3 □ APPLICATIONS OF DIFFERENTIATION

3.1 Maximum and Minimum Values

1. A function f has an **absolute minimum** at $x = c$ if $f(c)$ is the smallest function value on the entire domain of f, whereas f has a **local minimum** at c if $f(c)$ is the smallest function value when x is near c.

2. (a) The Extreme Value Theorem

(b) See the Closed Interval Method.

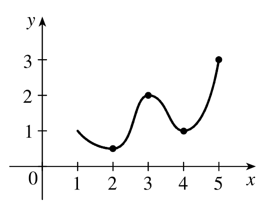
3. Absolute maximum at s, absolute minimum at r, local maximum at c, local minima at b and r, neither a maximum nor a minimum at a and d.

4. Absolute maximum at r; absolute minimum at a; local maxima at b and r; local minimum at d; neither a maximum nor a minimum at c and s.

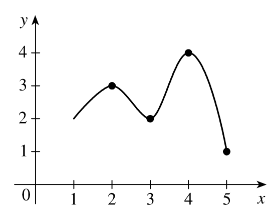
5. Absolute maximum value is $f(4) = 5$; there is no absolute minimum value; local maximum values are $f(4) = 5$ and $f(6) = 4$; local minimum values are $f(2) = 2$ and $f(1) = f(5) = 3$.

6. There is no absolute maximum value; absolute minimum value is $g(4) = 1$; local maximum values are $g(3) = 4$ and $g(6) = 3$; local minimum values are $g(2) = 2$ and $g(4) = 1$.

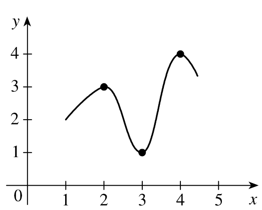
7. Absolute maximum at 5, absolute minimum at 2, local maximum at 3, local minima at 2 and 4



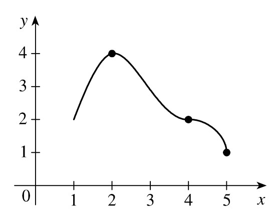
8. Absolute maximum at 4, absolute minimum at 5, local maximum at 2, local minimum at 3



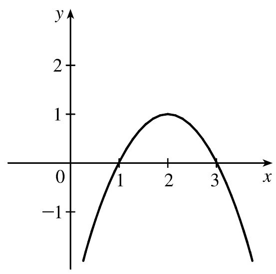
9. Absolute minimum at 3, absolute maximum at 4, local maximum at 2



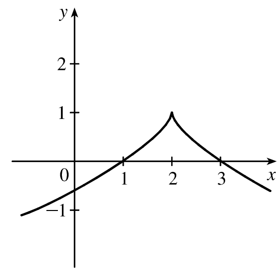
10. Absolute maximum at 2, absolute minimum at 5, 4 is a critial number but there is no local maximum or minimum there.



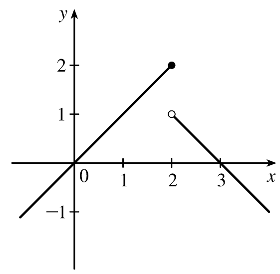
11. (a)



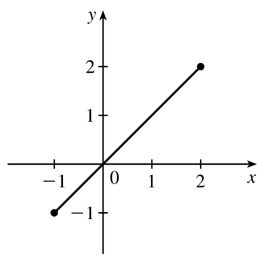
(b)



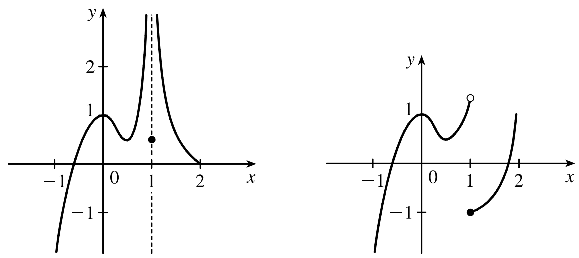
(c)



12. (a) Note that a local maximum cannot occur at an endpoint.

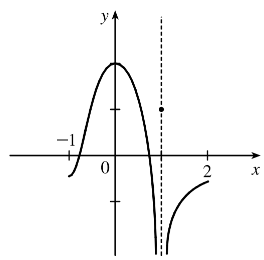


(b)

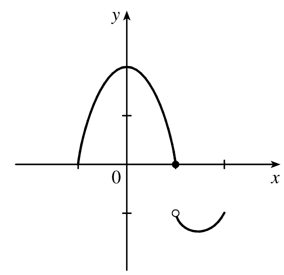


Note: By the Extreme Value Theorem, f must *not* be continuous.

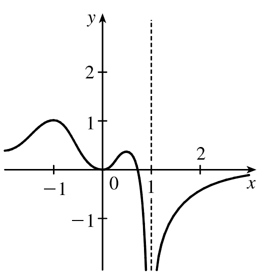
13. (a) *Note:* By the Extreme Value Theorem, f must *not* be continuous; because if it were, it would attain an absolute minimum.



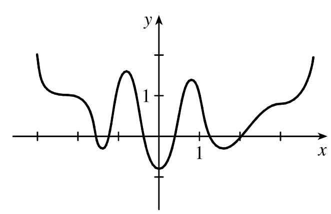
(b)



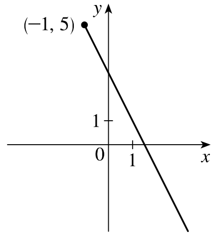
14. (a)



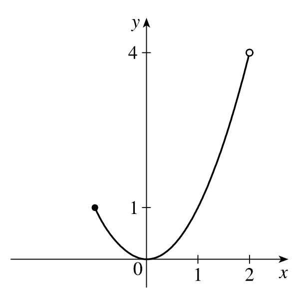
(b)



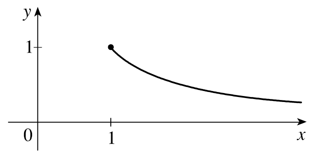
15. $f(x) = 3 - 2x$, $x \geq -1$. Absolute maximum $f(-1) = 5$; no local maximum. No absolute or local minimum.



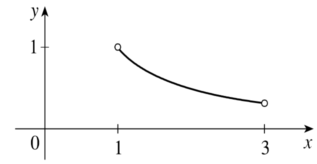
16. $f(x) = x^2$, $-1 \leq x < 2$. No absolute or local maximum. Absolute and local minimum $f(0) = 0$.



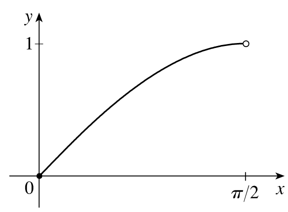
17. $f(x) = 1/x$, $x \geq 1$. Absolute maximum $f(1) = 1$; no local maximum. No absolute or local minimum.



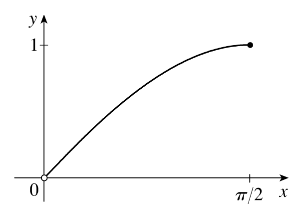
18. $f(x) = 1/x$, $1 < x < 3$. No absolute or local maximum. No absolute or local minimum.



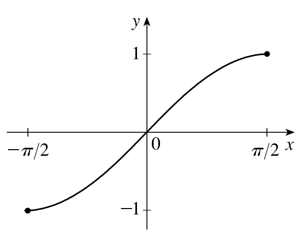
19. $f(x) = \sin x$, $0 \leq x < \pi/2$. No absolute or local maximum. Absolute minimum $f(0) = 0$; no local minimum.



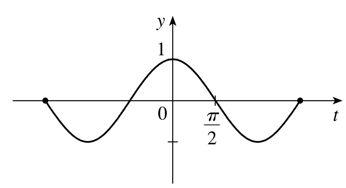
20. $f(x) = \sin x$, $0 < x \leq \pi/2$. Absolute maximum $f\left(\frac{\pi}{2}\right) = 1$; no local maximum. No absolute or local minimum.



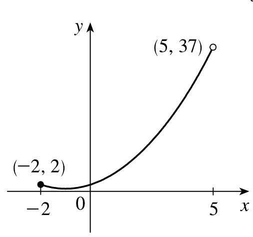
21. $f(x) = \sin x$, $-\pi/2 \leq x \leq \pi/2$. Absolute maximum $f\left(\frac{\pi}{2}\right) = 1$; no local maximum. Absolute minimum $f\left(-\frac{\pi}{2}\right) = -1$; no local minimum.



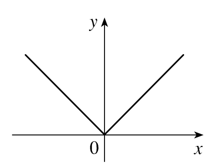
22. $f(t) = \cos t$, $-\frac{3\pi}{2} \leq t \leq \frac{3\pi}{2}$. Absolute and local maximum $f(0) = 1$; absolute and local minima $f(\pm\pi, -1)$.



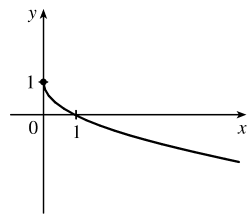
23. $f(x) = 1 + (x+1)^2$, $-2 \leq x < 5$. No absolute or local maximum. Absolute and local minimum $f(-1) = 1$.



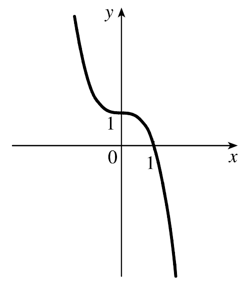
24. $f(x) = | x |$. No absolute or local maximum. Absolute and local minimum $f(0) = 0$.



25. $f(x) = 1 - \sqrt{x}$. Absolute maximum $f(0) = 1$; no local maximum. No absolute or local minimum.



26. $f(x) = 1 - x^3$. No absolute or local extreme values.

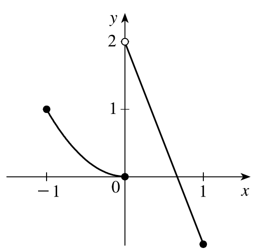


27. $f(x) = \begin{cases} x^2 & \text{if } -1 \le x \le 0 \\ 2 - 3x & \text{if } 0 < x \le 1 \end{cases}$

No absolute or local maximum.

Absolute minimum $f(1) = -1$.

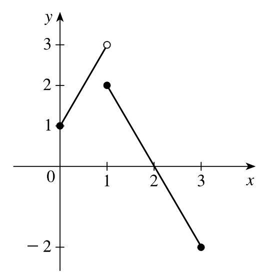
Local minimum $f(0) = 0$.



28. $f(x) = \begin{cases} 2x + 1 & \text{if } 0 \le x < 1 \\ 4 - 2x & \text{if } 1 \le x \le 3 \end{cases}$

No absolute or local maximum.

Absolute minimum $f(3) = -2$; no local minimum.



29. $f(x) = 3x^2 + x - 2 \quad \Rightarrow \quad f'(x) = 6x + 1$. $f'(x) = 0 \quad \Rightarrow \quad x = -\frac{1}{6}$. This is the only critical number.

30. $g(v) = v^3 - 12v + 4 \quad \Rightarrow \quad g'(v) = 3v^2 - 12 = 3(v^2 - 4) = 3(v + 2)(v - 2)$. $g'(v) = 0 \quad \Rightarrow \quad v = -2, 2$. These are the only critical numbers.

31. $f(x) = 3x^4 + 8x^3 - 48x^2 \quad \Rightarrow \quad f'(x) = 12x^3 + 24x^2 - 96x = 12x(x^2 + 2x - 8) = 12x(x + 4)(x - 2)$.

$f'(x) = 0 \quad \Rightarrow \quad x = -4, 0, 2$. These are the only critical numbers.

32. $f(x) = 2x^3 + x^2 + 8x \quad \Rightarrow \quad f'(x) = 6x^2 + 2x + 8 = 2(3x^2 + x + 4)$. Using the quadratic formula, $f'(x) = 0 \quad \Leftrightarrow$

$x = \dfrac{-1 \pm \sqrt{-47}}{6}$. Since the discriminant, -47, is negative, there are no real solutions, and hence, there are no critical numbers.

33. $g(t) = t^5 + 5t^3 + 50t \quad \Rightarrow \quad g'(t) = 5t^4 + 15t^2 + 50 = 5(t^4 + 3t^2 + 10)$. Using the quadratic formula to solve for t^2,

$g'(t) = 0 \quad \Leftrightarrow \quad t^2 = \dfrac{-3 \pm \sqrt{3^2 - 4(1)(10)}}{2(1)} = \dfrac{-3 \pm \sqrt{-31}}{2}$. Since the discriminant, -31, is negative, there are no real solutions, and hence, there are no critical numbers.

34. $A(x) = |3 - 2x| = \begin{cases} 3 - 2x & \text{if } 3 - 2x \geq 0 \\ -(3 - 2x) & \text{if } 3 - 2x < 0 \end{cases} = \begin{cases} 3 - 2x & \text{if } x \leq \frac{3}{2} \\ 2x - 3 & \text{if } x > \frac{3}{2} \end{cases}$

$A'(x) = \begin{cases} -2 & \text{if } x < \frac{3}{2} \\ 2 & \text{if } x > \frac{3}{2} \end{cases}$ and $A'(x)$ does not exist at $x = \dfrac{3}{2}$, so $x = \dfrac{3}{2}$ is a critical number.

35. $g(y) = \dfrac{y - 1}{y^2 - y + 1} \quad \Rightarrow$

$$g'(y) = \frac{(y^2 - y + 1)(1) - (y - 1)(2y - 1)}{(y^2 - y + 1)^2} = \frac{y^2 - y + 1 - (2y^2 - 3y + 1)}{(y^2 - y + 1)^2} = \frac{-y^2 + 2y}{(y^2 - y + 1)^2} = \frac{y(2 - y)}{(y^2 - y + 1)^2}.$$

$g'(y) = 0 \quad \Rightarrow \quad y = 0, 2$. The expression $y^2 - y + 1$ is never equal to 0, so $g'(y)$ exists for all real numbers. The critical numbers are 0 and 2.

36. $h(p) = \dfrac{p - 1}{p^2 + 4} \quad \Rightarrow \quad h'(p) = \dfrac{(p^2 + 4)(1) - (p - 1)(2p)}{(p^2 + 4)^2} = \dfrac{p^2 + 4 - 2p^2 + 2p}{(p^2 + 4)^2} = \dfrac{-p^2 + 2p + 4}{(p^2 + 4)^2}.$

$h'(p) = 0 \quad \Rightarrow \quad p = \dfrac{-2 \pm \sqrt{4 + 16}}{-2} = 1 \pm \sqrt{5}$. The critical numbers are $1 \pm \sqrt{5}$. [$h'(p)$ exists for all real numbers.]

37. $p(x) = \dfrac{x^2 + 2}{2x - 1} \quad \Rightarrow$

$$p'(x) = \frac{(2x - 1)(2x) - (x^2 + 2)(2)}{(2x - 1)^2} = \frac{4x^2 - 2x - 2x^2 - 4}{(2x - 1)^2} = \frac{2x^2 - 2x - 4}{(2x - 1)^2} = \frac{2(x^2 - x - 2)}{(2x - 1)^2} = \frac{2(x - 2)(x + 1)}{(2x - 1)^2}.$$

$p'(x) = 0 \quad \Rightarrow \quad x = -1$ or 2. $p'(x)$ does not exist at $x = \frac{1}{2}$, but $\frac{1}{2}$ is not in the domain of p, so the critical numbers are -1 and 2.

38. $q(t) = \dfrac{t^2 + 9}{t^2 - 9} \quad \Rightarrow \quad q'(t) = \dfrac{(t^2 - 9)(2t) - (t^2 + 9)(2t)}{(t^2 - 9)^2} = \dfrac{2t^3 - 18t - 2t^3 - 18t}{(t^2 - 9)^2} = -\dfrac{36t}{(t^2 - 9)^2}$. $q'(t) = 0 \quad \Rightarrow$

$36t = 0 \quad \Rightarrow \quad t = 0$. $q'(t)$ does not exist at $t = \pm 3$, but 3 and -3 are not in the domain of q, so 0 is the only critical number.

39. $h(t) = t^{3/4} - 2t^{1/4} \quad \Rightarrow \quad h'(t) = \frac{3}{4}t^{-1/4} - \frac{2}{4}t^{-3/4} = \frac{1}{4}t^{-3/4}(3t^{1/2} - 2) = \dfrac{3\sqrt{t} - 2}{4\sqrt[4]{t^3}}.$

$h'(t) = 0 \quad \Rightarrow \quad 3\sqrt{t} = 2 \quad \Rightarrow \quad \sqrt{t} = \frac{2}{3} \quad \Rightarrow \quad t = \frac{4}{9}$. $h'(t)$ does not exist at $t = 0$, so the critical numbers are 0 and $\frac{4}{9}$.

40. $g(x) = \sqrt[3]{4 - x^2} = (4 - x^2)^{1/3} \quad \Rightarrow \quad g'(x) = \frac{1}{3}(4 - x^2)^{-2/3}(-2x) = \dfrac{-2x}{3(4 - x^2)^{2/3}}$. $g'(x) = 0 \quad \Rightarrow \quad x = 0$.

$g'(\pm 2)$ do not exist. Thus, the three critical numbers are -2, 0, and 2.

41. $F(x) = x^{4/5}(x - 4)^2 \quad \Rightarrow$

$$F'(x) = x^{4/5} \cdot 2(x - 4) + (x - 4)^2 \cdot \tfrac{4}{5}x^{-1/5} = \tfrac{1}{5}x^{-1/5}(x - 4)[5 \cdot x \cdot 2 + (x - 4) \cdot 4]$$
$$= \frac{(x - 4)(14x - 16)}{5x^{1/5}} = \frac{2(x - 4)(7x - 8)}{5x^{1/5}}$$

$F'(x) = 0 \quad \Rightarrow \quad x = 4, \frac{8}{7}$. $F'(0)$ does not exist. Thus, the three critical numbers are 0, $\frac{8}{7}$, and 4.

42. $h(x) = x^{-1/3}(x-2) \quad \Rightarrow$

$$h'(x) = x^{-1/3} \cdot 1 + (x-2) \cdot \left(-\tfrac{1}{3}x^{-4/3}\right) = \tfrac{1}{3}x^{-4/3}\left[3x + (x-2)(-1)\right] = \frac{2x+2}{3x^{4/3}} = \frac{2(x+1)}{3x^{4/3}}. \;\; h'(x) = 0 \quad \Rightarrow$$

$2(x+1) = 0 \quad \Rightarrow \quad x = -1$. $h'(x)$ does not exist at $x = 0$, but 0 is not in the domain of h, so -1 is the only critical number.

43. $f(x) = x^{1/3}(4-x)^{2/3} \quad \Rightarrow$

$$f'(x) = x^{1/3} \cdot \tfrac{2}{3}(4-x)^{-1/3} \cdot (-1) + (4-x)^{2/3} \cdot \tfrac{1}{3}x^{-2/3} = \tfrac{1}{3}x^{-2/3}(4-x)^{-1/3}[-2x + (4-x)] = \frac{4-3x}{3x^{2/3}(4-x)^{1/3}}.$$

$f'(x) = 0 \quad \Rightarrow \quad 4 - 3x = 0 \quad \Rightarrow \quad x = \frac{4}{3}$. $f'(0)$ and $f'(4)$ are undefined. Thus, the three critical numbers are 0, $\frac{4}{3}$, and 4.

44. $f(\theta) = \theta + \sqrt{2}\cos\theta \quad \Rightarrow \quad f'(\theta) = 1 - \sqrt{2}\sin\theta$. $f'(0) = 0 \quad \Rightarrow \quad 1 - \sqrt{2}\sin\theta = 0 \quad \Rightarrow \quad \sin\theta = \dfrac{1}{\sqrt{2}} \quad \Rightarrow$

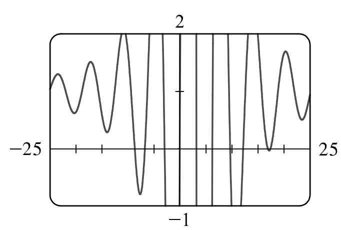
$\theta = \frac{\pi}{4} + 2n\pi$ [n an integer], $\theta = \frac{3\pi}{4} + 2n\pi$ are critical numbers. [$f'(\theta)$ exists for all real numbers.]

45. $f(\theta) = 2\cos\theta + \sin^2\theta \quad \Rightarrow \quad f'(\theta) = -2\sin\theta + 2\sin\theta\,\cos\theta$. $f'(\theta) = 0 \quad \Rightarrow \quad 2\sin\theta\,(\cos\theta - 1) = 0 \quad \Rightarrow \quad \sin\theta = 0$ or $\cos\theta = 1 \quad \Rightarrow \quad \theta = n\pi$ [n an integer] or $\theta = 2n\pi$. The solutions $\theta = n\pi$ include the solutions $\theta = 2n\pi$, so the critical numbers are $\theta = n\pi$.

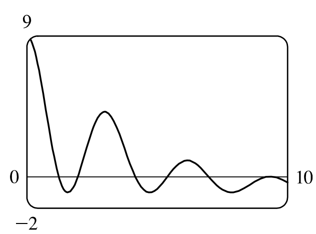
46. $g(x) = \sqrt{1-x^2} = (1-x^2)^{1/2} \quad \Rightarrow \quad g'(x) = \frac{1}{2}(1-x^2)^{-1/2}(-2x) = \dfrac{-x}{\sqrt{1-x^2}}$. $g'(x) = 0 \quad \Rightarrow \quad x = 0$.

$g'(x)$ does not exist $\quad \Rightarrow \quad 1 - x^2 = 0 \quad \Rightarrow \quad x = \pm 1$. The critical numbers are -1, 0, and 1.

47. A graph of $f'(x) = 1 + \dfrac{210\sin x}{x^2 - 6x + 10}$ is shown. There are 10 zeros between -25 and 25 (one is approximately -0.05). f' exists everywhere, so f has 10 critical numbers.



48. A graph of $f'(x) = \dfrac{100\cos^2 x}{10 + x^2} - 1$ is shown. There are 7 zeros between 0 and 10, and 7 more zeros since f' is an even function. f' exists everywhere, so f has 14 critical numbers.



49. $f(x) = 12 + 4x - x^2$, $[0, 5]$. $f'(x) = 4 - 2x = 0 \quad \Leftrightarrow \quad x = 2$. $f(0) = 12$, $f(2) = 16$, and $f(5) = 7$. So $f(2) = 16$ is the absolute maximum value and $f(5) = 7$ is the absolute minimum value.

50. $f(x) = 5 + 54x - 2x^3$, $[0, 4]$. $f'(x) = 54 - 6x^2 = 6(9 - x^2) = 6(3+x)(3-x) = 0 \quad \Leftrightarrow \quad x = -3, 3$. $f(0) = 5$, $f(3) = 113$, and $f(4) = 93$. So $f(3) = 113$ is the absolute maximum value and $f(0) = 5$ is the absolute minimum value.

51. $f(x) = 2x^3 - 3x^2 - 12x + 1,\ [-2, 3]$. $f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x-2)(x+1) = 0 \iff$ $x = 2, -1$. $f(-2) = -3$, $f(-1) = 8$, $f(2) = -19$, and $f(3) = -8$. So $f(-1) = 8$ is the absolute maximum value and $f(2) = -19$ is the absolute minimum value.

52. $f(x) = x^3 - 6x^2 + 5,\ [-3, 5]$. $f'(x) = 3x^2 - 12x = 3x(x-4) = 0 \iff x = 0, 4$. $f(-3) = -76$, $f(0) = 5$, $f(4) = -27$, and $f(5) = -20$. So $f(0) = 5$ is the absolute maximum value and $f(-3) = -76$ is the absolute minimum value.

53. $f(x) = 3x^4 - 4x^3 - 12x^2 + 1,\ [-2, 3]$. $f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2) = 12x(x+1)(x-2) = 0 \iff$ $x = -1, 0, 2$. $f(-2) = 33$, $f(-1) = -4$, $f(0) = 1$, $f(2) = -31$, and $f(3) = 28$. So $f(-2) = 33$ is the absolute maximum value and $f(2) = -31$ is the absolute minimum value.

54. $f(t) = (t^2 - 4)^3,\ [-2, 3]$. $f'(t) = 3(t^2 - 4)^2(2t) = 6t(t+2)^2(t-2)^2 = 0 \iff t = -2, 0, 2$. $f(\pm 2) = 0$, $f(0) = -64$, and $f(3) = 5^3 = 125$. So $f(3) = 125$ is the absolute maximum value and $f(0) = -64$ is the absolute minimum value.

55. $f(x) = x + \dfrac{1}{x},\ [0.2, 4]$. $f'(x) = 1 - \dfrac{1}{x^2} = \dfrac{x^2 - 1}{x^2} = \dfrac{(x+1)(x-1)}{x^2} = 0 \iff x = \pm 1$, but $x = -1$ is not in the given interval, $[0.2, 4]$. $f'(x)$ does not exist when $x = 0$, but 0 is not in the given interval, so 1 is the only critical nuumber. $f(0.2) = 5.2$, $f(1) = 2$, and $f(4) = 4.25$. So $f(0.2) = 5.2$ is the absolute maximum value and $f(1) = 2$ is the absolute minimum value.

56. $f(x) = \dfrac{x}{x^2 - x + 1},\ [0, 3]$.

$$f'(x) = \frac{(x^2 - x + 1) - x(2x - 1)}{(x^2 - x + 1)^2} = \frac{x^2 - x + 1 - 2x^2 + x}{(x^2 - x + 1)^2} = \frac{1 - x^2}{(x^2 - x + 1)^2} = \frac{(1+x)(1-x)}{(x^2 - x + 1)^2} = 0 \iff$$

$x = \pm 1$, but $x = -1$ is not in the given interval, $[0, 3]$. $f(0) = 0$, $f(1) = 1$, and $f(3) = \frac{3}{7}$. So $f(1) = 1$ is the absolute maximum value and $f(0) = 0$ is the absolute minimum value.

57. $f(t) = t - \sqrt[3]{t},\ [-1, 4]$. $f'(t) = 1 - \frac{1}{3}t^{-2/3} = 1 - \dfrac{1}{3t^{2/3}}$. $f'(t) = 0 \iff 1 = \dfrac{1}{3t^{2/3}} \iff t^{2/3} = \dfrac{1}{3} \iff$

$t = \pm\left(\dfrac{1}{3}\right)^{3/2} = \pm\sqrt{\dfrac{1}{27}} = \pm\dfrac{1}{3\sqrt{3}} = \pm\dfrac{\sqrt{3}}{9}$. $f'(t)$ does not exist when $t = 0$. $f(-1) = 0$, $f(0) = 0$,

$f\left(\dfrac{-1}{3\sqrt{3}}\right) = \dfrac{-1}{3\sqrt{3}} - \dfrac{-1}{\sqrt{3}} = \dfrac{-1+3}{3\sqrt{3}} = \dfrac{2\sqrt{3}}{9} \approx 0.3849$, $f\left(\dfrac{1}{3\sqrt{3}}\right) = \dfrac{1}{3\sqrt{3}} - \dfrac{1}{\sqrt{3}} = -\dfrac{2\sqrt{3}}{9}$, and

$f(4) = 4 - \sqrt[3]{4} \approx 2.413$. So $f(4) = 4 - \sqrt[3]{4}$ is the absolute maximum value and $f\left(\dfrac{\sqrt{3}}{9}\right) = -\dfrac{2\sqrt{3}}{9}$ is the absolute minimum value.

58. $f(t) = \dfrac{\sqrt{t}}{1+t^2}$, $[0, 2]$. $f'(t) = \dfrac{(1+t^2)(1/(2\sqrt{t})) - \sqrt{t}\,(2t)}{(1+t^2)^2} = \dfrac{(1+t^2) - 2\sqrt{t}\,\sqrt{t}\,(2t)}{2\sqrt{t}\,(1+t^2)^2} = \dfrac{1-3t^2}{2\sqrt{t}\,(1+t^2)^2}$.

$f'(t) = 0 \;\Leftrightarrow\; 1 - 3t^2 = 0 \;\Leftrightarrow\; t^2 = \dfrac{1}{3} \;\Leftrightarrow\; t = \pm\dfrac{1}{\sqrt{3}}$, but $t = -\dfrac{1}{\sqrt{3}}$ is not in the given interval, $[0, 2]$. $f'(t)$ does not exist when $t = 0$, which is an endpoint. $f(0) = 0$, $f\!\left(\dfrac{1}{\sqrt{3}}\right) = \dfrac{1/\sqrt[4]{3}}{1+1/3} = \dfrac{3^{-1/4}}{4/3} = \dfrac{3^{3/4}}{4} \approx 0.570$, and $f(2) = \dfrac{\sqrt{2}}{5} \approx 0.283$. So $f\!\left(\dfrac{1}{\sqrt{3}}\right) = \dfrac{3^{3/4}}{4}$ is the absolute maximum value and $f(0) = 0$ is the absolute minimum value.

59. $f(t) = 2\cos t + \sin 2t$, $[0, \pi/2]$.

$f'(t) = -2\sin t + \cos 2t \cdot 2 = -2\sin t + 2(1 - 2\sin^2 t) = -2(2\sin^2 t + \sin t - 1) = -2(2\sin t - 1)(\sin t + 1)$.

$f'(t) = 0 \;\Rightarrow\; \sin t = \frac{1}{2}$ or $\sin t = -1 \;\Rightarrow\; t = \frac{\pi}{6}$. $f(0) = 2$, $f(\frac{\pi}{6}) = \sqrt{3} + \frac{1}{2}\sqrt{3} = \frac{3}{2}\sqrt{3} \approx 2.60$, and $f(\frac{\pi}{2}) = 0$.

So $f(\frac{\pi}{6}) = \frac{3}{2}\sqrt{3}$ is the absolute maximum value and $f(\frac{\pi}{2}) = 0$ is the absolute minimum value.

60. $f(\theta) = 1 + \cos^2\theta$, $[\pi/4, \pi]$. $f'(\theta) = 2\cos\theta(-\sin\theta) = -2\sin\theta\,\cos\theta = -\sin 2\theta$. $f'(\theta) = 0 \;\Rightarrow\; -\sin 2\theta = 0 \;\Rightarrow\; 2\theta = n\pi \;\Rightarrow\; \theta = \frac{n\pi}{2}$. Only $\theta = \frac{\pi}{2}$ $[n = 1]$ is in the interval $(\pi/4, \pi)$. $f(\pi/4) = 1 + \left(\dfrac{\sqrt{2}}{2}\right)^2 = \dfrac{3}{2}$, $f(\pi/2) = 1 + 0^2 = 1$, and $f(\pi) = 1 + (-1)^2 = 2$. So $f(\pi) = 2$ is the absolute maximum value and $f(\pi/2) = 1$ is the absolute minimum value.

61. $f(x) = x^a(1-x)^b$, $0 \le x \le 1$, $a > 0$, $b > 0$.

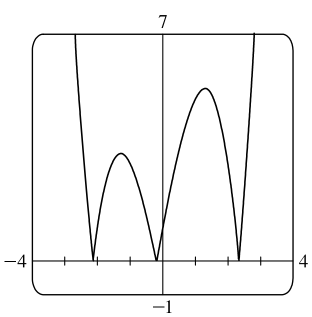
$$\begin{aligned} f'(x) &= x^a \cdot b(1-x)^{b-1}(-1) + (1-x)^b \cdot ax^{a-1} = x^{a-1}(1-x)^{b-1}[x \cdot b(-1) + (1-x)\cdot a] \\ &= x^{a-1}(1-x)^{b-1}(a - ax - bx) \end{aligned}$$

At the endpoints, we have $f(0) = f(1) = 0$ [the minimum value of f]. In the interval $(0, 1)$, $f'(x) = 0 \;\Leftrightarrow\; x = \dfrac{a}{a+b}$.

$$f\!\left(\frac{a}{a+b}\right) = \left(\frac{a}{a+b}\right)^a \left(1 - \frac{a}{a+b}\right)^b = \frac{a^a}{(a+b)^a}\left(\frac{a+b-a}{a+b}\right)^b = \frac{a^a}{(a+b)^a}\cdot\frac{b^b}{(a+b)^b} = \frac{a^a b^b}{(a+b)^{a+b}}.$$

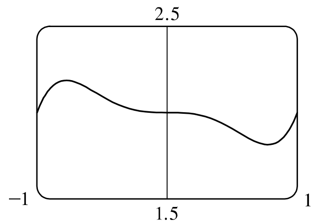
So $f\!\left(\dfrac{a}{a+b}\right) = \dfrac{a^a b^b}{(a+b)^{a+b}}$ is the absolute maximum value.

62.



The graph of $f(x) = |1 + 5x - x^3|$ indicates that $f'(x) = 0$ at $x \approx \pm 1.3$ and that $f'(x)$ does not exist at $x \approx -2.1$, -0.2, and 2.3. Those five values of x are the critical numbers of f.

63. (a)



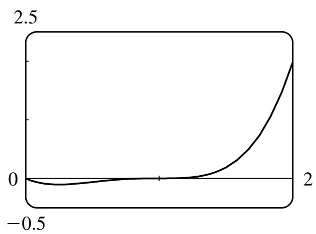
From the graph, it appears that the absolute maximum value is about $f(-0.77) = 2.19$, and the absolute minimum value is about $f(0.77) = 1.81$.

(b) $f(x) = x^5 - x^3 + 2 \quad\Rightarrow\quad f'(x) = 5x^4 - 3x^2 = x^2(5x^2 - 3)$. So $f'(x) = 0 \quad\Rightarrow\quad x = 0, \pm\sqrt{\frac{3}{5}}$.

$$f\left(-\sqrt{\tfrac{3}{5}}\right) = \left(-\sqrt{\tfrac{3}{5}}\right)^5 - \left(-\sqrt{\tfrac{3}{5}}\right)^3 + 2 = -\left(\tfrac{3}{5}\right)^2\sqrt{\tfrac{3}{5}} + \tfrac{3}{5}\sqrt{\tfrac{3}{5}} + 2$$
$$= \left(\tfrac{3}{5} - \tfrac{9}{25}\right)\sqrt{\tfrac{3}{5}} + 2 = \tfrac{6}{25}\sqrt{\tfrac{3}{5}} + 2 \quad \text{(maximum)}$$

and similarly, $f\left(\sqrt{\frac{3}{5}}\right) = -\frac{6}{25}\sqrt{\frac{3}{5}} + 2$ (minimum).

64. (a)



From the graph, it appears that the absolute maximum value is $f(2) = 2$, and that the absolute minimum value is about $f(0.25) = -0.11$.

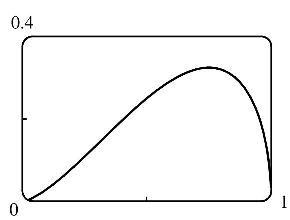
(b) $f(x) = x^4 - 3x^3 + 3x^2 - x \quad\Rightarrow\quad f'(x) = 4x^3 - 9x^2 + 6x - 1 = (4x - 1)(x - 1)^2$.

So $f'(x) = 0 \quad\Rightarrow\quad x = \frac{1}{4}$ or $x = 1$. Now $f(1) = 1^4 - 3 \cdot 1^3 + 3 \cdot 1^2 - 1 = 0$ (not an extremum)

and $f\left(\frac{1}{4}\right) = \left(\frac{1}{4}\right)^4 - 3\left(\frac{1}{4}\right)^3 + 3\left(\frac{1}{4}\right)^2 - \frac{1}{4} = -\frac{27}{256}$ (minimum). At the right endpoint we have

$f(2) = 2^4 - 3 \cdot 2^3 + 3 \cdot 2^2 - 2 = 2$ (maximum).

65. (a)



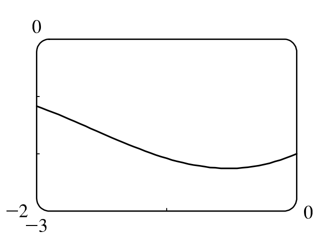
From the graph, it appears that the absolute maximum value is about $f(0.75) = 0.32$, and the absolute minimum value is $f(0) = f(1) = 0$; that is, at both endpoints.

(b) $f(x) = x\sqrt{x - x^2} \quad\Rightarrow\quad f'(x) = x \cdot \dfrac{1 - 2x}{2\sqrt{x - x^2}} + \sqrt{x - x^2} = \dfrac{(x - 2x^2) + (2x - 2x^2)}{2\sqrt{x - x^2}} = \dfrac{3x - 4x^2}{2\sqrt{x - x^2}}$.

So $f'(x) = 0 \quad\Rightarrow\quad 3x - 4x^2 = 0 \quad\Rightarrow\quad x(3 - 4x) = 0 \quad\Rightarrow\quad x = 0$ or $\frac{3}{4}$.

$f(0) = f(1) = 0$ (minimum), and $f\left(\frac{3}{4}\right) = \frac{3}{4}\sqrt{\frac{3}{4} - \left(\frac{3}{4}\right)^2} = \frac{3}{4}\sqrt{\frac{3}{16}} = \frac{3\sqrt{3}}{16}$ (maximum).

66. (a)



From the graph, it appears that the absolute maximum value is about $f(-2) = -1.17$, and the absolute minimum value is about $f(-0.52) = -2.26$.

(b) $f(x) = x - 2\cos x \quad \Rightarrow \quad f'(x) = 1 + 2\sin x$. So $f'(x) = 0 \quad \Rightarrow \quad \sin x = -\frac{1}{2} \quad \Rightarrow \quad x = -\frac{\pi}{6}$ on $[-2, 0]$.

$f(-2) = -2 - 2\cos(-2)$ (maximum) and $f\left(-\frac{\pi}{6}\right) = -\frac{\pi}{6} - 2\cos\left(-\frac{\pi}{6}\right) = -\frac{\pi}{6} - 2\left(\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{6} - \sqrt{3}$ (minimum).

67. The density is defined as $\rho = \dfrac{\text{mass}}{\text{volume}} = \dfrac{1000}{V(T)}$ (in g/cm^3). But a critical point of ρ will also be a critical point of V $\left[\text{since } \dfrac{d\rho}{dT} = -1000V^{-2}\dfrac{dV}{dT} \text{ and } V \text{ is never } 0\right]$, and V is easier to differentiate than ρ.

$V(T) = 999.87 - 0.06426T + 0.0085043T^2 - 0.0000679T^3 \quad \Rightarrow \quad V'(T) = -0.06426 + 0.0170086T - 0.0002037T^2$.

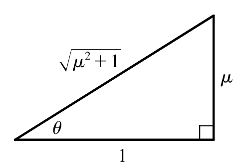
Setting this equal to 0 and using the quadratic formula to find T, we get

$T = \dfrac{-0.0170086 \pm \sqrt{0.0170086^2 - 4 \cdot 0.0002037 \cdot 0.06426}}{2(-0.0002037)} \approx 3.9665°\text{C}$ or $79.5318°\text{C}$. Since we are only interested in the region $0°\text{C} \le T \le 30°\text{C}$, we check the density ρ at the endpoints and at $3.9665°\text{C}$: $\rho(0) \approx \dfrac{1000}{999.87} \approx 1.00013$; $\rho(30) \approx \dfrac{1000}{1003.7628} \approx 0.99625$; $\rho(3.9665) \approx \dfrac{1000}{999.7447} \approx 1.000255$. So water has its maximum density at about $3.9665°\text{C}$.

68. $F = \dfrac{\mu W}{\mu\sin\theta + \cos\theta} \quad \Rightarrow \quad \dfrac{dF}{d\theta} = \dfrac{(\mu\sin\theta + \cos\theta)(0) - \mu W(\mu\cos\theta - \sin\theta)}{(\mu\sin\theta + \cos\theta)^2} = \dfrac{-\mu W(\mu\cos\theta - \sin\theta)}{(\mu\sin\theta + \cos\theta)^2}$.

So $\dfrac{dF}{d\theta} = 0 \quad \Rightarrow \quad \mu\cos\theta - \sin\theta = 0 \quad \Rightarrow \quad \mu = \dfrac{\sin\theta}{\cos\theta} = \tan\theta$. Substituting $\tan\theta$ for μ in F gives us

$$F = \frac{(\tan\theta)W}{(\tan\theta)\sin\theta + \cos\theta} = \frac{W\tan\theta}{\dfrac{\sin^2\theta}{\cos\theta} + \cos\theta} = \frac{W\tan\theta\,\cos\theta}{\sin^2\theta + \cos^2\theta} = \frac{W\sin\theta}{1} = W\sin\theta.$$



If $\tan\theta = \mu$, then $\sin\theta = \dfrac{\mu}{\sqrt{\mu^2+1}}$ (see the figure), so $F = \dfrac{\mu}{\sqrt{\mu^2+1}}W$.

We compare this with the value of F at the endpoints: $F(0) = \mu W$ and $F\left(\frac{\pi}{2}\right) = W$.

Now because $\dfrac{\mu}{\sqrt{\mu^2+1}} \le 1$ and $\dfrac{\mu}{\sqrt{\mu^2+1}} \le \mu$, we have that $\dfrac{\mu}{\sqrt{\mu^2+1}}W$ is less than or equal to each of $F(0)$ and $F\left(\frac{\pi}{2}\right)$.

Hence, $\dfrac{\mu}{\sqrt{\mu^2+1}}W$ is the absolute minimum value of $F(\theta)$, and it occurs when $\tan\theta = \mu$.

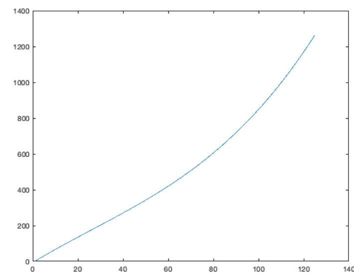
69. $L'(t) = 0 \quad \Rightarrow \quad 0.01317t^2 - 0.2546t + 0.8239 = 0$

$$\Rightarrow t = \frac{0.2546 \pm \sqrt{(-0.2546)^2 - 4(0.01317)(0.8239)}}{2(0.01317)} \approx 4.1,\ 15.2$$

For $0 \le t \le 12$, we have $L(0) = 323.1$, $L(4.1) \approx 324.6$, and $L(15.2) \approx 322.2$. Thus, the water level was highest 4.1 months after January 1.

70. (a) By Excel regression, we could get

$V(t) = 0.00045t^3 - 0.03520t^2 + 7.61427t - 6.48990$



(b) $V'(t) = 0.00135t^2 - 0.0704t + 7.61427$

$V''(t) = 0.0027t - 0.0704$

So $V''(t) = 0 \implies t \approx 26.1$

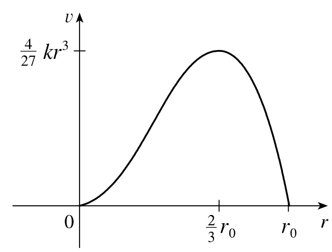
So we could get $V'(0) \approx 7.61$, $V'(26) \approx 6.70$, $V'(125) \approx 19.9$.

Therefore, the maximum acceleration is about 19.9 m/s^2 and the minimum acceleration is about 6.70 m/s^2.

71. (a) $v(r) = k(r_0 - r)r^2 = kr_0r^2 - kr^3 \Rightarrow v'(r) = 2kr_0r - 3kr^2$. $v'(r) = 0 \Rightarrow kr(2r_0 - 3r) = 0 \Rightarrow$ $r = 0$ or $\frac{2}{3}r_0$ (but 0 is not in the interval). Evaluating v at $\frac{1}{2}r_0$, $\frac{2}{3}r_0$, and r_0, we get $v\left(\frac{1}{2}r_0\right) = \frac{1}{8}kr_0^3$, $v\left(\frac{2}{3}r_0\right) = \frac{4}{27}kr_0^3$, and $v(r_0) = 0$. Since $\frac{4}{27} > \frac{1}{8}$, v attains its maximum value at $r = \frac{2}{3}r_0$. This supports the statement in the text.

(b) From part (a), the maximum value of v is $\frac{4}{27}kr_0^3$.

(c)



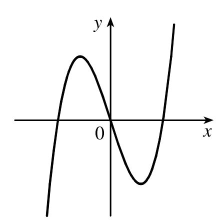
72. $f(x) = x^{101} + x^{51} + x + 1 \Rightarrow f'(x) = 101x^{100} + 51x^{50} + 1$. Since $f'(x) \geq 1$ for all x, $f'(x) = 0$ has no solution. Thus, f has no critical number, and the function f has neither a local maximum nor a local minimum.

73. (a) Suppose that f has a local minimum value at c, so $f(x) \geq f(c)$ for all x near c. Then $g(x) = -f(x) \leq -f(c) = g(c)$ for all x near c, so $g(x)$ has a local maximum value at c.

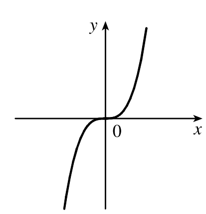
(b) If f has a local minimum at c, then $g(x) = -f(x)$ has a local maximum at c, so $g'(c) = 0$ by the case of Fermat's Theorem proved in the text. Thus, $f'(c) = -g'(c) = 0$.

74. (a) $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$. So $f'(x) = 3ax^2 + 2bx + c$ is a quadratic function and hence, the quadratic equation $f'(x) = 0$ has either 2, 1, or 0 real solutions. Thus, a cubic function can have two, one, or no critical number(s).

Case (i) [2 critical numbers]: $f(x) = x^3 - 3x \Rightarrow$
$f'(x) = 3x^2 - 3 = 3(x^2 - 1)$, so $x = -1, 1$
are critical numbers.



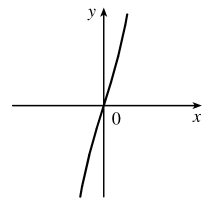
Case (ii) [1 critical number]: $f(x) = x^3 \Rightarrow$
$f'(x) = 3x^2$, so $x = 0$
is the only critical number.



Case (iii) [no critical number]: $f(x) = x^3 + 3x \quad \Rightarrow$

$f'(x) = 3x^2 + 3 = 3(x^2 + 1)$,

so there is no critical number.



(b) Since there are at most two critical numbers, a cubic function can have at most two local extreme values, and by (a)(i), this can occur. By (a)(ii) and (a)(iii), it can have no local extreme value. Thus, a cubic function can have zero or two local extreme values.

APPLIED PROJECT The Calculus of Rainbows

1. From Snell's Law, we have $\sin\alpha = \frac{4}{3}\sin\beta \quad \Rightarrow \quad \dfrac{d}{d\alpha}(\sin\alpha) = \dfrac{4}{3}\dfrac{d}{d\alpha}(\sin\beta) \quad \Rightarrow \quad \cos\alpha = \dfrac{4}{3}\cos\beta\,\dfrac{d\beta}{d\alpha} \quad \Rightarrow$

$\dfrac{d\beta}{d\alpha} = \dfrac{3\cos\alpha}{4\cos\beta}$. Now $D(\alpha) = \pi + 2\alpha - 4\beta \quad \Rightarrow \quad D'(\alpha) = 2 - 4\,\dfrac{d\beta}{d\alpha} = 2 - 3\dfrac{\cos\alpha}{\cos\beta}$. So $D'(\alpha) = 0 \quad \Leftrightarrow$

$2\cos\beta = 3\cos\alpha$. Thus, $4\cos^2\beta = 9\cos^2\alpha \quad \Rightarrow \quad 4 - 4\sin^2\beta = 9 - 9\sin^2\alpha$. Since $3\sin\alpha = 4\sin\beta$,

$\sin\beta = \frac{3}{4}\sin\alpha \quad \Rightarrow \quad 4 - 4\left(\frac{3}{4}\sin\alpha\right)^2 = 9 - 9\sin^2\alpha \quad \Rightarrow \quad \left(9 - \frac{9}{4}\right)\sin^2\alpha = 9 - 4 = 5 \quad \Rightarrow \quad \sin^2\alpha = \frac{20}{27} \quad \Rightarrow$

$\sin\alpha = \sqrt{\frac{20}{27}}$. So $\alpha \approx 1.037$ radians or 59.4°. We show that this α does give the minimum on $\left[0, \frac{\pi}{2}\right]$: When $\alpha = 0$,

$\sin\alpha = \frac{4}{3}\sin\beta \quad \Rightarrow \quad \beta = 0$, or $D(0) = \pi \approx 3.14$. When $\alpha = \frac{\pi}{2}$, $1 = \sin\frac{\pi}{2} = \frac{4}{3}\sin\beta \quad \Rightarrow \quad \sin\beta = \frac{3}{4} \quad \Rightarrow$

$\beta \approx 0.85$. So $D\left(\frac{\pi}{2}\right) \approx \pi + \pi - 4(0.85) \approx 2.88$. For $\alpha \approx 1.037$, $\sin\beta = \frac{3}{4}\sin\alpha = \frac{3}{4}\sqrt{\frac{20}{27}}$, so $\beta \approx 0.702 \quad \Rightarrow$

$D(\alpha) \approx \pi + 2(1.036) - 4(0.702) \approx 2.41$. So the minimum occurs when $\alpha \approx 1.04$ radians or 59.4°.

2. We repeat Problem 1 with k in place of $\frac{4}{3}$. So $\sin\alpha = k\sin\beta \quad \Rightarrow \quad \dfrac{d\beta}{d\alpha} = \dfrac{1}{k}\dfrac{\cos\alpha}{\cos\beta}$. Then

$D'(\alpha) = 2 - 4\,\dfrac{d\beta}{d\alpha} = 2 - \dfrac{4}{k}\dfrac{\cos\alpha}{\cos\beta}$ and $D'(\alpha) = 0 \quad \Leftrightarrow \quad k\cos\beta = 2\cos\alpha$. So $k^2\cos^2\beta = 4\cos^2\alpha \quad \Rightarrow$

$k^2 - k^2\sin^2\beta = 4 - 4\sin^2\alpha \quad \Rightarrow \quad k^2 - \sin^2\alpha = 4 - 4\sin^2\alpha \quad \Rightarrow \quad 3\sin^2\alpha = 4 - k^2 \quad \Rightarrow \quad \sin\alpha = \sqrt{\dfrac{4 - k^2}{3}}$.

So for $k \approx 1.3318$ (red light) the minimum occurs when $\sin\alpha_1 = \sqrt{\dfrac{4 - (1.3318)^2}{3}}$ or $\alpha_1 \approx 1.038$ radians, so the rainbow angle is about $\pi - D(\alpha_1) \approx 42.3^\circ$. For $k \approx 1.3435$ (violet light) the minimum occurs at $\alpha_2 \approx 1.026$ radians, and so the rainbow angle is about $\pi - D(\alpha_2) \approx 40.6^\circ$.

3. At each reflection or refraction, the light is bent in a counterclockwise direction: the bend at A is $\alpha - \beta$, the bend at B is $\pi - 2\beta$, the bend at C is again $\pi - 2\beta$, and the bend at D is $\alpha - \beta$. So the total bend is

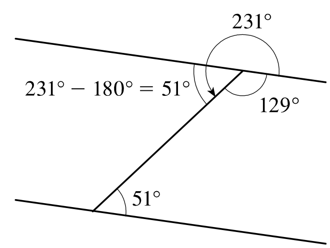
$D(\alpha) = 2(\alpha - \beta) + 2(\pi - 2\beta) = 2\alpha - 6\beta + 2\pi$, as required. Now $\sin\alpha = k\sin\beta \;\Rightarrow\; \dfrac{d\beta}{d\alpha} = \dfrac{1}{k}\dfrac{\cos\alpha}{\cos\beta}$. So

$D'(\alpha) = 2 - 6\,\dfrac{d\beta}{d\alpha} = 2 - \dfrac{6}{k}\dfrac{\cos\alpha}{\cos\beta}$ and $D'(\alpha) = 0 \;\Leftrightarrow\; k\cos\beta = 3\cos\alpha$. So $k^2\cos^2\beta = 9\cos^2\alpha \;\Rightarrow$

$k^2 - k^2\sin^2\beta = 9 - 9\sin^2\alpha \;\Rightarrow\; k^2 - \sin^2\alpha = 9 - 9\sin^2\alpha \;\Rightarrow\; \sin^2\alpha = \dfrac{9-k^2}{8} \;\Rightarrow\; \sin\alpha = \sqrt{\dfrac{9-k^2}{8}}$.

If $k = \frac{4}{3}$, then the minimum occurs when $\sin\alpha_1 = \sqrt{\dfrac{9-(4/3)^2}{8}}$ or $\alpha_1 \approx 1.254$ radians.

Thus, the minimum *counterclockwise* rotation is $D(\alpha_1) \approx 231^\circ$, which is equivalent to a *clockwise* rotation of $360^\circ - 231^\circ = 129^\circ$ (see the figure). So the rainbow angle for the secondary rainbow is about $180^\circ - 129^\circ = 51^\circ$, as required. In general, the rainbow angle for the secondary rainbow is $\pi - [2\pi - D(\alpha)] = D(\alpha) - \pi$.



4. In the primary rainbow, the rainbow angle gets smaller as k gets larger, as we found in Problem 2, so the colors appear from top to bottom in order of increasing k. But in the secondary rainbow, the rainbow angle gets larger as k gets larger. To see this, we find the minimum deviations for red light and for violet light in the secondary rainbow.

For $k \approx 1.3318$ (red light) the minimum occurs when $\sin\alpha_1 = \sqrt{\dfrac{9-(1.3318)^2}{8}}$ or $\alpha_1 \approx 1.254$ radians, and so the rainbow angle is $D(\alpha_1) - \pi \approx 50.6^\circ$. For $k \approx 1.3435$ (violet light) the minimum occurs when $\sin\alpha_2 = \sqrt{\dfrac{9-(1.3435)^2}{8}}$ or $\alpha_2 \approx 1.248$ radians, and so the rainbow angle is $D(\alpha_2) - \pi \approx 53.6^\circ$. Consequently, the rainbow angle is larger for colors with higher indices of refraction, and the colors appear from bottom to top in order of increasing k, the reverse of their order in the primary rainbow.

Note that our calculations above also explain why the secondary rainbow is more spread-out than the primary rainbow: in the primary rainbow, the difference between rainbow angles for red and violet light is about 1.7°, whereas in the secondary rainbow it is about 3°.

3.2 The Mean Value Theorem

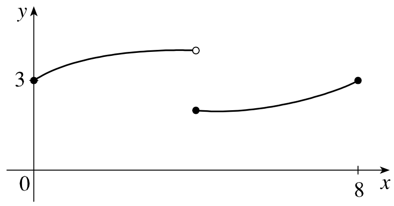
1. **(1)** f is continuous on the closed interval $[0, 8]$.

(2) f is differentiable on the open interval $(0, 8)$.

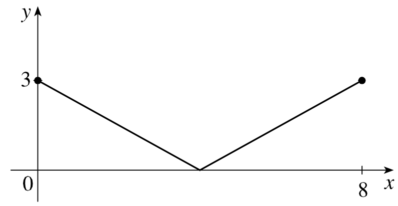
(3) $f(0) = 3$ and $f(8) = 3$

Thus, f satisfies the hypotheses of Rolle's Theorem. The numbers $c = 1$ and $c = 5$ satisfy the conclusion of Rolle's Theorem since $f'(1) = f'(5) = 0$.

2. The possible graphs fall into two general categories: **(1)** Not continuous and therefore not differentiable, **(2)** Continuous, but not differentiable.



Not continuous



Not differentiable

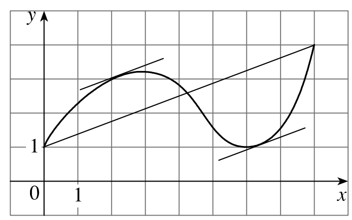
In either case, there is no number c such that $f'(c) = 0$.

3. (a) **(1)** g is continuous on the closed interval $[0, 8]$.

(2) g is differentiable on the open interval $(0, 8)$.

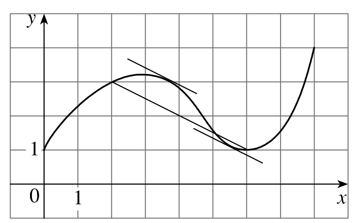
(b) $g'(c) = \dfrac{g(8) - g(0)}{8 - 0} = \dfrac{4 - 1}{8} = \dfrac{3}{8}$.

It appears that $g'(c) = \frac{3}{8}$ when $c \approx 2.2$ and 6.4.

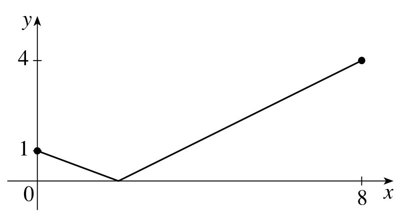


(c) $g'(c) = \dfrac{g(6) - g(2)}{6 - 2} = \dfrac{1 - 3}{4} = -\dfrac{1}{2}$.

It appears that $g'(c) = -\frac{1}{2}$ when $c \approx 3.7$ and 5.5.



4.



The function shown in the figure is continuous on $[0, 8]$ [but not differentiable on $(0, 8)$] with $f(0) = 1$ and $f(8) = 4$. The line passing through the two points has slope $\frac{3}{8}$. There is no number c in $(0, 8)$ such that $f'(c) = \frac{3}{8}$.

5. **(1)** f is continuous on the closed interval $[0, 5]$.

(2) f is not differentiable on the open interval $(0, 5)$ since f is not differentiable at 3.

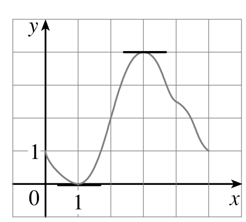
Thus, f does not satisfy the hypotheses of the Mean Value Theorem on the interval $[0, 5]$.

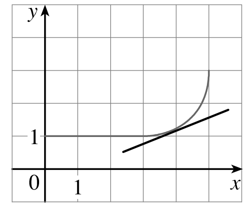
6. **(1)** f is continuous on the closed interval $[0, 5]$.

(2) f is differentiable on the open interval $(0, 5)$.

Thus, f satisfies the hypotheses of the Mean Value Theorem on the interval $[0, 5]$.

The line passing through $(0, f(0))$ and $(5, f(5))$ has slope 0. It appears that $f'(c) = 0$ for $c = 1$ and $c = 3$.





7. **(1)** f is continuous on the closed interval $[0, 5]$.

(2) f is differentiable on the open interval $(0, 5)$.

Thus, f satisfies the hypotheses of the Mean Value Theorem on the interval $[0, 5]$.

The line passing through $(0, f(0))$ and $(5, f(5))$ has slope

$\dfrac{f(5) - f(0)}{5 - 0} = \dfrac{3 - 1}{5} = \dfrac{2}{5}$. It appears that $f'(c) = \frac{2}{5}$ when $c \approx 3.8$.

8. **(1)** f is continuous on the closed interval $[0, 5]$.

(2) f is not differentiable on the open interval $(0, 5)$ since f is not differentiable at 4.

Thus, f does not satisfy the hypotheses of the Mean Value Theorem on the interval $[0, 5]$.

9. $f(x) = 2x^2 - 4x + 5,\ [-1, 3]$. f is a polynomial, so it's continuous and differentiable on $\mathbb{R}$, and hence, continuous on $[-1, 3]$ and differentiable on $(-1, 3)$. Since $f(-1) = 11$ and $f(3) = 11$, f satisfies all the hypotheses of Rolle's Theorem. $f'(c) = 4c - 4$ and $f'(c) = 0 \Leftrightarrow 4c - 4 = 0 \Leftrightarrow c = 1$. $c = 1$ is in the interval $(-1, 3)$, so 1 satisfies the conclusion of Rolle's Theorem.

