

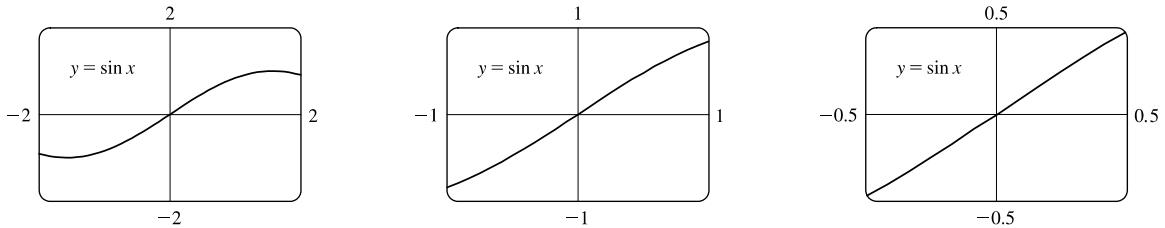
2 □ DERIVATIVES

2.1 Derivatives and Rates of Change

1. (a) This is just the slope of the line through two points: $m_{PQ} = \frac{\Delta y}{\Delta x} = \frac{f(x) - f(3)}{x - 3}$.

(b) This is the limit of the slope of the secant line PQ as Q approaches P : $m = \lim_{x \rightarrow 3} \frac{f(x) - f(3)}{x - 3}$.

2. The curve looks more like a line as the viewing rectangle gets smaller.



3. (a) (i) Using Definition 1 with $f(x) = x^2 + 3x$ and $P(-1, -2)$, the slope of the tangent line is

$$\begin{aligned} m &= \lim_{x \rightarrow -1} \frac{f(x) - f(-1)}{x - (-1)} = \lim_{x \rightarrow -1} \frac{(x^2 + 3x) - (-2)}{x - (-1)} = \lim_{x \rightarrow -1} \frac{x^2 + 3x + 2}{x + 1} = \lim_{x \rightarrow -1} \frac{(x+2)(x+1)}{x+1} \\ &= \lim_{x \rightarrow -1} (x+2) = -1 + 2 = 1 \end{aligned}$$

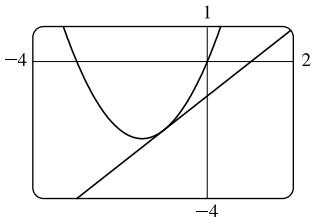
- (ii) Using Equation 2 with $f(x) = x^2 + 3x$ and $P(-1, -2)$, the slope of the tangent line is

$$\begin{aligned} m &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \rightarrow 0} \frac{f(-1+h) - f(-1)}{h} = \lim_{h \rightarrow 0} \frac{[(-1+h)^2 + 3(-1+h)] - (-2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1 - 2h + h^2 - 3 + 3h + 2}{h} = \lim_{h \rightarrow 0} \frac{h^2 + h}{h} = \lim_{h \rightarrow 0} \frac{h(h+1)}{h} = \lim_{h \rightarrow 0} (h+1) = 1 \end{aligned}$$

- (b) An equation of the tangent line is $y - f(a) = f'(a)(x - a)$ \Rightarrow $y - f(-1) = f'(-1)(x - (-1))$ \Rightarrow

$$y - (-2) = 1(x + 1) \Rightarrow y + 2 = x + 1, \text{ or } y = x - 1.$$

- (c)



The graph of $y = x - 1$ is tangent to the graph of $y = x^2 + 3x$ at the point $(-1, -2)$. Now zoom in toward the point $(-1, -2)$ until the parabola and the tangent line are indistinguishable.

4. (a) (i) Using Definition 1 with $f(x) = x^3 + 1$ and $P(1, 2)$, the slope of the tangent line is

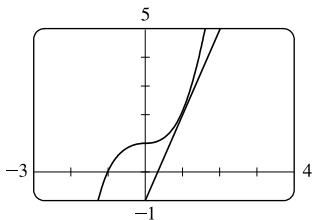
$$\begin{aligned} m &= \lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1} \frac{(x^3 + 1) - 2}{x - 1} = \lim_{x \rightarrow 1} \frac{x^3 - 1}{x - 1} = \lim_{x \rightarrow 1} \frac{(x-1)(x^2+x+1)}{x-1} \\ &= \lim_{x \rightarrow 1} (x^2 + x + 1) = 1^2 + 1 + 1 = 3 \end{aligned}$$

(ii) Using Equation 2 with $f(x) = x^3 + 1$ and $P(1, 2)$, the slope of the tangent line is

$$\begin{aligned} m &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \rightarrow 0} \frac{[(1+h)^3 + 1] - 2}{h} \\ &= \lim_{h \rightarrow 0} \frac{1 + 3h + 3h^2 + h^3 + 1 - 2}{h} = \lim_{h \rightarrow 0} \frac{h^3 + 3h^2 + 3h}{h} = \lim_{h \rightarrow 0} \frac{h(h^2 + 3h + 3)}{h} \\ &= \lim_{h \rightarrow 0} (h^2 + 3h + 3) = 3 \end{aligned}$$

(b) An equation of the tangent line is $y - f(a) = f'(a)(x - a) \Rightarrow y - f(1) = f'(1)(x - 1) \Rightarrow y - 2 = 3(x - 1)$, or $y = 3x - 1$.

(c)



The graph of $y = 3x - 1$ is tangent to the graph of $y = x^3 + 1$ at the point $(1, 2)$. Now zoom in toward the point $(1, 2)$ until the cubic and the tangent line appear to coincide.

5. Using (1) with $f(x) = 2x^2 - 5x + 1$ and $P(3, 4)$ [we could also use Equation (2)], the slope of the tangent line is

$$\begin{aligned} m &= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{x \rightarrow 3} \frac{(2x^2 - 5x + 1) - 4}{x - 3} = \lim_{x \rightarrow 3} \frac{2x^2 - 5x - 3}{x - 3} = \lim_{x \rightarrow 3} \frac{(2x+1)(x-3)}{x-3} \\ &= \lim_{x \rightarrow 3} (2x+1) = 2(3)+1 = 7 \end{aligned}$$

Tangent line: $y - 4 = 7(x - 3) \Leftrightarrow y - 4 = 7x - 21 \Leftrightarrow y = 7x - 17$

6. Using (2) with $f(x) = x^2 - 2x^3$ and $P(1, -1)$, the slope of the tangent line is

$$\begin{aligned} m &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \rightarrow 0} \frac{[(1+h)^2 - 2(1+h)^3] - (-1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1 + 2h + h^2 - 2 - 6h - 6h^2 - 2h^3 + 1}{h} = \lim_{h \rightarrow 0} \frac{-2h^3 - 5h^2 - 4h}{h} = \lim_{h \rightarrow 0} \frac{-h(2h^2 + 5h + 4)}{h} \\ &= \lim_{h \rightarrow 0} [-(2h^2 + 5h + 4)] = -4 \end{aligned}$$

Tangent line: $y - (-1) = -4(x - 1) \Leftrightarrow y + 1 = -4x + 4 \Leftrightarrow y = -4x + 3$

7. Using (1) with $f(x) = \frac{x+2}{x-3}$ and $P(2, -4)$, the slope of the tangent line is

$$\begin{aligned} m &= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{x \rightarrow 2} \frac{\frac{x+2}{x-3} - (-4)}{x - 2} = \lim_{x \rightarrow 2} \frac{\frac{x+2+4(x-3)}{x-3}}{x-2} = \lim_{x \rightarrow 2} \frac{5x-10}{(x-2)(x-3)} \\ &= \lim_{x \rightarrow 2} \frac{5(x-2)}{(x-2)(x-3)} = \lim_{x \rightarrow 2} \frac{5}{x-3} = \frac{5}{2-3} = -5 \end{aligned}$$

Tangent line: $y - (-4) = -5(x - 2) \Leftrightarrow y + 4 = -5x + 10 \Leftrightarrow y = -5x + 6$

8. Using (1) with $f(x) = \sqrt{1 - 3x}$ and $P(-1, 2)$, the slope of the tangent line is

$$\begin{aligned} m &= \lim_{x \rightarrow -1} \frac{f(x) - f(-1)}{x - (-1)} = \lim_{x \rightarrow -1} \frac{\sqrt{1 - 3x} - 2}{x - (-1)} = \lim_{x \rightarrow -1} \frac{(\sqrt{1 - 3x} - 2)(\sqrt{1 - 3x} + 2)}{(x + 1)(\sqrt{1 - 3x} + 2)} = \lim_{x \rightarrow -1} \frac{(1 - 3x) - 4}{(x + 1)(\sqrt{1 - 3x} + 2)} \\ &= \lim_{x \rightarrow -1} \frac{-3x - 3}{(x + 1)(\sqrt{1 - 3x} + 2)} = \lim_{x \rightarrow -1} \frac{-3(x + 1)}{(x + 1)(\sqrt{1 - 3x} + 2)} = \lim_{x \rightarrow -1} \frac{-3}{\sqrt{1 - 3x} + 2} = \frac{-3}{2 + 2} = -\frac{3}{4} \end{aligned}$$

$$\text{Tangent line: } y - 2 = -\frac{3}{4}[x - (-1)] \Leftrightarrow y - 2 = -\frac{3}{4}x - \frac{3}{4} \Leftrightarrow y = -\frac{3}{4}x + \frac{5}{4}$$

9. (a) Using (2) with $y = f(x) = 3 + 4x^2 - 2x^3$, the slope of the tangent line is

$$\begin{aligned} m &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \rightarrow 0} \frac{3 + 4(a+h)^2 - 2(a+h)^3 - (3 + 4a^2 - 2a^3)}{h} \\ &= \lim_{h \rightarrow 0} \frac{3 + 4(a^2 + 2ah + h^2) - 2(a^3 + 3a^2h + 3ah^2 + h^3) - 3 - 4a^2 + 2a^3}{h} \\ &= \lim_{h \rightarrow 0} \frac{3 + 4a^2 + 8ah + 4h^2 - 2a^3 - 6a^2h - 6ah^2 - 2h^3 - 3 - 4a^2 + 2a^3}{h} \\ &= \lim_{h \rightarrow 0} \frac{8ah + 4h^2 - 6a^2h - 6ah^2 - 2h^3}{h} = \lim_{h \rightarrow 0} \frac{h(8a + 4h - 6a^2 - 6ah - 2h^2)}{h} \\ &= \lim_{h \rightarrow 0} (8a + 4h - 6a^2 - 6ah - 2h^2) = 8a - 6a^2 \end{aligned}$$

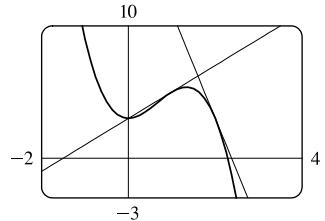
- (b) At $(1, 5)$: $m = 8(1) - 6(1)^2 = 2$, so an equation of the tangent line

$$\text{is } y - 5 = 2(x - 1) \Leftrightarrow y = 2x + 3.$$

- At $(2, 3)$: $m = 8(2) - 6(2)^2 = -8$, so an equation of the tangent

$$\text{line is } y - 3 = -8(x - 2) \Leftrightarrow y = -8x + 19.$$

(c)



10. (a) Using (1) with $f(x) = 2\sqrt{x}$ and $P(a, 2\sqrt{a})$, the slope of the tangent line is

$$\begin{aligned} m &= \lim_{x \rightarrow a} \frac{2\sqrt{x} - 2\sqrt{a}}{x - a} = \lim_{x \rightarrow a} \frac{(2\sqrt{x} - 2\sqrt{a})(2\sqrt{x} + 2\sqrt{a})}{(x - a)(2\sqrt{x} + 2\sqrt{a})} = \lim_{x \rightarrow a} \frac{4x - 4a}{(x - a)(2\sqrt{x} + 2\sqrt{a})} \\ &= \lim_{x \rightarrow a} \frac{4(x - a)}{(x - a)(2\sqrt{x} + 2\sqrt{a})} = \lim_{x \rightarrow a} \frac{4}{2\sqrt{x} + 2\sqrt{a}} = \frac{4}{2\sqrt{a} + 2\sqrt{a}} = \frac{4}{4\sqrt{a}} = \frac{1}{\sqrt{a}}, \text{ or } \frac{\sqrt{a}}{a} [a > 0] \end{aligned}$$

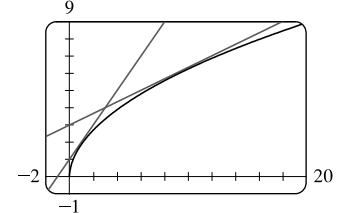
- (b) At $(1, 2)$: $m = \frac{1}{\sqrt{1}} = 1$, so an equation of the tangent line is

$$y - 2 = 1(x - 1) \Leftrightarrow y = x + 1.$$

- At $(9, 6)$: $m = \frac{1}{\sqrt{9}} = \frac{1}{3}$, so an equation of the tangent line is

$$y - 6 = \frac{1}{3}(x - 9) \Leftrightarrow y = \frac{1}{3}x + 3.$$

(c)



11. (a) We have $d(t) = 4.9t^2$. The diver will hit the water when $d(t) = 30 \iff 4.9t^2 = 30 \iff t^2 = \frac{30}{4.9} \iff t \approx 2.5(t > 0)$. The diver will hit the water after 2.5 seconds.

- (b) By Definition 3, the instantaneous velocity of an object with position function $d(t)$ at time $t = 2.5$ is

$$\begin{aligned} v(2.5) &= \lim_{h \rightarrow 0} \frac{d(2.5 + h) - d(2.5)}{h} = \lim_{h \rightarrow 0} \frac{4.9(2.5 + h)^2 - 30.625}{h} \\ &= \lim_{h \rightarrow 0} \frac{30.625 + 24.5h + h^2 - 30.625}{h} = \lim_{h \rightarrow 0} \frac{24.5h + h^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(24.5 + 4.9h)}{h} = \lim_{h \rightarrow 0} (24.5 + 4.9h) = 24.5 \end{aligned}$$

The diver will hit the water with a velocity of 24.5 m/s.

12. (a) Let $H(t) = 10t - 1.86t^2$.

$$\begin{aligned} v(1) &= \lim_{h \rightarrow 0} \frac{H(1 + h) - H(1)}{h} = \lim_{h \rightarrow 0} \frac{[10(1 + h) - 1.86(1 + h)^2] - (10 - 1.86)}{h} \\ &= \lim_{h \rightarrow 0} \frac{10 + 10h - 1.86(1 + 2h + h^2) - 10 + 1.86}{h} = \lim_{h \rightarrow 0} \frac{10 + 10h - 1.86 - 3.72h - 1.86h^2 - 10 + 1.86}{h} \\ &= \lim_{h \rightarrow 0} \frac{6.28h - 1.86h^2}{h} = \lim_{h \rightarrow 0} (6.28 - 1.86h) = 6.28 \end{aligned}$$

The velocity of the rock after one second is 6.28 m/s.

$$\begin{aligned} (b) v(a) &= \lim_{h \rightarrow 0} \frac{H(a + h) - H(a)}{h} = \lim_{h \rightarrow 0} \frac{[10(a + h) - 1.86(a + h)^2] - (10a - 1.86a^2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{10a + 10h - 1.86(a^2 + 2ah + h^2) - 10a + 1.86a^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{10a + 10h - 1.86a^2 - 3.72ah - 1.86h^2 - 10a + 1.86a^2}{h} = \lim_{h \rightarrow 0} \frac{10h - 3.72ah - 1.86h^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(10 - 3.72a - 1.86h)}{h} = \lim_{h \rightarrow 0} (10 - 3.72a - 1.86h) = 10 - 3.72a \end{aligned}$$

The velocity of the rock when $t = a$ is $(10 - 3.72a)$ m/s.

- (c) The rock will hit the surface when $H = 0 \iff 10t - 1.86t^2 = 0 \iff t(10 - 1.86t) = 0 \iff t = 0$ or $1.86t = 10$.

The rock hits the surface when $t = 10/1.86 \approx 5.4$ s.

- (d) The velocity of the rock when it hits the surface is $v\left(\frac{10}{1.86}\right) = 10 - 3.72\left(\frac{10}{1.86}\right) = 10 - 20 = -10$ m/s.

$$\begin{aligned} 13. v(a) &= \lim_{h \rightarrow 0} \frac{s(a + h) - s(a)}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{(a + h)^2} - \frac{1}{a^2}}{h} = \lim_{h \rightarrow 0} \frac{\frac{a^2 - (a + h)^2}{a^2(a + h)^2}}{h} \\ &= \lim_{h \rightarrow 0} \frac{a^2 - (a^2 + 2ah + h^2)}{ha^2(a + h)^2} = \lim_{h \rightarrow 0} \frac{-(2ah + h^2)}{ha^2(a + h)^2} \\ &= \lim_{h \rightarrow 0} \frac{-h(2a + h)}{ha^2(a + h)^2} = \lim_{h \rightarrow 0} \frac{-(2a + h)}{a^2(a + h)^2} = \frac{-2a}{a^2 \cdot a^2} = \frac{-2}{a^3} \text{ m/s} \end{aligned}$$

$$\text{So } v(1) = \frac{-2}{1^3} = -2 \text{ m/s}, v(2) = \frac{-2}{2^3} = -\frac{1}{4} \text{ m/s, and } v(3) = \frac{-2}{3^3} = -\frac{2}{27} \text{ m/s.}$$

14. (a) The average velocity between times t and $t + h$ is

$$\begin{aligned}\frac{s(t+h) - s(t)}{(t+h) - t} &= \frac{\frac{1}{2}(t+h)^2 - 6(t+h) + 23 - (\frac{1}{2}t^2 - 6t + 23)}{h} \\ &= \frac{\frac{1}{2}t^2 + th + \frac{1}{2}h^2 - 6t - 6h + 23 - \frac{1}{2}t^2 + 6t - 23}{h} \\ &= \frac{th + \frac{1}{2}h^2 - 6h}{h} = \frac{h(t + \frac{1}{2}h - 6)}{h} = (t + \frac{1}{2}h - 6) \text{ m/s}\end{aligned}$$

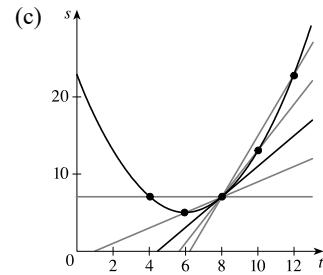
(i) [4, 8]: $t = 4$, $h = 8 - 4 = 4$, so the average velocity is $4 + \frac{1}{2}(4) - 6 = 0$ m/s.

(ii) [6, 8]: $t = 6$, $h = 8 - 6 = 2$, so the average velocity is $6 + \frac{1}{2}(2) - 6 = 1$ m/s.

(iii) [8, 10]: $t = 8$, $h = 10 - 8 = 2$, so the average velocity is $8 + \frac{1}{2}(2) - 6 = 3$ m/s.

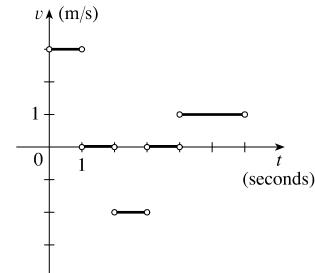
(iv) [8, 12]: $t = 8$, $h = 12 - 8 = 4$, so the average velocity is $8 + \frac{1}{2}(4) - 6 = 4$ m/s.

$$\begin{aligned}\text{(b)} \quad v(t) &= \lim_{h \rightarrow 0} \frac{s(t+h) - s(t)}{h} = \lim_{h \rightarrow 0} (t + \frac{1}{2}h - 6) \\ &= t - 6, \quad \text{so } v(8) = 2 \text{ m/s.}\end{aligned}$$



15. (a) The particle is moving to the right when s is increasing; that is, on the intervals $(0, 1)$ and $(4, 6)$. The particle is moving to the left when s is decreasing; that is, on the interval $(2, 3)$. The particle is standing still when s is constant; that is, on the intervals $(1, 2)$ and $(3, 4)$.

(b) The velocity of the particle is equal to the slope of the tangent line of the graph. Note that there is no slope at the corner points on the graph. On the interval $(0, 1)$, the slope is $\frac{3-0}{1-0} = 3$. On the interval $(2, 3)$, the slope is $\frac{1-3}{3-2} = -2$. On the interval $(4, 6)$, the slope is $\frac{3-1}{6-4} = 1$.



16. (a) **Runner A** runs the entire 100-meter race at the same velocity since the slope of the position function is constant.

Runner B starts the race at a slower velocity than runner A, but finishes the race at a faster velocity.

- (b) The distance between the runners is the greatest at the time when the largest vertical line segment fits between the two graphs—this appears to be somewhere between 9 and 10 seconds.
- (c) The runners had the same velocity when the slopes of their respective position functions are equal—this also appears to be at about 9.5 s. Note that the answers for parts (b) and (c) must be the same for these graphs because as soon as the velocity for runner B overtakes the velocity for runner A, the distance between the runners starts to decrease.

17. $g'(0)$ is the only negative value. The slope at $x = 4$ is smaller than the slope at $x = 2$ and both are smaller than the slope at $x = -2$. Thus, $g'(0) < 0 < g'(4) < g'(2) < g'(-2)$.

18. (a) On $[20, 60]$: $\frac{f(60) - f(20)}{60 - 20} = \frac{700 - 300}{40} = \frac{400}{40} = 10$

- (b) Pick any interval that has the same y -value at both endpoints. $[10, 50]$ is such an interval since $f(10) = 400$ and $f(50) = 400$.

(c) $\frac{f(40) - f(10)}{40 - 10} = \frac{200 - 400}{30} = \frac{-200}{30} = -\frac{20}{3}$

This value represents the slope of the line segment from $(10, f(10))$ to $(40, f(40))$.

- (d) The tangent line at $x = 50$ appears to pass through the points $(40, 200)$ and $(60, 700)$, so

$$f'(50) \approx \frac{700 - 200}{60 - 40} = \frac{500}{20} = 25.$$

- (e) The tangent line at $x = 10$ is steeper than the tangent line at $x = 30$, so it is larger in magnitude, but less in numerical value, that is, $f'(10) < f'(30)$.

- (f) The slope of the tangent line at $x = 60$, $f'(60)$, is greater than the slope of the line through $(40, f(40))$ and $(80, f(80))$.

So yes, $f'(60) > \frac{f(80) - f(40)}{80 - 40}$.

19. Using Definition 4 with $f(x) = \sqrt{4x + 1}$ and $a = 6$,

$$\begin{aligned} f'(6) &= \lim_{h \rightarrow 0} \frac{f(6+h) - f(6)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{4(6+h)+1} - 5}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{25+4h} - 5}{h} \\ &= \lim_{h \rightarrow 0} \frac{(\sqrt{25+4h} - 5)(\sqrt{25+4h} + 5)}{h(\sqrt{25+4h} + 5)} = \lim_{h \rightarrow 0} \frac{(25+4h) - 25}{h(\sqrt{25+4h} + 5)} = \lim_{h \rightarrow 0} \frac{4h}{h(\sqrt{25+4h} + 5)} \\ &= \lim_{h \rightarrow 0} \frac{4}{\sqrt{25+4h} + 5} = \frac{4}{5+5} = \frac{2}{5} \end{aligned}$$

20. Using Definition 4 with $f(x) = 5x^4$ and $a = -1$,

$$\begin{aligned} f'(-1) &= \lim_{h \rightarrow 0} \frac{f(-1+h) - f(-1)}{h} = \lim_{h \rightarrow 0} \frac{5(-1+h)^4 - 5}{h} = \lim_{h \rightarrow 0} \frac{5(1 - 4h + 6h^2 - 4h^3 + h^4) - 5}{h} \\ &= \lim_{h \rightarrow 0} \frac{-20h + 30h^2 - 20h^3 + 5h^4}{h} = \lim_{h \rightarrow 0} \frac{-h(20 - 30h + 20h^2 - 5h^3)}{h} \\ &= \lim_{h \rightarrow 0} [-(20 - 30h + 20h^2 - 5h^3)] = -20 \end{aligned}$$

21. Using Equation 5 with $f(x) = \frac{x^2}{x+6}$ and $a = 3$,

$$\begin{aligned} f'(3) &= \lim_{x \rightarrow 3} \frac{f(x) - f(3)}{x - 3} = \lim_{x \rightarrow 3} \frac{\frac{x^2}{x+6} - 1}{x - 3} = \lim_{x \rightarrow 3} \frac{\frac{x^2 - (x+6)}{x+6}}{x - 3} = \lim_{x \rightarrow 3} \frac{x^2 - x - 6}{(x+6)(x-3)} \\ &= \lim_{x \rightarrow 3} \frac{(x+2)(x-3)}{(x+6)(x-3)} = \lim_{x \rightarrow 3} \frac{x+2}{x+6} = \frac{3+2}{3+6} = \frac{5}{9} \end{aligned}$$

22. Using Equation 5 with $f(x) = \frac{1}{\sqrt{2x+2}}$ and $a = 1$,

$$\begin{aligned} f'(1) &= \lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1} \frac{\frac{1}{\sqrt{2x+2}} - \frac{1}{2}}{x - 1} = \lim_{x \rightarrow 1} \frac{\frac{2 - \sqrt{2x+2}}{2\sqrt{2x+2}}}{x - 1} = \lim_{x \rightarrow 1} \frac{2 - \sqrt{2x+2}}{2\sqrt{2x+2}(x-1)} \\ &= \lim_{x \rightarrow 1} \frac{(2 - \sqrt{2x+2})(2 + \sqrt{2x+2})}{2\sqrt{2x+2}(x-1)(2 + \sqrt{2x+2})} = \lim_{x \rightarrow 1} \frac{4 - (2x+2)}{2\sqrt{2x+2}(x-1)(2 + \sqrt{2x+2})} \\ &= \lim_{x \rightarrow 1} \frac{-2x+2}{2\sqrt{2x+2}(x-1)(2 + \sqrt{2x+2})} = \lim_{x \rightarrow 1} \frac{-2(x-1)}{2\sqrt{2x+2}(x-1)(2 + \sqrt{2x+2})} \\ &= \lim_{x \rightarrow 1} \frac{-1}{\sqrt{2x+2}(2 + \sqrt{2x+2})} = \frac{-1}{\sqrt{4}(2 + \sqrt{4})} = -\frac{1}{8} \end{aligned}$$

23. Using Definition 4 with $f(x) = 2x^2 - 5x + 3$,

$$\begin{aligned} f'(a) &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \rightarrow 0} \frac{[2(a+h)^2 - 5(a+h) + 3] - (2a^2 - 5a + 3)}{h} \\ &= \lim_{h \rightarrow 0} \frac{2a^2 + 4ah + 2h^2 - 5a - 5h + 3 - 2a^2 + 5a - 3}{h} = \lim_{h \rightarrow 0} \frac{4ah + 2h^2 - 5h}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(4a + 2h - 5)}{h} = \lim_{h \rightarrow 0} (4a + 2h - 5) = 4a - 5 \end{aligned}$$

24. Using Definition 4 with $f(t) = t^3 - 3t$,

$$\begin{aligned} f'(a) &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \rightarrow 0} \frac{[(a+h)^3 - 3(a+h)] - (a^3 - 3a)}{h} \\ &= \lim_{h \rightarrow 0} \frac{a^3 + 3a^2h + 3ah^2 + h^3 - 3a - 3h - a^3 + 3a}{h} = \lim_{h \rightarrow 0} \frac{3a^2h + 3ah^2 + h^3 - 3h}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(3a^2 + 3ah + h^2 - 3)}{h} = \lim_{h \rightarrow 0} (3a^2 + 3ah + h^2 - 3) = 3a^2 - 3 \end{aligned}$$

25. Using Definition 4 with $f(t) = \frac{1}{t^2 + 1}$.

$$\begin{aligned} f'(a) &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{(a+h)^2 + 1} - \frac{1}{a^2 + 1}}{h} = \lim_{h \rightarrow 0} \frac{\frac{(a^2 + 1) - [(a+h)^2 + 1]}{[(a+h)^2 + 1](a^2 + 1)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{(a^2 + 1) - (a^2 + 2ah + h^2 + 1)}{h[(a+h)^2 + 1](a^2 + 1)} = \lim_{h \rightarrow 0} \frac{-(2ah + h^2)}{h[(a+h)^2 + 1](a^2 + 1)} = \lim_{h \rightarrow 0} \frac{-h(2a + h)}{h[(a+h)^2 + 1](a^2 + 1)} \\ &= \lim_{h \rightarrow 0} \frac{-(2a + h)}{[(a+h)^2 + 1](a^2 + 1)} = \frac{-2a}{(a^2 + 1)(a^2 + 1)} = -\frac{2a}{(a^2 + 1)^2} \end{aligned}$$

26. Use Definition 4 with $f(x) = \frac{x}{1 - 4x}$.

$$\begin{aligned} f'(a) &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \rightarrow 0} \frac{\frac{a+h}{1 - 4(a+h)} - \frac{a}{1 - 4a}}{h} = \lim_{h \rightarrow 0} \frac{\frac{(a+h)(1 - 4a) - a[1 - 4(a+h)]}{[1 - 4(a+h)](1 - 4a)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{a - 4a^2 + h - 4ah - a + 4a^2 + 4ah}{h[1 - 4(a+h)](1 - 4a)} = \lim_{h \rightarrow 0} \frac{h}{h[1 - 4(a+h)](1 - 4a)} \\ &= \lim_{h \rightarrow 0} \frac{1}{[1 - 4(a+h)](1 - 4a)} = \frac{1}{(1 - 4a)(1 - 4a)} = \frac{1}{(1 - 4a)^2} \end{aligned}$$

27. Since $B(6) = 0$, the point $(6, 0)$ is on the graph of B . Since $B'(6) = -\frac{1}{2}$, the slope of the tangent line at $x = 6$ is $-\frac{1}{2}$.

Using the point-slope form of a line gives us $y - 0 = -\frac{1}{2}(x - 6)$, or $y = -\frac{1}{2}x + 3$.

28. Since $g(5) = -3$, the point $(5, -3)$ is on the graph of g . Since $g'(5) = 4$, the slope of the tangent line at $x = 5$ is 4.

Using the point-slope form of a line gives us $y - (-3) = 4(x - 5)$, or $y = 4x - 23$.

29. Using Definition 4 with $f(x) = 3x^2 - x^3$ and $a = 1$,

$$\begin{aligned} f'(1) &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \rightarrow 0} \frac{[3(1+h)^2 - (1+h)^3] - 2}{h} \\ &= \lim_{h \rightarrow 0} \frac{(3+6h+3h^2) - (1+3h+3h^2+h^3) - 2}{h} = \lim_{h \rightarrow 0} \frac{3h-h^3}{h} = \lim_{h \rightarrow 0} \frac{h(3-h^2)}{h} \\ &= \lim_{h \rightarrow 0} (3-h^2) = 3-0=3 \end{aligned}$$

Tangent line: $y - 2 = 3(x - 1) \Leftrightarrow y - 2 = 3x - 3 \Leftrightarrow y = 3x - 1$

30. Using Equation 5 with $g(x) = x^4 - 2$ and $a = 1$,

$$\begin{aligned} g'(1) &= \lim_{x \rightarrow 1} \frac{g(x) - g(1)}{x - 1} = \lim_{x \rightarrow 1} \frac{(x^4 - 2) - (-1)}{x - 1} = \lim_{x \rightarrow 1} \frac{x^4 - 1}{x - 1} = \lim_{x \rightarrow 1} \frac{(x^2 + 1)(x^2 - 1)}{x - 1} \\ &= \lim_{x \rightarrow 1} \frac{(x^2 + 1)(x + 1)(x - 1)}{x - 1} = \lim_{x \rightarrow 1} [(x^2 + 1)(x + 1)] = 2(2) = 4 \end{aligned}$$

Tangent line: $y - (-1) = 4(x - 1) \Leftrightarrow y + 1 = 4x - 4 \Leftrightarrow y = 4x - 5$

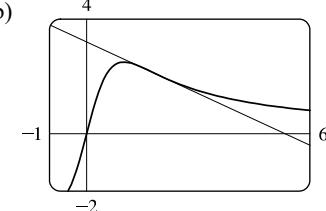
31. (a) Using Definition 4 with $F(x) = 5x/(1+x^2)$ and the point $(2, 2)$, we have

$$\begin{aligned} F'(2) &= \lim_{h \rightarrow 0} \frac{F(2+h) - F(2)}{h} = \lim_{h \rightarrow 0} \frac{\frac{5(2+h)}{1+(2+h)^2} - 2}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{5h+10}{h^2+4h+5} - 2}{h} = \lim_{h \rightarrow 0} \frac{\frac{5h+10-2(h^2+4h+5)}{h^2+4h+5}}{h} \\ &= \lim_{h \rightarrow 0} \frac{-2h^2-3h}{h(h^2+4h+5)} = \lim_{h \rightarrow 0} \frac{h(-2h-3)}{h(h^2+4h+5)} = \lim_{h \rightarrow 0} \frac{-2h-3}{h^2+4h+5} = \frac{-3}{5} \end{aligned}$$

So an equation of the tangent line at $(2, 2)$ is $y - 2 = -\frac{3}{5}(x - 2)$ or $y = -\frac{3}{5}x + \frac{16}{5}$.

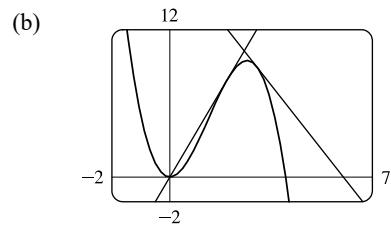
32. (a) Using Definition 4 with $G(x) = 4x^2 - x^3$, we have

$$\begin{aligned} G'(a) &= \lim_{h \rightarrow 0} \frac{G(a+h) - G(a)}{h} = \lim_{h \rightarrow 0} \frac{[4(a+h)^2 - (a+h)^3] - (4a^2 - a^3)}{h} \\ &= \lim_{h \rightarrow 0} \frac{4a^2 + 8ah + 4h^2 - (a^3 + 3a^2h + 3ah^2 + h^3) - 4a^2 + a^3}{h} \\ &= \lim_{h \rightarrow 0} \frac{8ah + 4h^2 - 3a^2h - 3ah^2 - h^3}{h} = \lim_{h \rightarrow 0} \frac{h(8a + 4h - 3a^2 - 3ah - h^2)}{h} \\ &= \lim_{h \rightarrow 0} (8a + 4h - 3a^2 - 3ah - h^2) = 8a - 3a^2 \end{aligned}$$



[continued]

At the point $(2, 8)$, $G'(2) = 16 - 12 = 4$, and an equation of the tangent line is $y - 8 = 4(x - 2)$, or $y = 4x$. At the point $(3, 9)$, $G'(3) = 24 - 27 = -3$, and an equation of the tangent line is $y - 9 = -3(x - 3)$, or $y = -3x + 18$.



33. For the tangent line $y = 4x - 5$: when $x = 2$, $y = 4(2) - 5 = 3$ and its slope is 4 (the coefficient of x). At the point of tangency, these values are shared with the curve $y = f(x)$; that is, $f(2) = 3$ and $f'(2) = 4$.

34. Since $(4, 3)$ is on $y = f(x)$, $f(4) = 3$. The slope of the tangent line between $(0, 2)$ and $(4, 3)$ is $\frac{1}{4}$, so $f'(4) = \frac{1}{4}$.

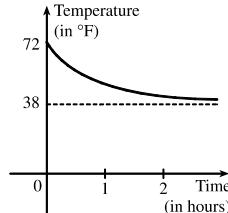
$$\begin{aligned} 35. v(4) = f'(4) &= \lim_{h \rightarrow 0} \frac{f(4+h) - f(4)}{h} = \lim_{h \rightarrow 0} \frac{[80(4+h) - 6(4+h)^2] - [80(4) - 6(4)^2]}{h} \\ &= \lim_{h \rightarrow 0} \frac{(320 + 80h - 96 - 48h - 6h^2) - (320 - 96)}{h} = \lim_{h \rightarrow 0} \frac{32h - 6h^2}{h} \\ &= \lim_{h \rightarrow 0} \frac{h(32 - 6h)}{h} = \lim_{h \rightarrow 0} (32 - 6h) = 32 \text{ m/s} \end{aligned}$$

The speed when $t = 4$ is $|32| = 32$ m/s.

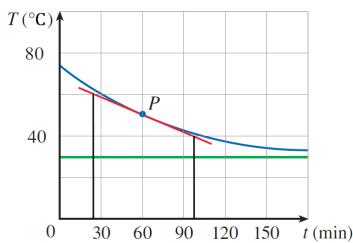
$$\begin{aligned} 36. v(4) = f'(4) &= \lim_{h \rightarrow 0} \frac{f(4+h) - f(4)}{h} = \lim_{h \rightarrow 0} \frac{\left(10 + \frac{45}{4+h+1}\right) - \left(10 + \frac{45}{4+1}\right)}{h} = \lim_{h \rightarrow 0} \frac{\frac{45}{5+h} - 9}{h} \\ &= \lim_{h \rightarrow 0} \frac{45 - 9(5+h)}{h(5+h)} = \lim_{h \rightarrow 0} \frac{-9h}{h(5+h)} = \lim_{h \rightarrow 0} \frac{-9}{5+h} = -\frac{9}{5} \text{ m/s.} \end{aligned}$$

The speed when $t = 4$ is $-\frac{9}{5} = \frac{9}{5}$ m/s.

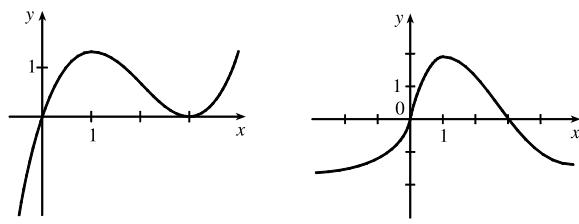
37. The sketch shows the graph for a room temperature of 22°F and a refrigerator temperature of 3°F . The initial rate of change is greater in magnitude than the rate of change after an hour.



38. The slope of the tangent (that is, the rate of change of temperature with respect to time) at $t = 1$ h seems to be about $\frac{40 - 60}{97 - 25} \approx -0.28^\circ\text{C/min}$.

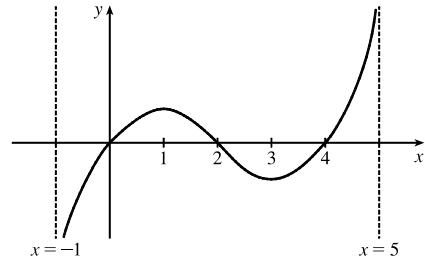


39. We begin by drawing a curve through the origin with a slope of 3 to satisfy $f(0) = 0$ and $f'(0) = 3$. Since $f'(1) = 0$, we will round off our figure so that there is a horizontal tangent directly over $x = 1$. Last, we make sure that the curve has a slope of -1 as we pass over $x = 2$. Two of the many possibilities are shown.

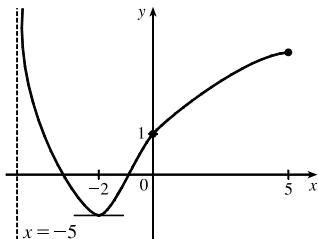


40. The condition $g(0) = g(2) = g(4) = 0$ means that the graph intersects the x -axis at $(0, 0)$, $(2, 0)$, and $(4, 0)$. The condition $g'(1) = g'(3) = 0$ means that the graph has horizontal tangents at $x = 1$ and $x = 3$. The conditions $g'(0) = g'(4) = 1$ and $g'(2) = -1$ mean that the tangents at $(0, 0)$ and $(4, 0)$ have slope 1, while the tangent at $(2, 0)$ has slope -1 . Finally,

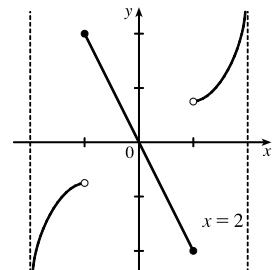
the conditions $\lim_{x \rightarrow 5^-} g(x) = \infty$ and $\lim_{x \rightarrow -1^+} g(x) = -\infty$ imply that $x = -1$ and $x = 5$ are vertical asymptotes. A sample graph is shown. Note that the function shown has domain $(-1, 5)$. That domain could easily be extended by drawing additional graph segments in $(-\infty, -1]$ and $[5, \infty)$ that satisfy the vertical line test.



41. We begin by drawing a curve through $(0, 1)$ with a slope of 1 to satisfy $g(0) = 1$ and $g'(0) = 1$. We round off our figure at $x = -2$ to satisfy $g'(-2) = 0$. As $x \rightarrow -5^+$, $y \rightarrow \infty$, so we draw a vertical asymptote at $x = -5$. As $x \rightarrow 5^-$, $y \rightarrow 3$, so we draw a dot at $(5, 3)$ [the dot could be open or closed].



42. We begin by drawing an odd function (symmetric with respect to the origin) through the origin with slope -2 to satisfy $f'(0) = -2$. Now draw a curve starting at $x = 1$ and increasing without bound as $x \rightarrow 2^-$ since $\lim_{x \rightarrow 2^-} f(x) = \infty$. Lastly, reflect the last curve through the origin (rotate 180°) since f is an odd function.



43. By Definition 4, $\lim_{h \rightarrow 0} \frac{\sqrt{9+h}-3}{h} = f'(9)$, where $f(x) = \sqrt{x}$ and $a = 9$.

44. By Definition 4, $\lim_{h \rightarrow 0} \frac{2^{3+h}-8}{h} = f'(3)$, where $f(x) = 2^x$ and $a = 3$.

45. By Equation 5, $\lim_{x \rightarrow 2} \frac{x^6 - 64}{x - 2} = f'(2)$, where $f(x) = x^6$ and $a = 2$.

- 46.** By Equation 5, $\lim_{x \rightarrow 1/4} \frac{\frac{1}{x} - 4}{x - \frac{1}{4}} = f'(4)$, where $f(x) = \frac{1}{x}$ and $a = \frac{1}{4}$.

- 47.** By Definition 4, $\lim_{h \rightarrow 0} \frac{\tan\left(\frac{\pi}{4} + h\right) - 1}{h} = f'\left(\frac{\pi}{4}\right)$, where $f(x) = \tan x$ and $a = \frac{\pi}{4}$.

- 48.** By Equation 5, $\lim_{\theta \rightarrow \pi/6} \frac{\sin \theta - \frac{1}{2}}{\theta - \frac{\pi}{6}} = f'\left(\frac{\pi}{6}\right)$, where $f(\theta) = \sin \theta$ and $a = \frac{\pi}{6}$.

- 49.** (a) (i) $\frac{\Delta C}{\Delta x} = \frac{C(105) - C(100)}{105 - 100} = \frac{6601.25 - 6500}{5} = \$20.25/\text{unit}$.

$$\text{(ii)} \quad \frac{\Delta C}{\Delta x} = \frac{C(101) - C(100)}{101 - 100} = \frac{6520.05 - 6500}{1} = \$20.05/\text{unit}.$$

$$\text{(b)} \quad \frac{C(100+h) - C(100)}{h} = \frac{[5000 + 10(100+h) + 0.05(100+h)^2] - 6500}{h} = \frac{20h + 0.05h^2}{h} \\ = 20 + 0.05h, h \neq 0$$

So the instantaneous rate of change is $\lim_{h \rightarrow 0} \frac{C(100+h) - C(100)}{h} = \lim_{h \rightarrow 0} (20 + 0.05h) = \$20/\text{unit}$.

- 50.** (a) $H'(14)$ is the rate at which the daily heating cost changes with respect to temperature when the outside temperature is 14°C . The units are dollars/ $^\circ\text{C}$.

- (b) If the outside temperature increases, the building should require less heating, so we would expect $H'(14)$ to be negative.

- 51.** (a) $f'(x)$ is the rate of change of the production cost with respect to the number of kilograms of gold produced. Its units are dollars per kilogram.

- (b) After 22 kg of gold have been produced, the rate at which the production cost is increasing is $\$17/\text{kg}$. So the cost of producing the 22nd (or 23rd) kg is about $\$17$.

- (c) In the short term, the values of $f'(x)$ will decrease because more efficient use is made of startup costs as x increases. But eventually $f'(x)$ might increase due to large-scale operations.

- 52.** (a) $f'(8)$ is the rate of change of the quantity of coffee sold with respect to the price per kilogram when the price is $\$8$ per kilogram. The units for $f'(8)$ are kilogram/(dollars/kilogram) or $\text{kg}^2/\$$.

- (b) $f'(8)$ is negative since the quantity of coffee sold will decrease as the price charged for it increases. People are generally less willing to buy a product when its price increases.

- 53.** (a) $S'(T)$ is the rate at which the oxygen solubility changes with respect to the water temperature. Its units are $(\text{mg/L})/{}^\circ\text{C}$.

- (b) For $T = 16^\circ\text{C}$, it appears that the tangent line to the curve goes through the points $(0, 14)$ and $(32, 6)$. So

$$S'(16) \approx \frac{6 - 14}{32 - 0} = -\frac{8}{32} = -0.25 \text{ (mg/L)}/{}^\circ\text{C}. \text{ This means that as the temperature increases past } 16^\circ\text{C}, \text{ the oxygen solubility is decreasing at a rate of } 0.25 \text{ (mg/L)}/{}^\circ\text{C}.$$

54. (a) $S'(T)$ is the rate of change of the maximum sustainable speed of Coho salmon with respect to the temperature. Its units are $(\text{cm/s})/\text{°C}$.

(b) For $T = 15^\circ\text{C}$, it appears the tangent line to the curve goes through the points $(10, 25)$ and $(20, 32)$. So

$S'(15) \approx \frac{32 - 25}{20 - 10} = 0.7 \text{ (cm/s)/}^\circ\text{C}$. This tells us that at $T = 15^\circ\text{C}$, the maximum sustainable speed of Coho salmon is changing at a rate of $0.7 \text{ (cm/s)/}^\circ\text{C}$. In a similar fashion for $T = 25^\circ\text{C}$, we can use the points $(20, 35)$ and $(25, 25)$ to obtain $S'(25) \approx \frac{25 - 35}{25 - 20} = -2 \text{ (cm/s)/}^\circ\text{C}$. As it gets warmer than 20°C , the maximum sustainable speed decreases rapidly.

55. (a) (i) $[1.0, 2.0]$: $\frac{C(2) - C(1)}{2 - 1} = \frac{0.018 - 0.033}{1} = -0.015 \frac{\text{g/dL}}{\text{h}}$

(ii) $[1.5, 2.0]$: $\frac{C(2) - C(1.5)}{2 - 1.5} = \frac{0.018 - 0.024}{0.5} = \frac{-0.006}{0.5} = -0.012 \frac{\text{g/dL}}{\text{h}}$

(iii) $[2.0, 2.5]$: $\frac{C(2.5) - C(2)}{2.5 - 2} = \frac{0.012 - 0.018}{0.5} = \frac{-0.006}{0.5} = -0.012 \frac{\text{g/dL}}{\text{h}}$

(iv) $[2.0, 3.0]$: $\frac{C(3) - C(2)}{3 - 2} = \frac{0.007 - 0.018}{1} = -0.011 \frac{\text{g/dL}}{\text{h}}$

(b) We estimate the instantaneous rate of change at $t = 2$ by averaging the average rates of change for $[1.5, 2.0]$ and $[2.0, 2.5]$:

$$\frac{-0.012 + (-0.012)}{2} = -0.012 \frac{\text{g/dL}}{\text{h}}. \text{ After two hours, the BAC is decreasing at a rate of } 0.012 \frac{\text{g/dL}}{\text{h}}.$$

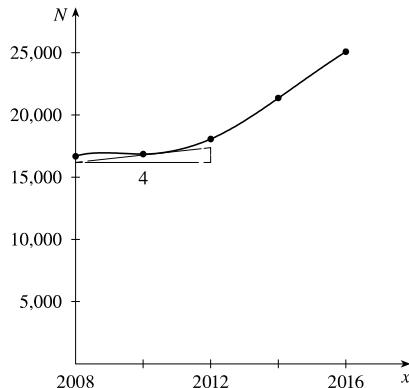
56. (a) (i) $[2008, 2010]$: $\frac{N(2010) - N(2008)}{2010 - 2008} = \frac{16,858 - 16,680}{2} = \frac{178}{2} = 89 \text{ locations/year.}$

(ii) $[2010, 2012]$: $\frac{N(2012) - N(2010)}{2012 - 2010} = \frac{18,066 - 16,858}{2} = \frac{1208}{2} = 604 \text{ locations/year.}$

The rate of growth increased over the period 2008 to 2012.

(b) Averaging the values from parts (i) and (ii) of (a), we have $\frac{89 + 604}{2} = \frac{693}{2} = 346.5 \text{ locations/year.}$

(c) We plot the function N and estimate the slope of the tangent line at $x = 2010$. The tangent segment has endpoints $(2008, 16,250)$ and $(2012, 17,500)$. An estimate of the instantaneous rate of growth in 2010 is $\frac{17,500 - 16,250}{2012 - 2008} = \frac{1250}{4} = 312.5 \text{ locations/year.}$



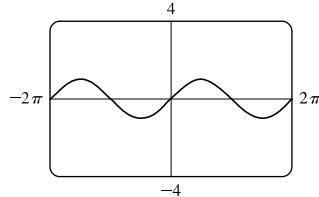
57. Since $f(x) = x \sin(1/x)$ when $x \neq 0$ and $f(0) = 0$, we have

$f'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{h \sin(1/h) - 0}{h} = \lim_{h \rightarrow 0} \sin(1/h)$. This limit does not exist since $\sin(1/h)$ takes the values -1 and 1 on any interval containing 0 . (Compare with Example 1.5.5.)

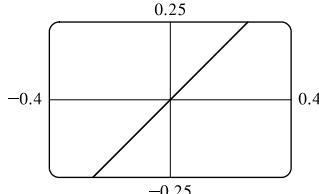
58. Since $f(x) = x^2 \sin(1/x)$ when $x \neq 0$ and $f(0) = 0$, we have

$f'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{h^2 \sin(1/h) - 0}{h} = \lim_{h \rightarrow 0} h \sin(1/h)$. Since $-1 \leq \sin \frac{1}{h} \leq 1$, we have
 $-|h| \leq |h| \sin \frac{1}{h} \leq |h| \Rightarrow -|h| \leq h \sin \frac{1}{h} \leq |h|$. Because $\lim_{h \rightarrow 0} (-|h|) = 0$ and $\lim_{h \rightarrow 0} |h| = 0$, we know that
 $\lim_{h \rightarrow 0} \left(h \sin \frac{1}{h} \right) = 0$ by the Squeeze Theorem. Thus, $f'(0) = 0$.

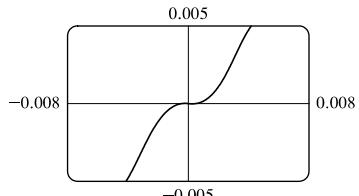
59. (a) The slope at the origin appears to be 1.



- (b) The slope at the origin still appears to be 1.



- (c) Yes, the slope at the origin now appears to be 0.



60. (a) The symmetric difference quotient on $[2004, 2012]$ is (with $a = 2008$ and $d = 4$)

$$\begin{aligned} \frac{f(2008+4) - f(2008-4)}{2(4)} &= \frac{f(2012) - f(2004)}{8} \\ &= \frac{16,432.7 - 7596.1}{8} \\ &= 1104.575 \approx 1105 \text{ billion dollars per year} \end{aligned}$$

This result agrees with the estimate for $D'(2008)$ computed in the example.

- (b) Averaging the average rates of change of f over the intervals $[a-d, a]$ and $[a, a+d]$ gives

$$\begin{aligned} \frac{\frac{f(a) - f(a-d)}{a - (a-d)} + \frac{f(a+d) - f(a)}{(a+d) - a}}{2} &= \frac{\frac{f(a) - f(a-d)}{d} + \frac{f(a+d) - f(a)}{d}}{2} \\ &= \frac{f(a+d) - f(a-d)}{2d} \end{aligned}$$

which is the symmetric difference quotient.

(c) For $f(x) = x^3 - 2x^2 + 2$, $a = 1$, and $d = 0.4$, we have

$$\begin{aligned} f'(1) &\approx \frac{f(1+0.4) - f(1-0.4)}{2(0.4)} = \frac{f(1.4) - f(0.6)}{0.8} \\ &= \frac{0.824 - 1.496}{0.8} = -0.84 \end{aligned}$$

On the graph, (i)–(v) correspond to:

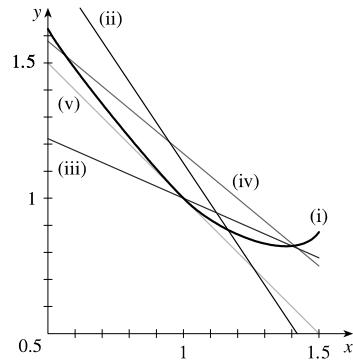
(i) $f(x) = x^3 - 2x^2 + 2$

(ii) secant line corresponding to average rate of change over $[1 - 0.4, 1] = [0.6, 1]$

(iii) secant line corresponding to average rate of change over $[1, 1 + 0.4] = [1, 1.4]$

(iv) secant line corresponding to average rate of change over $[1 - 0.4, 1 + 0.4] = [0.6, 1.4]$

(v) tangent line at $x = 1$



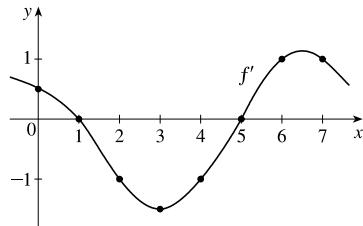
The secant line corresponding to the average rate of change over $[0.6, 1.4]$ —that is, graph (iv)—appears to have slope closest to that of the tangent line at $x = 1$.

2.2 The Derivative as a Function

1. We estimate the slopes of tangent lines on the graph of f to determine the derivative approximations that follow.

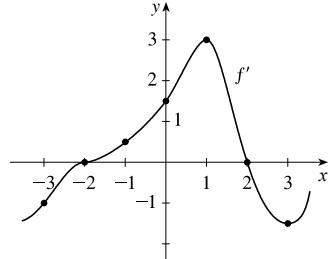
Your answers may vary depending on your estimates.

- | | |
|---------------------------|----------------------------------|
| (a) $f'(0) = \frac{1}{2}$ | (b) $f'(1) \approx 0$ |
| (c) $f'(2) \approx -1$ | (d) $f'(3) \approx -\frac{3}{2}$ |
| (e) $f'(4) \approx -1$ | (f) $f'(5) \approx 0$ |
| (g) $f'(6) \approx 1$ | (h) $f'(7) \approx 1$ |



2. We estimate the slopes of tangent lines on the graph of f to determine the derivative approximations that follow. Your answers may vary depending on your estimates.

- | | |
|----------------------------------|---------------------------------|
| (a) $f'(-3) \approx -1$ | (b) $f'(-2) \approx 0$ |
| (c) $f'(-1) \approx \frac{1}{2}$ | (d) $f'(0) \approx \frac{3}{2}$ |
| (e) $f'(1) \approx 3$ | (f) $f'(2) \approx 0$ |
| (g) $f'(3) \approx -\frac{3}{2}$ | |



3. (a)' = II, since from left to right, the slopes of the tangents to graph (a) start out negative, become 0, then positive, then 0, then negative again. The actual function values in graph II follow the same pattern.

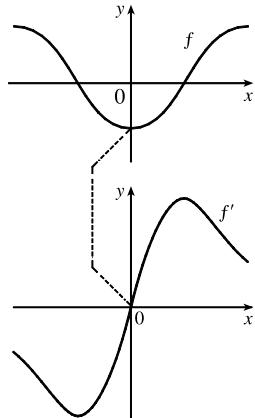
- (b)' = IV, since from left to right, the slopes of the tangents to graph (b) start out at a fixed positive quantity, then suddenly become negative, then positive again. The discontinuities in graph IV indicate sudden changes in the slopes of the tangents.

(c)' = I, since the slopes of the tangents to graph (c) are negative for $x < 0$ and positive for $x > 0$, as are the function values of graph I.

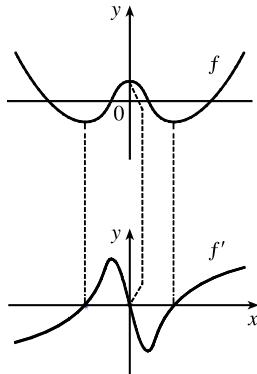
(d)' = III, since from left to right, the slopes of the tangents to graph (d) are positive, then 0, then negative, then 0, then positive, then 0, then negative again, and the function values in graph III follow the same pattern.

Hints for Exercises 4–11: First plot x -intercepts on the graph of f' for any horizontal tangents on the graph of f . Look for any corners on the graph of f —there will be a discontinuity on the graph of f' . On any interval where f has a tangent with positive (or negative) slope, the graph of f' will be positive (or negative). If the graph of the function is linear, the graph of f' will be a horizontal line.

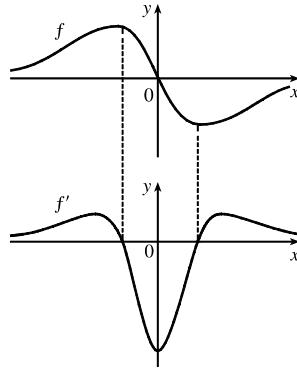
4.



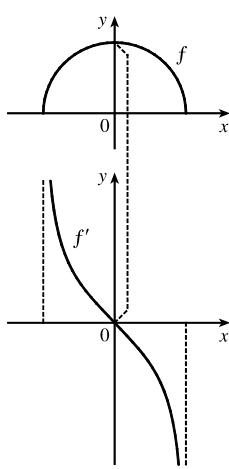
5.



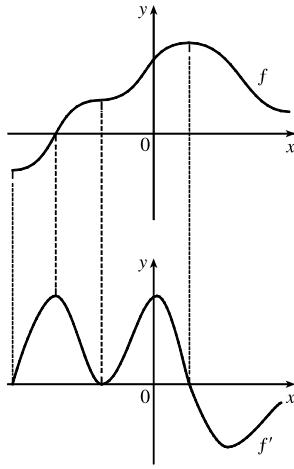
6.



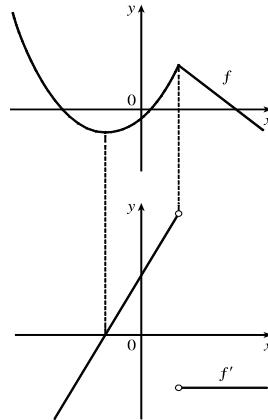
7.



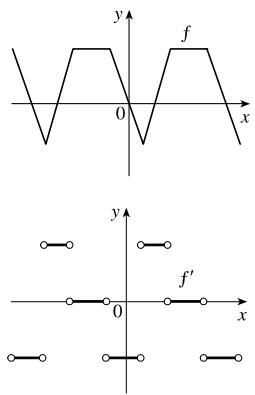
8.



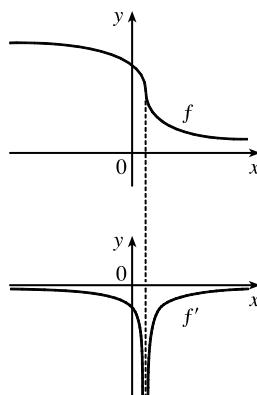
9.



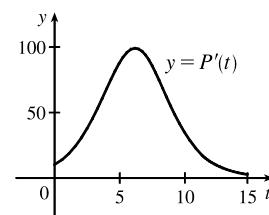
10.



11.

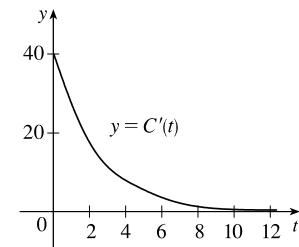


12. The slopes of the tangent lines on the graph of $y = P(t)$ are always positive, so the y -values of $y = P'(t)$ are always positive. These values start out relatively small and keep increasing, reaching a maximum at about $t = 6$. Then the y -values of $y = P'(t)$ decrease and get close to zero. The graph of P' tells us that the yeast culture grows most rapidly after 6 hours and then the growth rate declines.



13. (a) $C'(t)$ is the instantaneous rate of change of percentage of full capacity with respect to elapsed time in hours.

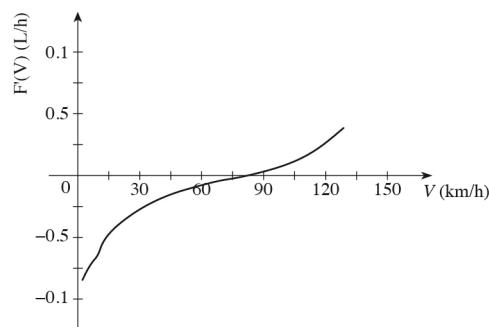
- (b) The graph of $C'(t)$ tells us that the rate of change of percentage of full capacity is decreasing and approaching 0.



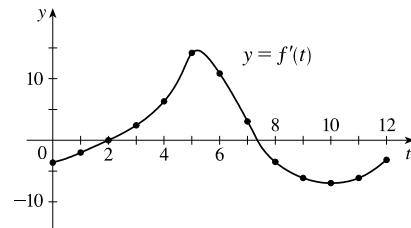
14. (a) $F'(v)$ is the instantaneous rate of change of fuel economy with respect to speed.

- (b) Graphs will vary depending on estimates of F' , but will change from negative to positive at about $v = 80$.

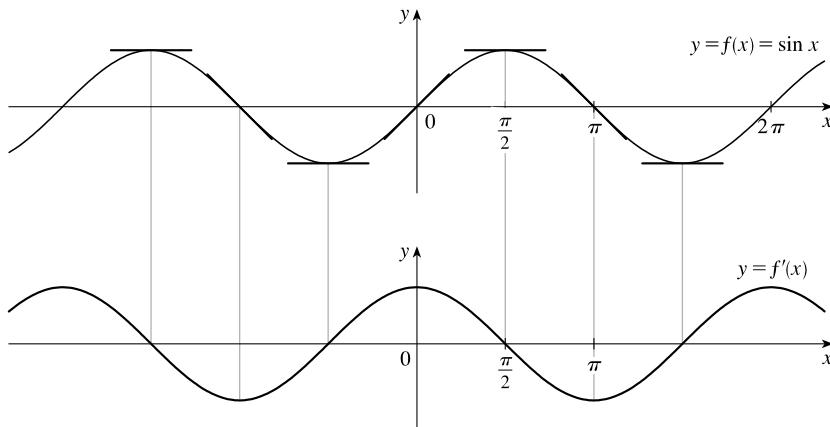
- (c) To save on gas, drive at the speed where F is a minimum and F' is 0, which is about 80 km/h.



15. It appears that there are horizontal tangents on the graph of f for $t = 2$ and for $t \approx 7.5$. Thus, there are zeros for those values of t on the graph of f' . The derivative is negative for values of t between 0 and 2 and for values of t between approximately 7.5 and 12. The value of $f'(t)$ appears to be largest at $t \approx 5.25$.



16.



The graph of the derivative looks like the graph of the cosine function.

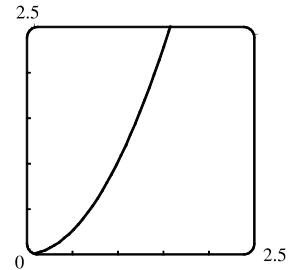
17. (a) By zooming in, we estimate that $f'(0) = 0$, $f'\left(\frac{1}{2}\right) = 1$, $f'(1) = 2$, and $f'(2) = 4$.

(b) By symmetry, $f'(-x) = -f'(x)$. So $f'\left(-\frac{1}{2}\right) = -1$, $f'(-1) = -2$, and $f'(-2) = -4$.

(c) It appears that $f'(x)$ is twice the value of x , so we guess that $f'(x) = 2x$.

$$(d) f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(x^2 + 2hx + h^2) - x^2}{h} = \lim_{h \rightarrow 0} \frac{2hx + h^2}{h} = \lim_{h \rightarrow 0} \frac{h(2x + h)}{h} = \lim_{h \rightarrow 0} (2x + h) = 2x$$

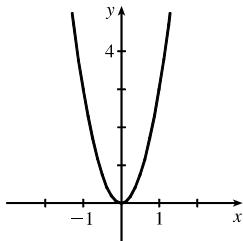


2.5

18. (a) By zooming in, we estimate that $f'(0) = 0$, $f'\left(\frac{1}{2}\right) \approx 0.75$, $f'(1) \approx 3$, $f'(2) \approx 12$, and $f'(3) \approx 27$.

(b) By symmetry, $f'(-x) = f'(x)$. So $f'\left(-\frac{1}{2}\right) \approx 0.75$, $f'(-1) \approx 3$, $f'(-2) \approx 12$, and $f'(-3) \approx 27$.

(c)



(d) Since $f'(0) = 0$, it appears that f' may have the form $f'(x) = ax^2$.

Using $f'(1) = 3$, we have $a = 3$, so $f'(x) = 3x^2$.

$$(e) f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h \rightarrow 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h}$$

$$= \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3}{h} = \lim_{h \rightarrow 0} \frac{h(3x^2 + 3xh + h^2)}{h} = \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2) = 3x^2$$

$$19. f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{[3(x+h) - 8] - (3x - 8)}{h} = \lim_{h \rightarrow 0} \frac{3x + 3h - 8 - 3x + 8}{h}$$

$$= \lim_{h \rightarrow 0} \frac{3h}{h} = \lim_{h \rightarrow 0} 3 = 3$$

Domain of f = domain of $f' = \mathbb{R}$.

$$20. f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{[m(x+h) + b] - (mx + b)}{h} = \lim_{h \rightarrow 0} \frac{mx + mh + b - mx - b}{h}$$

$$= \lim_{h \rightarrow 0} \frac{mh}{h} = \lim_{h \rightarrow 0} m = m$$

Domain of f = domain of $f' = \mathbb{R}$.

$$21. f'(t) = \lim_{h \rightarrow 0} \frac{f(t+h) - f(t)}{h} = \lim_{h \rightarrow 0} \frac{[2.5(t+h)^2 + 6(t+h)] - (2.5t^2 + 6t)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{2.5(t^2 + 2th + h^2) + 6t + 6h - 2.5t^2 - 6t}{h} = \lim_{h \rightarrow 0} \frac{2.5t^2 + 5th + 2.5h^2 + 6h - 2.5t^2}{h}$$

$$= \lim_{h \rightarrow 0} \frac{5th + 2.5h^2 + 6h}{h} = \lim_{h \rightarrow 0} \frac{h(5t + 2.5h + 6)}{h} = \lim_{h \rightarrow 0} (5t + 2.5h + 6)$$

$$= 5t + 6$$

Domain of f = domain of $f' = \mathbb{R}$.

$$\begin{aligned}
22. \quad f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{[4 + 8(x+h) - 5(x+h)^2] - (4 + 8x - 5x^2)}{h} \\
&= \lim_{h \rightarrow 0} \frac{4 + 8x + 8h - 5(x^2 + 2xh + h^2) - 4 - 8x + 5x^2}{h} = \lim_{h \rightarrow 0} \frac{8h - 5x^2 - 10xh - 5h^2 + 5x^2}{h} \\
&= \lim_{h \rightarrow 0} \frac{8h - 10xh - 5h^2}{h} = \lim_{h \rightarrow 0} \frac{h(8 - 10x - 5h)}{h} = \lim_{h \rightarrow 0} (8 - 10x - 5h) \\
&= 8 - 10x
\end{aligned}$$

Domain of f = domain of $f' = \mathbb{R}$.

$$\begin{aligned}
23. \quad A'(p) &= \lim_{h \rightarrow 0} \frac{A(p+h) - A(p)}{h} = \lim_{h \rightarrow 0} \frac{[4(p+h)^3 + 3(p+h)] - (4p^3 + 3p)}{h} \\
&= \lim_{h \rightarrow 0} \frac{4p^3 + 12p^2h + 12ph^2 + 4h^3 + 3p + 3h - 4p^3 - 3p}{h} = \lim_{h \rightarrow 0} \frac{12p^2h + 12ph^2 + 4h^3 + 3h}{h} \\
&= \lim_{h \rightarrow 0} \frac{h(12p^2 + 12ph + 4h^2 + 3)}{h} = \lim_{h \rightarrow 0} (12p^2 + 12ph + 4h^2 + 3) = 12p^2 + 3
\end{aligned}$$

Domain of A = Domain of $A' = \mathbb{R}$.

$$\begin{aligned}
24. \quad F'(t) &= \lim_{h \rightarrow 0} \frac{F(t+h) - F(t)}{h} = \lim_{h \rightarrow 0} \frac{[(t+h)^3 - 5(t+h)+1] - (t^3 - 5t+1)}{h} \\
&= \lim_{h \rightarrow 0} \frac{t^3 + 3t^2h + 3th^2 + h^3 - 5t - 5h + 1 - t^3 + 5t - 1}{h} = \lim_{h \rightarrow 0} \frac{3t^2h + 3th^2 + h^3 - 5h}{h} \\
&= \lim_{h \rightarrow 0} \frac{h(3t^2 + 3th + h^2 - 5)}{h} = \lim_{h \rightarrow 0} (3t^2 + 3th + h^2 - 5) = 3t^2 - 5
\end{aligned}$$

Domain of F = Domain of $F' = \mathbb{R}$.

$$\begin{aligned}
25. \quad f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{(x+h)^2 - 4} - \frac{1}{x^2 - 4}}{h} = \lim_{h \rightarrow 0} \frac{(x^2 - 4) - [(x+h)^2 - 4]}{h[(x+h)^2 - 4](x^2 - 4)} \\
&= \lim_{h \rightarrow 0} \frac{(x^2 - 4) - (x^2 + 2xh + h^2 - 4)}{h[(x+h)^2 - 4](x^2 - 4)} = \lim_{h \rightarrow 0} \frac{x^2 - 4 - x^2 - 2xh - h^2 + 4}{h[(x+h)^2 - 4](x^2 - 4)} = \lim_{h \rightarrow 0} \frac{-2xh - h^2}{h[(x+h)^2 - 4](x^2 - 4)} \\
&= \lim_{h \rightarrow 0} \frac{h(-2x - h)}{h[(x+h)^2 - 4](x^2 - 4)} = \lim_{h \rightarrow 0} \frac{-2x - h}{[(x+h)^2 - 4](x^2 - 4)} = \frac{-2x}{(x^2 - 4)(x^2 - 4)} = -\frac{2x}{(x^2 - 4)^2}
\end{aligned}$$

Domain of f = Domain of $f' = (-\infty, -2) \cup (-2, 2) \cup (2, \infty)$.

$$\begin{aligned}
26. \quad F'(v) &= \lim_{h \rightarrow 0} \frac{F(v+h) - F(v)}{h} = \lim_{h \rightarrow 0} \frac{\frac{v+h}{(v+h)+2} - \frac{v}{v+2}}{h} = \lim_{h \rightarrow 0} \frac{\frac{(v+h)(v+2) - v[(v+h)+2]}{[(v+h)+2](v+2)}}{h} \\
&= \lim_{h \rightarrow 0} \frac{v^2 + 2v + vh + 2h - v^2 - vh - 2v}{h[(v+h)+2](v+2)} = \lim_{h \rightarrow 0} \frac{2h}{h[(v+h)+2](v+2)} = \lim_{h \rightarrow 0} \frac{2}{[(v+h)+2](v+2)} \\
&= \frac{2}{(v+2)(v+2)} = \frac{2}{(v+2)^2}
\end{aligned}$$

Domain of F = Domain of $F' = (-\infty, -2) \cup (-2, \infty)$.

$$\begin{aligned}
27. \quad g'(u) &= \lim_{h \rightarrow 0} \frac{g(u+h) - g(u)}{h} = \lim_{h \rightarrow 0} \frac{\frac{(u+h)+1}{4(u+h)-1} - \frac{u+1}{4u-1}}{h} = \lim_{h \rightarrow 0} \frac{\frac{[(u+h)+1](4u-1) - (u+1)[4(u+h)-1]}{[4(u+h)-1](4u-1)}}{h} \\
&= \lim_{h \rightarrow 0} \frac{\frac{(u+h+1)(4u-1) - (u+1)(4u+4h-1)}{[4(u+h)-1](4u-1)}}{h} \\
&= \lim_{h \rightarrow 0} \frac{\frac{4u^2 + 4uh + 4u - u - h - 1 - 4u^2 - 4uh + u - 4u - 4h + 1}{h[4(u+h)-1](4u-1)}}{h} \\
&= \lim_{h \rightarrow 0} \frac{-5h}{h[4(u+h)-1](4u-1)} = \lim_{h \rightarrow 0} \frac{-5}{[4(u+h)-1](4u-1)} = \frac{-5}{(4u-1)(4u-1)} = -\frac{5}{(4u-1)^2}
\end{aligned}$$

Domain of g = Domain of $g' = (-\infty, \frac{1}{4}) \cup (\frac{1}{4}, \infty)$.

$$\begin{aligned}
28. \quad f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h)^4 - x^4}{h} = \lim_{h \rightarrow 0} \frac{(x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4) - x^4}{h} \\
&= \lim_{h \rightarrow 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h} = \lim_{h \rightarrow 0} \frac{h(4x^3 + 6x^2h + 4xh^2 + h^3)}{h} \\
&= \lim_{h \rightarrow 0} (4x^3 + 6x^2h + 4xh^2 + h^3) = 4x^3
\end{aligned}$$

Domain of f = domain of $f' = \mathbb{R}$.

