx$

$$= \tfrac{1}{2}\left(-\tfrac{1}{3}\cos 3x - \tfrac{1}{13}\cos 13x\right) + C = -\tfrac{1}{6}\cos 3x - \tfrac{1}{26}\cos 13x + C$$

44. $\int \sin 2\theta \sin 6\theta\,d\theta \overset{2b}{=} \int \frac{1}{2}[\cos(2\theta - 6\theta) - \cos(2\theta + 6\theta)]\,d\theta$

$$\begin{aligned}&= \tfrac{1}{2}\int[\cos(-4\theta) - \cos 8\theta]\,d\theta = \tfrac{1}{2}\int(\cos 4\theta - \cos 8\theta)\,d\theta \\ &= \tfrac{1}{2}\left(\tfrac{1}{4}\sin 4\theta - \tfrac{1}{8}\sin 8\theta\right) + C = \tfrac{1}{8}\sin 4\theta - \tfrac{1}{16}\sin 8\theta + C\end{aligned}$$

45. $\int_0^{\pi/2} \cos 5t \cos 10t\,dt \overset{2c}{=} \int_0^{\pi/2} \frac{1}{2}[\cos(5t - 10t) + \cos(5t + 10t)]\,dt$

$$\begin{aligned}&= \tfrac{1}{2}\int_0^{\pi/2}[\cos(-5t) + \cos 15t]\,dt = \tfrac{1}{2}\int_0^{\pi/2}(\cos 5t + \cos 15t)\,dt \\ &= \tfrac{1}{2}\left[\tfrac{1}{5}\sin 5t + \tfrac{1}{15}\sin 15t\right]_0^{\pi/2} = \tfrac{1}{2}\left(\tfrac{1}{5} - \tfrac{1}{15}\right) = \tfrac{1}{15}\end{aligned}$$

46. $\int t\cos^5(t^2)\,dt = \int t\cos^4(t^2)\cos(t^2)\,dt = \int t[1 - \sin^2(t^2)]^2\cos(t^2)\,dt$

$$\begin{aligned}&= \int \tfrac{1}{2}(1 - u^2)^2\,du \qquad \left[u = \sin(t^2),\ du = 2t\cos(t^2)\,dt\right] \\ &= \tfrac{1}{2}\int(u^4 - 2u^2 + 1)\,du = \tfrac{1}{2}\left(\tfrac{1}{5}u^5 - \tfrac{2}{3}u^3 + u\right) + C = \tfrac{1}{10}\sin^5(t^2) - \tfrac{1}{3}\sin^3(t^2) + \tfrac{1}{2}\sin(t^2) + C\end{aligned}$$

47. $\displaystyle\int \frac{\sin^2(1/t)}{t^2}\,dt = \int \sin^2 u\,(-du) \qquad \left[u = \tfrac{1}{t},\ du = -\tfrac{1}{t^2}\,dt\right]$

$$= -\int \frac{1}{2}(1 - \cos 2u)\,du = -\frac{1}{2}\left(u - \frac{1}{2}\sin 2u\right) + C = -\frac{1}{2t} + \frac{1}{4}\sin\left(\frac{2}{t}\right) + C$$

48. $\int \sec^2 y \cos^3(\tan y)\,dy = \int \cos^3 u\,du \qquad \left[u = \tan y,\ du = \sec^2 y\,dy\right]$

$$\begin{aligned}&= \sin u - \tfrac{1}{3}\sin^3 u + C \qquad [\text{by Example 1}] \\ &= \sin(\tan y) - \tfrac{1}{3}\sin^3(\tan y) + C\end{aligned}$$

49. $\int_0^{\pi/6} \sqrt{1 + \cos 2x}\,dx = \int_0^{\pi/6} \sqrt{1 + (2\cos^2 x - 1)}\,dx = \int_0^{\pi/6} \sqrt{2\cos^2 x}\,dx = \sqrt{2}\int_0^{\pi/6} \sqrt{\cos^2 x}\,dx$

$$\begin{aligned}&= \sqrt{2}\int_0^{\pi/6} |\cos x|\,dx = \sqrt{2}\int_0^{\pi/6} \cos x\,dx \qquad [\text{since } \cos x > 0 \text{ for } 0 \le x \le \pi/6] \\ &= \sqrt{2}\Big[\sin x\Big]_0^{\pi/6} = \sqrt{2}\left(\tfrac{1}{2} - 0\right) = \tfrac{1}{2}\sqrt{2}\end{aligned}$$

50. $\int_0^{\pi/4} \sqrt{1-\cos 4\theta}\,d\theta = \int_0^{\pi/4} \sqrt{1-(1-2\sin^2(2\theta))}\,d\theta = \int_0^{\pi/4} \sqrt{2\sin^2(2\theta)}\,d\theta = \sqrt{2}\int_0^{\pi/4} \sqrt{\sin^2(2\theta)}\,d\theta$

$$= \sqrt{2}\int_0^{\pi/4} |\sin 2\theta|\,d\theta = \sqrt{2}\int_0^{\pi/4} \sin 2\theta\,d\theta \qquad [\text{since } \sin 2\theta \geq 0 \text{ for } 0 \leq \theta \leq \pi/4]$$

$$= \sqrt{2}\left[-\tfrac{1}{2}\cos 2\theta\right]_0^{\pi/4} = -\tfrac{1}{2}\sqrt{2}\,(0-1) = \tfrac{1}{2}\sqrt{2}$$

51. $\int t\sin^2 t\,dt = \int t\left[\frac{1}{2}(1-\cos 2t)\right]\,dt = \frac{1}{2}\int (t - t\cos 2t)\,dt = \frac{1}{2}\int t\,dt - \frac{1}{2}\int t\cos 2t\,dt$

$$= \tfrac{1}{2}\left(\tfrac{1}{2}t^2\right) - \tfrac{1}{2}\left(\tfrac{1}{2}t\sin 2t - \int \tfrac{1}{2}\sin 2t\,dt\right) \qquad \begin{bmatrix} u = t, & dv = \cos 2t\,dt \\ du = dt, & v = \tfrac{1}{2}\sin 2t \end{bmatrix}$$

$$= \tfrac{1}{4}t^2 - \tfrac{1}{4}t\sin 2t + \tfrac{1}{2}\left(-\tfrac{1}{4}\cos 2t\right) + C = \tfrac{1}{4}t^2 - \tfrac{1}{4}t\sin 2t - \tfrac{1}{8}\cos 2t + C$$

52. Let $u = x, dv = \sec x\,\tan x\,dx \quad\Rightarrow\quad du = dx, v = \sec x$. Then

$\int x\,\sec x\,\tan x\,dx = x\sec x - \int \sec x\,dx = x\sec x - \ln|\sec x + \tan x| + C$.

53. $\int x\tan^2 x\,dx = \int x(\sec^2 x - 1)\,dx = \int x\sec^2 x\,dx - \int x\,dx$

$$= x\tan x - \int \tan x\,dx - \tfrac{1}{2}x^2 \qquad \begin{bmatrix} u = x, & dv = \sec^2 x\,dx \\ du = dx, & v = \tan x \end{bmatrix}$$

$$= x\tan x - \ln|\sec x| - \tfrac{1}{2}x^2 + C$$

54. $I = \int x\sin^3 x\,dx$. First, evaluate

$$\int \sin^3 x\,dx = \int (1-\cos^2 x)\,\sin x\,dx \overset{\text{c}}{=} \int (1-u^2)(-du) = \int (u^2-1)\,du$$

$$= \tfrac{1}{3}u^3 - u + C_1 = \tfrac{1}{3}\cos^3 x - \cos x + C_1.$$

Now for I, let $u = x$, $dv = \sin^3 x \quad\Rightarrow\quad du = dx$, $v = \frac{1}{3}\cos^3 x - \cos x$, so

$$I = \tfrac{1}{3}x\cos^3 x - x\cos x - \int \left(\tfrac{1}{3}\cos^3 x - \cos x\right)dx = \tfrac{1}{3}x\cos^3 x - x\cos x - \tfrac{1}{3}\int \cos^3 x\,dx + \sin x$$

$$= \tfrac{1}{3}x\cos^3 x - x\cos x - \tfrac{1}{3}(\sin x - \tfrac{1}{3}\sin^3 x) + \sin x + C \qquad [\text{by Example 1}]$$

$$= \tfrac{1}{3}x\cos^3 x - x\cos x + \tfrac{2}{3}\sin x + \tfrac{1}{9}\sin^3 x + C$$

55. $\displaystyle\int \frac{dx}{\cos x - 1} = \int \frac{1}{\cos x - 1}\cdot\frac{\cos x + 1}{\cos x + 1}\,dx = \int \frac{\cos x + 1}{\cos^2 x - 1}\,dx = \int \frac{\cos x + 1}{-\sin^2 x}\,dx$

$$= \int \left(-\cot x\,\csc x - \csc^2 x\right)dx = \csc x + \cot x + C$$

56. $\displaystyle\int \frac{1}{\sec\theta + 1}\,d\theta = \int \frac{1}{\sec\theta + 1}\cdot\frac{\sec\theta - 1}{\sec\theta - 1}\,d\theta = \int \frac{\sec\theta - 1}{\sec^2\theta - 1}\,d\theta = \int \frac{\sec\theta - 1}{\tan^2\theta}\,d\theta$

$$= \int \frac{\cos\theta}{\sin^2\theta}\,d\theta - \int \frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \int \frac{\cos\theta\,d\theta}{\sin^2\theta} - \int \frac{1-\sin^2\theta}{\sin^2\theta}\,d\theta \overset{\text{s}}{=} \int \frac{1}{u^2}\,du - \int \csc^2\theta\,d\theta + \int d\theta$$

$$= -\frac{1}{\sin\theta} + \cot\theta + \theta + C$$

Alternate solution:

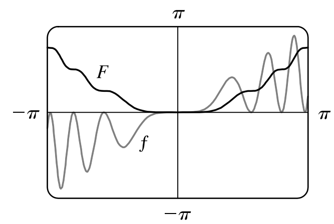
$$\int \frac{1}{\sec\theta + 1}\,d\theta = \int \frac{\cos\theta}{1+\cos\theta}\,d\theta = \int \frac{2\cos^2\left(\dfrac{\theta}{2}\right) - 1}{2\cos^2\left(\dfrac{\theta}{2}\right)}\,d\theta \qquad [\text{double-angle identities}]$$

$$= \int 1\,d\theta - \int \frac{1}{2}\sec^2\left(\frac{\theta}{2}\right)\,d\theta = \theta - \tan\left(\frac{\theta}{2}\right) + C$$

In Exercises 57–60, let $f(x)$ denote the integrand and $F(x)$ its antiderivative (with $C = 0$).

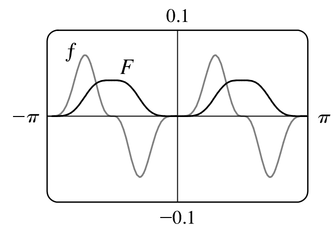
57. Let $u = x^2$, so that $du = 2x\,dx$. Then

$$\begin{aligned}\int x\sin^2(x^2)\,dx &= \int \sin^2 u\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\int \tfrac{1}{2}(1-\cos 2u)\,du\\ &= \tfrac{1}{4}\left(u - \tfrac{1}{2}\sin 2u\right) + C = \tfrac{1}{4}u - \tfrac{1}{4}\left(\tfrac{1}{2}\cdot 2\sin u\,\cos u\right) + C\\ &= \tfrac{1}{4}x^2 - \tfrac{1}{4}\sin(x^2)\,\cos(x^2) + C\end{aligned}$$



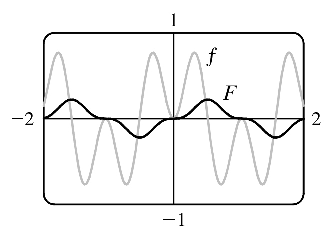
We see from the graph that this is reasonable, since F increases where f is positive and F decreases where f is negative. Note also that f is an odd function and F is an even function.

58. $$\begin{aligned}\int \sin^5 x\,\cos^3 x\,dx &= \int \sin^5 x\,\cos^2 x\,\cos x\,dx\\ &= \int \sin^5 x\,(1-\sin^2 x)\,\cos x\,dx\\ &\overset{s}{=} \int u^5(1-u^2)\,du = \int (u^5 - u^7)\,du\\ &= \tfrac{1}{6}\sin^6 x - \tfrac{1}{8}\sin^8 x + C\end{aligned}$$



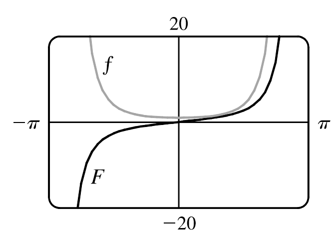
We see from the graph that this is reasonable, since F increases where f is positive and F decreases where f is negative. Note also that f is an odd function and F is an even function.

59. $$\begin{aligned}\int \sin 3x\,\sin 6x\,dx &= \int \tfrac{1}{2}[\cos(3x-6x) - \cos(3x+6x)]\,dx\\ &= \tfrac{1}{2}\int(\cos 3x - \cos 9x)\,dx\\ &= \tfrac{1}{6}\sin 3x - \tfrac{1}{18}\sin 9x + C\end{aligned}$$



Notice that $f(x) = 0$ whenever F has a horizontal tangent.

60. $$\begin{aligned}\int \sec^4\left(\tfrac{1}{2}x\right)\,dx &= \int\left(\tan^2\tfrac{x}{2} + 1\right)\sec^2\tfrac{x}{2}\,dx\\ &= \int (u^2+1)\,2\,du \quad \left[u = \tan\tfrac{x}{2},\, du = \tfrac{1}{2}\sec^2\tfrac{x}{2}\,dx\right]\\ &= \tfrac{2}{3}u^3 + 2u + C = \tfrac{2}{3}\tan^3\tfrac{x}{2} + 2\tan\tfrac{x}{2} + C\end{aligned}$$



Notice that F is increasing and f is positive on the intervals on which they are defined. Also, F has no horizontal tangent and f is never zero.

61. Let $u = \tan^7 x$, $dv = \sec x\,\tan x\,dx \quad\Rightarrow\quad du = 7\tan^6 x\,\sec^2 x\,dx$, $v = \sec x$. Then

$$\begin{aligned}\int \tan^8 x\,\sec x\,dx &= \int \tan^7 x\cdot\sec x\,\tan x\,dx = \tan^7 x\,\sec x - \int 7\tan^6 x\sec^2 x\sec x\,dx\\ &= \tan^7 x\,\sec x - 7\int \tan^6 x\,(\tan^2 x + 1)\,\sec x\,dx\\ &= \tan^7 x\,\sec x - 7\int \tan^8 x\sec x\,dx - 7\int \tan^6 x\sec x\,dx\end{aligned}$$

Thus, $8\int \tan^8 x\sec x\,dx = \tan^7 x\,\sec x - 7\int \tan^6 x\sec x\,dx$ and

$$\int_0^{\pi/4}\tan^8 x\sec x\,dx = \frac{1}{8}\Big[\tan^7 x\,\sec x\Big]_0^{\pi/4} - \frac{7}{8}\int_0^{\pi/4}\tan^6 x\,\sec x\,dx = \frac{\sqrt{2}}{8} - \frac{7}{8}I.$$

62. (a) $\int \tan^{2n} x\,dx = \int \tan^{2n-2} x \tan^2 x\,dx = \int \tan^{2n-2} x\,(\sec^2 x - 1)\,dx$

$$= \int \tan^{2n-2} x \sec^2 x\,dx - \int \tan^{2n-2} x\,dx$$

$$= \int u^{2n-2}\,du - \int \tan^{2n-2} x\,dx \qquad [u = \tan x,\, du = \sec^2 x\,dx]$$

$$= \frac{u^{2n-1}}{2n-1} - \int \tan^{2n-2} x\,dx = \frac{\tan^{2n-1} x}{2n-1} - \int \tan^{2n-2} x\,dx$$

(b) Starting with $n = 4$, repeated applications of the reduction formula in part (a) gives

$$\int \tan^8 x\,dx = \frac{\tan^7 x}{7} - \int \tan^6 x\,dx = \frac{\tan^7 x}{7} - \left(\frac{\tan^5 x}{5} - \int \tan^4 x\,dx\right)$$

$$= \frac{\tan^7 x}{7} - \frac{\tan^5 x}{5} + \left(\frac{\tan^3 x}{3} - \int \tan^2 x\,dx\right)$$

$$= \frac{\tan^7 x}{7} - \frac{\tan^5 x}{5} + \frac{\tan^3 x}{3} - \left(\frac{\tan x}{1} - \int 1\,dx\right)$$

$$= \frac{\tan^7 x}{7} - \frac{\tan^5 x}{5} + \frac{\tan^3 x}{3} - \tan x + x + C$$

63. $f_{\text{avg}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin^2 x\,\cos^3 x\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sin^2 x\,(1-\sin^2 x)\,\cos x\,dx$

$= \frac{1}{2\pi}\int_0^0 u^2(1-u^2)\,du$ [where $u = \sin x$] $= 0$

64. (a) Let $u = \cos x$. Then $du = -\sin x\,dx \;\Rightarrow\; \int \sin x\,\cos x\,dx = \int u(-du) = -\frac{1}{2}u^2 + C = -\frac{1}{2}\cos^2 x + C_1$.

(b) Let $u = \sin x$. Then $du = \cos x\,dx \;\Rightarrow\; \int \sin x\,\cos x\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2 x + C_2$.

(c) $\int \sin x\,\cos x\,dx = \int \frac{1}{2}\sin 2x\,dx = -\frac{1}{4}\cos 2x + C_3$

(d) Let $u = \sin x$, $dv = \cos x\,dx$. Then $du = \cos x\,dx$, $v = \sin x$, so $\int \sin x\,\cos x\,dx = \sin^2 x - \int \sin x\,\cos x\,dx$, by Equation 7.1.2, so $\int \sin x\,\cos x\,dx = \frac{1}{2}\sin^2 x + C_4$.

Using $\cos^2 x = 1 - \sin^2 x$ and $\cos 2x = 1 - 2\sin^2 x$, we see that the answers differ only by a constant.

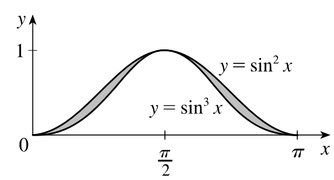
65. $A = \int_0^{\pi}(\sin^2 x - \sin^3 x)\,dx = \int_0^{\pi}\left[\frac{1}{2}(1-\cos 2x) - \sin x\,(1-\cos^2 x)\right]dx$

$= \int_0^{\pi}\left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)dx + \int_1^{-1}(1-u^2)\,du \qquad \begin{bmatrix} u = \cos x, \\ du = -\sin x\,dx \end{bmatrix}$

$= \left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{\pi} + 2\int_0^1 (u^2-1)\,du$

$= \left(\frac{1}{2}\pi - 0\right) - (0-0) + 2\left[\frac{1}{3}u^3 - u\right]_0^1$

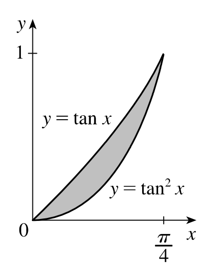
$= \frac{1}{2}\pi + 2\left(\frac{1}{3} - 1\right) = \frac{1}{2}\pi - \frac{4}{3}$



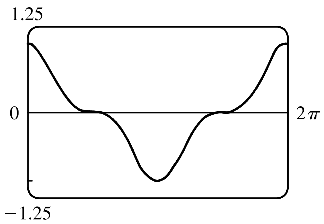
66. $A = \int_0^{\pi/4}(\tan x - \tan^2 x)\,dx = \int_0^{\pi/4}(\tan x - \sec^2 x + 1)\,dx$

$= \Big[\ln|\sec x| - \tan x + x\Big]_0^{\pi/4} = \left(\ln\sqrt{2} - 1 + \frac{\pi}{4}\right) - (\ln 1 - 0 + 0)$

$= \ln\sqrt{2} - 1 + \frac{\pi}{4}$

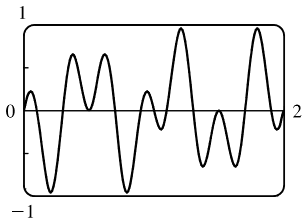


67.



It seems from the graph that $\int_0^{2\pi} \cos^3 x\,dx = 0$, since the area below the x-axis and above the graph looks about equal to the area above the axis and below the graph. By Example 1, the integral is $\left[\sin x - \frac{1}{3}\sin^3 x\right]_0^{2\pi} = 0$. Note that due to symmetry, the integral of any odd power of $\sin x$ or $\cos x$ between limits which differ by $2n\pi$ (n any integer) is 0.

68.



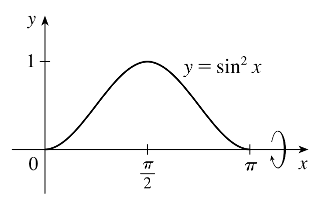
It seems from the graph that $\int_0^2 \sin 2\pi x\, \cos 5\pi x\,dx = 0$, since each bulge above the x-axis seems to have a corresponding depression below the x-axis. To evaluate the integral, we use a trigonometric identity:

$$\begin{aligned}\int_0^1 \sin 2\pi x\, \cos 5\pi x\,dx &= \tfrac{1}{2}\int_0^2 [\sin(2\pi x - 5\pi x) + \sin(2\pi x + 5\pi x)]\,dx \\ &= \tfrac{1}{2}\int_0^2 [\sin(-3\pi x) + \sin 7\pi x]\,dx \\ &= \tfrac{1}{2}\left[\tfrac{1}{3\pi}\cos(-3\pi x) - \tfrac{1}{7\pi}\cos 7\pi x\right]_0^2 \\ &= \tfrac{1}{2}\left[\tfrac{1}{3\pi}(1-1) - \tfrac{1}{7\pi}(1-1)\right] = 0\end{aligned}$$

69. Using disks, $V = \int_{\pi/2}^{\pi} \pi \sin^2 x\,dx = \pi \int_{\pi/2}^{\pi} \frac{1}{2}(1 - \cos 2x)\,dx = \pi\left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_{\pi/2}^{\pi} = \pi\left(\frac{\pi}{2} - 0 - \frac{\pi}{4} + 0\right) = \frac{\pi^2}{4}$

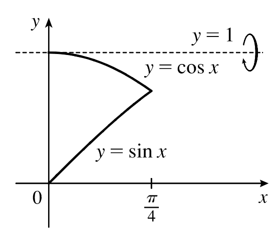
70. Using disks,

$$\begin{aligned}V &= \int_0^{\pi} \pi(\sin^2 x)^2\,dx = 2\pi\int_0^{\pi/2}\left[\tfrac{1}{2}(1-\cos 2x)\right]^2 dx \\ &= \tfrac{\pi}{2}\int_0^{\pi/2}(1 - 2\cos 2x + \cos^2 2x)\,dx \\ &= \tfrac{\pi}{2}\int_0^{\pi/2}\left[1 - 2\cos 2x + \tfrac{1}{2}(1-\cos 4x)\right]dx \\ &= \tfrac{\pi}{2}\int_0^{\pi/2}\left(\tfrac{3}{2} - 2\cos 2x - \tfrac{1}{2}\cos 4x\right)dx = \tfrac{\pi}{2}\left[\tfrac{3}{2}x - \sin 2x + \tfrac{1}{8}\sin 4x\right]_0^{\pi/2} \\ &= \tfrac{\pi}{2}\left[\left(\tfrac{3\pi}{4} - 0 + 0\right) - (0 - 0 + 0)\right] = \tfrac{3}{8}\pi^2\end{aligned}$$



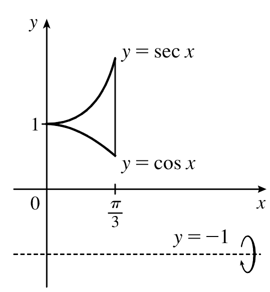
71. Using washers,

$$\begin{aligned}V &= \int_0^{\pi/4} \pi\left[(1-\sin x)^2 - (1-\cos x)^2\right]dx \\ &= \pi\int_0^{\pi/4}\left[(1 - 2\sin x + \sin^2 x) - (1 - 2\cos x + \cos^2 x)\right]dx \\ &= \pi\int_0^{\pi/4}(2\cos x - 2\sin x + \sin^2 x - \cos^2 x)\,dx \\ &= \pi\int_0^{\pi/4}(2\cos x - 2\sin x - \cos 2x)\,dx = \pi\left[2\sin x + 2\cos x - \tfrac{1}{2}\sin 2x\right]_0^{\pi/4} \\ &= \pi\left[\left(\sqrt{2} + \sqrt{2} - \tfrac{1}{2}\right) - (0 + 2 - 0)\right] = \pi\left(2\sqrt{2} - \tfrac{5}{2}\right)\end{aligned}$$



72. Using washers,

$$\begin{aligned}V &= \int_0^{\pi/3} \pi\left\{[\sec x - (-1)]^2 - [\cos x - (-1)]^2\right\}dx \\ &= \pi\int_0^{\pi/3}[(\sec^2 x + 2\sec x + 1) - (\cos^2 x + 2\cos x + 1)]\,dx \\ &= \pi\int_0^{\pi/3}\left[\sec^2 x + 2\sec x - \tfrac{1}{2}(1 + \cos 2x) - 2\cos x\right]dx \\ &= \pi\left[\tan x + 2\ln|\sec x + \tan x| - \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x - 2\sin x\right]_0^{\pi/3} \\ &= \pi\left[\left(\sqrt{3} + 2\ln\left(2 + \sqrt{3}\right) - \tfrac{\pi}{6} - \tfrac{1}{8}\sqrt{3} - \sqrt{3}\right) - 0\right] \\ &= 2\pi\ln\left(2 + \sqrt{3}\right) - \tfrac{1}{6}\pi^2 - \tfrac{1}{8}\pi\sqrt{3}\end{aligned}$$



73. $s = f(t) = \int_0^t \sin\omega u \, \cos^2\omega u \, du$. Let $y = \cos\omega u \quad \Rightarrow \quad dy = -\omega \sin\omega u \, du$. Then

$s = -\frac{1}{\omega}\int_1^{\cos\omega t} y^2 dy = -\frac{1}{\omega}\left[\frac{1}{3}y^3\right]_1^{\cos\omega t} = \frac{1}{3\omega}(1 - \cos^3\omega t)$.

74. (a) We want to calculate the square root of the average value of $[E(t)]^2 = [155\sin(120\pi t)]^2 = 155^2\sin^2(120\pi t)$. First, we calculate the average value itself, by integrating $[E(t)]^2$ over one cycle (between $t = 0$ and $t = \frac{1}{60}$, since there are 60 cycles per second) and dividing by $\left(\frac{1}{60} - 0\right)$:

$$\begin{aligned}[E(t)]^2_{\text{avg}} &= \frac{1}{1/60}\int_0^{1/60}[155^2\sin^2(120\pi t)]\,dt = 60\cdot 155^2\int_0^{1/60}\tfrac{1}{2}[1-\cos(240\pi t)]\,dt \\ &= 60\cdot 155^2\left(\tfrac{1}{2}\right)\left[t - \tfrac{1}{240\pi}\sin(240\pi t)\right]_0^{1/60} = 60\cdot 155^2\left(\tfrac{1}{2}\right)\left[\left(\tfrac{1}{60}-0\right)-(0-0)\right] = \tfrac{155^2}{2}\end{aligned}$$

The RMS value is just the square root of this quantity, which is $\frac{155}{\sqrt{2}} \approx 110$ V.

(b) $220 = \sqrt{[E(t)]^2_{\text{avg}}} \quad \Rightarrow$

$$\begin{aligned}220^2 = [E(t)]^2_{\text{avg}} &= \frac{1}{1/60}\int_0^{1/60} A^2\sin^2(120\pi t)\,dt = 60A^2\int_0^{1/60}\tfrac{1}{2}[1-\cos(240\pi t)]\,dt \\ &= 30A^2\left[t - \tfrac{1}{240\pi}\sin(240\pi t)\right]_0^{1/60} = 30A^2\left[\left(\tfrac{1}{60}-0\right)-(0-0)\right] = \tfrac{1}{2}A^2\end{aligned}$$

Thus, $220^2 = \frac{1}{2}A^2 \quad \Rightarrow \quad A = 220\sqrt{2} \approx 311$ V.

75. Just note that the integrand is odd $[f(-x) = -f(x)]$.

Or: If $m \neq n$, calculate

$$\int_{-\pi}^{\pi}\sin mx\,\cos nx\,dx = \int_{-\pi}^{\pi}\tfrac{1}{2}[\sin(m-n)x + \sin(m+n)x]\,dx = \tfrac{1}{2}\left[-\frac{\cos(m-n)x}{m-n} - \frac{\cos(m+n)x}{m+n}\right]_{-\pi}^{\pi} = 0$$

If $m = n$, then the first term in each set of brackets is zero.

76. $\int_{-\pi}^{\pi}\sin mx\,\sin nx\,dx = \int_{-\pi}^{\pi}\frac{1}{2}[\cos(m-n)x - \cos(m+n)x]\,dx$.

If $m \neq n$, this is equal to $\dfrac{1}{2}\left[\dfrac{\sin(m-n)x}{m-n} - \dfrac{\sin(m+n)x}{m+n}\right]_{-\pi}^{\pi} = 0$.

If $m = n$, we get $\int_{-\pi}^{\pi}\frac{1}{2}[1 - \cos(m+n)x]\,dx = \left[\frac{1}{2}x\right]_{-\pi}^{\pi} - \left[\dfrac{\sin(m+n)x}{2(m+n)}\right]_{-\pi}^{\pi} = \pi - 0 = \pi$.

77. $\int_{-\pi}^{\pi}\cos mx\,\cos nx\,dx = \int_{-\pi}^{\pi}\frac{1}{2}[\cos(m-n)x + \cos(m+n)x]\,dx$.

If $m \neq n$, this is equal to $\dfrac{1}{2}\left[\dfrac{\sin(m-n)x}{m-n} + \dfrac{\sin(m+n)x}{m+n}\right]_{-\pi}^{\pi} = 0$.

If $m = n$, we get $\int_{-\pi}^{\pi}\frac{1}{2}[1 + \cos(m+n)x]\,dx = \left[\frac{1}{2}x\right]_{-\pi}^{\pi} + \left[\dfrac{\sin(m+n)x}{2(m+n)}\right]_{-\pi}^{\pi} = \pi + 0 = \pi$.

78. $\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,\sin mx\,dx = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi}\left[\left(\sum_{n=1}^{m} a_n\sin nx\right)\sin mx\right]dx = \sum_{n=1}^{m}\dfrac{a_n}{\pi}\displaystyle\int_{-\pi}^{\pi}\sin mx\,\sin nx\,dx$. By Exercise 76, every term is zero except the mth one, and that term is $\dfrac{a_m}{\pi}\cdot\pi = a_m$.

7.3 Trigonometric Substitution

1. (a) Use $x = \tan\theta$, where $-\pi/2 < \theta < \pi/2$, since the integrand contains the expression $\sqrt{1+x^2}$.

(b) $x = \tan\theta \quad\Rightarrow\quad dx = \sec^2\theta\, d\theta$ and $\sqrt{1+x^2} = \sqrt{1+\tan^2\theta} = \sqrt{\sec^2\theta} = |\sec\theta| = \sec\theta$.

Then $\displaystyle\int \frac{x^3}{\sqrt{1+x^2}}\,dx = \int \frac{\tan^3\theta}{\sec\theta}\sec^2\theta\,d\theta = \int \tan^3\theta\sec\theta\,d\theta.$

2. (a) Use $x = 3\sin\theta$, where $-\pi/2 \le \theta \le \pi/2$, since the integrand contains the expression $\sqrt{3^2 - x^2}$.

(b) $x = 3\sin\theta \quad\Rightarrow\quad dx = 3\cos\theta\,d\theta$ and

$$\begin{aligned}\sqrt{9-x^2} &= \sqrt{9-9\sin^2\theta} = \sqrt{9(1-\sin^2\theta)} = \sqrt{9\cos^2\theta}\\ &= 3\,|\cos\theta| = 3\cos\theta\end{aligned}$$

Then $\displaystyle\int \frac{x^3}{\sqrt{9-x^2}}\,dx = \int \frac{27\sin^3\theta}{3\cos\theta}\,3\cos\theta\,d\theta = \int 27\sin^3\theta\,d\theta.$

3. (a) Use $x = \sqrt{2}\sec\theta$, where $0 < \theta < \pi/2$ or $\pi < \theta < 3\pi/2$, since the integrand contains the expression $\sqrt{x^2-2}$.

(b) $x = \sqrt{2}\sec\theta \quad\Rightarrow\quad dx = \sqrt{2}\sec\theta\tan\theta\,d\theta$ and

$\sqrt{x^2-2} = \sqrt{2\sec^2\theta - 2} = \sqrt{2(\sec^2\theta-1)} = \sqrt{2\tan^2\theta} = \sqrt{2}\,|\tan\theta| = \sqrt{2}\tan\theta.$

Then $\displaystyle\int \frac{x^2}{\sqrt{x^2-2}}\,dx = \int \frac{2\sec^2\theta}{\sqrt{2}\tan\theta}\,\sqrt{2}\sec\theta\tan\theta\,d\theta = \int 2\sec^3\theta\,d\theta.$

4. (a) Use $x = \frac{3}{2}\sin\theta$, where $-\pi/2 \le \theta \le \pi/2$, since the integrand contains the expression

$(9-4x^2)^{3/2} = 4^{3/2}\left(\frac{9}{4} - x^2\right)^{3/2}.$

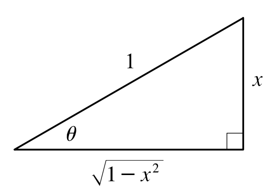
(b) $x = \frac{3}{2}\sin\theta \quad\Rightarrow\quad dx = \frac{3}{2}\cos\theta\,d\theta$ and

$(9-4x^2)^{3/2} = \left(\sqrt{9-9\sin^2\theta}\right)^3 = \left(\sqrt{9(1-\sin^2\theta)}\right)^3 = \left(\sqrt{9\cos^2\theta}\right)^3 = \left(3\,|\cos\theta|\right)^3 = 27\cos^3\theta.$

Then $\displaystyle\int \frac{x^3}{(9-4x^2)^{3/2}}\,dx = \int \frac{\frac{27}{8}\sin^3\theta}{27\cos^3\theta}\left(\tfrac{3}{2}\cos\theta\,d\theta\right) = \int \tfrac{3}{16}\sin^3\theta\sec^2\theta\,d\theta.$

5. Let $x = \sin\theta$, where $-\pi/2 \le \theta \le \pi/2$. Then $dx = \cos\theta\,d\theta$ and

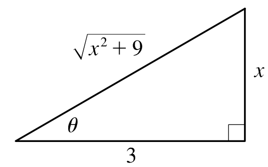
$\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = |\cos\theta| = \cos\theta$. Thus,



$$\begin{aligned}\int \frac{x^3}{\sqrt{1-x^2}}\,dx &= \int \frac{\sin^3\theta}{\cos\theta}\cos\theta\,d\theta = \int (1-\cos^2\theta)\sin\theta\,d\theta\\ &\overset{c}{=} \int (1-u^2)\,(-du) = \int (-1+u^2)\,du = -u + \frac{1}{3}u^3 + C\\ &= -\cos\theta + \tfrac{1}{3}\cos^3\theta + C = -\sqrt{1-x^2} + \tfrac{1}{3}\left(\sqrt{1-x^2}\right)^3 + C\end{aligned}$$

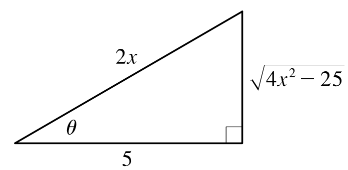
6. Let $x = 3\tan\theta$, where $-\pi/2 < \theta < \pi/2$. Then $dx = 3\sec^2\theta\, d\theta$ and

$\sqrt{9+x^2} = \sqrt{9+9\tan^2\theta} = \sqrt{9\,(1+\tan^2\theta)} = \sqrt{9\sec^2\theta} = 3\,|\sec\theta| = 3\sec\theta$. Thus,



$$\begin{aligned}\int \frac{x^3}{\sqrt{9+x^2}}\,dx &= \int \frac{27\tan^3\theta}{3\sec\theta}\left(3\sec^2\theta\,d\theta\right) = \int 27\tan^3\theta\sec\theta\,d\theta \\ &= 27\int\left(\sec^2\theta - 1\right)\tan\theta\sec\theta\,d\theta \\ &= 27\int(u^2-1)\,du \qquad [u=\sec\theta,\, du = \sec\theta\tan\theta\,d\theta] \\ &= 27\left(\tfrac{1}{3}u^3 - u\right) + C = 9\sec^3\theta - 27\sec\theta + C \\ &= 9\left(\frac{\sqrt{x^2+9}}{3}\right)^3 - \frac{27\sqrt{x^2+9}}{3} + C = \frac{1}{3}(x^2+9)^{3/2} - 9\sqrt{x^2+9} + C\end{aligned}$$

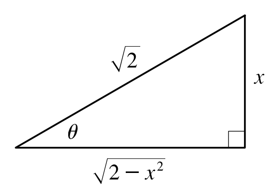
7. Let $x = \frac{5}{2}\sec\theta$, where $0 \le \theta \le \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then $dx = \frac{5}{2}\sec\theta\tan\theta\,d\theta$ and $\sqrt{4x^2-25} = \sqrt{25\sec^2\theta - 25} = \sqrt{25\tan^2\theta} = 5\,|\tan\theta| = 5\tan\theta$ for the relevant values of θ, so



$$\begin{aligned}\int \frac{\sqrt{4x^2-25}}{x}\,dx &= \int \frac{5\tan\theta}{\frac{5}{2}\sec\theta}\left(\frac{5}{2}\sec\theta\tan\theta\,d\theta\right) = 5\int \tan^2\theta\,d\theta \\ &= 5(\tan\theta - \theta) + C \qquad [\text{by Exercise 7.1.59 or integration by parts}] \\ &= 5\left(\frac{\sqrt{4x^2-25}}{5} - \sec^{-1}\left(\frac{2x}{5}\right)\right) + C = \sqrt{4x^2-25} - 5\sec^{-1}\left(\frac{2}{5}x\right) + C\end{aligned}$$

8. Let $x = \sqrt{2}\,\sin\theta$, where $-\pi/2 \le \theta \le \pi/2$. Then $dx = \sqrt{2}\cos\theta\,d\theta$ and

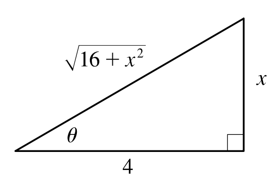
$\sqrt{2-x^2} = \sqrt{2-2\sin^2\theta} = \sqrt{2\cos^2\theta} = \sqrt{2}\,|\cos\theta| = \sqrt{2}\cos\theta$. Thus,



$$\begin{aligned}\int \frac{\sqrt{2-x^2}}{x^2}\,dx &= \int \frac{\sqrt{2}\cos\theta}{2\sin^2\theta}\sqrt{2}\,\cos\theta\,d\theta = \int \frac{\cos^2\theta}{\sin^2\theta}d\theta = \int \cot^2\theta\,d\theta \\ &= \int(\csc^2\theta - 1)\,d\theta = -\cot\theta - \theta + C \\ &= -\frac{\sqrt{2-x^2}}{x} - \sin^{-1}\left(\frac{x}{\sqrt{2}}\right) + C\end{aligned}$$

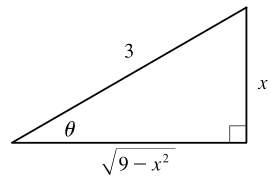
9. Let $x = 4\tan\theta$, where $-\pi/2 < \theta < \pi/2$. Then $dx = 4\sec^2\theta\,d\theta$ and

$\sqrt{16+x^2} = \sqrt{16+16\tan^2\theta} = \sqrt{16\sec^2\theta} = 4\,|\sec\theta| = 4\sec\theta$. Thus,



$$\begin{aligned}\int x^3\sqrt{16+x^2}\,dx &= \int 64\tan^3\theta\,(4\sec\theta)(4\sec^2\theta\,d\theta) = 1024\int\tan^3\theta\sec^3\theta\,d\theta \\ &= 1024\int\tan^2\theta\sec^2\theta\sec\theta\tan\theta\,d\theta = 1024\int(\sec^2\theta - 1)\,\sec^2\theta\sec\theta\,\tan\theta\,d\theta \\ &= 1024\int(u^2-1)u^2\,du \qquad [u=\sec\theta,\, du = \sec\theta\,\tan\theta\,d\theta] \\ &= 1024\left(\frac{1}{5}u^5 - \frac{1}{3}u^3\right) + C = \frac{1024}{5}\left(\frac{\sqrt{16+x^2}}{4}\right)^5 - \frac{1024}{3}\left(\frac{\sqrt{16+x^2}}{4}\right)^3 + C \\ &= \left(16+x^2\right)^{3/2}\left(\frac{1}{5}\left(16+x^2\right) - \frac{16}{3}\right) + C = \left(16+x^2\right)^{3/2}\left(\frac{1}{5}x^2 - \frac{32}{15}\right) + C\end{aligned}$$

10. Let $x = 3\sin\theta$, where $-\pi/2 \le \theta \le \pi/2$. Then $dx = 3\cos\theta\, d\theta$ and $\sqrt{9 - x^2} = \sqrt{9 - 9\sin^2\theta} = \sqrt{9\cos^2\theta} = 3\,|\cos\theta| = 3\cos\theta$.

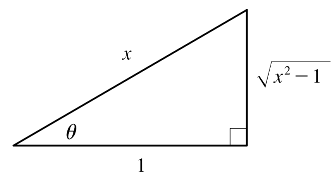


$$\int \frac{x^2}{\sqrt{9 - x^2}}\, dx = \int \frac{9\sin^2\theta}{3\cos\theta}\, 3\cos\theta\, d\theta = 9\int \sin^2\theta\, d\theta$$

$$= 9\int \tfrac{1}{2}(1 - \cos 2\theta)\, d\theta = \tfrac{9}{2}\left(\theta - \tfrac{1}{2}\sin 2\theta\right) + C = \tfrac{9}{2}\theta - \tfrac{9}{4}(2\sin\theta\,\cos\theta) + C$$

$$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{9}{2}\cdot\frac{x}{3}\cdot\frac{\sqrt{9 - x^2}}{3} + C = \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{1}{2}x\sqrt{9 - x^2} + C$$

11. Let $x = \sec\theta$, where $0 \le \theta \le \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then $dx = \sec\theta\,\tan\theta\, d\theta$ and $\sqrt{x^2 - 1} = \sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta} = |\tan\theta| = \tan\theta$ for the relevant values of θ, so



$$\int \frac{\sqrt{x^2 - 1}}{x^4}\, dx = \int \frac{\tan\theta}{\sec^4\theta}\,\sec\theta\,\tan\theta\, d\theta = \int \tan^2\theta\,\cos^3\theta\, d\theta$$

$$= \int \sin^2\theta\,\cos\theta\, d\theta \overset{s}{=} \int u^2\, du = \tfrac{1}{3}u^3 + C = \tfrac{1}{3}\sin^3\theta + C$$

$$= \frac{1}{3}\left(\frac{\sqrt{x^2 - 1}}{x}\right)^3 + C = \frac{1}{3}\frac{(x^2 - 1)^{3/2}}{x^3} + C$$

12. Let $u = 36 - x^2$, so $du = -2x\, dx$. When $x = 0$, $u = 36$; when $x = 3$, $u = 27$. Thus,

$$\int_0^3 \frac{x}{\sqrt{36 - x^2}}\, dx = \int_{36}^{27} \frac{1}{\sqrt{u}}\left(-\frac{1}{2}\, du\right) = -\frac{1}{2}\left[2\sqrt{u}\,\right]_{36}^{27} = -\left(\sqrt{27} - \sqrt{36}\right) = 6 - 3\sqrt{3}$$

Another method: Let $x = 6\sin\theta$, so $dx = 6\cos\theta\, d\theta$, $x = 0 \;\Rightarrow\; \theta = 0$, and $x = 3 \;\Rightarrow\; \theta = \frac{\pi}{6}$. Then

$$\int_0^3 \frac{x}{\sqrt{36 - x^2}}\, dx = \int_0^{\pi/6} \frac{6\sin\theta}{\sqrt{36(1 - \sin^2\theta)}}\, 6\cos\theta\, d\theta = \int_0^{\pi/6} \frac{6\sin\theta}{6\cos\theta}\, 6\cos\theta\, d\theta = 6\int_0^{\pi/6} \sin\theta\, d\theta$$

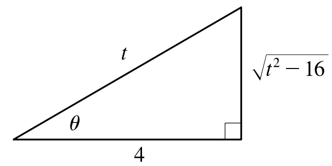
$$= 6\Big[-\cos\theta\Big]_0^{\pi/6} = 6\left(-\tfrac{\sqrt{3}}{2} + 1\right) = 6 - 3\sqrt{3}$$

13. Let $x = a\tan\theta$, where $a > 0$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = a\,\sec^2\theta\, d\theta$, $x = 0 \;\Rightarrow\; \theta = 0$, and $x = a \;\Rightarrow\; \theta = \frac{\pi}{4}$. Thus,

$$\int_0^a \frac{dx}{(a^2 + x^2)^{3/2}} = \int_0^{\pi/4} \frac{a\sec^2\theta\, d\theta}{\left[a^2(1 + \tan^2\theta)\right]^{3/2}} = \int_0^{\pi/4} \frac{a\sec^2\theta\, d\theta}{a^3\sec^3\theta} = \frac{1}{a^2}\int_0^{\pi/4} \cos\theta\, d\theta$$

$$= \frac{1}{a^2}\Big[\sin\theta\Big]_0^{\pi/4} = \frac{1}{a^2}\left(\frac{\sqrt{2}}{2} - 0\right) = \frac{1}{\sqrt{2}\,a^2}.$$

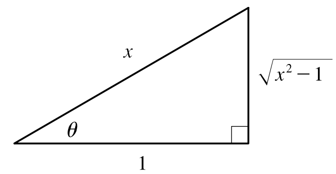
14. Let $t = 4\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then $dt = 4\sec\theta\,\tan\theta\,d\theta$ and $\sqrt{t^2-16} = \sqrt{16\sec^2\theta - 16} = \sqrt{16\tan^2\theta} = 4\tan\theta$ for the relevant values of θ, so



$$\int \frac{dt}{t^2\sqrt{t^2-16}} = \int \frac{4\sec\theta\,\tan\theta\,d\theta}{16\sec^2\theta\cdot 4\tan\theta} = \frac{1}{16}\int \frac{1}{\sec\theta}\,d\theta = \frac{1}{16}\int \cos\theta\,d\theta$$

$$= \frac{1}{16}\sin\theta + C = \frac{1}{16}\,\frac{\sqrt{t^2-16}}{t} + C = \frac{\sqrt{t^2-16}}{16t} + C$$

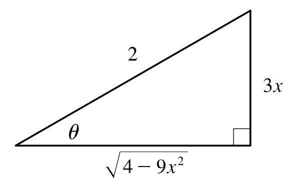
15. Let $x = \sec\theta$, so $dx = \sec\theta\,\tan\theta\,d\theta$, $x = 2 \;\Rightarrow\; \theta = \frac{\pi}{3}$, and $x = 3 \;\Rightarrow\; \theta = \sec^{-1} 3$. Then



$$\int_2^3 \frac{dx}{(x^2-1)^{3/2}} = \int_{\pi/3}^{\sec^{-1}3} \frac{\sec\theta\,\tan\theta\,d\theta}{\tan^3\theta} = \int_{\pi/3}^{\sec^{-1}3} \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

$$\overset{s}{=} \int_{\sqrt{3}/2}^{\sqrt{8}/3} \frac{1}{u^2}\,du = \left[-\frac{1}{u}\right]_{\sqrt{3}/2}^{\sqrt{8}/3} = \frac{-3}{\sqrt{8}} + \frac{2}{\sqrt{3}} = -\frac{3}{4}\sqrt{2} + \frac{2}{3}\sqrt{3}$$

16. Let $x = \frac{2}{3}\sin\theta$, so $dx = \frac{2}{3}\cos\theta\,d\theta$, $x = 0 \;\Rightarrow\; \theta = 0$, and $x = \frac{2}{3} \;\Rightarrow\; \theta = \frac{\pi}{2}$. Thus,



$$\int_0^{2/3} \sqrt{4-9x^2}\,dx = \int_0^{\pi/2} \sqrt{4 - 9\cdot\frac{4}{9}\sin^2\theta}\;\frac{2}{3}\cos\theta\,d\theta$$

$$= \int_0^{\pi/2} 2\cos\theta\cdot\frac{2}{3}\cos\theta\,d\theta = \frac{4}{3}\int_0^{\pi/2}\cos^2\theta\,d\theta$$

$$= \frac{4}{3}\int_0^{\pi/2}\frac{1}{2}(1+\cos 2\theta)\,d\theta = \frac{2}{3}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \frac{2}{3}\left[\left(\frac{\pi}{2}+0\right) - (0+0)\right] = \frac{\pi}{3}$$

17. $$\int_0^{1/2} x\sqrt{1-4x^2}\,dx = \int_1^0 u^{1/2}\left(-\frac{1}{8}\,du\right) \qquad \left[\begin{array}{l} u = 1-4x^2, \\ du = -8x\,dx \end{array}\right]$$

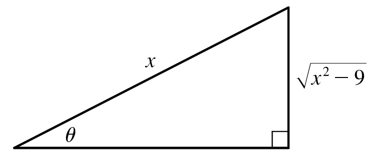
$$= \tfrac{1}{8}\left[\tfrac{2}{3}u^{3/2}\right]_0^1 = \tfrac{1}{12}(1-0) = \tfrac{1}{12}$$

18. Let $t = 2\tan\theta$, so $dt = 2\sec^2\theta\,d\theta$, $t = 0 \;\Rightarrow\; \theta = 0$, and $t = 2 \;\Rightarrow\; \theta = \frac{\pi}{4}$. Thus,

$$\int_0^2 \frac{dt}{\sqrt{4+t^2}} = \int_0^{\pi/4} \frac{2\sec^2\theta\,d\theta}{\sqrt{4+4\tan^2\theta}} = \int_0^{\pi/4}\frac{2\sec^2\theta\,d\theta}{2\sec\theta} = \int_0^{\pi/4}\sec\theta\,d\theta = \Big[\ln|\sec\theta+\tan\theta|\Big]_0^{\pi/4}$$

$$= \ln\left|\sqrt{2}+1\right| - \ln|1+0| = \ln(\sqrt{2}+1)$$

19. Let $x = 3\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then $dx = 3\sec\theta\,\tan\theta\,d\theta$ and $\sqrt{x^2-9} = 3\tan\theta$, so



$$\int \frac{\sqrt{x^2-9}}{x^3}\,dx = \int \frac{3\tan\theta}{27\sec^3\theta}\,3\sec\theta\,\tan\theta\,d\theta = \frac{1}{3}\int\frac{\tan^2\theta}{\sec^2\theta}d\theta$$

$$= \tfrac{1}{3}\int \sin^2\theta\,d\theta = \tfrac{1}{3}\int\tfrac{1}{2}(1-\cos 2\theta)\,d\theta = \tfrac{1}{6}\theta - \tfrac{1}{12}\sin 2\theta + C = \tfrac{1}{6}\theta - \tfrac{1}{6}\sin\theta\,\cos\theta + C$$

$$= \frac{1}{6}\sec^{-1}\!\left(\frac{x}{3}\right) - \frac{1}{6}\,\frac{\sqrt{x^2-9}}{x}\,\frac{3}{x} + C = \frac{1}{6}\sec^{-1}\!\left(\frac{x}{3}\right) - \frac{\sqrt{x^2-9}}{2x^2} + C$$

20. Let $x = \tan\theta$, so $dx = \sec^2\theta\, d\theta$, $x = 0 \;\Rightarrow\; \theta = 0$, and $x = 1 \;\Rightarrow\; \theta = \frac{\pi}{4}$. Then

$$\begin{aligned}\int_0^1 \frac{dx}{(x^2+1)^2} &= \int_0^{\pi/4} \frac{\sec^2\theta\, d\theta}{(\tan^2\theta+1)^2} = \int_0^{\pi/4} \frac{\sec^2\theta\, d\theta}{(\sec^2\theta)^2}\\ &= \int_0^{\pi/4} \cos^2\theta\, d\theta = \int_0^{\pi/4} \frac{1}{2}(1+\cos 2\theta)\, d\theta\\ &= \tfrac{1}{2}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/4} = \tfrac{1}{2}\left[\left(\tfrac{\pi}{4}+\tfrac{1}{2}\right)-0\right] = \tfrac{\pi}{8}+\tfrac{1}{4}\end{aligned}$$

21. Let $x = a\sin\theta$, $dx = a\cos\theta\, d\theta$, $x = 0 \;\Rightarrow\; \theta = 0$ and $x = a \;\Rightarrow\; \theta = \frac{\pi}{2}$. Then

$$\begin{aligned}\int_0^a x^2\sqrt{a^2-x^2}\, dx &= \int_0^{\pi/2} a^2\sin^2\theta\,(a\cos\theta)\,a\cos\theta\, d\theta = a^4\int_0^{\pi/2}\sin^2\theta\,\cos^2\theta\, d\theta\\ &= a^4\int_0^{\pi/2}\left[\tfrac{1}{2}(2\sin\theta\,\cos\theta)\right]^2 d\theta = \frac{a^4}{4}\int_0^{\pi/2}\sin^2 2\theta\, d\theta = \frac{a^4}{4}\int_0^{\pi/2}\tfrac{1}{2}(1-\cos 4\theta)\, d\theta\\ &= \frac{a^4}{8}\left[\theta - \tfrac{1}{4}\sin 4\theta\right]_0^{\pi/2} = \frac{a^4}{8}\left[\left(\frac{\pi}{2}-0\right)-0\right] = \frac{\pi}{16}a^4\end{aligned}$$

22. Let $x = \dfrac{1}{2}\sin\theta$, so $dx = \dfrac{1}{2}\cos\theta\, d\theta$, $x = \dfrac{1}{4} \;\Rightarrow\; \theta = \dfrac{\pi}{6}$, and $x = \dfrac{\sqrt{3}}{4} \;\Rightarrow\; \theta = \dfrac{\pi}{3}$. Then

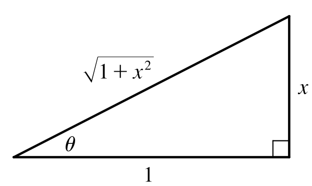
$$\begin{aligned}\int_{1/4}^{\sqrt{3}/4}\sqrt{1-4x^2}\, dx &= \int_{\pi/6}^{\pi/3}\sqrt{1-\sin^2\theta}\left(\frac{1}{2}\cos\theta\, d\theta\right) = \frac{1}{2}\int_{\pi/6}^{\pi/3}\cos^2\theta\, d\theta = \frac{1}{2}\int_{\pi/6}^{\pi/3}\frac{1}{2}(1+\cos 2\theta)\, d\theta\\ &= \frac{1}{4}\left[\theta + \frac{1}{2}\sin 2\theta\right]_{\pi/6}^{\pi/3} = \frac{1}{4}\left[\left(\frac{\pi}{3}+\frac{1}{2}\sin\frac{2\pi}{3}\right)-\left(\frac{\pi}{6}+\frac{1}{2}\sin\frac{\pi}{3}\right)\right]\\ &= \frac{1}{4}\left[\left(\frac{\pi}{3}+\frac{\sqrt{3}}{4}\right)-\left(\frac{\pi}{6}+\frac{\sqrt{3}}{4}\right)\right] = \frac{\pi}{24}\end{aligned}$$

23. Let $u = x^2 - 7$, so $du = 2x\, dx$. Then $\displaystyle\int \frac{x}{\sqrt{x^2-7}}\, dx = \frac{1}{2}\int\frac{1}{\sqrt{u}}\, du = \tfrac{1}{2}\cdot 2\sqrt{u} + C = \sqrt{x^2-7}+C.$

24. Let $u = 1 + x^2$, so $du = 2x\, dx$. Then

$$\int\frac{x}{\sqrt{1+x^2}}\, dx = \int\frac{1}{\sqrt{u}}\left(\frac{1}{2}\, du\right) = \frac{1}{2}\int u^{-1/2}\, du = \frac{1}{2}\cdot 2u^{1/2}+C = \sqrt{1+x^2}+C$$

25. Let $x = \tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = \sec^2\theta\, d\theta$ and $\sqrt{1+x^2} = \sec\theta$, so



$$\begin{aligned}\int\frac{\sqrt{1+x^2}}{x}\, dx &= \int\frac{\sec\theta}{\tan\theta}\sec^2\theta\, d\theta = \int\frac{\sec\theta}{\tan\theta}(1+\tan^2\theta)\, d\theta\\ &= \textstyle\int(\csc\theta + \sec\theta\,\tan\theta)\, d\theta\\ &= \ln|\csc\theta - \cot\theta| + \sec\theta + C \qquad \text{[by Exercise 7.2.41]}\\ &= \ln\left|\frac{\sqrt{1+x^2}}{x}-\frac{1}{x}\right| + \frac{\sqrt{1+x^2}}{1} + C = \ln\left|\frac{\sqrt{1+x^2}-1}{x}\right| + \sqrt{1+x^2}+C\end{aligned}$$

26. Let $x = \frac{3}{5}\sin\theta$, so $dx = \frac{3}{5}\cos\theta\,d\theta$, $x = 0 \;\Rightarrow\; \theta = 0$, and $x = 0.3 \;\Rightarrow\; \theta = \frac{\pi}{6}$. Then

$$\int_0^{0.3} \frac{x}{(9-25x^2)^{3/2}}\,dx = \int_0^{\pi/6} \frac{\frac{3}{5}\sin\theta}{\left(\sqrt{9-9\sin^2\theta}\right)^3}\left(\frac{3}{5}\cos\theta\,d\theta\right)$$

$$= \frac{9}{25}\int_0^{\pi/6} \frac{\sin\theta}{(3\cos\theta)^3}\cos\theta\,d\theta = \frac{1}{75}\int_0^{\pi/6}\frac{\sin\theta}{\cos^2\theta}\,d\theta \stackrel{c}{=} -\frac{1}{75}\int_1^{\sqrt{3}/2}\frac{1}{u^2}\,du$$

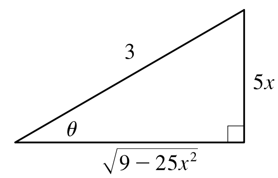
$$= -\frac{1}{75}\left[-\frac{1}{u}\right]_1^{\sqrt{3}/2} = \frac{1}{75}\left(\frac{2}{\sqrt{3}} - 1\right) = \frac{2\sqrt{3}}{225} - \frac{1}{75}$$

27. Let $x = \frac{3}{5}\sin\theta$, so $dx = \frac{3}{5}\cos\theta\,d\theta$, $x = 0 \;\Rightarrow\; \theta = 0$, and $x = 0.6 \;\Rightarrow\; \theta = \frac{\pi}{2}$. Then

$$\int_0^{0.6} \frac{x^2}{\sqrt{9-25x^2}}\,dx = \int_0^{\pi/2} \frac{\left(\frac{3}{5}\right)^2\sin^2\theta}{3\cos\theta}\left(\frac{3}{5}\cos\theta\,d\theta\right) = \frac{9}{125}\int_0^{\pi/2}\sin^2\theta\,d\theta$$

$$= \frac{9}{125}\int_0^{\pi/2}\tfrac{1}{2}(1-\cos 2\theta)\,d\theta = \frac{9}{250}\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2}$$

$$= \frac{9}{250}\left[\left(\tfrac{\pi}{2} - 0\right) - 0\right] = \frac{9}{500}\pi$$

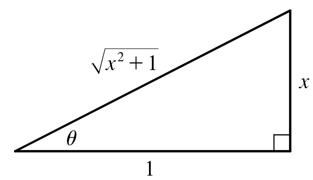


28. Let $x = \tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = \sec^2\theta\,d\theta$,
$\sqrt{x^2+1} = \sec\theta$ and $x = 0 \;\Rightarrow\; \theta = 0$, $x = 1 \;\Rightarrow\; \theta = \frac{\pi}{4}$, so

$$\int_0^1 \sqrt{x^2+1}\,dx = \int_0^{\pi/4}\sec\theta\,\sec^2\theta\,d\theta = \int_0^{\pi/4}\sec^3\theta\,d\theta$$

$$= \tfrac{1}{2}\Big[\sec\theta\,\tan\theta + \ln|\sec\theta + \tan\theta|\Big]_0^{\pi/4} \quad \text{[by Example 7.2.8]}$$

$$= \tfrac{1}{2}\left[\sqrt{2}\cdot 1 + \ln\left(1+\sqrt{2}\right) - 0 - \ln(1+0)\right] = \tfrac{1}{2}\left[\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right]$$

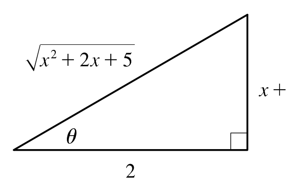


29. $$\int \frac{dx}{\sqrt{x^2+2x+5}} = \int \frac{dx}{\sqrt{(x+1)^2+4}} = \int \frac{2\sec^2\theta\,d\theta}{\sqrt{4\tan^2\theta+4}} \qquad \begin{bmatrix} x+1 = 2\tan\theta, \\ dx = 2\sec^2\theta\,d\theta \end{bmatrix}$$

$$= \int \frac{2\sec^2\theta\,d\theta}{2\sec\theta} = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C_1$$

$$= \ln\left|\frac{\sqrt{x^2+2x+5}}{2} + \frac{x+1}{2}\right| + C_1,$$

or $\ln\left|\sqrt{x^2+2x+5} + x + 1\right| + C$, where $C = C_1 - \ln 2$.



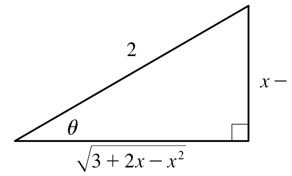
30. $$\int_0^1 \sqrt{x-x^2}\,dx = \int_0^1 \sqrt{\tfrac{1}{4} - \left(x^2 - x + \tfrac{1}{4}\right)}\,dx = \int_0^1 \sqrt{\tfrac{1}{4} - \left(x - \tfrac{1}{2}\right)^2}\,dx$$

$$= \int_{-\pi/2}^{\pi/2} \sqrt{\tfrac{1}{4} - \tfrac{1}{4}\sin^2\theta}\;\tfrac{1}{2}\cos\theta\,d\theta \qquad \begin{bmatrix} x - \frac{1}{2} = \frac{1}{2}\sin\theta, \\ dx = \frac{1}{2}\cos\theta\,d\theta \end{bmatrix}$$

$$= 2\int_0^{\pi/2}\tfrac{1}{2}\cos\theta\,\tfrac{1}{2}\cos\theta\,d\theta = \tfrac{1}{2}\int_0^{\pi/2}\cos^2\theta\,d\theta = \tfrac{1}{2}\int_0^{\pi/2}\tfrac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$= \tfrac{1}{4}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \tfrac{1}{4}\left(\tfrac{\pi}{2}\right) = \tfrac{\pi}{8}$$

31. $\int x^2\sqrt{3+2x-x^2}\,dx = \int x^2\sqrt{4-(x^2+2x+1)}\,dx = \int x^2\sqrt{2^2-(x-1)^2}\,dx$



$$= \int (1+2\sin\theta)^2\sqrt{4\cos^2\theta}\,2\cos\theta\,d\theta \qquad \begin{bmatrix} x-1=2\sin\theta, \\ dx = 2\cos\theta\,d\theta \end{bmatrix}$$

$$= \int (1+4\sin\theta+4\sin^2\theta)\,4\cos^2\theta\,d\theta$$

$$= 4\int(\cos^2\theta+4\sin\theta\,\cos^2\theta+4\sin^2\theta\,\cos^2\theta)\,d\theta$$

$$= 4\int\tfrac{1}{2}(1+\cos 2\theta)\,d\theta + 4\int 4\sin\theta\,\cos^2\theta\,d\theta + 4\int(2\sin\theta\,\cos\theta)^2\,d\theta$$

$$= 2\int(1+\cos 2\theta)\,d\theta + 16\int\sin\theta\,\cos^2\theta\,d\theta + 4\int\sin^2 2\theta\,d\theta$$

$$= 2\left(\theta+\tfrac{1}{2}\sin 2\theta\right) + 16\left(-\tfrac{1}{3}\cos^3\theta\right) + 4\int\tfrac{1}{2}(1-\cos 4\theta)\,d\theta$$

$$= 2\theta+\sin 2\theta - \tfrac{16}{3}\cos^3\theta + 2\left(\theta-\tfrac{1}{4}\sin 4\theta\right)+C$$

$$= 4\theta - \tfrac{1}{2}\sin 4\theta + \sin 2\theta - \tfrac{16}{3}\cos^3\theta + C$$

$$= 4\theta - \tfrac{1}{2}(2\sin 2\theta\,\cos 2\theta) + \sin 2\theta - \tfrac{16}{3}\cos^3\theta + C$$

$$= 4\theta + \sin 2\theta(1-\cos 2\theta) - \tfrac{16}{3}\cos^3\theta + C$$

$$= 4\theta + (2\sin\theta\,\cos\theta)(2\sin^2\theta) - \tfrac{16}{3}\cos^3\theta + C$$

$$= 4\theta + 4\sin^3\theta\,\cos\theta - \tfrac{16}{3}\cos^3\theta + C$$

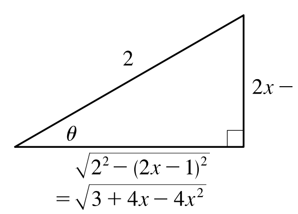
$$= 4\sin^{-1}\left(\frac{x-1}{2}\right) + 4\left(\frac{x-1}{2}\right)^3\frac{\sqrt{3+2x-x^2}}{2} - \frac{16}{3}\,\frac{(3+2x-x^2)^{3/2}}{2^3} + C$$

$$= 4\sin^{-1}\left(\frac{x-1}{2}\right) + \frac{1}{4}(x-1)^3\sqrt{3+2x-x^2} - \frac{2}{3}(3+2x-x^2)^{3/2} + C$$

32. $3+4x-4x^2 = -(4x^2-4x+1)+4 = 2^2-(2x-1)^2$.

Let $2x-1=2\sin\theta$, so $2\,dx = 2\cos\theta\,d\theta$ and $\sqrt{3+4x-4x^2} = 2\cos\theta$.

Then



$$\int\frac{x^2}{(3+4x-4x^2)^{3/2}}\,dx = \int\frac{\left[\frac{1}{2}(1+2\sin\theta)\right]^2}{(2\cos\theta)^3}\cos\theta\,d\theta$$

$$= \frac{1}{32}\int\frac{1+4\sin\theta+4\sin^2\theta}{\cos^2\theta}\,d\theta = \frac{1}{32}\int(\sec^2\theta+4\tan\theta\,\sec\theta+4\tan^2\theta)\,d\theta$$

$$= \tfrac{1}{32}\int[\sec^2\theta+4\tan\theta\,\sec\theta+4(\sec^2\theta-1)]\,d\theta$$

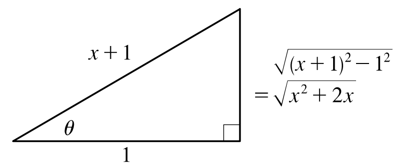
$$= \tfrac{1}{32}\int(5\sec^2\theta+4\tan\theta\,\sec\theta-4)\,d\theta = \tfrac{1}{32}(5\tan\theta+4\sec\theta-4\theta)+C$$

$$= \frac{1}{32}\left[5\cdot\frac{2x-1}{\sqrt{3+4x-4x^2}} + 4\cdot\frac{2}{\sqrt{3+4x-4x^2}} - 4\cdot\sin^{-1}\left(\frac{2x-1}{2}\right)\right]+C$$

$$= \frac{10x+3}{32\sqrt{3+4x-4x^2}} - \frac{1}{8}\sin^{-1}\left(\frac{2x-1}{2}\right)+C$$

33. $x^2 + 2x = (x^2 + 2x + 1) - 1 = (x+1)^2 - 1$. Let $x + 1 = 1\sec\theta$,

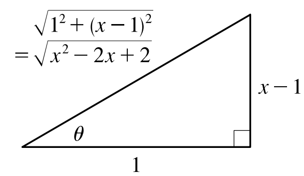
so $dx = \sec\theta\,\tan\theta\,d\theta$ and $\sqrt{x^2+2x} = \tan\theta$. Then



$$\begin{aligned}\int \sqrt{x^2+2x}\,dx &= \int \tan\theta\,(\sec\theta\,\tan\theta\,d\theta) = \int \tan^2\theta\,\sec\theta\,d\theta \\ &= \int (\sec^2\theta - 1)\,\sec\theta\,d\theta = \int \sec^3\theta\,d\theta - \int \sec\theta\,d\theta \\ &= \tfrac{1}{2}\sec\theta\,\tan\theta + \tfrac{1}{2}\ln|\sec\theta + \tan\theta| - \ln|\sec\theta + \tan\theta| + C \\ &= \tfrac{1}{2}\sec\theta\,\tan\theta - \tfrac{1}{2}\ln|\sec\theta + \tan\theta| + C = \tfrac{1}{2}(x+1)\sqrt{x^2+2x} - \tfrac{1}{2}\ln\left|x + 1 + \sqrt{x^2+2x}\,\right| + C\end{aligned}$$

34. $x^2 - 2x + 2 = (x^2 - 2x + 1) + 1 = (x-1)^2 + 1$. Let $x - 1 = 1\tan\theta$,

so $dx = \sec^2\theta\,d\theta$ and $\sqrt{x^2 - 2x + 2} = \sec\theta$. Then



$$\begin{aligned}\int \frac{x^2+1}{(x^2-2x+2)^2}\,dx &= \int \frac{(\tan\theta+1)^2+1}{\sec^4\theta}\,\sec^2\theta\,d\theta \\ &= \int \frac{\tan^2\theta + 2\tan\theta + 2}{\sec^2\theta}\,d\theta \\ &= \int (\sin^2\theta + 2\sin\theta\,\cos\theta + 2\cos^2\theta)\,d\theta = \int (1 + 2\sin\theta\,\cos\theta + \cos^2\theta)\,d\theta \\ &= \int \left[1 + 2\sin\theta\,\cos\theta + \tfrac{1}{2}(1+\cos 2\theta)\right]\,d\theta = \int \left(\tfrac{3}{2} + 2\sin\theta\,\cos\theta + \tfrac{1}{2}\cos 2\theta\right)\,d\theta \\ &= \tfrac{3}{2}\theta + \sin^2\theta + \tfrac{1}{4}\sin 2\theta + C = \tfrac{3}{2}\theta + \sin^2\theta + \tfrac{1}{2}\sin\theta\,\cos\theta + C \\ &= \frac{3}{2}\tan^{-1}\left(\frac{x-1}{1}\right) + \frac{(x-1)^2}{x^2-2x+2} + \frac{1}{2}\,\frac{x-1}{\sqrt{x^2-2x+2}}\,\frac{1}{\sqrt{x^2-2x+2}} + C \\ &= \frac{3}{2}\tan^{-1}(x-1) + \frac{2(x^2-2x+1)+x-1}{2(x^2-2x+2)} + C = \frac{3}{2}\tan^{-1}(x-1) + \frac{2x^2-3x+1}{2(x^2-2x+2)} + C\end{aligned}$$

We can write the answer as

$$\begin{aligned}\frac{3}{2}\tan^{-1}(x-1) + \frac{(2x^2-4x+4)+x-3}{2(x^2-2x+2)} + C &= \frac{3}{2}\tan^{-1}(x-1) + 1 + \frac{x-3}{2(x^2-2x+2)} + C \\ &= \frac{3}{2}\tan^{-1}(x-1) + \frac{x-3}{2(x^2-2x+2)} + C_1, \text{ where } C_1 = 1 + C\end{aligned}$$

35. Let $u = x^2$, $du = 2x\,dx$. Then

$$\begin{aligned}\int x\sqrt{1-x^4}\,dx &= \int \sqrt{1-u^2}\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\int \cos\theta\cdot\cos\theta\,d\theta \qquad \left[\begin{array}{c}\text{where } u = \sin\theta,\ du = \cos\theta\,d\theta, \\ \text{and } \sqrt{1-u^2} = \cos\theta\end{array}\right] \\ &= \tfrac{1}{2}\int \tfrac{1}{2}(1+\cos 2\theta)\,d\theta = \tfrac{1}{4}\theta + \tfrac{1}{8}\sin 2\theta + C = \tfrac{1}{4}\theta + \tfrac{1}{4}\sin\theta\,\cos\theta + C \\ &= \tfrac{1}{4}\sin^{-1}u + \tfrac{1}{4}u\sqrt{1-u^2} + C = \tfrac{1}{4}\sin^{-1}(x^2) + \tfrac{1}{4}x^2\sqrt{1-x^4} + C\end{aligned}$$

36. Let $u = \sin t$, $du = \cos t\,dt$. Then

$$\begin{aligned}\int_0^{\pi/2} \frac{\cos t}{\sqrt{1+\sin^2 t}}\,dt &= \int_0^1 \frac{1}{\sqrt{1+u^2}}\,du = \int_0^{\pi/4} \frac{1}{\sec\theta}\,\sec^2\theta\,d\theta \qquad \left[\begin{array}{c}\text{where } u = \tan\theta,\ du = \sec^2\theta\,d\theta, \\ \text{and } \sqrt{1+u^2} = \sec\theta\end{array}\right] \\ &= \int_0^{\pi/4} \sec\theta\,d\theta = \Big[\ln|\sec\theta + \tan\theta|\Big]_0^{\pi/4} \qquad \text{[by (1) in Section 7.2]} \\ &= \ln\left(\sqrt{2}+1\right) - \ln(1+0) = \ln\left(\sqrt{2}+1\right)\end{aligned}$$

37. (a) Let $x = a\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $\sqrt{x^2+a^2} = a\sec\theta$ and

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{a\sec^2\theta\,d\theta}{a\sec\theta} = \int \sec\theta\,d\theta = \ln|\sec\theta+\tan\theta| + C_1 = \ln\left|\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right| + C_1$$

$$= \ln\!\left(x+\sqrt{x^2+a^2}\right) + C \quad \text{where } C = C_1 - \ln|a|$$

(b) Let $x = a\sinh t$, so that $dx = a\cosh t\,dt$ and $\sqrt{x^2+a^2} = a\cosh t$. Then

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{a\cosh t\,dt}{a\cosh t} = t + C = \sinh^{-1}\frac{x}{a} + C.$$

38. (a) Let $x = a\tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then

$$I = \int \frac{x^2}{(x^2+a^2)^{3/2}}\,dx = \int \frac{a^2\tan^2\theta}{a^3\sec^3\theta}\,a\sec^2\theta\,d\theta = \int \frac{\tan^2\theta}{\sec\theta}\,d\theta = \int \frac{\sec^2\theta - 1}{\sec\theta}\,d\theta$$

$$= \int (\sec\theta - \cos\theta)\,d\theta = \ln|\sec\theta+\tan\theta| - \sin\theta + C$$

$$= \ln\left|\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right| - \frac{x}{\sqrt{x^2+a^2}} + C = \ln\!\left(x+\sqrt{x^2+a^2}\right) - \frac{x}{\sqrt{x^2+a^2}} + C_1$$

(b) Let $x = a\sinh t$. Then

$$I = \int \frac{a^2\sinh^2 t}{a^3\cosh^3 t}\,a\cosh t\,dt = \int \tanh^2 t\,dt = \int (1-\operatorname{sech}^2 t)\,dt = t - \tanh t + C$$

$$= \sinh^{-1}\!\left(\frac{x}{a}\right) - \frac{x}{\sqrt{a^2+x^2}} + C$$

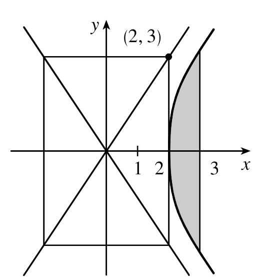
39. The average value of $f(x) = \sqrt{x^2-1}/x$ on the interval $[1,7]$ is

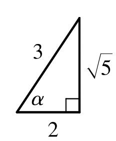
$$\frac{1}{7-1}\int_1^7 \frac{\sqrt{x^2-1}}{x}\,dx = \frac{1}{6}\int_0^\alpha \frac{\tan\theta}{\sec\theta}\cdot\sec\theta\tan\theta\,d\theta \qquad \left[\begin{array}{l}\text{where } x=\sec\theta,\ dx=\sec\theta\tan\theta\,d\theta,\\ \sqrt{x^2-1}=\tan\theta, \text{ and } \alpha=\sec^{-1}7\end{array}\right]$$

$$= \tfrac{1}{6}\int_0^\alpha \tan^2\theta\,d\theta = \tfrac{1}{6}\int_0^\alpha (\sec^2\theta - 1)\,d\theta = \tfrac{1}{6}\big[\tan\theta - \theta\big]_0^\alpha$$

$$= \tfrac{1}{6}(\tan\alpha - \alpha) = \tfrac{1}{6}\left(\sqrt{48} - \sec^{-1}7\right)$$

40. $9x^2 - 4y^2 = 36 \quad\Rightarrow\quad y = \pm\frac{3}{2}\sqrt{x^2-4} \quad\Rightarrow$





$$\text{area} = 2\int_2^3 \tfrac{3}{2}\sqrt{x^2-4}\,dx = 3\int_2^3 \sqrt{x^2-4}\,dx$$

$$= 3\int_0^\alpha 2\tan\theta\,2\sec\theta\,\tan\theta\,d\theta \qquad \left[\begin{array}{c}\text{where } x = 2\sec\theta,\\ dx = 2\sec\theta\,\tan\theta\,d\theta,\\ \alpha = \sec^{-1}\!\left(\frac{3}{2}\right)\end{array}\right]$$

$$= 12\int_0^\alpha \left(\sec^2\theta - 1\right)\sec\theta\,d\theta = 12\int_0^\alpha \left(\sec^3\theta - \sec\theta\right)d\theta$$

$$= 12\left[\tfrac{1}{2}(\sec\theta\,\tan\theta + \ln|\sec\theta+\tan\theta|) - \ln|\sec\theta+\tan\theta|\right]_0^\alpha$$

$$= 6\Big[\sec\theta\,\tan\theta - \ln|\sec\theta+\tan\theta|\Big]_0^\alpha = 6\left[\tfrac{3\sqrt{5}}{4} - \ln\!\left(\tfrac{3}{2} + \tfrac{\sqrt{5}}{2}\right)\right] = \tfrac{9\sqrt{5}}{2} - 6\ln\!\left(\tfrac{3+\sqrt{5}}{2}\right)$$

41. Area of $\triangle POQ = \frac{1}{2}(r\cos\theta)(r\sin\theta) = \frac{1}{2}r^2\sin\theta\,\cos\theta$. Area of region $PQR = \int_{r\cos\theta}^{r}\sqrt{r^2-x^2}\,dx$.