10. $f(x) = x^3 - 2x^2 - 4x + 2,\ [-2, 2]$. f is a polynomial, so it's continuous and differentiable on $\mathbb{R}$, and hence, continuous on $[-2, 2]$ and differentiable on $(-2, 2)$. Since $f(-2) = -6$ and $f(2) = -6$, f satisfies all the hypotheses of Rolle's Theorem. $f'(c) = 3c^2 - 4c - 4$ and $f'(c) = 0 \Leftrightarrow (3c + 2)(c - 2) = 0 \Leftrightarrow c = -\frac{2}{3}$ or 2. $c = -\frac{2}{3}$ is in the open interval $(-2, 2)$ (but 2 *isn't*), so only $-\frac{2}{3}$ satisfies the conclusion of Rolle's Theorem.

11. $f(x) = \sin(x/2),\ [\pi/2, 3\pi/2]$. f, being the composite of the sine function and the polynomial $x/2$, is continuous and differentiable on $\mathbb{R}$, so it is continuous on $[\pi/2, 3\pi/2]$ and differentiable on $(\pi/2, 3\pi/2)$. Also, $f\left(\frac{\pi}{2}\right) = \frac{1}{2}\sqrt{2} = f\left(\frac{3\pi}{2}\right)$.
$f'(c) = 0 \Leftrightarrow \frac{1}{2}\cos(c/2) = 0 \Leftrightarrow \cos(c/2) = 0 \Leftrightarrow c/2 = \frac{\pi}{2} + n\pi$ [n an integer] $\Leftrightarrow c = \pi + 2n\pi$.
Only $c = \pi$ [when $n = 0$] is in $(\pi/2, 3\pi/2)$, so π satisfies the conclusion of Rolle's Theorem.

12. $f(x) = x + 1/x,\ \left[\frac{1}{2}, 2\right]$. $f'(x) = 1 - 1/x^2 = \dfrac{x^2 - 1}{x^2}$. f is a rational function that is continuous on its domain, $(-\infty, 0) \cup (0, \infty)$, so it is continuous on $\left[\frac{1}{2}, 2\right]$. f' has the same domain and is differentiable on $\left(\frac{1}{2}, 2\right)$. Also, $f\left(\frac{1}{2}\right) = \frac{5}{2} = f(2)$. $f'(c) = 0 \Leftrightarrow \dfrac{c^2 - 1}{c^2} = 0 \Leftrightarrow c^2 - 1 = 0 \Leftrightarrow c = \pm 1$. Only 1 is in $\left(\frac{1}{2}, 2\right)$, so 1 satisfies the conclusion of Rolle's Theorem.

13. $f(x) = 1 - x^{2/3}$. $f(-1) = 1 - (-1)^{2/3} = 1 - 1 = 0 = f(1)$. $f'(x) = -\frac{2}{3}x^{-1/3}$, so $f'(c) = 0$ has no solution. This does not contradict Rolle's Theorem, since $f'(0)$ does not exist, and so f is not differentiable on $(-1, 1)$.

14. $f(x) = \tan x$. $f(0) = \tan 0 = 0 = \tan\pi = f(\pi)$. $f'(x) = \sec^2 x \geq 1$, so $f'(c) = 0$ has no solution. This does not contradict Rolle's Theorem, since $f'\left(\frac{\pi}{2}\right)$ does not exist, and so f is not differentiable on $(0, \pi)$. (Also, f is not continuous on $[0, \pi]$.)

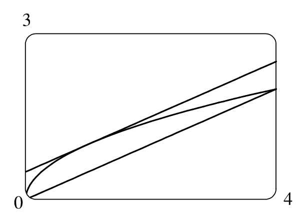
15. $f(x) = 2x^2 - 3x + 1,\ [0, 2]$. f is continuous on $[0, 2]$ and differentiable on $(0, 2)$ since polynomials are continuous and differentiable on $\mathbb{R}$. $f'(c) = \dfrac{f(b) - f(a)}{b - a} \iff 4c - 3 = \dfrac{f(2) - f(0)}{2 - 0} = \dfrac{3 - 1}{2} = 1 \iff 4c = 4 \iff c = 1$, which is in $(0, 2)$.

16. $f(x) = x^3 - 3x + 2,\ [-2, 2]$. f is continuous on $[-2, 2]$ and differentiable on $(-2, 2)$ since polynomials are continuous and differentiable on $\mathbb{R}$. $f'(c) = \dfrac{f(b) - f(a)}{b - a} \iff 3c^2 - 3 = \dfrac{f(2) - f(-2)}{2 - (-2)} = \dfrac{4 - 0}{4} = 1 \iff 3c^2 = 4 \iff c^2 = \dfrac{4}{3} \iff c = \pm\dfrac{2}{\sqrt{3}} \approx \pm 1.15$, which are both in $(-2, 2)$.

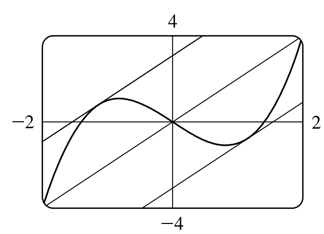
17. $f(x) = \sqrt[3]{x},\ [0, 1]$. f is continuous on $\mathbb{R}$ and differentiable on $(-\infty, 0) \cup (0, \infty)$, so f is continuous on $[0, 1]$ and differentiable on $(0, 1)$. $f'(c) = \dfrac{f(b) - f(a)}{b - a} \iff \dfrac{1}{3c^{2/3}} = \dfrac{f(1) - f(0)}{1 - 0} \iff \dfrac{1}{3c^{2/3}} = \dfrac{1 - 0}{1} \iff 3c^{2/3} = 1 \iff c^{2/3} = \frac{1}{3} \iff c^2 = \left(\frac{1}{3}\right)^3 = \frac{1}{27} \iff c = \pm\sqrt{\frac{1}{27}} = \pm\frac{\sqrt{3}}{9}$, but only $\frac{\sqrt{3}}{9}$ is in $(0, 1)$.

18. $f(x) = 1/x,\ [1, 3]$. f is continuous and differentiable on $(-\infty, 0) \cup (0, \infty)$, so f is continous on $[1, 3]$ and differentiable on $(1, 3)$. $f'(c) = \dfrac{f(b) - f(a)}{b - a} \iff -\dfrac{1}{c^2} = \dfrac{f(3) - f(1)}{3 - 1} = \dfrac{\frac{1}{3} - 1}{2} = -\dfrac{1}{3} \iff c^2 = 3 \iff c = \pm\sqrt{3} \approx \pm 1.73$, but only $\sqrt{3}$ is in $(1, 3)$.

19. $f(x) = \sqrt{x},\ [0, 4]$. $f'(c) = \dfrac{f(4) - f(0)}{4 - 0} \iff \dfrac{1}{2\sqrt{c}} = \dfrac{2 - 0}{4} \iff \dfrac{1}{2\sqrt{c}} = \dfrac{1}{2} \iff \sqrt{c} = 1 \iff c = 1$. The secant line and the tangent line are parallel.



20. $f(x) = x^3 - 2x, [-2, 2]$. $f'(c) = \dfrac{f(2) - f(-2)}{2 - (-2)} \iff 3c^2 - 2 = \dfrac{4 - (-4)}{4} \iff 3c^2 = 4 \iff c = \pm\frac{2\sqrt{3}}{3} \approx \pm 1.155$.
The secant line and the tangent lines are parallel.



21. $f(x) = (x - 3)^{-2} \Rightarrow f'(x) = -2(x - 3)^{-3}$. $f(4) - f(1) = f'(c)(4 - 1) \Rightarrow \dfrac{1}{1^2} - \dfrac{1}{(-2)^2} = \dfrac{-2}{(c - 3)^3} \cdot 3 \Rightarrow \dfrac{3}{4} = \dfrac{-6}{(c - 3)^3} \Rightarrow (c - 3)^3 = -8 \Rightarrow c - 3 = -2 \Rightarrow c = 1$, which is not in the *open* interval $(1, 4)$. This does not contradict the Mean Value Theorem since f is not continuous at $x = 3$, which is in the interval $[1, 4]$.

22. $f(x) = 2 - |2x - 1| = \begin{cases} 2 - (2x - 1) & \text{if } 2x - 1 \ge 0 \\ 2 - [-(2x - 1)] & \text{if } 2x - 1 < 0 \end{cases} = \begin{cases} 3 - 2x & \text{if } x \ge \frac{1}{2} \\ 1 + 2x & \text{if } x < \frac{1}{2} \end{cases} \Rightarrow f'(x) = \begin{cases} -2 & \text{if } x > \frac{1}{2} \\ 2 & \text{if } x < \frac{1}{2} \end{cases}$

$f(3) - f(0) = f'(c)(3 - 0) \Rightarrow -3 - 1 = f'(c) \cdot 3 \Rightarrow f'(c) = -\frac{4}{3}$ [not ± 2]. This does not contradict the Mean Value Theorem since f is not differentiable at $x = \frac{1}{2}$, which is in the interval $(0, 3)$.

23. Let $f(x) = 2x + \cos x$. Then $f(-\pi) = -2\pi - 1 < 0$ and $f(0) = 1 > 0$. Since f is the sum of the polynomial $2x$ and the trignometric function $\cos x$, f is continuous and differentiable for all x. By the Intermediate Value Theorem, there is a number c in $(-\pi, 0)$ such that $f(c) = 0$. Thus, the given equation has at least one real solution. If the equation has distinct real solutions a and b with $a < b$, then $f(a) = f(b) = 0$. Since f is continuous on $[a, b]$ and differentiable on (a, b), Rolle's Theorem implies that there is a number r in (a, b) such that $f'(r) = 0$. But $f'(r) = 2 - \sin r > 0$ since $\sin r \le 1$. This contradiction shows that the given equation can't have two distinct real solutions, so it has exactly one solution.

24. Let $f(x) = 2x - 1 - \sin x$. Then $f(0) = -1 < 0$ and $f(\pi/2) = \pi - 2 > 0$. f is the sum of the polynomial $2x - 1$ and the scalar multiple $(-1) \cdot \sin x$ of the trigonometric function $\sin x$, so f is continuous (and differentiable) for all x. By the Intermediate Value Theorem, there is a number c in $(0, \pi/2)$ such that $f(c) = 0$. Thus, the given equation has at least one real solution. If the equation has distinct real solutions a and b with $a < b$, then $f(a) = f(b) = 0$. Since f is continuous on $[a, b]$ and differentiable on (a, b), Rolle's Theorem implies that there is a number r in (a, b) such that $f'(r) = 0$. But $f'(r) = 2 - \cos r > 0$ since $\cos r \le 1$. This contradiction shows that the given equation can't have two distinct real solutions, so it has exactly one real solution.

25. Let $f(x) = x^3 - 15x + c$ for x in $[-2, 2]$. If f has two real solutions a and b in $[-2, 2]$, with $a < b$, then $f(a) = f(b) = 0$. Since the polynomial f is continuous on $[a, b]$ and differentiable on (a, b), Rolle's Theorem implies that there is a number r in (a, b) such that $f'(r) = 0$. Now $f'(r) = 3r^2 - 15$. Since r is in (a, b), which is contained in $[-2, 2]$, we have $|r| < 2$, so $r^2 < 4$. It follows that $3r^2 - 15 < 3 \cdot 4 - 15 = -3 < 0$. This contradicts $f'(r) = 0$, so the given equation, $x^3 - 15x + c = 0$, can't have two real solutions in $[-2, 2]$. Hence, it has at most one real solution in $[-2, 2]$.

26. $f(x) = x^4 + 4x + c$. Suppose that $f(x) = 0$ has three distinct real solutions a, b, d where $a < b < d$. Then $f(a) = f(b) = f(d) = 0$. By Rolle's Theorem there are numbers c_1 and c_2 with $a < c_1 < b$ and $b < c_2 < d$ and $0 = f'(c_1) = f'(c_2)$, so $f'(x) = 0$ must have at least two real solutions. However $0 = f'(x) = 4x^3 + 4 = 4(x^3 + 1) = 4(x + 1)(x^2 - x + 1)$ has as its only real solution $x = -1$. Thus, the given equation, $x^4 + 4x + c = 0$, can have at most two real solutions.

27. (a) Suppose that a cubic polynomial $P(x)$ has zeros $a_1 < a_2 < a_3 < a_4$, so $P(a_1) = P(a_2) = P(a_3) = P(a_4)$. By Rolle's Theorem there are numbers c_1, c_2, c_3 with $a_1 < c_1 < a_2$, $a_2 < c_2 < a_3$, and $a_3 < c_3 < a_4$ and $P'(c_1) = P'(c_2) = P'(c_3) = 0$. Thus, the second-degree polynomial $P'(x)$ has three distinct real zeros, which is impossible. This contradiction tells us that a polynomial of degree 3 has at most three real zeros.

(b) We prove by induction that a polynomial of degree n has at most n real zeros. This is certainly true for $n = 1$. Suppose that the result is true for all polynomials of degree n and let $P(x)$ be a polynomial of degree $n+1$. Suppose that $P(x)$ has more than $n+1$ real zeros, say $a_1 < a_2 < a_3 < \cdots < a_{n+1} < a_{n+2}$. Then $P(a_1) = P(a_2) = \cdots = P(a_{n+2}) = 0$. By Rolle's Theorem there are real numbers $c_1, \ldots, c_{n+1}$ with $a_1 < c_1 < a_2, \ldots, a_{n+1} < c_{n+1} < a_{n+2}$ and $P'(c_1) = \cdots = P'(c_{n+1}) = 0$. Thus, the nth-degree polynomial $P'(x)$ has at least $n+1$ zeros. This contradiction shows that $P(x)$ has at most $n+1$ real zeros and hence, a polynomial of degree n has at most n real zeros.

28. (a) Suppose that $f(a) = f(b) = 0$ where $a < b$. By Rolle's Theorem applied to f on $[a, b]$, there is a number c such that $a < c < b$ and $f'(c) = 0$.

(b) Suppose that $f(a) = f(b) = f(c) = 0$ where $a < b < c$. By Rolle's Theorem applied to f on $[a, b]$ and $[b, c]$, there are numbers d and e such that $a < d < b$ and $b < e < c$, with $f'(d) = 0$ and $f'(e) = 0$. By Rolle's Theorem applied to f' on $[d, e]$, there is a number g such that $d < g < e$ and $f''(g) = 0$.

(c) Suppose that f is n times differentiable on $\mathbb{R}$ and has $n+1$ distinct real zeros. Then $f^{(n)}$ has at least one real zero.

29. By the Mean Value Theorem, $f(4) - f(1) = f'(c)(4-1)$ for some $c \in (1, 4)$. (f is differentiable for all x, so, in particular, f is differentiable on $(1, 4)$ and continuous on $[1, 4]$. Thus, the hypotheses of the Mean Value Theorem are satisfied.) For every $c \in (1, 4)$, we have $f'(c) \geq 2$. Putting $f'(c) \geq 2$ into the above equation and substituting $f(1) = 10$, we get $f(4) = f(1) + f'(c)(4-1) = 10 + 3f'(c) \geq 10 + 3 \cdot 2 = 16$. So the smallest possible value of $f(4)$ is 16.

30. By the Mean Value Theorem, $f(8) - f(2) = f'(c)(8-2)$ for some $c \in (2, 8)$. (f is differentiable for all x, so, in particular, f is differentiable on $(2, 8)$ and continuous on $[2, 8]$. Thus, the hypotheses of the Mean Value Theorem are satisfied.) Since $f(8) - f(2) = 6f'(c)$ and $3 \leq f'(c) \leq 5$, it follows that $6 \cdot 3 \leq 6f'(c) \leq 6 \cdot 5$, so $18 \leq f(8) - f(2) \leq 30$.

31. Suppose that such a function f exists. By the Mean Value Theorem, there is a number c such that $0 < c < 2$ with $f'(c) = \dfrac{f(2) - f(0)}{2 - 0} = \dfrac{4 - (-1)}{2 - 0} = \dfrac{5}{2}$. This result, $f'(c) = \dfrac{5}{2}$, is impossible since $f'(x) \leq 2$ for all x, so no such function f exists.

32. Let $h = f - g$. Note that since $f(a) = g(a)$, $h(a) = f(a) - g(a) = 0$. Then since f and g are continuous on $[a, b]$ and differentiable on (a, b), so is h, and thus h satisfies the assumptions of the Mean Value Theorem. Therefore, there is a number c with $a < c < b$ such that $h(b) = h(b) - h(a) = h'(c)(b-a)$. Given $f'(x) < g'(x)$, we have $f' - g' < 0$ or, equivalently, $h' < 0$. Now since $h'(c) < 0$, $h'(c)(b-a) < 0$, so $h(b) = f(b) - g(b) < 0$ and hence $f(b) < g(b)$.

33. Consider the function $f(x) = \sin x$, which is continuous and differentiable on $\mathbb{R}$. Let a be a number such that $0 < a < 2\pi$. Then f is continuous on $[0, a]$ and differentiable on $(0, a)$. By the Mean Value Theorem, there is a number c in $(0, a)$ such that $f(a) - f(0) = f'(c)(a-0)$; that is, $\sin a - 0 = (\cos c)(a)$. Now $\cos c < 1$ for $0 < c < 2\pi$, so $\sin a < 1 \cdot a = a$. We took a to be an arbitrary number in $(0, 2\pi)$, so $\sin x < x$ for all x satisfying $0 < x < 2\pi$.

34. f satisfies the conditions for the Mean Value Theorem, so we use this theorem on the interval $[-b, b]$: $f'(c) = \dfrac{f(b) - f(-b)}{b - (-b)}$ for some $c \in (-b, b)$. Since f is odd, $f(-b) = -f(b)$. Substituting this into the above equation, we get $f'(c) = \dfrac{f(b) + f(b)}{2b}$, or, equivalently, $f'(c) = \dfrac{f(b)}{b}$.

35. Let $f(x) = \sin x$ on $[a, b]$. Then f is continuous on $[a, b]$ and differentiable on (a, b). By the Mean Value Theorem, there is a number $c \in (a, b)$ with $f(b) - f(a) = f'(c)(b - a)$ or, equivalently, $\sin b - \sin a = (\cos c)(b - a)$. Taking absolute values, $|\sin b - \sin a| \le |\cos c|\,|b - a|$ or, equivalently, $|\sin a - \sin b| \le 1\,|b - a|$. If $b < a$, then $|\sin a - \sin b| \le |a - b|$. If $a = b$, both sides of the inequality are 0, which proves the given inequality for all a and b.

36. Suppose that $f'(x) = c$. Let $g(x) = cx$, so $g'(x) = c$. Then, by Corollary 7, $f(x) = g(x) + d$, where d is a constant, so $f(x) = cx + d$.

37. For $x > 0$, $f(x) = g(x)$, so $f'(x) = g'(x)$. For $x < 0$, $f'(x) = (1/x)' = -1/x^2$ and $g'(x) = (1 + 1/x)' = -1/x^2$, so again $f'(x) = g'(x)$. However, the domain of $g(x)$ is not an interval [it is $(-\infty, 0) \cup (0, \infty)$], so we cannot conclude that $f - g$ is constant (in fact, it is not).

38. Let $v(t)$ be the velocity of the car t hours after 2:00 PM, then $\dfrac{v(1/6) - v(0)}{1/6 - 0} = \dfrac{65 - 50}{1/6} = 90$. Then by the Mean Value Theorem, there exists a number c such that $0 < c < \frac{1}{6}$ such that $v'(c) = 90$. Since $v'(t)$ is the acceleration at time t, the acceleration c hours after 2:00 PM is exactly 90 km/h^2.

39. Let $g(t)$ and $h(t)$ be the position functions of the two runners and let $f(t) = g(t) - h(t)$. By hypothesis, where b is the finishing time, $f(0) = g(0) - h(0) = 0$ and $f(b) = g(b) - h(b) = 0$. Then by the Mean Value Theorem, there is a time c, with $0 < c < b$, such that $f'(c) = \dfrac{f(b) - f(0)}{b - 0} = \dfrac{0 - 0}{b} = 0$. Since $f'(c) = g'(c) - h'(c) = 0$, we have $g'(c) = h'(c)$ [the velocities are equal]. So at time c, both runners have the same speed.

40. Assume that f is differentiable (and hence continuous) on $\mathbb{R}$ and that $f'(x) \ne 1$ for all x. Suppose f has more than one fixed point. Then there are numbers a and b such that $a < b$, $f(a) = a$, and $f(b) = b$. Applying the Mean Value Theorem to the function f on $[a, b]$, we find that there is a number c in (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. But then $f'(c) = \dfrac{b - a}{b - a} = 1$, contradicting our assumption that $f'(x) \ne 1$ for every real number x. This shows that our supposition was wrong, that is, that f cannot have more than one fixed point.

3.3 What Derivatives Tell Us about the Shape of a Graph

1. (a) f is increasing on $(1, 3)$ and $(4, 6)$.

(b) f is decreasing on $(0, 1)$ and $(3, 4)$.

(c) f is concave upward on $(0, 2)$.

(d) f is concave downward on $(2, 4)$ and $(4, 6)$.

(e) The point of inflection is $(2, 3)$.

2. (a) f is increasing on $(0, 1)$ and $(3, 7)$.

(b) f is decreasing on $(1, 3)$.

(c) f is concave upward on $(2, 4)$ and $(5, 7)$.

(d) f is concave downward on $(0, 2)$ and $(4, 5)$.

(e) The points of inflection are $(2, 2)$, $(4, 3)$, and $(5, 4)$.

3. (a) Use the Increasing/Decreasing (I/D) Test.

(b) Use the Concavity Test.

(c) At any value of x where the concavity changes, we have an inflection point at $(x, f(x))$.

4. (a) See the First Derivative Test.

(b) See the Second Derivative Test and the note that precedes Example 6.

5. (a) Since $f'(x) > 0$ on $(0, 1)$ and $(3, 5)$, f is increasing on these intervals. Since $f'(x) < 0$ on $(1, 3)$ and $(5, 6)$, f is decreasing on these intervals.

(b) Since $f'(x) = 0$ at $x = 1$ and $x = 5$, and f' changes from positive to negative at both values, f changes from increasing to decreasing and has local maxima at $x = 1$ and $x = 5$. Since $f'(x) = 0$ at $x = 3$, and f' changes from negative to positive there, f changes from decreasing to increasing and has a local minimum at $x = 3$.

6. (a) Since $f'(x) > 0$ on $(1, 4)$ and $(5, 6)$, f is increasing on these intervals. Since $f'(x) < 0$ on $(0, 1)$ and $(4, 5)$, f is decreasing on these intervals.

(b) Since $f'(x) = 0$ at $x = 4$, and f' changes from positive to negative there, f changes from increasing to decreasing and has a local maximum at $x = 4$. Since $f'(x) = 0$ at $x = 1$ and $x = 5$, and f' changes from negative to positive at both values, f changes from decreasing to increasing and has local minima at $x = 1$ and $x = 5$.

7. (a) There is an IP at $x = 3$ because the graph of f changes from CD to CU there. There is an IP at $x = 5$ because the graph of f changes from CU to CD there.

(b) There is an IP at $x = 2$ and at $x = 6$ because $f'(x)$ has a maximum value there, and so $f''(x)$ changes from positive to negative there. There is an IP at $x = 4$ because $f'(x)$ has a minimum value there and so $f''(x)$ changes from negative to positive there.

(c) There is an inflection point at $x = 1$ because $f''(x)$ changes from negative to positive there, and so the graph of f changes from concave downward to concave upward. There is an inflection point at $x = 7$ because $f''(x)$ changes from positive to negative there, and so the graph of f changes from concave upward to concave downward.

8. (a) f is increasing when f' is positive. This happens on the intervals $(0, 4)$ and $(6, 8)$.

(b) f has a local maximum where it changes from increasing to decreasing, that is, where f' changes from positive to negative (at $x = 4$ and $x = 8$). Similarly, f has a local minimum where f' changes from negative to positive (at $x = 6$).

(c) f is concave upward where f' is increasing (hence f'' is positive). This happens on $(0, 1)$, $(2, 3)$, and $(5, 7)$. Similarly, f is concave downward where f' is decreasing, that is, on $(1, 2)$, $(3, 5)$, and $(7, 9)$.

(d) f has an inflection point where the concavity changes. This happens at $x = 1$, 2, 3, 5, and 7.

9. $f(x) = 2x^3 - 15x^2 + 24x - 5 \quad\Rightarrow\quad f'(x) = 6x^2 - 30x + 24 = 6(x^2 - 5x + 4) = 6(x-1)(x-4)$.

Interval	$x-1$	$x-4$	$f'(x)$	f
$x < 1$	$-$	$-$	$+$	increasing on $(-\infty, 1)$
$1 < x < 4$	$+$	$-$	$-$	decreasing on $(1, 4)$
$x > 4$	$+$	$+$	$+$	increasing on $(4, \infty)$

f changes from increasing to decreasing at $x = 1$ and from decreasing to increasing at $x = 4$. Thus, $f(1) = 6$ is a local maximum value and $f(4) = -21$ is a local minimum value.

10. $f(x) = x^3 - 6x^2 - 135x \quad\Rightarrow\quad f'(x) = 3x^2 - 12x - 135 = 3(x^2 - 4x - 45) = 3(x+5)(x-9)$.

Interval	$x+5$	$x-9$	$f'(x)$	f
$x < -5$	$-$	$-$	$+$	increasing on $(-\infty, -5)$
$-5 < x < 9$	$+$	$-$	$-$	decreasing on $(-5, 9)$
$x > 9$	$+$	$+$	$+$	increasing on $(9, \infty)$

f changes from increasing to decreasing at $x = -5$ and from decreasing to increasing at $x = 9$. Thus, $f(-5) = 400$ is a local maximum value and $f(9) = -972$ is a local minimum value.

11. $f(x) = 6x^4 - 16x^3 + 1 \quad\Rightarrow\quad f'(x) = 24x^3 - 48x^2 = 24x^2(x-2)$.

Interval	x^2	$x-2$	$f'(x)$	f
$x < 0$	$+$	$-$	$-$	decreasing on $(-\infty, 0)$
$0 < x < 2$	$+$	$-$	$-$	decreasing on $(0, 2)$
$x > 2$	$+$	$+$	$+$	increasing on $(2, \infty)$

Note that f is differentiable and $f'(x) < 0$ on the interval $(-\infty, 2)$ except for the single number $x = 0$. By applying the result of Exercise 3.3.79, we can say that f is decreasing on the entire interval $(-\infty, 2)$. f changes from decreasing to increasing at $x = 2$. Thus, $f(2) = -31$ is a local minimum value.

12. $f(x) = x^{2/3}(x-3) \quad\Rightarrow\quad f'(x) = x^{2/3}(1) + (x-3)\cdot\frac{2}{3}x^{-1/3} = \frac{1}{3}x^{-1/3}\left[3x + (x-3)\cdot 2\right] = \frac{1}{3}x^{-1/3}(5x-6)$.

Interval	$x^{-1/3}$	$5x-6$	$f'(x)$	f
$x < 0$	$-$	$-$	$+$	increasing on $(-\infty, 0)$
$0 < x < \frac{6}{5}$	$+$	$-$	$-$	decreasing on $\left(0, \frac{6}{5}\right)$
$x > \frac{6}{5}$	$+$	$+$	$+$	increasing on $\left(\frac{6}{5}, \infty\right)$

f changes from increasing to decreasing at $x = 0$ and from decreasing to increasing at $x = \frac{6}{5}$. Thus, $f(0) = 0$ is a local maximum value and $f\left(\frac{6}{5}\right) = \left(\frac{6}{5}\right)^{2/3}\left(-\frac{9}{5}\right) \approx -2.03$ is a local minimum value.

13. $f(x) = \dfrac{x^2 - 24}{x - 5} \quad \Rightarrow$

$$f'(x) = \frac{(x-5)(2x) - (x^2 - 24)(1)}{(x-5)^2} = \frac{2x^2 - 10x - x^2 + 24}{(x-5)^2} = \frac{x^2 - 10x + 24}{(x-5)^2} = \frac{(x-4)(x-6)}{(x-5)^2}.$$

Interval	$x-4$	$x-6$	$(x-5)^2$	$f'(x)$	f
$x < 4$	$-$	$-$	$+$	$+$	increasing on $(-\infty, 4)$
$4 < x < 5$	$+$	$-$	$+$	$-$	decreasing on $(4, 5)$
$5 < x < 6$	$+$	$-$	$+$	$-$	decreasing on $(5, 6)$
$x > 6$	$+$	$+$	$+$	$+$	increasing on $(6, \infty)$

$x = 5$ is not in the domain of f. f changes from increasing to decreasing at $x = 4$ and from decreasing to increasing at $x = 6$. Thus, $f(4) = 8$ is a local maximum value and $f(6) = 12$ is a local minimum value.

14. $f(x) = x + \dfrac{4}{x^2} = x + 4x^{-2} \quad \Rightarrow \quad f'(x) = 1 - 8x^{-3} = 1 - \dfrac{8}{x^3} = \dfrac{x^3 - 8}{x^3} = \dfrac{(x-2)(x^2 + 2x + 4)}{x^3}$. The factor $x^2 + 2x + 4$ is always positive and does not affect the sign of $f'(x)$.

Interval	x^3	$x-2$	$f'(x)$	f
$x < 0$	$-$	$-$	$+$	increasing on $(-\infty, 0)$
$0 < x < 2$	$+$	$-$	$-$	decreasing on $(0, 2)$
$x > 2$	$+$	$+$	$+$	increasing on $(2, \infty)$

$x = 0$ is not in the domain of f. f changes from decreasing to increasing at $x = 2$. Thus, $f(2) = 3$ is a local minimum value.

15. $f(x) = x^3 - 3x^2 - 9x + 4 \quad \Rightarrow \quad f'(x) = 3x^2 - 6x - 9 \quad \Rightarrow \quad f''(x) = 6x - 6 = 6(x - 1)$. $\quad f''(x) > 0 \quad \Leftrightarrow \quad x > 1$ and $f''(x) < 0 \quad \Leftrightarrow \quad x < 1$. Thus, f is concave upward on $(1, \infty)$ and concave downward on $(-\infty, 1)$. There is an inflection point at $(1, f(1)) = (1, -7)$.

16. $f(x) = 2x^3 - 9x^2 + 12x - 3 \quad \Rightarrow \quad f'(x) = 6x^2 - 18x + 12 \quad \Rightarrow \quad f''(x) = 12x - 18 = 12\left(x - \frac{3}{2}\right)$. $\quad f''(x) > 0 \quad \Leftrightarrow$ $x > \frac{3}{2}$ and $f''(x) < 0 \quad \Leftrightarrow \quad x < \frac{3}{2}$. Thus, f is concave upward on $\left(\frac{3}{2}, \infty\right)$ and concave downward on $\left(-\infty, \frac{3}{2}\right)$. There is an inflection point at $\left(\frac{3}{2}, \frac{3}{2}\right)$.

17. $f(x) = \sin^2 x - \cos 2x,\ 0 \le x \le \pi$. $\quad f'(x) = 2\sin x\, \cos x + 2\sin 2x = \sin 2x + 2\sin 2x = 3\sin 2x$ and $f''(x) = 6\cos 2x$. $\quad f''(x) > 0 \quad \Leftrightarrow \quad \cos 2x > 0 \quad \Leftrightarrow \quad 0 < x < \frac{\pi}{4}$ and $\frac{3\pi}{4} < x < \pi$ and $f''(x) < 0 \quad \Leftrightarrow$ $\cos 2x < 0 \quad \Leftrightarrow \quad \frac{\pi}{4} < x < \frac{3\pi}{4}$. Thus, f is concave upward on $\left(0, \frac{\pi}{4}\right)$ and $\left(\frac{3\pi}{4}, \pi\right)$ and concave downward on $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$. There are inflection points at $\left(\frac{\pi}{4}, \frac{1}{2}\right)$ and $\left(\frac{3\pi}{4}, \frac{1}{2}\right)$.

18. $f(x) = x\,\sqrt[3]{x - 4} \quad \Rightarrow$

$$\begin{aligned} f'(x) &= x \cdot \tfrac{1}{3}(x-4)^{-2/3} + (x-4)^{1/3} \cdot 1 = \tfrac{1}{3}(x-4)^{-2/3}[x + 3(x-4)] \\ &= \tfrac{1}{3}(x-4)^{-2/3}(4x - 12) = \tfrac{4}{3}(x-4)^{-2/3}(x-3) \quad \Rightarrow \end{aligned}$$

[continued]

$$f''(x) = \tfrac{4}{3}\left[(x-4)^{-2/3} \cdot 1 + (x-3)\left(-\tfrac{2}{3}\right)(x-4)^{-5/3} \cdot 1\right]$$
$$= \tfrac{4}{3} \cdot \tfrac{1}{3}(x-4)^{-5/3}[3(x-4) + (x-3)(-2)] = \tfrac{4}{9}(x-4)^{-5/3}(x-6)$$

$f''(x) < 0 \iff 4 < x < 6$ and $f''(x) > 0 \iff x < 4$ or $x > 6$. Thus, f is concave downward on $(4, 6)$ and concave upward on $(-\infty, 4)$ and $(6, \infty)$. There are inflection points at $(4, 0)$ and $\left(6, 6\sqrt[3]{2}\right)$.

19. (a) $f(x) = x^4 - 2x^2 + 3 \Rightarrow f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 4x(x+1)(x-1)$.

Interval	$x+1$	$4x$	$x-1$	$f'(x)$	f
$x < -1$	$-$	$-$	$-$	$-$	decreasing on $(-\infty, -1)$
$-1 < x < 0$	$+$	$-$	$-$	$+$	increasing on $(-1, 0)$
$0 < x < 1$	$+$	$+$	$-$	$-$	decreasing on $(0, 1)$
$x > 1$	$+$	$+$	$+$	$+$	increasing on $(1, \infty)$

(b) f changes from increasing to decreasing at $x = 0$ and from decreasing to increasing at $x = -1$ and $x = 1$. Thus, $f(0) = 3$ is a local maximum value and $f(\pm 1) = 2$ are local minimum values.

(c) $f''(x) = 12x^2 - 4 = 12\left(x^2 - \frac{1}{3}\right) = 12\left(x + 1/\sqrt{3}\right)\left(x - 1/\sqrt{3}\right)$. $f''(x) > 0 \iff x < -1/\sqrt{3}$ or $x > 1/\sqrt{3}$ and $f''(x) < 0 \iff -1/\sqrt{3} < x < 1/\sqrt{3}$. Thus, f is concave upward on $\left(-\infty, -\sqrt{3}/3\right)$ and $\left(\sqrt{3}/3, \infty\right)$ and concave downward on $\left(-\sqrt{3}/3, \sqrt{3}/3\right)$. There are inflection points at $\left(\pm\sqrt{3}/3, \frac{22}{9}\right)$.

20. (a) $f(x) = \dfrac{x}{x^2+1} \Rightarrow f'(x) = \dfrac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2} = -\dfrac{(x+1)(x-1)}{(x^2+1)^2}$. Thus, $f'(x) > 0$ if $(x+1)(x-1) < 0 \iff -1 < x < 1$, and $f'(x) < 0$ if $x < -1$ or $x > 1$. So f is increasing on $(-1, 1)$ and f is decreasing on $(-\infty, -1)$ and $(1, \infty)$.

(b) f changes from decreasing to increasing at $x = -1$ and from increasing to decreasing at $x = 1$. Thus, $f(-1) = -\frac{1}{2}$ is a local minimum value and $f(1) = \frac{1}{2}$ is a local maximum value.

(c) $f''(x) = \dfrac{(x^2+1)^2(-2x) - (1-x^2)[2(x^2+1)(2x)]}{[(x^2+1)^2]^2} = \dfrac{(x^2+1)(-2x)[(x^2+1) + 2(1-x^2)]}{(x^2+1)^4} = \dfrac{2x(x^2-3)}{(x^2+1)^3}$.

$f''(x) > 0 \iff -\sqrt{3} < x < 0$ or $x > \sqrt{3}$, and $f''(x) < 0 \iff x < -\sqrt{3}$ or $0 < x < \sqrt{3}$. Thus, f is concave upward on $\left(-\sqrt{3}, 0\right)$ and $\left(\sqrt{3}, \infty\right)$ and concave downward on $\left(-\infty, -\sqrt{3}\right)$ and $\left(0, \sqrt{3}\right)$. There are inflection points at $\left(-\sqrt{3}, -\sqrt{3}/4\right)$, $(0, 0)$, and $\left(\sqrt{3}, \sqrt{3}/4\right)$.

21. (a) $f(x) = \sin x + \cos x,\ 0 \le x \le 2\pi$. $f'(x) = \cos x - \sin x = 0 \Rightarrow \cos x = \sin x \Rightarrow 1 = \dfrac{\sin x}{\cos x} \Rightarrow$ $\tan x = 1 \Rightarrow x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$. Thus, $f'(x) > 0 \iff \cos x - \sin x > 0 \iff \cos x > \sin x \iff 0 < x < \frac{\pi}{4}$ or $\frac{5\pi}{4} < x < 2\pi$ and $f'(x) < 0 \iff \cos x < \sin x \iff \frac{\pi}{4} < x < \frac{5\pi}{4}$. So f is increasing on $\left(0, \frac{\pi}{4}\right)$ and $\left(\frac{5\pi}{4}, 2\pi\right)$ and f is decreasing on $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$.

(b) f changes from increasing to decreasing at $x = \frac{\pi}{4}$ and from decreasing to increasing at $x = \frac{5\pi}{4}$. Thus, $f\left(\frac{\pi}{4}\right) = \sqrt{2}$ is a local maximum value and $f\left(\frac{5\pi}{4}\right) = -\sqrt{2}$ is a local minimum value.

(c) $f''(x) = -\sin x - \cos x = 0 \quad\Rightarrow\quad -\sin x = \cos x \quad\Rightarrow\quad \tan x = -1 \quad\Rightarrow\quad x = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$. Divide the interval $(0, 2\pi)$ into subintervals with these numbers as endpoints and complete a second derivative chart.

Interval	$f''(x) = -\sin x - \cos x$	Concavity
$\left(0, \frac{3\pi}{4}\right)$	$f''\left(\frac{\pi}{2}\right) = -1 < 0$	downward
$\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$	$f''(\pi) = 1 > 0$	upward
$\left(\frac{7\pi}{4}, 2\pi\right)$	$f''\left(\frac{11\pi}{6}\right) = \frac{1}{2} - \frac{1}{2}\sqrt{3} < 0$	downward

There are inflection points at $\left(\frac{3\pi}{4}, 0\right)$ and $\left(\frac{7\pi}{4}, 0\right)$.

22. (a) $f(x) = \cos^2 x - 2\sin x,\ \ 0 \le x \le 2\pi.\quad f'(x) = -2\cos x\,\sin x - 2\cos x = -2\cos x\,(1 + \sin x)$. Note that $1 + \sin x \ge 0\quad$ [since $\sin x \ge -1$], with equality $\quad\Leftrightarrow\quad \sin x = -1 \quad\Leftrightarrow\quad x = \frac{3\pi}{2}\quad$ [since $0 \le x \le 2\pi$] $\quad\Rightarrow$ $\cos x = 0$. Thus, $f'(x) > 0 \quad\Leftrightarrow\quad \cos x < 0 \quad\Leftrightarrow\quad \frac{\pi}{2} < x < \frac{3\pi}{2}$ and $f'(x) < 0 \quad\Leftrightarrow\quad \cos x > 0 \quad\Leftrightarrow\quad 0 < x < \frac{\pi}{2}$ or $\frac{3\pi}{2} < x < 2\pi$. Thus, f is increasing on $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ and f is decreasing on $\left(0, \frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2}, 2\pi\right)$.

(b) f changes from decreasing to increasing at $x = \frac{\pi}{2}$ and from increasing to decreasing at $x = \frac{3\pi}{2}$. Thus, $f\left(\frac{\pi}{2}\right) = -2$ is a local minimum value and $f\left(\frac{3\pi}{2}\right) = 2$ is a local maximum value.

(c) $f''(x) = 2\sin x\,(1 + \sin x) - 2\cos^2 x = 2\sin x + 2\sin^2 x - 2(1 - \sin^2 x)$

$= 4\sin^2 x + 2\sin x - 2 = 2(2\sin x - 1)(\sin x + 1)$

so $f''(x) > 0 \quad\Leftrightarrow\quad \sin x > \frac{1}{2} \quad\Leftrightarrow\quad \frac{\pi}{6} < x < \frac{5\pi}{6}$, and $f''(x) < 0 \quad\Leftrightarrow\quad \sin x < \frac{1}{2}$ and $\sin x \ne -1 \quad\Leftrightarrow$ $0 < x < \frac{\pi}{6}$ or $\frac{5\pi}{6} < x < \frac{3\pi}{2}$ or $\frac{3\pi}{2} < x < 2\pi$. Thus, f is concave upward on $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$ and concave downward on $\left(0, \frac{\pi}{6}\right)$, $\left(\frac{5\pi}{6}, \frac{3\pi}{2}\right)$, and $\left(\frac{3\pi}{2}, 2\pi\right)$. There are inflection points at $\left(\frac{\pi}{6}, -\frac{1}{4}\right)$ and $\left(\frac{5\pi}{6}, -\frac{1}{4}\right)$.

23. $f(x) = 1 + 3x^2 - 2x^3 \quad\Rightarrow\quad f'(x) = 6x - 6x^2 = 6x(1 - x)$.

First Derivative Test: $f'(x) > 0 \quad\Rightarrow\quad 0 < x < 1$ and $f'(x) < 0 \quad\Rightarrow\quad x < 0$ or $x > 1$. Since f' changes from negative to positive at $x = 0$, $f(0) = 1$ is a local minimum value; and since f' changes from positive to negative at $x = 1$, $f(1) = 2$ is a local maximum value.

Second Derivative Test: $f''(x) = 6 - 12x$. $f'(x) = 0 \quad\Leftrightarrow\quad x = 0, 1$. $f''(0) = 6 > 0 \quad\Rightarrow\quad f(0) = 1$ is a local minimum value. $f''(1) = -6 < 0 \quad\Rightarrow\quad f(1) = 2$ is a local maximum value.

Preference: For this function, the two tests are equally easy.

24. $f(x) = \dfrac{x^2}{x - 1} \quad\Rightarrow\quad f'(x) = \dfrac{(x - 1)(2x) - x^2(1)}{(x - 1)^2} = \dfrac{x^2 - 2x}{(x - 1)^2} = \dfrac{x(x - 2)}{(x - 1)^2}$.

First Derivative Test: $f'(x) > 0 \quad\Rightarrow\quad x < 0$ or $x > 2$ and $f'(x) < 0 \quad\Rightarrow\quad 0 < x < 1$ or $1 < x < 2$. Since f' changes from positive to negative at $x = 0$, $f(0) = 0$ is a local maximum value; and since f' changes from negative to positive at $x = 2$, $f(2) = 4$ is a local minimum value.

Second Derivative Test:

$$f''(x) = \frac{(x-1)^2(2x-2)-(x^2-2x)2(x-1)}{[(x-1)^2]^2} = \frac{2(x-1)[(x-1)^2-(x^2-2x)]}{(x-1)^4} = \frac{2}{(x-1)^3}.$$

$f'(x) = 0 \quad \Leftrightarrow \quad x = 0, 2.$ $f''(0) = -2 < 0 \quad \Rightarrow \quad f(0) = 0$ is a local maximum value. $f''(2) = 2 > 0 \quad \Rightarrow \quad f(2) = 4$ is a local minimum value.

Preference: Since calculating the second derivative is fairly difficult, the First Derivative Test is easier to use for this function.

25. $f'(x) = (x-4)^2(x+3)^7(x-5)^8$. The factors $(x-4)^2$ and $(x-5)^8$ are nonnegative. Hence, the sign of f' is determined by the sign of $(x+3)^7$, which is positive for $x > -3$. Thus, f increases on the intervals $(-3,4)$, $(4,5)$, and $(5,\infty)$. Note that f is differentiable and $f'(x) > 0$ on the interval $(-3,\infty)$ except for the numbers $x = 4$ and $x = 5$. By applying the result of Exercise 3.3.79, we can say that f is increasing on the entire interval $(-3,\infty)$.

26. (a) $f(x) = x^4(x-1)^3 \quad \Rightarrow \quad f'(x) = x^4 \cdot 3(x-1)^2 + (x-1)^3 \cdot 4x^3 = x^3(x-1)^2\,[3x + 4(x-1)] = x^3(x-1)^2(7x-4)$

The critical numbers are 0, 1, and $\frac{4}{7}$.

(b) $$\begin{aligned} f''(x) &= 3x^2(x-1)^2(7x-4) + x^3 \cdot 2(x-1)(7x-4) + x^3(x-1)^2 \cdot 7 \\ &= x^2(x-1)\,[3(x-1)(7x-4) + 2x(7x-4) + 7x(x-1)] \end{aligned}$$

Now $f''(0) = f''(1) = 0$, so the Second Derivative Test gives no information for $x = 0$ or $x = 1$.

$f''\left(\frac{4}{7}\right) = \left(\frac{4}{7}\right)^2\left(\frac{4}{7}-1\right)\left[0 + 0 + 7\left(\frac{4}{7}\right)\left(\frac{4}{7}-1\right)\right] = \left(\frac{4}{7}\right)^2\left(-\frac{3}{7}\right)(4)\left(-\frac{3}{7}\right) > 0$, so there is a local minimum at $x = \frac{4}{7}$.

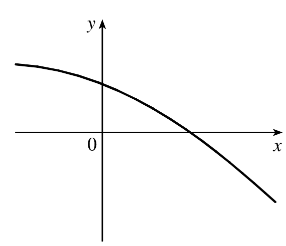
(c) f' is positive on $(-\infty,0)$, negative on $\left(0,\frac{4}{7}\right)$, positive on $\left(\frac{4}{7},1\right)$, and positive on $(1,\infty)$. So f has a local maximum at $x = 0$, a local minimum at $x = \frac{4}{7}$, and no local maximum or minimum at $x = 1$.

27. (a) By the Second Derivative Test, if $f'(2) = 0$ and $f''(2) = -5 < 0$, f has a local maximum at $x = 2$.

(b) If $f'(6) = 0$, we know that f has a horizontal tangent at $x = 6$. Knowing that $f''(6) = 0$ does not provide any additional information since the Second Derivative Test fails. For example, the first and second derivatives of $y = (x-6)^4$, $y = -(x-6)^4$, and $y = (x-6)^3$ all equal zero for $x = 6$, but the first has a local minimum at $x = 6$, the second has a local maximum at $x = 6$, and the third has an inflection point at $x = 6$.

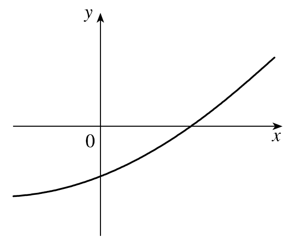
28. (a) $f'(x) < 0$ and $f''(x) < 0$ for all x

The function must be always decreasing (since the first derivative is always negative) and concave downward (since the second derivative is always negative).



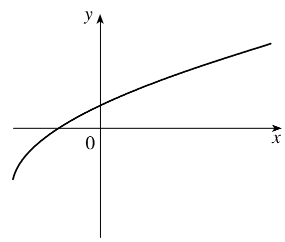
(b) $f'(x) > 0$ and $f''(x) > 0$ for all x

The function must be always increasing (since the first derivative is always positive) and concave upward (since the second derivative is always positive).



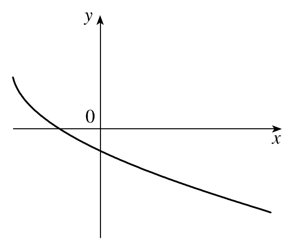
29. (a) $f'(x) > 0$ and $f''(x) < 0$ for all x

The function must be always increasing (since the first derivative is always positive) and concave downward (since the second derivative is always negative).



(b) $f'(x) < 0$ and $f''(x) > 0$ for all x

The function must be always decreasing (since the first derivative is always negative) and concave upward (since the second derivative is always positive).



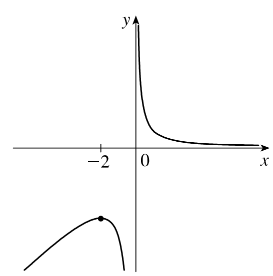
30. Vertical asymptote $x = 0$

$f'(x) > 0$ if $x < -2 \quad \Rightarrow \quad f$ is increasing on $(-\infty, -2)$.

$f'(x) < 0$ if $x > -2$ $(x \neq 0) \quad \Rightarrow \quad f$ is decreasing on $(-2, 0)$ and $(0, \infty)$.

$f''(x) < 0$ if $x < 0 \quad \Rightarrow \quad f$ is concave downward on $(-\infty, 0)$.

$f''(x) > 0$ if $x > 0 \quad \Rightarrow \quad f$ is concave upward on $(0, \infty)$.



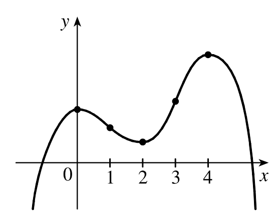
31. $f'(0) = f'(2) = f'(4) = 0 \quad \Rightarrow \quad$ horizontal tangents at $x = 0, 2, 4$.

$f'(x) > 0$ if $x < 0$ or $2 < x < 4 \quad \Rightarrow \quad f$ is increasing on $(-\infty, 0)$ and $(2, 4)$.

$f'(x) < 0$ if $0 < x < 2$ or $x > 4 \quad \Rightarrow \quad f$ is decreasing on $(0, 2)$ and $(4, \infty)$.

$f''(x) > 0$ if $1 < x < 3 \quad \Rightarrow \quad f$ is concave upward on $(1, 3)$.

$f''(x) < 0$ if $x < 1$ or $x > 3 \quad \Rightarrow \quad f$ is concave downward on $(-\infty, 1)$ and $(3, \infty)$. There are inflection points when $x = 1$ and 3.



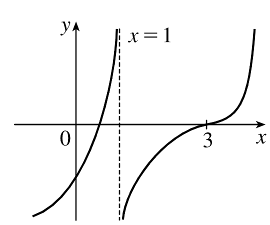
32. $f'(x) > 0$ for all $x \neq 1 \quad \Rightarrow \quad f$ is increasing on $(-\infty, 1)$ and $(1, \infty)$.

Vertical asymptote $x = 1$

$f''(x) > 0$ if $x < 1$ or $x > 3 \quad \Rightarrow \quad f$ is concave upward on $(-\infty, 1)$ and $(3, \infty)$.

$f''(x) < 0$ if $1 < x < 3 \quad \Rightarrow \quad f$ is concave downward on $(1, 3)$.

There is an inflection point at $x = 3$.

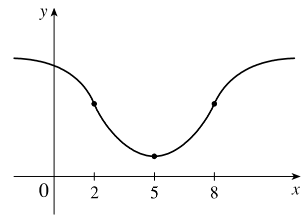


33. $f'(5) = 0 \quad \Rightarrow \quad$ horizontal tangent at $x = 5$.

$f'(x) < 0$ when $x < 5 \quad \Rightarrow \quad f$ is decreasing on $(-\infty, 5)$.

$f'(x) > 0$ when $x > 5 \quad \Rightarrow \quad f$ is increasing on $(5, \infty)$.

$f''(2) = 0, \quad f''(8) = 0, \quad f''(x) < 0$ when $x < 2$ or $x > 8$,



$f''(x) > 0$ for $2 < x < 8 \quad \Rightarrow \quad f$ is concave upward on $(2, 8)$ and concave downward on $(-\infty, 2)$ and $(8, \infty)$.

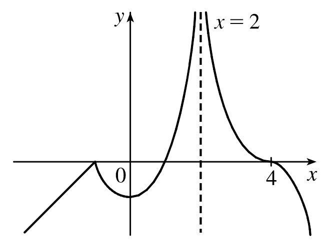
There are inflection points at $x = 2$ and $x = 8$.

34. $f'(0) = f'(4) = 0 \quad \Rightarrow \quad$ horizontal tangents at $x = 0$ and 4.

$f'(x) = 1$ if $x < -1 \quad \Rightarrow \quad f$ is a line with slope 1 on $(-\infty, -1)$.

$f'(x) > 0$ if $0 < x < 2 \quad \Rightarrow \quad f$ is increasing on $(0, 2)$.

$f'(x) < 0$ if $-1 < x < 0$ or $2 < x < 4$ or $x > 4 \quad \Rightarrow \quad f$ is decreasing on $(-1, 0)$, $(2, 4)$, and $(4, \infty)$.



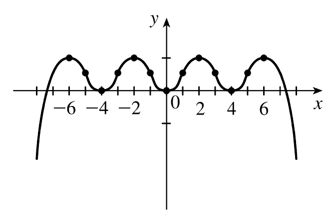
$\lim_{x \to 2^-} f'(x) = \infty \quad \Rightarrow \quad f'$ increases without bound as $x \to 2^-$.

$\lim_{x \to 2^+} f'(x) = -\infty \quad \Rightarrow \quad f'$ decreases without bound as $x \to 2^+$.

$f''(x) > 0$ if $-1 < x < 2$ or $2 < x < 4 \quad \Rightarrow \quad f$ is concave upward on $(-1, 2)$ and $(2, 4)$.

$f''(x) < 0$ if $x > 4 \quad \Rightarrow \quad f$ is concave downward on $(4, \infty)$.

35. $f(0) = f'(0) = 0 \quad \Rightarrow \quad$ the graph of f passes through the origin and has a horizontal tangent there. $\quad f'(2) = f'(4) = f'(6) = 0 \quad \Rightarrow \quad$ horizontal tangents at $x = 2, 4, 6$. $\quad f'(x) > 0$ if $0 < x < 2$ or $4 < x < 6 \quad \Rightarrow \quad f$ increasing on $(0, 2)$ and $(4, 6)$. $\quad f'(x) < 0$ if $2 < x < 4$ or $x > 6 \quad \Rightarrow \quad f$ decreasing on $(2, 4)$ and $(6, \infty)$. $f''(x) > 0$ if $0 < x < 1$ or $3 < x < 5 \quad \Rightarrow \quad f$ is CU on $(0, 1)$ and $(3, 5)$.



$f''(x) < 0$ if $1 < x < 3$ or $x > 5 \quad \Rightarrow \quad f$ is CD on $(1, 3)$ and $(5, \infty)$. $\quad f(-x) = f(x) \quad \Rightarrow \quad f$ is even and the graph is symmetric about the y-axis.

36. (a) $\dfrac{dy}{dx} > 0$ (f is increasing) and $\dfrac{d^2y}{dx^2} > 0$ (f is concave upward) at point B.

(b) $\dfrac{dy}{dx} < 0$ (f is decreasing) and $\dfrac{d^2y}{dx^2} < 0$ (f is concave downward) at point E.

(c) $\dfrac{dy}{dx} < 0$ (f is decreasing) and $\dfrac{d^2y}{dx^2} > 0$ (f is concave upward) at point A.

Note: At C, $\dfrac{dy}{dx} > 0$ and $\dfrac{d^2y}{dx^2} < 0$. At D, $\dfrac{dy}{dx} = 0$ and $\dfrac{d^2y}{dx^2} \leq 0$.

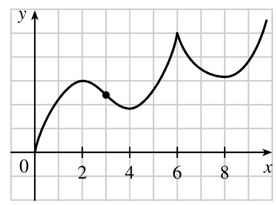
37. (a) f is increasing where f' is positive, that is, on $(0, 2)$, $(4, 6)$, and $(8, \infty)$; and decreasing where f' is negative, that is, on $(2, 4)$ and $(6, 8)$.

(b) f has local maxima where f' changes from positive to negative, at $x = 2$ and at $x = 6$, and local minima where f' changes from negative to positive, at $x = 4$ and at $x = 8$.

(c) f is concave upward (CU) where f' is increasing, that is, on $(3, 6)$ and $(6, \infty)$, and concave downward (CD) where f' is decreasing, that is, on $(0, 3)$.

(d) There is a point of inflection where f changes from being CD to being CU, that is, at $x = 3$.

(e)



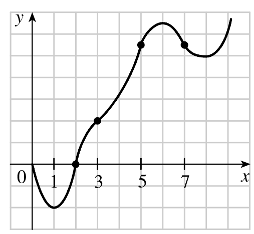
38. (a) f is increasing where f' is positive, on $(1, 6)$ and $(8, \infty)$, and decreasing where f' is negative, on $(0, 1)$ and $(6, 8)$.

(b) f has a local maximum where f' changes from positive to negative, at $x = 6$, and local minima where f' changes from negative to positive, at $x = 1$ and at $x = 8$.

(c) f is concave upward where f' is increasing, that is, on $(0, 2)$, $(3, 5)$, and $(7, \infty)$, and concave downward where f' is decreasing, that is, on $(2, 3)$ and $(5, 7)$.

(d) There are points of inflection where f changes its direction of concavity, at $x = 2$, $x = 3$, $x = 5$ and $x = 7$.

(e)



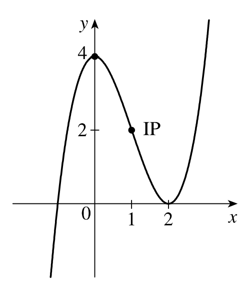
39. (a) $f(x) = x^3 - 3x^2 + 4 \quad \Rightarrow \quad f'(x) = 3x^2 - 6x = 3x(x - 2)$.

Interval	$3x$	$x - 2$	$f'(x)$	f
$x < 0$	$-$	$-$	$+$	increasing on $(-\infty, 0)$
$0 < x < 2$	$+$	$-$	$-$	decreasing on $(0, 2)$
$x > 2$	$+$	$+$	$+$	increasing on $(2, \infty)$

(b) f changes from increasing to decreasing at $x = 0$ and from decreasing to increasing at $x = 2$. Thus, $f(0) = 4$ is a local maximum value and $f(2) = 0$ is a local minimum value.

(c) $f''(x) = 6x - 6 = 6(x - 1)$. $\quad f''(x) = 0 \quad \Leftrightarrow \quad x = 1$. $\quad f''(x) > 0$ on $(1, \infty)$ and $f''(x) < 0$ on $(-\infty, 1)$. So f is concave upward on $(1, \infty)$ and f is concave downward on $(-\infty, 1)$. There is an inflection point at $(1, 2)$.

(d)

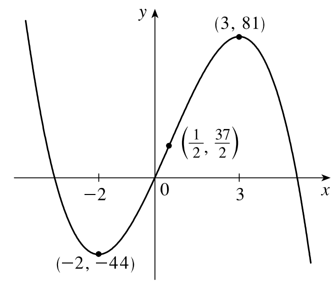


40. (a) $f(x) = 36x + 3x^2 - 2x^3 \quad \Rightarrow \quad f'(x) = 36 + 6x - 6x^2 = -6(x^2 - x - 6) = -6(x+2)(x-3)$. $f'(x) > 0 \quad \Leftrightarrow$ $-2 < x < 3$ and $f'(x) < 0 \quad \Leftrightarrow \quad x < -2$ or $x > 3$. So f is increasing on $(-2, 3)$ and f is decreasing on $(-\infty, -2)$ and $(3, \infty)$.