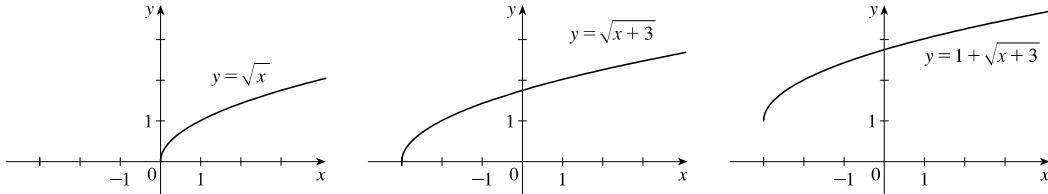
$$\begin{aligned}
29. \quad f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{\sqrt{1+(x+h)}} - \frac{1}{\sqrt{1+x}}}{h} \\
&= \lim_{h \rightarrow 0} \frac{\frac{1}{\sqrt{1+(x+h)}} - \frac{1}{\sqrt{1+x}}}{h} \cdot \frac{\sqrt{1+(x+h)}\sqrt{1+x}}{\sqrt{1+(x+h)}\sqrt{1+x}} \\
&= \lim_{h \rightarrow 0} \frac{\sqrt{1+x} - \sqrt{1+(x+h)}}{h\sqrt{1+(x+h)}\sqrt{1+x}} \cdot \frac{\sqrt{1+x} + \sqrt{1+(x+h)}}{\sqrt{1+x} + \sqrt{1+(x+h)}} \\
&= \lim_{h \rightarrow 0} \frac{(1+x) - [1+(x+h)]}{h\sqrt{1+(x+h)}\sqrt{1+x}(\sqrt{1+x} + \sqrt{1+(x+h)})} \\
&= \lim_{h \rightarrow 0} \frac{-h}{h\sqrt{1+x+h}\sqrt{1+x}(\sqrt{1+x} + \sqrt{1+x+h})} = \lim_{h \rightarrow 0} \frac{-1}{\sqrt{1+x+h}\sqrt{1+x}(\sqrt{1+x} + \sqrt{1+x+h})} \\
&= \frac{-1}{\sqrt{1+x}\sqrt{1+x}(\sqrt{1+x} + \sqrt{1+x})} = \frac{-1}{(1+x)(2\sqrt{1+x})} = -\frac{1}{2(1+x)^{3/2}}
\end{aligned}$$

Domain of f = Domain of $f' = (-1, \infty)$.

$$\begin{aligned}
30. \quad g'(x) &= \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{1}{1+\sqrt{x+h}} - \frac{1}{1+\sqrt{x}}}{h} = \lim_{h \rightarrow 0} \frac{\frac{(1+\sqrt{x}) - (1+\sqrt{x+h})}{(1+\sqrt{x+h})(1+\sqrt{x})}}{h} \\
&= \lim_{h \rightarrow 0} \frac{1+\sqrt{x}-1-\sqrt{x+h}}{h(1+\sqrt{x+h})(1+\sqrt{x})} = \lim_{h \rightarrow 0} \frac{\sqrt{x}-\sqrt{x+h}}{h(1+\sqrt{x+h})(1+\sqrt{x})} \cdot \frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}} \\
&= \lim_{h \rightarrow 0} \frac{x-(x+h)}{h(1+\sqrt{x+h})(1+\sqrt{x})(\sqrt{x}+\sqrt{x+h})} \\
&= \lim_{h \rightarrow 0} \frac{-h}{h(1+\sqrt{x+h})(1+\sqrt{x})(\sqrt{x}+\sqrt{x+h})} \\
&= \lim_{h \rightarrow 0} \frac{-1}{(1+\sqrt{x+h})(1+\sqrt{x})(\sqrt{x}+\sqrt{x+h})} \\
&= \frac{-1}{(1+\sqrt{x})(1+\sqrt{x})(\sqrt{x}+\sqrt{x})} = \frac{-1}{(1+\sqrt{x})^2 \cdot 2\sqrt{x}} = -\frac{1}{2\sqrt{x}(1+\sqrt{x})^2}
\end{aligned}$$

Domain of $g = [0, \infty)$, domain of $g' = (0, \infty)$.

31. (a)



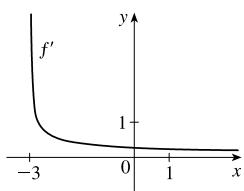
(b) Note that the third graph in part (a) generally has small positive values for its slope, f' ; but as $x \rightarrow -3^+$, $f' \rightarrow \infty$.

See the graph in part (d).

$$\begin{aligned}
(c) \quad f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{1 + \sqrt{(x+h)+3} - (1 + \sqrt{x+3})}{h} \\
&= \lim_{h \rightarrow 0} \frac{\sqrt{(x+h)+3} - \sqrt{x+3}}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{(x+h)+3} - \sqrt{x+3}}{h} \left[\frac{\sqrt{(x+h)+3} + \sqrt{x+3}}{\sqrt{(x+h)+3} + \sqrt{x+3}} \right] \\
&= \lim_{h \rightarrow 0} \frac{[(x+h)+3] - (x+3)}{h(\sqrt{(x+h)+3} + \sqrt{x+3})} = \lim_{h \rightarrow 0} \frac{x+h+3-x-3}{h(\sqrt{(x+h)+3} + \sqrt{x+3})} \\
&= \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{(x+h)+3} + \sqrt{x+3})} = \lim_{h \rightarrow 0} \frac{1}{\sqrt{(x+h)+3} + \sqrt{x+3}} = \frac{1}{2\sqrt{x+3}}
\end{aligned}$$

Domain of $f = [-3, \infty)$, Domain of $f' = (-3, \infty)$.

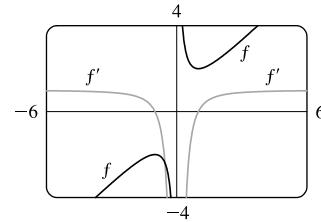
(d)



$$\begin{aligned}
 32. \text{ (a)} \quad f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{[(x+h) + 1/(x+h)] - (x+1/x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h)^2 + 1}{x+h} - \frac{x^2 + 1}{x} \\
 &= \lim_{h \rightarrow 0} \frac{x[(x+h)^2 + 1] - (x+h)(x^2 + 1)}{h(x+h)x} = \lim_{h \rightarrow 0} \frac{(x^3 + 2hx^2 + xh^2 + x) - (x^3 + x + hx^2 + h)}{h(x+h)x} \\
 &= \lim_{h \rightarrow 0} \frac{hx^2 + xh^2 - h}{h(x+h)x} = \lim_{h \rightarrow 0} \frac{h(x^2 + xh - 1)}{h(x+h)x} = \lim_{h \rightarrow 0} \frac{x^2 + xh - 1}{(x+h)x} = \frac{x^2 - 1}{x^2}, \text{ or } 1 - \frac{1}{x^2}
 \end{aligned}$$

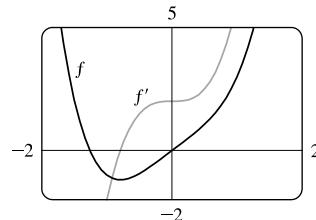
(b) Notice that $f'(x) = 0$ when f has a horizontal tangent, $f'(x)$ is

positive when the tangents have positive slope, and $f'(x)$ is
negative when the tangents have negative slope. Both functions
are discontinuous at $x = 0$.



$$\begin{aligned}
 33. \text{ (a)} \quad f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{[(x+h)^4 + 2(x+h)] - (x^4 + 2x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 + 2x + 2h - x^4 - 2x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4 + 2h}{h} = \lim_{h \rightarrow 0} \frac{h(4x^3 + 6x^2h + 4xh^2 + h^3 + 2)}{h} \\
 &= \lim_{h \rightarrow 0} (4x^3 + 6x^2h + 4xh^2 + h^3 + 2) = 4x^3 + 2
 \end{aligned}$$

(b) Notice that $f'(x) = 0$ when f has a horizontal tangent, $f'(x)$ is
positive when the tangents have positive slope, and $f'(x)$ is
negative when the tangents have negative slope.



34. (a) $N'(t)$ is the rate at which the number of minimally invasive cosmetic surgery procedures performed in the United States is changing with respect to time. Its units are thousands of surgeries per year.

(b) To find $N'(t)$, we use $\lim_{h \rightarrow 0} \frac{N(t+h) - N(t)}{(t+h) - t} \approx \frac{N(t+h) - N(t)}{h}$ for small values of h .

$$\text{For 2000: } N'(2000) \approx \frac{N(2002) - N(2000)}{2002 - 2000} = \frac{4897 - 5500}{2} = -301.5$$

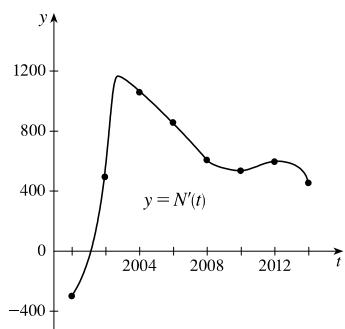
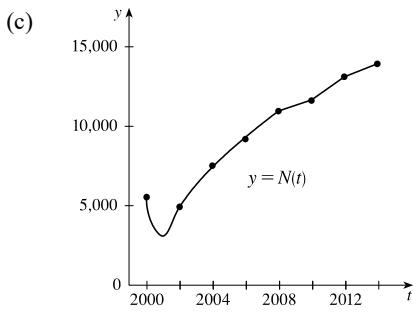
For 2002: We estimate $N'(2002)$ by using $h = -2$ and $h = 2$, and then average the two results to obtain a final estimate.

$$h = -2 \Rightarrow N'(2002) \approx \frac{N(2000) - N(2002)}{2000 - 2002} = \frac{5500 - 4897}{-2} = -301.5$$

$$h = 2 \Rightarrow N'(2002) \approx \frac{N(2004) - N(2002)}{2004 - 2002} = \frac{7470 - 4897}{2} = 1286.5$$

So we estimate that $N'(2002) \approx \frac{1}{2}[-301.5 + 1286.5] = 492.5$.

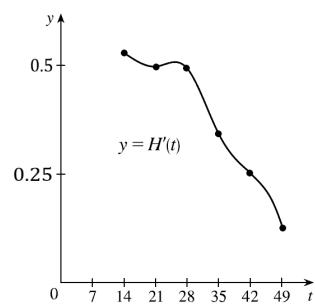
t	2000	2002	2004	2006	2008	2010	2012	2014
$N'(t)$	-301.5	492.5	1060.25	856.75	605.75	534.5	596	455



(d) We could get more accurate values for $N'(t)$ by obtaining data for more values of t .

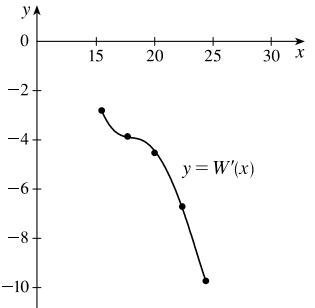
35. As in Exercise 34, we use one-sided difference quotients for the first and last values, and average two difference quotients for all other values.

t	14	21	28	35	42	49
$H(t)$	12	16	19	22	24	25
$H'(t)$	$\frac{4}{7}$	$\frac{1}{2}$	$\frac{3}{7}$	$\frac{5}{14}$	$\frac{3}{14}$	$\frac{1}{7}$



36. As in Exercise 34, we use one-sided difference quotients for the first and last values, and average two difference quotients for all other values. The units for $W'(x)$ are grams per degree ($\text{g}/^\circ\text{C}$).

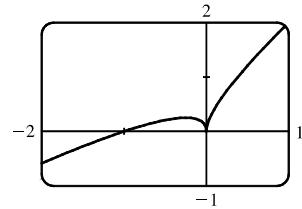
x	15.5	17.7	20.0	22.4	24.4
$W(x)$	37.2	31.0	19.8	9.7	-9.8
$W'(x)$	-2.82	-3.87	-4.53	-6.73	-9.75



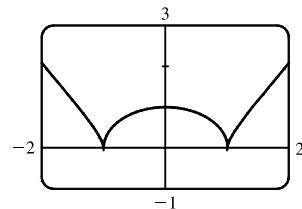
37. (a) dP/dt is the rate at which the percentage of the city's electrical power produced by solar panels changes with respect to time t , measured in percentage points per year.
- (b) 2 years after January 1, 2020 (January 1, 2022), the percentage of electrical power produced by solar panels was increasing at a rate of 3.5 percentage points per year.
38. dN/dp is the rate at which the number of people who travel by car to another state for a vacation changes with respect to the price of gasoline. If the price of gasoline goes up, we would expect fewer people to travel, so we would expect dN/dp to be negative.
39. f is not differentiable at $x = -4$, because the graph has a corner there, and at $x = 0$, because there is a discontinuity there.
40. f is not differentiable at $x = -1$, because there is a discontinuity there, and at $x = 2$, because the graph has a corner there.
41. f is not differentiable at $x = 1$, because f is not defined there, and at $x = 5$, because the graph has a vertical tangent there.

42. f is not differentiable at $x = -2$ and $x = 3$, because the graph has corners there, and at $x = 1$, because there is a discontinuity there.

43. As we zoom in toward $(-1, 0)$, the curve appears more and more like a straight line, so $f(x) = x + \sqrt{|x|}$ is differentiable at $x = -1$. But no matter how much we zoom in toward the origin, the curve doesn't straighten out—we can't eliminate the sharp point (a cusp). So f is not differentiable at $x = 0$.



44. As we zoom in toward $(0, 1)$, the curve appears more and more like a straight line, so $g(x) = (x^2 - 1)^{2/3}$ is differentiable at $x = 0$. But no matter how much we zoom in toward $(1, 0)$ or $(-1, 0)$, the curve doesn't straighten out—we can't eliminate the sharp point (a cusp). So g is not differentiable at $x = \pm 1$.



45. Call the curve with the positive y -intercept g and the other curve h . Notice that g has a maximum (horizontal tangent) at $x = 0$, but $h \neq 0$, so h cannot be the derivative of g . Also notice that where g is positive, h is increasing. Thus, $h = f$ and $g = f'$. Now $f'(-1)$ is negative since f' is below the x -axis there and $f''(1)$ is positive since f is concave upward at $x = 1$. Therefore, $f''(1)$ is greater than $f'(-1)$.

46. Call the curve with the smallest positive x -intercept g and the other curve h . Notice that where g is positive in the first quadrant, h is increasing. Thus, $h = f$ and $g = f'$. Now $f'(-1)$ is positive since f' is above the x -axis there and $f''(1)$ appears to be zero since f has an inflection point at $x = 1$. Therefore, $f'(1)$ is greater than $f''(-1)$.

47. $a = f$, $b = f'$, $c = f''$. We can see this because where a has a horizontal tangent, $b = 0$, and where b has a horizontal tangent, $c = 0$. We can immediately see that c can be neither f nor f' , since at the points where c has a horizontal tangent, neither a nor b is equal to 0.

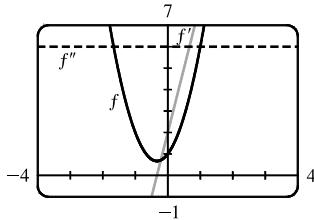
48. Where d has horizontal tangents, only c is 0, so $d' = c$. c has negative tangents for $x < 0$ and b is the only graph that is negative for $x < 0$, so $c' = b$. b has positive tangents on \mathbb{R} (except at $x = 0$), and the only graph that is positive on the same domain is a , so $b' = a$. We conclude that $d = f$, $c = f'$, $b = f''$, and $a = f'''$.

49. We can immediately see that a is the graph of the acceleration function, since at the points where a has a horizontal tangent, neither c nor b is equal to 0. Next, we note that $a = 0$ at the point where b has a horizontal tangent, so b must be the graph of the velocity function, and hence, $b' = a$. We conclude that c is the graph of the position function.

50. a must be the jerk since none of the graphs are 0 at its high and low points. a is 0 where b has a maximum, so $b' = a$. b is 0 where c has a maximum, so $c' = b$. We conclude that d is the position function, c is the velocity, b is the acceleration, and a is the jerk.

$$\begin{aligned}
 51. f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{[3(x+h)^2 + 2(x+h) + 1] - (3x^2 + 2x + 1)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(3x^2 + 6xh + 3h^2 + 2x + 2h + 1) - (3x^2 + 2x + 1)}{h} = \lim_{h \rightarrow 0} \frac{6xh + 3h^2 + 2h}{h} \\
 &= \lim_{h \rightarrow 0} \frac{h(6x + 3h + 2)}{h} = \lim_{h \rightarrow 0} (6x + 3h + 2) = 6x + 2
 \end{aligned}$$

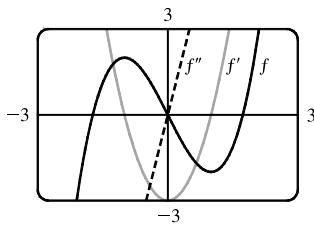
$$\begin{aligned}
 f''(x) &= \lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h \rightarrow 0} \frac{[6(x+h) + 2] - (6x + 2)}{h} = \lim_{h \rightarrow 0} \frac{(6x + 6h + 2) - (6x + 2)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{6h}{h} = \lim_{h \rightarrow 0} 6 = 6
 \end{aligned}$$



We see from the graph that our answers are reasonable because the graph of f' is that of a linear function and the graph of f'' is that of a constant function.

$$\begin{aligned}
 52. f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{[(x+h)^3 - 3(x+h)] - (x^3 - 3x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3 - 3x - 3h) - (x^3 - 3x)}{h} = \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3 - 3h}{h} \\
 &= \lim_{h \rightarrow 0} \frac{h(3x^2 + 3xh + h^2 - 3)}{h} = \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2 - 3) = 3x^2 - 3
 \end{aligned}$$

$$\begin{aligned}
 f''(x) &= \lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h \rightarrow 0} \frac{[3(x+h)^2 - 3] - (3x^2 - 3)}{h} = \lim_{h \rightarrow 0} \frac{(3x^2 + 6xh + 3h^2 - 3) - (3x^2 - 3)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{6xh + 3h^2}{h} = \lim_{h \rightarrow 0} \frac{h(6x + 3h)}{h} = \lim_{h \rightarrow 0} (6x + 3h) = 6x
 \end{aligned}$$



We see from the graph that our answers are reasonable because the graph of f' is that of an even function (f is an odd function) and the graph of f'' is that of an odd function. Furthermore, $f' = 0$ when f has a horizontal tangent and $f'' = 0$ when f' has a horizontal tangent.

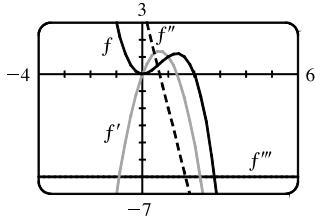
$$\begin{aligned}
 53. f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{[2(x+h)^2 - (x+h)^3] - (2x^2 - x^3)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{h(4x + 2h - 3x^2 - 3xh - h^2)}{h} = \lim_{h \rightarrow 0} (4x + 2h - 3x^2 - 3xh - h^2) = 4x - 3x^2
 \end{aligned}$$

$$\begin{aligned}
 f''(x) &= \lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h \rightarrow 0} \frac{[4(x+h) - 3(x+h)^2] - (4x - 3x^2)}{h} = \lim_{h \rightarrow 0} \frac{h(4 - 6x - 3h)}{h} \\
 &= \lim_{h \rightarrow 0} (4 - 6x - 3h) = 4 - 6x
 \end{aligned}$$

$$f'''(x) = \lim_{h \rightarrow 0} \frac{f''(x+h) - f''(x)}{h} = \lim_{h \rightarrow 0} \frac{[4 - 6(x+h)] - (4 - 6x)}{h} = \lim_{h \rightarrow 0} \frac{-6h}{h} = \lim_{h \rightarrow 0} (-6) = -6$$

[continued]

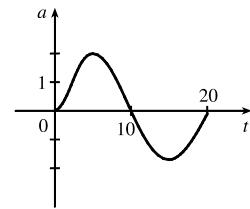
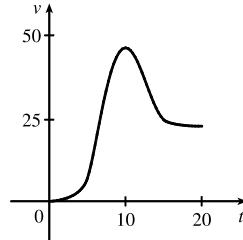
$$f^{(4)}(x) = \lim_{h \rightarrow 0} \frac{f'''(x+h) - f'''(x)}{h} = \lim_{h \rightarrow 0} \frac{-6 - (-6)}{h} = \lim_{h \rightarrow 0} \frac{0}{h} = \lim_{h \rightarrow 0} (0) = 0$$



The graphs are consistent with the geometric interpretations of the derivatives because f' has zeros where f has a local minimum and a local maximum, f'' has a zero where f' has a local maximum, and f''' is a constant function equal to the slope of f'' .

54. (a) Since we estimate the velocity to be a maximum

at $t = 10$, the acceleration is 0 at $t = 10$.



- (b) Drawing a tangent line at $t = 10$ on the graph of a , a appears to decrease by 10 m/s^2 over a period of 20 s.

So at $t = 10$ s, the jerk is approximately $-10/20 = -0.5 (\text{m/s}^2)/\text{s}$ or m/s^3 .

55. (a) Note that we have factored $x - a$ as the difference of two cubes in the third step.

$$\begin{aligned} f'(a) &= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{x \rightarrow a} \frac{x^{1/3} - a^{1/3}}{x - a} = \lim_{x \rightarrow a} \frac{x^{1/3} - a^{1/3}}{(x^{1/3} - a^{1/3})(x^{2/3} + x^{1/3}a^{1/3} + a^{2/3})} \\ &= \lim_{x \rightarrow a} \frac{1}{x^{2/3} + x^{1/3}a^{1/3} + a^{2/3}} = \frac{1}{3a^{2/3}} \text{ or } \frac{1}{3}a^{-2/3} \end{aligned}$$

- (b) $f'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt[3]{h} - 0}{h} = \lim_{h \rightarrow 0} \frac{1}{h^{2/3}}$. This function increases without bound, so the limit does not exist, and therefore $f'(0)$ does not exist.

- (c) $\lim_{x \rightarrow 0} |f'(x)| = \lim_{x \rightarrow 0} \frac{1}{3x^{2/3}} = \infty$ and f is continuous at $x = 0$ (root function), so f has a vertical tangent at $x = 0$.

56. (a) $g'(0) = \lim_{x \rightarrow 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{x^{2/3} - 0}{x} = \lim_{x \rightarrow 0} \frac{1}{x^{1/3}}$, which does not exist.

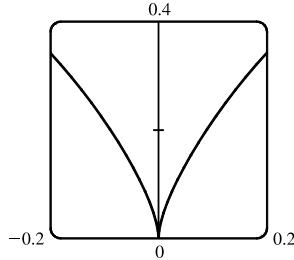
- (b) $g'(a) = \lim_{x \rightarrow a} \frac{g(x) - g(a)}{x - a} = \lim_{x \rightarrow a} \frac{x^{2/3} - a^{2/3}}{x - a} = \lim_{x \rightarrow a} \frac{(x^{1/3} - a^{1/3})(x^{1/3} + a^{1/3})}{(x^{1/3} - a^{1/3})(x^{2/3} + x^{1/3}a^{1/3} + a^{2/3})}$
 $= \lim_{x \rightarrow a} \frac{x^{1/3} + a^{1/3}}{x^{2/3} + x^{1/3}a^{1/3} + a^{2/3}} = \frac{2a^{1/3}}{3a^{2/3}} = \frac{2}{3a^{1/3}} \text{ or } \frac{2}{3}a^{-1/3}$

- (c) $g(x) = x^{2/3}$ is continuous at $x = 0$ and

$\lim_{x \rightarrow 0} |g'(x)| = \lim_{x \rightarrow 0} \frac{2}{3|x|^{1/3}} = \infty$. This shows that

g has a vertical tangent line at $x = 0$.

(d)



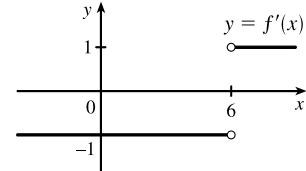
57. $f(x) = |x - 6| = \begin{cases} x - 6 & \text{if } x - 6 \geq 0 \\ -(x - 6) & \text{if } x - 6 < 0 \end{cases} = \begin{cases} x - 6 & \text{if } x \geq 6 \\ 6 - x & \text{if } x < 6 \end{cases}$

So the right-hand limit is $\lim_{x \rightarrow 6^+} \frac{f(x) - f(6)}{x - 6} = \lim_{x \rightarrow 6^+} \frac{|x - 6| - 0}{x - 6} = \lim_{x \rightarrow 6^+} \frac{x - 6}{x - 6} = \lim_{x \rightarrow 6^+} 1 = 1$, and the left-hand limit is $\lim_{x \rightarrow 6^-} \frac{f(x) - f(6)}{x - 6} = \lim_{x \rightarrow 6^-} \frac{|x - 6| - 0}{x - 6} = \lim_{x \rightarrow 6^-} \frac{6 - x}{x - 6} = \lim_{x \rightarrow 6^-} (-1) = -1$. Since these limits are not equal,

$f'(6) = \lim_{x \rightarrow 6} \frac{f(x) - f(6)}{x - 6}$ does not exist and f is not differentiable at 6.

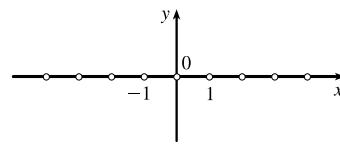
However, a formula for f' is $f'(x) = \begin{cases} -1 & \text{if } x < 6 \\ 1 & \text{if } x > 6 \end{cases}$

Another way of writing the formula is $f'(x) = \frac{x - 6}{|x - 6|}$.

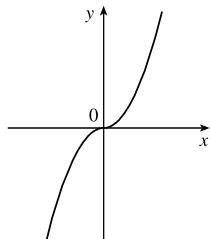


58. $f(x) = \llbracket x \rrbracket$ is not continuous at any integer n , so f is not differentiable at n by the contrapositive of Theorem 4. If a is not an integer, then f is constant on an open interval containing a , so $f'(a) = 0$. Thus,

$$f''(x) = 0, x \text{ not an integer.}$$



59. (a) $f(x) = x|x| = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$



(b) Since $f(x) = x^2$ for $x \geq 0$, we have $f'(x) = 2x$ for $x > 0$.

[See Exercise 17(d).] Similarly, since $f(x) = -x^2$ for $x < 0$, we have $f'(x) = -2x$ for $x < 0$. At $x = 0$, we have

$$f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{x|x|}{x} = \lim_{x \rightarrow 0} |x| = 0.$$

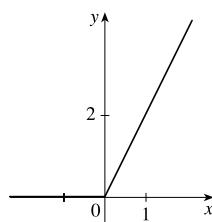
So f is differentiable at 0. Thus, f is differentiable for all x .

(c) From part (b), we have $f'(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ -2x & \text{if } x < 0 \end{cases} = 2|x|$.

60. (a) $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

$$\text{so } g(x) = x + |x| = \begin{cases} 2x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}.$$

Graph the line $y = 2x$ for $x \geq 0$ and graph $y = 0$ (the x -axis) for $x < 0$.



(b) g is not differentiable at $x = 0$ because the graph has a corner there, but

is differentiable at all other values; that is, g is differentiable on $(-\infty, 0) \cup (0, \infty)$.

(c) $g(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \Rightarrow g'(x) = \begin{cases} 2 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$

Another way of writing the formula is $g'(x) = 1 + \operatorname{sgn} x$ for $x \neq 0$.

61. (a) If f is even, then

$$\begin{aligned} f'(-x) &= \lim_{h \rightarrow 0} \frac{f(-x+h) - f(-x)}{h} = \lim_{h \rightarrow 0} \frac{f[-(x-h)] - f(-x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x-h) - f(x)}{h} = -\lim_{h \rightarrow 0} \frac{f(x-h) - f(x)}{-h} \quad [\text{let } \Delta x = -h] \\ &= -\lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = -f'(x) \end{aligned}$$

Therefore, f' is odd.

- (b) If f is odd, then

$$\begin{aligned} f'(-x) &= \lim_{h \rightarrow 0} \frac{f(-x+h) - f(-x)}{h} = \lim_{h \rightarrow 0} \frac{f[-(x-h)] - f(-x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{-f(x-h) + f(x)}{h} = \lim_{h \rightarrow 0} \frac{f(x-h) - f(x)}{-h} \quad [\text{let } \Delta x = -h] \\ &= \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = f'(x) \end{aligned}$$

Therefore, f' is even.

62. (a) $f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } x > 0 \end{cases}$

$$f'_-(0) = \lim_{h \rightarrow 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^-} \frac{0-0}{h} = \lim_{h \rightarrow 0^-} \frac{0}{h} = \lim_{h \rightarrow 0^-} 0 = 0$$

$$f'_+(0) = \lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^+} \frac{(0+h)-0}{h} = \lim_{h \rightarrow 0^+} \frac{h}{h} = \lim_{h \rightarrow 0^+} 1 = 1$$

Since these one-sided derivatives are not equal, $f'(0)$ does not exist, so f is not differentiable at 0.

(b) $f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x^2 & \text{if } x > 0 \end{cases}$

$$f'_-(0) = \lim_{h \rightarrow 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^-} \frac{0-0}{h} = \lim_{h \rightarrow 0^-} \frac{0}{h} = \lim_{h \rightarrow 0^-} 0 = 0$$

$$f'_+(0) = \lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^+} \frac{(0+h)^2 - 0^2}{h} = \lim_{h \rightarrow 0^+} \frac{h^2}{h} = \lim_{h \rightarrow 0^+} h = 0$$

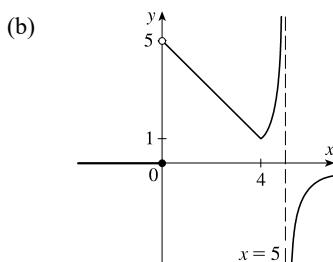
Since these one-sided derivatives are equal, $f'(0)$ exists (and equals 0), so f is differentiable at 0.

63. (a) $f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 5-x & \text{if } 0 < x < 4 \\ \frac{1}{5-x} & \text{if } x \geq 4 \end{cases}$

Note that as $h \rightarrow 0^-$, $4+h < 4$, so $f(4+h) = 5 - (4+h)$. As $h \rightarrow 0^+$, $4+h > 4$, so $f(4+h) = \frac{1}{5-(4+h)}$.

$$\begin{aligned} f'_-(4) &= \lim_{h \rightarrow 0^-} \frac{f(4+h) - f(4)}{h} = \lim_{h \rightarrow 0^-} \frac{[5-(4+h)] - (5-4)}{h} = \lim_{h \rightarrow 0^-} \frac{(5-4-h) - 1}{h} \\ &= \lim_{h \rightarrow 0^-} \frac{-h}{h} = \lim_{h \rightarrow 0^-} (-1) = -1 \end{aligned}$$

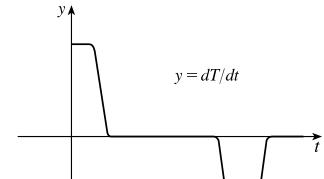
$$\begin{aligned} f'_+(4) &= \lim_{h \rightarrow 0^+} \frac{f(4+h) - f(4)}{h} = \lim_{h \rightarrow 0^+} \frac{\frac{1}{5-(4+h)} - \frac{1}{5-4}}{h} = \lim_{h \rightarrow 0^+} \frac{\frac{1}{1-h} - 1}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{\frac{1}{1-h} - \frac{1-h}{1-h}}{h} = \lim_{h \rightarrow 0^+} \frac{1 - (1-h)}{h(1-h)} = \lim_{h \rightarrow 0^+} \frac{h}{h(1-h)} = \lim_{h \rightarrow 0^+} \frac{1}{1-h} = 1 \end{aligned}$$



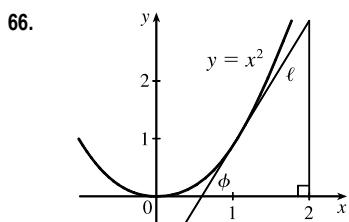
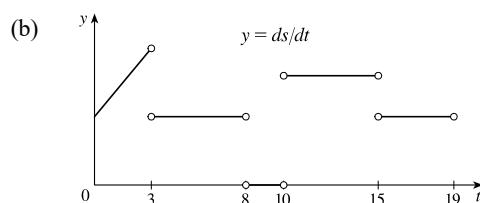
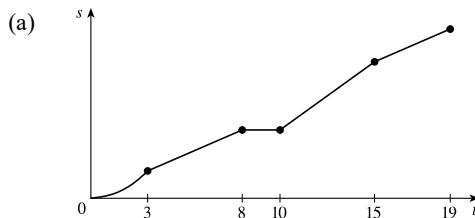
(c) f is discontinuous at $x = 0$ (jump discontinuity) and at $x = 5$ (infinite discontinuity).

(d) f is not differentiable at $x = 0$ [discontinuous, from part (c)], $x = 4$ [one-sided derivatives are not equal, from part (a)], and at $x = 5$ [discontinuous, from part (c)].

64. (a) The initial temperature of the water is close to room temperature because of the water that was in the pipes. When the water from the hot water tank starts coming out, dT/dt is large and positive as T increases to the temperature of the water in the tank. In the next phase, $dT/dt = 0$ as the water comes out at a constant, high temperature. After some time, dT/dt becomes small and negative as the contents of the hot water tank are exhausted. Finally, when the hot water has run out, dT/dt is once again 0 as the water maintains its (cold) temperature.



65. These graphs are idealizations conveying the spirit of the problem. In reality, changes in speed are not instantaneous, so the graph in (a) would not have corners and the graph in (b) would be continuous.



In the right triangle in the diagram, let Δy be the side opposite angle ϕ and Δx the side adjacent to angle ϕ . Then the slope of the tangent line ℓ is $m = \Delta y / \Delta x = \tan \phi$. Note that $0 < \phi < \frac{\pi}{2}$. We know (see Exercise 17) that the derivative of $f(x) = x^2$ is $f'(x) = 2x$. So the slope of the tangent to the curve at the point $(1, 1)$ is 2. Thus, ϕ is the angle between 0 and $\frac{\pi}{2}$ whose tangent is 2; that is, $\phi = \tan^{-1} 2 \approx 63^\circ$.

2.3 Differentiation Formulas

1. $g(x) = 4x + 7 \Rightarrow g'(x) = 4(1) + 0 = 4$

2. $g(t) = 5t + 4t^2 \Rightarrow g'(t) = 5(1) + 4(2t^{2-1}) = 5(1) + 4(2t) = 5 + 8t$

3. $f(x) = x^{75} - x + 3 \Rightarrow f'(x) = 75x^{75-1} - 1(1) + 0 = 75x^{74} - 1$

4. $g(x) = \frac{7}{4}x^2 - 3x + 12 \Rightarrow g'(x) = \frac{7}{4}(2x^{2-1}) - 3(1) + 0 = \frac{7}{4}(2x) - 3 = \frac{7}{2}x - 3$

5. $W(v) = 1.8v^{-3} \Rightarrow W'(v) = 1.8(-3v^{-3-1}) = 1.8(-3v^{-4}) = -5.4v^{-4}$

6. $r(z) = z^{-5} - z^{1/2} \Rightarrow r'(z) = -5z^{-6} - \frac{1}{2}z^{-1/2}$

7. $f(x) = x^{3/2} + x^{-3} \Rightarrow f'(x) = \frac{3}{2}x^{1/2} + (-3x^{-4}) = \frac{3}{2}x^{1/2} - 3x^{-4}$

8. $V(t) = t^{-3/5} + t^4 \Rightarrow V'(t) = -\frac{3}{5}t^{-8/5} + 4t^3$

9. $s(t) = \frac{1}{t} + \frac{1}{t^2} = t^{-1} + t^{-2} \Rightarrow s'(t) = -t^{-2} + (-2t^{-3}) = -t^{-2} - 2t^{-3} = -\frac{1}{t^2} - \frac{2}{t^3}$

10. $r(t) = \frac{a}{t^2} + \frac{b}{t^4} = at^{-2} + bt^{-4} \Rightarrow r'(t) = a(-2t^{-3}) + b(-4t^{-5}) = -2at^{-3} - 4bt^{-5} = -\frac{2a}{t^3} - \frac{4b}{t^5}$

11. $y = 2x + \sqrt{x} = 2x + x^{1/2} \Rightarrow y' = 2(1) + \frac{1}{2}x^{-1/2} = 2 + \frac{1}{2}x^{-1/2}$ or $2 + \frac{1}{2\sqrt{x}}$

12. $h(w) = \sqrt{2}w - \sqrt{2} \Rightarrow h'(w) = \sqrt{2}(1) - 0 = \sqrt{2}$

13. $g(x) = \frac{1}{\sqrt{x}} + \sqrt[4]{x} = x^{-1/2} + x^{1/4} \Rightarrow g'(x) = -\frac{1}{2}x^{-3/2} + \frac{1}{4}x^{-3/4}$ or $-\frac{1}{2x\sqrt{x}} + \frac{1}{4\sqrt[4]{x^3}}$

14. $S(R) = 4\pi R^2 \Rightarrow S'(R) = 4\pi(2R) = 8\pi R$

15. $f(x) = x^3(x+3) = x^4 + 3x^3 \Rightarrow f'(x) = 4x^3 + 3(3x^2) = 4x^3 + 9x^2$

16. $F(t) = (2t-3)^2 = 4t^2 - 12t + 9 \Rightarrow F'(t) = 4(2t) - 12(1) + 0 = 8t - 12$

17. $f(x) = \frac{3x^2 + x^3}{x} = \frac{3x^2}{x} + \frac{x^3}{x} = 3x + x^2 \Rightarrow f'(x) = 3(1) + 2x = 3 + 2x$

18. $y = \frac{\sqrt{x} + x}{x^2} = \frac{\sqrt{x}}{x^2} + \frac{x}{x^2} = x^{1/2-2} + x^{1-2} = x^{-3/2} + x^{-1} \Rightarrow y' = -\frac{3}{2}x^{-5/2} + (-1x^{-2}) = -\frac{3}{2}x^{-5/2} - x^{-2}$

19. $G(q) = (1+q^{-1})^2 = 1 + 2q^{-1} + q^{-2} \Rightarrow G'(q) = 0 + 2(-1q^{-2}) + (-2q^{-3}) = -2q^{-2} - 2q^{-3}$

20. $G(t) = \sqrt{5t} + \frac{\sqrt{7}}{t} = \sqrt{5}t^{1/2} + \sqrt{7}t^{-1} \Rightarrow G'(t) = \sqrt{5}\left(\frac{1}{2}t^{-1/2}\right) + \sqrt{7}(-1t^{-2}) = \frac{\sqrt{5}}{2\sqrt{t}} - \frac{\sqrt{7}}{t^2}$

21. $G(r) = \frac{3r^{3/2} + r^{5/2}}{r} = \frac{3r^{3/2}}{r} + \frac{r^{5/2}}{r} = 3r^{3/2-2/2} + r^{5/2-2/2} = 3r^{1/2} + r^{3/2} \Rightarrow$

$G'(r) = 3\left(\frac{1}{2}r^{-1/2}\right) + \frac{3}{2}r^{1/2} = \frac{3}{2}r^{-1/2} + \frac{3}{2}r^{1/2}$ or $\frac{3}{2\sqrt{r}} + \frac{3}{2}\sqrt{r}$

22. $F(z) = \frac{A+Bz+Cz^2}{z^2} = \frac{A}{z^2} + \frac{Bz}{z^2} + \frac{Cz^2}{z^2} = Az^{-2} + Bz^{-1} + C \Rightarrow$

$F'(z) = A(-2z^{-3}) + B(-1z^{-2}) + 0 = -2Az^{-3} - Bz^{-2} = -\frac{2A}{z^3} - \frac{B}{z^2}$ or $-\frac{2A+Bz}{z^3}$

23. $P(w) = \frac{2w^2 - w + 4}{\sqrt{w}} = \frac{2w^2}{\sqrt{w}} - \frac{w}{\sqrt{w}} + \frac{4}{\sqrt{w}} = 2w^{2-1/2} - w^{1-1/2} + 4w^{-1/2} = 2w^{3/2} - w^{1/2} + 4w^{-1/2} \Rightarrow$

$$P'(w) = 2\left(\frac{3}{2}w^{1/2}\right) - \frac{1}{2}w^{-1/2} + 4\left(-\frac{1}{2}w^{-3/2}\right) = 3w^{1/2} - \frac{1}{2}w^{-1/2} - 2w^{-3/2} \text{ or } 3\sqrt{w} - \frac{1}{2\sqrt{w}} - \frac{2}{w\sqrt{w}}$$

24. $D(t) = \frac{1+16t^2}{(4t)^3} = \frac{1+16t^2}{64t^3} = \frac{1}{64}t^{-3} + \frac{1}{4}t^{-1} \Rightarrow$

$$D'(t) = \frac{1}{64}(-3t^{-4}) + \frac{1}{4}(-1t^{-2}) = -\frac{3}{64}t^{-4} - \frac{1}{4}t^{-2} \text{ or } -\frac{3}{64t^4} - \frac{1}{4t^2} \text{ or } -\frac{3+16t^2}{64t^4}$$

25. $y = tx^2 + t^3x.$

To find dy/dx , we treat t as a constant and x as a variable to get $dy/dx = t(2x) + t^3(1) = 2tx + t^3$.

To find dy/dt , we treat x as a constant and t as a variable to get $dy/dt = (1)x^2 + (3t^2)x = x^2 + 3t^2x$.

26. $y = \frac{t}{x^2} + \frac{x}{t} = tx^{-2} + xt^{-1}.$

To find dy/dx , we treat t as a constant and x as a variable to get $dy/dx = t(-2x^{-3}) + (1)t^{-1} = -2tx^{-3} + t^{-1}$ or

$$-\frac{2t}{x^3} + \frac{1}{t}.$$

To find dy/dt , we treat x as a constant and t as a variable to get $dy/dt = (1)x^{-2} + x(-1t^{-2}) = x^{-2} - xt^{-2}$ or $\frac{1}{x^2} - \frac{x}{t^2}$.

27. Product Rule: $f(x) = (1+2x^2)(x-x^2) \Rightarrow$

$$f'(x) = (1+2x^2)(1-2x) + (x-x^2)(4x) = 1-2x+2x^2-4x^3+4x^2-4x^3 = 1-2x+6x^2-8x^3.$$

Multiplying first: $f(x) = (1+2x^2)(x-x^2) = x-x^2+2x^3-2x^4 \Rightarrow f'(x) = 1-2x+6x^2-8x^3$ (equivalent).

28. Quotient Rule: $F(x) = \frac{x^4 - 5x^3 + \sqrt{x}}{x^2} = \frac{x^4 - 5x^3 + x^{1/2}}{x^2} \Rightarrow$

$$\begin{aligned} F'(x) &= \frac{x^2(4x^3 - 15x^2 + \frac{1}{2}x^{-1/2}) - (x^4 - 5x^3 + x^{1/2})(2x)}{(x^2)^2} = \frac{4x^5 - 15x^4 + \frac{1}{2}x^{3/2} - 2x^5 + 10x^4 - 2x^{3/2}}{x^4} \\ &= \frac{2x^5 - 5x^4 - \frac{3}{2}x^{3/2}}{x^4} = 2x - 5 - \frac{3}{2}x^{-5/2} \end{aligned}$$

$$\text{Simplifying first: } F(x) = \frac{x^4 - 5x^3 + \sqrt{x}}{x^2} = x^2 - 5x + x^{-3/2} \Rightarrow F'(x) = 2x - 5 - \frac{3}{2}x^{-5/2} \text{ (equivalent).}$$

For this problem, simplifying first seems to be the better method.

29. By the Product Rule, $f(x) = (3x^2 - 5x)x^2 \Rightarrow$

$$\begin{aligned} f'(x) &= (3x^2 - 5x)(x^2)' + x^2(3x^2 - 5x)' = (3x^2 - 5x)(2x) + x^2(6x - 5) \\ &= 6x^3 - 10x^2 + 6x^3 - 5x^2 = 12x^3 - 15x^2 \end{aligned}$$

30. By the Product Rule, $y = (10x^2 + 7x - 2)(2 - x^2) \Rightarrow$

$$\begin{aligned} y' &= (10x^2 + 7x - 2)(2 - x^2)' + (2 - x^2)(10x^2 + 7x - 2)' = (10x^2 + 7x - 2)(-2x) + (2 - x^2)(20x + 7) \\ &= -20x^3 - 14x^2 + 4x + 40x + 14 - 20x^3 - 7x^2 = -40x^3 - 21x^2 + 44x + 14 \end{aligned}$$

31. By the Product Rule, $y = (4x^2 + 3)(2x + 5) \Rightarrow$

$$\begin{aligned}y' &= (4x^2 + 3)(2x + 5)' + (2x + 5)(4x^2 + 3)' = (4x^2 + 3)(2) + (2x + 5)(8x) \\&= 8x^2 + 6 + 16x^2 + 40x = 24x^2 + 40x + 6\end{aligned}$$

32. By the Product Rule, $g(x) = \sqrt{x}(x + 2\sqrt{x}) \Rightarrow$

$$\begin{aligned}g'(x) &= \sqrt{x}(x + 2\sqrt{x})' + (x + 2\sqrt{x})(\sqrt{x})' = \sqrt{x}\left(1 + 2 \cdot \frac{1}{2}x^{-1/2}\right) + (x + 2\sqrt{x})\left(\frac{1}{2}x^{-1/2}\right) \\&= \sqrt{x} + x^0 + \frac{1}{2}x^{1/2} + x^0 = \frac{3}{2}\sqrt{x} + 2\end{aligned}$$

33. By the Quotient Rule, $y = \frac{5x}{1+x} \Rightarrow$

$$y' = \frac{(1+x)(5x)' - 5x(1+x)'}{(1+x)^2} = \frac{(1+x)5 - 5x(1)}{(1+x)^2} = \frac{5 + 5x - 5x}{(1+x)^2} = \frac{5}{(1+x)^2}.$$

34. By the Quotient Rule, $y = \frac{x^2}{1-x} \Rightarrow y' = \frac{(1-x) \cdot 2x - x^2(-1)}{(1-x)^2} = \frac{x[2(1-x) + x]}{(1-x)^2} = \frac{x(2-x)}{(1-x)^2}.$

35. $g(t) = \frac{3-2t}{5t+1} \stackrel{\text{QR}}{\Rightarrow} g'(t) = \frac{(5t+1)(-2) - (3-2t)(5)}{(5t+1)^2} = \frac{-10t-2-15+10t}{(5t+1)^2} = -\frac{17}{(5t+1)^2}$

36. $G(u) = \frac{6u^4 - 5u}{u+1} \stackrel{\text{QR}}{\Rightarrow}$

$$G'(u) = \frac{(u+1)(24u^3 - 5) - (6u^4 - 5u)(1)}{(u+1)^2} = \frac{24u^4 - 5u + 24u^3 - 5 - 6u^4 + 5u}{(u+1)^2} = \frac{18u^4 + 24u^3 - 5}{(u+1)^2}$$

37. $f(t) = \frac{5t}{t^3 - t - 1} \stackrel{\text{QR}}{\Rightarrow} f'(t) = \frac{(t^3 - t - 1)(5) - (5t)(3t^2 - 1)}{(t^3 - t - 1)^2} = \frac{5t^3 - 5t - 5 - 15t^3 + 5t}{(t^3 - t - 1)^2} = -\frac{10t^3 + 5}{(t^3 - t - 1)^2}$

38. $F(x) = \frac{1}{2x^3 - 6x^2 + 5} \stackrel{\text{QR}}{\Rightarrow} F'(x) = \frac{(2x^3 - 6x^2 + 5)(0) - 1(6x^2 - 12x)}{(2x^3 - 6x^2 + 5)^2} = -\frac{6x^2 - 12x}{(2x^3 - 6x^2 + 5)^2}$

39. $y = \frac{s - \sqrt{s}}{s^2} = \frac{s}{s^2} - \frac{\sqrt{s}}{s^2} = s^{-1} - s^{-3/2} \Rightarrow y' = -s^{-2} + \frac{3}{2}s^{-5/2} = \frac{-1}{s^2} + \frac{3}{2s^{5/2}} = \frac{3 - 2\sqrt{s}}{2s^{5/2}}$

40. $y = \frac{\sqrt{x}}{\sqrt{x} + 1} \stackrel{\text{QR}}{\Rightarrow} y' = \frac{(\sqrt{x} + 1)\left(\frac{1}{2\sqrt{x}}\right) - \sqrt{x}\left(\frac{1}{2\sqrt{x}}\right)}{(\sqrt{x} + 1)^2} = \frac{\frac{1}{2} + \frac{1}{2\sqrt{x}} - \frac{1}{2}}{(\sqrt{x} + 1)^2} = \frac{1}{2\sqrt{x}(\sqrt{x} + 1)^2}$

41. $F(x) = \frac{2x^5 + x^4 - 6x}{x^3} = 2x^2 + x - 6x^{-2} \Rightarrow F'(x) = 4x + 1 + 12x^{-3} = 4x + 1 + \frac{12}{x^3}$ or $\frac{4x^4 + x^3 + 12}{x^3}$

42. $y = \frac{(u+2)^2}{1-u} = \frac{u^2 + 4u + 4}{1-u} \stackrel{\text{QR}}{\Rightarrow}$

$$y' = \frac{(1-u)(2u+4) - (u^2 + 4u + 4)(-1)}{(1-u)^2} = \frac{2u + 4 - 2u^2 - 4u + u^2 + 4u + 4}{(1-u)^2} = \frac{-u^2 + 2u + 8}{(1-u)^2}$$

43. $H(u) = (u - \sqrt{u})(u + \sqrt{u}) \stackrel{\text{PR}}{\Rightarrow}$

$$H'(u) = \left(u - \sqrt{u}\right) \left(1 + \frac{1}{2\sqrt{u}}\right) + \left(u + \sqrt{u}\right) \left(1 - \frac{1}{2\sqrt{u}}\right) = u + \frac{1}{2}\sqrt{u} - \sqrt{u} - \frac{1}{2} + u - \frac{1}{2}\sqrt{u} + \sqrt{u} - \frac{1}{2} = 2u - 1.$$

Alternate solution: An easier method is to simplify first and then differentiate as follows:

$$H(u) = (u - \sqrt{u})(u + \sqrt{u}) = u^2 - (\sqrt{u})^2 = u^2 - u \Rightarrow H'(u) = 2u - 1$$

44. $A(v) = v^{2/3}(2v^2 + 1 - v^{-2}) = 2v^{8/3} + v^{2/3} - v^{-4/3} \Rightarrow$

$$A'(v) = \frac{16}{3}v^{5/3} + \frac{2}{3}v^{-1/3} + \frac{4}{3}v^{-7/3} = \frac{2}{3}v^{-7/3}(8v^{12/3} + v^{6/3} + 2) = \frac{2(8v^4 + v^2 + 2)}{3v^{7/3}}$$

45. $J(u) = \left(\frac{1}{u} + \frac{1}{u^2}\right)(u + \frac{1}{u}) = (u^{-1} + u^{-2})(u + u^{-1}) \stackrel{\text{PR}}{\Rightarrow}$

$$\begin{aligned} J'(u) &= (u^{-1} + u^{-2})(u + u^{-1})' + (u + u^{-1})(u^{-1} + u^{-2})' = (u^{-1} + u^{-2})(1 - u^{-2}) + (u + u^{-1})(-u^{-2} - 2u^{-3}) \\ &= u^{-1} - u^{-3} + u^{-2} - u^{-4} - u^{-1} - 2u^{-2} - u^{-3} - 2u^{-4} = -u^{-2} - 2u^{-3} - 3u^{-4} = -\left(\frac{1}{u^2} + \frac{2}{u^3} + \frac{3}{u^4}\right) \end{aligned}$$

46. $h(w) = (w^2 + 3w)(w^{-1} - w^{-4}) \stackrel{\text{PR}}{\Rightarrow}$

$$\begin{aligned} h'(w) &= (w^2 + 3w)(-w^{-2} + 4w^{-5}) + (w^{-1} - w^{-4})(2w + 3) \\ &= -1 + 4w^{-3} - 3w^{-1} + 12w^{-4} + 2 + 3w^{-1} - 2w^{-3} - 3w^{-4} = 1 + 2w^{-3} + 9w^{-4} \end{aligned}$$

Alternate solution: An easier method is to simplify first and then differentiate as follows:

$$h(w) = (w^2 + 3w)(w^{-1} - w^{-4}) = w - w^{-2} + 3 - 3w^{-3} \Rightarrow h'(w) = 1 + 2w^{-3} + 9w^{-4}$$

47. $f(t) = \frac{\sqrt[3]{t}}{t-3} \stackrel{\text{QR}}{\Rightarrow}$

$$\begin{aligned} f'(t) &= \frac{(t-3)\left(\frac{1}{3}t^{-2/3}\right) - t^{1/3}(1)}{(t-3)^2} = \frac{\frac{1}{3}t^{1/3} - t^{-2/3} - t^{1/3}}{(t-3)^2} = \frac{-\frac{2}{3}t^{1/3} - t^{-2/3}}{(t-3)^2} \\ &= \frac{\frac{-2t}{3t^{2/3}} - \frac{3}{3t^{2/3}}}{(t-3)^2} = \frac{-2t-3}{3t^{2/3}(t-3)^2} \end{aligned}$$

48. $y = \frac{cx}{1+cx} \Rightarrow y' = \frac{(1+cx)(c) - (cx)(c)}{(1+cx)^2} = \frac{c + c^2x - c^2x}{(1+cx)^2} = \frac{c}{(1+cx)^2}$

49. $G(y) = \frac{B}{Ay^3 + B} \stackrel{\text{QR}}{\Rightarrow} G'(y) = \frac{(Ay^3 + B)(0) - B(3Ay^2)}{(Ay^3 + B)^2} = -\frac{3ABy^2}{(Ay^3 + B)^2}$

50. $F(t) = \frac{At}{Bt^2 + Ct^3} = \frac{A}{Bt + Ct^2} \stackrel{\text{QR}}{\Rightarrow}$

$$F'(t) = \frac{(Bt + Ct^2)(0) - A(B + 2Ct)}{(Bt + Ct^2)^2} = \frac{-A(B + 2Ct)}{(t)^2(B + Ct)^2} = -\frac{A(B + 2Ct)}{t^2(B + Ct)^2}$$

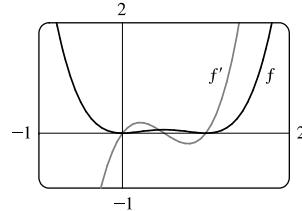
51. $f(x) = \frac{x}{x+\frac{c}{x}} \stackrel{\text{QR}}{\Rightarrow} f'(x) = \frac{(x+c/x)(1) - x(1-c/x^2)}{\left(x+\frac{c}{x}\right)^2} = \frac{x+c/x-x+c/x}{\left(\frac{x^2+c}{x}\right)^2} = \frac{\frac{2c/x}{(x^2+c)^2}}{\frac{x^2}{x^2}} \cdot \frac{x^2}{x^2} = \frac{2cx}{(x^2+c)^2}$

52. $f(x) = \frac{ax+b}{cx+d} \Rightarrow f'(x) = \frac{(cx+d)(a)-(ax+b)(c)}{(cx+d)^2} = \frac{acx+ad-acx-bc}{(cx+d)^2} = \frac{ad-bc}{(cx+d)^2}$

53. $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 \Rightarrow P'(x) = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \cdots + 2 a_2 x + a_1$

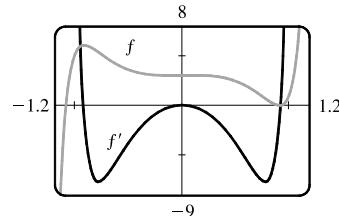
54. $f(x) = x^4 - 2x^3 + x^2 \Rightarrow f'(x) = 4x^3 - 6x^2 + 2x$

Note that $f'(x) = 0$ when f has a horizontal tangent, f' is positive when f is increasing, and f' is negative when f is decreasing.



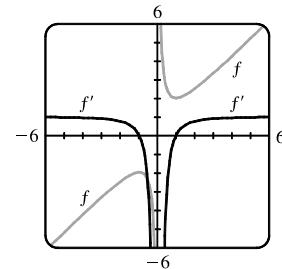
55. $f(x) = 3x^{15} - 5x^3 + 3 \Rightarrow f'(x) = 45x^{14} - 15x^2$.

Notice that $f'(x) = 0$ when f has a horizontal tangent, f' is positive when f is increasing, and f' is negative when f is decreasing.

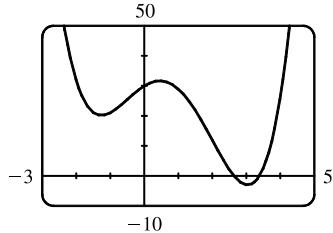


56. $f(x) = x + 1/x = x + x^{-1} \Rightarrow f'(x) = 1 - x^{-2} = 1 - 1/x^2$.

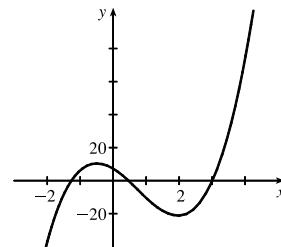
Notice that $f'(x) = 0$ when f has a horizontal tangent, f' is positive when f is increasing, and f' is negative when f is decreasing.



57. (a)

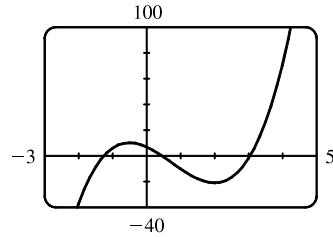


(b) From the graph in part (a), it appears that f' is zero at $x_1 \approx -1.25$, $x_2 \approx 0.5$, and $x_3 \approx 3$. The slopes are negative (so f' is negative) on $(-\infty, x_1)$ and (x_2, x_3) . The slopes are positive (so f' is positive) on (x_1, x_2) and (x_3, ∞) .

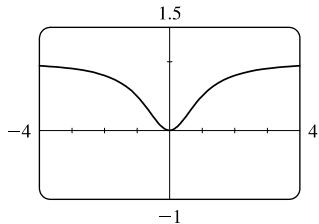


(c) $f(x) = x^4 - 3x^3 - 6x^2 + 7x + 30 \Rightarrow$

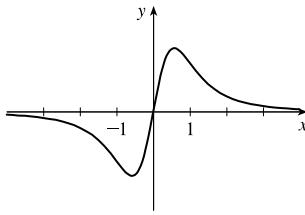
$$f'(x) = 4x^3 - 9x^2 - 12x + 7$$



58. (a)



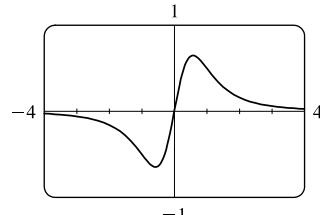
(b)



From the graph in part (a), it appears that g' is zero at $x = 0$. The slopes are negative (so g' is negative) on $(-\infty, 0)$. The slopes are positive (so g' is positive) on $(0, \infty)$.

$$(c) g(x) = \frac{x^2}{x^2 + 1} \Rightarrow$$

$$g'(x) = \frac{(x^2 + 1)(2x) - x^2(2x)}{(x^2 + 1)^2} = \frac{2x}{(x^2 + 1)^2}$$



$$59. y = \frac{2x}{x+1} \Rightarrow y' = \frac{(x+1)(2) - (2x)(1)}{(x+1)^2} = \frac{2}{(x+1)^2}.$$

At $(1, 1)$, $y' = \frac{1}{2}$, and an equation of the tangent line is $y - 1 = \frac{1}{2}(x - 1)$, or $y = \frac{1}{2}x + \frac{1}{2}$.

$$60. y = 2x^3 - x^2 + 2 \Rightarrow y' = 6x^2 - 2x. \text{ At } (1, 3), y' = 6(1)^2 - 2(1) = 4 \text{ and an equation of the tangent line is } y - 3 = 4(x - 1) \text{ or } y = 4x - 1.$$

$$61. y = x + \sqrt{x} \Rightarrow y' = 1 + \frac{1}{2}x^{-1/2} = 1 + 1/(2\sqrt{x}). \text{ At } (1, 2), y' = \frac{3}{2}, \text{ and an equation of the tangent line is } y - 2 = \frac{3}{2}(x - 1), \text{ or } y = \frac{3}{2}x + \frac{1}{2}. \text{ The slope of the normal line is } -\frac{2}{3}, \text{ so an equation of the normal line is } y - 2 = -\frac{2}{3}(x - 1), \text{ or } y = -\frac{2}{3}x + \frac{8}{3}.$$

$$62. y = x^{3/2} \Rightarrow y' = \frac{3}{2}x^{1/2}. \text{ At } (1, 1), y' = \frac{3}{2} \text{ and an equation of the tangent line is } y - 1 = \frac{3}{2}(x - 1) \text{ or } y = \frac{3}{2}x - \frac{1}{2}. \text{ The slope of the normal line is } -\frac{2}{3} \text{ (the negative reciprocal of } \frac{3}{2}) \text{ and an equation of the normal line is } y - 1 = -\frac{2}{3}(x - 1) \text{ or } y = -\frac{2}{3}x + \frac{5}{3}.$$

$$63. y = \frac{3x}{1+5x^2} \Rightarrow y' = \frac{(1+5x^2)(3) - 3x(10x)}{(1+5x^2)^2} = \frac{3+15x^2 - 30x^2}{(1+5x^2)^2} = \frac{3-15x^2}{(1+5x^2)^2}$$

$$\text{At } (1, \frac{1}{2}), y' = \frac{3-15(1^2)}{(1+5 \cdot 1^2)^2} = \frac{-12}{6^2} = -\frac{1}{3}, \text{ and an equation of the tangent line is } y - \frac{1}{2} = -\frac{1}{3}(x - 1), \text{ or } y = -\frac{1}{3}x + \frac{5}{6}.$$

The slope of the normal line is 3, so an equation of the normal line is $y - \frac{1}{2} = 3(x - 1)$, or $y = 3x - \frac{5}{2}$.

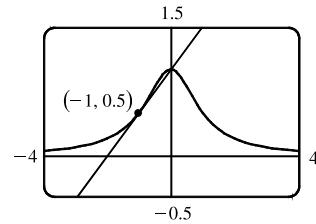
$$64. y = \frac{\sqrt{x}}{x+1} \Rightarrow y' = \frac{(x+1)\left(\frac{1}{2\sqrt{x}}\right) - \sqrt{x}(1)}{(x+1)^2} = \frac{(x+1) - (2x)}{2\sqrt{x}(x+1)^2} = \frac{1-x}{2\sqrt{x}(x+1)^2}.$$

$$\text{At } (4, 0.4), y' = \frac{-3}{100} = -0.03, \text{ and an equation of the tangent line is } y - 0.4 = -0.03(x - 4), \text{ or } y = -0.03x + 0.52. \text{ The slope of the normal line is } \frac{100}{3}, \text{ so an equation of the normal line is } y - 0.4 = \frac{100}{3}(x - 4) \Leftrightarrow y = \frac{100}{3}x - \frac{400}{3} + \frac{2}{5} \Leftrightarrow y = \frac{100}{3}x - \frac{1994}{15}.$$

65. (a) $y = f(x) = \frac{1}{1+x^2} \Rightarrow$

$f'(x) = \frac{(1+x^2)(0) - 1(2x)}{(1+x^2)^2} = \frac{-2x}{(1+x^2)^2}$. So the slope of the tangent line at the point $(-1, \frac{1}{2})$ is $f'(-1) = \frac{2}{2^2} = \frac{1}{2}$ and its equation is $y - \frac{1}{2} = \frac{1}{2}(x + 1)$ or $y = \frac{1}{2}x + 1$.

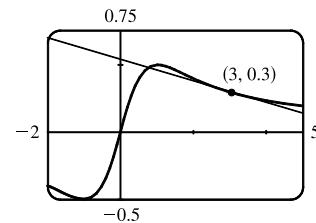
(b)



66. (a) $y = f(x) = \frac{x}{1+x^2} \Rightarrow$

$f'(x) = \frac{(1+x^2)1 - x(2x)}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$. So the slope of the tangent line at the point $(3, 0.3)$ is $f'(3) = \frac{-8}{100} = -0.08$ and its equation is $y - 0.3 = -0.08(x - 3)$ or $y = -0.08x + 0.54$.

(b)



67. $f(x) = 0.001x^5 - 0.02x^3 \Rightarrow f'(x) = 0.005x^4 - 0.06x^2 \Rightarrow f''(x) = 0.02x^3 - 0.12x$

68. $G(r) = \sqrt{r} + \sqrt[3]{r} \Rightarrow G'(r) = \frac{1}{2}r^{-1/2} + \frac{1}{3}r^{-2/3} \Rightarrow G''(r) = -\frac{1}{4}r^{-3/2} - \frac{2}{9}r^{-5/3}$

69. $f(x) = \frac{x^2}{1+2x} \Rightarrow f'(x) = \frac{(1+2x)(2x) - x^2(2)}{(1+2x)^2} = \frac{2x+4x^2-2x^2}{(1+2x)^2} = \frac{2x^2+2x}{(1+2x)^2} \Rightarrow$

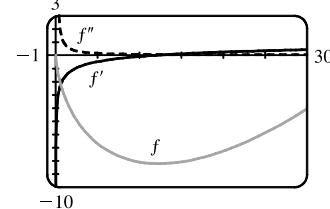
$$\begin{aligned} f''(x) &= \frac{(1+2x)^2(4x+2) - (2x^2+2x)(1+4x+4x^2)'}{[(1+2x)^2]^2} = \frac{2(1+2x)^2(2x+1) - 2x(x+1)(4+8x)}{(1+2x)^4} \\ &= \frac{2(1+2x)[(1+2x)^2 - 4x(x+1)]}{(1+2x)^4} = \frac{2(1+4x+4x^2 - 4x^2 - 4x)}{(1+2x)^3} = \frac{2}{(1+2x)^3} \end{aligned}$$

70. Using the Reciprocal Rule, $f(x) = \frac{1}{3-x} \Rightarrow f'(x) = -\frac{(3-x)'}{(3-x)^2} = -\frac{-1}{(3-x)^2} = \frac{1}{(3-x)^2} \Rightarrow$

$$f''(x) = -\frac{[(3-x)^2]'}{[(3-x)^2]^2} = -\frac{(9-6x+x^2)'}{(3-x)^4} = -\frac{-6+2x}{(3-x)^4} = -\frac{-2(3-x)}{(3-x)^4} = \frac{2}{(3-x)^3}.$$

71. $f(x) = 2x - 5x^{3/4} \Rightarrow f'(x) = 2 - \frac{15}{4}x^{-1/4} \Rightarrow f''(x) = \frac{15}{16}x^{-5/4}$

Note that f' is negative when f is decreasing and positive when f is increasing. f'' is always positive since f' is always increasing.



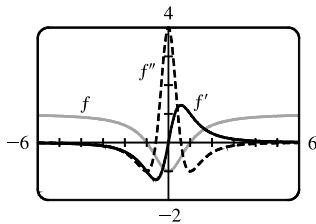
72. $f(x) = \frac{x^2 - 1}{x^2 + 1} \Rightarrow$

$$f'(x) = \frac{(x^2 + 1)(2x) - (x^2 - 1)(2x)}{(x^2 + 1)^2} = \frac{(2x)[(x^2 + 1) - (x^2 - 1)]}{(x^2 + 1)^2} = \frac{(2x)(2)}{(x^2 + 1)^2} = \frac{4x}{(x^2 + 1)^2} \Rightarrow$$

$$f''(x) = \frac{(x^2 + 1)^2(4) - 4x(x^4 + 2x^2 + 1)'}{[(x^2 + 1)^2]^2} = \frac{4(x^2 + 1)^2 - 4x(4x^3 + 4x)}{(x^2 + 1)^4}$$

$$= \frac{4(x^2 + 1)^2 - 16x^2(x^2 + 1)}{(x^2 + 1)^4} = \frac{4(x^2 + 1)[(x^2 + 1) - 4x^2]}{(x^2 + 1)^4} = \frac{4(1 - 3x^2)}{(x^2 + 1)^3}$$

[continued]



$f' = 0$ when f has a horizontal tangent and $f'' = 0$ when f' has a horizontal tangent. f' is negative when f is decreasing and positive when f is increasing. f'' is negative when f' is decreasing and positive when f' is increasing. f'' is negative when f is concave down and positive when f is concave up.

73. (a) $s = t^3 - 3t \Rightarrow v(t) = s'(t) = 3t^2 - 3 \Rightarrow a(t) = v'(t) = 6t$

(b) $a(2) = 6(2) = 12 \text{ m/s}^2$

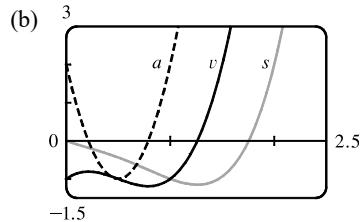
(c) $v(t) = 3t^2 - 3 = 0$ when $t^2 = 1$, that is, $t = 1$ [$t \geq 0$] and $a(1) = 6 \text{ m/s}^2$.

74. (a) $s = t^4 - 2t^3 + t^2 - t \Rightarrow$

$$v(t) = s'(t) = 4t^3 - 6t^2 + 2t - 1 \Rightarrow$$

$$a(t) = v'(t) = 12t^2 - 12t + 2$$

(c) $a(1) = 12(1)^2 - 12(1) + 2 = 2 \text{ m/s}^2$



75. $L = 0.0155A^3 - 0.372A^2 + 3.95A + 1.21 \Rightarrow \frac{dL}{dA} = 0.0465A^2 - 0.744A + 3.95$, so

$\frac{dL}{dA} \Big|_{A=12} = 0.117(12)^2 - 1.89(12) + 10.03 = 4.198$. The derivative is the instantaneous rate of change of the length (in inches) of an Alaskan rockfish with respect to its age when its age is 12 years. Its units are centimeters/year.

76. $S(A) = 0.882A^{0.842} \Rightarrow S'(A) = 0.882(0.842A^{-0.158}) = 0.742644A^{-0.158}$, so

$S'(100) = 0.742644(100)^{-0.158} \approx 0.36$. The derivative is the instantaneous rate of change of the number of tree species with respect to area. Its units are number of species per square meter.

77. (a) $P = \frac{k}{V}$ and $P = 50$ when $V = 0.106$, so $k = PV = 50(0.106) = 5.3$. Thus, $P = \frac{5.3}{V}$ and $V = \frac{5.3}{P}$.

(b) $V = 5.3P^{-1} \Rightarrow \frac{dV}{dP} = 5.3(-1P^{-2}) = -\frac{5.3}{P^2}$. When $P = 50$, $\frac{dV}{dP} = -\frac{5.3}{50^2} = -0.00212$. The derivative is the instantaneous rate of change of the volume with respect to the pressure at 25°C . Its units are m^3/kPa .

78. (a) $L = aP^2 + bP + c$, where $a \approx -0.00920$, $b \approx 4.5519$, and $c \approx -434.58914$.

(b) $\frac{dL}{dP} = 2aP + b$. When $P = 200$, $\frac{dL}{dP} \approx 0.88$, and when $P = 300$, $\frac{dL}{dP} \approx -0.96$. The derivative is the instantaneous rate of change of tire life with respect to pressure. Its units are (thousands of kilometers)/kPa. When $\frac{dL}{dP}$ is positive, tire life is increasing, and when $\frac{dL}{dP} < 0$, tire life is decreasing.

79. We are given that $f(5) = 1$, $f'(5) = 6$, $g(5) = -3$, and $g'(5) = 2$.

(a) $(fg)'(5) = f(5)g'(5) + g(5)f'(5) = (1)(2) + (-3)(6) = 2 - 18 = -16$

$$(b) \left(\frac{f}{g}\right)'(5) = \frac{g(5)f'(5) - f(5)g'(5)}{[g(5)]^2} = \frac{(-3)(6) - (1)(2)}{(-3)^2} = -\frac{20}{9}$$

$$(c) \left(\frac{g}{f}\right)'(5) = \frac{f(5)g'(5) - g(5)f'(5)}{[f(5)]^2} = \frac{(1)(2) - (-3)(6)}{(1)^2} = 20$$

80. We are given that $f(4) = 2$, $g(4) = 5$, $f'(4) = 6$, and $g'(4) = -3$.

$$(a) h(x) = 3f(x) + 8g(x) \Rightarrow h'(x) = 3f'(x) + 8g'(x), \text{ so}$$

$$h'(4) = 3f'(4) + 8g'(4) = 3(6) + 8(-3) = 18 - 24 = -6.$$

$$(b) h(x) = f(x)g(x) \Rightarrow h'(x) = f(x)g'(x) + g(x)f'(x), \text{ so}$$

$$h'(4) = f(4)g'(4) + g(4)f'(4) = 2(-3) + 5(6) = -6 + 30 = 24.$$

$$(c) h(x) = \frac{f(x)}{g(x)} \Rightarrow h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}, \text{ so}$$

$$h'(4) = \frac{g(4)f'(4) - f(4)g'(4)}{[g(4)]^2} = \frac{5(6) - 2(-3)}{5^2} = \frac{30 + 6}{25} = \frac{36}{25}.$$

$$(d) h(x) = \frac{g(x)}{f(x) + g(x)} \Rightarrow$$

$$h'(4) = \frac{[f(4) + g(4)]g'(4) - g(4)[f'(4) + g'(4)]}{[f(4) + g(4)]^2} = \frac{(2+5)(-3) - 5[6+(-3)]}{(2+5)^2} = \frac{-21 - 15}{7^2} = -\frac{36}{49}$$

$$81. f(x) = \sqrt{x}g(x) \Rightarrow f'(x) = \sqrt{x}g'(x) + g(x) \cdot \frac{1}{2}x^{-1/2}, \text{ so } f'(4) = \sqrt{4}g'(4) + g(4) \cdot \frac{1}{2\sqrt{4}} = 2 \cdot 7 + 8 \cdot \frac{1}{4} = 16.$$

$$82. \frac{d}{dx} \left[\frac{h(x)}{x} \right] = \frac{xh'(x) - h(x) \cdot 1}{x^2} \Rightarrow \frac{d}{dx} \left[\frac{h(x)}{x} \right]_{x=2} = \frac{2h'(2) - h(2)}{2^2} = \frac{2(-3) - (4)}{4} = \frac{-10}{4} = -2.5$$

83. (a) From the graphs of f and g , we obtain the following values: $f(1) = 2$ since the point $(1, 2)$ is on the graph of f ; $g(1) = 3$

since the point $(1, 3)$ is on the graph of g ; $f'(1) = \frac{1}{3}$ since the slope of the line segment between $(1, 2)$ and $(4, 3)$ is

$$\frac{3-2}{4-1} = \frac{1}{3}; g'(1) = 1 \text{ since the slope of the line segment between } (-2, 0) \text{ and } (2, 4) \text{ is } \frac{4-0}{2-(-2)} = \frac{4}{4} = 1.$$

Now $u(x) = f(x)g(x)$, so $u'(1) = f(1)g'(1) + g(1)f'(1) = 2 \cdot 1 + 3 \cdot \frac{1}{3} = 3$.