Let $x = r\cos u \;\Rightarrow\; dx = -r\sin u\,du$ for $\theta \le u \le \frac{\pi}{2}$. Then we obtain

$$\int\sqrt{r^2-x^2}\,dx = \int r\sin u\,(-r\sin u)\,du = -r^2\int\sin^2 u\,du = -\tfrac{1}{2}r^2(u-\sin u\,\cos u)+C$$
$$= -\tfrac{1}{2}r^2\cos^{-1}(x/r) + \tfrac{1}{2}x\sqrt{r^2-x^2}+C$$

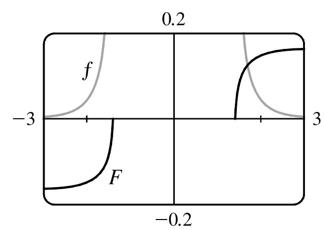
so

$$\text{area of region } PQR = \tfrac{1}{2}\left[-r^2\cos^{-1}(x/r) + x\sqrt{r^2-x^2}\,\right]_{r\cos\theta}^{r}$$
$$= \tfrac{1}{2}\left[0-(-r^2\theta + r\cos\theta\, r\sin\theta)\right] = \tfrac{1}{2}r^2\theta - \tfrac{1}{2}r^2\sin\theta\,\cos\theta$$

and thus, (area of sector POR) = (area of $\triangle POQ$) + (area of region PQR) $= \frac{1}{2}r^2\theta$.

42. Let $x = \sqrt{2}\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$, so $dx = \sqrt{2}\,\sec\theta\tan\theta\,d\theta$. Then

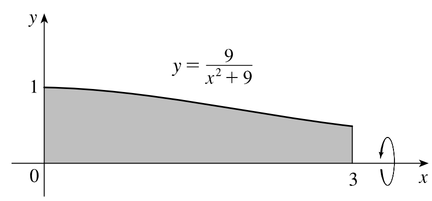
$$\int\frac{dx}{x^4\sqrt{x^2-2}} = \int\frac{\sqrt{2}\sec\theta\,\tan\theta\,d\theta}{4\sec^4\theta\,\sqrt{2}\,\tan\theta}$$
$$= \tfrac{1}{4}\int\cos^3\theta\,d\theta = \tfrac{1}{4}\int\left(1-\sin^2\theta\right)\cos\theta\,d\theta$$
$$= \tfrac{1}{4}\left[\sin\theta - \tfrac{1}{3}\sin^3\theta\right]+C \qquad [\text{substitute } u=\sin\theta]$$
$$= \frac{1}{4}\left[\frac{\sqrt{x^2-2}}{x} - \frac{\left(x^2-2\right)^{3/2}}{3x^3}\right]+C$$



From the graph, it appears that our answer is reasonable. [Notice that $f(x)$ is large when F increases rapidly and small when F levels out.]

43. Use disks about the x-axis:

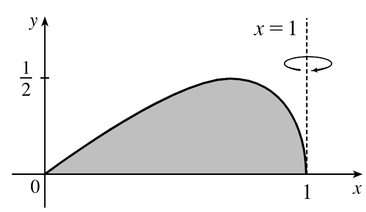
$$V = \int_0^3 \pi\left(\frac{9}{x^2+9}\right)^2 dx = 81\pi\int_0^3\frac{1}{\left(x^2+9\right)^2}\,dx$$



Let $x = 3\tan\theta$, so $dx = 3\sec^2\theta\,d\theta$, $x=0 \;\Rightarrow\; \theta = 0$ and $x=3 \;\Rightarrow\; \theta=\frac{\pi}{4}$. Thus,

$$V = 81\pi\int_0^{\pi/4}\frac{1}{(9\sec^2\theta)^2}\,3\sec^2\theta\,d\theta = 3\pi\int_0^{\pi/4}\cos^2\theta\,d\theta = 3\pi\int_0^{\pi/4}\frac{1}{2}(1+\cos 2\theta)\,d\theta$$
$$= \tfrac{3\pi}{2}\left[\theta+\tfrac{1}{2}\sin 2\theta\right]_0^{\pi/4} = \tfrac{3\pi}{2}\left[\left(\tfrac{\pi}{4}+\tfrac{1}{2}\right)-0\right] = \tfrac{3}{8}\pi^2+\tfrac{3}{4}\pi$$

44. Use shells about $x=1$:



$$V = \int_0^1 2\pi(1-x)\,x\sqrt{1-x^2}\,dx$$
$$= 2\pi\int_0^1 x\sqrt{1-x^2}\,dx - 2\pi\int_0^1 x^2\sqrt{1-x^2}\,dx = 2\pi V_1 - 2\pi V_2$$

For V_1, let $u = 1-x^2$, so $du = -2x\,dx$, and

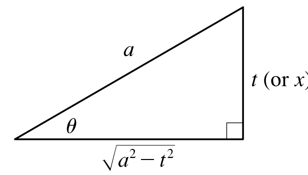
$$V_1 = \int_1^0\sqrt{u}\,\left(-\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\int_0^1 u^{1/2}\,du = \tfrac{1}{2}\left[\tfrac{2}{3}u^{3/2}\right]_0^1 = \tfrac{1}{2}\left(\tfrac{2}{3}\right) = \tfrac{1}{3}.$$

For V_2, let $x = \sin\theta$, so $dx = \cos\theta\,d\theta$, and

$$V_2 = \int_0^{\pi/2}\sin^2\theta\,\sqrt{\cos^2\theta}\,\cos\theta\,d\theta = \int_0^{\pi/2}\sin^2\theta\,\cos^2\theta\,d\theta = \int_0^{\pi/2}\tfrac{1}{4}(2\sin\theta\,\cos\theta)^2\,d\theta$$
$$= \tfrac{1}{4}\int_0^{\pi/2}\sin^2 2\theta\,d\theta = \tfrac{1}{4}\int_0^{\pi/2}\tfrac{1}{2}(1-\cos 2\theta)\,d\theta = \tfrac{1}{8}\left[\theta-\tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \tfrac{1}{8}\left(\tfrac{\pi}{2}\right) = \tfrac{\pi}{16}$$

Thus, $V = 2\pi\left(\frac{1}{3}\right) - 2\pi\left(\frac{\pi}{16}\right) = \frac{2}{3}\pi - \frac{1}{8}\pi^2$.

45. (a) Let $t = a\sin\theta$, $dt = a\cos\theta\, d\theta$, $t = 0 \;\Rightarrow\; \theta = 0$ and $t = x \;\Rightarrow$ $\theta = \sin^{-1}(x/a)$. Then



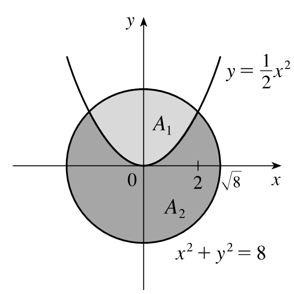
$$\int_0^x \sqrt{a^2 - t^2}\, dt = \int_0^{\sin^{-1}(x/a)} a\cos\theta\,(a\cos\theta\, d\theta) = a^2 \int_0^{\sin^{-1}(x/a)} \cos^2\theta\, d\theta$$

$$= \frac{a^2}{2}\int_0^{\sin^{-1}(x/a)} (1 + \cos 2\theta)\, d\theta = \frac{a^2}{2}\Big[\theta + \tfrac{1}{2}\sin 2\theta\Big]_0^{\sin^{-1}(x/a)} = \frac{a^2}{2}\Big[\theta + \sin\theta\,\cos\theta\Big]_0^{\sin^{-1}(x/a)}$$

$$= \frac{a^2}{2}\left[\left(\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{a}\cdot\frac{\sqrt{a^2 - x^2}}{a}\right) - 0\right] = \tfrac{1}{2}a^2 \sin^{-1}(x/a) + \tfrac{1}{2}x\sqrt{a^2 - x^2}$$

(b) The integral $\int_0^x \sqrt{a^2 - t^2}\, dt$ represents the area under the curve $y = \sqrt{a^2 - t^2}$ between the vertical lines $t = 0$ and $t = x$. The figure shows that this area consists of a triangular region and a sector of the circle $t^2 + y^2 = a^2$. The triangular region has base x and height $\sqrt{a^2 - x^2}$, so its area is $\frac{1}{2}x\sqrt{a^2 - x^2}$. The sector has area $\frac{1}{2}a^2\theta = \frac{1}{2}a^2 \sin^{-1}(x/a)$.

46. The curves intersect when $x^2 + \left(\frac{1}{2}x^2\right)^2 = 8 \;\Leftrightarrow\; x^2 + \frac{1}{4}x^4 = 8 \;\Leftrightarrow\; x^4 + 4x^2 - 32 = 0 \;\Leftrightarrow$ $(x^2 + 8)(x^2 - 4) = 0 \;\Leftrightarrow\; x = \pm 2$. The area inside the circle and above the parabola is given by



$$A_1 = \int_{-2}^{2}\left(\sqrt{8 - x^2} - \tfrac{1}{2}x^2\right) dx = 2\int_0^2 \sqrt{8 - x^2}\, dx - 2\int_0^2 \tfrac{1}{2}x^2\, dx$$

$$= 2\left[\tfrac{1}{2}(8)\sin^{-1}\left(\tfrac{2}{\sqrt{8}}\right) + \tfrac{1}{2}(2)\sqrt{8 - 2^2} - \tfrac{1}{2}\left[\tfrac{1}{3}x^3\right]_0^2\right] \qquad \text{[by Exercise 45]}$$

$$= 8\sin^{-1}\left(\tfrac{1}{\sqrt{2}}\right) + 2\sqrt{4} - \tfrac{8}{3} = 8\left(\tfrac{\pi}{4}\right) + 4 - \tfrac{8}{3} = 2\pi + \tfrac{4}{3}$$

Since the area of the disk is $\pi\left(\sqrt{8}\right)^2 = 8\pi$, the area inside the circle and below the parabola ia $A_2 = 8\pi - \left(2\pi + \frac{4}{3}\right) = 6\pi - \frac{4}{3}$.

47. We use cylindrical shells and assume that $R > r$. $x^2 = r^2 - (y - R)^2 \;\Rightarrow\; x = \pm\sqrt{r^2 - (y - R)^2}$, so $g(y) = 2\sqrt{r^2 - (y - R)^2}$ and

$$V = \int_{R-r}^{R+r} 2\pi y \cdot 2\sqrt{r^2 - (y - R)^2}\, dy = \int_{-r}^{r} 4\pi(u + R)\sqrt{r^2 - u^2}\, du \qquad [\text{where } u = y - R]$$

$$= 4\pi\int_{-r}^{r} u\sqrt{r^2 - u^2}\, du + 4\pi R\int_{-r}^{r} \sqrt{r^2 - u^2}\, du \qquad \left[\begin{array}{c}\text{where } u = r\sin\theta\text{ , } du = r\cos\theta\, d\theta \\ \text{in the second integral}\end{array}\right]$$

$$= 4\pi\left[-\tfrac{1}{3}(r^2 - u^2)^{3/2}\right]_{-r}^{r} + 4\pi R\int_{-\pi/2}^{\pi/2} r^2\cos^2\theta\, d\theta = -\tfrac{4\pi}{3}(0 - 0) + 4\pi R r^2\int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta$$

$$= 2\pi R r^2\int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta)\, d\theta = 2\pi R r^2\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_{-\pi/2}^{\pi/2} = 2\pi^2 R r^2$$

Another method: Use washers instead of shells, so $V = 8\pi R\int_0^r \sqrt{r^2 - y^2}\, dy$ as in Exercise 6.2.75(a), but evaluate the integral using $y = r\sin\theta$.

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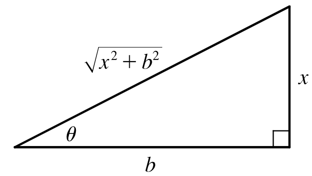
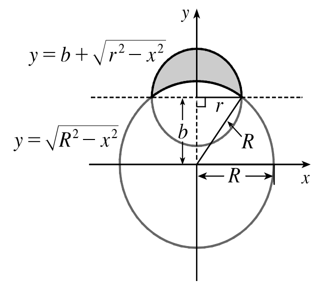
48. Let $x = b\tan\theta$, so that $dx = b\sec^2\theta\, d\theta$ and $\sqrt{x^2+b^2} = b\sec\theta$.



$$\begin{aligned} E(P) &= \int_{-a}^{L-a} \frac{\lambda b}{4\pi\varepsilon_0 (x^2+b^2)^{3/2}}\,dx = \frac{\lambda b}{4\pi\varepsilon_0}\int_{\theta_1}^{\theta_2} \frac{1}{(b\sec\theta)^3}\, b\sec^2\theta\, d\theta \\ &= \frac{\lambda}{4\pi\varepsilon_0 b}\int_{\theta_1}^{\theta_2} \frac{1}{\sec\theta}\,d\theta = \frac{\lambda}{4\pi\varepsilon_0 b}\int_{\theta_1}^{\theta_2}\cos\theta\,d\theta = \frac{\lambda}{4\pi\varepsilon_0 b}\Big[\sin\theta\Big]_{\theta_1}^{\theta_2} \\ &= \frac{\lambda}{4\pi\varepsilon_0 b}\left[\frac{x}{\sqrt{x^2+b^2}}\right]_{-a}^{L-a} = \frac{\lambda}{4\pi\varepsilon_0 b}\left(\frac{L-a}{\sqrt{(L-a)^2+b^2}} + \frac{a}{\sqrt{a^2+b^2}}\right) \end{aligned}$$

49. Let the equation of the large circle be $x^2+y^2=R^2$. Then the equation of the small circle is $x^2+(y-b)^2=r^2$, where $b=\sqrt{R^2-r^2}$ is the distance between the centers of the circles. The desired area is



$$\begin{aligned} A &= \textstyle\int_{-r}^{r}\left[\left(b+\sqrt{r^2-x^2}\right)-\sqrt{R^2-x^2}\right]dx \\ &= 2\textstyle\int_0^r \left(b+\sqrt{r^2-x^2}-\sqrt{R^2-x^2}\right)dx \\ &= 2\textstyle\int_0^r b\,dx + 2\int_0^r \sqrt{r^2-x^2}\,dx - 2\int_0^r \sqrt{R^2-x^2}\,dx \end{aligned}$$

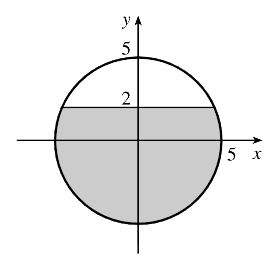
The first integral is just $2br = 2r\sqrt{R^2-r^2}$. The second integral represents the area of a quarter-circle of radius r, so its value is $\frac{1}{4}\pi r^2$. To evaluate the other integral, note that

$$\begin{aligned} \textstyle\int \sqrt{a^2-x^2}\,dx &= \textstyle\int a^2\cos^2\theta\,d\theta \quad [x=a\sin\theta,\ dx=a\cos\theta\,d\theta] \quad = \left(\frac{1}{2}a^2\right)\int(1+\cos 2\theta)\,d\theta \\ &= \tfrac{1}{2}a^2\left(\theta+\tfrac{1}{2}\sin 2\theta\right)+C = \tfrac{1}{2}a^2(\theta+\sin\theta\,\cos\theta)+C \\ &= \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right)+\frac{a^2}{2}\left(\frac{x}{a}\right)\frac{\sqrt{a^2-x^2}}{a}+C = \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right)+\frac{x}{2}\sqrt{a^2-x^2}+C \end{aligned}$$

Thus, the desired area is

$$\begin{aligned} A &= 2r\sqrt{R^2-r^2}+2\left(\tfrac{1}{4}\pi r^2\right)-\left[R^2\arcsin(x/R)+x\sqrt{R^2-x^2}\right]_0^r \\ &= 2r\sqrt{R^2-r^2}+\tfrac{1}{2}\pi r^2-\left[R^2\arcsin(r/R)+r\sqrt{R^2-r^2}\right] = r\sqrt{R^2-r^2}+\tfrac{\pi}{2}r^2-R^2\arcsin(r/R) \end{aligned}$$

50. Note that the circular cross-sections of the tank are the same everywhere, so the percentage of the total capacity that is being used is equal to the percentage of any cross-section that is under water. The underwater area is



$$\begin{aligned} A &= 2\textstyle\int_{-5}^{2}\sqrt{25-y^2}\,dy \\ &= \left[25\arcsin(y/5)+y\sqrt{25-y^2}\right]_{-5}^{2} \qquad \text{[substitute } y=5\sin\theta] \\ &= 25\arcsin\tfrac{2}{5}+2\sqrt{21}+\tfrac{25}{2}\pi \approx 58.72\ \text{m}^2 \end{aligned}$$

so the fraction of the total capacity in use is $\dfrac{A}{\pi(5)^2} \approx \dfrac{58.72}{25\pi} \approx 0.748$ or 74.8%.

7.4 Integration of Rational Functions by Partial Fractions

1. (a) $\dfrac{1}{(x-3)(x+5)} = \dfrac{A}{x-3} + \dfrac{B}{x+5}$

(b) $\dfrac{2x+5}{(x-2)^2(x^2+2)} = \dfrac{A}{x-2} + \dfrac{B}{(x-2)^2} + \dfrac{Cx+D}{x^2+2}$

2. (a) $\dfrac{x-6}{x^2+x-6} = \dfrac{x-6}{(x+3)(x-2)} = \dfrac{A}{x+3} + \dfrac{B}{x-2}$

(b) $\dfrac{1}{x^2+x^4} = \dfrac{1}{x^2(1+x^2)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{Cx+D}{1+x^2}$

3. (a) $\dfrac{x^2+4}{x^3-3x^2+2x} = \dfrac{x^2+4}{x(x^2-3x+2)} = \dfrac{x^2+4}{x(x-1)(x-2)} = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{C}{x-2}$

(b) $\dfrac{x^3+x}{x(2x-1)^2(x^2+3)^2} = \dfrac{A}{x} + \dfrac{B}{2x-1} + \dfrac{C}{(2x-1)^2} + \dfrac{Dx+E}{x^2+3} + \dfrac{Fx+G}{(x^2+3)^2}$

4. (a) $\dfrac{5}{x^4-1} = \dfrac{5}{(x^2+1)(x^2-1)} = \dfrac{5}{(x^2+1)(x+1)(x-1)} = \dfrac{A}{x+1} + \dfrac{B}{x-1} + \dfrac{Cx+D}{x^2+1}$

(b) $\dfrac{x^4+x+1}{(x^3-1)(x^2-1)} = \dfrac{x^4+x+1}{(x-1)(x^2+x+1)(x+1)(x-1)} = \dfrac{x^4+x+1}{(x+1)(x-1)^2(x^2+x+1)}$

$$= \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{Dx+E}{x^2+x+1}$$

5. (a) $\dfrac{x^5+1}{(x^2-x)(x^4+2x^2+1)} = \dfrac{x^5+1}{x(x-1)(x^2+1)^2} = \dfrac{A}{x} + \dfrac{B}{x-1} + \dfrac{Cx+D}{x^2+1} + \dfrac{Ex+F}{(x^2+1)^2}$

(b) $\dfrac{x^2}{x^2+x-6} = 1 + \dfrac{-x+6}{x^2+x-6} = 1 + \dfrac{-x+6}{(x-2)(x+3)} = 1 + \dfrac{A}{x-2} + \dfrac{B}{x+3}$

6. (a) $\dfrac{x^6}{x^2-4} = x^4 + 4x^2 + 16 + \dfrac{64}{(x+2)(x-2)}$ [by long division]

$$= x^4 + 4x^2 + 16 + \frac{A}{x+2} + \frac{B}{x-2}$$

(b) $\dfrac{x^4}{(x^2-x+1)(x^2+2)^2} = \dfrac{Ax+B}{x^2-x+1} + \dfrac{Cx+D}{x^2+2} + \dfrac{Ex+F}{(x^2+2)^2}$

7. $\dfrac{5}{(x-1)(x+4)} = \dfrac{A}{x-1} + \dfrac{B}{x+4}$. Multiply both sides by $(x-1)(x+4)$ to get $5 = A(x+4) + B(x-1) \Rightarrow$ $5 = (A+B)x + (4A-B)$. The coefficients of x must be equal and the constant terms are also equal, so $A+B=0$ and $4A-B=5$. Adding the equations together gives $5A=5 \Leftrightarrow A=1$, and hence $B=-1$. Thus,

$$\int \frac{5}{(x-1)(x+4)}\,dx = \int \left(\frac{1}{x-1} - \frac{1}{x+4}\right) dx = \ln|x-1| - \ln|x+4| + C.$$

8. $\dfrac{x-12}{x^2-4x} = \dfrac{x-12}{x(x-4)} = \dfrac{A}{x} + \dfrac{B}{x-4}$. Multiply both sides by $x(x-4)$ to get $x - 12 = A(x-4) + Bx \quad \Rightarrow$

$x - 12 = (A+B)x + (-4A)$. The coefficients of x must be equal and the constant terms are also equal, so $A + B = 1$ and $-4A = -12$. The second equation gives $A = 3$, which after substituting in the first equation gives $B = -2$. Thus,

$$\int \frac{x-12}{x^2-4x}\,dx = \int \left(\frac{3}{x} - \frac{2}{x-4}\right) dx = 3\ln|x| - 2\ln|x-4| + C.$$

9. $\dfrac{5x+1}{(2x+1)(x-1)} = \dfrac{A}{2x+1} + \dfrac{B}{x-1}$. Multiply both sides by $(2x+1)(x-1)$ to get $5x+1 = A(x-1) + B(2x+1) \quad \Rightarrow$

$5x + 1 = Ax - A + 2Bx + B \quad \Rightarrow \quad 5x + 1 = (A+2B)x + (-A+B)$.

The coefficients of x must be equal and the constant terms are also equal, so $A + 2B = 5$ and $-A + B = 1$. Adding these equations gives us $3B = 6 \quad \Leftrightarrow \quad B = 2$, and hence, $A = 1$. Thus,

$$\int \frac{5x+1}{(2x+1)(x-1)}\,dx = \int \left(\frac{1}{2x+1} + \frac{2}{x-1}\right) dx = \tfrac{1}{2}\ln|2x+1| + 2\ln|x-1| + C.$$

Another method: Substituting 1 for x in the equation $5x + 1 = A(x-1) + B(2x+1)$ gives $6 = 3B \quad \Leftrightarrow \quad B = 2$. Substituting $-\frac{1}{2}$ for x gives $-\frac{3}{2} = -\frac{3}{2}A \quad \Leftrightarrow \quad A = 1$.

10. $\dfrac{y}{(y+4)(2y-1)} = \dfrac{A}{y+4} + \dfrac{B}{2y-1}$. Multiply both sides by $(y+4)(2y-1)$ to get $y = A(2y-1) + B(y+4) \quad \Rightarrow$

$y = 2Ay - A + By + 4B \quad \Rightarrow \quad y = (2A+B)y + (-A+4B)$. The coefficients of y must be equal and the constant terms are also equal, so $2A + B = 1$ and $-A + 4B = 0$. Adding 2 times the second equation and the first equation gives us $9B = 1 \quad \Leftrightarrow \quad B = \frac{1}{9}$ and hence, $A = \frac{4}{9}$. Thus,

$$\begin{aligned}\int \frac{y}{(y+4)(2y-1)}\,dy &= \int \left(\frac{\frac{4}{9}}{y+4} + \frac{\frac{1}{9}}{2y-1}\right) dy = \frac{4}{9}\ln|y+4| + \frac{1}{9}\cdot\frac{1}{2}\ln|2y-1| + C \\ &= \tfrac{4}{9}\ln|y+4| + \tfrac{1}{18}\ln|2y-1| + C\end{aligned}$$

Another method: Substituting $\frac{1}{2}$ for y in the equation $y = A(2y-1) + B(y+4)$ gives $\frac{1}{2} = \frac{9}{2}B \quad \Leftrightarrow \quad B = \frac{1}{9}$. Substituting -4 for y gives $-4 = -9A \quad \Leftrightarrow \quad A = \frac{4}{9}$.

11. $\dfrac{2}{2x^2+3x+1} = \dfrac{2}{(2x+1)(x+1)} = \dfrac{A}{2x+1} + \dfrac{B}{x+1}$. Multiply both sides by $(2x+1)(x+1)$ to get

$2 = A(x+1) + B(2x+1)$. The coefficients of x must be equal and the constant terms are also equal, so $A + 2B = 0$ and $A + B = 2$. Subtracting the second equation from the first gives $B = -2$, and hence, $A = 4$. Thus,

$$\int_0^1 \frac{2}{2x^2+3x+1}\,dx = \int_0^1 \left(\frac{4}{2x+1} - \frac{2}{x+1}\right) dx = \left[\frac{4}{2}\ln|2x+1| - 2\ln|x+1|\right]_0^1 = (2\ln 3 - 2\ln 2) - 0 = 2\ln\frac{3}{2}.$$

Another method: Substituting -1 for x in the equation $2 = A(x+1) + B(2x+1)$ gives $2 = -B \quad \Leftrightarrow \quad B = -2$. Substituting $-\frac{1}{2}$ for x gives $2 = \frac{1}{2}A \quad \Leftrightarrow \quad A = 4$.

12. $\dfrac{x-4}{x^2-5x+6} = \dfrac{A}{x-2} + \dfrac{B}{x-3}$. Multiply both sides by $(x-2)(x-3)$ to get $x-4 = A(x-3)+B(x-2) \quad \Rightarrow$

$x-4 = Ax - 3A + Bx - 2B \quad \Rightarrow \quad x-4 = (A+B)x + (-3A-2B)$.

The coefficients of x must be equal and the constant terms are also equal, so $A+B=1$ and $-3A-2B=-4$.

Adding twice the first equation to the second gives us $-A=-2 \quad \Leftrightarrow \quad A=2$, and hence, $B=-1$.Thus,

$$\int_0^1 \frac{x-4}{x^2-5x+6}\,dx = \int_0^1 \left(\frac{2}{x-2} - \frac{1}{x-3}\right)dx = \left[2\ln|x-2| - \ln|x-3|\right]_0^1$$
$$= (0-\ln 2) - (2\ln 2 - \ln 3) = -3\ln 2 + \ln 3 \;\; [\text{or } \ln\tfrac{3}{8}]$$

Another method: Substituting 3 for x in the equation $x-4 = A(x-3)+B(x-2)$ gives $-1=B$. Substituting 2 for x gives $-2=-A \quad \Leftrightarrow \quad A=2$.

13. $\dfrac{1}{x(x-a)} = \dfrac{A}{x} + \dfrac{B}{x-a}$. Multiply both sides by $x(x-a)$ to get $1 = A(x-a)+Bx \quad \Rightarrow \quad 1 = (A+B)x + (-aA)$.

The coefficients of x must be equal and the constant terms are also equal, so $A+B=0$ and $-aA=1$. The second equation gives $A=-1/a$, which after substituting in the first equation gives $B=1/a$. Thus,

$$\int \frac{1}{x(x-a)}\,dx = \int\left(-\frac{1/a}{x} + \frac{1/a}{x-a}\right)dx = -\frac{1}{a}\ln|x| + \frac{1}{a}\ln|x-a| + C.$$

14. If $a \neq b$, $\dfrac{1}{(x+a)(x+b)} = \dfrac{1}{b-a}\left(\dfrac{1}{x+a} - \dfrac{1}{x+b}\right)$, so if $a \neq b$, then

$$\int \frac{1}{(x+a)(x+b)}\,dx = \frac{1}{b-a}\left(\ln|x+a| - \ln|x+b|\right) + C = \frac{1}{b-a}\ln\left|\frac{x+a}{x+b}\right| + C$$

If $a=b$, then $\displaystyle\int \frac{1}{(x+a)^2}\,dx = -\frac{1}{x+a} + C$.

15. $\dfrac{x^2}{x-1} = \dfrac{(x^2-1)+1}{x-1} = \dfrac{(x+1)(x-1)+1}{x-1} = x+1+\dfrac{1}{x-1}$. [This result can also be obtained using long division.]

Thus, $\displaystyle\int \frac{x^2}{x-1}\,dx = \int\left(x+1+\frac{1}{x-1}\right)dx = \frac{1}{2}x^2 + x + \ln|x-1| + C$.

16. $\dfrac{3t-2}{t+1} = \dfrac{3t+3-5}{t+1} = \dfrac{3(t+1)-5}{t+1} = 3 - \dfrac{5}{t+1}$. Thus, $\displaystyle\int \frac{3t-2}{t+1}\,dt = \int\left(3 - \frac{5}{t+1}\right)dt = 3t - 5\ln|t+1| + C$.

17. $\dfrac{4y^2-7y-12}{y(y+2)(y-3)} = \dfrac{A}{y} + \dfrac{B}{y+2} + \dfrac{C}{y-3} \quad \Rightarrow \quad 4y^2-7y-12 = A(y+2)(y-3) + By(y-3) + Cy(y+2)$. Setting $y=0$ gives $-12=-6A$, so $A=2$. Setting $y=-2$ gives $18=10B$, so $B=\frac{9}{5}$. Setting $y=3$ gives $3=15C$, so $C=\frac{1}{5}$.

Now

$$\int_1^2 \frac{4y^2-7y-12}{y(y+2)(y-3)}\,dy = \int_1^2\left(\frac{2}{y} + \frac{9/5}{y+2} + \frac{1/5}{y-3}\right)dy = \left[2\ln|y| + \tfrac{9}{5}\ln|y+2| + \tfrac{1}{5}\ln|y-3|\right]_1^2$$
$$= 2\ln 2 + \tfrac{9}{5}\ln 4 + \tfrac{1}{5}\ln 1 - 2\ln 1 - \tfrac{9}{5}\ln 3 - \tfrac{1}{5}\ln 2$$
$$= 2\ln 2 + \tfrac{18}{5}\ln 2 - \tfrac{1}{5}\ln 2 - \tfrac{9}{5}\ln 3 = \tfrac{27}{5}\ln 2 - \tfrac{9}{5}\ln 3 = \tfrac{9}{5}(3\ln 2 - \ln 3) = \tfrac{9}{5}\ln\tfrac{8}{3}$$

18. $\dfrac{3x^2+6x+2}{x^2+3x+2} = 3 + \dfrac{-3x-4}{(x+1)(x+2)}$. Write $\dfrac{-3x-4}{(x+1)(x+2)} = \dfrac{A}{x+1} + \dfrac{B}{x+2}$. Multiplying both sides by $(x+1)(x+2)$ gives $-3x-4 = A(x+2) + B(x+1)$. Substituting -2 for x gives $2 = -B \quad \Leftrightarrow \quad B = -2$. Substituting -1 for x gives $-1 = A$. Thus,

$$\int_1^2 \frac{3x^2+6x+2}{x^2+3x+2}\,dx = \int_1^2 \left(3 - \frac{1}{x+1} - \frac{2}{x+2}\right) dx = \Big[3x - \ln|x+1| - 2\ln|x+2|\Big]_1^2$$
$$= (6 - \ln 3 - 2\ln 4) - (3 - \ln 2 - 2\ln 3) = 3 + \ln 2 + \ln 3 - 2\ln 4, \text{ or } 3 + \ln\tfrac{3}{8}$$

19. $\dfrac{x^2+x+1}{(x+1)^2(x+2)} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{x+2}$. Multiplying both sides by $(x+1)^2(x+2)$ gives $x^2+x+1 = A(x+1)(x+2) + B(x+2) + C(x+1)^2$. Substituting -1 for x gives $1 = B$. Substituting -2 for x gives $3 = C$. Equating coefficients of x^2 gives $1 = A + C = A + 3$, so $A = -2$. Thus,

$$\int_0^1 \frac{x^2+x+1}{(x+1)^2(x+2)}\,dx = \int_0^1 \left(\frac{-2}{x+1} + \frac{1}{(x+1)^2} + \frac{3}{x+2}\right) dx = \left[-2\ln|x+1| - \frac{1}{x+1} + 3\ln|x+2|\right]_0^1$$
$$= \left(-2\ln 2 - \tfrac{1}{2} + 3\ln 3\right) - (0 - 1 + 3\ln 2) = \tfrac{1}{2} - 5\ln 2 + 3\ln 3, \text{ or } \tfrac{1}{2} + \ln\tfrac{27}{32}$$

20. $\dfrac{x(3-5x)}{(3x-1)(x-1)^2} = \dfrac{A}{3x-1} + \dfrac{B}{x-1} + \dfrac{C}{(x-1)^2}$. Multiplying both sides by $(3x-1)(x-1)^2$ gives $x(3-5x) = A(x-1)^2 + B(x-1)(3x-1) + C(3x-1)$. Substituting 1 for x gives $-2 = 2C \quad \Leftrightarrow \quad C = -1$. Substituting $\frac{1}{3}$ for x gives $\frac{4}{9} = \frac{4}{9}A \quad \Leftrightarrow \quad A = 1$. Substituting 0 for x gives $0 = A + B - C = 1 + B + 1$, so $B = -2$. Thus,

$$\int_2^3 \frac{x(3-5x)}{(3x-1)(x-1)^2}\,dx = \int_2^3 \left[\frac{1}{3x-1} - \frac{2}{x-1} - \frac{1}{(x-1)^2}\right] dx = \left[\frac{1}{3}\ln|3x-1| - 2\ln|x-1| + \frac{1}{x-1}\right]_2^3$$
$$= \left(\tfrac{1}{3}\ln 8 - 2\ln 2 + \tfrac{1}{2}\right) - \left(\tfrac{1}{3}\ln 5 - 0 + 1\right) = -\ln 2 - \tfrac{1}{3}\ln 5 - \tfrac{1}{2}$$

21. $\dfrac{1}{(t^2-1)^2} = \dfrac{1}{(t+1)^2(t-1)^2} = \dfrac{A}{t+1} + \dfrac{B}{(t+1)^2} + \dfrac{C}{t-1} + \dfrac{D}{(t-1)^2}$. Multiplying both sides by $(t+1)^2(t-1)^2$ gives $1 = A(t+1)(t-1)^2 + B(t-1)^2 + C(t-1)(t+1)^2 + D(t+1)^2$. Substituting 1 for t gives $1 = 4D \quad \Leftrightarrow \quad D = \frac{1}{4}$. Substituting -1 for t gives $1 = 4B \quad \Leftrightarrow \quad B = \frac{1}{4}$. Substituting 0 for t gives $1 = A + B - C + D = A + \frac{1}{4} - C + \frac{1}{4}$, so $\frac{1}{2} = A - C$. Equating coefficients of t^3 gives $0 = A + C$. Adding the last two equations gives $2A = \frac{1}{2} \quad \Leftrightarrow \quad A = \frac{1}{4}$, and so $C = -\frac{1}{4}$. Thus,

$$\int \frac{dt}{(t^2-1)^2} = \int \left[\frac{1/4}{t+1} + \frac{1/4}{(t+1)^2} - \frac{1/4}{t-1} + \frac{1/4}{(t-1)^2}\right] dt$$
$$= \frac{1}{4}\left[\ln|t+1| - \frac{1}{t+1} - \ln|t-1| - \frac{1}{t-1}\right] + C, \text{ or } \frac{1}{4}\left(\ln\left|\frac{t+1}{t-1}\right| + \frac{2t}{1-t^2}\right) + C$$

22. $\dfrac{3x^2+12x-20}{x^4-8x^2+16} = \dfrac{3x^2+12x-20}{(x^2-4)^2} = \dfrac{3x^2+12x-20}{(x-2)^2(x+2)^2} = \dfrac{A}{x-2} + \dfrac{B}{(x-2)^2} + \dfrac{C}{x+2} + \dfrac{D}{(x+2)^2}$. Multiply both sides by $(x-2)^2(x+2)^2$ to get $3x^2+12x-20 = A(x-2)(x+2)^2 + B(x+2)^2 + C(x-2)^2(x+2) + D(x-2)^2$. Setting $x = 2$ gives $16 = 16B$, so $B = 1$, and setting $x = -2$ gives $-32 = 16D$, so $D = -2$. Now, using these values of B and D and setting $x = 0$ gives $-20 = -8A + 4 + 8C - 8 \quad \Leftrightarrow \quad -2 = -A + C$ **(1)**. Also, setting $x = 1$ gives $-5 = -9A + 9 + 3C - 2 \quad \Leftrightarrow \quad -4 = -3A + C$ **(2)**. Subtracting **(2)** from **(1)** gives $2 = 2A \quad \Leftrightarrow \quad A = 1$, and hence $C = -1$. Thus,

$$\begin{aligned}\int \frac{3x^2+12x-20}{x^4-8x^2+16}\,dx &= \int \left(\frac{1}{x-2} + \frac{1}{(x-2)^2} - \frac{1}{x+2} - \frac{2}{(x+2)^2}\right) dx \\ &= \ln|x-2| - \frac{1}{x-2} - \ln|x+2| + \frac{2}{x+2} + C\end{aligned}$$

23. $\dfrac{10}{(x-1)(x^2+9)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+9}$. Multiply both sides by $(x-1)\big(x^2+9\big)$ to get $10 = A\big(x^2+9\big) + (Bx+C)(x-1)$ $(\star)$. Substituting 1 for x gives $10 = 10A \quad \Leftrightarrow \quad A = 1$. Substituting 0 for x gives $10 = 9A - C \quad \Rightarrow \quad C = 9(1) - 10 = -1$. The coefficients of the x^2-terms in $(\star)$ must be equal, so $0 = A + B \quad \Rightarrow$ $B = -1$. Thus,

$$\begin{aligned}\int \frac{10}{(x-1)(x^2+9)}\,dx &= \int \left(\frac{1}{x-1} + \frac{-x-1}{x^2+9}\right) dx = \int \left(\frac{1}{x-1} - \frac{x}{x^2+9} - \frac{1}{x^2+9}\right) dx \\ &= \ln|x-1| - \tfrac{1}{2}\ln(x^2+9) - \tfrac{1}{3}\tan^{-1}\!\big(\tfrac{x}{3}\big) + C\end{aligned}$$

In the second term we used the substitution $u = x^2 + 9$ and in the last term we used Formula 10.

24. $\dfrac{3x^2-x+8}{x^3+4x} = \dfrac{3x^2-x+8}{x(x^2+4)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+4}$. Multiply both sides by $x(x^2+4)$ to get $3x^2 - x + 8 = A(x^2+4) + x(Bx+C) \quad \Rightarrow \quad 3x^2 - x + 8 = (A+B)x^2 + Cx + 4A$. Equating constant terms, we get $4A = 8 \quad \Leftrightarrow \quad A = 2$. Equating coefficients of x gives $C = -1$. Now equating coefficients of x^2 gives $A + B = 3$, so $B = 1$. Thus,

$$\begin{aligned}\int \frac{3x^2-x+8}{x^3+4x}\,dx &= \int \left(\frac{2}{x} + \frac{x-1}{x^2+4}\right) dx = \int \frac{2}{x} + \frac{x}{x^2+4} - \frac{1}{x^2+4} dx \\ &= 2\,\ln(x) + \tfrac{1}{2}\ln(x^2+4) - \tfrac{1}{2}\tan^{-1}\!\big(\tfrac{x}{2}\big) + C\end{aligned}$$

25. $\dfrac{x^3-4x+1}{x^2-3x+2} = x + 3 + \dfrac{3x-5}{(x-1)(x-2)}$. Write $\dfrac{3x-5}{(x-1)(x-2)} = \dfrac{A}{x-1} + \dfrac{B}{x-2}$. Multiplying both sides by $(x-1)(x-2)$ gives $3x - 5 = A(x-2) + B(x-1)$. Substituting 2 for x gives $1 = B$. Substituting 1 for x gives $-2 = -A \quad \Leftrightarrow \quad A = 2$. Thus,

$$\begin{aligned}\int_{-1}^{0} \frac{x^3-4x+1}{x^2-3x+2}\,dx &= \int_{-1}^{0}\left(x + 3 + \frac{2}{x-1} + \frac{1}{x-2}\right) dx = \Big[\tfrac{1}{2}x^2 + 3x + 2\ln|x-1| + \ln|x-2|\Big]_{-1}^{0} \\ &= (0 + 0 + 0 + \ln 2) - \big(\tfrac{1}{2} - 3 + 2\ln 2 + \ln 3\big) = \tfrac{5}{2} - \ln 2 - \ln 3, \text{ or } \tfrac{5}{2} - \ln 6\end{aligned}$$

26. $\dfrac{x^3+4x^2+x-1}{x^3+x^2} = 1 + \dfrac{3x^2+x-1}{x^2(x+1)}$. Write $\dfrac{3x^2+x-1}{x^2(x+1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+1}$. Multiplying both sides by $x^2(x+1)$ gives $3x^2+x-1 = Ax(x+1)+B(x+1)+Cx^2$. Substituting 0 for x gives $-1 = B$. Substituting -1 for x gives $1 = C$. Equating coefficients of x^2 gives $3 = A + C = A + 1$, so $A = 2$. Thus,

$$\int_1^2 \frac{x^3+4x^2+x-1}{x^3+x^2}\,dx = \int_1^2 \left(1 + \frac{2}{x} - \frac{1}{x^2} + \frac{1}{x+1}\right)dx = \left[x + 2\ln|x| + \frac{1}{x} + \ln|x+1|\right]_1^2$$

$$= \left(2 + 2\ln 2 + \tfrac{1}{2} + \ln 3\right) - (1 + 0 + 1 + \ln 2) = \tfrac{1}{2} + \ln 2 + \ln 3, \text{ or } \tfrac{1}{2} + \ln 6.$$

27. $\dfrac{4x}{x^3+x^2+x+1} = \dfrac{4x}{x^2(x+1)+1(x+1)} = \dfrac{4x}{(x+1)(x^2+1)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+1}$. Multiply both sides by $(x+1)(x^2+1)$ to get $4x = A(x^2+1) + (Bx+C)(x+1) \quad\Leftrightarrow\quad 4x = Ax^2 + A + Bx^2 + Bx + Cx + C \quad\Leftrightarrow$ $4x = (A+B)x^2 + (B+C)x + (A+C)$. Comparing coefficients gives us the following system of equations:

$A + B = 0$ **(1)** $\qquad B + C = 4$ **(2)** $\qquad A + C = 0$ **(3)**

Subtracting equation **(1)** from equation **(2)** gives us $-A + C = 4$, and adding that equation to equation **(3)** gives us $2C = 4 \quad\Leftrightarrow\quad C = 2$, and hence $A = -2$ and $B = 2$. Thus,

$$\int \frac{4x}{x^3+x^2+x+1}\,dx = \int \left(\frac{-2}{x+1} + \frac{2x+2}{x^2+1}\right)dx = \int \left(\frac{-2}{x+1} + \frac{2x}{x^2+1} + \frac{2}{x^2+1}\right)dx$$

$$= -2\ln|x+1| + \ln(x^2+1) + 2\tan^{-1}x + C$$

28. $\displaystyle\int \frac{x^2+x+1}{(x^2+1)^2}\,dx = \int \frac{x^2+1}{(x^2+1)^2}\,dx + \int \frac{x}{(x^2+1)^2}\,dx = \int \frac{1}{x^2+1}\,dx + \frac{1}{2}\int \frac{1}{u^2}\,du \quad \left[u = x^2+1,\ du = 2x\,dx\right]$

$$= \tan^{-1}x + \frac{1}{2}\left(-\frac{1}{u}\right) + C = \tan^{-1}x - \frac{1}{2(x^2+1)} + C$$

29. $\dfrac{x^3+4x+3}{x^4+5x^2+4} = \dfrac{x^3+4x+3}{(x^2+1)(x^2+4)} = \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{x^2+4}$. Multiply both sides by $(x^2+1)(x^2+4)$ to get $x^3+4x+3 = (Ax+B)(x^2+4) + (Cx+D)(x^2+1) \quad\Leftrightarrow$

$x^3+4x+3 = Ax^3 + Bx^2 + 4Ax + 4B + Cx^3 + Dx^2 + Cx + D \quad\Leftrightarrow$

$x^3+4x+3 = (A+C)x^3 + (B+D)x^2 + (4A+C)x + (4B+D)$. Comparing coefficients gives us the following system of equations:

$A + C = 1$ **(1)** $\qquad B + D = 0$ **(2)** $\qquad 4A + C = 4$ **(3)** $\qquad 4B + D = 3$ **(4)**

Subtracting equation **(1)** from equation **(3)** gives us $A = 1$ and hence, $C = 0$. Subtracting equation **(2)** from equation **(4)** gives us $B = 1$ and hence, $D = -1$. Thus,

$$\int \frac{x^3+4x+3}{x^4+5x^2+4}\,dx = \int \left(\frac{x+1}{x^2+1} + \frac{-1}{x^2+4}\right)dx = \int \left(\frac{x}{x^2+1} + \frac{1}{x^2+1} - \frac{1}{x^2+4}\right)dx$$

$$= \frac{1}{2}\ln(x^2+1) + \tan^{-1}x - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

30. $\dfrac{x^3+6x-2}{x^4+6x^2} = \dfrac{x^3+6x-2}{x^2(x^2+6)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{Cx+D}{x^2+6}$. Multiply both sides by $x^2(x^2+6)$ to get

$x^3+6x-2 = Ax(x^2+6) + B(x^2+6) + (Cx+D)x^2 \quad \Leftrightarrow$

$x^3+6x-2 = Ax^3+6Ax+Bx^2+6B+Cx^3+Dx^2 \quad \Leftrightarrow \quad x^3+6x-2 = (A+C)x^3+(B+D)x^2+6Ax+6B.$

Substituting 0 for x gives $-2=6B \quad \Leftrightarrow \quad B=-\frac{1}{3}$. Equating coefficients of x^2 gives $0=B+D$, so $D=\frac{1}{3}$. Equating coefficients of x gives $6=6A \quad \Leftrightarrow \quad A=1$. Equating coefficients of x^3 gives $1=A+C$, so $C=0$. Thus,

$$\int \frac{x^3+6x-2}{x^4+6x^2}\,dx = \int \left(\frac{1}{x} + \frac{-1/3}{x^2} + \frac{1/3}{x^2+6}\right) dx = \ln|x| + \frac{1}{3x} + \frac{1}{3\sqrt{6}}\tan^{-1}\left(\frac{x}{\sqrt{6}}\right) + C.$$

31. $\displaystyle\int \frac{x+4}{x^2+2x+5}\,dx = \int \frac{x+1}{x^2+2x+5}\,dx + \int \frac{3}{x^2+2x+5}\,dx = \frac{1}{2}\int \frac{(2x+2)\,dx}{x^2+2x+5} + \int \frac{3\,dx}{(x+1)^2+4}$

$$= \frac{1}{2}\ln\left|x^2+2x+5\right| + 3\int \frac{2\,du}{4(u^2+1)} \qquad \left[\begin{array}{c}\text{where } x+1=2u,\\ \text{and } dx=2\,du\end{array}\right]$$

$$= \frac{1}{2}\ln(x^2+2x+5) + \frac{3}{2}\tan^{-1}u + C = \frac{1}{2}\ln(x^2+2x+5) + \frac{3}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C$$

32. $\displaystyle\int_0^1 \frac{x}{x^2+4x+13}\,dx = \int_0^1 \frac{\frac{1}{2}(2x+4)}{x^2+4x+13}\,dx - 2\int_0^1 \frac{dx}{(x+2)^2+9}$

$$= \frac{1}{2}\int_{13}^{18} \frac{dy}{y} - 2\int_{2/3}^{1} \frac{3\,du}{9u^2+9} \qquad \left[\begin{array}{c}\text{where } y=x^2+4x+13,\ dy=(2x+4)\,dx,\\ x+2=3u, \text{ and } dx=3\,du\end{array}\right]$$

$$= \tfrac{1}{2}\left[\ln y\right]_{13}^{18} - \tfrac{2}{3}\left[\tan^{-1}u\right]_{2/3}^{1} = \tfrac{1}{2}\ln\tfrac{18}{13} - \tfrac{2}{3}\left(\tfrac{\pi}{4} - \tan^{-1}\left(\tfrac{2}{3}\right)\right)$$

$$= \tfrac{1}{2}\ln\tfrac{18}{13} - \tfrac{\pi}{6} + \tfrac{2}{3}\tan^{-1}\left(\tfrac{2}{3}\right)$$

33. $\dfrac{1}{x^3-1} = \dfrac{1}{(x-1)(x^2+x+1)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+x+1} \quad \Rightarrow \quad 1 = A\left(x^2+x+1\right) + (Bx+C)(x-1).$

Take $x=1$ to get $A=\frac{1}{3}$. Equating coefficients of x^2 and then comparing the constant terms, we get $0=\frac{1}{3}+B$, $1=\frac{1}{3}-C$, so $B=-\frac{1}{3}$, $C=-\frac{2}{3} \quad \Rightarrow$

$$\int \frac{1}{x^3-1}dx = \int \frac{\frac{1}{3}}{x-1}\,dx + \int \frac{-\frac{1}{3}x-\frac{2}{3}}{x^2+x+1}\,dx = \tfrac{1}{3}\ln|x-1| - \frac{1}{3}\int \frac{x+2}{x^2+x+1}dx$$

$$= \tfrac{1}{3}\ln|x-1| - \frac{1}{3}\int \frac{x+1/2}{x^2+x+1}dx - \frac{1}{3}\int \frac{(3/2)\,dx}{(x+1/2)^2+3/4}$$

$$= \tfrac{1}{3}\ln|x-1| - \tfrac{1}{6}\ln\left(x^2+x+1\right) - \tfrac{1}{2}\left(\frac{2}{\sqrt{3}}\right)\tan^{-1}\left(\frac{x+\frac{1}{2}}{\sqrt{3}/2}\right) + K$$

$$= \tfrac{1}{3}\ln|x-1| - \tfrac{1}{6}\ln(x^2+x+1) - \tfrac{1}{\sqrt{3}}\tan^{-1}\left(\tfrac{1}{\sqrt{3}}(2x+1)\right) + K$$

34. $\dfrac{x^3-2x^2+2x-5}{x^4+4x^2+3}=\dfrac{x^3-2x^2+2x-5}{(x^2+1)(x^2+3)}=\dfrac{Ax+B}{x^2+1}+\dfrac{Cx+D}{x^2+3}$. Multiply both sides by $(x^2+1)(x^2+3)$ to get

$x^3-2x^2+2x-5=(Ax+B)(x^2+3)+(Cx+D)(x^2+1)\quad\Leftrightarrow$

$x^3-2x^2+2x-5=Ax^3+Bx^2+3Ax+3B+Cx^3+Dx^2+Cx+D\quad\Leftrightarrow$

$x^3-2x^2+2x-5=(A+C)x^3+(B+D)x^2+(3A+C)x+(3B+D)$. Comparing coefficients gives us the following system of equations:

$A+C=1$ **(1)** $\qquad B+D=-2$ **(2)** $\qquad 3A+C=2$ **(3)** $\qquad 3B+D=-5$ **(4)**

Subtracting equation **(1)** from equation **(3)** gives us $2A=1\quad\Leftrightarrow\quad A=\frac{1}{2}$, and hence, $C=\frac{1}{2}$. Subtracting equation **(2)** from equation **(4)** gives us $2B=-3\quad\Leftrightarrow\quad B=-\frac{3}{2}$, and hence, $D=-\frac{1}{2}$.

Thus,

$$\int\frac{x^3-2x^2+2x-5}{x^4+4x^2+3}\,dx=\int\left(\frac{\frac{1}{2}x-\frac{3}{2}}{x^2+1}+\frac{\frac{1}{2}x-\frac{1}{2}}{x^2+3}\right)dx=\int\left(\frac{\frac{1}{2}x}{x^2+1}-\frac{\frac{3}{2}}{x^2+1}+\frac{\frac{1}{2}x}{x^2+3}-\frac{\frac{1}{2}}{x^2+3}\right)dx$$

$$=\tfrac{1}{4}\ln(x^2+1)-\tfrac{3}{2}\tan^{-1}x+\tfrac{1}{4}\ln(x^2+3)-\frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)+C$$

35. Let $u=x^4+4x^2+3$, so that $du=(4x^3+8x)\,dx=4(x^3+2x)\,dx$, $x=0\quad\Rightarrow\quad u=3$, and $x=1\quad\Rightarrow\quad u=8$.

Then $\displaystyle\int_0^1\frac{x^3+2x}{x^4+4x^2+3}\,dx=\int_3^8\frac{1}{u}\left(\frac{1}{4}\,du\right)=\frac{1}{4}\Big[\ln|u|\Big]_3^8=\frac{1}{4}(\ln 8-\ln 3)=\frac{1}{4}\ln\frac{8}{3}$.

36. $\dfrac{x^5+x-1}{x^3+1}=x^2+\dfrac{-x^2+x-1}{x^3+1}=x^2+\dfrac{-x^2+x-1}{(x+1)(x^2-x+1)}=x^2+\dfrac{-1}{x+1}$, so

$$\int\frac{x^5+x-1}{x^3+1}\,dx=\int\left(x^2-\frac{1}{x+1}\right)dx=\frac{1}{3}x^3-\ln|x+1|+C$$

37. $\dfrac{5x^4+7x^2+x+2}{x(x^2+1)^2}=\dfrac{A}{x}+\dfrac{Bx+C}{x^2+1}+\dfrac{Dx+E}{(x^2+1)^2}$. Multiply by $x(x^2+1)^2$ to get

$5x^4+7x^2+x+2=A(x^2+1)^2+(Bx+C)x(x^2+1)+(Dx+E)x\quad\Leftrightarrow$

$5x^4+7x^2+x+2=A(x^4+2x^2+1)+(Bx^2+Cx)(x^2+1)+Dx^2+Ex\quad\Leftrightarrow$

$5x^4+7x^2+x+2=Ax^4+2Ax^2+A+Bx^4+Cx^3+Bx^2+Cx+Dx^2+Ex\quad\Leftrightarrow$

$5x^4+7x^2+x+2=(A+B)x^4+Cx^3+(2A+B+D)x^2+(C+E)x+A$. Equating coefficients gives us $C=0$, $A=2$, $A+B=5\quad\Rightarrow\quad B=3$, $C+E=1\quad\Rightarrow\quad E=1$, and $2A+B+D=7\quad\Rightarrow\quad D=0$. Thus,

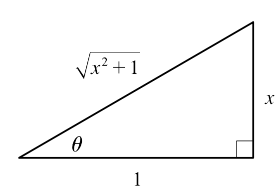
$$\int\frac{5x^4+7x^2+x+2}{x(x^2+1)^2}\,dx=\int\left[\frac{2}{x}+\frac{3x}{x^2+1}+\frac{1}{(x^2+1)^2}\right]dx=I.\text{ Now}$$

$$\int\frac{dx}{(x^2+1)^2}=\int\frac{\sec^2\theta\,d\theta}{(\tan^2\theta+1)^2}\qquad\begin{bmatrix}x=\tan\theta,\\ dx=\sec^2\theta\,d\theta\end{bmatrix}$$

$$=\int\frac{\sec^2\theta}{\sec^4\theta}\,d\theta=\int\cos^2\theta\,d\theta=\int\tfrac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$=\tfrac{1}{2}\theta+\tfrac{1}{4}\sin 2\theta+C=\tfrac{1}{2}\theta+\tfrac{1}{2}\sin\theta\,\cos\theta+C$$

$$=\frac{1}{2}\tan^{-1}x+\frac{1}{2}\frac{x}{\sqrt{x^2+1}}\frac{1}{\sqrt{x^2+1}}+C$$



Therefore, $I=2\ln|x|+\frac{3}{2}\ln(x^2+1)+\frac{1}{2}\tan^{-1}x+\dfrac{x}{2(x^2+1)}+C$.

38. Let $u = x^5 + 5x^3 + 5x$, so that $du = (5x^4 + 15x^2 + 5)dx = 5(x^4 + 3x^2 + 1)dx$. Then

$$\int \frac{x^4 + 3x^2 + 1}{x^5 + 5x^3 + 5x}\,dx = \int \frac{1}{u}\left(\frac{1}{5}\,du\right) = \frac{1}{5}\ln|u| + C = \frac{1}{5}\ln\left|x^5 + 5x^3 + 5x\right| + C$$

39. $\dfrac{x^2 - 3x + 7}{(x^2 - 4x + 6)^2} = \dfrac{Ax + B}{x^2 - 4x + 6} + \dfrac{Cx + D}{(x^2 - 4x + 6)^2} \quad \Rightarrow \quad x^2 - 3x + 7 = (Ax + B)(x^2 - 4x + 6) + Cx + D \quad \Rightarrow$

$x^2 - 3x + 7 = Ax^3 + (-4A + B)x^2 + (6A - 4B + C)x + (6B + D)$. So $A = 0$, $-4A + B = 1 \quad \Rightarrow \quad B = 1$,

$6A - 4B + C = -3 \quad \Rightarrow \quad C = 1$, $6B + D = 7 \quad \Rightarrow \quad D = 1$. Thus,

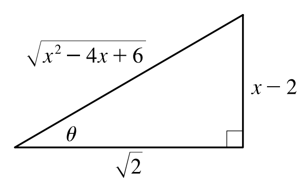
$$\begin{aligned} I &= \int \frac{x^2 - 3x + 7}{(x^2 - 4x + 6)^2}\,dx = \int \left(\frac{1}{x^2 - 4x + 6} + \frac{x + 1}{(x^2 - 4x + 6)^2}\right) dx \\ &= \int \frac{1}{(x - 2)^2 + 2}\,dx + \int \frac{x - 2}{(x^2 - 4x + 6)^2}\,dx + \int \frac{3}{(x^2 - 4x + 6)^2}\,dx \\ &= I_1 + I_2 + I_3. \end{aligned}$$

$$I_1 = \int \frac{1}{(x - 2)^2 + \left(\sqrt{2}\right)^2}\,dx = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x - 2}{\sqrt{2}}\right) + C_1$$

$$I_2 = \frac{1}{2}\int \frac{2x - 4}{(x^2 - 4x + 6)^2}\,dx = \tfrac{1}{2}\int \frac{1}{u^2}\,du = \tfrac{1}{2}\left(-\frac{1}{u}\right) + C_2 = -\frac{1}{2(x^2 - 4x + 6)} + C_2$$

$$I_3 = 3\int \frac{1}{\left[(x - 2)^2 + \left(\sqrt{2}\right)^2\right]^2}\,dx = 3\int \frac{1}{[2(\tan^2\theta + 1)]^2}\,\sqrt{2}\sec^2\theta\,d\theta \qquad \left[\begin{array}{l} x - 2 = \sqrt{2}\tan\theta, \\ dx = \sqrt{2}\,\sec^2\theta\,d\theta \end{array}\right]$$

$$\begin{aligned} &= \frac{3\sqrt{2}}{4}\int \frac{\sec^2\theta}{\sec^4\theta}\,d\theta = \frac{3\sqrt{2}}{4}\int \cos^2\theta\,d\theta = \frac{3\sqrt{2}}{4}\int \tfrac{1}{2}(1 + \cos 2\theta)\,d\theta \\ &= \frac{3\sqrt{2}}{8}\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + C_3 = \frac{3\sqrt{2}}{8}\tan^{-1}\left(\frac{x - 2}{\sqrt{2}}\right) + \frac{3\sqrt{2}}{8}\left(\tfrac{1}{2}\cdot 2\sin\theta\,\cos\theta\right) + C_3 \\ &= \frac{3\sqrt{2}}{8}\tan^{-1}\left(\frac{x - 2}{\sqrt{2}}\right) + \frac{3\sqrt{2}}{8}\cdot\frac{x - 2}{\sqrt{x^2 - 4x + 6}}\cdot\frac{\sqrt{2}}{\sqrt{x^2 - 4x + 6}} + C_3 \\ &= \frac{3\sqrt{2}}{8}\tan^{-1}\left(\frac{x - 2}{\sqrt{2}}\right) + \frac{3(x - 2)}{4(x^2 - 4x + 6)} + C_3 \end{aligned}$$



So $\quad I = I_1 + I_2 + I_3 \qquad [C = C_1 + C_2 + C_3]$

$$\begin{aligned} &= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x - 2}{\sqrt{2}}\right) + \frac{-1}{2(x^2 - 4x + 6)} + \frac{3\sqrt{2}}{8}\tan^{-1}\left(\frac{x - 2}{\sqrt{2}}\right) + \frac{3(x - 2)}{4(x^2 - 4x + 6)} + C \\ &= \left(\frac{4\sqrt{2}}{8} + \frac{3\sqrt{2}}{8}\right)\tan^{-1}\left(\frac{x - 2}{\sqrt{2}}\right) + \frac{3(x - 2) - 2}{4(x^2 - 4x + 6)} + C = \frac{7\sqrt{2}}{8}\tan^{-1}\left(\frac{x - 2}{\sqrt{2}}\right) + \frac{3x - 8}{4(x^2 - 4x + 6)} + C \end{aligned}$$

40. $\dfrac{x^3 + 2x^2 + 3x - 2}{(x^2 + 2x + 2)^2} = \dfrac{Ax + B}{x^2 + 2x + 2} + \dfrac{Cx + D}{(x^2 + 2x + 2)^2} \quad \Rightarrow$

$x^3 + 2x^2 + 3x - 2 = (Ax + B)(x^2 + 2x + 2) + Cx + D \quad \Rightarrow$

$x^3 + 2x^2 + 3x - 2 = Ax^3 + (2A + B)x^2 + (2A + 2B + C)x + 2B + D$.

So $A = 1$, $2A + B = 2 \quad \Rightarrow \quad B = 0$, $2A + 2B + C = 3 \quad \Rightarrow \quad C = 1$, and $2B + D = -2 \quad \Rightarrow \quad D = -2$. Thus,

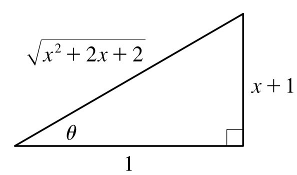
$$I = \int \frac{x^3 + 2x^2 + 3x - 2}{(x^2 + 2x + 2)^2}\,dx = \int \left(\frac{x}{x^2 + 2x + 2} + \frac{x - 2}{(x^2 + 2x + 2)^2}\right) dx$$
$$= \int \frac{x + 1}{x^2 + 2x + 2}\,dx + \int \frac{-1}{x^2 + 2x + 2}\,dx + \int \frac{x + 1}{(x^2 + 2x + 2)^2}\,dx + \int \frac{-3}{(x^2 + 2x + 2)^2}\,dx$$
$$= I_1 + I_2 + I_3 + I_4.$$

$$I_1 = \int \frac{x + 1}{x^2 + 2x + 2}\,dx = \int \frac{1}{u}\left(\frac{1}{2}\,du\right) \quad \begin{bmatrix} u = x^2 + 2x + 2, \\ du = 2(x + 1)\,dx \end{bmatrix} \quad = \frac{1}{2}\ln\left|x^2 + 2x + 2\right| + C_1$$

$$I_2 = -\int \frac{1}{(x + 1)^2 + 1}\,dx = -\frac{1}{1}\tan^{-1}\left(\frac{x + 1}{1}\right) + C_2 = -\tan^{-1}(x + 1) + C_2$$

$$I_3 = \int \frac{x + 1}{(x^2 + 2x + 2)^2}\,dx = \int \frac{1}{u^2}\left(\frac{1}{2}\,du\right) = -\frac{1}{2u} + C_3 = -\frac{1}{2(x^2 + 2x + 2)} + C_3$$

$$I_4 = -3\int \frac{1}{[(x + 1)^2 + 1]^2}\,dx = -3\int \frac{1}{(\tan^2\theta + 1)^2}\sec^2\theta\,d\theta \quad \begin{bmatrix} x + 1 = 1\tan\theta, \\ dx = \sec^2\theta\,d\theta \end{bmatrix}$$
$$= -3\int \frac{1}{\sec^2\theta}\,d\theta = -3\int \cos^2\theta\,d\theta = -\frac{3}{2}\int (1 + \cos 2\theta)\,d\theta$$
$$= -\tfrac{3}{2}\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + C_4 = -\tfrac{3}{2}\theta - \tfrac{3}{2}\left(\tfrac{1}{2}\cdot 2\sin\theta\,\cos\theta\right) + C_4$$
$$= -\frac{3}{2}\tan^{-1}\left(\frac{x + 1}{1}\right) - \frac{3}{2}\cdot\frac{x + 1}{\sqrt{x^2 + 2x + 2}}\cdot\frac{1}{\sqrt{x^2 + 2x + 2}} + C_4$$
$$= -\frac{3}{2}\tan^{-1}(x + 1) - \frac{3(x + 1)}{2(x^2 + 2x + 2)} + C_4$$



So $\quad I = I_1 + I_2 + I_3 + I_4 \qquad [C = C_1 + C_2 + C_3 + C_4]$

$$= \frac{1}{2}\ln(x^2 + 2x + 2) - \tan^{-1}(x + 1) - \frac{1}{2(x^2 + 2x + 2)} - \frac{3}{2}\tan^{-1}(x + 1) - \frac{3(x + 1)}{2(x^2 + 2x + 2)} + C$$
$$= \frac{1}{2}\ln(x^2 + 2x + 2) - \frac{5}{2}\tan^{-1}(x + 1) - \frac{3x + 4}{2(x^2 + 2x + 2)} + C$$

41. $$\int \frac{dx}{x\sqrt{x - 1}} = \int \frac{2u}{u(u^2 + 1)}\,du \qquad \begin{bmatrix} u = \sqrt{x - 1},\ x = u^2 + 1 \\ u^2 = x - 1,\ \ dx = 2u\,du \end{bmatrix}$$
$$= 2\int \frac{1}{u^2 + 1}\,du = 2\tan^{-1}u + C = 2\tan^{-1}\sqrt{x - 1} + C$$

42. Let $u = \sqrt{x + 3}$, so $u^2 = x + 3$ and $2u\,du = dx$. Then

$$\int \frac{dx}{2\sqrt{x + 3} + x} = \int \frac{2u\,du}{2u + (u^2 - 3)} = \int \frac{2u}{u^2 + 2u - 3}\,du = \int \frac{2u}{(u + 3)(u - 1)}\,du.$$ Now

$$\frac{2u}{(u + 3)(u - 1)} = \frac{A}{u + 3} + \frac{B}{u - 1} \quad \Rightarrow \quad 2u = A(u - 1) + B(u + 3).$$ Setting $u = 1$ gives $2 = 4B$, so $B = \frac{1}{2}$.

Setting $u = -3$ gives $-6 = -4A$, so $A = \frac{3}{2}$. Thus,

$$\int \frac{2u}{(u + 3)(u - 1)}\,du = \int \left(\frac{\frac{3}{2}}{u + 3} + \frac{\frac{1}{2}}{u - 1}\,du\right)$$
$$= \tfrac{3}{2}\ln|u + 3| + \tfrac{1}{2}\ln|u - 1| + C = \tfrac{3}{2}\ln\left(\sqrt{x + 3} + 3\right) + \tfrac{1}{2}\ln\left|\sqrt{x + 3} - 1\right| + C$$

43. Let $u = \sqrt{x}$, so $u^2 = x$ and $2u\,du = dx$. Then $\displaystyle\int \frac{dx}{x^2 + x\sqrt{x}} = \int \frac{2u\,du}{u^4 + u^3} = \int \frac{2\,du}{u^3 + u^2} = \int \frac{2\,du}{u^2(u+1)}$.