(b) f changes from increasing to decreasing at $x = 3$, so $f(3) = 81$ is a local maximum value. f changes from decreasing to increasing at $x = -2$, so $f(-2) = -44$ is a local minimum value.

(c) $f''(x) = 6 - 12x$. $f''(x) = 0 \quad \Leftrightarrow \quad x = \frac{1}{2}$. $f''(x) > 0$ on $\left(-\infty, \frac{1}{2}\right)$ and $f''(x) < 0$ on $\left(\frac{1}{2}, \infty\right)$. So f is CU on $\left(-\infty, \frac{1}{2}\right)$ and f is CD on $\left(\frac{1}{2}, \infty\right)$. There is an inflection point at $\left(\frac{1}{2}, \frac{37}{2}\right)$.

(d)

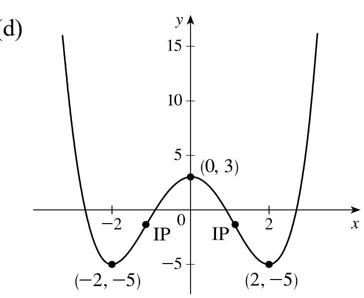


41. (a) $f(x) = \frac{1}{2}x^4 - 4x^2 + 3 \quad \Rightarrow \quad f'(x) = 2x^3 - 8x = 2x(x^2 - 4) = 2x(x+2)(x-2)$. $f'(x) > 0 \quad \Leftrightarrow \quad -2 < x < 0$ or $x > 2$, and $f'(x) < 0 \quad \Leftrightarrow \quad x < -2$ or $0 < x < 2$. So f is increasing on $(-2, 0)$ and $(2, \infty)$ and f is decreasing on $(-\infty, -2)$ and $(0, 2)$.

(b) f changes from increasing to decreasing at $x = 0$, so $f(0) = 3$ is a local maximum value.
f changes from decreasing to increasing at $x = \pm 2$, so $f(\pm 2) = -5$ is a local minimum value.

(c) $f''(x) = 6x^2 - 8 = 6\left(x^2 - \frac{4}{3}\right) = 6\left(x + \frac{2}{\sqrt{3}}\right)\left(x - \frac{2}{\sqrt{3}}\right)$.
$f''(x) = 0 \quad \Leftrightarrow \quad x = \pm\frac{2}{\sqrt{3}}$. $f''(x) > 0$ on $\left(-\infty, -\frac{2}{\sqrt{3}}\right)$ and $\left(\frac{2}{\sqrt{3}}, \infty\right)$ and $f''(x) < 0$ on $\left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$. So f is CU on $\left(-\infty, -\frac{2}{\sqrt{3}}\right)$ and $\left(\frac{2}{\sqrt{3}}, \infty\right)$, and f is CD on $\left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$. There are inflection points at $\left(\pm\frac{2}{\sqrt{3}}, -\frac{13}{9}\right)$.

(d)

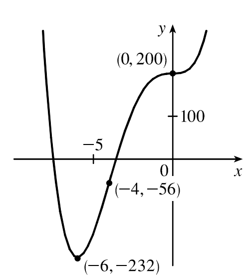


42. (a) $g(x) = 200 + 8x^3 + x^4 \quad \Rightarrow \quad g'(x) = 24x^2 + 4x^3 = 4x^2(6 + x) = 0$ when $x = -6$ and when $x = 0$.
$g'(x) > 0 \quad \Leftrightarrow \quad x > -6 \quad [x \neq 0]$ and $g'(x) < 0 \quad \Leftrightarrow \quad x < -6$, so g is decreasing on $(-\infty, -6)$ and g is increasing on $(-6, \infty)$, with a horizontal tangent at $x = 0$.

(b) $g(-6) = -232$ is a local minimum value. There is no local maximum value.

(c) $g''(x) = 48x + 12x^2 = 12x(4 + x) = 0$ when $x = -4$ and when $x = 0$.
$g''(x) > 0 \quad \Leftrightarrow \quad x < -4$ or $x > 0$ and $g''(x) < 0 \quad \Leftrightarrow \quad -4 < x < 0$, so g is CU on $(-\infty, -4)$ and $(0, \infty)$, and g is CD on $(-4, 0)$. There are inflection points at $(-4, -56)$ and $(0, 200)$.

(d)



43. (a) $g(t) = 3t^4 - 8t^3 + 12 \quad\Rightarrow\quad g'(t) = 12t^3 - 24t^2 = 12t^2(t-2)$.

Interval	$12t^2$	$t-2$	$g'(t)$	g
$t < 0$	$+$	$-$	$-$	decreasing on $(-\infty, 0)$
$0 < t < 2$	$+$	$-$	$-$	decreasing on $(0, 2)$
$t > 2$	$+$	$+$	$+$	increasing on $(2, \infty)$

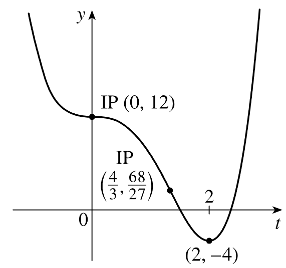
(b) g changes from decreasing to increasing at $x = 2$. Thus, $g(2) = -4$ is a local minimum value.

(c) $g''(t) = 36t^2 - 48t = 12t(3t-4)$. $\quad g''(t) = 0 \quad\Leftrightarrow\quad t = 0$ or $t = \frac{4}{3}$.

Interval	$12t$	$3t-4$	$g''(t)$	g
$t < 0$	$-$	$-$	$+$	concave up on $(-\infty, 0)$
$0 < t < \frac{4}{3}$	$+$	$-$	$-$	concave down on $\left(0, \frac{4}{3}\right)$
$t > \frac{4}{3}$	$+$	$+$	$+$	concave up on $\left(\frac{4}{3}, \infty\right)$

There are inflection points at $(0, 12)$ and $\left(\frac{4}{3}, \frac{68}{27}\right)$.

(d)

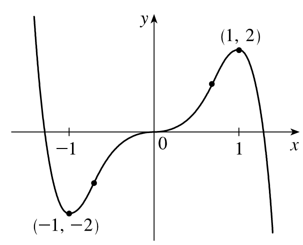


44. (a) $h(x) = 5x^3 - 3x^5 \quad\Rightarrow\quad h'(x) = 15x^2 - 15x^4 = 15x^2(1 - x^2) = 15x^2(1+x)(1-x)$. $\quad h'(x) > 0 \quad\Leftrightarrow$ $-1 < x < 0$ and $0 < x < 1$ [note that $h'(0) = 0$] and $h'(x) < 0 \quad\Leftrightarrow\quad x < -1$ or $x > 1$. So h is increasing on $(-1, 1)$ and h is decreasing on $(-\infty, -1)$ and $(1, \infty)$.

(b) h changes from decreasing to increasing at $x = -1$, so $h(-1) = -2$ is a local minimum value. h changes from increasing to decreasing at $x = 1$, so $h(1) = 2$ is a local maximum value.

(c) $h''(x) = 30x - 60x^3 = 30x(1 - 2x^2)$. $\quad h''(x) = 0 \quad\Leftrightarrow\quad x = 0$ or $1 - 2x^2 = 0 \quad\Leftrightarrow\quad x = 0$ or $x = \pm 1/\sqrt{2}$. $h''(x) > 0$ on $\left(-\infty, -1/\sqrt{2}\right)$ and $\left(0, 1/\sqrt{2}\right)$, and $h''(x) < 0$ on $\left(-1/\sqrt{2}, 0\right)$ and $\left(1/\sqrt{2}, \infty\right)$. So h is CU on $\left(-\infty, -1/\sqrt{2}\right)$ and $\left(0, 1/\sqrt{2}\right)$, and h is CD on $\left(-1/\sqrt{2}, 0\right)$ and $\left(1/\sqrt{2}, \infty\right)$. There are inflection points at $\left(-1/\sqrt{2}, -7/(4\sqrt{2})\right)$, $(0, 0)$, and $\left(1/\sqrt{2}, 7/(4\sqrt{2})\right)$.

(d)



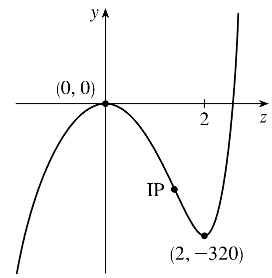
45. (a) $f(z) = z^7 - 112z^2 \quad\Rightarrow\quad f'(z) = 7z^6 - 224z = 7z(z^5 - 32)$. $\;f'(z) = 0 \quad\Rightarrow\quad z = 0, 2$.

Interval	$7z$	$z^5 - 32$	$f'(z)$	f
$z < 0$	$-$	$-$	$+$	increasing on $(-\infty, 0)$
$0 < z < 2$	$+$	$-$	$-$	decreasing on $(0, 2)$
$z > 2$	$+$	$+$	$+$	increasing on $(2, \infty)$

(b) f changes from increasing to decreasing at $x = 0$ and from decreasing to increasing at $x = 2$. Thus, $f(0) = 0$ is a local maximum value and $f(2) = -320$ is a local minimum value.

(c) $f''(z) = 42z^5 - 224 = 14\left(3z^5 - 16\right)$. $f''(z) = 0 \Leftrightarrow 3z^5 = 16 \Leftrightarrow$ $z^5 = \frac{16}{3} \Leftrightarrow z = \sqrt[5]{\frac{16}{3}}$ [call this value a]. $f''(z) > 0 \Leftrightarrow z > a$ and $f''(z) < 0 \Leftrightarrow z < a$. So, f is concave up on (a, ∞) and concave down on $(-\infty, a)$. There is an inflection point at $(a, f(a)) = \left(\sqrt[5]{\frac{16}{3}}, -\frac{320}{3}\sqrt[5]{\frac{256}{9}}\right) \approx (1.398, -208.4)$.

(d)



46. (a) $f(x) = \left(x^2 - 4\right)^3 \Rightarrow f'(x) = 3(x^2 - 4)^2(2x) = 6x(x^2 - 4)^2$. Since $(x^2 - 4)^2$ is nonnegative, the sign of $f'(x)$ is determined by the sign of $6x$. Thus, $f'(x) < 0 \Leftrightarrow x < 0$ $[x \neq -2]$ and $f'(x) > 0 \Leftrightarrow x > 0$ $[x \neq 2]$. So f is increasing on $(0, 2)$ and $(2, \infty)$, and f is decreasing on $(-\infty, -2)$ and $(-2, 0)$. By Exercise 3.3.79, we can say that f is increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$.

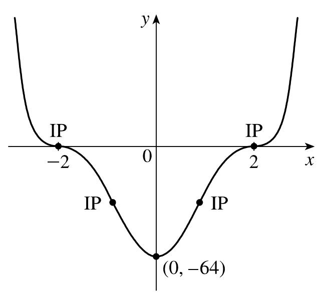
(b) f changes from decreasing to increasing at $x = 0$. Thus, $f(0) = -64$ is a local minimum value.

(c) $f''(x) = 6x \cdot 2(x^2 - 4)(2x) + (x^2 - 4)^2 \cdot 6 = 6(x^2 - 4)[4x^2 + (x^2 - 4)]$
$= 6(x^2 - 4)(5x^2 - 4) = 6(x + 2)(x - 2)\left(\sqrt{5}\,x + 2\right)\left(\sqrt{5}\,x - 2\right)$

Interval	$x+2$	$\sqrt{5}\,x+2$	$\sqrt{5}\,x-2$	$x-2$	$f''(x)$	f
$x < -2$	$-$	$-$	$-$	$-$	$+$	concave up on $(-\infty, -2)$
$-2 < x < -\frac{2}{\sqrt{5}}$	$+$	$-$	$-$	$-$	$-$	concave down on $\left(-2, -\frac{2}{\sqrt{5}}\right)$
$-\frac{2}{\sqrt{5}} < x < \frac{2}{\sqrt{5}}$	$+$	$+$	$-$	$-$	$+$	concave up on $\left(-\frac{2}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$
$\frac{2}{\sqrt{5}} < x < 2$	$+$	$+$	$+$	$-$	$-$	concave down on $\left(\frac{2}{\sqrt{5}}, 2\right)$
$x > 2$	$+$	$+$	$+$	$+$	$+$	concave up on $(2, \infty)$

There are inflection points at $(-2, 0)$, $\left(-\frac{2}{\sqrt{5}}, -\frac{4096}{125}\right)$, $\left(\frac{2}{\sqrt{5}}, -\frac{4096}{125}\right)$, and $(2, 0)$.

(d)



47. (a) $F(x) = x\sqrt{6 - x} \Rightarrow$

$F'(x) = x \cdot \frac{1}{2}(6 - x)^{-1/2}(-1) + (6 - x)^{1/2}(1) = \frac{1}{2}(6 - x)^{-1/2}[-x + 2(6 - x)] = \dfrac{-3x + 12}{2\sqrt{6 - x}}$.

$F'(x) > 0 \Leftrightarrow -3x + 12 > 0 \Leftrightarrow x < 4$ and $F'(x) < 0 \Leftrightarrow 4 < x < 6$. So F is increasing on $(-\infty, 4)$ and F is decreasing on $(4, 6)$.

(b) F changes from increasing to decreasing at $x = 4$, so $F(4) = 4\sqrt{2}$ is a local maximum value. There is no local minimum value.

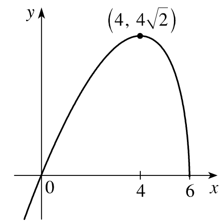
(c) $F'(x) = -\frac{3}{2}(x-4)(6-x)^{-1/2} \quad \Rightarrow$

$$F''(x) = -\tfrac{3}{2}\left[(x-4)\left(-\tfrac{1}{2}(6-x)^{-3/2}(-1)\right) + (6-x)^{-1/2}(1)\right]$$

$$= -\frac{3}{2}\cdot\frac{1}{2}(6-x)^{-3/2}[(x-4)+2(6-x)] = \frac{3(x-8)}{4(6-x)^{3/2}}$$

$F''(x) < 0$ on $(-\infty, 6)$, so F is CD on $(-\infty, 6)$. There is no inflection point.

(d)



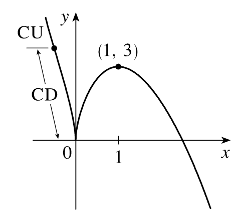
48. (a) $G(x) = 5x^{2/3} - 2x^{5/3} \quad \Rightarrow \quad G'(x) = \frac{10}{3}x^{-1/3} - \frac{10}{3}x^{2/3} = \frac{10}{3}x^{-1/3}(1-x) = \dfrac{10(1-x)}{3x^{1/3}}$.

$G'(x) > 0 \quad \Leftrightarrow \quad 0 < x < 1$ and $G'(x) < 0 \quad \Leftrightarrow \quad x < 0$ or $x > 1$. So G is increasing on $(0,1)$ and G is decreasing on $(-\infty, 0)$ and $(1, \infty)$.

(b) G changes from decreasing to increasing at $x = 0$, so $G(0) = 0$ is a local minimum value. G changes from increasing to decreasing at $x = 1$, so $G(1) = 3$ is a local maximum value. Note that the First Derivative Test applies at $x = 0$ even though G' is not defined at $x = 0$, since G is continuous at 0.

(c) $G''(x) = -\frac{10}{9}x^{-4/3} - \frac{20}{9}x^{-1/3} = -\frac{10}{9}x^{-4/3}(1+2x)$. $\quad G''(x) > 0 \quad \Leftrightarrow$ $x < -\frac{1}{2}$ and $G''(x) < 0 \quad \Leftrightarrow \quad -\frac{1}{2} < x < 0$ or $x > 0$. So G is CU on $\left(-\infty, -\frac{1}{2}\right)$ and G is CD on $\left(-\frac{1}{2}, 0\right)$ and $(0, \infty)$. The only change in concavity occurs at $x = -\frac{1}{2}$, so there is an inflection point at $\left(-\frac{1}{2}, 6/\sqrt[3]{4}\right)$.

(d)



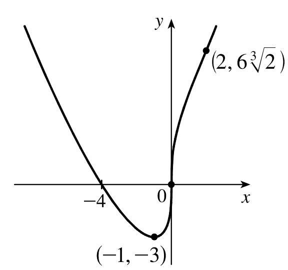
49. (a) $C(x) = x^{1/3}(x+4) = x^{4/3} + 4x^{1/3} \quad \Rightarrow \quad C'(x) = \frac{4}{3}x^{1/3} + \frac{4}{3}x^{-2/3} = \frac{4}{3}x^{-2/3}(x+1) = \dfrac{4(x+1)}{3\sqrt[3]{x^2}}$. $C'(x) > 0$ if $-1 < x < 0$ or $x > 0$ and $C'(x) < 0$ for $x < -1$, so C is increasing on $(-1, \infty)$ and C is decreasing on $(-\infty, -1)$.

(b) $C(-1) = -3$ is a local minimum value.

(c) $C''(x) = \frac{4}{9}x^{-2/3} - \frac{8}{9}x^{-5/3} = \frac{4}{9}x^{-5/3}(x-2) = \dfrac{4(x-2)}{9\sqrt[3]{x^5}}$.

$C''(x) < 0$ for $0 < x < 2$ and $C''(x) > 0$ for $x < 0$ and $x > 2$, so C is concave downward on $(0, 2)$ and concave upward on $(-\infty, 0)$ and $(2, \infty)$. There are inflection points at $(0, 0)$ and $\left(2, 6\sqrt[3]{2}\right) \approx (2, 7.56)$.

(d)

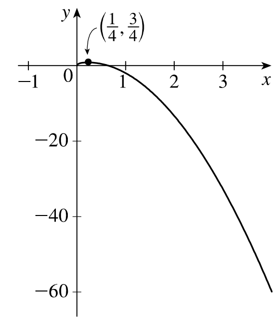


50. (a) $f(x) = 2\sqrt{x} - 4x^2 = 2x^{1/2} - 4x^2 \quad \Rightarrow \quad f'(x) = x^{-1/2} - 8x = \dfrac{1-8^{3/2}}{\sqrt{x}}$. $\quad f'(x) < 0 \quad \Leftrightarrow \quad 1 - 8x^{3/2} < 0 \quad \Leftrightarrow$

$x^{3/2} > \frac{1}{8} \quad \Leftrightarrow \quad x > \left(\frac{1}{8}\right)^{2/3} \quad \Leftrightarrow \quad x > \frac{1}{4}$, so f is increasing on $\left(0, \frac{1}{4}\right)$ and f is decreasing on $\left(\frac{1}{4}, \infty\right)$.

(b) f changes from increasing to decreasing at $x = \frac{1}{4}$, so $f\left(\frac{1}{4}\right) = 2 \cdot \frac{1}{2} - 4 \cdot \frac{1}{16} = \frac{3}{4}$ is a local maximum value. There is no local minimum value.

(c)



(d) $f''(x) = -\dfrac{1}{2}x^{-3/2} - 8 = -\dfrac{1}{2x^{3/2}} - 8$. $f''(x) < 0$ for all $x > 0$, so f is CD on $(0, \infty)$. There is no inflection point.

51. (a) $f(\theta) = 2\cos\theta + \cos^2\theta,\ 0 \le \theta \le 2\pi \quad \Rightarrow \quad f'(\theta) = -2\sin\theta + 2\cos\theta\,(-\sin\theta) = -2\sin\theta\,(1 + \cos\theta)$.
$f'(\theta) = 0 \quad \Leftrightarrow \quad \theta = 0, \pi$, and 2π. $f'(\theta) > 0 \quad \Leftrightarrow \quad \pi < \theta < 2\pi$ and $f'(\theta) < 0 \quad \Leftrightarrow \quad 0 < \theta < \pi$. So f is increasing on $(\pi, 2\pi)$ and f is decreasing on $(0, \pi)$.

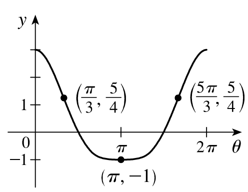
(b) $f(\pi) = -1$ is a local minimum value.

(c) $f'(\theta) = -2\sin\theta\,(1 + \cos\theta) \quad \Rightarrow$

$$\begin{aligned} f''(\theta) &= -2\sin\theta\,(-\sin\theta) + (1 + \cos\theta)(-2\cos\theta) = 2\sin^2\theta - 2\cos\theta - 2\cos^2\theta \\ &= 2(1 - \cos^2\theta) - 2\cos\theta - 2\cos^2\theta = -4\cos^2\theta - 2\cos\theta + 2 \\ &= -2(2\cos^2\theta + \cos\theta - 1) = -2(2\cos\theta - 1)(\cos\theta + 1) \end{aligned}$$

Since $-2(\cos\theta + 1) < 0$ [for $\theta \ne \pi$], $f''(\theta) > 0 \quad \Rightarrow \quad 2\cos\theta - 1 < 0 \quad \Rightarrow \quad \cos\theta < \frac{1}{2} \quad \Rightarrow \quad \frac{\pi}{3} < \theta < \frac{5\pi}{3}$ and $f''(\theta) < 0 \quad \Rightarrow \quad \cos\theta > \frac{1}{2} \quad \Rightarrow \quad 0 < \theta < \frac{\pi}{3}$ or $\frac{5\pi}{3} < \theta < 2\pi$. So f is CU on $\left(\frac{\pi}{3}, \frac{5\pi}{3}\right)$ and f is CD on $\left(0, \frac{\pi}{3}\right)$ and $\left(\frac{5\pi}{3}, 2\pi\right)$. There are points of inflection at $\left(\frac{\pi}{3}, f\left(\frac{\pi}{3}\right)\right) = \left(\frac{\pi}{3}, \frac{5}{4}\right)$ and $\left(\frac{5\pi}{3}, f\left(\frac{5\pi}{3}\right)\right) = \left(\frac{5\pi}{3}, \frac{5}{4}\right)$.

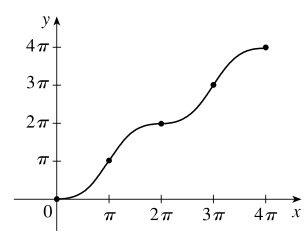
(d)



52. (a) $S(x) = x - \sin x, 0 \le x \le 4\pi \quad \Rightarrow \quad S'(x) = 1 - \cos x$. $S'(x) = 0 \quad \Leftrightarrow \quad \cos x = 1 \quad \Leftrightarrow \quad x = 0, 2\pi$, and 4π.
$S'(x) > 0 \quad \Leftrightarrow \quad \cos x < 1$, which is true for all x except integer multiples of 2π, so S is increasing on $(0, 4\pi)$ since $S'(2\pi) = 0$.

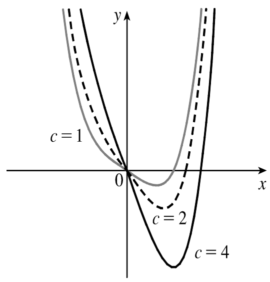
(b) There is no local maximum or minimum.

(c)



(d) $S''(x) = \sin x$. $S''(x) > 0$ if $0 < x < \pi$ or $2\pi < x < 3\pi$, and $S''(x) < 0$ if $\pi < x < 2\pi$ or $3\pi < x < 4\pi$. So S is CU on $(0, \pi)$ and $(2\pi, 3\pi)$, and S is CD on $(\pi, 2\pi)$ and $(3\pi, 4\pi)$. There are inflection points at (π, π), $(2\pi, 2\pi)$, and $(3\pi, 3\pi)$.

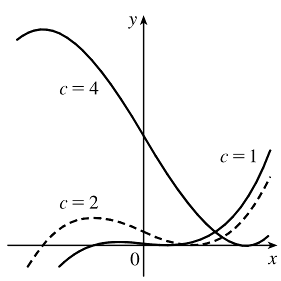
53. $f(x) = x^4 - cx$, $c > 0 \Rightarrow f'(x) = 4x^3 - c \Rightarrow f'(x) = 0 \Leftrightarrow x = \sqrt[3]{c/4}$. $f'(x) > 0 \Leftrightarrow x > \sqrt[3]{c/4}$ and $f'(x) < 0 \Leftrightarrow x < \sqrt[3]{c/4}$. Thus, f is increasing on $\left(\sqrt[3]{c/4}, \infty\right)$ and decreasing on $\left(-\infty, \sqrt[3]{c/4}\right)$. f changes from decreasing to increasing at $x = \sqrt[3]{c/4}$. Thus, $f\left(\sqrt[3]{c/4}\right)$ is a local minimum value. $f''(x) = 12x^2$ is positive except at $x = 0$, so f is concave up on $(-\infty, 0)$ and $(0, \infty)$. There are no inflection points.



The members of this family have one local minimum point with increasing x-coordinate and decreasing y-coordinate as c increases. The graphs are concave up on $(-\infty, 0)$ and $(0, \infty)$. Since the graphs are continuous at $x = 0$, and the graphs lie above their tangents, we can say that the graphs are concave up on $(-\infty, \infty)$. There is no inflection point.

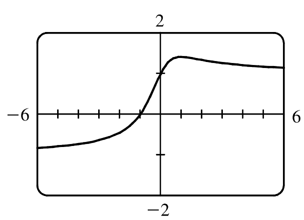
54. $f(x) = x^3 - 3c^2x + 2c^3$, $c > 0 \Rightarrow f'(x) = 3x^2 - 3c^2 = 3(x^2 - c^2) \Rightarrow f'(x) > 0 \Leftrightarrow |x| > c$ and $f'(x) < 0 \Leftrightarrow |x| < c$. Thus, f is increasing on $(-\infty, -c)$ and (c, ∞), and f is decreasing on $(-c, c)$. f changes from increasing to decreasing at $x = -c$ and from decreasing to increasing at $x = c$. Thus, $f(-c) = 4c^3$ is a local maximum value and $f(c) = 0$ is a local minimum value.

$f''(x) = 6x$, so $f''(x) > 0 \Leftrightarrow x > 0$ and $f''(x) < 0 \Leftrightarrow x < 0$. Thus, f is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$. There is an inflection point at $\left(0, 2c^3\right)$.



The members of this family have a local maximum point that moves higher and to the left as c increases, and there is a local minimum point on the x-axis that moves to the right as c increases. The graphs are concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$. There is an inflection point on the y-axis that moves higher as c increases.

55. (a)



From the graph, we get an estimate of $f(1) \approx 1.41$ as a local maximum value, and no local minimum value.

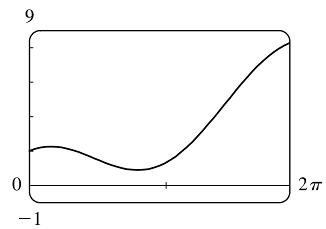
$f(x) = \dfrac{x+1}{\sqrt{x^2+1}} \Rightarrow f'(x) = \dfrac{1-x}{(x^2+1)^{3/2}}$.

$f'(x) = 0 \Leftrightarrow x = 1$. $f(1) = \frac{2}{\sqrt{2}} = \sqrt{2}$ is the exact value.

(b) From the graph in part (a), f increases most rapidly somewhere between $x = -\frac{1}{2}$ and $x = -\frac{1}{4}$. To find the exact value, we need to find the maximum value of f', which we can do by finding the critical numbers of f'.

$f''(x) = \dfrac{2x^2 - 3x - 1}{(x^2+1)^{5/2}} = 0 \Leftrightarrow x = \dfrac{3 \pm \sqrt{17}}{4}$. By the First Derivative Test applied to f', $x = \dfrac{3 + \sqrt{17}}{4}$ corresponds to the *minimum* value of f' and the maximum value of f' occurs at $x = \frac{3 - \sqrt{17}}{4} \approx -0.28$.

56. (a)

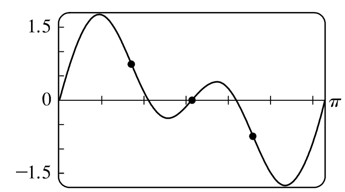


From the graph, we get estimates of $f(2.61) \approx 0.89$ as a local and absolute minimum, $f(0.53) \approx 2.26$ as a local maximum, and $f(2\pi) \approx 8.28$ as an absolute maximum. $f(x) = x + 2\cos x$ $(0 \le x \le 2\pi)$ $\Rightarrow$ $f'(x) = 1 - 2\sin x$. $f'(x) = 0$ $\Leftrightarrow$ $\sin x = \frac{1}{2}$ $\Leftrightarrow$ $x = \frac{\pi}{6}, \frac{5\pi}{6}$. $f\left(\frac{\pi}{6}\right) = \frac{\pi}{6} + \sqrt{3}$ is the exact value of the local maximum, $f\left(\frac{5\pi}{6}\right) = \frac{5\pi}{6} - \sqrt{3}$ is the exact value of the local and absolute minimum, and $f(2\pi) = 2\pi + 2$ is the exact value of the absolute maximum.

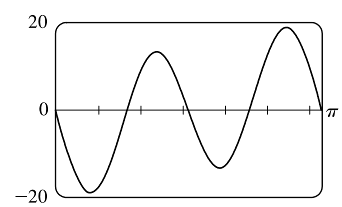
(b) From the graph in part (a), f increases most rapidly somewhere between $x = 4.5$ and $x = 5$. Now f increases most rapidly when $f'(x) = 1 - 2\sin x$ has its maximum value. $f''(x) = -2\cos x = 0$ $\Leftrightarrow$ $x = \frac{\pi}{2}, \frac{3\pi}{2}$. $f'(0) = f'(2\pi) = 1$, $f'\left(\frac{\pi}{2}\right) = -1$, and $f'\left(\frac{3\pi}{2}\right) = 3$. The maximum value of f' occurs at $\left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$.

57. $f(x) = \sin 2x + \sin 4x$ $\Rightarrow$ $f'(x) = 2\cos 2x + 4\cos 4x$ $\Rightarrow$ $f''(x) = -4\sin 2x - 16\sin 4x$

(a) From the graph of f, it seems that f is CD on $(0, 0.8)$, CU on $(0.8, 1.6)$, CD on $(1.6, 2.3)$, and CU on $(2.3, \pi)$. The inflection points appear to be at $(0.8, 0.7)$, $(1.6, 0)$, and $(2.3, -0.7)$.



(b) From the graph of f'' (and zooming in near the zeros), it seems that f is CD on $(0, 0.85)$, CU on $(0.85, 1.57)$, CD on $(1.57, 2.29)$, and CU on $(2.29, \pi)$. Refined estimates of the inflection points are $(0.85, 0.74)$, $(1.57, 0)$, and $(2.29, -0.74)$.

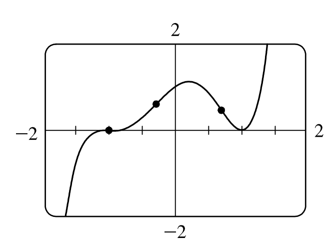


58. $f(x) = (x-1)^2(x+1)^3$ $\Rightarrow$

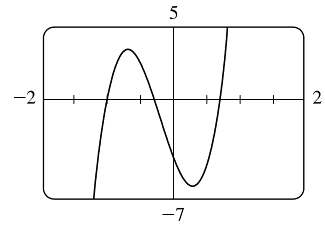
$$\begin{aligned} f'(x) &= (x-1)^2 3(x+1)^2 + (x+1)^3 2(x-1) \\ &= (x-1)(x+1)^2\left[3(x-1) + 2(x+1)\right] = (x-1)(x+1)^2(5x-1) \quad \Rightarrow \end{aligned}$$

$$\begin{aligned} f''(x) &= (1)(x+1)^2(5x-1) + (x-1)(2)(x+1)(5x-1) + (x-1)(x+1)^2(5) \\ &= (x+1)[(x+1)(5x-1) + 2(x-1)(5x-1) + 5(x-1)(x+1)] \\ &= (x+1)[5x^2 + 4x - 1 + 10x^2 - 12x + 2 + 5x^2 - 5] \\ &= (x+1)(20x^2 - 8x - 4) = 4(x+1)(5x^2 - 2x - 1) \end{aligned}$$

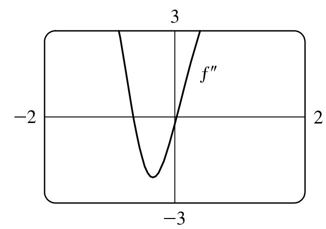
(a) From the graph of f, it seems that f is CD on $(-\infty, -1)$, CU on $(-1, -0.3)$, CD on $(-0.3, 0.7)$, and CU on $(0.7, \infty)$. The inflection points appear to be at $(-1, 0)$, $(-0.3, 0.6)$, and $(0.7, 0.5)$.



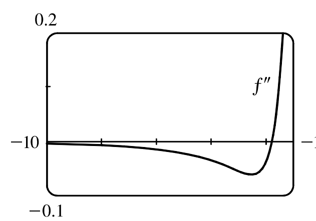
(b) From the graph of f'' (and zooming in near the zeros), it seems that f is CD on $(-1, 0)$, CU on $(-1, -0.29)$, CD on $(-0.29, 0.69)$, and CU on $(0.69, \infty)$. Refined estimates of the inflection points are $(-1, 0)$, $(-0.29, 0.60)$, and $(0.69, 0.46)$.

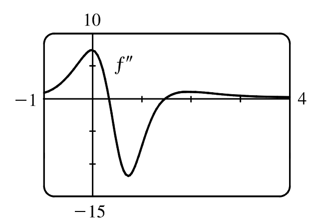


59. $f(x) = \dfrac{x^4 + x^3 + 1}{\sqrt{x^2 + x + 1}}$. In Maple, we define f and then use the command `plot(diff(diff(f,x),x),x=-2..2);`. In Mathematica, we define f and then use `Plot[Dt[Dt[f,x],x],{x,-2,2}]`. We see that $f'' > 0$ for $x < -0.6$ and $x > 0.0$ $[\approx 0.03]$ and $f'' < 0$ for $-0.6 < x < 0.0$. So f is CU on $(-\infty, -0.6)$ and $(0.0, \infty)$ and CD on $(-0.6, 0.0)$.



60. $f(x) = \dfrac{(x+1)^3(x^2+5)}{(x^3+1)(x^2+4)}$. It appears that f'' is positive (and thus f is concave up) on $(-1.8, 0.3)$ and $(1.5, \infty)$ and negative (so f is concave down) on $(-\infty, -1.8)$ and $(0.3, 1.5)$.





61. (a) The rate of increase of the population is initially very small, then gets larger until it reaches a maximum at about $t = 8$ hours, and decreases toward 0 as the population begins to level off.

(b) The rate of increase has its maximum value at $t = 8$ hours.

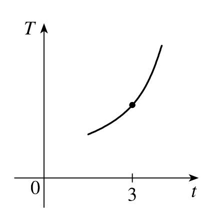
(c) The population function is concave upward on $(0, 8)$ and concave downward on $(8, 18)$.

(d) At $t = 8$, the population is about 350, so the inflection point is about $(8, 350)$.

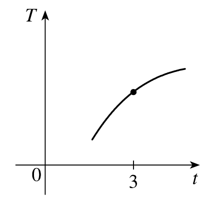
62. If $S(t)$ is the average SAT score as a function of time t, then $S'(t) < 0$ (since the SAT scores are declining) and $S''(t) > 0$ (since the rate of decrease of the scores is increasing—becoming less negative).

63. If $D(t)$ is the size of the national deficit as a function of time t, then at the time of the speech $D'(t) > 0$ (since the deficit is increasing), and $D''(t) < 0$ (since the rate of increase of the deficit is decreasing).

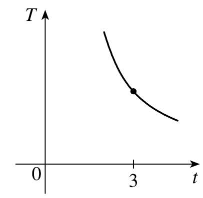
64. (a) I'm very unhappy. It's uncomfortably hot and $f'(3) = 2$ indicates that the temperature is increasing, and $f''(3) = 4$ indicates that the rate of increase is increasing. (The temperature is rapidly getting warmer.)



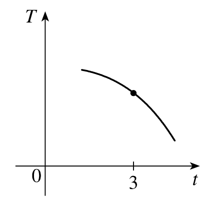
(b) I'm still unhappy, but not as unhappy as in part (a). It's uncomfortably hot and $f'(3) = 2$ indicates that the temperature is increasing, but $f''(3) = -4$ indicates that the rate of increase is decreasing. (The temperature is slowly getting warmer.)



(c) I'm somewhat happy. It's uncomfortably hot and $f'(3) = -2$ indicates that the temperature is decreasing, but $f''(3) = 4$ indicates that the rate of change is increasing. (The rate of change is negative but it's becoming less negative. The temperature is slowly getting cooler.)

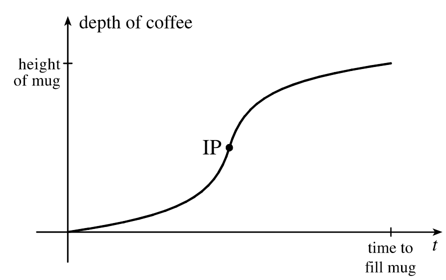


(d) I'm very happy. It's uncomfortably hot and $f'(3) = -2$ indicates that the temperature is decreasing, and $f''(3) = -4$ indicates that the rate of change is decreasing, that is, becoming more negative. (The temperature is rapidly getting cooler.)



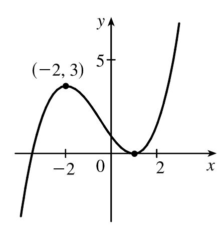
65. Most students learn more in the third hour of studying than in the eighth hour, so $K(3) - K(2)$ is larger than $K(8) - K(7)$. In other words, as you begin studying for a test, the rate of knowledge gain is large and then starts to taper off, so $K'(t)$ decreases and the graph of K is concave downward.

66. At first the depth increases slowly because the base of the mug is wide. But as the mug narrows, the coffee rises more quickly. Thus, the depth d increases at an increasing rate and its graph is concave upward. The rate of increase of d has a maximum where the mug is narrowest; that is, when the mug is half full. It is there that the inflection point (IP) occurs. Then the rate of increase of d starts to decrease as the mug widens and the graph becomes concave down.



67. $f(x) = ax^3 + bx^2 + cx + d \quad \Rightarrow \quad f'(x) = 3ax^2 + 2bx + c.$

We are given that $f(1) = 0$ and $f(-2) = 3$, so $f(1) = a + b + c + d = 0$ and $f(-2) = -8a + 4b - 2c + d = 3$. Also $f'(1) = 3a + 2b + c = 0$ and $f'(-2) = 12a - 4b + c = 0$ by Fermat's Theorem. Solving these four equations, we get $a = \frac{2}{9}$, $b = \frac{1}{3}$, $c = -\frac{4}{3}$, $d = \frac{7}{9}$, so the function is $f(x) = \frac{1}{9}\left(2x^3 + 3x^2 - 12x + 7\right)$.



68. $y = \dfrac{1+x}{1+x^2} \quad \Rightarrow \quad y' = \dfrac{(1+x^2)(1) - (1+x)(2x)}{(1+x^2)^2} = \dfrac{1 - 2x - x^2}{(1+x^2)^2} \quad \Rightarrow$

$$y'' = \frac{(1+x^2)^2(-2-2x) - (1-2x-x^2)\cdot 2(1+x^2)(2x)}{[(1+x^2)^2]^2} = \frac{2(1+x^2)[(1+x^2)(-1-x) - (1-2x-x^2)(2x)]}{(1+x^2)^4}$$

$$= \frac{2(-1-x-x^2-x^3-2x+4x^2+2x^3)}{(1+x^2)^3} = \frac{2(x^3+3x^2-3x-1)}{(1+x^2)^3} = \frac{2(x-1)(x^2+4x+1)}{(1+x^2)^3}$$

[continued]

So $y'' = 0 \Rightarrow x = 1, -2 \pm \sqrt{3}$. Let $a = -2 - \sqrt{3}$, $b = -2 + \sqrt{3}$, and $c = 1$. We can show that $f(a) = \frac{1}{4}\left(1 - \sqrt{3}\right)$, $f(b) = \frac{1}{4}\left(1 + \sqrt{3}\right)$, and $f(c) = 1$. To show that these three points of inflection lie on one straight line, we'll show that the slopes m_{ac} and m_{bc} are equal.

$$m_{ac} = \frac{f(c) - f(a)}{c - a} = \frac{1 - \frac{1}{4}\left(1 - \sqrt{3}\right)}{1 - \left(-2 - \sqrt{3}\right)} = \frac{\frac{3}{4} + \frac{1}{4}\sqrt{3}}{3 + \sqrt{3}} = \frac{1}{4}$$

$$m_{bc} = \frac{f(c) - f(b)}{c - b} = \frac{1 - \frac{1}{4}\left(1 + \sqrt{3}\right)}{1 - \left(-2 + \sqrt{3}\right)} = \frac{\frac{3}{4} - \frac{1}{4}\sqrt{3}}{3 - \sqrt{3}} = \frac{1}{4}$$

69. $y = x \sin x \Rightarrow y' = x \cos x + \sin x \Rightarrow y'' = -x \sin x + 2\cos x$. $y'' = 0 \Rightarrow 2\cos x = x \sin x$ [which is y] $\Rightarrow (2\cos x)^2 = (x \sin x)^2 \Rightarrow 4\cos^2 x = x^2 \sin^2 x \Rightarrow 4\cos^2 x = x^2(1 - \cos^2 x) \Rightarrow 4\cos^2 x + x^2\cos^2 x = x^2 \Rightarrow \cos^2 x(4 + x^2) = x^2 \Rightarrow 4\cos^2 x(x^2 + 4) = 4x^2 \Rightarrow y^2(x^2 + 4) = 4x^2$ since $y = 2\cos x$ when $y'' = 0$.

70. (a) We will make use of the converse of the Concavity Test (along with the stated assumptions); that is, if f is concave upward on I, then $f'' > 0$ on I. If f and g are CU on I, then $f'' > 0$ and $g'' > 0$ on I, so $(f + g)'' = f'' + g'' > 0$ on $I \Rightarrow f + g$ is CU on I.

(b) Since f is positive and CU on I, $f > 0$ and $f'' > 0$ on I. So $g(x) = [f(x)]^2 \Rightarrow g' = 2ff' \Rightarrow g'' = 2f'f' + 2ff'' = 2(f')^2 + 2ff'' > 0 \Rightarrow g$ is CU on I.

71. (a) Since f and g are positive, increasing, and CU on I with f'' and g'' never equal to 0, we have $f > 0$, $f' \geq 0$, $f'' > 0$, $g > 0$, $g' \geq 0$, $g'' > 0$ on I. Then $(fg)' = f'g + fg' \Rightarrow (fg)'' = f''g + 2f'g' + fg'' \geq f''g + fg'' > 0$ on $I \Rightarrow fg$ is CU on I.

(b) In part (a), if f and g are both decreasing instead of increasing, then $f' \leq 0$ and $g' \leq 0$ on I, so we still have $2f'g' \geq 0$ on I. Thus, $(fg)'' = f''g + 2f'g' + fg'' \geq f''g + fg'' > 0$ on $I \Rightarrow fg$ is CU on I as in part (a).

(c) Suppose f is increasing and g is decreasing [with f and g positive and CU]. Then $f' \geq 0$ and $g' \leq 0$ on I, so $2f'g' \leq 0$ on I and the argument in parts (a) and (b) fails.

Example 1. $I = (0, \infty)$, $f(x) = x^3$, $g(x) = 1/x$. Then $(fg)(x) = x^2$, so $(fg)'(x) = 2x$ and $(fg)''(x) = 2 > 0$ on I. Thus, fg is CU on I.

Example 2. $I = (0, \infty)$, $f(x) = 4x\sqrt{x}$, $g(x) = 1/x$. Then $(fg)(x) = 4\sqrt{x}$, so $(fg)'(x) = 2/\sqrt{x}$ and $(fg)''(x) = -1/\sqrt{x^3} < 0$ on I. Thus, fg is CD on I.

Example 3. $I = (0, \infty)$, $f(x) = x^2$, $g(x) = 1/x$. Thus, $(fg)(x) = x$, so fg is linear on I.

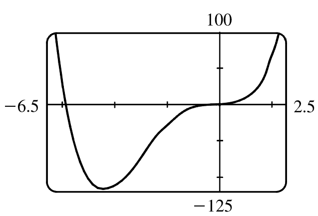
72. Since f and g are CU on $(-\infty, \infty)$, $f'' > 0$ and $g'' > 0$ on $(-\infty, \infty)$. $h(x) = f(g(x)) \Rightarrow h'(x) = f'(g(x))\,g'(x) \Rightarrow h''(x) = f''(g(x))\,g'(x)\,g'(x) + f'(g(x))\,g''(x) = f''(g(x))[g'(x)]^2 + f'(g(x))\,g''(x) > 0$ if $f' > 0$. So h is CU if f is increasing.

73. Let the cubic function be $f(x) = ax^3 + bx^2 + cx + d \quad \Rightarrow \quad f'(x) = 3ax^2 + 2bx + c \quad \Rightarrow \quad f''(x) = 6ax + 2b$.
So f is CU when $6ax + 2b > 0 \quad \Leftrightarrow \quad x > -b/(3a)$, CD when $x < -b/(3a)$, and so the only point of inflection occurs when $x = -b/(3a)$. If the graph has three x-intercepts x_1, x_2 and x_3, then the expression for $f(x)$ must factor as $f(x) = a(x - x_1)(x - x_2)(x - x_3)$. Multiplying these factors together gives us

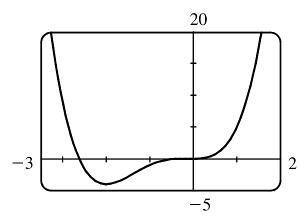
$$f(x) = a[x^3 - (x_1 + x_2 + x_3)x^2 + (x_1x_2 + x_1x_3 + x_2x_3)x - x_1x_2x_3]$$

Equating the coefficients of the x^2-terms for the two forms of f gives us $b = -a(x_1 + x_2 + x_3)$. Hence, the x-coordinate of the point of inflection is $-\dfrac{b}{3a} = -\dfrac{-a(x_1 + x_2 + x_3)}{3a} = \dfrac{x_1 + x_2 + x_3}{3}$.

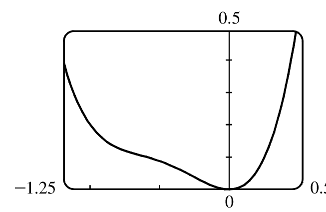
74. $P(x) = x^4 + cx^3 + x^2 \quad \Rightarrow \quad P'(x) = 4x^3 + 3cx^2 + 2x \quad \Rightarrow \quad P''(x) = 12x^2 + 6cx + 2$. The graph of $P''(x)$ is a parabola. If $P''(x)$ has two solutions, then it changes sign twice and so has two inflection points. This happens when the discriminant of $P''(x)$ is positive, that is, $(6c)^2 - 4 \cdot 12 \cdot 2 > 0 \quad \Leftrightarrow \quad 36c^2 - 96 > 0 \quad \Leftrightarrow \quad |c| > \frac{2\sqrt{6}}{3} \approx 1.63$. If $36c^2 - 96 = 0 \quad \Leftrightarrow \quad c = \pm\frac{2\sqrt{6}}{3}$, $P''(x)$ is 0 at one point, but there is still no inflection point since $P''(x)$ never changes sign, and if $36c^2 - 96 < 0 \quad \Leftrightarrow \quad |c| < \frac{2\sqrt{6}}{3}$, then $P''(x)$ never changes sign, and so there is no inflection point.



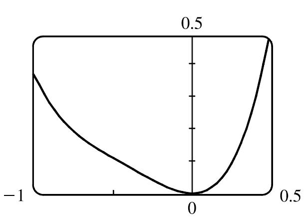
$c = 6$



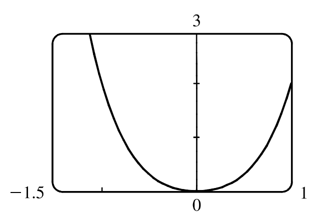
$c = 3$



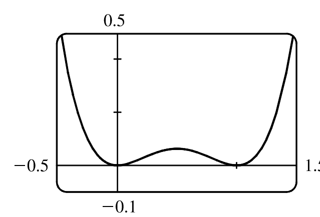
$c = 1.8$



$c = \frac{2\sqrt{6}}{3}$



$c = 0$



$c = -2$

For large positive c, the graph of f has two inflection points and a large dip to the left of the y-axis. As c decreases, the graph of f becomes flatter for $x < 0$, and eventually the dip rises above the x-axis, and then disappears entirely, along with the inflection points. As c continues to decrease, the dip and the inflection points reappear, to the right of the origin.

75. By hypothesis $g = f'$ is differentiable on an open interval containing c. Since $(c, f(c))$ is a point of inflection, the concavity changes at $x = c$, so $f''(x)$ changes signs at $x = c$. Hence, by the First Derivative Test, f' has a local extremum at $x = c$. Thus, by Fermat's Theorem $f''(c) = 0$.

76. $f(x) = x^4 \quad \Rightarrow \quad f'(x) = 4x^3 \quad \Rightarrow \quad f''(x) = 12x^2 \quad \Rightarrow \quad f''(0) = 0$. For $x < 0$, $f''(x) > 0$, so f is CU on $(-\infty, 0)$; for $x > 0$, $f''(x) > 0$, so f is also CU on $(0, \infty)$. Since f does not change concavity at 0, $(0, 0)$ is not an inflection point.

77. Using the fact that $|x| = \sqrt{x^2}$, we have that $g(x) = x\,|x| = x\sqrt{x^2} \quad \Rightarrow \quad g'(x) = \sqrt{x^2} + \sqrt{x^2} = 2\sqrt{x^2} = 2\,|x| \quad \Rightarrow$

$g''(x) = 2x\left(x^2\right)^{-1/2} = \dfrac{2x}{|x|} < 0$ for $x < 0$ and $g''(x) > 0$ for $x > 0$, so $(0, 0)$ is an inflection point. But $g''(0)$ does not exist.

78. There must exist some interval containing c on which f''' is positive, since $f'''(c)$ is positive and f''' is continuous. On this interval, f'' is increasing (since f''' is positive), so $f'' = (f')'$ changes from negative to positive at c. So by the First Derivative Test, f' has a local minimum at $x = c$ and thus cannot change sign there, so f has no maximum or minimum at c. But since f'' changes from negative to positive at c, f has a point of inflection at c (it changes from concave down to concave up).

79. Suppose that f is differentiable on an interval I and $f'(x) > 0$ for all x in I except $x = c$. To show that f is increasing on I, let x_1, x_2 be two numbers in I with $x_1 < x_2$.

Case 1 $x_1 < x_2 < c$. Let J be the interval $\{x \in I \mid x < c\}$. By applying the Increasing/Decreasing Test to f on J, we see that f is increasing on J, so $f(x_1) < f(x_2)$.

Case 2 $c < x_1 < x_2$. Apply the Increasing/Decreasing Test to f on $K = \{x \in I \mid x > c\}$.

Case 3 $x_1 < x_2 = c$. Apply the proof of the Increasing/Decreasing Test, using the Mean Value Theorem (MVT) on the interval $[x_1, x_2]$ and noting that the MVT does not require f to be differentiable at the endpoints of $[x_1, x_2]$.

Case 4 $c = x_1 < x_2$. Same proof as in Case 3.

Case 5 $x_1 < c < x_2$. By Cases 3 and 4, f is increasing on $[x_1, c]$ and on $[c, x_2]$, so $f(x_1) < f(c) < f(x_2)$.

In all cases, we have shown that $f(x_1) < f(x_2)$. Since x_1, x_2 were any numbers in I with $x_1 < x_2$, we have shown that f is increasing on I.

80. $f(x) = cx + \dfrac{1}{x^2 + 3} \quad \Rightarrow \quad f'(x) = c - \dfrac{2x}{(x^2 + 3)^2}$. $f'(x) > 0 \quad \Leftrightarrow \quad c > \dfrac{2x}{(x^2 + 3)^2}$ [call this $g(x)$]. Now f' is positive (and hence f increasing) if $c > g$, so we'll find the maximum value of g.

$$g'(x) = \frac{(x^2 + 3)^2 \cdot 2 - 2x \cdot 2(x^2 + 3) \cdot 2x}{[(x^2 + 3)^2]^2} = \frac{2(x^2 + 3)[(x^2 + 3) - 4x^2]}{(x^2 + 3)^4} = \frac{2(3 - 3x^2)}{(x^2 + 3)^3} = \frac{6(1 + x)(1 - x)}{(x^2 + 3)^3}.$$

$g'(x) = 0 \quad \Leftrightarrow \quad x = \pm 1$. $g'(x) > 0$ on $(0, 1)$ and $g'(x) < 0$ on $(1, \infty)$, so g is increasing on $(0, 1)$ and decreasing on $(1, \infty)$, and hence g has a maximum value on $(0, \infty)$ of $g(1) = \frac{2}{16} = \frac{1}{8}$. Also since $g(x) \leq 0$ if $x \leq 0$, the maximum value of g on $(-\infty, \infty)$ is $\frac{1}{8}$. Thus, when $c > \frac{1}{8}$, f is increasing. When $c = \frac{1}{8}$, $f'(x) > 0$ on $(-\infty, 1)$ and $(1, \infty)$, and hence f is increasing on these intervals. Since f is continuous, we may conclude that f is also increasing on $(-\infty, \infty)$ if $c = \frac{1}{8}$. Therefore, f is increasing on $(-\infty, \infty)$ if $c \geq \frac{1}{8}$.

81. (a) $f(x) = x^4 \sin \dfrac{1}{x} \quad \Rightarrow \quad f'(x) = x^4 \cos \dfrac{1}{x}\left(-\dfrac{1}{x^2}\right) + \sin \dfrac{1}{x}\,(4x^3) = 4x^3 \sin \dfrac{1}{x} - x^2 \cos \dfrac{1}{x}$.

$g(x) = x^4\left(2 + \sin \dfrac{1}{x}\right) = 2x^4 + f(x) \quad \Rightarrow \quad g'(x) = 8x^3 + f'(x)$.

[continued]

$h(x) = x^4\left(-2 + \sin\dfrac{1}{x}\right) = -2x^4 + f(x) \quad \Rightarrow \quad h'(x) = -8x^3 + f'(x).$

It is given that $f(0) = 0$, so $f'(0) = \lim\limits_{x \to 0} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x \to 0} \dfrac{x^4 \sin\frac{1}{x} - 0}{x} = \lim\limits_{x \to 0} x^3 \sin\dfrac{1}{x}$. Since $-\left|x^3\right| \le x^3 \sin\dfrac{1}{x} \le \left|x^3\right|$ and $\lim\limits_{x \to 0} \left|x^3\right| = 0$, we see that $f'(0) = 0$ by the Squeeze Theorem. Also, $g'(0) = 8(0)^3 + f'(0) = 0$ and $h'(0) = -8(0)^3 + f'(0) = 0$, so 0 is a critical number of f, g, and h.

For $x_{2n} = \dfrac{1}{2n\pi}$ [n a nonzero integer], $\sin\dfrac{1}{x_{2n}} = \sin 2n\pi = 0$ and $\cos\dfrac{1}{x_{2n}} = \cos 2n\pi = 1$, so $f'(x_{2n}) = -x_{2n}^2 < 0$.

For $x_{2n+1} = \dfrac{1}{(2n+1)\pi}$, $\sin\dfrac{1}{x_{2n+1}} = \sin(2n+1)\pi = 0$ and $\cos\dfrac{1}{x_{2n+1}} = \cos(2n+1)\pi = -1$, so $f'(x_{2n+1}) = x_{2n+1}^2 > 0$. Thus, f' changes sign infinitely often on both sides of 0.

Next, $g'(x_{2n}) = 8x_{2n}^3 + f'(x_{2n}) = 8x_{2n}^3 - x_{2n}^2 = x_{2n}^2(8x_{2n} - 1) < 0$ for $x_{2n} < \frac{1}{8}$, but $g'(x_{2n+1}) = 8x_{2n+1}^3 + x_{2n+1}^2 = x_{2n+1}^2(8x_{2n+1} + 1) > 0$ for $x_{2n+1} > -\frac{1}{8}$, so g' changes sign infinitely often on both sides of 0.

Last, $h'(x_{2n}) = -8x_{2n}^3 + f'(x_{2n}) = -8x_{2n}^3 - x_{2n}^2 = -x_{2n}^2(8x_{2n} + 1) < 0$ for $x_{2n} > -\frac{1}{8}$ and $h'(x_{2n+1}) = -8x_{2n+1}^3 + x_{2n+1}^2 = x_{2n+1}^2(-8x_{2n+1} + 1) > 0$ for $x_{2n+1} < \frac{1}{8}$, so h' changes sign infinitely often on both sides of 0.

(b) $f(0) = 0$ and since $\sin\dfrac{1}{x}$ and hence $x^4 \sin\dfrac{1}{x}$ is both positive and negative inifinitely often on both sides of 0, and arbitrarily close to 0, f has neither a local maximum nor a local minimum at 0.

Since $2 + \sin\dfrac{1}{x} \ge 1$, $g(x) = x^4\left(2 + \sin\dfrac{1}{x}\right) > 0$ for $x \ne 0$, so $g(0) = 0$ is a local minimum.

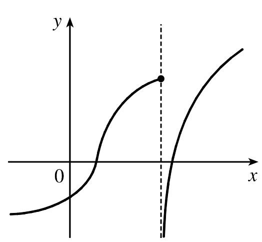
Since $-2 + \sin\dfrac{1}{x} \le -1$, $h(x) = x^4\left(-2 + \sin\dfrac{1}{x}\right) < 0$ for $x \ne 0$, so $h(0) = 0$ is a local maximum.

3.4 Limits at Infinity; Horizontal Asymptotes

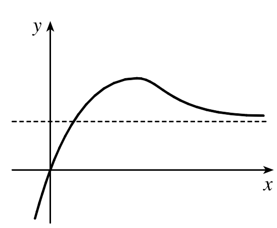
1. (a) As x becomes large, the values of $f(x)$ approach 5.

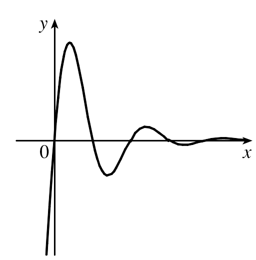
(b) As x becomes large negative, the values of $f(x)$ approach 3.

2. (a) The graph of a function can intersect a vertical asymptote in the sense that it can meet but not cross it.

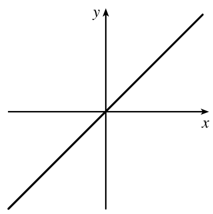


The graph of a function can intersect a horizontal asymptote. It can even intersect its horizontal asymptote an infinite number of times.

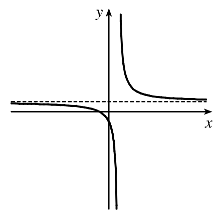




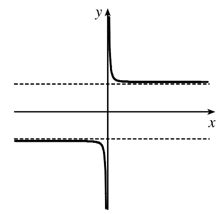
(b) The graph of a function can have 0, 1, or 2 horizontal asymptotes. Representative examples are shown.