(b) From the graphs of f and g , we obtain the following values: $f(4) = 3$ since the point $(4, 3)$ is on the graph of f ; $g(4) = 2$

since the point $(4, 2)$ is on the graph of g ; $f'(4) = f'(1) = \frac{1}{3}$ from the part (a); $g'(4) = 1$ since the slope of the line

segment between $(3, 1)$ and $(5, 3)$ is $\frac{3-1}{5-3} = \frac{2}{2} = 1$.

$$v(x) = \frac{f(x)}{g(x)}, \text{ so } v'(4) = \frac{g(4)f'(4) - f(4)g'(4)}{[g(4)]^2} = \frac{2 \cdot \frac{1}{3} - 3 \cdot 1}{2^2} = \frac{-\frac{7}{3}}{4} = -\frac{7}{12}$$

$$84. (a) P(x) = F(x)G(x), \text{ so } P'(2) = F(2)G'(2) + G(2)F'(2) = 3 \cdot \frac{2}{4} + 2 \cdot 0 = \frac{3}{2}.$$

$$(b) Q(x) = \frac{F(x)}{G(x)}, \text{ so } Q'(7) = \frac{G(7)F'(7) - F(7)G'(7)}{[G(7)]^2} = \frac{\frac{1}{4} \cdot \frac{1}{4} - 5 \cdot \left(-\frac{2}{3}\right)}{1^2} = \frac{1}{4} + \frac{10}{3} = \frac{43}{12}$$

85. (a) $y = xg(x) \Rightarrow y' = xg'(x) + g(x) \cdot 1 = xg'(x) + g(x)$

(b) $y = \frac{x}{g(x)} \Rightarrow y' = \frac{g(x) \cdot 1 - xg'(x)}{[g(x)]^2} = \frac{g(x) - xg'(x)}{[g(x)]^2}$

(c) $y = \frac{g(x)}{x} \Rightarrow y' = \frac{xg'(x) - g(x) \cdot 1}{(x)^2} = \frac{xg'(x) - g(x)}{x^2}$

86. (a) $y = x^2 f(x) \Rightarrow y' = x^2 f'(x) + f(x)(2x)$

(b) $y = \frac{f(x)}{x^2} \Rightarrow y' = \frac{x^2 f'(x) - f(x)(2x)}{(x^2)^2} = \frac{x f'(x) - 2f(x)}{x^3}$

(c) $y = \frac{x^2}{f(x)} \Rightarrow y' = \frac{f(x)(2x) - x^2 f'(x)}{[f(x)]^2}$

(d) $y = \frac{1 + xf(x)}{\sqrt{x}} \Rightarrow$

$$\begin{aligned} y' &= \frac{\sqrt{x}[xf'(x) + f(x)] - [1 + xf(x)] \frac{1}{2\sqrt{x}}}{(\sqrt{x})^2} \\ &= \frac{x^{3/2}f'(x) + x^{1/2}f(x) - \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{1/2}f(x)}{x} \cdot \frac{2x^{1/2}}{2x^{1/2}} = \frac{xf(x) + 2x^2f'(x) - 1}{2x^{3/2}} \end{aligned}$$

87. $y = x^3 + 3x^2 - 9x + 10 \Rightarrow y' = 3x^2 + 3(2x) - 9(1) + 0 = 3x^2 + 6x - 9$. Horizontal tangents occur where $y' = 0$.

Thus, $3x^2 + 6x - 9 = 0 \Rightarrow 3(x^2 + 2x - 3) = 0 \Rightarrow 3(x+3)(x-1) = 0 \Rightarrow x = -3$ or $x = 1$. The corresponding points are $(-3, 37)$ and $(1, 5)$.

88. $f(x) = x^3 + 3x^2 + x + 3$ has a horizontal tangent when $f'(x) = 3x^2 + 6x + 1 = 0 \Leftrightarrow$

$$x = \frac{-6 \pm \sqrt{36 - 12}}{6} = -1 \pm \frac{1}{3}\sqrt{6}.$$

89. $y = 6x^3 + 5x - 3 \Rightarrow m = y' = 18x^2 + 5$, but $x^2 \geq 0$ for all x , so $m \geq 5$ for all x .

90. $y = x^4 + 1 \Rightarrow y' = 4x^3$. The slope of the line $32x - y = 15$ (or $y = 32x - 15$) is 32, so the slope of any line parallel to it is also 32. Thus, $y' = 32 \Leftrightarrow 4x^3 = 32 \Leftrightarrow x^3 = 8 \Leftrightarrow x = 2$, which is the x -coordinate of the point on the curve at which the slope is 32. The y -coordinate is $2^4 + 1 = 17$, so an equation of the tangent line is $y - 17 = 32(x - 2)$ or $y = 32x - 47$.

91. The slope of the line $3x - y = 15$ (or $y = 3x - 15$) is 3, so the slope of both tangent lines to the curve is 3.

$$y = x^3 - 3x^2 + 3x - 3 \Rightarrow y' = 3x^2 - 6x + 3 = 3(x^2 - 2x + 1) = 3(x - 1)^2. \text{ Thus, } 3(x - 1)^2 = 3 \Rightarrow$$

$$(x - 1)^2 = 1 \Rightarrow x - 1 = \pm 1 \Rightarrow x = 0 \text{ or } 2, \text{ which are the } x\text{-coordinates at which the tangent lines have slope 3. The points on the curve are } (0, -3) \text{ and } (2, -1), \text{ so the tangent line equations are } y - (-3) = 3(x - 0) \text{ or } y = 3x - 3 \text{ and } y - (-1) = 3(x - 2) \text{ or } y = 3x - 7.$$

92. $y = \frac{x-1}{x+1} \Rightarrow y' = \frac{(x+1)(1)-(x-1)(1)}{(x+1)^2} = \frac{2}{(x+1)^2}$. If the tangent intersects

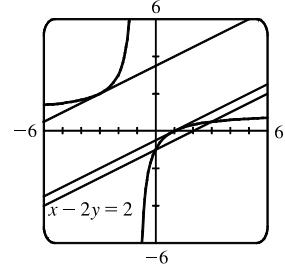
the curve when $x = a$, then its slope is $2/(a+1)^2$. But if the tangent is parallel to

$$x - 2y = 2, \text{ that is, } y = \frac{1}{2}x - 1, \text{ then its slope is } \frac{1}{2}. \text{ Thus, } \frac{2}{(a+1)^2} = \frac{1}{2} \Rightarrow$$

$$(a+1)^2 = 4 \Rightarrow a+1 = \pm 2 \Rightarrow a = 1 \text{ or } -3. \text{ When } a = 1, y = 0 \text{ and the}$$

$$\text{equation of the tangent is } y - 0 = \frac{1}{2}(x-1) \text{ or } y = \frac{1}{2}x - \frac{1}{2}. \text{ When } a = -3, y = 2 \text{ and}$$

$$\text{the equation of the tangent is } y - 2 = \frac{1}{2}(x+3) \text{ or } y = \frac{1}{2}x + \frac{7}{2}.$$



93. The slope of $y = \sqrt{x}$ is given by $y = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$. The slope of $2x + y = 1$ (or $y = -2x + 1$) is -2 , so the desired

$$\text{normal line must have slope } -2, \text{ and hence, the tangent line to the curve must have slope } \frac{1}{2}. \text{ This occurs if } \frac{1}{2\sqrt{x}} = \frac{1}{2} \Rightarrow$$

$$\sqrt{x} = 1 \Rightarrow x = 1. \text{ When } x = 1, y = \sqrt{1} = 1, \text{ and an equation of the normal line is } y - 1 = -2(x - 1) \text{ or}$$

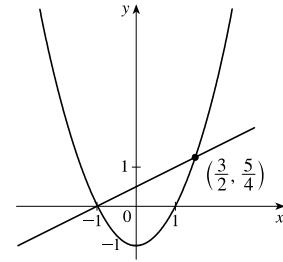
$$y = -2x + 3.$$

94. $y = f(x) = x^2 - 1 \Rightarrow f'(x) = 2x$. So $f'(-1) = -2$, and the slope of the normal line is $\frac{1}{2}$. The equation of the normal line at $(-1, 0)$ is

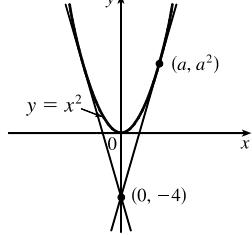
$$y - 0 = \frac{1}{2}[x - (-1)] \text{ or } y = \frac{1}{2}x + \frac{1}{2}.$$

$$\text{Substituting this into the equation of the parabola, we obtain } \frac{1}{2}x + \frac{1}{2} = x^2 - 1 \Leftrightarrow x + 1 = 2x^2 - 2 \Leftrightarrow 2x^2 - x - 3 = 0 \Leftrightarrow (2x-3)(x+1) = 0 \Leftrightarrow x = \frac{3}{2} \text{ or } -1. \text{ Substituting } \frac{3}{2}$$

$$\text{into the equation of the normal line gives us } y = \frac{5}{4}. \text{ Thus, the second point of intersection is } \left(\frac{3}{2}, \frac{5}{4}\right), \text{ as shown in the sketch.}$$



95.



Let (a, a^2) be a point on the parabola at which the tangent line passes through the point $(0, -4)$. The tangent line has slope $2a$ and equation $y - (-4) = 2a(x - 0) \Leftrightarrow y = 2ax - 4$. Since (a, a^2) also lies on the line, $a^2 = 2a(a) - 4$, or $a^2 = 4$. So $a = \pm 2$ and the points are $(2, 4)$ and $(-2, 4)$.

96. (a) If $y = x^2 + x$, then $y' = 2x + 1$. If the point at which a tangent meets the parabola is $(a, a^2 + a)$, then the slope of the

$$\text{tangent is } 2a + 1. \text{ But since it passes through } (2, -3), \text{ the slope must also be } \frac{\Delta y}{\Delta x} = \frac{a^2 + a + 3}{a - 2}.$$

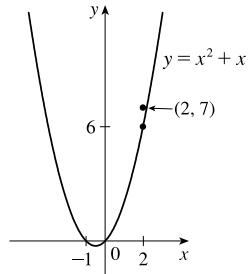
$$\text{Therefore, } 2a + 1 = \frac{a^2 + a + 3}{a - 2}. \text{ Solving this equation for } a \text{ we get } a^2 + a + 3 = 2a^2 - 3a - 2 \Leftrightarrow$$

$$a^2 - 4a - 5 = (a-5)(a+1) = 0 \Leftrightarrow a = 5 \text{ or } -1. \text{ If } a = -1, \text{ the point is } (-1, 0) \text{ and the slope is } -1, \text{ so the equation is } y - 0 = (-1)(x+1) \text{ or } y = -x - 1. \text{ If } a = 5, \text{ the point is } (5, 30) \text{ and the slope is } 11, \text{ so the equation is } y - 30 = 11(x-5) \text{ or } y = 11x - 25.$$

(b) As in part (a), but using the point $(2, 7)$, we get the equation

$$2a + 1 = \frac{a^2 + a - 7}{a - 2} \Rightarrow 2a^2 - 3a - 2 = a^2 + a - 7 \Leftrightarrow a^2 - 4a + 5 = 0.$$

The last equation has no real solution (discriminant $= -16 < 0$), so there is no line through the point $(2, 7)$ that is tangent to the parabola. The diagram shows that the point $(2, 7)$ is “inside” the parabola, but tangent lines to the parabola do not pass through points inside the parabola.



97. $y = f(x) = ax^2 \Rightarrow f'(x) = 2ax$. So the slope of the tangent to the parabola at $x = 2$ is $m = 2a(2) = 4a$. The slope of the given line, $2x + y = b \Leftrightarrow y = -2x + b$, is seen to be -2 , so we must have $4a = -2 \Leftrightarrow a = -\frac{1}{2}$. So when $x = 2$, the point in question has y -coordinate $-\frac{1}{2} \cdot 2^2 = -2$. Now we simply require that the given line, whose equation is $2x + y = b$, pass through the point $(2, -2)$: $2(2) + (-2) = b \Leftrightarrow b = 2$. So we must have $a = -\frac{1}{2}$ and $b = 2$.

98. (a) $f(x) = x^n \Rightarrow f'(x) = nx^{n-1} \Rightarrow f''(x) = n(n-1)x^{n-2} \Rightarrow \dots \Rightarrow f^{(n)}(x) = n(n-1)(n-2)\dots 2 \cdot 1x^{n-n} = n!$

(b) $f(x) = x^{-1} \Rightarrow f'(x) = (-1)x^{-2} \Rightarrow f''(x) = (-1)(-2)x^{-3} \Rightarrow \dots \Rightarrow$

$$f^{(n)}(x) = (-1)(-2)(-3)\dots(-n)x^{-(n+1)} = (-1)^n n! x^{-(n+1)} \text{ or } \frac{(-1)^n n!}{x^{n+1}}$$

99. Let $P(x) = ax^2 + bx + c$. Then $P'(x) = 2ax + b$ and $P''(x) = 2a$. $P''(2) = 2 \Rightarrow 2a = 2 \Rightarrow a = 1$.

$$P'(2) = 3 \Rightarrow 2(1)(2) + b = 3 \Rightarrow 4 + b = 3 \Rightarrow b = -1.$$

$$P(2) = 5 \Rightarrow 1(2)^2 + (-1)(2) + c = 5 \Rightarrow 2 + c = 5 \Rightarrow c = 3. \text{ So } P(x) = x^2 - x + 3.$$

100. $y = Ax^2 + Bx + C \Rightarrow y' = 2Ax + B \Rightarrow y'' = 2A$. We substitute these expressions into the equation

$y'' + y' - 2y = x^2$ to get

$$\begin{aligned} (2A) + (2Ax + B) - 2(Ax^2 + Bx + C) &= x^2 \\ 2A + 2Ax + B - 2Ax^2 - 2Bx - 2C &= x^2 \\ (-2A)x^2 + (2A - 2B)x + (2A + B - 2C) &= (1)x^2 + (0)x + (0) \end{aligned}$$

The coefficients of x^2 on each side must be equal, so $-2A = 1 \Rightarrow A = -\frac{1}{2}$. Similarly, $2A - 2B = 0 \Rightarrow$

$$A = B = -\frac{1}{2} \text{ and } 2A + B - 2C = 0 \Rightarrow -1 - \frac{1}{2} - 2C = 0 \Rightarrow C = -\frac{3}{4}.$$

101. $y = f(x) = ax^3 + bx^2 + cx + d \Rightarrow f'(x) = 3ax^2 + 2bx + c$. The point $(-2, 6)$ is on f , so $f(-2) = 6 \Rightarrow -8a + 4b - 2c + d = 6$ (1). The point $(2, 0)$ is on f , so $f(2) = 0 \Rightarrow 8a + 4b + 2c + d = 0$ (2). Since there are horizontal tangents at $(-2, 6)$ and $(2, 0)$, $f'(\pm 2) = 0$. $f'(-2) = 0 \Rightarrow 12a - 4b + c = 0$ (3) and $f'(2) = 0 \Rightarrow 12a + 4b + c = 0$ (4). Subtracting equation (3) from (4) gives $8b = 0 \Rightarrow b = 0$. Adding (1) and (2) gives $8b + 2d = 6$, so $d = 3$ since $b = 0$. From (3) we have $c = -12a$, so (2) becomes $8a + 4(0) + 2(-12a) + 3 = 0 \Rightarrow 3 = 16a \Rightarrow a = \frac{3}{16}$. Now $c = -12a = -12\left(\frac{3}{16}\right) = -\frac{9}{4}$ and the desired cubic function is $y = \frac{3}{16}x^3 - \frac{9}{4}x + 3$.

- 102.** $y = ax^2 + bx + c \Rightarrow y'(x) = 2ax + b$. The parabola has slope 4 at $x = 1$ and slope -8 at $x = -1$, so $y'(1) = 4 \Rightarrow 2a + b = 4$ (1) and $y'(-1) = -8 \Rightarrow -2a + b = -8$ (2). Adding (1) and (2) gives us $2b = -4 \Leftrightarrow b = -2$. From (1), $2a - 2 = 4 \Leftrightarrow a = 3$. Thus, the equation of the parabola is $y = 3x^2 - 2x + c$. Since it passes through the point $(2, 15)$, we have $15 = 3(2)^2 - 2(2) + c \Rightarrow c = 7$, so the equation is $y = 3x^2 - 2x + 7$.

- 103.** If $P(t)$ denotes the population at time t and $A(t)$ denotes the average annual income, then $T(t) = P(t)A(t)$ is the total personal income. The rate at which $T(t)$ is rising is given by $T'(t) = P(t)A'(t) + A(t)P'(t) \Rightarrow$

$$\begin{aligned} T'(2015) &= P(2015)A'(2015) + A(2015)P'(2015) = (107,350)(\$2250/\text{year}) + (\$60,220)(1960/\text{year}) \\ &= \$241,537,500/\text{year} + \$118,031,200/\text{year} = \$359,568,700/\text{year} \end{aligned}$$

So the total personal income in Boulder was rising by about \$360 million per year in 2015.

The term $P(t)A'(t) \approx \$242$ million represents the portion of the rate of change of total income due to the existing population's increasing income. The term $A(t)P'(t) \approx \$118$ million represents the portion of the rate of change of total income due to increasing population.

- 104.** (a) $f(20) = 10,000$ means that when the price of the fabric is \$20/meter, 10,000 yards will be sold.

$f'(20) = -350$ means that as the price of the fabric increases past \$20/meter, the amount of fabric which will be sold is decreasing at a rate of 350 yards per (dollar per meter).

- (b) $R(p) = pf(p) \Rightarrow R'(p) = pf'(p) + f(p) \cdot 1 \Rightarrow R'(20) = 20f'(20) + f(20) \cdot 1 = 20(-350) + 10,000 = 3000$. This means that as the price of the fabric increases past \$20/meter, the total revenue is increasing at \$3000/(\$/yard). Note that the Product Rule indicates that we will lose \$7000/(\$/meter) due to selling less fabric, but this loss is more than made up for by the additional revenue due to the increase in price.

105. $v = \frac{0.14[S]}{0.015 + [S]} \Rightarrow \frac{dv}{d[S]} = \frac{(0.015 + [S])(0.14) - (0.14[S])(1)}{(0.015 + [S])^2} = \frac{0.0021}{(0.015 + [S])^2}$.

$dv/d[S]$ represents the rate of change of the rate of an enzymatic reaction with respect to the concentration of a substrate S.

- 106.** $B(t) = N(t)M(t) \Rightarrow B'(t) = N(t)M'(t) + M(t)N'(t)$, so

$$B'(4) = N(4)M'(4) + M(4)N'(4) = 820(0.14) + 1.2(50) = 174.8 \text{ g/week.}$$

- 107.** (a) $(fgh)' = [(fg)h]' = (fg)'h + (fg)h' = (f'g + fg')h + (fg)h' = f'gh + fg'h + fgh'$

(b) Putting $f = g = h$ in part (a), we have $\frac{d}{dx}[f(x)]^3 = (fff)' = f'ff + ff'f + fff' = 3fff' = 3[f(x)]^2f'(x)$.

$$(c) \frac{d}{dx}(e^{3x}) = \frac{d}{dx}(e^x)^3 = 3(e^x)^2e^x = 3e^{2x}e^x = 3e^{3x}$$

108. (a) $\frac{d}{dx}\left(\frac{1}{g(x)}\right) = \frac{g(x) \cdot \frac{d}{dx}(1) - 1 \cdot \frac{d}{dx}[g(x)]}{[g(x)]^2} \quad [\text{Quotient Rule}] = \frac{g(x) \cdot 0 - 1 \cdot g'(x)}{[g(x)]^2} = \frac{0 - g'(x)}{[g(x)]^2} = -\frac{g'(x)}{[g(x)]^2}$

$$(b) \frac{d}{dx}\left(\frac{1}{2x^3 - 6x^2 + 5}\right) = -\frac{(2x^3 - 6x^2 + 5)'}{(2x^3 - 6x^2 + 5)^2} = -\frac{6x^2 - 12x}{(2x^3 - 6x^2 + 5)^2}$$

109. $F = f/g \Rightarrow f = Fg \Rightarrow f' = F'g + Fg' \Rightarrow F' = \frac{f' - Fg'}{g} = \frac{f' - (f/g)g'}{g} = \frac{f'g - fg'}{g^2}$

110. (a) We use the Product Rule repeatedly: $F = fg \Rightarrow F' = f'g + fg' \Rightarrow$

$$F'' = (f''g + f'g') + (f'g' + fg'') = f''g + 2f'g' + fg''.$$

(b) $F''' = f'''g + f''g' + 2(f''g' + f'g'') + f'g'' + fg''' = f'''g + 3f''g' + 3f'g'' + fg''' \Rightarrow$

$$F^{(4)} = f^{(4)}g + f'''g' + 3(f'''g' + f''g'') + 3(f''g'' + f'g''') + f'g''' + fg^{(4)}$$

$$= f^{(4)}g + 4f'''g' + 6f''g'' + 4f'g''' + fg^{(4)}$$

(c) By analogy with the Binomial Theorem, we make the guess:

$$F^{(n)} = f^{(n)}g + nf^{(n-1)}g' + \binom{n}{2}f^{(n-2)}g'' + \cdots + \binom{n}{k}f^{(n-k)}g^{(k)} + \cdots + nf'g^{(n-1)} + fg^{(n)},$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$.

111. $f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ x + 1 & \text{if } x \geq 1 \end{cases}$

Calculate the left- and right-hand derivatives as defined in Exercise 2.2.62:

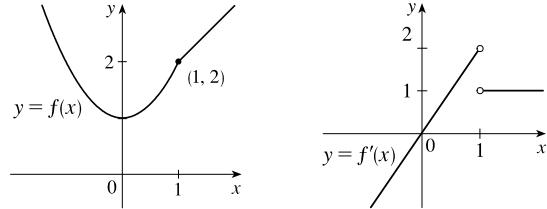
$$f'_-(1) = \lim_{h \rightarrow 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \rightarrow 0^-} \frac{[(1+h)^2 + 1] - (1+1)}{h} = \lim_{h \rightarrow 0^-} \frac{h^2 + 2h}{h} = \lim_{h \rightarrow 0^-} (h+2) = 2 \text{ and}$$

$$f'_+(1) = \lim_{h \rightarrow 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \rightarrow 0^+} \frac{[(1+h)+1] - (1+1)}{h} = \lim_{h \rightarrow 0^+} \frac{h}{h} = \lim_{h \rightarrow 0^+} 1 = 1.$$

Since the left and right limits are different,

$$\lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} \text{ does not exist, that is, } f'(1)$$

does not exist. Therefore, f is not differentiable at 1.



112. $g(x) = \begin{cases} 2x & \text{if } x \leq 0 \\ 2x - x^2 & \text{if } 0 < x < 2 \\ 2 - x & \text{if } x \geq 2 \end{cases}$

Investigate the left- and right-hand derivatives at $x = 0$ and $x = 2$:

$$g'_-(0) = \lim_{h \rightarrow 0^-} \frac{g(0+h) - g(0)}{h} = \lim_{h \rightarrow 0^-} \frac{2h - 2(0)}{h} = 2 \text{ and}$$

$$g'_+(0) = \lim_{h \rightarrow 0^+} \frac{g(0+h) - g(0)}{h} = \lim_{h \rightarrow 0^+} \frac{(2h - h^2) - 2(0)}{h} = \lim_{h \rightarrow 0^+} (2 - h) = 2, \text{ so } g \text{ is differentiable at } x = 0.$$

$$g'_-(2) = \lim_{h \rightarrow 0^-} \frac{g(2+h) - g(2)}{h} = \lim_{h \rightarrow 0^-} \frac{2(2+h) - (2+h)^2 - (2-2)}{h} = \lim_{h \rightarrow 0^-} \frac{-2h - h^2}{h} = \lim_{h \rightarrow 0^-} (-2 - h) = -2$$

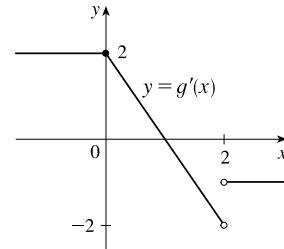
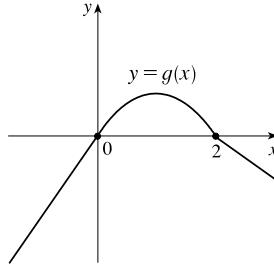
and

$$g'_+(2) = \lim_{h \rightarrow 0^+} \frac{g(2+h) - g(2)}{h} = \lim_{h \rightarrow 0^+} \frac{[2 - (2+h)] - (2-2)}{h} = \lim_{h \rightarrow 0^+} \frac{-h}{h} = \lim_{h \rightarrow 0^+} (-1) = -1,$$

[continued]

so g is not differentiable at $x = 2$. Thus, a formula for g' is

$$g'(x) = \begin{cases} 2 & \text{if } x \leq 0 \\ 2 - 2x & \text{if } 0 < x < 2 \\ -1 & \text{if } x > 2 \end{cases}$$



113. (a) Note that $x^2 - 9 < 0$ for $x^2 < 9 \Leftrightarrow |x| < 3 \Leftrightarrow -3 < x < 3$. So

$$f(x) = \begin{cases} x^2 - 9 & \text{if } x \leq -3 \\ -x^2 + 9 & \text{if } -3 < x < 3 \\ x^2 - 9 & \text{if } x \geq 3 \end{cases} \Rightarrow f'(x) = \begin{cases} 2x & \text{if } x < -3 \\ -2x & \text{if } -3 < x < 3 \\ 2x & \text{if } x > 3 \end{cases} = \begin{cases} 2x & \text{if } |x| > 3 \\ -2x & \text{if } |x| < 3 \end{cases}$$

To show that $f'(3)$ does not exist we investigate $\lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h}$ by computing the left- and right-hand derivatives defined in Exercise 2.2.62.

$$f'_-(3) = \lim_{h \rightarrow 0^-} \frac{f(3+h) - f(3)}{h} = \lim_{h \rightarrow 0^-} \frac{[-(3+h)^2 + 9] - 0}{h} = \lim_{h \rightarrow 0^-} (-6-h) = -6 \quad \text{and}$$

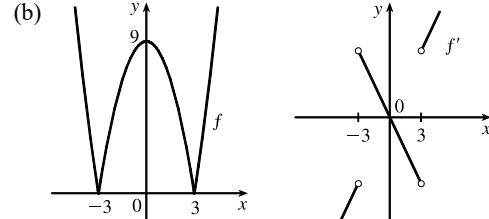
$$f'_+(3) = \lim_{h \rightarrow 0^+} \frac{f(3+h) - f(3)}{h} = \lim_{h \rightarrow 0^+} \frac{[(3+h)^2 - 9] - 0}{h} = \lim_{h \rightarrow 0^+} \frac{6h + h^2}{h} = \lim_{h \rightarrow 0^+} (6+h) = 6.$$

Since the left and right limits are different,

$\lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h}$ does not exist, that is, $f'(3)$

does not exist. Similarly, $f'(-3)$ does not exist.

Therefore, f is not differentiable at 3 or at -3.



114. If $x \geq 1$, then $h(x) = |x-1| + |x+2| = x-1+x+2 = 2x+1$.

If $-2 < x < 1$, then $h(x) = -(x-1) + x+2 = 3$.

If $x \leq -2$, then $h(x) = -(x-1) - (x+2) = -2x-1$. Therefore,

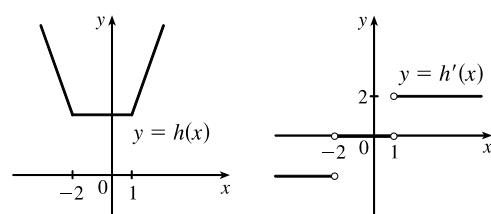
$$h(x) = \begin{cases} -2x-1 & \text{if } x \leq -2 \\ 3 & \text{if } -2 < x < 1 \\ 2x+1 & \text{if } x \geq 1 \end{cases} \Rightarrow h'(x) = \begin{cases} -2 & \text{if } x < -2 \\ 0 & \text{if } -2 < x < 1 \\ 2 & \text{if } x > 1 \end{cases}$$

To see that $h'(1) = \lim_{x \rightarrow 1} \frac{h(x) - h(1)}{x-1}$ does not exist,

observe that $\lim_{x \rightarrow 1^-} \frac{h(x) - h(1)}{x-1} = \lim_{x \rightarrow 1^-} \frac{3-3}{3-1} = 0$ but

$\lim_{x \rightarrow 1^+} \frac{h(x) - h(1)}{x-1} = \lim_{x \rightarrow 1^+} \frac{2x+1-3}{x-1} = 2$. Similarly,

$h'(-2)$ does not exist.



115. f is clearly differentiable for $x < 2$ and for $x > 2$. For $x < 2$, $f'(x) = 2x$, so $f'_-(2) = 4$. For $x > 2$, $f'(x) = m$, so

$f'_+(2) = m$. For f to be differentiable at $x = 2$, we need $4 = f'_-(2) = f'_+(2) = m$. So $f(x) = 4x + b$. We must also have continuity at $x = 2$, so $4 = f(2) = \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (4x + b) = 8 + b$. Hence, $b = -4$.

116. (a) $xy = c \Rightarrow y = \frac{c}{x}$. Let $P = \left(a, \frac{c}{a}\right)$. The slope of the tangent line at $x = a$ is $y'(a) = -\frac{c}{a^2}$. Its equation is

$$y - \frac{c}{a} = -\frac{c}{a^2}(x - a) \text{ or } y = -\frac{c}{a^2}x + \frac{2c}{a}, \text{ so its } y\text{-intercept is } \frac{2c}{a}. \text{ Setting } y = 0 \text{ gives } x = 2a, \text{ so the } x\text{-intercept is } 2a.$$

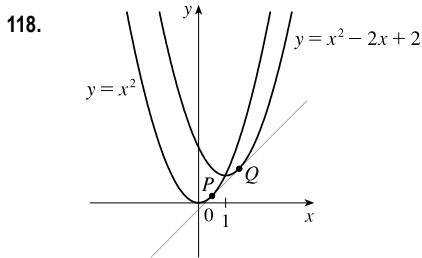
The midpoint of the line segment joining $\left(0, \frac{2c}{a}\right)$ and $(2a, 0)$ is $\left(a, \frac{c}{a}\right) = P$.

(b) We know the x - and y -intercepts of the tangent line from part (a), so the area of the triangle bounded by the axes and the tangent is $\frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}xy = \frac{1}{2}(2a)(2c/a) = 2c$, a constant.

117. $y = x^2 \Rightarrow y' = 2x$, so the slope of a tangent line at the point (a, a^2) is $y' = 2a$ and the slope of a normal line is $-1/(2a)$,

for $a \neq 0$. The slope of the normal line through the points (a, a^2) and $(0, c)$ is $\frac{a^2 - c}{a - 0}$, so $\frac{a^2 - c}{a} = -\frac{1}{2a} \Rightarrow$

$a^2 - c = -\frac{1}{2} \Rightarrow a^2 = c - \frac{1}{2}$. The last equation has two solutions if $c > \frac{1}{2}$, one solution if $c = \frac{1}{2}$, and no solution if $c < \frac{1}{2}$. Since the y -axis is normal to $y = x^2$ regardless of the value of c (this is the case for $a = 0$), we have *three* normal lines if $c > \frac{1}{2}$ and *one* normal line if $c \leq \frac{1}{2}$.



From the sketch, it appears that there may be a line that is tangent to both curves. The slope of the line through the points $P(a, a^2)$ and $Q(b, b^2 - 2b + 2)$ is $\frac{b^2 - 2b + 2 - a^2}{b - a}$. The slope of the tangent line at P is $2a$ [$y' = 2x$] and at Q is $2b - 2$ [$y' = 2x - 2$]. All three slopes are equal, so $2a = 2b - 2 \Leftrightarrow a = b - 1$.

$$\text{Also, } 2b - 2 = \frac{b^2 - 2b + 2 - a^2}{b - a} \Rightarrow 2b - 2 = \frac{b^2 - 2b + 2 - (b - 1)^2}{b - (b - 1)} \Rightarrow 2b - 2 = b^2 - 2b + 2 - b^2 + 2b - 1 \Rightarrow$$

$$2b = 3 \Rightarrow b = \frac{3}{2} \text{ and } a = \frac{3}{2} - 1 = \frac{1}{2}. \text{ Thus, an equation of the tangent line at } P \text{ is } y - \left(\frac{1}{2}\right)^2 = 2\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right) \text{ or}$$

$$y = x - \frac{1}{4}.$$

119. *Solution 1:* Let $f(x) = x^{1000}$. Then, by the definition of a derivative, $f'(1) = \lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1} \frac{x^{1000} - 1}{x - 1}$.

But this is just the limit we want to find, and we know (from the Power Rule) that $f'(x) = 1000x^{999}$, so

$$f'(1) = 1000(1)^{999} = 1000. \text{ So } \lim_{x \rightarrow 1} \frac{x^{1000} - 1}{x - 1} = 1000.$$

[continued]

Solution 2: Note that $(x^{1000} - 1) = (x - 1)(x^{999} + x^{998} + x^{997} + \dots + x^2 + x + 1)$. So

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{x^{1000} - 1}{x - 1} &= \lim_{x \rightarrow 1} \frac{(x - 1)(x^{999} + x^{998} + x^{997} + \dots + x^2 + x + 1)}{x - 1} = \lim_{x \rightarrow 1} (x^{999} + x^{998} + x^{997} + \dots + x^2 + x + 1) \\ &= \underbrace{1 + 1 + 1 + \dots + 1 + 1 + 1}_{1000 \text{ ones}} = 1000, \text{ as above.}\end{aligned}$$

APPLIED PROJECT Building a Better Roller Coaster

1. (a) $f(x) = ax^2 + bx + c \Rightarrow f'(x) = 2ax + b$.

The origin is at P : $f(0) = 0 \Rightarrow c = 0$

The slope of the ascent is 0.8: $f'(0) = 0.8 \Rightarrow b = 0.8$

The slope of the drop is -1.6: $f'(100) = -1.6 \Rightarrow 200a + b = -1.6$

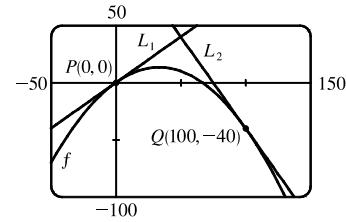
(b) $b = 0.8$, so $200a + b = -1.6 \Rightarrow 200a + 0.8 = -1.6 \Rightarrow 200a = -2.4 \Rightarrow a = -\frac{2.4}{200} = -0.012$.

Thus, $f(x) = -0.012x^2 + 0.8x$.

(c) Since L_1 passes through the origin with slope 0.8, it has equation $y = 0.8x$.

The horizontal distance between P and Q is 100, so the y -coordinate at Q is

$f(100) = -0.012(100)^2 + 0.8(100) = -40$. Since L_2 passes through the point $(100, -40)$ and has slope -1.6, it has equation $y + 40 = -1.6(x - 100)$ or $y = -1.6x + 120$.



(d) The difference in elevation between $P(0, 0)$ and $Q(100, -40)$ is $0 - (-40) = 40$ feet.

2. (a)

Interval	Function	First Derivative	Second Derivative
$(-\infty, 0)$	$L_1(x) = 0.8x$	$L'_1(x) = 0.8$	$L''_1(x) = 0$
$[0, 10]$	$g(x) = kx^3 + lx^2 + mx + n$	$g'(x) = 3kx^2 + 2lx + m$	$g''(x) = 6kx + 2l$
$[10, 90]$	$q(x) = ax^2 + bx + c$	$q'(x) = 2ax + b$	$q''(x) = 2a$
$(90, 100]$	$h(x) = px^3 + qx^2 + rx + s$	$h'(x) = 3px^2 + 2qx + r$	$h''(x) = 6px + 2q$
$(100, \infty)$	$L_2(x) = -1.6x + 120$	$L'_2(x) = -1.6$	$L''_2(x) = 0$

There are 4 values of x (0, 10, 90, and 100) for which we must make sure the function values are equal, the first derivative values are equal, and the second derivative values are equal. The third column in the following table contains the value of each side of the condition—these are found after solving the system in part (b).

[continued]

At $x =$	Condition	Value	Resulting Equation
0	$g(0) = L_1(0)$	0	$n = 0$
	$g'(0) = L'_1(0)$	$\frac{4}{5}$	$m = 0.8$
	$g''(0) = L''_1(0)$	0	$2l = 0$
10	$g(10) = q(10)$	$\frac{68}{9}$	$1000k + 100l + 10m + n = 100a + 10b + c$
	$g'(10) = q'(10)$	$\frac{2}{3}$	$300k + 20l + m = 20a + b$
	$g''(10) = q''(10)$	$-\frac{2}{75}$	$60k + 2l = 2a$
90	$h(90) = q(90)$	$-\frac{220}{9}$	$729,000p + 8100q + 90r + s = 8100a + 90b + c$
	$h'(90) = q'(90)$	$-\frac{22}{15}$	$24,300p + 180q + r = 180a + b$
	$h''(90) = q''(90)$	$-\frac{2}{75}$	$540p + 2q = 2a$
100	$h(100) = L_2(100)$	-40	$1,000,000p + 10,000q + 100r + s = -40$
	$h'(100) = L'_2(100)$	$-\frac{8}{5}$	$30,000p + 200q + r = -1.6$
	$h''(100) = L''_2(100)$	0	$600p + 2q = 0$

(b) We can arrange our work in a 12×12 matrix as follows.

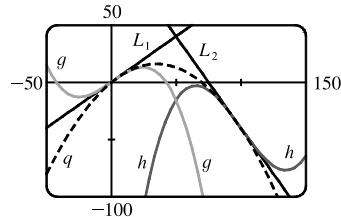
a	b	c	k	l	m	n	p	q	r	s	constant
0	0	0	0	0	0	1	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0.8
0	0	0	0	2	0	0	0	0	0	0	0
-100	-10	-1	1000	100	10	1	0	0	0	0	0
-20	-1	0	300	20	1	0	0	0	0	0	0
-2	0	0	60	2	0	0	0	0	0	0	0
-8100	-90	-1	0	0	0	0	729,000	8100	90	1	0
-180	-1	0	0	0	0	0	24,300	180	1	0	0
-2	0	0	0	0	0	0	540	2	0	0	0
0	0	0	0	0	0	0	1,000,000	10,000	100	1	-40
0	0	0	0	0	0	0	30,000	200	1	0	-1.6
0	0	0	0	0	0	0	600	2	0	0	0

Solving the system gives us the formulas for q , g , and h .

$$\left. \begin{aligned} a &= -0.01\bar{3} = -\frac{1}{75} \\ b &= 0.9\bar{3} = \frac{14}{15} \\ c &= -0.\bar{4} = -\frac{4}{9} \end{aligned} \right\} q(x) = -\frac{1}{75}x^2 + \frac{14}{15}x - \frac{4}{9}$$

$$\left. \begin{aligned} k &= -0.000\bar{4} = -\frac{1}{2250} \\ l &= 0 \\ m &= 0.8 = \frac{4}{5} \\ n &= 0 \end{aligned} \right\} g(x) = -\frac{1}{2250}x^3 + \frac{4}{5}x$$

$$\left. \begin{aligned} p &= 0.000\bar{4} = \frac{1}{2250} \\ q &= -0.1\bar{3} = -\frac{2}{15} \\ r &= 11.7\bar{3} = \frac{176}{15} \\ s &= -324.\bar{4} = -\frac{2920}{9} \end{aligned} \right\} h(x) = \frac{1}{2250}x^3 - \frac{2}{15}x^2 + \frac{176}{15}x - \frac{2920}{9}$$

(c) Graph of L_1 , q , g , h , and L_2 :

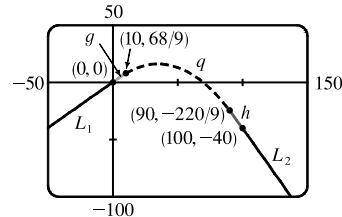
This is the piecewise-defined function assignment on a TI-83/4 Plus calculator, where $Y_2 = L_1$, $Y_6 = g$, $Y_5 = q$, $Y_7 = h$, and $Y_3 = L_2$.