$\displaystyle\frac{2}{u^2(u+1)} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u+1} \quad\Rightarrow\quad 2 = Au(u+1) + B(u+1) + Cu^2$. Setting $u = 0$ gives $B = 2$. Setting $u = -1$ gives $C = 2$. Equating coefficients of u^2, we get $0 = A + C$, so $A = -2$. Thus,

$$\int \frac{2\,du}{u^2(u+1)} = \int \left(\frac{-2}{u} + \frac{2}{u^2} + \frac{2}{u+1}\right) du = -2\ln|u| - \frac{2}{u} + 2\ln|u+1| + C = -2\ln\sqrt{x} - \frac{2}{\sqrt{x}} + 2\ln\left(\sqrt{x}+1\right) + C.$$

44. Let $u = \sqrt[3]{x}$. Then $x = u^3$, $dx = 3u^2\,du \quad\Rightarrow$

$$\int_0^1 \frac{1}{1+\sqrt[3]{x}}\,dx = \int_0^1 \frac{3u^2\,du}{1+u} = \int_0^1 \left(3u - 3 + \frac{3}{1+u}\right) du = \left[\tfrac{3}{2}u^2 - 3u + 3\ln(1+u)\right]_0^1 = 3\left(\ln 2 - \tfrac{1}{2}\right).$$

45. Let $u = \sqrt[3]{x^2+1}$. Then $x^2 = u^3 - 1$, $2x\,dx = 3u^2\,du \quad\Rightarrow$

$$\int \frac{x^3}{\sqrt[3]{x^2+1}}\,dx = \int \frac{(u^3-1)\frac{3}{2}u^2\,du}{u} = \frac{3}{2}\int (u^4 - u)\,du$$
$$= \tfrac{3}{10}u^5 - \tfrac{3}{4}u^2 + C = \tfrac{3}{10}(x^2+1)^{5/3} - \tfrac{3}{4}(x^2+1)^{2/3} + C$$

46. $$\int \frac{dx}{(1+\sqrt{x}\,)^2} = \int \frac{2(u-1)\,du}{u^2} \qquad \left[\begin{matrix} u = 1+\sqrt{x}, \\ x = (u-1)^2,\ dx = 2(u-1)\,du \end{matrix}\right]$$
$$= 2\int \left(\frac{1}{u} - \frac{1}{u^2}\right) du = 2\ln|u| + \frac{2}{u} + C = 2\ln(1+\sqrt{x}\,) + \frac{2}{1+\sqrt{x}} + C$$

47. If we were to substitute $u = \sqrt{x}$, then the square root would disappear but a cube root would remain. On the other hand, the substitution $u = \sqrt[3]{x}$ would eliminate the cube root but leave a square root. We can eliminate both roots by means of the substitution $u = \sqrt[6]{x}$. (Note that 6 is the least common multiple of 2 and 3.)

Let $u = \sqrt[6]{x}$. Then $x = u^6$, so $dx = 6u^5\,du$ and $\sqrt{x} = u^3$, $\sqrt[3]{x} = u^2$. Thus,

$$\int \frac{1}{\sqrt{x} - \sqrt[3]{x}}\,dx = \int \frac{6u^5\,du}{u^3 - u^2} = 6\int \frac{u^5}{u^2(u-1)}\,du = 6\int \frac{u^3}{u-1}\,du$$
$$= 6\int \left(u^2 + u + 1 + \frac{1}{u-1}\right) du \qquad \text{[by long division]}$$
$$= 6\left(\tfrac{1}{3}u^3 + \tfrac{1}{2}u^2 + u + \ln|u-1|\right) + C = 2\sqrt{x} + 3\sqrt[3]{x} + 6\sqrt[6]{x} + 6\ln\left|\sqrt[6]{x} - 1\right| + C$$

48. Let $u = x^{1/5} \quad\Rightarrow\quad x = u^5$, so $dx = 5u^4\,du$. This substitution gives

$$I = \int \frac{1}{x - x^{1/5}}\,dx = \int \frac{5u^4}{u^5 - u}\,du = \int \frac{5u^3}{u^4 - 1}\,du = \int \frac{5u^3}{(u^2-1)(u^2+1)}\,du = \int \frac{5u^3}{(u-1)(u+1)(u^2+1)}\,du.$$

Now $\displaystyle\frac{5u^3}{(u-1)(u+1)(u^2+1)} = \frac{A}{u-1} + \frac{B}{u+1} + \frac{Cu+D}{u^2+1}$. Multiply both sides by $(u-1)(u+1)(u^2+1)$ to get

$5u^3 = A(u+1)(u^2+1) + B(u-1)(u^2+1) + (Cu+D)(u-1)(u+1) \quad\Leftrightarrow$

$5u^3 = A(u^3+u^2+u+1) + B(u^3-u^2+u-1) + Cu(u^2-1) + D(u^2-1) \quad\Leftrightarrow$

$5u^3 = (A+B+C)u^3 + (A-B+D)u^2 + (A+B-C)u + (A-B-D)$

[continued]

Setting $u=1$ gives $5=4A$, so $A=\frac{5}{4}$. Now, comparing coefficients gives us the following system of equations:

$$A+B+C=5 \quad \textbf{(1)} \qquad A-B+D=0 \quad \textbf{(2)}$$

$$A+B-C=0 \quad \textbf{(3)} \qquad A-B-D=0 \quad \textbf{(4)}$$

Adding equations **(1)** and **(3)** gives $2A+2B=5$, so $B=\frac{5}{4}$. Subtracting equation **(4)** from equation **(2)** gives $D=0$. Finally, substituting the value of A and B in equation **(3)** gives $C=\frac{5}{2}$. Thus,

$$\begin{aligned} I &= \int \left(\frac{5/4}{u-1} + \frac{5/4}{u+1} + \frac{5u/2}{u^2+1} \right) du = \frac{5}{4}\ln|u-1| + \frac{5}{4}\ln|u+1| + \frac{5}{4}\ln(u^2+1) + C \\ &= \tfrac{5}{4}\ln\left|(u^2-1)(u^2+1)\right| + C = \tfrac{5}{4}\ln\left|u^4-1\right| + C = \tfrac{5}{4}\ln\left|x^{4/5}-1\right| + C \end{aligned}$$

49. Let $u=\sqrt{x} \quad \Rightarrow \quad x=u^2$, so $dx=2u\,du$. This substitution gives $I=\displaystyle\int \frac{1}{x-3\sqrt{x}+2}\,dx = \int \frac{2u}{u^2-3u+2}\,du$. Now $\dfrac{2u}{u^2-3u+2} = \dfrac{2u}{(u-2)(u-1)} = \dfrac{A}{u-2} + \dfrac{B}{u-1}$. Multiply both sides by $(u-2)(u-1)$ to get $2u = A(u-1)+B(u-2)$. Setting $u=1$ gives $2=-B$ or $B=-2$, and setting $u=2$ gives $A=4$. Thus,

$$I = \int \left(\frac{4}{u-2} - \frac{2}{u-1} \right) du = 4\ln|u-2| - 2\ln|u-1| + C = 4\ln\left|\sqrt{x}-2\right| - 2\ln\left|\sqrt{x}-1\right| + C.$$

50. Let $u=\sqrt{1+\sqrt{x}}$, so that $u^2=1+\sqrt{x}$, $x=(u^2-1)^2$, and $dx=2(u^2-1)\cdot 2u\,du = 4u(u^2-1)\,du$. Then

$$\int \frac{\sqrt{1+\sqrt{x}}}{x}\,dx = \int \frac{u}{(u^2-1)^2}\cdot 4u(u^2-1)\,du = \int \frac{4u^2}{u^2-1}\,du = \int \left(4+\frac{4}{u^2-1}\right) du.$$ Now

$\dfrac{4}{u^2-1} = \dfrac{A}{u+1} + \dfrac{B}{u-1} \quad \Rightarrow \quad 4 = A(u-1)+B(u+1)$. Setting $u=1$ gives $4=2B$, so $B=2$. Setting $u=-1$ gives $4=-2A$, so $A=-2$. Thus,

$$\begin{aligned} \int \left(4+\frac{4}{u^2-1}\right) du &= \int \left(4 - \frac{2}{u+1} + \frac{2}{u-1}\right) du = 4u - 2\ln|u+1| + 2\ln|u-1| + C \\ &= 4\sqrt{1+\sqrt{x}} - 2\ln\left(\sqrt{1+\sqrt{x}}+1\right) + 2\ln\left(\sqrt{1+\sqrt{x}}-1\right) + C \end{aligned}$$

51. Let $u=e^x$. Then $x=\ln u$, $dx=\dfrac{du}{u} \quad \Rightarrow$

$$\begin{aligned} \int \frac{e^{2x}\,dx}{e^{2x}+3e^x+2} &= \int \frac{u^2\,(du/u)}{u^2+3u+2} = \int \frac{u\,du}{(u+1)(u+2)} = \int \left[\frac{-1}{u+1} + \frac{2}{u+2}\right] du \\ &= 2\ln|u+2| - \ln|u+1| + C = \ln\frac{(e^x+2)^2}{e^x+1} + C \end{aligned}$$

52. Let $u=\cos x$, so that $du=-\sin x\,dx$. Then $\displaystyle\int \frac{\sin x}{\cos^2 x - 3\cos x}\,dx = \int \frac{1}{u^2-3u}\,(-du) = \int \frac{-1}{u(u-3)}\,du.$

$\dfrac{-1}{u(u-3)} = \dfrac{A}{u} + \dfrac{B}{u-3} \quad \Rightarrow \quad -1 = A(u-3)+Bu$. Setting $u=3$ gives $B=-\frac{1}{3}$. Setting $u=0$ gives $A=\frac{1}{3}$.

Thus, $\displaystyle\int \frac{-1}{u(u-3)}\,du = \int \left(\frac{\frac{1}{3}}{u} - \frac{\frac{1}{3}}{u-3}\right) du = \frac{1}{3}\ln|u| - \frac{1}{3}\ln|u-3| + C = \tfrac{1}{3}\ln|\cos x| - \tfrac{1}{3}\ln|\cos x - 3| + C.$

53. Let $u = \tan t$, so that $du = \sec^2 t\,dt$. Then $\displaystyle\int \frac{\sec^2 t}{\tan^2 t + 3\tan t + 2}\,dt = \int \frac{1}{u^2 + 3u + 2}\,du = \int \frac{1}{(u+1)(u+2)}\,du$.

Now $\dfrac{1}{(u+1)(u+2)} = \dfrac{A}{u+1} + \dfrac{B}{u+2} \quad\Rightarrow\quad 1 = A(u+2) + B(u+1)$.

Setting $u = -2$ gives $1 = -B$, so $B = -1$. Setting $u = -1$ gives $1 = A$.

Thus, $\displaystyle\int \frac{1}{(u+1)(u+2)}\,du = \int \left(\frac{1}{u+1} - \frac{1}{u+2}\right) du = \ln|u+1| - \ln|u+2| + C = \ln|\tan t + 1| - \ln|\tan t + 2| + C$.

54. Let $u = e^x$, so that $du = e^x\,dx$. Then $\displaystyle\int \frac{e^x}{(e^x - 2)(e^{2x} + 1)}\,dx = \int \frac{1}{(u-2)(u^2+1)}\,du$. Now

$\dfrac{1}{(u-2)(u^2+1)} = \dfrac{A}{u-2} + \dfrac{Bu + C}{u^2+1} \quad\Rightarrow\quad 1 = A(u^2+1) + (Bu + C)(u-2)$. Setting $u = 2$ gives $1 = 5A$, so $A = \frac{1}{5}$.

Setting $u = 0$ gives $1 = \frac{1}{5} - 2C$, so $C = -\frac{2}{5}$. Comparing coefficients of u^2 gives $0 = \frac{1}{5} + B$, so $B = -\frac{1}{5}$. Thus,

$$\begin{aligned}\int \frac{1}{(u-2)(u^2+1)}\,du &= \int \left(\frac{\frac{1}{5}}{u-2} + \frac{-\frac{1}{5}u - \frac{2}{5}}{u^2+1}\right) du = \tfrac{1}{5}\int \frac{1}{u-2}\,du - \tfrac{1}{5}\int \frac{u}{u^2+1}\,du - \tfrac{2}{5}\int \frac{1}{u^2+1}\,du \\ &= \tfrac{1}{5}\ln|u-2| - \tfrac{1}{5}\cdot\tfrac{1}{2}\ln\left|u^2+1\right| - \tfrac{2}{5}\tan^{-1} u + C \\ &= \tfrac{1}{5}\ln|e^x - 2| - \tfrac{1}{10}\ln(e^{2x}+1) - \tfrac{2}{5}\tan^{-1} e^x + C\end{aligned}$$

55. Let $u = e^x$, so that $du = e^x\,dx$ and $dx = \dfrac{du}{u}$. Then $\displaystyle\int \frac{dx}{1+e^x} = \int \frac{du}{(1+u)u}$. $\quad \dfrac{1}{u(u+1)} = \dfrac{A}{u} + \dfrac{B}{u+1} \quad\Rightarrow$

$1 = A(u+1) + Bu$. Setting $u = -1$ gives $B = -1$. Setting $u = 0$ gives $A = 1$. Thus,

$$\int \frac{du}{u(u+1)} = \int \left(\frac{1}{u} - \frac{1}{u+1}\right) du = \ln|u| - \ln|u+1| + C = \ln e^x - \ln(e^x+1) + C = x - \ln(e^x+1) + C.$$

56. Let $u = \sinh t$, so that $du = \cosh t\,dt$. Then $\displaystyle\int \frac{\cosh t}{\sinh^2 t + \sinh^4 t}\,dt = \int \frac{1}{u^2 + u^4}\,du = \int \frac{1}{u^2(u^2+1)}\,du$.

$\dfrac{1}{u^2(u^2+1)} = \dfrac{A}{u} + \dfrac{B}{u^2} + \dfrac{Cu + D}{u^2+1} \quad\Rightarrow\quad 1 = Au(u^2+1) + B(u^2+1) + (Cu+D)u^2$. Setting $u = 0$ gives $B = 1$.

Comparing coefficients of u^2, we get $0 = B + D$, so $D = -1$. Comparing coefficients of u, we get $0 = A$. Comparing coefficients of u^3, we get $0 = A + C$, so $C = 0$. Thus,

$$\begin{aligned}\int \frac{1}{u^2(u^2+1)}\,du &= \int \left(\frac{1}{u^2} - \frac{1}{u^2+1}\right) du = -\frac{1}{u} - \tan^{-1} u + C = -\frac{1}{\sinh t} - \tan^{-1}(\sinh t) + C \\ &= -\operatorname{csch} t - \tan^{-1}(\sinh t) + C\end{aligned}$$

57. Let $u = \ln(x^2 - x + 2)$, $dv = dx$. Then $du = \dfrac{2x-1}{x^2 - x + 2}\,dx$, $v = x$, and (by integration by parts)

$$\begin{aligned}\int \ln(x^2 - x + 2)\,dx &= x\ln(x^2 - x + 2) - \int \frac{2x^2 - x}{x^2 - x + 2}\,dx = x\ln(x^2 - x + 2) - \int \left(2 + \frac{x-4}{x^2 - x + 2}\right) dx \\ &= x\ln(x^2 - x + 2) - 2x - \int \frac{\frac{1}{2}(2x-1)}{x^2 - x + 2}\,dx + \frac{7}{2}\int \frac{dx}{(x - \frac{1}{2})^2 + \frac{7}{4}}\end{aligned}$$

[continued]

$$= x\ln(x^2-x+2) - 2x - \frac{1}{2}\ln(x^2-x+2) + \frac{7}{2}\int \frac{\frac{\sqrt{7}}{2}\,du}{\frac{7}{4}(u^2+1)} \qquad \left[\begin{array}{c}\text{where } x-\frac{1}{2}=\frac{\sqrt{7}}{2}u,\\ dx=\frac{\sqrt{7}}{2}\,du,\\ (x-\frac{1}{2})^2+\frac{7}{4}=\frac{7}{4}(u^2+1)\end{array}\right]$$

$$= (x-\tfrac{1}{2})\ln(x^2-x+2) - 2x + \sqrt{7}\tan^{-1}u + C$$

$$= (x-\tfrac{1}{2})\ln(x^2-x+2) - 2x + \sqrt{7}\tan^{-1}\frac{2x-1}{\sqrt{7}} + C$$

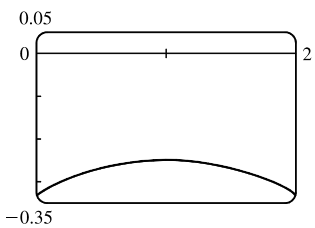
58. Let $u = \tan^{-1}x$, $dv = x\,dx \quad\Rightarrow\quad du = dx/(1+x^2)$, $v = \frac{1}{2}x^2$.

Then $\displaystyle\int x\tan^{-1}x\,dx = \tfrac{1}{2}x^2\tan^{-1}x - \frac{1}{2}\int\frac{x^2}{1+x^2}\,dx$. To evaluate the last integral, use long division or observe that

$\displaystyle\int\frac{x^2}{1+x^2}\,dx = \int\frac{(1+x^2)-1}{1+x^2}\,dx = \int 1\,dx - \int\frac{1}{1+x^2}\,dx = x - \tan^{-1}x + C_1$. So

$\int x\tan^{-1}x\,dx = \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}(x - \tan^{-1}x + C_1) = \frac{1}{2}(x^2\tan^{-1}x + \tan^{-1}x - x) + C$.

59.



From the graph, we see that the integral will be negative, and we guess that the area is about the same as that of a rectangle with width 2 and height 0.3, so we estimate the integral to be $-(2\cdot 0.3) = -0.6$. Now

$\displaystyle\frac{1}{x^2-2x-3} = \frac{1}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1} \quad\Leftrightarrow$

$1 = (A+B)x + A - 3B$, so $A = -B$ and $A - 3B = 1 \quad\Leftrightarrow\quad A = \frac{1}{4}$

and $B = -\frac{1}{4}$, so the integral becomes

$$\int_0^2\frac{dx}{x^2-2x-3} = \frac{1}{4}\int_0^2\frac{dx}{x-3} - \frac{1}{4}\int_0^2\frac{dx}{x+1} = \frac{1}{4}\Big[\ln|x-3| - \ln|x+1|\Big]_0^2 = \frac{1}{4}\left[\ln\left|\frac{x-3}{x+1}\right|\right]_0^2$$

$$= \tfrac{1}{4}\left(\ln\tfrac{1}{3} - \ln 3\right) = -\tfrac{1}{2}\ln 3 \approx -0.55$$

60. $\boldsymbol{k=0}$: $\displaystyle\int\frac{dx}{x^2+k} = \int\frac{dx}{x^2} = -\frac{1}{x} + C$

$\boldsymbol{k>0}$: $\displaystyle\int\frac{dx}{x^2+k} = \int\frac{dx}{x^2+(\sqrt{k}\,)^2} = \frac{1}{\sqrt{k}}\tan^{-1}\!\left(\frac{x}{\sqrt{k}}\right) + C$

$\boldsymbol{k<0}$: $\displaystyle\int\frac{dx}{x^2+k} = \int\frac{dx}{x^2-(-k)} = \int\frac{dx}{x^2-\left(\sqrt{-k}\right)^2} = \frac{1}{2\sqrt{-k}}\ln\left|\frac{x-\sqrt{-k}}{x+\sqrt{-k}}\right| + C$ [by Example 3]

61. $\displaystyle\int\frac{dx}{x^2-2x} = \int\frac{dx}{(x-1)^2-1} = \int\frac{du}{u^2-1}$ [put $u = x-1$]

$$= \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + C \quad \text{[by Equation 6]} \quad = \frac{1}{2}\ln\left|\frac{x-2}{x}\right| + C$$

62. $\displaystyle\int\frac{(2x+1)\,dx}{4x^2+12x-7} = \frac{1}{4}\int\frac{(8x+12)\,dx}{4x^2+12x-7} - \int\frac{2\,dx}{(2x+3)^2-16}$

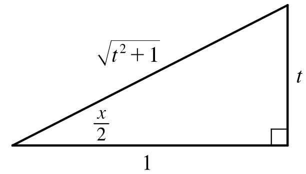
$$= \tfrac{1}{4}\ln\left|4x^2+12x-7\right| - \int\frac{du}{u^2-16} \qquad \text{[put } u = 2x+3\text{]}$$

$$= \tfrac{1}{4}\ln\left|4x^2+12x-7\right| - \tfrac{1}{8}\ln|(u-4)/(u+4)| + C \qquad \text{[by Equation 6]}$$

$$= \tfrac{1}{4}\ln\left|4x^2+12x-7\right| - \tfrac{1}{8}\ln|(2x-1)/(2x+7)| + C$$

63. (a) If $t = \tan\left(\frac{x}{2}\right)$, then $\frac{x}{2} = \tan^{-1} t$. The figure gives

$$\cos\left(\frac{x}{2}\right) = \frac{1}{\sqrt{1+t^2}} \text{ and } \sin\left(\frac{x}{2}\right) = \frac{t}{\sqrt{1+t^2}}.$$



(b) $$\cos x = \cos\left(2 \cdot \frac{x}{2}\right) = 2\cos^2\left(\frac{x}{2}\right) - 1$$

$$= 2\left(\frac{1}{\sqrt{1+t^2}}\right)^2 - 1 = \frac{2}{1+t^2} - 1 = \frac{1-t^2}{1+t^2}$$

$$\sin x = \sin\left(2 \cdot \frac{x}{2}\right) = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) = 2\frac{t}{\sqrt{t^2+1}} \cdot \frac{1}{\sqrt{t^2+1}} = \frac{2t}{t^2+1}$$

(c) $$\frac{x}{2} = \arctan t \quad \Rightarrow \quad x = 2\arctan t \quad \Rightarrow \quad dx = \frac{2}{1+t^2}\,dt$$

64. Let $t = \tan(x/2)$. Then, by using the expressions in Exercise 63, we have

$$\int \frac{dx}{1-\cos x} = \int \frac{2\,dt/(1+t^2)}{1-(1-t^2)/(1+t^2)} = \int \frac{2\,dt}{(1+t^2)-(1-t^2)} = \int \frac{2\,dt}{2t^2} = \int \frac{1}{t^2}\,dt$$

$$= -\frac{1}{t} + C = -\frac{1}{\tan(x/2)} + C = -\cot(x/2) + C$$

Another method: $$\int \frac{dx}{1-\cos x} = \int \left(\frac{1}{1-\cos x} \cdot \frac{1+\cos x}{1+\cos x}\right) dx = \int \frac{1+\cos x}{1-\cos^2 x}\,dx$$

$$= \int \frac{1+\cos x}{\sin^2 x}\,dx = \int \left(\frac{1}{\sin^2 x} + \frac{\cos x}{\sin^2 x}\right) dx$$

$$= \int (\csc^2 x + \csc x\,\cot x)\,dx = -\cot x - \csc x + C$$

65. Let $t = \tan(x/2)$. Then, using the expressions in Exercise 63, we have

$$\int \frac{1}{3\sin x - 4\cos x}\,dx = \int \frac{1}{3\left(\frac{2t}{1+t^2}\right) - 4\left(\frac{1-t^2}{1+t^2}\right)} \frac{2\,dt}{1+t^2} = 2\int \frac{dt}{3(2t)-4(1-t^2)} = \int \frac{dt}{2t^2+3t-2}$$

$$= \int \frac{dt}{(2t-1)(t+2)} = \int \left[\frac{2}{5}\frac{1}{2t-1} - \frac{1}{5}\frac{1}{t+2}\right] dt \qquad \text{[using partial fractions]}$$

$$= \tfrac{1}{5}\Big[\ln|2t-1| - \ln|t+2|\Big] + C = \frac{1}{5}\ln\left|\frac{2t-1}{t+2}\right| + C = \frac{1}{5}\ln\left|\frac{2\tan(x/2)-1}{\tan(x/2)+2}\right| + C$$

66. Let $t = \tan(x/2)$. Then, by Exercise 63,

$$\int_{\pi/3}^{\pi/2} \frac{dx}{1+\sin x - \cos x} = \int_{1/\sqrt{3}}^{1} \frac{2\,dt/(1+t^2)}{1+2t/(1+t^2)-(1-t^2)/(1+t^2)} = \int_{1/\sqrt{3}}^{1} \frac{2\,dt}{1+t^2+2t-1+t^2}$$

$$= \int_{1/\sqrt{3}}^{1} \left[\frac{1}{t} - \frac{1}{t+1}\right] dt = \Big[\ln t - \ln(t+1)\Big]_{1/\sqrt{3}}^{1}$$

$$= \ln\frac{1}{2} - \ln\frac{1}{\sqrt{3}+1} = \ln\frac{\sqrt{3}+1}{2}$$

67. Let $t = \tan(x/2)$. Then, by Exercise 63,

$$\int_0^{\pi/2} \frac{\sin 2x}{2+\cos x}\,dx = \int_0^{\pi/2} \frac{2\sin x\,\cos x}{2+\cos x}\,dx = \int_0^1 \frac{2\cdot\dfrac{2t}{1+t^2}\cdot\dfrac{1-t^2}{1+t^2}}{2+\dfrac{1-t^2}{1+t^2}}\,\frac{2}{1+t^2}\,dt = \int_0^1 \frac{\dfrac{8t(1-t^2)}{(1+t^2)^2}}{2(1+t^2)+(1-t^2)}\,dt$$

$$= \int_0^1 8t\cdot\frac{1-t^2}{(t^2+3)(t^2+1)^2}\,dt = I$$

If we now let $u = t^2$, then $\dfrac{1-t^2}{(t^2+3)(t^2+1)^2} = \dfrac{1-u}{(u+3)(u+1)^2} = \dfrac{A}{u+3} + \dfrac{B}{u+1} + \dfrac{C}{(u+1)^2} \quad\Rightarrow$

$1-u = A(u+1)^2 + B(u+3)(u+1) + C(u+3)$. Set $u=-1$ to get $2 = 2C$, so $C=1$. Set $u=-3$ to get $4=4A$, so $A=1$. Set $u=0$ to get $1 = 1+3B+3$, so $B=-1$. So

$$I = \int_0^1 \left[\frac{8t}{t^2+3} - \frac{8t}{t^2+1} + \frac{8t}{(t^2+1)^2}\right]dt = \left[4\ln(t^2+3) - 4\ln(t^2+1) - \frac{4}{t^2+1}\right]_0^1$$

$$= (4\ln 4 - 4\ln 2 - 2) - (4\ln 3 - 0 - 4) = 8\ln 2 - 4\ln 2 - 4\ln 3 + 2 = 4\ln\tfrac{2}{3} + 2$$

68. $\dfrac{1}{x^3+x} = \dfrac{1}{x(x^2+1)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1} \quad\Rightarrow\quad 1 = A(x^2+1) + (Bx+C)x$. Set $x=0$ to get $1=A$. So

$1 = (1+B)x^2 + Cx + 1 \quad\Rightarrow\quad B+1=0 \quad [B=-1]$ and $C=0$. Thus, the area is

$$\int_1^2 \frac{1}{x^3+x}\,dx = \int_1^2\left(\frac{1}{x} - \frac{x}{x^2+1}\right)dx = \left[\ln|x| - \tfrac{1}{2}\ln\left|x^2+1\right|\right]_1^2 = \left(\ln 2 - \tfrac{1}{2}\ln 5\right) - \left(0 - \tfrac{1}{2}\ln 2\right)$$

$$= \tfrac{3}{2}\ln 2 - \tfrac{1}{2}\ln 5 \quad \left[\text{or } \tfrac{1}{2}\ln\tfrac{8}{5}\right]$$

69. By long division, $\dfrac{x^2+1}{3x-x^2} = -1 + \dfrac{3x+1}{3x-x^2}$. Now

$\dfrac{3x+1}{3x-x^2} = \dfrac{3x+1}{x(3-x)} = \dfrac{A}{x} + \dfrac{B}{3-x} \quad\Rightarrow\quad 3x+1 = A(3-x) + Bx$. Set $x=3$ to get $10 = 3B$, so $B = \frac{10}{3}$. Set $x=0$ to get $1 = 3A$, so $A = \frac{1}{3}$. Thus, the area is

$$\int_1^2 \frac{x^2+1}{3x-x^2}\,dx = \int_1^2\left(-1 + \frac{\frac{1}{3}}{x} + \frac{\frac{10}{3}}{3-x}\right)dx = \left[-x + \tfrac{1}{3}\ln|x| - \tfrac{10}{3}\ln|3-x|\right]_1^2$$

$$= \left(-2 + \tfrac{1}{3}\ln 2 - 0\right) - \left(-1 + 0 - \tfrac{10}{3}\ln 2\right) = -1 + \tfrac{11}{3}\ln 2$$

70. (a) We use disks, so the volume is $V = \pi\displaystyle\int_0^1 \left[\frac{1}{x^2+3x+2}\right]^2 dx = \pi\int_0^1 \frac{dx}{(x+1)^2(x+2)^2}$. To evaluate the integral,

we use partial fractions: $\dfrac{1}{(x+1)^2(x+2)^2} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{x+2} + \dfrac{D}{(x+2)^2} \quad\Rightarrow$

$1 = A(x+1)(x+2)^2 + B(x+2)^2 + C(x+1)^2(x+2) + D(x+1)^2$. We set $x=-1$, giving $B=1$, then set $x=-2$, giving $D=1$. Now equating coefficients of x^3 gives $A=-C$, and then equating constants gives $1 = 4A + 4 + 2(-A) + 1 \quad\Rightarrow\quad A=-2 \quad\Rightarrow\quad C=2$. So the expression becomes

$$V = \pi\int_0^1 \left[\frac{-2}{x+1} + \frac{1}{(x+1)^2} + \frac{2}{(x+2)} + \frac{1}{(x+2)^2}\right]dx = \pi\left[2\ln\left|\frac{x+2}{x+1}\right| - \frac{1}{x+1} - \frac{1}{x+2}\right]_0^1$$

$$= \pi\left[\left(2\ln\tfrac{3}{2} - \tfrac{1}{2} - \tfrac{1}{3}\right) - \left(2\ln 2 - 1 - \tfrac{1}{2}\right)\right] = \pi\left(2\ln\tfrac{3/2}{2} + \tfrac{2}{3}\right) = \pi\left(\tfrac{2}{3} + \ln\tfrac{9}{16}\right)$$

(b) In this case, we use cylindrical shells, so the volume is $V = 2\pi \int_0^1 \frac{x\,dx}{x^2+3x+2} = 2\pi \int_0^1 \frac{x\,dx}{(x+1)(x+2)}$. We use partial fractions to simplify the integrand: $\frac{x}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} \Rightarrow x = (A+B)x + 2A + B$. So $A + B = 1$ and $2A + B = 0 \Rightarrow A = -1$ and $B = 2$. So the volume is

$$\begin{aligned}2\pi \int_0^1 \left[\frac{-1}{x+1} + \frac{2}{x+2}\right] dx &= 2\pi \Big[-\ln|x+1| + 2\ln|x+2|\Big]_0^1 \\ &= 2\pi(-\ln 2 + 2\ln 3 + \ln 1 - 2\ln 2) = 2\pi(2\ln 3 - 3\ln 2) = 2\pi \ln \tfrac{9}{8}\end{aligned}$$

71. $t = \int \frac{P+S}{P[(r-1)P-S]}\,dP = \int \frac{P+S}{P(0.1P-S)}\,dP \quad [r = 1.1]$. Now $\frac{P+S}{P(0.1P-S)} = \frac{A}{P} + \frac{B}{0.1P-S} \Rightarrow$ $P + S = A(0.1P - S) + BP$. Substituting 0 for P gives $S = -AS \Rightarrow A = -1$. Substituting $10S$ for P gives $11S = 10BS \Rightarrow B = \frac{11}{10}$. Thus, $t = \int \left(\frac{-1}{P} + \frac{11/10}{0.1P-S}\right) dP \Rightarrow t = -\ln P + 11\ln(0.1P - S) + C$.

When $t = 0$, $P = 10{,}000$ and $S = 900$, so $0 = -\ln 10{,}000 + 11\ln(1000 - 900) + C \Rightarrow$ $C = \ln 10{,}000 - 11\ln 100 \quad [= \ln 10^{-18} \approx -41.45]$.

Therefore, $t = -\ln P + 11\ln\left(\frac{1}{10}P - 900\right) + \ln 10{,}000 - 11\ln 100 \Rightarrow t = \ln\frac{10{,}000}{P} + 11\ln\frac{P-9000}{1000}$.

72. If we add and subtract $2x^2$ (because $2x^2$ completes the square for x^4+1), we get

$$\begin{aligned}x^4 + 1 = x^4 + 2x^2 + 1 - 2x^2 &= (x^2+1)^2 - 2x^2 = (x^2+1)^2 - (\sqrt{2}\,x)^2 \\ &= \left[(x^2+1) - \sqrt{2}\,x\right]\left[(x^2+1) + \sqrt{2}\,x\right] = \left(x^2 - \sqrt{2}\,x + 1\right)\left(x^2 + \sqrt{2}\,x + 1\right)\end{aligned}$$

So we can decompose $\frac{1}{x^4+1} = \frac{Ax+B}{x^2+\sqrt{2}\,x+1} + \frac{Cx+D}{x^2-\sqrt{2}\,x+1} \Rightarrow$ $1 = (Ax+B)\left(x^2 - \sqrt{2}\,x + 1\right) + (Cx+D)\left(x^2 + \sqrt{2}\,x + 1\right)$. Setting the constant terms equal gives $B + D = 1$, then from the coefficients of x^3 we get $A + C = 0$. Now from the coefficients of x we get $A + C + (B - D)\sqrt{2} = 0 \Leftrightarrow$ $[(1 - D) - D]\sqrt{2} = 0 \Rightarrow D = \frac{1}{2} \Rightarrow B = \frac{1}{2}$, and finally, from the coefficients of x^2 we get $\sqrt{2}\,(C - A) + B + D = 0 \Rightarrow C - A = -\frac{1}{\sqrt{2}} \Rightarrow C = -\frac{\sqrt{2}}{4}$ and $A = \frac{\sqrt{2}}{4}$. So we rewrite the integrand, splitting the terms into forms which we know how to integrate:

$$\begin{aligned}\frac{1}{x^4+1} &= \frac{\frac{\sqrt{2}}{4}x + \frac{1}{2}}{x^2+\sqrt{2}\,x+1} + \frac{-\frac{\sqrt{2}}{4}x + \frac{1}{2}}{x^2-\sqrt{2}\,x+1} = \frac{1}{4\sqrt{2}}\left[\frac{2x+2\sqrt{2}}{x^2+\sqrt{2}\,x+1} - \frac{2x-2\sqrt{2}}{x^2-\sqrt{2}\,x+1}\right] \\ &= \frac{\sqrt{2}}{8}\left[\frac{2x+\sqrt{2}}{x^2+\sqrt{2}\,x+1} - \frac{2x-\sqrt{2}}{x^2-\sqrt{2}\,x+1}\right] + \frac{1}{4}\left[\frac{1}{\left(x+\frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}} + \frac{1}{\left(x-\frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}}\right]\end{aligned}$$

Now we integrate:

$$\int \frac{dx}{x^4+1} = \frac{\sqrt{2}}{8}\ln\left(\frac{x^2+\sqrt{2}\,x+1}{x^2-\sqrt{2}\,x+1}\right) + \frac{\sqrt{2}}{4}\left[\tan^{-1}\left(\sqrt{2}\,x+1\right) + \tan^{-1}\left(\sqrt{2}\,x-1\right)\right] + C$$

73. (a) In Maple, we define $f(x)$, and then use `convert(f,parfrac,x);` to obtain

$$f(x) = \frac{24{,}110/4879}{5x+2} - \frac{668/323}{2x+1} - \frac{9438/80{,}155}{3x-7} + \frac{(22{,}098x + 48{,}935)/260{,}015}{x^2+x+5}$$

In Mathematica, we use the command `Apart`, and in Derive, we use `Expand`.

(b)
$$\begin{aligned}\int f(x)\,dx &= \tfrac{24{,}110}{4879} \cdot \tfrac{1}{5}\ln|5x+2| - \tfrac{668}{323}\cdot\tfrac{1}{2}\ln|2x+1| - \tfrac{9438}{80{,}155}\cdot\tfrac{1}{3}\ln|3x-7| \\ &\quad + \frac{1}{260{,}015}\int \frac{22{,}098\left(x+\frac{1}{2}\right)+37{,}886}{\left(x+\frac{1}{2}\right)^2+\frac{19}{4}}\,dx + C \\ &= \tfrac{24{,}110}{4879} \cdot \tfrac{1}{5}\ln|5x+2| - \tfrac{668}{323}\cdot\tfrac{1}{2}\ln|2x+1| - \tfrac{9438}{80{,}155}\cdot\tfrac{1}{3}\ln|3x-7| \\ &\quad + \tfrac{1}{260{,}015}\left[22{,}098\cdot\tfrac{1}{2}\ln\left(x^2+x+5\right) + 37{,}886\cdot\sqrt{\tfrac{4}{19}}\tan^{-1}\left(\tfrac{1}{\sqrt{19/4}}\left(x+\tfrac{1}{2}\right)\right)\right] + C \\ &= \tfrac{4822}{4879}\ln|5x+2| - \tfrac{334}{323}\ln|2x+1| - \tfrac{3146}{80{,}155}\ln|3x-7| + \tfrac{11{,}049}{260{,}015}\ln\left(x^2+x+5\right) \\ &\quad + \tfrac{75{,}772}{260{,}015\sqrt{19}}\tan^{-1}\left[\tfrac{1}{\sqrt{19}}(2x+1)\right] + C\end{aligned}$$

Using a CAS, we get

$$\frac{4822\ln(5x+2)}{4879} - \frac{334\ln(2x+1)}{323} - \frac{3146\ln(3x-7)}{80{,}155} + \frac{11{,}049\ln(x^2+x+5)}{260{,}015} + \frac{3988\sqrt{19}}{260{,}015}\tan^{-1}\left[\frac{\sqrt{19}}{19}(2x+1)\right]$$

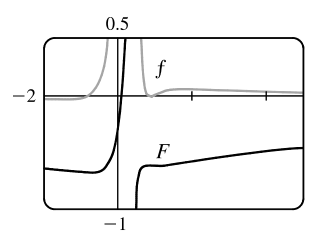
The main difference in this answer is that the absolute value signs and the constant of integration have been omitted. Also, the fractions have been reduced and the denominators rationalized.

74. (a) In Maple, we define $f(x)$, and then use `convert(f,parfrac,x);` to get

$$f(x) = \frac{5828/1815}{(5x-2)^2} - \frac{59{,}096/19{,}965}{5x-2} + \frac{2(2843x+816)/3993}{2x^2+1} + \frac{(313x-251)/363}{(2x^2+1)^2}.$$

In Mathematica, we use the command `Apart`, and in Derive, we use `Expand`.

(b) As we saw in Exercise 73, computer algebra systems omit the absolute value signs in $\int (1/y)\,dy = \ln|y|$. So we use the CAS to integrate the expression in part (a) and add the necessary absolute value signs and constant of integration to get



$$\begin{aligned}\int f(x)\,dx &= -\frac{5828}{9075(5x-2)} - \frac{59{,}096\ln|5x-2|}{99{,}825} + \frac{2843\ln\left(2x^2+1\right)}{7986} \\ &\quad + \frac{503}{15{,}972}\sqrt{2}\tan^{-1}\left(\sqrt{2}\,x\right) - \frac{1}{2904}\,\frac{1004x+626}{2x^2+1} + C\end{aligned}$$

(c) From the graph, we see that f goes from negative to positive at $x \approx -0.78$, then back to negative at $x \approx 0.8$, and finally back to positive at $x = 1$. Also, $\lim_{x\to 0.4} f(x) = \infty$. So we see (by the First Derivative Test) that $\int f(x)\,dx$ has minima at $x \approx -0.78$ and $x = 1$, and a maximum at $x \approx 0.80$, and that $\int f(x)\,dx$ is unbounded as $x \to 0.4$. Note also that just to the right of $x = 0.4$, f has large values, so $\int f(x)\,dx$ increases rapidly, but slows down as f drops toward 0. $\int f(x)\,dx$ decreases from about 0.8 to 1, then increases slowly since f stays small and positive.

75. There are only finitely many values of x where $Q(x) = 0$ (assuming that Q is not the zero polynomial). At all other values of x, $F(x)/Q(x) = G(x)/Q(x)$, so $F(x) = G(x)$. In other words, the values of F and G agree at all except perhaps finitely many values of x. By continuity of F and G, the polynomials F and G must agree at those values of x too.

More explicitly: if a is a value of x such that $Q(a) = 0$, then $Q(x) \neq 0$ for all x sufficiently close to a. Thus,

$$\begin{aligned} F(a) &= \lim_{x\to a} F(x) && \text{[by continuity of } F\text{]} \\ &= \lim_{x\to a} G(x) && \text{[whenever } Q(x) \neq 0\text{]} \\ &= G(a) && \text{[by continuity of } G\text{]} \end{aligned}$$

76. (a) Let $u = (x^2 + a^2)^{-n}$, $dv = dx \quad \Rightarrow \quad du = -n(x^2 + a^2)^{-n-1}\,2x\,dx$, $v = x$.

$$\begin{aligned} I_n = \int \frac{dx}{(x^2+a^2)^n} &= \frac{x}{(x^2+a^2)^n} - \int \frac{-2nx^2}{(x^2+a^2)^{n+1}}\,dx \qquad \text{[by parts]} \\ &= \frac{x}{(x^2+a^2)^n} + 2n \int \frac{(x^2+a^2) - a^2}{(x^2+a^2)^{n+1}}\,dx \\ &= \frac{x}{(x^2+a^2)^n} + 2n \int \frac{dx}{(x^2+a^2)^n} - 2na^2 \int \frac{dx}{(x^2+a^2)^{n+1}} \end{aligned}$$

Recognizing the last two integrals as I_n and I_{n+1}, we can solve for I_{n+1} in terms of I_n.

$$2na^2 I_{n+1} = \frac{x}{(x^2+a^2)^n} + 2nI_n - I_n \quad \Rightarrow \quad I_{n+1} = \frac{x}{2a^2n(x^2+a^2)^n} + \frac{2n-1}{2a^2n} I_n \quad \Rightarrow$$

$I_n = \dfrac{x}{2a^2(n-1)(x^2+a^2)^{n-1}} + \dfrac{2n-3}{2a^2(n-1)} I_{n-1}$ [decrease n-values by 1], which is the desired result.

(b) Using part (a) with $a = 1$ and $n = 2$, we get

$$\int \frac{dx}{(x^2+1)^2} = \frac{x}{2(x^2+1)} + \frac{1}{2}\int \frac{dx}{x^2+1} = \frac{x}{2(x^2+1)} + \frac{1}{2}\tan^{-1} x + C$$

Using part (a) with $a = 1$ and $n = 3$, we get

$$\begin{aligned} \int \frac{dx}{(x^2+1)^3} &= \frac{x}{2(2)(x^2+1)^2} + \frac{3}{2(2)} \int \frac{dx}{(x^2+1)^2} = \frac{x}{4(x^2+1)^2} + \frac{3}{4}\left[\frac{x}{2(x^2+1)} + \frac{1}{2}\tan^{-1} x\right] + C \\ &= \frac{x}{4(x^2+1)^2} + \frac{3x}{8(x^2+1)} + \frac{3}{8}\tan^{-1} x + C \end{aligned}$$

77. If $a \neq 0$ and n is a positive integer, then $f(x) = \dfrac{1}{x^n(x-a)} = \dfrac{A_1}{x} + \dfrac{A_2}{x^2} + \cdots + \dfrac{A_n}{x^n} + \dfrac{B}{x-a}$. Multiply both sides by $x^n(x-a)$ to get $1 = A_1x^{n-1}(x-a) + A_2x^{n-2}(x-a) + \cdots + A_n(x-a) + Bx^n$. Let $x = a$ in the last equation to get $1 = Ba^n \quad\Rightarrow\quad B = 1/a^n$. So

$$\begin{aligned} f(x) - \frac{B}{x-a} &= \frac{1}{x^n(x-a)} - \frac{1}{a^n(x-a)} = \frac{a^n - x^n}{x^n a^n (x-a)} = -\frac{x^n - a^n}{a^n x^n (x-a)} \\ &= -\frac{(x-a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + xa^{n-2} + a^{n-1})}{a^n x^n (x-a)} \\ &= -\left(\frac{x^{n-1}}{a^n x^n} + \frac{x^{n-2}a}{a^n x^n} + \frac{x^{n-3}a^2}{a^n x^n} + \cdots + \frac{xa^{n-2}}{a^n x^n} + \frac{a^{n-1}}{a^n x^n}\right) \\ &= -\frac{1}{a^n x} - \frac{1}{a^{n-1}x^2} - \frac{1}{a^{n-2}x^3} - \cdots - \frac{1}{a^2 x^{n-1}} - \frac{1}{ax^n} \end{aligned}$$

Thus, $f(x) = \dfrac{1}{x^n(x-a)} = -\dfrac{1}{a^n x} - \dfrac{1}{a^{n-1}x^2} - \cdots - \dfrac{1}{ax^n} + \dfrac{1}{a^n(x-a)}$.

78. Let $f(x) = ax^2 + bx + c$. We calculate the partial fraction decomposition of $\dfrac{f(x)}{x^2(x+1)^3}$. Since $f(0) = 1$, we must have $c = 1$, so $\dfrac{f(x)}{x^2(x+1)^3} = \dfrac{ax^2 + bx + 1}{x^2(x+1)^3} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+1} + \dfrac{D}{(x+1)^2} + \dfrac{E}{(x+1)^3}$. Now in order for the integral not to contain any logarithms (that is, in order for it to be a rational function), we must have $A = C = 0$, so $ax^2 + bx + 1 = B(x+1)^3 + Dx^2(x+1) + Ex^2$. Equating constant terms gives $B = 1$, then equating coefficients of x gives $3B = b \quad\Rightarrow\quad b = 3$. This is the quantity we are looking for, since $f'(0) = b$.

7.5 Strategy for Integration

1. (a) Let $u = 1 + x^2$, so that $du = 2x\,dx \quad\Rightarrow\quad \frac{1}{2}\,du = x\,dx$. Then,

$$\int \frac{x}{1+x^2}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(1+x^2) + C$$

Note the absolute value has been omitted in the last step since $1 + x^2 > 0$ for all $x \in \mathbb{R}$.

(b) $\displaystyle\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + C$

(c) $\displaystyle\int \frac{1}{1-x^2}\,dx = \int \frac{1}{(1+x)(1-x)}\,dx = \frac{1}{2}\int \left(\frac{1}{1+x} + \frac{1}{1-x}\right)dx$ [by partial fractions]

$$= \tfrac{1}{2}\ln|1+x| - \tfrac{1}{2}\ln|1-x| + C$$

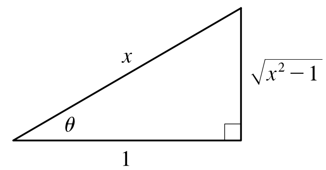
2. (a) Let $u = x^2 - 1$, so that $du = 2x\,dx \quad\Rightarrow\quad \frac{1}{2}\,du = x\,dx$. Then

$$\int x\sqrt{x^2-1}\,dx = \frac{1}{2}\int u^{1/2}\,du = \frac{1}{3}u^{3/2} + C = \frac{1}{3}(x^2-1)^{3/2} + C.$$

(b) Let $x = \sec\theta$ where $0 \le \theta < \pi/2$. Then $dx = \sec\theta\tan\theta\,d\theta$ and $\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta} = |\tan\theta| = \tan\theta$.

Thus, $\displaystyle\int \frac{1}{x\sqrt{x^2-1}}\,dx = \int \frac{\sec\theta\tan\theta}{\sec\theta\tan\theta}\,d\theta = \int d\theta = \theta + C = \sec^{-1}x + C.$

(c) Let $x = \sec\theta$ where $0 \le \theta < \pi/2$. Then $dx = \sec\theta\tan\theta\,d\theta$ and



$\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta} = |\tan\theta| = \tan\theta$. Thus,

$$\int \frac{\sqrt{x^2-1}}{x}\,dx = \int \frac{\tan\theta}{\sec\theta}\sec\theta\tan\theta\,d\theta = \int \tan^2\theta\,d\theta = \int (\sec^2\theta - 1)\,d\theta$$
$$= \tan\theta - \theta + C = \sqrt{x^2-1} - \sec^{-1}x + C$$

3. (a) Let $u = \ln x$, so that $du = \dfrac{1}{x}\,dx$. Then $\displaystyle\int \frac{\ln x}{x}dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln x)^2 + C$.

(b) Use integration by parts with $u = \ln(2x)$, $dv = dx \quad\Rightarrow\quad du = \dfrac{1}{2x}(2)\,dx = \dfrac{1}{x}\,dx$, $v = x$. Then

$$\int \ln(2x)\,dx = x\ln(2x) - \int x\left(\frac{1}{x}\,dx\right) = x\ln(2x) - \int dx = x\ln(2x) - x + C.$$

(c) Use integration by parts with $u = \ln x$, $dv = x\,dx \quad\Rightarrow\quad du = \dfrac{1}{x}\,dx$, $v = \dfrac{1}{2}x^2$. Then

$$\int x\ln x\,dx = \frac{1}{2}x^2\ln x - \int \frac{1}{2}x^2\left(\frac{1}{x}\,dx\right) = \frac{1}{2}x^2\ln x - \frac{1}{2}\int x\,dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C.$$

4. (a) $\displaystyle\int \sin^2 x\,dx = \int \tfrac{1}{2}(1 - \cos 2x)\,dx = \tfrac{1}{2}\left(x - \tfrac{1}{2}\sin 2x\right) + C = \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x + C$

(b) $\displaystyle\int \sin^3 x\,dx = \int \sin^2 x\sin x\,dx = \int \left(1 - \cos^2 x\right)\sin x\,dx \overset{c}{=} \int (1 - u^2)\,(-du)$
$$= -\left(u - \tfrac{1}{3}u^3\right) + C = -\cos x + \tfrac{1}{3}\cos^3 x + C$$

(c) Let $u = 2x$ so that $du = 2\,dx$. Then $\int \sin 2x\,dx = \int \sin u\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2}(-\cos u) + C = -\tfrac{1}{2}\cos 2x + C$.

5. (a) $\dfrac{1}{x^2 - 4x + 3} = \dfrac{1}{(x-3)(x-1)} = \dfrac{A}{x-3} + \dfrac{B}{x-1}$. Multiply both sides by $(x-3)(x-1)$ to get

$1 = A(x-1) + B(x-3)$. Setting $x = 3$ gives $1 = 2A$, so $A = \frac{1}{2}$. Now setting $x = 1$ gives $1 = -2B$, so $B = -\frac{1}{2}$.

Thus, $\displaystyle\int \frac{1}{x^2 - 4x + 3}\,dx = \frac{1}{2}\int \frac{1}{x-3} - \frac{1}{x-1}\,dx = \tfrac{1}{2}\ln|x-3| - \tfrac{1}{2}\ln|x-1| + C$.

(b) $\dfrac{1}{x^2 - 4x + 4} = \dfrac{1}{(x-2)^2}$. Let $u = x - 2$, so that $du = dx$. Thus,

$$\int \frac{1}{x^2 - 4x + 4}\,dx = \int \frac{1}{(x-2)^2}\,dx = \int \frac{1}{u^2}\,du = \int u^{-2}\,du = -u^{-1} + C = -\frac{1}{x-2} + C.$$

(c) $x^2 - 4x + 5$ is an irreducible quadratic, so it cannot be factored. Completing the square gives

$x^2 - 4x + 5 = (x^2 - 4x + 4) - 4 + 5 = (x-2)^2 + 1$. Now, use the substitution $u = x - 2$, so that $du = dx$. Thus,

$$\int \frac{1}{x^2 - 4x + 5}\,dx = \int \frac{1}{(x-2)^2 + 1}\,dx = \int \frac{1}{u^2 + 1}\,du = \tan^{-1}u + C = \tan^{-1}(x-2) + C.$$

6. (a) Let $u = x^2$, so that $du = 2x\,dx \quad \Rightarrow \quad \frac{1}{2}\,du = x\,dx$. Thus,

$$\int x\cos x^2\,dx = \frac{1}{2}\int \cos u\,du = \frac{1}{2}\sin u + C = \frac{1}{2}\sin x^2 + C.$$

(b) $\int x\cos^2 x\,dx = \frac{1}{2}\int x(1+\cos 2x)\,dx = \frac{1}{2}\int x\,dx + \frac{1}{2}\int x\cos 2x\,dx = \frac{1}{4}x^2 + \frac{1}{2}\int x\cos 2x\,dx.$

The remaining integral can be evaluated using integration by parts with $u = x$, $dv = \cos 2x\,dx \quad \Rightarrow$

$du = dx$, $v = \frac{1}{2}\sin 2x$. Thus,

$$\begin{aligned}\int x\cos^2 x\,dx &= \tfrac{1}{4}x^2 + \tfrac{1}{2}\left(\tfrac{1}{2}x\sin 2x - \int \tfrac{1}{2}\sin 2x\,dx\right) = \tfrac{1}{4}x^2 + \tfrac{1}{2}\left(\tfrac{1}{2}x\sin 2x + \tfrac{1}{4}\cos 2x\right) + C \\ &= \tfrac{1}{4}x^2 + \tfrac{1}{4}x\sin 2x + \tfrac{1}{8}\cos 2x + C\end{aligned}$$

(c) First, use integration by parts with $u = x^2$, $dv = \cos x\,dx \quad \Rightarrow \quad du = 2x\,dx$, $v = \sin x$. This gives $I = \int x^2\cos x\,dx = x^2\sin x - \int 2x\sin x\,dx$. Next, use integration by parts for the remaining integral with $U = 2x$, $dV = \sin x\,dx \quad \Rightarrow \quad dU = 2\,dx$, $V = -\cos x$. Thus, $I = x^2\sin x - (-2x\cos x + \int 2\cos x\,dx) = x^2\sin x + 2x\cos x - 2\sin x + C$.

7. (a) Let $u = x^3$, so that $du = 3x^2\,dx \quad \Rightarrow \quad \frac{1}{3}\,du = x^2\,dx$. Thus, $\displaystyle\int x^2 e^{x^3}\,dx = \frac{1}{3}\int e^u\,du = \frac{1}{3}e^u + C = \frac{1}{3}e^{x^3} + C.$

(b) First, use integration by parts with $u = x^2$, $dv = e^x\,dx \quad \Rightarrow \quad du = 2x\,dx$, $v = e^x$. This gives $I = \int x^2 e^x\,dx = x^2 e^x - \int 2xe^x\,dx$. Next, use integration by parts for the remaining integral with $U = 2x$, $dV = e^x\,dx \quad \Rightarrow \quad dU = 2\,dx$, $V = e^x$. Thus, $I = x^2e^x - \left(2xe^x - \int 2e^x\,dx\right) = x^2e^x - 2xe^x + 2e^x + C$.

(c) Let $y = x^2$, so that $dy = 2x\,dx$. Thus, $\int x^3 e^{x^2}\,dx = \int x^2 e^{x^2}x\,dx = \frac{1}{2}\int ye^y\,dy$. Now use integration by parts with $u = y$, $dv = e^y \quad \Rightarrow \quad du = dy$, $v = e^y$. This gives

$\int x^3 e^{x^2}dx = \frac{1}{2}\left(ye^y - \int e^y dy\right) = \frac{1}{2}ye^y - \frac{1}{2}e^y + C = \frac{1}{2}x^2e^{x^2} - \frac{1}{2}e^{x^2} + C.$

8. (a) Let $u = e^x - 1$, so that $du = e^x\,dx$. Thus, $\int e^x\sqrt{e^x - 1}\,dx = \int u^{1/2}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(e^x - 1)^{3/2} + C$.

(b) Let $u = e^x$, so that $du = e^x\,dx$. Thus, $\displaystyle\int \frac{e^x}{\sqrt{1 - e^{2x}}}\,dx = \int \frac{1}{\sqrt{1 - u^2}}\,du = \sin^{-1}u + C = \sin^{-1}(e^x) + C.$

(c) Let $u = \sqrt{e^x - 1}$, so that $u^2 = e^x - 1 \quad \Rightarrow \quad 2u\,du = e^x\,dx$, and $\dfrac{2u\,du}{u^2 + 1} = dx$. Then

$$\int \frac{1}{\sqrt{e^x - 1}}\,dx = \int \frac{1}{u}\,\frac{2u\,du}{u^2 + 1} = 2\int \frac{1}{u^2 + 1}\,du = 2\tan^{-1}u + C = 2\tan^{-1}\sqrt{e^x - 1} + C.$$

9. Let $u = 1 - \sin x$. Then $du = -\cos x\,dx \quad \Rightarrow$

$$\int \frac{\cos x}{1 - \sin x}\,dx = \int \frac{1}{u}\,(-du) = -\ln|u| + C = -\ln|1 - \sin x| + C = -\ln(1 - \sin x) + C.$$

10. Let $u = 3x + 1$. Then $du = 3\,dx \quad \Rightarrow$

$$\int_0^1 (3x+1)^{\sqrt{2}}\,dx = \int_1^4 u^{\sqrt{2}}\left(\frac{1}{3}\,du\right) = \frac{1}{3}\left[\frac{1}{\sqrt{2}+1}\,u^{\sqrt{2}+1}\right]_1^4 = \frac{1}{3(\sqrt{2}+1)}\left(4^{\sqrt{2}+1} - 1\right).$$

11. Let $u = \ln y$, $dv = \sqrt{y}\,dy \quad \Rightarrow \quad du = \dfrac{1}{y}\,dy$, $v = \dfrac{2}{3}y^{3/2}$. Then

$$\int_1^4 \sqrt{y}\ln y\,dy = \left[\frac{2}{3}y^{3/2}\ln y\right]_1^4 - \int_1^4 \frac{2}{3}y^{1/2}\,dy = \frac{2}{3}\cdot 8\ln 4 - 0 - \left[\frac{4}{9}y^{3/2}\right]_1^4$$
$$= \tfrac{16}{3}(2\ln 2) - \left(\tfrac{4}{9}\cdot 8 - \tfrac{4}{9}\right) = \tfrac{32}{3}\ln 2 - \tfrac{28}{9}$$

12. Let $u = \arcsin x$, so that $du = \dfrac{1}{\sqrt{1-x^2}}\,dx$. Thus, $\displaystyle\int \frac{e^{\arcsin x}}{\sqrt{1-x^2}}\,dx = \int e^u\,du = e^u + C = e^{\arcsin x} + C.$

13. Let $x = \ln y$, so that $dx = \dfrac{1}{y}\,dy$. Thus, $I = \displaystyle\int \frac{\ln(\ln y)}{y}dy = \int \ln x\,dx$. Now use integration by parts with

$u = \ln x$, $dv = dx \quad \Rightarrow \quad du = dx/x$, $v = x$. This gives

$I = x\ln x - \displaystyle\int \frac{x}{x}\,dx = x\ln x - \int dx = x\ln x - x + C = \ln y\,[\ln(\ln y)] - \ln y + C.$

14. Let $u = 2x + 1$. Then $du = 2\,dx \quad \Rightarrow$

$$\int_0^1 \frac{x}{(2x+1)^3}\,dx = \int_1^3 \frac{(u-1)/2}{u^3}\left(\frac{1}{2}\,du\right) = \frac{1}{4}\int_1^3\left(\frac{1}{u^2} - \frac{1}{u^3}\right)du = \frac{1}{4}\left[-\frac{1}{u} + \frac{1}{2u^2}\right]_1^3$$
$$= \tfrac{1}{4}\left[\left(-\tfrac{1}{3} + \tfrac{1}{18}\right) - \left(-1 + \tfrac{1}{2}\right)\right] = \tfrac{1}{4}\left(\tfrac{2}{9}\right) = \tfrac{1}{18}$$

15. Let $u = x^2$, so that $du = 2x\,dx$. Thus,

$$\int \frac{x}{x^4+9}\,dx = \frac{1}{2}\int \frac{1}{u^2+9}\,du = \frac{1}{2}\int \frac{(1/3)^2}{(u/3)^2+1}\,du = \frac{1}{2}\left(\frac{1}{3}\tan^{-1}\left(\frac{u}{3}\right)\right) + C = \frac{1}{6}\tan^{-1}\left(\frac{x^2}{3}\right) + C.$$

16. $\int t\sin t\,\cos t\,dt = \int t\cdot\tfrac{1}{2}(2\sin t\,\cos t)\,dt = \tfrac{1}{2}\int t\sin 2t\,dt$

$$= \tfrac{1}{2}\left(-\tfrac{1}{2}t\cos 2t - \int -\tfrac{1}{2}\cos 2t\,dt\right) \qquad \left[\begin{array}{ll} u = t, & dv = \sin 2t\,dt \\ du = dt, & v = -\frac{1}{2}\cos 2t \end{array}\right]$$
$$= -\tfrac{1}{4}t\cos 2t + \tfrac{1}{4}\int \cos 2t\,dt = -\tfrac{1}{4}t\cos 2t + \tfrac{1}{8}\sin 2t + C$$

17. $\dfrac{x+2}{x^2+3x-4} = \dfrac{x+2}{(x+4)(x-1)} = \dfrac{A}{x+4} + \dfrac{B}{x-1}$. Multiply by $(x+4)(x-1)$ to get $x + 2 = A(x-1) + B(x+4)$.

Substituting 1 for x gives $3 = 5B \quad \Leftrightarrow \quad B = \frac{3}{5}$. Substituting -4 for x gives $-2 = -5A \quad \Leftrightarrow \quad A = \frac{2}{5}$. Thus,

$$\int_2^4 \frac{x+2}{x^2+3x-4}\,dx = \int_2^4\left(\frac{2/5}{x+4} + \frac{3/5}{x-1}\right)dx = \left[\frac{2}{5}\ln|x+4| + \frac{3}{5}\ln|x-1|\right]_2^4$$
$$= \left(\tfrac{2}{5}\ln 8 + \tfrac{3}{5}\ln 3\right) - \left(\tfrac{2}{5}\ln 6 + 0\right) = \tfrac{2}{5}(3\ln 2) + \tfrac{3}{5}\ln 3 - \tfrac{2}{5}(\ln 2 + \ln 3)$$
$$= \tfrac{4}{5}\ln 2 + \tfrac{1}{5}\ln 3, \text{ or } \tfrac{1}{5}\ln 48$$

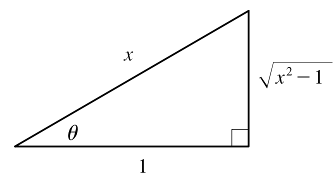
18. Let $u = \dfrac{1}{x}$, $dv = \dfrac{\cos(1/x)}{x^2} \quad \Rightarrow \quad du = -\dfrac{1}{x^2}\,dx$, $v = -\sin\left(\dfrac{1}{x}\right)$. Then

$$\int \frac{\cos(1/x)}{x^3}\,dx = -\frac{1}{x}\sin\left(\frac{1}{x}\right) - \int \frac{1}{x^2}\sin\left(\frac{1}{x}\right)dx = -\frac{1}{x}\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) + C.$$

19. Let $x = \sec\theta$, where $0 \le \theta \le \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then $dx = \sec\theta\,\tan\theta\,d\theta$ and

$\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta} = |\tan\theta| = \tan\theta$ for the relevant values of θ, so

$$\begin{aligned}\int \frac{1}{x^3\sqrt{x^2-1}}\,dx &= \int \frac{\sec\theta\,\tan\theta}{\sec^3\theta\,\tan\theta}\,d\theta = \int \cos^2\theta\,d\theta = \int \tfrac{1}{2}(1+\cos 2\theta)\,d\theta \\ &= \tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta + C = \tfrac{1}{2}\theta + \tfrac{1}{2}\sin\theta\,\cos\theta + C \\ &= \frac{1}{2}\sec^{-1}x + \frac{1}{2}\frac{\sqrt{x^2-1}}{x}\frac{1}{x} + C = \frac{1}{2}\sec^{-1}x + \frac{\sqrt{x^2-1}}{2x^2} + C\end{aligned}$$



20. $\dfrac{2x-3}{x^3+3x} = \dfrac{2x-3}{x(x^2+3)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+3}$. Multiply by $x(x^2+3)$ to get $2x-3 = A(x^2+3)+(Bx+C)x \quad\Leftrightarrow$

$2x-3 = (A+B)x^2 + Cx + 3A$. Equating coefficients gives us $C=2$, $3A=-3 \quad\Leftrightarrow\quad A=-1$, and $A+B=0$, so $B=1$. Thus,

$$\begin{aligned}\int \frac{2x-3}{x^3+3x}\,dx &= \int\left(\frac{-1}{x} + \frac{x+2}{x^2+3}\right)dx = \int\left(-\frac{1}{x} + \frac{x}{x^2+3} + \frac{2}{x^2+3}\right)dx \\ &= -\ln|x| + \frac{1}{2}\ln(x^2+3) + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C\end{aligned}$$

21. $\displaystyle\int \frac{\cos^3 x}{\csc x}\,dx = \int \cos^3 x\sin x\,dx \overset{c}{=} \int u^3\,(-du) = -\frac{1}{4}u^4 + C = -\frac{1}{4}\cos^4 x + C$

22. Let $u = \ln(1+x^2)$, $dv = dx \quad\Rightarrow\quad du = \dfrac{2x}{1+x^2}\,dx$, $v = x$. Then

$$\begin{aligned}\int \ln(1+x^2)\,dx &= x\ln(1+x^2) - \int \frac{2x^2}{1+x^2}\,dx = x\ln(1+x^2) - 2\int \frac{(x^2+1)-1}{1+x^2}\,dx \\ &= x\ln(1+x^2) - 2\int\left(1 - \frac{1}{1+x^2}\right)dx = x\ln(1+x^2) - 2x + 2\tan^{-1}x + C\end{aligned}$$

23. Let $u = x$, $dv = \sec x\,\tan x\,dx \quad\Rightarrow\quad du = dx$, $v = \sec x$. Then

$\int x\sec x\,\tan x\,dx = x\sec x - \int \sec x\,dx = x\sec x - \ln|\sec x + \tan x| + C$.

24. $\displaystyle\int_0^{\sqrt{2}/2} \frac{x^2}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/4} \frac{\sin^2\theta}{\cos\theta}\cos\theta\,d\theta \qquad \begin{bmatrix} u = \sin\theta, \\ du = \cos\theta\,d\theta \end{bmatrix}$

$$= \int_0^{\pi/4} \tfrac{1}{2}(1-\cos 2\theta)\,d\theta = \tfrac{1}{2}\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/4} = \tfrac{1}{2}\left[\left(\tfrac{\pi}{4} - \tfrac{1}{2}\right) - (0-0)\right] = \tfrac{\pi}{8} - \tfrac{1}{4}$$

25. $\int_0^\pi t\,\cos^2 t\,dt = \int_0^\pi t\left[\frac{1}{2}(1+\cos 2t)\right]dt = \frac{1}{2}\int_0^\pi t\,dt + \frac{1}{2}\int_0^\pi t\cos 2t\,dt$

$$= \tfrac{1}{2}\left[\tfrac{1}{2}t^2\right]_0^\pi + \tfrac{1}{2}\left[\tfrac{1}{2}t\sin 2t\right]_0^\pi - \tfrac{1}{2}\int_0^\pi \tfrac{1}{2}\sin 2t\,dt \qquad \begin{bmatrix} u = t, & dv = \cos 2t\,dt \\ du = dt, & v = \frac{1}{2}\sin 2t \end{bmatrix}$$

$$= \tfrac{1}{4}\pi^2 + 0 - \tfrac{1}{4}\left[-\tfrac{1}{2}\cos 2t\right]_0^\pi = \tfrac{1}{4}\pi^2 + \tfrac{1}{8}(1-1) = \tfrac{1}{4}\pi^2$$

26. Let $u = \sqrt{t}$. Then $du = \dfrac{1}{2\sqrt{t}}\,dt \quad\Rightarrow\quad \displaystyle\int_1^4 \frac{e^{\sqrt{t}}}{\sqrt{t}}\,dt = \int_1^2 e^u\,(2\,du) = 2\left[e^u\right]_1^2 = 2(e^2 - e)$.

27. Let $u = e^x$. Then $\int e^{x+e^x}\,dx = \int e^{e^x} e^x\,dx = \int e^u\,du = e^u + C = e^{e^x} + C$.

28. $\displaystyle\int \frac{e^x}{1+e^{2a}}\,dx = \frac{1}{1+e^{2a}} \int e^x\,dx = \frac{e^x}{1+e^{2a}} + C$

29. Let $t = \sqrt{x}$, so that $t^2 = x$ and $2t\,dt = dx$. Then $\int \arctan\sqrt{x}\,dx = \int \arctan t\,(2t\,dt) = I$. Now use parts with

$u = \arctan t$, $dv = 2t\,dt \quad\Rightarrow\quad du = \dfrac{1}{1+t^2}\,dt$, $v = t^2$. Thus,

$$I = t^2\arctan t - \int \frac{t^2}{1+t^2}\,dt = t^2\arctan t - \int\left(1 - \frac{1}{1+t^2}\right)dt = t^2\arctan t - t + \arctan t + C$$

$$= x\arctan\sqrt{x} - \sqrt{x} + \arctan\sqrt{x} + C \quad \left[\text{or } (x+1)\arctan\sqrt{x} - \sqrt{x} + C\right]$$

30. Let $u = 1 + (\ln x)^2$, so that $du = \dfrac{2\ln x}{x}\,dx$. Then

$$\int \frac{\ln x}{x\sqrt{1+(\ln x)^2}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = \frac{1}{2}\left(2\sqrt{u}\right) + C = \sqrt{1+(\ln x)^2} + C.$$

31. Let $u = 1 + \sqrt{x}$. Then $x = (u-1)^2$, $dx = 2(u-1)\,du \quad\Rightarrow$

$$\int_0^1 \left(1+\sqrt{x}\right)^8 dx = \int_1^2 u^8 \cdot 2(u-1)\,du = 2\int_1^2 (u^9 - u^8)\,du$$

$$= \left[\tfrac{1}{5}u^{10} - 2\cdot\tfrac{1}{9}u^9\right]_1^2 = \tfrac{1024}{5} - \tfrac{1024}{9} - \tfrac{1}{5} + \tfrac{2}{9} = \tfrac{4097}{45}$$

32. $\int (1+\tan x)^2 \sec x\,dx = \int (1 + 2\tan x + \tan^2 x)\sec x\,dx$

$$= \int [\sec x + 2\sec x\,\tan x + (\sec^2 x - 1)\sec x]\,dx = \int (2\sec x\,\tan x + \sec^3 x)\,dx$$

$$= 2\sec x + \tfrac{1}{2}(\sec x\,\tan x + \ln|\sec x + \tan x|) + C \quad \text{[by Example 7.2.8]}$$

33. $\displaystyle\int_0^1 \frac{1+12t}{1+3t}\,dt = \int_0^1 \frac{(12t+4)-3}{3t+1}\,dt = \int_0^1 \left(4 - \frac{3}{3t+1}\right)dt = \Big[4t - \ln|3t+1|\Big]_0^1$

$$= (4 - \ln 4) - (0 - 0) = 4 - \ln 4$$

34. $\dfrac{3x^2+1}{x^3+x^2+x+1} = \dfrac{3x^2+1}{(x^2+1)(x+1)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+1}$. Multiply by $(x+1)(x^2+1)$ to get

$3x^2 + 1 = A(x^2+1) + (Bx+C)(x+1) \quad\Leftrightarrow\quad 3x^2+1 = (A+B)x^2 + (B+C)x + (A+C)$. Substituting -1 for x

gives $4 = 2A \quad\Leftrightarrow\quad A = 2$. Equating coefficients of x^2 gives $3 = A + B = 2 + B \quad\Leftrightarrow\quad B = 1$. Equating coefficients of x

gives $0 = B + C = 1 + C \quad\Leftrightarrow\quad C = -1$. Thus,

$$\int_0^1 \frac{3x^2+1}{x^3+x^2+x+1}\,dx = \int_0^1 \left(\frac{2}{x+1} + \frac{x-1}{x^2+1}\right)dx = \int_0^1 \left(\frac{2}{x+1} + \frac{x}{x^2+1} - \frac{1}{x^2+1}\right)dx$$

$$= \Big[2\ln|x+1| + \tfrac{1}{2}\ln(x^2+1) - \tan^{-1}x\Big]_0^1 = \left(2\ln 2 + \tfrac{1}{2}\ln 2 - \tfrac{\pi}{4}\right) - (0 + 0 - 0)$$

$$= \tfrac{5}{2}\ln 2 - \tfrac{\pi}{4}$$

35. Let $u = 1 + e^x$, so that $du = e^x\,dx = (u-1)\,dx$. Then $\displaystyle\int \frac{1}{1+e^x}\,dx = \int \frac{1}{u}\cdot\frac{du}{u-1} = \int \frac{1}{u(u-1)}\,du = I$. Now

$\displaystyle\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1} \quad\Rightarrow\quad 1 = A(u-1) + Bu$. Set $u = 1$ to get $1 = B$. Set $u = 0$ to get $1 = -A$, so $A = -1$.

Thus, $\displaystyle I = \int\left(\frac{-1}{u} + \frac{1}{u-1}\right)du = -\ln|u| + \ln|u-1| + C = -\ln(1+e^x) + \ln e^x + C = x - \ln(1+e^x) + C$.

Another method: Multiply numerator and denominator by e^{-x} and let $u = e^{-x} + 1$. This gives the answer in the form $-\ln(e^{-x}+1) + C$.