No horizontal asymptote



One horizontal asymptote



Two horizontal asymptotes

3. (a) $\lim_{x\to\infty} f(x) = -2$ (b) $\lim_{x\to-\infty} f(x) = 2$ (c) $\lim_{x\to 1} f(x) = \infty$

(d) $\lim_{x\to 3} f(x) = -\infty$ (e) Vertical: $x = 1$, $x = 3$; horizontal: $y = -2$, $y = 2$

4. (a) $\lim_{x\to\infty} g(x) = 2$ (b) $\lim_{x\to-\infty} g(x) = -1$ (c) $\lim_{x\to 0} g(x) = -\infty$

(d) $\lim_{x\to 2^-} g(x) = -\infty$ (e) $\lim_{x\to 2^+} g(x) = \infty$ (f) Vertical: $x = 0$, $x = 2$;
horizontal: $y = -1$, $y = 2$

5. If $f(x) = x^2/2^x$, then a calculator gives $f(0) = 0$, $f(1) = 0.5$, $f(2) = 1$, $f(3) = 1.125$, $f(4) = 1$, $f(5) = 0.78125$, $f(6) = 0.5625$, $f(7) = 0.3828125$, $f(8) = 0.25$, $f(9) = 0.158203125$, $f(10) = 0.09765625$, $f(20) \approx 0.00038147$, $f(50) \approx 2.2204 \times 10^{-12}$, $f(100) \approx 7.8886 \times 10^{-27}$. It appears that $\lim_{x\to\infty} \left(x^2/2^x\right) = 0$.

6. (a) From a graph of $f(x) = (1 - 2/x)^x$ in a window of $[0, 10{,}000]$ by $[0, 0.2]$, we estimate that $\lim_{x\to\infty} f(x) = 0.14$ (to two decimal places).

(b)

x	$f(x)$
10,000	0.135 308
100,000	0.135 333
1,000,000	0.135 335

From the table, we estimate that $\lim_{x\to\infty} f(x) = 0.1353$ (to four decimal places).

7.
$$\begin{aligned}
\lim_{x\to\infty} \frac{2x^2 - 7}{5x^2 + x - 3} &= \lim_{x\to\infty} \frac{(2x^2 - 7)/x^2}{(5x^2 + x - 3)/x^2} && \text{[Divide both the numerator and denominator by } x^2 \\
& && \text{(the highest power of } x \text{ that appears in the denominator)]} \\
&= \frac{\lim_{x\to\infty} (2 - 7/x^2)}{\lim_{x\to\infty} (5 + 1/x - 3/x^2)} && \text{[Limit Law 5]} \\
&= \frac{\lim_{x\to\infty} 2 - \lim_{x\to\infty} (7/x^2)}{\lim_{x\to\infty} 5 + \lim_{x\to\infty} (1/x) - \lim_{x\to\infty} (3/x^2)} && \text{[Limit Laws 1 and 2]} \\
&= \frac{2 - 7 \lim_{x\to\infty} (1/x^2)}{5 + \lim_{x\to\infty} (1/x) - 3 \lim_{x\to\infty} (1/x^2)} && \text{[Limit Laws 8 and 3]} \\
&= \frac{2 - 7(0)}{5 + 0 + 3(0)} && \text{[Theorem 4]} \\
&= \frac{2}{5}
\end{aligned}$$

8. $\displaystyle \lim_{x\to\infty}\sqrt{\frac{9x^3+8x-4}{3-5x+x^3}}=\sqrt{\lim_{x\to\infty}\frac{9x^3+8x-4}{3-5x+x^3}}$ [Limit Law 7]

$\displaystyle =\sqrt{\lim_{x\to\infty}\frac{9+8/x^2-4/x^3}{3/x^3-5/x^2+1}}$ [Divide by x^3]

$\displaystyle =\sqrt{\frac{\lim\limits_{x\to\infty}(9+8/x^2-4/x^3)}{\lim\limits_{x\to\infty}(3/x^3-5/x^2+1)}}$ [Limit Law 5]

$\displaystyle =\sqrt{\frac{\lim\limits_{x\to\infty}9+\lim\limits_{x\to\infty}(8/x^2)-\lim\limits_{x\to\infty}(4/x^3)}{\lim\limits_{x\to\infty}(3/x^3)-\lim\limits_{x\to\infty}(5/x^2)+\lim\limits_{x\to\infty}1}}$ [Limit Laws 1 and 2]

$\displaystyle =\sqrt{\frac{9+8\lim\limits_{x\to\infty}(1/x^2)-4\lim\limits_{x\to\infty}(1/x^3)}{3\lim\limits_{x\to\infty}(1/x^3)-5\lim\limits_{x\to\infty}(1/x^2)+1}}$ [Limit Laws 8 and 3]

$\displaystyle =\sqrt{\frac{9+8(0)-4(0)}{3(0)-5(0)+1}}$ [Theorem 4]

$\displaystyle =\sqrt{\frac{9}{1}}=\sqrt{9}=3$

9. $\displaystyle \lim_{x\to\infty}\frac{4x+3}{5x-1}=\lim_{x\to\infty}\frac{(4x+3)/x}{(5x-1)/x}=\lim_{x\to\infty}\frac{4+3/x}{5-1/x}=\frac{\lim\limits_{x\to\infty}4+3\lim\limits_{x\to\infty}(1/x)}{\lim\limits_{x\to\infty}5-\lim\limits_{x\to\infty}(1/x)}=\frac{4+3(0)}{5-0}=\frac{4}{5}$

10. $\displaystyle \lim_{x\to\infty}\frac{-2}{3x+7}=\lim_{x\to\infty}\frac{-2/x}{(3x+7)/x}=\lim_{x\to\infty}\frac{-2/x}{3+7/x}=\frac{-2\lim\limits_{x\to\infty}(1/x)}{\lim\limits_{x\to\infty}3+7\lim\limits_{x\to\infty}(1/x)}=\frac{0}{3+0}=0$

11. $\displaystyle \lim_{t\to-\infty}\frac{3t^2+t}{t^3-4t+1}=\lim_{t\to-\infty}\frac{(3t^2+t)/t^3}{(t^3-4t+1)/t^3}=\lim_{t\to-\infty}\frac{3/t+1/t^2}{1-4/t^2+1/t^3}$

$\displaystyle =\frac{3\lim\limits_{t\to-\infty}(1/t)+\lim\limits_{t\to-\infty}(1/t^2)}{\lim\limits_{t\to-\infty}1-4\lim\limits_{t\to-\infty}(1/t^2)+\lim\limits_{t\to-\infty}(1/t^3)}=\frac{3(0)+0}{1-4(0)+0}=0$

12. $\displaystyle \lim_{t\to-\infty}\frac{6t^2+t-5}{9-2t^2}=\lim_{t\to-\infty}\frac{(6t^2+t-5)/t^2}{(9-2t^2)/t^2}=\lim_{t\to-\infty}\frac{6+1/t-5/t^2}{9/t^2-2}=\frac{6+0-0}{0-2}=-3$

13. $\displaystyle \lim_{r\to\infty}\frac{r-r^3}{2-r^2+3r^3}=\lim_{r\to\infty}\frac{(r-r^3)/r^3}{(2-r^2+3r^3)/r^3}=\lim_{r\to\infty}\frac{1/r^2-1}{2/r^3-1/r+3}=\frac{0-1}{0-0+3}=-\frac{1}{3}$

14. $\displaystyle \lim_{x\to\infty}\frac{3x^3-8x+2}{4x^3-5x^2-2}=\lim_{x\to\infty}\frac{(3x^3-8x+2)/x^3}{(4x^3-5x^2-2)/x^3}=\lim_{x\to\infty}\frac{3-8/x^2+2/x^3}{4-5/x-2/x^3}=\frac{3-0+0}{4-0-0}=\frac{3}{4}$

15. $\displaystyle \lim_{x\to\infty}\frac{4-\sqrt{x}}{2+\sqrt{x}}=\lim_{x\to\infty}\frac{(4-\sqrt{x})/\sqrt{x}}{(2+\sqrt{x})/\sqrt{x}}=\lim_{x\to\infty}\frac{4/\sqrt{x}-1}{2/\sqrt{x}+1}=\frac{0-1}{0+1}=-1$

16. $\displaystyle \lim_{u\to-\infty}\frac{(u^2+1)(2u^2-1)}{(u^2+2)^2}=\lim_{u\to-\infty}\frac{[(u^2+1)(2u^2-1)]/u^4}{(u^2+2)^2/u^4}=\lim_{u\to-\infty}\frac{[(u^2+1)/u^2][(2u^2-1)/u^2]}{(u^4+4u^2+4)/u^4}$

$\displaystyle =\lim_{u\to-\infty}\frac{(1+1/u^2)(2-1/u^2)}{(1+4/u^2+4/u^4)}=\frac{(1+0)(2-0)}{1+0+0}=2$

17. $\displaystyle\lim_{x\to\infty}\frac{\sqrt{x+3x^2}}{4x-1}=\lim_{x\to\infty}\frac{\sqrt{x+3x^2}/x}{(4x-1)/x}=\frac{\lim\limits_{x\to\infty}\sqrt{(x+3x^2)/x^2}}{\lim\limits_{x\to\infty}(4-1/x)}$ [since $x=\sqrt{x^2}$ for $x>0$]

$$=\frac{\lim\limits_{x\to\infty}\sqrt{1/x+3}}{\lim\limits_{x\to\infty}4-\lim\limits_{x\to\infty}(1/x)}=\frac{\sqrt{\lim\limits_{x\to\infty}(1/x)+\lim\limits_{x\to\infty}3}}{4-0}=\frac{\sqrt{0+3}}{4}=\frac{\sqrt{3}}{4}$$

18. $\displaystyle\lim_{t\to\infty}\frac{t+3}{\sqrt{2t^2-1}}=\lim_{t\to\infty}\frac{(t+3)/t}{\sqrt{2t^2-1}/t}=\lim_{t\to\infty}\frac{1+3/t}{\sqrt{2-1/t^2}}$ $\left[\text{since } t=\sqrt{t^2} \text{ for } t>0\right]$

$$=\frac{\lim\limits_{t\to\infty}(1+3/t)}{\lim\limits_{t\to\infty}\sqrt{2-1/t^2}}=\frac{\lim\limits_{t\to\infty}1+\lim\limits_{t\to\infty}(3/t)}{\sqrt{\lim\limits_{t\to\infty}2-\lim\limits_{t\to\infty}(1/t^2)}}=\frac{1+0}{\sqrt{2-0}}=\frac{1}{\sqrt{2}},\text{ or }\frac{\sqrt{2}}{2}$$

19. $\displaystyle\lim_{x\to\infty}\frac{\sqrt{1+4x^6}}{2-x^3}=\lim_{x\to\infty}\frac{\sqrt{1+4x^6}/x^3}{(2-x^3)/x^3}=\frac{\lim\limits_{x\to\infty}\sqrt{(1+4x^6)/x^6}}{\lim\limits_{x\to\infty}(2/x^3-1)}$ $\left[\text{since } x^3=\sqrt{x^6} \text{ for } x>0\right]$

$$=\frac{\lim\limits_{x\to\infty}\sqrt{1/x^6+4}}{\lim\limits_{x\to\infty}(2/x^3)-\lim\limits_{x\to\infty}1}=\frac{\sqrt{\lim\limits_{x\to\infty}(1/x^6)+\lim\limits_{x\to\infty}4}}{0-1}$$

$$=\frac{\sqrt{0+4}}{-1}=\frac{2}{-1}=-2$$

20. $\displaystyle\lim_{x\to-\infty}\frac{\sqrt{1+4x^6}}{2-x^3}=\lim_{x\to-\infty}\frac{\sqrt{1+4x^6}/x^3}{(2-x^3)/x^3}=\frac{\lim\limits_{x\to-\infty}-\sqrt{(1+4x^6)/x^6}}{\lim\limits_{x\to-\infty}(2/x^3-1)}$ $\left[\text{since } x^3=-\sqrt{x^6} \text{ for } x<0\right]$

$$=\frac{\lim\limits_{x\to-\infty}-\sqrt{1/x^6+4}}{2\lim\limits_{x\to-\infty}(1/x^3)-\lim\limits_{x\to-\infty}1}=\frac{-\sqrt{\lim\limits_{x\to-\infty}(1/x^6)+\lim\limits_{x\to-\infty}4}}{2(0)-1}$$

$$=\frac{-\sqrt{0+4}}{-1}=\frac{-2}{-1}=2$$

21. $\displaystyle\lim_{x\to-\infty}\frac{2x^5-x}{x^4+3}=\lim_{x\to-\infty}\frac{\left(2x^5-x\right)/x^4}{\left(x^4+3\right)/x^4}=\lim_{x\to-\infty}\frac{2x-1/x^3}{1+3/x^4}$

$=-\infty$ since $2x-1/x^3\to-\infty$ and $1+3/x^4\to1$ as $x\to-\infty$

22. $\displaystyle\lim_{q\to\infty}\frac{q^3+6q-4}{4q^2-3q+3}=\lim_{q\to\infty}\frac{\left(q^3+6q-4\right)/q^2}{\left(4q^2-3q+3\right)/q^2}=\lim_{q\to\infty}\frac{q+6/q-4/q^2}{4-3/q+3/q^2}$

$=\infty$ since $q+6/q-4/q^2\to\infty$ and $4-3/q+3/q+3/q^2\to4$ as $q\to\infty$

23. $\lim\limits_{x\to\infty}\cos x$ does not exist because as x increases $\cos x$ does not approach any one value, but oscillates between 1 and -1.

24. $\displaystyle\lim_{x\to-\infty}\frac{1+x^6}{x^4+1}=\lim_{x\to-\infty}\frac{(1+x^6)/x^4}{(x^4+1)/x^4}$ $\left[\begin{matrix}\text{divide by the highest power}\\ \text{of } x \text{ in the denominator}\end{matrix}\right]$ $\displaystyle=\lim_{x\to-\infty}\frac{1/x^4+x^2}{1+1/x^4}=\infty$

since the numerator increases without bound and the denominator approaches 1 as $x\to-\infty$.

25. $\displaystyle \lim_{t\to\infty}\left(\sqrt{25t^2+2}-5t\right)=\lim_{t\to\infty}\left(\sqrt{25t^2+2}-5t\right)\left(\frac{\sqrt{25t^2+2}+5t}{\sqrt{25t^2+2}+5t}\right)=\lim_{t\to\infty}\frac{\left(25t^2+2\right)-(5t)^2}{\sqrt{25t^2+2}+5t}$

$$=\lim_{t\to\infty}\frac{2}{\sqrt{25t^2+2}+5t}=\lim_{t\to\infty}\frac{2/t}{\left(\sqrt{25t^2+2}+5t\right)/t}$$

$$=\lim_{t\to\infty}\frac{2/t}{\sqrt{25+2/t^2}+5}\qquad\left[\text{since } t=\sqrt{t^2}\text{ for } t>0\right]$$

$$=\frac{0}{\sqrt{25+0}+5}=0$$

26. $\displaystyle \lim_{x\to-\infty}\left(\sqrt{4x^2+3x}+2x\right)=\lim_{x\to-\infty}\left(\sqrt{4x^2+3x}+2x\right)\left[\frac{\sqrt{4x^2+3x}-2x}{\sqrt{4x^2+3x}-2x}\right]=\lim_{x\to-\infty}\frac{\left(4x^2+3x\right)-(2x)^2}{\sqrt{4x^2+3x}-2x}$

$$=\lim_{x\to-\infty}\frac{3x}{\sqrt{4x^2+3x}-2x}=\lim_{x\to-\infty}\frac{3x/x}{\left(\sqrt{4x^2+3x}-2x\right)/x}$$

$$=\lim_{x\to-\infty}\frac{3}{-\sqrt{4+3/x}-2}\qquad\left[\text{since } x=-\sqrt{x^2}\text{ for } x<0\right]$$

$$=\frac{3}{-\sqrt{4+0}-2}=-\frac{3}{4}$$

27. $\displaystyle \lim_{x\to\infty}\left(\sqrt{x^2+ax}-\sqrt{x^2+bx}\right)=\lim_{x\to\infty}\frac{\left(\sqrt{x^2+ax}-\sqrt{x^2+bx}\right)\left(\sqrt{x^2+ax}+\sqrt{x^2+bx}\right)}{\sqrt{x^2+ax}+\sqrt{x^2+bx}}$

$$=\lim_{x\to\infty}\frac{\left(x^2+ax\right)-\left(x^2+bx\right)}{\sqrt{x^2+ax}+\sqrt{x^2+bx}}=\lim_{x\to\infty}\frac{[(a-b)x]/x}{\left(\sqrt{x^2+ax}+\sqrt{x^2+bx}\right)/\sqrt{x^2}}$$

$$=\lim_{x\to\infty}\frac{a-b}{\sqrt{1+a/x}+\sqrt{1+b/x}}=\frac{a-b}{\sqrt{1+0}+\sqrt{1+0}}=\frac{a-b}{2}$$

28. $\displaystyle \lim_{x\to\infty}\left(x-\sqrt{x}\right)=\lim_{x\to\infty}\left(x-\sqrt{x}\right)\left[\frac{x+\sqrt{x}}{x+\sqrt{x}}\right]=\lim_{x\to\infty}\frac{x^2-\left(\sqrt{x}\right)^2}{x+\sqrt{x}}=\lim_{x\to\infty}\frac{x^2-x}{x+\sqrt{x}}=\lim_{x\to\infty}\frac{\left(x^2-x\right)/x}{\left(x+\sqrt{x}\right)/x}$

$$=\lim_{x\to\infty}\frac{x-1}{1+1/\sqrt{x}}=\infty\quad\text{since } x-1\to\infty\text{ and } 1+1/\sqrt{x}\to 1\text{ as } x\to\infty$$

29. $\displaystyle \lim_{x\to-\infty}\left(x^2+2x^7\right)=\lim_{x\to-\infty}x^7\left(\frac{1}{x^5}+2\right)$ [factor out the largest power of x] $=-\infty$ because $x^7\to-\infty$ and $1/x^5+2\to 2$ as $x\to-\infty$.

Or: $\displaystyle \lim_{x\to-\infty}\left(x^2+2x^7\right)=\lim_{x\to-\infty}x^2\left(1+2x^5\right)=-\infty.$

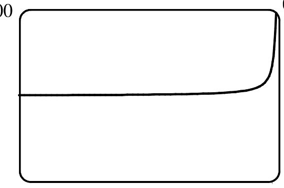
30. Since $0\le\sin^2x\le 1$, we have $0\le\dfrac{\sin^2x}{x^2+1}\le\dfrac{1}{x^2+1}$. We know that $\displaystyle\lim_{x\to\infty}0=0$ and $\displaystyle\lim_{x\to\infty}\frac{1}{x^2+1}=0$, so by the Squeeze Theorem, $\displaystyle\lim_{x\to\infty}\frac{\sin^2x}{x^2+1}=0.$

31. If $t=\dfrac{1}{x}$, then $\displaystyle\lim_{x\to\infty}x\sin\frac{1}{x}=\lim_{t\to0^+}\frac{1}{t}\sin t=\lim_{t\to0^+}\frac{\sin t}{t}=1.$

32. If $t=\dfrac{1}{x}$, then $\displaystyle\lim_{x\to\infty}\sqrt{x}\sin\frac{1}{x}=\lim_{t\to0^+}\frac{1}{\sqrt{t}}\sin t=\lim_{t\to0^+}\frac{t}{\sqrt{t}}\,\frac{\sin t}{t}=\lim_{t\to0^+}\sqrt{t}\cdot\lim_{t\to0^+}\frac{\sin t}{t}=0\cdot1=0.$

33. (a)

−100 0



−1

From the graph of $f(x) = \sqrt{x^2 + x + 1} + x$, we estimate the value of $\lim_{x \to -\infty} f(x)$ to be -0.5.

(b)

x	$f(x)$
$-10{,}000$	$-0.499\,962\,5$
$-100{,}000$	$-0.499\,996\,2$
$-1{,}000{,}000$	$-0.499\,999\,6$

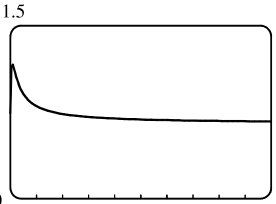
From the table, we estimate the limit to be -0.5.

(c)
$$\lim_{x \to -\infty} \left(\sqrt{x^2 + x + 1} + x\right) = \lim_{x \to -\infty} \left(\sqrt{x^2 + x + 1} + x\right)\left[\frac{\sqrt{x^2 + x + 1} - x}{\sqrt{x^2 + x + 1} - x}\right] = \lim_{x \to -\infty} \frac{\left(x^2 + x + 1\right) - x^2}{\sqrt{x^2 + x + 1} - x}$$
$$= \lim_{x \to -\infty} \frac{(x + 1)(1/x)}{\left(\sqrt{x^2 + x + 1} - x\right)(1/x)} = \lim_{x \to -\infty} \frac{1 + (1/x)}{-\sqrt{1 + (1/x) + (1/x^2)} - 1}$$
$$= \frac{1 + 0}{-\sqrt{1 + 0 + 0} - 1} = -\frac{1}{2}$$

Note that for $x < 0$, we have $\sqrt{x^2} = |x| = -x$, so when we divide the radical by x, with $x < 0$, we get
$\frac{1}{x}\sqrt{x^2 + x + 1} = -\frac{1}{\sqrt{x^2}}\sqrt{x^2 + x + 1} = -\sqrt{1 + (1/x) + (1/x^2)}$.

34. (a)

1.5



0 100
1.4

From the graph of $f(x) = \sqrt{3x^2 + 8x + 6} - \sqrt{3x^2 + 3x + 1}$, we estimate (to one decimal place) the value of $\lim_{x \to \infty} f(x)$ to be 1.4.

(b)

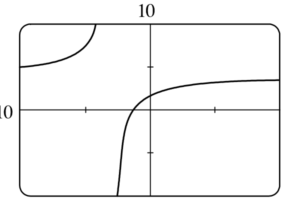
x	$f(x)$
10,000	1.443 39
100,000	1.443 38
1,000,000	1.443 38

From the table, we estimate (to four decimal places) the limit to be 1.4434.

(c)
$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{\left(\sqrt{3x^2 + 8x + 6} - \sqrt{3x^2 + 3x + 1}\right)\left(\sqrt{3x^2 + 8x + 6} + \sqrt{3x^2 + 3x + 1}\right)}{\sqrt{3x^2 + 8x + 6} + \sqrt{3x^2 + 3x + 1}}$$
$$= \lim_{x \to \infty} \frac{\left(3x^2 + 8x + 6\right) - \left(3x^2 + 3x + 1\right)}{\sqrt{3x^2 + 8x + 6} + \sqrt{3x^2 + 3x + 1}} = \lim_{x \to \infty} \frac{(5x + 5)(1/x)}{\left(\sqrt{3x^2 + 8x + 6} + \sqrt{3x^2 + 3x + 1}\right)(1/x)}$$
$$= \lim_{x \to \infty} \frac{5 + 5/x}{\sqrt{3 + 8/x + 6/x^2} + \sqrt{3 + 3/x + 1/x^2}} = \frac{5}{\sqrt{3} + \sqrt{3}} = \frac{5}{2\sqrt{3}} = \frac{5\sqrt{3}}{6} \approx 1.443376$$

35. $\lim_{x \to \pm\infty} \frac{5 + 4x}{x + 3} = \lim_{x \to \pm\infty} \frac{(5 + 4x)/x}{(x + 3)/x} = \lim_{x \to \pm\infty} \frac{5/x + 4}{1 + 3/x} = \frac{0 + 4}{1 + 0} = 4$, so $y = 4$ is a horizontal asymptote. $y = f(x) = \frac{5 + 4x}{x + 3}$, so $\lim_{x \to -3^+} f(x) = -\infty$ since $5 + 4x \to -7$ and $x + 3 \to 0^+$ as $x \to -3^+$. Thus, $x = -3$ is a vertical asymptote. The graph confirms our work.

10

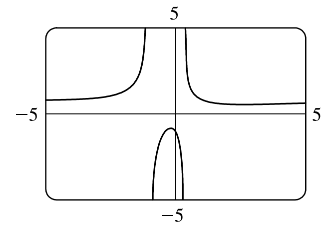


−10 10
−10

36. $\lim\limits_{x\to\pm\infty} \dfrac{2x^2+1}{3x^2+2x-1} = \lim\limits_{x\to\pm\infty} \dfrac{(2x^2+1)/x^2}{(3x^2+2x-1)/x^2}$

$= \lim\limits_{x\to\pm\infty} \dfrac{2+1/x^2}{3+2/x-1/x^2} = \dfrac{2}{3}$

so $y = \dfrac{2}{3}$ is a horizontal asymptote. $y = f(x) = \dfrac{2x^2+1}{3x^2+2x-1} = \dfrac{2x^2+1}{(3x-1)(x+1)}$.



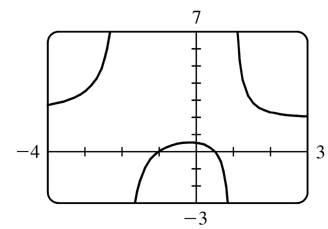
The denominator is zero when $x = \frac{1}{3}$ and -1, but the numerator is nonzero, so $x = \frac{1}{3}$ and $x = -1$ are vertical asymptotes. The graph confirms our work.

37. $\lim\limits_{x\to\pm\infty} \dfrac{2x^2+x-1}{x^2+x-2} = \lim\limits_{x\to\pm\infty} \dfrac{\dfrac{2x^2+x-1}{x^2}}{\dfrac{x^2+x-2}{x^2}} = \lim\limits_{x\to\pm\infty} \dfrac{2+\dfrac{1}{x}-\dfrac{1}{x^2}}{1+\dfrac{1}{x}-\dfrac{2}{x^2}} = \dfrac{\lim\limits_{x\to\pm\infty}\left(2+\dfrac{1}{x}-\dfrac{1}{x^2}\right)}{\lim\limits_{x\to\pm\infty}\left(1+\dfrac{1}{x}-\dfrac{2}{x^2}\right)}$

$= \dfrac{\lim\limits_{x\to\pm\infty} 2 + \lim\limits_{x\to\pm\infty} \dfrac{1}{x} - \lim\limits_{x\to\pm\infty} \dfrac{1}{x^2}}{\lim\limits_{x\to\pm\infty} 1 + \lim\limits_{x\to\pm\infty} \dfrac{1}{x} - 2\lim\limits_{x\to\pm\infty} \dfrac{1}{x^2}} = \dfrac{2+0-0}{1+0-2(0)} = 2,$ so $y = 2$ is a horizontal asymptote.

$y = f(x) = \dfrac{2x^2+x-1}{x^2+x-2} = \dfrac{(2x-1)(x+1)}{(x+2)(x-1)}$, so $\lim\limits_{x\to-2^-} f(x) = \infty$,

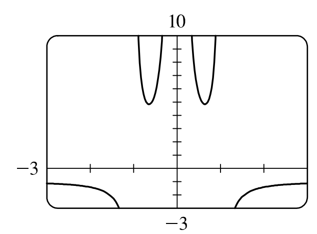
$\lim\limits_{x\to-2^+} f(x) = -\infty$, $\lim\limits_{x\to1^-} f(x) = -\infty$, and $\lim\limits_{x\to1^+} f(x) = \infty$. Thus, $x = -2$ and $x = 1$ are vertical asymptotes. The graph confirms our work.



38. $\lim\limits_{x\to\pm\infty} \dfrac{1+x^4}{x^2-x^4} = \lim\limits_{x\to\pm\infty} \dfrac{\dfrac{1+x^4}{x^4}}{\dfrac{x^2-x^4}{x^4}} = \lim\limits_{x\to\pm\infty} \dfrac{\dfrac{1}{x^4}+1}{\dfrac{1}{x^2}-1} = \dfrac{\lim\limits_{x\to\pm\infty}\left(\dfrac{1}{x^4}+1\right)}{\lim\limits_{x\to\pm\infty}\left(\dfrac{1}{x^2}-1\right)} = \dfrac{\lim\limits_{x\to\pm\infty} \dfrac{1}{x^4} + \lim\limits_{x\to\pm\infty} 1}{\lim\limits_{x\to\pm\infty} \dfrac{1}{x^2} - \lim\limits_{x\to\pm\infty} 1}$

$= \dfrac{0+1}{0-1} = -1,$ so $y = -1$ is a horizontal asymptote.

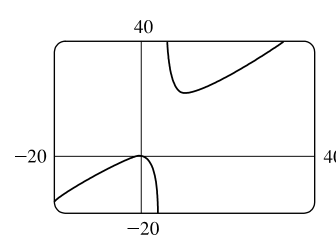
$y = f(x) = \dfrac{1+x^4}{x^2-x^4} = \dfrac{1+x^4}{x^2(1-x^2)} = \dfrac{1+x^4}{x^2(1+x)(1-x)}$. The denominator is zero when $x = 0, -1$, and 1, but the numerator is nonzero, so $x = 0$, $x = -1$, and $x = 1$ are vertical asymptotes. Notice that as $x \to 0$, the numerator and denominator are both positive, so $\lim\limits_{x\to0} f(x) = \infty$. The graph confirms our work.



39. $y = f(x) = \dfrac{x^3-x}{x^2-6x+5} = \dfrac{x(x^2-1)}{(x-1)(x-5)} = \dfrac{x(x+1)(x-1)}{(x-1)(x-5)} = \dfrac{x(x+1)}{x-5} = g(x)$ for $x \neq 1$.

The graph of g is the same as the graph of f with the exception of a hole in the graph of f at $x = 1$. By long division, $g(x) = \dfrac{x^2+x}{x-5} = x+6+\dfrac{30}{x-5}$.

As $x \to \pm\infty$, $g(x) \to \pm\infty$, so there is no horizontal asymptote. The denominator of g is zero when $x = 5$. $\lim\limits_{x\to5^-} g(x) = -\infty$ and $\lim\limits_{x\to5^+} g(x) = \infty$, so $x = 5$ is a vertical asymptote. The graph confirms our work.

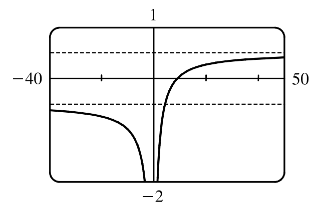


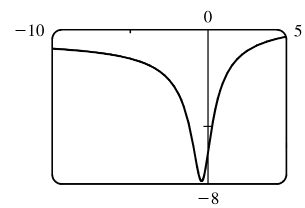
40. $\displaystyle\lim_{x\to\infty}\frac{x-9}{\sqrt{4x^2+3x+2}}=\lim_{x\to\infty}\frac{1-9/x}{\sqrt{4+(3/x)+(2/x^2)}}=\frac{1-0}{\sqrt{4+0+0}}=\frac{1}{2}.$

Using the fact that $\sqrt{x^2}=|x|=-x$ for $x<0$, we divide the numerator by $-x$ and the denominator by $\sqrt{x^2}$.

Thus, $\displaystyle\lim_{x\to-\infty}\frac{x-9}{\sqrt{4x^2+3x+2}}=\lim_{x\to-\infty}\frac{-1+9/x}{\sqrt{4+(3/x)+(2/x^2)}}=\frac{-1+0}{\sqrt{4+0+0}}=-\frac{1}{2}.$

The horizontal asymptotes are $y=\pm\frac{1}{2}$. The polynomial $4x^2+3x+2$ is positive for all x, so the denominator never approaches zero, and thus there is no vertical asymptote.





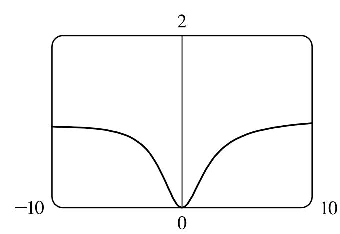
41. From the graph, it appears $y=1$ is a horizontal asymptote.

$$\lim_{x\to\pm\infty}\frac{3x^3+500x^2}{x^3+500x^2+100x+2000}=\lim_{x\to\pm\infty}\frac{\dfrac{3x^3+500x^2}{x^3}}{\dfrac{x^3+500x^2+100x+2000}{x^3}}$$

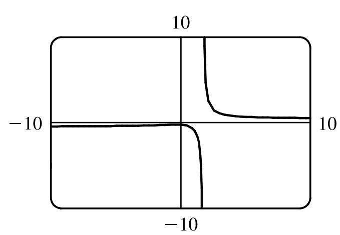
$$=\lim_{x\to\pm\infty}\frac{3+(500/x)}{1+(500/x)+(100/x^2)+(2000/x^3)}$$

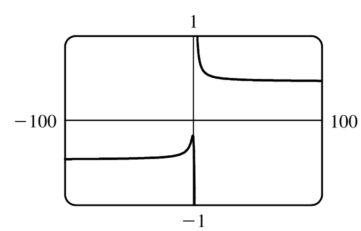
$$=\frac{3+0}{1+0+0+0}=3,\quad \text{so } y=3 \text{ is a horizontal asymptote.}$$

The discrepancy can be explained by the choice of the viewing window. Try $[-100{,}000,\ 100{,}000]$ by $[-1,4]$ to get a graph that lends credibility to our calculation that $y=3$ is a horizontal asymptote.



42. (a)





From the graph, it appears at first that there is only one horizontal asymptote, at $y\approx 0$, and a vertical asymptote at $x\approx 1.7$. However, if we graph the function with a wider and shorter viewing rectangle, we see that in fact there seem to be two horizontal asymptotes: one at $y\approx 0.5$ and one at $y\approx -0.5$. So we estimate that

$$\lim_{x\to\infty}\frac{\sqrt{2x^2+1}}{3x-5}\approx 0.5 \qquad \text{and} \qquad \lim_{x\to-\infty}\frac{\sqrt{2x^2+1}}{3x-5}\approx -0.5$$

(b) $f(1000)\approx 0.4722$ and $f(10{,}000)\approx 0.4715$, so we estimate that $\displaystyle\lim_{x\to\infty}\frac{\sqrt{2x^2+1}}{3x-5}\approx 0.47.$

$f(-1000)\approx -0.4706$ and $f(-10{,}000)\approx -0.4713$, so we estimate that $\displaystyle\lim_{x\to-\infty}\frac{\sqrt{2x^2+1}}{3x-5}\approx -0.47.$

(c) $\lim\limits_{x\to\infty} \dfrac{\sqrt{2x^2+1}}{3x-5} = \lim\limits_{x\to\infty} \dfrac{\sqrt{2+1/x^2}}{3-5/x}$ $\quad$ [since $\sqrt{x^2} = x$ for $x > 0$] $\quad = \dfrac{\sqrt{2}}{3} \approx 0.471404.$

For $x < 0$, we have $\sqrt{x^2} = |x| = -x$, so when we divide the numerator by x, with $x < 0$, we get $\dfrac{1}{x}\sqrt{2x^2+1} = -\dfrac{1}{\sqrt{x^2}}\sqrt{2x^2+1} = -\sqrt{2+1/x^2}$. Therefore,

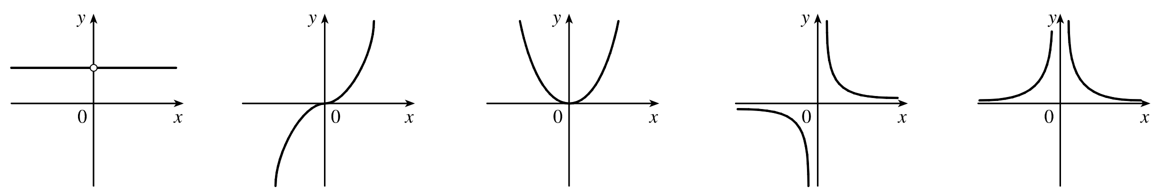
$$\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to-\infty} \frac{-\sqrt{2+1/x^2}}{3-5/x} = -\frac{\sqrt{2}}{3} \approx -0.471404.$$

43. Divide the numerator and the denominator by the highest power of x in $Q(x)$.

(a) If $\deg P < \deg Q$, then the numerator $\to 0$ but the denominator doesn't. So $\lim\limits_{x\to\infty} [P(x)/Q(x)] = 0$.

(b) If $\deg P > \deg Q$, then the numerator $\to \pm\infty$ but the denominator doesn't, so $\lim\limits_{x\to\infty} [P(x)/Q(x)] = \pm\infty$ (depending on the ratio of the leading coefficients of P and Q).

44.



(i) $n = 0$ $\quad$ (ii) $n > 0$ $\quad$ (n odd) $\quad$ (iii) $n > 0$ $\quad$ (n even) $\quad$ (iv) $n < 0$ $\quad$ (n odd) $\quad$ (v) $n < 0$ $\quad$ (n even)

From these sketches we see that

(a) $\lim\limits_{x\to 0^+} x^n = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \\ \infty & \text{if } n < 0 \end{cases}$

(b) $\lim\limits_{x\to 0^-} x^n = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n > 0 \\ -\infty & \text{if } n < 0,\ n \text{ odd} \\ \infty & \text{if } n < 0,\ n \text{ even} \end{cases}$

(c) $\lim\limits_{x\to\infty} x^n = \begin{cases} 1 & \text{if } n = 0 \\ \infty & \text{if } n > 0 \\ 0 & \text{if } n < 0 \end{cases}$

(d) $\lim\limits_{x\to-\infty} x^n = \begin{cases} 1 & \text{if } n = 0 \\ -\infty & \text{if } n > 0,\ n \text{ odd} \\ \infty & \text{if } n > 0,\ n \text{ even} \\ 0 & \text{if } n < 0 \end{cases}$

45. Let's look for a rational function.

(1) $\lim\limits_{x\to\pm\infty} f(x) = 0 \quad\Rightarrow\quad$ degree of numerator < degree of denominator

(2) $\lim\limits_{x\to 0} f(x) = -\infty \quad\Rightarrow\quad$ there is a factor of x^2 in the denominator (not just x, since that would produce a sign change at $x = 0$), and the function is negative near $x = 0$.

(3) $\lim\limits_{x\to 3^-} f(x) = \infty$ and $\lim\limits_{x\to 3^+} f(x) = -\infty \quad\Rightarrow\quad$ vertical asymptote at $x = 3$; there is a factor of $(x-3)$ in the denominator.

(4) $f(2) = 0 \quad\Rightarrow\quad$ 2 is an x-intercept; there is at least one factor of $(x-2)$ in the numerator.

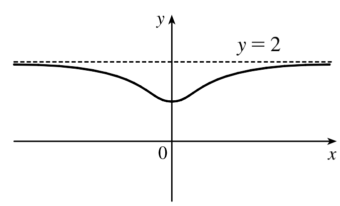
Combining all of this information and putting in a negative sign to give us the desired left- and right-hand limits gives us $f(x) = \dfrac{2-x}{x^2(x-3)}$ as one possibility.

46. Since the function has vertical asymptotes $x = 1$ and $x = 3$, the denominator of the rational function we are looking for must have factors $(x - 1)$ and $(x - 3)$. Because the horizontal asymptote is $y = 1$, the degree of the numerator must equal the degree of the denominator, and the ratio of the leading coefficients must be 1. One possibility is $f(x) = \dfrac{x^2}{(x-1)(x-3)}$.

47. (a) We must first find the function f. Since f has a vertical asymptote $x = 4$ and x-intercept $x = 1$, $x - 4$ is a factor of the denominator and $x - 1$ is a factor of the numerator. There is a removable discontinuity at $x = -1$, so $x - (-1) = x + 1$ is a factor of both the numerator and denominator. Thus, f now looks like this: $f(x) = \dfrac{a(x-1)(x+1)}{(x-4)(x+1)}$, where a is still to be determined. Then $\lim\limits_{x \to -1} f(x) = \lim\limits_{x \to -1} \dfrac{a(x-1)(x+1)}{(x-4)(x+1)} = \lim\limits_{x \to -1} \dfrac{a(x-1)}{x-4} = \dfrac{a(-1-1)}{(-1-4)} = \dfrac{2}{5}a$, so $\dfrac{2}{5}a = 2$, and $a = 5$. Thus $f(x) = \dfrac{5(x-1)(x+1)}{(x-4)(x+1)}$ is a ratio of quadratic functions satisfying all the given conditions and $f(0) = \dfrac{5(-1)(1)}{(-4)(1)} = \dfrac{5}{4}$.

(b) $\lim\limits_{x \to \infty} f(x) = 5 \lim\limits_{x \to \infty} \dfrac{x^2 - 1}{x^2 - 3x - 4} = 5 \lim\limits_{x \to \infty} \dfrac{(x^2/x^2) - (1/x^2)}{(x^2/x^2) - (3x/x^2) - (4/x^2)} = 5\dfrac{1-0}{1-0-0} = 5(1) = 5$

48. $y = \dfrac{1+2x^2}{1+x^2}$ has domain $\mathbb{R}$.

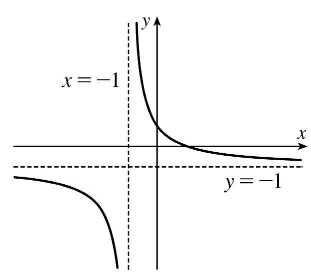


$\lim\limits_{x \to \pm\infty} \dfrac{1+2x^2}{1+x^2} = \lim\limits_{x \to \pm\infty} \dfrac{1/x^2 + 2}{1/x^2 + 1} = \dfrac{0+2}{0+1} = 2$, so $y = 2$ is a HA.

There is no VA. $y' = \dfrac{(1+x^2)(4x) - (1+2x^2)(2x)}{(1+x^2)^2} = \dfrac{2x}{(1+x^2)^2} > 0$ $\Leftrightarrow$ $x > 0$, and $y' < 0$ $\Leftrightarrow$ $x < 0$. Thus, y is increasing on $(0, \infty)$ and y is decreasing on $(-\infty, 0)$. There is a local (and absolute) minimum at $(0, 1)$. $y'' = \dfrac{(1+x^2)^2(2) - (2x) \cdot 2(1+x^2)(2x)}{[(1+x^2)^2]^2} = \dfrac{2-6x^2}{(1+x^2)^3} = 0$ $\Leftrightarrow$ $x = \pm\frac{1}{\sqrt{3}}$.

$y'' > 0$ $\Leftrightarrow$ $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$, so the curve is CU on $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and CD on $\left(-\infty, -\frac{1}{\sqrt{3}}\right)$ and $\left(\frac{1}{\sqrt{3}}, \infty\right)$. There are IP at $\left(\pm\frac{1}{\sqrt{3}}, \frac{5}{4}\right)$.

49. $y = \dfrac{1-x}{1+x}$ has domain $(-\infty, -1) \cup (-1, \infty)$.



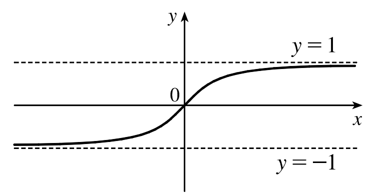
$\lim\limits_{x \to \pm\infty} \dfrac{1-x}{1+x} = \lim\limits_{x \to \pm\infty} \dfrac{1/x - 1}{1/x + 1} = \dfrac{0-1}{0+1} = -1$, so $y = -1$ is a HA.

The line $x = -1$ is a VA.

$y' = \dfrac{(1+x)(-1) - (1-x)(1)}{(1+x)^2} = \dfrac{-2}{(1+x)^2} < 0$ for $x \neq 1$. Thus, $(-\infty, -1)$ and $(-1, \infty)$ are intervals of decrease.

$y'' = -2 \cdot \dfrac{-2(1+x)}{[(1+x)^2]^2} = \dfrac{4}{(1+x)^3} < 0$ for $x < -1$ and $y'' > 0$ for $x > -1$, so the curve is CD on $(-\infty, -1)$ and CU on $(-1, \infty)$. Since $x = -1$ is not in the domain, there is no IP.

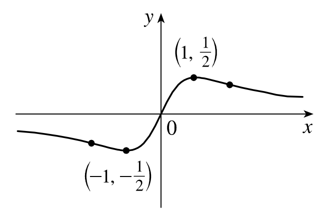
50. $y = \dfrac{x}{\sqrt{x^2+1}} = \dfrac{x/|x|}{\sqrt{1+1/x^2}}$ has domain $\mathbb{R}$. As $x \to \pm\infty$, $y \to \pm 1$, so $y = \pm 1$ are HA. There is no VA. $y = x\left(x^2+1\right)^{-1/2} \Rightarrow$



$$y' = x\left(-\tfrac{1}{2}\right)(x^2+1)^{-3/2}(2x) + (x^2+1)^{-1/2}(1)$$
$$= (x^2+1)^{-3/2}[-x^2 + (x^2+1)]$$
$$= (x^2+1)^{-3/2} > 0 \text{ for all } x$$

Thus, y is increasing for all x. $y'' = \left(-\frac{3}{2}\right)(x^2+1)^{-5/2}(2x) = \dfrac{-3x}{(x^2+1)^{5/2}} > 0$ for $x < 0$. So the curve is CU on $(-\infty, 0)$ and CD on $(0, \infty)$. There is an inflection point at $(0, 0)$.

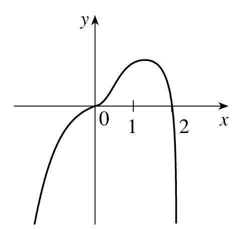
51. $\displaystyle\lim_{x\to\pm\infty} \frac{x}{x^2+1} = \lim_{x\to\pm\infty} \frac{1/x}{1+1/x^2} = \frac{0}{1+0} = 0$, so $y = 0$ is a horizontal asymptote.



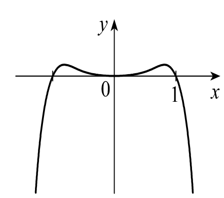
$y' = \dfrac{x^2+1-x(2x)}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2} = 0$ when $x = \pm 1$ and $y' > 0 \Leftrightarrow$ $x^2 < 1 \Leftrightarrow -1 < x < 1$, so y is increasing on $(-1, 1)$ and decreasing on $(-\infty, -1)$ and $(1, \infty)$.

$y'' = \dfrac{(1+x^2)^2(-2x) - (1-x^2)2(x^2+1)2x}{(1+x^2)^4} = \dfrac{2x(x^2-3)}{(1+x^2)^3} > 0 \Leftrightarrow x > \sqrt{3}$ or $-\sqrt{3} < x < 0$, so y is CU on $\left(\sqrt{3}, \infty\right)$ and $\left(-\sqrt{3}, 0\right)$ and CD on $\left(-\infty, -\sqrt{3}\right)$ and $\left(0, \sqrt{3}\right)$.

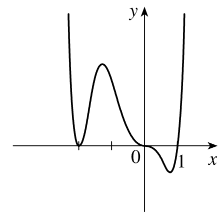
52. $y = f(x) = 2x^3 - x^4 = x^3(2-x)$. The y-intercept is $f(0) = 0$. The x-intercepts are 0 and 2. There are sign changes at 0 and 2 (odd exponents on x and $2-x$). As $x \to \infty$, $f(x) \to -\infty$ because $x^3 \to \infty$ and $2 - x \to -\infty$. As $x \to -\infty$, $f(x) \to -\infty$ because $x^3 \to -\infty$ and $2 - x \to \infty$. Note that the graph of f near $x = 0$ flattens out (looks like $y = x^3$).



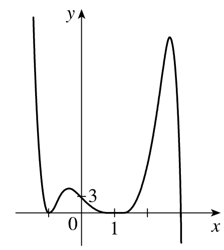
53. $y = f(x) = x^4 - x^6 = x^4(1-x^2) = x^4(1+x)(1-x)$. The y-intercept is $f(0) = 0$. The x-intercepts are 0, -1, and 1 [found by solving $f(x) = 0$ for x]. Since $x^4 > 0$ for $x \neq 0$, f doesn't change sign at $x = 0$. The function does change sign at $x = -1$ and $x = 1$. As $x \to \pm\infty$, $f(x) = x^4(1-x^2)$ approaches $-\infty$ because $x^4 \to \infty$ and $(1-x^2) \to -\infty$.



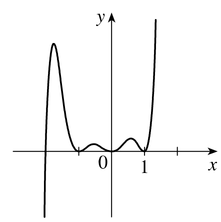
54. $y = f(x) = x^3(x+2)^2(x-1)$. The y-intercept is $f(0) = 0$. The x-intercepts are 0, -2, and 1. There are sign changes at 0 and 1 (odd exponents on x and $x - 1$). There is no sign change at -2. Also, $f(x) \to \infty$ as $x \to \infty$ because all three factors are large. And $f(x) \to \infty$ as $x \to -\infty$ because $x^3 \to -\infty$, $(x+2)^2 \to \infty$, and $(x-1) \to -\infty$. Note that the graph of f at $x = 0$ flattens out (looks like $y = -x^3$).



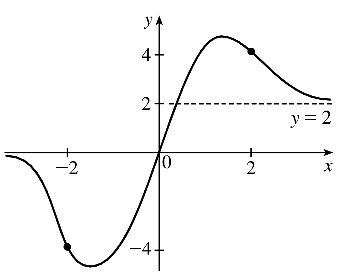
55. $y = f(x) = (3-x)(1+x)^2(1-x)^4$. The y-intercept is $f(0) = 3(1)^2(1)^4 = 3$. The x-intercepts are 3, -1, and 1. There is a sign change at 3, but not at -1 and 1. When x is large positive, $3 - x$ is negative and the other factors are positive, so $\lim_{x\to\infty} f(x) = -\infty$. When x is large negative, $3 - x$ is positive, so $\lim_{x\to-\infty} f(x) = \infty$.



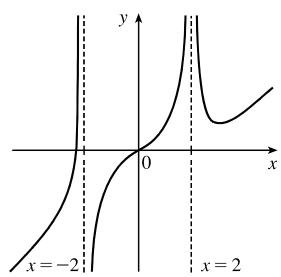
56. $y = f(x) = x^2(x^2-1)^2(x+2) = x^2(x+1)^2(x-1)^2(x+2)$. The y-intercept is $f(0) = 0$. The x-intercepts are 0, -1, 1, and -2. There is a sign change at -2, but not at 0, -1, and 1. When x is large positive, all the factors are positive, so $\lim_{x\to\infty} f(x) = \infty$. When x is large negative, only $x + 2$ is negative, so $\lim_{x\to-\infty} f(x) = -\infty$.



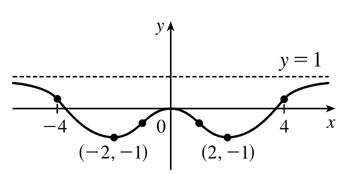
57. $f(2) = 4$, $f(-2) = -4$, $\lim_{x\to-\infty} f(x) = 0$, $\lim_{x\to\infty} f(x) = 2$



58. $\lim_{x\to-\infty} f(x) = -\infty$, $\lim_{x\to-2^-} f(x) = \infty$, $\lim_{x\to-2^+} f(x) = -\infty$, $\lim_{x\to2} f(x) = \infty$, $\lim_{x\to\infty} f(x) = \infty$

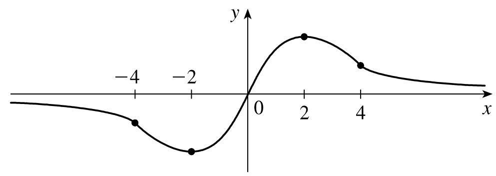


59. First we plot the points which are known to be on the graph: $(2, -1)$ and $(0, 0)$. We can also draw a short line segment of slope 0 at $x = 2$, since we are given that $f'(2) = 0$. Now we know that $f'(x) < 0$ (that is, the function is decreasing) on $(0, 2)$, and that $f''(x) < 0$ on $(0, 1)$ and $f''(x) > 0$ on $(1, 2)$. So we must join the points $(0, 0)$ and $(2, -1)$ in



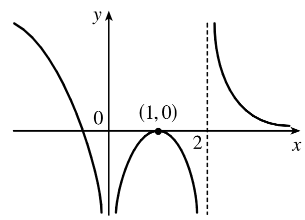
such a way that the curve is concave down on $(0, 1)$ and concave up on $(1, 2)$. The curve must be concave up and increasing on $(2, 4)$ and concave down and increasing toward $y = 1$ on $(4, \infty)$. Now we just need to reflect the curve in the y-axis, since we are given that f is an even function [the condition that $f(-x) = f(x)$ for all x].

60. The diagram shows one possible function satisfying all of the given conditions.

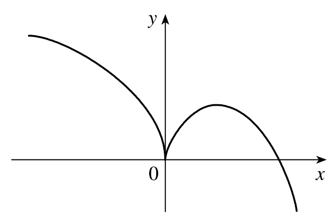


61. We are given that $f(1) = f'(1) = 0$. So we can draw a short horizontal line at the point $(1, 0)$ to represent this situation. We are given that $x = 0$ and $x = 2$ are vertical asymptotes, with $\lim_{x\to 0} f(x) = -\infty$, $\lim_{x\to 2^+} f(x) = \infty$ and $\lim_{x\to 2^-} f(x) = -\infty$, so we can draw the parts of the curve which approach these asymptotes.

On the interval $(-\infty, 0)$, the graph is concave down, and $f(x) \to \infty$ as $x \to -\infty$. Between the asymptotes the graph is concave down. On the interval $(2, \infty)$ the graph is concave up, and $f(x) \to 0$ as $x \to \infty$, so $y = 0$ is a horizontal asymptote. The diagram shows one possible function satisfying all of the given conditions.

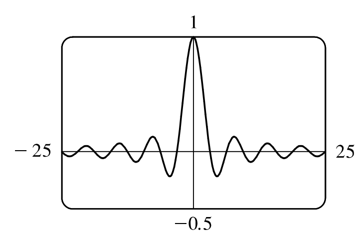


62. The diagram shows one possible function satisfying all of the given conditions.



63. (a) Since $-1 \le \sin x \le 1$ for all x, $-\dfrac{1}{x} \le \dfrac{\sin x}{x} \le \dfrac{1}{x}$ for $x > 0$. As $x \to \infty$, $-1/x \to 0$ and $1/x \to 0$, so by the Squeeze Theorem, $(\sin x)/x \to 0$. Thus, $\lim\limits_{x\to\infty} \dfrac{\sin x}{x} = 0$.

(b) From part (a), the horizontal asymptote is $y = 0$. The function $y = (\sin x)/x$ crosses the horizontal asymptote whenever $\sin x = 0$; that is, at $x = \pi n$ for every integer n. Thus, the graph crosses the asymptote *an infinite number of times.*

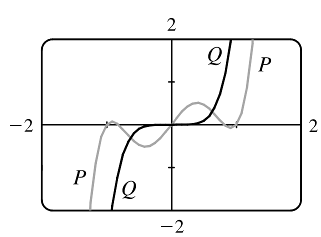


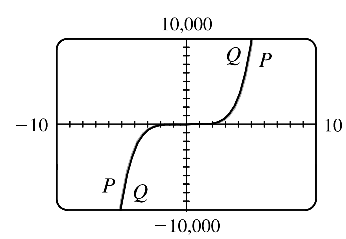
64. (a) In both viewing rectangles,

$\lim\limits_{x\to\infty} P(x) = \lim\limits_{x\to\infty} Q(x) = \infty$ and

$\lim\limits_{x\to-\infty} P(x) = \lim\limits_{x\to-\infty} Q(x) = -\infty$.

In the larger viewing rectangle, P and Q become less distinguishable.





(b) $\lim\limits_{x\to\infty} \dfrac{P(x)}{Q(x)} = \lim\limits_{x\to\infty} \dfrac{3x^5 - 5x^3 + 2x}{3x^5} = \lim\limits_{x\to\infty} \left(1 - \dfrac{5}{3} \cdot \dfrac{1}{x^2} + \dfrac{2}{3} \cdot \dfrac{1}{x^4}\right) = 1 - \frac{5}{3}(0) + \frac{2}{3}(0) = 1 \quad\Rightarrow$

P and Q have the same end behavior.

65. $\lim\limits_{x\to\infty} \dfrac{4x-1}{x} = \lim\limits_{x\to\infty} \left(4 - \dfrac{1}{x}\right) = 4$ and $\lim\limits_{x\to\infty} \dfrac{4x^2+3x}{x^2} = \lim\limits_{x\to\infty} \left(4 + \dfrac{3}{x}\right) = 4$. Therefore, by the Squeeze Theorem,

$\lim\limits_{x\to\infty} f(x) = 4.$

66. (a) After t minutes, $25t$ liters of brine with 30 g of salt per liter has been pumped into the tank, so it contains

$(5000 + 25t)$ liters of water and $25t \cdot 30 = 750t$ grams of salt. Therefore, the salt concentration at time t will be

$C(t) = \dfrac{750t}{5000+25t} = \dfrac{30t}{200+t} \dfrac{\text{g}}{\text{L}}.$

(b) $\lim\limits_{t\to\infty} C(t) = \lim\limits_{t\to\infty} \dfrac{30t}{200+t} = \lim\limits_{t\to\infty} \dfrac{30t/t}{200/t + t/t} = \dfrac{30}{0+1} = 30$. So the salt concentration approaches that of the brine

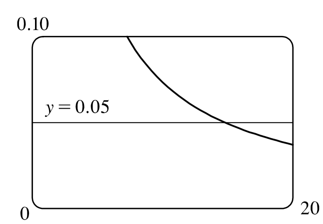
being pumped into the tank.

67. Let $g(x) = \dfrac{3x^2+1}{2x^2+x+1}$ and $f(x) = |g(x) - 1.5|$. Note that

$\lim\limits_{x\to\infty} g(x) = \frac{3}{2}$ and $\lim\limits_{x\to\infty} f(x) = 0$. We are interested in finding the

x-value at which $f(x) < 0.05$. From the graph, we find that $x \approx 14.804$,

so we choose $N = 15$ (or any larger number).

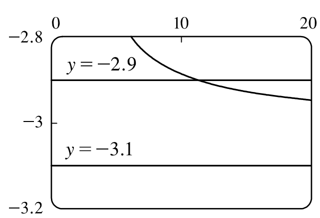


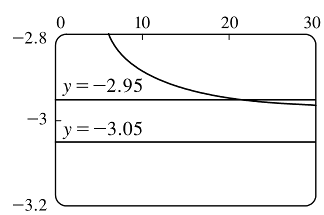
68. We want to find a value of N such that $x > N \quad\Rightarrow\quad \left|\dfrac{1-3x}{\sqrt{x^2+1}} - (-3)\right| < \varepsilon$, or equivalently,

$-3 - \varepsilon < \dfrac{1-3x}{\sqrt{x^2+1}} < -3 + \varepsilon$. When $\varepsilon = 0.1$, we graph $y = f(x) = \dfrac{1-3x}{\sqrt{x^2+1}}$, $y = -3.1$, and $y = -2.9$. From the graph,

we find that $f(x) = -2.9$ at about $x = 11.283$, so we choose $N = 12$ (or any larger number). Similarly for $\varepsilon = 0.05$, we find

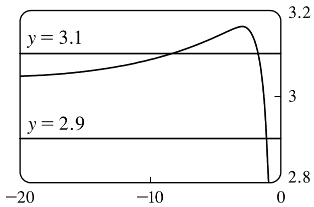
that $f(x) = -2.95$ at about $x = 21.379$, so we choose $N = 22$ (or any larger number).