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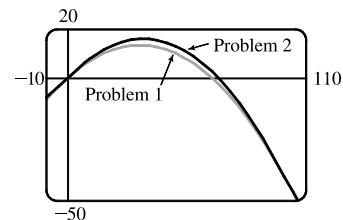
Plot1 Plot2 Plot3
:Y8=Y2*(X<0)+Y6*
(X≥0 and X<10)+Y
5*(X≥10 and X≤90
)+Y7*(X>90 and X
≤100)+Y3*(X>100)
:Y9=

```

The graph of the five functions as a piecewise-defined function:



A comparison of the graphs in part 1(c) and part 2(c):



2.4 Derivatives of Trigonometric Functions

1. $f(x) = 3 \sin x - 2 \cos x \Rightarrow f'(x) = 3(\cos x) - 2(-\sin x) = 3 \cos x + 2 \sin x$
2. $f(x) = \tan x - 4 \sin x \Rightarrow f'(x) = \sec^2 x - 4(\cos x) = \sec^2 x - 4 \cos x$
3. $y = x^2 + \cot x \Rightarrow y' = 2x + (-\csc^2 x) = 2x - \csc^2 x$
4. $y = 2 \sec x - \csc x \Rightarrow y' = 2(\sec x \tan x) - (-\csc x \cot x) = 2 \sec x \tan x + \csc x \cot x$
5. $h(\theta) = \theta^2 \sin \theta \stackrel{\text{PR}}{\Rightarrow} h'(\theta) = \theta^2(\cos \theta) + (\sin \theta)(2\theta) = \theta(\theta \cos \theta + 2 \sin \theta)$
6. $g(x) = 3x + x^2 \cos x \Rightarrow g'(x) = 3 + x^2(-\sin x) + (\cos x)(2x) = 3 - x^2 \sin x + 2x \cos x$
7. $y = \sec \theta \tan \theta \Rightarrow y' = \sec \theta (\sec^2 \theta) + \tan \theta (\sec \theta \tan \theta) = \sec \theta (\sec^2 \theta + \tan^2 \theta)$. Using the identity $1 + \tan^2 \theta = \sec^2 \theta$, we can write alternative forms of the answer as $\sec \theta (1 + 2 \tan^2 \theta)$ or $\sec \theta (2 \sec^2 \theta - 1)$.
8. $y = \sin \theta \cos \theta \Rightarrow y' = \sin \theta(-\sin \theta) + \cos \theta(\cos \theta) = \cos^2 \theta - \sin^2 \theta$ [or $\cos 2\theta$]
9. $f(\theta) = (\theta - \cos \theta) \sin \theta \stackrel{\text{PR}}{\Rightarrow} f'(\theta) = (\theta - \cos \theta)(\cos \theta) + (\sin \theta)(1 + \sin \theta) = \theta \cos \theta - \cos^2 \theta + \sin \theta + \sin^2 \theta$
10. $f(x) = x \cos x + 2 \tan x \Rightarrow f'(x) = x(-\sin x) + (\cos x)(1) + 2 \sec^2 x = \cos x - x \sin x + 2 \sec^2 x$
11. $H(t) = \cos^2 t = \cos t \cdot \cos t \stackrel{\text{PR}}{\Rightarrow} H'(t) = \cos t(-\sin t) + \cos t(-\sin t) = -2 \sin t \cos t$. Using the identity $\sin 2t = 2 \sin t \cos t$, we can write an alternative form of the answer as $-\sin 2t$.

12. $y = u(a \cos u + b \cot u) \Rightarrow$

$$y' = u(-a \sin u - b \csc^2 u) + (a \cos u + b \cot u) \cdot 1 = a \cos u + b \cot u - au \sin u - bu \csc^2 u$$

13. $f(\theta) = \frac{\sin \theta}{1 + \cos \theta} \Rightarrow$

$$f'(\theta) = \frac{(1 + \cos \theta) \cos \theta - (\sin \theta)(-\sin \theta)}{(1 + \cos \theta)^2} = \frac{\cos \theta + \cos^2 \theta + \sin^2 \theta}{(1 + \cos \theta)^2} = \frac{\cos \theta + 1}{(1 + \cos \theta)^2} = \frac{1}{1 + \cos \theta}$$

14. $y = \frac{\cos x}{1 - \sin x} \Rightarrow$

$$y' = \frac{(1 - \sin x)(-\sin x) - \cos x(-\cos x)}{(1 - \sin x)^2} = \frac{-\sin x + \sin^2 x + \cos^2 x}{(1 - \sin x)^2} = \frac{-\sin x + 1}{(1 - \sin x)^2} = \frac{1}{1 - \sin x}$$

15. $y = \frac{x}{2 - \tan x} \Rightarrow y' = \frac{(2 - \tan x)(1) - x(-\sec^2 x)}{(2 - \tan x)^2} = \frac{2 - \tan x + x \sec^2 x}{(2 - \tan x)^2}$

16. $f(t) = \frac{\cot t}{t^2} \stackrel{\text{QR}}{\Rightarrow} f'(t) = \frac{t^2(-\csc^2 t) - (\cot t)(2t)}{(t^2)^2} = \frac{t(-t \csc^2 t - 2 \cot t)}{t^4} = \frac{-t \csc^2 t - 2 \cot t}{t^3}$

17. $f(w) = \frac{1 + \sec w}{1 - \sec w} \stackrel{\text{QR}}{\Rightarrow}$

$$\begin{aligned} f'(w) &= \frac{(1 - \sec w)(\sec w \tan w) - (1 + \sec w)(-\sec w \tan w)}{(1 - \sec w)^2} \\ &= \frac{\sec w \tan w - \sec^2 w \tan w + \sec w \tan w + \sec^2 w \tan w}{(1 - \sec w)^2} = \frac{2 \sec w \tan w}{(1 - \sec w)^2} \end{aligned}$$

18. $y = \frac{\sin t}{1 + \tan t} \Rightarrow$

$$y' = \frac{(1 + \tan t) \cos t - (\sin t) \sec^2 t}{(1 + \tan t)^2} = \frac{\cos t + \sin t - \frac{\sin t}{\cos^2 t}}{(1 + \tan t)^2} = \frac{\cos t + \sin t - \tan t \sec t}{(1 + \tan t)^2}$$

19. $y = \frac{t \sin t}{1 + t} \Rightarrow$

$$y' = \frac{(1 + t)(t \cos t + \sin t) - t \sin t(1)}{(1 + t)^2} = \frac{t \cos t + \sin t + t^2 \cos t + t \sin t - t \sin t}{(1 + t)^2} = \frac{(t^2 + t) \cos t + \sin t}{(1 + t)^2}$$

20. $g(z) = \frac{z}{\sec z + \tan z} \stackrel{\text{QR}}{\Rightarrow}$

$$\begin{aligned} g'(z) &= \frac{(\sec z + \tan z)(1) - z(\sec z \tan z + \sec^2 z)}{(\sec z + \tan z)^2} = \frac{(\sec z + \tan z)(1) - z \sec z(\tan z + \sec z)}{(\sec z + \tan z)^2} \\ &= \frac{(1 - z \sec z)(\sec z + \tan z)}{(\sec z + \tan z)^2} = \frac{1 - z \sec z}{\sec z + \tan z} \end{aligned}$$

21. Using Exercise 2.3.107(a), $f(\theta) = \theta \cos \theta \sin \theta \Rightarrow$

$$f'(\theta) = 1 \cos \theta \sin \theta + \theta(-\sin \theta) \sin \theta + \theta \cos \theta(\cos \theta) = \cos \theta \sin \theta - \theta \sin^2 \theta + \theta \cos^2 \theta$$

$$= \sin \theta \cos \theta + \theta(\cos^2 \theta - \sin^2 \theta) = \frac{1}{2} \sin 2\theta + \theta \cos 2\theta \quad [\text{using double-angle formulas}]$$

22. Using Exercise 2.3.107(a), $f(x) = x^2 \sin x \tan x \Rightarrow$

$$\begin{aligned} f'(x) &= (x^2)' \sin x \tan x + x^2 (\sin x)' \tan x + x^2 \sin x (\tan x)' = 2x \sin x \tan x + x^2 \cos x \tan x + x^2 \sin x \sec^2 x \\ &= 2x \sin x \tan x + x^2 \sin x + x^2 \sin x \sec^2 x = x \sin x (2 \tan x + x + x \sec^2 x) \end{aligned}$$

$$23. \frac{d}{dx} (\csc x) = \frac{d}{dx} \left(\frac{1}{\sin x} \right) = \frac{(\sin x)(0) - 1(\cos x)}{\sin^2 x} = \frac{-\cos x}{\sin^2 x} = -\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} = -\csc x \cot x$$

$$24. \frac{d}{dx} (\sec x) = \frac{d}{dx} \left(\frac{1}{\cos x} \right) = \frac{(\cos x)(0) - 1(-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \tan x$$

$$25. \frac{d}{dx} (\cot x) = \frac{d}{dx} \left(\frac{\cos x}{\sin x} \right) = \frac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{\sin^2 x} = -\frac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x$$

26. $f(x) = \cos x \Rightarrow$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h \rightarrow 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\ &= \lim_{h \rightarrow 0} \left(\cos x \frac{\cos h - 1}{h} - \sin x \frac{\sin h}{h} \right) = \cos x \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} - \sin x \lim_{h \rightarrow 0} \frac{\sin h}{h} \\ &= (\cos x)(0) - (\sin x)(1) = -\sin x \end{aligned}$$

27. $y = \sin x + \cos x \Rightarrow y' = \cos x - \sin x$, so $y'(0) = \cos 0 - \sin 0 = 1 - 0 = 1$. An equation of the tangent line to the curve $y = \sin x + \cos x$ at the point $(0, 1)$ is $y - 1 = 1(x - 0)$ or $y = x + 1$.

28. $y = x + \sin x \Rightarrow y' = 1 + \cos x$, so $y'(\pi) = 1 + \cos \pi = 1 + (-1) = 0$. An equation of the tangent line to the curve $y = x + \sin x$ at the point (π, π) is $y - \pi = 0(x - \pi)$ or $y = \pi$.

29. $y = x + \tan x \Rightarrow y' = 1 + \sec^2 x$, so $y'(\pi) = 1 + (-1)^2 = 2$. An equation of the tangent line to the curve $y = x + \tan x$ at the point (π, π) is $y - \pi = 2(x - \pi)$ or $y = 2x - \pi$.

$$30. y = \frac{1 + \sin x}{\cos x} \stackrel{\text{QR}}{\Rightarrow} y' = \frac{\cos x(\cos x) - (1 + \sin x)(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin x + \sin^2 x}{\cos^2 x}$$

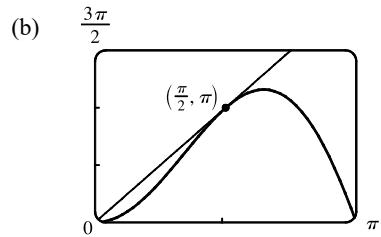
Using the identity $\cos^2 x + \sin^2 x = 1$, we can write $y' = \frac{1 + \sin x}{\cos^2 x}$.

$y'(\pi) = \frac{1 + \sin \pi}{\cos^2 \pi} = \frac{1 + 0}{(-1)^2} = 1$. An equation of the tangent line to the curve $y = \frac{1 + \sin x}{\cos x}$ at the point $(\pi, -1)$ is

$y - (-1) = 1(x - \pi)$ or $y = x - \pi - 1$.

31. (a) $y = 2x \sin x \Rightarrow y' = 2(x \cos x + \sin x \cdot 1)$. At $(\frac{\pi}{2}, \pi)$,

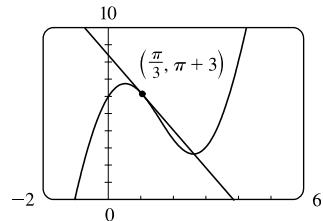
$y' = 2\left(\frac{\pi}{2} \cos \frac{\pi}{2} + \sin \frac{\pi}{2}\right) = 2(0 + 1) = 2$, and an equation of the tangent line is $y - \pi = 2(x - \frac{\pi}{2})$, or $y = 2x$.



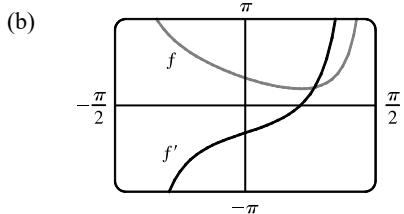
32. (a) $y = 3x + 6 \cos x \Rightarrow y' = 3 - 6 \sin x$. At $(\frac{\pi}{3}, \pi + 3)$,

$y' = 3 - 6 \sin \frac{\pi}{3} = 3 - 6 \frac{\sqrt{3}}{2} = 3 - 3\sqrt{3}$, and an equation of the tangent line is $y - (\pi + 3) = (3 - 3\sqrt{3})(x - \frac{\pi}{3})$, or $y = (3 - 3\sqrt{3})x + 3 + \pi\sqrt{3}$.

(b)

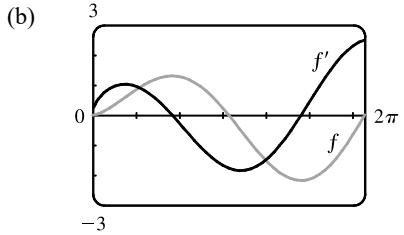


33. (a) $f(x) = \sec x - x \Rightarrow f'(x) = \sec x \tan x - 1$



Note that $f' = 0$ where f has a minimum. Also note that f' is negative when f is decreasing and f' is positive when f is increasing.

34. (a) $f(x) = \sqrt{x} \sin x \Rightarrow f'(x) = \sqrt{x} \cos x + (\sin x) \left(\frac{1}{2}x^{-1/2} \right) = \sqrt{x} \cos x + \frac{\sin x}{2\sqrt{x}}$



Notice that $f'(x) = 0$ when f has a horizontal tangent.

f' is positive when f is increasing and f' is negative when f is decreasing.

35. $g(\theta) = \frac{\sin \theta}{\theta} \stackrel{\text{QR}}{\Rightarrow} g'(\theta) = \frac{\theta(\cos \theta) - (\sin \theta)(1)}{\theta^2} = \frac{\theta \cos \theta - \sin \theta}{\theta^2}$

Using the Quotient Rule and $g'(\theta) = \frac{\theta \cos \theta - \sin \theta}{\theta^2}$, we get

$$\begin{aligned} g''(\theta) &= \frac{\theta^2 \{ [\theta(-\sin \theta) + (\cos \theta)(1)] - \cos \theta \} - (\theta \cos \theta - \sin \theta)(2\theta)}{(\theta^2)^2} \\ &= \frac{-\theta^3 \sin \theta + \theta^2 \cos \theta - \theta^2 \cos \theta - 2\theta^2 \cos \theta + 2\theta \sin \theta}{\theta^4} = \frac{\theta(-\theta^2 \sin \theta - 2\theta \cos \theta + 2 \sin \theta)}{\theta \cdot \theta^3} \\ &= \frac{-\theta^2 \sin \theta - 2\theta \cos \theta + 2 \sin \theta}{\theta^3} \end{aligned}$$

36. $f(t) = \sec t \Rightarrow f'(t) = \sec t \tan t \Rightarrow f''(t) = (\sec t) \sec^2 t + (\tan t) \sec t \tan t = \sec^3 t + \sec t \tan^2 t$, so

$$f''(\frac{\pi}{4}) = (\sqrt{2})^3 + \sqrt{2}(1)^2 = 2\sqrt{2} + \sqrt{2} = 3\sqrt{2}.$$

37. (a) $f(x) = \frac{\tan x - 1}{\sec x} \Rightarrow$

$$f'(x) = \frac{\sec x(\sec^2 x) - (\tan x - 1)(\sec x \tan x)}{(\sec x)^2} = \frac{\sec x(\sec^2 x - \tan^2 x + \tan x)}{\sec^2 x} = \frac{1 + \tan x}{\sec x}$$

$$(b) f(x) = \frac{\tan x - 1}{\sec x} = \frac{\frac{\sin x}{\cos x} - 1}{\frac{1}{\cos x}} = \frac{\sin x - \cos x}{\frac{1}{\cos x}} = \sin x - \cos x \Rightarrow f'(x) = \cos x - (-\sin x) = \cos x + \sin x$$

(c) From part (a), $f'(x) = \frac{1 + \tan x}{\sec x} = \frac{1}{\sec x} + \frac{\tan x}{\sec x} = \cos x + \sin x$, which is the expression for $f'(x)$ in part (b).

38. (a) $g(x) = f(x) \sin x \Rightarrow g'(x) = f(x) \cos x + \sin x \cdot f'(x)$, so

$$g'\left(\frac{\pi}{3}\right) = f\left(\frac{\pi}{3}\right) \cos \frac{\pi}{3} + \sin \frac{\pi}{3} \cdot f'\left(\frac{\pi}{3}\right) = 4 \cdot \frac{1}{2} + \frac{\sqrt{3}}{2} \cdot (-2) = 2 - \sqrt{3}$$

$$(b) h(x) = \frac{\cos x}{f(x)} \Rightarrow h'(x) = \frac{f(x) \cdot (-\sin x) - \cos x \cdot f'(x)}{[f(x)]^2}, \text{ so}$$

$$h'\left(\frac{\pi}{3}\right) = \frac{f\left(\frac{\pi}{3}\right) \cdot (-\sin \frac{\pi}{3}) - \cos \frac{\pi}{3} \cdot f'\left(\frac{\pi}{3}\right)}{[f\left(\frac{\pi}{3}\right)]^2} = \frac{4\left(-\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{2}\right)(-2)}{4^2} = \frac{-2\sqrt{3} + 1}{16} = \frac{1 - 2\sqrt{3}}{16}$$

39. $f(x) = x + 2 \sin x$ has a horizontal tangent when $f'(x) = 0 \Leftrightarrow 1 + 2 \cos x = 0 \Leftrightarrow \cos x = -\frac{1}{2} \Leftrightarrow$

$x = \frac{2\pi}{3} + 2\pi n$ or $\frac{4\pi}{3} + 2\pi n$, where n is an integer. Note that $\frac{4\pi}{3}$ and $\frac{2\pi}{3}$ are $\pm\frac{\pi}{3}$ units from π . This allows us to write the solutions in the more compact equivalent form $(2n+1)\pi \pm \frac{\pi}{3}$, n an integer.

$$40. y = \frac{\cos x}{2 + \sin x} \Rightarrow y' = \frac{(2 + \sin x)(-\sin x) - \cos x \cos x}{(2 + \sin x)^2} = \frac{-2 \sin x - \sin^2 x - \cos^2 x}{(2 + \sin x)^2} = \frac{-2 \sin x - 1}{(2 + \sin x)^2} = 0 \text{ when } -2 \sin x - 1 = 0 \Leftrightarrow \sin x = -\frac{1}{2} \Leftrightarrow x = \frac{11\pi}{6} + 2\pi n \text{ or } x = \frac{7\pi}{6} + 2\pi n, n \text{ an integer. So } y = \frac{1}{\sqrt{3}} \text{ or } y = -\frac{1}{\sqrt{3}} \text{ and the points on the curve with horizontal tangents are: } \left(\frac{11\pi}{6} + 2\pi n, \frac{1}{\sqrt{3}}\right), \left(\frac{7\pi}{6} + 2\pi n, -\frac{1}{\sqrt{3}}\right), n \text{ an integer.}$$

41. (a) $x(t) = 8 \sin t \Rightarrow v(t) = x'(t) = 8 \cos t \Rightarrow a(t) = x''(t) = -8 \sin t$

(b) The mass at time $t = \frac{2\pi}{3}$ has position $x\left(\frac{2\pi}{3}\right) = 8 \sin \frac{2\pi}{3} = 8\left(\frac{\sqrt{3}}{2}\right) = 4\sqrt{3}$, velocity $v\left(\frac{2\pi}{3}\right) = 8 \cos \frac{2\pi}{3} = 8\left(-\frac{1}{2}\right) = -4$, and acceleration $a\left(\frac{2\pi}{3}\right) = -8 \sin \frac{2\pi}{3} = -8\left(\frac{\sqrt{3}}{2}\right) = -4\sqrt{3}$. Since $v\left(\frac{2\pi}{3}\right) < 0$, the particle is moving to the left.

42. (a) $s(t) = 2 \cos t + 3 \sin t \Rightarrow v(t) = -2 \sin t + 3 \cos t \Rightarrow a(t) = -2 \cos t - 3 \sin t$

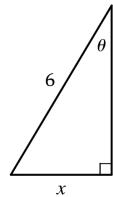
(c) $s = 0 \Rightarrow t_2 \approx 2.55$. So the mass passes through the equilibrium position for the first time when $t \approx 2.55$ s.

(d) $v = 0 \Rightarrow t_1 \approx 0.98, s(t_1) \approx 3.61$ cm. So the mass travels

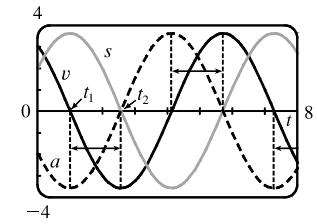
a maximum of about 3.6 cm (upward and downward) from its equilibrium position.

(e) The speed $|v|$ is greatest when $s = 0$, that is, when $t = t_2 + n\pi, n$ a positive integer.

43.



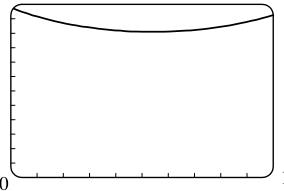
From the diagram we can see that $\sin \theta = x/6 \Leftrightarrow x = 6 \sin \theta$. We want to find the rate of change of x with respect to θ , that is, $dx/d\theta$. Taking the derivative of $x = 6 \sin \theta$, we get $dx/d\theta = 6(\cos \theta)$. So when $\theta = \frac{\pi}{3}$, $\frac{dx}{d\theta} = 6 \cos \frac{\pi}{3} = 6\left(\frac{1}{2}\right) = 3$ m/rad.



44. (a) $F = \frac{\mu mg}{\mu \sin \theta + \cos \theta} \Rightarrow \frac{dF}{d\theta} = \frac{(\mu \sin \theta + \cos \theta)(0) - \mu mg(\mu \cos \theta - \sin \theta)}{(\mu \sin \theta + \cos \theta)^2} = \frac{\mu mg(\sin \theta - \mu \cos \theta)}{(\mu \sin \theta + \cos \theta)^2}$

(b) $\frac{dF}{d\theta} = 0 \Leftrightarrow \mu W(\sin \theta - \mu \cos \theta) = 0 \Leftrightarrow \sin \theta = \mu \cos \theta \Leftrightarrow \tan \theta = \mu \Leftrightarrow \theta = \tan^{-1} \mu$

(c)



From the graph of $F = \frac{0.6(20)(9.8)}{0.6 \sin \theta + \cos \theta}$ for $0 \leq \theta \leq 1$, we see that

$$\frac{dF}{d\theta} = 0 \Rightarrow \theta \approx 0.54. \text{ Checking this with part (b) and } \mu = 0.6, \text{ we}$$

calculate $\theta = \tan^{-1} 0.6 \approx 0.54$. So the value from the graph is consistent with the value in part (b).

45. $\lim_{x \rightarrow 0} \frac{\sin 5x}{3x} = \lim_{x \rightarrow 0} \frac{5}{3} \left(\frac{\sin 5x}{5x} \right) = \frac{5}{3} \lim_{x \rightarrow 0} \frac{\sin 5x}{5x} = \frac{5}{3} \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \quad \begin{bmatrix} \text{where } \theta = 5x, \\ \text{using Equation 5} \end{bmatrix} = \frac{5}{3} \cdot 1 = \frac{5}{3}$

46. $\lim_{x \rightarrow 0} \frac{\sin x}{\sin \pi x} = \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \frac{\pi x}{\sin \pi x} \cdot \frac{1}{\pi} = \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \lim_{\theta \rightarrow 0} \frac{\theta}{\sin \theta} \cdot \frac{1}{\pi} \quad \begin{bmatrix} \text{where } \theta = \pi x, \\ \text{using Equation 5} \end{bmatrix}$
 $= 1 \cdot \lim_{\theta \rightarrow 0} \frac{1}{\sin \theta} \cdot \frac{1}{\pi} = 1 \cdot 1 \cdot \frac{1}{\pi} = \frac{1}{\pi}$

47. $\lim_{t \rightarrow 0} \frac{\sin 3t}{\sin t} = \lim_{t \rightarrow 0} \frac{\sin 3t}{3t} \cdot \frac{t}{\sin t} \cdot 3 = \lim_{t \rightarrow 0} \frac{\sin 3t}{3t} \cdot \lim_{t \rightarrow 0} \frac{t}{\sin t} \cdot \lim_{t \rightarrow 0} 3 = \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \cdot \lim_{t \rightarrow 0} \frac{1}{\sin t} \cdot \lim_{t \rightarrow 0} 3 \quad [\theta = 3t]$
 $= 1 \cdot 1 \cdot 3 = 3$

48. $\lim_{x \rightarrow 0} \frac{\sin^2 3x}{x} = \lim_{x \rightarrow 0} \frac{\sin 3x}{3x} \cdot \frac{\sin 3x}{3x} \cdot 3 \cdot 3 \cdot x = \lim_{x \rightarrow 0} \frac{\sin 3x}{3x} \cdot \lim_{x \rightarrow 0} \frac{\sin 3x}{3x} \cdot \lim_{x \rightarrow 0} 3 \cdot 3 \cdot x$
 $= \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \cdot \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \cdot \lim_{x \rightarrow 0} 9x \quad [\theta = 3x]$
 $= 1 \cdot 1 \cdot 0 = 0$

49. $\lim_{x \rightarrow 0} \frac{\sin x - \sin x \cos x}{x^2} = \lim_{x \rightarrow 0} \frac{\sin x (1 - \cos x)}{x^2} = \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \lim_{x \rightarrow 0} \frac{1 - \cos x}{x}$
 $= 1 \cdot 0 \quad [\text{by Equations 5 and 6}] = 0$

50. $\lim_{x \rightarrow 0} \frac{1 - \sec x}{2x} = \lim_{x \rightarrow 0} \frac{1 - (1/\cos x)}{2x} \cdot \frac{\cos x}{\cos x} = \lim_{x \rightarrow 0} \frac{\cos x - 1}{2x \cos x} = \lim_{x \rightarrow 0} \frac{\cos x - 1}{x} \cdot \lim_{x \rightarrow 0} \frac{1}{2 \cos x}$
 $= 0 \cdot \frac{1}{2} \quad [\text{using Equation 6}] = 0$

51. $\lim_{x \rightarrow 0} \frac{\tan 2x}{x} = \lim_{x \rightarrow 0} \frac{\frac{\sin 2x}{\cos 2x}}{x} = \lim_{x \rightarrow 0} \frac{\sin 2x}{x \cos 2x} = \lim_{x \rightarrow 0} \frac{\sin 2x}{2x} \cdot \frac{2}{\cos 2x}$
 $= \lim_{x \rightarrow 0} \frac{\sin 2x}{2x} \cdot \lim_{x \rightarrow 0} \frac{2}{\cos 2x} = \lim_{x \rightarrow 0} \frac{\sin \theta}{\theta} \cdot \lim_{x \rightarrow 0} \frac{2}{\cos 2x} \quad [\theta = 2x]$
 $= 1 \cdot \frac{2}{1} = 2$

$$\begin{aligned}
 52. \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\tan 7\theta} &= \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\frac{\sin 7\theta}{\cos 7\theta}} = \lim_{\theta \rightarrow 0} \sin \theta \cdot \frac{\cos 7\theta}{\sin 7\theta} = \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \cdot \cos 7\theta \cdot \frac{7\theta}{\sin 7\theta} \cdot \frac{1}{7} \\
 &= \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \cdot \lim_{\theta \rightarrow 0} \cos 7\theta \cdot \lim_{\theta \rightarrow 0} \frac{1}{\frac{\sin 7\theta}{7\theta}} \cdot \frac{1}{7} = 1 \cdot 1 \cdot \lim_{x \rightarrow 0} \frac{1}{\frac{\sin x}{x}} \cdot \frac{1}{7} \quad [x = 7\theta] \\
 &= 1 \cdot 1 \cdot 1 \cdot \frac{1}{7} = \frac{1}{7}
 \end{aligned}$$

$$53. \lim_{x \rightarrow 0} \frac{\sin 3x}{5x^3 - 4x} = \lim_{x \rightarrow 0} \left(\frac{\sin 3x}{3x} \cdot \frac{3}{5x^2 - 4} \right) = \lim_{x \rightarrow 0} \frac{\sin 3x}{3x} \cdot \lim_{x \rightarrow 0} \frac{3}{5x^2 - 4} = 1 \cdot \left(\frac{3}{-4} \right) = -\frac{3}{4}$$

$$\begin{aligned}
 54. \lim_{x \rightarrow 0} \frac{\sin 3x \sin 5x}{x^2} &= \lim_{x \rightarrow 0} \left(\frac{3 \sin 3x}{3x} \cdot \frac{5 \sin 5x}{5x} \right) = \lim_{x \rightarrow 0} \frac{3 \sin 3x}{3x} \cdot \lim_{x \rightarrow 0} \frac{5 \sin 5x}{5x} \\
 &= 3 \lim_{x \rightarrow 0} \frac{\sin 3x}{3x} \cdot 5 \lim_{x \rightarrow 0} \frac{\sin 5x}{5x} = 3(1) \cdot 5(1) = 15
 \end{aligned}$$

55. Divide numerator and denominator by θ . [sin θ also works.]

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta + \tan \theta} = \lim_{\theta \rightarrow 0} \frac{\frac{\sin \theta}{\theta}}{1 + \frac{\sin \theta}{\theta} \cdot \frac{1}{\cos \theta}} = \frac{\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta}}{1 + \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \lim_{\theta \rightarrow 0} \frac{1}{\cos \theta}} = \frac{1}{1 + 1 \cdot 1} = \frac{1}{2}$$

$$56. \lim_{x \rightarrow 0} \csc x \sin(\sin x) = \lim_{x \rightarrow 0} \frac{\sin(\sin x)}{\sin x} = \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \quad [\text{As } x \rightarrow 0, \theta = \sin x \rightarrow 0.] = 1$$

$$\begin{aligned}
 57. \lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{2\theta^2} &= \lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{2\theta^2} \cdot \frac{\cos \theta + 1}{\cos \theta + 1} = \lim_{\theta \rightarrow 0} \frac{\cos^2 \theta - 1}{2\theta^2(\cos \theta + 1)} = \lim_{\theta \rightarrow 0} \frac{-\sin^2 \theta}{2\theta^2(\cos \theta + 1)} \\
 &= -\frac{1}{2} \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \cdot \frac{\sin \theta}{\theta} \cdot \frac{1}{\cos \theta + 1} = -\frac{1}{2} \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \cdot \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \cdot \lim_{\theta \rightarrow 0} \frac{1}{\cos \theta + 1} \\
 &= -\frac{1}{2} \cdot 1 \cdot 1 \cdot \frac{1}{1 + 1} = -\frac{1}{4}
 \end{aligned}$$

$$58. \lim_{x \rightarrow 0} \frac{\sin(x^2)}{x} = \lim_{x \rightarrow 0} \left[x \cdot \frac{\sin(x^2)}{x \cdot x} \right] = \lim_{x \rightarrow 0} x \cdot \lim_{x \rightarrow 0} \frac{\sin(x^2)}{x^2} = 0 \cdot \lim_{y \rightarrow 0^+} \frac{\sin y}{y} \quad [\text{where } y = x^2] = 0 \cdot 1 = 0$$

$$59. \lim_{x \rightarrow \pi/4} \frac{1 - \tan x}{\sin x - \cos x} = \lim_{x \rightarrow \pi/4} \frac{\left(1 - \frac{\sin x}{\cos x}\right) \cdot \cos x}{(\sin x - \cos x) \cdot \cos x} = \lim_{x \rightarrow \pi/4} \frac{\cos x - \sin x}{(\sin x - \cos x) \cos x} = \lim_{x \rightarrow \pi/4} \frac{-1}{\cos x} = \frac{-1}{1/\sqrt{2}} = -\sqrt{2}$$

$$60. \lim_{x \rightarrow 1} \frac{\sin(x-1)}{x^2 + x - 2} = \lim_{x \rightarrow 1} \frac{\sin(x-1)}{(x+2)(x-1)} = \lim_{x \rightarrow 1} \frac{1}{x+2} \lim_{x \rightarrow 1} \frac{\sin(x-1)}{x-1} = \frac{1}{3} \cdot 1 = \frac{1}{3}$$

$$61. \frac{d}{dx} (\sin x) = \cos x \Rightarrow \frac{d^2}{dx^2} (\sin x) = -\sin x \Rightarrow \frac{d^3}{dx^3} (\sin x) = -\cos x \Rightarrow \frac{d^4}{dx^4} (\sin x) = \sin x.$$

The derivatives of $\sin x$ occur in a cycle of four. Since $99 = 4(24) + 3$, we have $\frac{d^{99}}{dx^{99}} (\sin x) = \frac{d^3}{dx^3} (\sin x) = -\cos x$.

62. Let $f(x) = x \sin x$ and $h(x) = \sin x$, so $f(x) = xh(x)$. Then $f'(x) = h(x) + xh'(x)$,

$$f''(x) = h'(x) + h'(x) + xh''(x) = 2h'(x) + xh''(x),$$

$$f'''(x) = 2h''(x) + h''(x) + xh'''(x) = 3h''(x) + xh'''(x), \dots, f^{(n)}(x) = nh^{(n-1)}(x) + xh^{(n)}(x).$$

Since $34 = 4(8) + 2$, we have $h^{(34)}(x) = h^{(2)}(x) = \frac{d^2}{dx^2}(\sin x) = -\sin x$ and $h^{(35)}(x) = -\cos x$.

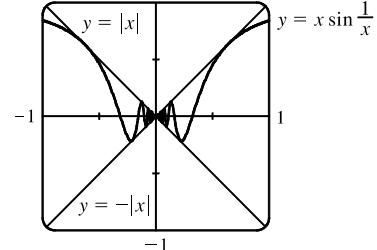
Thus, $\frac{d^{35}}{dx^{35}}(x \sin x) = 35h^{(34)}(x) + xh^{(35)}(x) = -35 \sin x - x \cos x$.

63. $y = A \sin x + B \cos x \Rightarrow y' = A \cos x - B \sin x \Rightarrow y'' = -A \sin x - B \cos x$. Substituting these expressions for y , y' , and y'' into the given differential equation $y'' + y' - 2y = \sin x$ gives us $(-A \sin x - B \cos x) + (A \cos x - B \sin x) - 2(A \sin x + B \cos x) = \sin x \Leftrightarrow -3A \sin x - B \sin x + A \cos x - 3B \cos x = \sin x \Leftrightarrow (-3A - B) \sin x + (A - 3B) \cos x = 1 \sin x$, so we must have $-3A - B = 1$ and $A - 3B = 0$ (since 0 is the coefficient of $\cos x$ on the right side). Solving for A and B , we add the first equation to three times the second to get $B = -\frac{1}{10}$ and $A = -\frac{3}{10}$.

64. Since $-1 \leq \sin(1/x) \leq 1$, we have (as illustrated in the figure)

$-|x| \leq x \sin(1/x) \leq |x|$. We know that $\lim_{x \rightarrow 0} (|x|) = 0$ and

$\lim_{x \rightarrow 0} (-|x|) = 0$; so by the Squeeze Theorem, $\lim_{x \rightarrow 0} x \sin(1/x) = 0$.



65. (a) $\frac{d}{dx} \tan x = \frac{d}{dx} \frac{\sin x}{\cos x} \Rightarrow \sec^2 x = \frac{\cos x \cos x - \sin x (-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$. So $\sec^2 x = \frac{1}{\cos^2 x}$.

(b) $\frac{d}{dx} \sec x = \frac{d}{dx} \frac{1}{\cos x} \Rightarrow \sec x \tan x = \frac{(\cos x)(0) - 1(-\sin x)}{\cos^2 x}$. So $\sec x \tan x = \frac{\sin x}{\cos^2 x}$.

(c) $\frac{d}{dx} (\sin x + \cos x) = \frac{d}{dx} \frac{1 + \cot x}{\csc x} \Rightarrow$

$$\begin{aligned} \cos x - \sin x &= \frac{\csc x (-\csc^2 x) - (1 + \cot x)(-\csc x \cot x)}{\csc^2 x} = \frac{\csc x [-\csc^2 x + (1 + \cot x) \cot x]}{\csc^2 x} \\ &= \frac{-\csc^2 x + \cot^2 x + \cot x}{\csc x} = \frac{-1 + \cot x}{\csc x} \end{aligned}$$

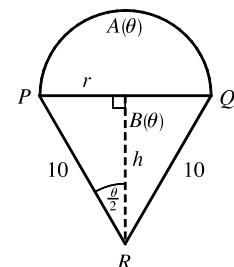
So $\cos x - \sin x = \frac{\cot x - 1}{\csc x}$.

66. We get the following formulas for r and h in terms of θ :

$$\sin \frac{\theta}{2} = \frac{r}{10} \Rightarrow r = 10 \sin \frac{\theta}{2} \text{ and } \cos \frac{\theta}{2} = \frac{h}{10} \Rightarrow h = 10 \cos \frac{\theta}{2}$$

Now $A(\theta) = \frac{1}{2}\pi r^2$ and $B(\theta) = \frac{1}{2}(2r)h = rh$. So

$$\begin{aligned} \lim_{\theta \rightarrow 0^+} \frac{A(\theta)}{B(\theta)} &= \lim_{\theta \rightarrow 0^+} \frac{\frac{1}{2}\pi r^2}{rh} = \frac{1}{2}\pi \lim_{\theta \rightarrow 0^+} \frac{r}{h} = \frac{1}{2}\pi \lim_{\theta \rightarrow 0^+} \frac{10 \sin(\theta/2)}{10 \cos(\theta/2)} \\ &= \frac{1}{2}\pi \lim_{\theta \rightarrow 0^+} \tan(\theta/2) = 0 \end{aligned}$$

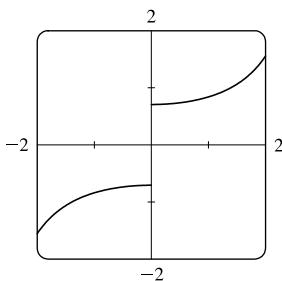


67. By the definition of radian measure, $s = r\theta$, where r is the radius of the circle. By drawing the bisector of the angle θ , we can

see that $\sin \frac{\theta}{2} = \frac{d/2}{r} \Rightarrow d = 2r \sin \frac{\theta}{2}$. So $\lim_{\theta \rightarrow 0^+} \frac{s}{d} = \lim_{\theta \rightarrow 0^+} \frac{r\theta}{2r \sin(\theta/2)} = \lim_{\theta \rightarrow 0^+} \frac{2 \cdot (\theta/2)}{2 \sin(\theta/2)} = \lim_{\theta \rightarrow 0} \frac{\theta/2}{\sin(\theta/2)} = 1$.

[This is just the reciprocal of the limit $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ combined with the fact that as $\theta \rightarrow 0$, $\frac{\theta}{2} \rightarrow 0$ also.]

68. (a)



It appears that $f(x) = \frac{x}{\sqrt{1 - \cos 2x}}$ has a jump discontinuity at $x = 0$.

- (b) Using the identity $\cos 2x = 1 - \sin^2 x$, we have $\frac{x}{\sqrt{1 - \cos 2x}} = \frac{x}{\sqrt{1 - (1 - 2\sin^2 x)}} = \frac{x}{\sqrt{2\sin^2 x}} = \frac{x}{\sqrt{2}|\sin x|}$.

Thus,

$$\begin{aligned} \lim_{x \rightarrow 0^-} \frac{x}{\sqrt{1 - \cos 2x}} &= \lim_{x \rightarrow 0^-} \frac{x}{\sqrt{2}|\sin x|} = \frac{1}{\sqrt{2}} \lim_{x \rightarrow 0^-} \frac{x}{-(\sin x)} \\ &= -\frac{1}{\sqrt{2}} \lim_{x \rightarrow 0^-} \frac{1}{\sin x/x} = -\frac{1}{\sqrt{2}} \cdot \frac{1}{1} = -\frac{\sqrt{2}}{2} \end{aligned}$$

Evaluating $\lim_{x \rightarrow 0^+} f(x)$ is similar, but $|\sin x| = +\sin x$, so we get $\frac{1}{2}\sqrt{2}$. These values appear to be reasonable values for the graph, so they confirm our answer to part (a).

Another method: Multiply numerator and denominator by $\sqrt{1 + \cos 2x}$.

2.5 The Chain Rule

- Let $u = g(x) = 5 - x^4$ and $y = f(u) = u^3$. Then $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = (3u^2)(-4x^3) = 3(5 - x^4)^2(-4x^3) = -12x^3(5 - x^4)^2$.
- Let $u = g(x) = x^3 + 2$ and $y = f(u) = \sqrt{u}$. Then $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = \frac{1}{2\sqrt{u}} \cdot 3x^2 = \frac{1}{2\sqrt{x^3 + 2}} \cdot 3x^2 = \frac{3x^2}{2\sqrt{x^3 + 2}}$.
- Let $u = g(x) = \cos x$ and $y = f(u) = \sin u$. Then $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = (\cos u)(-\sin x) = (\cos(\cos x))(-\sin x) = -\sin x \cos(\cos x)$.
- Let $u = g(x) = x^2$ and $y = f(u) = \tan u$. Then $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = (\sec^2 u)(2x) = (\sec^2(x^2))(2x) = 2x \sec^2(x^2)$.
- Let $u = g(x) = \sin x$ and $y = f(u) = \sqrt{u}$. Then $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = \frac{1}{2}u^{-1/2} \cos x = \frac{\cos x}{2\sqrt{u}} = \frac{\cos x}{2\sqrt{\sin x}}$.
- Let $u = g(x) = \sqrt{x}$ and $y = f(u) = \sin u$. Then $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = (\cos u)\left(\frac{1}{2}x^{-1/2}\right) = \frac{\cos u}{2\sqrt{x}} = \frac{\cos \sqrt{x}}{2\sqrt{x}}$.

7. $f(x) = (2x^3 - 5x^2 + 4)^5 \stackrel{\text{CR}}{\Rightarrow}$

$$\begin{aligned} f'(x) &= 5(2x^3 - 5x^2 + 4)^4 \cdot \frac{d}{dx}(2x^3 - 5x^2 + 4) = 5(2x^3 - 5x^2 + 4)^4(6x^2 - 10x) \\ &= 5(2x^3 - 5x^2 + 4)^4 \cdot 2x(3x - 5) = 10x(2x^3 - 5x^2 + 4)^4(3x - 5) \end{aligned}$$

8. $f(x) = (x^5 + 3x^2 - x)^{50} \stackrel{\text{CR}}{\Rightarrow}$

$$f'(x) = 50(x^5 + 3x^2 - x)^{49} \cdot \frac{d}{dx}(x^5 + 3x^2 - x) = 50(x^5 + 3x^2 - x)^{49}(5x^4 + 6x - 1)$$

9. $f(x) = \sqrt{5x+1} = (5x+1)^{1/2} \stackrel{\text{CR}}{\Rightarrow} f'(x) = \frac{1}{2}(5x+1)^{-1/2} \cdot \frac{d}{dx}(5x+1) = \frac{1}{2}(5x+1)^{-1/2}(5) = \frac{5}{2\sqrt{5x+1}}$

10. $f(x) = \frac{1}{\sqrt[3]{x^2 - 1}} = (x^2 - 1)^{-1/3} \stackrel{\text{CR}}{\Rightarrow} f'(x) = -\frac{1}{3}(x^2 - 1)^{-4/3}(2x) = \frac{-2x}{3(x^2 - 1)^{4/3}}$

11. $g(t) = \frac{1}{(2t+1)^2} = (2t+1)^{-2} \stackrel{\text{CR}}{\Rightarrow} g'(t) = -2(2t+1)^{-3} \cdot \frac{d}{dt}(2t+1) = -2(2t+1)^{-3}(2) = -\frac{4}{(2t+1)^3}$

12. $F(t) = \left(\frac{1}{2t+1}\right)^4 = [(2t+1)^{-1}]^4 = (2t+1)^{-4} \stackrel{\text{CR}}{\Rightarrow}$

$$F'(t) = -4(2t+1)^{-5} \cdot \frac{d}{dt}(2t+1) = -4(2t+1)^{-5}(2) = -\frac{8}{(2t+1)^5}$$

13. $A(t) = \frac{1}{(\cos t + \tan t)^2} = (\cos t + \tan t)^{-2} \stackrel{\text{CR}}{\Rightarrow} A'(t) = -2(\cos t + \tan t)^{-3}(-\sin t + \sec^2 t) = \frac{2(\sin t - \sec^2 t)}{(\cos t + \tan t)^3}$

14. $g(x) = (2 - \sin x)^{3/2} \stackrel{\text{CR}}{\Rightarrow}$

$$g'(x) = \frac{3}{2}(2 - \sin x)^{1/2} \cdot \frac{d}{dx}(2 - \sin x) = \frac{3}{2}(2 - \sin x)^{1/2}(-\cos x) = -\frac{3}{2}\cos x(2 - \sin x)^{1/2}$$

15. $f(\theta) = \cos(\theta^2) \stackrel{\text{CR}}{\Rightarrow} f'(\theta) = -\sin(\theta^2) \cdot (\theta^2)' = -\sin(\theta^2) \cdot (2\theta) = -2\theta\sin(\theta^2)$

16. $g(\theta) = \cos^2 \theta = (\cos \theta)^2 \stackrel{\text{CR}}{\Rightarrow} g'(\theta) = 2(\cos \theta)^1(-\sin \theta) = -2\sin \theta \cos \theta = -\sin 2\theta$

17. $h(v) = v\sqrt[3]{1+v^2} = v(1+v^2)^{1/3} \stackrel{\text{PR}}{\Rightarrow}$

$$h'(v) = v \cdot \frac{1}{3}(1+v^2)^{-2/3}(2v) + (1+v^2)^{1/3} \cdot 1 = \frac{1}{3}(1+v^2)^{-2/3}[2v^2 + 3(1+v^2)] = \frac{5v^2 + 3}{3(\sqrt[3]{1+v^2})^2}$$

18. Using the Product Rule and the Chain Rule, $f(t) = t \sin \pi t \Rightarrow f'(t) = t(\cos \pi t) \cdot \pi + (\sin \pi t) \cdot 1 = \pi t \cos \pi t + \sin \pi t$.

19. $F(x) = (4x+5)^3(x^2 - 2x + 5)^4 \Rightarrow$

$$\begin{aligned} F'(x) &= (4x+5)^3 \cdot 4(x^2 - 2x + 5)^3(2x - 2) + (x^2 - 2x + 5)^4 \cdot 3(4x+5)^2 \cdot 4 \\ &= 4(4x+5)^2(x^2 - 2x + 5)^3[(4x+5)(2x-2) + (x^2 - 2x + 5) \cdot 3] \\ &= 4(4x+5)^2(x^2 - 2x + 5)^3(8x^2 + 2x - 10 + 3x^2 - 6x + 15) \\ &= 4(4x+5)^2(x^2 - 2x + 5)^3(11x^2 - 4x + 5) \end{aligned}$$

20. $G(z) = (1 - 4z)^2 \sqrt{z^2 + 1} \Rightarrow$

$$\begin{aligned} G'(z) &= (1 - 4z)^2 \cdot \frac{1}{2\sqrt{z^2 + 1}} \cdot 2z + \sqrt{z^2 + 1} \cdot 2(1 - 4z)^1(-4) = 2(1 - 4z) \left[\frac{(1 - 4z)z}{2\sqrt{z^2 + 1}} - 4\sqrt{z^2 + 1} \right] \\ &= 2(1 - 4z) \left[\frac{(1 - 4z)z}{2\sqrt{z^2 + 1}} - \frac{8(z^2 + 1)}{2\sqrt{z^2 + 1}} \right] = 2(1 - 4z) \left(\frac{z - 4z^2 - 8z^2 - 8}{2\sqrt{z^2 + 1}} \right) \\ &= (1 - 4z) \left(\frac{-12z^2 + z - 8}{\sqrt{z^2 + 1}} \right) \text{ or } (4z - 1) \left(\frac{12z^2 - z + 8}{\sqrt{z^2 + 1}} \right) \end{aligned}$$

21. $h(t) = (t + 1)^{2/3}(2t^2 - 1)^3 \Rightarrow$

$$\begin{aligned} h'(t) &= (t + 1)^{2/3} \cdot 3(2t^2 - 1)^2 \cdot 4t + (2t^2 - 1)^3 \cdot \frac{2}{3}(t + 1)^{-1/3} = \frac{2}{3}(t + 1)^{-1/3}(2t^2 - 1)^2[18t(t + 1) + (2t^2 - 1)] \\ &= \frac{2}{3}(t + 1)^{-1/3}(2t^2 - 1)^2(20t^2 + 18t - 1) \end{aligned}$$

22. $F(t) = (3t - 1)^4(2t + 1)^{-3} \Rightarrow$

$$\begin{aligned} F'(t) &= (3t - 1)^4(-3)(2t + 1)^{-4}(2) + (2t + 1)^{-3} \cdot 4(3t - 1)^3(3) \\ &= 6(3t - 1)^3(2t + 1)^{-4}[-(3t - 1) + 2(2t + 1)] = 6(3t - 1)^3(2t + 1)^{-4}(t + 3) \end{aligned}$$

23. $y = \sqrt{\frac{x}{x+1}} = \left(\frac{x}{x+1}\right)^{1/2} \Rightarrow$

$$\begin{aligned} y' &= \frac{1}{2} \left(\frac{x}{x+1} \right)^{-1/2} \frac{d}{dx} \left(\frac{x}{x+1} \right) = \frac{1}{2} \frac{x^{-1/2}}{(x+1)^{-1/2}} \frac{(x+1)(1) - x(1)}{(x+1)^2} \\ &= \frac{1}{2} \frac{(x+1)^{1/2}}{x^{1/2}} \frac{1}{(x+1)^2} = \frac{1}{2\sqrt{x}(x+1)^{3/2}} \end{aligned}$$

24. $y = \left(x + \frac{1}{x}\right)^5 \Rightarrow y' = 5\left(x + \frac{1}{x}\right)^4 \frac{d}{dx} \left(x + \frac{1}{x}\right) = 5\left(x + \frac{1}{x}\right)^4 \left(1 - \frac{1}{x^2}\right).$

Another form of the answer is $\frac{5(x^2 + 1)^4(x^2 - 1)}{x^6}$.

25. $g(u) = \left(\frac{u^3 - 1}{u^3 + 1}\right)^8 \Rightarrow$

$$\begin{aligned} g'(u) &= 8\left(\frac{u^3 - 1}{u^3 + 1}\right)^7 \frac{d}{du} \frac{u^3 - 1}{u^3 + 1} = 8 \frac{(u^3 - 1)^7}{(u^3 + 1)^7} \frac{(u^3 + 1)(3u^2) - (u^3 - 1)(3u^2)}{(u^3 + 1)^2} \\ &= 8 \frac{(u^3 - 1)^7}{(u^3 + 1)^7} \frac{3u^2[(u^3 + 1) - (u^3 - 1)]}{(u^3 + 1)^2} = 8 \frac{(u^3 - 1)^7}{(u^3 + 1)^7} \frac{3u^2(2)}{(u^3 + 1)^2} = \frac{48u^2(u^3 - 1)^7}{(u^3 + 1)^9} \end{aligned}$$

26. $s(t) = \sqrt{\frac{1 + \sin t}{1 + \cos t}} = \left(\frac{1 + \sin t}{1 + \cos t}\right)^{1/2} \Rightarrow$

$$\begin{aligned} s'(t) &= \frac{1}{2} \left(\frac{1 + \sin t}{1 + \cos t} \right)^{-1/2} \frac{(1 + \cos t)\cos t - (1 + \sin t)(-\sin t)}{(1 + \cos t)^2} \\ &= \frac{1}{2} \frac{(1 + \sin t)^{-1/2}}{(1 + \cos t)^{-1/2}} \frac{\cos t + \cos^2 t + \sin t + \sin^2 t}{(1 + \cos t)^2} = \frac{\cos t + \sin t + 1}{2\sqrt{1 + \sin t}(1 + \cos t)^{3/2}} \end{aligned}$$

27. $H(r) = \frac{(r^2 - 1)^3}{(2r + 1)^5} \Rightarrow$

$$\begin{aligned} H'(r) &= \frac{(2r + 1)^5 \cdot 3(r^2 - 1)^2(2r) - (r^2 - 1)^3 \cdot 5(2r + 1)^4(2)}{[(2r + 1)^5]^2} = \frac{2(2r + 1)^4(r^2 - 1)^2[3r(2r + 1) - 5(r^2 - 1)]}{(2r + 1)^{10}} \\ &= \frac{2(r^2 - 1)^2(6r^2 + 3r - 5r^2 + 5)}{(2r + 1)^6} = \frac{2(r^2 - 1)^2(r^2 + 3r + 5)}{(2r + 1)^6} \end{aligned}$$

28. $F(t) = \frac{t^2}{\sqrt{t^3 + 1}} \Rightarrow$

$$F'(t) = \frac{(t^3 + 1)^{1/2}(2t) - t^2 \cdot \frac{1}{2}(t^3 + 1)^{-1/2}(3t^2)}{(\sqrt{t^3 + 1})^2} = \frac{t(t^3 + 1)^{-1/2}[2(t^3 + 1) - \frac{3}{2}t^3]}{(t^3 + 1)^1} = \frac{t(\frac{1}{2}t^3 + 2)}{(t^3 + 1)^{3/2}} = \frac{t(t^3 + 4)}{2(t^3 + 1)^{3/2}}$$

29. $y = \cos(\sec 4x) \Rightarrow$

$$y' = -\sin(\sec 4x) \frac{d}{dx} \sec 4x = -\sin(\sec 4x) \cdot \sec 4x \tan 4x \cdot 4 = -4 \sin(\sec 4x) \sec 4x \tan 4x$$

30. $J(\theta) = \tan^2(n\theta) = [\tan(n\theta)]^2 \Rightarrow$

$$J'(\theta) = 2[\tan(n\theta)]^1 \frac{d}{d\theta} \tan(n\theta) = 2 \tan(n\theta) \sec^2(n\theta) \cdot n = 2n \tan(n\theta) \sec^2(n\theta)$$

31. $y = \frac{\cos x}{\sqrt{1 + \sin x}} = (\cos x)(1 + \sin x)^{-1/2} \Rightarrow$

$$\begin{aligned} y' &= (\cos x) \cdot (-\frac{1}{2})(1 + \sin x)^{-3/2} \cos x + (1 + \sin x)^{-1/2}(-\sin x) \\ &= -\frac{1}{2}(1 + \sin x)^{-3/2}[\cos^2 x + 2(1 + \sin x) \sin x] = -\frac{1}{2}(1 + \sin x)^{-3/2}(\cos^2 x + 2 \sin x + 2 \sin^2 x) \\ &= -\frac{1}{2}(1 + \sin x)^{-3/2}(1 + 2 \sin x + \sin^2 x) = -\frac{1}{2}(1 + \sin x)^{-3/2}(1 + \sin x)^2 \\ &= -\frac{1}{2}(1 + \sin x)^{1/2} \text{ or } -\frac{1}{2}\sqrt{1 + \sin x} \end{aligned}$$

32. $h(\theta) = \tan(\theta^2 \sin \theta) \stackrel{\text{CR}}{\Rightarrow}$

$$h'(\theta) = \sec^2(\theta^2 \sin \theta) \cdot \frac{d}{d\theta}(\theta^2 \sin \theta) = \sec^2(\theta^2 \sin \theta) \cdot [\theta^2 \cos \theta + (\sin \theta)(2\theta)] = \theta \sec^2(\theta^2 \sin \theta)(\theta \cos \theta + 2 \sin \theta)$$

33. $y = \left(\frac{1 - \cos 2x}{1 + \cos 2x}\right)^4 \Rightarrow$

$$\begin{aligned} y' &= 4\left(\frac{1 - \cos 2x}{1 + \cos 2x}\right)^3 \cdot \frac{(1 + \cos 2x)(2 \sin 2x) + (1 - \cos 2x)(-2 \sin 2x)}{(1 + \cos 2x)^2} \\ &= 4\left(\frac{1 - \cos 2x}{1 + \cos 2x}\right)^3 \cdot \frac{2 \sin 2x(1 + \cos 2x + 1 - \cos 2x)}{(1 + \cos 2x)^2} = \frac{4(1 - \cos 2x)^3}{(1 + \cos 2x)^3} \frac{2 \sin 2x(2)}{(1 + \cos 2x)^2} = \frac{16 \sin 2x(1 - \cos 2x)^3}{(1 + \cos 2x)^5} \end{aligned}$$

34. $y = x \sin \frac{1}{x} \Rightarrow y' = \sin \frac{1}{x} + x \cos \frac{1}{x} \left(-\frac{1}{x^2}\right) = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x}$

35. $f(x) = \sin x \cos(1 - x^2) \Rightarrow$

$$f'(x) = \sin x [-\sin(1 - x^2)(-2x)] + \cos(1 - x^2) \cdot \cos x = 2x \sin x \sin(1 - x^2) + \cos x \cos(1 - x^2)$$

36. $y = \sin(t + \cos \sqrt{t}) \Rightarrow$

$$y' = \cos(t + \cos \sqrt{t}) \cdot \frac{d}{dt}(t + \cos \sqrt{t}) = \cos(t + \cos \sqrt{t}) \cdot \left(1 - \sin \sqrt{t} \cdot \frac{1}{2\sqrt{t}}\right) = \cos(t + \cos \sqrt{t}) \frac{2\sqrt{t} - \sin \sqrt{t}}{2\sqrt{t}}$$

37. $F(t) = \tan \sqrt{1+t^2} \Rightarrow F'(t) = \sec^2 \sqrt{1+t^2} \cdot \frac{1}{2\sqrt{1+t^2}} \cdot 2t = \frac{t \sec^2 \sqrt{1+t^2}}{\sqrt{1+t^2}}$

38. $G(z) = (1 + \cos^2 z)^3 \Rightarrow G'(z) = 3(1 + \cos^2 z)^2 [2(\cos z)(-\sin z)] = -6 \cos z \sin z (1 + \cos^2 z)^2$

39. $y = \sin^2(x^2 + 1) \Rightarrow y' = 2 \sin(x^2 + 1) \cdot \cos(x^2 + 1) \cdot 2x = 4x \sin(x^2 + 1) \cos(x^2 + 1)$

40. $g(u) = [(u^2 - 1)^6 - 3u]^4 \Rightarrow$

$$\begin{aligned} g'(u) &= 4[(u^2 - 1)^6 - 3u]^3 \cdot \frac{d}{du}[(u^2 - 1)^6 - 3u] = 4[(u^2 - 1)^6 - 3u]^3 \cdot [6(u^2 - 1)^5 \cdot 2u - 3] \\ &= 12[(u^2 - 1)^6 - 3u]^3[4u(u^2 - 1)^5 - 1] \end{aligned}$$

41. $y = \cos^4(\sin^3 x) = [\cos(\sin^3 x)]^4 \Rightarrow$

$$y' = 4[\cos(\sin^3 x)]^3(-\sin(\sin^3 x))3 \sin^2 x \cos x = -12 \cos^3(\sin^3 x) \sin(\sin^3 x) \sin^2 x \cos x$$

42. $y = \sin^3(\cos(x^2)) \Rightarrow$

$$y' = 3 \sin^2(\cos(x^2)) \cdot \cos(\cos(x^2)) \cdot [-\sin(x^2) \cdot 2x] = -6x \sin(x^2) \sin^2(\cos(x^2)) \cos(\cos(x^2))$$

43. $f(t) = \tan(\sec(\cos t)) \Rightarrow$

$$\begin{aligned} f'(t) &= \sec^2(\sec(\cos t)) \frac{d}{dt} \sec(\cos t) = \sec^2(\sec(\cos t))[\sec(\cos t) \tan(\cos t)] \frac{d}{dt} \cos t \\ &= -\sec^2(\sec(\cos t)) \sec(\cos t) \tan(\cos t) \sin t \end{aligned}$$

44. $y = \sqrt{x + \sqrt{x + \sqrt{x}}} \Rightarrow y' = \frac{1}{2} \left(x + \sqrt{x + \sqrt{x}} \right)^{-1/2} \left[1 + \frac{1}{2} \left(x + \sqrt{x} \right)^{-1/2} \left(1 + \frac{1}{2} x^{-1/2} \right) \right]$

45. $g(x) = (2r \sin rx + n)^p \Rightarrow g'(x) = p(2r \sin rx + n)^{p-1} (2r \cos rx \cdot r) = p(2r \sin rx + n)^{p-1} (2r^2 \cos rx)$

46. $y = \sin(\theta + \tan(\theta + \cos \theta)) \Rightarrow y' = \cos(\theta + \tan(\theta + \cos \theta)) \cdot [1 + \sec^2(\theta + \cos \theta) \cdot (1 - \sin \theta)]$

47. $y = \cos \sqrt{\sin(\tan \pi x)} = \cos(\sin(\tan \pi x))^{1/2} \Rightarrow$

$$\begin{aligned} y' &= -\sin(\sin(\tan \pi x))^{1/2} \cdot \frac{d}{dx} (\sin(\tan \pi x))^{1/2} = -\sin(\sin(\tan \pi x))^{1/2} \cdot \frac{1}{2} (\sin(\tan \pi x))^{-1/2} \cdot \frac{d}{dx} (\sin(\tan \pi x)) \\ &= \frac{-\sin \sqrt{\sin(\tan \pi x)}}{2 \sqrt{\sin(\tan \pi x)}} \cdot \cos(\tan \pi x) \cdot \frac{d}{dx} \tan \pi x = \frac{-\sin \sqrt{\sin(\tan \pi x)}}{2 \sqrt{\sin(\tan \pi x)}} \cdot \cos(\tan \pi x) \cdot \sec^2(\pi x) \cdot \pi \\ &= \frac{-\pi \cos(\tan \pi x) \sec^2(\pi x) \sin \sqrt{\sin(\tan \pi x)}}{2 \sqrt{\sin(\tan \pi x)}} \end{aligned}$$

48. $y = [x + (x + \sin^2 x)^3]^4 \Rightarrow y' = 4[x + (x + \sin^2 x)^3]^3 \cdot [1 + 3(x + \sin^2 x)^2 \cdot (1 + 2 \sin x \cos x)]$

49. $y = \cos(\sin 3\theta) \Rightarrow y' = -\sin(\sin 3\theta) \cdot (\cos 3\theta) \cdot 3 = -3 \cos 3\theta \sin(\sin 3\theta) \Rightarrow$
 $y'' = -3 [(\cos 3\theta) \cos(\sin 3\theta)(\cos 3\theta) \cdot 3 + \sin(\sin 3\theta)(-\sin 3\theta) \cdot 3] = -9 \cos^2(3\theta) \cos(\sin 3\theta) + 9(\sin 3\theta) \sin(\sin 3\theta)$

50. $y = (1 + \sqrt{x})^3 \Rightarrow y' = 3(1 + \sqrt{x})^2 \left(\frac{1}{2\sqrt{x}} \right) = \frac{3(1 + \sqrt{x})^2}{2\sqrt{x}} \Rightarrow$
 $y'' = \frac{2\sqrt{x} \cdot 3 \cdot 2(1 + \sqrt{x})^1 \cdot \frac{1}{2\sqrt{x}} - 3(1 + \sqrt{x})^2 \cdot 2 \cdot \frac{1}{2\sqrt{x}}}{(2\sqrt{x})^2}$
 $= \frac{6\sqrt{x}(1 + \sqrt{x}) \cdot \frac{1}{\sqrt{x}} - 3(1 + 2\sqrt{x} + x) \cdot \frac{1}{\sqrt{x}}}{4x} \cdot \frac{\sqrt{x}}{\sqrt{x}} = \frac{6\sqrt{x} + 6x - 3 - 6\sqrt{x} - 3x}{4x\sqrt{x}}$
 $= \frac{3x - 3}{4x\sqrt{x}} \text{ or } \frac{3(x - 1)}{4x^{3/2}}$

51. $y = \sqrt{\cos x} \Rightarrow y' = \frac{1}{2\sqrt{\cos x}}(-\sin x) = -\frac{\sin x}{2\sqrt{\cos x}}$. With $y' = \frac{-\sin x}{2\sqrt{\cos x}}$, we get

$$y'' = \frac{2\sqrt{\cos x} \cdot (-\cos x) - (-\sin x) \left(2 \cdot \frac{1}{2\sqrt{\cos x}} (-\sin x) \right)}{(2\sqrt{\cos x})^2} = \frac{-2 \cos x \sqrt{\cos x} - \frac{\sin^2 x}{\sqrt{\cos x}}}{4 \cos x} \cdot \frac{\sqrt{\cos x}}{\sqrt{\cos x}}$$

$$= \frac{-2 \cos x \cdot \cos x - \sin^2 x}{4 \cos x \sqrt{\cos x}} = -\frac{2 \cos^2 x + \sin^2 x}{4(\cos x)^{3/2}}$$

Using the identity $\sin^2 x + \cos^2 x = 1$, the answer may be written as $-\frac{1 + \cos^2 x}{4(\cos x)^{3/2}}$.

52. $y = \frac{4x}{\sqrt{x+1}} \Rightarrow$
 $y' = \frac{\sqrt{x+1} \cdot 4 - 4x \cdot \frac{1}{2}(x+1)^{-1/2}}{(\sqrt{x+1})^2} = \frac{4\sqrt{x+1} - 2x/\sqrt{x+1}}{x+1} = \frac{4(x+1) - 2x}{(x+1)^{3/2}} = \frac{2x+4}{(x+1)^{3/2}} \Rightarrow$
 $y'' = \frac{(x+1)^{3/2} \cdot 2 - (2x+4) \cdot \frac{3}{2}(x+1)^{1/2}}{[(x+1)^{3/2}]^2} = \frac{(x+1)^{1/2}[2(x+1) - 3(x+2)]}{(x+1)^3} = \frac{2x+2-3x-6}{(x+1)^{5/2}} = \frac{-x-4}{(x+1)^{5/2}}$

53. $y = (3x-1)^{-6} \Rightarrow y' = -6(3x-1)^{-7} \cdot 3 = -18(3x-1)^{-7}$. At $(0, 1)$, $y' = -18(-1)^{-7} = -18(-1) = 18$, and an equation of the tangent line is $y - 1 = 18(x - 0)$, or $y = 18x + 1$.

54. $y = \sqrt{1+x^3} = (1+x^3)^{1/2} \Rightarrow y' = \frac{1}{2}(1+x^3)^{-1/2} \cdot 3x^2 = \frac{3x^2}{2\sqrt{1+x^3}}$. At $(2, 3)$, $y' = \frac{3 \cdot 4}{2\sqrt{9}} = 2$, and an equation of the tangent line is $y - 3 = 2(x - 2)$, or $y = 2x - 1$.

55. $y = \sin(\sin x) \Rightarrow y' = \cos(\sin x) \cdot \cos x$. At $(\pi, 0)$, $y' = \cos(\sin \pi) \cdot \cos \pi = \cos(0) \cdot (-1) = 1(-1) = -1$, and an equation of the tangent line is $y - 0 = -1(x - \pi)$, or $y = -x + \pi$.

56. $y = \sin^2 x \cos x \Rightarrow y' = \sin^2 x(-\sin x) + \cos x(2 \sin x \cos x)$. At $(\pi/2, 0)$, $y' = 1(-1) + 0 = -1$, and an equation of the tangent line is $y - 0 = -1(x - \frac{\pi}{2})$, or $y = -x + \frac{\pi}{2}$.

57. (a) $y = f(x) = \tan\left(\frac{\pi}{4}x^2\right) \Rightarrow f'(x) = \sec^2\left(\frac{\pi}{4}x^2\right)(2 \cdot \frac{\pi}{4}x)$.

The slope of the tangent at $(1, 1)$ is thus

$$f'(1) = \sec^2\left(\frac{\pi}{4}\left(\frac{\pi}{2}\right)\right) = 2 \cdot \frac{\pi}{2} = \pi, \text{ and its equation}$$

is $y - 1 = \pi(x - 1)$ or $y = \pi x - \pi + 1$.

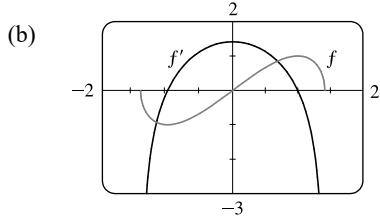
58. (a) For $x > 0$, $|x| = x$, and $y = f(x) = \frac{x}{\sqrt{2-x^2}} \Rightarrow$

$$\begin{aligned} f'(x) &= \frac{\sqrt{2-x^2}(1) - x\left(\frac{1}{2}\right)(2-x^2)^{-1/2}(-2x)}{(\sqrt{2-x^2})^2} \cdot \frac{(2-x^2)^{1/2}}{(2-x^2)^{1/2}} \\ &= \frac{(2-x^2)+x^2}{(2-x^2)^{3/2}} = \frac{2}{(2-x^2)^{3/2}} \end{aligned}$$

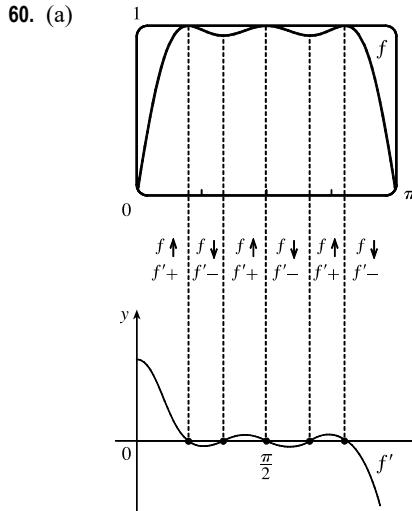
So at $(1, 1)$, the slope of the tangent line is $f'(1) = 2$ and its equation is $y - 1 = 2(x - 1)$ or $y = 2x - 1$.

59. (a) $f(x) = x\sqrt{2-x^2} = x(2-x^2)^{1/2} \Rightarrow$

$$f'(x) = x \cdot \frac{1}{2}(2-x^2)^{-1/2}(-2x) + (2-x^2)^{1/2} \cdot 1 = (2-x^2)^{-1/2}[-x^2 + (2-x^2)] = \frac{2-2x^2}{\sqrt{2-x^2}}$$



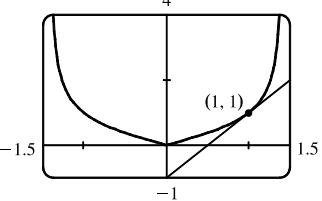
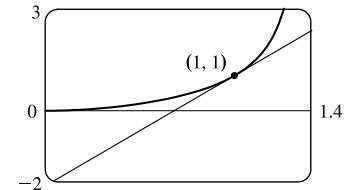
$f' = 0$ when f has a horizontal tangent line, f' is negative when f is decreasing, and f' is positive when f is increasing.



From the graph of f , we see that there are 5 horizontal tangents, so there must be 5 zeros on the graph of f' . From the symmetry of the graph of f , we must have the graph of f' as high at $x = 0$ as it is low at $x = \pi$. The intervals of increase and decrease as well as the signs of f' are indicated in the figure.

(b) $f(x) = \sin(x + \sin 2x) \Rightarrow$

$$f'(x) = \cos(x + \sin 2x) \cdot \frac{d}{dx}(x + \sin 2x) = \cos(x + \sin 2x)(1 + 2 \cos 2x)$$



- 61.** For the tangent line to be horizontal, $f'(x) = 0$. $f(x) = 2 \sin x + \sin^2 x \Rightarrow f'(x) = 2 \cos x + 2 \sin x \cos x = 0 \Leftrightarrow 2 \cos x(1 + \sin x) = 0 \Leftrightarrow \cos x = 0 \text{ or } \sin x = -1$, so $x = \frac{\pi}{2} + 2n\pi$ or $\frac{3\pi}{2} + 2n\pi$, where n is any integer. Now $f(\frac{\pi}{2}) = 3$ and $f(\frac{3\pi}{2}) = -1$, so the points on the curve with a horizontal tangent are $(\frac{\pi}{2} + 2n\pi, 3)$ and $(\frac{3\pi}{2} + 2n\pi, -1)$, where n is any integer.

- 62.** $y = \sqrt{1+2x} \Rightarrow y' = \frac{1}{2}(1+2x)^{-1/2} \cdot 2 = \frac{1}{\sqrt{1+2x}}$. The line $6x + 2y = 1$ (or $y = -3x + \frac{1}{2}$) has slope -3 , so the tangent line perpendicular to it must have slope $\frac{1}{3}$. Thus, $\frac{1}{3} = \frac{1}{\sqrt{1+2x}} \Leftrightarrow \sqrt{1+2x} = 3 \Rightarrow 1+2x=9 \Leftrightarrow 2x=8 \Leftrightarrow x=4$. When $x=4$, $y=\sqrt{1+2(4)}=3$, so the point is $(4, 3)$.

- 63.** $F(x) = f(g(x)) \Rightarrow F'(x) = f'(g(x)) \cdot g'(x)$, so $F'(5) = f'(g(5)) \cdot g'(5) = f'(-2) \cdot 6 = 4 \cdot 6 = 24$.

- 64.** $h(x) = \sqrt{4+3f(x)} \Rightarrow h'(x) = \frac{1}{2}(4+3f(x))^{-1/2} \cdot 3f'(x)$, so

$$h'(1) = \frac{1}{2}(4+3f(1))^{-1/2} \cdot 3f'(1) = \frac{1}{2}(4+3 \cdot 7)^{-1/2} \cdot 3 \cdot 4 = \frac{6}{\sqrt{25}} = \frac{6}{5}.$$

- 65.** (a) $h(x) = f(g(x)) \Rightarrow h'(x) = f'(g(x)) \cdot g'(x)$, so $h'(1) = f'(g(1)) \cdot g'(1) = f'(2) \cdot 6 = 5 \cdot 6 = 30$.

- (b) $H(x) = g(f(x)) \Rightarrow H'(x) = g'(f(x)) \cdot f'(x)$, so $H'(1) = g'(f(1)) \cdot f'(1) = g'(3) \cdot 4 = 9 \cdot 4 = 36$.

- 66.** (a) $F(x) = f(f(x)) \Rightarrow F'(x) = f'(f(x)) \cdot f'(x)$, so $F'(2) = f'(f(2)) \cdot f'(2) = f'(1) \cdot 5 = 4 \cdot 5 = 20$.

- (b) $G(x) = g(g(x)) \Rightarrow G'(x) = g'(g(x)) \cdot g'(x)$, so $G'(3) = g'(g(3)) \cdot g'(3) = g'(2) \cdot 9 = 7 \cdot 9 = 63$.

- 67.** (a) From the graphs of f and g , we obtain the following values: $g(1) = 4$ since the point $(1, 4)$ is on the graph of g ;

$f'(4) = -\frac{1}{4}$ since the slope of the line segment between $(2, 4)$ and $(6, 3)$ is $\frac{3-4}{6-2} = -\frac{1}{4}$; and $g'(1) = -1$ since the slope

of the line segment between $(0, 5)$ and $(3, 2)$ is $\frac{2-5}{3-0} = -1$. Now $u(x) = f(g(x))$, so

$$u'(1) = f'(g(1))g'(1) = f'(4)g'(1) = -\frac{1}{4}(-1) = \frac{1}{4}.$$

- (b) From the graphs of f and g , we obtain the following values: $f(1) = 2$ since the point $(1, 2)$ is on the graph of f ;

$g'(2) = g'(1) = -1$ [see part (a)]; and $f'(1) = 2$ since the slope of the line segment between $(0, 0)$ and $(2, 4)$ is

$\frac{4-0}{2-0} = 2$. Now $v(x) = g(f(x))$, so $v'(1) = g'(f(1))f'(1) = g'(2)f'(1) = -1(2) = -2$.

- (c) From part (a), we have $g(1) = 4$ and $g'(1) = -1$. From the graph of g we obtain $g'(4) = \frac{1}{2}$ since the slope of

the line segment between $(3, 2)$ and $(7, 4)$ is $\frac{4-2}{7-3} = \frac{1}{2}$. Now $w(x) = g(g(x))$, so

$$w'(1) = g'(g(1))g'(1) = g'(4)g'(1) = \frac{1}{2}(-1) = -\frac{1}{2}.$$

- 68.** (a) $h(x) = f(f(x)) \Rightarrow h'(x) = f'(f(x))f'(x)$. So $h'(2) = f'(f(2))f'(2) = f'(1)f'(2) \approx (-1)(-1) = 1$.

- (b) $g(x) = f(x^2) \Rightarrow g'(x) = f'(x^2) \cdot \frac{d}{dx}(x^2) = f'(x^2)(2x)$. So $g'(2) = f'(2^2)(2 \cdot 2) = 4f'(4) \approx 4(2) = 8$.

69. The point $(3, 2)$ is on the graph of f , so $f(3) = 2$. The tangent line at $(3, 2)$ has slope $\frac{\Delta y}{\Delta x} = \frac{-4}{6} = -\frac{2}{3}$.

$$\begin{aligned} g(x) &= \sqrt{f(x)} \Rightarrow g'(x) = \frac{1}{2}[f(x)]^{-1/2} \cdot f'(x) \Rightarrow \\ g'(3) &= \frac{1}{2}[f(3)]^{-1/2} \cdot f'(3) = \frac{1}{2}(2)^{-1/2}(-\frac{2}{3}) = -\frac{1}{3\sqrt{2}} \text{ or } -\frac{1}{6}\sqrt{2}. \end{aligned}$$

70. (a) $F(x) = f(x^\alpha) \Rightarrow F'(x) = f'(x^\alpha) \frac{d}{dx}(x^\alpha) = f'(x^\alpha)\alpha x^{\alpha-1}$

$$(b) G(x) = [f(x)]^\alpha \Rightarrow G'(x) = \alpha [f(x)]^{\alpha-1} f'(x)$$

71. $r(x) = f(g(h(x))) \Rightarrow r'(x) = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$, so

$$r'(1) = f'(g(h(1))) \cdot g'(h(1)) \cdot h'(1) = f'(g(2)) \cdot g'(2) \cdot 4 = f'(3) \cdot 5 \cdot 4 = 6 \cdot 5 \cdot 4 = 120$$

72. $f(x) = xg(x^2) \Rightarrow f'(x) = xg'(x^2)2x + g(x^2) \cdot 1 = 2x^2g'(x^2) + g(x^2) \Rightarrow$

$$f''(x) = 2x^2g''(x^2)2x + g'(x^2)4x + g'(x^2)2x = 4x^3g''(x^2) + 4xg'(x^2) + 2xg'(x^2) = 6xg'(x^2) + 4x^3g''(x^2)$$

73. $F(x) = f(3f(4f(x))) \Rightarrow$

$$\begin{aligned} F'(x) &= f'(3f(4f(x))) \cdot \frac{d}{dx}(3f(4f(x))) = f'(3f(4f(x))) \cdot 3f'(4f(x)) \cdot \frac{d}{dx}(4f(x)) \\ &= f'(3f(4f(x))) \cdot 3f'(4f(x)) \cdot 4f'(x), \text{ so} \end{aligned}$$

$$F'(0) = f'(3f(4f(0))) \cdot 3f'(4f(0)) \cdot 4f'(0) = f'(3f(4 \cdot 0)) \cdot 3f'(4 \cdot 0) \cdot 4 \cdot 2 = f'(3 \cdot 0) \cdot 3 \cdot 2 \cdot 4 \cdot 2 = 2 \cdot 3 \cdot 2 \cdot 4 \cdot 2 = 96.$$

74. $F(x) = f(xf(xf(x))) \Rightarrow$

$$\begin{aligned} F'(x) &= f'(xf(xf(x))) \cdot \frac{d}{dx}(xf(xf(x))) = f'(xf(xf(x))) \cdot \left[x \cdot f'(xf(x)) \cdot \frac{d}{dx}(xf(x)) + f(xf(x)) \cdot 1 \right] \\ &= f'(xf(xf(x))) \cdot [xf'(xf(x)) \cdot (xf'(x) + f(x) \cdot 1) + f(xf(x))], \text{ so} \end{aligned}$$

$$\begin{aligned} F'(1) &= f'(f(f(1))) \cdot [f'(f(1)) \cdot (f'(1) + f(1)) + f(f(1))] = f'(f(2)) \cdot [f'(2) \cdot (4+2) + f(2)] \\ &= f'(3) \cdot [5 \cdot 6 + 3] = 6 \cdot 33 = 198. \end{aligned}$$

75. Let $f(x) = \cos x$. Then $Df(2x) = 2f'(2x)$, $D^2f(2x) = 2^2f''(2x)$, $D^3f(2x) = 2^3f'''(2x)$, ...,