36. $\int \sin\sqrt{at}\,dt = \int \sin u\cdot \frac{2}{a}u\,du \quad [u = \sqrt{at},\ u^2 = at,\ 2u\,du = a\,dt] \quad = \frac{2}{a}\int u\sin u\,du$

$$= \tfrac{2}{a}[-u\cos u + \sin u] + C \quad \text{[integration by parts]} \quad = -\tfrac{2}{a}\sqrt{at}\,\cos\sqrt{at} + \tfrac{2}{a}\,\sin\sqrt{at} + C$$

$$= -2\sqrt{\tfrac{t}{a}}\,\cos\sqrt{at} + \tfrac{2}{a}\,\sin\sqrt{at} + C$$

37. Use integration by parts with $u = \ln\!\left(x + \sqrt{x^2-1}\right)$, $dv = dx \quad\Rightarrow$

$$du = \frac{1}{x+\sqrt{x^2-1}}\left(1 + \frac{x}{\sqrt{x^2-1}}\right)dx = \frac{1}{x+\sqrt{x^2-1}}\left(\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}}\right)dx = \frac{1}{\sqrt{x^2-1}}\,dx,\ v = x. \text{ Then}$$

$$\int \ln\!\left(x+\sqrt{x^2-1}\right)dx = x\ln\!\left(x+\sqrt{x^2-1}\right) - \int\frac{x}{\sqrt{x^2-1}}\,dx = x\ln\!\left(x+\sqrt{x^2-1}\right) - \sqrt{x^2-1} + C.$$

38. $$|e^x - 1| = \begin{cases} e^x - 1 & \text{if } e^x - 1 \ge 0 \\ -(e^x - 1) & \text{if } e^x - 1 < 0 \end{cases} = \begin{cases} e^x - 1 & \text{if } x \ge 0 \\ 1 - e^x & \text{if } x < 0 \end{cases}$$

Thus, $\int_{-1}^{2} |e^x - 1|\;dx = \int_{-1}^{0}(1-e^x)\,dx + \int_0^2(e^x-1)\,dx = \Big[x - e^x\Big]_{-1}^{0} + \Big[e^x - x\Big]_0^2$

$$= (0-1) - (-1-e^{-1}) + (e^2-2) - (1-0) = e^2 + e^{-1} - 3$$

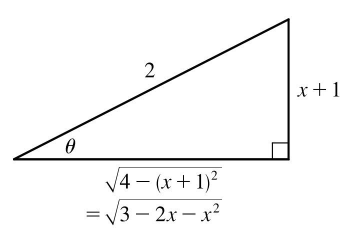
39. As in Example 5,

$$\int\sqrt{\frac{1+x}{1-x}}\,dx = \int\frac{\sqrt{1+x}}{\sqrt{1-x}}\cdot\frac{\sqrt{1+x}}{\sqrt{1+x}}\,dx = \int\frac{1+x}{\sqrt{1-x^2}}\,dx = \int\frac{dx}{\sqrt{1-x^2}} + \int\frac{x\,dx}{\sqrt{1-x^2}} = \sin^{-1}x - \sqrt{1-x^2} + C.$$

Another method: Substitute $u = \sqrt{(1+x)/(1-x)}$.

40. $$\int_1^3 \frac{e^{3/x}}{x^2}\,dx = \int_3^1 e^u\,\left(-\tfrac{1}{3}\,du\right) \quad \left[\begin{array}{l} u = 3/x, \\ du = -3/x^2\,dx \end{array}\right] \quad = -\tfrac{1}{3}\Big[e^u\Big]_3^1 = -\tfrac{1}{3}(e - e^3) = \tfrac{1}{3}(e^3 - e)$$

41. $3 - 2x - x^2 = -(x^2 + 2x + 1) + 4 = 4 - (x+1)^2$. Let $x + 1 = 2\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx = 2\cos\theta\,d\theta$ and



$$\int\sqrt{3-2x-x^2}\,dx = \int\sqrt{4-(x+1)^2}\,dx = \int\sqrt{4-4\sin^2\theta}\,2\cos\theta\,d\theta$$

$$= 4\int\cos^2\theta\,d\theta = 2\int(1+\cos 2\theta)\,d\theta$$

$$= 2\theta + \sin 2\theta + C = 2\theta + 2\sin\theta\cos\theta + C$$

$$= 2\sin^{-1}\!\left(\frac{x+1}{2}\right) + 2\cdot\frac{x+1}{2}\cdot\frac{\sqrt{3-2x-x^2}}{2} + C$$

$$= 2\sin^{-1}\!\left(\frac{x+1}{2}\right) + \frac{x+1}{2}\sqrt{3-2x-x^2} + C$$

42. $\displaystyle\int_{\pi/4}^{\pi/2} \frac{1+4\cot x}{4-\cot x}\,dx = \int_{\pi/4}^{\pi/2} \left[\frac{(1+4\cos x/\sin x)}{(4-\cos x/\sin x)} \cdot \frac{\sin x}{\sin x}\right] dx = \int_{\pi/4}^{\pi/2} \frac{\sin x + 4\cos x}{4\sin x - \cos x}\,dx$

$$= \int_{3/\sqrt{2}}^{4} \frac{1}{u}\,du \quad \left[\begin{array}{l} u = 4\sin x - \cos x, \\ du = (4\cos x + \sin x)\,dx \end{array}\right]$$

$$= \Big[\ln|u|\Big]_{3/\sqrt{2}}^{4} = \ln 4 - \ln\frac{3}{\sqrt{2}} = \ln\frac{4}{3/\sqrt{2}} = \ln\left(\frac{4}{3}\sqrt{2}\right)$$

43. The integrand is an odd function, so $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{x}{1+\cos^2 x}\,dx = 0$ [by 5.5.7].

44. $\displaystyle\int \frac{1+\sin x}{1+\cos x}\,dx = \int \frac{(1+\sin x)(1-\cos x)}{(1+\cos x)(1-\cos x)}\,dx = \int \frac{1-\cos x+\sin x-\sin x\,\cos x}{\sin^2 x}\,dx$

$$= \int \left(\csc^2 x - \frac{\cos x}{\sin^2 x} + \csc x - \frac{\cos x}{\sin x}\right) dx$$

$$\overset{s}{=} -\cot x + \frac{1}{\sin x} + \ln|\csc x - \cot x| - \ln|\sin x| + C \quad \text{[by Exercise 7.2.41]}$$

The answer can be written as $\dfrac{1-\cos x}{\sin x} - \ln(1+\cos x) + C$.

45. Let $u = \tan\theta$. Then $du = \sec^2\theta\,d\theta \;\Rightarrow\; \int_0^{\pi/4} \tan^3\theta\,\sec^2\theta\,d\theta = \int_0^1 u^3\,du = \left[\frac{1}{4}u^4\right]_0^1 = \frac{1}{4}$.

46. $\displaystyle\int_{\pi/6}^{\pi/3} \frac{\sin\theta\,\cot\theta}{\sec\theta}\,d\theta = \int_{\pi/6}^{\pi/3} \cos^2\theta\,d\theta = \frac{1}{2}\int_{\pi/6}^{\pi/3} (1+\cos 2\theta)\,d\theta = \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_{\pi/6}^{\pi/3}$

$$= \frac{1}{2}\left[\left(\frac{\pi}{3}+\frac{\sqrt{3}}{4}\right) - \left(\frac{\pi}{6}+\frac{\sqrt{3}}{4}\right)\right] = \frac{1}{2}\left(\frac{\pi}{6}\right) = \frac{\pi}{12}$$

47. Let $u = \sec\theta$, so that $du = \sec\theta\,\tan\theta\,d\theta$. Then $\displaystyle\int \frac{\sec\theta\,\tan\theta}{\sec^2\theta - \sec\theta}\,d\theta = \int \frac{1}{u^2-u}\,du = \int \frac{1}{u(u-1)}\,du = I$. Now

$\dfrac{1}{u(u-1)} = \dfrac{A}{u} + \dfrac{B}{u-1} \;\Rightarrow\; 1 = A(u-1) + Bu$. Set $u = 1$ to get $1 = B$. Set $u = 0$ to get $1 = -A$, so $A = -1$.

Thus, $I = \displaystyle\int \left(\frac{-1}{u} + \frac{1}{u-1}\right) du = -\ln|u| + \ln|u-1| + C = \ln|\sec\theta - 1| - \ln|\sec\theta| + C$ [or $\ln|1-\cos\theta| + C$].

48. Using product formula 2(a) in Section 7.2, $\sin 6x\,\cos 3x = \frac{1}{2}[\sin(6x-3x) + \sin(6x+3x)] = \frac{1}{2}(\sin 3x + \sin 9x)$. Thus,

$$\int_0^\pi \sin 6x\,\cos 3x\,dx = \int_0^\pi \tfrac{1}{2}(\sin 3x + \sin 9x)\,dx = \tfrac{1}{2}\left[-\tfrac{1}{3}\cos 3x - \tfrac{1}{9}\cos 9x\right]_0^\pi$$

$$= \tfrac{1}{2}\left[\left(\tfrac{1}{3}+\tfrac{1}{9}\right) - \left(-\tfrac{1}{3}-\tfrac{1}{9}\right)\right] = \tfrac{1}{2}\left(\tfrac{4}{9}+\tfrac{4}{9}\right) = \tfrac{4}{9}$$

49. Let $u = \theta$, $dv = \tan^2\theta\,d\theta = \left(\sec^2\theta - 1\right)d\theta \;\Rightarrow\; du = d\theta$ and $v = \tan\theta - \theta$. So

$$\int \theta\tan^2\theta\,d\theta = \theta(\tan\theta - \theta) - \int(\tan\theta - \theta)\,d\theta = \theta\tan\theta - \theta^2 - \ln|\sec\theta| + \tfrac{1}{2}\theta^2 + C$$

$$= \theta\tan\theta - \tfrac{1}{2}\theta^2 - \ln|\sec\theta| + C$$

50. Let $u = \sqrt{x-1}$, so that $x = u^2 + 1$ and $dx = 2u\,du$. Thus,

$$\int \frac{1}{x\sqrt{x-1}}\,dx = \int \frac{2u}{(u^2+1)u}\,du = 2\int \frac{1}{u^2+1}\,du = 2\tan^{-1}u + C = 2\tan^{-1}\left(\sqrt{x-1}\right) + C.$$

51. Let $u = \sqrt{x}$, so that $du = \dfrac{1}{2\sqrt{x}}\,dx$. Then

$$\int \frac{\sqrt{x}}{1+x^3}\,dx = \int \frac{u}{1+u^6}\,(2u\,du) = 2\int \frac{u^2}{1+(u^3)^2}\,du = 2\int \frac{1}{1+t^2}\left(\frac{1}{3}\,dt\right) \qquad \begin{bmatrix} t = u^3 \\ dt = 3u^2\,du \end{bmatrix}$$

$$= \tfrac{2}{3}\tan^{-1}t + C = \tfrac{2}{3}\tan^{-1}u^3 + C = \tfrac{2}{3}\tan^{-1}(x^{3/2}) + C$$

Another method: Let $u = x^{3/2}$ so that $u^2 = x^3$ and $du = \frac{3}{2}x^{1/2}\,dx \;\Rightarrow\; \sqrt{x}\,dx = \frac{2}{3}\,du$. Then

$$\int \frac{\sqrt{x}}{1+x^3}\,dx = \int \frac{\frac{2}{3}}{1+u^2}\,du = \frac{2}{3}\tan^{-1}u + C = \frac{2}{3}\tan^{-1}(x^{3/2}) + C.$$

52. Let $u = \sqrt{1+e^x}$. Then $u^2 = 1 + e^x$, $2u\,du = e^x\,dx = (u^2-1)\,dx$, and $dx = \dfrac{2u}{u^2-1}du$, so

$$\int \sqrt{1+e^x}\,dx = \int u \cdot \frac{2u}{u^2-1}\,du = \int \frac{2u^2}{u^2-1}\,du = \int\left(2 + \frac{2}{u^2-1}\right)du = \int\left(2 + \frac{1}{u-1} - \frac{1}{u+1}\right)du$$

$$= 2u + \ln|u-1| - \ln|u+1| + C = 2\sqrt{1+e^x} + \ln\left(\sqrt{1+e^x}-1\right) - \ln\left(\sqrt{1+e^x}+1\right) + C$$

53. Let $u = \sqrt{x}$, so that $x = u^2$ and $dx = 2u\,du$. Thus,

$$\int \frac{x}{1+\sqrt{x}}\,dx = \int \frac{u^2}{1+u}\,(2u\,du) = 2\int \frac{u^3}{1+u}\,du = 2\int \frac{(u^3+1)-1}{u+1}\,du \qquad \text{[or use long division]}$$

$$= 2\int \frac{(u+1)(u^2-u+1)-1}{u+1}\,du = 2\int\left(u^2 - u + 1 - \frac{1}{u+1}\right)du$$

$$= 2\left(\tfrac{1}{3}u^3 - \tfrac{1}{2}u^2 + u - \ln|u+1|\right) + C = \tfrac{2}{3}u^3 - u^2 + 2u - 2\ln|u+1| + C$$

$$= \tfrac{2}{3}x^{3/2} - x + 2\sqrt{x} - 2\ln\left(\sqrt{x}+1\right) + C$$

54. Use integration by parts with $u = (x-1)e^x$, $dv = \dfrac{1}{x^2}\,dx \;\Rightarrow\; du = [(x-1)e^x + e^x]\,dx = xe^x\,dx$, $v = -\dfrac{1}{x}$. Then

$$\int \frac{(x-1)e^x}{x^2}\,dx = (x-1)e^x\left(-\frac{1}{x}\right) - \int -e^x\,dx = -e^x + \frac{e^x}{x} + e^x + C = \frac{e^x}{x} + C.$$

55. Let $u = x - 1$, so that $du = dx$. Then

$$\int x^3(x-1)^{-4}\,dx = \int (u+1)^3u^{-4}\,du = \int (u^3+3u^2+3u+1)u^{-4}\,du = \int (u^{-1}+3u^{-2}+3u^{-3}+u^{-4})\,du$$

$$= \ln|u| - 3u^{-1} - \tfrac{3}{2}u^{-2} - \tfrac{1}{3}u^{-3} + C = \ln|x-1| - 3(x-1)^{-1} - \tfrac{3}{2}(x-1)^{-2} - \tfrac{1}{3}(x-1)^{-3} + C$$

56. Let $u = \sqrt{1-x^2}$, so $u^2 = 1 - x^2$, and $2u\,du = -2x\,dx$. Then $\int_0^1 x\sqrt{2-\sqrt{1-x^2}}\,dx = \int_1^0 \sqrt{2-u}\,(-u\,du)$.

Now let $v = \sqrt{2-u}$, so $v^2 = 2 - u$, and $2v\,dv = -du$. Thus,

$$\int_1^0 \sqrt{2-u}\,(-u\,du) = \int_1^{\sqrt{2}} v(2-v^2)\,(2v\,dv) = \int_1^{\sqrt{2}} (4v^2 - 2v^4)\,dv = \left[\tfrac{4}{3}v^3 - \tfrac{2}{5}v^5\right]_1^{\sqrt{2}}$$

$$= \left(\tfrac{8}{3}\sqrt{2} - \tfrac{8}{5}\sqrt{2}\right) - \left(\tfrac{4}{3} - \tfrac{2}{5}\right) = \tfrac{16}{15}\sqrt{2} - \tfrac{14}{15}$$

57. Let $u = \sqrt{4x+1} \;\Rightarrow\; u^2 = 4x+1 \;\Rightarrow\; 2u\,du = 4\,dx \;\Rightarrow\; dx = \frac{1}{2}u\,du$. So

$$\int \frac{1}{x\sqrt{4x+1}}\,dx = \int \frac{\frac{1}{2}u\,du}{\frac{1}{4}(u^2-1)\,u} = 2\int \frac{du}{u^2-1} = 2\left(\tfrac{1}{2}\right)\ln\left|\frac{u-1}{u+1}\right| + C \qquad \text{[by Formula 19]}$$

$$= \ln\left|\frac{\sqrt{4x+1}-1}{\sqrt{4x+1}+1}\right| + C$$

58. As in Exercise 57, let $u = \sqrt{4x+1}$. Then $\displaystyle\int \frac{dx}{x^2\sqrt{4x+1}} = \int \frac{\frac{1}{2}u\,du}{\left[\frac{1}{4}(u^2-1)\right]^2 u} = 8\int \frac{du}{(u^2-1)^2}$. Now

$$\frac{1}{(u^2-1)^2} = \frac{1}{(u+1)^2(u-1)^2} = \frac{A}{u+1} + \frac{B}{(u+1)^2} + \frac{C}{u-1} + \frac{D}{(u-1)^2} \quad\Rightarrow$$

$1 = A(u+1)(u-1)^2 + B(u-1)^2 + C(u-1)(u+1)^2 + D(u+1)^2$. $u=1 \;\Rightarrow\; D = \frac{1}{4}$, $u=-1 \;\Rightarrow\; B = \frac{1}{4}$.

Equating coefficients of u^3 gives $A + C = 0$, and equating coefficients of 1 gives $1 = A + B - C + D \;\Rightarrow$

$1 = A + \frac{1}{4} - C + \frac{1}{4} \;\Rightarrow\; \frac{1}{2} = A - C$. So $A = \frac{1}{4}$ and $C = -\frac{1}{4}$. Therefore,

$$\int \frac{dx}{x^2\sqrt{4x+1}} = 8\int \left[\frac{1/4}{u+1} + \frac{1/4}{(u+1)^2} + \frac{-1/4}{u-1} + \frac{1/4}{(u-1)^2}\right] du$$

$$= \int \left[\frac{2}{u+1} + 2(u+1)^{-2} - \frac{2}{u-1} + 2(u-1)^{-2}\right] du$$

$$= 2\ln|u+1| - \frac{2}{u+1} - 2\ln|u-1| - \frac{2}{u-1} + C$$

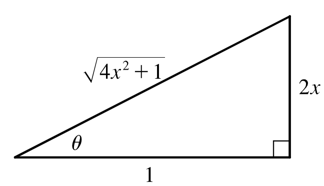
$$= 2\ln\left(\sqrt{4x+1}+1\right) - \frac{2}{\sqrt{4x+1}+1} - 2\ln\left|\sqrt{4x+1}-1\right| - \frac{2}{\sqrt{4x+1}-1} + C$$

59. Let $2x = \tan\theta \;\Rightarrow\; x = \frac{1}{2}\tan\theta$, $dx = \frac{1}{2}\sec^2\theta\,d\theta$, $\sqrt{4x^2+1} = \sec\theta$, so

$$\int \frac{dx}{x\sqrt{4x^2+1}} = \int \frac{\frac{1}{2}\sec^2\theta\,d\theta}{\frac{1}{2}\tan\theta\,\sec\theta} = \int \frac{\sec\theta}{\tan\theta}\,d\theta = \int \csc\theta\,d\theta$$

$$= -\ln|\csc\theta + \cot\theta| + C \qquad [\text{or } \ln|\csc\theta - \cot\theta| + C]$$

$$= -\ln\left|\frac{\sqrt{4x^2+1}}{2x} + \frac{1}{2x}\right| + C \qquad \left[\text{or } \ln\left|\frac{\sqrt{4x^2+1}}{2x} - \frac{1}{2x}\right| + C\right]$$



60. Let $u = x^2$. Then $du = 2x\,dx \;\Rightarrow$

$$\int \frac{dx}{x(x^4+1)} = \int \frac{x\,dx}{x^2(x^4+1)} = \frac{1}{2}\int \frac{du}{u(u^2+1)} = \frac{1}{2}\int \left[\frac{1}{u} - \frac{u}{u^2+1}\right] du = \tfrac{1}{2}\ln|u| - \tfrac{1}{4}\ln(u^2+1) + C$$

$$= \tfrac{1}{2}\ln(x^2) - \tfrac{1}{4}\ln(x^4+1) + C = \tfrac{1}{4}\left[\ln(x^4) - \ln(x^4+1)\right] + C = \frac{1}{4}\ln\left(\frac{x^4}{x^4+1}\right) + C$$

Or: Write $I = \displaystyle\int \frac{x^3\,dx}{x^4(x^4+1)}$ and let $u = x^4$.

61. $\displaystyle\int x^2\sinh mx\,dx = \frac{1}{m}x^2\cosh mx - \frac{2}{m}\int x\cosh mx\,dx \qquad \begin{bmatrix} u = x^2, & dv = \sinh mx\,dx, \\ du = 2x\,dx & v = \frac{1}{m}\cosh mx \end{bmatrix}$

$$= \frac{1}{m}x^2\cosh\, mx - \frac{2}{m}\left(\frac{1}{m}x\sinh mx - \frac{1}{m}\int \sinh mx\,dx\right) \qquad \begin{bmatrix} U = x, & dV = \cosh mx\,dx, \\ dU = dx & V = \frac{1}{m}\sinh mx \end{bmatrix}$$

$$= \frac{1}{m}x^2\cosh mx - \frac{2}{m^2}x\sinh mx + \frac{2}{m^3}\cosh mx + C$$

62. $\int (x+\sin x)^2\,dx = \int \left(x^2+2x\sin x+\sin^2 x\right)dx = \frac{1}{3}x^3 + 2(\sin x - x\cos x) + \frac{1}{2}(x - \sin x\cos x) + C$
$= \frac{1}{3}x^3 + \frac{1}{2}x + 2\sin x - \frac{1}{2}\sin x\cos x - 2x\cos x + C$

63. Let $u = \sqrt{x}$, so that $x = u^2$ and $dx = 2u\,du$. Then $\displaystyle\int \frac{dx}{x + x\sqrt{x}} = \int \frac{2u\,du}{u^2 + u^2\cdot u} = \int \frac{2}{u(1+u)}\,du = I$.

Now $\dfrac{2}{u(1+u)} = \dfrac{A}{u} + \dfrac{B}{1+u} \;\Rightarrow\; 2 = A(1+u) + Bu$. Set $u = -1$ to get $2 = -B$, so $B = -2$. Set $u = 0$ to get $2 = A$.

Thus, $I = \displaystyle\int \left(\frac{2}{u} - \frac{2}{1+u}\right)du = 2\ln|u| - 2\ln|1+u| + C = 2\ln\sqrt{x} - 2\ln\left(1+\sqrt{x}\right) + C$.

64. Let $u = \sqrt{x}$, so that $x = u^2$ and $dx = 2u\,du$. Then

$$\int \frac{dx}{\sqrt{x} + x\sqrt{x}} = \int \frac{2u\,du}{u + u^2\cdot u} = \int \frac{2}{1+u^2}\,du = 2\tan^{-1}u + C = 2\tan^{-1}\sqrt{x} + C.$$

65. Let $u = \sqrt[3]{x+c}$. Then $x = u^3 - c \;\Rightarrow$

$$\int x\sqrt[3]{x+c}\,dx = \int (u^3 - c)u\cdot 3u^2\,du = 3\int (u^6 - cu^3)\,du = \tfrac{3}{7}u^7 - \tfrac{3}{4}cu^4 + C = \tfrac{3}{7}(x+c)^{7/3} - \tfrac{3}{4}c(x+c)^{4/3} + C$$

66. Let $t = \sqrt{x^2-1}$. Then $dt = \left(x/\sqrt{x^2-1}\right)dx$, $x^2 - 1 = t^2$, $x = \sqrt{t^2+1}$, so

$I = \displaystyle\int \frac{x\ln x}{\sqrt{x^2-1}}\,dx = \int \ln\sqrt{t^2+1}\,dt = \frac{1}{2}\int \ln(t^2+1)\,dt$. Now use parts with $u = \ln(t^2+1)$, $dv = dt$:

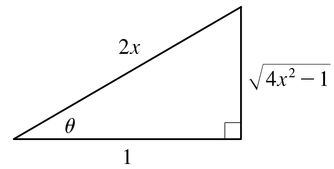
$$\begin{aligned} I &= \tfrac{1}{2}t\ln\left(t^2+1\right) - \int \frac{t^2}{t^2+1}\,dt = \tfrac{1}{2}t\ln\left(t^2+1\right) - \int \left[1 - \frac{1}{t^2+1}\right]dt \\ &= \tfrac{1}{2}t\ln\left(t^2+1\right) - t + \tan^{-1}t + C = \sqrt{x^2-1}\,\ln x - \sqrt{x^2-1} + \tan^{-1}\sqrt{x^2-1} + C \end{aligned}$$

Another method: First integrate by parts with $u = \ln x$, $dv = \left(x/\sqrt{x^2-1}\right)dx$ and then use substitution $\left(x = \sec\theta \text{ or } u = \sqrt{x^2-1}\right)$.

67. $\dfrac{1}{x^4-16} = \dfrac{1}{(x^2-4)(x^2+4)} = \dfrac{1}{(x-2)(x+2)(x^2+4)} = \dfrac{A}{x-2} + \dfrac{B}{x+2} + \dfrac{Cx+D}{x^2+4}$. Multiply by $(x-2)(x+2)(x^2+4)$ to get $1 = A(x+2)(x^2+4) + B(x-2)(x^2+4) + (Cx+D)(x-2)(x+2)$. Substituting 2 for x gives $1 = 32A \;\Leftrightarrow\; A = \frac{1}{32}$. Substituting -2 for x gives $1 = -32B \;\Leftrightarrow\; B = -\frac{1}{32}$. Equating coefficients of x^3 gives $0 = A + B + C = \frac{1}{32} - \frac{1}{32} + C$, so $C = 0$. Equating constant terms gives $1 = 8A - 8B - 4D = \frac{1}{4} + \frac{1}{4} - 4D$, so $\frac{1}{2} = -4D \;\Leftrightarrow\; D = -\frac{1}{8}$. Thus,

$$\begin{aligned} \int \frac{dx}{x^4-16} &= \int \left(\frac{1/32}{x-2} - \frac{1/32}{x+2} - \frac{1/8}{x^2+4}\right)dx = \frac{1}{32}\ln|x-2| - \frac{1}{32}\ln|x+2| - \frac{1}{8}\cdot\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C \\ &= \frac{1}{32}\ln\left|\frac{x-2}{x+2}\right| - \frac{1}{16}\tan^{-1}\left(\frac{x}{2}\right) + C \end{aligned}$$

68. Let $2x = \sec\theta$, so that $2\,dx = \sec\theta\,\tan\theta\,d\theta$. Then



$$\int \frac{dx}{x^2\sqrt{4x^2-1}} = \int \frac{\frac{1}{2}\sec\theta\,\tan\theta\,d\theta}{\frac{1}{4}\sec^2\theta\sqrt{\sec^2\theta-1}} = \int \frac{2\tan\theta\,d\theta}{\sec\theta\,\tan\theta}$$

$$= 2\textstyle\int \cos\theta\,d\theta = 2\sin\theta + C$$

$$= 2\cdot\frac{\sqrt{4x^2-1}}{2x} + C = \frac{\sqrt{4x^2-1}}{x} + C$$

69. $$\int \frac{d\theta}{1+\cos\theta} = \int \left(\frac{1}{1+\cos\theta}\cdot\frac{1-\cos\theta}{1-\cos\theta}\right)d\theta = \int \frac{1-\cos\theta}{1-\cos^2\theta}\,d\theta = \int \frac{1-\cos\theta}{\sin^2\theta}\,d\theta = \int\left(\frac{1}{\sin^2\theta} - \frac{\cos\theta}{\sin^2\theta}\right)d\theta$$

$$= \textstyle\int (\csc^2\theta - \cot\theta\,\csc\theta)\,d\theta = -\cot\theta + \csc\theta + C$$

Another method: Use the substitutions in Exercise 7.4.63.

$$\int \frac{d\theta}{1+\cos\theta} = \int \frac{2/(1+t^2)\,dt}{1+(1-t^2)/(1+t^2)} = \int \frac{2\,dt}{(1+t^2)+(1-t^2)} = \int dt = t + C = \tan\left(\frac{\theta}{2}\right) + C$$

70. $$\int \frac{d\theta}{1+\cos^2\theta} = \int \frac{(1/\cos^2\theta)\,d\theta}{(1+\cos^2\theta)/\cos^2\theta} = \int \frac{\sec^2\theta}{\sec^2\theta+1}\,d\theta = \int \frac{\sec^2\theta}{\tan^2\theta+2}\,d\theta = \int \frac{1}{u^2+2}\,du \quad \left[\begin{matrix} u = \tan\theta, \\ du = \sec^2\theta\,d\theta \end{matrix}\right]$$

$$= \int \frac{1}{u^2+\left(\sqrt{2}\right)^2}\,du = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + C = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan\theta}{\sqrt{2}}\right) + C$$

71. Let $y = \sqrt{x}$ so that $dy = \dfrac{1}{2\sqrt{x}}\,dx \;\Rightarrow\; dx = 2\sqrt{x}\,dy = 2y\,dy$. Then

$$\int \sqrt{x}\,e^{\sqrt{x}}\,dx = \int ye^y(2y\,dy) = \int 2y^2e^y\,dy \qquad \left[\begin{matrix} u = 2y^2, & dv = e^y\,dy, \\ du = 4y\,dy & v = e^y \end{matrix}\right]$$

$$= 2y^2e^y - \int 4ye^y\,dy \qquad \left[\begin{matrix} U = 4y, & dV = e^y\,dy, \\ dU = 4\,dy & V = e^y \end{matrix}\right]$$

$$= 2y^2e^y - \left(4ye^y - \textstyle\int 4e^y\,dy\right) = 2y^2e^y - 4ye^y + 4e^y + C$$

$$= 2(y^2 - 2y + 2)e^y + C = 2\left(x - 2\sqrt{x} + 2\right)e^{\sqrt{x}} + C$$

72. Let $u = \sqrt{x} + 1$, so that $x = (u-1)^2$ and $dx = 2(u-1)\,du$. Then

$$\int \frac{1}{\sqrt{\sqrt{x}+1}}\,dx = \int \frac{2(u-1)\,du}{\sqrt{u}} = \int (2u^{1/2} - 2u^{-1/2})\,du = \frac{4}{3}u^{3/2} - 4u^{1/2} + C = \frac{4}{3}(\sqrt{x}+1)^{3/2} - 4\sqrt{\sqrt{x}+1} + C.$$

73. Let $u = \cos^2 x$, so that $du = 2\cos x\,(-\sin x)\,dx$. Then

$$\int \frac{\sin 2x}{1+\cos^4 x}\,dx = \int \frac{2\sin x\,\cos x}{1+(\cos^2 x)^2}\,dx = \int \frac{1}{1+u^2}\,(-du) = -\tan^{-1}u + C = -\tan^{-1}(\cos^2 x) + C.$$

74. Let $u = \tan x$. Then

$$\int_{\pi/4}^{\pi/3} \frac{\ln(\tan x)\,dx}{\sin x\,\cos x} = \int_{\pi/4}^{\pi/3} \frac{\ln(\tan x)}{\tan x}\sec^2 x\,dx = \int_1^{\sqrt{3}} \frac{\ln u}{u}\,du = \left[\tfrac{1}{2}(\ln u)^2\right]_1^{\sqrt{3}} = \tfrac{1}{2}\left(\ln\sqrt{3}\right)^2 = \tfrac{1}{8}(\ln 3)^2.$$

75. $\displaystyle\int \frac{1}{\sqrt{x+1}+\sqrt{x}}\,dx = \int\left(\frac{1}{\sqrt{x+1}+\sqrt{x}}\cdot\frac{\sqrt{x+1}-\sqrt{x\sqrt{x}}}{\sqrt{x+1}-\sqrt{x}}\right)dx = \int\left(\sqrt{x+1}-\sqrt{x}\right)dx$

$$= \tfrac{2}{3}\left[(x+1)^{3/2}-x^{3/2}\right]+C$$

76. $\displaystyle\int \frac{x^2}{x^6+3x^3+2}\,dx = \int\frac{x^2\,dx}{(x^3+1)(x^3+2)} = \int\frac{\frac{1}{3}\,du}{(u+1)(u+2)}\quad\left[\begin{array}{l} u=x^3, \\ du=3x^2\,dx\end{array}\right].$

Now $\displaystyle\frac{1}{(u+1)(u+2)} = \frac{A}{u+1}+\frac{B}{u+2}\quad\Rightarrow\quad 1 = A(u+2)+B(u+1)$. Setting $u=-2$ gives $B=-1$. Setting $u=-1$ gives $A=1$. Thus,

$$\frac{1}{3}\int\frac{du}{(u+1)(u+2)} = \frac{1}{3}\int\left(\frac{1}{u+1}-\frac{1}{u+2}\right)du = \frac{1}{3}\ln|u+1|-\frac{1}{3}\ln|u+2|+C$$

$$= \tfrac{1}{3}\ln\left|x^3+1\right|-\tfrac{1}{3}\ln\left|x^3+2\right|+C$$

77. Let $x=\tan\theta$, so that $dx=\sec^2\theta\,d\theta$, $x=\sqrt{3}\quad\Rightarrow\quad\theta=\frac{\pi}{3}$, and $x=1\quad\Rightarrow\quad\theta=\frac{\pi}{4}$. Then

$$\int_1^{\sqrt{3}}\frac{\sqrt{1+x^2}}{x^2}\,dx = \int_{\pi/4}^{\pi/3}\frac{\sec\theta}{\tan^2\theta}\sec^2\theta\,d\theta = \int_{\pi/4}^{\pi/3}\frac{\sec\theta\,(\tan^2\theta+1)}{\tan^2\theta}\,d\theta = \int_{\pi/4}^{\pi/3}\left(\frac{\sec\theta\,\tan^2\theta}{\tan^2\theta}+\frac{\sec\theta}{\tan^2\theta}\right)d\theta$$

$$= \int_{\pi/4}^{\pi/3}(\sec\theta+\csc\theta\,\cot\theta)\,d\theta = \Big[\ln|\sec\theta+\tan\theta|-\csc\theta\Big]_{\pi/4}^{\pi/3}$$

$$= \left(\ln\left|2+\sqrt{3}\right|-\tfrac{2}{\sqrt{3}}\right)-\left(\ln\left|\sqrt{2}+1\right|-\sqrt{2}\right) = \sqrt{2}-\tfrac{2}{\sqrt{3}}+\ln\left(2+\sqrt{3}\right)-\ln\left(1+\sqrt{2}\right)$$

78. Let $u=e^x$. Then $x=\ln u$, $dx=du/u\quad\Rightarrow$

$$\int\frac{1}{1+2e^x-e^{-x}}\,dx = \int\frac{du/u}{1+2u-1/u} = \int\frac{du}{2u^2+u-1} = \int\left[\frac{2/3}{2u-1}-\frac{1/3}{u+1}\right]du$$

$$= \tfrac{1}{3}\ln|2u-1|-\tfrac{1}{3}\ln|u+1|+C = \tfrac{1}{3}\ln|(2e^x-1)/(e^x+1)|+C$$

79. Let $u=e^x$. Then $x=\ln u$, $dx=du/u\quad\Rightarrow$

$$\int\frac{e^{2x}}{1+e^x}\,dx = \int\frac{u^2}{1+u}\frac{du}{u} = \int\frac{u}{1+u}\,du = \int\left(1-\frac{1}{1+u}\right)du = u-\ln|1+u|+C = e^x-\ln(1+e^x)+C.$$

80. Use parts with $u=\ln(x+1)$, $dv=dx/x^2$:

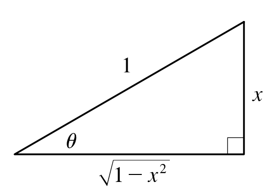
$$\int\frac{\ln(x+1)}{x^2}\,dx = -\frac{1}{x}\ln(x+1)+\int\frac{dx}{x(x+1)} = -\frac{1}{x}\ln(x+1)+\int\left[\frac{1}{x}-\frac{1}{x+1}\right]dx$$

$$= -\frac{1}{x}\ln(x+1)+\ln|x|-\ln(x+1)+C = -\left(1+\frac{1}{x}\right)\ln(x+1)+\ln|x|+C$$

81. Let $\theta=\arcsin x$, so that $d\theta=\dfrac{1}{\sqrt{1-x^2}}\,dx$ and $x=\sin\theta$. Then

$$\int\frac{x+\arcsin x}{\sqrt{1-x^2}}\,dx = \int(\sin\theta+\theta)\,d\theta = -\cos\theta+\tfrac{1}{2}\theta^2+C$$

$$= -\sqrt{1-x^2}+\tfrac{1}{2}(\arcsin x)^2+C$$

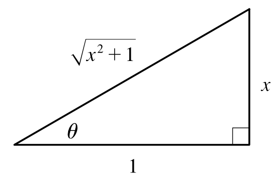


82. $\displaystyle\int \frac{4^x + 10^x}{2^x}\,dx = \int \left(\frac{4^x}{2^x} + \frac{10^x}{2^x}\right) dx = \int (2^x + 5^x)\,dx = \frac{2^x}{\ln 2} + \frac{5^x}{\ln 5} + C$

83. $\displaystyle\int \frac{dx}{x\ln x - x} = \int \frac{dx}{x(\ln x - 1)} = \int \frac{du}{u} \qquad \begin{bmatrix} u = \ln x - 1, \\ du = (1/x)\,dx \end{bmatrix}$

$= \ln|u| + C = \ln|\ln x - 1| + C$

84. $\displaystyle\int \frac{x^2}{\sqrt{x^2+1}}\,dx = \int \frac{\tan^2\theta}{\sec\theta}\sec^2\theta\,d\theta \qquad \begin{bmatrix} x = \tan\theta, \\ dx = \sec^2\theta\,d\theta \end{bmatrix}$



$= \int \tan^2\theta\,\sec\theta\,d\theta = \int (\sec^2\theta - 1)\sec\theta\,d\theta$

$= \int (\sec^3\theta - \sec\theta)\,d\theta$

$= \frac{1}{2}(\sec\theta\,\tan\theta + \ln|\sec\theta + \tan\theta|) - \ln|\sec\theta + \tan\theta| + C$ [by (7.2.1) and Example 7.2.8]

$= \frac{1}{2}(\sec\theta\,\tan\theta - \ln|\sec\theta + \tan\theta|) + C = \frac{1}{2}\left[x\sqrt{x^2+1} - \ln(\sqrt{x^2+1} + x)\right] + C$

85. Let $y = \sqrt{1+e^x}$, so that $y^2 = 1 + e^x$, $2y\,dy = e^x\,dx$, $e^x = y^2 - 1$, and $x = \ln(y^2 - 1)$. Then

$$\int \frac{xe^x}{\sqrt{1+e^x}}\,dx = \int \frac{\ln(y^2-1)}{y}(2y\,dy) = 2\int [\ln(y+1) + \ln(y-1)]\,dy$$

$= 2[(y+1)\ln(y+1) - (y+1) + (y-1)\ln(y-1) - (y-1)] + C$ [by Example 7.1.2]

$= 2[y\ln(y+1) + \ln(y+1) - y - 1 + y\ln(y-1) - \ln(y-1) - y + 1] + C$

$= 2[y(\ln(y+1) + \ln(y-1)) + \ln(y+1) - \ln(y-1) - 2y] + C$

$$= 2\left[y\ln(y^2-1) + \ln\frac{y+1}{y-1} - 2y\right] + C = 2\left[\sqrt{1+e^x}\,\ln(e^x) + \ln\frac{\sqrt{1+e^x}+1}{\sqrt{1+e^x}-1} - 2\sqrt{1+e^x}\right] + C$$

$$= 2x\sqrt{1+e^x} + 2\ln\frac{\sqrt{1+e^x}+1}{\sqrt{1+e^x}-1} - 4\sqrt{1+e^x} + C = 2(x-2)\sqrt{1+e^x} + 2\ln\frac{\sqrt{1+e^x}+1}{\sqrt{1+e^x}-1} + C$$

86. $\displaystyle\frac{1+\sin x}{1-\sin x} = \frac{1+\sin x}{1-\sin x}\cdot\frac{1+\sin x}{1+\sin x} = \frac{1+2\sin x+\sin^2 x}{1-\sin^2 x} = \frac{1+2\sin x+\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} + \frac{2\sin x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x}$

$= \sec^2 x + 2\sec x\,\tan x + \tan^2 x = \sec^2 x + 2\sec x\,\tan x + \sec^2 x - 1 = 2\sec^2 x + 2\sec x\,\tan x - 1$

Thus, $\displaystyle\int \frac{1+\sin x}{1-\sin x}\,dx = \int (2\sec^2 x + 2\sec x\,\tan x - 1)\,dx = 2\tan x + 2\sec x - x + C$

87. Let $u = x$, $dv = \sin^2 x\cos x\,dx \quad\Rightarrow\quad du = dx$, $v = \frac{1}{3}\sin^3 x$. Then

$\int x\sin^2 x\,\cos x\,dx = \frac{1}{3}x\sin^3 x - \int \frac{1}{3}\sin^3 x\,dx = \frac{1}{3}x\sin^3 x - \frac{1}{3}\int(1-\cos^2 x)\sin x\,dx$

$\displaystyle= \frac{1}{3}x\sin^3 x + \frac{1}{3}\int (1-y^2)\,dy \qquad \begin{bmatrix} u = \cos x, \\ du = -\sin x\,dx \end{bmatrix}$

$= \frac{1}{3}x\sin^3 x + \frac{1}{3}y - \frac{1}{9}y^3 + C = \frac{1}{3}x\sin^3 x + \frac{1}{3}\cos x - \frac{1}{9}\cos^3 x + C$

88. $$\int \frac{\sec x\,\cos 2x}{\sin x + \sec x}\,dx = \int \frac{\sec x\,\cos 2x}{\sin x + \sec x}\cdot\frac{2\cos x}{2\cos x}\,dx = \int \frac{2\cos 2x}{2\sin x\cos x + 2}\,dx$$

$$= \int \frac{2\cos 2x}{\sin 2x + 2}\,dx = \int \frac{1}{u}\,du \qquad \left[\begin{array}{l} u = \sin 2x + 2, \\ du = 2\cos 2x\,dx \end{array}\right]$$

$$= \ln|u| + C = \ln|\sin 2x + 2| + C = \ln(\sin 2x + 2) + C$$

89. $$\int \sqrt{1-\sin x}\,dx = \int \sqrt{\frac{1-\sin x}{1}\cdot\frac{1+\sin x}{1+\sin x}}\,dx = \int \sqrt{\frac{1-\sin^2 x}{1+\sin x}}\,dx$$

$$= \int \sqrt{\frac{\cos^2 x}{1+\sin x}}\,dx = \int \frac{\cos x\,dx}{\sqrt{1+\sin x}} \qquad [\text{assume } \cos x > 0]$$

$$= \int \frac{du}{\sqrt{u}} \qquad \left[\begin{array}{l} u = 1 + \sin x, \\ du = \cos x\,dx \end{array}\right]$$

$$= 2\sqrt{u} + C = 2\sqrt{1+\sin x} + C$$

Another method: Let $u = \sin x$ so that $du = \cos x\,dx = \sqrt{1-\sin^2 x}\,dx = \sqrt{1-u^2}\,dx$. Then

$$\int \sqrt{1-\sin x}\,dx = \int \sqrt{1-u}\left(\frac{du}{\sqrt{1-u^2}}\right) = \int \frac{1}{\sqrt{1+u}}\,du = 2\sqrt{1+u} + C = 2\sqrt{1+\sin x} + C.$$

90. $$\int \frac{\sin x\,\cos x}{\sin^4 x + \cos^4 x}\,dx = \int \frac{\sin x\,\cos x}{(\sin^2 x)^2 + (\cos^2 x)^2}\,dx = \int \frac{\sin x\,\cos x}{(\sin^2 x)^2 + (1-\sin^2 x)^2}\,dx$$

$$= \int \frac{1}{u^2 + (1-u)^2}\left(\frac{1}{2}\,du\right) \qquad \left[\begin{array}{l} u = \sin^2 x, \\ du = 2\sin x\,\cos x\,dx \end{array}\right]$$

$$= \int \frac{1}{4u^2 - 4u + 2}\,du = \int \frac{1}{(4u^2 - 4u + 1) + 1}\,du$$

$$= \int \frac{1}{(2u-1)^2 + 1}\,du = \frac{1}{2}\int \frac{1}{y^2+1}\,dy \qquad \left[\begin{array}{l} y = 2u - 1, \\ dy = 2\,du \end{array}\right]$$

$$= \tfrac{1}{2}\tan^{-1} y + C = \tfrac{1}{2}\tan^{-1}(2u-1) + C = \tfrac{1}{2}\tan^{-1}(2\sin^2 x - 1) + C$$

Another solution:

$$\int \frac{\sin x\,\cos x}{\sin^4 x + \cos^4 x}\,dx = \int \frac{(\sin x\,\cos x)/\cos^4 x}{(\sin^4 x + \cos^4 x)/\cos^4 x}\,dx = \int \frac{\tan x\,\sec^2 x}{\tan^4 x + 1}\,dx$$

$$= \int \frac{1}{u^2+1}\left(\frac{1}{2}\,du\right) \qquad \left[\begin{array}{l} u = \tan^2 x, \\ du = 2\tan x\,\sec^2 x\,dx \end{array}\right]$$

$$= \tfrac{1}{2}\tan^{-1} u + C = \tfrac{1}{2}\tan^{-1}(\tan^2 x) + C$$

91. $$\int_1^3 \left(\sqrt{\frac{9-x}{x}} - \sqrt{\frac{x}{9-x}}\right)dx = \int_1^3 \left(\frac{\sqrt{9-x}}{\sqrt{x}} - \frac{\sqrt{x}}{\sqrt{9-x}}\right)dx = \int_1^3 \left(\frac{9-x-x}{\sqrt{x}\sqrt{9-x}}\right)dx = \int_1^3 \left(\frac{9-2x}{\sqrt{9x-x^2}}\right)dx$$

$$= \int_8^{18} \left(\frac{1}{\sqrt{u}}\right)du \quad \left[\begin{array}{l} u = 9x - x^2 \\ du = (9-2x)\,dx \end{array}\right] \quad = \int_8^{18} u^{-1/2}\,du = \left[2u^{1/2}\right]_8^{18}$$

$$= 2\sqrt{18} - 2\sqrt{8} = 6\sqrt{2} - 4\sqrt{2} = 2\sqrt{2}$$

92. $$\int \frac{1}{(\sin x + \cos x)^2}\,dx = \int \frac{1}{\sin^2 x + 2\sin x\cos x + \cos^2 x}\,dx = \int \frac{1}{\cos^2 x\left(\dfrac{\sin^2 x}{\cos^2 x} + \dfrac{2\sin x}{\cos x} + 1\right)}\,dx$$

$$= \int \frac{1}{\cos^2 x\,(\tan^2 x + 2\tan x + 1)}\,dx = \int \frac{1}{\cos^2 x\,(\tan x + 1)^2}\,dx = \int \frac{\sec^2 x}{(\tan x + 1)^2}\,dx$$

$$= \int \frac{1}{u^2}\,du \quad \left[\begin{matrix} u = \tan x + 1 \\ du = \sec^2 x\,dx \end{matrix}\right] \quad = -\frac{1}{u} + C = -\frac{1}{\tan x + 1} + C$$

93. $$\int_0^{\pi/6} \sqrt{1 + \sin 2\theta}\,d\theta = \int_0^{\pi/6} \sqrt{(\sin^2\theta + \cos^2\theta) + 2\sin\theta\cos\theta}\,d\theta = \int_0^{\pi/6} \sqrt{(\sin\theta + \cos\theta)^2}\,d\theta$$

$$= \int_0^{\pi/6} |\sin\theta + \cos\theta|\,d\theta = \int_0^{\pi/6} (\sin\theta + \cos\theta)\,d\theta \qquad \left[\begin{matrix}\text{since integrand is} \\ \text{positive on } [0, \pi/6]\end{matrix}\right]$$

$$= \Big[-\cos\theta + \sin\theta\Big]_0^{\pi/6} = \left(-\frac{\sqrt{3}}{2} + \frac{1}{2}\right) - (-1 + 0) = \frac{3 - \sqrt{3}}{2}$$

Alternate solution:

$$\int_0^{\pi/6} \sqrt{1 + \sin 2\theta}\,d\theta = \int_0^{\pi/6} \sqrt{1 + \sin 2\theta} \cdot \frac{\sqrt{1 - \sin 2\theta}}{\sqrt{1 - \sin 2\theta}}\,d\theta = \int_0^{\pi/6} \frac{\sqrt{1 - \sin^2 2\theta}}{\sqrt{1 - \sin 2\theta}}\,d\theta$$

$$= \int_0^{\pi/6} \frac{\sqrt{\cos^2 2\theta}}{\sqrt{1 - \sin 2\theta}}\,d\theta = \int_0^{\pi/6} \frac{|\cos 2\theta|}{\sqrt{1 - \sin 2\theta}}\,d\theta = \int_0^{\pi/6} \frac{\cos 2\theta}{\sqrt{1 - \sin 2\theta}}\,d\theta$$

$$= -\frac{1}{2}\int_1^{1 - \sqrt{3}/2} u^{-1/2}\,du \qquad [u = 1 - \sin 2\theta,\ du = -2\cos 2\theta\,d\theta]$$

$$= -\tfrac{1}{2}\Big[2u^{1/2}\Big]_1^{1 - \sqrt{3}/2} = 1 - \sqrt{1 - \left(\sqrt{3}/2\right)}$$

94. (a) $$\int_1^2 \frac{e^x}{x}\,dx = \int_0^{\ln 2} \frac{e^{e^t}}{e^t}\,e^t\,dt \quad \left[\begin{matrix} x = e^t, \\ dx = e^t\,dt \end{matrix}\right] \quad = \int_0^{\ln 2} e^{e^t}\,dt = F(\ln 2)$$

(b) $$\int_2^3 \frac{1}{\ln x}\,dx = \int_{\ln 2}^{\ln 3} \frac{1}{u}\,(e^u\,du) \quad \left[\begin{matrix} u = \ln x, \\ du = \dfrac{1}{x}\,dx \end{matrix}\right] \quad = \int_{\ln\ln 2}^{\ln\ln 3} \frac{e^{e^v}}{e^v}\,e^v\,dv \quad \left[\begin{matrix} u = e^v, \\ du = e^v\,dv \end{matrix}\right]$$

$$= \int_{\ln\ln 2}^{0} e^{e^v}\,dv + \int_0^{\ln\ln 3} e^{e^v}\,dv \quad [\text{note that } \ln\ln 2 < 0]$$

$$= \int_0^{\ln\ln 3} e^{e^v}\,dv - \int_0^{\ln\ln 2} e^{e^v}\,dv = F(\ln\ln 3) - F(\ln\ln 2)$$

Another method: Substitute $x = e^{e^t}$ in the original integral.

95. The function $y = 2xe^{x^2}$ *does* have an elementary antiderivative, so we'll use this fact to help evaluate the integral.

$$\textstyle\int (2x^2 + 1)e^{x^2}\,dx = \int 2x^2 e^{x^2}\,dx + \int e^{x^2}\,dx = \int x\left(2xe^{x^2}\right)dx + \int e^{x^2}\,dx$$

$$\textstyle= xe^{x^2} - \int e^{x^2}\,dx + \int e^{x^2}\,dx \quad \left[\begin{matrix} u = x, & dv = 2xe^{x^2}\,dx, \\ du = dx & v = e^{x^2} \end{matrix}\right] \quad = xe^{x^2} + C$$

7.6 Integration Using Tables and Computer Algebra Systems

Keep in mind that there are several ways to approach many of these exercises, and different methods can lead to different forms of the answer.

1. $$\int_0^{\pi/2} \cos 5x \cos 2x\,dx \overset{80}{=} \left[\frac{\sin(5-2)x}{2(5-2)} + \frac{\sin(5+2)x}{2(5+2)}\right]_0^{\pi/2} \qquad \begin{bmatrix} a=5, \\ b=2 \end{bmatrix}$$
$$= \left[\frac{\sin 3x}{6} + \frac{\sin 7x}{14}\right]_0^{\pi/2} = \left(-\frac{1}{6} - \frac{1}{14}\right) - 0 = \frac{-7-3}{42} = -\frac{5}{21}$$

2. $$\int_0^1 \sqrt{x-x^2}\,dx = \int_0^1 \sqrt{2\left(\tfrac{1}{2}\right)x - x^2}\,dx \overset{113}{=} \left[\frac{x-\frac{1}{2}}{2}\sqrt{2\left(\tfrac{1}{2}\right)x - x^2} + \frac{\left(\frac{1}{2}\right)^2}{2}\cos^{-1}\left(\frac{\frac{1}{2}-x}{\frac{1}{2}}\right)\right]_0^1$$
$$= \left[\frac{2x-1}{4}\sqrt{x-x^2} + \frac{1}{8}\cos^{-1}(1-2x)\right]_0^1 = \left(0 + \frac{1}{8}\cdot\pi\right) - \left(0 + \frac{1}{8}\cdot 0\right) = \frac{1}{8}\pi$$

3. Let $u = x^2$, so that $du = 2x\,dx$. Thus,
$$\int x \arcsin(x^2)\,dx = \tfrac{1}{2}\int \sin^{-1} u\,du \overset{87}{=} \tfrac{1}{2}\left[u\sin^{-1}u + \sqrt{1-u^2}\right] + C = \tfrac{1}{2}x^2\sin^{-1}(x^2) + \tfrac{1}{2}\sqrt{1-x^4} + C.$$

4. $$\int \frac{\tan\theta}{\sqrt{2+\cos\theta}}\,d\theta = \int \frac{\sin\theta}{\cos\theta\sqrt{2+\cos\theta}}\,d\theta \overset{c}{=} \int \frac{1}{u\sqrt{2+u}}\,(-du)$$
$$\overset{57}{=} -\frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2+u}-\sqrt{2}}{\sqrt{2+u}+\sqrt{2}}\right| + C = -\frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2+\cos\theta}-\sqrt{2}}{\sqrt{2+\cos\theta}+\sqrt{2}}\right| + C$$

5. Let $u = y^2$, so that $du = 2y\,dy$. Then,
$$\int \frac{y^5}{\sqrt{4+y^4}}\,dy = \int \frac{y^4}{\sqrt{4+y^4}}\,y\,dy = \frac{1}{2}\int \frac{u^2}{\sqrt{4+u^2}}\,du \overset{26}{=} \frac{1}{2}\left[\frac{u}{2}\sqrt{4+u^2} - \frac{4}{2}\ln\left(u + \sqrt{4+u^2}\right)\right] + C$$
$$= \frac{y^2}{4}\sqrt{4+y^4} - \ln\left(y^2 + \sqrt{4+y^4}\right) + C$$

6. Let $u = t^3$, so that $du = 3t^2\,dt$. Thus,
$$\int \frac{\sqrt{t^6-5}}{t}\,dt = \int \frac{\sqrt{t^6-5}}{3t^3}\,3t^2\,dt = \frac{1}{3}\int \frac{\sqrt{u^2-5}}{u}\,du \overset{41}{=} \frac{1}{3}\left[\sqrt{u^2-5} - \sqrt{5}\cos^{-1}\left(\frac{\sqrt{5}}{|u|}\right)\right] + C$$
$$= \frac{1}{3}\sqrt{t^6-5} - \frac{\sqrt{5}}{3}\cos^{-1}\left(\frac{\sqrt{5}}{|t^3|}\right) + C$$

7. $$\int_0^{\pi/8} \arctan 2x\,dx = \tfrac{1}{2}\int_0^{\pi/4} \arctan u\,du \qquad [u = 2x,\ du = 2\,dx]$$
$$\overset{89}{=} \frac{1}{2}\left[u\arctan u - \frac{1}{2}\ln(1+u^2)\right]_0^{\pi/4} = \frac{1}{2}\left\{\left[\frac{\pi}{4}\arctan\frac{\pi}{4} - \frac{1}{2}\ln\left(1 + \frac{\pi^2}{16}\right)\right] - 0\right\}$$
$$= \frac{\pi}{8}\arctan\frac{\pi}{4} - \frac{1}{4}\ln\left(1 + \frac{\pi^2}{16}\right)$$

8. $$\int_0^2 x^2\sqrt{4-x^2}\,dx \overset{31}{=} \left[\frac{x}{8}(2x^2-4)\sqrt{4-x^2} + \frac{16}{8}\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2 = \left(0 + 2\cdot\frac{\pi}{2}\right) - 0 = \pi$$

9. $$\int \frac{\cos x}{\sin^2 x - 9}\,dx = \int \frac{1}{u^2-9}\,du \quad \begin{bmatrix} u = \sin x, \\ du = \cos x\,dx \end{bmatrix} \quad \overset{20}{=} \frac{1}{2(3)}\ln\left|\frac{u-3}{u+3}\right| + C = \frac{1}{6}\ln\left|\frac{\sin x - 3}{\sin x + 3}\right| + C$$

10. $\displaystyle\int \frac{e^x}{4-e^{2x}}\,dx = \int \frac{1}{4-u^2}\,du \quad \begin{bmatrix} u=e^x, \\ du=e^x\,dx \end{bmatrix} \quad \overset{19}{=} \frac{1}{2(2)}\ln\left|\frac{u+2}{u-2}\right| + C = \frac{1}{4}\ln\left|\frac{e^x+2}{e^x-2}\right| + C$

11. $$\int \frac{\sqrt{9x^2+4}}{x^2}\,dx = \int \frac{\sqrt{u^2+4}}{u^2/9}\left(\frac{1}{3}du\right) \quad \begin{bmatrix} u=3x, \\ du=3\,dx \end{bmatrix}$$
$$= 3\int \frac{\sqrt{4+u^2}}{u^2}\,du \overset{24}{=} 3\left[-\frac{\sqrt{4+u^2}}{u} + \ln(u+\sqrt{4+u^2})\right] + C$$
$$= -\frac{3\sqrt{4+9x^2}}{3x} + 3\ln(3x+\sqrt{4+9x^2}) + C = -\frac{\sqrt{9x^2+4}}{x} + 3\ln(3x+\sqrt{9x^2+4}) + C$$

12. Let $u=\sqrt{2}\,y$ and $a=\sqrt{3}$. Then $du=\sqrt{2}\,dy$ and
$$\int \frac{\sqrt{2y^2-3}}{y^2}\,dy = \int \frac{\sqrt{u^2-a^2}}{\frac{1}{2}u^2}\,\frac{du}{\sqrt{2}} = \sqrt{2}\int \frac{\sqrt{u^2-a^2}}{u^2}\,du$$
$$\overset{42}{=} \sqrt{2}\left(-\frac{\sqrt{u^2-a^2}}{u} + \ln\left|u+\sqrt{u^2-a^2}\right|\right) + C$$
$$= \sqrt{2}\left(-\frac{\sqrt{2y^2-3}}{\sqrt{2}\,y} + \ln\left|\sqrt{2}\,y+\sqrt{2y^2-3}\right|\right) + C$$
$$= -\frac{\sqrt{2y^2-3}}{y} + \sqrt{2}\ln\left|\sqrt{2}\,y+\sqrt{2y^2-3}\right| + C$$

13. $\int_0^\pi \cos^6\theta\,d\theta \overset{74}{=} \left[\frac{1}{6}\cos^5\theta\,\sin\theta\right]_0^\pi + \frac{5}{6}\int_0^\pi \cos^4\theta\,d\theta \overset{74}{=} 0 + \frac{5}{6}\left\{\left[\frac{1}{4}\cos^3\theta\,\sin\theta\right]_0^\pi + \frac{3}{4}\int_0^\pi \cos^2\theta\,d\theta\right\}$

$\overset{64}{=} \frac{5}{6}\left\{0 + \frac{3}{4}\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_0^\pi\right\} = \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{\pi}{2} = \frac{5\pi}{16}$

14. $\int x\sqrt{2+x^4}\,dx = \int \sqrt{2+u^2}\,(\frac{1}{2}\,du) \quad \begin{bmatrix} u=x^2, \\ du=2x\,dx \end{bmatrix}$
$$\overset{21}{=} \frac{1}{2}\left[\frac{u}{2}\sqrt{2+u^2} + \frac{2}{2}\ln\left(u+\sqrt{2+u^2}\,\right)\right] + C = \frac{x^2}{4}\sqrt{2+x^4} + \frac{1}{2}\ln(x^2+\sqrt{2+x^4}\,) + C$$

15. $\displaystyle\int \frac{\arctan\sqrt{x}}{\sqrt{x}}\,dx = \int \arctan u\,(2\,du) \quad \begin{bmatrix} u=\sqrt{x}, \\ du=1/(2\sqrt{x})\,dx \end{bmatrix}$
$$\overset{89}{=} 2\left[u\arctan u - \tfrac{1}{2}\ln(1+u^2)\right] + C = 2\sqrt{x}\arctan\sqrt{x} - \ln(1+x) + C$$

16. $\int_0^\pi x^3\sin x\,dx \overset{84}{=} \left[-x^3\cos x\right]_0^\pi + 3\int_0^\pi x^2\cos x\,dx \overset{85}{=} -\pi^3(-1) + 3\left\{\left[x^2\sin x\right]_0^\pi - 2\int_0^\pi x\sin x\,dx\right\}$

$= \pi^3 - 6\int_0^\pi x\sin x\,dx \overset{84}{=} \pi^3 - 6\left\{\left[-x\cos x\right]_0^\pi + \int_0^\pi \cos x\,dx\right\} = \pi^3 - 6[\pi] - 6\left[\sin x\right]_0^\pi$

$= \pi^3 - 6\pi$

17. $\displaystyle\int \frac{\coth(1/y)}{y^2}\,dy = \int \coth u\,(-du) \quad \begin{bmatrix} u=1/y, \\ du=-1/y^2\,dy \end{bmatrix}$
$$\overset{106}{=} -\ln|\sinh u| + C = -\ln|\sinh(1/y)| + C$$

18. $\displaystyle\int \frac{e^{3t}}{\sqrt{e^{2t}-1}}\,dt = \int \frac{e^{2t}}{\sqrt{e^{2t}-1}}\,(e^t\,dt) = \int \frac{u^2}{\sqrt{u^2-1}}\,du \quad \begin{bmatrix} u=e^t, \\ du=e^t\,dt \end{bmatrix}$
$$\overset{44}{=} \frac{u}{2}\sqrt{u^2-1} + \tfrac{1}{2}\ln\left|u+\sqrt{u^2-1}\right| + C = \tfrac{1}{2}e^t\sqrt{e^{2t}-1} + \tfrac{1}{2}\ln\left(e^t+\sqrt{e^{2t}-1}\,\right) + C$$

19. Let $z = 6 + 4y - 4y^2 = 6 - (4y^2 - 4y + 1) + 1 = 7 - (2y - 1)^2$, $u = 2y - 1$, and $a = \sqrt{7}$.

Then $z = a^2 - u^2$, $du = 2\,dy$, and

$$\begin{aligned}
\int y\sqrt{6 + 4y - 4y^2}\,dy &= \int y\sqrt{z}\,dy = \int \tfrac{1}{2}(u+1)\sqrt{a^2 - u^2}\,\tfrac{1}{2}\,du = \tfrac{1}{4}\int u\sqrt{a^2 - u^2}\,du + \tfrac{1}{4}\int \sqrt{a^2 - u^2}\,du \\
&= \tfrac{1}{4}\int \sqrt{a^2 - u^2}\,du - \tfrac{1}{8}\int (-2u)\sqrt{a^2 - u^2}\,du \\
&\overset{30}{=} \frac{u}{8}\sqrt{a^2 - u^2} + \frac{a^2}{8}\sin^{-1}\!\left(\frac{u}{a}\right) - \frac{1}{8}\int \sqrt{w}\,dw \qquad \begin{bmatrix} w = a^2 - u^2, \\ dw = -2u\,du \end{bmatrix} \\
&= \frac{2y - 1}{8}\sqrt{6 + 4y - 4y^2} + \frac{7}{8}\sin^{-1}\frac{2y - 1}{\sqrt{7}} - \frac{1}{8}\cdot\frac{2}{3}w^{3/2} + C \\
&= \frac{2y - 1}{8}\sqrt{6 + 4y - 4y^2} + \frac{7}{8}\sin^{-1}\frac{2y - 1}{\sqrt{7}} - \frac{1}{12}(6 + 4y - 4y^2)^{3/2} + C
\end{aligned}$$

This can be rewritten as

$$\begin{aligned}
&\sqrt{6 + 4y - 4y^2}\left[\frac{1}{8}(2y - 1) - \frac{1}{12}(6 + 4y - 4y^2)\right] + \frac{7}{8}\sin^{-1}\frac{2y - 1}{\sqrt{7}} + C \\
&\qquad = \left(\frac{1}{3}y^2 - \frac{1}{12}y - \frac{5}{8}\right)\sqrt{6 + 4y - 4y^2} + \frac{7}{8}\sin^{-1}\left(\frac{2y - 1}{\sqrt{7}}\right) + C \\
&\qquad = \frac{1}{24}(8y^2 - 2y - 15)\sqrt{6 + 4y - 4y^2} + \frac{7}{8}\sin^{-1}\left(\frac{2y - 1}{\sqrt{7}}\right) + C
\end{aligned}$$

20. $\displaystyle\int \frac{dx}{2x^3 - 3x^2} = \int \frac{dx}{x^2(-3 + 2x)} \overset{50}{=} -\frac{1}{-3x} + \frac{2}{(-3)^2}\ln\left|\frac{-3 + 2x}{x}\right| + C = \frac{1}{3x} + \tfrac{2}{9}\ln\left|\frac{2x - 3}{x}\right| + C$

21. Let $u = \sin x$. Then $du = \cos x\,dx$, so

$$\begin{aligned}
\int \sin^2 x\,\cos x\,\ln(\sin x)\,dx &= \int u^2 \ln u\,du \overset{101}{=} \frac{u^{2+1}}{(2+1)^2}\left[(2+1)\ln u - 1\right] + C = \tfrac{1}{9}u^3(3\ln u - 1) + C \\
&= \tfrac{1}{9}\sin^3 x\,[3\ln(\sin x) - 1] + C
\end{aligned}$$

22. Let $u = \sin\theta$, so that $du = \cos\theta\,d\theta$. Then

$$\begin{aligned}
\int \frac{\sin 2\theta}{\sqrt{5 - \sin\theta}}\,d\theta &= \int \frac{2\sin\theta\,\cos\theta}{\sqrt{5 - \sin\theta}}\,d\theta = 2\int \frac{u}{\sqrt{5 - u}}\,du \overset{55}{=} 2\cdot\frac{2}{3(-1)^2}\left[-1u - 2(5)\right]\sqrt{5 - u} + C \\
&= \tfrac{4}{3}(-u - 10)\sqrt{5 - u} + C = -\tfrac{4}{3}(\sin\theta + 10)\sqrt{5 - \sin\theta} + C
\end{aligned}$$

23.
$$\begin{aligned}
\int \frac{\sin 2\theta}{\sqrt{\cos^4\theta + 4}}\,d\theta &= \int \frac{2\sin\theta\cos\theta}{\sqrt{\cos^4\theta + 4}}\,d\theta = -\int \frac{1}{\sqrt{u^2 + 4}}\,du \qquad \begin{bmatrix} u = \cos^2\theta \\ du = -2\sin\theta\cos\theta\,d\theta \end{bmatrix} \\
&\overset{25}{=} \ln\!\left(u + \sqrt{u^2 + 4}\right) + C = -\ln\!\left(\cos^2\theta + \sqrt{\cos^4\theta + 4}\right) + C
\end{aligned}$$

24. Let $u = x^2$ and $a = 2$. Then $du = 2x\,dx$ and

$$\begin{aligned}
\int_0^2 x^3\sqrt{4x^2 - x^4}\,dx &= \tfrac{1}{2}\int_0^2 x^2\sqrt{2\cdot 2\cdot x^2 - (x^2)^2}\cdot 2x\,dx = \tfrac{1}{2}\int_0^4 u\sqrt{2au - u^2}\,du \\
&\overset{114}{=} \left[\frac{2u^2 - au - 3a^2}{12}\sqrt{2au - u^2} + \frac{a^3}{4}\cos^{-1}\left(\frac{a - u}{a}\right)\right]_0^4 \\
&= \left[\frac{2u^2 - 2u - 12}{12}\sqrt{4u - u^2} + \frac{8}{4}\cos^{-1}\left(\frac{2 - u}{2}\right)\right]_0^4 \\
&= \left[\frac{u^2 - u - 6}{6}\sqrt{4u - u^2} + 2\cos^{-1}\left(\frac{2 - u}{2}\right)\right]_0^4 \\
&= \left[0 + 2\cos^{-1}(-1)\right] - \left(0 + 2\cos^{-1} 1\right) = 2\cdot\pi - 2\cdot 0 = 2\pi
\end{aligned}$$

25. $\int x^3e^{2x}\,dx \overset{97}{=} \frac{1}{2}x^3e^{2x} - \frac{3}{2}\int x^2e^{2x}\,dx \overset{97}{=} \frac{1}{2}x^3e^{2x} - \frac{3}{2}\left(\frac{1}{2}x^2e^{2x} - \int xe^{2x}\,dx\right)$

$\overset{96}{=} \frac{1}{2}x^3e^{2x} - \frac{3}{2}\left(\frac{1}{2}x^2e^{2x} - \frac{1}{4}(2x-1)e^{2x}\right) + C$

$= \frac{1}{2}x^3e^{2x} - \frac{3}{4}x^2e^{2x} + \frac{3}{4}xe^{2x} - \frac{3}{8}e^{2x} + C$

$= \frac{1}{2}e^{2x}\left(x^3 - \frac{3}{2}x^2 + \frac{3}{2}x - \frac{3}{4}\right) + C$

26. $\int x^3\arcsin(x^2)\,dx = \int u\arcsin u\,\left(\frac{1}{2}du\right) \quad \left[\begin{array}{c} u = x^2, \\ du = 2x\,dx \end{array}\right]$

$$\overset{90}{=} \frac{1}{2}\left[\frac{2u^2-1}{4}\arcsin u + \frac{u\sqrt{1-u^2}}{4}\right] + C = \frac{2x^4-1}{8}\arcsin(x^2) + \frac{x^2\sqrt{1-x^4}}{8} + C$$

27. $\int \cos^5 y\,dy \overset{74}{=} \frac{1}{5}\cos^4 y\sin y + \frac{4}{5}\int\cos^3 y\,dy \overset{68}{=} \frac{1}{5}\cos^4 y\sin y + \frac{4}{5}\left[\frac{1}{3}(2+\cos^2 y)\sin y\right] + C$

$= \frac{1}{5}\cos^4 y\sin y + \frac{8}{15}\sin y + \frac{4}{15}\cos^2 y\sin y + C = \frac{1}{5}\sin y\left(\cos^4 y + \frac{4}{3}\cos^2 y + \frac{8}{3}\right) + C$

28. Let $u = \ln x$, so that $du = (1/x)\,dx$. Thus,

$$\int \frac{\sqrt{(\ln x)^2 - 9}}{x\ln x}\,dx = \int \frac{\sqrt{u^2-9}}{u}\,du \overset{41}{=} \sqrt{u^2-9} - 3\cos^{-1}\left(\frac{3}{|u|}\right) + C$$

$$= \sqrt{(\ln x)^2 - 9} - 3\cos^{-1}\left(\frac{3}{|\ln x|}\right) + C$$

29. $\displaystyle\int \frac{\cos^{-1}(x^{-2})}{x^3}\,dx = -\frac{1}{2}\int \cos^{-1}u\,du \quad \left[\begin{array}{c} u = x^{-2}, \\ du = -2x^{-3}\,dx \end{array}\right]$

$$\overset{88}{=} -\tfrac{1}{2}\left(u\cos^{-1}u - \sqrt{1-u^2}\right) + C = -\tfrac{1}{2}x^{-2}\cos^{-1}(x^{-2}) + \tfrac{1}{2}\sqrt{1-x^{-4}} + C$$

30. $\displaystyle\int \frac{dx}{\sqrt{1-e^{2x}}} = \int \frac{1}{\sqrt{1-u^2}}\left(\frac{du}{u}\right) \quad \left[\begin{array}{c} u = e^x, \\ du = e^x\,dx,\ dx = du/u \end{array}\right]$

$$\overset{35}{=} -\frac{1}{1}\ln\left|\frac{1+\sqrt{1-u^2}}{u}\right| + C = -\ln\left|\frac{1+\sqrt{1-e^{2x}}}{e^x}\right| + C = -\ln\left(\frac{1+\sqrt{1-e^{2x}}}{e^x}\right) + C$$

31. Let $u = e^x$. Then $x = \ln u$, $dx = du/u$, so

$$\int \sqrt{e^{2x}-1}\,dx = \int \frac{\sqrt{u^2-1}}{u}\,du \overset{41}{=} \sqrt{u^2-1} - \cos^{-1}(1/u) + C = \sqrt{e^{2x}-1} - \cos^{-1}(e^{-x}) + C.$$

32. $\displaystyle\int \sin 2\theta\arctan(\sin\theta)\,d\theta = \int 2\sin\theta\cos\theta\tan^{-1}(\sin\theta)\,d\theta \overset{s}{=} 2\int u\tan^{-1}u\,du$

$$\overset{92}{=} 2\left(\frac{u^2+1}{2}\tan^{-1}u - \frac{u}{2}\right) + C = (\sin^2\theta + 1)\tan^{-1}(\sin\theta) - \sin\theta + C$$

33. $\displaystyle\int \frac{x^4}{\sqrt{x^{10}-2}}\,dx = \int \frac{x^4}{\sqrt{(x^5)^2-2}}\,dx = \frac{1}{5}\int \frac{1}{\sqrt{u^2-2}}\,du \quad \left[\begin{array}{c} u = x^5, \\ du = 5x^4\,dx \end{array}\right]$

$$\overset{43}{=} \tfrac{1}{5}\ln\left|u + \sqrt{u^2-2}\right| + C = \tfrac{1}{5}\ln\left|x^5 + \sqrt{x^{10}-2}\right| + C$$

34. Let $u = \tan\theta$ and $a = 3$. Then $du = \sec^2\theta\, d\theta$ and

$$\int \frac{\sec^2\theta\,\tan^2\theta}{\sqrt{9-\tan^2\theta}}\,d\theta = \int \frac{u^2}{\sqrt{a^2-u^2}}\,du \overset{34}{=} -\frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{u}{a}\right) + C$$
$$= -\frac{1}{2}\tan\theta\,\sqrt{9-\tan^2\theta} + \frac{9}{2}\sin^{-1}\left(\frac{\tan\theta}{3}\right) + C$$

35. Use disks about the x-axis:

$$V = \int_0^\pi \pi(\sin^2 x)^2\,dx = \pi\int_0^\pi \sin^4 x\,dx \overset{73}{=} \pi\left\{\left[-\tfrac{1}{4}\sin^3 x\,\cos x\right]_0^\pi + \tfrac{3}{4}\int_0^\pi \sin^2 x\,dx\right\}$$

$$\overset{63}{=} \pi\left\{0 + \tfrac{3}{4}\left[\tfrac{1}{2}x - \tfrac{1}{4}\sin 2x\right]_0^\pi\right\} = \pi\left[\tfrac{3}{4}\left(\tfrac{1}{2}\pi - 0\right)\right] = \tfrac{3}{8}\pi^2$$

36. Use shells about the y-axis:

$$V = \int_0^1 2\pi x\arcsin x\,dx \overset{90}{=} 2\pi\left[\frac{2x^2-1}{4}\sin^{-1}x + \frac{x\sqrt{1-x^2}}{4}\right]_0^1 = 2\pi\left[\left(\frac{1}{4}\cdot\frac{\pi}{2}+0\right)-0\right] = \frac{1}{4}\pi^2$$

37. (a) $$\frac{d}{du}\left[\frac{1}{b^3}\left(a+bu-\frac{a^2}{a+bu}-2a\ln|a+bu|\right)+C\right] = \frac{1}{b^3}\left[b+\frac{ba^2}{(a+bu)^2}-\frac{2ab}{(a+bu)}\right]$$
$$= \frac{1}{b^3}\left[\frac{b(a+bu)^2+ba^2-(a+bu)2ab}{(a+bu)^2}\right]$$
$$= \frac{1}{b^3}\left[\frac{b^3u^2}{(a+bu)^2}\right] = \frac{u^2}{(a+bu)^2}$$

(b) Let $t = a+bu \;\Rightarrow\; dt = b\,du$. Note that $u = \dfrac{t-a}{b}$ and $du = \dfrac{1}{b}\,dt$.

$$\int\frac{u^2\,du}{(a+bu)^2} = \frac{1}{b^3}\int\frac{(t-a)^2}{t^2}\,dt = \frac{1}{b^3}\int\frac{t^2-2at+a^2}{t^2}\,dt = \frac{1}{b^3}\int\left(1-\frac{2a}{t}+\frac{a^2}{t^2}\right)dt$$
$$= \frac{1}{b^3}\left(t-2a\ln|t|-\frac{a^2}{t}\right)+C = \frac{1}{b^3}\left(a+bu-\frac{a^2}{a+bu}-2a\ln|a+bu|\right)+C$$

38. (a) $$\frac{d}{du}\left[\frac{u}{8}(2u^2-a^2)\sqrt{a^2-u^2}+\frac{a^4}{8}\sin^{-1}\frac{u}{a}+C\right]$$
$$= \frac{u}{8}(2u^2-a^2)\frac{-u}{\sqrt{a^2-u^2}}+\sqrt{a^2-u^2}\left[\frac{u}{8}(4u)+(2u^2-a^2)\tfrac{1}{8}\right]+\frac{a^4}{8}\frac{1/a}{\sqrt{1-u^2/a^2}}$$
$$= -\frac{u^2(2u^2-a^2)}{8\sqrt{a^2-u^2}}+\sqrt{a^2-u^2}\left[\frac{u^2}{2}+\frac{2u^2-a^2}{8}\right]+\frac{a^4}{8\sqrt{a^2-u^2}}$$
$$= \tfrac{1}{2}(a^2-u^2)^{-1/2}\left[-\frac{u^2}{4}(2u^2-a^2)+u^2(a^2-u^2)+\tfrac{1}{4}(a^2-u^2)(2u^2-a^2)+\frac{a^4}{4}\right]$$
$$= \tfrac{1}{2}(a^2-u^2)^{-1/2}[2u^2a^2-2u^4]$$
$$= \frac{u^2(a^2-u^2)}{\sqrt{a^2-u^2}} = u^2\sqrt{a^2-u^2}$$

(b) Let $u = a\sin\theta \quad \Rightarrow \quad du = a\cos\theta\,d\theta$. Then

$$\begin{aligned}
\int u^2\sqrt{a^2-u^2}\,du &= \int a^2\sin^2\theta\, a\sqrt{1-\sin^2\theta}\,a\cos\theta\,d\theta = a^4\int\sin^2\theta\,\cos^2\theta\,d\theta \\
&= a^4\int\tfrac{1}{2}(1+\cos 2\theta)\tfrac{1}{2}(1-\cos 2\theta)\,d\theta = \tfrac{1}{4}a^4\int(1-\cos^2 2\theta)\,d\theta \\
&= \tfrac{1}{4}a^4\int\left[1-\tfrac{1}{2}(1+\cos 4\theta)\right]d\theta = \tfrac{1}{4}a^4\left(\tfrac{1}{2}\theta-\tfrac{1}{8}\sin 4\theta\right)+C \\
&= \tfrac{1}{4}a^4\left(\tfrac{1}{2}\theta-\tfrac{1}{8}\cdot 2\sin 2\theta\,\cos 2\theta\right)+C = \tfrac{1}{4}a^4\left[\tfrac{1}{2}\theta-\tfrac{1}{2}\sin\theta\,\cos\theta(1-2\sin^2\theta)\right]+C \\
&= \frac{a^4}{8}\left[\sin^{-1}\frac{u}{a}-\frac{u}{a}\frac{\sqrt{a^2-u^2}}{a}\left(1-\frac{2u^2}{a^2}\right)\right]+C = \frac{a^4}{8}\left[\sin^{-1}\frac{u}{a}-\frac{u}{a}\frac{\sqrt{a^2-u^2}}{a}\frac{a^2-2u^2}{a^2}\right]+C \\
&= \frac{u}{8}(2u^2-a^2)\sqrt{a^2-u^2}+\frac{a^4}{8}\sin^{-1}\frac{u}{a}+C
\end{aligned}$$

39. Maple and Mathematica both give $\int\sec^4 x\,dx = \frac{2}{3}\tan x+\frac{1}{3}\tan x\,\sec^2 x$. Using Formula 77, we get $\int\sec^4 x\,dx = \frac{1}{3}\tan x\,\sec^2 x+\frac{2}{3}\int\sec^2 x\,dx = \frac{1}{3}\tan x\,\sec^2 x+\frac{2}{3}\tan x+C$.