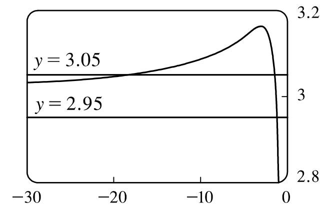




69. We want a value of N such that $x < N \Rightarrow \left|\dfrac{1-3x}{\sqrt{x^2+1}} - 3\right| < \varepsilon$, or equivalently, $3 - \varepsilon < \dfrac{1-3x}{\sqrt{x^2+1}} < 3 + \varepsilon$. When $\varepsilon = 0.1$,

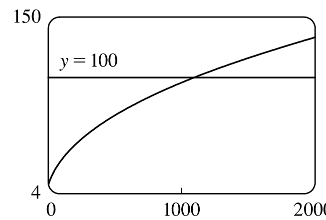
we graph $y = f(x) = \dfrac{1-3x}{\sqrt{x^2+1}}$, $y = 3.1$, and $y = 2.9$. From the graph, we find that $f(x) = 3.1$ at about $x = -8.092$, so we choose $N = -9$ (or any lesser number). Similarly for $\varepsilon = 0.05$, we find that $f(x) = 3.05$ at about $x = -18.338$, so we choose $N = -19$ (or any lesser number).





70. We want to find a value of N such that $x > N \quad \Rightarrow \quad \dfrac{3x}{\sqrt{x-3}} > 100.$

We graph $y = f(x) = \dfrac{3x}{\sqrt{x-3}}$ and $y = 100$. From the graph, we find that $f(x) = 100$ at about $x = 1108.103$, so we choose $N = 1109$ (or any larger number).



71. (a) $1/x^2 < 0.0001 \quad \Leftrightarrow \quad x^2 > 1/0.0001 = 10\,000 \quad \Leftrightarrow \quad x > 100 \quad (x > 0)$

(b) If $\varepsilon > 0$ is given, then $1/x^2 < \varepsilon \quad \Leftrightarrow \quad x^2 > 1/\varepsilon \quad \Leftrightarrow \quad x > 1/\sqrt{\varepsilon}$. Let $N = 1/\sqrt{\varepsilon}$.

Then $x > N \quad \Rightarrow \quad x > \dfrac{1}{\sqrt{\varepsilon}} \quad \Rightarrow \quad \left|\dfrac{1}{x^2} - 0\right| = \dfrac{1}{x^2} < \varepsilon$, so $\displaystyle\lim_{x\to\infty} \frac{1}{x^2} = 0.$

72. (a) $1/\sqrt{x} < 0.0001 \quad \Leftrightarrow \quad \sqrt{x} > 1/0.0001 = 10^4 \quad \Leftrightarrow \quad x > 10^8$

(b) If $\varepsilon > 0$ is given, then $1/\sqrt{x} < \varepsilon \quad \Leftrightarrow \quad \sqrt{x} > 1/\varepsilon \quad \Leftrightarrow \quad x > 1/\varepsilon^2$. Let $N = 1/\varepsilon^2$.

Then $x > N \quad \Rightarrow \quad x > \dfrac{1}{\varepsilon^2} \quad \Rightarrow \quad \left|\dfrac{1}{\sqrt{x}} - 0\right| = \dfrac{1}{\sqrt{x}} < \varepsilon$, so $\displaystyle\lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0.$

73. For $x < 0$, $|1/x - 0| = -1/x$. If $\varepsilon > 0$ is given, then $-1/x < \varepsilon \quad \Leftrightarrow \quad x < -1/\varepsilon$.

Take $N = -1/\varepsilon$. Then $x < N \quad \Rightarrow \quad x < -1/\varepsilon \quad \Rightarrow \quad |(1/x) - 0| = -1/x < \varepsilon$, so $\displaystyle\lim_{x\to-\infty} (1/x) = 0.$

74. Given $M > 0$, we need $N > 0$ such that $x > N \quad \Rightarrow \quad x^3 > M$. Now $x^3 > M \quad \Leftrightarrow \quad x > \sqrt[3]{M}$, so take $N = \sqrt[3]{M}$. Then $x > N = \sqrt[3]{M} \quad \Rightarrow \quad x^3 > M$, so $\displaystyle\lim_{x\to\infty} x^3 = \infty.$

75. (a) Suppose that $\displaystyle\lim_{x\to\infty} f(x) = L$. Then for every $\varepsilon > 0$ there is a corresponding positive number N such that $|f(x) - L| < \varepsilon$ whenever $x > N$. If $t = 1/x$, then $x > N \quad \Leftrightarrow \quad 0 < 1/x < 1/N \quad \Leftrightarrow \quad 0 < t < 1/N$. Thus, for every $\varepsilon > 0$ there is a corresponding $\delta > 0$ (namely $1/N$) such that $|f(1/t) - L| < \varepsilon$ whenever $0 < t < \delta$. This proves that $\displaystyle\lim_{t\to0^+} f(1/t) = L = \lim_{x\to\infty} f(x).$

Now suppose that $\displaystyle\lim_{x\to-\infty} f(x) = L$. Then for every $\varepsilon > 0$ there is a corresponding negative number N such that $|f(x) - L| < \varepsilon$ whenever $x < N$. If $t = 1/x$, then $x < N \quad \Leftrightarrow \quad 1/N < 1/x < 0 \quad \Leftrightarrow \quad 1/N < t < 0$. Thus, for every

$\varepsilon > 0$ there is a corresponding $\delta > 0$ (namely $-1/N$) such that $|f(1/t) - L| < \varepsilon$ whenever $-\delta < t < 0$. This proves that $\lim\limits_{t \to 0^-} f(1/t) = L = \lim\limits_{x \to -\infty} f(x)$.

(b) $\lim\limits_{x \to 0^+} x \sin \dfrac{1}{x} = \lim\limits_{t \to 0^+} t \sin \dfrac{1}{t}$ [let $x = t$]

$= \lim\limits_{y \to \infty} \dfrac{1}{y} \sin y$ [part (a) with $y = 1/t$]

$= \lim\limits_{x \to \infty} \dfrac{\sin x}{x}$ [let $y = x$]

$= 0$ [by Exercise 63]

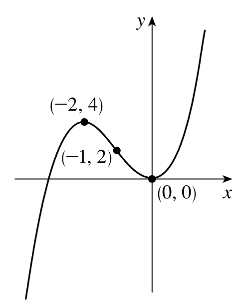
76. Definition Let f be a function defined on some interval $(-\infty, a)$. Then $\lim\limits_{x \to -\infty} f(x) = -\infty$ means that for every negative number M there is a corresponding negative number N such that $f(x) < M$ whenever $x < N$.

Now we use the definition to prove that $\lim\limits_{x \to -\infty} \left(1 + x^3\right) = -\infty$. Given a negative number M, we need a negative number N such that $x < N \quad \Rightarrow \quad 1 + x^3 < M$. Now $1 + x^3 < M \quad \Leftrightarrow \quad x^3 < M - 1 \quad \Leftrightarrow \quad x < \sqrt[3]{M - 1}$. Thus, we take $N = \sqrt[3]{M - 1}$ and find that $x < N \quad \Rightarrow \quad 1 + x^3 < M$. This proves that $\lim\limits_{x \to -\infty} \left(1 + x^3\right) = -\infty$.

3.5 Summary of Curve Sketching

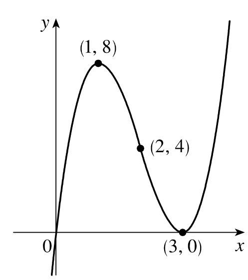
1. $y = f(x) = x^3 + 3x^2 = x^2(x + 3)$ **A.** f is a polynomial, so $D = \mathbb{R}$.
B. y-intercept $= f(0) = 0$, x-intercepts are 0 and -3 **C.** No symmetry
D. No asympote **E.** $f'(x) = 3x^2 + 6x = 3x(x + 2) > 0 \quad \Leftrightarrow \quad x < -2$ or $x > 0$, so f is increasing on $(-\infty, -2)$ and $(0, \infty)$, and decreasing on $(-2, 0)$.
F. Local maximum value $f(-2) = 4$, local minimum value $f(0) = 0$
G. $f''(x) = 6x + 6 = 6(x + 1) > 0 \quad \Leftrightarrow \quad x > -1$, so f is CU on $(-1, \infty)$ and CD on $(-\infty, -1)$. IP at $(-1, 2)$

H.



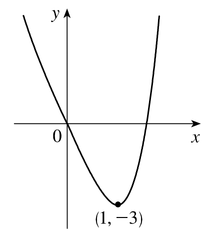
2. $y = f(x) = 2x^3 - 12x^2 + 18x = 2x(x^2 - 6x + 9) = 2x(x - 3)^2$
A. $D = \mathbb{R}$ **B**. x-intercepts are 0 and 3, y-intercept $f(0) = 0$ **C**. No symmetry
D. No asymptote
E. $f'(x) = 6x^2 - 24x + 18 = 6(x^2 - 4x + 3)$
$= 6(x - 1)(x - 3) > 0 \quad \Leftrightarrow \quad x < 1$ or $x > 3$
and $f'(x) < 0 \quad \Leftrightarrow \quad 1 < x < 3$, so f is increasing on $(-\infty, 1)$ and $(3, \infty)$, and decreasing on $(1, 3)$. **F**. Local maximum value $f(1) = 8$, local minimum value $f(3) = 0$ **G**. $f''(x) = 12x - 24 = 12(x - 2) > 0 \quad \Leftrightarrow \quad x > 2$, so f is CU on $(2, \infty)$, and f is CD on $(-\infty, 2)$. IP at $(2, 4)$

H.



3. $y = f(x) = x^4 - 4x = x(x^3 - 4)$ **A.** $D = \mathbb{R}$ **B.** x-intercepts are 0 and $\sqrt[3]{4}$, y-intercept $= f(0) = 0$ **C.** No symmetry **D.** No asymptote **E.** $f'(x) = 4x^3 - 4 = 4(x^3 - 1) = 4(x-1)(x^2+x+1) > 0 \Leftrightarrow x > 1$, so f is increasing on $(1, \infty)$ and decreasing on $(-\infty, 1)$. **F.** Local minimum value $f(1) = -3$, no local maximum **G.** $f''(x) = 12x^2 > 0$ for all x, so f is CU on $(-\infty, \infty)$. No IP

H.

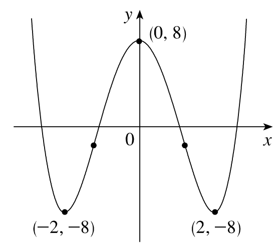


4. $y = f(x) = x^4 - 8x^2 + 8$ **A.** $D = \mathbb{R}$ **B.** y-intercept $f(0) = 8$; x-intercepts: $f(x) = 0 \Rightarrow$ [by the quadratic formula] $x = \pm\sqrt{4 \pm 2\sqrt{2}} \approx \pm 2.61, \pm 1.08$ **C.** $f(-x) = f(x)$, so f is even and symmetric about the y-axis **D.** No asymptote **E.** $f'(x) = 4x^3 - 16x = 4x(x^2 - 4) = 4x(x+2)(x-2) > 0 \Leftrightarrow -2 < x < 0$ or $x > 2$, so f is increasing on $(-2, 0)$ and $(2, \infty)$, and f is decreasing on $(-\infty, -2)$ and $(0, 2)$.

F. Local maximum value $f(0) = 8$, local minimum values $f(\pm 2) = -8$

G. $f''(x) = 12x^2 - 16 = 4(3x^2 - 4) > 0 \Rightarrow |x| > 2/\sqrt{3}$ [≈ 1.15], so f is CU on $\left(-\infty, -2/\sqrt{3}\right)$ and $\left(2/\sqrt{3}, \infty\right)$, and f is CD on $\left(-2/\sqrt{3}, 2/\sqrt{3}\right)$. IP at $\left(\pm 2/\sqrt{3}, -\frac{8}{9}\right)$

H.



5. $y = f(x) = x(x-4)^3$ **A.** $D = \mathbb{R}$ **B.** x-intercepts are 0 and 4, y-intercept $f(0) = 0$ **C.** No symmetry

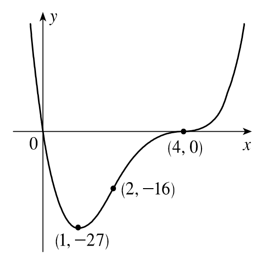
D. No asymptote

E.
$$\begin{aligned} f'(x) &= x \cdot 3(x-4)^2 + (x-4)^3 \cdot 1 = (x-4)^2[3x + (x-4)] \\ &= (x-4)^2(4x-4) = 4(x-1)(x-4)^2 > 0 \quad \Leftrightarrow \end{aligned}$$
$x > 1$, so f is increasing on $(1, \infty)$ and decreasing on $(-\infty, 1)$.

F. Local minimum value $f(1) = -27$, no local maximum value

G.
$$\begin{aligned} f''(x) &= 4[(x-1) \cdot 2(x-4) + (x-4)^2 \cdot 1] = 4(x-4)[2(x-1) + (x-4)] \\ &= 4(x-4)(3x-6) = 12(x-4)(x-2) < 0 \quad \Leftrightarrow \end{aligned}$$
$2 < x < 4$, so f is CD on $(2, 4)$ and CU on $(-\infty, 2)$ and $(4, \infty)$. IPs at $(2, -16)$ and $(4, 0)$

H.

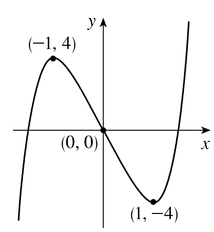


6. $y = f(x) = x^5 - 5x = x(x^4 - 5)$ **A.** $D = \mathbb{R}$ **B.** x-intercepts $\pm\sqrt[4]{5}$ and 0, y-intercept $= f(0) = 0$

C. $f(-x) = -f(x)$, so f is odd; the curve is symmetric about the origin. **D.** No asymptote

E.
$$\begin{aligned} f'(x) &= 5x^4 - 5 = 5(x^4 - 1) = 5(x^2 - 1)(x^2 + 1) \\ &= 5(x+1)(x-1)(x^2+1) > 0 \quad \Leftrightarrow \end{aligned}$$
$x < -1$ or $x > 1$, so f is increasing on $(-\infty, -1)$ and $(1, \infty)$, and f is decreasing on $(-1, 1)$. **F.** Local maximum value $f(-1) = 4$, local minimum value $f(1) = -4$ **G.** $f''(x) = 20x^3 > 0 \Leftrightarrow x > 0$, so f is CU on $(0, \infty)$ and CD on $(-\infty, 0)$. IP at $(0, 0)$

H.



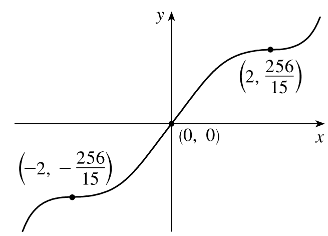
7. $y = f(x) = \frac{1}{5}x^5 - \frac{8}{3}x^3 + 16x = x\left(\frac{1}{5}x^4 - \frac{8}{3}x^2 + 16\right)$ **A.** $D = \mathbb{R}$ **B.** x-intercept 0, y-intercept $= f(0) = 0$

C. $f(-x) = -f(x)$, so f is odd; the curve is symmetric about the origin. **D.** No asymptote

E. $f'(x) = x^4 - 8x^2 + 16 = (x^2-4)^2 = (x+2)^2(x-2)^2 > 0$ for all x except ± 2, so f is increasing on $\mathbb{R}$. **F.** There is no local maximum or minimum value.

G. $f''(x) = 4x^3 - 16x = 4x(x^2-4) = 4x(x+2)(x-2) > 0 \quad\Leftrightarrow$ $-2 < x < 0$ or $x > 2$, so f is CU on $(-2, 0)$ and $(2, \infty)$, and f is CD on $(-\infty, -2)$ and $(0, 2)$. IP at $\left(-2, -\frac{256}{15}\right)$, $(0, 0)$, and $\left(2, \frac{256}{15}\right)$

H.



8. $y = f(x) = (4 - x^2)^5$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 4^5 = 1024$; x-intercepts: ± 2 **C.** $f(-x) = f(x) \quad\Rightarrow$ f is even; the curve is symmetric about the y-axis. **D.** No asymptote **E.** $f'(x) = 5(4-x^2)^4(-2x) = -10x(4-x^2)^4$, so for $x \neq \pm 2$ we have $f'(x) > 0 \quad\Leftrightarrow\quad x < 0$ and $f'(x) < 0 \quad\Leftrightarrow\quad x > 0$. Thus, f is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$. **F.** Local maximum value $f(0) = 1024$

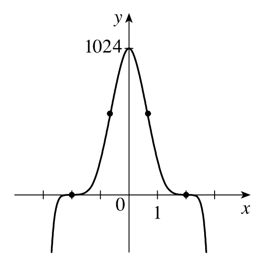
G. $f''(x) = -10x \cdot 4(4-x^2)^3(-2x) + (4-x^2)^4(-10)$

$$= -10(4-x^2)^3[-8x^2 + 4 - x^2] = -10(4-x^2)^3(4-9x^2)$$

so $f''(x) = 0 \quad\Leftrightarrow\quad x = \pm 2, \pm\frac{2}{3}$. $f''(x) > 0 \quad\Leftrightarrow\quad -2 < x < -\frac{2}{3}$ and $\frac{2}{3} < x < 2$ and $f''(x) < 0 \quad\Leftrightarrow\quad x < -2, -\frac{2}{3} < x < \frac{2}{3}$, and $x > 2$, so f is CU on $(-\infty, 2)$, $\left(-\frac{2}{3}, \frac{2}{3}\right)$, and $(2, \infty)$, and CD on $\left(-2, -\frac{2}{3}\right)$ and $\left(\frac{2}{3}, 2\right)$.

IP at $(\pm 2, 0)$ and $\left(\pm\frac{2}{3}, \left(\frac{32}{9}\right)^5\right) \approx (\pm 0.67, 568.25)$

H.

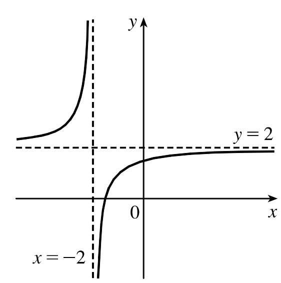


9. $y = f(x) = \dfrac{2x+3}{x+2}$ **A.** $D = \{x \mid x \neq -2\} = (-\infty, -2) \cup (-2, \infty)$ **B.** x-intercept $= -\frac{3}{2}$, y-intercept $= f(0) = \frac{3}{2}$

C. No symmetry **D.** $\displaystyle\lim_{x\to\pm\infty} \frac{2x+3}{x+2} = 2$, so $y = 2$ is a HA. $\displaystyle\lim_{x\to -2^-} \frac{2x+3}{x+2} = \infty$, $\displaystyle\lim_{x\to -2^+} \frac{2x+3}{x+2} = -\infty$, so $x = -2$ is a VA. **E.** $f'(x) = \dfrac{(x+2)\cdot 2 - (2x+3)\cdot 1}{(x+2)^2} = \dfrac{1}{(x+2)^2} > 0$ for $x \neq -2$, so f is increasing on $(-\infty, -2)$ and $(-2, \infty)$. **F.** No extreme values

G. $f''(x) = \dfrac{-2}{(x+2)^3} > 0 \quad\Leftrightarrow\quad x < -2$, so f is CU on $(-\infty, -2)$ and CD on $(-2, \infty)$. No IP

H.



10. $y = f(x) = \dfrac{x^2 + 5x}{25 - x^2} = \dfrac{x(x+5)}{(5+x)(5-x)} = \dfrac{x}{5-x}$ for $x \neq -5$. There is a hole in the graph at $\left(-5, -\frac{1}{2}\right)$.

A. $D = \{x \mid x \neq \pm 5\} = (-\infty, -5) \cup (-5, 5) \cup (5, \infty)$ **B.** x-intercept $= 0$, y-intercept $= f(0) = 0$ **C.** No symmetry

D. $\displaystyle\lim_{x\to\pm\infty} \frac{x}{5-x} = -1$, so $y = -1$ is a HA. $\displaystyle\lim_{x\to 5^-} \frac{x}{5-x} = \infty$, $\displaystyle\lim_{x\to 5^+} \frac{x}{5-x} = -\infty$, so $x = 5$ is a VA.

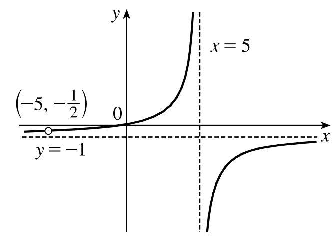
[continued]

E. $f'(x) = \dfrac{(5-x)(1) - x(-1)}{(5-x)^2} = \dfrac{5}{(5-x)^2} > 0$ for all x in D, so f is increasing on $(-\infty, -5)$, $(-5, 5)$, and $(5, \infty)$. **F.** No extreme values

G. $f'(x) = 5(5-x)^{-2} \;\Rightarrow$

$f''(x) = -10(5-x)^{-3}(-1) = \dfrac{10}{(5-x)^3} > 0 \;\Leftrightarrow\; x < 5$, so f is CU on $(-\infty, -5)$ and $(-5, 5)$, and f is CD on $(5, \infty)$. No IP

H.



11. $y = f(x) = \dfrac{x - x^2}{2 - 3x + x^2} = \dfrac{x(1-x)}{(1-x)(2-x)} = \dfrac{x}{2-x}$ for $x \neq 1$. There is a hole in the graph at $(1, 1)$.

A. $D = \{x \mid x \neq 1, 2\} = (-\infty, 1) \cup (1, 2) \cup (2, \infty)$ **B.** x-intercept $= 0$, y-intercept $= f(0) = 0$ **C.** No symmetry

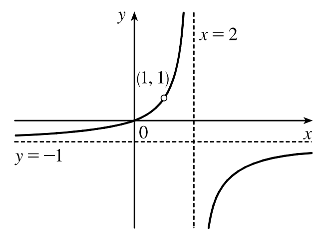
D. $\lim\limits_{x\to\pm\infty} \dfrac{x}{2-x} = -1$, so $y = -1$ is a HA. $\lim\limits_{x\to 2^-} \dfrac{x}{2-x} = \infty$, $\lim\limits_{x\to 2^+} \dfrac{x}{2-x} = -\infty$, so $x = 2$ is a VA.

E. $f'(x) = \dfrac{(2-x)(1) - x(-1)}{(2-x)^2} = \dfrac{2}{(2-x)^2} > 0$ $[x \neq 1, 2]$, so f is increasing on $(-\infty, 1)$, $(1, 2)$, and $(2, \infty)$. **F.** No extreme values

G. $f'(x) = 2(2-x)^{-2} \;\Rightarrow$

$f''(x) = -4(2-x)^{-3}(-1) = \dfrac{4}{(2-x)^3} > 0 \;\Leftrightarrow\; x < 2$, so f is CU on $(-\infty, 1)$ and $(1, 2)$, and f is CD on $(2, \infty)$. No IP

H.

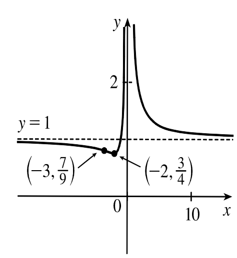


12. $y = f(x) = 1 + \dfrac{1}{x} + \dfrac{1}{x^2} = \dfrac{x^2 + x + 1}{x^2}$ **A.** $D = (-\infty, 0) \cup (0, \infty)$ **B.** y-intercept: none $[x \neq 0]$; x-intercepts: $f(x) = 0 \;\Leftrightarrow\; x^2 + x + 1 = 0$, there is no real solution, and hence, no x-intercept **C.** No symmetry

D. $\lim\limits_{x\to\pm\infty} \left(1 + \dfrac{1}{x} + \dfrac{1}{x^2}\right) = 1$, so $y = 1$ is a HA. $\lim\limits_{x\to 0} f(x) = \infty$, so $x = 0$ is a VA. **E.** $f'(x) = -\dfrac{1}{x^2} - \dfrac{2}{x^3} = \dfrac{-x-2}{x^3}$.

$f'(x) > 0 \;\Leftrightarrow\; -2 < x < 0$ and $f'(x) < 0 \;\Leftrightarrow\; x < -2$ or $x > 0$, so f is increasing on $(-2, 0)$ and decreasing on $(-\infty, -2)$ and $(0, \infty)$. **F.** Local minimum value $f(-2) = \frac{3}{4}$; no local maximum **G.** $f''(x) = \dfrac{2}{x^3} + \dfrac{6}{x^4} = \dfrac{2x + 6}{x^4}$. $f''(x) < 0 \;\Leftrightarrow\; x < -3$ and $f''(x) > 0 \;\Leftrightarrow\; -3 < x < 0$ and $x > 0$, so f is CD on $(-\infty, -3)$ and CU on $(-3, 0)$ and $(0, \infty)$. IP at $\left(-3, \frac{7}{9}\right)$

H.



13. $y = f(x) = \dfrac{x}{x^2 - 4} = \dfrac{x}{(x+2)(x-2)}$ **A.** $D = (-\infty, -2) \cup (-2, 2) \cup (2, \infty)$ **B.** x-intercept $= 0$, y-intercept $= f(0) = 0$ **C.** $f(-x) = -f(x)$, so f is odd; the graph is symmetric about the origin.

D. $\lim\limits_{x\to 2^+} \dfrac{x}{x^2 - 4} = \infty$, $\lim\limits_{x\to 2^-} f(x) = -\infty$, $\lim\limits_{x\to -2^+} f(x) = \infty$, $\lim\limits_{x\to -2^-} f(x) = -\infty$, so $x = \pm 2$ are VAs.

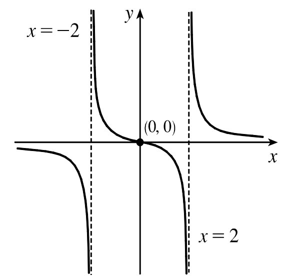
[continued]

$\lim\limits_{x\to\pm\infty} \dfrac{x}{x^2-4} = 0$, so $y = 0$ is a HA. **E.** $f'(x) = \dfrac{(x^2-4)(1) - x(2x)}{(x^2-4)^2} = -\dfrac{x^2+4}{(x^2-4)^2} < 0$ for all x in D, so f is decreasing on $(-\infty,-2)$, $(-2,2)$, and $(2,\infty)$. **F.** No extreme values

G. $$f''(x) = -\frac{(x^2-4)^2(2x) - (x^2+4)2(x^2-4)(2x)}{[(x^2-4)^2]^2} = -\frac{2x(x^2-4)[(x^2-4)-2(x^2+4)]}{(x^2-4)^4} = -\frac{2x(-x^2-12)}{(x^2-4)^3} = \frac{2x(x^2+12)}{(x+2)^3(x-2)^3}.$$

$f''(x) < 0$ if $x < -2$ or $0 < x < 2$, so f is CD on $(-\infty,-2)$ and $(0,2)$, and CU on $(-2,0)$ and $(2,\infty)$. IP at $(0,0)$

H.



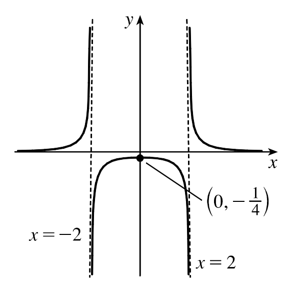
14. $y = f(x) = \dfrac{1}{x^2-4} = \dfrac{1}{(x+2)(x-2)}$ **A.** $D = (-\infty,-2)\cup(-2,2)\cup(2,\infty)$ **B.** No x-intercept, y-intercept $= f(0) = -\frac{1}{4}$ **C.** $f(-x) = f(x)$, so f is even; the graph is symmetric about the y-axis.

D. $\lim\limits_{x\to2^+} \dfrac{1}{x^2-4} = \infty$, $\lim\limits_{x\to2^-} f(x) = -\infty$, $\lim\limits_{x\to-2^+} f(x) = -\infty$, $\lim\limits_{x\to-2^-} f(x) = \infty$, so $x = \pm2$ are VAs. $\lim\limits_{x\to\pm\infty} f(x) = 0$, so $y = 0$ is a HA. **E.** $f'(x) = -\dfrac{2x}{(x^2-4)^2}$ [Reciprocal Rule] > 0 if $x < 0$ and x is in D, so f is increasing on $(-\infty,-2)$ and $(-2,0)$. f is decreasing on $(0,2)$ and $(2,\infty)$. **F.** Local maximum value $f(0) = -\frac{1}{4}$, no local minimum value

G. $$f''(x) = \frac{(x^2-4)^2(-2) - (-2x)2(x^2-4)(2x)}{[(x^2-4)^2]^2} = \frac{-2(x^2-4)\left[(x^2-4)-4x^2\right]}{(x^2-4)^4} = \frac{-2(-3x^2-4)}{(x^2-4)^3} = \frac{2(3x^2+4)}{(x^2-4)^3}$$

$f''(x) < 0 \iff -2 < x < 2$, so f is CD on $(-2,2)$ and CU on $(-\infty,-2)$ and $(2,\infty)$. No IP

H.



15. $y = f(x) = \dfrac{x^2}{x^2+3} = \dfrac{(x^2+3)-3}{x^2+3} = 1 - \dfrac{3}{x^2+3}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercept: $f(x) = 0 \iff x = 0$ **C.** $f(-x) = f(x)$, so f is even; the graph is symmetric about the y-axis.

D. $\lim\limits_{x\to\pm\infty} \dfrac{x^2}{x^2+3} = 1$, so $y = 1$ is a HA. No VA. **E.** Using the Reciprocal Rule, $f'(x) = -3\cdot\dfrac{-2x}{(x^2+3)^2} = \dfrac{6x}{(x^2+3)^2}$.

$f'(x) > 0 \iff x > 0$ and $f'(x) < 0 \iff x < 0$, so f is decreasing on $(-\infty,0)$ and increasing on $(0,\infty)$.

F. Local minimum value $f(0) = 0$, no local maximum.

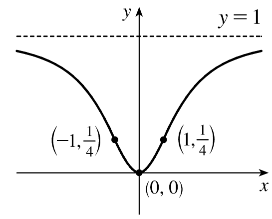
[continued]

G. $f''(x) = \dfrac{(x^2+3)^2 \cdot 6 - 6x \cdot 2(x^2+3) \cdot 2x}{[(x^2+3)^2]^2}$

$= \dfrac{6(x^2+3)[(x^2+3) - 4x^2]}{(x^2+3)^4} = \dfrac{6(3-3x^2)}{(x^2+3)^3} = \dfrac{-18(x+1)(x-1)}{(x^2+3)^3}$

$f''(x)$ is negative on $(-\infty, -1)$ and $(1, \infty)$ and positive on $(-1, 1)$, so f is CD on $(-\infty, -1)$ and $(1, \infty)$ and CU on $(-1, 1)$. IP at $\left(\pm 1, \frac{1}{4}\right)$

H.



16. $y = f(x) = \dfrac{(x-1)^2}{x^2+1} \geq 0$ with equality $\Leftrightarrow$ $x = 1$. **A.** $D = \mathbb{R}$ **B.** y-intercept $= f(0) = 1$; x-intercept 1 **C.** No symmetry **D.** $\lim\limits_{x\to\pm\infty} f(x) = \lim\limits_{x\to\pm\infty} \dfrac{x^2-2x+1}{x^2+1} = \lim\limits_{x\to\pm\infty} \dfrac{1-2/x+1/x^2}{1+1/x^2} = 1$, so $y = 1$ is a HA. No VA

E. $f'(x) = \dfrac{(x^2+1)2(x-1) - (x-1)^2(2x)}{(x^2+1)^2} = \dfrac{2(x-1)\left[(x^2+1) - x(x-1)\right]}{(x^2+1)^2} = \dfrac{2(x-1)(x+1)}{(x^2+1)^2} < 0 \quad \Leftrightarrow$

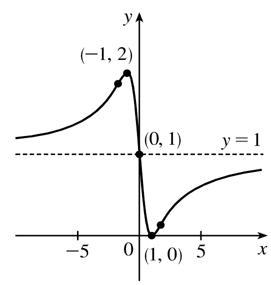
$-1 < x < 1$, so f is decreasing on $(-1, 1)$ and increasing on $(-\infty, -1)$ and $(1, \infty)$ **F.** Local maximum value $f(-1) = 2$, local minimum value $f(1) = 0$

G. $f''(x) = \dfrac{(x^2+1)^2(4x) - (2x^2-2)2(x^2+1)(2x)}{[(x^2+1)^2]^2} = \dfrac{4x(x^2+1)\left[(x^2+1) - (2x^2-2)\right]}{(x^2+1)^4} = \dfrac{4x(3-x^2)}{(x^2+1)^3}$.

$f''(x) > 0 \quad \Leftrightarrow \quad x < -\sqrt{3}$ or $0 < x < \sqrt{3}$, so f is CU on $\left(-\infty, -\sqrt{3}\right)$ and $\left(0, \sqrt{3}\right)$, and f is CD on $\left(-\sqrt{3}, 0\right)$ and $\left(\sqrt{3}, \infty\right)$.

$f\left(\pm\sqrt{3}\right) = \frac{1}{4}\left(\sqrt{3} \mp 1\right)^2 = \frac{1}{4}\left(4 \mp 2\sqrt{3}\right) = 1 \mp \frac{1}{2}\sqrt{3}$ [$\approx 0.13, 1.87$], so there are IPs at $\left(-\sqrt{3}, 1 + \frac{1}{2}\sqrt{3}\right)$, $(0, 1)$, and $\left(\sqrt{3}, 1 - \frac{1}{2}\sqrt{3}\right)$. Note that the graph is symmetric about the point $(0, 1)$.

H.



17. $y = f(x) = \dfrac{x-1}{x^2}$ **A.** $D = \{x \mid x \neq 0\} = (-\infty, 0) \cup (0, \infty)$ **B.** No y-intercept; x-intercept: $f(x) = 0 \quad \Leftrightarrow \quad x = 1$

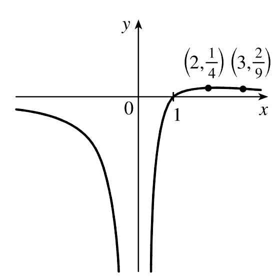
C. No symmetry **D.** $\lim\limits_{x\to\pm\infty} \dfrac{x-1}{x^2} = 0$, so $y = 0$ is a HA. $\lim\limits_{x\to 0} \dfrac{x-1}{x^2} = -\infty$, so $x = 0$ is a VA.

E. $f'(x) = \dfrac{x^2 \cdot 1 - (x-1) \cdot 2x}{(x^2)^2} = \dfrac{-x^2+2x}{x^4} = \dfrac{-(x-2)}{x^3}$, so $f'(x) > 0 \quad \Leftrightarrow \quad 0 < x < 2$ and $f'(x) < 0 \quad \Leftrightarrow$ $x < 0$ or $x > 2$. Thus, f is increasing on $(0, 2)$ and decreasing on $(-\infty, 0)$ and $(2, \infty)$. **F.** No local minimum, local maximum value $f(2) = \frac{1}{4}$.

G. $f''(x) = \dfrac{x^3 \cdot (-1) - [-(x-2)] \cdot 3x^2}{(x^3)^2} = \dfrac{2x^3 - 6x^2}{x^6} = \dfrac{2(x-3)}{x^4}$.

$f''(x)$ is negative on $(-\infty, 0)$ and $(0, 3)$ and positive on $(3, \infty)$, so f is CD on $(-\infty, 0)$ and $(0, 3)$ and CU on $(3, \infty)$. IP at $\left(3, \frac{2}{9}\right)$

H.



18. $y = f(x) = \dfrac{x}{x^3 - 1}$ **A.** $D = (-\infty, 1) \cup (1, \infty)$ **B.** y-intercept: $f(0) = 0$; x-intercept: $f(x) = 0 \iff x = 0$

C. No symmetry **D.** $\displaystyle\lim_{x \to \pm\infty} \frac{x}{x^3 - 1} = 0$, so $y = 0$ is a HA. $\displaystyle\lim_{x \to 1^-} f(x) = -\infty$ and $\displaystyle\lim_{x \to 1^+} f(x) = \infty$, so $x = 1$ is a VA.

E. $f'(x) = \dfrac{(x^3 - 1)(1) - x(3x^2)}{(x^3 - 1)^2} = \dfrac{-2x^3 - 1}{(x^3 - 1)^2}$. $f'(x) = 0 \Rightarrow x = -\sqrt[3]{1/2}$. $f'(x) > 0 \iff x < -\sqrt[3]{1/2}$ and

$f'(x) < 0 \iff -\sqrt[3]{1/2} < x < 1$ and $x > 1$, so f is increasing on $\left(-\infty, -\sqrt[3]{1/2}\right)$ and decreasing on $\left(-\sqrt[3]{1/2}, 1\right)$

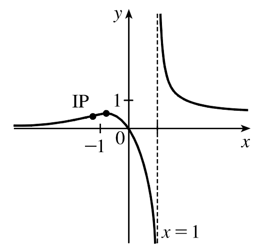
and $(1, \infty)$. **F.** Local maximum value $f\left(-\sqrt[3]{1/2}\right) = \frac{2}{3}\sqrt[3]{1/2}$; no local minimum

G. $$f''(x) = \frac{(x^3 - 1)^2(-6x^2) - (-2x^3 - 1)2(x^3 - 1)(3x^2)}{[(x^3 - 1)^2]^2}$$
$$= \frac{-6x^2(x^3 - 1)[(x^3 - 1) - (2x^3 + 1)]}{(x^3 - 1)^4} = \frac{6x^2(x^3 + 2)}{(x^3 - 1)^3}.$$

$f''(x) > 0 \iff x < -\sqrt[3]{2}$ and $x > 1$, $f''(x) < 0 \iff -\sqrt[3]{2} < x < 0$ and $0 < x < 1$, so f is CU on $\left(-\infty, -\sqrt[3]{2}\right)$ and $(1, \infty)$ and CD on $\left(-\sqrt[3]{2}, 1\right)$.

IP at $\left(-\sqrt[3]{2}, \frac{1}{3}\sqrt[3]{2}\right)$

H.



19. $y = f(x) = \dfrac{x^3}{x^3 + 1} = \dfrac{x^3}{(x + 1)(x^2 - x + 1)}$ **A.** $D = (-\infty, -1) \cup (-1, \infty)$ **B.** y-intercept: $f(0) = 0$; x-intercept:

$f(x) = 0 \iff x = 0$ **C.** No symmetry **D.** $\displaystyle\lim_{x \to \pm\infty} \frac{x^3}{x^3 + 1} = \frac{1}{1 + 1/x^3} = 1$, so $y = 1$ is a HA. $\displaystyle\lim_{x \to -1^-} f(x) = \infty$ and

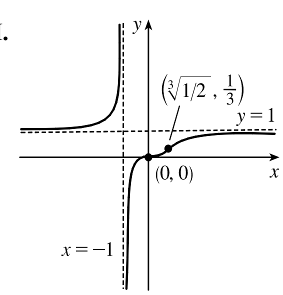
$\displaystyle\lim_{x \to -1^+} f(x) = -\infty$, so $x = -1$ is a VA. **E.** $f'(x) = \dfrac{(x^3 + 1)(3x^2) - x^3(3x^2)}{(x^3 + 1)^2} = \dfrac{3x^2}{(x^3 + 1)^2}$. $f'(x) > 0$ for $x \neq -1$

(not in the domain) and $x \neq 0$ ($f' = 0$), so f is increasing on $(-\infty, -1)$, $(-1, 0)$, and $(0, \infty)$, and furthermore, by Exercise 3.3.79, f is increasing on $(-\infty, -1)$, and $(-1, \infty)$. **F.** No extreme values

G. $$f''(x) = \frac{(x^3 + 1)^2(6x) - 3x^2[2(x^3 + 1)(3x^2)]}{[(x^3 + 1)^2]^2}$$
$$= \frac{(x^3 + 1)(6x)[(x^3 + 1) - 3x^3]}{(x^3 + 1)^4} = \frac{6x(1 - 2x^3)}{(x^3 + 1)^3}$$

$f''(x) > 0 \iff x < -1$ or $0 < x < \sqrt[3]{\frac{1}{2}}$ [≈ 0.79], so f is CU on $(-\infty, -1)$ and $\left(0, \sqrt[3]{\frac{1}{2}}\right)$ and CD on $(-1, 0)$ and $\left(\sqrt[3]{\frac{1}{2}}, \infty\right)$. There are IPs at $(0, 0)$ and $\left(\sqrt[3]{\frac{1}{2}}, \frac{1}{3}\right)$.

H.



20. $y = f(x) = \dfrac{x^3}{x - 2} = x^2 + 2x + 4 + \dfrac{8}{x - 2}$ [by long division] **A.** $D = (-\infty, 2) \cup (2, \infty)$ **B.** x-intercept $= 0$,

y-intercept $= f(0) = 0$ **C.** No symmetry **D.** $\displaystyle\lim_{x \to 2^-} \frac{x^3}{x - 2} = -\infty$ and $\displaystyle\lim_{x \to 2^+} \frac{x^3}{x - 2} = \infty$, so $x = 2$ is a VA.

[continued]

There are no horizontal or slant asymptotes. *Note:* Since $\lim_{x\to\pm\infty} \frac{8}{x-2} = 0$, the parabola $y = x^2 + 2x + 4$ is approached asymptotically as $x \to \pm\infty$.

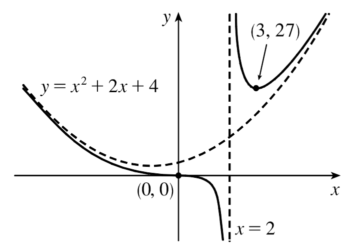
E. $f'(x) = \frac{(x-2)(3x^2) - x^3(1)}{(x-2)^2} = \frac{x^2[3(x-2) - x]}{(x-2)^2} = \frac{x^2(2x-6)}{(x-2)^2} = \frac{2x^2(x-3)}{(x-2)^2} > 0 \quad\Leftrightarrow\quad x > 3$ and

$f'(x) < 0 \quad\Leftrightarrow\quad x < 0$ or $0 < x < 2$ or $2 < x < 3$, so f is increasing on $(3, \infty)$ and f is decreasing on $(-\infty, 2)$ and $(2, 3)$.

F. Local minimum value $f(3) = 27$, no local maximum value **G.** $f'(x) = 2\frac{x^3 - 3x^2}{(x-2)^2} \quad\Rightarrow$

$$\begin{aligned} f''(x) &= 2\frac{(x-2)^2(3x^2 - 6x) - (x^3 - 3x^2)2(x-2)}{[(x-2)^2]^2} \\ &= 2\frac{(x-2)x[(x-2)(3x-6) - (x^2 - 3x)2]}{(x-2)^4} \\ &= \frac{2x(3x^2 - 12x + 12 - 2x^2 + 6x)}{(x-2)^3} \\ &= \frac{2x(x^2 - 6x + 12)}{(x-2)^3} > 0 \quad\Leftrightarrow \end{aligned}$$

H.



$x < 0$ or $x > 2$, so f is CU on $(-\infty, 0)$ and $(2, \infty)$, and f is CD on $(0, 2)$. IP at $(0, 0)$

21. $y = f(x) = (x-3)\sqrt{x} = x^{3/2} - 3x^{1/2}$ **A.** $D = [0, \infty)$ **B.** x-intercepts: $0, 3$; y-intercept $= f(0) = 0$ **C.** No symmetry **D.** No asymptote **E.** $f'(x) = \frac{3}{2}x^{1/2} - \frac{3}{2}x^{-1/2} = \frac{3}{2}x^{-1/2}(x-1) = \frac{3(x-1)}{2\sqrt{x}} > 0 \quad\Leftrightarrow\quad x > 1$,

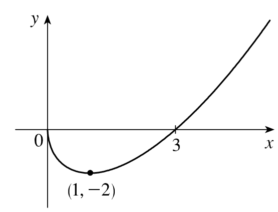
so f is increasing on $(1, \infty)$ and decreasing on $(0, 1)$.

F. Local minimum value $f(1) = -2$, no local maximum value

G. $f''(x) = \frac{3}{4}x^{-1/2} + \frac{3}{4}x^{-3/2} = \frac{3}{4}x^{-3/2}(x+1) = \frac{3(x+1)}{4x^{3/2}} > 0$ for $x > 0$,

so f is CU on $(0, \infty)$. No IP

H.



22. $y = f(x) = (x-4)\sqrt[3]{x} = x^{4/3} - 4x^{1/3}$ **A.** $D = \mathbb{R}$ **B.** y-intercept $= f(0) = 0$; x-intercepts: 0 and 4

C. No symmetry **D.** No asymptote

E. $f'(x) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3} = \frac{4}{3}x^{-2/3}(x-1) = \frac{4(x-1)}{3x^{2/3}}$. $\quad f'(x) > 0 \quad\Leftrightarrow$

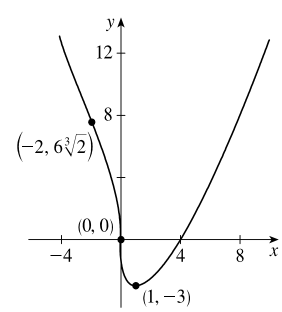
$x > 1$, so f is increasing on $(1, \infty)$ and f is decreasing on $(-\infty, 1)$.

F. Local minimum value $f(1) = -3$

G. $f''(x) = \frac{4}{9}x^{-2/3} + \frac{8}{9}x^{-5/3} = \frac{4}{9}x^{-5/3}(x+2) = \frac{4(x+2)}{9x^{5/3}}$.

$f''(x) < 0 \quad\Leftrightarrow\quad -2 < x < 0$, so f is CD on $(-2, 0)$, and f is CU on $(-\infty, -2)$ and $(0, \infty)$. There are IPs at $\left(-2, 6\sqrt[3]{2}\right)$ and $(0, 0)$.

H.



23. $y = f(x) = \sqrt{x^2 + x - 2} = \sqrt{(x+2)(x-1)}$ **A.** $D = \{x \mid (x+2)(x-1) \geq 0\} = (-\infty, -2] \cup [1, \infty)$

B. y-intercept: none; x-intercepts: -2 and 1 **C.** No symmetry **D.** No asymptote

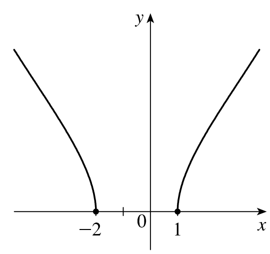
E. $f'(x) = \frac{1}{2}(x^2 + x - 2)^{-1/2}(2x+1) = \dfrac{2x+1}{2\sqrt{x^2+x-2}}$, $f'(x) = 0$ if $x = -\frac{1}{2}$, but $-\frac{1}{2}$ is not in the domain.

$f'(x) > 0 \;\Rightarrow\; x > -\frac{1}{2}$ and $f'(x) < 0 \;\Rightarrow\; x < -\frac{1}{2}$, so (considering the domain) f is increasing on $(1, \infty)$ and f is decreasing on $(-\infty, -2)$. **F.** No extreme values

G.
$$f''(x) = \frac{2(x^2+x-2)^{1/2}(2) - (2x+1)\cdot 2\cdot \frac{1}{2}(x^2+x-2)^{-1/2}(2x+1)}{\left(2\sqrt{x^2+x-2}\right)^2}$$
$$= \frac{(x^2+x-2)^{-1/2}\left[4(x^2+x-2) - (4x^2+4x+1)\right]}{4(x^2+x-2)}$$
$$= \frac{-9}{4(x^2+x-2)^{3/2}} < 0$$

so f is CD on $(-\infty, -2)$ and $(1, \infty)$. No IP

H.



24. $y = f(x) = \sqrt{x^2 + x} - x = \sqrt{x(x+1)} - x$ **A.** $D = (-\infty, -1] \cup [0, \infty)$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \;\Rightarrow\; \sqrt{x^2+x} = x \;\Rightarrow\; x^2 + x = x^2 \;\Rightarrow\; x = 0$ **C.** No symmetry

D.
$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(\sqrt{x^2+x} - x\right)\frac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x} = \lim_{x\to\infty}\frac{x^2+x-x^2}{\sqrt{x^2+x}+x}$$
$$= \lim_{x\to\infty}\frac{x/x}{\left(\sqrt{x^2+x}+x\right)/x} = \lim_{x\to\infty}\frac{1}{\sqrt{1+1/x}+1} = \frac{1}{2},$$
so $y = \frac{1}{2}$ is a HA. No VA

E. $f'(x) = \frac{1}{2}(x^2+x)^{-1/2}(2x+1) - 1 = \dfrac{2x+1}{2\sqrt{x^2+x}} - 1 > 0 \;\Leftrightarrow\; 2x + 1 > 2\sqrt{x^2+x} \;\Leftrightarrow$

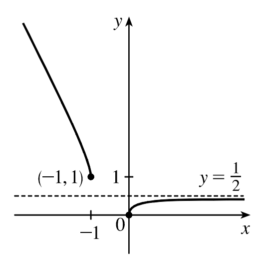
$x + \frac{1}{2} > \sqrt{\left(x+\frac{1}{2}\right)^2 - \frac{1}{4}}$. Keep in mind that the domain excludes the interval $(-1, 0)$. When $x + \frac{1}{2}$ is positive (for $x \geq 0$), the last inequality is *true* since the value of the radical is less than $x + \frac{1}{2}$. When $x + \frac{1}{2}$ is negative (for $x \leq -1$), the last inequality is *false* since the value of the radical is positive. So f is increasing on $(0, \infty)$ and decreasing on $(-\infty, -1)$.

F. No extreme values

G.
$$f''(x) = \frac{2(x^2+x)^{1/2}(2) - (2x+1)\cdot 2\cdot\frac{1}{2}(x^2+x)^{-1/2}(2x+1)}{\left(2\sqrt{x^2+x}\right)^2}$$
$$= \frac{(x^2+x)^{-1/2}\left[4(x^2+x) - (2x+1)^2\right]}{4(x^2+x)} = \frac{-1}{4(x^2+x)^{3/2}}.$$

$f''(x) < 0$ when it is defined, so f is CD on $(-\infty, -1)$ and $(0, \infty)$. No IP

H.



25. $y = f(x) = x/\sqrt{x^2+1}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \;\Rightarrow\; x = 0$

C. $f(-x) = -f(x)$, so f is odd; the graph is symmetric about the origin.

D.
$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty}\frac{x/x}{\sqrt{x^2+1}/x} = \lim_{x\to\infty}\frac{x/x}{\sqrt{x^2+1}/\sqrt{x^2}} = \lim_{x\to\infty}\frac{1}{\sqrt{1+1/x^2}} = \frac{1}{\sqrt{1+0}} = 1$$

and

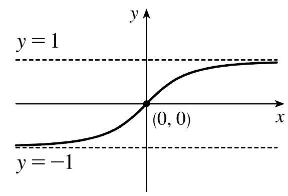
$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to-\infty} \frac{x/x}{\sqrt{x^2+1}/x} = \lim_{x\to-\infty} \frac{x/x}{\sqrt{x^2+1}/\left(-\sqrt{x^2}\right)} = \lim_{x\to-\infty} \frac{1}{-\sqrt{1+1/x^2}}$$

$$= \frac{1}{-\sqrt{1+0}} = -1 \quad \text{so } y = \pm 1 \text{ are HA. No VA}$$

E. $f'(x) = \dfrac{\sqrt{x^2+1} - x \cdot \dfrac{2x}{2\sqrt{x^2+1}}}{[(x^2+1)^{1/2}]^2} = \dfrac{x^2+1-x^2}{(x^2+1)^{3/2}} = \dfrac{1}{(x^2+1)^{3/2}} > 0$ for all x, so f is increasing on $\mathbb{R}$.

F. No extreme values **G.** $f''(x) = -\frac{3}{2}(x^2+1)^{-5/2} \cdot 2x = \dfrac{-3x}{(x^2+1)^{5/2}}$, so $f''(x) > 0$ for $x < 0$ and $f''(x) < 0$ for $x > 0$. Thus, f is CU on $(-\infty, 0)$ and CD on $(0, \infty)$. IP at $(0, 0)$

H.



26. $y = f(x) = x\sqrt{2-x^2}$ **A.** $D = \left[-\sqrt{2}, \sqrt{2}\right]$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \Rightarrow x = 0, \pm\sqrt{2}$. **C.** $f(-x) = -f(x)$, so f is odd; the graph is symmetric about the origin. **D.** No asymptote

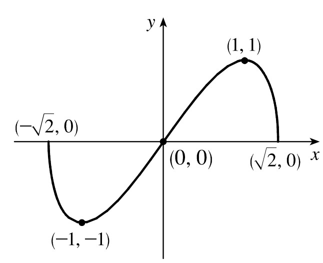
E. $f'(x) = x \cdot \dfrac{-x}{\sqrt{2-x^2}} + \sqrt{2-x^2} = \dfrac{-x^2+2-x^2}{\sqrt{2-x^2}} = \dfrac{2(1+x)(1-x)}{\sqrt{2-x^2}}$. $f'(x)$ is negative for $-\sqrt{2} < x < -1$ and $1 < x < \sqrt{2}$, and positive for $-1 < x < 1$, so f is decreasing on $\left(-\sqrt{2}, -1\right)$ and $\left(1, \sqrt{2}\right)$ and increasing on $(-1, 1)$.

F. Local minimum value $f(-1) = -1$, local maximum value $f(1) = 1$.

G. $$f''(x) = \frac{\sqrt{2-x^2}(-4x) - (2-2x^2)\dfrac{-x}{\sqrt{2-x^2}}}{[(2-x^2)^{1/2}]^2}$$
$$= \frac{(2-x^2)(-4x) + (2-2x^2)x}{(2-x^2)^{3/2}} = \frac{2x^3-6x}{(2-x^2)^{3/2}} = \frac{2x(x^2-3)}{(2-x^2)^{3/2}}$$

Since $x^2 - 3 < 0$ for x in $\left[-\sqrt{2}, \sqrt{2}\right]$, $f''(x) > 0$ for $-\sqrt{2} < x < 0$ and $f''(x) < 0$ for $0 < x < \sqrt{2}$. Thus, f is CU on $\left(-\sqrt{2}, 0\right)$ and CD on $\left(0, \sqrt{2}\right)$. The only IP is $(0, 0)$.

H.

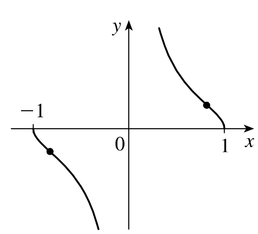


27. $y = f(x) = \sqrt{1-x^2}/x$ **A.** $D = \{x \mid |x| \le 1, x \ne 0\} = [-1, 0) \cup (0, 1]$ **B.** x-intercepts ± 1, no y-intercept

C. $f(-x) = -f(x)$, so the curve is symmetric about $(0, 0)$. **D.** $\displaystyle\lim_{x\to 0^+} \frac{\sqrt{1-x^2}}{x} = \infty$, $\displaystyle\lim_{x\to 0^-} \frac{\sqrt{1-x^2}}{x} = -\infty$, so $x = 0$ is a VA. **E.** $f'(x) = \dfrac{\left(-x^2/\sqrt{1-x^2}\right) - \sqrt{1-x^2}}{x^2} = -\dfrac{1}{x^2\sqrt{1-x^2}} < 0$, so f is decreasing on $(-1, 0)$ and $(0, 1)$. **F.** No extreme values

G. $f''(x) = \dfrac{2-3x^2}{x^3(1-x^2)^{3/2}} > 0 \quad \Leftrightarrow \quad -1 < x < -\sqrt{\frac{2}{3}}$ or $0 < x < \sqrt{\frac{2}{3}}$, so f is CU on $\left(-1, -\sqrt{\frac{2}{3}}\right)$ and $\left(0, \sqrt{\frac{2}{3}}\right)$ and CD on $\left(-\sqrt{\frac{2}{3}}, 0\right)$ and $\left(\sqrt{\frac{2}{3}}, 1\right)$. IP at $\left(\pm\sqrt{\frac{2}{3}}, \pm\frac{1}{\sqrt{2}}\right)$

H.



28. $y = f(x) = x/\sqrt{x^2-1}$ **A.** $D = (-\infty, -1) \cup (1, \infty)$ **B.** No intercepts **C.** $f(-x) = -f(x)$, so f is odd; the graph is symmetric about the origin. **D.** $\lim\limits_{x\to\infty} \dfrac{x}{\sqrt{x^2-1}} = 1$ and $\lim\limits_{x\to-\infty} \dfrac{x}{\sqrt{x^2-1}} = -1$, so $y = \pm 1$ are HA.

$\lim\limits_{x\to 1^+} f(x) = +\infty$ and $\lim\limits_{x\to -1^-} f(x) = -\infty$, so $x = \pm 1$ are VA.

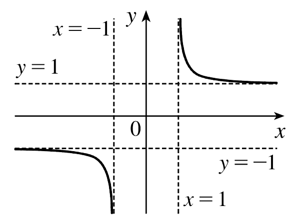
E. $f'(x) = \dfrac{\sqrt{x^2-1} - x \cdot \dfrac{x}{\sqrt{x^2-1}}}{[(x^2-1)^{1/2}]^2} = \dfrac{x^2-1-x^2}{(x^2-1)^{3/2}} = \dfrac{-1}{(x^2-1)^{3/2}} < 0$, so f is decreasing on $(-\infty, -1)$ and $(1, \infty)$.

F. No extreme values

G. $f''(x) = (-1)\left(-\frac{3}{2}\right)(x^2-1)^{-5/2} \cdot 2x = \dfrac{3x}{(x^2-1)^{5/2}}$.

$f''(x) < 0$ on $(-\infty, -1)$ and $f''(x) > 0$ on $(1, \infty)$, so f is CD on $(-\infty, -1)$ and CU on $(1, \infty)$. No IP

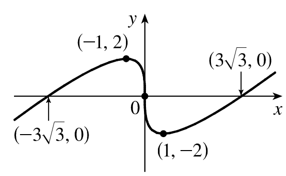
H.



29. $y = f(x) = x - 3x^{1/3}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \Rightarrow x = 3x^{1/3} \Rightarrow x^3 = 27x \Rightarrow x^3 - 27x = 0 \Rightarrow x(x^2 - 27) = 0 \Rightarrow x = 0, \pm 3\sqrt{3}$ **C.** $f(-x) = -f(x)$, so f is odd; the graph is symmetric about the origin. **D.** No asymptote **E.** $f'(x) = 1 - x^{-2/3} = 1 - \dfrac{1}{x^{2/3}} = \dfrac{x^{2/3}-1}{x^{2/3}}$.

$f'(x) > 0$ when $|x| > 1$ and $f'(x) < 0$ when $0 < |x| < 1$, so f is increasing on $(-\infty, -1)$ and $(1, \infty)$, and decreasing on $(-1, 0)$ and $(0, 1)$ [hence decreasing on $(-1, 1)$ since f is continuous on $(-1, 1)$]. **F.** Local maximum value $f(-1) = 2$, local minimum value $f(1) = -2$ **G.** $f''(x) = \frac{2}{3}x^{-5/3} < 0$ when $x < 0$ and $f''(x) > 0$ when $x > 0$, so f is CD on $(-\infty, 0)$ and CU on $(0, \infty)$. IP at $(0, 0)$

H.



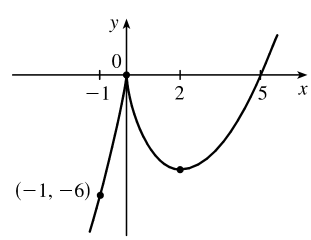
30. $y = f(x) = x^{5/3} - 5x^{2/3} = x^{2/3}(x-5)$ **A.** $D = \mathbb{R}$ **B.** x-intercepts 0, 5; y-intercept 0 **C.** No symmetry

D. $\lim\limits_{x\to\pm\infty} x^{2/3}(x-5) = \pm\infty$, so there is no asymptote. **E.** $f'(x) = \frac{5}{3}x^{2/3} - \frac{10}{3}x^{-1/3} = \frac{5}{3}x^{-1/3}(x-2) > 0 \Leftrightarrow$ $x < 0$ or $x > 2$, so f is increasing on $(-\infty, 0)$, $(2, \infty)$ and decreasing on $(0, 2)$.