$D^{(n)}f(2x) = 2^n f^{(n)}(2x)$. Since the derivatives of $\cos x$ occur in a cycle of four, and since $103 = 4(25) + 3$, we have

$$f^{(103)}(x) = f^{(3)}(x) = \sin x \text{ and } D^{103} \cos 2x = 2^{103} f^{(103)}(2x) = 2^{103} \sin 2x.$$

76. Let $f(x) = x \sin \pi x$ and $h(x) = \sin \pi x$, so $f(x) = xh(x)$. Then $Df(x) = xh'(x) + h(x)$,

$$D^2f(x) = xh''(x) + h'(x) + h'(x) = xh''(x) + 2h'(x)$$

$D^n f(x) = xh^{(n)}(x) + nh^{(n-1)}(x)$. We now find a pattern for the derivatives of h : $h'(x) = \pi \cos \pi x$, $h''(x) = -\pi^2 \sin \pi x$,

$h'''(x) = -\pi^3 \cos \pi x$, $h^4(x) = \pi^4 \sin \pi x$, and so on. Since $34 = 4(8) + 2$, we have $h^{(34)}(x) = -\pi^{34} \sin \pi x$ and

$h^{(35)}(x) = -\pi^{35} \cos \pi x$. Thus,

$$D^{35}f(x) = xh^{(35)}(x) + 35h^{(34)}(x) = x(-\pi^{35} \cos \pi x) + 35(-\pi^{34} \sin \pi x) = -\pi^{35}x \cos \pi x - 35\pi^{34} \sin \pi x.$$

77. $s(t) = 10 + \frac{1}{4} \sin(10\pi t) \Rightarrow$ the velocity after t seconds is $v(t) = s'(t) = \frac{1}{4} \cos(10\pi t)(10\pi) = \frac{5\pi}{2} \cos(10\pi t)$ cm/s.

78. (a) $s = A \cos(\omega t + \delta) \Rightarrow$ velocity $= s' = -\omega A \sin(\omega t + \delta)$.

(b) If $A \neq 0$ and $\omega \neq 0$, then $s' = 0 \Leftrightarrow \sin(\omega t + \delta) = 0 \Leftrightarrow \omega t + \delta = n\pi \Leftrightarrow t = \frac{n\pi - \delta}{\omega}$, n an integer.

79. (a) $B(t) = 4.0 + 0.35 \sin\left(\frac{2\pi t}{5.4}\right) \Rightarrow \frac{dB}{dt} = \left(0.35 \cos\frac{2\pi t}{5.4}\right)\left(\frac{2\pi}{5.4}\right) = \frac{0.7\pi}{5.4} \cos\frac{2\pi t}{5.4} = \frac{7\pi}{54} \cos\frac{2\pi t}{5.4}$

(b) At $t = 1$, $\frac{dB}{dt} = \frac{7\pi}{54} \cos\frac{2\pi}{5.4} \approx 0.16$.

80. $L(t) = 12 + 2.8 \sin\left[\frac{2\pi}{365}(t - 80)\right] \Rightarrow L'(t) = 2.8 \cos\left[\frac{2\pi}{365}(t - 80)\right]\left(\frac{2\pi}{365}\right)$.

On March 21, $t = 80$, and $L'(80) \approx 0.0482$ hours per day. On May 21, $t = 141$, and $L'(141) \approx 0.02398$, which is approximately one-half of $L'(80)$.

81. By the Chain Rule, $a(t) = \frac{dv}{dt} = \frac{dv}{ds} \frac{ds}{dt} = \frac{dv}{ds} v(t) = v(t) \frac{dv}{ds}$. The derivative dv/dt is the rate of change of the velocity with respect to time (in other words, the acceleration) whereas the derivative dv/ds is the rate of change of the velocity with respect to the displacement.

82. (a) The derivative dV/dr represents the rate of change of the volume with respect to the radius and the derivative dV/dt represents the rate of change of the volume with respect to time.

(b) Since $V = \frac{4}{3}\pi r^3$, $\frac{dV}{dt} = \frac{dV}{dr} \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$.

83. (a) If f is even, then $f(x) = f(-x)$. Using the Chain Rule to differentiate this equation, we get

$$f'(x) = f'(-x) \frac{d}{dx}(-x) = -f'(-x). \text{ Thus, } f'(-x) = -f'(x), \text{ so } f' \text{ is odd.}$$

(b) If f is odd, then $f(x) = -f(-x)$. Differentiating this equation, we get $f'(x) = -f'(-x)(-1) = f'(-x)$, so f' is even.

$$\begin{aligned} 84. \left[\frac{f(x)}{g(x)} \right]' &= \{f(x)[g(x)]^{-1}\}' = f'(x)[g(x)]^{-1} + (-1)[g(x)]^{-2}g'(x)f(x) \\ &= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{[g(x)]^2} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} \end{aligned}$$

This is an alternative derivation of the formula in the Quotient Rule. But part of the purpose of the Quotient Rule is to show that if f and g are differentiable, so is f/g . The proof in Section 2.3 does that; this one doesn't.

85. Since $\theta^\circ = (\frac{\pi}{180})\theta$ rad, we have $\frac{d}{d\theta}(\sin \theta^\circ) = \frac{d}{d\theta}(\sin \frac{\pi}{180}\theta) = \frac{\pi}{180} \cos \frac{\pi}{180}\theta = \frac{\pi}{180} \cos \theta^\circ$.

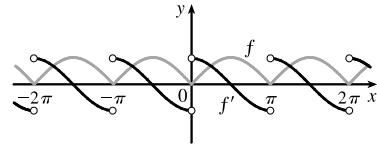
86. (a) $f(x) = |x| = \sqrt{x^2} = (x^2)^{1/2} \Rightarrow f'(x) = \frac{1}{2}(x^2)^{-1/2}(2x) = x/\sqrt{x^2} = x/|x|$ for $x \neq 0$.

f is not differentiable at $x = 0$.

(b) $f(x) = |\sin x| = \sqrt{\sin^2 x} \Rightarrow$

$$\begin{aligned} f'(x) &= \frac{1}{2}(\sin^2 x)^{-1/2} 2 \sin x \cos x = \frac{\sin x}{|\sin x|} \cos x \\ &= \begin{cases} \cos x & \text{if } \sin x > 0 \\ -\cos x & \text{if } \sin x < 0 \end{cases} \end{aligned}$$

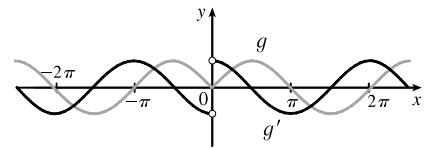
f is not differentiable when $x = n\pi$, n an integer.



(c) $g(x) = \sin|x| = \sin\sqrt{x^2} \Rightarrow$

$$g'(x) = \cos|x| \cdot \frac{x}{|x|} = \frac{x}{|x|} \cos x = \begin{cases} \cos x & \text{if } x > 0 \\ -\cos x & \text{if } x < 0 \end{cases}$$

g is not differentiable at 0.



87. Let $j(x) = g(h(x))$ so that $F(x) = f(g(h(x))) = f(j(x))$. By the Chain Rule, we have $j'(x) = g'(h(x)) \cdot h'(x)$ and, by the Chain Rule and substitution, we have $F'(x) = f'(j(x)) \cdot j'(x) = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$.

88. $F(x) = f(g(x)) \Rightarrow F'(x) = f'(g(x)) \cdot g'(x)$ by the Chain Rule. By the Product Rule and Chain Rule we have

$$\begin{aligned} F''(x) &= f'(g(x)) \cdot g''(x) + g'(x) \cdot \frac{d}{dx} f'(g(x)) = f'(g(x)) \cdot g''(x) + g'(x) \cdot f''(g(x)) g'(x) \\ &= f''(g(x))[g'(x)]^2 + f'(g(x)) \cdot g''(x) \end{aligned}$$

APPLIED PROJECT Where Should a Pilot Start Descent?

1. Condition (i) will hold if and only if all of the following four conditions hold:

(α) $P(0) = 0$

(β) $P'(0) = 0$ (for a smooth landing)

(γ) $P'(\ell) = 0$ (since the plane is cruising horizontally when it begins its descent)

(δ) $P(\ell) = h$.

First of all, condition α implies that $P(0) = d = 0$, so $P(x) = ax^3 + bx^2 + cx \Rightarrow P'(x) = 3ax^2 + 2bx + c$. But

$$P'(0) = c = 0 \text{ by condition } \beta. \text{ So } P'(\ell) = 3a\ell^2 + 2b\ell = \ell(3a\ell + 2b). \text{ Now by condition } \gamma, 3a\ell + 2b = 0 \Rightarrow a = -\frac{2b}{3\ell}.$$

Therefore, $P(x) = -\frac{2b}{3\ell}x^3 + bx^2$. Setting $P(\ell) = h$ for condition δ, we get $P(\ell) = -\frac{2b}{3\ell}\ell^3 + b\ell^2 = h \Rightarrow$

$$-\frac{2}{3}b\ell^2 + b\ell^2 = h \Rightarrow \frac{1}{3}b\ell^2 = h \Rightarrow b = \frac{3h}{\ell^2} \Rightarrow a = -\frac{2h}{\ell^3}. \text{ So } y = P(x) = -\frac{2h}{\ell^3}x^3 + \frac{3h}{\ell^2}x^2.$$

2. By condition (ii), $\frac{dx}{dt} = -v$ for all t , so $x(t) = \ell - vt$. Condition (iii) states that $\left| \frac{d^2y}{dt^2} \right| \leq k$. By the Chain Rule,

$$\text{we have } \frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} = -\frac{2h}{\ell^3}(3x^2) \frac{dx}{dt} + \frac{3h}{\ell^2}(2x) \frac{dx}{dt} = \frac{6hx^2v}{\ell^3} - \frac{6hvx}{\ell^2} \quad (\text{for } x \leq \ell) \Rightarrow$$

$$\frac{d^2y}{dt^2} = \frac{6hv}{\ell^3}(2x) \frac{dx}{dt} - \frac{6hv}{\ell^2} \frac{dx}{dt} = -\frac{12hv^2}{\ell^3}x + \frac{6hv^2}{\ell^2}. \text{ In particular, when } t = 0, x = \ell \text{ and so}$$

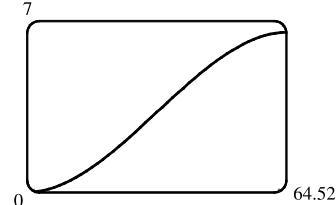
$$\left. \frac{d^2y}{dt^2} \right|_{t=0} = -\frac{12hv^2}{\ell^3}\ell + \frac{6hv^2}{\ell^2} = -\frac{6hv^2}{\ell^2}. \text{ Thus, } \left. \frac{d^2y}{dt^2} \right|_{t=0} = \frac{6hv^2}{\ell^2} \leq k. \text{ (This condition also follows from taking } x = 0).$$

3. We substitute $k = 1385 \text{ km/h}^2$, $h = 11,000 \text{ m} = 11 \text{ km}$, and $v = 480 \text{ km/h}$ into the result of part (b): $\frac{6(11)(480)^2}{l^2} \leq 1385 \implies l \geq 480\sqrt{\frac{66}{1385}} \approx 104.8 \text{ kilometers.}$

4. Substituting the values of h and ℓ in Problem 3 into

$$P(x) = -\frac{2h}{\ell^3}x^3 + \frac{3h}{\ell^2}x^2 \text{ gives us } P(x) = ax^3 + bx^2,$$

where $a \approx -4.937 \times 10^{-5}$ and $b \approx 4.78 \times 10^{-3}$.



2.6 Implicit Differentiation

1. (a) $\frac{d}{dx}(5x^2 - y^3) = \frac{d}{dx}(7) \Rightarrow 10x - 3y^2 y' = 0 \Rightarrow 3y^2 y' = 10x \Rightarrow y' = \frac{10x}{3y^2}$

(b) $5x^2 - y^3 = 7 \Rightarrow y^3 = 5x^2 - 7 \Rightarrow y = \sqrt[3]{5x^2 - 7}$, so $y' = \frac{1}{3}(5x^2 - 7)^{-2/3}(10x) = \frac{10x}{3(5x^2 - 7)^{2/3}}$

(c) From part (a), $y' = \frac{10x}{3y^2} = \frac{10x}{3(y^3)^{2/3}} = \frac{10x}{3(5x^2 - 7)^{2/3}}$, which agrees with part (b).

2. (a) $\frac{d}{dx}(6x^4 + y^5) = \frac{d}{dx}(2x) \Rightarrow 24x^3 + 5y^4 y' = 2 \Rightarrow 5y^4 y' = 2 - 24x^3 \Rightarrow y' = \frac{2 - 24x^3}{5y^4}$

(b) $6x^4 + y^5 = 2x \Rightarrow y^5 = 2x - 6x^4 \Rightarrow y = \sqrt[5]{2x - 6x^4}$, so $y' = \frac{1}{5}(2x - 6x^4)^{-4/5}(2 - 24x^3) = \frac{2 - 24x^3}{5(2x - 6x^4)^{4/5}}$

(c) From part (a), $y' = \frac{2 - 24x^3}{5y^4} = \frac{2 - 24x^3}{5(y^5)^{4/5}} = \frac{2 - 24x^3}{5(2x - 6x^4)^{4/5}}$, which agrees with part (b).

3. (a) $\frac{d}{dx}(\sqrt{x} + \sqrt{y}) = \frac{d}{dx}(1) \Rightarrow \frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}y' = 0 \Rightarrow \frac{1}{2\sqrt{y}}y' = -\frac{1}{2\sqrt{x}} \Rightarrow y' = -\frac{\sqrt{y}}{\sqrt{x}}$

(b) $\sqrt{x} + \sqrt{y} = 1 \Rightarrow \sqrt{y} = 1 - \sqrt{x} \Rightarrow y = (1 - \sqrt{x})^2 \Rightarrow y = 1 - 2\sqrt{x} + x$, so

$$y' = -2 \cdot \frac{1}{2}x^{-1/2} + 1 = 1 - \frac{1}{\sqrt{x}}.$$

(c) From part (a), $y' = -\frac{\sqrt{y}}{\sqrt{x}} = -\frac{1 - \sqrt{x}}{\sqrt{x}}$ [from part (b)] $= -\frac{1}{\sqrt{x}} + 1$, which agrees with part (b).

4. (a) $\frac{d}{dx}\left(\frac{2}{x} - \frac{1}{y}\right) = \frac{d}{dx}(4) \Rightarrow -2x^{-2} + y^{-2}y' = 0 \Rightarrow \frac{1}{y^2}y' = \frac{2}{x^2} \Rightarrow y' = \frac{2y^2}{x^2}$

(b) $\frac{2}{x} - \frac{1}{y} = 4 \Rightarrow \frac{1}{y} = \frac{2}{x} - 4 \Rightarrow \frac{1}{y} = \frac{2 - 4x}{x} \Rightarrow y = \frac{x}{2 - 4x}$, so

$$y' = \frac{(2 - 4x)(1) - x(-4)}{(2 - 4x)^2} = \frac{2}{(2 - 4x)^2} \quad \left[\text{or } \frac{1}{2(1 - 2x)^2}\right].$$

(c) From part (a), $y' = \frac{2y^2}{x^2} = \frac{2\left(\frac{x}{2-4x}\right)^2}{x^2}$ [from part (b)] $= \frac{2x^2}{x^2(2-4x)^2} = \frac{2}{(2-4x)^2}$, which agrees with part (b).

5. $\frac{d}{dx}(x^2 - 4xy + y^2) = \frac{d}{dx}(4) \Rightarrow 2x - 4[xy' + y(1)] + 2y y' = 0 \Rightarrow 2y y' - 4xy' = 4y - 2x \Rightarrow$

$$y'(y - 2x) = 2y - x \Rightarrow y' = \frac{2y - x}{y - 2x}$$

6. $\frac{d}{dx}(2x^2 + xy - y^2) = \frac{d}{dx}(2) \Rightarrow 4x + xy' + y(1) - 2y y' = 0 \Rightarrow xy' - 2y y' = -4x - y \Rightarrow$

$$(x - 2y)y' = -4x - y \Rightarrow y' = \frac{-4x - y}{x - 2y}$$

7. $\frac{d}{dx}(x^4 + x^2y^2 + y^3) = \frac{d}{dx}(5) \Rightarrow 4x^3 + x^2 \cdot 2y y' + y^2 \cdot 2x + 3y^2 y' = 0 \Rightarrow 2x^2 y y' + 3y^2 y' = -4x^3 - 2xy^2 \Rightarrow$

$$(2x^2 y + 3y^2)y' = -4x^3 - 2xy^2 \Rightarrow y' = \frac{-4x^3 - 2xy^2}{2x^2 y + 3y^2} = -\frac{2x(2x^2 + y^2)}{y(2x^2 + 3y)}$$

8. $\frac{d}{dx}(x^3 - xy^2 + y^3) = \frac{d}{dx}(1) \Rightarrow 3x^2 - x \cdot 2y y' - y^2 \cdot 1 + 3y^2 y' = 0 \Rightarrow 3y^2 y' - 2x y y' = y^2 - 3x^2 \Rightarrow$

$$(3y^2 - 2xy)y' = y^2 - 3x^2 \Rightarrow y' = \frac{y^2 - 3x^2}{3y^2 - 2xy} = \frac{y^2 - 3x^2}{y(3y - 2x)}$$

9. $\frac{d}{dx}\left(\frac{x^2}{x+y}\right) = \frac{d}{dx}(y^2 + 1) \Rightarrow \frac{(x+y)(2x) - x^2(1+y')}{(x+y)^2} = 2y y' \Rightarrow$

$$2x^2 + 2xy - x^2 - x^2 y' = 2y(x+y)^2 y' \Rightarrow x^2 + 2xy = 2y(x+y)^2 y' + x^2 y' \Rightarrow$$

$$x(x+2y) = [2y(x^2 + 2xy + y^2) + x^2]y' \Rightarrow y' = \frac{x(x+2y)}{2x^2 y + 4xy^2 + 2y^3 + x^2}$$

Or: Start by clearing fractions and then differentiate implicitly.

10. $\frac{d}{dx}(y^5 + x^2y^3) = \frac{d}{dx}(1 + x^4y) \Rightarrow 5y^4 y' + x^2 \cdot 3y^2 y' + y^3 \cdot 2x = 0 + x^4 y' + y \cdot 4x^3 \Rightarrow$

$$y'(5y^4 + 3x^2y^2 - x^4) = 4x^3y - 2xy^3 \Rightarrow y' = \frac{4x^3y - 2xy^3}{5y^4 + 3x^2y^2 - x^4}$$

11. $\frac{d}{dx}(\sin x + \cos y) = \frac{d}{dx}(2x - 3y) \Rightarrow \cos x - \sin y \cdot y' = 2 - 3y' \Rightarrow 3y' - \sin y \cdot y' = 2 - \cos x \Rightarrow$

$$y'(3 - \sin y) = 2 - \cos x \Rightarrow y' = \frac{2 - \cos x}{3 - \sin y}$$

12. $\frac{d}{dx}[y \sin(x^2)] = \frac{d}{dx}[x \sin(y^2)] \Rightarrow y \cos(x^2) \cdot 2x + \sin(x^2) \cdot y' = x \cos(y^2) \cdot 2y y' + \sin(y^2) \cdot 1 \Rightarrow$

$$y' [\sin(x^2) - 2xy \cos(y^2)] = \sin(y^2) - 2xy \cos(x^2) \Rightarrow y' = \frac{\sin(y^2) - 2xy \cos(x^2)}{\sin(x^2) - 2xy \cos(y^2)}$$

13. $\frac{d}{dx} \sin(x+y) = \frac{d}{dx}(\cos x + \cos y) \Rightarrow \cos(x+y) \cdot (1+y') = -\sin x - \sin y \cdot y' \Rightarrow$

$$\cos(x+y) + y' \cos(x+y) = -\sin x - \sin y \cdot y' \Rightarrow y' \cos(x+y) + \sin y \cdot y' = -\sin x - \cos(x+y) \Rightarrow$$

$$y' [\cos(x+y) + \sin y] = -[\sin x + \cos(x+y)] \Rightarrow y' = -\frac{\cos(x+y) + \sin x}{\cos(x+y) + \sin y}$$

14. $\frac{d}{dx} \tan(x-y) = \frac{d}{dx} (2xy^3 + 1) \Rightarrow \sec^2(x-y) \cdot (1-y') = 2x(3y^2 y') + y^3 \cdot 2 \Rightarrow$
 $\sec^2(x-y) - y' \sec^2(x-y) = 6xy^2 y' + 2y^3 \Rightarrow 6xy^2 y' + y' \sec^2(x-y) = \sec^2(x-y) - 2y^3 \Rightarrow$
 $y' [6xy^2 + \sec^2(x-y)] = \sec^2(x-y) - 2y^3 \Rightarrow y' = \frac{\sec^2(x-y) - 2y^3}{\sec^2(x-y) + 6xy^2}$

15. $\frac{d}{dx} \tan(x/y) = \frac{d}{dx} (x+y) \Rightarrow \sec^2(x/y) \cdot \frac{y \cdot 1 - x \cdot y'}{y^2} = 1 + y' \Rightarrow$
 $y \sec^2(x/y) - x \sec^2(x/y) \cdot y' = y^2 + y^2 y' \Rightarrow y \sec^2(x/y) - y^2 = y^2 y' + x \sec^2(x/y) \Rightarrow$
 $y \sec^2(x/y) - y^2 = [y^2 + x \sec^2(x/y)] \cdot y' \Rightarrow y' = \frac{y \sec^2(x/y) - y^2}{y^2 + x \sec^2(x/y)}$

16. $\frac{d}{dx} \sin(xy) = \frac{d}{dx} \cos(x+y) \Rightarrow \cos(xy) \cdot (xy' + y \cdot 1) = -\sin(x+y) \cdot (1+y') \Rightarrow$
 $x \cos(xy) y' + y \cos(xy) = -\sin(x+y) - y' \sin(x+y) \Rightarrow$
 $x \cos(xy) y' + y' \sin(x+y) = -y \cos(xy) - \sin(x+y) \Rightarrow$
 $[x \cos(xy) + \sin(x+y)] y' = -1 [y \cos(xy) + \sin(x+y)] \Rightarrow y' = -\frac{y \cos(xy) + \sin(x+y)}{x \cos(xy) + \sin(x+y)}$

17. $\frac{d}{dx} \sqrt{x+y} = \frac{d}{dx} (x^4 + y^4) \Rightarrow \frac{1}{2}(x+y)^{-1/2} (1+y') = 4x^3 + 4y^3 y' \Rightarrow$
 $\frac{1}{2\sqrt{x+y}} + \frac{1}{2\sqrt{x+y}} y' = 4x^3 + 4y^3 y' \Rightarrow \frac{1}{2\sqrt{x+y}} - 4x^3 = 4y^3 y' - \frac{1}{2\sqrt{x+y}} y' \Rightarrow$
 $\frac{1 - 8x^3 \sqrt{x+y}}{2\sqrt{x+y}} = \frac{8y^3 \sqrt{x+y} - 1}{2\sqrt{x+y}} y' \Rightarrow y' = \frac{1 - 8x^3 \sqrt{x+y}}{8y^3 \sqrt{x+y} - 1}$

18. $\frac{d}{dx} (\sin x \cos y) = \frac{d}{dx} (x^2 - 5y) \Rightarrow \sin x (-\sin y) \cdot y' + \cos y (\cos x) = 2x - 5y' \Rightarrow$
 $5y' - \sin x \sin y \cdot y' = 2x - \cos x \cos y \Rightarrow y' (5 - \sin x \sin y) = 2x - \cos x \cos y \Rightarrow y' = \frac{2x - \cos x \cos y}{5 - \sin x \sin y}$

19. $\frac{d}{dx} \sqrt{xy} = \frac{d}{dx} (1+x^2 y) \Rightarrow \frac{1}{2}(xy)^{-1/2} (xy' + y \cdot 1) = 0 + x^2 y' + y \cdot 2x \Rightarrow$
 $\frac{x}{2\sqrt{xy}} y' + \frac{y}{2\sqrt{xy}} = x^2 y' + 2xy \Rightarrow y' \left(\frac{x}{2\sqrt{xy}} - x^2 \right) = 2xy - \frac{y}{2\sqrt{xy}} \Rightarrow$
 $y' \left(\frac{x - 2x^2 \sqrt{xy}}{2\sqrt{xy}} \right) = \frac{4xy \sqrt{xy} - y}{2\sqrt{xy}} \Rightarrow y' = \frac{4xy \sqrt{xy} - y}{x - 2x^2 \sqrt{xy}}$

20. $\frac{d}{dx} (xy) = \frac{d}{dx} \sqrt{x^2 + y^2} \Rightarrow xy' + y(1) = \frac{1}{2}(x^2 + y^2)^{-1/2} (2x + 2y y') \Rightarrow$
 $xy' + y = \frac{x}{\sqrt{x^2 + y^2}} + \frac{y}{\sqrt{x^2 + y^2}} y' \Rightarrow xy' - \frac{y}{\sqrt{x^2 + y^2}} y' = \frac{x}{\sqrt{x^2 + y^2}} - y \Rightarrow$
 $\frac{x\sqrt{x^2 + y^2} - y}{\sqrt{x^2 + y^2}} y' = \frac{x - y\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2}} \Rightarrow y' = \frac{x - y\sqrt{x^2 + y^2}}{x\sqrt{x^2 + y^2} - y}$

21. $\frac{d}{dx} \{f(x) + x^2[f(x)]^3\} = \frac{d}{dx} (10) \Rightarrow f'(x) + x^2 \cdot 3[f(x)]^2 \cdot f'(x) + [f(x)]^3 \cdot 2x = 0.$ If $x = 1$, we have

$$f'(1) + 1^2 \cdot 3[f(1)]^2 \cdot f'(1) + [f(1)]^3 \cdot 2(1) = 0 \Rightarrow f'(1) + 1 \cdot 3 \cdot 2^2 \cdot f'(1) + 2^3 \cdot 2 = 0 \Rightarrow$$

$$f'(1) + 12f'(1) = -16 \Rightarrow 13f'(1) = -16 \Rightarrow f'(1) = -\frac{16}{13}.$$

22. $\frac{d}{dx} [g(x) + x \sin g(x)] = \frac{d}{dx} (x^2) \Rightarrow g'(x) + x \cos g(x) \cdot g'(x) + \sin g(x) \cdot 1 = 2x.$ If $x = 0$, we have

$$g'(0) + 0 + \sin g(0) = 2(0) \Rightarrow g'(0) + \sin 0 = 0 \Rightarrow g'(0) + 0 = 0 \Rightarrow g'(0) = 0.$$

23. $\frac{d}{dy} (x^4y^2 - x^3y + 2xy^3) = \frac{d}{dy} (0) \Rightarrow x^4 \cdot 2y + y^2 \cdot 4x^3 \cdot x' - (x^3 \cdot 1 + y \cdot 3x^2 \cdot x') + 2(x \cdot 3y^2 + y^3 \cdot x') = 0 \Rightarrow$

$$4x^3y^2 \cdot x' - 3x^2y \cdot x' + 2y^3 \cdot x' = -2x^4y + x^3 - 6xy^2 \Rightarrow (4x^3y^2 - 3x^2y + 2y^3) \cdot x' = -2x^4y + x^3 - 6xy^2 \Rightarrow$$

$$x' = \frac{dx}{dy} = \frac{-2x^4y + x^3 - 6xy^2}{4x^3y^2 - 3x^2y + 2y^3}$$

24. $\frac{d}{dy} (y \sec x) = \frac{d}{dy} (x \tan y) \Rightarrow y \cdot \sec x \tan x \cdot x' + \sec x \cdot 1 = x \cdot \sec^2 y + \tan y \cdot x' \Rightarrow$

$$y \sec x \tan x \cdot x' - \tan y \cdot x' = x \sec^2 y - \sec x \Rightarrow (y \sec x \tan x - \tan y) \cdot x' = x \sec^2 y - \sec x \Rightarrow$$

$$x' = \frac{dx}{dy} = \frac{x \sec^2 y - \sec x}{y \sec x \tan x - \tan y}$$

25. $y \sin 2x = x \cos 2y \Rightarrow y \cdot \cos 2x \cdot 2 + \sin 2x \cdot y' = x(-\sin 2y \cdot 2y') + \cos(2y) \cdot 1 \Rightarrow$

$$\sin 2x \cdot y' + 2x \sin 2y \cdot y' = -2y \cos 2x + \cos 2y \Rightarrow y'(\sin 2x + 2x \sin 2y) = -2y \cos 2x + \cos 2y \Rightarrow$$

$y' = \frac{-2y \cos 2x + \cos 2y}{\sin 2x + 2x \sin 2y}.$ When $x = \frac{\pi}{2}$ and $y = \frac{\pi}{4}$, we have $y' = \frac{(-\pi/2)(-1) + 0}{0 + \pi \cdot 1} = \frac{\pi/2}{\pi} = \frac{1}{2}$, so an equation of the

tangent line is $y - \frac{\pi}{4} = \frac{1}{2}(x - \frac{\pi}{2})$, or $y = \frac{1}{2}x$.

26. $\tan(x+y) + \sec(x-y) = 2 \Rightarrow \sec^2(x+y) \cdot (1+y') + \sec(x-y) \tan(x-y) \cdot (1-y') = 0 \Rightarrow$

$$\sec^2(x+y) + y' \cdot \sec^2(x+y) + \sec(x-y) \tan(x-y) - y' \cdot \sec(x-y) \tan(x-y) = 0 \Rightarrow$$

$$\sec^2(x+y) + \sec(x-y) \tan(x-y) = y' \cdot \sec(x-y) \tan(x-y) - y' \cdot \sec^2(x+y) \Rightarrow$$

$$\sec^2(x+y) + \sec(x-y) \tan(x-y) = y' [\sec(x-y) \tan(x-y) - \sec^2(x+y)] \Rightarrow$$

$$y' = \frac{\sec(x-y) \tan(x-y) + \sec^2(x+y)}{\sec(x-y) \tan(x-y) - \sec^2(x+y)}.$$

When $x = \frac{\pi}{8}$ and $y = \frac{\pi}{8}$, we have

$$y' = \frac{\sec\left(\frac{\pi}{8} - \frac{\pi}{8}\right) \tan\left(\frac{\pi}{8} - \frac{\pi}{8}\right) + \sec^2\left(\frac{\pi}{8} + \frac{\pi}{8}\right)}{\sec\left(\frac{\pi}{8} - \frac{\pi}{8}\right) \tan\left(\frac{\pi}{8} - \frac{\pi}{8}\right) - \sec^2\left(\frac{\pi}{8} + \frac{\pi}{8}\right)} = \frac{\sec 0 \tan 0 + \sec^2\left(\frac{\pi}{4}\right)}{\sec 0 \tan 0 - \sec^2\left(\frac{\pi}{4}\right)} = \frac{0 + (\sqrt{2})^2}{0 - (\sqrt{2})^2} = -1,$$
 so an equation of the

tangent line is $y - \frac{\pi}{8} = -1(x - \frac{\pi}{8})$, or $y = -x + \frac{\pi}{4}$.

27. $x^{2/3} + y^{2/3} = 4 \Rightarrow \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0 \Rightarrow \frac{1}{\sqrt[3]{x}} + \frac{y'}{\sqrt[3]{y}} = 0 \Rightarrow y' = -\frac{\sqrt[3]{y}}{\sqrt[3]{x}}.$

When $x = -3\sqrt{3}$ and $y = 1$, we have $y' = -\frac{1}{(-3\sqrt{3})^{1/3}} = -\frac{1}{(-3^{3/2})^{1/3}} = \frac{1}{3^{1/2}} = \frac{1}{\sqrt{3}}$, so an equation of the tangent line is $y - 1 = \frac{1}{\sqrt{3}}(x + 3\sqrt{3})$ or $y = \frac{1}{\sqrt{3}}x + 4$.

28. $y^2(6-x) = x^3 \Rightarrow y^2(-1) + (6-x) \cdot 2yy' = 3x^2 \Rightarrow y' \cdot 2y(6-x) = 3x^2 + y^2 \Rightarrow y' = \frac{3x^2 + y^2}{2y(6-x)}.$

When $x = 2$ and $y = \sqrt{2}$, we have $y' = \frac{3(2)^2 + (\sqrt{2})^2}{2\sqrt{2}(6-2)} = \frac{12+2}{2\sqrt{2} \cdot 4} = \frac{14}{8\sqrt{2}} = \frac{7}{4\sqrt{2}}$, so an equation of the tangent line is $y - \sqrt{2} = \frac{7}{4\sqrt{2}}(x - 2)$, or $y = \frac{7}{4\sqrt{2}}x - \frac{7}{2\sqrt{2}} + \frac{4}{2\sqrt{2}} = \frac{7}{4\sqrt{2}}x - \frac{3}{2\sqrt{2}}$.

29. $x^2 - xy - y^2 = 1 \Rightarrow 2x - (xy' + y \cdot 1) - 2y y' = 0 \Rightarrow 2x - xy' - y - 2y y' = 0 \Rightarrow 2x - y = xy' + 2y y' \Rightarrow 2x - y = (x + 2y) y' \Rightarrow y' = \frac{2x - y}{x + 2y}.$

When $x = 2$ and $y = 1$, we have $y' = \frac{4-1}{2+2} = \frac{3}{4}$, so an equation of the tangent line is $y - 1 = \frac{3}{4}(x - 2)$, or $y = \frac{3}{4}x - \frac{1}{2}$.

30. $x^2 + 2xy + 4y^2 = 12 \Rightarrow 2x + 2x y' + 2y + 8y y' = 0 \Rightarrow 2x y' + 8y y' = -2x - 2y \Rightarrow (x + 4y) y' = -x - y \Rightarrow y' = -\frac{x + y}{x + 4y}.$

When $x = 2$ and $y = 1$, we have $y' = -\frac{2+1}{2+4} = -\frac{1}{2}$, so an equation of the tangent line is $y - 1 = -\frac{1}{2}(x - 2)$ or $y = -\frac{1}{2}x + 2$.

31. $x^2 + y^2 = (2x^2 + 2y^2 - x)^2 \Rightarrow 2x + 2y y' = 2(2x^2 + 2y^2 - x)(4x + 4y y' - 1).$

When $x = 0$ and $y = \frac{1}{2}$, we have $0 + y' = 2(\frac{1}{2})(2y' - 1) \Rightarrow y' = 2y' - 1 \Rightarrow y' = 1$, so an equation of the tangent line is $y - \frac{1}{2} = 1(x - 0)$ or $y = x + \frac{1}{2}$.

32. $x^2 y^2 = (y+1)^2(4-y^2) \Rightarrow x^2 \cdot 2y y' + y^2 \cdot 2x = (y+1)^2(-2y y') + (4-y^2) \cdot 2(y+1) y' \Rightarrow 2x^2 y y' + 2x y^2 = -2y y'(y+1)^2 + 2y'(4-y^2)(y+1) \Rightarrow 2x y^2 = 2y'(4-y^2)(y+1) - 2y y'(y+1)^2 - 2x^2 y y' \Rightarrow 2x y^2 = y'[2(4-y^2)(y+1) - 2y(y+1)^2 - 2x^2 y] \Rightarrow y' = \frac{2x y^2}{2(4-y^2)(y+1) - 2y(y+1)^2 - 2x^2 y} = \frac{x y^2}{(4-y^2)(y+1) - y(y+1)^2 - x^2 y}$

When $x = 2\sqrt{3}$ and $y = 1$, we have $y' = \frac{2\sqrt{3} \cdot 1^2}{(4-1^2)(1+1) - 1(1+1)^2 - (2\sqrt{3})^2 \cdot 1} = \frac{2\sqrt{3}}{6-4-12} = -\frac{2\sqrt{3}}{10} = -\frac{\sqrt{3}}{5}$, so an equation of the tangent line is $y - 1 = -\frac{\sqrt{3}}{5}(x - 2\sqrt{3})$, or $y = -\frac{\sqrt{3}}{5}x + \frac{11}{5}$.

$$\begin{aligned}
 33. \quad 2(x^2 + y^2)^2 = 25(x^2 - y^2) &\Rightarrow 4(x^2 + y^2)(2x + 2y y') = 25(2x - 2y y') \Rightarrow \\
 4(x + y y')(x^2 + y^2) = 25(x - y y') &\Rightarrow 4y y'(x^2 + y^2) + 25y y' = 25x - 4x(x^2 + y^2) \Rightarrow \\
 y' = \frac{25x - 4x(x^2 + y^2)}{25y + 4y(x^2 + y^2)}.
 \end{aligned}$$

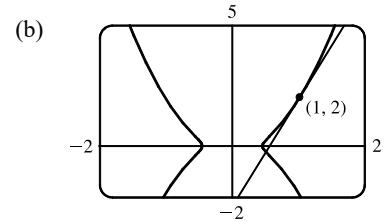
When $x = 3$ and $y = 1$, we have $y' = \frac{75 - 120}{25 + 40} = -\frac{45}{65} = -\frac{9}{13}$, so an equation of the tangent line is $y - 1 = -\frac{9}{13}(x - 3)$ or $y = -\frac{9}{13}x + \frac{40}{13}$.

$$34. \quad y^2(y^2 - 4) = x^2(x^2 - 5) \Rightarrow y^4 - 4y^2 = x^4 - 5x^2 \Rightarrow 4y^3 y' - 8y y' = 4x^3 - 10x.$$

When $x = 0$ and $y = -2$, we have $-32y' + 16y' = 0 \Rightarrow -16y' = 0 \Rightarrow y' = 0$, so an equation of the tangent line is $y + 2 = 0(x - 0)$ or $y = -2$.

$$35. \quad (a) \quad y^2 = 5x^4 - x^2 \Rightarrow 2y y' = 5(4x^3) - 2x \Rightarrow y' = \frac{10x^3 - x}{y}.$$

So at the point $(1, 2)$ we have $y' = \frac{10(1)^3 - 1}{2} = \frac{9}{2}$, and an equation of the tangent line is $y - 2 = \frac{9}{2}(x - 1)$ or $y = \frac{9}{2}x - \frac{5}{2}$.



$$36. \quad (a) \quad y^2 = x^3 + 3x^2 \Rightarrow 2y y' = 3x^2 + 3(2x) \Rightarrow y' = \frac{3x^2 + 6x}{2y}. \quad \text{So at the point } (1, -2) \text{ we have}$$

$$y' = \frac{3(1)^2 + 6(1)}{2(-2)} = -\frac{9}{4}, \text{ and an equation of the tangent line is } y + 2 = -\frac{9}{4}(x - 1) \text{ or } y = -\frac{9}{4}x + \frac{1}{4}.$$

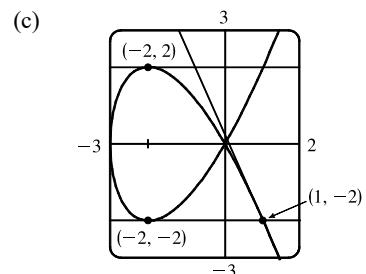
(b) The curve has a horizontal tangent where $y' = 0 \Leftrightarrow$

$$3x^2 + 6x = 0 \Leftrightarrow 3x(x + 2) = 0 \Leftrightarrow x = 0 \text{ or } x = -2.$$

But note that at $x = 0, y = 0$ also, so the derivative does not exist.

At $x = -2, y^2 = (-2)^3 + 3(-2)^2 = -8 + 12 = 4$, so $y = \pm 2$.

So the two points at which the curve has a horizontal tangent are $(-2, -2)$ and $(-2, 2)$.



$$37. \quad x^2 + 4y^2 = 4 \Rightarrow 2x + 8y y' = 0 \Rightarrow y' = -x/(4y) \Rightarrow$$

$$y'' = -\frac{1}{4} \frac{y \cdot 1 - x \cdot y'}{y^2} = -\frac{1}{4} \frac{y - x[-x/(4y)]}{y^2} = -\frac{1}{4} \frac{4y^2 + x^2}{4y^3} = -\frac{1}{4} \frac{4}{4y^3} \quad \left[\text{since } x \text{ and } y \text{ must satisfy the original equation } x^2 + 4y^2 = 4 \right]$$

$$\text{Thus, } y'' = -\frac{1}{4y^3}.$$

$$38. \quad x^2 + xy + y^2 = 3 \Rightarrow 2x + xy' + y + 2y y' = 0 \Rightarrow (x + 2y)y' = -2x - y \Rightarrow y' = \frac{-2x - y}{x + 2y}.$$

Differentiating $2x + xy' + y + 2y y' = 0$ to find y'' gives $2 + xy'' + y' + y' + 2y y'' + 2y' y' = 0 \Rightarrow$

$$(x+2y)y'' = -2 - 2y' - 2(y')^2 = -2 \left[1 - \frac{2x+y}{x+2y} + \left(\frac{2x+y}{x+2y} \right)^2 \right] \Rightarrow$$

$$y'' = -\frac{2}{x+2y} \left[\frac{(x+2y)^2 - (2x+y)(x+2y) + (2x+y)^2}{(x+2y)^2} \right]$$

$$= -\frac{2}{(x+2y)^3} (x^2 + 4xy + 4y^2 - 2x^2 - 4xy - xy - 2y^2 + 4x^2 + 4xy + y^2)$$

$$= -\frac{2}{(x+2y)^3} (3x^2 + 3xy + 3y^2) = -\frac{2}{(x+2y)^3} (9) \quad \begin{array}{l} \text{since } x \text{ and } y \text{ must satisfy the} \\ \text{original equation } x^2 + xy + y^2 = 3 \end{array}$$

Thus, $y'' = -\frac{18}{(x+2y)^3}$.

39. $\sin y + \cos x = 1 \Rightarrow \cos y \cdot y' - \sin x = 0 \Rightarrow y' = \frac{\sin x}{\cos y} \Rightarrow$

$$y'' = \frac{\cos y \cos x - \sin x(-\sin y) y'}{(\cos y)^2} = \frac{\cos y \cos x + \sin x \sin y (\sin x / \cos y)}{\cos^2 y}$$

$$= \frac{\cos^2 y \cos x + \sin^2 x \sin y}{\cos^2 y \cos y} = \frac{\cos^2 y \cos x + \sin^2 x \sin y}{\cos^3 y}$$

40. $x^3 - y^3 = 7 \Rightarrow 3x^2 - 3y^2 y' = 0 \Rightarrow y' = \frac{x^2}{y^2} \Rightarrow$

$$y'' = \frac{y^2(2x) - x^2(2y y')}{(y^2)^2} = \frac{2xy[y - x(x^2/y^2)]}{y^4} = \frac{2x(y - x^3/y^2)}{y^3} = \frac{2x(y^3 - x^3)}{y^3 y^2} = \frac{2x(-7)}{y^5} = \frac{-14x}{y^5}$$

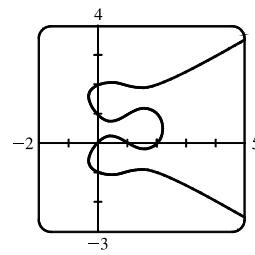
41. If $x = 0$ in $xy + y^3 = 1$, then we get $y^3 = 1 \Rightarrow y = 1$, so the point where $x = 0$ is $(0, 1)$. Differentiating implicitly with respect to x gives us $xy' + y \cdot 1 + 3y^2 y' = 0$. Substituting 0 for x and 1 for y gives us $1 + 3y' = 0 \Rightarrow y' = -\frac{1}{3}$. Differentiating $xy' + y + 3y^2 y' = 0$ implicitly with respect to x gives us $xy'' + y' + y' + 3(y^2 y'' + y' \cdot 2y y') = 0$. Now substitute 0 for x , 1 for y , and $-\frac{1}{3}$ for y' . $0 - \frac{1}{3} - \frac{1}{3} + 3[y'' + (-\frac{1}{3}) \cdot 2(-\frac{1}{3})] = 0 \Rightarrow 3(y'' + \frac{2}{9}) = \frac{2}{3} \Rightarrow y'' + \frac{2}{9} = \frac{2}{9} \Rightarrow y'' = 0$.

42. If $x = 1$ in $x^2 + xy + y^3 = 1$, then we get $1 + y + y^3 = 1 \Rightarrow y^3 + y = 0 \Rightarrow y(y^2 + 1) \Rightarrow y = 0$, so the point where $x = 1$ is $(1, 0)$. Differentiating implicitly with respect to x gives us $2x + xy' + y \cdot 1 + 3y^2 \cdot y' = 0$. Substituting 1 for x and 0 for y gives us $2 + y' + 0 + 0 = 0 \Rightarrow y' = -2$. Differentiating $2x + xy' + y + 3y^2 y' = 0$ implicitly with respect to x gives us $2 + xy'' + y' \cdot 1 + y' + 3(y^2 y'' + y' \cdot 2yy') = 0$. Now substitute 1 for x , 0 for y , and -2 for y' . $2 + y'' + (-2) + 3(0 + 0) = 0 \Rightarrow y'' = 2$. Differentiating $2 + xy'' + 2y' + 3y^2 y'' + 6y(y')^2 = 0$ implicitly with respect to x gives us $xy''' + y'' \cdot 1 + 2y'' + 3(y^2 y''' + y'' \cdot 2y y') + 6[y \cdot 2y' y'' + (y')^2 y'] = 0$. Now substitute 1 for x , 0 for y , -2 for y' , and 2 for y'' . $y''' + 2 + 4 + 3(0 + 0) + 6[0 + (-8)] = 0 \Rightarrow y''' = -2 - 4 + 48 = 42$.

43. (a) There are eight points with horizontal tangents: four at $x \approx 1.57735$ and four at $x \approx 0.42265$.

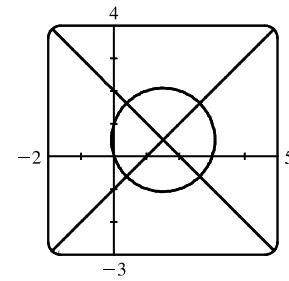
$$(b) y' = \frac{3x^2 - 6x + 2}{2(2y^3 - 3y^2 - y + 1)} \Rightarrow y' = -1 \text{ at } (0, 1) \text{ and } y' = \frac{1}{3} \text{ at } (0, 2).$$

Equations of the tangent lines are $y = -x + 1$ and $y = \frac{1}{3}x + 2$.

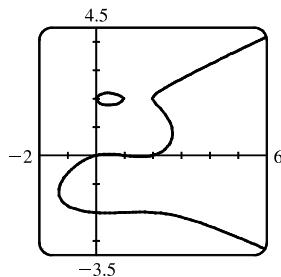


$$(c) y' = 0 \Rightarrow 3x^2 - 6x + 2 = 0 \Rightarrow x = 1 \pm \frac{1}{3}\sqrt{3}$$

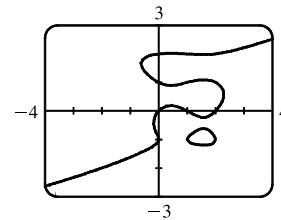
- (d) By multiplying the right side of the equation by $x - 3$, we obtain the first graph. By modifying the equation in other ways, we can generate the other graphs.



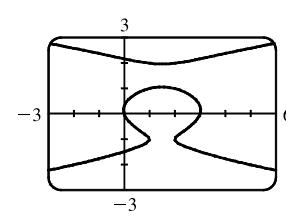
$$\begin{aligned} y(y^2 - 1)(y - 2) \\ = x(x - 1)(x - 2)(x - 3) \end{aligned}$$



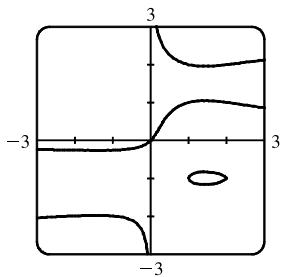
$$\begin{aligned} y(y^2 - 4)(y - 2) \\ = x(x - 1)(x - 2) \end{aligned}$$



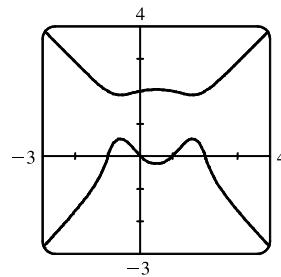
$$\begin{aligned} y(y + 1)(y^2 - 1)(y - 2) \\ = x(x - 1)(x - 2) \end{aligned}$$



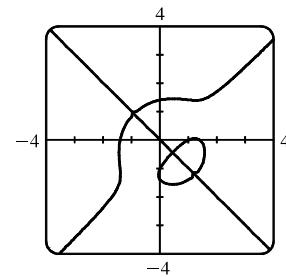
$$\begin{aligned} (y + 1)(y^2 - 1)(y - 2) \\ = (x - 1)(x - 2) \end{aligned}$$



$$\begin{aligned} x(y + 1)(y^2 - 1)(y - 2) \\ = y(x - 1)(x - 2) \end{aligned}$$

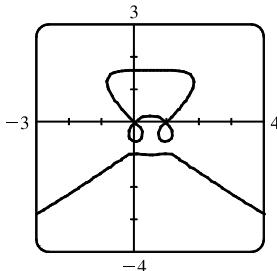


$$\begin{aligned} y(y^2 + 1)(y - 2) \\ = x(x^2 - 1)(x - 2) \end{aligned}$$



$$\begin{aligned} y(y + 1)(y^2 - 2) \\ = x(x - 1)(x^2 - 2) \end{aligned}$$

44. (a)



$$(b) \frac{d}{dx}(2y^3 + y^2 - y^5) = \frac{d}{dx}(x^4 - 2x^3 + x^2) \Rightarrow$$

$$6y^2y' + 2yy' - 5y^4y' = 4x^3 - 6x^2 + 2x \Rightarrow$$

$$y' = \frac{2x(2x^2 - 3x + 1)}{6y^2 + 2y - 5y^4} = \frac{2x(2x - 1)(x - 1)}{y(6y + 2 - 5y^3)}. \text{ From the graph and the}$$

values for which $y' = 0$, we speculate that there are 9 points with horizontal tangents: 3 at $x = 0$, 3 at $x = \frac{1}{2}$, and 3 at $x = 1$. The three horizontal tangents along the top of the wagon are hard to find, but by limiting the y -range of the graph (to $[1.6, 1.7]$, for example) they are distinguishable.

45. From Exercise 33, a tangent to the lemniscate will be horizontal if $y' = 0 \Rightarrow 25x - 4x(x^2 + y^2) = 0 \Rightarrow$

$$x[25 - 4(x^2 + y^2)] = 0 \Rightarrow x^2 + y^2 = \frac{25}{4} \quad (1). \text{ (Note that when } x \text{ is 0, } y \text{ is also 0, and there is no horizontal tangent at the origin.) Substituting } \frac{25}{4} \text{ for } x^2 + y^2 \text{ in the equation of the lemniscate, } 2(x^2 + y^2)^2 = 25(x^2 - y^2), \text{ we get } x^2 - y^2 = \frac{25}{8} \quad (2).$$

Solving (1) and (2), we have $x^2 = \frac{75}{16}$ and $y^2 = \frac{25}{16}$, so the four points are $\left(\pm\frac{5\sqrt{3}}{4}, \pm\frac{5}{4}\right)$.

46. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{2x}{a^2} + \frac{2yy'}{b^2} = 0 \Rightarrow y' = -\frac{b^2x}{a^2y} \Rightarrow$ an equation of the tangent line at (x_0, y_0) is

$$y - y_0 = \frac{-b^2x_0}{a^2y_0}(x - x_0). \text{ Multiplying both sides by } \frac{y_0}{b^2} \text{ gives } \frac{y_0y}{b^2} - \frac{y_0^2}{b^2} = -\frac{x_0x}{a^2} + \frac{x_0^2}{a^2}. \text{ Since } (x_0, y_0) \text{ lies on the ellipse,}$$

$$\text{we have } \frac{x_0x}{a^2} + \frac{y_0y}{b^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1.$$

47. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow \frac{2x}{a^2} - \frac{2yy'}{b^2} = 0 \Rightarrow y' = \frac{b^2x}{a^2y} \Rightarrow$ an equation of the tangent line at (x_0, y_0) is

$$y - y_0 = \frac{b^2x_0}{a^2y_0}(x - x_0). \text{ Multiplying both sides by } \frac{y_0}{b^2} \text{ gives } \frac{y_0y}{b^2} - \frac{y_0^2}{b^2} = \frac{x_0x}{a^2} - \frac{x_0^2}{a^2}. \text{ Since } (x_0, y_0) \text{ lies on the hyperbola,}$$

$$\text{we have } \frac{x_0x}{a^2} - \frac{y_0y}{b^2} = \frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} = 1.$$

48. $\sqrt{x} + \sqrt{y} = \sqrt{c} \Rightarrow \frac{1}{2\sqrt{x}} + \frac{y'}{2\sqrt{y}} = 0 \Rightarrow y' = -\frac{\sqrt{y}}{\sqrt{x}} \Rightarrow$ an equation of the tangent line at (x_0, y_0)

$$\text{is } y - y_0 = -\frac{\sqrt{y_0}}{\sqrt{x_0}}(x - x_0). \text{ Now } x = 0 \Rightarrow y = y_0 - \frac{\sqrt{y_0}}{\sqrt{x_0}}(-x_0) = y_0 + \sqrt{x_0}\sqrt{y_0}, \text{ so the } y\text{-intercept is}$$

$$y_0 + \sqrt{x_0}\sqrt{y_0}. \text{ And } y = 0 \Rightarrow -y_0 = -\frac{\sqrt{y_0}}{\sqrt{x_0}}(x - x_0) \Rightarrow x - x_0 = \frac{y_0\sqrt{x_0}}{\sqrt{y_0}} \Rightarrow x = x_0 + \sqrt{x_0}\sqrt{y_0},$$

so the x -intercept is $x_0 + \sqrt{x_0}\sqrt{y_0}$. The sum of the intercepts is

$$\left(y_0 + \sqrt{x_0}\sqrt{y_0}\right) + \left(x_0 + \sqrt{x_0}\sqrt{y_0}\right) = x_0 + 2\sqrt{x_0}\sqrt{y_0} + y_0 = \left(\sqrt{x_0} + \sqrt{y_0}\right)^2 = (\sqrt{c})^2 = c.$$

49. If the circle has radius r , its equation is $x^2 + y^2 = r^2 \Rightarrow 2x + 2yy' = 0 \Rightarrow y' = -\frac{x}{y}$, so the slope of the tangent line

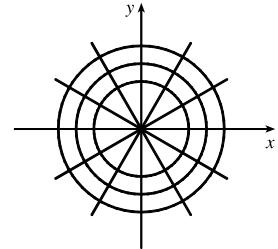
at $P(x_0, y_0)$ is $-\frac{x_0}{y_0}$. The negative reciprocal of that slope is $\frac{-1}{-x_0/y_0} = \frac{y_0}{x_0}$, which is the slope of OP , so the tangent line at

P is perpendicular to the radius OP .

$$50. y^q = x^p \Rightarrow qy^{q-1}y' = px^{p-1} \Rightarrow y' = \frac{px^{p-1}}{qy^{q-1}} = \frac{px^{p-1}y}{qy^q} = \frac{px^{p-1}x^{p/q}}{qx^p} = \frac{p}{q}x^{(p/q)-1}$$

51. $x^2 + y^2 = r^2$ is a circle with center O and $ax + by = 0$ is a line through O [assume a

and b are not both zero]. $x^2 + y^2 = r^2 \Rightarrow 2x + 2yy' = 0 \Rightarrow y' = -x/y$, so the slope of the tangent line at $P_0(x_0, y_0)$ is $-x_0/y_0$. The slope of the line OP_0 is y_0/x_0 , which is the negative reciprocal of $-x_0/y_0$. Hence, the curves are orthogonal, and the families of curves are orthogonal trajectories of each other.



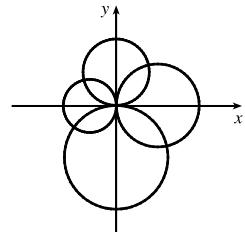
52. The circles $x^2 + y^2 = ax$ and $x^2 + y^2 = by$ intersect at the origin where the tangents are vertical and horizontal [assume a and b are both nonzero]. If (x_0, y_0) is the other point of intersection, then $x_0^2 + y_0^2 = ax_0$ (1) and $x_0^2 + y_0^2 = by_0$ (2).

Now $x^2 + y^2 = ax \Rightarrow 2x + 2yy' = a \Rightarrow y' = \frac{a - 2x}{2y}$ and $x^2 + y^2 = by \Rightarrow$

$2x + 2yy' = by' \Rightarrow y' = \frac{2x}{b - 2y}$. Thus, the curves are orthogonal at $(x_0, y_0) \Leftrightarrow$

$$\frac{a - 2x_0}{2y_0} = -\frac{b - 2y_0}{2x_0} \Leftrightarrow 2ax_0 - 4x_0^2 = 4y_0^2 - 2by_0 \Leftrightarrow ax_0 + by_0 = 2(x_0^2 + y_0^2),$$

which is true by (1) and (2).

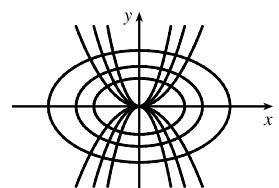


53. $y = cx^2 \Rightarrow y' = 2cx$ and $x^2 + 2y^2 = k$ [assume $k > 0$] $\Rightarrow 2x + 4yy' = 0 \Rightarrow$

$2yy' = -x \Rightarrow y' = -\frac{x}{2(y)} = -\frac{x}{2(cx^2)} = -\frac{1}{2cx}$, so the curves are orthogonal if

$c \neq 0$. If $c = 0$, then the horizontal line $y = cx^2 = 0$ intersects $x^2 + 2y^2 = k$ orthogonally

at $(\pm\sqrt{k}, 0)$, since the ellipse $x^2 + 2y^2 = k$ has vertical tangents at those two points.

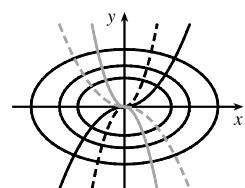


54. $y = ax^3 \Rightarrow y' = 3ax^2$ and $x^2 + 3y^2 = b$ [assume $b > 0$] $\Rightarrow 2x + 6yy' = 0 \Rightarrow$

$3yy' = -x \Rightarrow y' = -\frac{x}{3(y)} = -\frac{x}{3(ax^3)} = -\frac{1}{3ax^2}$, so the curves are orthogonal if

$a \neq 0$. If $a = 0$, then the horizontal line $y = ax^3 = 0$ intersects $x^2 + 3y^2 = b$ orthogonally

at $(\pm\sqrt{b}, 0)$, since the ellipse $x^2 + 3y^2 = b$ has vertical tangents at those two points.



55. Since $A^2 < a^2$, we are assured that there are four points of intersection.

$$(1) \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{2x}{a^2} + \frac{2yy'}{b^2} = 0 \Rightarrow \frac{yy'}{b^2} = -\frac{x}{a^2} \Rightarrow$$

$$y' = m_1 = -\frac{xb^2}{ya^2}.$$

$$(2) \frac{x^2}{A^2} - \frac{y^2}{B^2} = 1 \Rightarrow \frac{2x}{A^2} - \frac{2yy'}{B^2} = 0 \Rightarrow \frac{yy'}{B^2} = \frac{x}{A^2} \Rightarrow$$

$$y' = m_2 = \frac{xB^2}{yA^2}.$$

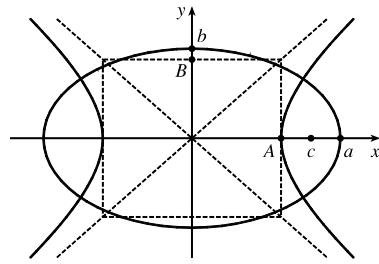
Now $m_1m_2 = -\frac{xb^2}{ya^2} \cdot \frac{xB^2}{yA^2} = -\frac{b^2B^2}{a^2A^2} \cdot \frac{x^2}{y^2}$ (3). Subtracting equations, (1) – (2), gives us $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{x^2}{A^2} + \frac{y^2}{B^2} = 0 \Rightarrow$

$$\frac{y^2}{b^2} + \frac{y^2}{B^2} = \frac{x^2}{A^2} - \frac{x^2}{a^2} \Rightarrow \frac{y^2(B^2 + b^2)}{b^2B^2} = \frac{x^2a^2 - x^2A^2}{A^2a^2} \Rightarrow \frac{y^2(b^2 + B^2)}{b^2B^2} = \frac{x^2(a^2 - A^2)}{a^2A^2}$$

$a^2 - b^2 = A^2 + B^2$, we have $a^2 - A^2 = b^2 + B^2$. Thus, equation (4) becomes $\frac{y^2}{b^2B^2} = \frac{x^2}{A^2a^2} \Rightarrow \frac{x^2}{y^2} = \frac{A^2a^2}{b^2B^2}$, and

substituting for $\frac{x^2}{y^2}$ in equation (3) gives us $m_1m_2 = -\frac{b^2B^2}{a^2A^2} \cdot \frac{a^2A^2}{b^2B^2} = -1$. Hence, the ellipse and hyperbola are orthogonal

trajectories.



56. $y = (x+c)^{-1} \Rightarrow y' = -(x+c)^{-2}$ and $y = a(x+k)^{1/3} \Rightarrow y' = \frac{1}{3}a(x+k)^{-2/3}$, so the curves are orthogonal if the

product of the slopes is -1 , that is, $\frac{-1}{(x+c)^2} \cdot \frac{a}{3(x+k)^{2/3}} = -1 \Rightarrow a = 3(x+c)^2(x+k)^{2/3} \Rightarrow$

$$a = 3\left(\frac{1}{y}\right)^2 \left(\frac{y}{a}\right)^2 \text{ [since } y^2 = (x+c)^{-2} \text{ and } y^2 = a^2(x+k)^{2/3}] \Rightarrow a = 3\left(\frac{1}{a^2}\right) \Rightarrow a^3 = 3 \Rightarrow a = \sqrt[3]{3}.$$

57. (a) $\left(P + \frac{n^2a}{V^2}\right)(V - nb) = nRT \Rightarrow PV - Pnb + \frac{n^2a}{V} - \frac{n^3ab}{V^2} = nRT \Rightarrow$

$$\frac{d}{dP}(PV - Pnb + n^2aV^{-1} - n^3abV^{-2}) = \frac{d}{dP}(nRT) \Rightarrow$$

$$PV' + V \cdot 1 - nb - n^2aV^{-2} \cdot V' + 2n^3abV^{-3} \cdot V' = 0 \Rightarrow V'(P - n^2aV^{-2} + 2n^3abV^{-3}) = nb - V \Rightarrow$$

$$V' = \frac{nb - V}{P - n^2aV^{-2} + 2n^3abV^{-3}} \text{ or } \frac{dV}{dP} = \frac{V^3(nb - V)}{PV^3 - n^2aV + 2n^3ab}$$

- (b) Using the last expression for dV/dP from part (a), we get

$$\begin{aligned} \frac{dV}{dP} &= \frac{(10 \text{ L})^3[(1 \text{ mole})(0.04267 \text{ L/mole}) - 10 \text{ L}]}{\left[(2.5 \text{ atm})(10 \text{ L})^3 - (1 \text{ mole})^2(3.592 \text{ L}^2 \cdot \text{atm/mole}^2)(10 \text{ L})\right.} \\ &\quad \left.+ 2(1 \text{ mole})^3(3.592 \text{ L}^2 \cdot \text{atm/mole}^2)(0.04267 \text{ L/mole})\right]} \\ &= \frac{-9957.33 \text{ L}^4}{2464.386541 \text{ L}^3 \cdot \text{atm}} \approx -4.04 \text{ L/atm.} \end{aligned}$$

58. (a) $x^2 + xy + y^2 + 1 = 0 \Rightarrow 2x + xy' + y \cdot 1 + 2yy' + 0 = 0 \Rightarrow y'(x + 2y) = -2x - y \Rightarrow y' = \frac{-2x - y}{x + 2y}$

(b) Plotting the curve in part (a) gives us an empty graph, that is, there are no points that satisfy the equation. If there were any points that satisfied the equation, then x and y would have opposite signs; otherwise, all the terms are positive and their sum can not equal 0. $x^2 + xy + y^2 + 1 = 0 \Rightarrow x^2 + 2xy + y^2 - xy + 1 = 0 \Rightarrow (x + y)^2 = xy - 1$. The left side of the last equation is nonnegative, but the right side is at most -1 , so that proves there are no points that satisfy the equation.

$$\begin{aligned} \text{Another solution: } x^2 + xy + y^2 + 1 &= \frac{1}{2}x^2 + xy + \frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{1}{2}y^2 + 1 = \frac{1}{2}(x^2 + 2xy + y^2) + \frac{1}{2}(x^2 + y^2) + 1 \\ &= \frac{1}{2}(x + y)^2 + \frac{1}{2}(x^2 + y^2) + 1 \geq 1 \end{aligned}$$

Another solution: Regarding $x^2 + xy + y^2 + 1 = 0$ as a quadratic in x , the discriminant is $y^2 - 4(y^2 + 1) = -3y^2 - 4$. This is negative, so there are no real solutions.

(c) The expression for y' in part (a) is meaningless; that is, since the equation in part (a) has no solution, it does not implicitly define a function y of x , and therefore it is meaningless to consider y' .

59. To find the points at which the ellipse $x^2 - xy + y^2 = 3$ crosses the x -axis, let $y = 0$ and solve for x .

$$y = 0 \Rightarrow x^2 - x(0) + 0^2 = 3 \Leftrightarrow x = \pm\sqrt{3}. \text{ So the graph of the ellipse crosses the } x\text{-axis at the points } (\pm\sqrt{3}, 0).$$

$$\text{Using implicit differentiation to find } y', \text{ we get } 2x - xy' - y + 2yy' = 0 \Rightarrow y'(2y - x) = y - 2x \Leftrightarrow y' = \frac{y - 2x}{2y - x}.$$

So y' at $(\sqrt{3}, 0)$ is $\frac{0 - 2\sqrt{3}}{2(0) - \sqrt{3}} = 2$ and y' at $(-\sqrt{3}, 0)$ is $\frac{0 + 2\sqrt{3}}{2(0) + \sqrt{3}} = 2$. Thus, the tangent lines at these points are parallel.

60. (a) We use implicit differentiation to find $y' = \frac{y - 2x}{2y - x}$ as in Exercise 59. The slope (b)

$$\text{of the tangent line at } (-1, 1) \text{ is } m = \frac{1 - 2(-1)}{2(1) - (-1)} = \frac{3}{3} = 1, \text{ so the slope of the}$$

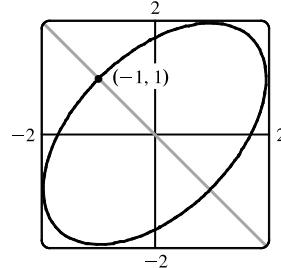
$$\text{normal line is } -\frac{1}{m} = -1, \text{ and its equation is } y - 1 = -1(x + 1) \Leftrightarrow$$

$y = -x$. Substituting $-x$ for y in the equation of the ellipse, we get

$$x^2 - x(-x) + (-x)^2 = 3 \Rightarrow 3x^2 = 3 \Leftrightarrow x = \pm 1. \text{ So the normal line}$$

must intersect the ellipse again at $x = 1$, and since the equation of the line is

$y = -x$, the other point of intersection must be $(1, -1)$.



61. $x^2y^2 + xy = 2 \Rightarrow x^2 \cdot 2yy' + y^2 \cdot 2x + x \cdot y' + y \cdot 1 = 0 \Leftrightarrow y'(2x^2y + x) = -2xy^2 - y \Leftrightarrow$

$$y' = -\frac{2xy^2 + y}{2x^2y + x}. \text{ So } -\frac{2xy^2 + y}{2x^2y + x} = -1 \Leftrightarrow 2xy^2 + y = 2x^2y + x \Leftrightarrow y(2xy + 1) = x(2xy + 1) \Leftrightarrow$$

$$y(2xy + 1) - x(2xy + 1) = 0 \Leftrightarrow (2xy + 1)(y - x) = 0 \Leftrightarrow xy = -\frac{1}{2} \text{ or } y = x. \text{ But } xy = -\frac{1}{2} \Rightarrow$$

$$x^2y^2 + xy = \frac{1}{4} - \frac{1}{2} \neq 2, \text{ so we must have } x = y. \text{ Then } x^2y^2 + xy = 2 \Rightarrow x^4 + x^2 = 2 \Leftrightarrow x^4 + x^2 - 2 = 0 \Leftrightarrow$$

$$(x^2 + 2)(x^2 - 1) = 0. \text{ So } x^2 = -2, \text{ which is impossible, or } x^2 = 1 \Leftrightarrow x = \pm 1. \text{ Since } x = y, \text{ the points on the curve}$$

where the tangent line has a slope of -1 are $(-1, -1)$ and $(1, 1)$.

62. $x^2 + 4y^2 = 36 \Rightarrow 2x + 8yy' = 0 \Rightarrow y' = -\frac{x}{4y}$. Let (a, b) be a point on $x^2 + 4y^2 = 36$ whose tangent line passes

through $(12, 3)$. The tangent line is then $y - 3 = -\frac{a}{4b}(x - 12)$, so $b - 3 = -\frac{a}{4b}(a - 12)$. Multiplying both sides by $4b$ gives $4b^2 - 12b = -a^2 + 12a$, so $4b^2 + a^2 = 12(a + b)$. But $4b^2 + a^2 = 36$, so $36 = 12(a + b) \Rightarrow a + b = 3 \Rightarrow b = 3 - a$. Substituting $3 - a$ for b into $a^2 + 4b^2 = 36$ gives $a^2 + 4(3 - a)^2 = 36 \Leftrightarrow a^2 + 36 - 24a + 4a^2 = 36 \Leftrightarrow 5a^2 - 24a = 0 \Leftrightarrow a(5a - 24) = 0$, so $a = 0$ or $a = \frac{24}{5}$. If $a = 0$, $b = 3 - 0 = 3$, and if $a = \frac{24}{5}$, $b = 3 - \frac{24}{5} = -\frac{9}{5}$.

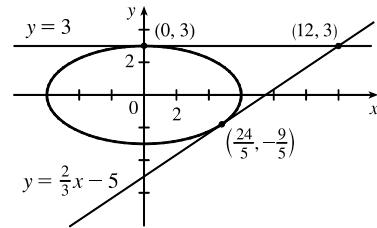
So the two points on the ellipse are $(0, 3)$ and $(\frac{24}{5}, -\frac{9}{5})$. Using

$y - 3 = -\frac{a}{4b}(x - 12)$ with $(a, b) = (0, 3)$ gives us the tangent line

$y - 3 = 0$ or $y = 3$. With $(a, b) = (\frac{24}{5}, -\frac{9}{5})$, we have

$$y - 3 = -\frac{24/5}{4(-9/5)}(x - 12) \Leftrightarrow y - 3 = \frac{2}{3}(x - 12) \Leftrightarrow y = \frac{2}{3}x - 5.$$

A graph of the ellipse and the tangent lines confirms our results.



63. For $\frac{x}{y} = y^2 + 1$, $y \neq 0$, we have $\frac{d}{dx}\left(\frac{x}{y}\right) = \frac{d}{dx}(y^2 + 1) \Rightarrow \frac{y \cdot 1 - x \cdot y'}{y^2} = 2y y' \Rightarrow y - x y' = 2y^3 y' \Rightarrow 2y^3 y' + x y' = y \Rightarrow y'(2y^3 + x) = y \Rightarrow y' = \frac{y}{2y^3 + x}$.

For $x = y^3 + y$, $y \neq 0$, we have $\frac{d}{dx}(x) = \frac{d}{dx}(y^3 + y) \Rightarrow 1 = 3y^2 y' + y' \Rightarrow 1 = y'(3y^2 + 1) \Rightarrow$

$$y' = \frac{1}{3y^2 + 1}.$$

From part (a), $y' = \frac{y}{2y^3 + x}$. Since $y \neq 0$, we substitute $y^3 + y$ for x to get

$$\frac{y}{2y^3 + x} = \frac{y}{2y^3 + (y^3 + y)} = \frac{y}{3y^3 + y} = \frac{y}{y(3y^2 + 1)} = \frac{1}{3y^2 + 1}, \text{ which agrees with part (b).}$$

64. (a) $y = J(x)$ and $xy'' + y' + xy = 0 \Rightarrow xJ''(x) + J'(x) + xJ(x) = 0$. If $x = 0$, we have $0 + J'(0) + 0 = 0$,

$$\text{so } J'(0) = 0.$$

(b) Differentiating $xy'' + y' + xy = 0$ implicitly, we get $xy''' + y'' \cdot 1 + y'' + xy' + y \cdot 1 = 0 \Rightarrow$

$xy''' + 2y'' + xy' + y = 0$, so $xJ'''(x) + 2J''(x) + xJ'(x) + J(x) = 0$. If $x = 0$, we have

$$0 + 2J''(0) + 0 + 1 \quad [J(0) = 1 \text{ is given}] = 0 \Rightarrow 2J''(0) = -1 \Rightarrow J''(0) = -\frac{1}{2}.$$

65. $x^2 + 4y^2 = 5 \Rightarrow 2x + 4(2yy') = 0 \Rightarrow y' = -\frac{x}{4y}$. Now let h be the height of the lamp, and let (a, b) be the point of

tangency of the line passing through the points $(3, h)$ and $(-5, 0)$. This line has slope $(h - 0)/[3 - (-5)] = \frac{1}{8}h$. But the

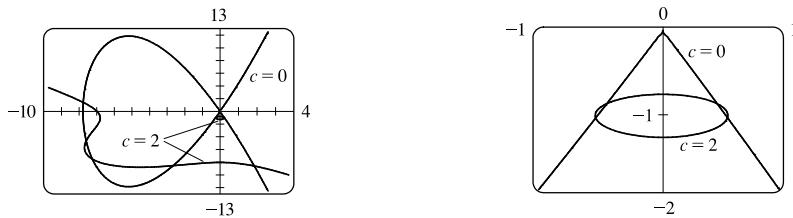
slope of the tangent line through the point (a, b) can be expressed as $y' = -\frac{a}{4b}$, or as $\frac{b - 0}{a - (-5)} = \frac{b}{a + 5}$ [since the line

passes through $(-5, 0)$ and (a, b)], so $-\frac{a}{4b} = \frac{b}{a + 5} \Leftrightarrow 4b^2 = -a^2 - 5a \Leftrightarrow a^2 + 4b^2 = -5a$. But $a^2 + 4b^2 = 5$

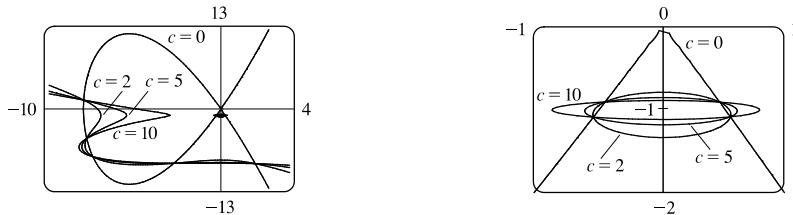
[since (a, b) is on the ellipse], so $5 = -5a \Leftrightarrow a = -1$. Then $4b^2 = -a^2 - 5a = -1 - 5(-1) = 4 \Rightarrow b = 1$, since the point is on the top half of the ellipse. So $\frac{h}{8} = \frac{b}{a+5} = \frac{1}{-1+5} = \frac{1}{4} \Rightarrow h = 2$. So the lamp is located 2 units above the x -axis.

DISCOVERY PROJECT Families of Implicit Curves

1. (a) There appear to be nine points of intersection. The “inner four” near the origin are about $(\pm 0.2, -0.9)$ and $(\pm 0.3, -1.1)$. The “outer five” are about $(2.0, -8.9)$, $(-2.8, -8.8)$, $(-7.5, -7.7)$, $(-7.8, -4.7)$, and $(-8.0, 1.5)$.



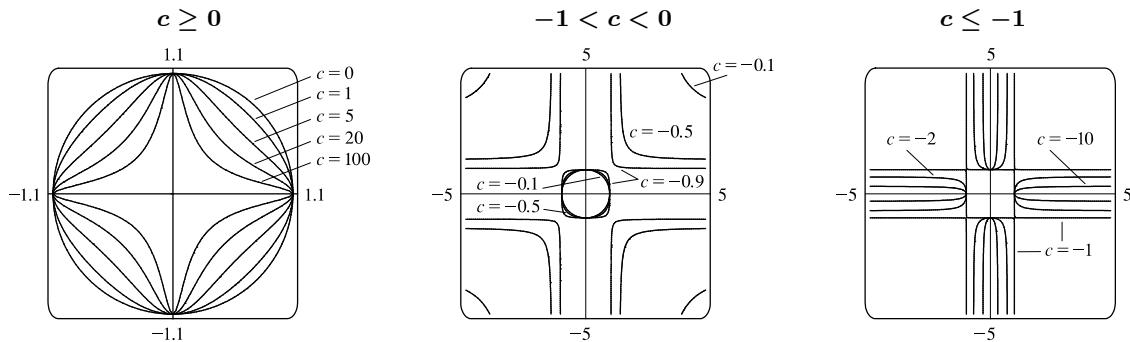
1. (b) We see from the graphs with $c = 5$ and $c = 10$, and for other values of c , that the curves change shape but the nine points of intersection are the same.



2. (a) If $c = 0$, the graph is the unit circle. As c increases, the graph looks more diamondlike and then more crosslike (see the graph for $c \geq 0$).

For $-1 < c < 0$ (see the graph), there are four hyperbolicalike branches as well as an ellipticalike curve bounded by $|x| \leq 1$ and $|y| \leq 1$ for values of c close to 0. As c gets closer to -1 , the branches and the curve become more rectangular, approaching the lines $|x| = 1$ and $|y| = 1$.

For $c = -1$, we get the lines $x = \pm 1$ and $y = \pm 1$. As c decreases, we get four test-tubelike curves (see the graph) that are bounded by $|x| = 1$ and $|y| = 1$, and get thinner as $|c|$ gets larger.



(b) The curve for $c = -1$ is described in part (a). When $c = -1$, we get

$x^2 + y^2 - x^2y^2 = 1 \Leftrightarrow 0 = x^2y^2 - x^2 - y^2 + 1 \Leftrightarrow 0 = (x^2 - 1)(y^2 - 1) \Leftrightarrow x = \pm 1$ or $y = \pm 1$, which algebraically proves that the graph consists of the stated lines.

$$(c) \frac{d}{dx}(x^2 + y^2 + cx^2y^2) = \frac{d}{dx}(1) \Rightarrow 2x + 2y y' + c(x^2 \cdot 2y y' + y^2 \cdot 2x) = 0 \Rightarrow$$

$$2y y' + 2cx^2y y' = -2x - 2cxy^2 \Rightarrow 2y(1 + cx^2)y' = -2x(1 + cy^2) \Rightarrow y' = -\frac{x(1 + cy^2)}{y(1 + cx^2)}.$$

$$\text{For } c = -1, y' = -\frac{x(1 - y^2)}{y(1 - x^2)} = -\frac{x(1 + y)(1 - y)}{y(1 + x)(1 - x)}, \text{ so } y' = 0 \text{ when } y = \pm 1 \text{ or } x = 0 \text{ (which leads to } y = \pm 1)$$

and y' is undefined when $x = \pm 1$ or $y = 0$ (which leads to $x = \pm 1$). Since the graph consists of the lines $x = \pm 1$ and $y = \pm 1$, the slope at any point on the graph is undefined or 0, which is consistent with the expression found for y' .

2.7 Rates of Change in the Natural and Social Sciences

1. (a) $s = f(t) = t^3 - 9t^2 + 24t$ (in m) $\Rightarrow v(t) = f'(t) = 3t^2 - 18t + 24$ (in m/s)

(b) $v(1) = 3(1)^2 - 18(1) + 24 = 9$ m/s

(c) The particle is at rest when $v(t) = 0$: $3t^2 - 18t + 24 = 0 \Leftrightarrow 3(t - 2)(t - 4) = 0 \Leftrightarrow t = 2$ s or $t = 4$ s.

(d) The particle is moving in the positive direction when $v(t) > 0$. $3(t - 2)(t - 4) > 0 \Rightarrow 0 \leq t < 2$ or $t > 4$.

(e) Because the particle changes direction when $t = 2$ and $t = 4$, we need to calculate the distances traveled in the intervals $[0, 2]$, $[2, 4]$, and $[4, 6]$ separately.

$$|f(2) - f(0)| = |20 - 0| = 20$$

$$|f(4) - f(2)| = |16 - 20| = 4$$

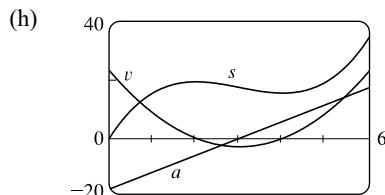
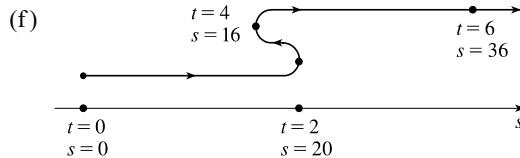
$$|f(6) - f(4)| = |36 - 16| = 20$$

The total distance is $20 + 4 + 20 = 44$ m.

(g) $v(t) = 3t^2 - 18t + 24 \Rightarrow$

$$a(t) = v'(t) = 6t - 18 \text{ (in (m/s)/s or m/s}^2\text{).}$$

$$a(1) = 6(1) - 18 = -12 \text{ m/s}^2$$



- (i) The particle is speeding up when v and a have the same sign. This occurs when $2 < t < 3$ (v and a are both negative) and when $t > 4$ (v and a are both positive). It is slowing down when v and a have opposite signs; that is, when $0 \leq t < 2$ and when $3 < t < 4$.

2. (a) $s = f(t) = 0.01t^4 - 0.04t^3$ (in feet) $\Rightarrow v(t) = f'(t) = 0.04t^3 - 0.12t^2$ (in ft/s)

(b) $v(3) = 0.04(3)^3 - 0.12(3)^2 = 0$ ft/s

(c) The particle is at rest when $v(t) = 0$. $0.04t^3 - 0.12t^2 = 0 \Leftrightarrow 0.04t^2(t - 3) = 0 \Leftrightarrow t = 0$ s or 3 s.

(d) The particle is moving in the positive direction when $v(t) > 0$. $0.04t^2(t - 3) > 0 \Leftrightarrow t > 3$.

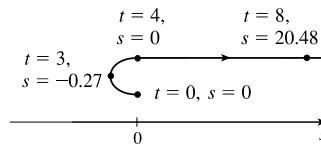
(e) See Exercise 1(e).

$$|f(3) - f(0)| = |-0.27 - 0| = 0.27.$$

$$|f(8) - f(3)| = |20.48 - (-0.27)| = 20.75.$$

The total distance is $0.27 + 20.75 = 21.02$ ft.

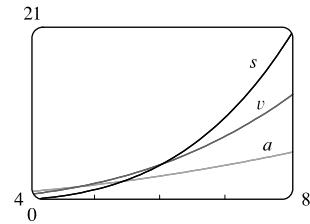
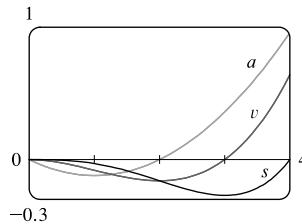
(f)



(g) $v(t) = 0.04t^3 - 0.12t^2 \Rightarrow a(t) = v'(t) = 0.12t^2 - 0.24t$. $a(3) = 0.12(3)^2 - 0.24(3) = 0.36$ (ft/s)/s or ft/s².