40. Maple gives $I = \displaystyle\int\csc^5 x\,dx = -\frac{1}{4}\frac{\cos x}{\sin^4 x}-\frac{3}{8}\frac{\cos x}{\sin^2 x}+\frac{3}{8}\ln(\csc x-\cot x)$. Mathematica gives

$$\begin{aligned}
I &= -\frac{3}{32}\csc^2\frac{x}{2}-\frac{1}{64}\csc^4\frac{x}{2}-\frac{3}{8}\log\cos\frac{x}{2}+\frac{3}{8}\log\sin\frac{x}{2}+\frac{3}{32}\sec^2\frac{x}{2}+\frac{1}{64}\sec^4\frac{x}{2} \\
&= \frac{3}{8}\left(\log\sin\frac{x}{2}-\log\cos\frac{x}{2}\right)+\frac{3}{32}\left(\sec^2\frac{x}{2}-\csc^2\frac{x}{2}\right)+\frac{1}{64}\left(\sec^4\frac{x}{2}-\csc^4\frac{x}{2}\right) \\
&= \frac{3}{8}\log\frac{\sin(x/2)}{\cos(x/2)}+\frac{3}{32}\left[\frac{1}{\cos^2(x/2)}-\frac{1}{\sin^2(x/2)}\right]+\frac{1}{64}\left[\frac{1}{\cos^4(x/2)}-\frac{1}{\sin^4(x/2)}\right] \\
&= \frac{3}{8}\log\tan\frac{x}{2}+\frac{3}{32}\left[\frac{\sin^2(x/2)-\cos^2(x/2)}{\cos^2(x/2)\,\sin^2(x/2)}\right]+\frac{1}{64}\left[\frac{\sin^4(x/2)-\cos^4(x/2)}{\cos^4(x/2)\,\sin^4(x/2)}\right]
\end{aligned}$$

Now
$$\frac{\sin^2(x/2)-\cos^2(x/2)}{\cos^2(x/2)\,\sin^2(x/2)} = \frac{\dfrac{1-\cos x}{2}-\dfrac{1+\cos x}{2}}{\dfrac{1+\cos x}{2}\cdot\dfrac{1-\cos x}{2}} = \frac{-\dfrac{2\cos x}{2}}{\dfrac{1-\cos^2 x}{4}} = \frac{-4\cos x}{\sin^2 x}$$

and
$$\begin{aligned}
\frac{\sin^4(x/2)-\cos^4(x/2)}{\cos^4(x/2)\,\sin^4(x/2)} &= \frac{\sin^2(x/2)-\cos^2(x/2)}{\cos^2(x/2)\,\sin^2(x/2)}\,\frac{\sin^2(x/2)+\cos^2(x/2)}{\cos^2(x/2)\,\sin^2(x/2)} \\
&= \frac{-4\cos x}{\sin^2 x}\,\frac{1}{\dfrac{1+\cos x}{2}\cdot\dfrac{1-\cos x}{2}} = -\frac{4\cos x}{\sin^2 x}\,\frac{4}{1-\cos^2 x} = -\frac{16\cos x}{\sin^4 x}
\end{aligned}$$

Returning to the expression for I, we get

$$I = \frac{3}{8}\log\tan\frac{x}{2}+\frac{3}{32}\left(\frac{-4\cos x}{\sin^2 x}\right)+\frac{1}{64}\left(\frac{-16\cos x}{\sin^4 x}\right) = \frac{3}{8}\log\tan\frac{x}{2}-\frac{3}{8}\frac{\cos x}{\sin^2 x}-\frac{1}{4}\frac{\cos x}{\sin^4 x},$$

so all are equivalent.

Now use Formula 78 to get

$$\begin{aligned}
\int\csc^5 x\,dx &= \frac{-1}{4}\cot x\,\csc^3 x+\frac{3}{4}\int\csc^3 x\,dx = -\frac{1}{4}\frac{\cos x}{\sin x}\frac{1}{\sin^3 x}+\frac{3}{4}\left(\frac{-1}{2}\cot x\csc x+\frac{1}{2}\int\csc x\,dx\right) \\
&= -\frac{1}{4}\frac{\cos x}{\sin^4 x}-\frac{3}{8}\frac{\cos x}{\sin x}\frac{1}{\sin x}+\frac{3}{8}\int\csc x\,dx = -\frac{1}{4}\frac{\cos x}{\sin^4 x}-\frac{3}{8}\frac{\cos x}{\sin^2 x}+\frac{3}{8}\ln|\csc x-\cot x|+C
\end{aligned}$$

41. Maple gives $\int x^2\sqrt{2^2+x^2}\,dx = \frac{1}{4}x(x^2+4)^{3/2} - \frac{1}{2}x\sqrt{x^2+4} - 2\,\text{arcsinh}\left(\frac{1}{2}x\right)$. Applying the command

`convert(%,ln);` yields

$$\begin{aligned}\tfrac{1}{4}x(x^2+4)^{3/2} - \tfrac{1}{2}x\sqrt{x^2+4} - 2\ln\left(\tfrac{1}{2}x + \tfrac{1}{2}\sqrt{x^2+4}\right) &= \tfrac{1}{4}x(x^2+4)^{1/2}\left[(x^2+4)-2\right] - 2\ln\left[\left(x+\sqrt{x^2+4}\right)/2\right]\\ &= \tfrac{1}{4}x(x^2+2)\sqrt{x^2+4} - 2\ln\left(\sqrt{x^2+4}+x\right) + 2\ln 2\end{aligned}$$

Mathematica gives $\frac{1}{4}x(2+x^2)\sqrt{3+x^2} - 2\,\text{arcsinh}(x/2)$. Applying the `TrigToExp` and `Simplify` commands gives $\frac{1}{4}\left[x(2+x^2)\sqrt{4+x^2} - 8\log\left(\frac{1}{2}\left(x+\sqrt{4+x^2}\right)\right)\right] = \frac{1}{4}x(x^2+2)\sqrt{x^2+4} - 2\ln\left(x+\sqrt{4+x^2}\right) + 2\ln 2$, so all are equivalent (without constant).

Now use Formula 22 to get

$$\begin{aligned}\int x^2\sqrt{2^2+x^2}\,dx &= \frac{x}{8}(2^2+2x^2)\sqrt{2^2+x^2} - \frac{2^4}{8}\ln\left(x+\sqrt{2^2+x^2}\right) + C\\ &= \frac{x}{8}(2)(2+x^2)\sqrt{4+x^2} - 2\ln\left(x+\sqrt{4+x^2}\right) + C\\ &= \tfrac{1}{4}x(x^2+2)\sqrt{x^2+4} - 2\ln\left(\sqrt{x^2+4}+x\right) + C\end{aligned}$$

42. Maple gives $\displaystyle\int \frac{1}{e^x(3e^x+2)}\,dx = \frac{3}{4}\ln(3e^x+2) - \frac{1}{2e^x} - \frac{3}{4}\ln(e^x)$, whereas Mathematica gives

$$-\frac{e^{-x}}{2} + \frac{3}{4}\log(3+2e^{-x}) = -\frac{e^{-x}}{2} + \frac{3}{4}\log\left(\frac{3e^x+2}{e^x}\right) = -\frac{e^{-x}}{2} + \frac{3}{4}\frac{\ln(3e^x+2)}{\ln e^x} = -\frac{e^{-x}}{2} + \frac{3}{4}\ln(3e^x+2) - \frac{3}{4}x,$$

so both are equivalent. Now let $u = e^x$, so $du = e^x\,dx$ and $dx = du/u$. Then

$$\begin{aligned}\int \frac{1}{e^x(3e^x+2)}\,dx &= \int \frac{1}{u(3u+2)}\,\frac{du}{u} = \int \frac{1}{u^2(2+3u)}\,du \overset{50}{=} -\frac{1}{2u} + \frac{3}{2^2}\ln\left|\frac{2+3u}{u}\right| + C\\ &= -\frac{1}{2e^x} + \frac{3}{4}\ln(2+3e^x) - \frac{3}{4}\ln e^x + C = -\frac{1}{2e^x} + \frac{3}{4}\ln(3e^x+2) - \frac{3}{4}x + C\end{aligned}$$

43. Maple gives $\displaystyle\int \cos^4 x\,dx = \frac{\sin x\,\cos^3 x}{4} + \frac{3\sin x\,\cos x}{8} + \frac{3x}{8}$, whereas Mathematica gives

$$\begin{aligned}\frac{3x}{8} + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) &= \frac{3x}{8} + \frac{1}{4}(2\sin x\,\cos x) + \frac{1}{32}(2\sin 2x\,\cos 2x)\\ &= \frac{3x}{8} + \frac{1}{2}\sin x\,\cos x + \frac{1}{16}[2\sin x\,\cos x\,(2\cos^2 x - 1)]\\ &= \frac{3x}{8} + \frac{1}{2}\sin x\,\cos x + \frac{1}{4}\sin x\,\cos^3 x - \frac{1}{8}\sin x\,\cos x,\end{aligned}$$

so both are equivalent.

Using tables,

$$\begin{aligned}\int \cos^4 x\,dx &\overset{74}{=} \tfrac{1}{4}\cos^3 x\,\sin x + \tfrac{3}{4}\int \cos^2 x\,dx \overset{64}{=} \tfrac{1}{4}\cos^3 x\,\sin x + \tfrac{3}{4}\left(\tfrac{1}{2}x + \tfrac{1}{4}\sin 2x\right) + C\\ &= \tfrac{1}{4}\cos^3 x\,\sin x + \tfrac{3}{8}x + \tfrac{3}{16}(2\sin x\,\cos x) + C = \tfrac{1}{4}\cos^3 x\,\sin x + \tfrac{3}{8}x + \tfrac{3}{8}\sin x\,\cos x + C\end{aligned}$$

44. Maple gives

$$\int x^2\sqrt{1-x^2}\,dx = -\frac{x}{4}(1-x^2)^{3/2} + \frac{x}{8}\sqrt{1-x^2} + \frac{1}{8}\arcsin x = \frac{x}{8}(1-x^2)^{1/2}[-2(1-x^2)+1] + \frac{1}{8}\arcsin x$$
$$= \frac{x}{8}(1-x^2)^{1/2}(2x^2-1) + \frac{1}{8}\arcsin x,$$

and Mathematica gives $\frac{1}{8}\left(x\sqrt{1-x^2}(-1+2x^2)+\arcsin x\right)$, so both are equivalent.

Now use Formula 31 to get

$$\int x^2\sqrt{1-x^2}\,dx = \frac{x}{8}(2x^2-1)\sqrt{1-x^2} + \frac{1}{8}\sin^{-1}x + C$$

45. Maple gives $\int \tan^5 x\,dx = \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x + \frac{1}{2}\ln(1+\tan^2 x)$, and Mathematica gives $\int \tan^5 x\,dx = \frac{1}{4}[-1-2\cos(2x)]\sec^4 x - \ln(\cos x)$. These expressions are equivalent, and neither includes absolute value bars or a constant of integration. Note that Mathematica's expression suggests that the integral is undefined where $\cos x < 0$, which is not the case. Using Formula 75, $\int \tan^5 x\,dx = \frac{1}{5-1}\tan^{5-1}x - \int \tan^{5-2}x\,dx = \frac{1}{4}\tan^4 x - \int \tan^3 x\,dx$. Using Formula 69, $\int \tan^3 x\,dx = \frac{1}{2}\tan^2 x + \ln|\cos x| + C$, so $\int \tan^5 x\,dx = \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \ln|\cos x| + C$.

46. Maple and Mathematica both give $\displaystyle\int \frac{1}{\sqrt{1+\sqrt[3]{x}}}\,dx = \frac{2}{5}\sqrt{\sqrt[3]{x}+1}\left(3\sqrt[3]{x^2}-4\sqrt[3]{x}+8\right)$. [Maple adds a constant of $-\frac{16}{5}$.] We'll change the form of the integral by letting $u = \sqrt[3]{x}$, so that $u^3 = x$ and $3u^2\,du = dx$. Then

$$\int \frac{1}{\sqrt{1+\sqrt[3]{x}}}\,dx = \int \frac{3u^2\,du}{\sqrt{1+u}} \overset{56}{=} 3\left[\frac{2}{15(1)^3}\left(8(1)^2+3(1)^2u^2-4(1)(1)u\right)\sqrt{1+u}\right] + C$$
$$= \tfrac{2}{5}(8+3u^2-4u)\sqrt{1+u} + C = \tfrac{2}{5}\left(8+3\sqrt[3]{x^2}-4\sqrt[3]{x}\right)\sqrt{1+\sqrt[3]{x}} + C$$

47. (a) $\displaystyle F(x) = \int f(x)\,dx = \int \frac{1}{x\sqrt{1-x^2}}\,dx \overset{35}{=} -\frac{1}{1}\ln\left|\frac{1+\sqrt{1-x^2}}{x}\right| + C = -\ln\left|\frac{1+\sqrt{1-x^2}}{x}\right| + C.$

f has domain $\left\{x \mid x \neq 0, 1-x^2 > 0\right\} = \{x \mid x \neq 0,\ |x| < 1\} = (-1,0)\cup(0,1)$. F has the same domain.

(b) Mathematica gives $F(x) = \ln x - \ln\left(1+\sqrt{1-x^2}\right)$. Maple gives $F(x) = -\operatorname{arctanh}\left(1/\sqrt{1-x^2}\right)$. This function has domain $\left\{x \mid |x| < 1, -1 < 1/\sqrt{1-x^2} < 1\right\} = \left\{x \mid |x| < 1, 1/\sqrt{1-x^2} < 1\right\} = \left\{x \mid |x| < 1, \sqrt{1-x^2} > 1\right\} = \emptyset$, the empty set! If we apply the command `convert(%,ln);` to Maple's answer, we get

$-\dfrac{1}{2}\ln\left(\dfrac{1}{\sqrt{1-x^2}}+1\right) + \dfrac{1}{2}\ln\left(1-\dfrac{1}{\sqrt{1-x^2}}\right)$, which has the same domain, $\emptyset$.

48. Neither Maple nor Mathematica is able to evaluate $\int (1+\ln x)\sqrt{1+(x\ln x)^2}\,dx$. However, if we let $u = x\ln x$, then $du = (1+\ln x)\,dx$ and the integral is simply $\int \sqrt{1+u^2}\,du$, which any CAS can evaluate. The antiderivative is $\frac{1}{2}\ln\left(x\ln x+\sqrt{1+(x\ln x)^2}\right) + \frac{1}{2}x\ln x\sqrt{1+(x\ln x)^2} + C$.

DISCOVERY PROJECT Patterns in Integrals

1. (a) The CAS results are listed. Note that the absolute value symbols are missing, as is the familiar "$+C$".

(i) $\displaystyle\int \frac{1}{(x+2)(x+3)}\,dx = \ln(x+2) - \ln(x+3)$ (ii) $\displaystyle\int \frac{1}{(x+1)(x+5)}\,dx = \frac{\ln(x+1)}{4} - \frac{\ln(x+5)}{4}$

(iii) $\displaystyle\int \frac{1}{(x+2)(x-5)}\,dx = \frac{\ln(x-5)}{7} - \frac{\ln(x+2)}{7}$ (iv) $\displaystyle\int \frac{1}{(x+2)^2}\,dx = -\frac{1}{x+2}$

(b) If $a \neq b$, it appears that $\ln(x+a)$ is divided by $b-a$ and $\ln(x+b)$ is divided by $a-b$, so we guess that $\displaystyle\int \frac{1}{(x+a)(x+b)}\,dx = \frac{\ln(x+a)}{b-a} + \frac{\ln(x+b)}{a-b} + C$. If $a=b$, as in part (a)(iv), it appears that $\displaystyle\int \frac{1}{(x+a)^2}\,dx = -\frac{1}{x+a} + C$.

(c) The CAS verifies our guesses. Now $\displaystyle\frac{1}{(x+a)(x+b)} = \frac{A}{x+a} + \frac{B}{x+b} \quad\Rightarrow\quad 1 = A(x+b) + B(x+a)$.

Setting $x=-b$ gives $B = 1/(a-b)$ and setting $x=-a$ gives $A = 1/(b-a)$. So

$$\int \frac{1}{(x+a)(x+b)}\,dx = \int \left[\frac{1/(b-a)}{x+a} + \frac{1/(a-b)}{x+b}\right] dx = \frac{\ln|x+a|}{b-a} + \frac{\ln|x+b|}{a-b} + C$$

and our guess for $a \neq b$ is correct. If $a=b$, then $\displaystyle\frac{1}{(x+a)(x+b)} = \frac{1}{(x+a)^2} = (x+a)^{-2}$. Letting $u = x+a \quad\Rightarrow$ $du = dx$, we have $\displaystyle\int (x+a)^{-2}\,dx = \int u^{-2}\,du = -\frac{1}{u} + C = -\frac{1}{x+a} + C$, and our guess for $a=b$ is also correct.

2. (a) (i) $\displaystyle\int \sin x\,\cos 2x\,dx = \frac{\cos x}{2} - \frac{\cos 3x}{6}$ (ii) $\displaystyle\int \sin 3x\,\cos 7x\,dx = \frac{\cos 4x}{8} - \frac{\cos 10x}{20}$

(iii) $\displaystyle\int \sin 8x\,\cos 3x\,dx = -\frac{\cos 11x}{22} - \frac{\cos 5x}{10}$

(b) Looking at the sums and differences of a and b in part (a), we guess that

$$\int \sin ax\,\cos bx\,dx = \frac{\cos((a-b)x)}{2(b-a)} - \frac{\cos((a+b)x)}{2(a+b)} + C$$

Note that $\cos((a-b)x) = \cos((b-a)x)$.

(c) The CAS verifies our guess. Again, we can prove that the guess is correct by differentiating:

$$\begin{aligned}\frac{d}{dx}\left[\frac{\cos((a-b)x)}{2(b-a)} - \frac{\cos((a+b)x)}{2(a+b)}\right] &= \frac{1}{2(b-a)}\,[-\sin((a-b)x)](a-b) - \frac{1}{2(a+b)}\,[-\sin((a+b)x)](a+b)\\ &= \tfrac{1}{2}\sin(ax-bx) + \tfrac{1}{2}\sin(ax+bx)\\ &= \tfrac{1}{2}(\sin ax\,\cos bx - \cos ax\,\sin bx) + \tfrac{1}{2}(\sin ax\,\cos bx + \cos ax\,\sin bx)\\ &= \sin ax\,\cos bx\end{aligned}$$

Our formula is valid for $a \neq b$.

3. (a) (i) $\int \ln x\,dx = x\ln x - x$ (ii) $\int x\ln x\,dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2$

(iii) $\int x^2\ln x\,dx = \frac{1}{3}x^3\ln x - \frac{1}{9}x^3$ (iv) $\int x^3\ln x\,dx = \frac{1}{4}x^4\ln x - \frac{1}{16}x^4$

(v) $\int x^7\ln x\,dx = \frac{1}{8}x^8\ln x - \frac{1}{64}x^8$

(b) We guess that $\int x^n \ln x\,dx = \dfrac{1}{n+1}x^{n+1}\ln x - \dfrac{1}{(n+1)^2}x^{n+1}$.

(c) Let $u = \ln x$, $dv = x^n\,dx \quad\Rightarrow\quad du = \dfrac{dx}{x}$, $v = \dfrac{1}{n+1}x^{n+1}$. Then

$$\int x^n \ln x\,dx = \frac{1}{n+1}x^{n+1}\ln x - \frac{1}{n+1}\int x^n\,dx = \frac{1}{n+1}x^{n+1}\ln x - \frac{1}{n+1}\cdot\frac{1}{n+1}x^{n+1},$$

which verifies our guess. We must have $n+1 \neq 0 \quad\Leftrightarrow\quad n \neq -1$.

4. (a) (i) $\int xe^x\,dx = e^x(x-1)$ (ii) $\int x^2e^x\,dx = e^x(x^2-2x+2)$

(iii) $\int x^3e^x\,dx = e^x(x^3-3x^2+6x-6)$ (iv) $\int x^4e^x\,dx = e^x(x^4-4x^3+12x^2-24x+24)$

(v) $\int x^5e^x\,dx = e^x\left(x^5-5x^4+20x^3-60x^2+120x-120\right)$

(b) Notice from part (a) that we can write

$$\int x^4e^x\,dx = e^x(x^4-4x^3+4\cdot 3x^2-4\cdot 3\cdot 2x+4\cdot 3\cdot 2\cdot 1)$$

and $$\int x^5e^x\,dx = e^x(x^5-5x^4+5\cdot 4x^3-5\cdot 4\cdot 3x^2+5\cdot 4\cdot 3\cdot 2x-5\cdot 4\cdot 3\cdot 2\cdot 1)$$

So we guess that

$$\begin{aligned}\int x^6e^x\,dx &= e^x(x^6-6x^5+6\cdot 5x^4-6\cdot 5\cdot 4x^3+6\cdot 5\cdot 4\cdot 3x^2-6\cdot 5\cdot 4\cdot 3\cdot 2x+6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1)\\ &= e^x(x^6-6x^5+30x^4-120x^3+360x^2-720x+720)\end{aligned}$$

The CAS verifies our guess.

(c) From the results in part (a), as well as our prediction in part (b), we speculate that

$$\int x^ne^x\,dx = e^x\left[x^n-nx^{n-1}+n(n-1)x^{n-2}-n(n-1)(n-2)x^{n-3}+\cdots\pm n!x\mp n!\right] = e^x\sum_{i=0}^{n}(-1)^{n-i}\frac{n!}{i!}\,x^i.$$

(We have reversed the order of the polynomial's terms.)

(d) Let S_n be the statement that $\int x^ne^x\,dx = e^x\sum\limits_{i=0}^{n}(-1)^{n-i}\dfrac{n!}{i!}\,x^i$.

S_1 is true by part (a)(i). Suppose S_k is true for some k, and consider S_{k+1}. Integrating by parts with $u = x^{k+1}$, $dv = e^x\,dx \quad\Rightarrow\quad du = (k+1)x^k\,dx$, $v = e^x$, we get

$$\begin{aligned}\int x^{k+1}e^x\,dx &= x^{k+1}e^x-(k+1)\int x^ke^x\,dx = x^{k+1}e^x-(k+1)\left[e^x\sum_{i=0}^{k}(-1)^{k-i}\frac{k!}{i!}\,x^i\right]\\ &= e^x\left[x^{k+1}-(k+1)\sum_{i=0}^{k}(-1)^{k-i}\frac{k!}{i!}\,x^i\right] = e^x\left[x^{k+1}+\sum_{i=0}^{k}(-1)^{k-i+1}\frac{(k+1)k!}{i!}\,x^i\right]\\ &= e^x\sum_{i=0}^{k+1}(-1)^{(k+1)-i}\frac{(k+1)!}{i!}\,x^i\end{aligned}$$

This verifies S_n for $n = k+1$. Thus, by mathematical induction, S_n is true for all n, where n is a positive integer.

7.7 Approximate Integration

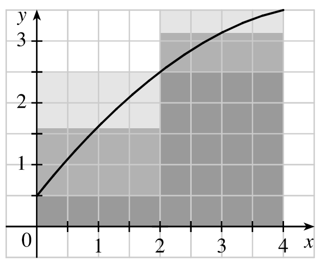
1. (a) $\Delta x = (b-a)/n = (4-0)/2 = 2$

$$L_2 = \sum_{i=1}^{2} f(x_{i-1})\,\Delta x = f(x_0)\cdot 2 + f(x_1)\cdot 2 = 2\,[f(0)+f(2)] = 2(0.5+2.5) = 6$$

$$R_2 = \sum_{i=1}^{2} f(x_i)\,\Delta x = f(x_1)\cdot 2 + f(x_2)\cdot 2 = 2\,[f(2)+f(4)] = 2(2.5+3.5) = 12$$

$$M_2 = \sum_{i=1}^{2} f(\overline{x}_i)\Delta x = f(\overline{x}_1)\cdot 2 + f(\overline{x}_2)\cdot 2 = 2\,[f(1)+f(3)] \approx 2(1.6+3.2) = 9.6$$

(b)



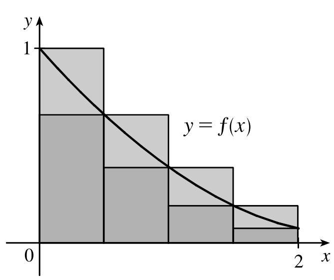
L_2 is an underestimate, since the area under the small rectangles is less than the area under the curve, and R_2 is an overestimate, since the area under the large rectangles is greater than the area under the curve. It appears that M_2 is an overestimate, though it is fairly close to I. See the solution to Exercise 47 for a proof of the fact that if f is concave down on $[a,b]$, then the Midpoint Rule is an overestimate of $\int_a^b f(x)\,dx$.

(c) $T_2 = \left(\frac{1}{2}\,\Delta x\right)[f(x_0) + 2f(x_1) + f(x_2)] = \frac{2}{2}[f(0) + 2f(2) + f(4)] = 0.5 + 2(2.5) + 3.5 = 9.$

This approximation is an underestimate, since the graph is concave down. Thus, $T_2 = 9 < I$. See the solution to Exercise 47 for a general proof of this conclusion.

(d) For any n, we will have $L_n < T_n < I < M_n < R_n$.

2.



The diagram shows that $L_4 > T_4 > \int_0^2 f(x)\,dx > R_4$, and it appears that M_4 is a bit less than $\int_0^2 f(x)\,dx$. In fact, for any function that is concave upward, it can be shown that $L_n > T_n > \int_0^2 f(x)\,dx > M_n > R_n$.

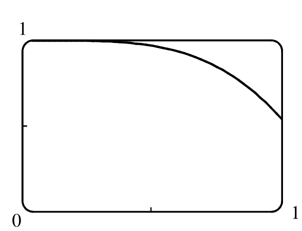
(a) Since $0.9540 > 0.8675 > 0.8632 > 0.7811$, it follows that $L_n = 0.9540$, $T_n = 0.8675$, $M_n = 0.8632$, and $R_n = 0.7811$.

(b) Since $M_n < \int_0^2 f(x)\,dx < T_n$, we have $0.8632 < \int_0^2 f(x)\,dx < 0.8675$.

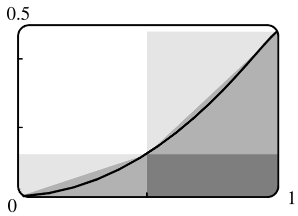
3. $f(x) = \cos(x^2)$, $\Delta x = \frac{1-0}{4} = \frac{1}{4}$

(a) $T_4 = \frac{1}{4\cdot 2}\left[f(0) + 2f\left(\frac{1}{4}\right) + 2f\left(\frac{2}{4}\right) + 2f\left(\frac{3}{4}\right) + f(1)\right] \approx 0.895759$

(b) $M_4 = \frac{1}{4}\left[f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + f\left(\frac{5}{8}\right) + f\left(\frac{7}{8}\right)\right] \approx 0.908907$



The graph shows that f is concave down on $[0,1]$. So T_4 is an underestimate and M_4 is an overestimate. We can conclude that $0.895759 < \int_0^1 \cos(x^2)\,dx < 0.908907$.

4. 

(a) $f(x) = \sin\left(\frac{1}{2}x^2\right)$. Since f is increasing on $[0, 1]$, L_2 will underestimate I (since the area of the darkest rectangle is less than the area under the curve), and R_2 will overestimate I. Since f is concave upward on $[0, 1]$, M_2 will underestimate I and T_2 will overestimate I (the area under the straight line segments is greater than the area under the curve).

(b) For any n, we will have $L_n < M_n < I < T_n < R_n$.

(c) $L_5 = \sum\limits_{i=1}^{5} f(x_{i-1})\,\Delta x = \frac{1}{5}[f(0.0) + f(0.2) + f(0.4) + f(0.6) + f(0.8)] \approx 0.1187$

$R_5 = \sum\limits_{i=1}^{5} f(x_i)\,\Delta x = \frac{1}{5}[f(0.2) + f(0.4) + f(0.6) + f(0.8) + f(1)] \approx 0.2146$

$M_5 = \sum\limits_{i=1}^{5} f(\overline{x}_i)\,\Delta x = \frac{1}{5}[f(0.1) + f(0.3) + f(0.5) + f(0.7) + f(0.9)] \approx 0.1622$

$T_5 = \left(\frac{1}{2}\,\Delta x\right)[f(0) + 2f(0.2) + 2f(0.4) + 2f(0.6) + 2f(0.8) + f(1)] \approx 0.1666$

From the graph, it appears that the Midpoint Rule gives the best approximation. (This is in fact the case, since $I \approx 0.16371405$.)

5. (a) $f(x) = x\sin x,\ \Delta x = \dfrac{b-a}{n} = \dfrac{\pi - 0}{6} = \dfrac{\pi}{6}$

$$M_6 = \frac{\pi}{6}\left[f\left(\frac{\pi}{12}\right) + f\left(\frac{3\pi}{12}\right) + f\left(\frac{5\pi}{12}\right) + f\left(\frac{7\pi}{12}\right) + f\left(\frac{9\pi}{12}\right) + f\left(\frac{11\pi}{12}\right)\right] \approx 3.177769$$

(b) $$S_6 = \frac{\pi}{6\cdot 3}\left[f(0) + 4f\left(\frac{\pi}{6}\right) + 2f\left(\frac{2\pi}{6}\right) + 4f\left(\frac{3\pi}{6}\right) + 2f\left(\frac{4\pi}{6}\right) + 4f\left(\frac{5\pi}{6}\right) + f\left(\frac{6\pi}{6}\right)\right] \approx 3.142949$$

Actual: $I = \int_0^\pi x\sin x\,dx = \Big[-x\cos x + \sin x\Big]_0^\pi$ $\quad$ [use parts with $u = x$ and $dv = \sin x\,dx$]

$= (-\pi(-1) - 0) - (0 + 0) = \pi \approx 3.141593$

Errors: $E_M = \text{actual} - M_6 \approx 3.141593 - 3.177769 \approx -0.036176$

$Es = \text{actual} - S_6 \approx 3.141593 - 3.142949 \approx -0.001356$

6. (a) $f(x) = \dfrac{x}{\sqrt{1+x^2}},\ \Delta x = \dfrac{b-a}{n} = \dfrac{2-0}{8} = \dfrac{1}{4}$

$M_8 = \frac{1}{4}\left[f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + f\left(\frac{5}{8}\right) + f\left(\frac{7}{8}\right) + f\left(\frac{9}{8}\right) + f\left(\frac{11}{8}\right) + f\left(\frac{13}{8}\right) + f\left(\frac{15}{8}\right)\right] \approx 1.238455$

(b) $$S_8 = \frac{1}{4\cdot 3}\left[f(0) + 4f\left(\frac{1}{4}\right) + 2f\left(\frac{2}{4}\right) + 4f\left(\frac{3}{4}\right) + 2f\left(\frac{4}{4}\right) + 4f\left(\frac{5}{4}\right) + 2f\left(\frac{6}{4}\right) + 4f\left(\frac{7}{4}\right) + f\left(\frac{8}{4}\right)\right]$$

≈ 1.236147

Actual: $I = \displaystyle\int_0^2 \frac{x}{\sqrt{1+x^2}}\,dx = \left[\sqrt{1+x^2}\right]_0^2$ $\quad$ $[u = 1 + x^2,\ du = 2x\,dx]$

$= \sqrt{1+4} - \sqrt{1} = \sqrt{5} - 1 \approx 1.236068$

Errors: $E_M = \text{actual} - M_8 \approx 1.236068 - 1.238455 \approx -0.002387$

$Es = \text{actual} - S_8 \approx 1.236068 - 1.236147 \approx -0.000079$

7. $f(x) = \sqrt{1+x^3}$, $\Delta x = \dfrac{b-a}{n} = \dfrac{1-0}{4} = \dfrac{1}{4}$

(a) $T_4 = \frac{1}{4 \cdot 2}[f(0) + 2f(0.25) + 2f(0.5) + 2f(0.75) + f(1)] \approx 1.116993$

(b) $M_4 = \frac{1}{4}[f(0.125) + f(0.375) + f(0.625) + f(0.875)] \approx 1.108667$

(c) $S_4 = \frac{1}{4 \cdot 3}[f(0) + 4f(0.25) + 2f(0.5) + 4f(0.75) + f(1)] \approx 1.111363$

8. $f(x) = \sin\sqrt{x}$, $\Delta x = \dfrac{b-a}{n} = \dfrac{4-1}{6} = \dfrac{3}{6} = \dfrac{1}{2}$

(a) $T_6 = \frac{1}{2 \cdot 2}[f(1) + 2f(1.5) + 2f(2) + 2f(2.5) + 2f(3) + 2f(3.5) + f(4)] \approx 2.873085$

(b) $M_6 = \frac{1}{2}[f(1.25) + f(1.75) + f(2.25) + f(2.75) + f(3.25) + f(3.75)] \approx 2.884712$

(c) $S_6 = \frac{1}{2 \cdot 3}[f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + 2f(3) + 4f(3.5) + f(4)] \approx 2.880721$

9. $f(x) = \sqrt{e^x - 1}$, $\Delta x = \dfrac{b-a}{n} = \dfrac{1-0}{10} = \dfrac{1}{10}$

(a) $$T_{10} = \tfrac{1}{10 \cdot 2}[f(0) + 2f(0.1) + 2f(0.2) + 2f(0.3) + 2f(0.4) + 2f(0.5) + 2f(0.6) + 2f(0.7) + 2f(0.8) + 2f(0.9) + f(1)] \approx 0.777722$$

(b) $$M_{10} = \tfrac{1}{10}[f(0.05) + f(0.15) + f(0.25) + f(0.35) + f(0.45) + f(0.55) + f(0.65) + f(0.75) + f(0.85) + f(0.95)] \approx 0.784958$$

(c) $$S_{10} = \tfrac{1}{10 \cdot 3}[f(0) + 4f(0.1) + 2f(0.2) + 4f(0.3) + 2f(0.4) + 4f(0.5) + 2f(0.6) + 4f(0.7) + 2f(0.8) + 4f(0.9) + f(1)] \approx 0.780895$$

10. $f(x) = \sqrt[3]{1-x^2}$, $\Delta x = \dfrac{b-a}{n} = \dfrac{2-0}{10} = \dfrac{2}{10} = \dfrac{1}{5}$

(a) $$T_{10} = \tfrac{1}{5 \cdot 2}[f(0) + 2f(0.2) + 2f(0.4) + 2f(0.6) + 2f(0.8) + 2f(1) + 2f(1.2) + 2f(1.4) + 2f(1.6) + 2f(1.8) + f(2)] \approx -0.186646$$

(b) $$M_{10} = \tfrac{1}{5}[f(0.1) + f(0.3) + f(0.5) + f(0.7) + f(0.9) + f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9)] \approx -0.184073$$

(c) $$S_{10} = \tfrac{1}{5 \cdot 3}[f(0) + 4f(0.2) + 2f(0.4) + 4f(0.6) + 2f(0.8) + 4f(1) + 2f(1.2) + 4f(1.4) + 2f(1.6) + 4f(1.8) + f(2)] \approx -0.183984$$

11. $f(x) = e^{x+\cos x}$, $\Delta x = \frac{2-(-1)}{6} = \frac{1}{2}$

(a) $T_6 = \frac{1}{2}\,[f(-1.0) + 2f(-0.5) + 2f(0) + 2f(0.5) + 2f(1) + 2f(1.5) + f(2.0)] \approx 10.185560$

(b) $M_6 = \frac{1}{2}\,[f(-0.75) + f(-0.25) + f(0.25) + f(0.75) + f(1.25) + f(1.75)] \approx 10.208618$

(c) $S_6 = \frac{1}{2 \cdot 3}\,[f(-1.0) + 4f(-0.5) + 2f(0) + 4f(0.5) + 2f(1.0) + 4f(1.5) + f(2.0)] \approx 10.201790$

12. $f(x) = e^{1/x}$, $\Delta x = \frac{3-1}{8} = \frac{1}{4}$

(a) $T_8 = \frac{1}{4 \cdot 2}\left[f(1) + 2f\left(\frac{5}{4}\right) + 2f\left(\frac{3}{2}\right) + 2f\left(\frac{7}{4}\right) + 2f(2) + 2f\left(\frac{9}{4}\right) + 2f\left(\frac{5}{2}\right) + 2f\left(\frac{11}{4}\right) + f(3)\right] \approx 3.534934$

(b) $M_8 = \frac{1}{4}\left[f\left(\frac{9}{8}\right) + f\left(\frac{11}{8}\right) + f\left(\frac{13}{8}\right) + f\left(\frac{15}{8}\right) + f\left(\frac{17}{8}\right) + f\left(\frac{19}{8}\right) + f\left(\frac{21}{8}\right) + f\left(\frac{23}{8}\right)\right] \approx 3.515248$

(c) $S_8 = \frac{1}{4 \cdot 3}\left[f(1) + 4f\left(\frac{5}{4}\right) + 2f\left(\frac{3}{2}\right) + 4f\left(\frac{7}{4}\right) + 2f(2) + 4f\left(\frac{9}{4}\right) + 2f\left(\frac{5}{2}\right) + 4f\left(\frac{11}{4}\right) + f(3)\right] \approx 3.522375$

13. $f(y) = \sqrt{y}\cos y$, $\Delta y = \frac{4-0}{8} = \frac{1}{2}$

(a) $T_8 = \frac{1}{2 \cdot 2}\left[f(0) + 2f\left(\frac{1}{2}\right) + 2f(1) + 2f\left(\frac{3}{2}\right) + 2f(2) + 2f\left(\frac{5}{2}\right) + 2f(3) + 2f\left(\frac{7}{2}\right) + f(4)\right] \approx -2.364034$

(b) $M_8 = \frac{1}{2}\left[f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right) + f\left(\frac{5}{4}\right) + f\left(\frac{7}{4}\right) + f\left(\frac{9}{4}\right) + f\left(\frac{11}{4}\right) + f\left(\frac{13}{4}\right) + f\left(\frac{15}{4}\right)\right] \approx -2.310690$

(c) $S_8 = \frac{1}{2 \cdot 3}\left[f(0) + 4f\left(\frac{1}{2}\right) + 2f(1) + 4f\left(\frac{3}{2}\right) + 2f(2) + 4f\left(\frac{5}{2}\right) + 2f(3) + 4f\left(\frac{7}{2}\right) + f(4)\right] \approx -2.346520$

14. $f(t) = \dfrac{1}{\ln t}$, $\Delta t = \dfrac{3-2}{10} = \dfrac{1}{10}$

(a) $T_{10} = \frac{1}{10 \cdot 2}\{f(2) + 2[f(2.1) + f(2.2) + \cdots + f(2.9)] + f(3)\} \approx 1.119061$

(b) $M_{10} = \frac{1}{10}[f(2.05) + f(2.15) + \cdots + f(2.85) + f(2.95)] \approx 1.118107$

(c) $S_{10} = \frac{1}{10 \cdot 3}[f(2) + 4f(2.1) + 2f(2.2) + 4f(2.3) + 2f(2.4) + 4f(2.5) + 2f(2.6)$
$+ 4f(2.7) + 2f(2.8) + 4f(2.9) + f(3)] \approx 1.118428$

15. $f(x) = \dfrac{x^2}{1+x^4}$, $\Delta x = \dfrac{1-0}{10} = \dfrac{1}{10}$

(a) $T_{10} = \frac{1}{10 \cdot 2}\{f(0) + 2[f(0.1 + f(0.2) + \cdots + f(0.9)] + f(1)\} \approx 0.243747$

(b) $M_{10} = \frac{1}{10}[f(0.05) + f(0.15) + \cdots + f(0.85) + f(0.95)] \approx 0.243748$

(c) $S_{10} = \frac{1}{10 \cdot 3}[f(0) + 4f(0.1) + 2f(0.2) + 4f(0.3) + 2f(0.4) + 4f(0.5) + 2f(0.6)$
$+ 4f(0.7) + 2f(0.8) + 4f(0.9) + f(1)] \approx 0.243751$

Note: $\int_0^1 f(x)\,dx \approx 0.24374775$. This is a rare case where the Trapezoidal and Midpoint Rules give better approximations than Simpson's Rule.

16. $f(t) = \dfrac{\sin t}{t}$, $\Delta t = \dfrac{3-1}{4} = \dfrac{1}{2}$

(a) $T_4 = \frac{1}{2 \cdot 2}[f(1) + 2f(1.5) + 2f(2) + 2f(2.5) + f(3)] \approx 0.901645$

(b) $M_4 = \frac{1}{2}[f(1.25) + f(1.75) + f(2.25) + f(2.75)] \approx 0.903031$

(c) $S_4 = \frac{1}{2 \cdot 3}[f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + f(3)] \approx 0.902558$

17. $f(x) = \ln(1 + e^x)$, $\Delta x = \frac{4-0}{8} = \frac{1}{2}$

(a) $T_8 = \frac{1}{2 \cdot 2}\{f(0) + 2[f(0.5) + f(1) + \cdots + f(3) + f(3.5)] + f(4)\} \approx 8.814278$

(b) $M_8 = \frac{1}{2}[f(0.25) + f(0.75) + \cdots + f(3.25) + f(3.75)] \approx 8.799212$

(c) $S_8 = \frac{1}{2 \cdot 3}[f(0) + 4f(0.5) + 2f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + 2f(3) + 4f(3.5) + f(4)] \approx 8.804229$

18. $f(x) = \sqrt{x + x^3}$, $\Delta x = \frac{1-0}{10} = \frac{1}{10}$

(a) $T_{10} = \frac{1}{2 \cdot 2}\{f(0) + 2[f(0.1) + f(0.2) + \cdots + f(0.8) + f(0.9)] + f(1)\} \approx 0.787092$

(b) $M_{10} = \frac{1}{2}[f(0.05) + f(0.15) + \cdots + f(0.85) + f(0.95)] \approx 0.793821$

(c) $S_{10} = \frac{1}{2 \cdot 3}[f(0) + 4f(0.1) + 2f(0.2) + 4f(0.3) + 2f(0.4) + 4f(0.5) + 2f(0.6)$
$+ 4f(0.7) + 2f(0.8) + 4f(0.9) + f(1)]$
≈ 0.789915

19. $f(x) = \cos(x^2)$, $\Delta x = \frac{1-0}{8} = \frac{1}{8}$

(a) $T_8 = \frac{1}{8 \cdot 2}\{f(0) + 2\left[f\left(\frac{1}{8}\right) + f\left(\frac{2}{8}\right) + \cdots + f\left(\frac{7}{8}\right)\right] + f(1)\} \approx 0.902333$

$M_8 = \frac{1}{8}\left[f\left(\frac{1}{16}\right) + f\left(\frac{3}{16}\right) + f\left(\frac{5}{16}\right) + \cdots + f\left(\frac{15}{16}\right)\right] = 0.905620$

(b) $f(x) = \cos(x^2)$, $f'(x) = -2x\sin(x^2)$, $f''(x) = -2\sin(x^2) - 4x^2\cos(x^2)$. For $0 \le x \le 1$, sin and cos are positive, so $|f''(x)| = 2\sin(x^2) + 4x^2\cos(x^2) \le 2 \cdot 1 + 4 \cdot 1 \cdot 1 = 6$ since $\sin(x^2) \le 1$ and $\cos(x^2) \le 1$ for all x, and $x^2 \le 1$ for $0 \le x \le 1$. So for $n = 8$, we take $K = 6$, $a = 0$, and $b = 1$ in Theorem 3, to get $|E_T| \le 6 \cdot 1^3/(12 \cdot 8^2) = \frac{1}{128} = 0.0078125$ and $|E_M| \le \frac{1}{256} = 0.00390625$. [A better estimate is obtained by noting from a graph of f'' that $|f''(x)| \le 4$ for $0 \le x \le 1$.]

(c) Take $K = 6$ [as in part (b)] in Theorem 3. $|E_T| \le \dfrac{K(b-a)^3}{12n^2} \le 0.0001 \quad \Leftrightarrow \quad \dfrac{6(1-0)^3}{12n^2} \le 10^{-4} \quad \Leftrightarrow$ $\dfrac{1}{2n^2} \le \dfrac{1}{10^4} \quad \Leftrightarrow \quad 2n^2 \ge 10^4 \quad \Leftrightarrow \quad n^2 \ge 5000 \quad \Leftrightarrow \quad n \ge 71$. Take $n = 71$ for T_n. For E_M, again take $K = 6$ in Theorem 3 to get $|E_M| \le 10^{-4} \quad \Leftrightarrow \quad 4n^2 \ge 10^4 \quad \Leftrightarrow \quad n^2 \ge 2500 \quad \Leftrightarrow \quad n \ge 50$. Take $n = 50$ for M_n.

20. $f(x) = e^{1/x}$, $\Delta x = \frac{2-1}{10} = \frac{1}{10}$

(a) $T_{10} = \frac{1}{10 \cdot 2}[f(1) + 2f(1.1) + 2f(1.2) + \cdots + 2f(1.9) + f(2)] \approx 2.021976$

$M_{10} = \frac{1}{10}[f(1.05) + f(1.15) + f(1.25) + \cdots + f(1.95)] \approx 2.019102$

(b) $f(x) = e^{1/x}$, $f'(x) = -\dfrac{1}{x^2}e^{1/x}$, $f''(x) = \dfrac{2x+1}{x^4}\,e^{1/x}$. Now f'' is decreasing on $[1, 2]$, so let $x = 1$ to take $K = 3e$.

$|E_T| \le \dfrac{3e(2-1)^3}{12(10)^2} = \dfrac{e}{400} \approx 0.006796$. $\quad |E_M| \le \dfrac{|E_T|}{2} = \dfrac{e}{800} \approx 0.003398$.

(c) Take $K = 3e$ [as in part (b)] in Theorem 3. $|E_T| \le \dfrac{K(b-a)^3}{12n^2} \le 0.0001 \quad \Leftrightarrow \quad \dfrac{3e(2-1)^3}{12n^2} \le 10^{-4} \quad \Leftrightarrow$ $\dfrac{e}{4n^2} \le \dfrac{1}{10^4} \quad \Leftrightarrow \quad n^2 \ge \dfrac{10^4 e}{4} \quad \Leftrightarrow \quad n \ge 83$. Take $n = 83$ for T_n. For E_M, again take $K = 3e$ in Theorem 3 to get $|E_M| \le 10^{-4} \quad \Leftrightarrow \quad n^2 \ge \dfrac{10^4 e}{8} \quad \Leftrightarrow \quad n \ge 59$. Take $n = 59$ for M_n.

21. $f(x) = \sin x$, $\Delta x = \frac{\pi - 0}{10} = \frac{\pi}{10}$

(a) $T_{10} = \frac{\pi}{10 \cdot 2}\left[f(0) + 2f\left(\frac{\pi}{10}\right) + 2f\left(\frac{2\pi}{10}\right) + \cdots + 2f\left(\frac{9\pi}{10}\right) + f(\pi)\right] \approx 1.983524$

$M_{10} = \frac{\pi}{10}\left[f\left(\frac{\pi}{20}\right) + f\left(\frac{3\pi}{20}\right) + f\left(\frac{5\pi}{20}\right) + \cdots + f\left(\frac{19\pi}{20}\right)\right] \approx 2.008248$

$S_{10} = \frac{\pi}{10 \cdot 3}\left[f(0) + 4f\left(\frac{\pi}{10}\right) + 2f\left(\frac{2\pi}{10}\right) + 4f\left(\frac{3\pi}{10}\right) + \cdots + 4f\left(\frac{9\pi}{10}\right) + f(\pi)\right] \approx 2.000110$

Since $I = \int_0^\pi \sin x\, dx = \left[-\cos x\right]_0^\pi = 1 - (-1) = 2$, $E_T = I - T_{10} \approx 0.016476$, $E_M = I - M_{10} \approx -0.008248$, and $E_S = I - S_{10} \approx -0.000110$.

(b) $f(x) = \sin x \quad \Rightarrow \quad \left|f^{(n)}(x)\right| \leq 1$, so take $K = 1$ for all error estimates.

$|E_T| \leq \dfrac{K(b-a)^3}{12n^2} = \dfrac{1(\pi - 0)^3}{12(10)^2} = \dfrac{\pi^3}{1200} \approx 0.025839.$ $\quad |E_M| \leq \dfrac{|E_T|}{2} = \dfrac{\pi^3}{2400} \approx 0.012919.$

$|E_S| \leq \dfrac{K(b-a)^5}{180n^4} = \dfrac{1(\pi - 0)^5}{180(10)^4} = \dfrac{\pi^5}{1{,}800{,}000} \approx 0.000170.$

The actual error is about 64% of the error estimate in all three cases.

(c) $|E_T| \leq 0.00001 \quad \Leftrightarrow \quad \dfrac{\pi^3}{12n^2} \leq \dfrac{1}{10^5} \quad \Leftrightarrow \quad n^2 \geq \dfrac{10^5\pi^3}{12} \quad \Rightarrow \quad n \geq 508.3$. Take $n = 509$ for T_n.

$|E_M| \leq 0.00001 \quad \Leftrightarrow \quad \dfrac{\pi^3}{24n^2} \leq \dfrac{1}{10^5} \quad \Leftrightarrow \quad n^2 \geq \dfrac{10^5\pi^3}{24} \quad \Rightarrow \quad n \geq 359.4$. Take $n = 360$ for M_n.

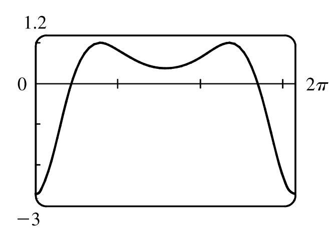
$|E_S| \leq 0.00001 \quad \Leftrightarrow \quad \dfrac{\pi^5}{180n^4} \leq \dfrac{1}{10^5} \quad \Leftrightarrow \quad n^4 \geq \dfrac{10^5\pi^5}{180} \quad \Rightarrow \quad n \geq 20.3.$

Take $n = 22$ for S_n (since n must be even).

22. From Example 7(b), we take $K = 76e$ to get $|E_S| \leq \dfrac{76e(1)^5}{180n^4} \leq 0.00001 \quad \Rightarrow \quad n^4 \geq \dfrac{76e}{180(0.00001)} \quad \Rightarrow \quad n \geq 18.4.$

Take $n = 20$ (since n must be even).

23. (a) Using a CAS, we differentiate $f(x) = e^{\cos x}$ twice, and find that $f''(x) = e^{\cos x}(\sin^2 x - \cos x)$. From the graph, we see that the maximum value of $|f''(x)|$ occurs at the endpoints of the interval $[0, 2\pi]$. Since $f''(0) = -e$, we can use $K = e$ or $K = 2.8$.



(b) A CAS gives $M_{10} \approx 7.954926518$. (In Maple, use `Student[Calculus1][RiemannSum]` or `Student[Calculus1][ApproximateInt]`.)

(c) Using Theorem 3 for the Midpoint Rule, with $K = e$, we get $|E_M| \leq \dfrac{e(2\pi - 0)^3}{24 \cdot 10^2} \approx 0.280945995.$

With $K = 2.8$, we get $|E_M| \leq \dfrac{2.8(2\pi - 0)^3}{24 \cdot 10^2} = 0.289391916.$

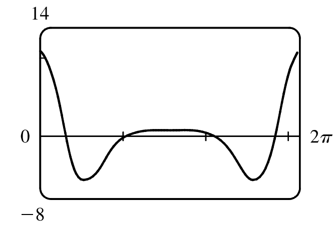
(d) A CAS gives $I \approx 7.954926521$.

(e) The actual error is only about 3×10^{-9}, much less than the estimate in part (c).

(f) We use the CAS to differentiate twice more, and then graph

$f^{(4)}(x) = e^{\cos x}(\sin^4 x - 6\sin^2 x\, \cos x + 3 - 7\sin^2 x + \cos x)$.

From the graph, we see that the maximum value of $\left|f^{(4)}(x)\right|$ occurs at the endpoints of the interval $[0, 2\pi]$. Since $f^{(4)}(0) = 4e$, we can use $K = 4e$ or $K = 10.9$.



(g) A CAS gives $S_{10} \approx 7.953789422$. (In Maple, use `Student[Calculus1][ApproximateInt]`.)

(h) Using Theorem 4 with $K = 4e$, we get $|E_S| \le \dfrac{4e(2\pi - 0)^5}{180 \cdot 10^4} \approx 0.059153618$.

With $K = 10.9$, we get $|E_S| \le \dfrac{10.9(2\pi - 0)^5}{180 \cdot 10^4} \approx 0.059299814$.

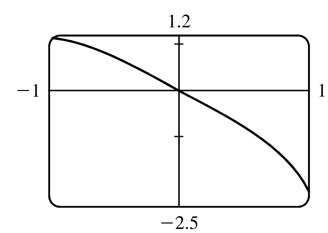
(i) The actual error is about $7.954926521 - 7.953789422 \approx 0.00114$. This is quite a bit smaller than the estimate in part (h), though the difference is not nearly as great as it was in the case of the Midpoint Rule.

(j) To ensure that $|E_S| \le 0.0001$, we use Theorem 4: $|E_S| \le \dfrac{4e(2\pi)^5}{180 \cdot n^4} \le 0.0001 \quad\Rightarrow\quad \dfrac{4e(2\pi)^5}{180 \cdot 0.0001} \le n^4 \quad\Rightarrow$

$n^4 \ge 5{,}915{,}362 \quad\Leftrightarrow\quad n \ge 49.3$. So we must take $n \ge 50$ to ensure that $|I - S_n| \le 0.0001$. ($K = 10.9$ leads to the same value of n.)

24. (a) Using the CAS, we differentiate $f(x) = \sqrt{4 - x^3}$ twice, and find that $f''(x) = -\dfrac{9x^4}{4(4 - x^3)^{3/2}} - \dfrac{3x}{(4 - x^3)^{1/2}}$.

From the graph, we see that $|f''(x)| < 2.2$ on $[-1, 1]$.



(b) A CAS gives $M_{10} \approx 3.995804152$. (In Maple, use `Student[Calculus1][RiemannSum]` or `Student[Calculus1][ApproximateInt]`.)

(c) Using Theorem 3 for the Midpoint Rule, with $K = 2.2$, we get $|E_M| \le \dfrac{2.2\,[1 - (-1)]^3}{24 \cdot 10^2} \approx 0.00733$.

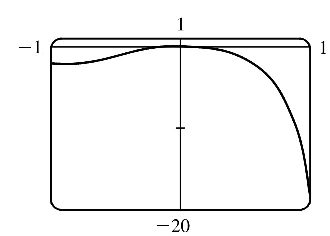
(d) A CAS gives $I \approx 3.995487677$.

(e) The actual error is about -0.0003165, much less than the estimate in part (c).

(f) We use the CAS to differentiate twice more, and then graph

$f^{(4)}(x) = \dfrac{9}{16}\,\dfrac{x^2(x^6 - 224x^3 - 1280)}{(4 - x^3)^{7/2}}$.

From the graph, we see that $\left|f^{(4)}(x)\right| < 18.1$ on $[-1, 1]$.



(g) A CAS gives $S_{10} \approx 3.995449790$. (In Maple, use `Student[Calculus1][ApproximateInt]`.)

(h) Using Theorem 4 with $K = 18.1$, we get $|E_S| \le \dfrac{18.1\,[1 - (-1)]^5}{180 \cdot 10^4} \approx 0.000322$.

(i) The actual error is about $3.995487677 - 3.995449790 \approx 0.0000379$. This is quite a bit smaller than the estimate in part (h).

(j) To ensure that $|E_S| \le 0.0001$, we use Theorem 4: $|E_S| \le \dfrac{18.1(2)^5}{180 \cdot n^4} \le 0.0001 \quad\Rightarrow\quad \dfrac{18.1(2)^5}{180 \cdot 0.0001} \le n^4 \quad\Rightarrow$

$n^4 \ge 32{,}178 \quad\Rightarrow n \ge 13.4$. So we must take $n \ge 14$ to ensure that $|I - S_n| \le 0.0001$.

25. $I = \int_0^1 xe^x\,dx = [(x-1)e^x]_0^1$ [parts or Formula 96] $= 0 - (-1) = 1$, $f(x) = xe^x$, $\Delta x = 1/n$

$n = 5$: $L_5 = \frac{1}{5}[f(0) + f(0.2) + f(0.4) + f(0.6) + f(0.8)] \approx 0.742943$

$R_5 = \frac{1}{5}[f(0.2) + f(0.4) + f(0.6) + f(0.8) + f(1)] \approx 1.286599$

$T_5 = \frac{1}{5 \cdot 2}[f(0) + 2f(0.2) + 2f(0.4) + 2f(0.6) + 2f(0.8) + f(1)] \approx 1.014771$

$M_5 = \frac{1}{5}[f(0.1) + f(0.3) + f(0.5) + f(0.7) + f(0.9)] \approx 0.992621$

$E_L = I - L_5 \approx 1 - 0.742943 = 0.257057$
$E_R \approx 1 - 1.286599 = -0.286599$
$E_T \approx 1 - 1.014771 = -0.014771$
$E_M \approx 1 - 0.992621 = 0.007379$

$n = 10$: $L_{10} = \frac{1}{10}[f(0) + f(0.1) + f(0.2) + \cdots + f(0.9)] \approx 0.867782$

$R_{10} = \frac{1}{10}[f(0.1) + f(0.2) + \cdots + f(0.9) + f(1)] \approx 1.139610$

$T_{10} = \frac{1}{10 \cdot 2}\{f(0) + 2[f(0.1) + f(0.2) + \cdots + f(0.9)] + f(1)\} \approx 1.003696$

$M_{10} = \frac{1}{10}[f(0.05) + f(0.15) + \cdots + f(0.85) + f(0.95)] \approx 0.998152$

$E_L = I - L_{10} \approx 1 - 0.867782 = 0.132218$
$E_R \approx 1 - 1.139610 = -0.139610$
$E_T \approx 1 - 1.003696 = -0.003696$
$E_M \approx 1 - 0.998152 = 0.001848$

$n = 20$: $L_{20} = \frac{1}{20}[f(0) + f(0.05) + f(0.10) + \cdots + f(0.95)] \approx 0.932967$

$R_{20} = \frac{1}{20}[f(0.05) + f(0.10) + \cdots + f(0.95) + f(1)] \approx 1.068881$

$T_{20} = \frac{1}{20 \cdot 2}\{f(0) + 2[f(0.05) + f(0.10) + \cdots + f(0.95)] + f(1)\} \approx 1.000924$

$M_{20} = \frac{1}{20}[f(0.025) + f(0.075) + f(0.125) + \cdots + f(0.975)] \approx 0.999538$

$E_L = I - L_{20} \approx 1 - 0.932967 = 0.067033$
$E_R \approx 1 - 1.068881 = -0.068881$
$E_T \approx 1 - 1.000924 = -0.000924$
$E_M \approx 1 - 0.999538 = 0.000462$

n	L_n	R_n	T_n	M_n
5	0.742943	1.286599	1.014771	0.992621
10	0.867782	1.139610	1.003696	0.998152
20	0.932967	1.068881	1.000924	0.999538

n	E_L	E_R	E_T	E_M
5	0.257057	−0.286599	−0.014771	0.007379
10	0.132218	−0.139610	−0.003696	0.001848
20	0.067033	−0.068881	−0.000924	0.000462

Observations:

1. E_L and E_R are always opposite in sign, as are E_T and E_M.
2. As n is doubled, E_L and E_R are decreased by about a factor of 2, and E_T and E_M are decreased by a factor of about 4.
3. The Midpoint approximation is about twice as accurate as the Trapezoidal approximation.
4. All the approximations become more accurate as the value of n increases.
5. The Midpoint and Trapezoidal approximations are much more accurate than the endpoint approximations.

26. $I = \int_1^2 \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^2 = -\frac{1}{2} - (-1) = \frac{1}{2}, f(x) = \frac{1}{x^2}, \Delta x = \frac{1}{n}$

$n = 5$: $L_5 = \frac{1}{5}[f(1) + f(1.2) + f(1.4) + f(1.6) + f(1.8)] \approx 0.580783$

$R_5 = \frac{1}{5}[f(1.2) + f(1.4) + f(1.6) + f(1.8) + f(2)] \approx 0.430783$

$T_5 = \frac{1}{5 \cdot 2}[f(1) + 2f(1.2) + 2f(1.4) + 2f(1.6) + 2f(1.8) + f(2)] \approx 0.505783$

$M_5 = \frac{1}{5}[f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9)] \approx 0.497127$

$E_L = I - L_5 \approx \frac{1}{2} - 0.580783 = -0.080783$

$E_R \approx \frac{1}{2} - 0.430783 = 0.069217$

$E_T \approx \frac{1}{2} - 0.505783 = -0.005783$

$E_M \approx \frac{1}{2} - 0.497127 = 0.002873$

$n = 10$: $L_{10} = \frac{1}{10}[f(1) + f(1.1) + f(1.2) + \cdots + f(1.9)] \approx 0.538955$

$R_{10} = \frac{1}{10}[f(1.1) + f(1.2) + \cdots + f(1.9) + f(2)] \approx 0.463955$

$T_{10} = \frac{1}{10 \cdot 2}\{f(1) + 2[f(1.1) + f(1.2) + \cdots + f(1.9)] + f(2)\} \approx 0.501455$

$M_{10} = \frac{1}{10}[f(1.05) + f(1.15) + \cdots + f(1.85) + f(1.95)] \approx 0.499274$

$E_L = I - L_{10} \approx \frac{1}{2} - 0.538955 = -0.038955$

$E_R \approx \frac{1}{2} - 0.463955 = 0.036049$

$E_T \approx \frac{1}{2} - 0.501455 = -0.001455$

$E_M \approx \frac{1}{2} - 0.499274 = 0.000726$

$n = 20$: $L_{20} = \frac{1}{20}[f(1) + f(1.05) + f(1.10) + \cdots + f(1.95)] \approx 0.519114$

$R_{20} = \frac{1}{20}[f(1.05) + f(1.10) + \cdots + f(1.95) + f(2)] \approx 0.481614$

$T_{20} = \frac{1}{20 \cdot 2}\{f(1) + 2[f(1.05) + f(1.10) + \cdots + f(1.95)] + f(2)\} \approx 0.500364$

$M_{20} = \frac{1}{20}[f(1.025) + f(1.075) + f(1.125) + \cdots + f(1.975)] \approx 0.499818$

$E_L = I - L_{20} \approx \frac{1}{2} - 0.519114 = -0.019114$

$E_R \approx \frac{1}{2} - 0.481614 = 0.018386$

$E_T \approx \frac{1}{2} - 0.500364 = -0.000364$

$E_M \approx \frac{1}{2} - 0.499818 = 0.000182$

n	L_n	R_n	T_n	M_n
5	0.580783	0.430783	0.505783	0.497127
10	0.538955	0.463955	0.501455	0.499274
20	0.519114	0.481614	0.500364	0.499818

n	E_L	E_R	E_T	E_M
5	−0.080783	0.069217	−0.005783	0.002873
10	−0.038955	0.036049	−0.001455	0.000726
20	−0.019114	0.018386	−0.000364	0.000182

[continued]

Observations:

1. E_L and E_R are always opposite in sign, as are E_T and E_M.
2. As n is doubled, E_L and E_R are decreased by about a factor of 2, and E_T and E_M are decreased by a factor of about 4.
3. The Midpoint approximation is about twice as accurate as the Trapezoidal approximation.
4. All the approximations become more accurate as the value of n increases.
5. The Midpoint and Trapezoidal approximations are much more accurate than the endpoint approximations.

27. $I = \int_0^2 x^4\,dx = \left[\frac{1}{5}x^5\right]_0^2 = \frac{32}{5} - 0 = 6.4$, $f(x) = x^4$, $\Delta x = \frac{2-0}{n} = \frac{2}{n}$

$n = 6$:
$$T_6 = \tfrac{2}{6\cdot 2}\left\{f(0) + 2\left[f\left(\tfrac{1}{3}\right) + f\left(\tfrac{2}{3}\right) + f\left(\tfrac{3}{3}\right) + f\left(\tfrac{4}{3}\right) + f\left(\tfrac{5}{3}\right)\right] + f(2)\right\} \approx 6.695473$$
$$M_6 = \tfrac{2}{6}\left[f\left(\tfrac{1}{6}\right) + f\left(\tfrac{3}{6}\right) + f\left(\tfrac{5}{6}\right) + f\left(\tfrac{7}{6}\right) + f\left(\tfrac{9}{6}\right) + f\left(\tfrac{11}{6}\right)\right] \approx 6.252572$$
$$S_6 = \tfrac{2}{6\cdot 3}\left[f(0) + 4f\left(\tfrac{1}{3}\right) + 2f\left(\tfrac{2}{3}\right) + 4f\left(\tfrac{3}{3}\right) + 2f\left(\tfrac{4}{3}\right) + 4f\left(\tfrac{5}{3}\right) + f(2)\right] \approx 6.403292$$
$$E_T = I - T_6 \approx 6.4 - 6.695473 = -0.295473$$
$$E_M \approx 6.4 - 6.252572 = 0.147428$$
$$E_S \approx 6.4 - 6.403292 = -0.003292$$

$n = 12$:
$$T_{12} = \tfrac{2}{12\cdot 2}\left\{f(0) + 2\left[f\left(\tfrac{1}{6}\right) + f\left(\tfrac{2}{6}\right) + f\left(\tfrac{3}{6}\right) + \cdots + f\left(\tfrac{11}{6}\right)\right] + f(2)\right\} \approx 6.474023$$
$$M_6 = \tfrac{2}{12}\left[f\left(\tfrac{1}{12}\right) + f\left(\tfrac{3}{12}\right) + f\left(\tfrac{5}{12}\right) + \cdots + f\left(\tfrac{23}{12}\right)\right] \approx 6.363008$$
$$S_6 = \tfrac{2}{12\cdot 3}\left[f(0) + 4f\left(\tfrac{1}{6}\right) + 2f\left(\tfrac{2}{6}\right) + 4f\left(\tfrac{3}{6}\right) + 2f\left(\tfrac{4}{6}\right) + \cdots + 4f\left(\tfrac{11}{6}\right) + f(2)\right] \approx 6.400206$$
$$E_T = I - T_{12} \approx 6.4 - 6.474023 = -0.074023$$
$$E_M \approx 6.4 - 6.363008 = 0.036992$$
$$E_S \approx 6.4 - 6.400206 = -0.000206$$

n	T_n	M_n	S_n
6	6.695473	6.252572	6.403292
12	6.474023	6.363008	6.400206

n	E_T	E_M	E_S
6	-0.295473	0.147428	-0.003292
12	-0.074023	0.036992	-0.000206

Observations:

1. E_T and E_M are opposite in sign and decrease by a factor of about 4 as n is doubled.
2. The Simpson's approximation is much more accurate than the Midpoint and Trapezoidal approximations, and E_S seems to decrease by a factor of about 16 as n is doubled.