F. Local maximum value $f(0) = 0$, local minimum value $f(2) = -3\sqrt[3]{4}$

G. $f''(x) = \frac{10}{9}x^{-1/3} + \frac{10}{9}x^{-4/3} = \frac{10}{9}x^{-4/3}(x+1) > 0 \Leftrightarrow x > -1$, so f is CU on $(-1, 0)$ and $(0, \infty)$, CD on $(-\infty, -1)$. IP at $(-1, -6)$

H.



31. $y = f(x) = \sqrt[3]{x^2-1}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = -1$; x-intercepts: $f(x) = 0 \Leftrightarrow x^2 - 1 = 0 \Leftrightarrow x = \pm 1$ **C.** $f(-x) = f(x)$, so the curve is symmetric about the y-axis. **D.** No asymptote

E. $f'(x) = \frac{1}{3}(x^2-1)^{-2/3}(2x) = \dfrac{2x}{3\sqrt[3]{(x^2-1)^2}}$. $f'(x) > 0 \Leftrightarrow x > 0$ and $f'(x) < 0 \Leftrightarrow x < 0$, so f is increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$. **F.** Local minimum value $f(0) = -1$

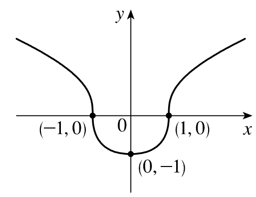
[continued]

G. $f''(x) = \dfrac{2}{3} \cdot \dfrac{(x^2-1)^{2/3}(1) - x \cdot \frac{2}{3}(x^2-1)^{-1/3}(2x)}{[(x^2-1)^{2/3}]^2}$

$= \dfrac{2}{9} \cdot \dfrac{(x^2-1)^{-1/3}[3(x^2-1) - 4x^2]}{(x^2-1)^{4/3}} = -\dfrac{2(x^2+3)}{9(x^2-1)^{5/3}}$

$f''(x) > 0 \quad \Leftrightarrow \quad -1 < x < 1$ and $f''(x) < 0 \quad \Leftrightarrow \quad x < -1$ or $x > 1$, so f is CU on $(-1, 1)$ and f is CD on $(-\infty, -1)$ and $(1, \infty)$. IP at $(\pm 1, 0)$

H.



32. $y = f(x) = \sqrt[3]{x^3+1}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 1$; x-intercept: $f(x) = 0 \quad \Leftrightarrow \quad x^3 + 1 = 0 \quad \Rightarrow \quad x = -1$

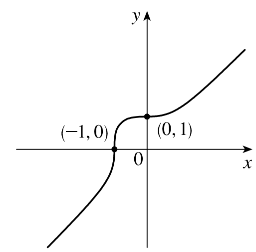
C. No symmetry **D.** No asymptote **E.** $f'(x) = \frac{1}{3}(x^3+1)^{-2/3}(3x^2) = \dfrac{x^2}{\sqrt[3]{(x^3+1)^2}}$. $f'(x) > 0$ if $x < -1$, $-1 < x < 0$, and $x > 0$, so f is increasing on $\mathbb{R}$. **F.** No extreme values

G. $f''(x) = \dfrac{(x^3+1)^{2/3}(2x) - x^2 \cdot \frac{2}{3}(x^3+1)^{-1/3}(3x^2)}{[(x^3+1)^{2/3}]^2}$

$= \dfrac{x(x^3+1)^{-1/3}[2(x^3+1) - 2x^3]}{(x^3+1)^{4/3}} = \dfrac{2x}{(x^3+1)^{5/3}}$

$f''(x) > 0 \quad \Leftrightarrow \quad x < -1$ or $x > 0$ and $f''(x) < 0 \quad \Leftrightarrow \quad -1 < x < 0$, so f is CU on $(-\infty, -1)$ and $(0, \infty)$ and CD on $(-1, 0)$. IP at $(-1, 0)$ and $(0, 1)$

H.



33. $y = f(x) = \sin^3 x$ **A.** $D = \mathbb{R}$ **B.** x-intercepts: $f(x) = 0 \quad \Leftrightarrow \quad x = n\pi$, n an integer; y-intercept $= f(0) = 0$

C. $f(-x) = -f(x)$, so f is odd and the curve is symmetric about the origin. Also, $f(x + 2\pi) = f(x)$, so f is periodic with period 2π, and we determine **E–G** for $0 \le x \le \pi$. Since f is odd, we can reflect the graph of f on $[0, \pi]$ about the origin to obtain the graph of f on $[-\pi, \pi]$, and then since f has period 2π, we can extend the graph of f for all real numbers.

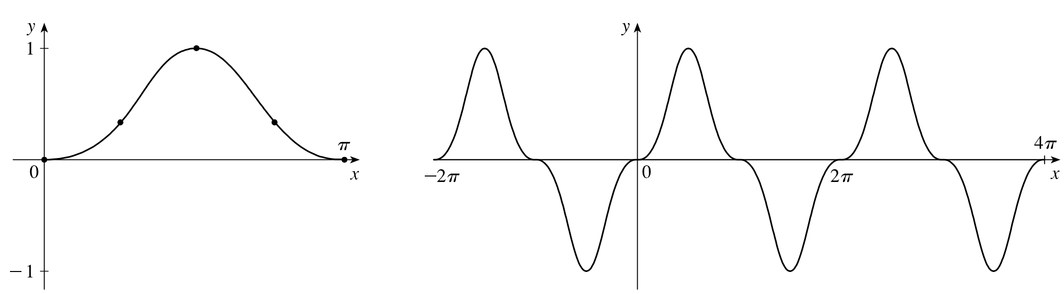
D. No asymptote **E.** $f'(x) = 3\sin^2 x \cos x > 0 \quad \Leftrightarrow \quad \cos x > 0$ and $\sin x \ne 0 \quad \Leftrightarrow \quad 0 < x < \frac{\pi}{2}$, so f is increasing on $\left(0, \frac{\pi}{2}\right)$ and f is decreasing on $\left(\frac{\pi}{2}, \pi\right)$. **F.** Local maximum value $f\left(\frac{\pi}{2}\right) = 1$ $\left[\text{local minimum value } f\left(-\frac{\pi}{2}\right) = -1\right]$

G. $f''(x) = 3\sin^2 x\,(-\sin x) + 3\cos x\,(2\sin x\,\cos x) = 3\sin x\,(2\cos^2 x - \sin^2 x)$

$= 3\sin x\,[2(1 - \sin^2 x) - \sin^2 x] = 3\sin x(2 - 3\sin^2 x) > 0 \quad \Leftrightarrow$

$\sin x > 0$ and $\sin^2 x < \frac{2}{3} \quad \Leftrightarrow \quad 0 < x < \pi$ and $0 < \sin x < \sqrt{\frac{2}{3}} \quad \Leftrightarrow \quad 0 < x < \sin^{-1}\sqrt{\frac{2}{3}}$ $\left[\text{let } \alpha = \sin^{-1}\sqrt{\frac{2}{3}}\right]$ or $\pi - \alpha < x < \pi$, so f is CU on $(0, \alpha)$ and $(\pi - \alpha, \pi)$, and f is CD on $(\alpha, \pi - \alpha)$. There are inflection points at $x = 0$, π, α, and $x = \pi - \alpha$.

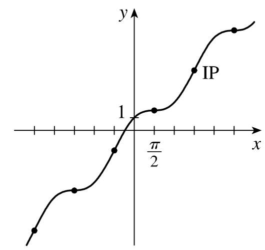
H.



34. $y = f(x) = x + \cos x$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 1$; the x-intercept is about -0.74 and can be found using Newton's method **C.** No symmetry **D.** No asymptote **E.** $f'(x) = 1 - \sin x > 0$ except for $x = \frac{\pi}{2} + 2n\pi$, so f is increasing on $\mathbb{R}$. **F.** No extreme values

G. $f''(x) = -\cos x$. $f''(x) > 0 \Rightarrow -\cos x > 0 \Rightarrow \cos x < 0 \Rightarrow$ x is in $\left(\frac{\pi}{2} + 2n\pi, \frac{3\pi}{2} + 2n\pi\right)$ and $f''(x) < 0 \Rightarrow$ x is in $\left(-\frac{\pi}{2} + 2n\pi, \frac{\pi}{2} + 2n\pi\right)$, so f is CU on $\left(\frac{\pi}{2} + 2n\pi, \frac{3\pi}{2} + 2n\pi\right)$ and CD on $\left(-\frac{\pi}{2} + 2n\pi, \frac{\pi}{2} + 2n\pi\right)$. IP at $\left(\frac{\pi}{2} + n\pi, f\left(\frac{\pi}{2} + n\pi\right)\right) = \left(\frac{\pi}{2} + n\pi, \frac{\pi}{2} + n\pi\right)$ [on the line $y = x$]

H.

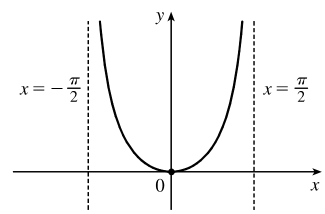


35. $y = f(x) = x\tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$ **A.** $D = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ **B.** Intercepts are 0 **C.** $f(-x) = f(x)$, so the curve is symmetric about the y-axis. **D.** $\lim\limits_{x \to (\pi/2)^-} x\tan x = \infty$ and $\lim\limits_{x \to -(\pi/2)^+} x\tan x = \infty$, so $x = \frac{\pi}{2}$ and $x = -\frac{\pi}{2}$ are VA.

E. $f'(x) = \tan x + x\sec^2 x > 0 \Leftrightarrow 0 < x < \frac{\pi}{2}$, so f increases on $\left(0, \frac{\pi}{2}\right)$ and decreases on $\left(-\frac{\pi}{2}, 0\right)$. **F.** Absolute and local minimum value $f(0) = 0$.

G. $y'' = 2\sec^2 x + 2x\tan x\,\sec^2 x > 0$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, so f is CU on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. No IP

H.



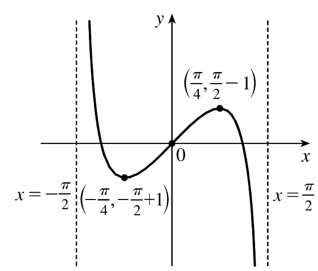
36. $y = f(x) = 2x - \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$ **A.** $D = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \Leftrightarrow 2x = \tan x \Leftrightarrow x = 0$ or $x \approx \pm 1.17$ **C.** $f(-x) = -f(x)$, so f is odd; the graph is symmetric about the origin.

D. $\lim\limits_{x \to (-\pi/2)^+} (2x - \tan x) = \infty$ and $\lim\limits_{x \to (\pi/2)^-} (2x - \tan x) = -\infty$, so $x = \pm\frac{\pi}{2}$ are VA. No HA.

E. $f'(x) = 2 - \sec^2 x < 0 \Leftrightarrow |\sec x| > \sqrt{2}$ and $f'(x) > 0 \Leftrightarrow |\sec x| < \sqrt{2}$, so f is decreasing on $\left(-\frac{\pi}{2}, -\frac{\pi}{4}\right)$, increasing on $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$, and decreasing again on $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ **F.** Local maximum value $f\left(\frac{\pi}{4}\right) = \frac{\pi}{2} - 1$, local minimum value $f\left(-\frac{\pi}{4}\right) = -\frac{\pi}{2} + 1$

G. $f''(x) = -2\sec x \cdot \sec x\,\tan x = -2\tan x\,\sec^2 x = -2\tan x(\tan^2 x + 1)$ so $f''(x) > 0 \Leftrightarrow \tan x < 0 \Leftrightarrow -\frac{\pi}{2} < x < 0$, and $f''(x) < 0 \Leftrightarrow \tan x > 0 \Leftrightarrow 0 < x < \frac{\pi}{2}$. Thus, f is CU on $\left(-\frac{\pi}{2}, 0\right)$ and CD on $\left(0, \frac{\pi}{2}\right)$. IP at $(0, 0)$

H.



37. $y = f(x) = \sin x + \sqrt{3}\cos x$, $-2\pi \le x \le 2\pi$ **A.** $D = [-2\pi, 2\pi]$ **B.** y-intercept: $f(0) = \sqrt{3}$; x-intercepts: $f(x) = 0 \Leftrightarrow \sin x = -\sqrt{3}\cos x \Leftrightarrow \tan x = -\sqrt{3} \Leftrightarrow x = -\frac{4\pi}{3}, -\frac{\pi}{3}, \frac{2\pi}{3}$, or $\frac{5\pi}{3}$ **C.** f is periodic with period 2π. **D.** No asymptote **E.** $f'(x) = \cos x - \sqrt{3}\sin x$. $f'(x) = 0 \Leftrightarrow \cos x = \sqrt{3}\sin x \Leftrightarrow \tan x = \dfrac{1}{\sqrt{3}} \Leftrightarrow$

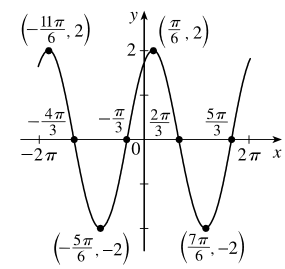
$x = -\frac{11\pi}{6}, -\frac{5\pi}{6}, \frac{\pi}{6}$, or $\frac{7\pi}{6}$. $f'(x) < 0 \quad \Leftrightarrow \quad -\frac{11\pi}{6} < x < -\frac{5\pi}{6}$ or $\frac{\pi}{6} < x < \frac{7\pi}{6}$, so f is decreasing on $\left(-\frac{11\pi}{6}, -\frac{5\pi}{6}\right)$ and $\left(\frac{\pi}{6}, \frac{7\pi}{6}\right)$, and f is increasing on $\left(-2\pi, -\frac{11\pi}{6}\right)$, $\left(-\frac{5\pi}{6}, \frac{\pi}{6}\right)$, and $\left(\frac{7\pi}{6}, 2\pi\right)$. **F.** Local maximum value $f\left(-\frac{11\pi}{6}\right) = f\left(\frac{\pi}{6}\right) = \frac{1}{2} + \sqrt{3}\left(\frac{1}{2}\sqrt{3}\right) = 2$, local minimum value $f\left(-\frac{5\pi}{6}\right) = f\left(\frac{7\pi}{6}\right) = -\frac{1}{2} + \sqrt{3}\left(-\frac{1}{2}\sqrt{3}\right) = -2$

G. $f''(x) = -\sin x - \sqrt{3}\cos x$. $f''(x) = 0 \quad \Leftrightarrow \quad \sin x = -\sqrt{3}\cos x \quad \Leftrightarrow$

$\tan x = -\dfrac{1}{\sqrt{3}} \quad \Leftrightarrow \quad x = -\frac{4\pi}{3}, -\frac{\pi}{3}, \frac{2\pi}{3}$, or $\frac{5\pi}{3}$. $f''(x) > 0 \quad \Leftrightarrow$

$-\frac{4\pi}{3} < x < -\frac{\pi}{3}$ or $\frac{2\pi}{3} < x < \frac{5\pi}{3}$, so f is CU on $\left(-\frac{4\pi}{3}, -\frac{\pi}{3}\right)$ and $\left(\frac{2\pi}{3}, \frac{5\pi}{3}\right)$, and f is CD on $\left(-2\pi, -\frac{4\pi}{3}\right)$, $\left(-\frac{\pi}{3}, \frac{2\pi}{3}\right)$, and $\left(\frac{5\pi}{3}, 2\pi\right)$. There are IPs at $\left(-\frac{4\pi}{3}, 0\right)$, $\left(-\frac{\pi}{3}, 0\right)$, $\left(\frac{2\pi}{3}, 0\right)$, and $\left(\frac{5\pi}{3}, 0\right)$.

H.



38. $y = f(x) = \csc x - 2\sin x$, $0 < x < \pi$ **A.** $D = (0, \pi)$ **B.** No y-intercept; x-intercept: $f(x) = 0 \quad \Leftrightarrow$ $\csc x = 2\sin x \quad \Leftrightarrow \quad \frac{1}{2} = \sin^2 x \quad \Leftrightarrow \quad \sin x = \pm\frac{1}{2}\sqrt{2} \quad \Leftrightarrow \quad x = \frac{\pi}{4}$ or $\frac{3\pi}{4}$ **C.** No symmetry

D. $\lim\limits_{x\to 0^+} f(x) = \infty$ and $\lim\limits_{x\to \pi^-} f(x) = \infty$, so $x = 0$ and $x = \pi$ are VAs.

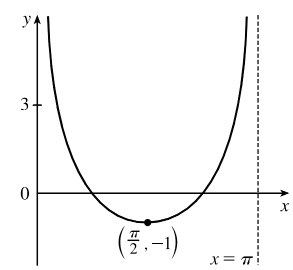
E. $f'(x) = -\csc x \cot x - 2\cos x = -\dfrac{\cos x}{\sin^2 x} - 2\cos x = -\cos x\left(\dfrac{1}{\sin^2 x} + 2\right)$. $f'(x) > 0$ when $-\cos x > 0 \quad \Leftrightarrow$ $\cos x < 0 \quad \Leftrightarrow \quad \frac{\pi}{2} < x < \pi$, so f' is increasing on $\left(\frac{\pi}{2}, \pi\right)$, and f is decreasing on $\left(0, \frac{\pi}{2}\right)$. **F.** Local minimum value $f\left(\frac{\pi}{2}\right) = -1$

G. $f''(x) = (-\csc x)(-\csc^2 x) + (\cot x)(\csc x \ \cot x) + 2\sin x$

$$= \frac{1 + \cos^2 x + 2\sin^4 x}{\sin^3 x}$$

f'' has the same sign as $\sin x$, which is positive on $(0, \pi)$, so f is CU on $(0, \pi)$. No IP

H.



39. $y = f(x) = \dfrac{\sin x}{1 + \cos x}$ $\left[\overset{\text{when } \cos x \neq 1}{=} \dfrac{\sin x}{1 + \cos x} \cdot \dfrac{1 - \cos x}{1 - \cos x} = \dfrac{\sin x\,(1 - \cos x)}{\sin^2 x} = \dfrac{1 - \cos x}{\sin x} = \csc x - \cot x\right]$

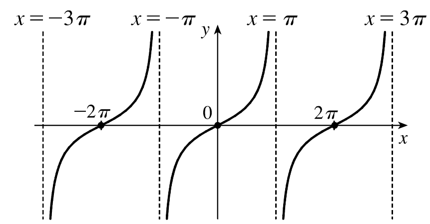
A. The domain of f is the set of all real numbers except odd integer multiples of π; that is, all reals except $(2n+1)\pi$, where n is an integer. **B.** y-intercept: $f(0) = 0$; x-intercepts: $x = 2n\pi$, n an integer. **C.** $f(-x) = -f(x)$, so f is an odd function; the graph is symmetric about the origin and has period 2π. **D.** When n is an odd integer, $\lim\limits_{x\to(n\pi)^-} f(x) = \infty$ and $\lim\limits_{x\to(n\pi)^+} f(x) = -\infty$, so $x = n\pi$ is a VA for each odd integer n.

No HA. **E.** $f'(x) = \dfrac{(1 + \cos x)\cdot\cos x - \sin x(-\sin x)}{(1 + \cos x)^2} = \dfrac{1 + \cos x}{(1 + \cos x)^2} = \dfrac{1}{1 + \cos x}$. $f'(x) > 0$ for all x except odd multiples of π, so f is increasing on $((2k-1)\pi, (2k+1)\pi)$ for each integer k. **F.** No extreme values

[continued]

G. $f''(x) = \dfrac{\sin x}{(1+\cos x)^2} > 0 \;\Rightarrow\; \sin x > 0 \;\Rightarrow$ $x \in (2k\pi, (2k+1)\pi)$ and $f''(x) < 0$ on $((2k-1)\pi, 2k\pi)$ for each integer k. f is CU on $(2k\pi, (2k+1)\pi)$ and CD on $((2k-1)\pi, 2k\pi)$ for each integer k. f has IPs at $(2k\pi, 0)$ for each integer k.

H.



40. $y = f(x) = \dfrac{\sin x}{2+\cos x}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \;\Leftrightarrow\; \sin x = 0 \;\Leftrightarrow\; x = n\pi$

C. $f(-x) = -f(x)$, so the curve is symmetric about the origin. f is periodic with period 2π, so we determine **E–G** for $0 \le x \le 2\pi$. **D.** No asymptote

E. $f'(x) = \dfrac{(2+\cos x)\cos x - \sin x(-\sin x)}{(2+\cos x)^2} = \dfrac{2\cos x + \cos^2 x + \sin^2 x}{(2+\cos x)^2} = \dfrac{2\cos x + 1}{(2+\cos x)^2}$.

$f'(x) > 0 \;\Leftrightarrow\; 2\cos x + 1 > 0 \;\Leftrightarrow\; \cos x > -\frac{1}{2} \;\Leftrightarrow\; x$ is in $\left(0, \frac{2\pi}{3}\right)$ or $\left(\frac{4\pi}{3}, 2\pi\right)$, so f is increasing on $\left(0, \frac{2\pi}{3}\right)$ and $\left(\frac{4\pi}{3}, 2\pi\right)$, and f is decreasing on $\left(\frac{2\pi}{3}, \frac{4\pi}{3}\right)$.

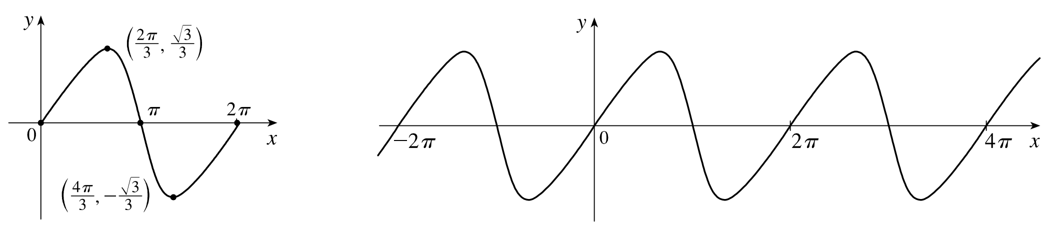
F. Local maximum value $f\left(\frac{2\pi}{3}\right) = \dfrac{\sqrt{3}/2}{2-(1/2)} = \dfrac{\sqrt{3}}{3}$ and local minimum value $f\left(\frac{4\pi}{3}\right) = \dfrac{-\sqrt{3}/2}{2-(1/2)} = -\dfrac{\sqrt{3}}{3}$

G. $f''(x) = \dfrac{(2+\cos x)^2(-2\sin x) - (2\cos x + 1)2(2+\cos x)(-\sin x)}{[(2+\cos x)^2]^2}$

$= \dfrac{-2\sin x\,(2+\cos x)[(2+\cos x) - (2\cos x + 1)]}{(2+\cos x)^4} = \dfrac{-2\sin x\,(1-\cos x)}{(2+\cos x)^3}$

$f''(x) > 0 \;\Leftrightarrow\; -2\sin x > 0 \;\Leftrightarrow\; \sin x < 0 \;\Leftrightarrow\; x$ is in $(\pi, 2\pi)$ [f is CU] and $f''(x) < 0 \;\Leftrightarrow$ x is in $(0, \pi)$ [f is CD]. The inflection points are $(0, 0)$, $(\pi, 0)$, and $(2\pi, 0)$.

H.



41. $g(x) = \sqrt{f(x)}$

(a) The domain of g consists of all x such that $f(x) \ge 0$, so g has domain $(-\infty, 7]$. $g'(x) = \dfrac{1}{2\sqrt{f(x)}} \cdot f'(x) = \dfrac{f'(x)}{2\sqrt{f(x)}}$.

Since $f'(3)$ does not exist, $g'(3)$ does not exist. (Note that $f(7) = 0$, but 7 is an endpoint of the domain of g.) The domain of g' is $(-\infty, 3) \cup (3, 7)$.

(b) $g'(x) = 0 \;\Rightarrow\; f'(x) = 0$ on the domain of $g \;\Rightarrow\; x = 5$ [there is a horizontal tangent line there]. From part (a), $g'(3)$ does not exist. So the critical numbers of g are 3 and 5.

(c) From part (a), $g'(x) = \dfrac{f'(x)}{2\sqrt{f(x)}}$. $g'(6) = \dfrac{f'(6)}{2\sqrt{f(6)}} \approx \dfrac{-2}{2\sqrt{3}} = -\dfrac{1}{\sqrt{3}} \approx -0.58$.

(d) $\lim\limits_{x\to-\infty} g(x) = \lim\limits_{x\to-\infty} \sqrt{f(x)} = \sqrt{2}$, so $y = \sqrt{2}$ is a horizontal asymptote. No VA

42. $g(x) = \sqrt[3]{f(x)}$

(a) Since the cube-root function is defined for all reals, the domain of g equals the domain of f, $(-\infty, \infty)$.

$g'(x) = \dfrac{1}{3\left(\sqrt[3]{f(x)}\right)^2} \cdot f'(x) = \dfrac{f'(x)}{3\left(\sqrt[3]{f(x)}\right)^2}$. Since $f'(3)$ does not exist and $f(7) = 0$, $g'(3)$ and $g'(7)$ do not exist.

The domain of g' is $(-\infty, 3) \cup (3, 7) \cup (7, \infty)$.

(b) $g'(x) = 0 \quad\Rightarrow\quad f'(x) = 0 \quad\Rightarrow\quad x = 5$ or $x = 9$ [there are horizontal tangent lines there]. From part (a), $g'(x)$ does not exist at $x = 3$ and $x = 7$. So the critical numbers of g are 3, 5, 7, and 9.

(c) From part (a), $g'(x) = \dfrac{f'(x)}{3\left(\sqrt[3]{f(x)}\right)^2}$. $g'(6) = \dfrac{f'(6)}{3\left(\sqrt[3]{f(6)}\right)^2} \approx \dfrac{-2}{3\left(\sqrt[3]{3}\right)^2} = -\dfrac{2}{3^{5/3}} \approx -0.32$.

(d) $\lim\limits_{x\to-\infty} g(x) = \lim\limits_{x\to-\infty} \sqrt[3]{f(x)} = \sqrt[3]{2}$ and $\lim\limits_{x\to\infty} g(x) = \lim\limits_{x\to\infty} \sqrt[3]{f(x)} = \sqrt[3]{-1} = -1$, so $y = \sqrt[3]{2}$ and $y = -1$ are horizontal asymptotes. No VA

43. $g(x) = |f(x)|$

(a) Since the absolute-value function is defined for all reals, the domain of g equals the domain of f, $(-\infty, \infty)$. The domain of g' equals the domain of f' except for any values of x such that both $f(x) = 0$ and $f'(x) \neq 0$. $f'(3)$ does not exist, $f(7) = 0$, and $f'(7) \neq 0$. Thus, the domain of g' is $(-\infty, 3) \cup (3, 7) \cup (7, \infty)$.

(b) $g'(x) = 0 \quad\Rightarrow\quad f'(x) = 0 \quad\Rightarrow\quad x = 5$ or $x = 9$ [there are horizontal tangent lines there]. From part (a), g' does not exist at $x = 3$ and $x = 7$. So the critical numbers of g are 3, 5, 7, and 9.

(c) Since f is positive near $x = 6$, $g(x) = |f(x)| = f(x)$ near 6, so $g'(6) = f'(6) \approx -2$.

(d) $\lim\limits_{x\to-\infty} g(x) = \lim\limits_{x\to-\infty} |f(x)| = |2| = 2$ and $\lim\limits_{x\to\infty} g(x) = \lim\limits_{x\to\infty} |f(x)| = |-1| = 1$, so $y = 2$ and $y = 1$ are horizontal asymptotes. No VA

44. $g(x) = 1/f(x)$

(a) The domain of g consists of all x such that $f(x) \neq 0$, so g has domain $(-\infty, 7) \cup (7, \infty)$. $g(x) = \dfrac{1}{f(x)} = [f(x)]^{-1} \quad\Rightarrow$

$g'(x) = -1[f(x)]^{-2} \cdot f'(x) = -\dfrac{f'(x)}{[f(x)]^2}$. The domain of g' will equal the domain of f except for any values of f such that $f(x) = 0$. $f'(3)$ does not exist, and $f(7) = 0$. Thus, the domain of g' is $(-\infty, 3) \cup (3, 7) \cup (7, \infty)$.

(b) $g'(x) = 0 \;\Rightarrow\; f'(x) = 0 \;\Rightarrow\; x = 5$ or $x = 9$ [there are horizontal tangent lines there]. From part (a), g' does not exist at $x = 3$ and $x = 7$. So the critical numbers of g are 3, 5, 7, and 9.

(c) From part (a), $g'(x) = -\dfrac{f'(x)}{[f(x)]^2}$. $g'(6) = -\dfrac{f'(6)}{[f(6)]^2} \approx -\left(\dfrac{-2}{3^2}\right) = \dfrac{2}{9}$.

(d) $\displaystyle\lim_{x\to-\infty} g(x) = \lim_{x\to-\infty} \frac{1}{f(x)} = \frac{1}{2}$ and $\displaystyle\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{1}{f(x)} = \frac{1}{-1} = -1$, so $y = \frac{1}{2}$ and $y = -1$ are horizontal asymptotes. $\displaystyle\lim_{x\to7^-} \frac{1}{f(x)} = \frac{1}{\lim\limits_{x\to7^-} f(x)} = \infty$ and $\displaystyle\lim_{x\to7^+} g(x) = \lim_{x\to7^+} \frac{1}{f(x)} = \frac{1}{\lim\limits_{x\to7^+} f(x)} = -\infty$, so $x = 7$ is a vertical asymptote.

45. $m = f(v) = \dfrac{m_0}{\sqrt{1 - v^2/c^2}}$. The m-intercept is $f(0) = m_0$. There are no v-intercepts. $\displaystyle\lim_{v\to c^-} f(v) = \infty$, so $v = c$ is a VA.

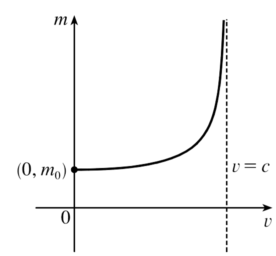
$$f'(v) = -\tfrac{1}{2}m_0(1 - v^2/c^2)^{-3/2}(-2v/c^2) = \frac{m_0 v}{c^2(1 - v^2/c^2)^{3/2}} = \frac{m_0 v}{\dfrac{c^2(c^2 - v^2)^{3/2}}{c^3}} = \frac{m_0 c v}{(c^2 - v^2)^{3/2}} > 0, \text{ so } f \text{ is}$$

increasing on $(0, c)$. There are no local extreme values.

$$f''(v) = \frac{(c^2 - v^2)^{3/2}(m_0 c) - m_0 c v \cdot \frac{3}{2}(c^2 - v^2)^{1/2}(-2v)}{[(c^2 - v^2)^{3/2}]^2}$$

$$= \frac{m_0 c(c^2 - v^2)^{1/2}[(c^2 - v^2) + 3v^2]}{(c^2 - v^2)^3} = \frac{m_0 c(c^2 + 2v^2)}{(c^2 - v^2)^{5/2}} > 0,$$

so f is CU on $(0, c)$. There are no inflection points.



46. Let $a = m_0^2c^4$ and $b = h^2c^2$, so the equation can be written as $E = f(\lambda) = \sqrt{a + b/\lambda^2} = \sqrt{\dfrac{a\lambda^2 + b}{\lambda^2}} = \dfrac{\sqrt{a\lambda^2 + b}}{\lambda}$.

$\displaystyle\lim_{\lambda\to0^+} \frac{\sqrt{a\lambda^2 + b}}{\lambda} = \infty$, so $\lambda = 0$ is a VA.

$\displaystyle\lim_{\lambda\to\infty} \frac{\sqrt{a\lambda^2 + b}}{\lambda} = \lim_{\lambda\to\infty} \frac{\sqrt{a\lambda^2 + b}/\lambda}{\lambda/\lambda} = \lim_{\lambda\to\infty} \frac{\sqrt{a + b/\lambda^2}}{1} = \sqrt{a}$, so $E = \sqrt{a} = m_0c^2$ is a HA.

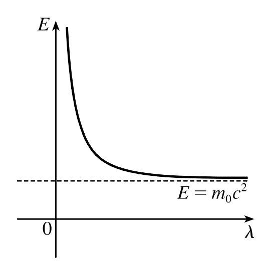
$$f'(\lambda) = \frac{\lambda \cdot \frac{1}{2}(a\lambda^2 + b)^{-1/2}(2a\lambda) - (a\lambda^2 + b)^{1/2}(1)}{\lambda^2} = \frac{(a\lambda^2 + b)^{-1/2}[a\lambda^2 - (a\lambda^2 + b)]}{\lambda^2} = \frac{-b}{\lambda^2\sqrt{a\lambda^2 + b}} < 0,$$

so f is decreasing on $(0, \infty)$. Using the Reciprocal Rule,

$$f''(\lambda) = b \cdot \frac{\lambda^2 \cdot \frac{1}{2}(a\lambda^2 + b)^{-1/2}(2a\lambda) + (a\lambda^2 + b)^{1/2}(2\lambda)}{\left(\lambda^2\sqrt{a\lambda^2 + b}\right)^2}$$

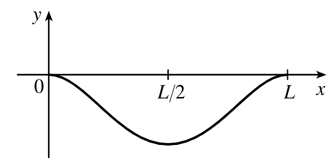
$$= \frac{b\lambda(a\lambda^2 + b)^{-1/2}[a\lambda^2 + 2(a\lambda^2 + b)]}{\left(\lambda^2\sqrt{a\lambda^2 + b}\right)^2} = \frac{b(3a\lambda^2 + 2b)}{\lambda^3(a\lambda^2 + b)^{3/2}} > 0,$$

so f is CU on $(0, \infty)$. There are no extrema or inflection points. The graph shows that as λ decreases, the energy increases and as λ increases, the energy decreases. For large wavelengths, the energy is very close to the energy at rest.



47. $y = -\dfrac{W}{24EI}x^4 + \dfrac{WL}{12EI}x^3 - \dfrac{WL^2}{24EI}x^2 = -\dfrac{W}{24EI}x^2\left(x^2 - 2Lx + L^2\right)$

$= \dfrac{-W}{24EI}x^2(x-L)^2 = cx^2(x-L)^2$



where $c = -\dfrac{W}{24EI}$ is a negative constant and $0 \le x \le L$. We sketch

$f(x) = cx^2(x-L)^2$ for $c = -1$. $f(0) = f(L) = 0$.

$f'(x) = cx^2[2(x-L)] + (x-L)^2(2cx) = 2cx(x-L)[x + (x-L)] = 2cx(x-L)(2x-L)$. So for $0 < x < L$,

$f'(x) > 0 \iff x(x-L)(2x-L) < 0$ [since $c < 0$] $\iff L/2 < x < L$ and $f'(x) < 0 \iff 0 < x < L/2$.

Thus, f is increasing on $(L/2, L)$ and decreasing on $(0, L/2)$, and there is a local and absolute minimum at the point $(L/2, f(L/2)) = \left(L/2, cL^4/16\right)$. $f'(x) = 2c[x(x-L)(2x-L)] \Rightarrow$

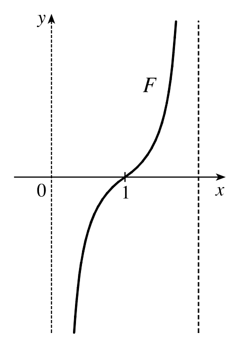
$f''(x) = 2c[1(x-L)(2x-L) + x(1)(2x-L) + x(x-L)(2)] = 2c(6x^2 - 6Lx + L^2) = 0 \iff$

$x = \dfrac{6L \pm \sqrt{12L^2}}{12} = \frac{1}{2}L \pm \frac{\sqrt{3}}{6}L$, and these are the x-coordinates of the two inflection points.

48. $F(x) = -\dfrac{k}{x^2} + \dfrac{k}{(x-2)^2}$, where $k > 0$ and $0 < x < 2$. For $0 < x < 2$, $x - 2 < 0$, so

$F'(x) = \dfrac{2k}{x^3} - \dfrac{2k}{(x-2)^3} > 0$ and F is increasing. $\lim\limits_{x\to 0^+} F(x) = -\infty$ and

$\lim\limits_{x\to 2^-} F(x) = \infty$, so $x = 0$ and $x = 2$ are vertical asymptotes. Notice that when the middle particle is at $x = 1$, the net force acting on it is 0. When $x > 1$, the net force is positive, meaning that it acts to the right. And if the particle approaches $x = 2$, the force on it rapidly becomes very large. When $x < 1$, the net force is negative, so it acts to the left. If the particle approaches 0, the force becomes very large to the left.



49. $y = \dfrac{x^2+1}{x+1}$. Long division gives us:

$$\begin{array}{r|l} & \quad\;\; x - 1 \\ \cline{2-2} x+1 & x^2 \qquad + 1 \\ & \underline{x^2 + x \qquad} \\ & \quad -x + 1 \\ & \quad \underline{-x - 1} \\ & \qquad\quad 2 \end{array}$$

Thus, $y = f(x) = \dfrac{x^2+1}{x+1} = x - 1 + \dfrac{2}{x+1}$ and $f(x) - (x-1) = \dfrac{2}{x+1} = \dfrac{\dfrac{2}{x}}{1 + \dfrac{1}{x}}$ [for $x \ne 0$] $\to 0$ as $x \to \pm\infty$.

So the line $y = x - 1$ is a slant asymptote (SA).

50. $y = \dfrac{4x^3 - 10x^2 - 11x + 1}{x^2 - 3x}$. Long division gives us:

$$\begin{array}{r|l} & \quad\quad\quad\quad\quad 4x + 2 \\ \hline x^2 - 3x & 4x^3 - 10x^2 - 11x + 1 \\ & 4x^3 - 12x^2 \\ \hline & \quad\quad 2x^2 - 11x \\ & \quad\quad 2x^2 - 6x \\ \hline & \quad\quad\quad\quad -5x + 1 \end{array}$$

Thus, $y = f(x) = \dfrac{4x^3 - 10x^2 - 11x + 1}{x^2 - 3x} = 4x + 2 + \dfrac{-5x + 1}{x^2 - 3x}$ and $f(x) - (4x + 2) = \dfrac{-5x + 1}{x^2 - 3x} = \dfrac{-\frac{5}{x} + \frac{1}{x^2}}{1 - \frac{3}{x}}$

[for $x \neq 0$] $\to \frac{0}{1} = 0$ as $x \to \pm\infty$. So the line $y = 4x + 2$ is a slant asymptote (SA).

51. $y = \dfrac{2x^3 - 5x^2 + 3x}{x^2 - x - 2}$. Long division gives us:

$$\begin{array}{r|l} & \quad\quad\quad\quad 2x - 3 \\ \hline x^2 - x - 2 & 2x^3 - 5x^2 + 3x \\ & 2x^3 - 2x^2 - 4x \\ \hline & \quad -3x^2 + 7x \\ & \quad -3x^2 + 3x + 6 \\ \hline & \quad\quad\quad 4x - 6 \end{array}$$

Thus, $y = f(x) = \dfrac{2x^3 - 5x^2 + 3x}{x^2 - x - 2} = 2x - 3 + \dfrac{4x - 6}{x^2 - x - 2}$ and $f(x) - (2x - 3) = \dfrac{4x - 6}{x^2 - x - 2} = \dfrac{\frac{4}{x} - \frac{6}{x^2}}{1 - \frac{1}{x} - \frac{1}{x^2}}$

[for $x \neq 0$] $\to \frac{0}{1} = 0$ as $x \to \pm\infty$. So the line $y = 2x - 3$ is a slant asymptote (SA).

52. $y = \dfrac{-6x^4 + 2x^3 + 3}{2x^3 - x}$. Long division gives us:

$$\begin{array}{r|l} & \quad\quad\quad\quad\quad -3x + 1 \\ \hline 2x^3 - x & -6x^4 + 2x^3 \quad\quad\quad\quad + 3 \\ & -6x^4 \quad\quad + 3x^2 \\ \hline & \quad\quad 2x^3 - 3x^2 \\ & \quad\quad 2x^3 \quad\quad\quad - x \\ \hline & \quad\quad\quad -3x^2 + x + 3 \end{array}$$

Thus, $y = f(x) = \dfrac{-6x^4 + 2x^3 + 3}{2x^3 - x} = -3x + 1 + \dfrac{-3x^2 + x + 3}{2x^3 - x}$ and

$f(x) - (-3x + 1) = \dfrac{-3x^2 + x + 3}{2x^3 - x} = \dfrac{-\frac{3}{x} + \frac{1}{x^2} + \frac{3}{x^3}}{2 - \frac{1}{x^2}}$ [for $x \neq 0$] $\to \frac{0}{2} = 0$ as $x \to \pm\infty$. So the line $y = -3x + 1$ is a slant asymptote (SA).

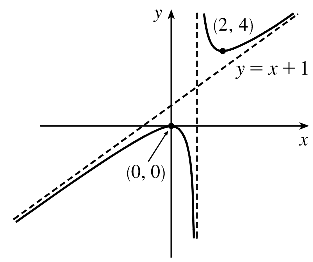
53. $y = f(x) = \dfrac{x^2}{x - 1} = x + 1 + \dfrac{1}{x - 1}$ **A.** $D = (-\infty, 1) \cup (1, \infty)$ **B.** x-intercept: $f(x) = 0 \iff x = 0$; y-intercept: $f(0) = 0$ **C.** No symmetry **D.** $\lim\limits_{x \to 1^-} f(x) = -\infty$ and $\lim\limits_{x \to 1^+} f(x) = \infty$, so $x = 1$ is a VA.

$\lim\limits_{x \to \pm\infty} [f(x) - (x + 1)] = \lim\limits_{x \to \pm\infty} \dfrac{1}{x - 1} = 0$, so the line $y = x + 1$ is a SA.

[continued]

E. $f'(x) = 1 - \dfrac{1}{(x-1)^2} = \dfrac{(x-1)^2 - 1}{(x-1)^2} = \dfrac{x^2 - 2x}{(x-1)^2} = \dfrac{x(x-2)}{(x-1)^2} > 0$ for $x < 0$ or $x > 2$, so f is increasing on $(-\infty, 0)$ and $(2, \infty)$, and f is decreasing on $(0, 1)$ and $(1, 2)$. **F.** Local maximum value $f(0) = 0$, local minimum value $f(2) = 4$ **G.** $f''(x) = \dfrac{2}{(x-1)^3} > 0$ for $x > 1$, so f is CU on $(1, \infty)$ and f is CD on $(-\infty, 1)$. No IP

H.



54. $y = f(x) = \dfrac{1 + 5x - 2x^2}{x - 2} = -2x + 1 + \dfrac{3}{x-2}$ **A.** $D = (-\infty, 2) \cup (2, \infty)$ **B.** x-intercepts: $f(x) = 0 \Leftrightarrow$ $1 + 5x - 2x^2 = 0 \Rightarrow x = \dfrac{-5 \pm \sqrt{33}}{-4} \Rightarrow x \approx -0.19, 2.69$; y-intercept: $f(0) = -\frac{1}{2}$ **C.** No symmetry

D. $\lim\limits_{x \to 2^-} f(x) = -\infty$ and $\lim\limits_{x \to 2^+} f(x) = \infty$, so $x = 2$ is a VA. $\lim\limits_{x \to \pm\infty} [f(x) - (-2x + 1)] = \lim\limits_{x \to \pm\infty} \dfrac{3}{x - 2} = 0$, so $y = -2x + 1$ is a SA.

E. $f'(x) = -2 - \dfrac{3}{(x-2)^2} = \dfrac{-2(x^2 - 4x + 4) - 3}{(x-2)^2}$

$= \dfrac{-2x^2 + 8x - 11}{(x-2)^2} < 0$

for $x \neq 2$, so f is decreasing on $(-\infty, 2)$ and $(2, \infty)$. **F.** No extreme values

G. $f''(x) = \dfrac{6}{(x-2)^3} > 0$ for $x > 2$, so f is CU on $(2, \infty)$ and CD on $(-\infty, 2)$. No IP

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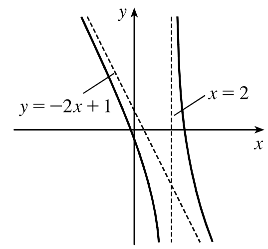
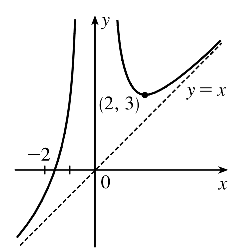


55. $y = f(x) = \dfrac{x^3 + 4}{x^2} = x + \dfrac{4}{x^2}$ **A.** $D = (-\infty, 0) \cup (0, \infty)$ **B.** x-intercept: $f(x) = 0 \Leftrightarrow x = -\sqrt[3]{4}$; no y-intercept

C. No symmetry **D.** $\lim\limits_{x \to 0} f(x) = \infty$, so $x = 0$ is a VA. $\lim\limits_{x \to \pm\infty} [f(x) - x] = \lim\limits_{x \to \pm\infty} \dfrac{4}{x^2} = 0$, so $y = x$ is a SA.

E. $f'(x) = 1 - \dfrac{8}{x^3} = \dfrac{x^3 - 8}{x^3} > 0$ for $x < 0$ or $x > 2$, so f is increasing on $(-\infty, 0)$ and $(2, \infty)$, and f is decreasing on $(0, 2)$. **F.** Local minimum value $f(2) = 3$, no local maximum value **G.** $f''(x) = \dfrac{24}{x^4} > 0$ for $x \neq 0$, so f is CU on $(-\infty, 0)$ and $(0, \infty)$. No IP

H.



56. $y = f(x) = \dfrac{x^3}{(x+1)^2} = x - 2 + \dfrac{3x + 2}{(x+1)^2}$ **A.** $D = (-\infty, -1) \cup (-1, \infty)$ **B.** x-intercept: 0; y-intercept: $f(0) = 0$

C. No symmetry **D.** $\lim\limits_{x \to -1^-} f(x) = -\infty$ and $\lim\limits_{x \to -1^+} f(x) = -\infty$, so $x = -1$ is a VA.

$\lim\limits_{x \to \pm\infty} [f(x) - (x - 2)] = \lim\limits_{x \to \pm\infty} \dfrac{3x + 2}{(x+1)^2} = 0$, so $y = x - 2$ is a SA.

E. $f'(x) = \dfrac{(x+1)^2(3x^2) - x^3 \cdot 2(x+1)}{[(x+1)^2]^2} = \dfrac{x^2(x+1)[3(x+1) - 2x]}{(x+1)^4} = \dfrac{x^2(x+3)}{(x+1)^3} > 0 \Leftrightarrow x < -3$ or

[continued]

$x > -1$ $[x \neq 0]$, so f is increasing on $(-\infty, -3)$ and $(-1, \infty)$, and f is decreasing on $(-3, -1)$.

F. Local maximum value $f(-3) = -\frac{27}{4}$, no local minimum

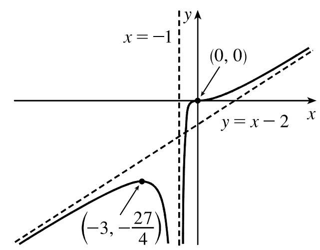
G. $f''(x) = \dfrac{(x+1)^3(3x^2+6x) - (x^3+3x^2)\cdot 3(x+1)^2}{[(x+1)^3]^2}$

$= \dfrac{3x(x+1)^2[(x+1)(x+2) - (x^2+3x)]}{(x+1)^6}$

$= \dfrac{3x(x^2+3x+2-x^2-3x)}{(x+1)^4} = \dfrac{6x}{(x+1)^4} > 0 \quad \Leftrightarrow$

$x > 0$, so f is CU on $(0, \infty)$ and f is CD on $(-\infty, -1)$ and $(-1, 0)$. IP at $(0, 0)$

H.



57. $y = f(x) = \dfrac{2x^3 + x^2 + 1}{x^2+1} = 2x + 1 + \dfrac{-2x}{x^2+1}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 1$; x-intercept: $f(x) = 0 \;\Rightarrow$

$0 = 2x^3 + x^2 + 1 = (x+1)(2x^2 - x + 1) \;\Rightarrow\; x = -1$ **C.** No symmetry **D.** No VA

$\displaystyle\lim_{x\to\pm\infty} [f(x) - (2x+1)] = \lim_{x\to\pm\infty} \frac{-2x}{x^2+1} = \lim_{x\to\pm\infty} \frac{-2/x}{1+1/x^2} = 0$, so the line $y = 2x+1$ is a slant asymptote.

E. $f'(x) = 2 + \dfrac{(x^2+1)(-2) - (-2x)(2x)}{(x^2+1)^2} = \dfrac{2(x^4+2x^2+1) - 2x^2 - 2 + 4x^2}{(x^2+1)^2} = \dfrac{2x^4+6x^2}{(x^2+1)^2} = \dfrac{2x^2(x^2+3)}{(x^2+1)^2}$

so $f'(x) > 0$ if $x \neq 0$. Thus, f is increasing on $(-\infty, 0)$ and $(0, \infty)$. Since f is continuous at 0, f is increasing on $\mathbb{R}$.

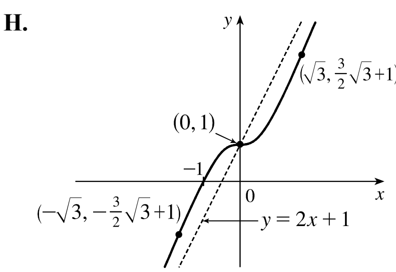
F. No extreme values

G. $f''(x) = \dfrac{(x^2+1)^2 \cdot (8x^3+12x) - (2x^4+6x^2)\cdot 2(x^2+1)(2x)}{[(x^2+1)^2]^2}$

$= \dfrac{4x(x^2+1)[(x^2+1)(2x^2+3) - 2x^4 - 6x^2]}{(x^2+1)^4} = \dfrac{4x(-x^2+3)}{(x^2+1)^3}$

so $f''(x) > 0$ for $x < -\sqrt{3}$ and $0 < x < \sqrt{3}$, and $f''(x) < 0$ for $-\sqrt{3} < x < 0$ and $x > \sqrt{3}$. f is CU on $\left(-\infty, -\sqrt{3}\right)$ and $\left(0, \sqrt{3}\right)$, and CD on $\left(-\sqrt{3}, 0\right)$ and $\left(\sqrt{3}, \infty\right)$. There are three IPs: $(0, 1)$, $\left(-\sqrt{3}, -\frac{3}{2}\sqrt{3}+1\right) \approx (-1.73, -1.60)$, and $\left(\sqrt{3}, \frac{3}{2}\sqrt{3}+1\right) \approx (1.73, 3.60)$.

H.



58. $y = f(x) = \dfrac{(x+1)^3}{(x-1)^2} = \dfrac{x^3+3x^2+3x+1}{x^2-2x+1} = x + 5 + \dfrac{12x-4}{(x-1)^2}$

A. $D = \{x \in \mathbb{R} \mid x \neq 1\} = (-\infty, 1) \cup (1, \infty)$ **B.** y-intercept: $f(0) = 1$; x-intercept: $f(x) = 0 \;\Rightarrow$

$x = -1$ **C.** No symmetry **D.** $\displaystyle\lim_{x\to 1} f(x) = \infty$, so $x = 1$ is a VA.

$\displaystyle\lim_{x\to\pm\infty} [f(x) - (x+5)] = \lim_{x\to\pm\infty} \frac{12x-4}{x^2-2x+1} = \lim_{x\to\pm\infty} \frac{\dfrac{12}{x} - \dfrac{4}{x^2}}{1 - \dfrac{2}{x} + \dfrac{1}{x^2}} = 0$, so the line $y = x+5$ is a SA.

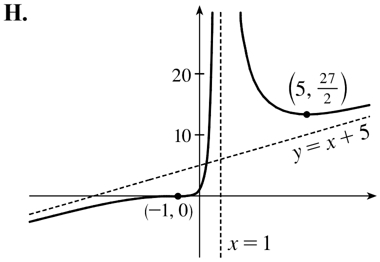
[continued]

E. $f'(x) = \dfrac{(x-1)^2 \cdot 3(x+1)^2 - (x+1)^3 \cdot 2(x-1)}{[(x-1)^2]^2}$

$= \dfrac{(x-1)(x+1)^2[3(x-1) - 2(x+1)]}{(x-1)^4} = \dfrac{(x+1)^2(x-5)}{(x-1)^3}$

so $f'(x) > 0$ when $x < -1$, $-1 < x < 1$, or $x > 5$, and $f'(x) < 0$ when $1 < x < 5$. f is increasing on $(-\infty, 1)$ and $(5, \infty)$ and decreasing on $(1, 5)$.

H.



F. Local minimum value $f(5) = \frac{216}{16} = \frac{27}{2}$, no local maximum

G. $f''(x) = \dfrac{(x-1)^3[(x-1)^2 + (x-5) \cdot 2(x+1)] - (x+1)^2(x-5) \cdot 3(x-1)^2}{[(x-1)^3]^2}$

$= \dfrac{(x-1)^2(x+1)\,\{(x-1)[(x+1) + 2(x-5)] - 3(x+1)(x-5)\}}{(x-1)^6}$

$= \dfrac{(x+1)\{(x-1)[3x-9] - 3(x^2-4x-5)\}}{(x-1)^4} = \dfrac{(x+1)(24)}{(x-1)^4}$

so $f''(x) > 0$ if $-1 < x < 1$ or $x > 1$, and $f''(x) < 0$ if $x < -1$. Thus, f is CU on $(-1, 1)$ and $(1, \infty)$ and CD on $(-\infty, -1)$. IP at $(-1, 0)$

59. $y = f(x) = \sqrt{4x^2+9} \quad \Rightarrow \quad f'(x) = \dfrac{4x}{\sqrt{4x^2+9}} \quad \Rightarrow$

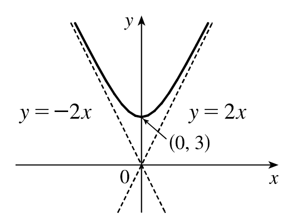
$f''(x) = \dfrac{\sqrt{4x^2+9} \cdot 4 - 4x \cdot 4x/\sqrt{4x^2+9}}{4x^2+9} = \dfrac{4(4x^2+9) - 16x^2}{(4x^2+9)^{3/2}} = \dfrac{36}{(4x^2+9)^{3/2}}$. f is defined on $(-\infty, \infty)$.

$f(-x) = f(x)$, so f is even, which means its graph is symmetric about the y-axis. The y-intercept is $f(0) = 3$. There are no x-intercepts since $f(x) > 0$ for all x.

$\lim\limits_{x\to\infty} \left(\sqrt{4x^2+9} - 2x\right) = \lim\limits_{x\to\infty} \dfrac{\left(\sqrt{4x^2+9} - 2x\right)\left(\sqrt{4x^2+9} + 2x\right)}{\sqrt{4x^2+9} + 2x}$

$= \lim\limits_{x\to\infty} \dfrac{(4x^2+9) - 4x^2}{\sqrt{4x^2+9} + 2x} = \lim\limits_{x\to\infty} \dfrac{9}{\sqrt{4x^2+9} + 2x} = 0$

and, similarly, $\lim\limits_{x\to-\infty} \left(\sqrt{4x^2+9} + 2x\right) = \lim\limits_{x\to-\infty} \dfrac{9}{\sqrt{4x^2+9} - 2x} = 0$,



so $y = \pm 2x$ are slant asymptotes. f is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$ with local minimum $f(0) = 3$. $f''(x) > 0$ for all x, so f is CU on $\mathbb{R}$.

60. $y = f(x) = \sqrt{x^2+4x} = \sqrt{x(x+4)}$. $\quad x(x+4) \geq 0 \quad \Leftrightarrow \quad x \leq -4$ or $x \geq 0$, so $D = (-\infty, -4] \cup [0, \infty)$.

y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \quad \Rightarrow \quad x = -4, 0$.

$\sqrt{x^2+4x} \mp (x+2) = \dfrac{\sqrt{x^2+4x} \mp (x+2)}{1} \cdot \dfrac{\sqrt{x^2+4x} \pm (x+2)}{\sqrt{x^2+4x} \pm (x+2)} = \dfrac{(x^2+4x) - (x^2+4x+4)}{\sqrt{x^2+4x} \pm (x+2)}$

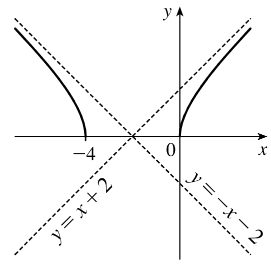
$= \dfrac{-4}{\sqrt{x^2+4x} \pm (x+2)}$

[continued]

so $\lim\limits_{x\to\pm\infty} [f(x) \mp (x+2)] = 0$. Thus, the graph of f approaches the slant asymptote $y = x+2$ as $x \to \infty$ and it approaches the slant asymptote $y = -(x+2)$ as $x \to -\infty$. $f'(x) = \dfrac{x+2}{\sqrt{x^2+4x}}$, so $f'(x) < 0$ for $x < -4$ and $f'(x) > 0$ for $x > 0$; that is, f is decreasing on $(-\infty, -4)$ and increasing on $(0, \infty)$. There are no local extreme values. $f'(x) = (x+2)(x^2+4x)^{-1/2} \quad \Rightarrow$

$$\begin{aligned} f''(x) &= (x+2)\cdot\left(-\tfrac{1}{2}\right)(x^2+4x)^{-3/2}\cdot(2x+4) + (x^2+4x)^{-1/2} \\ &= (x^2+4x)^{-3/2}\left[-(x+2)^2 + (x^2+4x)\right] = -4(x^2+4x)^{-3/2} < 0 \text{ on } D \end{aligned}$$

so f is CD on $(-\infty, -4)$ and $(0, \infty)$. No IP



61. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1 \quad \Rightarrow \quad y = \pm\dfrac{b}{a}\sqrt{x^2-a^2}$. Now

$$\lim_{x\to\infty}\left[\frac{b}{a}\sqrt{x^2-a^2} - \frac{b}{a}x\right] = \frac{b}{a}\cdot\lim_{x\to\infty}\left(\sqrt{x^2-a^2}-x\right)\frac{\sqrt{x^2-a^2}+x}{\sqrt{x^2-a^2}+x} = \frac{b}{a}\cdot\lim_{x\to\infty}\frac{-a^2}{\sqrt{x^2-a^2}+x} = 0,$$

which shows that $y = \dfrac{b}{a}x$ is a slant asymptote. Similarly,

$$\lim_{x\to\infty}\left[-\frac{b}{a}\sqrt{x^2-a^2} - \left(-\frac{b}{a}x\right)\right] = -\frac{b}{a}\cdot\lim_{x\to\infty}\frac{-a^2}{\sqrt{x^2-a^2}+x} = 0, \text{ so } y = -\frac{b}{a}x \text{ is a slant asymptote.}$$

62. $f(x) - x^2 = \dfrac{x^3+1}{x} - x^2 = \dfrac{x^3+1-x^3}{x} = \dfrac{1}{x}$, and $\lim\limits_{x\to\pm\infty}\dfrac{1}{x} = 0$. Therefore, $\lim\limits_{x\to\pm\infty}[f(x) - x^2] = 0$, and so the graph of f is asymptotic to that of $y = x^2$. For purposes of differentiation, we will use $f(x) = x^2 + 1/x$.