(h) Here we show the graph of s , v , and a

for $0 \leq t \leq 4$ and $4 \leq t \leq 8$.



(i) The particle is speeding up when v and a have the same sign. This occurs when $0 < t < 2$ [v and a are both negative] and when $t > 3$ [v and a are both positive]. It is slowing down when v and a have opposite signs; that is, when $2 < t < 3$.

3. (a) $s = f(t) = \sin(\pi t/2)$ (in m) $\Rightarrow v(t) = f'(t) = \cos(\pi t/2) \cdot (\pi/2) = \frac{\pi}{2} \cos(\pi t/2)$ (in m/s)

(b) $v(1) = \frac{\pi}{2} \cos \frac{\pi}{2} = \frac{\pi}{2}(0) = 0$ m/s

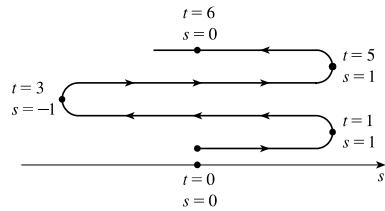
(c) The particle is at rest when $v(t) = 0$. $\frac{\pi}{2} \cos \frac{\pi}{2}t = 0 \Leftrightarrow \cos \frac{\pi}{2}t = 0 \Leftrightarrow \frac{\pi}{2}t = \frac{\pi}{2} + n\pi \Leftrightarrow t = 1 + 2n$, where n is a nonnegative integer since $t \geq 0$.

(d) The particle is moving in the positive direction when $v(t) > 0$. From part (c), we see that v changes sign at every positive odd integer. v is positive when $0 < t < 1$, $3 < t < 5$, $7 < t < 9$, and so on.

(e) v changes sign at $t = 1$, 3 , and 5 in the interval $[0, 6]$. The total distance traveled during the first 6 seconds is

$$\begin{aligned} |f(1) - f(0)| + |f(3) - f(1)| + |f(5) - f(3)| + |f(6) - f(5)| &= |1 - 0| + |-1 - 1| + |1 - (-1)| + |0 - 1| \\ &= 1 + 2 + 2 + 1 = 6 \text{ m} \end{aligned}$$

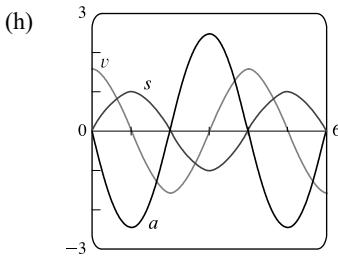
(f)



(g) $v(t) = \frac{\pi}{2} \cos(\pi t/2) \Rightarrow$

$$\begin{aligned} a(t) &= v'(t) = \frac{\pi}{2} [-\sin(\pi t/2) \cdot (\pi/2)] \\ &= (-\pi^2/4) \sin(\pi t/2) \text{ m/s}^2 \end{aligned}$$

$$a(1) = (-\pi^2/4) \sin(\pi/2) = -\pi^2/4 \text{ m/s}^2$$



(i) The particle is speeding up when v and a have the same sign. From the figure in part (h), we see that v and a are both positive when $3 < t < 4$ and both negative when $1 < t < 2$ and $5 < t < 6$. Thus, the particle is speeding up when $1 < t < 2$, $3 < t < 4$, and $5 < t < 6$. The particle is slowing down when v and a have opposite signs; that is, when $0 < t < 1$, $2 < t < 3$, and $4 < t < 5$.

4. (a) $s = f(t) = \frac{9t}{t^2 + 9}$ (in m) $\Rightarrow v(t) = f'(t) = \frac{(t^2 + 9)(9) - 9t(2t)}{(t^2 + 9)^2} = \frac{-9t^2 + 81}{(t^2 + 9)^2} = \frac{-9(t^2 - 9)}{(t^2 + 9)^2}$ (in m/s)

(b) $v(1) = \frac{-9(1 - 9)}{(1 + 9)^2} = \frac{72}{100} = 0.72$ m/s

(c) The particle is at rest when $v(t) = 0$. $\frac{-9(t^2 - 9)}{(t^2 + 9)^2} = 0 \Leftrightarrow t^2 - 9 = 0 \Rightarrow t = 3$ s [since $t \geq 0$].

(d) The particle is moving in the positive direction when $v(t) > 0$.

$$\frac{-9(t^2 - 9)}{(t^2 + 9)^2} > 0 \Rightarrow -9(t^2 - 9) > 0 \Rightarrow t^2 - 9 < 0 \Rightarrow t^2 < 9 \Rightarrow 0 \leq t < 3.$$

(e) Since the particle is moving in the positive direction and in the negative direction, we need to calculate the distance traveled in the intervals $[0, 3]$ and $[3, 6]$, respectively.

$$|f(3) - f(0)| = \left| \frac{27}{18} - 0 \right| = \frac{3}{2}$$

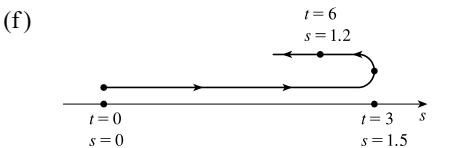
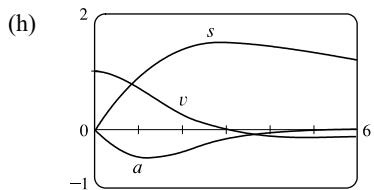
$$|f(6) - f(3)| = \left| \frac{54}{45} - \frac{27}{18} \right| = \frac{3}{10}$$

The total distance is $\frac{3}{2} + \frac{3}{10} = \frac{9}{5}$ or 1.8 m.

(g) $v(t) = -9 \frac{t^2 - 9}{(t^2 + 9)^2} \Rightarrow$

$$a(t) = v'(t) = -9 \frac{(t^2 + 9)^2(2t) - (t^2 - 9)2(t^2 + 9)(2t)}{[(t^2 + 9)^2]^2} = -9 \frac{2t(t^2 + 9)[(t^2 + 9) - 2(t^2 - 9)]}{(t^2 + 9)^4} = \frac{18t(t^2 - 27)}{(t^2 + 9)^3}.$$

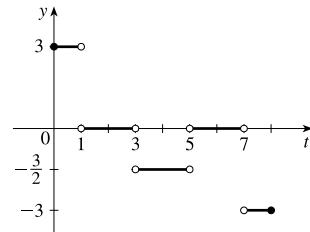
$$a(1) = \frac{18(-26)}{10^3} = -0.468 \text{ m/s}^2$$



(i) The particle is speeding up when v and a have the same sign. a is negative for $0 < t < \sqrt{27} \approx 5.2$, so from the figure in part (h), we see that v and a are both negative for $3 < t < 3\sqrt{3}$. The particle is slowing down when v and a have opposite signs. This occurs when $0 < t < 3$ and when $t > 3\sqrt{3}$.

5. (a) From the figure, the velocity v is positive on the interval $(0, 2)$ and negative on the interval $(2, 3)$. The acceleration a is positive (negative) when the slope of the tangent line is positive (negative), so the acceleration is positive on the interval $(0, 1)$, and negative on the interval $(1, 3)$. The particle is speeding up when v and a have the same sign, that is, on the interval $(0, 1)$ when $v > 0$ and $a > 0$, and on the interval $(2, 3)$ when $v < 0$ and $a < 0$. The particle is slowing down when v and a have opposite signs, that is, on the interval $(1, 2)$ when $v > 0$ and $a < 0$.
- (b) $v > 0$ on $(0, 3)$ and $v < 0$ on $(3, 4)$. $a > 0$ on $(1, 2)$ and $a < 0$ on $(0, 1)$ and $(2, 4)$. The particle is speeding up on $(1, 2)$ [$v > 0, a > 0$] and on $(3, 4)$ [$v < 0, a < 0$]. The particle is slowing down on $(0, 1)$ and $(2, 3)$ [$v > 0, a < 0$].
6. (a) The velocity v is positive when s is increasing, that is, on the intervals $(0, 1)$ and $(3, 4)$; and it is negative when s is decreasing, that is, on the interval $(1, 3)$. The acceleration a is positive when the graph of s is concave upward (v is increasing), that is, on the interval $(2, 4)$; and it is negative when the graph of s is concave downward (v is decreasing), that is, on the interval $(0, 2)$. The particle is speeding up on the interval $(1, 2)$ [$v < 0, a < 0$] and on $(3, 4)$ [$v > 0, a > 0$]. The particle is slowing down on the interval $(0, 1)$ [$v > 0, a < 0$] and on $(2, 3)$ [$v < 0, a > 0$].
- (b) The velocity v is positive on $(3, 4)$ and negative on $(0, 3)$. The acceleration a is positive on $(0, 1)$ and $(2, 4)$ and negative on $(1, 2)$. The particle is speeding up on the interval $(1, 2)$ [$v < 0, a < 0$] and on $(3, 4)$ [$v > 0, a > 0$]. The particle is slowing down on the interval $(0, 1)$ [$v < 0, a > 0$] and on $(2, 3)$ [$v < 0, a > 0$].
7. The particle is traveling forward when its velocity is positive. From the graph, this occurs when $0 < t < 5$. The particle is traveling backward when its velocity is negative. From the graph, this occurs when $7 < t < 8$. When $5 < t < 7$, its velocity is zero and the particle is not moving.
8. The graph of the acceleration function is the graph of the derivative of the velocity function. Since the velocity function is piecewise linear, its derivative, where it exists, equals the slope of the corresponding piece of the velocity graph. Thus, we obtain the following table and graph of the acceleration function.

Interval	Acceleration (slope of velocity graph)
$0 < t < 1$	3
$1 < t < 3$	0
$3 < t < 5$	$-\frac{3}{2}$
$5 < t < 7$	0
$7 < t < 8$	-3



The particle speeds up when its velocity and acceleration have the same sign. This occurs when $0 < t < 1$ ($v > 0$ and $a > 0$) and when $7 < t < 8$ ($v < 0$ and $a < 0$). The particle slows down when its velocity and acceleration have opposite signs. This occurs when $3 < t < 5$ ($v > 0$ and $a < 0$). The particle travels at a constant speed when its acceleration is zero. This occurs when $1 < t < 3$ and when $5 < t < 7$.

9. (a) $h(t) = 2 + 24.5t - 4.9t^2 \Rightarrow v(t) = h'(t) = 24.5 - 9.8t$. The velocity after 2 s is $v(2) = 24.5 - 9.8(2) = 4.9$ m/s and after 4 s is $v(4) = 24.5 - 9.8(4) = -14.7$ m/s.

(b) The projectile reaches its maximum height when the velocity is zero. $v(t) = 0 \Leftrightarrow 24.5 - 9.8t = 0 \Leftrightarrow$

$$t = \frac{24.5}{9.8} = 2.5 \text{ s.}$$

(c) The maximum height occurs when $t = 2.5$. $h(2.5) = 2 + 24.5(2.5) - 4.9(2.5)^2 = 32.625 \text{ m } [\text{or } 32\frac{5}{8} \text{ m}].$

(d) The projectile hits the ground when $h = 0 \Leftrightarrow 2 + 24.5t - 4.9t^2 = 0 \Leftrightarrow$

$$t = \frac{-24.5 \pm \sqrt{24.5^2 - 4(-4.9)(2)}}{2(-4.9)} \Rightarrow t = t_f \approx 5.08 \text{ s } [\text{since } t \geq 0].$$

(e) The projectile hits the ground when $t = t_f$. Its velocity is $v(t_f) = 24.5 - 9.8t_f \approx -25.3 \text{ m/s } [\text{downward}].$

- 10.** (a) At maximum height, the velocity of the ball is 0 m/s. $v(t) = s'(t) = 24.5 - 9.8t \Leftrightarrow 9.8t = 24.5 \Leftrightarrow t = \frac{5}{2}$. So the maximum height is $s(\frac{5}{2}) = 24.5(\frac{5}{2}) - 4.9(\frac{5}{2})^2 = 30.625 \text{ m.}$

(b) $s(t) = 24.5t - 4.9t^2 = 29.4 \Leftrightarrow 4.9t^2 - 24.5 + 29.4 = 0 \Leftrightarrow 4.9(t^2 - 5t + 6) = 0 \Leftrightarrow 4.9(t-2)(t-3) = 0.$

So the ball has a height of 29.4 m on the way up when $t = 2$ and on the way down at $t = 3$. At these times the velocities are $v(2) = 24.5 - 9.8(2) = 4.9 \text{ m/s}$ and $v(3) = 24.5 - 9.8(3) = -4.9 \text{ m/s}$, respectively.

- 11.** (a) $h(t) = 15t - 1.86t^2 \Rightarrow v(t) = h'(t) = 15 - 3.72t$. The velocity after 2 s is $v(2) = 15 - 3.72(2) = 7.56 \text{ m/s.}$

(b) $25 = h \Leftrightarrow 1.86t^2 - 15t + 25 = 0 \Leftrightarrow t = \frac{15 \pm \sqrt{15^2 - 4(1.86)(25)}}{2(1.86)} \Leftrightarrow t = t_1 \approx 2.35 \text{ or } t = t_2 \approx 5.71.$

The velocities are $v(t_1) = 15 - 3.72t_1 \approx 6.24 \text{ m/s } [\text{upward}]$ and $v(t_2) = 15 - 3.72t_2 \approx -6.24 \text{ m/s } [\text{downward}].$

- 12.** (a) $s(t) = t^4 - 4t^3 - 20t^2 + 20t \Rightarrow v(t) = s'(t) = 4t^3 - 12t^2 - 40t + 20$. $v = 20 \Leftrightarrow$

$$4t^3 - 12t^2 - 40t + 20 = 20 \Leftrightarrow 4t^3 - 12t^2 - 40t = 0 \Leftrightarrow 4t(t^2 - 3t - 10) = 0 \Leftrightarrow$$

$$4t(t-5)(t+2) = 0 \Leftrightarrow t = 0 \text{ s or } 5 \text{ s } [\text{for } t \geq 0].$$

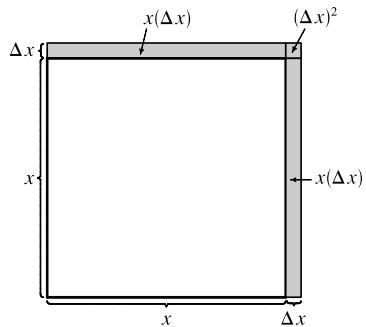
(b) $a(t) = v'(t) = 12t^2 - 24t - 40$. $a = 0 \Leftrightarrow 12t^2 - 24t - 40 = 0 \Leftrightarrow 4(3t^2 - 6t - 10) = 0 \Leftrightarrow$

$$t = \frac{6 \pm \sqrt{6^2 - 4(3)(-10)}}{2(3)} = 1 \pm \frac{1}{3}\sqrt{39} \approx 3.08 \text{ s } [\text{for } t \geq 0]. \text{ At this time, the acceleration changes from negative to}$$

positive and the velocity attains its minimum value.

- 13.** (a) $A(x) = x^2 \Rightarrow A'(x) = 2x$. $A'(15) = 30 \text{ mm}^2/\text{mm}$ is the rate at which the area is increasing with respect to the side length as x reaches 15 mm.

- (b) The perimeter is $P(x) = 4x$, so $A'(x) = 2x = \frac{1}{2}(4x) = \frac{1}{2}P(x)$. The figure suggests that if Δx is small, then the change in the area of the square is approximately half of its perimeter (2 of the 4 sides) times Δx . From the figure, $\Delta A = 2x(\Delta x) + (\Delta x)^2$. If Δx is small, then $\Delta A \approx 2x(\Delta x)$ and so $\Delta A/\Delta x \approx 2x$.

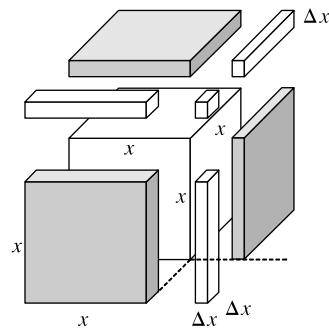


14. (a) $V(x) = x^3 \Rightarrow \frac{dV}{dx} = 3x^2$. $\left. \frac{dV}{dx} \right|_{x=3} = 3(3)^2 = 27 \text{ mm}^3/\text{mm}$ is the rate at which the volume is increasing as x increases past 3 mm.

(b) The surface area is $S(x) = 6x^2$, so $V'(x) = 3x^2 = \frac{1}{2}(6x^2) = \frac{1}{2}S(x)$.

The figure suggests that if Δx is small, then the change in the volume of the cube is approximately half of its surface area (the area of 3 of the 6 faces) times Δx . From the figure, $\Delta V = 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3$.

If Δx is small, then $\Delta V \approx 3x^2(\Delta x)$ and so $\Delta V/\Delta x \approx 3x^2$.



15. (a) Using $A(r) = \pi r^2$, we find that the average rate of change is:

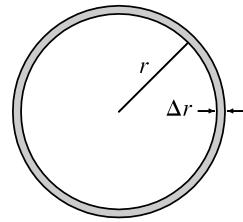
$$(i) \frac{A(3) - A(2)}{3 - 2} = \frac{9\pi - 4\pi}{1} = 5\pi$$

$$(ii) \frac{A(2.5) - A(2)}{2.5 - 2} = \frac{6.25\pi - 4\pi}{0.5} = 4.5\pi$$

$$(iii) \frac{A(2.1) - A(2)}{2.1 - 2} = \frac{4.41\pi - 4\pi}{0.1} = 4.1\pi$$

(b) $A(r) = \pi r^2 \Rightarrow A'(r) = 2\pi r$, so $A'(2) = 4\pi$.

(c) The circumference is $C(r) = 2\pi r = A'(r)$. The figure suggests that if Δr is small, then the change in the area of the circle (a ring around the outside) is approximately equal to its circumference times Δr . Straightening out this ring gives us a shape that is approximately rectangular with length $2\pi r$ and width Δr , so $\Delta A \approx 2\pi r(\Delta r)$.



Algebraically, $\Delta A = A(r + \Delta r) - A(r) = \pi(r + \Delta r)^2 - \pi r^2 = 2\pi r(\Delta r) + \pi(\Delta r)^2$.

So we see that if Δr is small, then $\Delta A \approx 2\pi r(\Delta r)$ and therefore, $\Delta A/\Delta r \approx 2\pi r$.

16. After t seconds the radius is $r = 60t$, so the area is $A(t) = \pi(60t)^2 = 3600\pi t^2 \Rightarrow A'(t) = 7200\pi t \Rightarrow$

$$(a) A'(1) = 7200\pi \text{ cm}^2/\text{s}$$

$$(b) A'(3) = 21,600\pi \text{ cm}^2/\text{s}$$

$$(c) A'(5) = 36,000\pi \text{ cm}^2/\text{s}$$

As time goes by, the area grows at an increasing rate. In fact, the rate of change is linear with respect to time.

17. $S(r) = 4\pi r^2 \Rightarrow S'(r) = 8\pi r \Rightarrow$

$$(a) S'(20) = 160\pi \text{ m}^2/\text{m}$$

$$(b) S'(40) = 320\pi \text{ m}^2/\text{m}$$

$$(c) S'(60) = 480\pi \text{ m}^2/\text{m}$$

As the radius increases, the surface area grows at an increasing rate. In fact, the rate of change is linear with respect to the radius.

18. (a) Using $V(r) = \frac{4}{3}\pi r^3$, we find that the average rate of change is:

$$(i) \frac{V(8) - V(5)}{8 - 5} = \frac{\frac{4}{3}\pi(512) - \frac{4}{3}\pi(125)}{3} = 172\pi \mu\text{m}^3/\mu\text{m}$$

$$(ii) \frac{V(6) - V(5)}{6 - 5} = \frac{\frac{4}{3}\pi(216) - \frac{4}{3}\pi(125)}{1} = 121.\overline{3}\pi \mu\text{m}^3/\mu\text{m}$$

$$(iii) \frac{V(5.1) - V(5)}{5.1 - 5} = \frac{\frac{4}{3}\pi(5.1)^3 - \frac{4}{3}\pi(5)^3}{0.1} = 102.01\overline{3}\pi \mu\text{m}^3/\mu\text{m}$$

(b) $V'(r) = 4\pi r^2$, so $V'(5) = 100\pi \mu\text{m}^3/\mu\text{m}$.

(c) $V(r) = \frac{4}{3}\pi r^3 \Rightarrow V'(r) = 4\pi r^2 = S(r)$. By analogy with Exercise 15(c), we can say that the change in the volume of the spherical shell, ΔV , is approximately equal to its thickness, Δr , times the surface area of the inner sphere. Thus, $\Delta V \approx 4\pi r^2(\Delta r)$ and so $\Delta V/\Delta r \approx 4\pi r^2$.

- 19.** The mass is $f(x) = 3x^2$, so the linear density at x is $\rho(x) = f'(x) = 6x$.

(a) $\rho(1) = 6 \text{ kg/m}$ (b) $\rho(2) = 12 \text{ kg/m}$ (c) $\rho(3) = 18 \text{ kg/m}$

Since ρ is an increasing function, the density will be the highest at the right end of the rod and lowest at the left end.

- 20.** $V(t) = 5000\left(1 - \frac{1}{40}t\right)^2 \Rightarrow V'(t) = 5000 \cdot 2\left(1 - \frac{1}{40}t\right)\left(-\frac{1}{40}\right) = -250\left(1 - \frac{1}{40}t\right)$

<p>(a) $V'(5) = -250\left(1 - \frac{5}{40}\right) = -218.75 \text{ L/min}$</p>	<p>(b) $V'(10) = -250\left(1 - \frac{10}{40}\right) = -187.5 \text{ L/min}$</p>
<p>(c) $V'(20) = -250\left(1 - \frac{20}{40}\right) = -125 \text{ L/min}$</p>	<p>(d) $V'(40) = -250\left(1 - \frac{40}{40}\right) = 0 \text{ L/min}$</p>

The water is flowing out the fastest at the beginning—when $t = 0$, $V'(t) = -250 \text{ L/min}$. The water is flowing out the slowest at the end—when $t = 40$, $V'(t) = 0$. As the tank empties, the water flows out more slowly.

- 21.** The quantity of charge is $Q(t) = t^3 - 2t^2 + 6t + 2$, so the current is $Q'(t) = 3t^2 - 4t + 6$.

(a) $Q'(0.5) = 3(0.5)^2 - 4(0.5) + 6 = 4.75 \text{ A}$ (b) $Q'(1) = 3(1)^2 - 4(1) + 6 = 5 \text{ A}$

The current is lowest when Q' has a minimum. $Q''(t) = 6t - 4 < 0$ when $t < \frac{2}{3}$. So the current decreases when $t < \frac{2}{3}$ and increases when $t > \frac{2}{3}$. Thus, the current is lowest at $t = \frac{2}{3} \text{ s}$.

- 22.** (a) $F = \frac{GmM}{r^2} = (GmM)r^{-2} \Rightarrow \frac{dF}{dr} = -2(GmM)r^{-3} = -\frac{2GmM}{r^3}$, which is the rate of change of the force with respect to the distance between the bodies. The minus sign indicates that as the distance r between the bodies increases, the magnitude of the force F exerted by the body of mass m on the body of mass M is decreasing.

(b) Given $F'(20,000) = -2$, find $F'(10,000)$. $-2 = -\frac{2GmM}{20,000^3} \Rightarrow GmM = 20,000^3$.

$$F'(10,000) = -\frac{2(20,000^3)}{10,000^3} = -2 \cdot 2^3 = -16 \text{ N/km}$$

- 23.** With $m = m_0 \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$,

$$\begin{aligned} F &= \frac{d}{dt}(mv) = m \frac{d}{dt}(v) + v \frac{d}{dt}(m) = m_0 \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \cdot a + v \cdot m_0 \left[-\frac{1}{2} \left(1 - \frac{v^2}{c^2}\right)^{-3/2}\right] \left(-\frac{2v}{c^2}\right) \frac{d}{dt}(v) \\ &= m_0 \left(1 - \frac{v^2}{c^2}\right)^{-3/2} \cdot a \left[\left(1 - \frac{v^2}{c^2}\right) + \frac{v^2}{c^2}\right] = \frac{m_0 a}{(1 - v^2/c^2)^{3/2}} \end{aligned}$$

Note that we factored out $(1 - v^2/c^2)^{-3/2}$ since $-3/2$ was the lesser exponent. Also note that $\frac{d}{dt}(v) = a$.

24. (a) $D(t) = 7 + 5 \cos[0.503(t - 6.75)] \Rightarrow D'(t) = -5 \sin[0.503(t - 6.75)](0.503) = -2.515 \sin[0.503(t - 6.75)].$

At 3:00 AM, $t = 3$, and $D'(3) = -2.515 \sin[0.503(-3.75)] \approx 2.39$ m/h (rising).

(b) At 6:00 AM, $t = 6$, and $D'(6) = -2.515 \sin[0.503(-0.75)] \approx 0.93$ m/h (rising).

(c) At 9:00 AM, $t = 9$, and $D'(9) = -2.515 \sin[0.503(2.25)] \approx -2.28$ m/h (falling).

(d) At noon, $t = 12$, and $D'(12) = -2.515 \sin[0.503(5.25)] \approx -1.21$ m/h (falling).

25. (a) To find the rate of change of volume with respect to pressure, we first solve for V in terms of P .

$$PV = C \Rightarrow V = \frac{C}{P} \Rightarrow \frac{dV}{dP} = -\frac{C}{P^2}.$$

(b) From the formula for dV/dP in part (a), we see that as P increases, the absolute value of dV/dP decreases.

Thus, the volume is decreasing more rapidly at the beginning of the 10 minutes.

$$(c) \beta = -\frac{1}{V} \frac{dV}{dP} = -\frac{1}{V} \left(-\frac{C}{P^2} \right) \quad [\text{from part (a)}] = \frac{C}{(PV)P} = \frac{C}{CP} = \frac{1}{P}$$

$$26. (a) [C] = \frac{a^2 kt}{akt + 1} \Rightarrow \text{rate of reaction} = \frac{d[C]}{dt} = \frac{(akt + 1)(a^2 k) - (a^2 kt)(ak)}{(akt + 1)^2} = \frac{a^2 k(akt + 1 - akt)}{(akt + 1)^2} = \frac{a^2 k}{(akt + 1)^2}$$

$$(b) \text{If } x = [C], \text{ then } a - x = a - \frac{a^2 kt}{akt + 1} = \frac{a^2 kt + a - a^2 kt}{akt + 1} = \frac{a}{akt + 1}.$$

$$\text{So } k(a - x)^2 = k \left(\frac{a}{akt + 1} \right)^2 = \frac{a^2 k}{(akt + 1)^2} = \frac{d[C]}{dt} \quad [\text{from part (a)}] = \frac{dx}{dt}.$$

$$27. (a) \mathbf{1920:} m_1 = \frac{1860 - 1750}{1920 - 1910} = \frac{110}{10} = 11, m_2 = \frac{2070 - 1860}{1930 - 1920} = \frac{210}{10} = 21,$$

$$(m_1 + m_2)/2 = (11 + 21)/2 = 16 \text{ million/year}$$

$$\mathbf{1980:} m_1 = \frac{4450 - 3710}{1980 - 1970} = \frac{740}{10} = 74, m_2 = \frac{5280 - 4450}{1990 - 1980} = \frac{830}{10} = 83,$$

$$(m_1 + m_2)/2 = (74 + 83)/2 = 78.5 \text{ million/year}$$

(b) $P(t) = at^3 + bt^2 + ct + d$ (in millions of people), where $a \approx -0.0002849003$, $b \approx 0.52243312243$,

$c \approx -6.395641396$, and $d \approx 1720.586081$.

(c) $P(t) = at^3 + bt^2 + ct + d \Rightarrow P'(t) = 3at^2 + 2bt + c$ (in millions of people per year)

(d) 1920 corresponds to $t = 20$ and $P'(20) \approx 14.16$ million/year. 1980 corresponds to $t = 80$ and

$P'(80) \approx 71.72$ million/year. These estimates are smaller than the estimates in part (a).

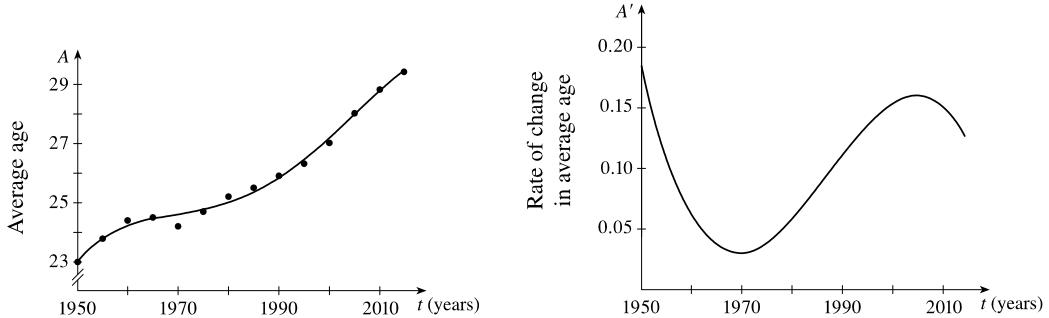
(e) $f(t) = pq^t$ (where $p = 1.43653 \times 10^9$ and $q = 1.01395$) $\Rightarrow f'(t) = pq^t \ln q$ (in millions of people per year)

28. (a) $A(t) = at^4 + bt^3 + ct^2 + dt + e$ years of age, where $a \approx -1.404\ 771\ 699 \times 10^{-6}$, $b \approx 0.011\ 167\ 331\ 7$,
 $c \approx -33.288\ 096\ 21$, $d \approx 44,097.25101$, $e \approx -21,904,396.36$.

(b) $A(t) = at^4 + bt^3 + ct^2 + dt + e \Rightarrow A'(t) = 4at^3 + 3bt^2 + 2ct + d$ (in years of age per year).

(c) $A'(1990) \approx 0.11$, so the rate of change of first marriage age for Japanese women in 1990 was approximately 0.11 years of age per year.

(d) The model for A and the data points are shown on the left, and a graph for $A'(t)$ is shown on the right:



29. (a) Using $v = \frac{P}{4\eta l}(R^2 - r^2)$ with $R = 0.01$, $l = 3$, $P = 3000$, and $\eta = 0.027$, we have v as a function of r :

$$v(r) = \frac{3000}{4(0.027)3}(0.01^2 - r^2). v(0) = 0.925 \text{ cm/s}, v(0.005) = 0.694 \text{ cm/s}, v(0.01) = 0.$$

(b) $v(r) = \frac{P}{4\eta l}(R^2 - r^2) \Rightarrow v'(r) = \frac{P}{4\eta l}(-2r) = -\frac{Pr}{2\eta l}$. When $l = 3$, $P = 3000$, and $\eta = 0.027$, we have

$$v'(r) = -\frac{3000r}{2(0.027)3}. v'(0) = 0, v'(0.005) = -92.592 \text{ (cm/s)/cm}, \text{ and } v'(0.01) = -185.185 \text{ (cm/s)/cm}.$$

(c) The velocity is greatest where $r = 0$ (at the center) and the velocity is changing most where $r = R = 0.01$ cm (at the edge).

30. (a) (i) $f = \frac{1}{2L} \sqrt{\frac{T}{\rho}} = \left(\frac{1}{2} \sqrt{\frac{T}{\rho}}\right) L^{-1} \Rightarrow \frac{df}{dL} = -\left(\frac{1}{2} \sqrt{\frac{T}{\rho}}\right) L^{-2} = -\frac{1}{2L^2} \sqrt{\frac{T}{\rho}}$

(ii) $f = \frac{1}{2L} \sqrt{\frac{T}{\rho}} = \left(\frac{1}{2L\sqrt{\rho}}\right) T^{1/2} \Rightarrow \frac{df}{dT} = \frac{1}{2} \left(\frac{1}{2L\sqrt{\rho}}\right) T^{-1/2} = \frac{1}{4L\sqrt{T\rho}}$

(iii) $f = \frac{1}{2L} \sqrt{\frac{T}{\rho}} = \left(\frac{\sqrt{T}}{2L}\right) \rho^{-1/2} \Rightarrow \frac{df}{d\rho} = -\frac{1}{2} \left(\frac{\sqrt{T}}{2L}\right) \rho^{-3/2} = -\frac{\sqrt{T}}{4L\rho^{3/2}}$

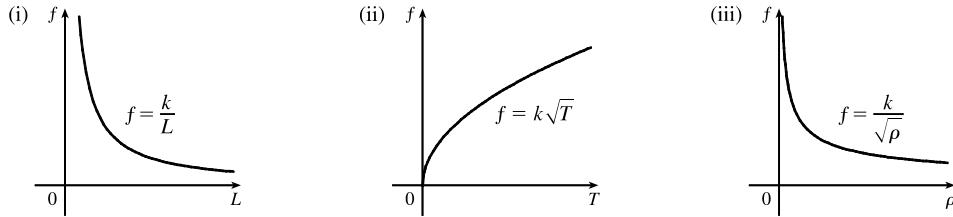
(b) Note: Illustrating tangent lines on the generic figures may help to explain the results.

(i) $\frac{df}{dL} < 0$ and L is decreasing $\Rightarrow f$ is increasing \Rightarrow higher note

(ii) $\frac{df}{dT} > 0$ and T is increasing $\Rightarrow f$ is increasing \Rightarrow higher note

[continued]

(iii) $\frac{df}{d\rho} < 0$ and ρ is increasing $\Rightarrow f$ is decreasing \Rightarrow lower note



31. (a) $C(x) = 2000 + 3x + 0.01x^2 + 0.0002x^3 \Rightarrow C'(x) = 0 + 3(1) + 0.01(2x) + 0.0002(3x^2) = 3 + 0.02x + 0.0006x^2$

(b) $C''(100) = 3 + 0.02(100) + 0.0006(100)^2 = 3 + 2 + 6 = \$11/\text{pair}$. $C'(100)$ is the rate at which the cost is increasing as the 100th pair of jeans is produced. It predicts the (approximate) cost of the 101st pair.

(c) The cost of manufacturing the 101st pair of jeans is

$$C(101) - C(100) = 2611.0702 - 2600 = 11.0702 \approx \$11.07. \text{ This is close to the marginal cost from part (b).}$$

32. (a) $C(q) = 84 + 0.16q - 0.0006q^2 + 0.000003q^3 \Rightarrow C'(q) = 0.16 - 0.0012q + 0.000009q^2$, and

$C''(100) = 0.16 - 0.0012(100) + 0.000009(100)^2 = 0.13$. This is the rate at which the cost is increasing as the 100th item is produced.

(b) The actual cost of producing the 101st item is $C(101) - C(100) = 97.13030299 - 97 \approx \0.13

33. (a) $A(x) = \frac{p(x)}{x} \Rightarrow A'(x) = \frac{xp'(x) - p(x) \cdot 1}{x^2} = \frac{xp'(x) - p(x)}{x^2}$.

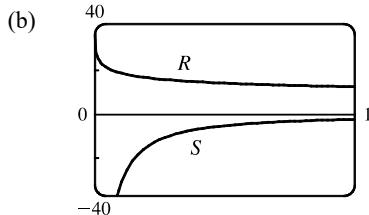
$A'(x) > 0 \Rightarrow A(x)$ is increasing; that is, the average productivity increases as the size of the workforce increases.

(b) $p'(x)$ is greater than the average productivity $\Rightarrow p'(x) > A(x) \Rightarrow p'(x) > \frac{p(x)}{x} \Rightarrow xp'(x) > p(x) \Rightarrow$

$$xp'(x) - p(x) > 0 \Rightarrow \frac{xp'(x) - p(x)}{x^2} > 0 \Rightarrow A'(x) > 0.$$

34. (a) $R = \frac{40 + 24x^{0.4}}{1 + 4x^{0.4}} \Rightarrow S = \frac{dR}{dx} = \frac{(1 + 4x^{0.4})(9.6x^{-0.6}) - (40 + 24x^{0.4})(1.6x^{-0.6})}{(1 + 4x^{0.4})^2}$

$$= \frac{9.6x^{-0.6} + 38.4x^{-0.2} - 64x^{-0.6} - 38.4x^{-0.2}}{(1 + 4x^{0.4})^2} = -\frac{54.4x^{-0.6}}{(1 + 4x^{0.4})^2}$$



At low levels of brightness, R is quite large [$R(0) = 40$] and is quickly decreasing, that is, S is negative with large absolute value. This is to be expected: at low levels of brightness, the eye is more sensitive to slight changes than it is at higher levels of brightness.

35. $PV = nRT \Rightarrow T = \frac{PV}{nR} = \frac{PV}{(10)(0.0821)} = \frac{1}{0.821}(PV)$. Using the Product Rule, we have

$$\frac{dT}{dt} = \frac{1}{0.821} [P(t)V'(t) + V(t)P'(t)] = \frac{1}{0.821} [(8)(-0.15) + (10)(0.10)] \approx -0.2436 \text{ K/min.}$$

36. $f(r) = 2\sqrt{Dr} \Rightarrow f'(r) = 2 \cdot \frac{1}{2}(Dr)^{-1/2} \cdot D = \frac{D}{\sqrt{Dr}} = \sqrt{\frac{D}{r}}$. $f'(r)$ is the rate of change of the wave speed with respect to the reproductive rate.

37. (a) If the populations are stable, then the growth rates are neither positive nor negative; that is, $\frac{dC}{dt} = 0$ and $\frac{dW}{dt} = 0$.

(b) “The caribou go extinct” means that the population is zero, or mathematically, $C = 0$.

(c) We have the equations $\frac{dC}{dt} = aC - bCW$ and $\frac{dW}{dt} = -cW + dCW$. Let $dC/dt = dW/dt = 0$, $a = 0.05$, $b = 0.001$, $c = 0.05$, and $d = 0.0001$ to obtain $0.05C - 0.001CW = 0$ (1) and $-0.05W + 0.0001CW = 0$ (2). Adding 10 times (2) to (1) eliminates the CW -terms and gives us $0.05C - 0.5W = 0 \Rightarrow C = 10W$. Substituting $C = 10W$ into (1) results in $0.05(10W) - 0.001(10W)W = 0 \Leftrightarrow 0.5W - 0.01W^2 = 0 \Leftrightarrow 50W - W^2 = 0 \Leftrightarrow W(50 - W) = 0 \Leftrightarrow W = 0 \text{ or } 50$. Since $C = 10W$, $C = 0$ or 500 . Thus, the population pairs (C, W) that lead to stable populations are $(0, 0)$ and $(500, 50)$. So it is possible for the two species to live in harmony.

38. (a) The rate of change of retention t days after a task is learned is given by $R'(t)$. $R(t) = a + b(1 + ct)^{-\beta} \Rightarrow$

$$R'(t) = b \cdot (-\beta)(1 + ct)^{-\beta-1} \cdot c = -\beta bc(1 + ct)^{-\beta-1} \text{ (expressed as a fraction of memory per day).}$$

(b) We may write the rate of change as $R'(t) = -\frac{\beta bc}{(1 + ct)^{\beta+1}}$. The magnitude of this quantity decreases as t increases.

Thus, you forget how to perform a task faster soon after learning it than a long time after you have learned it.

$$(c) \lim_{t \rightarrow \infty} R(t) = \lim_{t \rightarrow \infty} \left[a + \frac{b}{(1 + ct)^\beta} \right] = a + 0 = a, \text{ so the fraction of memory that is permanent is } a.$$

2.8 Related Rates

$$1. (a) V = x^3 \Rightarrow \frac{dV}{dt} = \frac{dV}{dx} \frac{dx}{dt} = 3x^2 \frac{dx}{dt}$$

$$(b) \text{With } \frac{dx}{dt} = 4 \text{ cm/s and } x = 15 \text{ cm, we have } \frac{dV}{dt} = 3(15)^2 \cdot 4 = 2700 \text{ cm}^3/\text{s.}$$

$$2. (a) A = \pi r^2 \Rightarrow \frac{dA}{dt} = \frac{dA}{dr} \frac{dr}{dt} = 2\pi r \frac{dr}{dt}$$

$$(b) \text{With } \frac{dr}{dt} = 2 \text{ m/s and } r = 30 \text{ m, we have } \frac{dA}{dt} = 2\pi \cdot 30 \cdot 2 = 120\pi \text{ m}^2/\text{s.}$$

3. Let s denote the side of a square. The square’s area A is given by $A = s^2$. Differentiating with respect to t gives us

$$\frac{dA}{dt} = 2s \frac{ds}{dt}. \text{ When } A = 16, s = 4. \text{ Substituting 4 for } s \text{ and 6 for } \frac{ds}{dt} \text{ gives us } \frac{dA}{dt} = 2(4)(6) = 48 \text{ cm}^2/\text{s.}$$

4. $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \frac{dr}{dt} \Rightarrow \frac{dV}{dt} = 4\pi \left(\frac{1}{2} \cdot 80\right)^2 (4) = 25,600\pi \text{ mm}^3/\text{s}.$

5. $S = 4\pi r^2 \Rightarrow \frac{dS}{dt} = 4\pi \cdot 2r \frac{dr}{dt} \Rightarrow \frac{dS}{dt} = 4\pi \cdot 2 \cdot 8 \cdot 2 = 128\pi \text{ cm}^2/\text{min}.$

6. $A = \ell w \Rightarrow \frac{dA}{dt} = \ell \cdot \frac{dw}{dt} + w \cdot \frac{d\ell}{dt} = 20(3) + 10(8) = 140 \text{ cm}^2/\text{s}.$

7. $V = \pi r^2 h = \pi(5)^2 h = 25\pi h \Rightarrow \frac{dV}{dt} = 25\pi \frac{dh}{dt} \Rightarrow 3 = 25\pi \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{3}{25\pi} \text{ m/min}.$

8. (a) $A = \frac{1}{2}ab \sin \theta \Rightarrow \frac{dA}{dt} = \frac{1}{2}ab \cos \theta \frac{d\theta}{dt} = \frac{1}{2}(2)(3)(\cos \frac{\pi}{3})(0.2) = 3(\frac{1}{2})(0.2) = 0.3 \text{ cm}^2/\text{min}.$

(b) $A = \frac{1}{2}ab \sin \theta \Rightarrow$

$$\begin{aligned} \frac{dA}{dt} &= \frac{1}{2}a \left(b \cos \theta \frac{d\theta}{dt} + \sin \theta \frac{db}{dt} \right) = \frac{1}{2}(2)[3(\cos \frac{\pi}{3})(0.2) + (\sin \frac{\pi}{3})(1.5)] \\ &= 3(\frac{1}{2})(0.2) + \frac{1}{2}\sqrt{3}(\frac{3}{2}) = 0.3 + \frac{3}{4}\sqrt{3} \text{ cm}^2/\text{min} [\approx 1.6] \end{aligned}$$

(c) $A = \frac{1}{2}ab \sin \theta \Rightarrow$

$$\begin{aligned} \frac{dA}{dt} &= \frac{1}{2} \left(\frac{da}{dt} b \sin \theta + a \frac{db}{dt} \sin \theta + ab \cos \theta \frac{d\theta}{dt} \right) \quad [\text{by Exercise 2.3.107(a)}] \\ &= \frac{1}{2}[(2.5)(3)(\frac{1}{2}\sqrt{3}) + (2)(1.5)(\frac{1}{2}\sqrt{3}) + (2)(3)(\frac{1}{2})(0.2)] \\ &= (\frac{15}{8}\sqrt{3} + \frac{3}{4}\sqrt{3} + 0.3) = (\frac{21}{8}\sqrt{3} + 0.3) \text{ cm}^2/\text{min} [\approx 4.85] \end{aligned}$$

Note how this answer relates to the answer in part (a) [θ changing] and part (b) [b and θ changing].

9. (a) $\frac{d}{dt}(4x^2 + 9y^2) = \frac{d}{dt}(25) \Rightarrow 8x \frac{dx}{dt} + 18y \frac{dy}{dt} = 0 \Rightarrow 4x \frac{dx}{dt} + 9y \frac{dy}{dt} = 0 \Rightarrow$

$$4(2) \frac{dx}{dt} + 9(1) \cdot \frac{1}{3} = 0 \Rightarrow 8 \frac{dx}{dt} + 3 = 0 \Rightarrow \frac{dx}{dt} = -\frac{3}{8}$$

(b) $4x \frac{dx}{dt} + 9y \frac{dy}{dt} = 0 \Rightarrow 4(-2)(3) + 9(1) \cdot \frac{dy}{dt} = 0 \Rightarrow -24 + 9 \frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = \frac{24}{9} = \frac{8}{3}$

10. $\frac{d}{dt}(x^2 + y^2 + z^2) = \frac{d}{dt}(9) \Rightarrow 2x \frac{dx}{dt} + 2y \frac{dy}{dt} + 2z \frac{dz}{dt} = 0 \Rightarrow x \frac{dx}{dt} + y \frac{dy}{dt} + z \frac{dz}{dt} = 0.$

If $\frac{dx}{dt} = 5$, $\frac{dy}{dt} = 4$ and $(x, y, z) = (2, 2, 1)$, then $2(5) + 2(4) + 1 \frac{dz}{dt} = 0 \Rightarrow \frac{dz}{dt} = -18$.

11. $w = w_0 \left(\frac{6370}{6370+h} \right)^2 = w_0 \cdot 6370^2(6370+h)^{-2} \Rightarrow \frac{dw}{dt} = w_0 \cdot 6370^2(-2)(6370+h)^{-3} \cdot \frac{dh}{dt}$. Then $w_0 = 580 \text{ N}$, $h = 60 \text{ km}$,

and $dh/dt = 19 \text{ km/s} \Rightarrow \frac{dw}{dt} = 580 \cdot 6370^2(-2)(6370+60)^{-3}(19) = -3.364012 \approx -3.364 \text{ N/s}$.

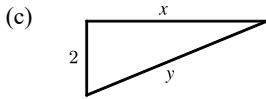
12. $\frac{d}{dt}(xy) = \frac{d}{dt}(8) \Rightarrow x \frac{dy}{dt} + y \frac{dx}{dt} = 0$. If $\frac{dy}{dt} = -3 \text{ cm/s}$ and $(x, y) = (4, 2)$, then $4(-3) + 2 \frac{dx}{dt} = 0 \Rightarrow$

$\frac{dx}{dt} = 6$. Thus, the x -coordinate is increasing at a rate of 6 cm/s.

13. (a) Given: a plane flying horizontally at an altitude of 2 km and a speed of 800 km/h passes directly over a radar station.

If we let t be time (in hours) and x be the horizontal distance traveled by the plane (in km), then we are given that $dx/dt = 800$ km/h.

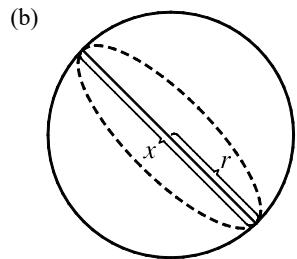
- (b) Unknown: the rate at which the distance from the plane to the station is increasing when it is 3 km from the station. If we let y be the distance from the plane to the station, then we want to find dy/dt when $y = 3$ km.



- (d) By the Pythagorean Theorem, $y^2 = x^2 + 2^2 \Rightarrow 2y(dy/dt) = 2x(dx/dt)$.

(e) $\frac{dy}{dt} = \frac{x}{y} \frac{dx}{dt} = \frac{x}{y}(800)$. Since $y^2 = x^2 + 4$, when $y = 3$, $x = \sqrt{5}$, so $\frac{dy}{dt} = \frac{\sqrt{5}}{3}(800) \approx 596$ km/h.

14. (a) Given: the rate of decrease of the surface area is 1 cm²/min. If we let t be time (in minutes) and S be the surface area (in cm²), then we are given that $dS/dt = -1$ cm²/s.



- (c) Unknown: the rate of decrease of the diameter when the diameter is 10 cm. If we let x be the diameter, then we want to find dx/dt when $x = 10$ cm.

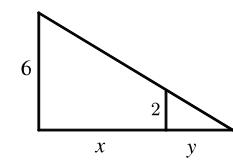
- (d) If the radius is r and the diameter $x = 2r$, then $r = \frac{1}{2}x$ and $S = 4\pi r^2 = 4\pi(\frac{1}{2}x)^2 = \pi x^2 \Rightarrow$

$$\frac{dS}{dt} = \frac{dS}{dx} \frac{dx}{dt} = 2\pi x \frac{dx}{dt}.$$

(e) $-1 = \frac{dS}{dt} = 2\pi x \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = -\frac{1}{2\pi x}$. When $x = 10$, $\frac{dx}{dt} = -\frac{1}{20\pi}$. So the rate of decrease is $\frac{1}{20\pi}$ cm/min.

15. (a) Given: a man 2 m tall walks away from a street light mounted on a 6 m tall pole at a rate of 1.5 m/s. If we let t be time (in s) and x be the distance from the pole to the man (in m), then we are given that $dx/dt = 1.5$ m/s.

- (b) Unknown: the rate at which the tip of his shadow is moving when he is 10 m from the pole. If we let y be the distance from the man to the tip of his shadow (in m), then we want to find $\frac{d}{dt}(x+y)$ when $x = 10$ m.



(d) By similar triangles, $\frac{6}{2} = \frac{x+y}{y} \Rightarrow 3y = x+y \Rightarrow 2y = x \Rightarrow y = \frac{x}{2}$.

(e) The tip of the shadow moves at a rate of $\frac{d}{dt}(x+y) = \frac{d}{dt}(x + \frac{x}{2}) = \frac{3}{2} \frac{dx}{dt} = \frac{3}{2}(1.5) = \frac{9}{4} = 2.25$ m/s

16. (a) Given: at noon, ship A is 150 km west of ship B; ship A is sailing east at 35 km/h, and ship B is sailing north at 25 km/h.

If we let t be time (in hours), x be the distance traveled by ship A (in km), and y be the distance traveled by ship B (in km), then we are given that $dx/dt = 35$ km/h and $dy/dt = 25$ km/h.

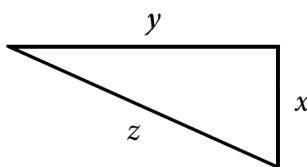
- (b) Unknown: the rate at which the distance between the ships is changing at 4:00 PM. If we let z be the distance between the ships, then we want to find dz/dt when $t = 4$ h.

$$(d) z^2 = (150 - x)^2 + y^2 \Rightarrow 2z \frac{dz}{dt} = 2(150 - x) \left(-\frac{dx}{dt} \right) + 2y \frac{dy}{dt}$$

(e) At 4:00 PM, $x = 4(35) = 140$ and $y = 4(25) = 100 \Rightarrow z = \sqrt{(150 - 140)^2 + 100^2} = \sqrt{10,100}$.

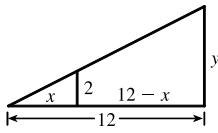
$$\text{So } \frac{dz}{dt} = \frac{1}{z} \left[(x - 150) \frac{dx}{dt} + y \frac{dy}{dt} \right] = \frac{-10(35) + 100(25)}{\sqrt{10,100}} = \frac{215}{\sqrt{101}} \approx 21.4 \text{ km/h.}$$

17.



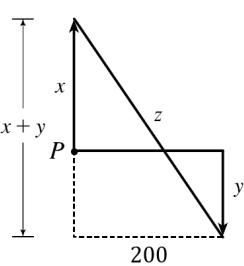
We are given that $\frac{dx}{dt} = 30$ km/h and $\frac{dy}{dt} = 72$ km/h. $z^2 = x^2 + y^2 \Rightarrow 2z \frac{dz}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \Rightarrow z \frac{dz}{dt} = x \frac{dx}{dt} + y \frac{dy}{dt} \Rightarrow \frac{dz}{dt} = \frac{1}{z} (x \frac{dx}{dt} + y \frac{dy}{dt})$. After 2 hours, $x = 2(30) = 60$, and $y = 2(72) = 144 \Rightarrow z = \sqrt{60^2 + 144^2} = 156$, so $\frac{dz}{dt} = \frac{1}{z} (x \frac{dx}{dt} + y \frac{dy}{dt}) = \frac{60(30) + 144(72)}{156} = 78$ km/h.

18.



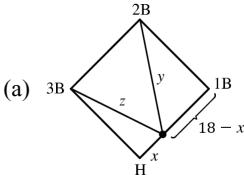
We are given that $\frac{dx}{dt} = 1.6$ m/s. By similar triangles, $\frac{y}{12} = \frac{2}{x} \Rightarrow y = \frac{24}{x} \Rightarrow \frac{dy}{dt} = -\frac{24}{x^2} \frac{dx}{dt} = -\frac{24}{x^2}(1.6)$. When $x = 8$, $\frac{dy}{dt} = -\frac{24(1.6)}{64} = -0.6$ m/s, so the shadow is decreasing at a rate of 0.6 m/s.

19.



We are given that $\frac{dx}{dt} = 1.2$ m/s and $\frac{dy}{dt} = 1.6$ m/s. $z^2 = (x + y)^2 + 200^2 \Rightarrow 2z \frac{dz}{dt} = 2(x + y) \left(\frac{dx}{dt} + \frac{dy}{dt} \right)$. 15 minutes after the woman starts, we have $x = (1.2 \text{ m/s})(20 \text{ min})(60 \text{ s/min}) = 1440$ m and $y = 1.6 \cdot 15 \cdot 60 = 1440 \Rightarrow z = \sqrt{(2 \cdot 1440)^2 + 200^2} = \sqrt{8,334,400}$, so $\frac{dz}{dt} = \frac{x+y}{z} \left(\frac{dx}{dt} + \frac{dy}{dt} \right) = \frac{1440+1440}{\sqrt{8,334,400}} (1.2 + 1.6) = \frac{8064}{10\sqrt{83,344}} \approx 2.79$ m/s.

20. We are given that $\frac{dx}{dt} = 7.5$ m/s.

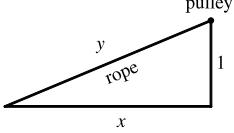


$y^2 = (18-x)^2 + 18^2 \Rightarrow 2y \frac{dy}{dt} = 2(18-x) \left(-\frac{dx}{dt} \right)$. When $x = 9$, $y = \sqrt{9^2 + 18^2} = 9\sqrt{5}$, so $\frac{dy}{dt} = \frac{18-x}{y} \left(-\frac{dx}{dt} \right) = \frac{9}{9\sqrt{5}} (-7.5) = -\frac{7.5}{\sqrt{5}}$, so the distance from second base is decreasing at a rate of $\frac{7.5}{\sqrt{5}} \approx 3.35$ m/s.

(b) Due to the symmetric nature of the problem in part (a), we expect to get the same answer—and we do.

$$z^2 = x^2 + 18 \Rightarrow 2z \frac{dz}{dt} = 2x \frac{dx}{dt}. \text{ When } x = 9, z = 9\sqrt{5}, \text{ so } \frac{dz}{dt} = \frac{9}{9\sqrt{5}}(7.5) = \frac{7.5}{\sqrt{5}} \approx 3.35 \text{ m/s.}$$

21. $A = \frac{1}{2}bh$, where b is the base and h is the altitude. We are given that $\frac{dh}{dt} = 1$ cm/min and $\frac{dA}{dt} = 2$ cm²/min. Using the Product Rule, we have $\frac{dA}{dt} = \frac{1}{2}\left(b \frac{dh}{dt} + h \frac{db}{dt}\right)$. When $h = 10$ and $A = 100$, we have $100 = \frac{1}{2}b(10) \Rightarrow \frac{1}{2}b = 10 \Rightarrow b = 20$, so $2 = \frac{1}{2}\left(20 \cdot 1 + 10 \frac{db}{dt}\right) \Rightarrow 4 = 20 + 10 \frac{db}{dt} \Rightarrow \frac{db}{dt} = \frac{4 - 20}{10} = -1.6$ cm/min.

22.  Given $\frac{dy}{dt} = -1$ m/s, find $\frac{dx}{dt}$ when $x = 8$ m. $y^2 = x^2 + 1 \Rightarrow 2y \frac{dy}{dt} = 2x \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{y}{x} \frac{dy}{dt} = -\frac{y}{x}$. When $x = 8$, $y = \sqrt{65}$, so $\frac{dx}{dt} = -\frac{\sqrt{65}}{8}$. Thus, the boat approaches the dock at $\frac{\sqrt{65}}{8} \approx 1.01$ m/s.

23. Let x be the distance (in meters) the first dropped stone has traveled, and let y be the distance (in meters) the stone dropped one second later has traveled. Let t be the time (in seconds) since the woman drops the second stone. Using $d = 4.9t^2$, we have $x = 4.9(t+1)^2$ and $y = 4.9t^2$. Let z be the distance between the stones. Then $z = x - y$ and we have

$$\frac{dz}{dt} = \frac{dx}{dt} - \frac{dy}{dt} \Rightarrow \frac{dz}{dt} = 9.8(t+1) - 9.8t = 9.8 \text{ m/s.}$$

24. Given: Two men 10 m apart each drop a stone, the second one, one minute after the first. Let x be the distance (in meters) the first dropped stone has traveled, and let y be the distance (in meters) the second stone has traveled. Let t be the time (in seconds) since the man drops the second stone. Using $d = 4.9t^2$, we have

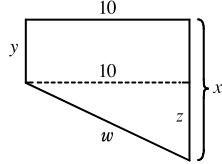
$$x = 4.9(t+1)^2 \text{ and } y = 4.9t^2. \text{ Let } z \text{ be the vertical distance between the stones. Then } z = x - y \Rightarrow$$

$$\frac{dz}{dt} = \frac{dx}{dt} - \frac{dy}{dt} \Rightarrow \frac{dz}{dt} = 9.8(t+1) - 9.8t = 9.8 \text{ m/s.}$$

By the Pythagorean Theorem, $w^2 = 10^2 + z^2$. Differentiating with respect to t , we obtain

$$2w \frac{dw}{dt} = 2z \frac{dz}{dt} \Rightarrow \frac{dw}{dt} = \frac{z(dz/dt)}{w}. \text{ One second after the second stone is dropped, } t = 1, \text{ so}$$

$$z = x - y = 4.9(1+1)^2 - 4.9(1)^2 = 14.7 \text{ m, and } w = \sqrt{10^2 + (14.7)^2} = \sqrt{316.09}, \text{ so } \frac{dw}{dt} = \frac{14.7(9.8)}{\sqrt{316.09}} \approx 8.10 \text{ m/s.}$$

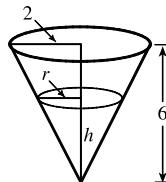


25. If $C =$ the rate at which water is pumped in, then $\frac{dV}{dt} = C - 10,000$, where

$$V = \frac{1}{3}\pi r^2 h \text{ is the volume at time } t. \text{ By similar triangles, } \frac{r}{2} = \frac{h}{6} \Rightarrow r = \frac{1}{3}h \Rightarrow$$

$$V = \frac{1}{3}\pi\left(\frac{1}{3}h\right)^2 h = \frac{\pi}{27}h^3 \Rightarrow \frac{dV}{dt} = \frac{\pi}{9}h^2 \frac{dh}{dt}. \text{ When } h = 200 \text{ cm,}$$

$$\frac{dh}{dt} = 20 \text{ cm/min, so } C - 10,000 = \frac{\pi}{9}(200)^2(20) \Rightarrow C = 10,000 + \frac{800,000}{9}\pi \approx 289,253 \text{ cm}^3/\text{min.}$$



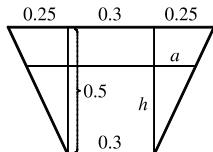
26. The distance z of the particle to the origin is given by $z = \sqrt{x^2 + y^2}$, so $z^2 = x^2 + [2 \sin(\pi x/2)]^2 \Rightarrow$

$$2z \frac{dz}{dt} = 2x \frac{dx}{dt} + 4 \cdot 2 \sin\left(\frac{\pi}{2}x\right) \cos\left(\frac{\pi}{2}x\right) \cdot \frac{\pi}{2} \frac{dx}{dt} \Rightarrow z \frac{dz}{dt} = x \frac{dx}{dt} + 2\pi \sin\left(\frac{\pi}{2}x\right) \cos\left(\frac{\pi}{2}x\right) \frac{dx}{dt}. \text{ When}$$

$$(x, y) = \left(\frac{1}{3}, 1\right), z = \sqrt{\left(\frac{1}{3}\right)^2 + 1^2} = \sqrt{\frac{10}{9}} = \frac{1}{3}\sqrt{10}, \text{ so } \frac{1}{3}\sqrt{10} \frac{dz}{dt} = \frac{1}{3}\sqrt{10} + 2\pi \sin \frac{\pi}{6} \cos \frac{\pi}{6} \cdot \sqrt{10} \Rightarrow$$

$$\frac{1}{3} \frac{dz}{dt} = \frac{1}{3} + 2\pi \left(\frac{1}{2}\right) \left(\frac{1}{2}\sqrt{3}\right) \Rightarrow \frac{dz}{dt} = 1 + \frac{3\sqrt{3}\pi}{2} \text{ cm/s.}$$

27.



The figure is labeled in meters. The area A of a trapezoid is

$$\frac{1}{2}(\text{base}_1 + \text{base}_2)(\text{height}), \text{ and the volume } V \text{ of the 10-meter-long trough is } 10A.$$

$$\text{Thus, the volume of the trapezoid with height } h \text{ is } V = (10)\frac{1}{2}[0.3 + (0.3 + 2a)]h.$$

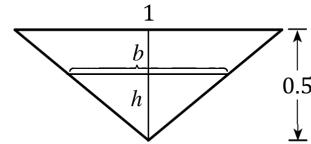
$$\text{By similar triangles, } \frac{a}{h} = \frac{0.25}{0.5} = \frac{1}{2}, \text{ so } 2a = h \Rightarrow V = 5(0.6 + h)h = 3h + 5h^2.$$

$$\text{Now } \frac{dV}{dt} = \frac{dV}{dh} \frac{dh}{dt} \Rightarrow 0.2 = (3 + 10h) \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{0.2}{3 + 10h}. \text{ When } h = 0.3,$$

$$\frac{dh}{dt} = \frac{0.2}{3 + 10(0.3)} = \frac{0.2}{6} \text{ m/min} = \frac{1}{30} \text{ m/min or } \frac{10}{3} \text{ cm/min.}$$

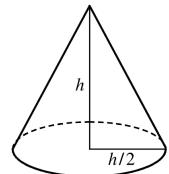
28. By similar triangles, $\frac{1}{0.5} = \frac{b}{h}$, so $b = 2h$. The trough has volume $V = \frac{1}{2}bh(6) =$

$$3(2h)h = 6h^2 \Rightarrow 1.2 = \frac{dV}{dt} = 12h \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{1}{10h}. \text{ When } h = 0.3, \frac{dh}{dt} = \frac{1}{10 \cdot 0.3} = \frac{1}{3} \text{ m/min.}$$



29. We are given that $\frac{dV}{dt} = 3 \text{ m}^3/\text{min}$. $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h = \frac{\pi h^3}{12} \Rightarrow \frac{dV}{dt} = \frac{dV}{dh} \frac{dh}{dt} \Rightarrow$

$$3 = \frac{\pi h^2}{4} \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{12}{\pi h^2}. \text{ When } h = 3 \text{ m}, \frac{dh}{dt} = \frac{12}{3^2\pi} = \frac{4}{3\pi} \approx 0.4246 \text{ m/min.}$$



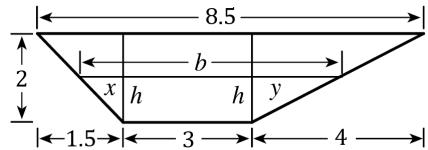
30. The figure is drawn without the top 1 meter.

$$V = \frac{1}{2}(b+3)h(5) = \frac{5}{2}(b+3)h \text{ and, from similar triangles, } \frac{x}{h} = \frac{1.5}{2} =$$

$$\frac{3}{4} \text{ and } \frac{y}{h} = \frac{4}{2} = 2, \text{ so } b = x + 3 + y = \frac{3h}{4} + 3 + 2h = 3 + \frac{11h}{4}. \text{ Thus,}$$

$$V = \frac{5}{2} \left(6 + \frac{11h}{4}\right) h = 15h + \frac{55}{8}h^2 \text{ and so } 0.1 = \frac{dV}{dt} = \left(15 + \frac{55}{4}h\right) \frac{dh}{dt}.$$

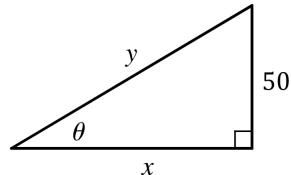
$$\text{When } h = 1, \frac{dh}{dt} = \frac{0.1}{(15 + \frac{55}{4})} = \frac{2}{575} \approx 0.00348 \text{ m/min.}$$



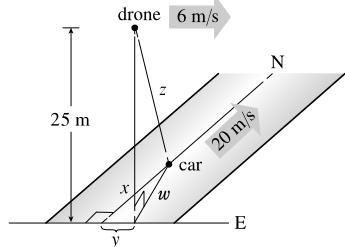
31. The area A of an equilateral triangle with side s is given by $A = \frac{1}{4}\sqrt{3}s^2$.

$$\frac{dA}{dt} = \frac{1}{4}\sqrt{3} \cdot 2s \frac{ds}{dt} = \frac{1}{4}\sqrt{3} \cdot 2(30)(10) = 150\sqrt{3} \text{ cm}^2/\text{min}.$$

32. We are given $dx/dt = 2$ m/s. $\cot \theta = \frac{x}{50} \Rightarrow x = 50 \cot \theta \Rightarrow \frac{dx}{dt} = -50 \csc^2 \theta \frac{d\theta}{dt} \Rightarrow \frac{d\theta}{dt} = -\frac{\sin^2 \theta}{50} \cdot 2$. When $y = 100$, $\sin \theta = \frac{50}{100} = \frac{1}{2} \Rightarrow \frac{d\theta}{dt} = -\frac{(1/2)^2}{50} \cdot 2 = -\frac{1}{100}$ rad/s. The angle is decreasing at a rate of $\frac{1}{100}$ rad/s.



33.



Let t be the time, in seconds, after the drone passes directly over the car. Given

$$\frac{dx}{dt} = 20 \text{ m/s}, x = 20t \text{ m}, \frac{dy}{dt} = 6 \text{ m/s}, \text{ and } y = 6t \text{ m, find } \frac{dz}{dt} \text{ when } t = 5.$$

By the Pythagorean Theorem, $w^2 = x^2 + y^2$ and $z^2 = 25^2 + w^2$. This gives

$$\begin{aligned} z^2 &= 25^2 + x^2 + y^2 \Rightarrow \\ 2z \frac{dz}{dt} &= 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \Rightarrow \frac{dz}{dt} = \frac{x(dx/dt) + y(dy/dt)}{z} \end{aligned}$$

When $t = 5$, $x = 20(5) = 100$ and $y = 6(5) = 30$, so $z^2 = 25^2 + 100^2 + 30^2 \Rightarrow z = \sqrt{11,525}$ m.

$$\frac{dz}{dt} = \frac{100(20) + 30(6)}{\sqrt{11,525}} \approx 20.3 \text{ m/s.}$$

34. The area A of a sector of a circle with radius r and angle θ is given by $A = \frac{1}{2}r^2\theta$. Here r is constant and θ varies, so

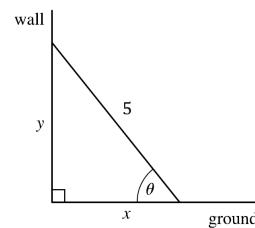
$$\frac{dA}{dt} = \frac{1}{2}r^2 \frac{d\theta}{dt}. \text{ The minute hand rotates through } 360^\circ = 2\pi \text{ radians each hour, so } \frac{d\theta}{dt} = 2\pi \text{ and}$$

$$\frac{dA}{dt} = \frac{1}{2}r^2(2\pi) = \pi r^2 \text{ cm}^2/\text{h. This answer makes sense because the minute hand sweeps through the full area of a circle, } \pi r^2, \text{ each hour.}$$

35. $\cos \theta = \frac{x}{5} \Rightarrow -\sin \theta \frac{d\theta}{dt} = \frac{1}{5} \frac{dx}{dt}$. From Example 2, $\frac{dx}{dt} = 1$ and

$$\text{when } x = 3, y = 4, \text{ so } \sin \theta = \frac{4}{5}. \text{ Thus, } -\frac{4}{5} \frac{d\theta}{dt} = \frac{1}{5}(1) \Rightarrow \frac{d\theta}{dt} =$$

$$-\frac{1}{4} \text{ rad/s.}$$



36. According to the model in Example 2, $\frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt} \rightarrow -\infty$ as $y \rightarrow 0$, which doesn't make physical sense. For example, the

model predicts that for sufficiently small y , the tip of the ladder moves at a speed greater than the speed of light. Therefore the model is not appropriate for small values of y . What actually happens is that the tip of the ladder leaves the wall at some point in its descent. For a discussion of the true situation see the article "The Falling Ladder Paradox" by Paul Scholten and Andrew Simoson in *The College Mathematics Journal*, 27, (1), January 1996, pages 49–54. Also see "On Mathematical and Physical Ladders" by M. Freeman and P. Palffy-Muhoray in the *American Journal of Physics*, 53 (3), March 1985, pages 276–277.

37. Differentiating both sides of $PV = C$ with respect to t and using the Product Rule gives us $P \frac{dV}{dt} + V \frac{dP}{dt} = 0 \Rightarrow$

$\frac{dV}{dt} = -\frac{V}{P} \frac{dP}{dt}$. When $V = 600$, $P = 150$ and $\frac{dP}{dt} = 20$, so we have $\frac{dV}{dt} = -\frac{600}{150}(20) = -80$. Thus, the volume is decreasing at a rate of $80 \text{ cm}^3/\text{min}$.

38. The volume of a hemisphere is $\frac{2}{3}\pi r^3$, so the volume of a hemispherical basin of radius 30 cm is $\frac{2}{3}\pi(30)^3 = 18,000\pi \text{ cm}^3$.

If the basin is half full, then $V = \pi(rh^2 - \frac{1}{3}h^3) \Rightarrow 9000\pi = \pi(30h^2 - \frac{1}{3}h^3) \Rightarrow \frac{1}{3}h^3 - 30h^2 + 9000 = 0 \Rightarrow h = H \approx 19.58$ [from a graph or numerical rootfinder; the other two solutions are less than 0 and greater than 30].

$$V = \pi(30h^2 - \frac{1}{3}h^3) \Rightarrow \frac{dV}{dt} = \pi\left(60h \frac{dh}{dt} - h^2 \frac{dh}{dt}\right) \Rightarrow \left(2 \frac{\text{L}}{\text{min}}\right)\left(1000 \frac{\text{cm}^3}{\text{L}}\right) = \pi(60h - h^2) \frac{dh}{dt} \Rightarrow$$

$$\frac{dh}{dt} = \frac{2000}{\pi(60H - H^2)} \approx 0.804 \text{ cm/min.}$$

39. With $R_1 = 80$ and $R_2 = 100$, $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{80} + \frac{1}{100} = \frac{180}{8000} = \frac{9}{400}$, so $R = \frac{400}{9}$. Differentiating $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$

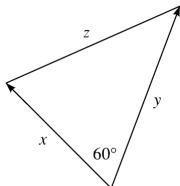
with respect to t , we have $-\frac{1}{R^2} \frac{dR}{dt} = -\frac{1}{R_1^2} \frac{dR_1}{dt} - \frac{1}{R_2^2} \frac{dR_2}{dt} \Rightarrow \frac{dR}{dt} = R^2 \left(\frac{1}{R_1^2} \frac{dR_1}{dt} + \frac{1}{R_2^2} \frac{dR_2}{dt} \right)$. When $R_1 = 80$ and

$$R_2 = 100, \frac{dR}{dt} = \frac{400^2}{9^2} \left[\frac{1}{80^2}(0.3) + \frac{1}{100^2}(0.2) \right] = \frac{107}{810} \approx 0.132 \Omega/\text{s.}$$

40. $PV^{1.4} = C \Rightarrow P \cdot 1.4V^{0.4} \frac{dV}{dt} + V^{1.4} \frac{dP}{dt} = 0 \Rightarrow \frac{dV}{dt} = -\frac{V^{1.4}}{P \cdot 1.4V^{0.4}} \frac{dP}{dt} = -\frac{V}{1.4P} \frac{dP}{dt}$.

When $V = 400$, $P = 80$ and $\frac{dP}{dt} = -10$, so we have $\frac{dV}{dt} = -\frac{400}{1.4(80)}(-10) = \frac{250}{7}$. Thus, the volume is increasing at a rate of $\frac{250}{7} \approx 36 \text{ cm}^3/\text{min}$.

- 41.



We are given that $\frac{dx}{dt} = 60 \text{ km/h}$ and $\frac{dy}{dt} = 100 \text{ km/h}$. By the Law of Cosines,

$$z^2 = x^2 + y^2 - 2xy \cos 60^\circ = x^2 + y^2 - xy \Rightarrow 2z \frac{dz}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} - x \frac{dy}{dt} - y \frac{dx}{dt}$$

At $t = \frac{1}{2} \text{ h}$, we have $x = 60(\frac{1}{2}) = 30$ and $y = 100(\frac{1}{2}) = 50 \Rightarrow z^2 = 30^2 + 50^2 - (30)(50) = 1900 \Rightarrow$

$$z = \sqrt{1900} \text{ and } \frac{dz}{dt} = \frac{2(30)(60) + 2(50)(100) - (30)(100) - (50)(60)}{2\sqrt{1900}} = \frac{7600}{2\sqrt{1900}} = 20\sqrt{19} \approx 87.2 \text{ km/h.}$$

42. We want to find $\frac{dB}{dt}$ when $L = 18$ using $B = 0.007W^{2/3}$ and $W = 0.12L^{2.53}$.

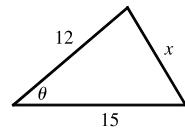
$$\begin{aligned} \frac{dB}{dt} &= \frac{dB}{dW} \frac{dW}{dL} \frac{dL}{dt} = \left(0.007 \cdot \frac{2}{3}W^{-1/3}\right)(0.12 \cdot 2.53 \cdot L^{1.53}) \left(\frac{20 - 15}{10,000,000}\right) \\ &= \left[0.007 \cdot \frac{2}{3}(0.12 \cdot 18^{2.53})^{-1/3}\right](0.12 \cdot 2.53 \cdot 18^{1.53}) \left(\frac{5}{10^7}\right) \approx 1.045 \times 10^{-8} \text{ g/yr} \end{aligned}$$

43. We are given $d\theta/dt = 2^\circ/\text{min} = \frac{\pi}{90}$ rad/min. By the Law of Cosines,

$$x^2 = 12^2 + 15^2 - 2(12)(15) \cos \theta = 369 - 360 \cos \theta \Rightarrow$$