28. $I = \displaystyle\int_1^4 \frac{1}{\sqrt{x}}\,dx = \left[2\sqrt{x}\right]_1^4 = 4 - 2 = 2$, $f(x) = \dfrac{1}{\sqrt{x}}$, $\Delta x = \dfrac{4-1}{n} = \dfrac{3}{n}$

$n = 6$:
$$T_6 = \tfrac{3}{6\cdot 2}\left\{f(1) + 2\left[f\left(\tfrac{3}{2}\right) + f\left(\tfrac{4}{2}\right) + f\left(\tfrac{5}{2}\right) + f\left(\tfrac{6}{2}\right) + f\left(\tfrac{7}{2}\right)\right] + f(4)\right\} \approx 2.008966$$
$$M_6 = \tfrac{3}{6}\left[f\left(\tfrac{5}{4}\right) + f\left(\tfrac{7}{4}\right) + f\left(\tfrac{9}{4}\right) + f\left(\tfrac{11}{4}\right) + f\left(\tfrac{13}{4}\right) + f\left(\tfrac{15}{4}\right)\right] \approx 1.995572$$
$$S_6 = \tfrac{3}{6\cdot 3}\left[f(1) + 4f\left(\tfrac{3}{2}\right) + 2f\left(\tfrac{4}{2}\right) + 4f\left(\tfrac{5}{2}\right) + 2f\left(\tfrac{6}{2}\right) + 4f\left(\tfrac{7}{2}\right) + f(4)\right] \approx 2.000469$$
$$E_T = I - T_6 \approx 2 - 2.008966 = -0.008966,$$
$$E_M \approx 2 - 1.995572 = 0.004428,$$
$$E_S \approx 2 - 2.000469 = -0.000469$$

[continued]

$n = 12$: $T_{12} = \frac{3}{12 \cdot 2}\left\{f(1) + 2\left[f\left(\frac{5}{4}\right) + f\left(\frac{6}{4}\right) + f\left(\frac{7}{4}\right) + \cdots + f\left(\frac{15}{4}\right)\right] + f(4)\right\} \approx 2.002269$

$M_{12} = \frac{3}{12}\left[f\left(\frac{9}{8}\right) + f\left(\frac{11}{8}\right) + f\left(\frac{13}{8}\right) + \cdots + f\left(\frac{31}{8}\right)\right] \approx 1.998869$

$S_{12} = \frac{3}{12 \cdot 3}\left[f(1) + 4f\left(\frac{5}{4}\right) + 2f\left(\frac{6}{4}\right) + 4f\left(\frac{7}{4}\right) + 2f\left(\frac{8}{4}\right) + \cdots + 4f\left(\frac{15}{4}\right) + f(4)\right] \approx 2.000036$

$E_T = I - T_{12} \approx 2 - 2.002269 = -0.002269$

$E_M \approx 2 - 1.998869 = 0.001131$

$E_S \approx 2 - 2.000036 = -0.000036$

n	T_n	M_n	S_n
6	2.008966	1.995572	2.000469
12	2.002269	1.998869	2.000036

n	E_T	E_M	E_S
6	-0.008966	0.004428	-0.000469
12	-0.002269	0.001131	-0.000036

Observations:

1. E_T and E_M are opposite in sign and decrease by a factor of about 4 as n is doubled.
2. The Simpson's approximation is much more accurate than the Midpoint and Trapezoidal approximations, and E_S seems to decrease by a factor of about 16 as n is doubled.

29. (a) $\Delta x = (b - a)/n = (6 - 0)/6 = 1$

$T_6 = \frac{1}{2}[f(0) + 2f(1) + 2f(2) + 2f(3) + 2f(4) + 2f(5) + f(6)]$

$\approx \frac{1}{2}[2 + 2(1) + 2(3) + 2(5) + 2(4) + 2(3) + 4] = \frac{1}{2}(38) = 19$

(b) $M_6 = 1[f(0.5) + f(1.5) + f(2.5) + f(3.5) + f(4.5) + f(5.5)] \approx 1.3 + 1.5 + 4.6 + 4.7 + 3.3 + 3.2 = 18.6$

(c) $S_6 = \frac{1}{3}[f(0) + 4f(1) + 2f(2) + 4f(3) + 2f(4) + 4f(5) + f(6)]$

$\approx \frac{1}{3}[2 + 4(1) + 2(3) + 4(5) + 2(4) + 4(3) + 4] = \frac{1}{3}(56) = 18.\overline{6}$

30. If x = distance from left end of pool and $w = w(x)$ = width at x, then Simpson's Rule with $n = 8$ and $\Delta x = 2$ gives

Area $= \int_0^{16} w\,dx \approx \frac{2}{3}[0 + 4(6.2) + 2(7.2) + 4(6.8) + 2(5.6) + 4(5.0) + 2(4.8) + 4(4.8) + 0] \approx 84\text{ m}^2$.

31. (a) $\int_1^5 f(x)\,dx \approx M_4 = \frac{5-1}{4}[f(1.5) + f(2.5) + f(3.5) + f(4.5)] = 1(2.9 + 3.6 + 4.0 + 3.9) = 14.4$

(b) $-2 \le f''(x) \le 3 \;\Rightarrow\; |f''(x)| \le 3 \;\Rightarrow\; K = 3$, since $|f''(x)| \le K$. The error estimate for the Midpoint Rule is

$$|E_M| \le \frac{K(b-a)^3}{24n^2} = \frac{3(5-1)^3}{24(4)^2} = \frac{1}{2}.$$

32. (a) $\int_0^{1.6} g(x)\,dx \approx S_8 = \frac{1.6-0}{8 \cdot 3}[g(0) + 4g(0.2) + 2g(0.4) + 4g(0.6) + 2g(0.8) + 4g(1.0) + 2g(1.2) + 4g(1.4) + g(1.6)]$

$= \frac{1}{15}[12.1 + 4(11.6) + 2(11.3) + 4(11.1) + 2(11.7) + 4(12.2) + 2(12.6) + 4(13.0) + 13.2]$

$= \frac{1}{15}(288.1) = \frac{2881}{150} \approx 19.2$

(b) $-5 \le g^{(4)}(x) \le 2 \;\Rightarrow\; \left|g^{(4)}(x)\right| \le 5 \;\Rightarrow\; K = 5$, since $\left|g^{(4)}(x)\right| \le K$. The error estimate for Simpson's Rule is

$$|E_S| \le \frac{K(b-a)^5}{180n^4} = \frac{5(1.6-0)^5}{180(8)^4} = \frac{2}{28{,}125} = 7.\overline{1} \times 10^{-5}.$$

33. We use Simpson's Rule with $n = 12$ and $\Delta t = \frac{24-0}{12} = 2$.

$$\begin{aligned} S_{12} &= \frac{2}{3}[T(0) + 4T(2) + 2T(4) + 4T(6) + 2T(8) + 4T(10) + 2T(12) \\ &\quad +4T(14) + 2T(16) + 4T(18) + 2T(20) + 4T(22) + T(24)] \\ &\approx \frac{2}{3}[18.5 + 4 \times 18 + 2 \times 17 + 4 \times 16 + 2 \times 18.75 + 4 \times 21.25 + 2 \times 22.5 \\ &\quad +4 \times 23.75 + 2 \times 24 + 4 \times 23 + 2 \times 23 + 4 \times 21 + 18.75] \approx 493.17 \end{aligned}$$

Thus, $\int_0^{24} T(t)dt \approx S_{12}$ and $T_{ave} = \frac{1}{24-0}\int_0^{24} T(t)dt \approx 20.5°\text{C}$

34. We use Simpson's Rule with $n = 10$ and $\Delta x = \frac{1}{2}$:

$$\begin{aligned} \text{distance} = \int_0^5 v(t)\,dt \approx S_{10} &= \tfrac{1}{2 \cdot 3}[f(0) + 4f(0.5) + 2f(1) + \cdots + 4f(4.5) + f(5)] \\ &= \tfrac{1}{6}[0 + 4(4.67) + 2(7.34) + 4(8.86) + 2(9.73) + 4(10.22) \\ &\qquad + 2(10.51) + 4(10.67) + 2(10.76) + 4(10.81) + 10.81] \\ &= \tfrac{1}{6}(268.41) = 44.735 \text{ m} \end{aligned}$$

35. By the Net Change Theorem, the increase in velocity is equal to $\int_0^6 a(t)dt$. We use Simpson's Rule with $n = 6$ and $\Delta t = \frac{6-0}{6} = 1$ to estimate this integral:

$$\begin{aligned} \int_0^t a(t)dt \approx S_6 &= \frac{1}{3}[a(0) + 4a(1) + 2a(2) + 4a(3) + 2a(4) + 4a(5) + a(6)] \\ &\approx \frac{1}{3}[0 + 4 \times 0.25 + 2 \times 2.1 + 4 \times 5 + 2 \times 6.5 + 4 \times 4.75 + 0] = \frac{1}{3}(57.2) \approx 19.1 \text{ m/s} \end{aligned}$$

36. By the Net Change Theorem, the total amount of water that leaked out during the first six hours is equal to $\int_0^6 r(t)\,dt$.

We use Simpson's Rule with $n = 6$ and $\Delta t = \frac{6-0}{6} = 1$ to estimate this integral:

$$\begin{aligned} \int_0^6 r(t)\,dt \approx S_6 &= \tfrac{1}{3}[r(0) + 4r(1) + 2r(2) + 4r(3) + 2r(4) + 4r(5) + r(6)] \\ &\approx \tfrac{1}{3}[4 + 4(3) + 2(2.4) + 4(1.9) + 2(1.4) + 4(1.1) + 1] = \tfrac{1}{3}(36.6) = 12.2 \text{ liters} \end{aligned}$$

37. By the Net Change Theorem, the energy used is equal to $\int_0^6 P(t)\,dt$. We use Simpson's Rule with $n = 12$ and $\Delta t = \frac{6-0}{12} = \frac{1}{2}$ to estimate this integral:

$$\begin{aligned} \int_0^6 P(t)\,dt \approx S_{12} &= \tfrac{1/2}{3}[P(0) + 4P(0.5) + 2P(1) + 4P(1.5) + 2P(2) + 4P(2.5) + 2P(3) \\ &\qquad + 4P(3.5) + 2P(4) + 4P(4.5) + 2P(5) + 4P(5.5) + P(6)] \\ &= \tfrac{1}{6}[1814 + 4(1735) + 2(1686) + 4(1646) + 2(1637) + 4(1609) + 2(1604) \\ &\qquad + 4(1611) + 2(1621) + 4(1666) + 2(1745) + 4(1886) + 2052] \\ &= \tfrac{1}{6}(61{,}064) = 10{,}177.\overline{3} \text{ megawatt-hours} \end{aligned}$$

38. By the Net Change Theorem, the total amount of data transmitted is equal to $\int_0^8 D(t)\,dt \times 3600$ [since $D(t)$ is measured in megabits per second and t is in hours]. We use Simpson's Rule with $n = 8$ and $\Delta t = (8-0)/8 = 1$ to estimate this integral:

$$\begin{aligned} \int_0^8 D(t)\,dt \approx S_8 &= \tfrac{1}{3}[D(0) + 4D(1) + 2D(2) + 4D(3) + 2D(4) + 4D(5) + 2D(6) + 4D(7) + D(8)] \\ &\approx \tfrac{1}{3}[0.35 + 4(0.32) + 2(0.41) + 4(0.50) + 2(0.51) + 4(0.56) + 2(0.56) + 4(0.83) + 0.88] \\ &= \tfrac{1}{3}(13.03) = 4.34\overline{3} \end{aligned}$$

Now multiply by 3600 to obtain 15,636 megabits.

39. (a) Let $y = f(x)$ denote the curve. Using disks, $V = \int_2^{10} \pi[f(x)]^2\,dx = \pi\int_2^{10} g(x)\,dx = \pi I_1$.

Now use Simpson's Rule to approximate I_1:

$$\begin{aligned} I_1 \approx S_8 &= \tfrac{10-2}{3(8)}[g(2) + 4g(3) + 2g(4) + 4g(5) + 2g(6) + 4g(7) + g(8)] \\ &\approx \tfrac{1}{3}[0^2 + 4(1.5)^2 + 2(1.9)^2 + 4(2.2)^2 + 2(3.0)^2 + 4(3.8)^2 + 2(4.0)^2 + 4(3.1)^2 + 0^2] \\ &= \tfrac{1}{3}(181.78) \end{aligned}$$

Thus, $V \approx \pi \cdot \frac{1}{3}(181.78) \approx 190.4$ or 190 cubic units.

(b) Using cylindrical shells, $V = \int_2^{10} 2\pi x f(x)\,dx = 2\pi\int_2^{10} x f(x)\,dx = 2\pi I_1$.

Now use Simpson's Rule to approximate I_1:

$$\begin{aligned} I_1 \approx S_8 &= \tfrac{10-2}{3(8)}[2f(2) + 4\cdot 3f(3) + 2\cdot 4f(4) + 4\cdot 5f(5) + 2\cdot 6f(6) \\ &\qquad + 4\cdot 7f(7) + 2\cdot 8f(8) + 4\cdot 9f(9) + 10f(10)] \\ &\approx \tfrac{1}{3}[2(0) + 12(1.5) + 8(1.9) + 20(2.2) + 12(3.0) + 28(3.8) + 16(4.0) + 36(3.1) + 10(0)] \\ &= \tfrac{1}{3}(395.2) \end{aligned}$$

Thus, $V \approx 2\pi \cdot \frac{1}{3}(395.2) \approx 827.7$ or 828 cubic units.

40. $$\begin{aligned} \text{Work} = \int_0^{18} f(x)\,dx \approx S_6 &= \tfrac{18-0}{6\cdot 3}\,[f(0) + 4f(3) + 2f(6) + 4f(9) + 2f(12) + 4f(15) + f(18)] \\ &= 1\cdot[9.8 + 4(9.1) + 2(8.5) + 4(8.0) + 2(7.7) + 4(7.5) + 7.4] = 148 \text{ joules} \end{aligned}$$

41. The curve is $y = f(x) = 1/(1 + e^{-x})$. Using disks, $V = \int_0^{10} \pi[f(x)]^2\,dx = \pi\int_0^{10} g(x)\,dx = \pi I_1$. Now use Simpson's Rule to approximate I_1:

$$\begin{aligned} I_1 \approx S_{10} &= \tfrac{10-0}{10\cdot 3}[g(0) + 4g(1) + 2g(2) + 4g(3) + 2g(4) + 4g(5) + 2g(6) + 4g(7) + 2g(8) + 4g(9) + g(10)] \\ &\approx 8.80825 \end{aligned}$$

Thus, $V \approx \pi I_1 \approx 27.7$ or 28 cubic units.

42. Using Simpson's Rule with $n = 10$, $\Delta x = \frac{\pi/2}{10}$, $L = 1$, $\theta_0 = \frac{42\pi}{180}$ radians, $g = 9.8\text{ m/s}^2$, $k^2 = \sin^2\left(\frac{1}{2}\theta_0\right)$, and $f(x) = 1/\sqrt{1 - k^2\sin^2 x}$, we get

$$\begin{aligned} T &= 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\frac{dx}{\sqrt{1 - k^2\sin^2 x}} \approx 4\sqrt{\frac{L}{g}}\,S_{10} \\ &= 4\sqrt{\tfrac{1}{9.8}}\left(\tfrac{\pi/2}{10\cdot 3}\right)\left[f(0) + 4f\left(\tfrac{\pi}{20}\right) + 2f\left(\tfrac{2\pi}{20}\right) + \cdots + 4f\left(\tfrac{9\pi}{20}\right) + f\left(\tfrac{\pi}{2}\right)\right] \approx 2.07665 \end{aligned}$$

43. $I(\theta) = \dfrac{N^2\sin^2 k}{k^2}$, where $k = \dfrac{\pi N d\sin\theta}{\lambda}$, $N = 10{,}000$, $d = 10^{-4}$, and $\lambda = 632.8\times 10^{-9}$. So $I(\theta) = \dfrac{(10^4)^2\sin^2 k}{k^2}$, where $k = \dfrac{\pi(10^4)(10^{-4})\sin\theta}{632.8\times 10^{-9}}$. Now $n = 10$ and $\Delta\theta = \dfrac{10^{-6} - (-10^{-6})}{10} = 2\times 10^{-7}$, so

$M_{10} = 2\times 10^{-7}[I(-0.0000009) + I(-0.0000007) + \cdots + I(0.0000009)] \approx 59.4$.

44. $f(x) = \cos(\pi x)$, $\Delta x = \frac{20-0}{10} = 2 \quad \Rightarrow$

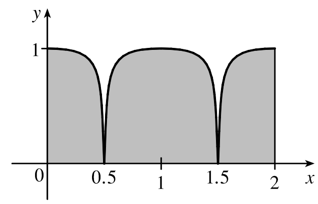
$$\begin{aligned} T_{10} &= \tfrac{2}{2}\{f(0) + 2[f(2) + f(4) + \cdots + f(18)] + f(20)\} = 1[\cos 0 + 2(\cos 2\pi + \cos 4\pi + \cdots + \cos 18\pi) + \cos 20\pi] \\ &= 1 + 2(1+1+1+1+1+1+1+1+1) + 1 = 20 \end{aligned}$$

The actual value is $\int_0^{20} \cos(\pi x)\,dx = \frac{1}{\pi}\big[\sin \pi x\big]_0^{20} = \frac{1}{\pi}(\sin 20\pi - \sin 0) = 0$. The discrepancy is due to the fact that the function is sampled only at points of the form $2n$, where its value is $f(2n) = \cos(2n\pi) = 1$.

45. Consider the function f whose graph is shown. The area $\int_0^2 f(x)\,dx$ is close to 2. The Trapezoidal Rule gives

$T_2 = \frac{2-0}{2\cdot 2}\,[f(0) + 2f(1) + f(2)] = \frac{1}{2}\,[1 + 2\cdot 1 + 1] = 2.$

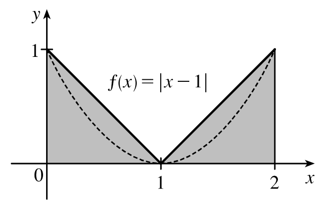
The Midpoint Rule gives $M_2 = \frac{2-0}{2}\,[f(0.5) + f(1.5)] = 1[0+0] = 0$, so the Trapezoidal Rule is more accurate.



46. Consider the function $f(x) = |x-1|$, $0 \le x \le 2$. The area $\int_0^2 f(x)\,dx$ is exactly 1. So is the right endpoint approximation:

$R_2 = f(1)\,\Delta x + f(2)\,\Delta x = 0\cdot 1 + 1\cdot 1 = 1$. But Simpson's Rule approximates f with the parabola $y = (x-1)^2$, shown dashed, and

$$S_2 = \frac{\Delta x}{3}\,[f(0) + 4f(1) + f(2)] = \frac{1}{3}\,[1 + 4\cdot 0 + 1] = \frac{2}{3}.$$



47. Since the Trapezoidal and Midpoint approximations on the interval $[a,b]$ are the sums of the Trapezoidal and Midpoint approximations on the subintervals $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$, we can focus our attention on one such interval. The condition $f''(x) < 0$ for $a \le x \le b$ means that the graph of f is concave down as in Figure 5. In that figure, T_n is the area of the trapezoid $AQRD$, $\int_a^b f(x)\,dx$ is the area of the region $AQPRD$, and M_n is the area of the trapezoid $ABCD$, so $T_n < \int_a^b f(x)\,dx < M_n$. In general, the condition $f'' < 0$ implies that the graph of f on $[a,b]$ lies above the chord joining the points $(a, f(a))$ and $(b, f(b))$. Thus, $\int_a^b f(x)\,dx > T_n$. Since M_n is the area under a tangent to the graph, and since $f'' < 0$ implies that the tangent lies above the graph, we also have $M_n > \int_a^b f(x)\,dx$. Thus, $T_n < \int_a^b f(x)\,dx < M_n$.

48. (a) Let f be a polynomial of degree ≤ 3; say $f(x) = Ax^3 + Bx^2 + Cx + D$. It will suffice to show that Simpson's estimate is exact when there are two subintervals ($n = 2$), because for a larger even number of subintervals the sum of exact estimates is exact. As in the derivation of Simpson's Rule, we can assume that $x_0 = -h$, $x_1 = 0$, and $x_2 = h$. Then Simpson's approximation is

$$\begin{aligned} \int_{-h}^{h} f(x)\,dx &\approx \tfrac{1}{3}\cdot\tfrac{2h}{2}[f(-h) + 4f(0) + f(h)] = \tfrac{1}{3}h\big[\big(-Ah^3 + Bh^2 - Ch + D\big) + 4D + \big(Ah^3 + Bh^2 + Ch + D\big)\big] \\ &= \tfrac{1}{3}h[2Bh^2 + 6D] = \tfrac{2}{3}Bh^3 + 2Dh \end{aligned}$$

The exact value of the integral is

$$\begin{aligned} \int_{-h}^{h} (Ax^3 + Bx^2 + Cx + D)\,dx &= 2\int_0^h (Bx^2 + D)\,dx \qquad \text{[by Theorem 4.5.6]} \\ &= 2\big[\tfrac{1}{3}Bx^3 + Dx\big]_0^h = \tfrac{2}{3}Bh^3 + 2Dh \end{aligned}$$

Thus, Simpson's Rule is exact.

(b) Using Simpson's Rule with $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{8-0}{4} = 2$, and $f(x) = x^3 - 6x^2 + 4x$, we get

$S_4 = \frac{2}{3}[f(0) + 4f(2) + 2f(4) + 4f(6) + f(8)] = \frac{2}{3}(192) = 128$. The exact value of the integral is

$\int_0^8 (x^3 - 6x^2 + 4x)\,dx = \left[\frac{1}{4}x^4 - 2x^3 + 2x^2\right]_0^8 = (1024 - 1024 + 128) - 0 = 128.$

Thus, $S_4 = \int_0^8 (x^3 - 6x^2 + 4x)\,dx$.

(c) $f(x) = Ax^3 + Bx^2 + Cx + D \quad \Rightarrow \quad f'(x) = 3Ax^2 + 2Bx + C \quad \Rightarrow \quad f''(x) = 6Ax + 2B \quad \Rightarrow \quad f'''(x) = 6A \quad \Rightarrow$

$f^{(4)}(x) = 0$. Since $\left|f^{(4)}(x)\right| = 0$ for all x, the error bound in (4) gives $|E_S| \leq \dfrac{(0)(b-a)^5}{180n^4} = 0$, indicating the error in using Simpson's Rule is zero. Hence, Simpson's Rule gives the exact value of the integral for a polynomial of degree 3 or lower.

49. $T_n = \frac{1}{2}\,\Delta x\,[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)]$ and

$M_n = \Delta x\,[f(\overline{x}_1) + f(\overline{x}_2) + \cdots + f(\overline{x}_{n-1}) + f(\overline{x}_n)]$, where $\overline{x}_i = \frac{1}{2}(x_{i-1} + x_i)$. Now

$T_{2n} = \frac{1}{2}\left(\frac{1}{2}\Delta x\right)[f(x_0) + 2f(\overline{x}_1) + 2f(x_1) + 2f(\overline{x}_2) + 2f(x_2) + \cdots + 2f(\overline{x}_{n-1}) + 2f(x_{n-1}) + 2f(\overline{x}_n) + f(x_n)]$, so

$$\begin{aligned}
\tfrac{1}{2}(T_n + M_n) &= \tfrac{1}{2}T_n + \tfrac{1}{2}M_n \\
&= \tfrac{1}{4}\Delta x[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)] \\
&\qquad + \tfrac{1}{4}\Delta x[2f(\overline{x}_1) + 2f(\overline{x}_2) + \cdots + 2f(\overline{x}_{n-1}) + 2f(\overline{x}_n)] \\
&= T_{2n}
\end{aligned}$$

50. $T_n = \dfrac{\Delta x}{2}\left[f(x_0) + 2\sum\limits_{i=1}^{n-1} f(x_i) + f(x_n)\right]$ and $M_n = \Delta x \sum\limits_{i=1}^{n} f\left(x_i - \dfrac{\Delta x}{2}\right)$, so

$$\tfrac{1}{3}T_n + \tfrac{2}{3}M_n = \tfrac{1}{3}(T_n + 2M_n) = \frac{\Delta x}{3 \cdot 2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n) + 4\sum_{i=1}^{n} f\left(x_i - \frac{\Delta x}{2}\right)\right]$$

where $\Delta x = \dfrac{b-a}{n}$. Let $\delta x = \dfrac{b-a}{2n}$. Then $\Delta x = 2\delta x$, so

$$\begin{aligned}
\tfrac{1}{3}T_n + \tfrac{2}{3}M_n &= \frac{\delta x}{3}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n) + 4\sum_{i=1}^{n} f(x_i - \delta x)\right] \\
&= \tfrac{1}{3}\delta x[f(x_0) + 4f(x_1 - \delta x) + 2f(x_1) + 4f(x_2 - \delta x) \\
&\qquad + 2f(x_2) + \cdots + 2f(x_{n-1}) + 4f(x_n - \delta x) + f(x_n)]
\end{aligned}$$

Since $x_0, x_1 - \delta x, x_1, x_2 - \delta x, x_2, \ldots, x_{n-1}, x_n - \delta x, x_n$ are the subinterval endpoints for S_{2n}, and since $\delta x = \dfrac{b-a}{2n}$ is the width of the subintervals for S_{2n}, the last expression for $\frac{1}{3}T_n + \frac{2}{3}M_n$ is the usual expression for S_{2n}. Therefore, $\frac{1}{3}T_n + \frac{2}{3}M_n = S_{2n}$.

7.8 Improper Integrals

1. (a) Since $y = \dfrac{1}{x-3}$ has an infinite discontinuity at $x = 3$, $\displaystyle\int_1^4 \frac{dx}{x-3}$ is a Type 2 improper integral.

(b) Since $\displaystyle\int_3^\infty \frac{dx}{x^2-4}$ has an infinite interval of integration, it is an improper integral of Type 1.

(c) Since $y = \tan \pi x$ has an infinite discontinuity at $x = \frac{1}{2}$, $\int_0^1 \tan \pi x\, dx$ is a Type 2 improper integral.

(d) Since $\displaystyle\int_{-\infty}^{-1} \frac{e^x}{x}\, dx$ has an infinite interval of integration, it is an improper integral of Type 1.

2. (a) Since $y = \sec x$ has an infinite discontinuity at $x = \frac{\pi}{2}$, $\int_0^\pi \sec x\, dx$ is a Type 2 improper integral.

(b) Since $y = \dfrac{1}{x-5}$ is defined and continuous on the interval $[0, 4]$, $\displaystyle\int_0^4 \frac{dx}{x-5}$ is a proper integral.

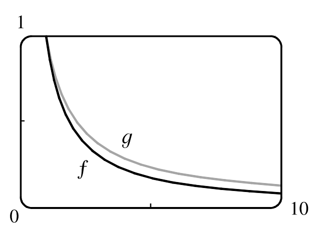
(c) Since $y = \dfrac{1}{x+x^3} = \dfrac{1}{x\,(1+x^2)}$ has an infinite discontinuity at $x = 0$, $\displaystyle\int_{-1}^3 \frac{1}{x+x^3}\, dx$ is a Type 2 improper integral.

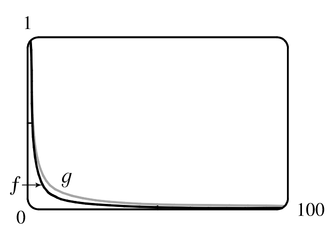
(d) Since $\displaystyle\int_1^\infty \frac{1}{x+x^3}\, dx$ has an infinite interval of integration, it is an improper integral of Type 1.

3. The area under the graph of $y = 1/x^3 = x^{-3}$ between $x = 1$ and $x = t$ is

$A(t) = \int_1^t x^{-3}\, dx = \left[-\frac{1}{2}x^{-2}\right]_1^t = -\frac{1}{2}t^{-2} - \left(-\frac{1}{2}\right) = \frac{1}{2} - 1/(2t^2)$. So the area for $1 \le x \le 10$ is $A(10) = 0.5 - 0.005 = 0.495$, the area for $1 \le x \le 100$ is $A(100) = 0.5 - 0.00005 = 0.49995$, and the area for $1 \le x \le 1000$ is $A(1000) = 0.5 - 0.0000005 = 0.4999995$. The total area under the curve for $x \ge 1$ is $\lim\limits_{t\to\infty} A(t) = \lim\limits_{t\to\infty} \left[\frac{1}{2} - 1/(2t^2)\right] = \frac{1}{2}$.

4. (a)





(b) The area under the graph of f from $x = 1$ to $x = t$ is

$$F(t) = \int_1^t f(x)\, dx = \int_1^t x^{-1.1}\, dx = \left[-\tfrac{1}{0.1}x^{-0.1}\right]_1^t = -10(t^{-0.1} - 1) = 10(1 - t^{-0.1})$$

and the area under the graph of g is

$$G(t) = \int_1^t g(x)\, dx = \int_1^t x^{-0.9}\, dx = \left[\tfrac{1}{0.1}x^{0.1}\right]_1^t = 10(t^{0.1} - 1).$$

t	$F(t)$	$G(t)$
10	2.06	2.59
100	3.69	5.85
10^4	6.02	15.12
10^6	7.49	29.81
10^{10}	9	90
10^{20}	9.9	990

(c) The total area under the graph of f is $\lim\limits_{t\to\infty} F(t) = \lim\limits_{t\to\infty} 10(1 - t^{-0.1}) = 10$.

The total area under the graph of g does not exist, since $\lim\limits_{t\to\infty} G(t) = \lim\limits_{t\to\infty} 10(t^{0.1} - 1) = \infty$.

5. $\int_1^\infty 2x^{-3}\,dx = \lim\limits_{t\to\infty}\int_1^t 2x^{-3}\,dx = \lim\limits_{t\to\infty}\left[-x^{-2}\right]_1^t = \lim\limits_{t\to\infty}\left[-t^{-2}+1\right] = 0+1=1.$ Convergent

6. $\displaystyle\int_{-\infty}^{-1}\frac{1}{\sqrt[3]{x}}\,dx = \lim_{t\to-\infty}\int_t^{-1}x^{-1/3}\,dx = \lim_{t\to-\infty}\left[\tfrac{3}{2}x^{2/3}\right]_t^{-1} = \lim_{t\to-\infty}\left[\tfrac{3}{2}-\tfrac{3}{2}t^{2/3}\right] = -\infty.$ Divergent

7. $\int_0^\infty e^{-2x}\,dx = \lim\limits_{t\to\infty}\int_0^t e^{-2x}\,dx = \lim\limits_{t\to\infty}\left[-\tfrac{1}{2}e^{-2x}\right]_0^t = \lim\limits_{t\to\infty}\left[-\tfrac{1}{2}e^{-2t}+\tfrac{1}{2}\right] = 0+\tfrac{1}{2}=\tfrac{1}{2}.$ Convergent

8. $\displaystyle\int_1^\infty\left(\frac{1}{3}\right)^x dx = \lim_{t\to\infty}\int_1^t 3^{-x}\,dx = \lim_{t\to\infty}\left[-\frac{3^{-x}}{\ln 3}\right]_1^t = \lim_{t\to\infty}\left[-\frac{3^{-t}}{\ln 3}+\frac{3^{-1}}{\ln 3}\right] = 0+\frac{1}{3\ln 3} = \frac{1}{\ln 27}.$ Convergent

9. $\displaystyle\int_{-2}^\infty\frac{1}{x+4}\,dx = \lim_{t\to\infty}\int_{-2}^t\frac{1}{x+4}\,dx = \lim_{t\to\infty}\Big[\ln|x+4|\Big]_{-2}^t = \lim_{t\to\infty}\Big[\ln|t+4|-\ln 2\Big] = \infty$ since $\lim\limits_{t\to\infty}\ln|x+4| = \infty.$
Divergent

10. $\displaystyle\int_1^\infty\frac{1}{x^2+4}\,dx = \lim_{t\to\infty}\int_1^t\frac{1}{x^2+4}\,dx = \lim_{t\to\infty}\left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_1^t = \lim_{t\to\infty}\left[\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)-\frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right)\right]$

$\displaystyle= \frac{\pi}{4}-\frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right).$ Convergent

11. $\displaystyle\int_3^\infty\frac{1}{(x-2)^{3/2}}\,dx = \lim_{t\to\infty}\int_3^t(x-2)^{-3/2}\,dx = \lim_{t\to\infty}\left[-2\,(x-2)^{-1/2}\right]_3^t$ $[u=x-2,\,du=dx]$

$\displaystyle= \lim_{t\to\infty}\left(\frac{-2}{\sqrt{t-2}}+\frac{2}{\sqrt{1}}\right) = 0+2=2.$ Convergent

12. $\displaystyle\int_0^\infty\frac{1}{\sqrt[4]{1+x}}\,dx = \lim_{t\to\infty}\int_0^t(1+x)^{-1/4}\,dx = \lim_{t\to\infty}\left[\tfrac{4}{3}(1+x)^{3/4}\right]_0^t$ $[u=1+x,\,du=dx]$

$\displaystyle= \lim_{t\to\infty}\left[\tfrac{4}{3}(1+t)^{3/4}-\tfrac{4}{3}\right] = \infty.$ Divergent

13. $\displaystyle\int_{-\infty}^0\frac{x}{(x^2+1)^3}\,dx = \lim_{t\to-\infty}\int_t^0\frac{x}{(x^2+1)^3}\,dx = \lim_{t\to-\infty}\left[-\frac{1}{4}(x^2+1)^{-2}\right]_t^0 = \lim_{t\to-\infty}\left[-\frac{1}{4\,(x^2+1)^2}\right]_t^0$

$\displaystyle= \lim_{t\to-\infty}\left[-\frac{1}{4}+\frac{1}{4(t^2+1)^2}\right] = -\frac{1}{4}+0 = -\frac{1}{4}.$ Convergent

14. $\displaystyle\int_{-\infty}^{-3}\frac{x}{4-x^2}\,dx = \lim_{t\to-\infty}\int_t^{-3}\frac{x}{4-x^2}\,dx = \lim_{t\to-\infty}\left[-\frac{1}{2}\ln\left|4-x^2\right|\right]_t^{-3}$

$= \lim\limits_{t\to-\infty}\left[-\tfrac{1}{2}\ln 5+\tfrac{1}{2}\ln\left|4-t^2\right|\right] = \infty$ since $\lim\limits_{t\to-\infty}\ln\left|4-t^2\right| = \infty.$ Divergent

15. $\displaystyle\int_1^\infty\frac{x^2+x+1}{x^4}\,dx = \lim_{t\to\infty}\int_1^t\left(x^{-2}+x^{-3}+x^{-4}\right)dx$

$\displaystyle= \lim_{t\to\infty}\left[-x^{-1}-\frac{1}{2}x^{-2}-\frac{1}{3}x^{-3}\right]_1^t = \lim_{t\to\infty}\left[-\frac{1}{x}-\frac{1}{2x^2}-\frac{1}{3x^3}\right]_1^t$

$\displaystyle= \lim_{t\to\infty}\left[\left(-\frac{1}{t}-\frac{1}{2t^2}-\frac{1}{3t^3}\right)-\left(-1-\frac{1}{2}-\frac{1}{3}\right)\right] = 0+\frac{11}{6} = \frac{11}{6}.$ Convergent

16. $\displaystyle\int_2^\infty\frac{x}{\sqrt{x^2-1}}\,dx = \lim_{t\to\infty}\int_2^t\frac{x}{\sqrt{x^2-1}}\,dx = \lim_{t\to\infty}\left[\sqrt{x^2-1}\,\right]_2^t = \lim_{t\to\infty}\left[\sqrt{t^2-1}-\sqrt{3}\right] = \infty.$ Divergent

17. $\displaystyle\int_0^\infty \frac{e^x}{(1+e^x)^2}\,dx = \lim_{t\to\infty}\int_0^t \frac{e^x}{(1+e^x)^2}\,dx = \lim_{t\to\infty}\Big[-(1+e^x)^{-1}\Big]_0^t = \lim_{t\to\infty}\left[-\frac{1}{1+e^x}\right]_0^t$

$$= \lim_{t\to\infty}\left[-\frac{1}{1+e^t}+\frac{1}{2}\right] = 0+\frac{1}{2} = \frac{1}{2}. \qquad \text{Convergent}$$

18. $\displaystyle I = \int_{-\infty}^{-1}\frac{x^2+x}{x^3}\,dx = \int_{-\infty}^{-1}\frac{1}{x}\,dx + \int_{-\infty}^{-1}\frac{1}{x^2}\,dx = I_1 + I_2.$

Now, $\displaystyle\int_{-\infty}^{-1}\frac{1}{x}\,dx = \lim_{t\to-\infty}\int_t^{-1}\frac{1}{x}\,dx = \lim_{t\to-\infty}\Big[\ln|x|\Big]_t^{-1} = \lim_{t\to-\infty}\Big[\ln 1 - \ln|t|\Big] = -\infty.$

Since I_1 is divergent, I is divergent. Divergent

19. $\int_{-\infty}^{\infty} xe^{-x^2}\,dx = \int_{-\infty}^{0} xe^{-x^2}\,dx + \int_0^\infty xe^{-x^2}\,dx.$

$\int_{-\infty}^{0} xe^{-x^2}\,dx = \lim\limits_{t\to-\infty}\left(-\frac{1}{2}\right)\Big[e^{-x^2}\Big]_t^0 = \lim\limits_{t\to-\infty}\left(-\frac{1}{2}\right)\left(1-e^{-t^2}\right) = -\frac{1}{2}\cdot(1-0) = -\frac{1}{2}$, and

$\int_0^\infty xe^{-x^2}\,dx = \lim\limits_{t\to\infty}\left(-\frac{1}{2}\right)\Big[e^{-x^2}\Big]_0^t = \lim\limits_{t\to\infty}\left(-\frac{1}{2}\right)\left(e^{-t^2}-1\right) = -\frac{1}{2}\cdot(0-1) = \frac{1}{2}.$

Therefore, $\int_{-\infty}^{\infty} xe^{-x^2}\,dx = -\frac{1}{2}+\frac{1}{2} = 0.$ Convergent

20. $\displaystyle I = \int_{-\infty}^{\infty}\frac{x}{x^2+1}\,dx = \int_{-\infty}^{0}\frac{x}{x^2+1}\,dx + \int_0^\infty\frac{x}{x^2+1}\,dx = I_1+I_2$, but

$$I_2 = \lim_{t\to\infty}\int_0^t\frac{x}{x^2+1}\,dx = \lim_{t\to\infty}\int_1^{t^2+1}\frac{1/2}{u}\,du \quad \begin{bmatrix} u = x^2+1, \\ du = 2x\,dx \end{bmatrix} \quad = \frac{1}{2}\lim_{t\to\infty}\Big[\ln|u|\Big]_1^{t^2+1}$$

$$= \frac{1}{2}\lim_{t\to\infty}\Big[\ln\left|t^2+1\right| - 0\Big] = \infty$$

Since I_2 is divergent, I is divergent, and there is no need to evaluate I_1. Divergent

21. $I = \int_{-\infty}^{\infty}\cos 2t\,dt = \int_{-\infty}^{0}\cos 2t\,dt + \int_0^\infty \cos 2t\,dt = I_1 + I_2$, but $I_1 = \lim\limits_{s\to-\infty}\Big[\frac{1}{2}\sin 2t\Big]_s^0 = \lim\limits_{s\to-\infty}\left(-\frac{1}{2}\sin 2s\right)$, and this limit does not exist. Since I_1 is divergent, I is divergent, and there is no need to evaluate I_2. Divergent

22. $\displaystyle\int_1^\infty \frac{e^{-1/x}}{x^2}\,dx = \lim_{t\to\infty}\int_1^t\frac{e^{-1/x}}{x^2}\,dx = \lim_{t\to\infty}\Big[e^{-1/x}\Big]_1^t = \lim_{t\to\infty}\left(e^{-1/t}-e^{-1}\right) = 1-\frac{1}{e}.$ Convergent

23. $\int_0^\infty \sin^2\alpha\,d\alpha = \lim\limits_{t\to\infty}\int_0^t \frac{1}{2}(1-\cos 2\alpha)\,d\alpha = \lim\limits_{t\to\infty}\left[\frac{1}{2}\left(\alpha-\frac{1}{2}\sin 2\alpha\right)\right]_0^t = \lim\limits_{t\to\infty}\left[\frac{1}{2}\left(t-\frac{1}{2}\sin 2t\right)-0\right] = \infty.$

Divergent

24. $\int_0^\infty \sin\theta\, e^{\cos\theta}\,d\theta = \lim\limits_{t\to\infty}\int_0^t \sin\theta\, e^{\cos\theta}\,d\theta = \lim\limits_{t\to\infty}\Big[-e^{\cos\theta}\Big]_0^t = \lim\limits_{t\to\infty}\left(-e^{\cos t}+e\right)$

This limit does not exist since $\cos t$ oscillates in value between -1 and 1, so $e^{\cos t}$ oscillates in value between e^{-1} and e^1. Divergent

25. $\displaystyle\int_1^\infty\frac{1}{x^2+x}\,dx = \lim_{t\to\infty}\int_1^t\frac{1}{x(x+1)}\,dx = \lim_{t\to\infty}\int_1^t\left(\frac{1}{x}-\frac{1}{x+1}\right)dx \qquad \text{[partial fractions]}$

$$= \lim_{t\to\infty}\Big[\ln|x| - \ln|x+1|\Big]_1^t = \lim_{t\to\infty}\left[\ln\left|\frac{x}{x+1}\right|\right]_1^t = \lim_{t\to\infty}\left(\ln\frac{t}{t+1}-\ln\frac{1}{2}\right) = 0 - \ln\frac{1}{2} = \ln 2.$$

Convergent

26. $\displaystyle\int_2^\infty \frac{dv}{v^2+2v-3} = \lim_{t\to\infty}\int_2^t \frac{dv}{(v+3)(v-1)} = \lim_{t\to\infty}\int_2^t \left(\frac{-\frac14}{v+3}+\frac{\frac14}{v-1}\right)dv = \lim_{t\to\infty}\left[-\tfrac14\ln|v+3|+\tfrac14\ln|v-1|\right]_2^t$

$\displaystyle = \frac14\lim_{t\to\infty}\left[\ln\frac{v-1}{v+3}\right]_2^t = \frac14\lim_{t\to\infty}\left(\ln\frac{t-1}{t+3}-\ln\frac15\right) = \frac14(0+\ln 5) = \frac14\ln 5.$ Convergent

27. $\int_{-\infty}^0 ze^{2z}\,dz = \lim\limits_{t\to-\infty}\int_t^0 ze^{2z}\,dz = \lim\limits_{t\to-\infty}\left[\frac12 ze^{2z}-\frac14 e^{2z}\right]_t^0$ $\left[\begin{array}{c}\text{integration by parts with}\\ u=z,\ dv=e^{2z}\,dz\end{array}\right]$

$= \lim\limits_{t\to-\infty}\left[\left(0-\frac14\right)-\left(\frac12 te^{2t}-\frac14 e^{2t}\right)\right] = -\frac14-0+0$ [by l'Hospital's Rule] $= -\frac14.$ Convergent

28. $\int_2^\infty ye^{-3y}\,dy = \lim\limits_{t\to\infty}\int_2^t ye^{-3y}\,dy = \lim\limits_{t\to\infty}\left[-\frac13 ye^{-3y}-\frac19 e^{-3y}\right]_2^t$ $\left[\begin{array}{c}\text{integration by parts with}\\ u=y,\ dv=e^{-3y}\,dy\end{array}\right]$

$= \lim\limits_{t\to\infty}\left[\left(-\frac13 te^{-3t}-\frac19 e^{-3t}\right)-\left(-\frac23 e^{-6}-\frac19 e^{-6}\right)\right] = 0-0+\frac79 e^{-6}$ [by l'Hospital's Rule] $= \frac79 e^{-6}.$

Convergent

29. $\displaystyle\int_1^\infty \frac{\ln x}{x}\,dx = \lim_{t\to\infty}\left[\frac{(\ln x)^2}{2}\right]_1^t$ $\left[\begin{array}{l}\text{by substitution with}\\ u=\ln x,\ du=dx/x\end{array}\right]$ $\displaystyle = \lim_{t\to\infty}\frac{(\ln t)^2}{2} = \infty.$ Divergent

30. $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty}\left[-\frac{\ln x}{x}-\frac1x\right]_1^t$ $\left[\begin{array}{c}\text{integration by parts with}\\ u=\ln x,\ dv=(1/x^2)\,dx\end{array}\right]$

$\displaystyle = \lim_{t\to\infty}\left(-\frac{\ln t}{t}-\frac1t+1\right) \overset{\text{H}}{=} \lim_{t\to\infty}\left(-\frac{1/t}{1}\right)-\lim_{t\to\infty}\frac1t+\lim_{t\to\infty}1 = 0-0+1 = 1.$ Convergent

31. $\displaystyle\int_{-\infty}^0 \frac{z}{z^4+4}\,dz = \lim_{t\to-\infty}\int_t^0 \frac{z}{z^4+4}\,dz = \lim_{t\to-\infty}\frac12\left[\frac12\tan^{-1}\left(\frac{z^2}{2}\right)\right]_t^0$ $\left[\begin{array}{c}u=z^2,\\ du=2z\,dz\end{array}\right]$

$\displaystyle = \lim_{t\to-\infty}\left[0-\frac14\tan^{-1}\left(\frac{t^2}{2}\right)\right] = -\frac14\left(\frac{\pi}{2}\right) = -\frac{\pi}{8}.$ Convergent

32. $\displaystyle\int_e^\infty \frac{1}{x(\ln x)^2}\,dx = \lim_{t\to\infty}\int_e^t \frac{1}{x(\ln x)^2}\,dx = \lim_{t\to\infty}\left[-\frac{1}{\ln x}\right]_e^t$ $\left[\begin{array}{c}u=\ln x,\\ du=(1/x)\,dx\end{array}\right]$

$\displaystyle = \lim_{t\to\infty}\left(-\frac{1}{\ln t}+1\right) = 0+1 = 1.$ Convergent

33. $\displaystyle\int_0^\infty e^{-\sqrt{y}}\,dy = \lim_{t\to\infty}\int_0^t e^{-\sqrt{y}}\,dy = \lim_{t\to\infty}\int_0^{\sqrt{t}} e^{-x}\,(2x\,dx)$ $\left[\begin{array}{c}x=\sqrt{y},\\ dx=1/(2\sqrt{y}\,)\,dy\end{array}\right]$

$\displaystyle = \lim_{t\to\infty}\left\{\left[-2xe^{-x}\right]_0^{\sqrt{t}}+\int_0^{\sqrt{t}} 2e^{-x}\,dx\right\}$ $\left[\begin{array}{ll}u=2x, & dv=e^{-x}\,dx\\ du=2\,dx, & v=-e^{-x}\end{array}\right]$

$\displaystyle = \lim_{t\to\infty}\left(-2\sqrt{t}\,e^{-\sqrt{t}}+\left[-2e^{-x}\right]_0^{\sqrt{t}}\right) = \lim_{t\to\infty}\left(\frac{-2\sqrt{t}}{e^{\sqrt{t}}}-\frac{2}{e^{\sqrt{t}}}+2\right) = 0-0+2 = 2.$

Convergent

Note: $\displaystyle\lim_{t\to\infty}\frac{\sqrt{t}}{e^{\sqrt{t}}} \overset{\text{H}}{=} \lim_{t\to\infty}\frac{2\sqrt{t}}{2\sqrt{t}\,e^{\sqrt{t}}} = \lim_{t\to\infty}\frac{1}{e^{\sqrt{t}}} = 0$

34. $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x}+x\sqrt{x}}\,dx = \lim_{t\to\infty}\int_1^t \frac{dx}{\sqrt{x}\,(1+x)} = \lim_{t\to\infty}\int_1^{\sqrt{t}} \frac{1}{u(1+u^2)}\,(2u\,du) \qquad \left[\begin{array}{l} u=\sqrt{x}, \\ du = 1/(2\sqrt{x}\,)\,dx \end{array}\right]$

$\displaystyle = \lim_{t\to\infty}\int_1^{\sqrt{t}} \frac{2}{1+u^2}\,du = \lim_{t\to\infty}\left[2\tan^{-1}u\right]_1^{\sqrt{t}} = \lim_{t\to\infty} 2(\tan^{-1}\sqrt{t}-\tan^{-1}1)$

$= 2(\frac{\pi}{2}-\frac{\pi}{4}) = \frac{\pi}{2}.$ Convergent

35. $\displaystyle\int_0^1 \frac{1}{x}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{x}\,dx = \lim_{t\to 0^+}\Big[\ln|x|\Big]_t^1 = \lim_{t\to 0^+}(-\ln t) = \infty.$ Divergent

36. $\displaystyle\int_0^5 \frac{1}{\sqrt[3]{5-x}}\,dx = \lim_{t\to 5^-}\int_0^t (5-x)^{-1/3}\,dx = \lim_{t\to 5^-}\left[-\tfrac{3}{2}(5-x)^{2/3}\right]_0^t = \lim_{t\to 5^-}\left\{-\tfrac{3}{2}[(5-t)^{2/3}-5^{2/3}]\right\}$

$= \frac{3}{2}5^{2/3}.$ Convergent

37. $\displaystyle\int_{-2}^{14} \frac{dx}{\sqrt[4]{x+2}} = \lim_{t\to -2^+}\int_t^{14}(x+2)^{-1/4}\,dx = \lim_{t\to -2^+}\left[\frac{4}{3}(x+2)^{3/4}\right]_t^{14} = \frac{4}{3}\lim_{t\to -2^+}\left[16^{3/4}-(t+2)^{3/4}\right]$

$= \frac{4}{3}(8-0) = \frac{32}{3}.$ Convergent

38. $\displaystyle\int_{-1}^2 \frac{x}{(x+1)^2}\,dx = \lim_{t\to -1^+}\int_t^2 \frac{x}{(x+1)^2}\,dx = \lim_{t\to -1^+}\int_t^2\left[\frac{1}{x+1}-\frac{1}{(x+1)^2}\right]dx$ [partial fractions]

$\displaystyle = \lim_{t\to -1^+}\left[\ln|x+1|+\frac{1}{x+1}\right]_t^2 = \lim_{t\to -1^+}\left[\ln 3+\frac{1}{3}-\left(\ln(t+1)+\frac{1}{t+1}\right)\right] = -\infty.$ Divergent

Note: To justify the last step, $\displaystyle\lim_{t\to -1^+}\left[\ln(t+1)+\frac{1}{t+1}\right] = \lim_{x\to 0^+}\left(\ln x+\frac{1}{x}\right) \quad \left[\begin{array}{c}\text{substitute} \\ x \text{ for } t+1\end{array}\right] = \lim_{x\to 0^+}\frac{x\ln x+1}{x} = \infty$

since $\displaystyle\lim_{x\to 0^+}(x\ln x) = \lim_{x\to 0^+}\frac{\ln x}{1/x} \stackrel{\text{H}}{=} \lim_{x\to 0^+}\frac{1/x}{-1/x^2} = \lim_{x\to 0^+}(-x) = 0.$

39. $\displaystyle\int_{-2}^3 \frac{1}{x^4}\,dx = \int_{-2}^0 \frac{dx}{x^4}+\int_0^3 \frac{dx}{x^4}$, but $\displaystyle\int_{-2}^0 \frac{dx}{x^4} = \lim_{t\to 0^-}\left[-\frac{x^{-3}}{3}\right]_{-2}^t = \lim_{t\to 0^-}\left[-\frac{1}{3t^3}-\frac{1}{24}\right] = \infty.$ Divergent

40. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{t\to 1^-}\int_0^t \frac{dx}{\sqrt{1-x^2}} = \lim_{t\to 1^-}\left[\sin^{-1}x\right]_0^t = \lim_{t\to 1^-}\sin^{-1}t = \frac{\pi}{2}.$ Convergent

41. There is an infinite discontinuity at $x=1$. $\displaystyle\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx = \int_0^1 (x-1)^{-1/3}\,dx+\int_1^9 (x-1)^{-1/3}\,dx.$

Here $\int_0^1 (x-1)^{-1/3}\,dx = \lim\limits_{t\to 1^-}\int_0^t (x-1)^{-1/3}\,dx = \lim\limits_{t\to 1^-}\left[\frac{3}{2}(x-1)^{2/3}\right]_0^t = \lim\limits_{t\to 1^-}\left[\frac{3}{2}(t-1)^{2/3}-\frac{3}{2}\right] = -\frac{3}{2}$

and $\int_1^9 (x-1)^{-1/3}\,dx = \lim\limits_{t\to 1^+}\int_t^9 (x-1)^{-1/3}\,dx = \lim\limits_{t\to 1^+}\left[\frac{3}{2}(x-1)^{2/3}\right]_t^9 = \lim\limits_{t\to 1^+}\left[6-\frac{3}{2}(t-1)^{2/3}\right] = 6.$ Thus,

$\displaystyle\int_0^9 \frac{1}{\sqrt[3]{x-1}}\,dx = -\frac{3}{2}+6 = \frac{9}{2}.$ Convergent

42. There is an infinite discontinuity at $w=2$.

$\displaystyle\int_0^2 \frac{w}{w-2}\,dw = \lim_{t\to 2^-}\int_0^t\left(1+\frac{2}{w-2}\right)dw = \lim_{t\to 2^-}\Big[w+2\ln|w-2|\Big]_0^t = \lim_{t\to 2^-}(t+2\ln|t-2|-2\ln 2) = -\infty$, so

$\displaystyle\int_0^2 \frac{w}{w-2}\,dw$ diverges, and hence, $\displaystyle\int_0^5 \frac{w}{w-2}\,dw$ diverges. Divergent

43. $\int_0^{\pi/2} \tan^2\theta\, d\theta = \lim\limits_{t\to(\pi/2)^-} \int_0^t \tan^2\theta\, d\theta = \lim\limits_{t\to(\pi/2)^-} \int_0^t (\sec^2\theta - 1)\, d\theta = \lim\limits_{t\to(\pi/2)^-} \Big[\tan\theta - \theta\Big]_0^t$

$= \lim\limits_{t\to(\pi/2)^-} (\tan t - t) = \infty$ since $\tan t \to \infty$ as $t \to \frac{\pi}{2}^-$. Divergent

44. $\displaystyle\int_0^4 \frac{dx}{x^2 - x - 2} = \int_0^4 \frac{dx}{(x-2)(x+1)} = \int_0^2 \frac{dx}{(x-2)(x+1)} + \int_2^4 \frac{dx}{(x-2)(x+1)}$

Considering only $\displaystyle\int_0^2 \frac{dx}{(x-2)(x+1)}$ and using partial fractions, we have

$\displaystyle\int_0^2 \frac{dx}{(x-2)(x+1)} = \lim_{t\to 2^-} \int_0^t \left(\frac{\frac{1}{3}}{x-2} - \frac{\frac{1}{3}}{x+1}\right) dx = \lim_{t\to 2^-} \Big[\tfrac{1}{3}\ln|x-2| - \tfrac{1}{3}\ln|x+1|\Big]_0^t$

$= \lim\limits_{t\to 2^-} \big[\frac{1}{3}\ln|t-2| - \frac{1}{3}\ln|t+1| - \frac{1}{3}\ln 2 + 0\big] = -\infty$ since $\ln|t-2| \to -\infty$ as $t \to 2^-$.

Thus, $\displaystyle\int_0^2 \frac{dx}{x^2 - x - 2}$ is divergent, and hence, $\displaystyle\int_0^4 \frac{dx}{x^2 - x - 2}$ is divergent as well.

45. $\int_0^1 r \ln r\, dr = \lim\limits_{t\to 0^+} \int_t^1 r \ln r\, dr = \lim\limits_{t\to 0^+} \Big[\frac{1}{2}r^2 \ln r - \frac{1}{4}r^2\Big]_t^1$ $\quad\begin{bmatrix} u = \ln r, & dv = r\,dr \\ du = (1/r)\,dr, & v = \frac{1}{2}r^2 \end{bmatrix}$

$= \lim\limits_{t\to 0^+} \big[\big(0 - \frac{1}{4}\big) - \big(\frac{1}{2}t^2 \ln t - \frac{1}{4}t^2\big)\big] = -\frac{1}{4} - 0 = -\frac{1}{4}$

since $\displaystyle\lim_{t\to 0^+} t^2 \ln t = \lim_{t\to 0^+} \frac{\ln t}{1/t^2} \overset{\text{H}}{=} \lim_{t\to 0^+} \frac{1/t}{-2/t^3} = \lim_{t\to 0^+} \left(-\tfrac{1}{2}t^2\right) = 0.$ Convergent

46. $\displaystyle\int_0^{\pi/2} \frac{\cos\theta}{\sqrt{\sin\theta}}\, d\theta = \lim_{t\to 0^+} \int_t^{\pi/2} \frac{\cos\theta}{\sqrt{\sin\theta}}\, d\theta = \lim_{t\to 0^+} \Big[2\sqrt{\sin\theta}\Big]_t^{\pi/2}$ $\quad\begin{bmatrix} u = \sin\theta, \\ du = \cos\theta\, d\theta \end{bmatrix}$

$= \lim\limits_{t\to 0^+} (2 - 2\sqrt{\sin t}\,) = 2 - 0 = 2.$ Convergent

47. $\displaystyle\int_{-1}^0 \frac{e^{1/x}}{x^3}\, dx = \lim_{t\to 0^-} \int_{-1}^t \frac{1}{x} e^{1/x} \cdot \frac{1}{x^2}\, dx = \lim_{t\to 0^-} \int_{-1}^{1/t} ue^u\,(-du)$ $\quad\begin{bmatrix} u = 1/x, \\ du = -dx/x^2 \end{bmatrix}$

$\displaystyle = \lim_{t\to 0^-} \Big[(u-1)e^u\Big]_{1/t}^{-1}$ $\begin{bmatrix} \text{use parts} \\ \text{or Formula 96} \end{bmatrix}$ $\displaystyle = \lim_{t\to 0^-} \left[-2e^{-1} - \left(\frac{1}{t} - 1\right)e^{1/t}\right]$

$\displaystyle = -\frac{2}{e} - \lim_{s\to -\infty} (s-1)e^s \quad [s = 1/t] \quad = -\frac{2}{e} - \lim_{s\to -\infty} \frac{s-1}{e^{-s}} \overset{\text{H}}{=} -\frac{2}{e} - \lim_{s\to -\infty} \frac{1}{-e^{-s}}$

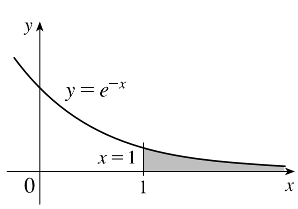
$\displaystyle = -\frac{2}{e} - 0 = -\frac{2}{e}.$ Convergent

48. $\displaystyle\int_0^1 \frac{e^{1/x}}{x^3}\, dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{x} e^{1/x} \cdot \frac{1}{x^2}\, dx = \lim_{t\to 0^+} \int_{1/t}^1 ue^u\,(-du)$ $\quad\begin{bmatrix} u = 1/x, \\ du = -dx/x^2 \end{bmatrix}$

$\displaystyle = \lim_{t\to 0^+} \Big[(u-1)e^u\Big]_1^{1/t}$ $\begin{bmatrix} \text{use parts} \\ \text{or Formula 96} \end{bmatrix}$ $\displaystyle = \lim_{t\to 0^+} \left[\left(\frac{1}{t} - 1\right)e^{1/t} - 0\right]$

$= \lim\limits_{s\to\infty} (s-1)e^s \quad [s = 1/t] \quad = \infty.$ Divergent

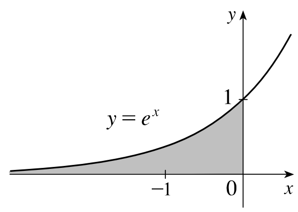
49.



Area $= \int_1^\infty e^{-x}\, dx = \lim\limits_{t\to\infty} \int_1^t e^{-x}\, dx = \lim\limits_{t\to\infty} \Big[-e^{-x}\Big]_1^t$

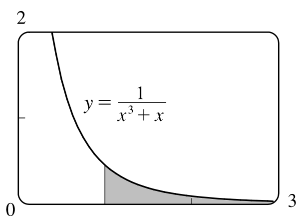
$= \lim\limits_{t\to\infty} (-e^{-t} + e^{-1}) = 0 + e^{-1} = 1/e$

50.



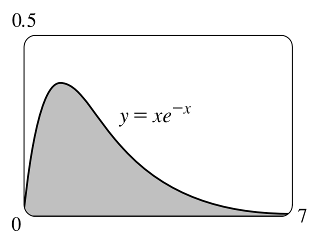
$$\text{Area} = \int_{-\infty}^{0} e^x\,dx = \lim_{t\to-\infty}\int_t^0 e^x\,dx = \lim_{t\to-\infty}\Big[e^x\Big]_t^0$$
$$= \lim_{t\to-\infty}(e^0 - e^t) = 1 - 0 = 1$$

51.



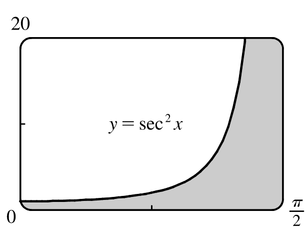
$$\text{Area} = \int_1^\infty \frac{1}{x^3+x}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x(x^2+1)}\,dx$$
$$= \lim_{t\to\infty}\int_1^t \left(\frac{1}{x} - \frac{x}{x^2+1}\right)dx \qquad \text{[partial fractions]}$$
$$= \lim_{t\to\infty}\left[\ln|x| - \frac{1}{2}\ln\left|x^2+1\right|\right]_1^t = \lim_{t\to\infty}\left[\ln\frac{x}{\sqrt{x^2+1}}\right]_1^t$$
$$= \lim_{t\to\infty}\left(\ln\frac{t}{\sqrt{t^2+1}} - \ln\frac{1}{\sqrt{2}}\right) = \ln 1 - \ln 2^{-1/2} = \frac{1}{2}\ln 2$$

52.



$$\text{Area} = \int_0^\infty xe^{-x}\,dx = \lim_{t\to\infty}\int_0^t xe^{-x}\,dx$$
$$= \lim_{t\to\infty}\Big[-xe^{-x} - e^{-x}\Big]_0^t \qquad [\text{use parts wtih } u = x \text{ and } dv = e^{-x}\,dx]$$
$$= \lim_{t\to\infty}\left[(-te^{-t} - e^{-t}) - (-1)\right]$$
$$= 0 \quad [\text{use l'Hospital's Rule}] \quad - 0 + 1 = 1$$

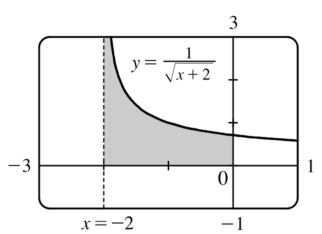
53.



$$\text{Area} = \int_0^{\pi/2}\sec^2 x\,dx = \lim_{t\to(\pi/2)^-}\int_0^t \sec^2 x\,dx = \lim_{t\to(\pi/2)^-}\Big[\tan x\Big]_0^t$$
$$= \lim_{t\to(\pi/2)^-}(\tan t - 0) = \infty$$

Infinite area

54.



$$\text{Area} = \int_{-2}^{0}\frac{1}{\sqrt{x+2}}\,dx = \lim_{t\to-2^+}\int_t^0 \frac{1}{\sqrt{x+2}}\,dx = \lim_{t\to-2^+}\left[2\sqrt{x+2}\right]_t^0$$
$$= \lim_{t\to-2^+}\left(2\sqrt{2} - 2\sqrt{t+2}\right) = 2\sqrt{2} - 0 = 2\sqrt{2}$$

55. (a)

t	$\int_1^t g(x)\,dx$
2	0.447453
5	0.577101
10	0.621306
100	0.668479
1000	0.672957
10,000	0.673407

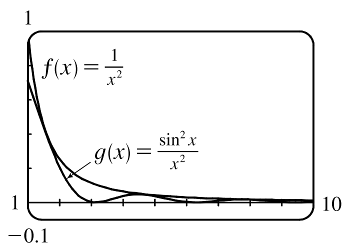
$g(x) = \dfrac{\sin^2 x}{x^2}$.

It appears that the integral is convergent.

(b) $-1 \le \sin x \le 1 \quad \Rightarrow \quad 0 \le \sin^2 x \le 1 \quad \Rightarrow \quad 0 \le \dfrac{\sin^2 x}{x^2} \le \dfrac{1}{x^2}$. Since $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ is convergent

[Theorem 2 with $p = 2 > 1$], $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ is convergent by the Comparison Theorem.

(c)



Since $\int_1^\infty f(x)\,dx$ is finite and the area under $g(x)$ is less than the area under $f(x)$ on any interval $[1, t]$, $\int_1^\infty g(x)\,dx$ must be finite; that is, the integral is convergent.

56. (a)

t	$\int_2^t g(x)dx$
5	3.830327
10	6.801200
100	23.328769
1000	69.023361
10,000	208.124560

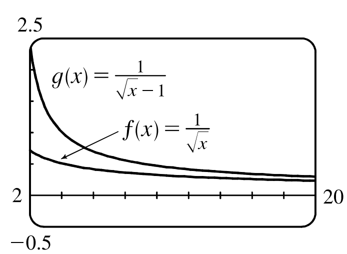
$g(x) = \dfrac{1}{\sqrt{x} - 1}$.

It appears that the integral is divergent.

(b) For $x \ge 2$, $\sqrt{x} > \sqrt{x} - 1 \quad \Rightarrow \quad \dfrac{1}{\sqrt{x}} < \dfrac{1}{\sqrt{x} - 1}$. Since $\displaystyle\int_2^\infty \frac{1}{\sqrt{x}}\,dx$ is divergent [Theorem 2 with $p = \frac{1}{2} \le 1$],

$\displaystyle\int_2^\infty \frac{1}{\sqrt{x} - 1}\,dx$ is divergent by the Comparison Theorem.

(c)



Since $\int_2^\infty f(x)\,dx$ is infinite and the area under $g(x)$ is greater than the area under $f(x)$ on any interval $[2, t]$, $\int_2^\infty g(x)\,dx$ must be infinite; that is, the integral is divergent.

57. For $x > 0$, $\dfrac{x}{x^3 + 1} < \dfrac{x}{x^3} = \dfrac{1}{x^2}$. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ is convergent by Theorem 2 with $p = 2 > 1$, so $\displaystyle\int_1^\infty \frac{x}{x^3 + 1}\,dx$ is convergent by the Comparison Theorem. $\displaystyle\int_0^1 \frac{x}{x^3 + 1}\,dx$ is a constant, so $\displaystyle\int_0^\infty \frac{x}{x^3 + 1}\,dx = \int_0^1 \frac{x}{x^3 + 1}\,dx + \int_1^\infty \frac{x}{x^3 + 1}\,dx$ is also convergent.

58. For $x \ge 1$, $\dfrac{1 + \sin^2 x}{\sqrt{x}} \ge \dfrac{1}{\sqrt{x}}$. $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ is divergent by Theorem 2 with $p = \frac{1}{2} \le 1$, so $\displaystyle\int_1^\infty \frac{1 + \sin^2 x}{\sqrt{x}}\,dx$ is divergent by the Comparison Theorem.

59. For $x \ge 1$, $\dfrac{1}{x - \ln x} \ge \dfrac{1}{x}$. $\displaystyle\int_2^\infty \frac{1}{x}\,dx$ is divergent by Equation 2 with $p = 1 \le 1$, so $\displaystyle\int_2^\infty \frac{1}{x - \ln x}\,dx$ is divergent by the Comparison Theorem.

60. For $x \geq 0$, $\arctan x < \dfrac{\pi}{2} < 2$, so $\dfrac{\arctan x}{2+e^x} < \dfrac{2}{2+e^x} < \dfrac{2}{e^x} = 2e^{-x}$. Now

$I = \displaystyle\int_0^\infty 2e^{-x}\,dx = \lim_{t\to\infty}\int_0^t 2e^{-x}\,dx = \lim_{t\to\infty}\left[-2e^{-x}\right]_0^t = \lim_{t\to\infty}\left(-\frac{2}{e^t}+2\right) = 2$, so I is convergent, and by comparison,

$\displaystyle\int_0^\infty \frac{\arctan x}{2+e^x}\,dx$ is convergent.

61. For $x > 1$, $f(x) = \dfrac{x+1}{\sqrt{x^4-x}} > \dfrac{x+1}{\sqrt{x^4}} > \dfrac{x}{x^2} = \dfrac{1}{x}$, so $\displaystyle\int_2^\infty f(x)\,dx$ diverges by comparison with $\displaystyle\int_2^\infty \frac{1}{x}\,dx$, which diverges

by Theorem 2 with $p = 1 \leq 1$. Thus, $\int_1^\infty f(x)\,dx = \int_1^2 f(x)\,dx + \int_2^\infty f(x)\,dx$ also diverges.

62. For $x > 1$, $\dfrac{2+\cos x}{\sqrt{x^4+x^2}} \leq \dfrac{2+1}{\sqrt{x^4+x^2}} < \dfrac{3}{\sqrt{x^4}} = \dfrac{3}{x^2}$. $\displaystyle\int_1^\infty \frac{3}{x^2}\,dx = 3\int_1^\infty \frac{1}{x^2}\,dx$ is convergent by Equation 2 with

$p = 2 > 1$, so $\displaystyle\int_1^\infty \frac{2+\cos x}{\sqrt{x^4+x^2}}\,dx$ is convergent by the Comparison Theorem.

63. For $0 < x \leq 1$, $\dfrac{\sec^2 x}{x\sqrt{x}} > \dfrac{1}{x^{3/2}}$. Now

$I = \displaystyle\int_0^1 x^{-3/2}\,dx = \lim_{t\to 0^+}\int_t^1 x^{-3/2}\,dx = \lim_{t\to 0^+}\left[-2x^{-1/2}\right]_t^1 = \lim_{t\to 0^+}\left(-2+\frac{2}{\sqrt{t}}\right) = \infty$, so I is divergent, and by

comparison, $\displaystyle\int_0^1 \frac{\sec^2 x}{x\sqrt{x}}$ is divergent.

64. For $0 < x \leq 1$, $\dfrac{\sin^2 x}{\sqrt{x}} \leq \dfrac{1}{\sqrt{x}}$. Now

$I = \displaystyle\int_0^\pi \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\int_t^\pi x^{-1/2}\,dx = \lim_{t\to 0^+}\left[2x^{1/2}\right]_t^\pi = \lim_{t\to 0^+}\left(2\pi - 2\sqrt{t}\right) = 2\pi - 0 = 2\pi$, so I is convergent, and by

comparison, $\displaystyle\int_0^\pi \frac{\sin^2 x}{\sqrt{x}}\,dx$ is convergent.

65. $I = \displaystyle\int_0^\infty \frac{1}{x^2}\,dx = \int_0^1 \frac{1}{x^2}\,dx + \int_1^\infty \frac{1}{x^2}\,dx = I_1 + I_2$. Now,

$I_1 = \displaystyle\lim_{t\to 0^+}\int_t^1 \frac{1}{x^2}\,dx = \lim_{t\to 0^+}\left[-x^{-1}\right]_t^1 = \lim_{t\to 0^+}\left[-1+\frac{1}{t}\right] = \infty$. Since I_1 is divergent, I is divergent, and there is no need

to evaluate I_2.

66. $I = \displaystyle\int_0^\infty \frac{1}{\sqrt{x}}\,dx = \int_0^1 \frac{1}{x^{1/2}}\,dx + \int_1^\infty \frac{1}{x^{1/2}}\,dx = I_1 + I_2$. Since I_2 is divergent $\left[\text{Equation 2 with } p = \tfrac{1}{2} \leq 1\right]$,

I is divergent, and there is no need to evaluate I_1.