A. $D = \{x \mid x \neq 0\}$ **B.** No y-intercept; to find the x-intercept, we set $y = 0 \quad \Leftrightarrow \quad x = -1$.

C. No symmetry **D.** $\lim\limits_{x\to0^+}\dfrac{x^3+1}{x} = \infty$ and $\lim\limits_{x\to0^-}\dfrac{x^3+1}{x} = -\infty$, so $x = 0$ is a vertical asymptote. Also, the graph is asymptotic to the parabola $y = x^2$, as shown above. **E.** $f'(x) = 2x - 1/x^2 > 0 \quad \Leftrightarrow \quad x > \frac{1}{\sqrt[3]{2}}$, so f is increasing on $\left(\dfrac{1}{\sqrt[3]{2}}, \infty\right)$ and decreasing on $(-\infty, 0)$ and $\left(0, \dfrac{1}{\sqrt[3]{2}}\right)$.

F. Local minimum value $f\left(\dfrac{1}{\sqrt[3]{2}}\right) = \dfrac{3\sqrt[3]{3}}{2}$, no local maximum

G. $f''(x) = 2 + 2/x^3 > 0 \quad \Leftrightarrow \quad x < -1$ or $x > 0$, so f is CU on $(-\infty, -1)$ and $(0, \infty)$, and CD on $(-1, 0)$. IP at $(-1, 0)$

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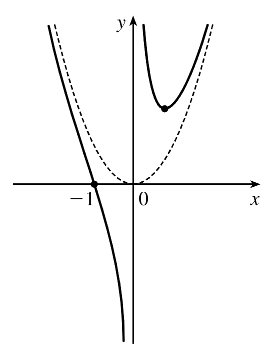


63. $\lim\limits_{x\to\pm\infty}\left[f(x) - x^3\right] = \lim\limits_{x\to\pm\infty}\dfrac{x^4+1}{x} - \dfrac{x^4}{x} = \lim\limits_{x\to\pm\infty}\dfrac{1}{x} = 0$, so the graph of f is asymptotic to that of $y = x^3$.

A. $D = \{x \mid x \neq 0\}$ **B.** No intercept **C.** f is symmetric about the origin. **D.** $\lim\limits_{x\to0^-}\left(x^3 + \dfrac{1}{x}\right) = -\infty$ and

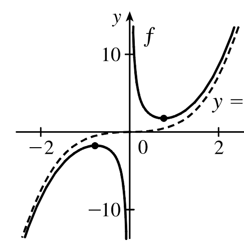
[continued]

$\lim\limits_{x \to 0^+} \left(x^3 + \dfrac{1}{x} \right) = \infty$, so $x = 0$ is a vertical asymptote, and as shown above, the graph of f is asymptotic to that of $y = x^3$.

E. $f'(x) = 3x^2 - 1/x^2 > 0 \quad \Leftrightarrow \quad x^4 > \frac{1}{3} \quad \Leftrightarrow \quad |x| > \frac{1}{\sqrt[4]{3}}$, so f is increasing on $\left(-\infty, -\dfrac{1}{\sqrt[4]{3}} \right)$ and $\left(\dfrac{1}{\sqrt[4]{3}}, \infty \right)$ and decreasing on $\left(-\dfrac{1}{\sqrt[4]{3}}, 0 \right)$ and $\left(0, \dfrac{1}{\sqrt[4]{3}} \right)$. **F.** Local maximum value $f\left(-\dfrac{1}{\sqrt[4]{3}} \right) = -4 \cdot 3^{-5/4}$, local minimum value $f\left(\dfrac{1}{\sqrt[4]{3}} \right) = 4 \cdot 3^{-5/4}$

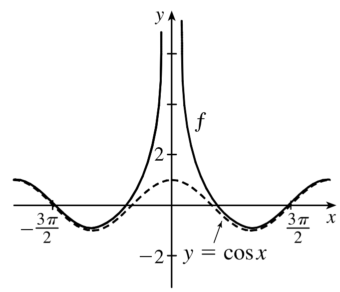
G. $f''(x) = 6x + 2/x^3 > 0 \quad \Leftrightarrow \quad x > 0$, so f is CU on $(0, \infty)$ and CD on $(-\infty, 0)$. No IP

H.



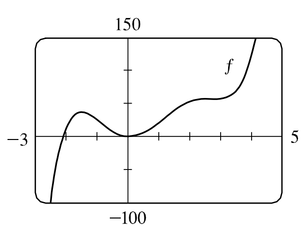
64. $\lim\limits_{x \to \pm\infty} [f(x) - \cos x] = \lim\limits_{x \to \pm\infty} 1/x^2 = 0$, so the graph of f is asymptotic to that of $\cos x$. The intercepts can only be found approximately.

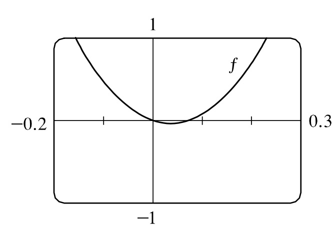
$f(x) = f(-x)$, so f is even. $\lim\limits_{x \to 0} \left(\cos x + \dfrac{1}{x^2} \right) = \infty$, so $x = 0$ is a vertical asymptote. We don't need to calculate the derivatives, since we know the asymptotic behavior of the curve.

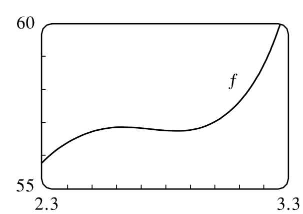


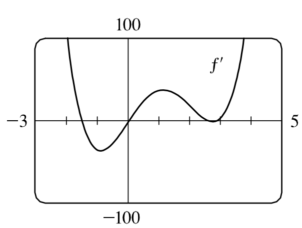
3.6 Graphing with Calculus and Technology

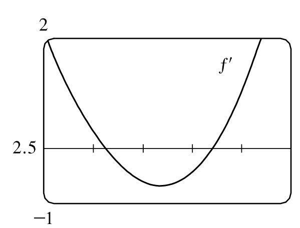
1. $f(x) = x^5 - 5x^4 - x^3 + 28x^2 - 2x \quad \Rightarrow \quad f'(x) = 5x^4 - 20x^3 - 3x^2 + 56x - 2 \quad \Rightarrow \quad f''(x) = 20x^3 - 60x^2 - 6x + 56$. $f(x) = 0 \quad \Leftrightarrow \quad x = 0$ or $x \approx -2.09, 0.07$; $f'(x) = 0 \quad \Leftrightarrow \quad x \approx -1.50, 0.04, 2.62, 2.84$; $f''(x) = 0 \quad \Leftrightarrow \quad x \approx -0.89, 1.15, 2.74$.







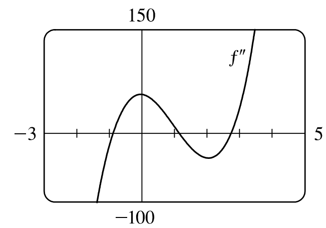




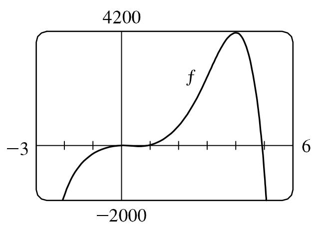
[continued]

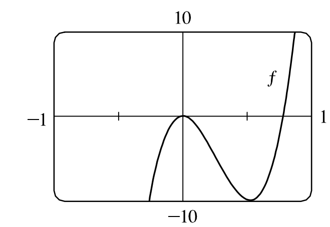
From the graphs of f', we estimate that $f' < 0$ and that f is decreasing on $(-1.50, 0.04)$ and $(2.62, 2.84)$, and that $f' > 0$ and f is increasing on $(-\infty, -1.50)$, $(0.04, 2.62)$, and $(2.84, \infty)$ with local minimum values $f(0.04) \approx -0.04$ and $f(2.84) \approx 56.73$ and local maximum values $f(-1.50) \approx 36.47$ and $f(2.62) \approx 56.83$.

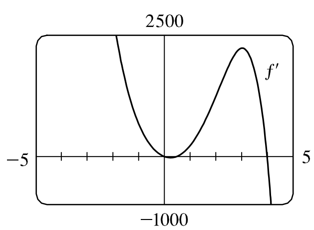
From the graph of f'', we estimate that $f'' > 0$ and that f is CU on $(-0.89, 1.15)$ and $(2.74, \infty)$, and that $f'' < 0$ and f is CD on $(-\infty, -0.89)$ and $(1.15, 2.74)$. There are inflection points at about $(-0.89, 20.90)$, $(1.15, 26.57)$, and $(2.74, 56.78)$.

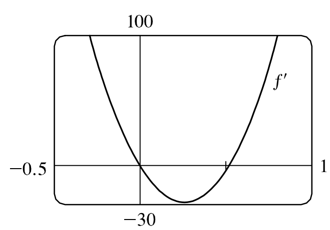


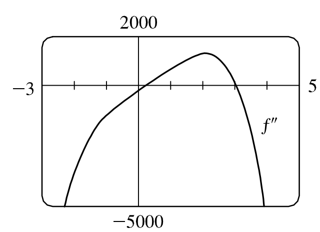
2. $f(x) = -2x^6 + 5x^5 + 140x^3 - 110x^2 \quad\Rightarrow\quad f'(x) = -12x^5 + 25x^4 + 420x^2 - 220x \quad\Rightarrow$
$f''(x) = -60x^4 + 100x^3 + 840x - 220.$ $\;f(x) = 0 \quad\Leftrightarrow\quad x = 0$ or $x \approx 0.77, 4.93$; $\;f'(x) = 0 \quad\Leftrightarrow\quad x = 0$ or $x \approx 0.52, 3.99$; $\;f''(x) = 0 \quad\Leftrightarrow\quad x \approx 0.26, 3.05$.





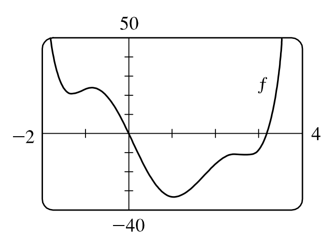






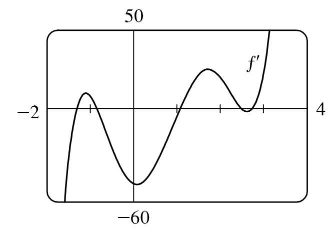
From the graphs of f', we estimate that $f' > 0$ and that f is increasing on $(-\infty, 0)$ and $(0.52, 3.99)$, and that $f' < 0$ and that f is decreasing on $(0, 0.52)$ and $(3.99, \infty)$. f has local maximum values $f(0) = 0$ and $f(3.99) \approx 4128.20$, and f has local minimum value $f(0.52) \approx -9.91$. From the graph of f'', we estimate that $f'' > 0$ and f is CU on $(0.26, 3.05)$, and that $f'' < 0$ and f is CD on $(-\infty, 0.26)$ and $(3.05, \infty)$. There are inflection points at about $(0.26, -4.97)$ and $(3.05, 2649.46)$.

3. $f(x) = x^6 - 5x^5 + 25x^3 - 6x^2 - 48x \quad\Rightarrow$
$f'(x) = 6x^5 - 25x^4 + 75x^2 - 12x - 48 \quad\Rightarrow$
$f''(x) = 30x^4 - 100x^3 + 150x - 12.$ $\;f(x) = 0 \quad\Leftrightarrow\quad x = 0$ or $x \approx 3.20$;
$f'(x) = 0 \quad\Leftrightarrow\quad x \approx -1.31, -0.84, 1.06, 2.50, 2.75$; $\;f''(x) = 0 \quad\Leftrightarrow$
$x \approx -1.10, 0.08, 1.72, 2.64$.

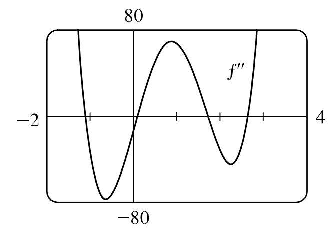


[continued]

From the graph of f', we estimate that f is decreasing on $(-\infty, -1.31)$, increasing on $(-1.31, -0.84)$, decreasing on $(-0.84, 1.06)$, increasing on $(1.06, 2.50)$, decreasing on $(2.50, 2.75)$, and increasing on $(2.75, \infty)$. f has local minimum values $f(-1.31) \approx 20.72$, $f(1.06) \approx -33.12$, and $f(2.75) \approx -11.33$. f has local maximum values $f(-0.84) \approx 23.71$ and $f(2.50) \approx -11.02$.

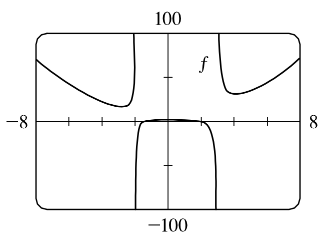


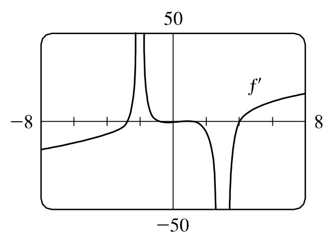
From the graph of f'', we estimate that f is CU on $(-\infty, -1.10)$, CD on $(-1.10, 0.08)$, CU on $(0.08, 1.72)$, CD on $(1.72, 2.64)$, and CU on $(2.64, \infty)$. There are inflection points at about $(-1.10, 22.09)$, $(0.08, -3.88)$, $(1.72, -22.53)$, and $(2.64, -11.18)$.

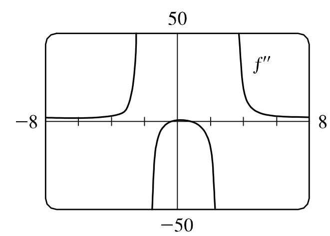


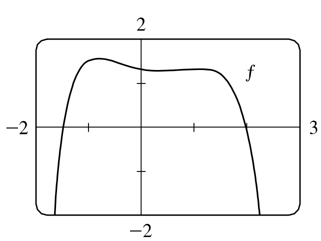
4. $f(x) = \dfrac{x^4 - x^3 - 8}{x^2 - x - 6} \quad \Rightarrow \quad f'(x) = \dfrac{2(x^5 - 2x^4 - 11x^3 + 9x^2 + 8x - 4)}{(x^2 - x - 6)^2} \quad \Rightarrow$

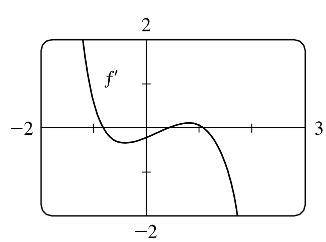
$f''(x) = \dfrac{2(x^6 - 3x^5 - 15x^4 + 41x^3 + 174x^2 - 84x - 56)}{(x^2 - x - 6)^3}$. $f(x) = 0 \;\Leftrightarrow\; x \approx -1.48$ or $x = 2$; $f'(x) = 0 \;\Leftrightarrow$ $x \approx -2.74, -0.81, 0.41, 1.08, 4.06$; $f''(x) = 0 \;\Leftrightarrow\; x \approx -0.39, 0.79$. The VAs are $x = -2$ and $x = 3$.

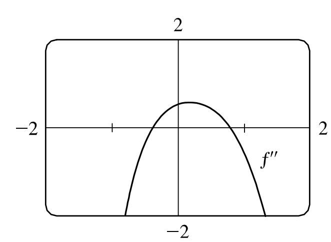








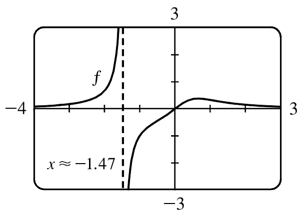


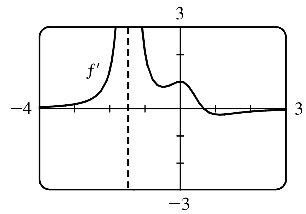


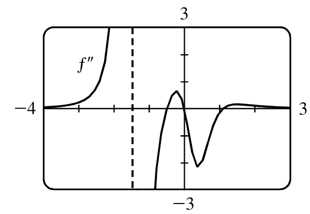
From the graphs of f', we estimate that f is decreasing on $(-\infty, -2.74)$, increasing on $(-2.74, -2)$, increasing on $(-2, -0.81)$, decreasing on $(-0.81, 0.41)$, increasing on $(0.41, 1.08)$, decreasing on $(1.08, 3)$, decreasing on $(3, 4.06)$, and increasing on $(4.06, \infty)$. f has local minimum values $f(-2.74) \approx 16.23$, $f(0.41) \approx 1.29$, and $f(4.06) \approx 30.63$. f has local maximum values $f(-0.81) \approx 1.55$ and $f(1.08) \approx 1.34$.

From the graphs of f'', we estimate that f is CU on $(-\infty, -2)$, CD on $(-2, -0.39)$, CU on $(-0.39, 0.79)$, CD on $(0.79, 3)$, and CU on $(3, \infty)$. There are inflection points at about $(-0.39, 1.45)$ and $(0.79, 1.31)$.

5. $f(x) = \dfrac{x}{x^3 + x^2 + 1} \quad\Rightarrow\quad f'(x) = -\dfrac{2x^3 + x^2 - 1}{(x^3 + x^2 + 1)^2} \quad\Rightarrow\quad f''(x) = \dfrac{2x(3x^4 + 3x^3 + x^2 - 6x - 3)}{(x^3 + x^2 + 1)^3}$



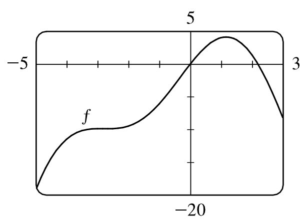


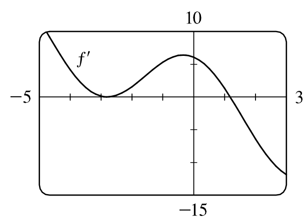


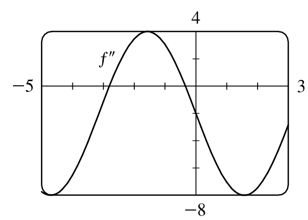
From the graph of f, we see that there is a VA at $x \approx -1.47$. From the graph of f', we estimate that f is increasing on $(-\infty, -1.47)$, increasing on $(-1.47, 0.66)$, and decreasing on $(0.66, \infty)$, with local maximum value $f(0.66) \approx 0.38$.

From the graph of f'', we estimate that f is CU on $(-\infty, -1.47)$, CD on $(-1.47, -0.49)$, CU on $(-0.49, 0)$, CD on $(0, 1.10)$, and CU on $(1.10, \infty)$. There is an inflection point at $(0, 0)$ and at about $(-0.49, -0.44)$ and $(1.10, 0.31)$.

6. $f(x) = 6\sin x - x^2$, $-5 \le x \le 3 \quad\Rightarrow\quad f'(x) = 6\cos x - 2x \quad\Rightarrow\quad f''(x) = -6\sin x - 2$



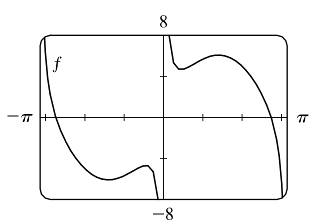


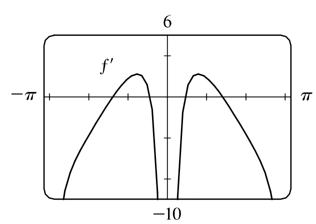


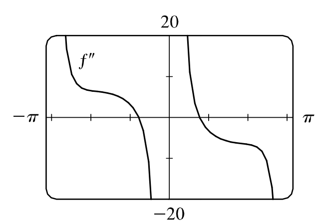
From the graph of f', which has two negative zeros, we estimate that f is increasing on $(-5, -2.94)$, decreasing on $(-2.94, -2.66)$, increasing on $(-2.66, 1.17)$, and decreasing on $(1.17, 3)$, with local maximum values $f(-2.94) \approx -9.84$ and $f(1.17) \approx 4.16$, and local minimum value $f(-2.66) \approx -9.85$.

From the graph of f'', we estimate that f is CD on $(-5, -2.80)$, CU on $(-2.80, -0.34)$, and CD on $(-0.34, 3)$. There are inflection points at about $(-2.80, -9.85)$ and $(-0.34, -2.12)$.

7. $f(x) = 6\sin x + \cot x$, $-\pi \le x \le \pi \quad\Rightarrow\quad f'(x) = 6\cos x - \csc^2 x \quad\Rightarrow\quad f''(x) = -6\sin x + 2\csc^2 x\,\cot x$





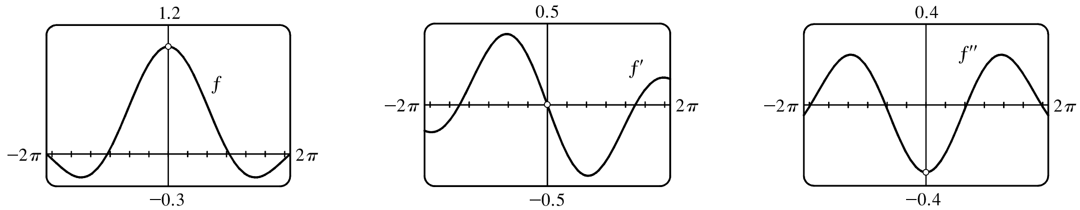


From the graph of f, we see that there are VAs at $x = 0$ and $x = \pm\pi$. f is an odd function, so its graph is symmetric about the origin. From the graph of f', we estimate that f is decreasing on $(-\pi, -1.40)$, increasing on $(-1.40, -0.44)$, decreasing on $(-0.44, 0)$, decreasing on $(0, 0.44)$, increasing on $(0.44, 1.40)$, and decreasing on $(1.40, \pi)$, with local minimum values $f(-1.40) \approx -6.09$ and $f(0.44) \approx 4.68$, and local maximum values $f(-0.44) \approx -4.68$ and $f(1.40) \approx 6.09$.

From the graph of f'', we estimate that f is CU on $(-\pi, -0.77)$, CD on $(-0.77, 0)$, CU on $(0, 0.77)$, and CD on $(0.77, \pi)$. There are IPs at about $(-0.77, -5.22)$ and $(0.77, 5.22)$.

8. $f(x) = \dfrac{\sin x}{x}$, $-2\pi \le x \le 2\pi$. $f'(x) = \dfrac{x\cos x - \sin x}{x^2} \Rightarrow$

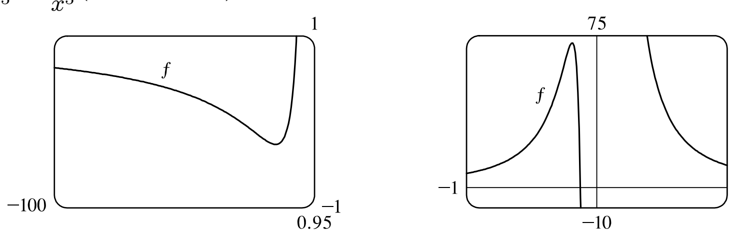
$$f''(x) = \frac{x^2(\cos x - x\sin x - \cos x) - (x\cos x - \sin x)(2x)}{(x^2)^2} = \frac{-x^2\sin x - 2x\cos x + 2\sin x}{x^3}$$



f is an even function with domain $(-\infty, 0) \cup (0, \infty)$. There is no y-intercept, but $\lim\limits_{x\to 0} f(x) = 1$. The x-intercepts are -2π, $-\pi$, 0, π, and 2π. From the graph of f', we estimate that f is decreasing on $(-2\pi, -4.49)$, increasing on $(-4.49, 0)$, decreasing on $(0, 4.49)$, and increasing on $(4.49, 2\pi)$. Thus, f has local minima of $f(\pm 4.49) \approx -0.22$. From the graph of f'', we estimate that f is CD on $(-2\pi, -5.94)$, CU on $(-5.94, -2.08)$, CD on $(-2.08, 0)$ and $(0, 2.08)$, CU on $(2.08, 5.94)$, and CD on $(5.94, 2\pi)$. f has IPs at approximately $(\pm 5.94, -0.06)$ and $(\pm 2.08, 0.42)$.

9. $f(x) = 1 + \dfrac{1}{x} + \dfrac{8}{x^2} + \dfrac{1}{x^3} \;\Rightarrow\; f'(x) = -\dfrac{1}{x^2} - \dfrac{16}{x^3} - \dfrac{3}{x^4} = -\dfrac{1}{x^4}(x^2 + 16x + 3) \;\Rightarrow$

$f''(x) = \dfrac{2}{x^3} + \dfrac{48}{x^4} + \dfrac{12}{x^5} = \dfrac{2}{x^5}(x^2 + 24x + 6)$.



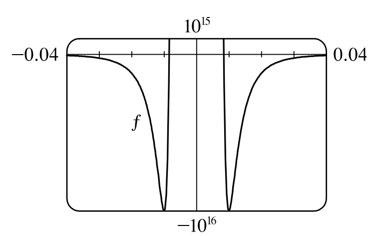
From the graphs, it appears that f increases on $(-15.8, -0.2)$ and decreases on $(-\infty, -15.8)$, $(-0.2, 0)$, and $(0, \infty)$; that f has a local minimum value of $f(-15.8) \approx 0.97$ and a local maximum value of $f(-0.2) \approx 72$; that f is CD on $(-\infty, -24)$ and $(-0.25, 0)$ and is CU on $(-24, -0.25)$ and $(0, \infty)$; and that f has IPs at $(-24, 0.97)$ and $(-0.25, 60)$.

To find the exact values, note that $f' = 0 \;\Rightarrow\; x = \dfrac{-16 \pm \sqrt{256 - 12}}{2} = -8 \pm \sqrt{61}$ [≈ -0.19 and -15.81]. f' is positive (f is increasing) on $\left(-8 - \sqrt{61}, -8 + \sqrt{61}\right)$ and f' is negative (f is decreasing) on $\left(-\infty, -8 - \sqrt{61}\right)$, $\left(-8 + \sqrt{61}, 0\right)$, and $(0, \infty)$. $f'' = 0 \;\Rightarrow\; x = \dfrac{-24 \pm \sqrt{576 - 24}}{2} = -12 \pm \sqrt{138}$ [≈ -0.25 and -23.75]. f'' is positive (f is CU) on $\left(-12 - \sqrt{138}, -12 + \sqrt{138}\right)$ and $(0, \infty)$ and f'' is negative (f is CD) on $\left(-\infty, -12 - \sqrt{138}\right)$ and $\left(-12 + \sqrt{138}, 0\right)$.

10. $f(x) = \dfrac{1}{x^8} - \dfrac{c}{x^4}$ [$c = 2 \times 10^8$] $\Rightarrow$

$f'(x) = -\dfrac{8}{x^9} + \dfrac{4c}{x^5} = -\dfrac{4}{x^9}(2 - cx^4) \;\Rightarrow$

$f''(x) = \dfrac{72}{x^{10}} - \dfrac{20c}{x^6} = \dfrac{4}{x^{10}}(18 - 5cx^4)$.



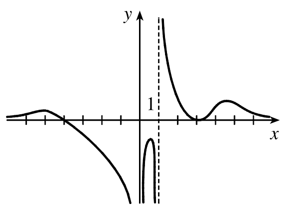
[continued]

From the graph, it appears that f increases on $(-0.01, 0)$ and $(0.01, \infty)$ and decreases on $(-\infty, -0.01)$ and $(0, 0.01)$; that f has a local minimum value of $f(\pm 0.01) = -10^{16}$; and that f is CU on $(-0.012, 0)$ and $(0, 0.012)$ and f is CD on $(-\infty, -0.012)$ and $(0.012, \infty)$.

To find the exact values, note that $f' = 0 \quad \Rightarrow \quad x^4 = \frac{2}{c} \quad \Rightarrow \quad x \pm \sqrt[4]{\frac{2}{c}} = \pm\frac{1}{100} \quad [c = 2 \times 10^8]$. f' is positive (f is increasing) on $(-0.01, 0)$ and $(0.01, \infty)$ and f' is negative (f is decreasing) on $(-\infty, -0.01)$ and $(0, 0.01)$.

$f'' = 0 \quad \Rightarrow \quad x^4 = \dfrac{18}{5c} \quad \Rightarrow \quad x = \pm\sqrt[4]{\dfrac{18}{5c}} = \pm\dfrac{1}{100}\sqrt[4]{1.8} \quad [\approx \pm 0.0116]$. f'' is positive (f is CU) on $\left(-\frac{1}{100}\sqrt[4]{1.8}, 0\right)$ and $\left(0, \frac{1}{100}\sqrt[4]{1.8}\right)$ and f'' is negative (f is CD) on $\left(-\infty, -\frac{1}{100}\sqrt[4]{1.8}\right)$ and $\left(\frac{1}{100}\sqrt[4]{1.8}, \infty\right)$.

11.

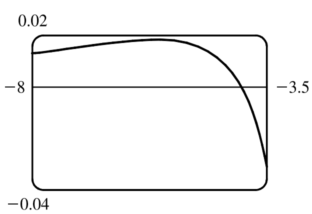


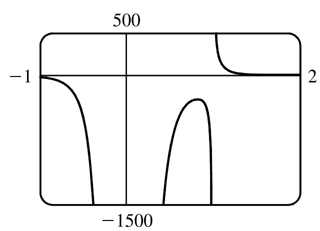
$f(x) = \dfrac{(x+4)(x-3)^2}{x^4(x-1)}$ has VA at $x = 0$ and at $x = 1$ since $\lim\limits_{x\to 0} f(x) = -\infty$, $\lim\limits_{x\to 1^-} f(x) = -\infty$ and $\lim\limits_{x\to 1^+} f(x) = \infty$.

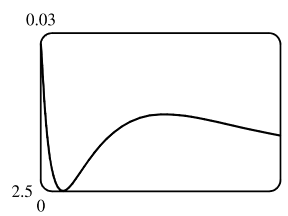
$$f(x) = \frac{\dfrac{x+4}{x} \cdot \dfrac{(x-3)^2}{x^2}}{\dfrac{x^4}{x^3} \cdot (x-1)} \quad \left[\begin{array}{c}\text{dividing numerator}\\ \text{and denominator by } x^3\end{array}\right] = \frac{(1+4/x)(1-3/x)^2}{x(x-1)} \to 0$$

as $x \to \pm\infty$, so f is asymptotic to the x-axis.

Since f is undefined at $x = 0$, it has no y-intercept. $f(x) = 0 \quad \Rightarrow \quad (x+4)(x-3)^2 = 0 \quad \Rightarrow \quad x = -4$ or $x = 3$, so f has x-intercepts -4 and 3. Note, however, that the graph of f is only tangent to the x-axis and does not cross it at $x = 3$, since f is positive as $x \to 3^-$ and as $x \to 3^+$.

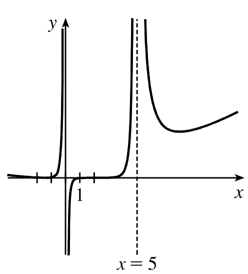






From these graphs, it appears that f has three maximum values and one minimum value. The maximum values are approximately $f(-5.6) = 0.0182$, $f(0.82) = -281.5$ and $f(5.2) = 0.0145$ and we know (since the graph is tangent to the x-axis at $x = 3$) that the minimum value is $f(3) = 0$.

12.



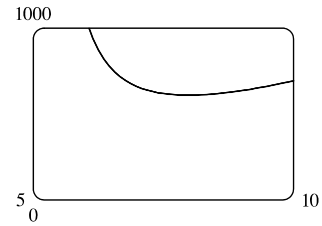
$f(x) = \dfrac{(2x+3)^2(x-2)^5}{x^3(x-5)^2}$ has VAs at $x = 0$ and $x = 5$ since $\lim\limits_{x\to 0^-} f(x) = \infty$, $\lim\limits_{x\to 0^+} f(x) = -\infty$, and $\lim\limits_{x\to 5} f(x) = \infty$. No HA since $\lim\limits_{x\to\pm\infty} f(x) = \infty$.

Since f is undefined at $x = 0$, it has no y-intercept.

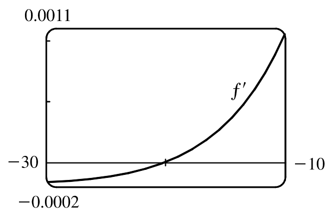
$f(x) = 0 \quad \Leftrightarrow \quad (2x+3)^2\,(x-2)^5 = 0 \quad \Leftrightarrow \quad x = -\frac{3}{2}$ or $x = 2$, so f has x-intercepts at $-\frac{3}{2}$ and 2. Note, however, that the graph of f is only tangent to the x-axis and does not cross it at $x = -\frac{3}{2}$, since f is positive as $x \to \left(-\frac{3}{2}\right)^-$ and as $x \to \left(-\frac{3}{2}\right)^+$. There is a local minimum value of $f\left(-\frac{3}{2}\right) = 0$.

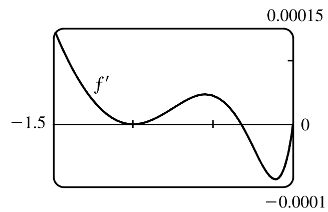
[continued]

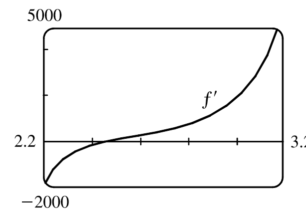
The only "mystery" feature is the local minimum to the right of the VA $x = 5$. From the graph, we see that $f(7.98) \approx 609$ is a local minimum value.



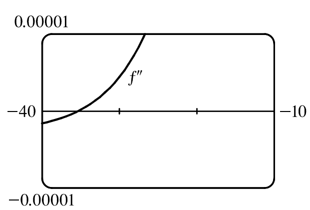
13. $f(x) = \dfrac{x^2(x+1)^3}{(x-2)^2(x-4)^4} \quad \Rightarrow \quad f'(x) = -\dfrac{x(x+1)^2(x^3+18x^2-44x-16)}{(x-2)^3(x-4)^5}$ [from CAS].

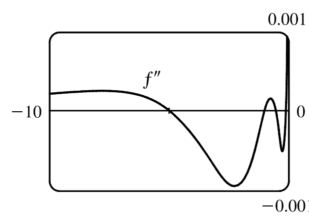


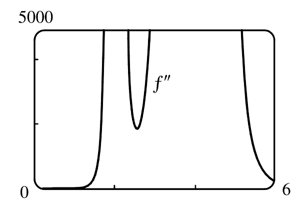




From the graphs of f', it seems that the critical points which indicate extrema occur at $x \approx -20, -0.3$, and 2.5, as estimated in Example 3. (There is another critical point at $x = -1$, but the sign of f' does not change there.) We differentiate again, obtaining $f''(x) = 2\dfrac{(x+1)(x^6+36x^5+6x^4-628x^3+684x^2+672x+64)}{(x-2)^4(x-4)^6}$.







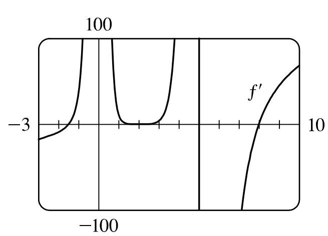
From the graphs of f'', it appears that f is CU on $(-35.3, -5.0)$, $(-1, -0.5)$, $(-0.1, 2)$, $(2, 4)$, and $(4, \infty)$ and CD on $(-\infty, -35.3)$, $(-5.0, -1)$ and $(-0.5, -0.1)$. We check back on the graphs of f to find the y-coordinates of the inflection points, and find that these points are approximately $(-35.3, -0.015)$, $(-5.0, -0.005)$, $(-1, 0)$, $(-0.5, 0.00001)$, and $(-0.1, 0.0000066)$.

14. From a CAS,

$$f'(x) = \frac{2(x-2)^4(2x+3)(2x^3-14x^2-10x-45)}{x^4(x-5)^3}$$

and

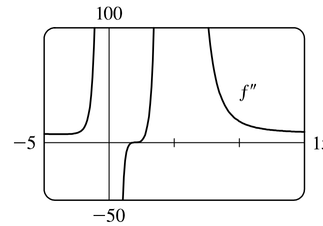
$$f''(x) = \frac{2(x-2)^3(4x^6-56x^5+216x^4+460x^3+805x^2+1710x+5400)}{x^5(x-5)^4}$$



From Exercise 12 and $f'(x)$ above, we know that the zeros of f' are -1.5, 2, and 7.98. From the graph of f', we conclude that f is decreasing on $(-\infty, -1.5)$, increasing on $(-1.5, 0)$ and $(0, 5)$, decreasing on $(5, 7.98)$, and increasing on $(7.98, \infty)$.

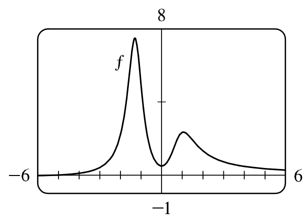
[continued]

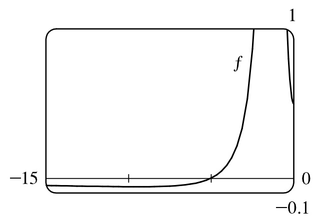
From $f''(x)$, we know that $x = 2$ is a zero, and the graph of f'' shows us that $x = 2$ is the only zero of f''. Thus, f is CU on $(-\infty, 0)$, CD on $(0, 2)$, CU on $(2, 5)$, and CU on $(5, \infty)$.



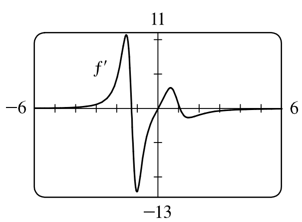
15. $f(x) = \dfrac{x^3 + 5x^2 + 1}{x^4 + x^3 - x^2 + 2}$. From a CAS, $f'(x) = \dfrac{-x(x^5 + 10x^4 + 6x^3 + 4x^2 - 3x - 22)}{(x^4 + x^3 - x^2 + 2)^2}$ and

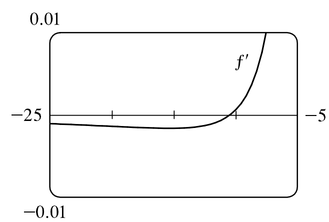
$$f''(x) = \frac{2(x^9 + 15x^8 + 18x^7 + 21x^6 - 9x^5 - 135x^4 - 76x^3 + 21x^2 + 6x + 22)}{(x^4 + x^3 - x^2 + 2)^3}$$



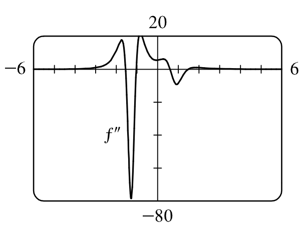


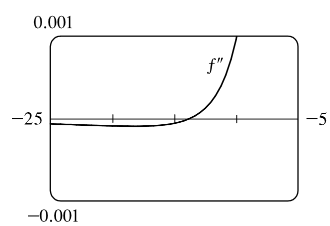
The first graph of f shows that $y = 0$ is a HA. As $x \to \infty$, $f(x) \to 0$ through positive values. As $x \to -\infty$, it is not clear if $f(x) \to 0$ through positive or negative values. The second graph of f shows that f has an x-intercept near -5, and will have a local minimum and inflection point to the left of -5.





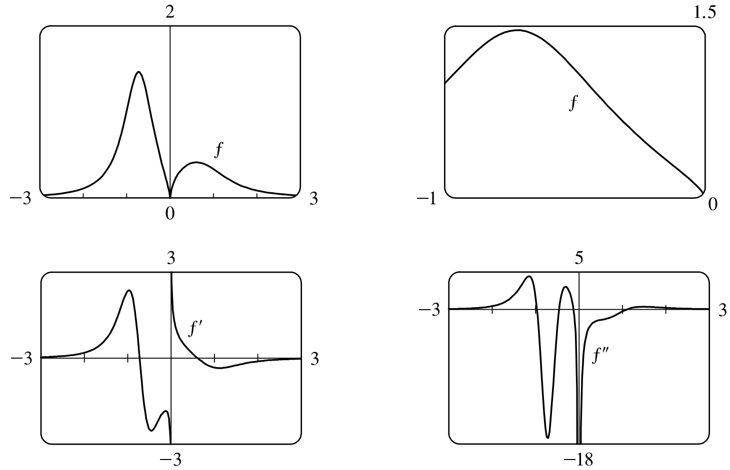
From the two graphs of f', we see that f' has four zeros. We conclude that f is decreasing on $(-\infty, -9.41)$, increasing on $(-9.41, -1.29)$, decreasing on $(-1.29, 0)$, increasing on $(0, 1.05)$, and decreasing on $(1.05, \infty)$. We have local minimum values $f(-9.41) \approx -0.056$ and $f(0) = 0.5$, and local maximum values $f(-1.29) \approx 7.49$ and $f(1.05) \approx 2.35$.





From the two graphs of f'', we see that f'' has five zeros. We conclude that f is CD on $(-\infty, -13.81)$, CU on $(-13.81, -1.55)$, CD on $(-1.55, -1.03)$, CU on $(-1.03, 0.60)$, CD on $(0.60, 1.48)$, and CU on $(1.48, \infty)$. There are five inflection points: $(-13.81, -0.05)$, $(-1.55, 5.64)$, $(-1.03, 5.39)$, $(0.60, 1.52)$, and $(1.48, 1.93)$.

16. $y = f(x) = \dfrac{x^{2/3}}{1 + x + x^4}$. From a CAS, $y' = -\dfrac{10x^4 + x - 2}{3x^{1/3}(x^4 + x + 1)^2}$ and $y'' = \dfrac{2(65x^8 - 14x^5 - 80x^4 + 2x^2 - 8x - 1)}{9x^{4/3}(x^4 + x + 1)^3}$

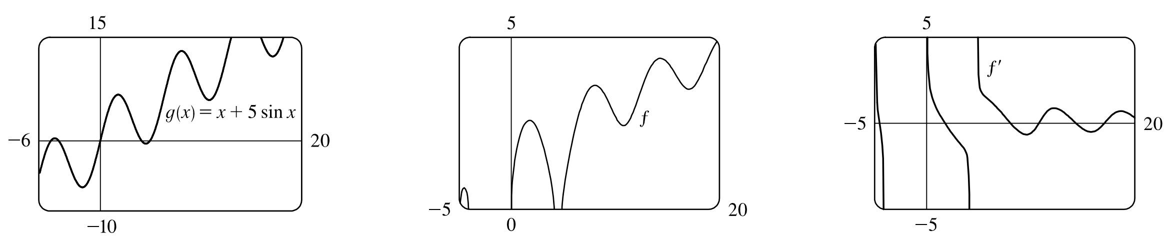


$f'(x)$ does not exist at $x = 0$ and $f'(x) = 0 \iff x \approx -0.72$ and 0.61, so f is increasing on $(-\infty, -0.72)$, decreasing on $(-0.72, 0)$, increasing on $(0, 0.61)$, and decreasing on $(0.61, \infty)$. There is a local maximum value of $f(-0.72) \approx 1.46$ and a local minimum value of $f(0.61) \approx 0.41$. $f''(x)$ does not exist at $x = 0$ and $f''(x) = 0 \iff x \approx -0.97, -0.46, -0.12$, and 1.11, so f is CU on $(-\infty, -0.97)$, CD on $(-0.97, -0.46)$, CU on $(-0.46, -0.12)$, CD on $(-0.12, 0)$, CD on $(0, 1.11)$, and CU on $(1.11, \infty)$. There are inflection points at $(-0.97, 1.08)$, $(-0.46, 1.01)$, $(-0.12, 0.28)$, and $(1.11, 0.29)$.

17. $y = f(x) = \sqrt{x + 5\sin x}$, $x \le 20$.

From a CAS, $y' = \dfrac{5\cos x + 1}{2\sqrt{x + 5\sin x}}$ and $y'' = -\dfrac{10\cos x + 25\sin^2 x + 10x\sin x + 26}{4(x + 5\sin x)^{3/2}}$.

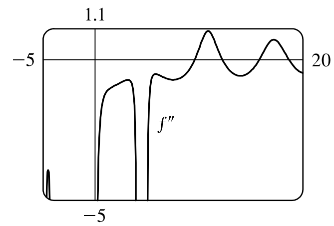
We'll start with a graph of $g(x) = x + 5\sin x$. Note that $f(x) = \sqrt{g(x)}$ is only defined if $g(x) \ge 0$. $g(x) = 0 \iff x = 0$ or $x \approx -4.91, -4.10, 4.10$, and 4.91. Thus, the domain of f is $[-4.91, -4.10] \cup [0, 4.10] \cup [4.91, 20]$.



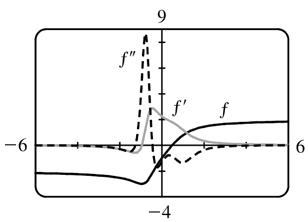
From the expression for y', we see that $y' = 0 \iff 5\cos x + 1 = 0 \Rightarrow x_1 = \cos^{-1}\left(-\frac{1}{5}\right) \approx 1.77$ and $x_2 = 2\pi - x_1 \approx -4.51$ (not in the domain of f). The leftmost zero of f' is $x_1 - 2\pi \approx -4.51$. Moving to the right, the zeros of f' are x_1, $x_1 + 2\pi$, $x_2 + 2\pi$, $x_1 + 4\pi$, and $x_2 + 4\pi$. Thus, f is increasing on $(-4.91, -4.51)$, decreasing on $(-4.51, -4.10)$, increasing on $(0, 1.77)$, decreasing on $(1.77, 4.10)$, increasing on $(4.91, 8.06)$, decreasing on $(8.06, 10.79)$, increasing on $(10.79, 14.34)$, decreasing on $(14.34, 17.08)$, and increasing on $(17.08, 20)$. The local maximum values are

$f(-4.51) \approx 0.62$, $f(1.77) \approx 2.58$, $f(8.06) \approx 3.60$, and $f(14.34) \approx 4.39$. The local minimum values are $f(10.79) \approx 2.43$ and $f(17.08) \approx 3.49$.

f is CD on $(-4.91, -4.10)$, $(0, 4.10)$, $(4.91, 9.60)$, CU on $(9.60, 12.25)$, CD on $(12.25, 15.81)$, CU on $(15.81, 18.65)$, and CD on $(18.65, 20)$. There are inflection points at $(9.60, 2.95)$, $(12.25, 3.27)$, $(15.81, 3.91)$, and $(18.65, 4.20)$.

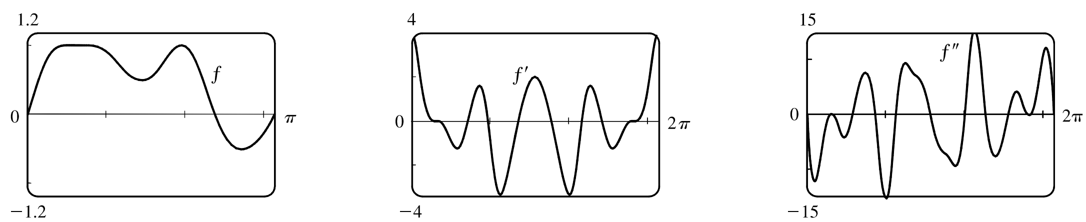


18. $f(x) = \dfrac{2x-1}{\sqrt[4]{x^4+x+1}} \quad\Rightarrow\quad f'(x) = \dfrac{4x^3+6x+9}{4(x^4+x+1)^{5/4}} \quad\Rightarrow\quad f''(x) = -\dfrac{32x^6+96x^4+152x^3-48x^2+6x+21}{16(x^4+x+1)^{9/4}}$



From the graph of f', f appears to be decreasing on $(-\infty, -0.94)$ and increasing on $(-0.94, \infty)$. There is a local minimum value of $f(-0.94) \approx -3.01$. From the graph of f'', f appears to be CU on $(-1.25, -0.44)$ and CD on $(-\infty, -1.25)$ and $(-0.44, \infty)$. There are inflection points at $(-1.25, -2.87)$ and $(-0.44, -2.14)$.

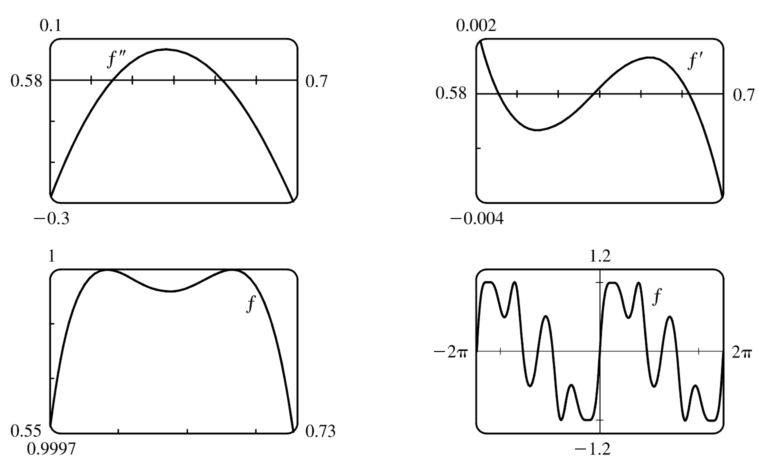
19.



From the graph of $f(x) = \sin(x + \sin 3x)$ in the viewing rectangle $[0, \pi]$ by $[-1.2, 1.2]$, it looks like f has two maxima and two minima. If we calculate and graph $f'(x) = [\cos(x + \sin 3x)]\,(1 + 3\cos 3x)$ on $[0, 2\pi]$, we see that the graph of f' appears to be almost tangent to the x-axis at about $x = 0.7$. The graph of

$$f'' = -\,[\sin(x + \sin 3x)]\,(1 + 3\cos 3x)^2 + \cos(x + \sin 3x)(-9\sin 3x)$$

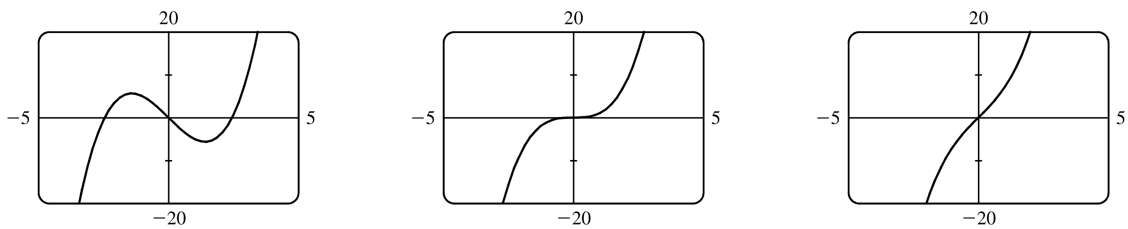
is even more interesting near this x-value: it seems to just touch the x-axis.



[continued]

If we zoom in on this place on the graph of f'', we see that f'' actually does cross the axis twice near $x = 0.65$, indicating a change in concavity for a very short interval. If we look at the graph of f' on the same interval, we see that it changes sign three times near $x = 0.65$, indicating that what we had thought was a broad extremum at about $x = 0.7$ actually consists of three extrema (two maxima and a minimum). These maximum values are roughly $f(0.59) = 1$ and $f(0.68) = 1$, and the minimum value is roughly $f(0.64) = 0.99996$. There are also a maximum value of about $f(1.96) = 1$ and minimum values of about $f(1.46) = 0.49$ and $f(2.73) = -0.51$. The points of inflection on $(0, \pi)$ are about $(0.61, 0.99998)$, $(0.66, 0.99998)$, $(1.17, 0.72)$, $(1.75, 0.77)$, and $(2.28, 0.34)$. On $(\pi, 2\pi)$, they are about $(4.01, -0.34)$, $(4.54, -0.77)$, $(5.11, -0.72)$, $(5.62, -0.99998)$, and $(5.67, -0.99998)$. There are also IP at $(0, 0)$ and $(\pi, 0)$. Note that the function is odd and periodic with period 2π, and it is also rotationally symmetric about all points of the form $((2n + 1)\pi, 0)$, n an integer.

20. $f(x) = x^3 + cx = x(x^2 + c) \quad \Rightarrow \quad f'(x) = 3x^2 + c \quad \Rightarrow \quad f''(x) = 6x$



$c = -6$ $\qquad$ $c = 0$ $\qquad$ $c = 6$

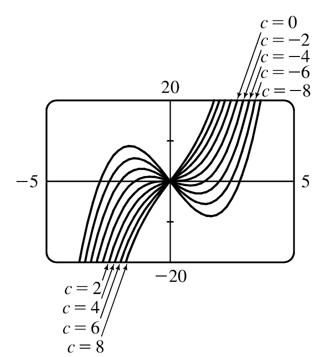
x-intercepts: When $c \geq 0$, 0 is the only x-intercept. When $c < 0$, the x-intercepts are 0 and $\pm\sqrt{-c}$.

y-intercept $= f(0) = 0$. f is odd, so the graph is symmetric with respect to the origin. $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$, so f is CD on $(-\infty, 0)$ and CU on $(0, \infty)$. The origin is the only inflection point.

If $c > 0$, then $f'(x) > 0$ for all x, so f is increasing and has no local maximum or minimum.

If $c = 0$, then $f'(x) \geq 0$ with equality at $x = 0$, so again f is increasing and has no local maximum or minimum.

If $c < 0$, then $f'(x) = 3[x^2 - (-c/3)] = 3\left(x + \sqrt{-c/3}\right)\left(x - \sqrt{-c/3}\right)$, so $f'(x) > 0$ on $\left(-\infty, -\sqrt{-c/3}\right)$ and $\left(\sqrt{-c/3}, \infty\right)$; $f'(x) < 0$ on $\left(-\sqrt{-c/3}, \sqrt{-c/3}\right)$. It follows that $f\left(-\sqrt{-c/3}\right) = -\frac{2}{3}c\sqrt{-c/3}$ is a local maximum value and $f\left(\sqrt{-c/3}\right) = \frac{2}{3}c\sqrt{-c/3}$ is a local minimum value. As c decreases (toward more negative values), the local maximum and minimum move further apart.



There is no absolute maximum or minimum value. The only transitional value of c corresponding to a change in character of the graph is $c = 0$.

21. $f(x) = x^2 + 6x + c/x \quad \Rightarrow \quad f'(x) = 2x + 6 - c/x^2 \quad \Rightarrow \quad f''(x) = 2 + 2c/x^3$

$\boldsymbol{c = 0}$: The graph is the parabola $y = x^2 + 6x$, which has x-intercepts -6 and 0, vertex $(-3, -9)$, and opens upward.

[continued]

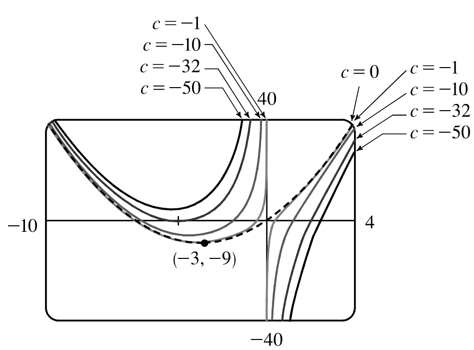
$\boldsymbol{c \neq 0}$: The parabola $y = x^2 + 6x$ is an asymptote that the graph of f approaches as $x \to \pm\infty$. The y-axis is a vertical asymptote.

$\boldsymbol{c < 0}$**:** The x-intercepts are found by solving $f(x) = 0 \quad \Leftrightarrow \quad x^3 + 6x^2 + c = g(x) = 0$. Now $g'(x) = 0 \quad \Leftrightarrow \quad x = -4$ or 0, and g (*not* f) has a local maximum at $x = -4$. $g(-4) = 32 + c$, so if $c < -32$, the maximum is negative and there are no negative x-intercepts; if $c = -32$, the maximum is 0 and there is one negative x-intercept; if $-32 < c < 0$, the maximum is positive and there are two negative x-intercepts. In all cases, there is one positive x-intercept.

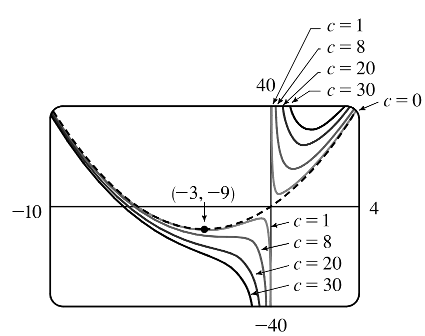
As $c \to 0^-$, the local minimum point moves down and right, approaching $(-3, -9)$. [Note that since $f'(x) = \dfrac{2x^3 + 6x^2 - c}{x^2}$, Descartes' Rule of Signs implies that f' has no positive solutions and one negative solution when $c < 0$. $f''(x) = \dfrac{2(x^3 + c)}{x^3} > 0$ at that negative solution, so that critical point yields a local minimum value. This tells us that there are no local maximums when $c < 0$.] $f'(x) > 0$ for $x > 0$, so f is increasing on $(0, \infty)$. From $f''(x) = \dfrac{2(x^3 + c)}{x^3}$, we see that f has an inflection point at $\left(\sqrt[3]{-c}, 6\sqrt[3]{-c}\right)$. This inflection point moves down and left, approaching the origin as $c \to 0^-$.

f is CU on $(-\infty, 0)$, CD on $\left(0, \sqrt[3]{-c}\right)$, and CU on $\left(\sqrt[3]{-c}, \infty\right)$.

$\boldsymbol{c > 0}$**:** The inflection point $\left(\sqrt[3]{-c}, 6\sqrt[3]{-c}\right)$ is now in the third quadrant and moves up and right, approaching the origin as $c \to 0^+$. f is CU on $\left(-\infty, \sqrt[3]{-c}\right)$, CD on $\left(\sqrt[3]{-c}, 0\right)$, and CU on $(0, \infty)$. f has a local minimum point in the first quadrant. It moves down and left, approaching the origin as $c \to 0^+$. $f'(x) = 0 \quad \Leftrightarrow \quad 2x^3 + 6x^2 - c = h(x) = 0$. Now $h'(x) = 0 \quad \Leftrightarrow \quad x = -2$ or 0, and h (*not* f) has a local maximum at $x = -2$. $h(-2) = 8 - c$, so $c = 8$ makes $h(x) = 0$, and hence, $f'(x) = 0$. When $c > 8$, $f'(x) < 0$ and f is decreasing on $(-\infty, 0)$. For $0 < c < 8$, there is a local minimum that moves toward $(-3, -9)$ and a local maximum that moves toward the origin as c decreases.