$$2x \frac{dx}{dt} = 360 \sin \theta \frac{d\theta}{dt} \Rightarrow \frac{dx}{dt} = \frac{180 \sin \theta}{x} \frac{d\theta}{dt}. \text{ When } \theta = 60^\circ,$$

$$x = \sqrt{369 - 360 \cos 60^\circ} = \sqrt{189} = 3\sqrt{21}, \text{ so } \frac{dx}{dt} = \frac{180 \sin 60^\circ}{3\sqrt{21}} \frac{\pi}{90} = \frac{\pi \sqrt{3}}{3\sqrt{21}} = \frac{\sqrt{7}\pi}{21} \approx 0.396 \text{ m/min.}$$



44. Using Q for the origin, we are given $\frac{dx}{dt} = -0.5$ m/s and need to find $\frac{dy}{dt}$ when $x = -5$.

Using the Pythagorean Theorem twice, we have $\sqrt{x^2 + 4^2} + \sqrt{y^2 + 4^2} = 12$, the

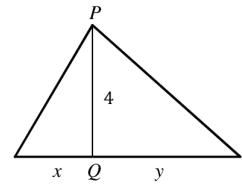
total length of the rope. Differentiating with respect to t , we get

$$\frac{x}{\sqrt{x^2+4^2}} \frac{dx}{dt} + \frac{y}{\sqrt{y^2+4^2}} \frac{dy}{dt} = 0, \text{ so } \frac{dy}{dt} = -\frac{x\sqrt{y^2+4^2}}{y\sqrt{x^2+4^2}} \frac{dx}{dt}.$$

$$\text{Now, when } x = -3, 12 = \sqrt{(-3)^2 + 4^2} + \sqrt{y^2 + 4^2} = 5 + \sqrt{y^2 + 4^2} \Leftrightarrow \sqrt{y^2 + 4^2} = 7, \text{ and } y = \sqrt{7^2 - 4^2} = \sqrt{33}.$$

$$\text{So when } x = -3, \frac{dy}{dt} = -\frac{(-3)(7)}{\sqrt{33}(5)}(-0.5) \approx -0.37 \text{ m/s.}$$

So cart B is moving towards Q at about 0.37 m/s.

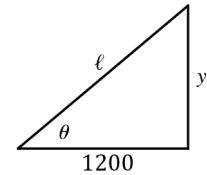


45. (a) By the Pythagorean Theorem, $1200^2 + y^2 = l^2$. Differentiating with respect to

t , we obtain $2y \frac{dy}{dt} = 2l \frac{dl}{dt}$. We know that $\frac{dy}{dt} = 200$ m/s, so when $y = 900$ m,

$$l = \sqrt{1200^2 + 900^2} = \sqrt{2,250,000} = 1500 \text{ m}$$

$$\text{and } \frac{dl}{dt} = \frac{y}{l} \frac{dy}{dt} = \frac{900}{1500}(200) = \frac{600}{5} = 120 \text{ m/s.}$$



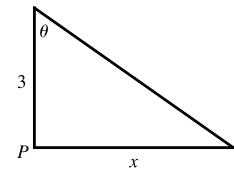
(b) Here $\tan \theta = \frac{y}{1200} \Rightarrow \frac{d}{dt}(\tan \theta) = \frac{d}{dt}\left(\frac{y}{1200}\right) \Rightarrow \sec^2 \theta \frac{d\theta}{dt} = \frac{1}{1200} \frac{dy}{dt} \Rightarrow \frac{d\theta}{dt} = \frac{\cos^2 \theta}{1200} \frac{dy}{dt}$. When $y = 900$ m, $\frac{dy}{dt} =$

$$200 \text{ m/s, } l = 1500 \text{ and } \cos \theta = \frac{1200}{l} = \frac{1200}{1500} = \frac{4}{5}, \text{ so } \frac{d\theta}{dt} = \frac{(4/5)^2}{1200}(200) = 0.106 \text{ rad/s.}$$

46. We are given that $\frac{d\theta}{dt} = 4(2\pi) = 8\pi$ rad/min. $x = 3 \tan \theta \Rightarrow$

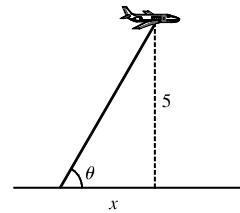
$$\frac{dx}{dt} = 3 \sec^2 \theta \frac{d\theta}{dt}. \text{ When } x = 1, \tan \theta = \frac{1}{3}, \text{ so } \sec^2 \theta = 1 + \left(\frac{1}{3}\right)^2 = \frac{10}{9}$$

$$\text{and } \frac{dx}{dt} = 3\left(\frac{10}{9}\right)(8\pi) = \frac{80}{3}\pi \approx 83.8 \text{ km/min.}$$



47. $\cot \theta = \frac{x}{5} \Rightarrow -\csc^2 \theta \frac{d\theta}{dt} = \frac{1}{5} \frac{dx}{dt} \Rightarrow -\left(\csc \frac{\pi}{3}\right)^2 \left(-\frac{\pi}{6}\right) = \frac{1}{5} \frac{dx}{dt} \Rightarrow$

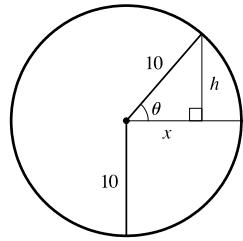
$$\frac{dx}{dt} = \frac{5\pi}{6} \left(\frac{2}{\sqrt{3}}\right)^2 = \frac{10}{9}\pi \text{ km/min } [\approx 130 \text{ mi/h}]$$



48. We are given that $\frac{d\theta}{dt} = \frac{2\pi \text{ rad}}{2 \text{ min}} = \pi \text{ rad/min}$. By the Pythagorean Theorem, when

$$h = 6, x = 8, \text{ so } \sin \theta = \frac{6}{10} \text{ and } \cos \theta = \frac{8}{10}. \text{ From the figure, } \sin \theta = \frac{h}{10} \Rightarrow$$

$$h = 10 \sin \theta, \text{ so } \frac{dh}{dt} = 10 \cos \theta \frac{d\theta}{dt} = 10 \left(\frac{8}{10} \right) \pi = 8\pi \text{ m/min.}$$

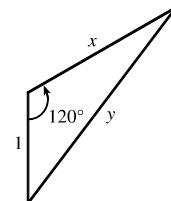


49. We are given that $\frac{dx}{dt} = 300 \text{ km/h}$. By the Law of Cosines,

$$y^2 = x^2 + 1^2 - 2(1)(x) \cos 120^\circ = x^2 + 1 - 2x(-\frac{1}{2}) = x^2 + x + 1, \text{ so}$$

$$2y \frac{dy}{dt} = 2x \frac{dx}{dt} + \frac{dx}{dt} \Rightarrow \frac{dy}{dt} = \frac{2x+1}{2y} \frac{dx}{dt}. \text{ After 1 minute, } x = \frac{300}{60} = 5 \text{ km} \Rightarrow$$

$$y = \sqrt{5^2 + 5 + 1} = \sqrt{31} \text{ km} \Rightarrow \frac{dy}{dt} = \frac{2(5)+1}{2\sqrt{31}}(300) = \frac{1650}{\sqrt{31}} \approx 296 \text{ km/h.}$$

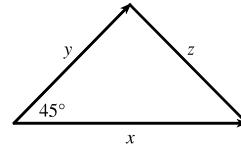


50. We are given that $\frac{dx}{dt} = 4 \text{ km/h}$ and $\frac{dy}{dt} = 2 \text{ km/h}$. By the Law of Cosines, $z^2 =$

$$x^2 + y^2 - 2xy \cos 45^\circ = x^2 + y^2 - \sqrt{2}xy \Rightarrow 2z \frac{dz}{dt} = 2x \frac{dx}{dt} + 2 \frac{dy}{dt} - \sqrt{2}x \frac{dy}{dt} - \sqrt{2}y \frac{dx}{dt}.$$

After 15 minutes [$= \frac{1}{4} \text{ h}$],

$$\text{we have } x = \frac{4}{4} = 1 \text{ and } y = \frac{2}{4} = \frac{1}{2} \Rightarrow z^2 = (1)^2 + (\frac{1}{2})^2 - \sqrt{2}(1)(\frac{1}{2}) \Rightarrow z = \frac{\sqrt{5-2\sqrt{2}}}{2} \text{ and } \frac{dz}{dt} = \frac{1}{\sqrt{5-2\sqrt{2}}} [2(1)4 + 2(\frac{1}{2})2 - \sqrt{2}(1)2 - \sqrt{2}(\frac{1}{2})4] = \frac{1}{\sqrt{5-2\sqrt{2}}} (10 - 4\sqrt{2}) = 2\sqrt{5-2\sqrt{2}} \approx 2.947 \text{ km/h.}$$



51. Let the distance between the runner and the friend be ℓ . Then by the Law of Cosines,

$$\ell^2 = 200^2 + 100^2 - 2 \cdot 200 \cdot 100 \cdot \cos \theta = 50,000 - 40,000 \cos \theta \quad (*).$$

Differentiating implicitly with respect to t , we obtain $2\ell \frac{d\ell}{dt} = -40,000(-\sin \theta) \frac{d\theta}{dt}$. Now if D is the

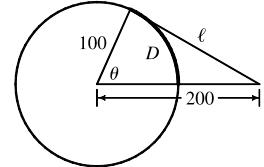
distance run when the angle is θ radians, then by the formula for the length of an arc

on a circle, $s = r\theta$, we have $D = 100\theta$, so $\theta = \frac{1}{100}D \Rightarrow \frac{d\theta}{dt} = \frac{1}{100} \frac{dD}{dt} = \frac{7}{100}$. To substitute into the expression for

$\frac{d\ell}{dt}$, we must know $\sin \theta$ at the time when $\ell = 200$, which we find from (*): $200^2 = 50,000 - 40,000 \cos \theta \Leftrightarrow$

$$\cos \theta = \frac{1}{4} \Rightarrow \sin \theta = \sqrt{1 - (\frac{1}{4})^2} = \frac{\sqrt{15}}{4}. \text{ Substituting, we get } 2(200) \frac{d\ell}{dt} = 40,000 \frac{\sqrt{15}}{4} \left(\frac{7}{100} \right) \Rightarrow$$

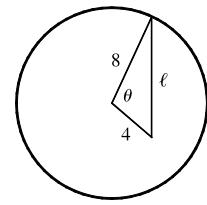
$d\ell/dt = \frac{7\sqrt{15}}{4} \approx 6.78 \text{ m/s. Whether the distance between them is increasing or decreasing depends on the direction in which the runner is running.}$



52. The hour hand of a clock goes around once every 12 hours or, in radians per hour,

$\frac{2\pi}{12} = \frac{\pi}{6}$ rad/h. The minute hand goes around once an hour, or at the rate of 2π rad/h.

So the angle θ between them (measuring clockwise from the minute hand to the hour hand) is changing at the rate of $d\theta/dt = \frac{\pi}{6} - 2\pi = -\frac{11\pi}{6}$ rad/h. Now, to relate θ to ℓ , we use the Law of Cosines: $\ell^2 = 4^2 + 8^2 - 2 \cdot 4 \cdot 8 \cdot \cos \theta = 80 - 64 \cos \theta$ (*).



Differentiating implicitly with respect to t , we get $2\ell \frac{d\ell}{dt} = -64(-\sin \theta) \frac{d\theta}{dt}$. At 1:00, the angle between the two hands is

one-twelfth of the circle, that is, $\frac{2\pi}{12} = \frac{\pi}{6}$ radians. We use (*) to find ℓ at 1:00: $\ell = \sqrt{80 - 64 \cos \frac{\pi}{6}} = \sqrt{80 - 32\sqrt{3}}$.

$$\text{Substituting, we get } 2\ell \frac{d\ell}{dt} = 64 \sin \frac{\pi}{6} \left(-\frac{11\pi}{6} \right) \Rightarrow \frac{d\ell}{dt} = \frac{64 \left(\frac{1}{2} \right) \left(-\frac{11\pi}{6} \right)}{2\sqrt{80 - 32\sqrt{3}}} = -\frac{88\pi}{3\sqrt{80 - 32\sqrt{3}}} \approx -18.6.$$

So at 1:00, the distance between the tips of the hands is decreasing at a rate of 18.6 mm/h ≈ 0.005 mm/s.

53. The volume of the snowball is given by $V = \frac{4}{3}\pi r^3$, so $\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$. Since the volume is proportional to the surface area S , with $S = 4\pi r^2$, we also have $\frac{dV}{dt} = k \cdot 4\pi r^2$ for some constant k . Equating the two expressions for $\frac{dV}{dt}$ gives $4\pi r^2 \frac{dr}{dt} = k \cdot 4\pi r^2 \Rightarrow \frac{dr}{dt} = k$, that is, dr/dt is constant.

2.9 Linear Approximations and Differentials

1. $f(x) = x^3 - x^2 + 3 \Rightarrow f'(x) = 3x^2 - 2x$, so $f(-2) = -9$ and $f'(-2) = 16$. Thus,

$$L(x) = f(-2) + f'(-2)(x - (-2)) = -9 + 16(x + 2) = 16x + 23.$$

2. $f(x) = \cos 2x \Rightarrow f'(x) = -2 \sin 2x$, so $f(\frac{\pi}{6}) = \frac{1}{2}$ and $f'(\frac{\pi}{6}) = -\sqrt{3}$. Thus,

$$L(x) = f(\frac{\pi}{6}) + f'(\frac{\pi}{6})(x - \frac{\pi}{6}) = \frac{1}{2} - \sqrt{3}(x - \frac{\pi}{6}) = -\sqrt{3}x + (\sqrt{3}\pi)/6 + \frac{1}{2}.$$

3. $f(x) = \sqrt[3]{x} \Rightarrow f'(x) = \frac{1}{3\sqrt[3]{x^2}}$, so $f(8) = 2$ and $f'(8) = \frac{1}{12}$. Thus,

$$L(x) = f(8) + f'(8)(x - 8) = 2 + \frac{1}{12}(x - 8) = \frac{1}{12}x + \frac{4}{3}.$$

4. $f(x) = 2/\sqrt{x^2 - 5} = 2(x^2 - 5)^{-1/2} \Rightarrow f'(x) = 2(-\frac{1}{2})(x^2 - 5)^{-3/2}(2x) = -\frac{2x}{(x^2 - 5)^{3/2}}$, so $f(3) = 1$ and

$$f'(3) = -\frac{3}{4}. \text{ Thus, } L(x) = f(3) + f'(3)(x - 3) = 1 - \frac{3}{4}(x - 3) = -\frac{3}{4}x + \frac{13}{4}.$$

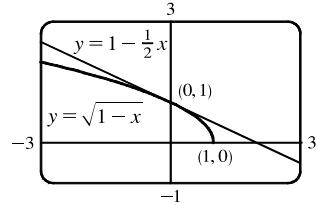
5. $f(x) = \sqrt{1-x} \Rightarrow f'(x) = \frac{-1}{2\sqrt{1-x}}$, so $f(0) = 1$ and $f'(0) = -\frac{1}{2}$.

Therefore,

$$\sqrt{1-x} = f(x) \approx f(0) + f'(0)(x - 0) = 1 + (-\frac{1}{2})(x - 0) = 1 - \frac{1}{2}x.$$

$$\text{So } \sqrt{0.9} = \sqrt{1 - 0.1} \approx 1 - \frac{1}{2}(0.1) = 0.95$$

$$\text{and } \sqrt{0.99} = \sqrt{1 - 0.01} \approx 1 - \frac{1}{2}(0.01) = 0.995.$$

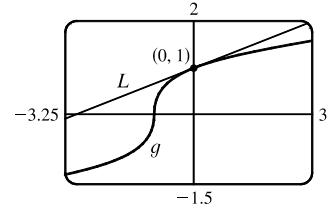


6. $g(x) = \sqrt[3]{1+x} = (1+x)^{1/3} \Rightarrow g'(x) = \frac{1}{3}(1+x)^{-2/3}$, so $g(0) = 1$ and

$g'(0) = \frac{1}{3}$. Therefore, $\sqrt[3]{1+x} = g(x) \approx g(0) + g'(0)(x-0) = 1 + \frac{1}{3}x$.

So $\sqrt[3]{0.95} = \sqrt[3]{1+(-0.05)} \approx 1 + \frac{1}{3}(-0.05) = 0.98\bar{3}$,

and $\sqrt[3]{1.1} = \sqrt[3]{1+0.1} \approx 1 + \frac{1}{3}(0.1) = 1.0\bar{3}$.

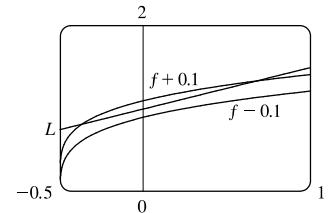


7. $f(x) = \sqrt[4]{1+2x} \Rightarrow f'(x) = \frac{1}{4}(1+2x)^{-3/4}(2) = \frac{1}{2}(1+2x)^{-3/4}$, so

$f(0) = 1$ and $f'(0) = \frac{1}{2}$. Thus, $f(x) \approx f(0) + f'(0)(x-0) = 1 + \frac{1}{2}x$.

We need $\sqrt[4]{1+2x} - 0.1 < 1 + \frac{1}{2}x < \sqrt[4]{1+2x} + 0.1$, which is true when

$-0.368 < x < 0.677$. Note that to ensure the accuracy, we have rounded the smaller value up and the larger value down.

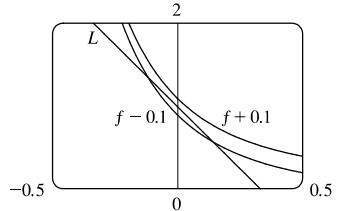


8. $f(x) = (1+x)^{-3} \Rightarrow f'(x) = -3(1+x)^{-4}$, so $f(0) = 1$ and

$f'(0) = -3$. Thus, $f(x) \approx f(0) + f'(0)(x-0) = 1 - 3x$. We need

$(1+x)^{-3} - 0.1 < 1 - 3x < (1+x)^{-3} + 0.1$, which is true when

$-0.116 < x < 0.144$. Note that to ensure the accuracy, we have rounded the smaller value up and the larger value down.

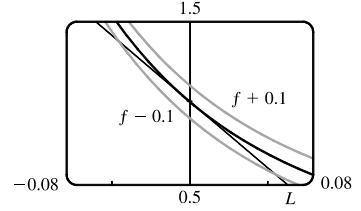


9. $f(x) = \frac{1}{(1+2x)^4} = (1+2x)^{-4} \Rightarrow$

$f'(x) = -4(1+2x)^{-5}(2) = \frac{-8}{(1+2x)^5}$, so $f(0) = 1$ and $f'(0) = -8$.

Thus, $f(x) \approx f(0) + f'(0)(x-0) = 1 + (-8)(x-0) = 1 - 8x$.

We need $\frac{1}{(1+2x)^4} - 0.1 < 1 - 8x < \frac{1}{(1+2x)^4} + 0.1$, which is true when $-0.045 < x < 0.055$.

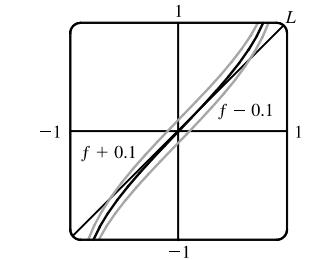


10. $f(x) = \tan x \Rightarrow f'(x) = \sec^2 x$, so $f(0) = 0$ and $f'(0) = 1$.

Thus, $f(x) \approx f(0) + f'(0)(x-0) = 0 + 1(x-0) = x$.

We need $\tan x - 0.1 < x < \tan x + 0.1$, which is true when

$-0.63 < x < 0.63$.



11. The differential dy is defined in terms of dx by the equation $dy = f'(x) dx$. For $y = f(x) = (x^2 - 3)^{-2}$,

$$f'(x) = -2(x^2 - 3)^{-3}(2x) = -\frac{4x}{(x^2 - 3)^3}, \text{ so } dy = -\frac{4x}{(x^2 - 3)^3} dx.$$

12. For $y = f(t) = \sqrt{1-t^4}$, $f'(t) = \frac{1}{2}(1-t^4)^{-1/2}(-4t^3) = -\frac{2t^3}{\sqrt{1-t^4}}$, so $dy = -\frac{2t^3}{\sqrt{1-t^4}} dt$.

13. For $y = f(u) = \frac{1+2u}{1+3u}$, $f'(u) = \frac{(1+3u)(2)-(1+2u)(3)}{(1+3u)^2} = \frac{-1}{(1+3u)^2}$, so $dy = \frac{-1}{(1+3u)^2} du$.

14. For $y = f(\theta) = \theta^2 \sin 2\theta$, $f'(\theta) = \theta^2(\cos 2\theta)(2) + (\sin 2\theta)(2\theta)$, so $dy = 2\theta(\theta \cos 2\theta + \sin 2\theta) d\theta$.

15. For $y = f(x) = \frac{1}{x^2 - 3x} = (x^2 - 3x)^{-1}$, $f'(x) = -(x^2 - 3x)^{-2} \cdot (2x - 3) = -\frac{2x - 3}{(x^2 - 3x)^2}$, so $dy = -\frac{2x - 3}{(x^2 - 3x)^2} dx$.

16. For $y = f(\theta) = \sqrt{1 + \cos \theta}$, $f'(\theta) = \frac{1}{2}(1 + \cos \theta)^{-1/2}(-\sin \theta) = -\frac{\sin \theta}{2\sqrt{1 + \cos \theta}}$, so $dy = -\frac{\sin \theta}{2\sqrt{1 + \cos \theta}} d\theta$.

17. For $y = f(t) = \sqrt{t - \cos t}$, $f'(t) = \frac{1}{2}(t - \cos t)^{-1/2}(1 + \sin t) = \frac{1 + \sin t}{2\sqrt{t - \cos t}}$, so $dy = \frac{1 + \sin t}{2\sqrt{t - \cos t}} dt$.

18. For $y = f(x) = \frac{1}{x} \sin x$, $f'(x) = \frac{1}{x} \cos x - \frac{1}{x^2} \sin x = \frac{x \cos x - \sin x}{x^2}$, so $dy = \frac{x \cos x - \sin x}{x^2} dx$.

19. (a) $y = \tan x \Rightarrow dy = \sec^2 x dx$

(b) When $x = \pi/4$ and $dx = -0.1$, $dy = [\sec(\pi/4)]^2(-0.1) = (\sqrt{2})^2(-0.1) = -0.2$.

20. (a) $y = \cos \pi x \Rightarrow dy = -\sin \pi x \cdot \pi dx = -\pi \sin \pi x dx$

(b) $x = \frac{1}{3}$ and $dx = -0.02 \Rightarrow dy = -\pi \sin \frac{\pi}{3}(-0.02) = \pi(\sqrt{3}/2)(0.02) = 0.01\pi\sqrt{3} \approx 0.054$.

21. (a) $y = \sqrt{3+x^2} \Rightarrow dy = \frac{1}{2}(3+x^2)^{-1/2}(2x) dx = \frac{x}{\sqrt{3+x^2}} dx$

(b) $x = 1$ and $dx = -0.1 \Rightarrow dy = \frac{1}{\sqrt{3+1^2}}(-0.1) = \frac{1}{2}(-0.1) = -0.05$.

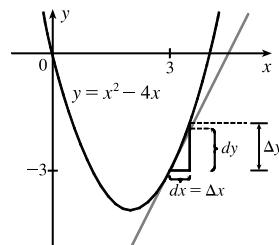
22. (a) $y = \frac{x+1}{x-1} \Rightarrow dy = \frac{(x-1)(1) - (x+1)(1)}{(x-1)^2} dx = \frac{-2}{(x-1)^2} dx$

(b) $x = 2$ and $dx = 0.05 \Rightarrow dy = \frac{-2}{(2-1)^2}(0.05) = -2(0.05) = -0.1$.

23. $y = f(x) = x^2 - 4x$, $x = 3$, $\Delta x = 0.5 \Rightarrow$

$\Delta y = f(3.5) - f(3) = -1.75 - (-3) = 1.25$

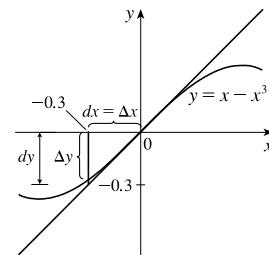
$dy = f'(x) dx = (2x-4) dx = (6-4)(0.5) = 1$



24. $y = f(x) = x - x^3$, $x = 0$, $\Delta x = -0.3 \Rightarrow$

$\Delta y = f(-0.3) - f(0) = -0.273 - 0 = -0.273$

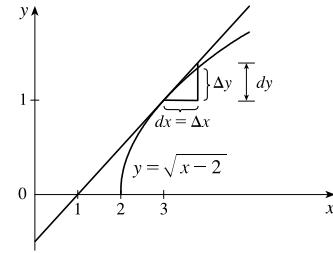
$dy = f'(x) dx = (1-3x^2) dx = (1-0)(-0.3) = -0.3$



25. $y = f(x) = \sqrt{x-2}$, $x = 3$, $\Delta x = 0.8 \Rightarrow$

$$\Delta y = f(3.8) - f(3) = \sqrt{1.8} - 1 \approx 0.34$$

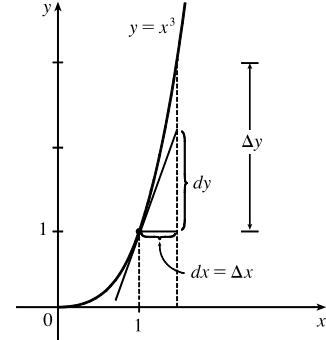
$$dy = f'(x) dx = \frac{1}{2\sqrt{x-2}} dx = \frac{1}{2(1)}(0.8) = 0.4$$



26. $y = x^3$, $x = 1$, $\Delta x = 0.5 \Rightarrow$

$$\Delta y = (1.5)^3 - 1^3 = 3.375 - 1 = 2.375.$$

$$dy = 3x^2 dx = 3(1)^2(0.5) = 1.5$$



27. $y = f(x) = x^4 - x + 1$. If x changes from 1 to 1.05, $dx = \Delta x = 1.05 - 1 = 0.05 \Rightarrow$

$$\Delta y = f(1.05) - f(1) = 1.16550625 - 1 \approx 0.1655 \text{ and } dy = (4x^3 - 1) dx = (4 \cdot 1^3 - 1)(0.05) = 3(0.05) = 0.15.$$

If x changes from 1 to 1.01, $dx = \Delta x = 1.01 - 1 = 0.01 \Rightarrow \Delta y = f(1.01) - f(1) = 1.03060401 - 1 \approx 0.0306$

and $dy = (4x^3 - 1) dx = 3(0.01) = 0.03$.

With $\Delta x = 0.05$, $|dy - \Delta y| \approx |0.15 - 0.1655| = 0.0155$. With $\Delta x = 0.01$, $|dy - \Delta y| \approx |0.03 - 0.0306| = 0.0006$.

Since $|dy - \Delta y|$ is smaller for $\Delta x = 0.01$ than for $\Delta x = 0.05$, yes, the approximation $\Delta y \approx dy$ becomes better as Δx gets smaller.

28. $y = f(x) = (x^3 + 3)^2$. If x changes from 1 to 1.05, $dx = \Delta x = 1.05 - 1 = 0.05 \Rightarrow$

$$\Delta y = f(1.05) - f(1) \approx 17.28585 - 16 = 1.28585 \text{ and } dy = 6x^2(x^3 + 3) dx = 6(1)^2(1^3 + 3)(0.05) = 24(0.05) = 1.2.$$

If x changes from 1 to 1.01, $dx = \Delta x = 1.01 - 1 = 0.01 \Rightarrow \Delta y = f(1.01) - f(1) \approx 16.24333 - 16 = 0.24333$ and

$$dy = 6x^2(x^3 + 3) dx = 24(0.01) = 0.24. \text{ With } \Delta x = 0.05, |dy - \Delta y| \approx |1.2 - 1.28585| = 0.08585. \text{ With } \Delta x = 0.01, |dy - \Delta y| \approx |0.24 - 0.24333| = 0.00333. \text{ Since } |dy - \Delta y| \text{ is smaller for } \Delta x = 0.01 \text{ than for } \Delta x = 0.05, \text{ yes, the approximation } \Delta y \approx dy \text{ becomes better as } \Delta x \text{ gets smaller.}$$

29. $y = f(x) = \sqrt{5-x}$. If x changes from 1 to 1.05, $dx = \Delta x = 1.05 - 1 = 0.05 \Rightarrow$

$$\Delta y = f(1.05) - f(1) = \sqrt{3.95} - 2 \approx -0.012539 \text{ and } dy = -\frac{1}{2\sqrt{5-x}} dx = -\frac{1}{2\sqrt{5-1}} dx = -\frac{1}{4}(0.05) = -0.0125.$$

If x changes from 1 to 1.01, $dx = \Delta x = 1.01 - 1 = 0.01 \Rightarrow \Delta y = f(1.01) - f(1) = \sqrt{3.99} - 2 \approx -0.002502$

and $dy = -\frac{1}{4}(0.01) = -0.0025$.

[continued]

With $\Delta x = 0.05$, $|dy - \Delta y| \approx |-0.0125 - (-0.012539)| = 0.000039$. With $\Delta x = 0.01$,

$|dy - \Delta y| \approx |-0.0025 - (-0.002502)| = 0.000002$. Since $|dy - \Delta y|$ is smaller for $\Delta x = 0.01$ than for $\Delta x = 0.05$, yes, the approximation $\Delta y \approx dy$ becomes better as Δx gets smaller.

30. $y = f(x) = \frac{1}{x^2 + 1}$. If x changes from 1 to 1.05, $dx = \Delta x = 1.05 - 1 = 0.05 \Rightarrow$

$$\Delta y = f(1.05) - f(1) = \frac{1}{(1.05)^2 - 1} - \frac{1}{2} \approx -0.02438 \text{ and}$$

$$dy = -\frac{2x}{(x^2 + 1)^2} dx = -\frac{2(1)}{(1^2 + 1)^2} (0.05) = -\frac{1}{2} (0.05) = -0.025.$$

$$\text{If } x \text{ changes from 1 to 1.01, } dx = \Delta x = 1.01 - 1 = 0.01 \Rightarrow \Delta y = f(1.01) - f(1) = \frac{1}{1.01^2 - 1} - \frac{1}{2} \approx -0.00498$$

$$\text{and } dy = -\frac{1}{2}(0.01) = -0.005.$$

With $\Delta x = 0.05$, $|dy - \Delta y| \approx |-0.025 - (-0.02438)| = 0.00038$. With $\Delta x = 0.01$,

$|dy - \Delta y| \approx |-0.005 - (-0.00498)| = 0.00002$. Since $|dy - \Delta y|$ is smaller for $\Delta x = 0.01$ than for $\Delta x = 0.05$, yes, the approximation $\Delta y \approx dy$ becomes better as Δx gets smaller.

31. To estimate $(1.999)^4$, we'll find the linearization of $f(x) = x^4$ at $a = 2$. Since $f'(x) = 4x^3$, $f(2) = 16$, and

$f'(2) = 32$, we have $L(x) = 16 + 32(x - 2)$. Thus, $x^4 \approx 16 + 32(x - 2)$ when x is near 2, so

$$(1.999)^4 \approx 16 + 32(1.999 - 2) = 16 - 0.032 = 15.968.$$

32. $y = f(x) = 1/x \Rightarrow dy = -1/x^2 dx$. When $x = 4$ and $dx = 0.002$, $dy = -\frac{1}{16}(0.002) = -\frac{1}{8000}$, so

$$\frac{1}{4.002} \approx f(4) + dy = \frac{1}{4} - \frac{1}{8000} = \frac{1999}{8000} = 0.249875.$$

33. $y = f(x) = \sqrt[3]{x} \Rightarrow dy = \frac{1}{3}x^{-2/3} dx$. When $x = 1000$ and $dx = 1$, $dy = \frac{1}{3}(1000)^{-2/3}(1) = \frac{1}{300}$, so

$$\sqrt[3]{1001} = f(1001) \approx f(1000) + dy = 10 + \frac{1}{300} = 10.00\overline{3} \approx 10.003.$$

34. $y = f(x) = \sqrt{x} \Rightarrow dy = \frac{1}{2}x^{-1/2} dx$. When $x = 100$ and $dx = 0.5$, $dy = \frac{1}{2}(100)^{-1/2}(\frac{1}{2}) = \frac{1}{40}$, so

$$\sqrt{100.5} = f(100.5) \approx f(100) + dy = 10 + \frac{1}{40} = 10.025.$$

35. $y = f(x) = \tan x \Rightarrow dy = \sec^2 x dx$. When $x = 0^\circ$ [i.e., 0 radians] and $dx = 2^\circ$ [i.e., $\frac{\pi}{90}$ radians],

$$dy = (\sec^2 0)(\frac{\pi}{90}) = 1^2(\frac{\pi}{90}) = \frac{\pi}{90}, \text{ so } \tan 2^\circ = f(2^\circ) \approx f(0^\circ) + dy = 0 + \frac{\pi}{90} = \frac{\pi}{90} \approx 0.0349.$$

36. $y = f(x) = \cos x \Rightarrow dy = -\sin x dx$. When $x = 30^\circ$ [$\pi/6$] and $dx = -1^\circ$ [$-\pi/180$],

$$dy = (-\sin \frac{\pi}{6})(-\frac{\pi}{180}) = -\frac{1}{2}(-\frac{\pi}{180}) = \frac{\pi}{360}, \text{ so } \cos 29^\circ = f(29^\circ) \approx f(30^\circ) + dy = \frac{1}{2}\sqrt{3} + \frac{\pi}{360} \approx 0.875.$$

37. $y = f(x) = \sec x \Rightarrow f'(x) = \sec x \tan x$, so $f(0) = 1$ and $f'(0) = 1 \cdot 0 = 0$. The linear approximation of f at 0 is

$$f(0) + f'(0)(x - 0) = 1 + 0(x) = 1. \text{ Since } 0.08 \text{ is close to } 0, \text{ approximating } \sec 0.08 \text{ with } 1 \text{ is reasonable.}$$

38. $y = f(x) = \sqrt{x} \Rightarrow f'(x) = 1/(2\sqrt{x})$, so $f(4) = 2$ and $f'(4) = \frac{1}{4}$. The linear approximation of f at 4 is $f(4) + f'(4)(x - 4) = 2 + \frac{1}{4}(x - 4)$. Now $f(4.02) = \sqrt{4.02} \approx 2 + \frac{1}{4}(0.02) = 2 + 0.005 = 2.005$, so the approximation is reasonable.

39. (a) If x is the edge length, then $V = x^3 \Rightarrow dV = 3x^2 dx$. When $x = 30$ and $dx = 0.1$, $dV = 3(30)^2(0.1) = 270$, so the maximum possible error in computing the volume of the cube is about 270 cm^3 . The relative error is calculated by dividing the change in V , ΔV , by V . We approximate ΔV with dV .

$$\text{Relative error} = \frac{\Delta V}{V} \approx \frac{dV}{V} = \frac{3x^2 dx}{x^3} = 3 \frac{dx}{x} = 3 \left(\frac{0.1}{30} \right) = 0.01.$$

Percentage error = relative error $\times 100\% = 0.01 \times 100\% = 1\%$.

- (b) $S = 6x^2 \Rightarrow dS = 12x dx$. When $x = 30$ and $dx = 0.1$, $dS = 12(30)(0.1) = 36$, so the maximum possible error in computing the surface area of the cube is about 36 cm^2 .

$$\text{Relative error} = \frac{\Delta S}{S} \approx \frac{dS}{S} = \frac{12x dx}{6x^2} = 2 \frac{dx}{x} = 2 \left(\frac{0.1}{30} \right) = 0.00\bar{6}.$$

Percentage error = relative error $\times 100\% = 0.00\bar{6} \times 100\% = 0.\bar{6}\%$.

40. (a) $A = \pi r^2 \Rightarrow dA = 2\pi r dr$. When $r = 24$ and $dr = 0.2$, $dA = 2\pi(24)(0.2) = 9.6\pi$, so the maximum possible error in the calculated area of the disk is about $9.6\pi \approx 30 \text{ cm}^2$.

$$(b) \text{Relative error} = \frac{\Delta A}{A} \approx \frac{dA}{A} = \frac{2\pi r dr}{\pi r^2} = \frac{2 dr}{r} = \frac{2(0.2)}{24} = \frac{0.2}{12} = \frac{1}{60} = 0.01\bar{6}.$$

Percentage error = relative error $\times 100\% = 0.01\bar{6} \times 100\% = 1.\bar{6}\%$.

41. (a) For a sphere of radius r , the circumference is $C = 2\pi r$ and the surface area is $S = 4\pi r^2$, so

$$r = \frac{C}{2\pi} \Rightarrow S = 4\pi \left(\frac{C}{2\pi} \right)^2 = \frac{C^2}{\pi} \Rightarrow dS = \frac{2}{\pi} C dC. \text{ When } C = 84 \text{ and } dC = 0.5, dS = \frac{2}{\pi}(84)(0.5) = \frac{84}{\pi},$$

so the maximum error is about $\frac{84}{\pi} \approx 27 \text{ cm}^2$. Relative error $\approx \frac{dS}{S} = \frac{84/\pi}{84^2/\pi} = \frac{1}{84} \approx 0.012 = 1.2\%$

$$(b) V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \left(\frac{C}{2\pi} \right)^3 = \frac{C^3}{6\pi^2} \Rightarrow dV = \frac{1}{2\pi^2} C^2 dC. \text{ When } C = 84 \text{ and } dC = 0.5,$$

$$dV = \frac{1}{2\pi^2}(84)^2(0.5) = \frac{1764}{\pi^2}, \text{ so the maximum error is about } \frac{1764}{\pi^2} \approx 179 \text{ cm}^3.$$

$$\text{The relative error is approximately } \frac{dV}{V} = \frac{1764/\pi^2}{(84)^3/(6\pi^2)} = \frac{1}{56} \approx 0.018 = 1.8\%.$$

42. For a hemispherical dome, $V = \frac{2}{3}\pi r^3 \Rightarrow dV = 2\pi r^2 dr$. When $r = \frac{1}{2}(50) = 25 \text{ m}$ and $dr = 0.05 \text{ cm} = 0.0005 \text{ m}$,

$$dV = 2\pi(25)^2(0.0005) = \frac{5\pi}{8}, \text{ so the amount of paint needed is about } \frac{5\pi}{8} \approx 2 \text{ m}^3.$$

43. (a) $V = \pi r^2 h \Rightarrow \Delta V \approx dV = 2\pi r h dr = 2\pi r h \Delta r$

(b) The error is

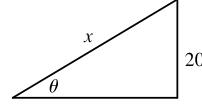
$$\Delta V - dV = [\pi(r + \Delta r)^2 h - \pi r^2 h] - 2\pi r h \Delta r = \pi r^2 h + 2\pi r h \Delta r + \pi(\Delta r)^2 h - \pi r^2 h - 2\pi r h \Delta r = \pi(\Delta r)^2 h.$$

44. (a) $\sin \theta = \frac{20}{x} \Rightarrow x = 20 \csc \theta \Rightarrow$

$$dx = 20(-\csc \theta \cot \theta) d\theta = -20 \csc 30^\circ \cot 30^\circ (\pm 1^\circ)$$

$$= -20(2)(\sqrt{3})\left(\pm \frac{\pi}{180}\right) = \pm \frac{2\sqrt{3}}{9}\pi$$

So the maximum error is about $\pm \frac{2}{9}\sqrt{3}\pi \approx \pm 1.21$ cm.



(b) The relative error is $\frac{\Delta x}{x} \approx \frac{dx}{x} = \frac{\pm \frac{2}{9}\sqrt{3}\pi}{20(2)} = \pm \frac{\sqrt{3}}{180}\pi \approx \pm 0.03$, so the percentage error is approximately $\pm 3\%$.

45. $V = RI \Rightarrow I = \frac{V}{R} \Rightarrow dI = -\frac{V}{R^2} dR$. The relative error in calculating I is $\frac{\Delta I}{I} \approx \frac{dI}{I} = \frac{-(V/R^2) dR}{V/R} = -\frac{dR}{R}$.

Hence, the relative error in calculating I is approximately the same (in magnitude) as the relative error in R .

46. $F = kR^4 \Rightarrow dF = 4kR^3 dR \Rightarrow \frac{dF}{F} = \frac{4kR^3 dR}{kR^4} = 4\left(\frac{dR}{R}\right)$. Thus, the relative change in F is about 4 times the

relative change in R . So a 5% increase in the radius corresponds to a 20% increase in blood flow.

47. (a) $dc = \frac{dc}{dx} dx = 0 \Rightarrow dx = 0$

(b) $d(cu) = \frac{d}{dx}(cu) dx = c \frac{du}{dx} dx = c du$

(c) $d(u+v) = \frac{d}{dx}(u+v) dx = \left(\frac{du}{dx} + \frac{dv}{dx}\right) dx = \frac{du}{dx} dx + \frac{dv}{dx} dx = du + dv$

(d) $d(uv) = \frac{d}{dx}(uv) dx = \left(u \frac{dv}{dx} + v \frac{du}{dx}\right) dx = u \frac{dv}{dx} dx + v \frac{du}{dx} dx = u dv + v du$

(e) $d\left(\frac{u}{v}\right) = \frac{d}{dx}\left(\frac{u}{v}\right) dx = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} dx = \frac{v \frac{du}{dx} dx - u \frac{dv}{dx} dx}{v^2} = \frac{v du - u dv}{v^2}$

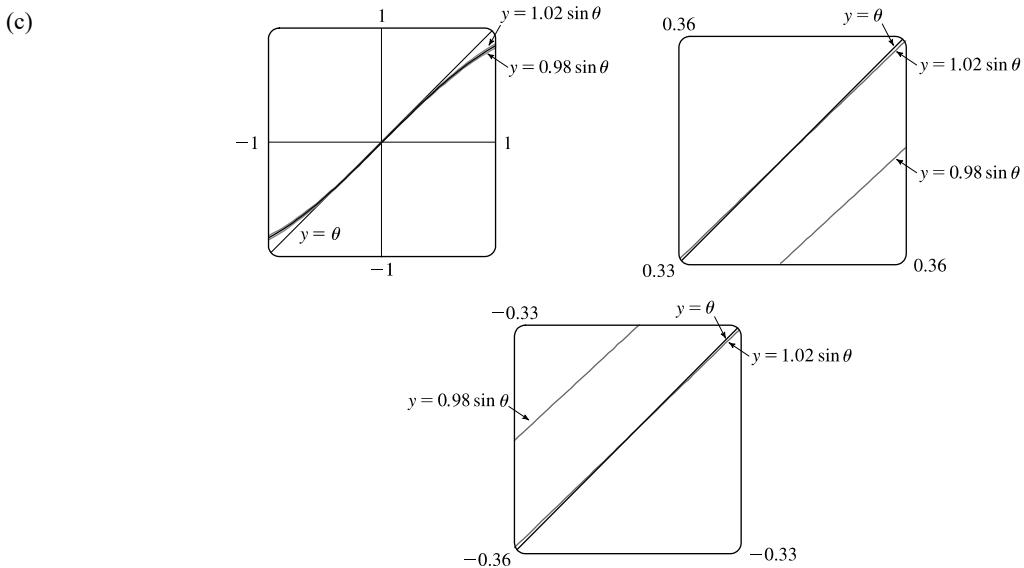
(f) $d(x^n) = \frac{d}{dx}(x^n) dx = nx^{n-1} dx$

48. (a) $f(\theta) = \sin \theta \Rightarrow f'(\theta) = \cos \theta$, so $f(0) = 0$ and $f'(0) = 1$.

Thus, $f(\theta) \approx f(0) + f'(0)(\theta - 0) = 0 + 1(\theta - 0) = \theta$.

(b) The relative error in approximating $\sin \theta$ by θ for $\theta = \pi/18$ is $\frac{\pi/18 - \sin(\pi/18)}{\sin(\pi/18)} \approx 0.0051$, so the percentage error

is 0.51%.



We want to know the values of θ for which $y = \theta$ approximates $y = \sin \theta$ with less than a 2% difference; that is, the values of θ for which

$$\left| \frac{\theta - \sin \theta}{\sin \theta} \right| < 0.02 \Leftrightarrow -0.02 < \frac{\theta - \sin \theta}{\sin \theta} < 0.02 \Leftrightarrow$$

$$\begin{cases} -0.02 \sin \theta < \theta - \sin \theta < 0.02 \sin \theta & \text{if } \sin \theta > 0 \\ -0.02 \sin \theta > \theta - \sin \theta > 0.02 \sin \theta & \text{if } \sin \theta < 0 \end{cases} \Leftrightarrow \begin{cases} 0.98 \sin \theta < \theta < 1.02 \sin \theta & \text{if } \sin \theta > 0 \\ 1.02 \sin \theta < \theta < 0.98 \sin \theta & \text{if } \sin \theta < 0 \end{cases}$$

In the first figure, we see that the graphs are very close to each other near $\theta = 0$. Changing the viewing rectangle and using an intersect feature (see the second figure) we find that $y = \theta$ intersects $y = 1.02 \sin \theta$ at $\theta \approx 0.344$.

By symmetry, they also intersect at $\theta \approx -0.344$ (see the third figure).

Thus, with θ measured in radians, $\sin \theta$ and θ differ by less than 2% for $-0.344 < \theta < 0.344$. Converting 0.344 radians to degrees, we get $0.344(180^\circ/\pi) \approx 19.7^\circ$, so the corresponding interval in degrees is approximately $-19.7^\circ < \theta < 19.7^\circ$.

- 49.** (a) The graph shows that $f'(1) = 2$, so $L(x) = f(1) + f'(1)(x - 1) = 5 + 2(x - 1) = 2x + 3$.

$f(0.9) \approx L(0.9) = 4.8$ and $f(1.1) \approx L(1.1) = 5.2$.

- (b) From the graph, we see that $f'(x)$ is positive and decreasing. This means that the slopes of the tangent lines are positive, but the tangents are becoming less steep. So the tangent lines lie *above* the curve. Thus, the estimates in part (a) are too large.

- 50.** (a) $g'(x) = \sqrt{x^2 + 5} \Rightarrow g'(2) = \sqrt{9} = 3$. $g(1.95) \approx g(2) + g'(2)(1.95 - 2) = -4 + 3(-0.05) = -4.15$.

$g(2.05) \approx g(2) + g'(2)(2.05 - 2) = -4 + 3(0.05) = -3.85$.

- (b) The formula $g'(x) = \sqrt{x^2 + 5}$ shows that $g'(x)$ is positive and increasing. This means that the slopes of the tangent lines are positive and the tangents are getting steeper. So the tangent lines lie *below* the graph of g . Hence, the estimates in part (a) are too small.

DISCOVERY PROJECT Polynomial Approximations

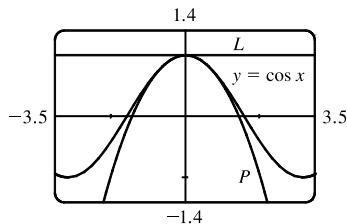
1. We first write the functions described in conditions (i), (ii), and (iii):

$$\begin{array}{ll} P(x) = A + Bx + Cx^2 & f(x) = \cos x \\ P'(x) = B + 2Cx & f'(x) = -\sin x \\ P''(x) = 2C & f''(x) = -\cos x \end{array}$$

So, taking $a = 0$, our three conditions become

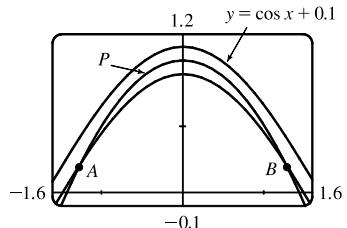
$$\begin{aligned} P(0) = f(0): \quad A &= \cos 0 = 1 \\ P'(0) = f'(0): \quad B &= -\sin 0 = 0 \\ P''(0) = f''(0): \quad 2C &= -\cos 0 = -1 \Rightarrow C = -\frac{1}{2} \end{aligned}$$

The desired quadratic function is $P(x) = 1 - \frac{1}{2}x^2$, so the quadratic approximation is $\cos x \approx 1 - \frac{1}{2}x^2$.



The figure shows a graph of the cosine function together with its linear approximation $L(x) = 1$ and quadratic approximation $P(x) = 1 - \frac{1}{2}x^2$ near 0. You can see that the quadratic approximation is much better than the linear one.

2. Accuracy to within 0.1 means that $|\cos x - (1 - \frac{1}{2}x^2)| < 0.1 \Leftrightarrow -0.1 < \cos x - (1 - \frac{1}{2}x^2) < 0.1 \Leftrightarrow 0.1 > (1 - \frac{1}{2}x^2) - \cos x > -0.1 \Leftrightarrow \cos x + 0.1 > 1 - \frac{1}{2}x^2 > \cos x - 0.1 \Leftrightarrow \cos x - 0.1 < 1 - \frac{1}{2}x^2 < \cos x + 0.1$.



From the figure we see that this is true between A and B . Zooming in or using an intersect feature, we find that the x -coordinates of B and A are about ± 1.26 . Thus, the approximation $\cos x \approx 1 - \frac{1}{2}x^2$ is accurate to within 0.1 when $-1.26 < x < 1.26$.

3. If $P(x) = A + B(x - a) + C(x - a)^2$, then $P'(x) = B + 2C(x - a)$ and $P''(x) = 2C$. Applying the conditions (i), (ii), and (iii), we get

$$\begin{aligned} P(a) = f(a): \quad A &= f(a) \\ P'(a) = f'(a): \quad B &= f'(a) \\ P''(a) = f''(a): \quad 2C = f''(a) &\Rightarrow C = \frac{1}{2}f''(a) \end{aligned}$$

Thus, $P(x) = A + B(x - a) + C(x - a)^2$ can be written in the form $P(x) = f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2$.

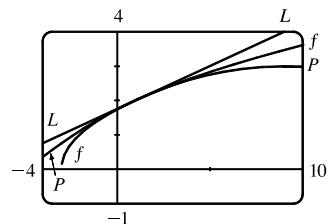
4. From Example 2.9.1, we have $f(1) = 2$, $f'(1) = \frac{1}{4}$, and $f'(x) = \frac{1}{2}(x + 3)^{-1/2}$.

So $f''(x) = -\frac{1}{4}(x + 3)^{-3/2} \Rightarrow f''(1) = -\frac{1}{32}$.

From Problem 3, the quadratic approximation $P(x)$ is

$$\sqrt{x+3} \approx f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2 = 2 + \frac{1}{4}(x-1) - \frac{1}{64}(x-1)^2.$$

The figure shows the function $f(x) = \sqrt{x+3}$ together with its linear



approximation $L(x) = \frac{1}{4}x + \frac{7}{4}$ and its quadratic approximation $P(x)$. You can see that $P(x)$ is a better approximation than $L(x)$ and this is borne out by the numerical values in the following chart.

	from $L(x)$	actual value	from $P(x)$
$\sqrt{3.98}$	1.9950	1.99499373...	1.99499375
$\sqrt{4.05}$	2.0125	2.01246118...	2.01246094
$\sqrt{4.2}$	2.0500	2.04939015...	2.04937500

5. $T_n(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + \dots + c_n(x - a)^n$. If we put $x = a$ in this equation,

then all terms after the first are 0 and we get $T_n(a) = c_0$. Now we differentiate $T_n(x)$ and obtain

$$T'_n(x) = c_1 + 2c_2(x - a) + 3c_3(x - a)^2 + 4c_4(x - a)^3 + \dots + nc_n(x - a)^{n-1}. \text{ Substituting } x = a \text{ gives } T'_n(a) = c_1.$$

Differentiating again, we have $T''_n(x) = 2c_2 + 2 \cdot 3c_3(x - a) + 3 \cdot 4c_4(x - a)^2 + \dots + (n-1)nc_n(x - a)^{n-2}$ and so

$$T''_n(a) = 2c_2. \text{ Continuing in this manner, we get } T'''_n(x) = 2 \cdot 3c_3 + 2 \cdot 3 \cdot 4c_4(x - a) + \dots + (n-2)(n-1)nc_n(x - a)^{n-3}$$

and $T'''_n(a) = 2 \cdot 3c_3$. By now we see the pattern. If we continue to differentiate and substitute $x = a$, we obtain

$$T^{(4)}_n(a) = 2 \cdot 3 \cdot 4c_4 \text{ and in general, for any integer } k \text{ between 1 and } n, T_n^{(k)}(a) = 2 \cdot 3 \cdot 4 \cdot 5 \cdots kc_k = k!c_k \Rightarrow$$

$$c_k = \frac{T_n^{(k)}(a)}{k!}. \text{ Because we want } T_n \text{ and } f \text{ to have the same derivatives at } a, \text{ we require that } c_k = \frac{f^{(k)}(a)}{k!} \text{ for}$$

$$k = 1, 2, \dots, n.$$

6. $T_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x - a)^n$. To compute the coefficients in this equation we need to calculate the derivatives of f at 0:

$$\begin{array}{ll} f(x) = \cos x & f(0) = \cos 0 = 1 \\ f'(x) = -\sin x & f'(0) = -\sin 0 = 0 \\ f''(x) = -\cos x & f''(0) = -1 \\ f'''(x) = \sin x & f'''(0) = 0 \\ f^{(4)}(x) = \cos x & f^{(4)}(0) = 1 \end{array}$$

We see that the derivatives repeat in a cycle of length 4, so $f^{(5)}(0) = 0$, $f^{(6)}(0) = -1$, $f^{(7)}(0) = 0$, and $f^{(8)}(0) = 1$.

From the original expression for $T_n(x)$, with $n = 8$ and $a = 0$, we have

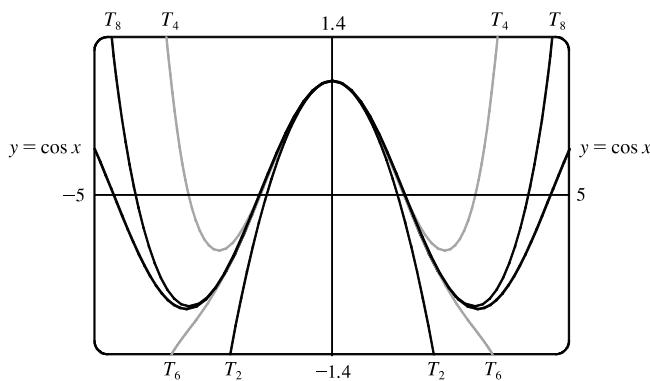
$$\begin{aligned} T_8(x) &= f(0) + f'(0)(x - 0) + \frac{f''(0)}{2!}(x - 0)^2 + \frac{f'''(0)}{3!}(x - 0)^3 + \dots + \frac{f^{(8)}(0)}{8!}(x - 0)^8 \\ &= 1 + 0 \cdot x + \frac{-1}{2!}x^2 + 0 \cdot x^3 + \frac{1}{4!}x^4 + 0 \cdot x^5 + \frac{-1}{6!}x^6 + 0 \cdot x^7 + \frac{1}{8!}x^8 = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \end{aligned}$$

and the desired approximation is $\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}$. The Taylor polynomials T_2 , T_4 , and T_6 consist of the

initial terms of T_8 up through degree 2, 4, and 6, respectively. Therefore, $T_2(x) = 1 - \frac{x^2}{2!}$, $T_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$, and

$$T_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}. \text{ We graph } T_2, T_4, T_6, T_8, \text{ and } f:$$

[continued]



Notice that $T_2(x)$ is a good approximation to $\cos x$ near 0, $T_4(x)$ is a good approximation on a larger interval, $T_6(x)$ is a better approximation, and $T_8(x)$ is better still. Each successive Taylor polynomial is a good approximation on a larger interval than the previous one.

2 Review

TRUE-FALSE QUIZ

1. False. See the note after Theorem 2.2.4.
2. True. This is the Sum Rule.
3. False. See the warning before the Product Rule.
4. True. This is the Chain Rule.
5. True. $\frac{d}{dx} \sqrt{f(x)} = \frac{d}{dx} [f(x)]^{1/2} = \frac{1}{2} [f(x)]^{-1/2} f'(x) = \frac{f'(x)}{2\sqrt{f(x)}}$
6. False. $\frac{d}{dx} f(\sqrt{x}) = f'(\sqrt{x}) \cdot \frac{1}{2}x^{-1/2} = \frac{f'(\sqrt{x})}{2\sqrt{x}}$, which is not $\frac{f'(x)}{2\sqrt{x}}$.
7. False. $f(x) = |x^2 + x| = x^2 + x$ for $x \geq 0$ or $x \leq -1$ and $|x^2 + x| = -(x^2 + x)$ for $-1 < x < 0$. So $f'(x) = 2x + 1$ for $x > 0$ or $x < -1$ and $f'(x) = -(2x + 1)$ for $-1 < x < 0$. But $|2x + 1| = 2x + 1$ for $x \geq -\frac{1}{2}$ and $|2x + 1| = -2x - 1$ for $x < -\frac{1}{2}$.
8. True. $f'(r)$ exists \Rightarrow f is differentiable at $r \Rightarrow f$ is continuous at $r \Rightarrow \lim_{x \rightarrow r} f(x) = f(r)$.
9. True. $g(x) = x^5 \Rightarrow g'(x) = 5x^4 \Rightarrow g'(2) = 5(2)^4 = 80$, and by the definition of the derivative, $\lim_{x \rightarrow 2} \frac{g(x) - g(2)}{x - 2} = g'(2) = 5(2)^4 = 80$.
10. False. $\frac{d^2y}{dx^2}$ is the second derivative while $\left(\frac{dy}{dx}\right)^2$ is the first derivative squared. For example, if $y = x$, then $\frac{d^2y}{dx^2} = 0$, but $\left(\frac{dy}{dx}\right)^2 = 1$.
11. False. A tangent line to the parabola $y = x^2$ has slope $dy/dx = 2x$, so at $(-2, 4)$ the slope of the tangent is $2(-2) = -4$ and an equation of the tangent line is $y - 4 = -4(x + 2)$. [The given equation, $y - 4 = 2x(x + 2)$, is not even linear!]

12. True. $\frac{d}{dx}(\tan^2 x) = 2 \tan x \sec^2 x$, and $\frac{d}{dx}(\sec^2 x) = 2 \sec x (\sec x \tan x) = 2 \tan x \sec^2 x$.

$$\text{Or: } \frac{d}{dx}(\sec^2 x) = \frac{d}{dx}(1 + \tan^2 x) = \frac{d}{dx}(\tan^2 x).$$

13. True. If $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, then $p'(x) = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \cdots + a_1$, which is a polynomial.

14. True. If $r(x) = \frac{p(x)}{q(x)}$, then $r'(x) = \frac{q(x)p'(x) - p(x)q'(x)}{[q(x)]^2}$, which is a quotient of polynomials, that is, a rational function.

15. False. For example, let $f(x) = x$ and $a = 0$. Then f is differentiable at a , but $|f| = |x|$ is not.

16. True. $f(x) = (x^6 - x^4)^5$ is a polynomial of degree 30, so its 31st derivative, $f^{(31)}(x)$, is 0.

EXERCISES

1. (a) $s = s(t) = 1 + 2t + t^2/4$. The average velocity over the time interval $[1, 1+h]$ is

$$v_{\text{avg}} = \frac{s(1+h) - s(1)}{(1+h) - 1} = \frac{1 + 2(1+h) + (1+h)^2/4 - 13/4}{h} = \frac{10h + h^2}{4h} = \frac{10+h}{4}$$

So for the following intervals the average velocities are:

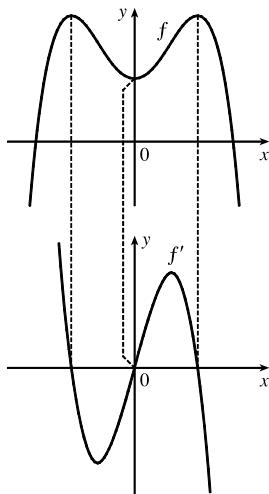
(i) $[1, 3]$: $h = 2$, $v_{\text{avg}} = (10+2)/4 = 3$ m/s (ii) $[1, 2]$: $h = 1$, $v_{\text{avg}} = (10+1)/4 = 2.75$ m/s

(iii) $[1, 1.5]$: $h = 0.5$, $v_{\text{avg}} = (10+0.5)/4 = 2.625$ m/s (iv) $[1, 1.1]$: $h = 0.1$, $v_{\text{avg}} = (10+0.1)/4 = 2.525$ m/s

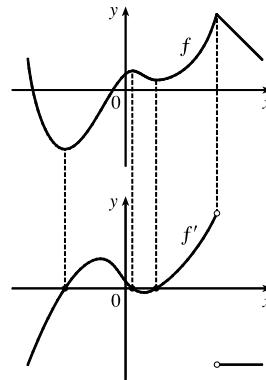
(b) When $t = 1$, the instantaneous velocity is $\lim_{h \rightarrow 0} \frac{s(1+h) - s(1)}{h} = \lim_{h \rightarrow 0} \frac{10+h}{4} = \frac{10}{4} = 2.5$ m/s.

2. f is not differentiable: at $x = -4$ because f is not continuous, at $x = -1$ because f has a corner, at $x = 2$ because f is not continuous, and at $x = 5$ because f has a vertical tangent.

3.



4.



5. The graph of a has tangent lines with positive slope for $x < 0$ and negative slope for $x > 0$, and the values of c fit this pattern, so c must be the graph of the derivative of the function for a . The graph of c has horizontal tangent lines to the left and right of the x -axis and b has zeros at these points. Hence, b is the graph of the derivative of the function for c . Therefore, a is the graph of f , c is the graph of f' , and b is the graph of f'' .

6. $2^6 = 64$, so $f(x) = x^6$ and $a = 2$.

7. (a) $f'(r)$ is the rate at which the total cost changes with respect to the interest rate. Its units are dollars/(percent per year).

(b) The total cost of paying off the loan is increasing by \$1200/(percent per year) as the interest rate reaches 10%. So if the interest rate goes up from 10% to 11%, the cost goes up approximately \$1200.

(c) As r increases, C increases. So $f'(r)$ will always be positive.

8. (a) Drawing slope triangles, we obtain the following estimates: $F'(1950) \approx \frac{1.1}{10} = 0.11$, $F'(1965) \approx \frac{-1.6}{10} = -0.16$, and $F'(1987) \approx \frac{0.2}{10} = 0.02$.

(b) The rate of change of the average number of children born to each woman was increasing by 0.11 in 1950, decreasing by 0.16 in 1965, and increasing by 0.02 in 1987.

(c) There are many possible reasons:

- In the baby-boom era (post-WWII), there was optimism about the economy and family size was rising.
- In the baby-bust era, there was less economic optimism, and it was considered less socially responsible to have a large family.
- In the baby-boomlet era, there was increased economic optimism and a return to more conservative attitudes.

9. (a) $P'(t)$ is the rate at which the percentage of Americans under the age of 18 is changing with respect to time. Its units are percent per year (%/yr).

(b) To find $P'(t)$, we use $\lim_{h \rightarrow 0} \frac{P(t+h) - P(t)}{h} \approx \frac{P(t+h) - P(t)}{h}$ for small values of h .

$$\text{For 1950: } P'(1950) \approx \frac{P(1960) - P(1950)}{1960 - 1950} = \frac{35.7 - 31.1}{10} = 0.46$$

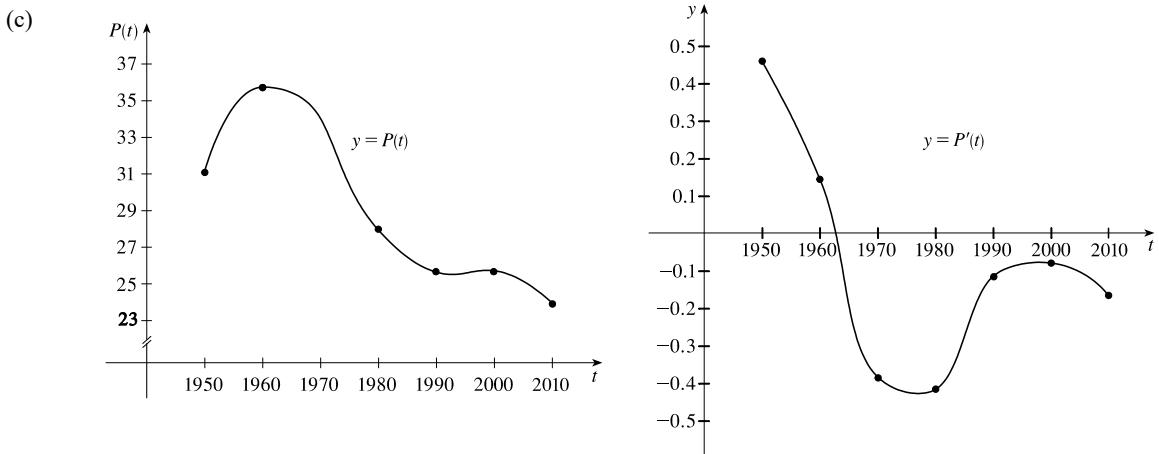
For 1960: We estimate $P'(1960)$ by using $h = -10$ and $h = 10$, and then average the two results to obtain a final estimate.

$$h = -10 \Rightarrow P'(1960) \approx \frac{P(1950) - P(1960)}{1950 - 1960} = \frac{31.1 - 35.7}{-10} = 0.46$$

$$h = 10 \Rightarrow P'(1960) \approx \frac{P(1970) - P(1960)}{1970 - 1960} = \frac{34.0 - 35.7}{10} = -0.17$$

So we estimate that $P'(1960) \approx \frac{1}{2}[0.46 + (-0.17)] = 0.145$.

t	1950	1960	1970	1980	1990	2000	2010
$P'(t)$	0.460	0.145	-0.385	-0.415	-0.115	-0.085	-0.170



(d) We could get more accurate values for $P'(t)$ by obtaining data for the mid-decade years 1955, 1965, 1975, 1985, 1995, and 2005.

10. $f(x) = \frac{2}{x^2} \Rightarrow$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{2}{(x+h)^2} - \frac{2}{x^2}}{h} = \lim_{h \rightarrow 0} \frac{2x^2 - 2(x+h)^2}{hx^2(x+h)^2} = \lim_{h \rightarrow 0} \frac{2x^2 - 2x^2 - 4xh - 2h^2}{hx^2(x+h)^2} \\ &= \lim_{h \rightarrow 0} \frac{-2h(2x+h)}{hx^2(x+h)^2} = \lim_{h \rightarrow 0} \frac{-2(2x+h)}{x^2(x+h)^2} = \frac{-2 \cdot 2x}{x^2 \cdot x^2} = \frac{-4}{x^3} \end{aligned}$$

11. $f(x) = x^3 + 5x + 4 \Rightarrow$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h)^3 + 5(x+h) + 4 - (x^3 + 5x + 4)}{h} \\ &= \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3 + 5h}{h} = \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2 + 5) = 3x^2 + 5 \end{aligned}$$

12. (a) $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{3-5(x+h)} - \sqrt{3-5x}}{h} \frac{\sqrt{3-5(x+h)} + \sqrt{3-5x}}{\sqrt{3-5(x+h)} + \sqrt{3-5x}}$
 $= \lim_{h \rightarrow 0} \frac{[3-5(x+h)] - (3-5x)}{h(\sqrt{3-5(x+h)} + \sqrt{3-5x})} = \lim_{h \rightarrow 0} \frac{-5}{\sqrt{3-5(x+h)} + \sqrt{3-5x}} = \frac{-5}{2\sqrt{3-5x}}$

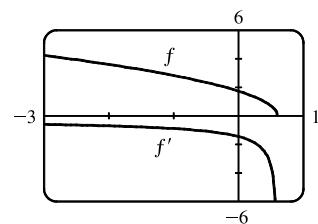
(b) Domain of f : (the radicand must be nonnegative) $3-5x \geq 0 \Rightarrow$

$$5x \leq 3 \Rightarrow x \in (-\infty, \frac{3}{5}]$$

Domain of f' : exclude $\frac{3}{5}$ because it makes the denominator zero;

$$x \in (-\infty, \frac{3}{5})$$

(c) Our answer to part (a) is reasonable because $f'(x)$ is always negative and f is always decreasing.



13. $y = (x^2 + x^3)^4 \Rightarrow y' = 4(x^2 + x^3)^3(2x + 3x^2) = 4(x^2)^3(1+x)^3x(2+3x) = 4x^7(x+1)^3(3x+2)$

14. $y = \frac{1}{\sqrt{x}} - \frac{1}{\sqrt[5]{x^3}} = x^{-1/2} - x^{-3/5} \Rightarrow y' = -\frac{1}{2}x^{-3/2} + \frac{3}{5}x^{-8/5}$ or $\frac{3}{5x\sqrt[5]{x^3}} - \frac{1}{2x\sqrt{x}}$ or $\frac{1}{10}x^{-8/5}(-5x^{1/10} + 6)$

15. $y = \frac{x^2 - x + 2}{\sqrt{x}} = x^{3/2} - x^{1/2} + 2x^{-1/2} \Rightarrow y' = \frac{3}{2}x^{1/2} - \frac{1}{2}x^{-1/2} - x^{-3/2} = \frac{3}{2}\sqrt{x} - \frac{1}{2\sqrt{x}} - \frac{1}{\sqrt{x^3}}$

16. $y = \frac{\tan x}{1 + \cos x} \Rightarrow y' = \frac{(1 + \cos x)\sec^2 x - \tan x(-\sin x)}{(1 + \cos x)^2} = \frac{(1 + \cos x)\sec^2 x + \tan x \sin x}{(1 + \cos x)^2}$

17. $y = x^2 \sin \pi x \Rightarrow y' = x^2(\cos \pi x)\pi + (\sin \pi x)(2x) = x(\pi x \cos \pi x + 2 \sin \pi x)$

18. $y = \left(x + \frac{1}{x^2}\right)^{\sqrt{7}} \Rightarrow y' = \sqrt{7} \left(x + \frac{1}{x^2}\right)^{\sqrt{7}-1} \left(1 - \frac{2}{x^3}\right)$

19. $y = \frac{t^4 - 1}{t^4 + 1} \Rightarrow y' = \frac{(t^4 + 1)4t^3 - (t^4 - 1)4t^3}{(t^4 + 1)^2} = \frac{4t^3[(t^4 + 1) - (t^4 - 1)]}{(t^4 + 1)^2} = \frac{8t^3}{(t^4 + 1)^2}$

20. $y = \sin(\cos x) \Rightarrow y' = \cos(\cos x)(-\sin x) = -\sin x \cos(\cos x)$

21. $y = \tan \sqrt{1-x} \Rightarrow y' = (\sec^2 \sqrt{1-x}) \left(\frac{1}{2\sqrt{1-x}}\right)(-1) = -\frac{\sec^2 \sqrt{1-x}}{2\sqrt{1-x}}$

22. Using the Reciprocal Rule, $g(x) = \frac{1}{f(x)} \Rightarrow g'(x) = -\frac{f'(x)}{[f(x)]^2}$, we have $y = \frac{1}{\sin(x - \sin x)} \Rightarrow y' = -\frac{\cos(x - \sin x)(1 - \cos x)}{\sin^2(x - \sin x)}$.

23. $\frac{d}{dx}(xy^4 + x^2y) = \frac{d}{dx}(x + 3y) \Rightarrow x \cdot 4y^3y' + y^4 \cdot 1 + x^2 \cdot y' + y \cdot 2x = 1 + 3y' \Rightarrow y'(4xy^3 + x^2 - 3) = 1 - y^4 - 2xy \Rightarrow y' = \frac{1 - y^4 - 2xy}{4xy^3 + x^2 - 3}$

24. $y = \sec(1 + x^2) \Rightarrow y' = 2x \sec(1 + x^2) \tan(1 + x^2)$

25. $y = \frac{\sec 2\theta}{1 + \tan 2\theta} \Rightarrow$
 $y' = \frac{(1 + \tan 2\theta)(\sec 2\theta \tan 2\theta \cdot 2) - (\sec 2\theta)(\sec^2 2\theta \cdot 2)}{(1 + \tan 2\theta)^2} = \frac{2 \sec 2\theta [(1 + \tan 2\theta) \tan 2\theta - \sec^2 2\theta]}{(1 + \tan 2\theta)^2}$
 $= \frac{2 \sec 2\theta (\tan 2\theta + \tan^2 2\theta - \sec^2 2\theta)}{(1 + \tan 2\theta)^2} = \frac{2 \sec 2\theta (\tan 2\theta - 1)}{(1 + \tan 2\theta)^2} \quad [1 + \tan^2 x = \sec^2 x]$

26. $\frac{d}{dx}(y + x \cos y) = \frac{d}{dx}(x^2y) \Rightarrow y' + x(-\sin y \cdot y') + \cos y \cdot 1 = x^2y' + y \cdot 2x \Rightarrow y' - x \sin y \cdot y' - x^2y' = 2xy - \cos y \Rightarrow (1 - x \sin y - x^2)y' = 2xy - \cos y \Rightarrow y' = \frac{2xy - \cos y}{1 - x \sin y - x^2}$

27. $y = (1 - x^{-1})^{-1} \Rightarrow$
 $y' = -1(1 - x^{-1})^{-2}[-(-1x^{-2})] = -(1 - 1/x)^{-2}x^{-2} = -((x - 1)/x)^{-2}x^{-2} = -(x - 1)^{-2}$

28. $y = \frac{1}{\sqrt[3]{x+\sqrt{x}}} = (x+\sqrt{x})^{-1/3} \Rightarrow y' = -\frac{1}{3}(x+\sqrt{x})^{-4/3}\left(1 + \frac{1}{2\sqrt{x}}\right)$

29. $\sin(xy) = x^2 - y \Rightarrow \cos(xy)(xy' + y \cdot 1) = 2x - y' \Rightarrow x \cos(xy)y' + y' = 2x - y \cos(xy) \Rightarrow$

$$y'[x \cos(xy) + 1] = 2x - y \cos(xy) \Rightarrow y' = \frac{2x - y \cos(xy)}{x \cos(xy) + 1}$$

30. $y = \sqrt{\sin\sqrt{x}} \Rightarrow y' = \frac{1}{2}(\sin\sqrt{x})^{-1/2}(\cos\sqrt{x})\left(\frac{1}{2\sqrt{x}}\right) = \frac{\cos\sqrt{x}}{4\sqrt{x}\sin\sqrt{x}}$

31. $y = \cot(3x^2 + 5) \Rightarrow y' = -\csc^2(3x^2 + 5)(6x) = -6x \csc^2(3x^2 + 5)$

32. $y = \frac{(x+\lambda)^4}{x^4+\lambda^4} \Rightarrow y' = \frac{(x^4+\lambda^4)(4)(x+\lambda)^3 - (x+\lambda)^4(4x^3)}{(x^4+\lambda^4)^2} = \frac{4(x+\lambda)^3(\lambda^4 - \lambda x^3)}{(x^4+\lambda^4)^2}$

33. $y = \sqrt{x} \cos\sqrt{x} \Rightarrow$

$$\begin{aligned} y' &= \sqrt{x} \left(\cos\sqrt{x}\right)' + \cos\sqrt{x} \left(\sqrt{x}\right)' = \sqrt{x} \left[-\sin\sqrt{x} \left(\frac{1}{2}x^{-1/2}\right)\right] + \cos\sqrt{x} \left(\frac{1}{2}x^{-1/2}\right) \\ &= \frac{1}{2}x^{-1/2} \left(-\sqrt{x} \sin\sqrt{x} + \cos\sqrt{x}\right) = \frac{\cos\sqrt{x} - \sqrt{x} \sin\sqrt{x}}{2\sqrt{x}} \end{aligned}$$

34. $y = \frac{\sin mx}{x} \Rightarrow y' = \frac{x \cdot \cos mx \cdot m - \sin mx \cdot 1}{(x)^2} = \frac{mx \cos mx - \sin mx}{x^2}$

35. $y = \tan^2(\sin\theta) = [\tan(\sin\theta)]^2 \Rightarrow y' = 2[\tan(\sin\theta)] \cdot \sec^2(\sin\theta) \cdot \cos\theta$

36. $x \tan y = y - 1 \Rightarrow \tan y + (x \sec^2 y) y' = y' \Rightarrow y' = \frac{\tan y}{1 - x \sec^2 y}$

37. $y = (x \tan x)^{1/5} \Rightarrow y' = \frac{1}{5}(x \tan x)^{-4/5}(\tan x + x \sec^2 x)$

38. $y = \frac{(x-1)(x-4)}{(x-2)(x-3)} = \frac{x^2 - 5x + 4}{x^2 - 5x + 6} \Rightarrow y' = \frac{(x^2 - 5x + 6)(2x-5) - (x^2 - 5x + 4)(2x-5)}{(x^2 - 5x + 6)^2} = \frac{2(2x-5)}{(x-2)^2(x-3)^2}$

39. $y = \sin(\tan\sqrt{1+x^3}) \Rightarrow y' = \cos(\tan\sqrt{1+x^3})(\sec^2\sqrt{1+x^3})[3x^2/(2\sqrt{1+x^3})]$

$$\begin{aligned} 40. \quad y &= \sin^2(\cos\sqrt{\sin\pi x}) = \left[\sin(\cos\sqrt{\sin\pi x})\right]^2 \Rightarrow \\ y' &= 2\left[\sin(\cos\sqrt{\sin\pi x})\right]\left[\sin(\cos\sqrt{\sin\pi x})\right]' = 2\sin(\cos\sqrt{\sin\pi x})\cos(\cos\sqrt{\sin\pi x})(\cos\sqrt{\sin\pi x})' \\ &= 2\sin(\cos\sqrt{\sin\pi x})\cos(\cos\sqrt{\sin\pi x})(-\sin\sqrt{\sin\pi x})(\sqrt{\sin\pi x})' \\ &= -2\sin(\cos\sqrt{\sin\pi x})\cos(\cos\sqrt{\sin\pi x})\sin\sqrt{\sin\pi x} \cdot \frac{1}{2}(\sin\pi x)^{-1/2}(\sin\pi x)' \\ &\quad - \sin(\cos\sqrt{\sin\pi x})\cos(\cos\sqrt{\sin\pi x})\sin\sqrt{\sin\pi x} \\ &= \frac{-\pi\sin(\cos\sqrt{\sin\pi x})\cos(\cos\sqrt{\sin\pi x})\sin\sqrt{\sin\pi x}\cos\pi x}{\sqrt{\sin\pi x}} \end{aligned}$$

- 41.** $f(t) = \sqrt{4t+1} \Rightarrow f'(t) = \frac{1}{2}(4t+1)^{-1/2} \cdot 4 = 2(4t+1)^{-1/2} \Rightarrow$
 $f''(t) = 2(-\frac{1}{2})(4t+1)^{-3/2} \cdot 4 = -4/(4t+1)^{3/2}$, so $f''(2) = -4/9^{3/2} = -\frac{4}{27}$.
- 42.** $g(\theta) = \theta \sin \theta \Rightarrow g'(\theta) = \theta \cos \theta + \sin \theta \cdot 1 \Rightarrow g''(\theta) = \theta(-\sin \theta) + \cos \theta \cdot 1 + \cos \theta = 2 \cos \theta - \theta \sin \theta$,
so $g''(\pi/6) = 2 \cos(\pi/6) - (\pi/6) \sin(\pi/6) = 2(\sqrt{3}/2) - (\pi/6)(1/2) = \sqrt{3} - \pi/12$.
- 43.** $x^6 + y^6 = 1 \Rightarrow 6x^5 + 6y^5y' = 0 \Rightarrow y' = -x^5/y^5 \Rightarrow$
 $y'' = -\frac{y^5(5x^4) - x^5(5y^4y')}{(y^5)^2} = -\frac{5x^4y^4[y - x(-x^5/y^5)]}{y^{10}} = -\frac{5x^4[(y^6 + x^6)/y^5]}{y^6} = -\frac{5x^4}{y^{11}}$
- 44.** $f(x) = (2-x)^{-1} \Rightarrow f'(x) = (2-x)^{-2} \Rightarrow f''(x) = 2(2-x)^{-3} \Rightarrow f'''(x) = 2 \cdot 3(2-x)^{-4} \Rightarrow$
 $f^{(4)}(x) = 2 \cdot 3 \cdot 4(2-x)^{-5}$. In general, $f^{(n)}(x) = 2 \cdot 3 \cdot 4 \cdots n(2-x)^{-(n+1)} = \frac{n!}{(2-x)^{(n+1)}}$.
- 45.** $\lim_{x \rightarrow 0} \frac{\sec x}{1 - \sin x} = \frac{\sec 0}{1 - \sin 0} = \frac{1}{1 - 0} = 1$
- 46.** $\lim_{t \rightarrow 0} \frac{t^3}{\tan^3 2t} = \lim_{t \rightarrow 0} \frac{t^3 \cos^3 2t}{\sin^3 2t} = \lim_{t \rightarrow 0} \cos^3 2t \cdot \frac{1}{8 \frac{\sin^3 2t}{(2t)^3}} = \lim_{t \rightarrow 0} \frac{\cos^3 2t}{8 \left(\lim_{t \rightarrow 0} \frac{\sin 2t}{2t} \right)^3} = \frac{1}{8 \cdot 1^3} = \frac{1}{8}$
- 47.** $y = 4 \sin^2 x \Rightarrow y' = 4 \cdot 2 \sin x \cos x$. At $(\frac{\pi}{6}, 1)$, $y' = 8 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3}$, so an equation of the tangent line
is $y - 1 = 2\sqrt{3}(x - \frac{\pi}{6})$, or $y = 2\sqrt{3}x + 1 - \pi\sqrt{3}/3$.
- 48.** $y = \frac{x^2 - 1}{x^2 + 1} \Rightarrow y' = \frac{(x^2 + 1)(2x) - (x^2 - 1)(2x)}{(x^2 + 1)^2} = \frac{4x}{(x^2 + 1)^2}$.
At $(0, -1)$, $y' = 0$, so an equation of the tangent line is $y + 1 = 0(x - 0)$, or $y = -1$.
- 49.** $y = \sqrt{1 + 4 \sin x} \Rightarrow y' = \frac{1}{2}(1 + 4 \sin x)^{-1/2} \cdot 4 \cos x = \frac{2 \cos x}{\sqrt{1 + 4 \sin x}}$.