67. $\displaystyle\int_0^\infty \frac{1}{\sqrt{x}\,(1+x)}\,dx = \int_0^1 \frac{dx}{\sqrt{x}\,(1+x)} + \int_1^\infty \frac{dx}{\sqrt{x}\,(1+x)} = \lim_{t\to 0^+}\int_t^1 \frac{dx}{\sqrt{x}\,(1+x)} + \lim_{t\to\infty}\int_1^t \frac{dx}{\sqrt{x}\,(1+x)}$. Now

$\displaystyle\int \frac{dx}{\sqrt{x}\,(1+x)} = \int \frac{2u\,du}{u(1+u^2)}\quad\left[\begin{array}{c} u = \sqrt{x},\, x = u^2, \\ dx = 2u\,du \end{array}\right] = 2\int \frac{du}{1+u^2} = 2\tan^{-1}u + C = 2\tan^{-1}\sqrt{x} + C$, so

$$\int_0^\infty \frac{dx}{\sqrt{x}\,(1+x)} = \lim_{t\to 0^+}\left[2\tan^{-1}\sqrt{x}\right]_t^1 + \lim_{t\to\infty}\left[2\tan^{-1}\sqrt{x}\right]_1^t$$

$$= \lim_{t\to 0^+}\left[2\left(\tfrac{\pi}{4}\right) - 2\tan^{-1}\sqrt{t}\right] + \lim_{t\to\infty}\left[2\tan^{-1}\sqrt{t} - 2\left(\tfrac{\pi}{4}\right)\right] = \tfrac{\pi}{2} - 0 + 2\left(\tfrac{\pi}{2}\right) - \tfrac{\pi}{2} = \pi.$$

68. $\displaystyle\int_2^\infty \frac{1}{x\sqrt{x^2-4}}\,dx = \int_2^3 \frac{dx}{x\sqrt{x^2-4}} + \int_3^\infty \frac{dx}{x\sqrt{x^2-4}} = \lim_{t\to 2^+}\int_t^3 \frac{dx}{x\sqrt{x^2-4}} + \lim_{t\to\infty}\int_3^t \frac{dx}{x\sqrt{x^2-4}}$. Now

$$\int \frac{dx}{x\sqrt{x^2-4}} = \int \frac{2\sec\theta\,\tan\theta\,d\theta}{2\sec\theta\,2\tan\theta}\quad\left[\begin{array}{c}x=2\sec\theta,\text{ where}\\ 0\le\theta<\pi/2\text{ or }\pi\le\theta<3\pi/2\end{array}\right] = \tfrac{1}{2}\theta + C = \tfrac{1}{2}\sec^{-1}\left(\tfrac{1}{2}x\right) + C, \text{ so}$$

$$\int_2^\infty \frac{dx}{x\sqrt{x^2-4}} = \lim_{t\to 2^+}\left[\tfrac{1}{2}\sec^{-1}\left(\tfrac{1}{2}x\right)\right]_t^3 + \lim_{t\to\infty}\left[\tfrac{1}{2}\sec^{-1}\left(\tfrac{1}{2}x\right)\right]_3^t = \tfrac{1}{2}\sec^{-1}\left(\tfrac{3}{2}\right) - 0 + \tfrac{1}{2}\left(\tfrac{\pi}{2}\right) - \tfrac{1}{2}\sec^{-1}\left(\tfrac{3}{2}\right) = \tfrac{\pi}{4}.$$

69. If $p=1$, then $\displaystyle\int_0^1 \frac{1}{x^p}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{dx}{x} = \lim_{t\to 0^+}\left[\ln x\right]_t^1 = \infty.$ Divergent

If $p\neq 1$, then $\displaystyle\int_0^1 \frac{dx}{x^p} = \lim_{t\to 0^+}\int_t^1 \frac{dx}{x^p}$ [note that the integral is not improper if $p<0$]

$$= \lim_{t\to 0^+}\left[\frac{x^{-p+1}}{-p+1}\right]_t^1 = \lim_{t\to 0^+}\frac{1}{1-p}\left[1-\frac{1}{t^{p-1}}\right]$$

If $p>1$, then $p-1>0$, so $\dfrac{1}{t^{p-1}}\to\infty$ as $t\to 0^+$, and the integral diverges.

If $p<1$, then $p-1<0$, so $\dfrac{1}{t^{p-1}}\to 0$ as $t\to 0^+$ and $\displaystyle\int_0^1 \frac{dx}{x^p} = \frac{1}{1-p}\left[\lim_{t\to 0^+}\left(1-t^{1-p}\right)\right] = \frac{1}{1-p}.$

Thus, the integral converges if and only if $p<1$, and in that case its value is $\dfrac{1}{1-p}$.

70. Let $u=\ln x$. Then $du = dx/x \Rightarrow \displaystyle\int_e^\infty \frac{1}{x\,(\ln x)^p}\,dx = \int_1^\infty \frac{du}{u^p}$. By Example 4, this converges to $\dfrac{1}{p-1}$ if $p>1$ and diverges otherwise.

71. First suppose $p=-1$. Then

$\displaystyle\int_0^1 x^p\ln x\,dx = \int_0^1 \frac{\ln x}{x}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{\ln x}{x}\,dx = \lim_{t\to 0^+}\left[\tfrac{1}{2}(\ln x)^2\right]_t^1 = -\tfrac{1}{2}\lim_{t\to 0^+}(\ln t)^2 = -\infty$, so the integral diverges. Now suppose $p\neq -1$. Then integration by parts gives

$\displaystyle\int x^p\ln x\,dx = \frac{x^{p+1}}{p+1}\ln x - \int \frac{x^p}{p+1}\,dx = \frac{x^{p+1}}{p+1}\ln x - \frac{x^{p+1}}{(p+1)^2} + C$. If $p<-1$, then $p+1<0$, so

$$\int_0^1 x^p\ln x\,dx = \lim_{t\to 0^+}\left[\frac{x^{p+1}}{p+1}\ln x - \frac{x^{p+1}}{(p+1)^2}\right]_t^1 = \frac{-1}{(p+1)^2} - \left(\frac{1}{p+1}\right)\lim_{t\to 0^+}\left[t^{p+1}\left(\ln t - \frac{1}{p+1}\right)\right] = \infty.$$

If $p>-1$, then $p+1>0$ and

$$\int_0^1 x^p\ln x\,dx = \frac{-1}{(p+1)^2} - \left(\frac{1}{p+1}\right)\lim_{t\to 0^+}\frac{\ln t - 1/(p+1)}{t^{-(p+1)}} \overset{\text{H}}{=} \frac{-1}{(p+1)^2} - \left(\frac{1}{p+1}\right)\lim_{t\to 0^+}\frac{1/t}{-(p+1)t^{-(p+2)}}$$

$$= \frac{-1}{(p+1)^2} + \frac{1}{(p+1)^2}\lim_{t\to 0^+} t^{p+1} = \frac{-1}{(p+1)^2}$$

Thus, the integral converges to $-\dfrac{1}{(p+1)^2}$ if $p>-1$ and diverges otherwise.

72. (a) $\boldsymbol{n=0}$: $\displaystyle\int_0^\infty x^n e^{-x}\,dx = \lim_{t\to\infty}\int_0^t e^{-x}\,dx = \lim_{t\to\infty}\left[-e^{-x}\right]_0^t = \lim_{t\to\infty}\left[-e^{-t}+1\right] = 0+1=1$

$\boldsymbol{n=1}$: $\displaystyle\int_0^\infty x^n e^{-x}\,dx = \lim_{t\to\infty}\int_0^t xe^{-x}\,dx$. To evaluate $\displaystyle\int xe^{-x}\,dx$, we'll use integration by parts with $u=x$, $dv=e^{-x}\,dx \;\Rightarrow\; du=dx$, $v=-e^{-x}$.

So $\displaystyle\int xe^{-x}\,dx = -xe^{-x} - \int -e^{-x}\,dx = -xe^{-x}-e^{-x}+C = (-x-1)e^{-x}+C$ and

$$\begin{aligned}\lim_{t\to\infty}\int_0^t xe^{-x}\,dx &= \lim_{t\to\infty}\left[(-x-1)e^{-x}\right]_0^t = \lim_{t\to\infty}\left[(-t-1)e^{-t}+1\right] = \lim_{t\to\infty}\left[-te^{-t}-e^{-t}+1\right]\\ &= 0-0+1 \quad [\text{use l'Hospital's Rule}] \quad = 1\end{aligned}$$

$\boldsymbol{n=2}$: $\displaystyle\int_0^\infty x^n e^{-x}\,dx = \lim_{t\to\infty}\int_0^t x^2e^{-x}\,dx$. To evaluate $\displaystyle\int x^2e^{-x}\,dx$, we could use integration by parts again or Formula 97. Thus,

$$\begin{aligned}\lim_{t\to\infty}\int_0^t x^2e^{-x}\,dx &= \lim_{t\to\infty}\left[-x^2e^{-x}\right]_0^t + 2\lim_{t\to\infty}\int_0^t xe^{-x}\,dx\\ &= 0+0+2(1) \quad [\text{use l'Hospital's Rule and the result for } n=1] \quad = 2\end{aligned}$$

$\boldsymbol{n=3}$:
$$\begin{aligned}\int_0^\infty x^n e^{-x}dx &= \lim_{t\to\infty}\int_0^t x^3e^{-x}\,dx \overset{97}{=} \lim_{t\to\infty}\left[-x^3e^{-x}\right]_0^t + 3\lim_{t\to\infty}\int_0^t x^2e^{-x}\,dx\\ &= 0+0+3(2) \quad [\text{use l'Hospital's Rule and the result for } n=2] \quad = 6\end{aligned}$$

(b) For $n=1$, 2, and 3, we have $\int_0^\infty x^ne^{-x}\,dx = 1$, 2, and 6. The values for the integral are equal to the factorials for n, so we guess $\int_0^\infty x^ne^{-x}\,dx = n!$.

(c) Suppose that $\int_0^\infty x^ke^{-x}\,dx = k!$ for some positive integer k. Then $\int_0^\infty x^{k+1}e^{-x}\,dx = \lim\limits_{t\to\infty}\int_0^t x^{k+1}e^{-x}\,dx$.

To evaluate $\int x^{k+1}e^{-x}\,dx$, we use parts with $u=x^{k+1}$, $dv=e^{-x}\,dx \;\Rightarrow\; du=(k+1)x^k\,dx$, $v=-e^{-x}$.

So $\int x^{k+1}e^{-x}dx = -x^{k+1}e^{-x} - \int -(k+1)x^ke^{-x}\,dx = -x^{k+1}e^{-x} + (k+1)\int x^ke^{-x}\,dx$ and

$$\begin{aligned}\lim_{t\to\infty}\int_0^t x^{k+1}e^{-x}\,dx &= \lim_{t\to\infty}\left[-x^{k+1}e^{-x}\right]_0^t + (k+1)\lim_{t\to\infty}\int_0^t x^ke^{-x}\,dx\\ &= \lim_{t\to\infty}\left[-t^{k+1}e^{-t}+0\right] + (k+1)k! = 0+0+(k+1)! = (k+1)!,\end{aligned}$$

so the formula holds for $k+1$. By induction, the formula holds for all positive integers. (Since $0!=1$, the formula holds for $n=0$, too.)

73. $\displaystyle I=\int_{-\infty}^\infty x\,dx = \int_{-\infty}^0 x\,dx + \int_0^\infty x\,dx$ and $\displaystyle\int_0^\infty x\,dx = \lim_{t\to\infty}\int_0^t x\,dx = \lim_{t\to\infty}\left[\tfrac12x^2\right]_0^t = \lim_{t\to\infty}\left[\tfrac12t^2-0\right]=\infty$,

so I is divergent. The Cauchy principal value of I is given by

$\displaystyle\lim_{t\to\infty}\int_{-t}^t x\,dx = \lim_{t\to\infty}\left[\tfrac12x^2\right]_{-t}^t = \lim_{t\to\infty}\left[\tfrac12t^2-\tfrac12(-t)^2\right] = \lim_{t\to\infty}[0] = 0$. Hence, I is divergent, but its Cauchy principal value is 0.

74. Let $k = \dfrac{M}{2RT}$ so that $\overline{v} = \dfrac{4}{\sqrt{\pi}}\, k^{3/2} \displaystyle\int_0^\infty v^3 e^{-kv^2}\,dv$. Let I denote the integral and use parts to integrate I. Let $\alpha = v^2$,

$d\beta = ve^{-kv^2}\,dv \quad \Rightarrow \quad d\alpha = 2v\,dv$, $\beta = -\dfrac{1}{2k}e^{-kv^2}$:

$$I = \lim_{t\to\infty}\left[-\frac{1}{2k}v^2 e^{-kv^2}\right]_0^t + \frac{1}{k}\int_0^\infty ve^{-kv^2}\,dv_0^t = -\frac{1}{2k}\lim_{t\to\infty}\left(t^2 e^{-kt^2}\right) + \frac{1}{k}\lim_{t\to\infty}\left[-\frac{1}{2k}e^{-kv^2}\right]$$

$$\overset{\text{H}}{=} -\frac{1}{2k}\cdot 0 - \frac{1}{2k^2}(0-1) = \frac{1}{2k^2}$$

Thus, $\overline{v} = \dfrac{4}{\sqrt{\pi}}k^{3/2}\cdot\dfrac{1}{2k^2} = \dfrac{2}{(k\pi)^{1/2}} = \dfrac{2}{[\pi M/(2RT)]^{1/2}} = \dfrac{2\sqrt{2}\sqrt{RT}}{\sqrt{\pi M}} = \sqrt{\dfrac{8RT}{\pi M}}$.

75. Volume $= \displaystyle\int_1^\infty \pi\left(\frac{1}{x}\right)^2 dx = \pi\lim_{t\to\infty}\int_1^t \frac{dx}{x^2} = \pi\lim_{t\to\infty}\left[-\frac{1}{x}\right]_1^t = \pi\lim_{t\to\infty}\left(1-\frac{1}{t}\right) = \pi < \infty$.

76. Work $= \displaystyle\int_R^\infty \frac{GMm}{r^2}\,dr = \lim_{t\to\infty}\int_R^t \frac{GMm}{r^2}\,dr = \lim_{t\to\infty} GMm\left[\frac{-1}{r}\right]_R^t = GMm\lim_{t\to\infty}\left(\frac{-1}{t}+\frac{1}{R}\right) = \frac{GMm}{R}$, where

M = mass of the earth = 5.98×10^{24} kg, m = mass of satellite = 10^3 kg, R = radius of the earth = 6.37×10^6 m, and

G = gravitational constant = 6.67×10^{-11} N·m²/kg.

Therefore, Work $= \dfrac{6.67\times 10^{-11}\cdot 5.98\times 10^{24}\cdot 10^3}{6.37\times 10^6} \approx 6.26\times 10^{10}$ J.

77. Work $= \displaystyle\int_R^\infty F\,dr = \lim_{t\to\infty}\int_R^t \frac{GmM}{r^2}\,dr = \lim_{t\to\infty} GmM\left(\frac{1}{R}-\frac{1}{t}\right) = \frac{GmM}{R}$. The initial kinetic energy provides the work,

so $\frac{1}{2}mv_0^2 = \dfrac{GmM}{R} \quad\Rightarrow\quad v_0 = \sqrt{\dfrac{2GM}{R}}$.

78. $y(s) = \displaystyle\int_s^R \frac{2r}{\sqrt{r^2-s^2}}\,x(r)\,dr$ and $x(r) = \frac{1}{2}(R-r)^2 \quad\Rightarrow$

$$y(s) = \lim_{t\to s^+}\int_t^R \frac{r(R-r)^2}{\sqrt{r^2-s^2}}\,dr = \lim_{t\to s^+}\int_t^R \frac{r^3-2Rr^2+R^2r}{\sqrt{r^2-s^2}}\,dr$$

$$= \lim_{t\to s^+}\left[\int_t^R \frac{r^3\,dr}{\sqrt{r^2-s^2}} - 2R\int_t^R \frac{r^2\,dr}{\sqrt{r^2-s^2}} + R^2\int_t^R \frac{r\,dr}{\sqrt{r^2-s^2}}\right] = \lim_{t\to s^+}\left(I_1 - 2RI_2 + R^2 I_3\right) = L$$

For I_1: Let $u = \sqrt{r^2-s^2} \quad\Rightarrow\quad u^2 = r^2-s^2$, $r^2 = u^2+s^2$, $2r\,dr = 2u\,du$, so, omitting limits and constant of integration,

$$I_1 = \int \frac{(u^2+s^2)u}{u}\,du = \int (u^2+s^2)\,du = \tfrac{1}{3}u^3 + s^2u = \tfrac{1}{3}u(u^2+3s^2)$$

$$= \tfrac{1}{3}\sqrt{r^2-s^2}\,(r^2-s^2+3s^2) = \tfrac{1}{3}\sqrt{r^2-s^2}\,(r^2+2s^2)$$

For I_2: Using Formula 44, $I_2 = \dfrac{r}{2}\sqrt{r^2-s^2} + \dfrac{s^2}{2}\ln\left|r+\sqrt{r^2-s^2}\right|$.

For I_3: Let $u = r^2-s^2 \quad\Rightarrow\quad du = 2r\,dr$. Then $I_3 = \dfrac{1}{2}\displaystyle\int\frac{du}{\sqrt{u}} = \tfrac{1}{2}\cdot 2\sqrt{u} = \sqrt{r^2-s^2}$.

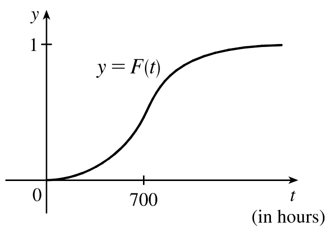
[continued]

Thus,

$$\begin{aligned}
L &= \lim_{t\to s^+}\left[\tfrac{1}{3}\sqrt{r^2-s^2}\,(r^2+2s^2) - 2R\left(\frac{r}{2}\sqrt{r^2-s^2} + \frac{s^2}{2}\ln\left|r+\sqrt{r^2-s^2}\,\right|\right) + R^2\sqrt{r^2-s^2}\right]_t^R \\
&= \lim_{t\to s^+}\left[\tfrac{1}{3}\sqrt{R^2-s^2}(R^2+2s^2) - 2R\left(\frac{R}{2}\sqrt{R^2-s^2} + \frac{s^2}{2}\ln\left|R+\sqrt{R^2-s^2}\,\right|\right) + R^2\sqrt{R^2-s^2}\right] \\
&\qquad - \lim_{t\to s^+}\left[\tfrac{1}{3}\sqrt{t^2-s^2}\,(t^2+2s^2) - 2R\left(\frac{t}{2}\sqrt{t^2-s^2} + \frac{s^2}{2}\ln\left|t+\sqrt{t^2-s^2}\,\right|\right) + R^2\sqrt{t^2-s^2}\right] \\
&= \left[\tfrac{1}{3}\sqrt{R^2-s^2}\,(R^2+2s^2) - Rs^2\ln\left|R+\sqrt{R^2-s^2}\,\right|\right] - \left[-Rs^2\ln|s|\right] \\
&= \tfrac{1}{3}\sqrt{R^2-s^2}\,(R^2+2s^2) - Rs^2\ln\left(\frac{R+\sqrt{R^2-s^2}}{s}\right)
\end{aligned}$$

79. We would expect a small percentage of bulbs to burn out in the first few hundred hours, most of the bulbs to burn out after close to 700 hours, and a few overachievers to burn on and on.

(a)



(b) $r(t) = F'(t)$ is the rate at which the fraction $F(t)$ of burnt-out bulbs increases as t increases. This could be interpreted as a fractional burnout rate.

(c) $\int_0^\infty r(t)\,dt = \lim\limits_{x\to\infty} F(x) = 1$, since all of the bulbs will eventually burn out.

80. $I = \displaystyle\int_0^\infty te^{kt}\,dt = \lim_{s\to\infty}\left[\frac{1}{k^2}(kt-1)\,e^{kt}\right]_0^s$ [Formula 96, or parts] $= \displaystyle\lim_{s\to\infty}\left[\left(\frac{1}{k}se^{ks} - \frac{1}{k^2}e^{ks}\right) - \left(-\frac{1}{k^2}\right)\right]$.

Since $k < 0$, the first two terms approach 0 (you can verify that the first term does so with l'Hospital's Rule), so the limit is equal to $1/k^2$. Thus, $M = -kI = -k\left(1/k^2\right) = -1/k = -1/(-0.000121) \approx 8264.5$ years.

81. $$\gamma = \int_0^\infty \frac{cN(1-e^{-kt})}{k}\,e^{-\lambda t}\,dt = \frac{cN}{k}\lim_{x\to\infty}\int_0^x \left[e^{-\lambda t} - e^{(-k-\lambda)t}\right]dt$$

$$= \frac{cN}{k}\lim_{x\to\infty}\left[\frac{1}{-\lambda}e^{-\lambda t} - \frac{1}{-k-\lambda}e^{(-k-\lambda)t}\right]_0^x = \frac{cN}{k}\lim_{x\to\infty}\left[\frac{1}{-\lambda e^{\lambda x}} + \frac{1}{(k+\lambda)e^{(k+\lambda)x}} - \left(\frac{1}{-\lambda} + \frac{1}{k+\lambda}\right)\right]$$

$$= \frac{cN}{k}\left(\frac{1}{\lambda} - \frac{1}{k+\lambda}\right) = \frac{cN}{k}\left(\frac{k+\lambda-\lambda}{\lambda(k+\lambda)}\right) = \frac{cN}{\lambda(k+\lambda)}$$

82. $$\int_0^\infty u(t)\,dt = \lim_{x\to\infty}\int_0^x \frac{r}{V}C_0\,e^{-rt/V}\,dt = \frac{r}{V}C_0\lim_{x\to\infty}\left[\frac{e^{-rt/V}}{-r/V}\right]_0^x = \frac{r}{V}C_0\left(-\frac{V}{r}\right)\lim_{x\to\infty}\left(e^{-rx/V}-1\right)$$

$$= -C_0(0-1) = C_0.$$

$\int_0^\infty u(t)\,dt$ represents the total amount of urea removed from the blood if dialysis is continued indefinitely. The fact that $\int_0^\infty u(t)\,dt = C_0$ means that, in the limit, as $t\to\infty$, all the urea in the blood at time $t=0$ is removed. The calculation says nothing about how rapidly that limit is approached.

83. $I = \displaystyle\int_a^\infty \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \int_a^t \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \left[\tan^{-1} x\right]_a^t = \lim_{t\to\infty} \left(\tan^{-1} t - \tan^{-1} a\right) = \frac{\pi}{2} - \tan^{-1} a.$

$I < 0.001 \quad\Rightarrow\quad \frac{\pi}{2} - \tan^{-1} a < 0.001 \quad\Rightarrow\quad \tan^{-1} a > \frac{\pi}{2} - 0.001 \quad\Rightarrow\quad a > \tan\left(\frac{\pi}{2} - 0.001\right) \approx 1000.$

84. $f(x) = e^{-x^2}$ and $\Delta x = \frac{4-0}{8} = \frac{1}{2}$.

$$\int_0^4 f(x)\,dx \approx S_8 = \tfrac{1}{2\cdot 3}[f(0) + 4f(0.5) + 2f(1) + \cdots + 2f(3) + 4f(3.5) + f(4)] \approx \tfrac{1}{6}(5.31717808) \approx 0.8862$$

Now $x > 4 \quad\Rightarrow\quad -x\cdot x < -x\cdot 4 \quad\Rightarrow\quad e^{-x^2} < e^{-4x} \quad\Rightarrow\quad \int_4^\infty e^{-x^2}\,dx < \int_4^\infty e^{-4x}\,dx.$

$\int_4^\infty e^{-4x}\,dx = \lim\limits_{t\to\infty} \left[-\frac{1}{4}e^{-4x}\right]_4^t = -\frac{1}{4}\left(0 - e^{-16}\right) = 1/(4e^{16}) \approx 0.0000000281 < 0.0000001$, as desired.

85. (a) $F(s) = \displaystyle\int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty e^{-st}\,dt = \lim_{n\to\infty} \left[-\frac{e^{-st}}{s}\right]_0^n = \lim_{n\to\infty} \left(\frac{e^{-sn}}{-s} + \frac{1}{s}\right)$. This converges to $\dfrac{1}{s}$ only if $s > 0$.

Therefore $F(s) = \dfrac{1}{s}$ with domain $\{s \mid s > 0\}$.

(b) $F(s) = \displaystyle\int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty e^t e^{-st}\,dt = \lim_{n\to\infty} \int_0^n e^{t(1-s)}\,dt = \lim_{n\to\infty} \left[\frac{1}{1-s}e^{t(1-s)}\right]_0^n$

$\quad = \displaystyle\lim_{n\to\infty} \left(\frac{e^{(1-s)n}}{1-s} - \frac{1}{1-s}\right)$

This converges only if $1 - s < 0 \quad\Rightarrow\quad s > 1$, in which case $F(s) = \dfrac{1}{s-1}$ with domain $\{s \mid s > 1\}$.

(c) $F(s) = \int_0^\infty f(t)e^{-st}\,dt = \lim\limits_{n\to\infty} \int_0^n te^{-st}\,dt$. Use integration by parts: let $u = t$, $dv = e^{-st}\,dt \quad\Rightarrow\quad du = dt$,

$v = -\dfrac{e^{-st}}{s}$. Then $F(s) = \displaystyle\lim_{n\to\infty} \left[-\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\right]_0^n = \lim_{n\to\infty} \left(\frac{-n}{se^{sn}} - \frac{1}{s^2 e^{sn}} + 0 + \frac{1}{s^2}\right) = \frac{1}{s^2}$ only if $s > 0$.

Therefore, $F(s) = \dfrac{1}{s^2}$ and the domain of F is $\{s \mid s > 0\}$.

86. $0 \le f(t) \le Me^{at} \quad\Rightarrow\quad 0 \le f(t)e^{-st} \le Me^{at}e^{-st}$ for $t \ge 0$. Now use the Comparison Theorem:

$$\int_0^\infty Me^{at}e^{-st}\,dt = \lim_{n\to\infty} M \int_0^n e^{t(a-s)}\,dt = M\cdot \lim_{n\to\infty} \left[\frac{1}{a-s}e^{t(a-s)}\right]_0^n = M\cdot \lim_{n\to\infty} \frac{1}{a-s}\left[e^{n(a-s)} - 1\right]$$

This is convergent only when $a - s < 0 \quad\Rightarrow\quad s > a$. Therefore, by the Comparison Theorem, $F(s) = \int_0^\infty f(t)\,e^{-st}\,dt$ is also convergent for $s > a$.

87. $G(s) = \int_0^\infty f'(t)e^{-st}\,dt$. Integrate by parts with $u = e^{-st}$, $dv = f'(t)\,dt \quad\Rightarrow\quad du = -se^{-st}$, $v = f(t)$:

$$G(s) = \lim_{n\to\infty} \left[f(t)e^{-st}\right]_0^n + s\int_0^\infty f(t)e^{-st}\,dt = \lim_{n\to\infty} f(n)e^{-sn} - f(0) + sF(s)$$

But $0 \le f(t) \le Me^{at} \quad\Rightarrow\quad 0 \le f(t)e^{-st} \le Me^{at}e^{-st}$ and $\lim\limits_{t\to\infty} Me^{t(a-s)} = 0$ for $s > a$. So by the Squeeze Theorem,

$\lim\limits_{t\to\infty} f(t)e^{-st} = 0$ for $s > a \quad\Rightarrow\quad G(s) = 0 - f(0) + sF(s) = sF(s) - f(0)$ for $s > a$.

88. Assume without loss of generality that $a < b$. Then

$$\begin{aligned}\int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx &= \lim_{t\to-\infty}\int_{t}^{a} f(x)\,dx + \lim_{u\to\infty}\int_{a}^{u} f(x)\,dx \\ &= \lim_{t\to-\infty}\int_{t}^{a} f(x)\,dx + \lim_{u\to\infty}\left[\int_{a}^{b} f(x)\,dx + \int_{b}^{u} f(x)\,dx\right] \\ &= \lim_{t\to-\infty}\int_{t}^{a} f(x)\,dx + \int_{a}^{b} f(x)\,dx + \lim_{u\to\infty}\int_{b}^{u} f(x)\,dx \\ &= \lim_{t\to-\infty}\left[\int_{t}^{a} f(x)\,dx + \int_{a}^{b} f(x)\,dx\right] + \int_{b}^{\infty} f(x)\,dx \\ &= \lim_{t\to-\infty}\int_{t}^{b} f(x)\,dx + \int_{b}^{\infty} f(x)\,dx = \int_{-\infty}^{b} f(x)\,dx + \int_{b}^{\infty} f(x)\,dx\end{aligned}$$

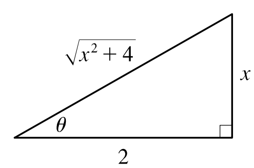
89. We use integration by parts: let $u = x$, $dv = xe^{-x^2}\,dx \quad\Rightarrow\quad du = dx$, $v = -\frac{1}{2}e^{-x^2}$. So

$$\int_0^\infty x^2 e^{-x^2}\,dx = \lim_{t\to\infty}\left[-\frac{1}{2}xe^{-x^2}\right]_0^t + \frac{1}{2}\int_0^\infty e^{-x^2}\,dx = \lim_{t\to\infty}\left[-\frac{t}{2e^{t^2}}\right] + \frac{1}{2}\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\int_0^\infty e^{-x^2}\,dx$$

(The limit is 0 by l'Hospital's Rule.)

90. $\int_0^\infty e^{-x^2}\,dx$ is the area under the curve $y = e^{-x^2}$ for $0 \le x < \infty$ and $0 < y \le 1$. Solving $y = e^{-x^2}$ for x, we get $y = e^{-x^2} \quad\Rightarrow\quad \ln y = -x^2 \quad\Rightarrow\quad -\ln y = x^2 \quad\Rightarrow\quad x = \pm\sqrt{-\ln y}$. Since x is positive, choose $x = \sqrt{-\ln y}$, and the area is represented by $\int_0^1 \sqrt{-\ln y}\,dy$. Therefore, each integral represents the same area, so the integrals are equal.

91. For the first part of the integral, let $x = 2\tan\theta \quad\Rightarrow\quad dx = 2\sec^2\theta\,d\theta$.



$$\int \frac{1}{\sqrt{x^2+4}}\,dx = \int \frac{2\sec^2\theta}{2\sec\theta}\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta|.$$

From the figure, $\tan\theta = \dfrac{x}{2}$, and $\sec\theta = \dfrac{\sqrt{x^2+4}}{2}$. So

$$\begin{aligned} I = \int_0^\infty \left(\frac{1}{\sqrt{x^2+4}} - \frac{C}{x+2}\right)dx &= \lim_{t\to\infty}\left[\ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| - C\ln|x+2|\right]_0^t \\ &= \lim_{t\to\infty}\left[\ln\frac{\sqrt{t^2+4}+t}{2} - C\ln(t+2) - (\ln 1 - C\ln 2)\right] \\ &= \lim_{t\to\infty}\left[\ln\left(\frac{\sqrt{t^2+4}+t}{2\,(t+2)^C}\right) + \ln 2^C\right] = \ln\left(\lim_{t\to\infty}\frac{t+\sqrt{t^2+4}}{(t+2)^C}\right) + \ln 2^{C-1}\end{aligned}$$

Now $L = \displaystyle\lim_{t\to\infty}\frac{t+\sqrt{t^2+4}}{(t+2)^C} \overset{\text{H}}{=} \lim_{t\to\infty}\frac{1+t/\sqrt{t^2+4}}{C\,(t+2)^{C-1}} = \frac{2}{C\displaystyle\lim_{t\to\infty}(t+2)^{C-1}}$.

If $C < 1$, $L = \infty$ and I diverges.

If $C = 1$, $L = 2$ and I converges to $\ln 2 + \ln 2^0 = \ln 2$.

If $C > 1$, $L = 0$ and I diverges to $-\infty$.

92. $I = \int_0^\infty \left(\frac{x}{x^2+1} - \frac{C}{3x+1}\right) dx = \lim_{t\to\infty} \left[\frac{1}{2}\ln(x^2+1) - \frac{1}{3}C\ln(3x+1)\right]_0^t = \lim_{t\to\infty} \left[\ln(t^2+1)^{1/2} - \ln(3t+1)^{C/3}\right]$

$$= \lim_{t\to\infty} \left(\ln \frac{(t^2+1)^{1/2}}{(3t+1)^{C/3}}\right) = \ln\left(\lim_{t\to\infty} \frac{\sqrt{t^2+1}}{(3t+1)^{C/3}}\right)$$

For $C \le 0$, the integral diverges. For $C > 0$, we have

$$L = \lim_{t\to\infty} \frac{\sqrt{t^2+1}}{(3t+1)^{C/3}} \stackrel{\text{H}}{=} \lim_{t\to\infty} \frac{t/\sqrt{t^2+1}}{C(3t+1)^{(C/3)-1}} = \frac{1}{C} \lim_{t\to\infty} \frac{1}{(3t+1)^{(C/3)-1}}$$

For $C/3 < 1 \quad\Leftrightarrow\quad C < 3$, $L = \infty$ and I diverges.

For $C = 3$, $L = \frac{1}{3}$ and $I = \ln\frac{1}{3}$.

For $C > 3$, $L = 0$ and I diverges to $-\infty$.

93. No, $I = \int_0^\infty f(x)\,dx$ must be *divergent*. Since $\lim_{x\to\infty} f(x) = 1$, there must exist an N such that if $x \ge N$, then $f(x) \ge \frac{1}{2}$.

Thus, $I = I_1 + I_2 = \int_0^N f(x)\,dx + \int_N^\infty f(x)\,dx$, where I_1 is an ordinary definite integral that has a finite value, and I_2 is improper and diverges by comparison with the divergent integral $\int_N^\infty \frac{1}{2}\,dx$.

94. As in Exercises 65–68, we let $I = \int_0^\infty \frac{x^a}{1+x^b}\,dx = I_1 + I_2$, where $I_1 = \int_0^1 \frac{x^a}{1+x^b}\,dx$ and $I_2 = \int_1^\infty \frac{x^a}{1+x^b}\,dx$. We will show that I_1 converges for $a > -1$ and I_2 converges for $b > a+1$, so that I converges when $a > -1$ and $b > a+1$.

I_1 is improper only when $a < 0$. When $0 \le x \le 1$, we have $\frac{1}{1+x^b} \le 1 \quad\Rightarrow\quad \frac{1}{x^{-a}(1+x^b)} \le \frac{1}{x^{-a}}$. The integral $\int_0^1 \frac{1}{x^{-a}}\,dx$ converges for $-a < 1$ [or $a > -1$] by Exercise 69, so by the Comparison Theorem, $\int_0^1 \frac{1}{x^{-a}(1+x^b)}\,dx$ converges for $-1 < a < 0$. I_1 is not improper when $a \ge 0$, so it has a finite real value in that case. Therefore, I_1 has a finite real value (converges) when $a > -1$.

I_2 is always improper. When $x \ge 1$, $\frac{x^a}{1+x^b} = \frac{1}{x^{-a}(1+x^b)} = \frac{1}{x^{-a}+x^{b-a}} < \frac{1}{x^{b-a}}$. By (2), $\int_1^\infty \frac{1}{x^{b-a}}\,dx$ converges for $b - a > 1$ (or $b > a+1$), so by the Comparison Theorem, $\int_1^\infty \frac{x^a}{1+x^b}\,dx$ converges for $b > a+1$.

Thus, I converges if $a > -1$ and $b > a+1$.

7 Review

TRUE-FALSE QUIZ

1. True. See Example 5 in Section 7.1.

2. True. Integration by parts can be used to show that $\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx$, so that the power of x in the new integrand is reduced by 1. Hence, when $n = 5$, repeatedly applying integration by parts five times will reduce the final integral to $\int e^x\,dx$, which evaluates to e^x.

3. False. Substituting $x = 5\sin\theta$ into $\sqrt{25 + x^2}$ gives $\sqrt{25 + 25\sin^2\theta}$. This expression cannot be further simplified using a trigonometric identity. A more useful substitiution would be $x = 5\tan\theta$.

4. False. To use entry 25, we need to first write $\displaystyle\int \frac{dx}{\sqrt{9 + e^{2x}}}$ in the form $\displaystyle\int \frac{du}{\sqrt{9 + u^2}}$, which suggests making the substitution $u = e^x$, so that $du = e^x\,dx$, or $du/u = dx$. Thus, $\displaystyle\int \frac{dx}{\sqrt{9 + e^{2x}}} = \int \frac{du}{u\sqrt{9 + u^2}}$, however, entry 25 cannot be used to evaluate this new integral. Instead, entry 27 would be needed.

5. False. Since the numerator has a higher degree than the denominator, $\dfrac{x(x^2 + 4)}{x^2 - 4} = x + \dfrac{8x}{x^2 - 4} = x + \dfrac{A}{x + 2} + \dfrac{B}{x - 2}$.

6. True. $\dfrac{x^2 + 4}{x(x^2 - 4)} = \dfrac{-1}{x} + \dfrac{1}{x + 2} + \dfrac{1}{x - 2}$

7. False. $\dfrac{x^2 + 4}{x^2(x - 4)}$ can be put in the form $\dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x - 4}$.

8. False. $\dfrac{x^2 - 4}{x(x^2 + 4)}$ can be put into the form $\dfrac{A}{x} + \dfrac{Bx + C}{x^2 + 4}$.

9. False. This is an improper integral, since the denominator vanishes at $x = 1$.

$$\int_0^4 \frac{x}{x^2 - 1}\,dx = \int_0^1 \frac{x}{x^2 - 1}\,dx + \int_1^4 \frac{x}{x^2 - 1}\,dx \text{ and}$$

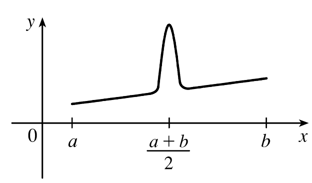
$$\int_0^1 \frac{x}{x^2 - 1}\,dx = \lim_{t \to 1^-} \int_0^t \frac{x}{x^2 - 1}\,dx = \lim_{t \to 1^-} \left[\tfrac{1}{2}\ln\left|x^2 - 1\right|\right]_0^t = \lim_{t \to 1^-} \tfrac{1}{2}\ln\left|t^2 - 1\right| = \infty$$

So the integral diverges.

10. True by Theorem 7.8.2 with $p = \sqrt{2} > 1$.

11. True. $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_0^{\infty} f(x)\,dx = I_1 + I_2$. If $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$ is convergent, it follows that both I_1 and I_2 must be convergent.

12. False. For example, with $n = 1$ the Trapezoidal Rule is much more accurate than the Midpoint Rule for the function in the diagram.



13. (a) True. See the end of Section 7.5.

(b) False. Examples include the functions $f(x) = e^{x^2}$, $g(x) = \sin(x^2)$, and $h(x) = \dfrac{\sin x}{x}$.

14. True. If f is continuous on $[0, \infty)$, then $\int_0^1 f(x)\,dx$ is finite. Since $\int_1^{\infty} f(x)\,dx$ is finite, so is $\int_0^{\infty} f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^{\infty} f(x)\,dx$.

15. False. If $f(x) = 1/x$, then f is continuous and decreasing on $[1, \infty)$ with $\lim\limits_{x \to \infty} f(x) = 0$, but $\int_1^{\infty} f(x)\,dx$ is divergent.

16. True. $\int_a^\infty [f(x)+g(x)]\,dx = \lim\limits_{t\to\infty}\int_a^t [f(x)+g(x)]\,dx = \lim\limits_{t\to\infty}\left(\int_a^t f(x)\,dx + \int_a^t g(x)\,dx\right)$

$$= \lim_{t\to\infty}\int_a^t f(x)\,dx + \lim_{t\to\infty}\int_a^t g(x)\,dx \quad \left[\begin{array}{c}\text{since both limits}\\ \text{in the sum exist}\end{array}\right]$$

$$= \int_a^\infty f(x)\,dx + \int_a^\infty g(x)\,dx$$

Since the two integrals are finite, so is their sum.

17. False. Take $f(x)=1$ for all x and $g(x)=-1$ for all x. Then $\int_a^\infty f(x)\,dx = \infty$ [divergent] and $\int_a^\infty g(x)\,dx = -\infty$ [divergent], but $\int_a^\infty [f(x)+g(x)]\,dx = 0$ [convergent].

18. False. $\int_0^\infty f(x)\,dx$ could converge or diverge. For example, if $g(x)=1$, then $\int_0^\infty f(x)\,dx$ diverges if $f(x)=1$ and converges if $f(x)=0$.

EXERCISES

1. $$\int_1^2 \frac{(x+1)^2}{x}\,dx = \int_1^2 \frac{x^2+2x+1}{x}\,dx = \int_1^2 \left(x+2+\frac{1}{x}\right)dx = \left[\frac{1}{2}x^2+2x+\ln|x|\right]_1^2$$
$$= (2+4+\ln 2) - \left(\tfrac{1}{2}+2+0\right) = \tfrac{7}{2}+\ln 2$$

2. $$\int_1^2 \frac{x}{(x+1)^2}\,dx = \int_2^3 \frac{u-1}{u^2}\,du \qquad \left[\begin{array}{l} u=x+1,\\ du=dx\end{array}\right]$$
$$= \int_2^3\left(\frac{1}{u}-\frac{1}{u^2}\right)du = \left[\ln|u|+\frac{1}{u}\right]_2^3 = \left(\ln 3+\frac{1}{3}\right)-\left(\ln 2+\frac{1}{2}\right) = \ln\frac{3}{2}-\frac{1}{6}$$

3. $$\int \frac{e^{\sin x}}{\sec x}\,dx = \int \cos x\, e^{\sin x}\,dx = \int e^u\,du \qquad \left[\begin{array}{l} u=\sin x,\\ du=\cos x\,dx\end{array}\right]$$
$$= e^u + C = e^{\sin x}+C$$

4. $$\int_0^{\pi/6} t\sin 2t\,dt = \left[-\tfrac{1}{2}t\cos 2t\right]_0^{\pi/6} - \int_0^{\pi/6}\left(-\tfrac{1}{2}\cos 2t\right)dt \qquad \left[\begin{array}{ll} u=t, & dv=\sin 2t\\ du=dt, & v=-\tfrac{1}{2}\cos 2t\end{array}\right]$$
$$= \left(-\tfrac{\pi}{12}\cdot\tfrac{1}{2}\right) - (0) + \left[\tfrac{1}{4}\sin 2t\right]_0^{\pi/6} = -\tfrac{\pi}{24}+\tfrac{1}{8}\sqrt{3}$$

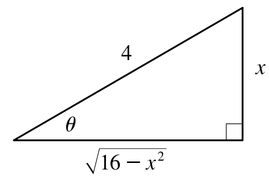
5. $$\int \frac{dt}{2t^2+3t+1} = \int \frac{1}{(2t+1)(t+1)}\,dt = \int\left(\frac{2}{2t+1}-\frac{1}{t+1}\right)dt \quad \text{[partial fractions]} \quad = \ln|2t+1|-\ln|t+1|+C$$

6. $$\int_1^2 x^5\ln x\,dx = \left[\tfrac{1}{6}x^6\ln x\right]_1^2 - \int_1^2 \tfrac{1}{6}x^5\,dx \qquad \left[\begin{array}{ll} u=\ln x, & dv=x^5\,dx\\ du=\tfrac{1}{x}\,dx, & v=\tfrac{1}{6}x^6\end{array}\right]$$
$$= \tfrac{64}{6}\ln 2 - 0 - \left[\tfrac{1}{36}x^6\right]_1^2 = \tfrac{32}{3}\ln 2 - \left(\tfrac{64}{36}-\tfrac{1}{36}\right) = \tfrac{32}{3}\ln 2-\tfrac{7}{4}$$

7. $$\int_0^{\pi/2}\sin^3\theta\,\cos^2\theta\,d\theta = \int_0^{\pi/2}(1-\cos^2\theta)\cos^2\theta\,\sin\theta\,d\theta = \int_1^0(1-u^2)u^2\,(-du) \qquad \left[\begin{array}{l} u=\cos\theta,\\ du=-\sin\theta\,d\theta\end{array}\right]$$
$$= \int_0^1(u^2-u^4)\,du = \left[\tfrac{1}{3}u^3-\tfrac{1}{5}u^5\right]_0^1 = \left(\tfrac{1}{3}-\tfrac{1}{5}\right)-0 = \tfrac{2}{15}$$

8. Let $x = 4\sin\theta$, where $-\pi/2 \le \theta \le \pi/2$. Then $dx = 4\cos\theta\, d\theta$ and

$\sqrt{16 - x^2} = \sqrt{16 - 16\sin^2\theta} = \sqrt{16\cos^2\theta} = 4\,|\cos\theta\,| = 4\cos\theta$. Thus,



$$\int \frac{dx}{x^2\sqrt{16 - x^2}} = \int \frac{4\cos\theta}{16\sin^2\theta\,(4\cos\theta)}\, d\theta = \frac{1}{16}\int \csc^2\theta\, d\theta$$

$$= -\frac{1}{16}\cot\theta + C = -\frac{\sqrt{16 - x^2}}{16x} + C$$

9. Let $u = \ln t$, $du = dt/t$. Then $\displaystyle\int \frac{\sin(\ln t)}{t}\, dt = \int \sin u\, du = -\cos u + C = -\cos(\ln t) + C$.

10. Let $u = \arctan x$, $du = dx/(1 + x^2)$. Then

$$\int_0^1 \frac{\sqrt{\arctan x}}{1 + x^2}\, dx = \int_0^{\pi/4} \sqrt{u}\, du = \frac{2}{3}\Big[u^{3/2}\Big]_0^{\pi/4} = \frac{2}{3}\left[\frac{\pi^{3/2}}{4^{3/2}} - 0\right] = \frac{2}{3}\cdot\frac{1}{8}\pi^{3/2} = \frac{1}{12}\pi^{3/2}.$$

11. First let $u = (\ln x)^2$, $dv = x\, dx \;\Rightarrow\; du = \dfrac{2\ln x}{x}\, dx$, $v = \dfrac{1}{2}x^2$. Then $I = \displaystyle\int x\,(\ln x)^2\, dx = \frac{1}{2}x^2\,(\ln x)^2 - \int x\ln x\, dx$.

Next, let $U = \ln x$, $dV = x\, dx \;\Rightarrow\; dU = \dfrac{1}{x}\, dx$, $V = \dfrac{1}{2}x^2$, so $\displaystyle\int x\ln x\, dx = \frac{1}{2}x^2\ln x - \frac{1}{2}\int x\, dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2$.

Substituting in the previous formula gives $I = \frac{1}{2}x^2\,(\ln x)^2 - \left(\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right) + C = \frac{1}{4}x^2[2(\ln x)^2 - 2\ln x + 1] + C$.

12. Let $t = \cos x$, so that $dt = -\sin x\, dx$. Then

$$\int \sin x\cos x\ln(\cos x)\, dx = \int t\ln t\,(-dt) = -\tfrac{1}{2}t^2\ln t + \int \tfrac{1}{2}t\, dt \qquad \left[\begin{array}{c} u = \ln t,\ dv = -t\, dt \\ du = \dfrac{1}{t}\, dt,\ v = -\frac{1}{2}t^2 \end{array}\right]$$

$$= -\tfrac{1}{2}t^2\ln t + \tfrac{1}{4}t^2 + C$$

$$= -\tfrac{1}{2}\cos^2 x\ln(\cos x) + \tfrac{1}{4}\cos^2 x + C$$

13. Let $x = \sec\theta$. Then

$$\int_1^2 \frac{\sqrt{x^2 - 1}}{x}\, dx = \int_0^{\pi/3} \frac{\tan\theta}{\sec\theta}\,\sec\theta\,\tan\theta\, d\theta = \int_0^{\pi/3} \tan^2\theta\, d\theta = \int_0^{\pi/3} (\sec^2\theta - 1)\, d\theta = \big[\tan\theta - \theta\big]_0^{\pi/3} = \sqrt{3} - \tfrac{\pi}{3}.$$

14. $$\int \frac{e^{2x}}{1 + e^{4x}}\, dx = \int \frac{1}{1 + u^2}\left(\tfrac{1}{2}\, du\right) \qquad \left[\begin{array}{c} u = e^{2x}, \\ du = 2e^{2x}\, dx \end{array}\right]$$

$$= \tfrac{1}{2}\tan^{-1} u + C = \tfrac{1}{2}\tan^{-1} e^{2x} + C$$

15. Let $w = \sqrt[3]{x}$. Then $w^3 = x$ and $3w^2\, dw = dx$, so $\int e^{\sqrt[3]{x}}\, dx = \int e^w\cdot 3w^2\, dw = 3I$. To evaluate I, let $u = w^2$,

$dv = e^w\, dw \;\Rightarrow\; du = 2w\, dw$, $v = e^w$, so $I = \int w^2e^w\, dw = w^2e^w - \int 2we^w\, dw$. Now let $U = w$, $dV = e^w\, dw \;\Rightarrow$

$dU = dw$, $V = e^w$. Thus, $I = w^2e^w - 2\big[we^w - \int e^w\, dw\big] = w^2e^w - 2we^w + 2e^w + C_1$, and hence

$3I = 3e^w(w^2 - 2w + 2) + C = 3e^{\sqrt[3]{x}}(x^{2/3} - 2x^{1/3} + 2) + C$.

16. $$\int \frac{x^2 + 2}{x + 2}\, dx = \int \left(x - 2 + \frac{6}{x + 2}\right) dx = \tfrac{1}{2}x^2 - 2x + 6\ln|x + 2| + C$$

17. Integrate by parts with $u = \tan^{-1} x$, $dv = x^2\,dx$, so that $du = \dfrac{1}{1+x^2}\,dx$, $v = \dfrac{1}{3}x^3$. Then

$$\begin{aligned}\int x^2 \tan^{-1} x\,dx &= \frac{1}{3}x^3\tan^{-1}x - \frac{1}{3}\int \frac{x^3}{1+x^2}\,dx = \frac{1}{3}x^3\tan^{-1}x - \frac{1}{3}\cdot\frac{1}{2}\int\frac{y-1}{y}\,dy \qquad \begin{bmatrix} y = 1+x^2, \\ dy = 2x\,dx \end{bmatrix} \\ &= \frac{1}{3}x^3\tan^{-1}x - \frac{1}{6}\int\left(1-\frac{1}{y}\right)dy = \frac{1}{3}x^3\tan^{-1}x - \frac{1}{6}\left(y - \ln|y|\right) + C_1 \\ &= \tfrac{1}{3}x^3\tan^{-1}x - \tfrac{1}{6}\left[1 + x^2 - \ln(1+x^2)\right] + C_1 \\ &= \tfrac{1}{6}\left[2x^3\tan^{-1}x - x^2 + \ln(1+x^2)\right] + C, \text{ where } C = C_1 - \tfrac{1}{6}\end{aligned}$$

18. Let $u = x+1$ so that $u+1 = x+2$ and $du = dx$. Thus,

$$\begin{aligned}\int (x+2)^2(x+1)^{20}\,dx &= \int (u+1)^2u^{20}\,du = \int (u^2+2u+1)\,u^{20}\,du = \int (u^{22}+2u^{21}+u^{20})\,du \\ &= \tfrac{1}{23}u^{23} + \tfrac{2}{22}u^{22} + \tfrac{1}{21}u^{21} + C = \tfrac{1}{23}(x+1)^{23} + \tfrac{1}{11}(x+1)^{22} + \tfrac{1}{21}(x+1)^{21} + C\end{aligned}$$

19. $\dfrac{x-1}{x^2+2x} = \dfrac{x-1}{x(x+2)} = \dfrac{A}{x} + \dfrac{B}{x+2} \quad\Rightarrow\quad x - 1 = A(x+2) + Bx$. Set $x = -2$ to get $-3 = -2B$, so $B = \frac{3}{2}$. Set $x = 0$ to get $-1 = 2A$, so $A = -\frac{1}{2}$. Thus, $\displaystyle\int \frac{x-1}{x^2+2x}\,dx = \int\left(\frac{-\frac{1}{2}}{x} + \frac{\frac{3}{2}}{x+2}\right)dx = -\tfrac{1}{2}\ln|x| + \tfrac{3}{2}\ln|x+2| + C.$

20.
$$\begin{aligned}\int \frac{\sec^6\theta}{\tan^2\theta}\,d\theta &= \int \frac{(\tan^2\theta+1)^2\sec^2\theta}{\tan^2\theta}\,d\theta \quad \begin{bmatrix} u = \tan\theta, \\ du = -\sec^2\theta\,d\theta \end{bmatrix} = \int\frac{(u^2+1)^2}{u^2}\,du = \int\frac{u^4+2u^2+1}{u^2}\,du \\ &= \int\left(u^2 + 2 + \frac{1}{u^2}\right)du = \frac{u^3}{3} + 2u - \frac{1}{u} + C = \tfrac{1}{3}\tan^3\theta + 2\tan\theta - \cot\theta + C\end{aligned}$$

21.
$$\begin{aligned}\textstyle\int x\cosh x\,dx &= x\sinh x - \textstyle\int \sinh x\,dx \qquad \begin{bmatrix} u = x, & dv = \cosh x\,dx \\ du = dx, & v = \sinh x \end{bmatrix} \\ &= x\sinh x - \cosh x + C\end{aligned}$$

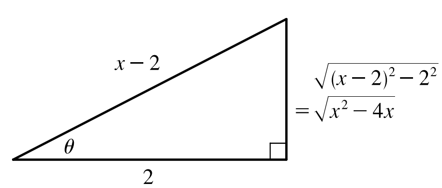
22. $\dfrac{x^2+8x-3}{x^3+3x^2} = \dfrac{x^2+8x-3}{x^2(x+3)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+3} \quad\Rightarrow\quad x^2 + 8x - 3 = Ax(x+3) + B(x+3) + Cx^2.$

Taking $x = 0$, we get $-3 = 3B$, so $B = -1$. Taking $x = -3$, we get $-18 = 9C$, so $C = -2$.

Taking $x = 1$, we get $6 = 4A + 4B + C = 4A - 4 - 2$, so $4A = 12$ and $A = 3$. Now

$$\int\frac{x^2+8x-3}{x^3+3x^2}\,dx = \int\left(\frac{3}{x} - \frac{1}{x^2} - \frac{2}{x+3}\right)dx = 3\ln|x| + \frac{1}{x} - 2\ln|x+3| + C.$$

23.
$$\begin{aligned}\int\frac{dx}{\sqrt{x^2-4x}} &= \int\frac{dx}{\sqrt{(x^2-4x+4)-4}} = \int\frac{dx}{\sqrt{(x-2)^2-2^2}} \\ &= \int\frac{2\sec\theta\,\tan\theta\,d\theta}{2\tan\theta} \qquad \begin{bmatrix} x - 2 = 2\sec\theta, \\ dx = 2\sec\theta\,\tan\theta\,d\theta \end{bmatrix} \\ &= \textstyle\int\sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C_1 \\ &= \ln\left|\frac{x-2}{2} + \frac{\sqrt{x^2-4x}}{2}\right| + C_1 \\ &= \ln\left|x - 2 + \sqrt{x^2-4x}\,\right| + C, \text{ where } C = C_1 - \ln 2\end{aligned}$$



24. $\displaystyle\int \frac{2^{\sqrt{x}}}{\sqrt{x}}\,dx = \int 2^u\,(2\,du) \quad \begin{bmatrix} u = \sqrt{x}, \\ du = 1/(2\sqrt{x}\,)\,dx \end{bmatrix} \quad = 2\cdot\frac{2^u}{\ln 2} + C = \frac{2^{\sqrt{x}+1}}{\ln 2} + C$

25. $\displaystyle\int \frac{x+1}{9x^2+6x+5}\,dx = \int \frac{x+1}{(9x^2+6x+1)+4}\,dx = \int \frac{x+1}{(3x+1)^2+4}\,dx \quad \begin{bmatrix} u = 3x+1, \\ du = 3\,dx \end{bmatrix}$

$$= \int \frac{\left[\frac{1}{3}(u-1)\right]+1}{u^2+4}\left(\frac{1}{3}\,du\right) = \frac{1}{3}\cdot\frac{1}{3}\int\frac{(u-1)+3}{u^2+4}\,du$$

$$= \frac{1}{9}\int\frac{u}{u^2+4}\,du + \frac{1}{9}\int\frac{2}{u^2+2^2}\,du = \frac{1}{9}\cdot\frac{1}{2}\ln(u^2+4) + \frac{2}{9}\cdot\frac{1}{2}\tan^{-1}\left(\frac{1}{2}u\right) + C$$

$$= \tfrac{1}{18}\ln(9x^2+6x+5) + \tfrac{1}{9}\tan^{-1}\left[\tfrac{1}{2}(3x+1)\right] + C$$

26. $\int \tan^5\theta\,\sec^3\theta\,d\theta = \int \tan^4\theta\,\sec^2\theta\,\sec\theta\,\tan\theta\,d\theta = \int(\sec^2\theta-1)^2\sec^2\theta\,\sec\theta\,\tan\theta\,d\theta \quad \begin{bmatrix} u = \sec\theta, \\ du = \sec\theta\,\tan\theta\,d\theta \end{bmatrix}$

$$= \int(u^2-1)^2u^2\,du = \int(u^6-2u^4+u^2)\,du$$

$$= \tfrac{1}{7}u^7 - \tfrac{2}{5}u^5 + \tfrac{1}{3}u^3 + C = \tfrac{1}{7}\sec^7\theta - \tfrac{2}{5}\sec^5\theta + \tfrac{1}{3}\sec^3\theta + C$$

27. $\sqrt{x^2-2x+2} = \sqrt{x^2-2x+1+1} = \sqrt{(x-1)^2+1}$. Since this is a sum of squares, we try the substitution $x-1=\tan\theta$, where $-\pi/2<\theta<\pi/2$. Then $dx = \sec^2\theta\,d\theta$ and $\sqrt{(x-1)^2+1} = \sqrt{\tan^2\theta+1} = \sqrt{\sec^2\theta} = |\sec\theta| = \sec\theta$. Also, $x=0 \;\Rightarrow\; \theta=-\pi/4$ and $x=2 \;\Rightarrow\; \theta=\pi/4$. Thus,

$$\int_0^2\sqrt{x^2-2x+2}\,dx = \int_{-\pi/4}^{\pi/4}\sec\theta\,(\sec^2\theta\,d\theta) = \int_{-\pi/4}^{\pi/4}\sec^3\theta\,d\theta$$

$$= \frac{1}{2}\Big[\sec\theta\,\tan\theta + \ln|\sec\theta+\tan\theta|\Big]_{-\pi/4}^{\pi/4} \qquad \text{[by Example 8 in Section 7.2]}$$

$$= \frac{1}{2}\left[\left(\sqrt{2}+\ln(\sqrt{2}+1)\right) - \left(-\sqrt{2}+\ln(\sqrt{2}-1)\right)\right] = \frac{1}{2}\left[2\sqrt{2}+\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\right]$$

$$= \frac{1}{2}\left[2\sqrt{2}+\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\cdot\frac{\sqrt{2}+1}{\sqrt{2}+1}\right)\right] = \frac{1}{2}\left[2\sqrt{2}+\ln\left(\sqrt{2}+1\right)^2\right]$$

$$= \tfrac{1}{2}\left[2\sqrt{2}+2\ln(\sqrt{2}+1)\right] = \sqrt{2}+\ln(\sqrt{2}+1)$$

28. $\int\cos\sqrt{t}\,dt = \int 2x\,\cos x\,dx \quad \begin{bmatrix} x = \sqrt{t}, \\ x^2 = t, \quad 2x\,dx = dt \end{bmatrix}$

$$= 2x\,\sin x - \int 2\sin x\,dx \quad \begin{bmatrix} u = x, & dv = \cos x\,dx \\ du = dx, & v = \sin x \end{bmatrix}$$

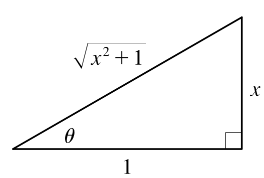
$$= 2x\,\sin x + 2\cos x + C = 2\sqrt{t}\,\sin\sqrt{t} + 2\cos\sqrt{t} + C$$

29. Let $x=\tan\theta$, so that $dx = \sec^2\theta\,d\theta$. Then

$$\int\frac{dx}{x\sqrt{x^2+1}} = \int\frac{\sec^2\theta\,d\theta}{\tan\theta\,\sec\theta} = \int\frac{\sec\theta}{\tan\theta}\,d\theta$$

$$= \int\csc\theta\,d\theta = \ln|\csc\theta-\cot\theta| + C$$

$$= \ln\left|\frac{\sqrt{x^2+1}}{x} - \frac{1}{x}\right| + C = \ln\left|\frac{\sqrt{x^2+1}-1}{x}\right| + C$$



30. Let $u = \cos x$, $dv = e^x\,dx \quad \Rightarrow \quad du = -\sin x\,dx$, $v = e^x$: (∗) $I = \int e^x \cos x\,dx = e^x \cos x + \int e^x \sin x\,dx$.

To integrate $\int e^x \sin x\,dx$, let $U = \sin x$, $dV = e^x\,dx \quad \Rightarrow \quad dU = \cos x\,dx$, $V = e^x$. Then

$\int e^x \sin x\,dx = e^x \sin x - \int e^x \cos x\,dx = e^x \sin x - I$. By substitution in (∗), $I = e^x \cos x + e^x \sin x - I \quad \Rightarrow$

$2I = e^x(\cos x + \sin x) \quad \Rightarrow \quad I = \frac{1}{2}e^x(\cos x + \sin x) + C$.

31. Let $u = \sqrt{1 + x^2}$, so that $du = \dfrac{x}{\sqrt{1 + x^2}}\,dx$. Thus,

$$\int \frac{x \sin\left(\sqrt{1+x^2}\right)}{\sqrt{1+x^2}}\,dx = \int \sin u\,du = -\cos u + C = -\cos\left(\sqrt{1+x^2}\right) + C.$$

32. Let $u = x^{1/4} \quad \Rightarrow \quad x = u^4$, so that $dx = 4u^3\,du$. Thus,

$$\int \frac{1}{x^{1/2} + x^{1/4}}\,dx = \int \frac{4u^3}{u^2 + u}\,du = 4\int \frac{u^2}{u+1}\,du = 4\int \left(u - 1 + \frac{1}{u+1}\right) du \qquad \text{[using long division]}$$

$$= 4\left(\tfrac{1}{2}u^2 - u + \ln|u+1|\right) + C = 2x^{1/2} - 4x^{1/4} + 4\ln(x^{1/4} + 1) + C$$

33. $\dfrac{3x^3 - x^2 + 6x - 4}{(x^2+1)(x^2+2)} = \dfrac{Ax + B}{x^2 + 1} + \dfrac{Cx + D}{x^2 + 2} \quad \Rightarrow \quad 3x^3 - x^2 + 6x - 4 = (Ax + B)\left(x^2 + 2\right) + (Cx + D)\left(x^2 + 1\right)$.

Equating the coefficients gives $A + C = 3$, $B + D = -1$, $2A + C = 6$, and $2B + D = -4 \quad \Rightarrow$

$A = 3$, $C = 0$, $B = -3$, and $D = 2$. Now

$$\int \frac{3x^3 - x^2 + 6x - 4}{(x^2+1)(x^2+2)}\,dx = 3\int \frac{x-1}{x^2+1}\,dx + 2\int \frac{dx}{x^2+2} = \frac{3}{2}\ln\left(x^2+1\right) - 3\tan^{-1}x + \sqrt{2}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C.$$

34. $\int x \sin x\,\cos x\,dx = \int \frac{1}{2}x \sin 2x\,dx \qquad \begin{bmatrix} u = \frac{1}{2}x, & dv = \sin 2x\,dx, \\ du = \frac{1}{2}\,dx & v = -\frac{1}{2}\cos 2x \end{bmatrix}$

$$= -\tfrac{1}{4}x\cos 2x + \int \tfrac{1}{4}\cos 2x\,dx = -\tfrac{1}{4}x\cos 2x + \tfrac{1}{8}\sin 2x + C$$

35. $\int_0^{\pi/2} \cos^3 x\,\sin 2x\,dx = \int_0^{\pi/2} \cos^3 x\,(2\sin x\,\cos x)\,dx = \int_0^{\pi/2} 2\cos^4 x\,\sin x\,dx = \left[-\frac{2}{5}\cos^5 x\right]_0^{\pi/2} = \frac{2}{5}$

36. Let $u = \sqrt[3]{x}$. Then $x = u^3$, $dx = 3u^2\,du \quad \Rightarrow$

$$\int \frac{\sqrt[3]{x} + 1}{\sqrt[3]{x} - 1}\,dx = \int \frac{u+1}{u-1}\,3u^2 du = 3\int \left(u^2 + 2u + 2 + \frac{2}{u-1}\right) du$$

$$= u^3 + 3u^2 + 6u + 6\ln|u-1| + C = x + 3x^{2/3} + 6\sqrt[3]{x} + 6\ln|\sqrt[3]{x} - 1| + C$$

37. The integrand is an odd function, so $\displaystyle\int_{-3}^{3} \frac{x}{1 + |x|}\,dx = 0$ [by 5.5.7(b)].

38. Let $u = e^{-x}$, $du = -e^{-x}\,dx$. Then

$$\int \frac{dx}{e^x\sqrt{1 - e^{-2x}}} = \int \frac{e^{-x}\,dx}{\sqrt{1 - (e^{-x})^2}} = \int \frac{-du}{\sqrt{1 - u^2}} = -\sin^{-1}u + C = -\sin^{-1}(e^{-x}) + C.$$

39. Let $u = \sqrt{e^x - 1}$. Then $u^2 = e^x - 1$ and $2u\,du = e^x\,dx$. Also, $e^x + 8 = u^2 + 9$. Thus,

$$\int_0^{\ln 10} \frac{e^x\sqrt{e^x-1}}{e^x+8}\,dx = \int_0^3 \frac{u \cdot 2u\,du}{u^2+9} = 2\int_0^3 \frac{u^2}{u^2+9}\,du = 2\int_0^3 \left(1 - \frac{9}{u^2+9}\right) du$$

$$= 2\left[u - \frac{9}{3}\tan^{-1}\!\left(\frac{u}{3}\right)\right]_0^3 = 2\left[(3 - 3\tan^{-1}1) - 0\right] = 2\left(3 - 3 \cdot \frac{\pi}{4}\right) = 6 - \frac{3\pi}{2}$$

40. $$\int_0^{\pi/4} \frac{x\sin x}{\cos^3 x}\,dx = \int_0^{\pi/4} x\tan x\,\sec^2 x\,dx \qquad \left[\begin{array}{ll} u = x, & dv = \tan x\,\sec^2 x\,dx, \\ du = dx & v = \frac{1}{2}\tan^2 x \end{array}\right]$$

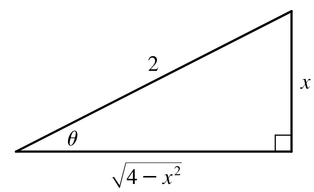
$$= \left[\frac{x}{2}\tan^2 x\right]_0^{\pi/4} - \frac{1}{2}\int_0^{\pi/4}\tan^2 x\,dx = \frac{\pi}{8}\cdot 1^2 - 0 - \frac{1}{2}\int_0^{\pi/4}(\sec^2 x - 1)\,dx$$

$$= \frac{\pi}{8} - \frac{1}{2}\Big[\tan x - x\Big]_0^{\pi/4} = \frac{\pi}{8} - \frac{1}{2}\left(1 - \frac{\pi}{4}\right) = \frac{\pi}{4} - \frac{1}{2}$$

41. Let $x = 2\sin\theta \;\Rightarrow\; \left(4 - x^2\right)^{3/2} = (2\cos\theta)^3$, $dx = 2\cos\theta\,d\theta$, so

$$\int \frac{x^2}{(4-x^2)^{3/2}}\,dx = \int \frac{4\sin^2\theta}{8\cos^3\theta}\,2\cos\theta\,d\theta = \int \tan^2\theta\,d\theta = \int \left(\sec^2\theta - 1\right) d\theta$$

$$= \tan\theta - \theta + C = \frac{x}{\sqrt{4-x^2}} - \sin^{-1}\!\left(\frac{x}{2}\right) + C$$



42. Integrate by parts twice, first with $u = (\arcsin x)^2$, $dv = dx$:

$$I = \int (\arcsin x)^2\,dx = x(\arcsin x)^2 - \int 2x\arcsin x\left(\frac{dx}{\sqrt{1-x^2}}\right)$$

Now let $U = \arcsin x$, $dV = \dfrac{x}{\sqrt{1-x^2}}\,dx \;\Rightarrow\; dU = \dfrac{1}{\sqrt{1-x^2}}\,dx$, $V = -\sqrt{1-x^2}$. So

$$I = x(\arcsin x)^2 - 2\left[\arcsin x\left(-\sqrt{1-x^2}\right) + \textstyle\int dx\right] = x(\arcsin x)^2 + 2\sqrt{1-x^2}\arcsin x - 2x + C$$

43. $$\int \frac{1}{\sqrt{x + x^{3/2}}}\,dx = \int \frac{dx}{\sqrt{x\left(1+\sqrt{x}\right)}} = \int \frac{dx}{\sqrt{x}\sqrt{1+\sqrt{x}}} \qquad \left[\begin{array}{l} u = 1 + \sqrt{x}, \\ du = \dfrac{dx}{2\sqrt{x}} \end{array}\right] \qquad = \int \frac{2\,du}{\sqrt{u}} = \int 2u^{-1/2}\,du$$

$$= 4\sqrt{u} + C = 4\sqrt{1+\sqrt{x}} + C$$

44. $$\int \frac{1 - \tan\theta}{1 + \tan\theta}\,d\theta = \int \frac{\dfrac{\cos\theta}{\cos\theta} - \dfrac{\sin\theta}{\cos\theta}}{\dfrac{\cos\theta}{\cos\theta} + \dfrac{\sin\theta}{\cos\theta}}\,d\theta = \int \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}\,d\theta = \ln|\cos\theta + \sin\theta| + C$$

45. $\int(\cos x + \sin x)^2\cos 2x\,dx = \int\left(\cos^2 x + 2\sin x\cos x + \sin^2 x\right)\cos 2x\,dx = \int(1 + \sin 2x)\cos 2x\,dx$

$$= \int \cos 2x\,dx + \tfrac{1}{2}\int \sin 4x\,dx = \tfrac{1}{2}\sin 2x - \tfrac{1}{8}\cos 4x + C$$

Or: $\int(\cos x + \sin x)^2\cos 2x\,dx = \int(\cos x + \sin x)^2(\cos^2 x - \sin^2 x)\,dx$

$$= \int(\cos x + \sin x)^3(\cos x - \sin x)\,dx = \tfrac{1}{4}(\cos x + \sin x)^4 + C_1$$

46. Let $u = \sin(x^2)$, so that $du = 2x\cos(x^2)\,dx$ and $\cos^2(x^2) = 1 - u^2$. Thus,

$$\begin{aligned}\int x\cos^3(x^2)\sqrt{\sin(x^2)}\,dx &= \int x\cos(x^2)\cos^2(x^2)\sqrt{\sin(x^2)}\,dx = \frac{1}{2}\int\left(1-u^2\right)\sqrt{u}\,du \\ &= \frac{1}{2}\int\left(u^{1/2} - u^{5/2}\right)du = \frac{1}{2}\left(\frac{2}{3}u^{3/2} - \frac{2}{7}u^{7/2}\right) + C \\ &= \tfrac{1}{3}\sqrt{\sin^3(x^2)} - \tfrac{1}{7}\sqrt{\sin^7(x^2)} + C\end{aligned}$$

47. We'll integrate $I = \displaystyle\int \frac{xe^{2x}}{(1+2x)^2}\,dx$ by parts with $u = xe^{2x}$ and $dv = \dfrac{dx}{(1+2x)^2}$. Then $du = (x \cdot 2e^{2x} + e^{2x} \cdot 1)\,dx$ and $v = -\dfrac{1}{2} \cdot \dfrac{1}{1+2x}$, so

$$I = -\frac{1}{2}\cdot\frac{xe^{2x}}{1+2x} - \int\left[-\frac{1}{2}\cdot\frac{e^{2x}(2x+1)}{1+2x}\right]dx = -\frac{xe^{2x}}{4x+2} + \frac{1}{2}\cdot\frac{1}{2}e^{2x} + C = e^{2x}\left(\frac{1}{4} - \frac{x}{4x+2}\right) + C$$

Thus, $\displaystyle\int_0^{1/2}\frac{xe^{2x}}{(1+2x)^2}\,dx = \left[e^{2x}\left(\frac{1}{4} - \frac{x}{4x+2}\right)\right]_0^{1/2} = e\left(\frac{1}{4} - \frac{1}{8}\right) - 1\left(\frac{1}{4} - 0\right) = \frac{1}{8}e - \frac{1}{4}.$

48.
$$\begin{aligned}\int_{\pi/4}^{\pi/3}\frac{\sqrt{\tan\theta}}{\sin 2\theta}\,d\theta &= \int_{\pi/4}^{\pi/3}\frac{\sqrt{\frac{\sin\theta}{\cos\theta}}}{2\sin\theta\cos\theta}\,d\theta = \int_{\pi/4}^{\pi/3}\frac{1}{2}(\sin\theta)^{-1/2}(\cos\theta)^{-3/2}\,d\theta = \int_{\pi/4}^{\pi/3}\frac{1}{2}\left(\frac{\sin\theta}{\cos\theta}\right)^{-1/2}(\cos\theta)^{-2}d\theta \\ &= \int_{\pi/4}^{\pi/3}\frac{1}{2}(\tan\theta)^{-1/2}\sec^2\theta\,d\theta = \left[\sqrt{\tan\theta}\,\right]_{\pi/4}^{\pi/3} = \sqrt{\sqrt{3}} - \sqrt{1} = \sqrt[4]{3} - 1\end{aligned}$$

49. Let $u = \sqrt{e^x - 4}$, so that $e^x = u^2 + 4$ and $2u\,du = e^x\,dx = (u^2+4)\,dx \quad\Leftrightarrow\quad \dfrac{2u}{u^2+4}\,du = dx$. Thus,

$$\int\frac{1}{\sqrt{e^x-4}}\,dx = \int\frac{1}{u}\cdot\frac{2u}{u^2+4}\,du = 2\int\frac{1}{u^2+4}\,du = 2\left(\frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right)\right) + C = \tan^{-1}\left(\frac{\sqrt{e^x-4}}{2}\right) + C.$$

50. Let $y = \sqrt{1+x^2}$, so that $y^2 = 1 + x^2$ and $2y\,dy = 2x\,dx \quad\Rightarrow\quad dx = \dfrac{y}{x}\,dy$. Thus,

$$\begin{aligned}\textstyle\int x\sin\left(\sqrt{1+x^2}\,\right)dx &= \textstyle\int y\sin y\,dy \qquad \left[\begin{array}{l}u = y,\ dv = \sin y\,dy \\ du = dy,\ v = -\cos y\end{array}\right] \\ &= -y\cos y + \textstyle\int\cos y\,dy = -y\cos y + \sin y + C \\ &= -\sqrt{1+x^2}\cos\sqrt{1+x^2} + \sin\sqrt{1+x^2} + C\end{aligned}$$

51.
$$\begin{aligned}\int_1^\infty\frac{1}{(2x+1)^3}\,dx &= \lim_{t\to\infty}\int_1^t\frac{1}{(2x+1)^3}\,dx = \lim_{t\to\infty}\int_1^t\tfrac{1}{2}(2x+1)^{-3}\,2\,dx = \lim_{t\to\infty}\left[-\frac{1}{4(2x+1)^2}\right]_1^t \\ &= -\frac{1}{4}\lim_{t\to\infty}\left[\frac{1}{(2t+1)^2} - \frac{1}{9}\right] = -\frac{1}{4}\left(0 - \frac{1}{9}\right) = \frac{1}{36}\end{aligned}$$

52.
$$\begin{aligned}\int_1^\infty\frac{\ln x}{x^4}\,dx &= \lim_{t\to\infty}\int_1^t\frac{\ln x}{x^4}\,dx \qquad \left[\begin{array}{ll}u = \ln x, & dv = dx/x^4, \\ du = dx/x & v = -1/(3x^3)\end{array}\right] \\ &= \lim_{t\to\infty}\left[-\frac{\ln x}{3x^3}\right]_1^t + \int_1^t\frac{1}{3x^4}\,dx = \lim_{t\to\infty}\left(-\frac{\ln t}{3t^3} + 0 + \left[\frac{-1}{9x^3}\right]_1^t\right) \overset{\text{H}}{=} \lim_{t\to\infty}\left(-\frac{1}{9t^3} + \left[\frac{-1}{9t^3} + \frac{1}{9}\right]\right) \\ &= 0 + 0 + \tfrac{1}{9} = \tfrac{1}{9}\end{aligned}$$

53. $\displaystyle\int \frac{dx}{x\ln x} \quad \begin{bmatrix} u=\ln x, \\ du = dx/x \end{bmatrix} \quad = \int \frac{du}{u} = \ln|u| + C = \ln|\ln x| + C$, so

$\displaystyle\int_2^\infty \frac{dx}{x\ln x} = \lim_{t\to\infty}\int_2^t \frac{dx}{x\ln x} = \lim_{t\to\infty}\Big[\ln|\ln x|\Big]_2^t = \lim_{t\to\infty}\left[\ln(\ln t) - \ln(\ln 2)\right] = \infty$, so the integral is divergent.