At $(0, 1)$, $y' = \frac{2}{\sqrt{1}} = 2$, so an equation of the tangent line is $y - 1 = 2(x - 0)$, or $y = 2x + 1$.
The slope of the normal line is $-\frac{1}{2}$, so an equation of the normal line is $y - 1 = -\frac{1}{2}(x - 0)$, or $y = -\frac{1}{2}x + 1$.
- 50.** $x^2 + 4xy + y^2 = 13 \Rightarrow 2x + 4(xy' + y \cdot 1) + 2yy' = 0 \Rightarrow x + 2xy' + 2y + yy' = 0 \Rightarrow$
 $2xy' + yy' = -x - 2y \Rightarrow y'(2x + y) = -x - 2y \Rightarrow y' = \frac{-x - 2y}{2x + y}$.
At $(2, 1)$, $y' = \frac{-2 - 2}{4 + 1} = -\frac{4}{5}$, so an equation of the tangent line is $y - 1 = -\frac{4}{5}(x - 2)$, or $y = -\frac{4}{5}x + \frac{13}{5}$.
The slope of the normal line is $\frac{5}{4}$, so an equation of the normal line is $y - 1 = \frac{5}{4}(x - 2)$, or $y = \frac{5}{4}x - \frac{3}{2}$.

51. (a) $f(x) = x\sqrt{5-x} \Rightarrow$

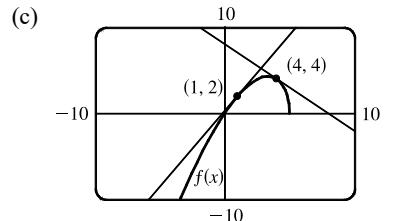
$$\begin{aligned} f'(x) &= x \left[\frac{1}{2}(5-x)^{-1/2}(-1) \right] + \sqrt{5-x} = \frac{-x}{2\sqrt{5-x}} + \sqrt{5-x} \cdot \frac{2\sqrt{5-x}}{2\sqrt{5-x}} = \frac{-x}{2\sqrt{5-x}} + \frac{2(5-x)}{2\sqrt{5-x}} \\ &= \frac{-x+10-2x}{2\sqrt{5-x}} = \frac{10-3x}{2\sqrt{5-x}} \end{aligned}$$

(b) At $(1, 2)$: $f'(1) = \frac{7}{4}$.

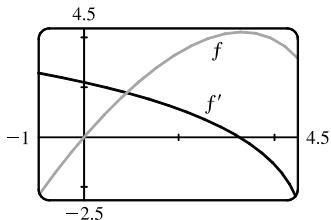
So an equation of the tangent line is $y - 2 = \frac{7}{4}(x - 1)$ or $y = \frac{7}{4}x + \frac{1}{4}$.

At $(4, 4)$: $f'(4) = -\frac{2}{2} = -1$.

So an equation of the tangent line is $y - 4 = -1(x - 4)$ or $y = -x + 8$.



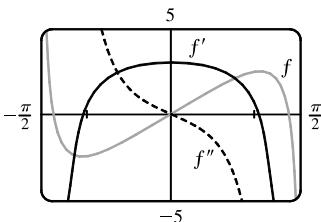
(d)



The graphs look reasonable, since f' is positive where f has tangents with positive slope, and f' is negative where f has tangents with negative slope.

52. (a) $f(x) = 4x - \tan x \Rightarrow f'(x) = 4 - \sec^2 x \Rightarrow f''(x) = -2 \sec x (\sec x \tan x) = -2 \sec^2 x \tan x$.

(b)



We can see that our answers are reasonable, since the graph of f' is 0 where f has a horizontal tangent, and the graph of f' is positive where f has tangents with positive slope and negative where f has tangents with negative slope. The same correspondence holds between the graphs of f' and f'' .

53. $y = \sin x + \cos x \Rightarrow y' = \cos x - \sin x = 0 \Leftrightarrow \cos x = \sin x \text{ and } 0 \leq x \leq 2\pi \Leftrightarrow x = \frac{\pi}{4} \text{ or } \frac{5\pi}{4}$, so the points are $(\frac{\pi}{4}, \sqrt{2})$ and $(\frac{5\pi}{4}, -\sqrt{2})$.

54. $x^2 + 2y^2 = 1 \Rightarrow 2x + 4yy' = 0 \Rightarrow y' = -x/(2y) = 1 \Leftrightarrow x = -2y$. Since the points lie on the ellipse,

we have $(-2y)^2 + 2y^2 = 1 \Rightarrow 6y^2 = 1 \Rightarrow y = \pm \frac{1}{\sqrt{6}}$. The points are $(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}})$ and $(\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}})$.

55. $y = f(x) = ax^2 + bx + c \Rightarrow f'(x) = 2ax + b$. We know that $f'(-1) = 6$ and $f'(5) = -2$, so $-2a + b = 6$ and

$10a + b = -2$. Subtracting the first equation from the second gives $12a = -8 \Rightarrow a = -\frac{2}{3}$. Substituting $-\frac{2}{3}$ for a in the first equation gives $b = \frac{14}{3}$. Now $f(1) = 4 \Rightarrow 4 = a + b + c$, so $c = 4 + \frac{2}{3} - \frac{14}{3} = 0$ and hence, $f(x) = -\frac{2}{3}x^2 + \frac{14}{3}x$.

56. If $y = f(x) = \frac{x}{x+1}$, then $f'(x) = \frac{(x+1)(1)-x(1)}{(x+1)^2} = \frac{1}{(x+1)^2}$. When $x = a$, the equation of the tangent line is

$y - \frac{a}{a+1} = \frac{1}{(a+1)^2}(x - a)$. This line passes through $(1, 2)$ when $2 - \frac{a}{a+1} = \frac{1}{(a+1)^2}(1-a) \Leftrightarrow$

$$2(a+1)^2 - a(a+1) = 1 - a \Leftrightarrow 2a^2 + 4a + 2 - a^2 - a - 1 + a = 0 \Leftrightarrow a^2 + 4a + 1 = 0.$$

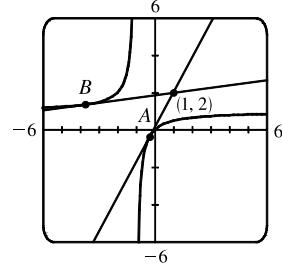
The quadratic formula gives the solutions of this equation as $a = \frac{-4 \pm \sqrt{4^2 - 4(1)(1)}}{2(1)} = \frac{-4 \pm \sqrt{12}}{2} = -2 \pm \sqrt{3}$,

so there are two such tangent lines. Since

$$\begin{aligned} f(-2 \pm \sqrt{3}) &= \frac{-2 \pm \sqrt{3}}{-2 \pm \sqrt{3} + 1} = \frac{-2 \pm \sqrt{3}}{-1 \pm \sqrt{3}} \cdot \frac{-1 \mp \sqrt{3}}{-1 \mp \sqrt{3}} \\ &= \frac{2 \pm 2\sqrt{3} \mp \sqrt{3} - 3}{1 - 3} = \frac{-1 \pm \sqrt{3}}{-2} = \frac{1 \mp \sqrt{3}}{2}, \end{aligned}$$

the lines touch the curve at $A\left(-2 + \sqrt{3}, \frac{1-\sqrt{3}}{2}\right) \approx (-0.27, -0.37)$

and $B\left(-2 - \sqrt{3}, \frac{1+\sqrt{3}}{2}\right) \approx (-3.73, 1.37)$.



57. $f(x) = (x-a)(x-b)(x-c) \Rightarrow f'(x) = (x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b).$

$$\text{So } \frac{f'(x)}{f(x)} = \frac{(x-b)(x-c) + (x-a)(x-c) + (x-a)(x-b)}{(x-a)(x-b)(x-c)} = \frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c}.$$

$$\text{Or: } f(x) = (x-a)(x-b)(x-c) \Rightarrow \ln|f(x)| = \ln|x-a| + \ln|x-b| + \ln|x-c| \Rightarrow$$

$$\frac{f'(x)}{f(x)} = \frac{1}{x-a} + \frac{1}{x-b} + \frac{1}{x-c}$$

58. (a) $\cos 2x = \cos^2 x - \sin^2 x \Rightarrow -2 \sin 2x = -2 \cos x \sin x - 2 \sin x \cos x \Leftrightarrow \sin 2x = 2 \sin x \cos x$

(b) $\sin(x+a) = \sin x \cos a + \cos x \sin a \Rightarrow \cos(x+a) = \cos x \cos a - \sin x \sin a.$

59. (a) $S(x) = f(x) + g(x) \Rightarrow S'(x) = f'(x) + g'(x) \Rightarrow S'(1) = f'(1) + g'(1) = 3 + 1 = 4$

(b) $P(x) = f(x)g(x) \Rightarrow P'(x) = f(x)g'(x) + g(x)f'(x) \Rightarrow$

$$P'(2) = f(2)g'(2) + g(2)f'(2) = 1(4) + 1(2) = 4 + 2 = 6$$

(c) $Q(x) = \frac{f(x)}{g(x)} \Rightarrow Q'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} \Rightarrow$

$$Q'(1) = \frac{g(1)f'(1) - f(1)g'(1)}{[g(1)]^2} = \frac{3(3) - 2(1)}{3^2} = \frac{9 - 2}{9} = \frac{7}{9}$$

(d) $C(x) = f(g(x)) \Rightarrow C'(x) = f'(g(x))g'(x) \Rightarrow C'(2) = f'(g(2))g'(2) = f'(1) \cdot 4 = 3 \cdot 4 = 12$

60. (a) $P(x) = f(x)g(x) \Rightarrow P'(x) = f(x)g'(x) + g(x)f'(x) \Rightarrow$

$$P'(2) = f(2)g'(2) + g(2)f'(2) = (1)\left(\frac{6-0}{3-0}\right) + (4)\left(\frac{0-3}{3-0}\right) = (1)(2) + (4)(-1) = 2 - 4 = -2$$

(b) $Q(x) = \frac{f(x)}{g(x)} \Rightarrow Q'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} \Rightarrow$

$$Q'(2) = \frac{g(2)f'(2) - f(2)g'(2)}{[g(2)]^2} = \frac{(4)(-1) - (1)(2)}{4^2} = \frac{-6}{16} = -\frac{3}{8}$$

(c) $C(x) = f(g(x)) \Rightarrow C'(x) = f'(g(x))g'(x) \Rightarrow$

$$C'(2) = f'(g(2))g'(2) = f'(4)g'(2) = \left(\frac{6-0}{5-3}\right)(2) = (3)(2) = 6$$

61. $f(x) = x^2 g(x) \Rightarrow f'(x) = x^2 g'(x) + g(x)(2x)$ or $x[xg'(x) + 2g(x)]$

62. $f(x) = g(x^2) \Rightarrow f'(x) = g'(x^2)(2x) = 2xg'(x^2)$

63. $f(x) = [g(x)]^2 \Rightarrow f'(x) = 2[g(x)] \cdot g'(x) = 2g(x)g'(x)$

64. $f(x) = x^a g(x^b) \Rightarrow f'(x) = ax^{a-1}g(x^b) + x^a g'(x^b)(bx^{b-1}) = ax^{a-1}g(x^b) + bx^{a+b-1}g'(x^b)$

65. $f(x) = g(g(x)) \Rightarrow f'(x) = g'(g(x))g'(x)$

66. $f(x) = \sin(g(x)) \Rightarrow f'(x) = \cos(g(x)) \cdot g'(x)$

67. $f(x) = g(\sin x) \Rightarrow f'(x) = g'(\sin x) \cdot \cos x$

68. $f(x) = g(\tan \sqrt{x}) \Rightarrow$

$$f'(x) = g'(\tan \sqrt{x}) \cdot \frac{d}{dx} (\tan \sqrt{x}) = g'(\tan \sqrt{x}) \cdot \sec^2 \sqrt{x} \cdot \frac{d}{dx} (\sqrt{x}) = \frac{g'(\tan \sqrt{x}) \sec^2 \sqrt{x}}{2\sqrt{x}}$$

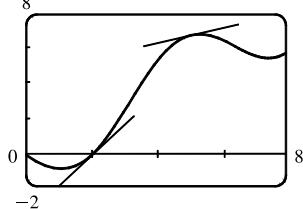
69. $h(x) = \frac{f(x)g(x)}{f(x)+g(x)} \Rightarrow$

$$\begin{aligned} h'(x) &= \frac{[f(x)+g(x)][f(x)g'(x)+g(x)f'(x)] - f(x)g(x)[f'(x)+g'(x)]}{[f(x)+g(x)]^2} \\ &= \frac{[f(x)]^2 g'(x) + f(x)g(x)f'(x) + f(x)g(x)g'(x) + [g(x)]^2 f'(x) - f(x)g(x)f'(x) - f(x)g(x)g'(x)}{[f(x)+g(x)]^2} \\ &= \frac{f'(x)[g(x)]^2 + g'(x)[f(x)]^2}{[f(x)+g(x)]^2} \end{aligned}$$

70. Using the Chain Rule repeatedly, $h(x) = f(g(\sin 4x)) \Rightarrow$

$$h'(x) = f'(g(\sin 4x)) \cdot \frac{d}{dx} (g(\sin 4x)) = f'(g(\sin 4x)) \cdot g'(\sin 4x) \cdot \frac{d}{dx} (\sin 4x) = f'(g(\sin 4x))g'(\sin 4x)(\cos 4x)(4).$$

71. (a)



(b) The average rate of change is larger on $[2, 3]$.

(c) The instantaneous rate of change (the slope of the tangent) is larger at $x = 2$.

(d) $f(x) = x - 2 \sin x \Rightarrow f'(x) = 1 - 2 \cos x,$

$$\text{so } f'(2) = 1 - 2 \cos 2 \approx 1.8323 \text{ and } f'(5) = 1 - 2 \cos 5 \approx 0.4327.$$

So $f'(2) > f'(5)$, as predicted in part (c).

72. (a) $x = \sqrt{b^2 + c^2 t^2} \Rightarrow v(t) = x' = [1/(2\sqrt{b^2 + c^2 t^2})] 2c^2 t = c^2 t / \sqrt{b^2 + c^2 t^2} \Rightarrow$

$$a(t) = v'(t) = \frac{c^2 \sqrt{b^2 + c^2 t^2} - c^2 t (c^2 t / \sqrt{b^2 + c^2 t^2})}{b^2 + c^2 t^2} = \frac{b^2 c^2}{(b^2 + c^2 t^2)^{3/2}}$$

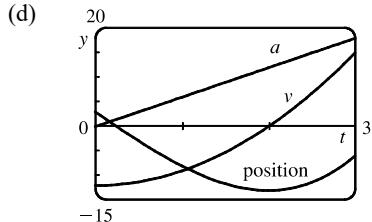
(b) $v(t) > 0$ for $t > 0$, so the particle always moves in the positive direction.

73. (a) $y = t^3 - 12t + 3 \Rightarrow v(t) = y' = 3t^2 - 12 \Rightarrow a(t) = v'(t) = 6t$

(b) $v(t) = 3(t^2 - 4) > 0$ when $t > 2$, so it moves upward when $t > 2$ and downward when $0 \leq t < 2$.

(c) Distance upward = $y(3) - y(2) = -6 - (-13) = 7$,

Distance downward = $y(0) - y(2) = 3 - (-13) = 16$. Total distance = $7 + 16 = 23$.



(e) The particle is speeding up when v and a have the same sign, that is, when $t > 2$. The particle is slowing down when v and a have opposite signs; that is, when $0 < t < 2$.

74. (a) $V = \frac{1}{3}\pi r^2 h \Rightarrow dV/dh = \frac{1}{3}\pi r^2$ [r constant]

(b) $V = \frac{1}{3}\pi r^2 h \Rightarrow dV/dr = \frac{2}{3}\pi rh$ [h constant]

75. The linear density ρ is the rate of change of mass m with respect to length x .

$$m = x(1 + \sqrt{x}) = x + x^{3/2} \Rightarrow \rho = dm/dx = 1 + \frac{3}{2}\sqrt{x}, \text{ so the linear density when } x = 4 \text{ is } 1 + \frac{3}{2}\sqrt{4} = 4 \text{ kg/m.}$$

76. (a) $C(x) = 920 + 2x - 0.02x^2 + 0.00007x^3 \Rightarrow C'(x) = 2 - 0.04x + 0.00021x^2$

(b) $C'(100) = 2 - 4 + 2.1 = \$0.10/\text{unit}$. This value represents the rate at which costs are increasing as the hundredth unit is produced, and is the approximate cost of producing the 101st unit.

(c) The cost of producing the 101st item is $C(101) - C(100) = 990.10107 - 990 = \0.10107 , slightly larger than $C'(100)$.

77. If x = edge length, then $V = x^3 \Rightarrow dV/dt = 3x^2 dx/dt = 10 \Rightarrow dx/dt = 10/(3x^2)$ and $S = 6x^2 \Rightarrow$

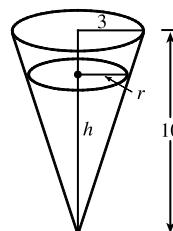
$$dS/dt = (12x) dx/dt = 12x[10/(3x^2)] = 40/x. \text{ When } x = 30, dS/dt = \frac{40}{30} = \frac{4}{3} \text{ cm}^2/\text{min.}$$

78. Given $dV/dt = 2$, find dh/dt when $h = 5$. $V = \frac{1}{3}\pi r^2 h$ and, from similar

$$\text{triangles, } \frac{r}{h} = \frac{3}{10} \Rightarrow V = \frac{\pi}{3} \left(\frac{3h}{10}\right)^2 h = \frac{3\pi}{100} h^3, \text{ so}$$

$$2 = \frac{dV}{dt} = \frac{9\pi}{100} h^2 \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{200}{9\pi h^2} = \frac{200}{9\pi (5)^2} = \frac{8}{9\pi} \text{ cm/s}$$

when $h = 5$.

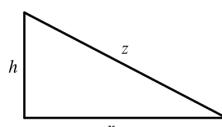


79. Given $dh/dt = 2$ and $dx/dt = 5$, find dz/dt . $z^2 = x^2 + h^2 \Rightarrow 2z \frac{dz}{dt} = 2x \frac{dx}{dt} +$

$$2h \frac{dh}{dt} - x \frac{dh}{dt} - h \frac{dx}{dt} \Rightarrow \frac{dz}{dt} = \frac{1}{z}(5x + 2h). \text{ When } t = 3, h = 15 + 3(2) = 21 \text{ and}$$

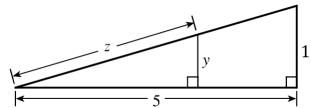
$$x = 5(3) = 15 \Rightarrow z = \sqrt{21^2 + 15^2} = \sqrt{666}, \text{ so } \frac{dz}{dt} = \frac{1}{\sqrt{666}}[5(15) + 2(21)] =$$

$$\frac{117}{\sqrt{666}} \approx 4.5 \text{ m/s.}$$



80. We are given $dz/dt = 10$ m/s. By similar triangles, $\frac{y}{z} = \frac{1}{\sqrt{26}} \Rightarrow y = \frac{1}{\sqrt{26}}z$, so

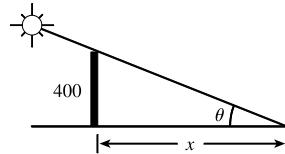
$$\frac{dy}{dt} = \frac{1}{\sqrt{26}} \frac{dz}{dt} = \frac{10}{\sqrt{26}} \approx 2.0 \text{ m/s.}$$



81. We are given $d\theta/dt = -0.25$ rad/h. $\tan \theta = 400/x \Rightarrow$

$$x = 400 \cot \theta \Rightarrow \frac{dx}{dt} = -400 \csc^2 \theta \frac{d\theta}{dt}. \text{ When } \theta = \frac{\pi}{6},$$

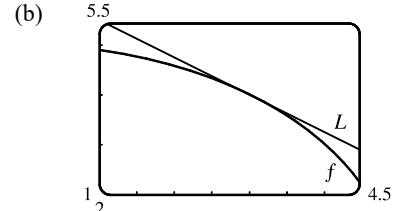
$$\frac{dx}{dt} = -400(2)^2(-0.25) = 400 \text{ m/h.}$$



82. (a) $f(x) = \sqrt{25 - x^2} \Rightarrow f'(x) = \frac{-2x}{2\sqrt{25 - x^2}} = -x(25 - x^2)^{-1/2}.$

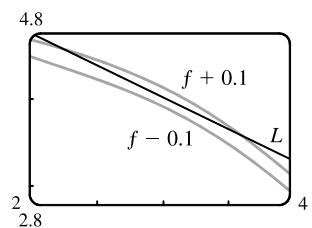
So the linear approximation to $f(x)$ near 3

$$\text{is } f(x) \approx f(3) + f'(3)(x - 3) = 4 - \frac{3}{4}(x - 3).$$



- (c) For the required accuracy, we want $\sqrt{25 - x^2} - 0.1 < 4 - \frac{3}{4}(x - 3)$ and

$4 - \frac{3}{4}(x - 3) < \sqrt{25 - x^2} + 0.1$. From the graph, it appears that these both hold for $2.24 < x < 3.66$.



83. (a) $f(x) = \sqrt[3]{1+3x} = (1+3x)^{1/3} \Rightarrow f'(x) = (1+3x)^{-2/3}$, so the linearization of f at $a = 0$ is

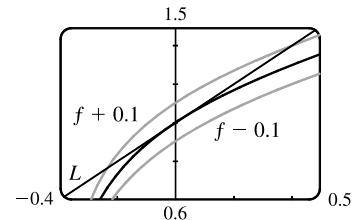
$$L(x) = f(0) + f'(0)(x - 0) = 1^{1/3} + 1^{-2/3}x = 1 + x. \text{ Thus, } \sqrt[3]{1+3x} \approx 1 + x \Rightarrow$$

$$\sqrt[3]{1.03} = \sqrt[3]{1+3(0.01)} \approx 1 + (0.01) = 1.01.$$

- (b) The linear approximation is $\sqrt[3]{1+3x} \approx 1 + x$, so for the required accuracy

we want $\sqrt[3]{1+3x} - 0.1 < 1 + x < \sqrt[3]{1+3x} + 0.1$. From the graph,

it appears that this is true when $-0.235 < x < 0.401$.

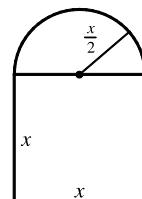


84. $y = x^3 - 2x^2 + 1 \Rightarrow dy = (3x^2 - 4x)dx$. When $x = 2$ and $dx = 0.2$, $dy = [3(2)^2 - 4(2)](0.2) = 0.8$.

85. $A = x^2 + \frac{1}{2}\pi(\frac{1}{2}x)^2 = (1 + \frac{\pi}{8})x^2 \Rightarrow dA = (2 + \frac{\pi}{4})x dx$. When $x = 60$

and $dx = 0.1$, $dA = (2 + \frac{\pi}{4})60(0.1) = 12 + \frac{3\pi}{2}$, so the maximum error is

approximately $12 + \frac{3\pi}{2} \approx 16.7 \text{ cm}^2$.



86. $\lim_{x \rightarrow 1} \frac{x^{17} - 1}{x - 1} = \left[\frac{d}{dx} x^{17} \right]_{x=1} = 17(1)^{16} = 17$

87. $\lim_{h \rightarrow 0} \frac{\sqrt[4]{16+h} - 2}{h} = \left[\frac{d}{dx} \sqrt[4]{x} \right]_{x=16} = \frac{1}{4}x^{-3/4} \Big|_{x=16} = \frac{1}{4(\sqrt[4]{16})^3} = \frac{1}{32}$

88. $\lim_{\theta \rightarrow \pi/3} \frac{\cos \theta - 0.5}{\theta - \pi/3} = \left[\frac{d}{d\theta} \cos \theta \right]_{\theta=\pi/3} = -\sin \frac{\pi}{3} = -\frac{\sqrt{3}}{2}$

89.
$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sqrt{1+\tan x} - \sqrt{1+\sin x}}{x^3} &= \lim_{x \rightarrow 0} \frac{(\sqrt{1+\tan x} - \sqrt{1+\sin x})(\sqrt{1+\tan x} + \sqrt{1+\sin x})}{x^3 (\sqrt{1+\tan x} + \sqrt{1+\sin x})} \\ &= \lim_{x \rightarrow 0} \frac{(1+\tan x) - (1+\sin x)}{x^3 (\sqrt{1+\tan x} + \sqrt{1+\sin x})} = \lim_{x \rightarrow 0} \frac{\sin x (1/\cos x - 1)}{x^3 (\sqrt{1+\tan x} + \sqrt{1+\sin x})} \cdot \frac{\cos x}{\cos x} \\ &= \lim_{x \rightarrow 0} \frac{\sin x (1-\cos x)}{x^3 (\sqrt{1+\tan x} + \sqrt{1+\sin x}) \cos x} \cdot \frac{1+\cos x}{1+\cos x} \\ &= \lim_{x \rightarrow 0} \frac{\sin x \cdot \sin^2 x}{x^3 (\sqrt{1+\tan x} + \sqrt{1+\sin x}) \cos x (1+\cos x)} \\ &= \left(\lim_{x \rightarrow 0} \frac{\sin x}{x} \right)^3 \lim_{x \rightarrow 0} \frac{1}{(\sqrt{1+\tan x} + \sqrt{1+\sin x}) \cos x (1+\cos x)} \\ &= 1^3 \cdot \frac{1}{(\sqrt{1+\sqrt{1}}) \cdot 1 \cdot (1+1)} = \frac{1}{4} \end{aligned}$$

90. Differentiating the first given equation implicitly with respect to x and using the Chain Rule, we obtain $f(g(x)) = x \Rightarrow$

$f'(g(x)) g'(x) = 1 \Rightarrow g'(x) = \frac{1}{f'(g(x))}$. Using the second given equation to expand the denominator of this expression

gives $g'(x) = \frac{1}{1 + [f(g(x))]^2}$. But the first given equation states that $f(g(x)) = x$, so $g'(x) = \frac{1}{1+x^2}$.

91. $\frac{d}{dx} [f(2x)] = x^2 \Rightarrow f'(2x) \cdot 2 = x^2 \Rightarrow f'(2x) = \frac{1}{2}x^2$. Let $t = 2x$. Then $f'(t) = \frac{1}{2}(\frac{1}{2}t)^2 = \frac{1}{8}t^2$, so $f'(x) = \frac{1}{8}x^2$.

92. Let (b, c) be on the curve, that is, $b^{2/3} + c^{2/3} = a^{2/3}$. Now $x^{2/3} + y^{2/3} = a^{2/3} \Rightarrow \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3} \frac{dy}{dx} = 0$, so

$\frac{dy}{dx} = -\frac{y^{1/3}}{x^{1/3}} = -\left(\frac{y}{x}\right)^{1/3}$, so at (b, c) the slope of the tangent line is $-(c/b)^{1/3}$ and an equation of the tangent line is

$y - c = -(c/b)^{1/3}(x - b)$ or $y = -(c/b)^{1/3}x + (c + b^{2/3}c^{1/3})$. Setting $y = 0$, we find that the x -intercept is

$b^{1/3}c^{2/3} + b = b^{1/3}(c^{2/3} + b^{2/3}) = b^{1/3}a^{2/3}$ and setting $x = 0$ we find that the y -intercept is

$c + b^{2/3}c^{1/3} = c^{1/3}(c^{2/3} + b^{2/3}) = c^{1/3}a^{2/3}$. So the length of the tangent line between these two points is

$$\begin{aligned} \sqrt{(b^{1/3}a^{2/3})^2 + (c^{1/3}a^{2/3})^2} &= \sqrt{b^{2/3}a^{4/3} + c^{2/3}a^{4/3}} = \sqrt{(b^{2/3} + c^{2/3})a^{4/3}} \\ &= \sqrt{a^{2/3}a^{4/3}} = \sqrt{a^2} = a = \text{constant} \end{aligned}$$

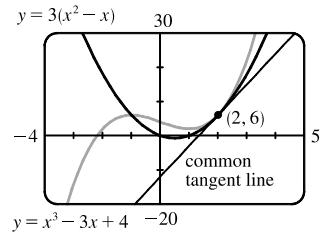
□ PROBLEMS PLUS

1. Let a be the x -coordinate of Q . Since the derivative of $y = 1 - x^2$ is $y' = -2x$, the slope at Q is $-2a$. But since the triangle is equilateral, $\overline{AO}/\overline{OC} = \sqrt{3}/1$, so the slope at Q is $-\sqrt{3}$. Therefore, we must have that $-2a = -\sqrt{3} \Rightarrow a = \frac{\sqrt{3}}{2}$.

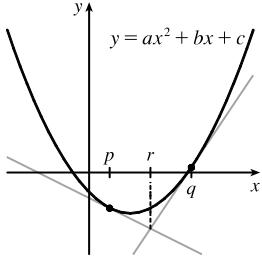
Thus, the point Q has coordinates $\left(\frac{\sqrt{3}}{2}, 1 - \left(\frac{\sqrt{3}}{2}\right)^2\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{4}\right)$ and by symmetry, P has coordinates $\left(-\frac{\sqrt{3}}{2}, \frac{1}{4}\right)$.

$$2. y = x^3 - 3x + 4 \Rightarrow y' = 3x^2 - 3, \text{ and } y = 3(x^2 - x) \Rightarrow y' = 6x - 3.$$

The slopes of the tangents of the two curves are equal when $3x^2 - 3 = 6x - 3$; that is, when $x = 0$ or 2 . At $x = 0$, both tangents have slope -3 , but the curves do not intersect. At $x = 2$, both tangents have slope 9 and the curves intersect at $(2, 6)$. So there is a common tangent line at $(2, 6)$, $y = 9x - 12$.



3.



We must show that r (in the figure) is halfway between p and q , that is,

$r = (p + q)/2$. For the parabola $y = ax^2 + bx + c$, the slope of the tangent line is given by $y' = 2ax + b$. An equation of the tangent line at $x = p$ is $y - (ap^2 + bp + c) = (2ap + b)(x - p)$. Solving for y gives us

$$y = (2ap + b)x - 2ap^2 - bp + (ap^2 + bp + c)$$

$$\text{or} \quad y = (2ap + b)x + c - ap^2 \quad (1)$$

Similarly, an equation of the tangent line at $x = q$ is

$$y = (2aq + b)x + c - aq^2 \quad (2)$$

We can eliminate y and solve for x by subtracting equation (1) from equation (2).

$$\begin{aligned} [(2aq + b) - (2ap + b)]x - aq^2 + ap^2 &= 0 \\ (2aq - 2ap)x &= aq^2 - ap^2 \\ 2a(q - p)x &= a(q^2 - p^2) \\ x &= \frac{a(q + p)(q - p)}{2a(q - p)} = \frac{p + q}{2} \end{aligned}$$

Thus, the x -coordinate of the point of intersection of the two tangent lines, namely r , is $(p + q)/2$.

4. We could differentiate and then simplify or we can simplify and then differentiate. The latter seems to be the simpler method.

$$\begin{aligned} \frac{\sin^2 x}{1 + \cot x} + \frac{\cos^2 x}{1 + \tan x} &= \frac{\sin^2 x}{1 + \frac{\cos x}{\sin x}} \cdot \frac{\sin x}{\sin x} + \frac{\cos^2 x}{1 + \frac{\sin x}{\cos x}} \cdot \frac{\cos x}{\cos x} = \frac{\sin^3 x}{\sin x + \cos x} + \frac{\cos^3 x}{\cos x + \sin x} \\ &= \frac{\sin^3 x + \cos^3 x}{\sin x + \cos x} \quad [\text{factor sum of cubes}] = \frac{(\sin x + \cos x)(\sin^2 x - \sin x \cos x + \cos^2 x)}{\sin x + \cos x} \\ &= \sin^2 x - \sin x \cos x + \cos^2 x = 1 - \sin x \cos x = 1 - \frac{1}{2}(2 \sin x \cos x) = 1 - \frac{1}{2} \sin 2x \end{aligned}$$

$$\text{Thus, } \frac{d}{dx} \left(\frac{\sin^2 x}{1 + \cot x} + \frac{\cos^2 x}{1 + \tan x} \right) = \frac{d}{dx} (1 - \frac{1}{2} \sin 2x) = -\frac{1}{2} \cos 2x \cdot 2 = -\cos 2x.$$

5. Using $f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$, we recognize the given expression, $f(x) = \lim_{t \rightarrow x} \frac{\sec t - \sec x}{t - x}$, as $g'(x)$

with $g(x) = \sec x$. Now $f'(\frac{\pi}{4}) = g''(\frac{\pi}{4})$, so we will find $g''(x)$. $g'(x) = \sec x \tan x \Rightarrow$

$$g''(x) = \sec x \sec^2 x + \tan x \sec x \tan x = \sec x (\sec^2 x + \tan^2 x), \text{ so } g''(\frac{\pi}{4}) = \sqrt{2}(\sqrt{2}^2 + 1^2) = \sqrt{2}(2 + 1) = 3\sqrt{2}.$$

6. Using $f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0}$, we see that for the given equation, $\lim_{x \rightarrow 0} \frac{\sqrt[3]{ax+b}-2}{x} = \frac{5}{12}$, we have $f(x) = \sqrt[3]{ax+b}$,

$f(0) = 2$, and $f'(0) = \frac{5}{12}$. Now $f(0) = 2 \Leftrightarrow \sqrt[3]{b} = 2 \Leftrightarrow b = 8$. Also $f'(x) = \frac{1}{3}(ax+b)^{-2/3} \cdot a$, so

$$f'(0) = \frac{5}{12} \Leftrightarrow \frac{1}{3}(8)^{-2/3} \cdot a = \frac{5}{12} \Leftrightarrow \frac{1}{3}(\frac{1}{4})a = \frac{5}{12} \Leftrightarrow a = 5.$$

7. We use mathematical induction. Let S_n be the statement that $\frac{d^n}{dx^n}(\sin^4 x + \cos^4 x) = 4^{n-1} \cos(4x + n\pi/2)$.

S_1 is true because

$$\begin{aligned} \frac{d}{dx}(\sin^4 x + \cos^4 x) &= 4 \sin^3 x \cos x - 4 \cos^3 x \sin x = 4 \sin x \cos x (\sin^2 x - \cos^2 x) x \\ &= -4 \sin x \cos x \cos 2x = -2 \sin 2x \cos 2 = -\sin 4x = \sin(-4x) \\ &= \cos(\frac{\pi}{2} - (-4x)) = \cos(\frac{\pi}{2} + 4x) = 4^{n-1} \cos(4x + n\frac{\pi}{2}) \text{ when } n = 1 \end{aligned}$$

Now assume S_k is true, that is, $\frac{d^k}{dx^k}(\sin^4 x + \cos^4 x) = 4^{k-1} \cos(4x + k\frac{\pi}{2})$. Then

$$\begin{aligned} \frac{d^{k+1}}{dx^{k+1}}(\sin^4 x + \cos^4 x) &= \frac{d}{dx} \left[\frac{d^k}{dx^k}(\sin^4 x + \cos^4 x) \right] = \frac{d}{dx} [4^{k-1} \cos(4x + k\frac{\pi}{2})] \\ &= -4^{k-1} \sin(4x + k\frac{\pi}{2}) \cdot \frac{d}{dx}(4x + k\frac{\pi}{2}) = -4^k \sin(4x + k\frac{\pi}{2}) \\ &= 4^k \sin(-4x - k\frac{\pi}{2}) = 4^k \cos(\frac{\pi}{2} - (-4x - k\frac{\pi}{2})) = 4^k \cos(4x + (k+1)\frac{\pi}{2}) \end{aligned}$$

which shows that S_{k+1} is true.

Therefore, $\frac{d^n}{dx^n}(\sin^4 x + \cos^4 x) = 4^{n-1} \cos(4x + n\frac{\pi}{2})$ for every positive integer n , by mathematical induction.

Another proof: First write

$$\sin^4 x + \cos^4 x = (\sin^2 x + \cos^2 x)^2 - 2 \sin^2 x \cos^2 x = 1 - \frac{1}{2} \sin^2 2x = 1 - \frac{1}{4}(1 - \cos 4x) = \frac{3}{4} + \frac{1}{4} \cos 4x$$

Then we have $\frac{d^n}{dx^n}(\sin^4 x + \cos^4 x) = \frac{d^n}{dx^n} \left(\frac{3}{4} + \frac{1}{4} \cos 4x \right) = \frac{1}{4} \cdot 4^n \cos \left(4x + n\frac{\pi}{2} \right) = 4^{n-1} \cos \left(4x + n\frac{\pi}{2} \right)$.

8. $\lim_{x \rightarrow a} \frac{f(x) - f(a)}{\sqrt{x} - \sqrt{a}} = \lim_{x \rightarrow a} \left[\frac{f(x) - f(a)}{\sqrt{x} - \sqrt{a}} \cdot \frac{\sqrt{x} + \sqrt{a}}{\sqrt{x} + \sqrt{a}} \right] = \lim_{x \rightarrow a} \left[\frac{f(x) - f(a)}{x - a} \cdot (\sqrt{x} + \sqrt{a}) \right]$
 $= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x \rightarrow a} (\sqrt{x} + \sqrt{a}) = f'(a) \cdot (\sqrt{a} + \sqrt{a}) = 2\sqrt{a} f'(a)$

9. We must find a value x_0 such that the normal lines to the parabola $y = x^2$ at $x = \pm x_0$ intersect at a point one unit from the

points $(\pm x_0, x_0^2)$. The normals to $y = x^2$ at $x = \pm x_0$ have slopes $-\frac{1}{\pm 2x_0}$ and pass through $(\pm x_0, x_0^2)$ respectively, so the

normals have the equations $y - x_0^2 = -\frac{1}{2x_0}(x - x_0)$ and $y - x_0^2 = \frac{1}{2x_0}(x + x_0)$. The common y -intercept is $x_0^2 + \frac{1}{2}$.

We want to find the value of x_0 for which the distance from $(0, x_0^2 + \frac{1}{2})$ to (x_0, x_0^2) equals 1. The square of the distance is

$$(x_0 - 0)^2 + [x_0^2 - (x_0^2 + \frac{1}{2})]^2 = x_0^2 + \frac{1}{4} = 1 \Leftrightarrow x_0 = \pm \frac{\sqrt{3}}{2}. \text{ For these values of } x_0, \text{ the } y\text{-intercept is } x_0^2 + \frac{1}{2} = \frac{5}{4}, \text{ so the center of the circle is at } (0, \frac{5}{4}).$$

Another solution: Let the center of the circle be $(0, a)$. Then the equation of the circle is $x^2 + (y - a)^2 = 1$.

Solving with the equation of the parabola, $y = x^2$, we get $x^2 + (x^2 - a)^2 = 1 \Leftrightarrow x^2 + x^4 - 2ax^2 + a^2 = 1 \Leftrightarrow$

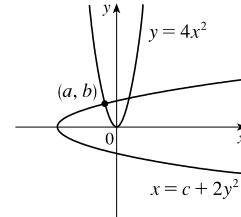
$$x^4 + (1 - 2a)x^2 + a^2 - 1 = 0. \text{ The parabola and the circle will be tangent to each other when this quadratic equation in } x^2 \text{ has equal roots; that is, when the discriminant is 0. Thus, } (1 - 2a)^2 - 4(a^2 - 1) = 0 \Leftrightarrow 1 - 4a + 4a^2 - 4a^2 + 4 = 0 \Leftrightarrow 4a = 5, \text{ so } a = \frac{5}{4}. \text{ The center of the circle is } (0, \frac{5}{4}).$$

10. See the figure. The parabolas $y = 4x^2$ and $x = c + 2y^2$ intersect each other

at right angles at the point (a, b) if and only if (a, b) satisfies both equations

and the tangent lines at (a, b) are perpendicular. $y = 4x^2 \Rightarrow y' = 8x$

and $x = c + 2y^2 \Rightarrow 1 = 4y y' \Rightarrow y' = \frac{1}{4y}$, so at (a, b) we must



have $8a = -\frac{1}{1/(4b)} \Rightarrow 8a = -4b \Rightarrow b = -2a$. Since (a, b) is on both parabolas, we have (1) $b = 4a^2$ and (2)

$a = c + 2b^2$. Substituting $-2a$ for b in (1) gives us $-2a = 4a^2 \Rightarrow 4a^2 + 2a = 0 \Rightarrow 2a(2a + 1) = 0 \Rightarrow a = 0$ or $a = -\frac{1}{2}$.

If $a = 0$, then $b = 0$ and $c = 0$, and the tangent lines at $(0, 0)$ are $y = 0$ and $x = 0$.

If $a = -\frac{1}{2}$, then $b = -2(-\frac{1}{2}) = 1$ and $-\frac{1}{2} = c + 2(1)^2 \Rightarrow c = -\frac{5}{2}$, and the tangent lines at $(-\frac{1}{2}, 1)$ are

$$y - 1 = -4(x + \frac{1}{2}) \quad [\text{or } y = -4x - 1] \quad \text{and} \quad y - 1 = \frac{1}{4}(x + \frac{1}{2}) \quad [\text{or } y = \frac{1}{4}x + \frac{9}{8}].$$

11. See the figure. Clearly, the line $y = 2$ is tangent to both circles at the point

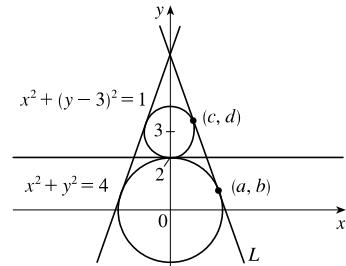
$(0, 2)$. We'll look for a tangent line L through the points (a, b) and (c, d) , and if

such a line exists, then its reflection through the y -axis is another such line. The

slope of L is the same at (a, b) and (c, d) . Find those slopes: $x^2 + y^2 = 4 \Rightarrow$

$$2x + 2y y' = 0 \Rightarrow y' = -\frac{x}{y} \quad \left[= -\frac{a}{b} \right] \quad \text{and} \quad x^2 + (y - 3)^2 = 1 \Rightarrow$$

$$2x + 2(y - 3)y' = 0 \Rightarrow y' = -\frac{x}{y - 3} \quad \left[= -\frac{c}{d - 3} \right].$$



Now an equation for L can be written using either point-slope pair, so we get $y - b = -\frac{a}{b}(x - a)$ $\left[\text{or } y = -\frac{a}{b}x + \frac{a^2}{b} + b \right]$

and $y - d = -\frac{c}{d - 3}(x - c)$ $\left[\text{or } y = -\frac{c}{d - 3}x + \frac{c^2}{d - 3} + d \right]$. The slopes are equal, so $-\frac{a}{b} = -\frac{c}{d - 3} \Leftrightarrow$

$d - 3 = \frac{bc}{a}$. Since (c, d) is a solution of $x^2 + (y - 3)^2 = 1$, we have $c^2 + (d - 3)^2 = 1$, so $c^2 + \left(\frac{bc}{a}\right)^2 = 1 \Rightarrow a^2 c^2 + b^2 c^2 = a^2 \Rightarrow c^2(a^2 + b^2) = a^2 \Rightarrow 4c^2 = a^2$ [since (a, b) is a solution of $x^2 + y^2 = 4$] $\Rightarrow a = 2c$.

Now $d - 3 = \frac{bc}{a} \Rightarrow d = 3 + \frac{bc}{2c}$, so $d = 3 + \frac{b}{2}$. The y -intercepts are equal, so $\frac{a^2}{b} + b = \frac{c^2}{d-3} + d \Leftrightarrow \frac{a^2}{b} + b = \frac{(a/2)^2}{b/2} + \left(3 + \frac{b}{2}\right) \Leftrightarrow \left[\frac{a^2}{b} + b = \frac{a^2}{2b} + 3 + \frac{b}{2}\right] (2b) \Leftrightarrow 2a^2 + 2b^2 = a^2 + 6b + b^2 \Leftrightarrow a^2 + b^2 = 6b \Leftrightarrow 4 = 6b \Leftrightarrow b = \frac{2}{3}$. It follows that $d = 3 + \frac{b}{2} = \frac{10}{3}$, $a^2 = 4 - b^2 = 4 - \frac{4}{9} = \frac{32}{9} \Rightarrow a = \frac{4}{3}\sqrt{2}$,

and $c^2 = 1 - (d - 3)^2 = 1 - \left(\frac{1}{3}\right)^2 = \frac{8}{9} \Rightarrow c = \frac{2}{3}\sqrt{2}$. Thus, L has equation $y - \frac{2}{3} = -\frac{(4/3)\sqrt{2}}{2/3} \left(x - \frac{4}{3}\sqrt{2}\right) \Leftrightarrow y - \frac{2}{3} = -2\sqrt{2}(x - \frac{4}{3}\sqrt{2}) \Leftrightarrow y = -2\sqrt{2}x + 6$. Its reflection has equation $y = 2\sqrt{2}x + 6$.

In summary, there are three lines tangent to both circles: $y = 2$ touches at $(0, 2)$, L touches at $(\frac{4}{3}\sqrt{2}, \frac{2}{3})$ and $(\frac{2}{3}\sqrt{2}, \frac{10}{3})$, and its reflection through the y -axis touches at $(-\frac{4}{3}\sqrt{2}, \frac{2}{3})$ and $(-\frac{2}{3}\sqrt{2}, \frac{10}{3})$.

12. $f(x) = \frac{x^{46} + x^{45} + 2}{1+x} = \frac{x^{45}(x+1) + 2}{x+1} = \frac{x^{45}(x+1)}{x+1} + \frac{2}{x+1} = x^{45} + 2(x+1)^{-1}$, so

$f^{(46)}(x) = (x^{45})^{(46)} + 2[(x+1)^{-1}]^{(46)}$. The forty-sixth derivative of any forty-fifth degree polynomial is 0, so

$(x^{45})^{46} = 0$. Thus, $f^{(46)}(x) = 2[(-1)(-2)(-3) \cdots (-46)(x+1)^{-47}] = 2(46!)(x+1)^{-47}$ and $f^{(46)}(3) = 2(46!)(4)^{-47}$ or $(46!)2^{-93}$.

13. We can assume without loss of generality that $\theta = 0$ at time $t = 0$, so that $\theta = 12\pi t$ rad. [The angular velocity of the wheel is 360 rpm = $360 \cdot (2\pi \text{ rad})/(60 \text{ s}) = 12\pi \text{ rad/s}$.] Then the position of A as a function of time is

$$A = (40 \cos \theta, 40 \sin \theta) = (40 \cos 12\pi t, 40 \sin 12\pi t), \text{ so } \sin \alpha = \frac{y}{1.2 \text{ m}} = \frac{40 \sin \theta}{120} = \frac{\sin \theta}{3} = \frac{1}{3} \sin 12\pi t.$$

(a) Differentiating the expression for $\sin \alpha$, we get $\cos \alpha \cdot \frac{d\alpha}{dt} = \frac{1}{3} \cdot 12\pi \cdot \cos 12\pi t = 4\pi \cos \theta$. When $\theta = \frac{\pi}{3}$, we have

$$\sin \alpha = \frac{1}{3} \sin \theta = \frac{\sqrt{3}}{6}, \text{ so } \cos \alpha = \sqrt{1 - \left(\frac{\sqrt{3}}{6}\right)^2} = \sqrt{\frac{11}{12}} \text{ and } \frac{d\alpha}{dt} = \frac{4\pi \cos \frac{\pi}{3}}{\cos \alpha} = \frac{2\pi}{\sqrt{11/12}} = \frac{4\pi \sqrt{3}}{\sqrt{11}} \approx 6.56 \text{ rad/s.}$$

(b) By the Law of Cosines, $|AP|^2 = |OA|^2 + |OP|^2 - 2|OA||OP|\cos \theta \Rightarrow$

$$120^2 = 40^2 + |OP|^2 - 2 \cdot 40 |OP| \cos \theta \Rightarrow |OP|^2 - (80 \cos \theta) |OP| - 12,800 = 0 \Rightarrow$$

$$|OP| = \frac{1}{2}(80 \cos \theta \pm \sqrt{6400 \cos^2 \theta + 51,200}) = 40 \cos \theta \pm 40 \sqrt{\cos^2 \theta + 8} = 40(\cos \theta + \sqrt{8 + \cos^2 \theta}) \text{ cm}$$

[since $|OP| > 0$]. As a check, note that $|OP| = 160$ cm when $\theta = 0$ and $|OP| = 80\sqrt{2}$ cm when $\theta = \frac{\pi}{2}$.

(c) By part (b), the x -coordinate of P is given by $x = 40(\cos \theta + \sqrt{8 + \cos^2 \theta})$, so

$$\frac{dx}{dt} = \frac{dx}{d\theta} \frac{d\theta}{dt} = 40 \left(-\sin \theta - \frac{2 \cos \theta \sin \theta}{2\sqrt{8 + \cos^2 \theta}} \right) \cdot 12\pi = -480\pi \sin \theta \left(1 + \frac{\cos \theta}{\sqrt{8 + \cos^2 \theta}} \right) \text{ cm/s.}$$

In particular, $dx/dt = 0$ cm/s when $\theta = 0$ and $dx/dt = -480\pi$ cm/s when $\theta = \frac{\pi}{2}$.

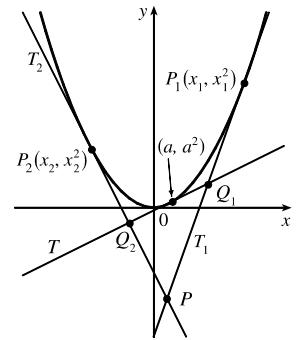
14. The equation of T_1 is $y - x_1^2 = 2x_1(x - x_1) = 2x_1x - 2x_1^2$ or $y = 2x_1x - x_1^2$.

The equation of T_2 is $y = 2x_2x - x_2^2$. Solving for the point of intersection, we get $2x(x_1 - x_2) = x_1^2 - x_2^2 \Rightarrow x = \frac{1}{2}(x_1 + x_2)$. Therefore, the coordinates of P are $(\frac{1}{2}(x_1 + x_2), x_1x_2)$. So if the point of contact of T is (a, a^2) , then Q_1 is $(\frac{1}{2}(a + x_1), ax_1)$ and Q_2 is $(\frac{1}{2}(a + x_2), ax_2)$. Therefore,

$$|PQ_1|^2 = \frac{1}{4}(a - x_2)^2 + x_1^2(a - x_2)^2 = (a - x_2)^2(\frac{1}{4} + x_1^2) \text{ and}$$

$$|PP_1|^2 = \frac{1}{4}(x_1 - x_2)^2 + x_1^2(x_1 - x_2)^2 = (x_1 - x_2)^2(\frac{1}{4} + x_1^2).$$

So $\frac{|PQ_1|^2}{|PP_1|^2} = \frac{(a - x_2)^2}{(x_1 - x_2)^2}$, and similarly $\frac{|PQ_2|^2}{|PP_2|^2} = \frac{(x_1 - a)^2}{(x_1 - x_2)^2}$. Finally, $\frac{|PQ_1|}{|PP_1|} + \frac{|PQ_2|}{|PP_2|} = \frac{a - x_2}{x_1 - x_2} + \frac{x_1 - a}{x_1 - x_2} = 1$.



15. It seems from the figure that as P approaches the point $(0, 2)$ from the right, $x_T \rightarrow \infty$ and $y_T \rightarrow 2^+$. As P approaches the point $(3, 0)$ from the left, it appears that $x_T \rightarrow 3^+$ and $y_T \rightarrow \infty$. So we guess that $x_T \in (3, \infty)$ and $y_T \in (2, \infty)$. It is more difficult to estimate the range of values for x_N and y_N . We might perhaps guess that $x_N \in (0, 3)$, and $y_N \in (-\infty, 0)$ or $(-2, 0)$.

In order to actually solve the problem, we implicitly differentiate the equation of the ellipse to find the equation of the

tangent line: $\frac{x^2}{9} + \frac{y^2}{4} = 1 \Rightarrow \frac{2x}{9} + \frac{2y}{4}y' = 0$, so $y' = -\frac{4}{9}\frac{x}{y}$. So at the point (x_0, y_0) on the ellipse, an equation of the tangent line is $y - y_0 = -\frac{4}{9}\frac{x_0}{y_0}(x - x_0)$ or $4x_0x + 9y_0y = 4x_0^2 + 9y_0^2$. This can be written as $\frac{x_0x}{9} + \frac{y_0y}{4} = \frac{x_0^2}{9} + \frac{y_0^2}{4} = 1$, because (x_0, y_0) lies on the ellipse. So an equation of the tangent line is $\frac{x_0x}{9} + \frac{y_0y}{4} = 1$.

Therefore, the x -intercept x_T for the tangent line is given by $\frac{x_0x_T}{9} = 1 \Leftrightarrow x_T = \frac{9}{x_0}$, and the y -intercept y_T is given by $\frac{y_0y_T}{4} = 1 \Leftrightarrow y_T = \frac{4}{y_0}$.

So as x_0 takes on all values in $(0, 3)$, x_T takes on all values in $(3, \infty)$, and as y_0 takes on all values in $(0, 2)$, y_T takes on all values in $(2, \infty)$. At the point (x_0, y_0) on the ellipse, the slope of the normal line is $-\frac{1}{y'(x_0, y_0)} = \frac{9}{4}\frac{y_0}{x_0}$, and its equation is $y - y_0 = \frac{9}{4}\frac{y_0}{x_0}(x - x_0)$. So the x -intercept x_N for the normal line is given by $0 - y_0 = \frac{9}{4}\frac{y_0}{x_0}(x_N - x_0) \Rightarrow x_N = -\frac{4x_0}{9} + x_0 = \frac{5x_0}{9}$, and the y -intercept y_N is given by $y_N - y_0 = \frac{9}{4}\frac{y_0}{x_0}(0 - x_0) \Rightarrow y_N = -\frac{9y_0}{4} + y_0 = -\frac{5y_0}{4}$.

So as x_0 takes on all values in $(0, 3)$, x_N takes on all values in $(0, \frac{5}{3})$, and as y_0 takes on all values in $(0, 2)$, y_N takes on all values in $(-\frac{5}{2}, 0)$.

16. $\lim_{x \rightarrow 0} \frac{\sin(3+x)^2 - \sin 9}{x} = f'(3)$ where $f(x) = \sin x^2$. Now $f'(x) = (\cos x^2)(2x)$, so $f'(3) = 6 \cos 9$.

17. (a) If the two lines L_1 and L_2 have slopes m_1 and m_2 and angles of

inclination ϕ_1 and ϕ_2 , then $m_1 = \tan \phi_1$ and $m_2 = \tan \phi_2$. The triangle in the figure shows that $\phi_1 + \alpha + (180^\circ - \phi_2) = 180^\circ$ and so $\alpha = \phi_2 - \phi_1$. Therefore, using the identity for $\tan(x - y)$, we have

$$\tan \alpha = \tan(\phi_2 - \phi_1) = \frac{\tan \phi_2 - \tan \phi_1}{1 + \tan \phi_2 \tan \phi_1} \text{ and so } \tan \alpha = \frac{m_2 - m_1}{1 + m_1 m_2}.$$

- (b) (i) The parabolas intersect when $x^2 = (x - 2)^2 \Rightarrow x = 1$. If $y = x^2$, then $y' = 2x$, so the slope of the tangent

to $y = x^2$ at $(1, 1)$ is $m_1 = 2(1) = 2$. If $y = (x - 2)^2$, then $y' = 2(x - 2)$, so the slope of the tangent to

$$y = (x - 2)^2 \text{ at } (1, 1) \text{ is } m_2 = 2(1 - 2) = -2. \text{ Therefore, } \tan \alpha = \frac{m_2 - m_1}{1 + m_1 m_2} = \frac{-2 - 2}{1 + 2(-2)} = \frac{4}{3} \text{ and}$$

$$\text{so } \alpha = \tan^{-1}\left(\frac{4}{3}\right) \approx 53^\circ \text{ [or } 127^\circ\text{].}$$

- (ii) $x^2 - y^2 = 3$ and $x^2 - 4x + y^2 + 3 = 0$ intersect when $x^2 - 4x + (x^2 - 3) + 3 = 0 \Leftrightarrow 2x(x - 2) = 0 \Rightarrow$

$x = 0$ or 2 , but 0 is extraneous. If $x = 2$, then $y = \pm 1$. If $x^2 - y^2 = 3$ then $2x - 2yy' = 0 \Rightarrow y' = x/y$ and

$$x^2 - 4x + y^2 + 3 = 0 \Rightarrow 2x - 4 + 2yy' = 0 \Rightarrow y' = \frac{2-x}{y}. \text{ At } (2, 1) \text{ the slopes are } m_1 = 2 \text{ and}$$

$$m_2 = 0, \text{ so } \tan \alpha = \frac{0 - (-2)}{1 + (-2)(0)} = -2 \Rightarrow \alpha \approx 117^\circ. \text{ At } (2, -1) \text{ the slopes are } m_1 = -2 \text{ and } m_2 = 0,$$

$$\text{so } \tan \alpha = \frac{0 - (-2)}{1 + (-2)(0)} = 2 \Rightarrow \alpha \approx 63^\circ \text{ [or } 117^\circ\text{].}$$

18. $y^2 = 4px \Rightarrow 2yy' = 4p \Rightarrow y' = 2p/y \Rightarrow$ slope of tangent at $P(x_1, y_1)$ is $m_1 = 2p/y_1$. The slope of FP is

$$m_2 = \frac{y_1}{x_1 - p}, \text{ so by the formula from Problem 17(a),}$$

$$\begin{aligned} \tan \alpha &= \frac{\frac{y_1}{x_1 - p} - \frac{2p}{y_1}}{1 + \left(\frac{2p}{y_1}\right)\left(\frac{y_1}{x_1 - p}\right)} \cdot \frac{y_1(x_1 - p)}{y_1(x_1 - p)} = \frac{y_1^2 - 2p(x_1 - p)}{y_1(x_1 - p) + 2py_1} = \frac{4px_1 - 2px_1 + 2p^2}{x_1y_1 - py_1 + 2py_1} \\ &= \frac{2p(p + x_1)}{y_1(p + x_1)} = \frac{2p}{y_1} = \text{slope of tangent at } P = \tan \beta \end{aligned}$$

Since $0 \leq \alpha, \beta \leq \frac{\pi}{2}$, this proves that $\alpha = \beta$.

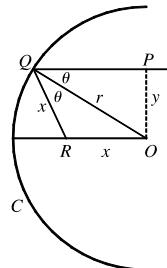
19. Since $\angle ROQ = \angle OQP = \theta$, the triangle QOR is isosceles, so

$|QR| = |RO| = x$. By the Law of Cosines, $x^2 = x^2 + r^2 - 2rx \cos \theta$. Hence,

$2rx \cos \theta = r^2$, so $x = \frac{r^2}{2r \cos \theta} = \frac{r}{2 \cos \theta}$. Note that as $y \rightarrow 0^+$, $\theta \rightarrow 0^+$ (since

$\sin \theta = y/r$), and hence $x \rightarrow \frac{r}{2 \cos 0} = \frac{r}{2}$. Thus, as P is taken closer and closer

to the x -axis, the point R approaches the midpoint of the radius AO .



20. $\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow 0} \frac{f(x) - 0}{g(x) - 0} = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{g(x) - g(0)} = \lim_{x \rightarrow 0} \frac{\frac{f(x) - f(0)}{x - 0}}{\frac{g(x) - g(0)}{x - 0}} = \frac{\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0}}{\lim_{x \rightarrow 0} \frac{g(x) - g(0)}{x - 0}} = \frac{f'(0)}{g'(0)}$

$$\begin{aligned}
21. \lim_{x \rightarrow 0} & \frac{\sin(a+2x) - 2\sin(a+x) + \sin a}{x^2} \\
&= \lim_{x \rightarrow 0} \frac{\sin a \cos 2x + \cos a \sin 2x - 2\sin a \cos x - 2\cos a \sin x + \sin a}{x^2} \\
&= \lim_{x \rightarrow 0} \frac{\sin a (\cos 2x - 2\cos x + 1) + \cos a (\sin 2x - 2\sin x)}{x^2} \\
&= \lim_{x \rightarrow 0} \frac{\sin a (2\cos^2 x - 1 - 2\cos x + 1) + \cos a (2\sin x \cos x - 2\sin x)}{x^2} \\
&= \lim_{x \rightarrow 0} \frac{\sin a (2\cos x)(\cos x - 1) + \cos a (2\sin x)(\cos x - 1)}{x^2} \\
&= \lim_{x \rightarrow 0} \frac{2(\cos x - 1)[\sin a \cos x + \cos a \sin x](\cos x + 1)}{x^2(\cos x + 1)} \\
&= \lim_{x \rightarrow 0} \frac{-2\sin^2 x [\sin(a+x)]}{x^2(\cos x + 1)} = -2 \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)^2 \cdot \frac{\sin(a+x)}{\cos x + 1} = -2(1)^2 \frac{\sin(a+0)}{\cos 0 + 1} = -\sin a
\end{aligned}$$

22. Suppose that $y = mx + c$ is a tangent line to the ellipse. Then it intersects the ellipse at only one point, so the discriminant

of the equation $\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1 \Leftrightarrow (b^2 + a^2m^2)x^2 + 2mca^2x + a^2c^2 - a^2b^2 = 0$ must be 0; that is,

$$\begin{aligned}
0 &= (2mca^2)^2 - 4(b^2 + a^2m^2)(a^2c^2 - a^2b^2) = 4a^4c^2m^2 - 4a^2b^2c^2 + 4a^2b^4 - 4a^4c^2m^2 + 4a^4b^2m^2 \\
&= 4a^2b^2(a^2m^2 + b^2 - c^2)
\end{aligned}$$

Therefore, $a^2m^2 + b^2 - c^2 = 0$.

Now if a point (α, β) lies on the line $y = mx + c$, then $c = \beta - m\alpha$, so from above,

$$0 = a^2m^2 + b^2 - (\beta - m\alpha)^2 = (a^2 - \alpha^2)m^2 + 2\alpha\beta m + b^2 - \beta^2 \Leftrightarrow m^2 + \frac{2\alpha\beta}{a^2 - \alpha^2}m + \frac{b^2 - \beta^2}{a^2 - \alpha^2} = 0.$$

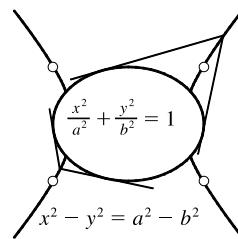
(a) Suppose that the two tangent lines from the point (α, β) to the ellipse

have slopes m and $\frac{1}{m}$. Then m and $\frac{1}{m}$ are roots of the equation

$$z^2 + \frac{2\alpha\beta}{a^2 - \alpha^2}z + \frac{b^2 - \beta^2}{a^2 - \alpha^2} = 0. \text{ This implies that } (z - m)\left(z - \frac{1}{m}\right) = 0 \Leftrightarrow$$

$$z^2 - \left(m + \frac{1}{m}\right)z + m\left(\frac{1}{m}\right) = 0, \text{ so equating the constant terms in the two}$$

quadratic equations, we get $\frac{b^2 - \beta^2}{a^2 - \alpha^2} = m\left(\frac{1}{m}\right) = 1$, and hence $b^2 - \beta^2 = a^2 - \alpha^2$. So (α, β) lies on the hyperbola $x^2 - y^2 = a^2 - b^2$.

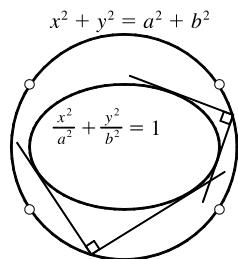


(b) If the two tangent lines from the point (α, β) to the ellipse have slopes m

and $-\frac{1}{m}$, then m and $-\frac{1}{m}$ are roots of the quadratic equation, and so

$$(z - m)\left(z + \frac{1}{m}\right) = 0, \text{ and equating the constant terms as in part (a), we get}$$

$\frac{b^2 - \beta^2}{a^2 - \alpha^2} = -1$, and hence $b^2 - \beta^2 = \alpha^2 - a^2$. So the point (α, β) lies on the circle $x^2 + y^2 = a^2 + b^2$.



23. $y = x^4 - 2x^2 - x \Rightarrow y' = 4x^3 - 4x - 1$. The equation of the tangent line at $x = a$ is

$y - (a^4 - 2a^2 - a) = (4a^3 - 4a - 1)(x - a)$ or $y = (4a^3 - 4a - 1)x + (-3a^4 + 2a^2)$ and similarly for $x = b$. So if at $x = a$ and $x = b$ we have the same tangent line, then $4a^3 - 4a - 1 = 4b^3 - 4b - 1$ and $-3a^4 + 2a^2 = -3b^4 + 2b^2$. The first equation gives $a^3 - b^3 = a - b \Rightarrow (a - b)(a^2 + ab + b^2) = (a - b)$. Assuming $a \neq b$, we have $1 = a^2 + ab + b^2$. The second equation gives $3(a^4 - b^4) = 2(a^2 - b^2) \Rightarrow 3(a^2 - b^2)(a^2 + b^2) = 2(a^2 - b^2)$ which is true if $a = -b$. Substituting into $1 = a^2 + ab + b^2$ gives $1 = a^2 - a^2 + a^2 \Rightarrow a = \pm 1$ so that $a = 1$ and $b = -1$ or vice versa. Thus, the points $(1, -2)$ and $(-1, 0)$ have a common tangent line.

As long as there are only two such points, we are done. So we show that these are in fact the only two such points.

Suppose that $a^2 - b^2 \neq 0$. Then $3(a^2 - b^2)(a^2 + b^2) = 2(a^2 - b^2)$ gives $3(a^2 + b^2) = 2$ or $a^2 + b^2 = \frac{2}{3}$.

Thus, $ab = (a^2 + ab + b^2) - (a^2 + b^2) = 1 - \frac{2}{3} = \frac{1}{3}$, so $b = \frac{1}{3a}$. Hence, $a^2 + \frac{1}{9a^2} = \frac{2}{3}$, so $9a^4 + 1 = 6a^2 \Rightarrow 0 = 9a^4 - 6a^2 + 1 = (3a^2 - 1)^2$. So $3a^2 - 1 = 0 \Rightarrow a^2 = \frac{1}{3} \Rightarrow b^2 = \frac{1}{9a^2} = \frac{1}{3} = a^2$, contradicting our assumption that $a^2 \neq b^2$.

24. Suppose that the normal lines at the three points (a_1, a_1^2) , (a_2, a_2^2) , and (a_3, a_3^2) intersect at a common point. Now if one of the a_i is 0 (suppose $a_1 = 0$) then by symmetry $a_2 = -a_3$, so $a_1 + a_2 + a_3 = 0$. So we can assume that none of the a_i is 0.

The slope of the tangent line at (a_i, a_i^2) is $2a_i$, so the slope of the normal line is $-\frac{1}{2a_i}$ and its equation is

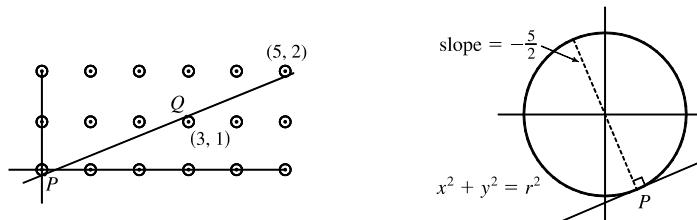
$y - a_i^2 = -\frac{1}{2a_i}(x - a_i)$. We solve for the x -coordinate of the intersection of the normal lines from (a_1, a_1^2) and (a_2, a_2^2) :

$$\begin{aligned} y = a_1^2 - \frac{1}{2a_1}(x - a_1) &= a_2^2 - \frac{1}{2a_2}(x - a_2) \Rightarrow x\left(\frac{1}{2a_2} - \frac{1}{2a_1}\right) = a_2^2 - a_1^2 \Rightarrow \\ x\left(\frac{a_1 - a_2}{2a_1 a_2}\right) &= (-a_1 - a_2)(a_1 + a_2) \Leftrightarrow x = -2a_1 a_2(a_1 + a_2) \quad (1). \end{aligned}$$

Similarly, solving for the x -coordinate of the intersections of the normal lines from (a_1, a_1^2) and (a_3, a_3^2) gives $x = -2a_1 a_3(a_1 + a_3)$ (2).

$$\begin{aligned} \text{Equating (1) and (2) gives } a_2(a_1 + a_2) &= a_3(a_1 + a_3) \Leftrightarrow a_1(a_2 - a_3) = a_3^2 - a_2^2 = -(a_2 + a_3)(a_2 - a_3) \Leftrightarrow \\ a_1 = -(a_2 + a_3) &\Leftrightarrow a_1 + a_2 + a_3 = 0. \end{aligned}$$

25. Because of the periodic nature of the lattice points, it suffices to consider the points in the 5×2 grid shown. We can see that the minimum value of r occurs when there is a line with slope $\frac{2}{5}$ which touches the circle centered at $(3, 1)$ and the circles centered at $(0, 0)$ and $(5, 2)$.



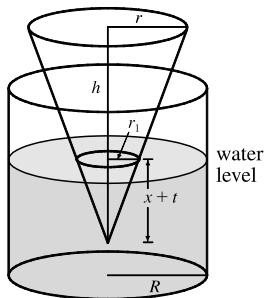
To find P , the point at which the line is tangent to the circle at $(0, 0)$, we simultaneously solve $x^2 + y^2 = r^2$ and

$y = -\frac{5}{2}x \Rightarrow x^2 + \frac{25}{4}x^2 = r^2 \Rightarrow x^2 = \frac{4}{29}r^2 \Rightarrow x = \frac{2}{\sqrt{29}}r, y = -\frac{5}{\sqrt{29}}r$. To find Q , we either use symmetry or solve $(x-3)^2 + (y-1)^2 = r^2$ and $y-1 = -\frac{5}{2}(x-3)$. As above, we get $x = 3 - \frac{2}{\sqrt{29}}r, y = 1 + \frac{5}{\sqrt{29}}r$. Now the slope of

$$\text{the line } PQ \text{ is } \frac{2}{5}, \text{ so } m_{PQ} = \frac{1 + \frac{5}{\sqrt{29}}r - \left(-\frac{5}{\sqrt{29}}r\right)}{3 - \frac{2}{\sqrt{29}}r - \frac{2}{\sqrt{29}}r} = \frac{1 + \frac{10}{\sqrt{29}}r}{3 - \frac{4}{\sqrt{29}}r} = \frac{\sqrt{29} + 10r}{3\sqrt{29} - 4r} = \frac{2}{5} \Rightarrow$$

$5\sqrt{29} + 50r = 6\sqrt{29} - 8r \Leftrightarrow 58r = \sqrt{29} \Leftrightarrow r = \frac{\sqrt{29}}{58}$. So the minimum value of r for which any line with slope $\frac{2}{5}$ intersects circles with radius r centered at the lattice points on the plane is $r = \frac{\sqrt{29}}{58} \approx 0.093$.

26.



Assume the axes of the cone and the cylinder are parallel. Let H denote the initial height of the water. When the cone has been dropping for t seconds, the water level has risen x centimeters, so the tip of the cone is $x + t$ centimeters below the water line. We want to find dx/dt when $x + t = h$ (when the cone is completely submerged).

Using similar triangles, $\frac{r_1}{x+t} = \frac{r}{h} \Rightarrow r_1 = \frac{r}{h}(x+t)$.

volume of water and cone at time t = original volume of water + volume of submerged part of cone

$$\pi R^2(H+x) = \pi R^2H + \frac{1}{3}\pi r_1^2(x+t)$$

$$\pi R^2H + \pi R^2x = \pi R^2H + \frac{1}{3}\pi \frac{r^2}{h^2}(x+t)^3$$

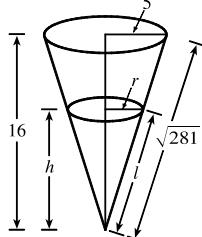
$$3h^2R^2x = r^2(x+t)^3$$

Differentiating implicitly with respect to t gives us $3h^2R^2 \frac{dx}{dt} = r^2 \left[3(x+t)^2 \frac{dx}{dt} + 3(x+t)^2 \frac{dt}{dt} \right] \Rightarrow$

$$\frac{dx}{dt} = \frac{r^2(x+t)^2}{h^2R^2 - r^2(x+t)^2} \Rightarrow \left. \frac{dx}{dt} \right|_{x+t=h} = \frac{r^2h^2}{h^2R^2 - r^2h^2} = \frac{r^2}{R^2 - r^2}. \text{ Thus, the water level is rising at a rate of}$$

$$\frac{r^2}{R^2 - r^2} \text{ cm/s at the instant the cone is completely submerged.}$$

27.



By similar triangles, $\frac{r}{5} = \frac{h}{16} \Rightarrow r = \frac{5h}{16}$. The volume of the cone is

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \left(\frac{5h}{16}\right)^2 h = \frac{25\pi}{768}h^3, \text{ so } \frac{dV}{dt} = \frac{25\pi}{256}h^2 \frac{dh}{dt}. \text{ Now the rate of}$$

change of the volume is also equal to the difference of what is being added

($2 \text{ cm}^3/\text{min}$) and what is oozing out ($k\pi rl$, where πrl is the area of the cone and k

is a proportionality constant). Thus, $\frac{dV}{dt} = 2 - k\pi rl$.

Equating the two expressions for $\frac{dV}{dt}$ and substituting $h = 10, \frac{dh}{dt} = -0.3, r = \frac{5(10)}{16} = \frac{25}{8}$, and $\frac{l}{\sqrt{281}} = \frac{10}{16} \Leftrightarrow$

$$l = \frac{5}{8}\sqrt{281}, \text{ we get } \frac{25\pi}{256}(10)^2(-0.3) = 2 - k\pi \frac{25}{8} \cdot \frac{5}{8}\sqrt{281} \Leftrightarrow \frac{125k\pi\sqrt{281}}{64} = 2 + \frac{750\pi}{256}. \text{ Solving for } k \text{ gives us}$$

$k = \frac{256 + 375\pi}{250\pi\sqrt{281}}$. To maintain a certain height, the rate of oozing, $k\pi rl$, must equal the rate of the liquid being poured in;

that is, $\frac{dV}{dt} = 0$. Thus, the rate at which we should pour the liquid into the container is

$$k\pi rl = \frac{256 + 375\pi}{250\pi\sqrt{281}} \cdot \pi \cdot \frac{25}{8} \cdot \frac{5\sqrt{281}}{8} = \frac{256 + 375\pi}{128} \approx 11.204 \text{ cm}^3/\text{min}$$

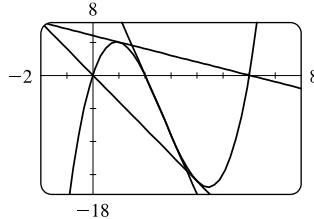
28. (a) $f(x) = x(x - 2)(x - 6) = x^3 - 8x^2 + 12x \Rightarrow f'(x) = 3x^2 - 16x + 12$. The average of the first pair of zeros is

$(0 + 2)/2 = 1$. At $x = 1$, the slope of the tangent line is $f'(1) = -1$, so an equation of the tangent line has the form

$y = -1x + b$. Since $f(1) = 5$, we have $5 = -1 + b \Rightarrow b = 6$ and the tangent has equation $y = -x + 6$.

Similarly, at $x = \frac{0+6}{2} = 3$, $y = -9x + 18$; at $x = \frac{2+6}{2} = 4$, $y = -4x$. From the graph, we see that each tangent

line drawn at the average of two zeros intersects the graph of f at the third zero.



(b) Using Exercise 2.3.107(a), $f(x) = (x - a)(x - b)(x - c) \Rightarrow$

$$f'(x) = (x - b)(x - c) + (x - a)(x - c) + (x - a)(x - b).$$

$$\begin{aligned} f'\left(\frac{a+b}{2}\right) &= \left(\frac{a+b}{2} - b\right)\left(\frac{a+b}{2} - c\right) + \left(\frac{a+b}{2} - a\right)\left(\frac{a+b}{2} - c\right) + \left(\frac{a+b}{2} - a\right)\left(\frac{a+b}{2} - b\right) \\ &= \left(\frac{a-b}{2}\right)\left(\frac{a+b-2c}{2}\right) + \left(\frac{-a+b}{2}\right)\left(\frac{a+b-2c}{2}\right) + \left(\frac{-a+b}{2}\right)\left(\frac{a-b}{2}\right) \\ &= \left(\frac{a-b}{2}\right)\left(\frac{a+b-2c}{2}\right) - \left(\frac{a-b}{2}\right)\left(\frac{a+b-2c}{2}\right) - \left(\frac{a-b}{2}\right)\left(\frac{a-b}{2}\right) \\ &= -\frac{(a-b)^2}{4} \end{aligned}$$

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= \left(\frac{a+b}{2} - a\right)\left(\frac{a+b}{2} - b\right)\left(\frac{a+b}{2} - c\right) \\ &= \left(\frac{b-a}{2}\right)\left(\frac{a-b}{2}\right)\left(\frac{a+b-2c}{2}\right) \\ &= -\frac{(a-b)^2}{8}(a+b-2c) \end{aligned}$$

An equation of the tangent line at $x = \frac{a+b}{2}$ is $y = -\frac{(a-b)^2}{4}\left(x - \frac{a+b}{2}\right) - \frac{(a-b)^2}{8}(a+b-2c)$.

To find the x -intercept, let $y = 0$ and solve for x .

[continued]

$$\begin{aligned}\frac{(a-b)^2}{4} \left(x - \frac{a+b}{2} \right) &= -\frac{(a-b)^2}{8}(a+b-2c) \\ x - \frac{a+b}{2} &= -\frac{1}{2}(a+b-2c) \\ x = \frac{1}{2}a + \frac{1}{2}b - \frac{1}{2}a - \frac{1}{2}b + c &= c\end{aligned}$$

The result is $x = c$, which is the third zero.