54. Let $u = \sqrt{y-2}$. Then $y = u^2 + 2$ and $dy = 2u\,du$, so

$$\int \frac{y\,dy}{\sqrt{y-2}} = \int \frac{(u^2+2)2u\,du}{u} = 2\int (u^2+2)\,du = 2\left[\tfrac{1}{3}u^3 + 2u\right] + C$$

Thus, $\displaystyle\int_2^6 \frac{y}{\sqrt{y-2}}\,dy = \lim_{t\to 2^+}\int_t^6 \frac{y}{\sqrt{y-2}}\,dy = \lim_{t\to 2^+}\left[\tfrac{2}{3}(y-2)^{3/2} + 4\sqrt{y-2}\right]_t^6$

$\displaystyle\qquad = \lim_{t\to 2^+}\left[\tfrac{16}{3} + 8 - \tfrac{2}{3}(t-2)^{3/2} - 4\sqrt{t-2}\right] = \tfrac{40}{3}.$

55. $\displaystyle\int_0^4 \frac{\ln x}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\int_t^4 \frac{\ln x}{\sqrt{x}}\,dx \overset{\star}{=} \lim_{t\to 0^+}\left[2\sqrt{x}\,\ln x - 4\sqrt{x}\right]_t^4$

$\displaystyle\qquad = \lim_{t\to 0^+}\left[(2\cdot 2\ln 4 - 4\cdot 2) - \left(2\sqrt{t}\,\ln t - 4\sqrt{t}\right)\right] \overset{\star\star}{=} (4\ln 4 - 8) - (0-0) = 4\ln 4 - 8$

($\star$) Let $u = \ln x$, $dv = \dfrac{1}{\sqrt{x}}\,dx \;\Rightarrow\; du = \dfrac{1}{x}\,dx$, $v = 2\sqrt{x}$. Then

$$\int \frac{\ln x}{\sqrt{x}}\,dx = 2\sqrt{x}\,\ln x - 2\int \frac{dx}{\sqrt{x}} = 2\sqrt{x}\,\ln x - 4\sqrt{x} + C$$

($\star\star$) $\displaystyle\lim_{t\to 0^+}\left(2\sqrt{t}\,\ln t\right) = \lim_{t\to 0^+}\frac{2\ln t}{t^{-1/2}} \overset{\text{H}}{=} \lim_{t\to 0^+}\frac{2/t}{-\frac{1}{2}t^{-3/2}} = \lim_{t\to 0^+}\left(-4\sqrt{t}\right) = 0$

56. Note that $f(x) = 1/(2-3x)$ has an infinite discontinuity at $x = \frac{2}{3}$. Now

$$\int_0^{2/3}\frac{1}{2-3x}\,dx = \lim_{t\to(2/3)^-}\int_0^t \frac{1}{2-3x}\,dx = \lim_{t\to(2/3)^-}\Big[-\tfrac{1}{3}\ln|2-3x|\Big]_0^t = -\tfrac{1}{3}\lim_{t\to(2/3)^-}\Big[\ln|2-3t| - \ln 2\Big] = \infty$$

Since $\displaystyle\int_0^{2/3}\frac{1}{2-3x}\,dx$ diverges, so does $\displaystyle\int_0^1 \frac{1}{2-3x}\,dx$.

57. $\displaystyle\int_0^1 \frac{x-1}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\int_t^1\left(\frac{x}{\sqrt{x}} - \frac{1}{\sqrt{x}}\right)dx = \lim_{t\to 0^+}\int_t^1\left(x^{1/2} - x^{-1/2}\right)dx = \lim_{t\to 0^+}\left[\tfrac{2}{3}x^{3/2} - 2x^{1/2}\right]_t^1$

$\displaystyle\qquad = \lim_{t\to 0^+}\left[\left(\tfrac{2}{3} - 2\right) - \left(\tfrac{2}{3}t^{3/2} - 2t^{1/2}\right)\right] = -\tfrac{4}{3} - 0 = -\tfrac{4}{3}$

58. $\displaystyle I = \int_{-1}^1 \frac{dx}{x^2-2x} = \int_{-1}^1 \frac{dx}{x(x-2)} = \int_{-1}^0 \frac{dx}{x(x-2)} + \int_0^1 \frac{dx}{x(x-2)} = I_1 + I_2$. Now

$\dfrac{1}{x(x-2)} = \dfrac{A}{x} + \dfrac{B}{x-2} \quad\Rightarrow\quad 1 = A(x-2) + Bx$. Set $x = 2$ to get $1 = 2B$, so $B = \frac{1}{2}$. Set $x = 0$ to get $1 = -2A$,

$A = -\frac{1}{2}$. Thus,

$$I_2 = \lim_{t\to 0^+} \int_t^1 \left(\frac{-\frac{1}{2}}{x} + \frac{\frac{1}{2}}{x-2} \right) dx = \lim_{t\to 0^+} \left[-\tfrac{1}{2}\ln|x| + \tfrac{1}{2}\ln|x-2| \right]_t^1 = \lim_{t\to 0^+} \left[(0+0) - \left(-\tfrac{1}{2}\ln t + \tfrac{1}{2}\ln|t-2| \right) \right]$$

$$= -\tfrac{1}{2}\ln 2 + \tfrac{1}{2}\lim_{t\to 0^+} \ln t = -\infty.$$

Since I_2 diverges, I is divergent.

59. Let $u = 2x + 1$. Then

$$\int_{-\infty}^{\infty} \frac{dx}{4x^2+4x+5} = \int_{-\infty}^{\infty} \frac{\frac{1}{2}\,du}{u^2+4} = \frac{1}{2}\int_{-\infty}^{0} \frac{du}{u^2+4} + \frac{1}{2}\int_0^{\infty} \frac{du}{u^2+4}$$

$$= \tfrac{1}{2}\lim_{t\to-\infty}\left[\tfrac{1}{2}\tan^{-1}\left(\tfrac{1}{2}u\right)\right]_t^0 + \tfrac{1}{2}\lim_{t\to\infty}\left[\tfrac{1}{2}\tan^{-1}\left(\tfrac{1}{2}u\right)\right]_0^t = \tfrac{1}{4}\left[0 - \left(-\tfrac{\pi}{2}\right)\right] + \tfrac{1}{4}\left[\tfrac{\pi}{2} - 0\right] = \tfrac{\pi}{4}.$$

60. $\displaystyle\int_1^{\infty} \frac{\tan^{-1}x}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{\tan^{-1}x}{x^2}\,dx$. Integrate by parts:

$$\int \frac{\tan^{-1}x}{x^2}\,dx = \frac{-\tan^{-1}x}{x} + \int \frac{1}{x}\frac{dx}{1+x^2} = \frac{-\tan^{-1}x}{x} + \int\left[\frac{1}{x} - \frac{x}{x^2+1}\right]dx$$

$$= \frac{-\tan^{-1}x}{x} + \ln|x| - \tfrac{1}{2}\ln(x^2+1) + C = \frac{-\tan^{-1}x}{x} + \frac{1}{2}\ln\frac{x^2}{x^2+1} + C$$

Thus,

$$\int_1^{\infty} \frac{\tan^{-1}x}{x^2}\,dx = \lim_{t\to\infty}\left[-\frac{\tan^{-1}x}{x} + \frac{1}{2}\ln\frac{x^2}{x^2+1}\right]_1^t = \lim_{t\to\infty}\left[-\frac{\tan^{-1}t}{t} + \frac{1}{2}\ln\frac{t^2}{t^2+1} + \frac{\pi}{4} - \frac{1}{2}\ln\frac{1}{2}\right]$$

$$= 0 + \tfrac{1}{2}\ln 1 + \tfrac{\pi}{4} + \tfrac{1}{2}\ln 2 = \tfrac{\pi}{4} + \tfrac{1}{2}\ln 2$$

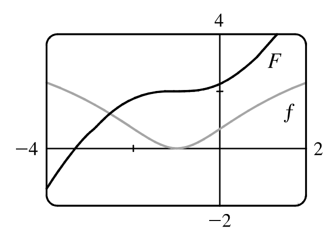
61. We first make the substitution $t = x + 1$, so $\ln(x^2+2x+2) = \ln\left[(x+1)^2+1\right] = \ln(t^2+1)$. Then we use parts with $u = \ln(t^2+1)$, $dv = dt$:

$$\int \ln(t^2+1)\,dt = t\,\ln(t^2+1) - \int\frac{t(2t)\,dt}{t^2+1} = t\,\ln(t^2+1) - 2\int\frac{t^2\,dt}{t^2+1} = t\,\ln(t^2+1) - 2\int\left(1 - \frac{1}{t^2+1}\right)dt$$

$$= t\,\ln(t^2+1) - 2t + 2\arctan t + C$$

$$= (x+1)\,\ln(x^2+2x+2) - 2x + 2\arctan(x+1) + K, \text{ where } K = C - 2$$

[Alternatively, we could have integrated by parts immediately with $u = \ln(x^2+2x+2)$.] Notice from the graph that $f = 0$ where F has a horizontal tangent. Also, F is always increasing, and $f \geq 0$.

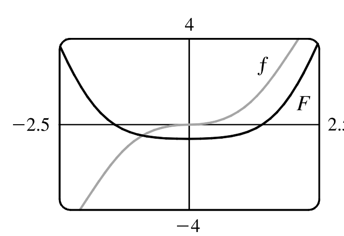


62. Let $u = x^2 + 1$. Then $x^2 = u - 1$ and $x\,dx = \frac{1}{2}\,du$, so

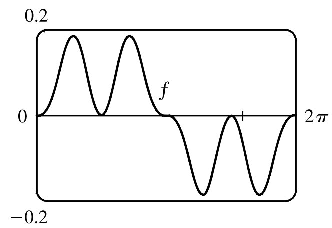
$$\int\frac{x^3}{\sqrt{x^2+1}}\,dx = \int\frac{(u-1)}{\sqrt{u}}\left(\tfrac{1}{2}\,du\right) = \frac{1}{2}\int\left(u^{1/2} - u^{-1/2}\right)du$$

$$= \tfrac{1}{2}\left(\tfrac{2}{3}u^{3/2} - 2u^{1/2}\right) + C = \tfrac{1}{3}(x^2+1)^{3/2} - (x^2+1)^{1/2} + C$$

$$= \tfrac{1}{3}(x^2+1)^{1/2}\left[(x^2+1) - 3\right] + C = \tfrac{1}{3}\sqrt{x^2+1}\,(x^2-2) + C$$



63. From the graph, it seems as though $\int_0^{2\pi} \cos^2 x \, \sin^3 x \, dx$ is equal to 0.



To evaluate the integral, we write the integral as

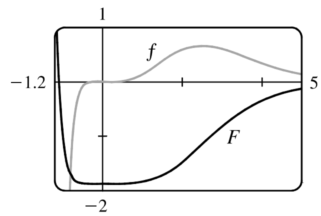
$I = \int_0^{2\pi} \cos^2 x \, (1 - \cos^2 x) \, \sin x \, dx$ and let $u = \cos x \quad \Rightarrow$

$du = -\sin x \, dx$. Thus, $I = \int_1^1 u^2(1 - u^2)(-du) = 0$.

64. (a) To evaluate $\int x^5 e^{-2x} \, dx$ by hand, we would integrate by parts repeatedly, always taking $dv = e^{-2x}$ and starting with $u = x^5$. Each time we would reduce the degree of the x-factor by 1.

(b) To evaluate the integral using tables, we would use Formula 97 (which is proved using integration by parts) until the exponent of x was reduced to 1, and then we would use Formula 96.

(d)



(c) $\int x^5 e^{-2x} \, dx = -\frac{1}{8} e^{-2x} \left(4x^5 + 10x^4 + 20x^3 + 30x^2 + 30x + 15\right) + C$

65. $\int \sqrt{4x^2 - 4x - 3} \, dx = \int \sqrt{(2x-1)^2 - 4} \, dx \quad \begin{bmatrix} u = 2x - 1, \\ du = 2\,dx \end{bmatrix} \quad = \int \sqrt{u^2 - 2^2} \left(\frac{1}{2} \, du\right)$

$$\overset{39}{=} \frac{1}{2}\left(\frac{u}{2}\sqrt{u^2 - 2^2} - \frac{2^2}{2} \ln\left|u + \sqrt{u^2 - 2^2}\right|\right) + C = \frac{1}{4} u \sqrt{u^2 - 4} - \ln\left|u + \sqrt{u^2 - 4}\right| + C$$

$$= \frac{1}{4}(2x - 1)\sqrt{4x^2 - 4x - 3} - \ln\left|2x - 1 + \sqrt{4x^2 - 4x - 3}\right| + C$$

66. $\int \csc^5 t \, dt \overset{78}{=} -\frac{1}{4} \cot t \, \csc^3 t + \frac{3}{4} \int \csc^3 t \, dt \overset{72}{=} -\frac{1}{4} \cot t \, \csc^3 t + \frac{3}{4}\left[-\frac{1}{2} \csc t \, \cot t + \frac{1}{2} \ln|\csc t - \cot t|\right] + C$

$$= -\frac{1}{4} \cot t \, \csc^3 t - \frac{3}{8} \csc t \, \cot t + \frac{3}{8} \ln|\csc t - \cot t| + C$$

67. Let $u = \sin x$, so that $du = \cos x \, dx$. Then

$$\int \cos x \sqrt{4 + \sin^2 x} \, dx = \int \sqrt{2^2 + u^2} \, du \overset{21}{=} \frac{u}{2}\sqrt{2^2 + u^2} + \frac{2^2}{2} \ln\left(u + \sqrt{2^2 + u^2}\right) + C$$

$$= \frac{1}{2} \sin x \sqrt{4 + \sin^2 x} + 2 \ln\left(\sin x + \sqrt{4 + \sin^2 x}\right) + C$$

68. Let $u = \sin x$. Then $du = \cos x \, dx$, so

$$\int \frac{\cot x \, dx}{\sqrt{1 + 2\sin x}} = \int \frac{du}{u\sqrt{1 + 2u}} \overset{\substack{57 \text{ with} \\ a=1,\, b=2}}{=} \ln\left|\frac{\sqrt{1 + 2u} - 1}{\sqrt{1 + 2u} + 1}\right| + C = \ln\left|\frac{\sqrt{1 + 2\sin x} - 1}{\sqrt{1 + 2\sin x} + 1}\right| + C$$

69. (a) $\dfrac{d}{du}\left[-\dfrac{1}{u}\sqrt{a^2 - u^2} - \sin^{-1}\left(\dfrac{u}{a}\right) + C\right] = \dfrac{1}{u^2}\sqrt{a^2 - u^2} + \dfrac{1}{\sqrt{a^2 - u^2}} - \dfrac{1}{\sqrt{1 - u^2/a^2}} \cdot \dfrac{1}{a}$

$$= \left(a^2 - u^2\right)^{-1/2}\left[\frac{1}{u^2}\left(a^2 - u^2\right) + 1 - 1\right] = \frac{\sqrt{a^2 - u^2}}{u^2}$$

(b) Let $u = a \sin\theta \quad \Rightarrow \quad du = a\cos\theta \, d\theta$, $a^2 - u^2 = a^2\left(1 - \sin^2\theta\right) = a^2 \cos^2\theta$.

$$\int \frac{\sqrt{a^2 - u^2}}{u^2} \, du = \int \frac{a^2 \cos^2\theta}{a^2 \sin^2\theta} \, d\theta = \int \frac{1 - \sin^2\theta}{\sin^2\theta} \, d\theta = \int (\csc^2\theta - 1) \, d\theta = -\cot\theta - \theta + C$$

$$= -\frac{\sqrt{a^2 - u^2}}{u} - \sin^{-1}\left(\frac{u}{a}\right) + C$$

70. Work backward, and use integration by parts with $U = u^{-(n-1)}$ and $dV = (a+bu)^{-1/2}\,du \;\Rightarrow$

$dU = \dfrac{-(n-1)\,du}{u^n}$ and $V = \dfrac{2}{b}\sqrt{a+bu}$, to get

$$\begin{aligned}\int \frac{du}{u^{n-1}\sqrt{a+bu}} &= \int U\,dV = UV - \int V\,dU = \frac{2\sqrt{a+bu}}{bu^{n-1}} + \frac{2(n-1)}{b}\int \frac{\sqrt{a+bu}}{u^n}\,du \\ &= \frac{2\sqrt{a+bu}}{bu^{n-1}} + \frac{2(n-1)}{b}\int \frac{a+bu}{u^n\sqrt{a+bu}}\,du \\ &= \frac{2\sqrt{a+bu}}{bu^{n-1}} + 2(n-1)\int \frac{du}{u^{n-1}\sqrt{a+bu}} + \frac{2a(n-1)}{b}\int \frac{du}{u^n\sqrt{a+bu}}\end{aligned}$$

Rearranging the equation gives $\dfrac{2a(n-1)}{b}\displaystyle\int \frac{du}{u^n\sqrt{a+bu}} = -\frac{2\sqrt{a+bu}}{bu^{n-1}} - (2n-3)\int \frac{du}{u^{n-1}\sqrt{a+bu}} \;\Rightarrow$

$$\int \frac{du}{u^n\sqrt{a+bu}} = \frac{-\sqrt{a+bu}}{a(n-1)u^{n-1}} - \frac{b(2n-3)}{2a(n-1)}\int \frac{du}{u^{n-1}\sqrt{a+bu}}$$

71. For $n \geq 0$, $\int_0^\infty x^n\,dx = \lim\limits_{t\to\infty}\left[x^{n+1}/(n+1)\right]_0^t = \infty$. For $n < 0$, $\int_0^\infty x^n\,dx = \int_0^1 x^n\,dx + \int_1^\infty x^n\,dx$. Both integrals are improper. By (7.8.2), the second integral diverges if $-1 \leq n < 0$. By Exercise 7.8.69, the first integral diverges if $n \leq -1$. Thus, $\int_0^\infty x^n\,dx$ is divergent for all values of n.

72. $I = \displaystyle\int_0^\infty e^{ax}\cos x\,dx = \lim_{t\to\infty}\int_0^t e^{ax}\cos x\,dx \overset{\substack{99\text{ with}\\ b=1}}{=} \lim_{t\to\infty}\left[\frac{e^{ax}}{a^2+1}(a\cos x + \sin x)\right]_0^t$

$= \displaystyle\lim_{t\to\infty}\left[\frac{e^{at}}{a^2+1}(a\cos t + \sin t) - \frac{1}{a^2+1}(a)\right] = \frac{1}{a^2+1}\lim_{t\to\infty}\left[e^{at}(a\cos t + \sin t) - a\right].$

For $a \geq 0$, the limit does not exist due to oscillation. For $a < 0$, $\lim\limits_{t\to\infty}\left[e^{at}(a\cos t + \sin t)\right] = 0$ by the Squeeze Theorem, because $\left|e^{at}(a\cos t + \sin t)\right| \leq e^{at}(|a|+1)$, so $I = \dfrac{1}{a^2+1}(-a) = -\dfrac{a}{a^2+1}$.

73. $f(x) = \dfrac{1}{\ln x}$, $\Delta x = \dfrac{b-a}{n} = \dfrac{4-2}{10} = \dfrac{1}{5}$

(a) $T_{10} = \frac{1}{5\cdot 2}\{f(2) + 2[f(2.2) + f(2.4) + \cdots + f(3.8)] + f(4)\} \approx 1.925444$

(b) $M_{10} = \frac{1}{5}[f(2.1) + f(2.3) + f(2.5) + \cdots + f(3.9)] \approx 1.920915$

(c) $S_{10} = \frac{1}{5\cdot 3}[f(2) + 4f(2.2) + 2f(2.4) + \cdots + 2f(3.6) + 4f(3.8) + f(4)] \approx 1.922470$

74. $f(x) = \sqrt{x}\cos x$, $\Delta x = \dfrac{b-a}{n} = \dfrac{4-1}{10} = \dfrac{3}{10}$

(a) $T_{10} = \frac{3}{10\cdot 2}\{f(1) + 2[f(1.3) + f(1.6) + \cdots + f(3.7)] + f(4)\} \approx -2.835151$

(b) $M_{10} = \frac{3}{10}[f(1.15) + f(1.45) + f(1.75) + \cdots + f(3.85)] \approx -2.856809$

(c) $S_{10} = \frac{3}{10\cdot 3}[f(1) + 4f(1.3) + 2f(1.6) + \cdots + 2f(3.4) + 4f(3.7) + f(4)] \approx -2.849672$

75. $f(x) = \dfrac{1}{\ln x} \;\Rightarrow\; f'(x) = -\dfrac{1}{x(\ln x)^2} \;\Rightarrow\; f''(x) = \dfrac{2+\ln x}{x^2(\ln x)^3} = \dfrac{2}{x^2(\ln x)^3} + \dfrac{1}{x^2(\ln x)^2}$. Note that each term of $f''(x)$ decreases on $[2,4]$, so we'll take $K = f''(2) \approx 2.022$. $|E_T| \leq \dfrac{K(b-a)^3}{12n^2} \approx \dfrac{2.022(4-2)^3}{12(10)^2} = 0.01348$ and

$|E_M| \le \dfrac{K(b-a)^3}{24n^2} = 0.00674.$ $\quad |E_T| \le 0.00001 \quad \Leftrightarrow \quad \dfrac{2.022(8)}{12n^2} \le \dfrac{1}{10^5} \quad \Leftrightarrow \quad n^2 \ge \dfrac{10^5(2.022)(8)}{12} \quad \Rightarrow \quad n \ge 367.2.$

Take $n = 368$ for T_n. $\quad |E_M| \le 0.00001 \quad \Leftrightarrow \quad n^2 \ge \dfrac{10^5(2.022)(8)}{24} \quad \Rightarrow \quad n \ge 259.6$. Take $n = 260$ for M_n.

76. $\displaystyle\int_1^4 \frac{e^x}{x}\,dx \approx S_6 = \frac{(4-1)/6}{3}\,[f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + 2f(3) + 4f(3.5) + f(4)] \approx 17.739438$

77. $\Delta t = \left(\frac{10}{60} - 0\right)/\,10 = \frac{1}{60}$.

$$\begin{aligned}\text{Distance traveled } &= \textstyle\int_0^{10} v\,dt \approx S_{10}\\ &= \tfrac{1}{60\cdot 3}[64 + 4(67) + 2(72) + 4(78) + 2(83) + 4(86) + 2(90) + 4(91) + 2(91) + 4(88) + 90]\\ &= \tfrac{1}{180}(2466) = 13.\overline{7}\text{ km}\end{aligned}$$

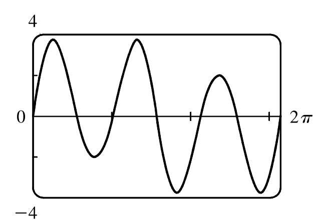
78. We use Simpson's Rule with $n = 6$ and $\Delta t = \frac{24-0}{6} = 4$:

$$\begin{aligned}\text{Increase in bee population } &= \textstyle\int_0^{24} r(t)\,dt \approx S_6\\ &= \tfrac{4}{3}[r(0) + 4r(4) + 2r(8) + 4r(12) + 2r(16) + 4r(20) + r(24)]\\ &= \tfrac{4}{3}[0 + 4(300) + 2(3000) + 4(11{,}000) + 2(4000) + 4(400) + 0]\\ &= \tfrac{4}{3}(60{,}800) \approx 81{,}067 \text{ bees}\end{aligned}$$

79. (a) $f(x) = \sin(\sin x)$. A CAS gives

$$\begin{aligned}f^{(4)}(x) &= \sin(\sin x)[\cos^4 x + 7\cos^2 x - 3]\\ &\quad + \cos(\sin x)\left[6\cos^2 x \sin x + \sin x\right]\end{aligned}$$

From the graph, we see that $\left|f^{(4)}(x)\right| < 3.8$ for $x \in [0, \pi]$.



(b) We use Simpson's Rule with $f(x) = \sin(\sin x)$ and $\Delta x = \frac{\pi}{10}$:

$$\textstyle\int_0^{\pi} f(x)\,dx \approx \frac{\pi}{10\cdot 3}\left[f(0) + 4f\left(\frac{\pi}{10}\right) + 2f\left(\frac{2\pi}{10}\right) + \cdots + 4f\left(\frac{9\pi}{10}\right) + f(\pi)\right] \approx 1.786721$$

From part (a), we know that $\left|f^{(4)}(x)\right| < 3.8$ on $[0, \pi]$, so we use Theorem 7.7.4 with $K = 3.8$, and estimate the error as $|E_S| \le \dfrac{3.8(\pi - 0)^5}{180(10)^4} \approx 0.000646$.

(c) If we want the error to be less than 0.00001, we must have $|E_S| \le \dfrac{3.8\pi^5}{180n^4} \le 0.00001$,

so $n^4 \ge \dfrac{3.8\pi^5}{180(0.00001)} \approx 646{,}041.6 \quad \Rightarrow \quad n \ge 28.35$. Since n must be even for Simpson's Rule, we must have $n \ge 30$ to ensure the desired accuracy.

80. With an x-axis in the normal position, at $x = 7$ we have $C = 2\pi r = 45 \quad \Rightarrow \quad r(7) = \frac{2\pi}{45}$.

Using Simpson's Rule with $n = 4$ and $\Delta x = 7$, we have

$$V = \textstyle\int_0^{28} \pi[r(x)]^2\,dx \approx S_4 = \frac{7}{3}\left[0 + 4\pi\left(\frac{45}{2\pi}\right)^2 + 2\pi\left(\frac{53}{2\pi}\right)^2 + 4\pi\left(\frac{45}{2\pi}\right)^2 + 0\right] = \frac{7}{3}\left(\frac{21{,}818}{4\pi}\right) \approx 4051\text{ cm}^3.$$

81. (a) $\dfrac{2+\sin x}{\sqrt{x}} \geq \dfrac{1}{\sqrt{x}}$ for x in $[1, \infty)$. $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ is divergent by (7.8.2) with $p = \frac{1}{2} \leq 1$. Therefore, $\displaystyle\int_1^\infty \frac{2+\sin x}{\sqrt{x}}\,dx$ is divergent by the Comparison Theorem.

(b) $\dfrac{1}{\sqrt{1+x^4}} < \dfrac{1}{\sqrt{x^4}} = \dfrac{1}{x^2}$ for x in $[1, \infty)$. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ is convergent by (7.8.2) with $p = 2 > 1$. Therefore, $\displaystyle\int_1^\infty \frac{1}{\sqrt{1+x^4}}\,dx$ is convergent by the Comparison Theorem.

82. The line $y = 3$ intersects the hyperbola $y^2 - x^2 = 1$ at two points on its upper branch, namely $\left(-2\sqrt{2}, 3\right)$ and $\left(2\sqrt{2}, 3\right)$. The desired area is

$$A = \int_{-2\sqrt{2}}^{2\sqrt{2}} \left(3 - \sqrt{x^2+1}\right) dx = 2\int_0^{2\sqrt{2}} \left(3 - \sqrt{x^2+1}\right) dx \overset{21}{=} 2\left[3x - \tfrac{1}{2}x\sqrt{x^2+1} - \tfrac{1}{2}\ln\left(x + \sqrt{x^2+1}\right)\right]_0^{2\sqrt{2}}$$

$$= \left[6x - x\sqrt{x^2+1} - \ln\left(x + \sqrt{x^2+1}\right)\right]_0^{2\sqrt{2}} = 12\sqrt{2} - 2\sqrt{2}\cdot 3 - \ln\left(2\sqrt{2}+3\right) = 6\sqrt{2} - \ln\left(3 + 2\sqrt{2}\right)$$

Another method: $A = 2\int_1^3 \sqrt{y^2-1}\,dy$ and use Formula 39.

83. For x in $\left[0, \frac{\pi}{2}\right]$, $0 \leq \cos^2 x \leq \cos x$. For x in $\left[\frac{\pi}{2}, \pi\right]$, $\cos x \leq 0 \leq \cos^2 x$. Thus,

$$\text{area} = \int_0^{\pi/2} (\cos x - \cos^2 x)\,dx + \int_{\pi/2}^{\pi} (\cos^2 x - \cos x)\,dx$$

$$= \left[\sin x - \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x\right]_0^{\pi/2} + \left[\tfrac{1}{2}x + \tfrac{1}{4}\sin 2x - \sin x\right]_{\pi/2}^{\pi} = \left[\left(1 - \tfrac{\pi}{4}\right) - 0\right] + \left[\tfrac{\pi}{2} - \left(\tfrac{\pi}{4} - 1\right)\right] = 2$$

84. The curves $y = \dfrac{1}{2 \pm \sqrt{x}}$ are defined for $x \geq 0$. For $x > 0$, $\dfrac{1}{2-\sqrt{x}} > \dfrac{1}{2+\sqrt{x}}$. Thus, the required area is

$$\int_0^1 \left(\frac{1}{2-\sqrt{x}} - \frac{1}{2+\sqrt{x}}\right) dx = \int_0^1 \left(\frac{1}{2-u} - \frac{1}{2+u}\right) 2u\,du \quad \left[u = \sqrt{x}\right] \quad = 2\int_0^1 \left(-\frac{u}{u-2} - \frac{u}{u+2}\right) du$$

$$= 2\int_0^1 \left(-1 - \frac{2}{u-2} - 1 + \frac{2}{u+2}\right) du = 2\left[2\ln\left|\frac{u+2}{u-2}\right| - 2u\right]_0^1 = 4\ln 3 - 4.$$

85. Using the formula for disks, the volume is

$$V = \int_0^{\pi/2} \pi\,[f(x)]^2\,dx = \pi\int_0^{\pi/2} (\cos^2 x)^2\,dx = \pi\int_0^{\pi/2} \left[\tfrac{1}{2}(1 + \cos 2x)\right]^2\,dx$$

$$= \tfrac{\pi}{4}\int_0^{\pi/2} (1 + \cos^2 2x + 2\cos 2x)\,dx = \tfrac{\pi}{4}\int_0^{\pi/2} \left[1 + \tfrac{1}{2}(1 + \cos 4x) + 2\cos 2x\right] dx$$

$$= \tfrac{\pi}{4}\left[\tfrac{3}{2}x + \tfrac{1}{2}\left(\tfrac{1}{4}\sin 4x\right) + 2\left(\tfrac{1}{2}\sin 2x\right)\right]_0^{\pi/2} = \tfrac{\pi}{4}\left[\left(\tfrac{3\pi}{4} + \tfrac{1}{8}\cdot 0 + 0\right) - 0\right] = \tfrac{3}{16}\pi^2$$

86. Using the formula for cylindrical shells, the volume is

$$V = \int_0^{\pi/2} 2\pi x f(x)\,dx = 2\pi\int_0^{\pi/2} x\cos^2 x\,dx = 2\pi\int_0^{\pi/2} x\left[\tfrac{1}{2}(1 + \cos 2x)\right] dx = 2\left(\tfrac{1}{2}\right)\pi\int_0^{\pi/2} (x + x\cos 2x)\,dx$$

$$= \pi\left(\left[\tfrac{1}{2}x^2\right]_0^{\pi/2} + \left[x\left(\tfrac{1}{2}\sin 2x\right)\right]_0^{\pi/2} - \int_0^{\pi/2} \tfrac{1}{2}\sin 2x\,dx\right) \quad \left[\begin{array}{l}\text{parts with } u = x,\\ dv = \cos 2x\,dx\end{array}\right]$$

$$= \pi\left[\tfrac{1}{2}\left(\tfrac{\pi}{2}\right)^2 + 0 - \tfrac{1}{2}\left[-\tfrac{1}{2}\cos 2x\right]_0^{\pi/2}\right] = \tfrac{\pi^3}{8} + \tfrac{\pi}{4}(-1 - 1) = \tfrac{1}{8}(\pi^3 - 4\pi)$$

87. By the Fundamental Theorem of Calculus,

$\int_0^\infty f'(x)\,dx = \lim\limits_{t\to\infty} \int_0^t f'(x)\,dx = \lim\limits_{t\to\infty} [f(t) - f(0)] = \lim\limits_{t\to\infty} f(t) - f(0) = 0 - f(0) = -f(0).$

88. (a) $$(\tan^{-1} x)_{\text{avg}} = \lim_{t\to\infty} \frac{1}{t-0}\int_0^t \tan^{-1} x\,dx \overset{89}{=} \lim_{t\to\infty} \left\{ \frac{1}{t}\left[x\tan^{-1} x - \tfrac{1}{2}\ln(1+x^2)\right]_0^t \right\}$$

$$= \lim_{t\to\infty}\left[\frac{1}{t}\left(t\tan^{-1} t - \tfrac{1}{2}\ln(1+t^2)\right)\right] = \lim_{t\to\infty}\left[\tan^{-1} t - \frac{\ln(1+t^2)}{2t}\right]$$

$$\overset{\text{H}}{=} \frac{\pi}{2} - \lim_{t\to\infty}\frac{2t/(1+t^2)}{2} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

(b) $f(x) \ge 0$ and $\int_a^\infty f(x)\,dx$ is divergent $\Rightarrow$ $\lim\limits_{t\to\infty}\int_a^t f(x)\,dx = \infty$.

$$f_{\text{avg}} = \lim_{t\to\infty}\frac{\int_a^t f(x)\,dx}{t-a}\,dx \overset{\text{H}}{=} \lim_{t\to\infty}\frac{f(t)}{1} \quad \text{[by FTC1]} \quad = \lim_{x\to\infty} f(x),\ \text{if this limit exists.}$$

(c) Suppose $\int_a^\infty f(x)\,dx$ converges; that is, $\lim\limits_{t\to\infty}\int_a^t f(x)\,dx = L < \infty$. Then

$$f_{\text{avg}} = \lim_{t\to\infty}\left[\frac{1}{t-a}\int_a^t f(x)\,dx\right] = \lim_{t\to\infty}\frac{1}{t-a}\cdot\lim_{t\to\infty}\int_a^t f(x)\,dx = 0\cdot L = 0.$$

(d) $$(\sin x)_{\text{avg}} = \lim_{t\to\infty}\frac{1}{t}\int_0^t \sin x\,dx = \lim_{t\to\infty}\left(\frac{1}{t}\left[-\cos x\right]_0^t\right) = \lim_{t\to\infty}\left(-\frac{\cos t}{t} + \frac{1}{t}\right) = \lim_{t\to\infty}\frac{1-\cos t}{t} = 0$$

89. Let $u = 1/x \;\Rightarrow\; x = 1/u \;\Rightarrow\; dx = -(1/u^2)\,du$.

$$\int_0^\infty \frac{\ln x}{1+x^2}\,dx = \int_\infty^0 \frac{\ln(1/u)}{1+1/u^2}\left(-\frac{du}{u^2}\right) = \int_\infty^0 \frac{-\ln u}{u^2+1}(-du) = \int_\infty^0 \frac{\ln u}{1+u^2}\,du = -\int_0^\infty \frac{\ln u}{1+u^2}\,du$$

Therefore, $\displaystyle\int_0^\infty \frac{\ln x}{1+x^2}\,dx = -\int_0^\infty \frac{\ln x}{1+x^2}\,dx = 0.$

90. If the distance between P and the point charge is d, then the potential V at P is

$$V = W = \int_\infty^d F\,dr = \int_\infty^d \frac{q}{4\pi\varepsilon_0 r^2}\,dr = \lim_{t\to\infty}\frac{q}{4\pi\varepsilon_0}\left[-\frac{1}{r}\right]_t^d = \frac{q}{4\pi\varepsilon_0}\lim_{t\to\infty}\left(-\frac{1}{d}+\frac{1}{t}\right) = -\frac{q}{4\pi\varepsilon_0 d}.$$

PROBLEMS PLUS

1. By symmetry, the problem can be reduced to find the line $x = c$ such that the shaded area is one-third of the area of the quarter-circle. An equation of the semicircle is $y = \sqrt{18^2 - x^2}$, so we require that $\int_0^c \sqrt{18^2 - x^2}dx = \frac{1}{3} \cdot \frac{1}{4}\pi(18)^2 \Leftrightarrow$ $[\frac{1}{2}x\sqrt{18^2 - x^2} + \frac{18^2}{2} sin^{-1}(x/18)]|_0^c = \frac{18^2}{12}\pi \Leftrightarrow \frac{1}{2}c\sqrt{18^2 - c^2} + \frac{18^2}{2} sin^{-1}(c/18) = \frac{18^2}{12}\pi$.

By the computer's calculation, we could find that the equation holds for $c \approx 4.77$. So the cuts should be made at distances of about 4.77 cm from the center of the pizza.

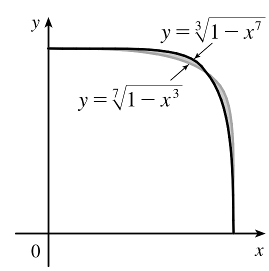
2.
$$\int \frac{1}{x^7 - x}\,dx = \int \frac{dx}{x(x^6 - 1)} = \int \frac{x^5}{x^6(x^6 - 1)}\,dx = \frac{1}{6}\int \frac{1}{u(u-1)}\,du \qquad \begin{bmatrix} u = x^6, \\ du = 6x^5\,dx \end{bmatrix}$$
$$= \frac{1}{6}\int \left(\frac{1}{u-1} - \frac{1}{u}\right) du = \frac{1}{6}(\ln|u-1| - \ln|u|) + C$$
$$= \frac{1}{6}\ln\left|\frac{u-1}{u}\right| + C = \frac{1}{6}\ln\left|\frac{x^6 - 1}{x^6}\right| + C$$

Alternate method:
$$\int \frac{1}{x^7 - x}\,dx = \int \frac{x^{-7}}{1 - x^{-6}}\,dx \qquad \begin{bmatrix} u = 1 - x^{-6}, \\ du = 6x^{-7}\,dx \end{bmatrix}$$
$$= \tfrac{1}{6}\int du/u = \tfrac{1}{6}\ln|u| + C = \tfrac{1}{6}\ln\left|1 - x^{-6}\right| + C$$

Other methods: Substitute $u = x^3$ or $x^3 = \sec\theta$.

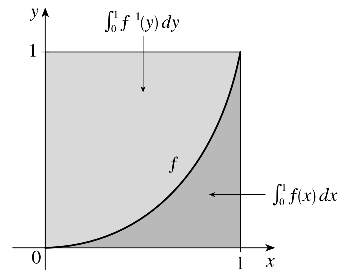
3. The given integral represents the difference of the shaded areas, which appears to be 0. It can be calculated by integrating with respect to either x or y, so we find x in terms of y for each curve: $y = \sqrt[3]{1 - x^7} \quad\Rightarrow\quad x = \sqrt[7]{1 - y^3}$ and $y = \sqrt[7]{1 - x^3} \quad\Rightarrow\quad x = \sqrt[3]{1 - y^7}$, so

$\int_0^1 \left(\sqrt[3]{1 - y^7} - \sqrt[7]{1 - y^3}\right) dy = \int_0^1 \left(\sqrt[7]{1 - x^3} - \sqrt[3]{1 - x^7}\right) dx$. But this equation is of the form $z = -z$. So $\int_0^1 \left(\sqrt[3]{1 - x^7} - \sqrt[7]{1 - x^3}\right) dx = 0$.



4. First note that since f is increasing, it is one-to-one and hence has an inverse. Now

$$\begin{aligned}\textstyle\int_0^1 f(x) + f^{-1}(x)\,dx &= \textstyle\int_0^1 f(x)\,dx + \int_0^1 f^{-1}(x)\,dx\\ &= \textstyle\int_0^1 f(x)\,dx + \int_0^1 f^{-1}(y)\,dy\\ &= 1\end{aligned}$$



The last equality is true because, viewing f^{-1} as a function of y and using the interpretation of the integral as the area under a graph, we see from the figure that the integral gives the area of the unit square, which is 1.

5. $I = \displaystyle\int_{-r}^{r} \frac{f(x)}{1+a^x}\,dx = \int_{-r}^{0} \frac{f(x)}{1+a^x}\,dx + \int_{0}^{r} \frac{f(x)}{1+a^x}\,dx = I_1 + I_2$

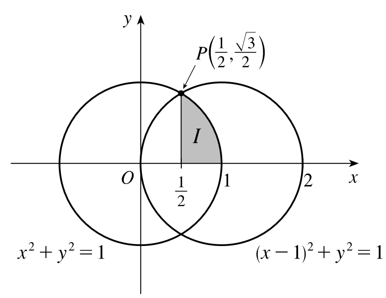
Using the substitution $u = -x$, $du = -dx$ to evaluate I_1 gives

$$\begin{aligned}\int_{-r}^{0} \frac{f(x)}{1+a^x}\,dx = \int_{r}^{0} \frac{f(-u)}{1+a^{-u}}\,(-du) &= \int_0^r f(-u)\left[\frac{1}{1+a^{-u}}\right] du\\ &= \int_0^r f(u)\left[\frac{1}{1+a^{-u}}\right] du \quad \text{[since } f(x) \text{ is even]}\\ &= \int_0^r f(u)\left[1-\frac{1}{1+a^{u}}\right] du \qquad \text{[using the provided hint]}\\ &= \int_0^r f(u)\,du - \int_0^r \frac{f(u)}{1+a^u}\,du\end{aligned}$$

Thus, $I = I_1 + I_2 = \displaystyle\left(\int_0^r f(u)\,du - \int_0^r \frac{f(u)}{1+a^u}\,du\right) + \int_0^r \frac{f(x)}{1+a^x}\,dx = \int_0^r f(u)\,du.$

6. The area of each circle is $\pi(1)^2 = \pi$. By symmetry, the area of the union of the two disks is $A = \pi + \pi - 4I$.

$$\begin{aligned}I &= \textstyle\int_{1/2}^{1} \sqrt{1-x^2}\,dx\\ &\overset{30}{=} \left[\tfrac{x}{2}\sqrt{1-x^2} + \tfrac{1}{2}\sin^{-1}\left(\tfrac{x}{1}\right)\right]_{1/2}^{1} \quad \text{[or substitute } x = \sin\theta\text{]}\\ &= \left(0 + \tfrac{\pi}{4}\right) - \left(\tfrac{1}{4}\tfrac{\sqrt{3}}{2} + \tfrac{1}{2}\tfrac{\pi}{6}\right) = \tfrac{\pi}{4} - \tfrac{\sqrt{3}}{8} - \tfrac{\pi}{12} = \tfrac{\pi}{6} - \tfrac{\sqrt{3}}{8}\end{aligned}$$

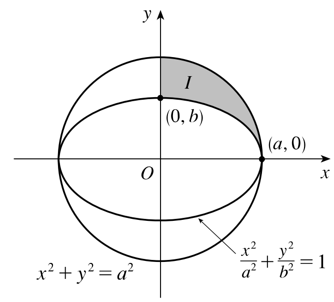


Thus, $A = 2\pi - 4\left(\frac{\pi}{6} - \frac{\sqrt{3}}{8}\right) = 2\pi - \frac{2\pi}{3} + \frac{\sqrt{3}}{2} = \frac{4\pi}{3} + \frac{\sqrt{3}}{2}$.

Alternate solution (no calculus): The area of the sector, with central angle at the origin, containing I is $\frac{1}{2}r^2\theta = \frac{1}{2}(1)^2\left(\frac{\pi}{3}\right) = \frac{\pi}{6}$. The area of the triangle with hypotenuse OP is $\frac{1}{2}\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{8}$.

Thus, the area of I is $\frac{\pi}{6} - \frac{\sqrt{3}}{8}$, as calculated above.

7. The area A of the remaining part of the circle is given by



$$A = 4I = 4\int_0^a \left(\sqrt{a^2 - x^2} - \frac{b}{a}\sqrt{a^2 - x^2}\right) dx = 4\left(1 - \frac{b}{a}\right)\int_0^a \sqrt{a^2 - x^2}\, dx$$

$$\stackrel{30}{=} \frac{4}{a}(a - b)\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

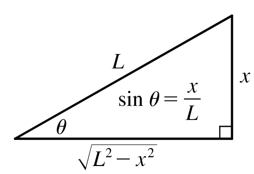
$$= \frac{4}{a}(a - b)\left[\left(0 + \frac{a^2}{2}\frac{\pi}{2}\right) - 0\right] = \frac{4}{a}(a - b)\left(\frac{a^2\pi}{4}\right) = \pi a(a - b),$$

which is the area of an ellipse with semiaxes a and $a - b$.

Alternate solution: Subtracting the area of the ellipse from the area of the circle gives us $\pi a^2 - \pi ab = \pi a\,(a - b)$, as calculated above. (The formula for the area of an ellipse was derived in Example 2 in Section 7.3.)

8. (a) The tangent to the curve $y = f(x)$ at $x = x_0$ has the equation $y - f(x_0) = f'(x_0)(x - x_0)$. The y-intercept of this tangent line is $f(x_0) - f'(x_0)x_0$. Thus, L is the distance from the point $(0, f(x_0) - f'(x_0)x_0)$ to the point $(x_0, f(x_0))$; that is, $L^2 = x_0^2 + [f'(x_0)]^2\, x_0^2$, so $[f'(x_0)]^2 = \dfrac{L^2 - x_0^2}{x_0^2}$ and $f'(x_0) = -\dfrac{\sqrt{L^2 - x_0^2}}{x_0}$ for $0 < x_0 < L$.

(b) $\dfrac{dy}{dx} = -\dfrac{\sqrt{L^2 - x^2}}{x} \quad\Rightarrow\quad y = \displaystyle\int\left(-\frac{\sqrt{L^2 - x^2}}{x}\right) dx.$



Let $x = L\sin\theta$. Then $dx = L\cos\theta\, d\theta$ and

$$y = \int \frac{-L\cos\theta\, L\cos\theta\, d\theta}{L\sin\theta} = L\int \frac{\sin^2\theta - 1}{\sin\theta}\, d\theta = L\int(\sin\theta - \csc\theta)\, d\theta$$

$$= -L\cos\theta - L\ln|\csc\theta - \cot\theta| + C = -\sqrt{L^2 - x^2} - L\ln\left(\frac{L}{x} - \frac{\sqrt{L^2 - x^2}}{x}\right) + C$$

When $x = L$, $y = 0$, and $0 = -0 - L\ln(1 - 0) + C$, so $C = 0$. Therefore, $y = -\sqrt{L^2 - x^2} - L\ln\left(\dfrac{L - \sqrt{L^2 - x^2}}{x}\right)$.

9. Recall that $\cos A\cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)]$. So

$$f(x) = \textstyle\int_0^\pi \cos t\cos(x - t)\, dt = \frac{1}{2}\int_0^\pi [\cos(t + x - t) + \cos(t - x + t)]\, dt = \frac{1}{2}\int_0^\pi [\cos x + \cos(2t - x)]\, dt$$

$$= \tfrac{1}{2}\left[t\cos x + \tfrac{1}{2}\sin(2t - x)\right]_0^\pi = \tfrac{\pi}{2}\cos x + \tfrac{1}{4}\sin(2\pi - x) - \tfrac{1}{4}\sin(-x)$$

$$= \tfrac{\pi}{2}\cos x + \tfrac{1}{4}\sin(-x) - \tfrac{1}{4}\sin(-x) = \tfrac{\pi}{2}\cos x$$

The minimum of $\cos x$ on this domain is -1, so the minimum value of $f(x)$ is $f(\pi) = -\frac{\pi}{2}$.

10. n is a positive integer, so

$$\textstyle\int(\ln x)^n\, dx = x(\ln x)^n - \int x\cdot n(\ln x)^{n-1}\,(dx/x) \quad [\text{by parts}] \;= x(\ln x)^n - n\int(\ln x)^{n-1}\, dx$$

Thus,
$$\textstyle\int_0^1(\ln x)^n\, dx = \displaystyle\lim_{t\to 0^+}\textstyle\int_t^1(\ln x)^n\, dx = \displaystyle\lim_{t\to 0^+}\left[x(\ln x)^n\right]_t^1 - n\lim_{t\to 0^+}\textstyle\int_t^1(\ln x)^{n-1}\, dx$$

$$= -\lim_{t\to 0^+}\frac{(\ln t)^n}{1/t} - n\int_0^1(\ln x)^{n-1}\, dx = -n\int_0^1(\ln x)^{n-1}\, dx$$

by repeated application of l'Hospital's Rule. We want to prove that $\int_0^1(\ln x)^n\, dx = (-1)^n n!$ for every positive integer n. For

[continued]

$n = 1$, we have

$$\int_0^1 (\ln x)^1\,dx = (-1)\int_0^1 (\ln x)^0\,dx = -\int_0^1 dx = -1 \qquad \left[\text{or } \int_0^1 \ln x\,dx = \lim_{t\to 0^+}\left[x\ln x - x\right]_t^1 = -1\right]$$

Assuming that the formula holds for n, we find that

$$\int_0^1 (\ln x)^{n+1}\,dx = -(n+1)\int_0^1 (\ln x)^n\,dx = -(n+1)(-1)^n\,n! = (-1)^{n+1}(n+1)!$$

This is the formula for $n+1$. Thus, the formula holds for all positive integers n by induction.

11. In accordance with the hint, we let $I_k = \int_0^1 (1-x^2)^k\,dx$, and we find an expression for I_{k+1} in terms of I_k. We integrate I_{k+1} by parts with $u = (1-x^2)^{k+1} \;\Rightarrow\; du = (k+1)(1-x^2)^k(-2x)$, $dv = dx \;\Rightarrow\; v = x$, and then split the remaining integral into identifiable quantities:

$$\begin{aligned} I_{k+1} &= x(1-x^2)^{k+1}\Big|_0^1 + 2(k+1)\int_0^1 x^2(1-x^2)^k\,dx = (2k+2)\int_0^1 (1-x^2)^k[1-(1-x^2)]\,dx \\ &= (2k+2)(I_k - I_{k+1}) \end{aligned}$$

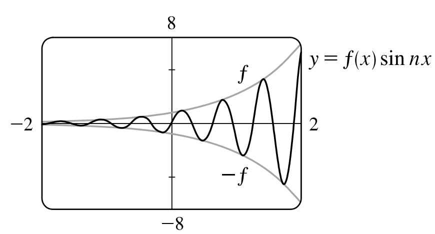
So $I_{k+1}[1+(2k+2)] = (2k+2)I_k \;\Rightarrow\; I_{k+1} = \dfrac{2k+2}{2k+3}I_k$. Now to complete the proof, we use induction:

$I_0 = 1 = \dfrac{2^0(0!)^2}{1!}$, so the formula holds for $n = 0$. Now suppose it holds for $n = k$. Then

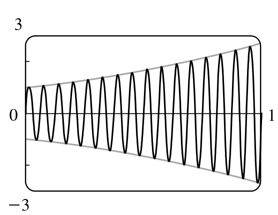
$$\begin{aligned} I_{k+1} &= \frac{2k+2}{2k+3}I_k = \frac{2k+2}{2k+3}\left[\frac{2^{2k}(k!)^2}{(2k+1)!}\right] = \frac{2(k+1)2^{2k}(k!)^2}{(2k+3)(2k+1)!} = \frac{2(k+1)}{2k+2}\cdot\frac{2(k+1)2^{2k}(k!)^2}{(2k+3)(2k+1)!} \\ &= \frac{[2(k+1)]^2\,2^{2k}(k!)^2}{(2k+3)(2k+2)(2k+1)!} = \frac{2^{2(k+1)}\,[(k+1)!]^2}{[2(k+1)+1]!} \end{aligned}$$

So by induction, the formula holds for all integers $n \ge 0$.

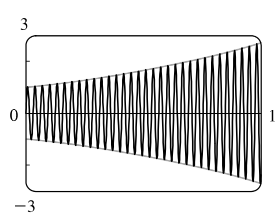
12. (a) Since $-1 \le \sin \le 1$, we have $-f(x) \le f(x)\sin nx \le f(x)$, and the graph of $y = f(x)\,\sin nx$ oscillates between $f(x)$ and $-f(x)$. (The diagram shows the case $f(x) = e^x$ and $n = 10$.) As $n \to \infty$, the graph oscillates more and more frequently; see the graphs in part (b).



(b) From the graphs of the integrand, it seems that $\lim_{n\to\infty}\int_0^1 f(x)\,\sin nx\,dx = 0$, since as n increases, the integrand oscillates more and more rapidly, and thus (since f' is continuous) it makes sense that the areas above the x-axis and below it during each oscillation approach equality.



$n = 100$



$n = 200$

(c) We integrate by parts with $u = f(x) \quad \Rightarrow \quad du = f'(x)\,dx$, $dv = \sin nx\,dx \quad \Rightarrow \quad v = -\dfrac{\cos nx}{n}$:

$$\int_0^1 f(x)\,\sin nx\,dx = \left[-\frac{f(x)\cos nx}{n}\right]_0^1 + \int_0^1 \frac{\cos nx}{n}\,f'(x)\,dx = \frac{1}{n}\left(\int_0^1 \cos nx f'(x)\,dx - \big[f(x)\,\cos nx\big]_0^1\right)$$

$$= \tfrac{1}{n}\left[\int_0^1 \cos nx f'(x)\,dx + f(0) - f(1)\,\cos n\right]$$

Taking absolute values of the first and last terms in this equality, and using the facts that $|\alpha \pm \beta| \le |\alpha| + |\beta|$, $\int_0^1 f(x)\,dx \le \int_0^1 |f(x)|\,dx$, $|f(0)| = f(0)$ [f is positive], $|f'(x)| \le M$ for $0 \le x \le 1$, and $|\cos nx| \le 1$,

$$\left|\int_0^1 f(x)\,\sin nx\,dx\right| \le \tfrac{1}{n}\left[\left|\int_0^1 M\,dx\right| + |f(0)| + |f(1)|\right] = \tfrac{1}{n}\left[M + |f(0)| + |f(1)|\right]$$

which approaches 0 as $n \to \infty$. The result follows by the Squeeze Theorem.

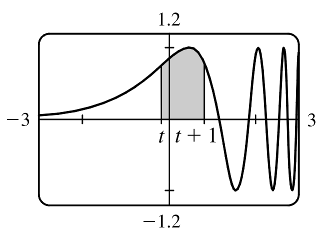
13. $0 < a < b$. Now

$$\int_0^1 [bx + a(1-x)]^t\,dx = \int_a^b \frac{u^t}{(b-a)}\,du \quad [u = bx + a(1-x)] \quad = \left[\frac{u^{t+1}}{(t+1)(b-a)}\right]_a^b = \frac{b^{t+1} - a^{t+1}}{(t+1)(b-a)}.$$

Now let $y = \lim\limits_{t\to 0}\left[\dfrac{b^{t+1} - a^{t+1}}{(t+1)(b-a)}\right]^{1/t}$. Then $\ln y = \lim\limits_{t\to 0}\left[\dfrac{1}{t}\ln\dfrac{b^{t+1} - a^{t+1}}{(t+1)(b-a)}\right]$. This limit is of the form $0/0$, so we can apply l'Hospital's Rule to get

$$\ln y = \lim_{t\to 0}\left[\frac{b^{t+1}\ln b - a^{t+1}\ln a}{b^{t+1} - a^{t+1}} - \frac{1}{t+1}\right] = \frac{b\ln b - a\ln a}{b-a} - 1 = \frac{b\ln b}{b-a} - \frac{a\ln a}{b-a} - \ln e = \ln\frac{b^{b/(b-a)}}{ea^{a/(b-a)}}.$$

Therefore, $y = e^{-1}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}$.

14. 

From the graph, it appears that the area under the graph of $f(x) = \sin(e^x)$ on the interval $[t, t+1]$ is greatest when $t \approx -0.2$. To find the exact value, we write the integral as $I = \int_t^{t+1} f(x)\,dx = \int_0^{t+1} f(x)\,dx - \int_0^t f(x)\,dx$, and use FTC1 to find $dI/dt = f(t+1) - f(t) = \sin\!\left(e^{t+1}\right) - \sin\!\left(e^t\right) = 0$ when $\sin\!\left(e^{t+1}\right) = \sin\!\left(e^t\right)$.

Now we have $\sin x = \sin y$ whenever $x - y = 2k\pi$ and also whenever x and y are the same distance from $\left(k + \frac{1}{2}\right)\pi$, k any integer, since $\sin x$ is symmetric about the line $x = \left(k + \frac{1}{2}\right)\pi$. The first possibility is the more obvious one, but if we calculate $e^{t+1} - e^t = 2k\pi$, we get $t = \ln(2k\pi/(e-1))$, which is about 1.3 for $k = 1$ (the least possible value of k). From the graph, this looks unlikely to give the maximum we are looking for. So instead we set $e^{t+1} - \left(k + \frac{1}{2}\right)\pi = \left(k + \frac{1}{2}\right)\pi - e^t \quad \Leftrightarrow$ $e^{t+1} + e^t = (2k+1)\pi \quad \Leftrightarrow \quad e^t(e+1) = (2k+1)\pi \quad \Leftrightarrow \quad t = \ln((2k+1)\pi/(e+1))$. Now $k = 0 \quad \Rightarrow$ $t = \ln(\pi/(e+1)) \approx -0.16853$, which does give the maximum value, as we have seen from the graph of f.

15. Write $I = \displaystyle\int \frac{x^8}{(1+x^6)^2}\,dx = \int x^3 \cdot \frac{x^5}{(1+x^6)^2}\,dx$. Integrate by parts with $u = x^3$, $dv = \dfrac{x^5}{(1+x^6)^2}\,dx$. Then $du = 3x^2\,dx$, $v = -\dfrac{1}{6(1+x^6)} \quad \Rightarrow \quad I = -\dfrac{x^3}{6(1+x^6)} + \dfrac{1}{2}\displaystyle\int \frac{x^2}{1+x^6}\,dx$. Substitute $t = x^3$ in this latter integral. $\displaystyle\int \frac{x^2}{1+x^6}\,dx = \frac{1}{3}\int \frac{dt}{1+t^2} = \frac{1}{3}\tan^{-1} t + C = \frac{1}{3}\tan^{-1}(x^3) + C$. Therefore,

$I = -\dfrac{x^3}{6(1+x^6)} + \dfrac{1}{6}\tan^{-1}(x^3) + C$. Returning to the improper integral,

$$\begin{aligned}\int_{-1}^{\infty}\left(\frac{x^4}{1+x^6}\right)^2 dx &= \lim_{t\to\infty}\int_{-1}^{t}\frac{x^8}{(1+x^6)^2}\,dx = \lim_{t\to\infty}\left[-\frac{x^3}{6(1+x^6)} + \frac{1}{6}\tan^{-1}(x^3)\right]_{-1}^{t} \\ &= \lim_{t\to\infty}\left(-\frac{t^3}{6(1+t^6)} + \frac{1}{6}\tan^{-1}(t^3) + \frac{-1}{6(1+1)} - \frac{1}{6}\tan^{-1}(-1)\right) \\ &= 0 + \frac{1}{6}\left(\frac{\pi}{2}\right) - \frac{1}{12} - \frac{1}{6}\left(-\frac{\pi}{4}\right) = \frac{\pi}{12} - \frac{1}{12} + \frac{\pi}{24} = \frac{\pi}{8} - \frac{1}{12}\end{aligned}$$

16. $\displaystyle\int \sqrt{\tan x}\,dx = \int u\left(\frac{2u}{u^4+1}\,du\right) \qquad \begin{bmatrix} u = \sqrt{\tan x}, & u^2 = \tan x \\ 2u\,du = \sec^2 x\,dx, & (\tan^2 x + 1)\,dx = (u^4+1)\,dx \end{bmatrix}$

Factoring the denominator, we get

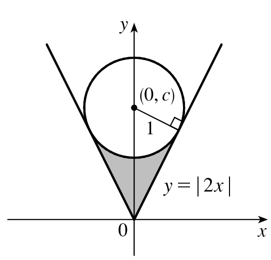
$u^4 + 1 = u^4 + 2u^2 + 1 - 2u^2 = (u^2+1)^2 - \left(\sqrt{2}\,u\right)^2 = \left(u^2 + \sqrt{2}\,u + 1\right)\left(u^2 - \sqrt{2}\,u + 1\right)$. So

$\dfrac{2u^2}{u^4+1} = \dfrac{Au+B}{u^2+\sqrt{2}\,u+1} + \dfrac{Cu+D}{u^2-\sqrt{2}\,u+1} \quad\Rightarrow\quad 2u^2 = (Au+B)\left(u^2-\sqrt{2}\,u+1\right) + (Cu+D)\left(u^2+\sqrt{2}\,u+1\right)$.

Equating coefficients of powers of u, we get $A + C = 0$ (u^3), $B - \sqrt{2}\,A + D + \sqrt{2}\,C = 2$ (u^2), $A - \sqrt{2}\,B + C + \sqrt{2}\,D = 0$ (u), $B + D = 0$ (constants). Substituting $-A$ for C and $-B$ for D in the u-equation leads to $B = 0$ and $D = 0$, and then substituting those values in the u^2-equation gives us $A = -1/\sqrt{2}$ and $C = 1/\sqrt{2}$. Thus,

$$\begin{aligned}\int \frac{2u^2}{u^4+1}\,du &= -\frac{1}{\sqrt{2}}\int \frac{u}{u^2+\sqrt{2}\,u+1}\,du + \frac{1}{\sqrt{2}}\int \frac{u}{u^2-\sqrt{2}\,u+1}\,du \\ &= \frac{1}{\sqrt{2}}\int \frac{\frac{1}{2}\left(2u-\sqrt{2}\right)+1/\sqrt{2}}{u^2-\sqrt{2}\,u+1}\,du - \frac{1}{\sqrt{2}}\int \frac{\frac{1}{2}\left(2u+\sqrt{2}\right)-1/\sqrt{2}}{u^2+\sqrt{2}\,u+1}\,du \\ &= \frac{1}{2\sqrt{2}}\int \frac{2u-\sqrt{2}}{u^2-\sqrt{2}\,u+1}\,du + \frac{1}{2}\int \frac{du}{u^2-\sqrt{2}\,u+1} - \frac{1}{2\sqrt{2}}\int \frac{2u+\sqrt{2}}{u^2+\sqrt{2}\,u+1}\,du + \frac{1}{2}\int \frac{du}{u^2+\sqrt{2}\,u+1} \\ &= \frac{1}{2\sqrt{2}}\ln\left(u^2-\sqrt{2}\,u+1\right) + \frac{1}{2}\int \frac{du}{\left(u-1/\sqrt{2}\right)^2+\frac{1}{2}} - \frac{1}{2\sqrt{2}}\ln\left(u^2+\sqrt{2}\,u+1\right) + \frac{1}{2}\int \frac{du}{\left(u+1/\sqrt{2}\right)^2+\frac{1}{2}} \\ &= \frac{1}{2\sqrt{2}}\ln\frac{u^2-\sqrt{2}\,u+1}{u^2+\sqrt{2}\,u+1} + \frac{1}{2}\,\frac{1}{1/\sqrt{2}}\tan^{-1}\frac{u-1/\sqrt{2}}{1/\sqrt{2}} + \frac{1}{2}\,\frac{1}{1/\sqrt{2}}\tan^{-1}\frac{u+1/\sqrt{2}}{1/\sqrt{2}} + C \\ &= \frac{\sqrt{2}}{4}\ln\frac{\tan x - \sqrt{2\tan x}+1}{\tan x + \sqrt{2\tan x}+1} + \frac{\sqrt{2}}{2}\tan^{-1}\left(\sqrt{2\tan x}-1\right) + \frac{\sqrt{2}}{2}\tan^{-1}\left(\sqrt{2\tan x}+1\right) + C\end{aligned}$$

17.



An equation of the circle with center $(0, c)$ and radius 1 is $x^2 + (y-c)^2 = 1^2$, so an equation of the lower semicircle is $y = c - \sqrt{1-x^2}$. At the points of tangency, the slopes of the line and semicircle must be equal. For $x \geq 0$, we must have

$y' = 2 \quad\Rightarrow\quad \dfrac{x}{\sqrt{1-x^2}} = 2 \quad\Rightarrow\quad x = 2\sqrt{1-x^2} \quad\Rightarrow\quad x^2 = 4(1-x^2) \quad\Rightarrow$

$5x^2 = 4 \quad\Rightarrow\quad x^2 = \frac{4}{5} \quad\Rightarrow\quad x = \frac{2}{5}\sqrt{5}$ and so $y = 2\left(\frac{2}{5}\sqrt{5}\right) = \frac{4}{5}\sqrt{5}$.

[continued]

The slope of the perpendicular line segment is $-\frac{1}{2}$, so an equation of the line segment is $y - \frac{4}{5}\sqrt{5} = -\frac{1}{2}\left(x - \frac{2}{5}\sqrt{5}\right) \iff$ $y = -\frac{1}{2}x + \frac{1}{5}\sqrt{5} + \frac{4}{5}\sqrt{5} \iff y = -\frac{1}{2}x + \sqrt{5}$, so $c = \sqrt{5}$ and an equation of the lower semicircle is $y = \sqrt{5} - \sqrt{1 - x^2}$. Thus, the shaded area is

$$\begin{aligned} 2\int_0^{(2/5)\sqrt{5}} \left[\left(\sqrt{5} - \sqrt{1-x^2}\right) - 2x\right] dx &\overset{30}{=} 2\left[\sqrt{5}\,x - \frac{x}{2}\sqrt{1-x^2} - \frac{1}{2}\sin^{-1} x - x^2\right]_0^{(2/5)\sqrt{5}} \\ &= 2\left[2 - \frac{\sqrt{5}}{5}\cdot\frac{1}{\sqrt{5}} - \frac{1}{2}\sin^{-1}\left(\frac{2}{\sqrt{5}}\right) - \frac{4}{5}\right] - 2(0) \\ &= 2\left[1 - \frac{1}{2}\sin^{-1}\left(\frac{2}{\sqrt{5}}\right)\right] = 2 - \sin^{-1}\left(\frac{2}{\sqrt{5}}\right) \end{aligned}$$

18. (a) $M\dfrac{dv}{dt} - ub = -Mg \;\Rightarrow\; (M_0 - bt)\dfrac{dv}{dt} = ub - (M_0 - bt)g \;\Rightarrow\; \dfrac{dv}{dt} = \dfrac{ub}{M_0 - bt} - g \;\Rightarrow$

$v(t) = -u\ln(M_0 - bt) - gt + C$. Now $0 = v(0) = -u\ln M_0 + C$, so $C = u\ln M_0$. Thus,

$v(t) = u\ln M_0 - u\ln(M_0 - bt) - gt = u\ln\dfrac{M_0}{M_0 - bt} - gt.$

(b) Burnout velocity $= v\left(\dfrac{M_2}{b}\right) = u\ln\dfrac{M_0}{M_0 - M_2} - g\dfrac{M_2}{b} = u\ln\dfrac{M_0}{M_1} - g\dfrac{M_2}{b}.$

Note: The reason for the term "burnout velocity" is that M_2 kilograms of fuel is used in M_2/b seconds, so $v(M_2/b)$ is the rocket's velocity when the fuel is used up.

(c) Height at burnout time $= y\left(\dfrac{M_2}{b}\right)$. Now $\dfrac{dy}{dt} = v(t) = u\ln M_0 - gt - u\ln(M_0 - bt)$, so

$y(t) = (u\ln M_0)t - \dfrac{gt^2}{2} - \dfrac{u}{b}(M_0 - bt)\ln(M_0 - bt) + ut + C$. Since $0 = y(0) = \dfrac{u}{b}M_0\ln M_0 + C$, we get

$C = -\dfrac{u}{b}M_0\ln M_0$ and $y(t) = u(1 + \ln M_0)t - \dfrac{gt^2}{2} + \dfrac{u}{b}(M_0 - bt)\ln(M_0 - bt) - \dfrac{u}{b}M_0\ln M_0.$

Therefore, the height at burnout is

$$\begin{aligned} y\left(\frac{M_2}{b}\right) &= u(1 + \ln M_0)\frac{M_2}{b} - \frac{g}{2}\left(\frac{M_2}{b}\right)^2 + \frac{u}{b}M_1\ln M_1 - \frac{u}{b}M_0\ln M_0 \\ &= \frac{u}{b}M_2 - \frac{u}{b}M_1\ln M_0 + \frac{u}{b}M_1\ln M_1 - \frac{g}{2}\left(\frac{M_2}{b}\right)^2 = \frac{u}{b}M_2 + \frac{u}{b}M_1\ln\frac{M_1}{M_0} - \frac{g}{2}\left(\frac{M_2}{b}\right)^2 \end{aligned}$$

[In the calculation of $y(M_2/b)$, repeated use was made of the relation $M_0 = M_1 + M_2$. In particular, $t = M_2/b \;\Rightarrow\; M_0 - bt = M_1$.]

(d) The formula for $y(t)$ in part (c) holds while there is still fuel. Once the fuel is used up, gravity is the only force acting on the rocket. $-M_1 g = M_1\dfrac{dv}{dt} \;\Rightarrow\; \dfrac{dv}{dt} = -g \;\Rightarrow\; v(t) = -gt + c_1$, where $c_1 = v\left(\dfrac{M_2}{b}\right) + \dfrac{gM_2}{b} \;\Rightarrow$

$v(t) = v\left(\dfrac{M_2}{b}\right) - g\left(t - \dfrac{M_2}{b}\right) \;\Rightarrow\; y(t) = v\left(\dfrac{M_2}{b}\right)\left(t - \dfrac{M_2}{b}\right) - \dfrac{g}{2}\left(t - \dfrac{M_2}{b}\right)^2 + c_2$, where $c_2 = y\left(\dfrac{M_2}{b}\right)$, so $y(t) = y\left(\dfrac{M_2}{b}\right) + v\left(\dfrac{M_2}{b}\right)\left(t - \dfrac{M_2}{b}\right) - \dfrac{g}{2}\left(t - \dfrac{M_2}{b}\right)^2,\ t \geq \dfrac{M_2}{b}$.

To summarize: For $0 \leq t \leq \dfrac{M_2}{b}$, $y(t) = u(1 + \ln M_0)t - \dfrac{gt^2}{2} + \dfrac{u}{b}\,(M_0 - bt)\ln(M_0 - bt) - \dfrac{u}{b}M_0 \ln M_0$

[from part (c)], and for $t \geq \dfrac{M_2}{b}$, $y(t) = y\left(\dfrac{M_2}{b}\right) + v\left(\dfrac{M_2}{b}\right)\left(t - \dfrac{M_2}{b}\right) - \dfrac{g}{2}\left(t - \dfrac{M_2}{b}\right)^2$ [from above].

$y\left(\dfrac{M_2}{b}\right)$ and $v\left(\dfrac{M_2}{b}\right)$ are given in parts (c) and (b), respectively.