$\boldsymbol{c \leq 0}$



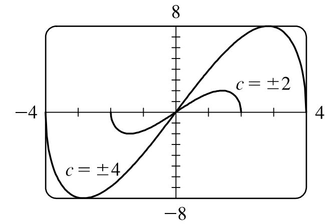
$\boldsymbol{c \geq 0}$

22. With $c = 0$ in $y = f(x) = x\sqrt{c^2 - x^2}$, the graph of f is just the point $(0, 0)$. Since $(-c)^2 = c^2$, we only consider $c > 0$. Since $f(-x) = -f(x)$, the graph is symmetric about the origin. The domain of f is found by solving $c^2 - x^2 \geq 0 \quad \Leftrightarrow \quad x^2 \leq c^2 \quad \Leftrightarrow \quad |x| \leq c$, which gives us $[-c, c]$.

$$f'(x) = x \cdot \tfrac{1}{2}(c^2 - x^2)^{-1/2}(-2x) + (c^2 - x^2)^{1/2}(1) = (c^2 - x^2)^{-1/2}[-x^2 + (c^2 - x^2)] = \frac{c^2 - 2x^2}{\sqrt{c^2 - x^2}}.$$

$f'(x) > 0 \quad \Leftrightarrow \quad c^2 - 2x^2 > 0 \quad \Leftrightarrow \quad x^2 < c^2/2 \quad \Leftrightarrow \quad |x| < c/\sqrt{2}$, so f is increasing on $\left(-c/\sqrt{2}, c/\sqrt{2}\right)$ and decreasing on $\left(-c, -c/\sqrt{2}\right)$ and $\left(c/\sqrt{2}, c\right)$. There is a local minimum value of $f\left(-c/\sqrt{2}\right) = \left(-c/\sqrt{2}\right)\sqrt{c^2 - c^2/2} = \left(-c/\sqrt{2}\right)\left(c/\sqrt{2}\right) = -c^2/2$ and a local maximum value of $f\left(c/\sqrt{2}\right) = c^2/2$.

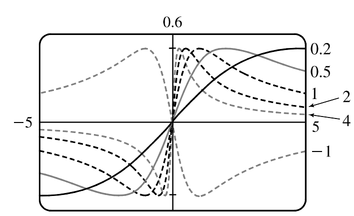
$$\begin{aligned} f''(x) &= \frac{(c^2 - x^2)^{1/2}(-4x) - (c^2 - 2x^2)\frac{1}{2}\left(c^2 - x^2\right)^{-1/2}(-2x)}{[(c^2 - x^2)^{1/2}]^2} \\ &= \frac{x(c^2 - x^2)^{-1/2}[(c^2 - x^2)(-4) + (c^2 - 2x^2)]}{(c^2 - x^2)^1} = \frac{2x\left(2x^2 - 3c^2\right)}{(c^2 - x^2)^{3/2}}, \end{aligned}$$



so $f''(x) = 0 \quad \Leftrightarrow \quad x = 0$ or $x = \pm\sqrt{\frac{3}{2}}\,c$, but only 0 is in the domain of f. $f''(x) < 0$ for $0 < x < c$ and $f''(x) > 0$ for $-c < x < 0$, so f is CD on $(0, c)$ and CU on $(-c, 0)$. There is an IP at $(0, 0)$. So as $|c|$ gets larger, the maximum and minimum values increase in magnitude. The value of c does not affect the concavity of f.

23. Note that $c = 0$ is a transitional value at which the graph consists of the x-axis. Also, we can see that if we substitute $-c$ for c, the function $f(x) = \dfrac{cx}{1 + c^2x^2}$ will be reflected in the x-axis, so we investigate only positive values of c (except $c = -1$, as a demonstration of this reflective property). Also, f is an odd function. $\lim\limits_{x \to \pm\infty} f(x) = 0$, so $y = 0$ is a horizontal asymptote for all c. We calculate $f'(x) = \dfrac{(1 + c^2x^2)c - cx(2c^2x)}{(1 + c^2x^2)^2} = -\dfrac{c(c^2x^2 - 1)}{(1 + c^2x^2)^2}$. $f'(x) = 0 \quad \Leftrightarrow \quad c^2x^2 - 1 = 0 \quad \Leftrightarrow$ $x = \pm 1/c$. So there is an absolute maximum value of $f(1/c) = \frac{1}{2}$ and an absolute minimum value of $f(-1/c) = -\frac{1}{2}$. These extrema have the same value regardless of c, but the maximum points move closer to the y-axis as c increases.

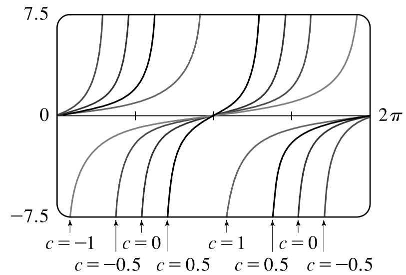
$$\begin{aligned} f''(x) &= \frac{(-2c^3x)(1 + c^2x^2)^2 - (-c^3x^2 + c)[2(1 + c^2x^2)(2c^2x)]}{(1 + c^2x^2)^4} \\ &= \frac{(-2c^3x)(1 + c^2x^2) + (c^3x^2 - c)(4c^2x)}{(1 + c^2x^2)^3} = \frac{2c^3x(c^2x^2 - 3)}{(1 + c^2x^2)^3} \end{aligned}$$



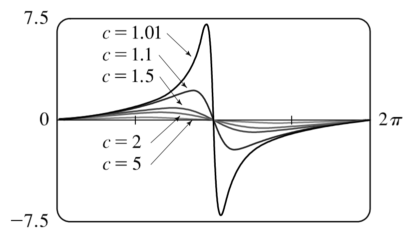
$f''(x) = 0 \quad \Leftrightarrow \quad x = 0$ or $\pm\sqrt{3}/c$, so there are inflection points at $(0, 0)$ and at $\left(\pm\sqrt{3}/c, \pm\sqrt{3}/4\right)$. Again, the y-coordinate of the inflection points does not depend on c, but as c increases, both inflection points approach the y-axis.

24. $f(x) = \dfrac{\sin x}{c + \cos x} \quad \Rightarrow \quad f'(x) = \dfrac{1 + c\cos x}{\cos^2 x + 2c\cos x + c^2} \quad \Rightarrow \quad f''(x) = \dfrac{\sin x(c\cos x - c^2 + 2)}{\cos^3 x + 3c\cos^2 x + 3c^2\cos x + c^3}$. Notice that f is an odd function and has period 2π. We will graph f for $0 \le x \le 2\pi$.

$|\mathbf{c}| \le \mathbf{1}$**:** See the first figure. f has VAs when the denominator is zero, that is, at $x = \cos^{-1}(-c)$ and $x = 2\pi - \cos^{-1}(-c)$. So for $c = -1$, there are VAs at $x = 0$ and $x = 2\pi$, and as c increases, they move closer to $x = \pi$, which is the single VA when $c = 1$. Note that if $c = 0$, then $f(x) = \tan x$. There are no extreme points (on the entire domain) and inflection points occur at multiples of π.



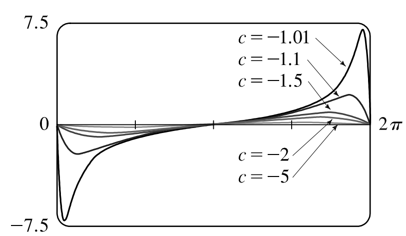
$\mathbf{c} > \mathbf{1}$**:** See the second figure. $f'(x) = 0 \quad \Leftrightarrow \quad x = \cos^{-1}\left(\dfrac{-1}{c}\right)$ or $x = 2\pi - \cos^{-1}\left(\dfrac{-1}{c}\right)$. The VA disappears and there is now a local maximum and a local minimum. As $c \to 1^+$, the coordinates of the local maximum approach π and ∞, and the coordinates of the local minimum approach π and $-\infty$.



As $c \to \infty$, the graph of f looks like a graph of $y = \sin x$ that is vertically compressed, and the local maximum and local minimum approach $\left(\frac{\pi}{2}, 0\right)$ and $\left(\frac{3\pi}{2}, 0\right)$, respectively.

$f''(x) = 0 \quad \Leftrightarrow \quad \sin x = 0 \quad$ (IPs at $x = n\pi$) or $c\cos x - c^2 + 2 = 0$. The second condition is true if $\cos x = \dfrac{c^2 - 2}{c}$ $[c \ne 0]$. The last equation has two solutions if $-1 < \dfrac{c^2 - 2}{c} < 1 \quad \Rightarrow \quad -c < c^2 - 2 < c \quad \Rightarrow \quad -c < c^2 - 2$ and $c^2 - 2 < c \quad \Rightarrow \quad c^2 + c - 2 > 0$ and $c^2 - c - 2 < 0 \quad \Rightarrow \quad (c + 2)(c - 1) > 0$ and $(c - 2)(c + 1) < 0 \quad \Rightarrow$ $c - 1 > 0 \quad [\text{since } c > 1]$ and $c - 2 < 0 \quad \Rightarrow \quad c > 1$ and $c < 2$. Thus, for $1 < c < 2$, we have 2 nontrivial IPs at $x = \cos^{-1}\left(\dfrac{c^2 - 2}{c}\right)$ and $x = 2\pi - \cos^{-1}\left(\dfrac{c^2 - 2}{c}\right)$.

$\mathbf{c} < -\mathbf{1}$**:** See the third figure. The VAs for $c = -1$ at $x = 0$ and $x = 2\pi$ in the first figure disappear and we now have a local minimum and a local maximum. As $c \to -1^+$, the coordinates of the local minimum approach 0 and $-\infty$, and the coordinates of the local maximum approach 2π and ∞. As $c \to -\infty$, the graph of f looks like a graph of $y = \sin x$ that is vertically compressed, and the local minimum and local maximum approach $\left(\frac{\pi}{2}, 0\right)$ and $\left(\frac{3\pi}{2}, 0\right)$, respectively. As above, we have two nontrivial IPs for $-2 < c < -1$.



25. $f(x) = cx + \sin x \quad \Rightarrow \quad f'(x) = c + \cos x \quad \Rightarrow \quad f''(x) = -\sin x$

$f(-x) = -f(x)$, so f is an odd function and its graph is symmetric with respect to the origin.

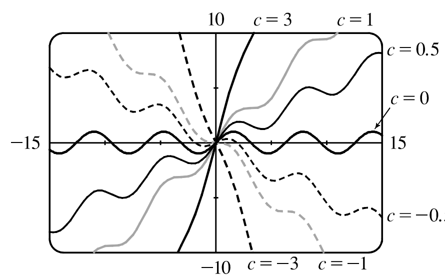
$f(x) = 0 \quad \Leftrightarrow \quad \sin x = -cx$, so 0 is always an x-intercept.

$f'(x) = 0 \quad \Leftrightarrow \quad \cos x = -c$, so there is no critical number when $|c| > 1$. If $|c| \le 1$, then there are infinitely many critical numbers. If x_1 is the unique solution of $\cos x = -c$ in the interval $[0, \pi]$, then the critical numbers are $2n\pi \pm x_1$, where n ranges over the integers. (Special cases: When $c = -1$, $x_1 = 0$; when $c = 0$, $x = \frac{\pi}{2}$; and when $c = 1$, $x_1 = \pi$.)

$f''(x) < 0 \quad \Leftrightarrow \quad \sin x > 0$, so f is CD on intervals of the form $(2n\pi, (2n+1)\pi)$. f is CU on intervals of the form $((2n-1)\pi, 2n\pi)$. The inflection points of f are the points $(n\pi, n\pi c)$, where n is an integer.

If $c \ge 1$, then $f'(x) \ge 0$ for all x, so f is increasing and has no extremum. If $c \le -1$, then $f'(x) \le 0$ for all x, so f is decreasing and has no extremum. If $|c| < 1$, then $f'(x) > 0 \quad \Leftrightarrow \quad \cos x > -c \quad \Leftrightarrow \quad x$ is in an interval of the form $(2n\pi - x_1, 2n\pi + x_1)$ for some integer n. These are the intervals on which f is increasing. Similarly, we find that f is decreasing on the intervals of the form $(2n\pi + x_1, 2(n+1)\pi - x_1)$. Thus, f has local maxima at the points $2n\pi + x_1$, where f has the values $c(2n\pi + x_1) + \sin x_1 = c(2n\pi + x_1) + \sqrt{1 - c^2}$, and f has local minima at the points $2n\pi - x_1$, where we have $f(2n\pi - x_1) = c(2n\pi - x_1) - \sin x_1 = c(2n\pi - x_1) - \sqrt{1 - c^2}$.

The transitional values of c are -1 and 1. The inflection points move vertically, but not horizontally, when c changes. When $|c| \ge 1$, there is no extremum. For $|c| < 1$, the maxima are spaced 2π apart horizontally, as are the minima. The horizontal spacing between maxima and adjacent minima is regular (and equals π) when $c = 0$, but the horizontal space between a local maximum and the nearest local minimum shrinks as $|c|$ approaches 1.

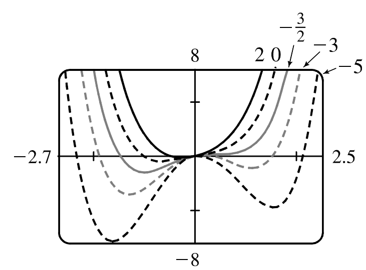


26. (a) $f(x) = cx^4 - 4x^2 + 1 \quad \Rightarrow \quad f'(x) = 4cx^3 - 8x = 4x(cx^2 - 2)$. If $c \le 0$, then the only real solution of $f'(x) = 0$ is $x = 0$. f' changes from positive to negative at $x = 0$, so f has only a maximum point in this case. If $c > 0$, then $f'(x) = 4x(cx^2 - 2) = 4x\left(\sqrt{c}\,x + \sqrt{2}\right)\left(\sqrt{c}\,x - \sqrt{2}\right)$, and f changes from negative to positive at $x = \pm\sqrt{2}/\sqrt{c}$. Thus, if $c > 0$, the curve has minimum points.

(b) $f\left(\pm\dfrac{\sqrt{2}}{\sqrt{c}}\right) = c\left(\pm\dfrac{\sqrt{2}}{c}\right)^4 - 4\left(\pm\dfrac{\sqrt{2}}{\sqrt{c}}\right)^2 + 1 = \dfrac{4}{c} - \dfrac{8}{c} + 1 = -\dfrac{4}{c} + 1$. For $y = g(x) = -2x^2 + 1$, we have $g\left(\pm\dfrac{\sqrt{2}}{\sqrt{c}}\right) = -2\left(\pm\dfrac{\sqrt{2}}{c}\right)^2 + 1 = -\dfrac{4}{c} + 1$. Also, $f(0) = 1$ and $g(0) = 1$. Thus, the minimum points $\left(\pm\dfrac{\sqrt{2}}{\sqrt{c}}, -\dfrac{4}{c} + 1\right)$ and the maximum point (0,1) of every curve in the family $f(x) = cx^4 - 4x^2 + 1$ lie on the parabola $y = -2x^2 + 1$.

27. For $c = 0$, there is no inflection point; the curve is CU everywhere. If c increases, the curve simply becomes steeper, and there are still no inflection points. If c starts at 0 and decreases, a slight upward bulge appears near $x = 0$, so that there are two inflection points for any $c < 0$. This can be seen algebraically by calculating the second derivative:
$f(x) = x^4 + cx^2 + x \;\Rightarrow\; f'(x) = 4x^3 + 2cx + 1 \;\Rightarrow\; f''(x) = 12x^2 + 2c$. Thus, $f''(x) > 0$ when $c > 0$. For $c < 0$, there are inflection points when $x = \pm\sqrt{-\frac{1}{6}c}$. For $c = 0$, the graph has one critical number, at the absolute minimum somewhere around $x = -0.6$. As c increases, the number of critical points does not change. If c instead decreases from 0, we see that the graph eventually sprouts another local minimum, to the right of the origin, somewhere between $x = 1$ and $x = 2$. Consequently, there is also a maximum near $x = 0$.

After a bit of experimentation, we find that at $c = -1.5$, there appear to be two critical numbers: the absolute minimum at about $x = -1$, and a horizontal tangent with no extremum at about $x = 0.5$. For any c smaller than this there will be 3 critical points, as shown in the graphs with $c = -3$ and with $c = -5$.



To prove this algebraically, we calculate $f'(x) = 4x^3 + 2cx + 1$. Now if we substitute our value of $c = -1.5$, the formula for $f'(x)$ becomes $4x^3 - 3x + 1 = (x + 1)(2x - 1)^2$. This has a double solution at $x = \frac{1}{2}$, indicating that the function has two critical points: $x = -1$ and $x = \frac{1}{2}$, just as we had guessed from the graph.

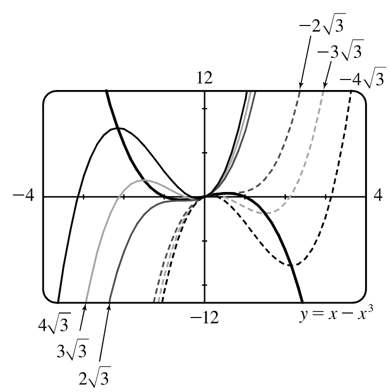
28. (a) $f(x) = 2x^3 + cx^2 + 2x \;\Rightarrow\; f'(x) = 6x^2 + 2cx + 2 = 2(3x^2 + cx + 1)$. $f'(x) = 0 \;\Leftrightarrow\; x = \dfrac{-c \pm \sqrt{c^2 - 12}}{6}$.

So f has critical points $\;\Leftrightarrow\; c^2 - 12 \geq 0 \;\Leftrightarrow\; |c| \geq 2\sqrt{3}$. For $c = \pm 2\sqrt{3}$, $f'(x) \geq 0$ on $(-\infty, \infty)$, so f' does not change signs at $-c/6$, and there is no extremum. If $c^2 - 12 > 0$, then f' changes from positive to negative at $x = \dfrac{-c - \sqrt{c^2 - 12}}{6}$ and from negative to positive at $x = \dfrac{-c + \sqrt{c^2 - 12}}{6}$. So f has a local maximum at $x = \dfrac{-c - \sqrt{c^2 - 12}}{6}$ and a local minimum at $x = \dfrac{-c + \sqrt{c^2 - 12}}{6}$.

(b) Let x_0 be a critical number for $f(x)$. Then $f'(x_0) = 0 \;\Rightarrow\;$
$3x_0^2 + cx_0 + 1 = 0 \;\Leftrightarrow\; c = \dfrac{-1 - 3x_0^2}{x_0}$. Now

$$f(x_0) = 2x_0^3 + cx_0^2 + 2x_0 = 2x_0^3 + x_0^2\left(\frac{-1 - 3x_0^2}{x_0}\right) + 2x_0$$
$$= 2x_0^3 - x_0 - 3x_0^3 + 2x_0 = x_0 - x_0^3$$

So the point is $(x_0, y_0) = \left(x_0, x_0 - x_0^3\right)$; that is, the point lies on the curve $y = x - x^3$.



3.7 Optimization Problems

1. (a)

First Number	Second Number	Product
1	22	22
2	21	42
3	20	60
4	19	76
5	18	90
6	17	102
7	16	112
8	15	120
9	14	126
10	13	130
11	12	132

We needn't consider pairs where the first number is larger than the second, since we can just interchange the numbers in such cases. The answer appears to be 11 and 12, but we have considered only integers in the table.

(b) Call the two numbers x and y. Then $x + y = 23$, so $y = 23 - x$. Call the product P. Then $P = xy = x(23 - x) = 23x - x^2$, so we wish to maximize the function $P(x) = 23x - x^2$. Since $P'(x) = 23 - 2x$, we see that $P'(x) = 0 \quad \Leftrightarrow \quad x = \frac{23}{2} = 11.5$. Thus, the maximum value of P is $P(11.5) = (11.5)^2 = 132.25$ and it occurs when $x = y = 11.5$.

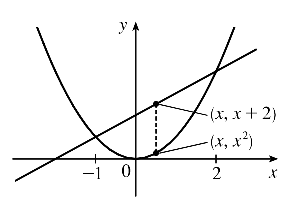
Or: Note that $P''(x) = -2 < 0$ for all x, so P is everywhere concave downward and the local maximum at $x = 11.5$ must be an absolute maximum.

2. The two numbers are $x + 100$ and x. Minimize $f(x) = (x + 100)x = x^2 + 100x$. $f'(x) = 2x + 100 = 0 \quad \Rightarrow \quad x = -50$. Since $f''(x) = 2 > 0$, there is an absolute minimum at $x = -50$. The two numbers are 50 and -50.

3. The two numbers are x and $\dfrac{100}{x}$, where $x > 0$. Minimize $f(x) = x + \dfrac{100}{x}$. $f'(x) = 1 - \dfrac{100}{x^2} = \dfrac{x^2 - 100}{x^2}$. The critical number is $x = 10$. Since $f'(x) < 0$ for $0 < x < 10$ and $f'(x) > 0$ for $x > 10$, there is an absolute minimum at $x = 10$. The numbers are 10 and 10.

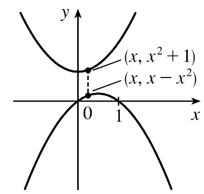
4. Call the two numbers x and y. Then $x + y = 16$, so $y = 16 - x$. Call the sum of their squares S. Then $S = x^2 + y^2 = x^2 + (16 - x)^2 \quad \Rightarrow \quad S' = 2x + 2(16 - x)(-1) = 2x - 32 + 2x = 4x - 32$. $S' = 0 \quad \Rightarrow \quad x = 8$. Since $S'(x) < 0$ for $0 < x < 8$ and $S'(x) > 0$ for $x > 8$, there is an absolute minimum at $x = 8$. Thus, $y = 16 - 8 = 8$ and $S = 8^2 + 8^2 = 128$.

5. Let the vertical distance be given by $v(x) = (x + 2) - x^2$, $-1 \le x \le 2$. $v'(x) = 1 - 2x = 0 \quad \Leftrightarrow \quad x = \frac{1}{2}$. $v(-1) = 0$, $v\left(\frac{1}{2}\right) = \frac{9}{4}$, and $v(2) = 0$, so there is an absolute maximum at $x = \frac{1}{2}$. The maximum distance is $v\left(\frac{1}{2}\right) = \frac{1}{2} + 2 - \frac{1}{4} = \frac{9}{4}$.



6. Let the vertical distance be given by

$v(x) = (x^2+1) - (x - x^2) = 2x^2 - x + 1$. $v'(x) = 4x - 1 = 0 \Leftrightarrow$ $x = \frac{1}{4}$. $v'(x) < 0$ for $x < \frac{1}{4}$ and $v'(x) > 0$ for $x > \frac{1}{4}$, so there is an absolute minimum at $x = \frac{1}{4}$. The minimum distance is $v\left(\frac{1}{4}\right) = \frac{1}{8} - \frac{1}{4} + 1 = \frac{7}{8}$.



7. If the rectangle has dimensions x and y, then its perimeter is $2x + 2y = 100$ m, so $y = 50 - x$. Thus, the area is $A = xy = x(50 - x)$. We wish to maximize the function $A(x) = x(50 - x) = 50x - x^2$, where $0 < x < 50$. Since $A'(x) = 50 - 2x = -2(x - 25)$, $A'(x) > 0$ for $0 < x < 25$ and $A'(x) < 0$ for $25 < x < 50$. Thus, A has an absolute maximum at $x = 25$, and $A(25) = 25^2 = 625$ m^2. The dimensions of the rectangle that maximize its area are $x = y = 25$ m. (The rectangle is a square.)

8. If the rectangle has dimensions x and y, then its area is $xy = 1000$ m^2, so $y = 1000/x$. The perimeter $P = 2x + 2y = 2x + 2000/x$. We wish to minimize the function $P(x) = 2x + 2000/x$ for $x > 0$. $P'(x) = 2 - 2000/x^2 = (2/x^2)(x^2 - 1000)$, so the only critical number in the domain of P is $x = \sqrt{1000}$. $P''(x) = 4000/x^3 > 0$, so P is concave upward throughout its domain and $P\left(\sqrt{1000}\right) = 4\sqrt{1000}$ is an absolute minimum value. The dimensions of the rectangle with minimal perimeter are $x = y = \sqrt{1000} = 10\sqrt{10}$ m. (The rectangle is a square.)

9. We need to maximize Y for $N \geq 0$. $Y(N) = \dfrac{kN}{1+N^2} \Rightarrow$

$$Y'(N) = \frac{(1+N^2)k - kN(2N)}{(1+N^2)^2} = \frac{k(1-N^2)}{(1+N^2)^2} = \frac{k(1+N)(1-N)}{(1+N^2)^2}.$$

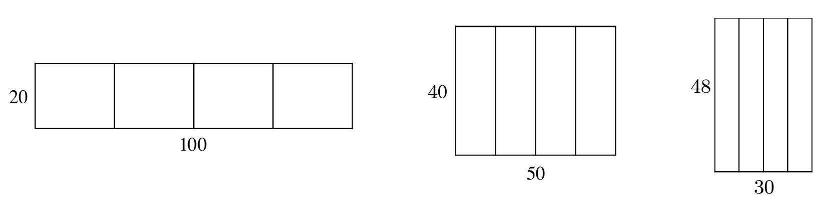
$Y'(N) > 0$ for $0 < N < 1$ and $Y'(N) < 0$ for $N > 1$. Thus, Y has an absolute maximum of $Y(1) = \frac{1}{2}k$ at $N = 1$.

10. We need to maximize P for $I \geq 0$. $P(I) = \dfrac{100I}{I^2 + I + 4} \Rightarrow$

$$P'(I) = \frac{(I^2+I+4)(100) - 100I(2I+1)}{(I^2+I+4)^2} = \frac{100(I^2+I+4-2I^2-I)}{(I^2+I+4)^2} = \frac{-100(I^2-4)}{(I^2+I+4)^2} = \frac{-100(I+2)(I-2)}{(I^2+I+4)^2}.$$

$P'(I) > 0$ for $0 < I < 2$ and $P'(I) < 0$ for $I > 2$. Thus, P has an absolute maximum of $P(2) = 20$ at $I = 2$.

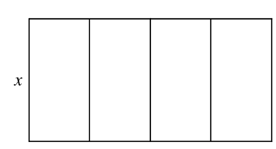
11. (a)



The areas of the three figures are 2000, 2000, and 1440 m^2. There appears to be a maximum area of at least 2000 m^2.

(b) Let x denote the length of each of two sides and three dividers.

Let y denote the length of the other two sides.



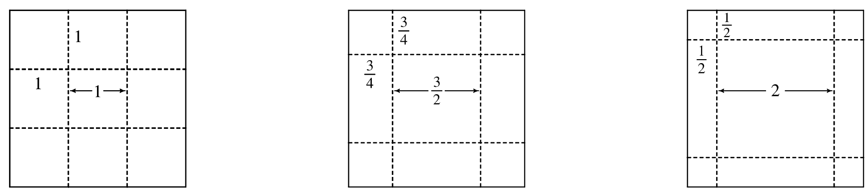
(c) Area $A = \text{length} \times \text{width} = y \cdot x$

(d) Length of fencing $= 300 \quad \Rightarrow \quad 5x + 2y = 300$

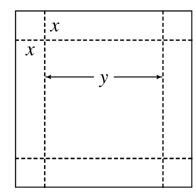
(e) $5x + 2y = 300 \quad \Rightarrow \quad y = 150 - \frac{5}{2}x \quad \Rightarrow \quad A(x) = \left(150 - \frac{5}{2}x\right)x = 150x - \frac{5}{2}x^2$

(f) $A'(x) = 150 - 5x = 0 \quad \Rightarrow \quad x = 30$. Since $A''(x) = -5 < 0$ there is an absolute maximum when $x = 30$. Then $y = \frac{150}{2} = 75$. The largest area is $y \cdot x = (75)(30) = 2250\ \text{m}^2$. These values of x and y are between the values in the first and second figures in part (a). Our original estimate was low.

12. (a)



The volumes of the resulting boxes are 1, 1.6875, and 2 m^3. There appears to be a maximum volume of at least 2 m^3.



(b) Let x denote the length of the side of the square being cut out. Let y denote the length of the base.

(c) Volume V = length × width × height $\quad \Rightarrow \quad V = y \cdot y \cdot x = xy^2$

(d) Length of cardboard $= 3 \quad \Rightarrow \quad x + y + x = 3 \quad \Rightarrow \quad y + 2x = 3$

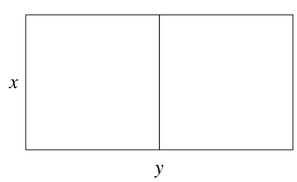
(e) $y + 2x = 3 \quad \Rightarrow \quad y = 3 - 2x \quad \Rightarrow \quad V(x) = x(3 - 2x)^2$

(f) $V(x) = x(3 - 2x)^2 \quad \Rightarrow$

$V'(x) = x \cdot 2(3 - 2x)(-2) + (3 - 2x)^2 \cdot 1 = (3 - 2x)[-4x + (3 - 2x)] = (3 - 2x)(-6x + 3)$,

so the critical numbers are $x = \frac{3}{2}$ and $x = \frac{1}{2}$. Now $0 \leq x \leq \frac{3}{2}$ and $V(0) = V\left(\frac{3}{2}\right) = 0$, so the maximum is $V\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)(2)^2 = 2\ \text{m}^3$, which is the value found from our third figure in part (a).

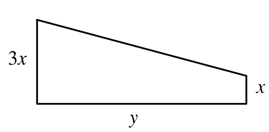
13.



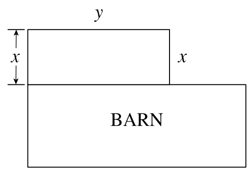
$xy = 1.5 \times 10^4$, so $y = 1.5 \times 10^4/x$. Minimize the amount of fencing, which is $3x + 2y = 3x + 2(1.5 \times 10^4/x) = 3x + 3 \times 10^4/x = F(x)$.
$F'(x) = 3 - 3 \times 10^4/x^2 = 3(x^2 - 10^4)/x^2$. The critical number is $x = 10^2$ and $F'(x) < 0$ for $0 < x < 10^2$ and $F'(x) > 0$ if $x > 10^2$, so the absolute minimum occurs when $x = 10^2$ and $y = 1.5 \times 10^2$.

The field should be 1000 feet by 1500 feet with the middle fence parallel to the short side of the field.

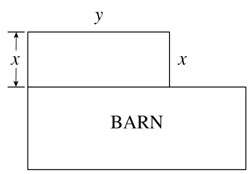
14.



From the figure, we see that the constraint is given by $4x + y = 400$, or $y = 400 - 4x$. The area of the trapezoid is given by $A(x) = \frac{1}{2}(3x + x)(y) = \frac{1}{2}(4x)(400 - 4x) = 800x - 8x^2$ for $0 < x < 100$. Solving $A'(x) = 0 \Rightarrow 0 = 800 - 16x \Rightarrow x = 50$.
Since $A'(x) > 0$ for $0 < x < 50$ and $A'(x) < 0$ for $50 < x < 100$, there is an absolute maximum when $x = 50$ by the First Derivative Test for Absolute Extreme Values. Thus, when $x = 50$, we have $y = 400 - 4(50) = 200$. The maximum area is given by $2(50)(200) = 20.000\text{m}^2$.

15. 

See the figure. The fencing cost \$30 per linear meter to install and the cost of the fencing on the west side will be split with the neighbor, so the farmer's cost C will be $C = \frac{1}{2}(30x) + 30y + 30x = 30y + 45x$. The area A will be maximized when $C = 1800$, so $1800 = 30y + 45x \Leftrightarrow 30y = 1800 - 45x \Leftrightarrow$ $y = 60 - \frac{3}{2}x$. Now $A = xy = x\left(60 - \frac{3}{2}x\right) = 60x - \frac{3}{2}x^2 \Rightarrow A' = 60 - 3x$. $A' = 0 \Leftrightarrow x = 20$ and since $A'' = -3 < 0$, we have a maximum for A when $x = 20$ m and $y = 60 - \frac{3}{2}(20) = 30$ m. [The maximum area is $30(20) = 600\text{ m}^2$.]

16. 

See the figure. The fencing cost \$30 per linear meter to install and the cost of the fencing on the west side will be split with the neighbor, so the farmer's cost C will be $C = \frac{1}{2}(30x) + 30y + 30x = 30y + 45x$. The area A to be enclosed is 750 m^2, so $A = xy = 750 \Rightarrow y = \dfrac{750}{x}$.

Now $C = 30y + 45x = 30\left(\dfrac{750}{x}\right) + 45x = \dfrac{22{,}500}{x} + 45x \Rightarrow C' = -\dfrac{22{,}500}{x^2} + 45$. $C' = 0 \Leftrightarrow$

$45 = \dfrac{22{,}500}{x^2} \Leftrightarrow x^2 = 500 \Rightarrow x = \sqrt{500} = 10\sqrt{5}$. Since $C'' = \dfrac{45{,}000}{x^3} > 0$ [for $x > 0$],

we have a minimum for C when $x = 10\sqrt{5}$ m and $y = \dfrac{750}{x} = \dfrac{750}{10\sqrt{5}} \cdot \dfrac{\sqrt{5}}{\sqrt{5}} = 15\sqrt{5}$ m. [The minimum cost is $30(15\sqrt{5}) + 45(10\sqrt{5}) = 900\sqrt{5} \approx \$2012.46$.]

17. (a) Let the rectangle have sides x and y and area A, so $A = xy$ or $y = A/x$. The problem is to minimize the perimeter $= 2x + 2y = 2x + 2A/x = P(x)$. Now $P'(x) = 2 - 2A/x^2 = 2\left(x^2 - A\right)/x^2$. So the critical number is $x = \sqrt{A}$. Since $P'(x) < 0$ for $0 < x < \sqrt{A}$ and $P'(x) > 0$ for $x > \sqrt{A}$, there is an absolute minimum at $x = \sqrt{A}$. The sides of the rectangle are $\sqrt{A}$ and $A/\sqrt{A} = \sqrt{A}$, so the rectangle is a square.

(b) Let p be the perimeter and x and y the lengths of the sides, so $p = 2x + 2y \Rightarrow 2y = p - 2x \Rightarrow y = \frac{1}{2}p - x$. The area is $A(x) = x\left(\frac{1}{2}p - x\right) = \frac{1}{2}px - x^2$. Now $A'(x) = 0 \Rightarrow \frac{1}{2}p - 2x = 0 \Rightarrow 2x = \frac{1}{2}p \Rightarrow x = \frac{1}{4}p$. Since $A''(x) = -2 < 0$, there is an absolute maximum for A when $x = \frac{1}{4}p$ by the Second Derivative Test. The sides of the rectangle are $\frac{1}{4}p$ and $\frac{1}{2}p - \frac{1}{4}p = \frac{1}{4}p$, so the rectangle is a square.

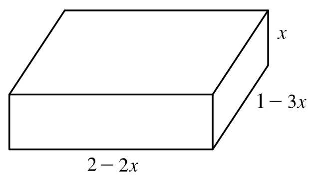
18. Let b be the length of the base of the box and h the height. The volume is $32{,}000 = b^2h \Rightarrow h = 32{,}000/b^2$. The surface area of the open box is $S = b^2 + 4hb = b^2 + 4(32{,}000/b^2)b = b^2 + 4(32{,}000)/b$. So $S'(b) = 2b - 4(32{,}000)/b^2 = 2\left(b^3 - 64{,}000\right)/b^2 = 0 \Leftrightarrow b = \sqrt[3]{64{,}000} = 40$. This gives an absolute minimum since $S'(b) < 0$ if $0 < b < 40$ and $S'(b) > 0$ if $b > 40$. The box should be $40 \times 40 \times 20$.

19. Let b be the length of the base of the box and h the height. The surface area is $1200 = b^2 + 4hb \quad \Rightarrow \quad h = (1200 - b^2)/(4b)$.

The volume is $V = b^2 h = b^2(1200 - b^2)/4b = 300b - b^3/4 \quad \Rightarrow \quad V'(b) = 300 - \frac{3}{4}b^2$.

$V'(b) = 0 \quad \Rightarrow \quad 300 = \frac{3}{4}b^2 \quad \Rightarrow \quad b^2 = 400 \quad \Rightarrow \quad b = \sqrt{400} = 20$. Since $V'(b) > 0$ for $0 < b < 20$ and $V'(b) < 0$ for $b > 20$, there is an absolute maximum when $b = 20$ by the First Derivative Test for Absolute Extreme Values. If $b = 20$, then $h = (1200 - 20^2)/(4 \cdot 20) = 10$, so the largest possible volume is $b^2 h = (20)^2(10) = 4000 \text{ cm}^3$.

20.



$V = x(2 - 2x)(1 - 3x) = 6x^3 - 8x^2 + 2x,\ 0 \le x \le 1/3$.

$V'(x) = 0 \Rightarrow 0 = 18x^2 - 16x + 2 = 9x^2 - 8x + 1$.

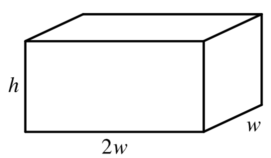
By the quadratic formula, we have that $x = \dfrac{8 \pm \sqrt{(-8)^2 - 4(9)(1)}}{2(9)} = \dfrac{4 \pm \sqrt{7}}{9}$.

Since $V''(x) = 18x - 8$ and $V''\left(\dfrac{4 - \sqrt{7}}{9}\right) < 0$,

there is an absolute maximum at $x = \frac{4-\sqrt{7}}{9}$. Indeed, since $V(0) = V(1/3) = 0$. Thus the maximum volume is

$V\left(\frac{4-\sqrt{7}}{9}\right) = \frac{4}{243}(7\sqrt{7} - 10) \approx 0.14m^3$

21.



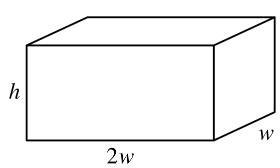
$V = lwh \quad \Rightarrow \quad 10 = (2w)(w)h = 2w^2 h$, so $h = 5/w^2$.

The cost is $10(2w^2) + 6[2(2wh) + 2(hw)] = 20w^2 + 36wh$, so

$C(w) = 20w^2 + 36w\big(5/w^2\big) = 20w^2 + 180/w$.

$C'(w) = 40w - 180/w^2 = (40w^3 - 180)/w^2 = 40\left(w^3 - \frac{9}{2}\right)/w^2 \quad \Rightarrow \quad w = \sqrt[3]{\frac{9}{2}}$ is the critical number. There is an absolute minimum for C when $w = \sqrt[3]{\frac{9}{2}}$ since $C'(w) < 0$ for $0 < w < \sqrt[3]{\frac{9}{2}}$ and $C'(w) > 0$ for $w > \sqrt[3]{\frac{9}{2}}$. The minimum cost is $C\left(\sqrt[3]{\frac{9}{2}}\right) = 20\left(\sqrt[3]{\frac{9}{2}}\right)^2 + \dfrac{180}{\sqrt[3]{9/2}} \approx \$163.54$.

22.



$V = lwh \quad \Rightarrow \quad 10 = (2w)(w)h = 2w^2 h$, so $h = 5/w^2$.

The cost is $10(2w^2) + 6[2(2wh) + 2(hw)] + 6(2w^2) = 32w^2 + 36wh$, so

$C(w) = 32w^2 + 36w(5/w^2) = 32w^2 + 180/w$.

$C'(w) = 64w - 180/w^2 = (64w^3 - 180)/w^2 = 4(16w^3 - 45)/w^2 \quad \Rightarrow \quad w = \sqrt[3]{\frac{45}{16}}$ is the critical number. There is an absolute minimum for C when $w = \sqrt[3]{\frac{45}{16}}$ since $C'(w) < 0$ for $0 < w < \sqrt[3]{\frac{45}{16}}$ and $C'(w) > 0$ for $w > \sqrt[3]{\frac{45}{16}}$. The minimum cost is $C\left(\sqrt[3]{\frac{45}{16}}\right) = 32\left(\sqrt[3]{\frac{45}{16}}\right)^2 + \dfrac{180}{\sqrt[3]{45/16}} \approx \$191.28$.

23. x : the length of the package y : the length of the square base V : the volume $x + 4y = 274 \implies x = 274 - 4y$ $V = x \cdot y^2 = (274 - 4y) \cdot y^2 = 274y^2 - 4y^3$ Therefore, $V^{'}(y) = 548y - 12y^2 = 0 \implies y = \frac{137}{3}$ cm so $V^{'}(y) >$ $0,\ when\ 0 < y < \frac{137}{3}$ $V^{'}(y) < 0,\ when\ \frac{137}{3} < y < \frac{274}{4}$ Therefore, the greatest volume that may be mailed should be at $y = \frac{137}{3}$, $x = 274 - 4 \cdot \frac{137}{3} = \frac{274}{3}$. And the volume will be $V = \frac{274}{3} \times \frac{137}{3} \times \frac{137}{3} \approx 190{,}470.5926$ cm^3

24. Let $x > 0$ be the length of the package and $r > 0$ be the radius of the circluar base. We have $x + 2\pi r = 274 \implies$ $x = 274 - 2\pi r$. The volume is $V = \pi r^2 x = \pi r^2(274 - 2\pi r) = \pi(274r^2 - 2\pi r^3)$[for $0 < r < \frac{137}{\pi}$].

$$V^{'}(r) = \pi(548r - 6\pi r) = \pi r(548 - 6\pi r) = 0 \implies r = \frac{274}{3\pi}[\text{since } r \neq 0]$$

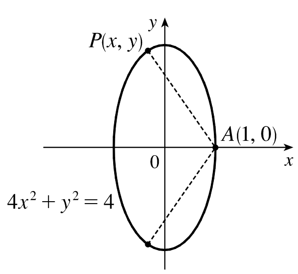
Since $V'(r) > 0$ for $0 < r < \frac{274}{3\pi}$ and $V'(r) < 0$ for $r > \frac{274}{3\pi}$, there is an abosolute maximum when $r = \frac{274}{3\pi}$ by the First Derivative Test for Absolute Extreme Values.

If $r = \frac{274}{3\pi}$, then $x = 274 - 2\pi(\frac{274}{3\pi}) = 274 - \frac{548}{3} = \frac{274}{3}$, so the dimensions that give the greatest volume are a length of $\frac{274}{3}$ cm and a base radius of $\frac{274}{3\pi} \approx 29.09$ cm, giving a greatest possible volume of $\frac{20570824}{27\pi}$ cm^3 [242,637.7 cm^3]

25. The distance d from the origin $(0,0)$ to a point $(x, 2x+3)$ on the line is given by $d = \sqrt{(x-0)^2 + (2x+3-0)^2}$ and the square of the distance is $S = d^2 = x^2 + (2x+3)^2$. $S' = 2x + 2(2x+3)2 = 10x + 12$ and $S' = 0 \Leftrightarrow x = -\frac{6}{5}$. Now $S'' = 10 > 0$, so we know that S has a minimum at $x = -\frac{6}{5}$. Thus, the y-value is $2\left(-\frac{6}{5}\right) + 3 = \frac{3}{5}$ and the point is $\left(-\frac{6}{5}, \frac{3}{5}\right)$.

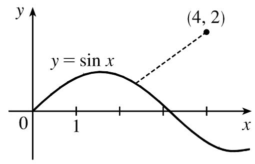
26. The distance d from the point $(3,0)$ to a point $(x, \sqrt{x}\,)$ on the curve is given by $d = \sqrt{(x-3)^2 + (\sqrt{x} - 0)^2}$ and the square of the distance is $S = d^2 = (x-3)^2 + x$. $S' = 2(x-3) + 1 = 2x - 5$ and $S' = 0 \Leftrightarrow x = \frac{5}{2}$. Now $S'' = 2 > 0$, so we know that S has a minimum at $x = \frac{5}{2}$. Thus, the y-value is $\sqrt{\frac{5}{2}}$ and the point is $\left(\frac{5}{2}, \sqrt{\frac{5}{2}}\right)$.

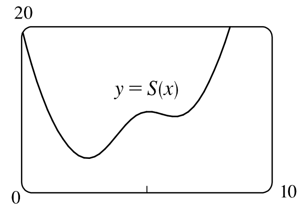
27.



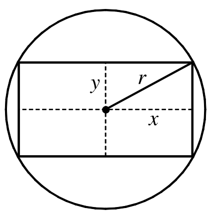
From the figure, we see that there are two points that are farthest away from $A(1,0)$. The distance d from A to an arbitrary point $P(x,y)$ on the ellipse is $d = \sqrt{(x-1)^2 + (y-0)^2}$ and the square of the distance is $S = d^2 = x^2 - 2x + 1 + y^2 = x^2 - 2x + 1 + (4 - 4x^2) = -3x^2 - 2x + 5$. $S' = -6x - 2$ and $S' = 0 \Rightarrow x = -\frac{1}{3}$. Now $S'' = -6 < 0$, so we know that S has a maximum at $x = -\frac{1}{3}$. Since $-1 \leq x \leq 1$, $S(-1) = 4$, $S\left(-\frac{1}{3}\right) = \frac{16}{3}$, and $S(1) = 0$, we see that the maximum distance is $\sqrt{\frac{16}{3}}$. The corresponding y-values are $y = \pm\sqrt{4 - 4\left(-\frac{1}{3}\right)^2} = \pm\sqrt{\frac{32}{9}} = \pm\frac{4}{3}\sqrt{2} \approx \pm 1.89$. The points are $\left(-\frac{1}{3}, \pm\frac{4}{3}\sqrt{2}\right)$.

28. The distance d from the point $(4, 2)$ to a point $(x, \sin x)$ on the curve is given by $d = \sqrt{(x-4)^2 + (\sin x - 2)^2}$ and the square of the distance is $S = d^2 = (x-4)^2 + (\sin x - 2)^2$. $S' = 2(x-4) + 2(\sin x - 2)\cos x$. Using a calculator, it is clear that S has a minimum between 0 and 5, and from a graph of S', we find that $S' = 0 \quad \Rightarrow \quad x \approx 2.65$, so the point is about $(2.65, 0.47)$.





29.

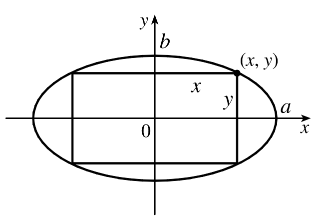


The area of the rectangle is $(2x)(2y) = 4xy$. Also $r^2 = x^2 + y^2$ so $y = \sqrt{r^2 - x^2}$, so the area is $A(x) = 4x\sqrt{r^2 - x^2}$. Now

$$A'(x) = 4\left(\sqrt{r^2 - x^2} - \frac{x^2}{\sqrt{r^2 - x^2}}\right) = 4\,\frac{r^2 - 2x^2}{\sqrt{r^2 - x^2}}.$$ The critical number is $x = \frac{1}{\sqrt{2}}r$. Clearly this gives a maximum.

$y = \sqrt{r^2 - \left(\frac{1}{\sqrt{2}}r\right)^2} = \sqrt{\frac{1}{2}r^2} = \frac{1}{\sqrt{2}}r = x$, which tells us that the rectangle is a square. The dimensions are $2x = \sqrt{2}\,r$ and $2y = \sqrt{2}\,r$.

30.

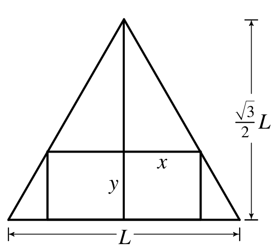


The area of the rectangle is $(2x)(2y) = 4xy$. Now $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ gives $y = \dfrac{b}{a}\sqrt{a^2 - x^2}$, so we maximize $A(x) = 4\dfrac{b}{a}x\sqrt{a^2 - x^2}$.

$$\begin{aligned} A'(x) &= \frac{4b}{a}\left[x \cdot \tfrac{1}{2}(a^2 - x^2)^{-1/2}(-2x) + (a^2 - x^2)^{1/2} \cdot 1\right] \\ &= \frac{4b}{a}\,(a^2 - x^{2-1/2}[-x^2 + a^2 - x^2] = \frac{4b}{a\sqrt{a^2 - x^2}}[a^2 - 2x^2] \end{aligned}$$

So the critical number is $x = \frac{1}{\sqrt{2}}\,a$, and this clearly gives a maximum. Then $y = \frac{1}{\sqrt{2}}\,b$, so the maximum area is $4\left(\frac{1}{\sqrt{2}}\,a\right)\left(\frac{1}{\sqrt{2}}\,b\right) = 2ab$.

31.



The height h of the equilateral triangle with sides of length L is $\frac{\sqrt{3}}{2}\,L$, since $h^2 + (L/2)^2 = L^2 \quad \Rightarrow \quad h^2 = L^2 - \frac{1}{4}L^2 = \frac{3}{4}L^2 \quad \Rightarrow$

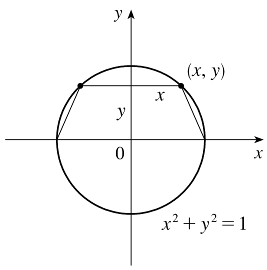
$h = \frac{\sqrt{3}}{2}L$. Using similar triangles, $\dfrac{\frac{\sqrt{3}}{2}L - y}{x} = \dfrac{\frac{\sqrt{3}}{2}L}{L/2} = \sqrt{3} \quad \Rightarrow$

$\sqrt{3}\,x = \frac{\sqrt{3}}{2}L - y \quad \Rightarrow \quad y = \frac{\sqrt{3}}{2}L - \sqrt{3}\,x \quad \Rightarrow \quad y = \frac{\sqrt{3}}{2}(L - 2x)$.

The area of the inscribed rectangle is $A(x) = (2x)y = \sqrt{3}\,x(L - 2x) = \sqrt{3}\,Lx - 2\sqrt{3}\,x^2$, where $0 \le x \le L/2$. Now

$0 = A'(x) = \sqrt{3}\,L - 4\sqrt{3}\,x \quad \Rightarrow \quad x = \sqrt{3}\,L/\big(4\sqrt{3}\,\big) = L/4$. Since $A(0) = A(L/2) = 0$, the maximum occurs when $x = L/4$, and $y = \frac{\sqrt{3}}{2}L - \frac{\sqrt{3}}{4}L = \frac{\sqrt{3}}{4}L$, so the dimensions are $L/2$ and $\frac{\sqrt{3}}{4}L$.

32.

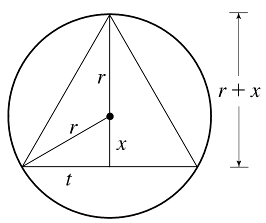


The area A of a trapezoid is given by $A = \frac{1}{2}h(B + b)$. From the diagram, $h = y$, $B = 2$, and $b = 2x$, so $A = \frac{1}{2}y(2 + 2x) = y(1 + x)$. Since it's easier to substitute for y^2, we'll let $T = A^2 = y^2(1 + x)^2 = (1 - x^2)(1 + x)^2$. Now

$$\begin{aligned} T' &= (1 - x^2)2(1 + x) + (1 + x)^2(-2x) = -2(1 + x)[-(1 - x^2) + (1 + x)x] \\ &= -2(1 + x)(2x^2 + x - 1) = -2(1 + x)(2x - 1)(x + 1) \end{aligned}$$

$T' = 0 \quad \Leftrightarrow \quad x = -1$ or $x = \frac{1}{2}$. $T' > 0$ if $x < \frac{1}{2}$ and $T' < 0$ if $x > \frac{1}{2}$, so we get a maximum at $x = \frac{1}{2}$ $\;[x = -1$ gives us $A = 0]$. Thus, $y = \sqrt{1 - \left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2}$ and the maximum area is $A = y(1 + x) = \frac{\sqrt{3}}{2}\left(1 + \frac{1}{2}\right) = \frac{3\sqrt{3}}{4}$.

33.



The area of the triangle is

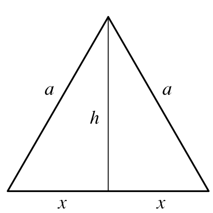
$$A(x) = \tfrac{1}{2}(2t)(r + x) = t(r + x) = \sqrt{r^2 - x^2}\,(r + x). \text{ Then}$$

$$\begin{aligned} 0 = A'(x) &= r\frac{-2x}{2\sqrt{r^2 - x^2}} + \sqrt{r^2 - x^2} + x\frac{-2x}{2\sqrt{r^2 - x^2}} \\ &= -\frac{x^2 + rx}{\sqrt{r^2 - x^2}} + \sqrt{r^2 - x^2} \quad \Rightarrow \end{aligned}$$

$$\frac{x^2 + rx}{\sqrt{r^2 - x^2}} = \sqrt{r^2 - x^2} \quad \Rightarrow \quad x^2 + rx = r^2 - x^2 \quad \Rightarrow \quad 0 = 2x^2 + rx - r^2 = (2x - r)(x + r) \quad \Rightarrow$$

$x = \frac{1}{2}r$ or $x = -r$. Now $A(r) = 0 = A(-r) \quad \Rightarrow \quad$ the maximum occurs where $x = \frac{1}{2}r$, so the triangle has height $r + \frac{1}{2}r = \frac{3}{2}r$ and base $2\sqrt{r^2 - \left(\frac{1}{2}r\right)^2} = 2\sqrt{\frac{3}{4}r^2} = \sqrt{3}\,r$.

34.

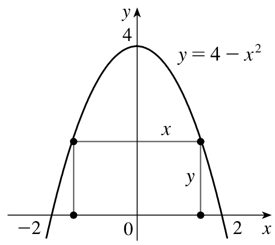


From the figure, we have $x^2 + h^2 = a^2 \quad \Rightarrow \quad h = \sqrt{a^2 - x^2}$. The area of the isosceles triangle is $A = \frac{1}{2}(2x)h = xh = x\sqrt{a^2 - x^2}$ with $0 \le x \le a$. Now

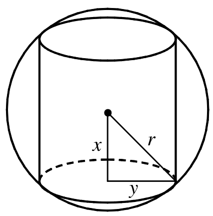
$$\begin{aligned} A' &= x \cdot \tfrac{1}{2}(a^2 - x^2)^{-1/2}(-2x) + (a^2 - x^2)^{1/2}(1) \\ &= (a^2 - x^2)^{-1/2}[-x^2 + (a^2 - x^2)] = \frac{a^2 - 2x^2}{\sqrt{a^2 - x^2}} \end{aligned}$$

$A' = 0 \quad \Leftrightarrow \quad x^2 = \frac{1}{2}a^2 \quad \Rightarrow \quad x = a/\sqrt{2}$. Since $A(0) = 0$, $A(a) = 0$, and $A(a/\sqrt{2}\,) = (a/\sqrt{2}\,)\sqrt{a^2/2} = \frac{1}{2}a^2$, we see that $x = a/\sqrt{2}$ gives us the maximum area and the length of the base is $2x = 2(a/\sqrt{2}\,) = \sqrt{2}\,a$. Note that the triangle has sides a, a, and $\sqrt{2}\,a$, which form a *right* triangle, with the right angle between the two sides of equal length.

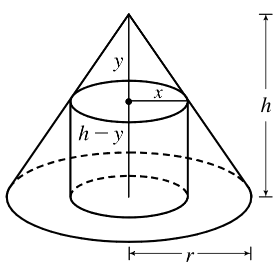
35. The area of the triangle is $A = \frac{1}{2}a(2a)\sin\theta$ for $0 < \theta < \pi$. $A'(\theta) = a^2\cos\theta = 0 \quad \Rightarrow \quad \cos\theta = 0 \quad \Rightarrow \quad \theta = \frac{\pi}{2}$. Since $A'(\theta) > 0$ for $0 < \theta < \frac{\pi}{2}$ and $A'(\theta) < 0$ for $\frac{\pi}{2} < \theta < \pi$, there is an absolute maximum when $\theta = \frac{\pi}{2}$ by the First Derivative Test for Absolute Extreme Values. (The maximum area of $\frac{1}{2}a(2a)\sin\frac{\pi}{2} = a^2$ results from the triangle being a right triangle.)

36. 

The area of the rectangle is $A = 2xy = 2x(4 - x^2) = 8x - 2x^3,\ 0 < x < 2$. $A'(x) = 8 - 6x^2 = 2(4 - 3x^2) = 0 \quad\Rightarrow\quad x = \dfrac{2}{\sqrt{3}}$ [since $x > 0$]. Since $A'(x) > 0$ for $0 < x < \dfrac{2}{\sqrt{3}}$ and $A'(x) < 0$ for $\dfrac{2}{\sqrt{3}} < x < 2$, there is an absolute maximum when $x = \dfrac{2}{\sqrt{3}}$ by the First Derivative Test for Absolute Extreme Values. Thus, the largest possible area of the rectangle is $A = 2\left(\dfrac{2}{\sqrt{3}}\right)\left[4 - \left(\dfrac{2}{\sqrt{3}}\right)^2\right] = \left(\dfrac{4}{\sqrt{3}}\right)\left(\dfrac{8}{3}\right) = \dfrac{32}{3\sqrt{3}} \approx 6.16$.

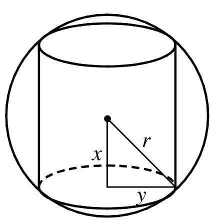
37. 

The cylinder has volume $V = \pi y^2(2x)$. Also $x^2 + y^2 = r^2 \quad\Rightarrow\quad y^2 = r^2 - x^2$, so $V(x) = \pi(r^2 - x^2)(2x) = 2\pi(r^2x - x^3)$, where $0 \le x \le r$. $V'(x) = 2\pi\left(r^2 - 3x^2\right) = 0 \quad\Rightarrow\quad x = r/\sqrt{3}$. Now $V(0) = V(r) = 0$, so there is a maximum when $x = r/\sqrt{3}$ and $V\left(r/\sqrt{3}\right) = \pi(r^2 - r^2/3)\left(2r/\sqrt{3}\right) = 4\pi r^3/\left(3\sqrt{3}\right)$.

38. 

By similar triangles, $y/x = h/r$, so $y = hx/r$. The volume of the cylinder is $\pi x^2(h - y) = \pi hx^2 - (\pi h/r)x^3 = V(x)$. Now $V'(x) = 2\pi hx - (3\pi h/r)x^2 = \pi hx(2 - 3x/r)$. So $V'(x) = 0 \quad\Rightarrow\quad x = 0$ or $x = \frac{2}{3}r$. The maximum clearly occurs when $x = \frac{2}{3}r$ and then the volume is

$\pi hx^2 - (\pi h/r)x^3 = \pi hx^2(1 - x/r) = \pi\left(\frac{2}{3}r\right)^2 h\left(1 - \frac{2}{3}\right) = \frac{4}{27}\pi r^2 h$.

39. 

The cylinder has surface area

$$\begin{aligned} 2(\text{area of the base}) + (\text{lateral surface area}) &= 2\pi(\text{radius})^2 + 2\pi(\text{radius})(\text{height}) \\ &= 2\pi y^2 + 2\pi y(2x) \end{aligned}$$

Now $x^2 + y^2 = r^2 \quad\Rightarrow\quad y^2 = r^2 - x^2 \quad\Rightarrow\quad y = \sqrt{r^2 - x^2}$, so the surface area is

$$\begin{aligned} S(x) &= 2\pi(r^2 - x^2) + 4\pi x\sqrt{r^2 - x^2},\quad 0 \le x \le r \\ &= 2\pi r^2 - 2\pi x^2 + 4\pi\left(x\sqrt{r^2 - x^2}\right) \end{aligned}$$

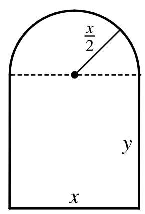
Thus,
$$\begin{aligned} S'(x) &= 0 - 4\pi x + 4\pi\left[x \cdot \tfrac{1}{2}(r^2 - x^2)^{-1/2}(-2x) + (r^2 - x^2)^{1/2} \cdot 1\right] \\ &= 4\pi\left[-x - \frac{x^2}{\sqrt{r^2 - x^2}} + \sqrt{r^2 - x^2}\right] = 4\pi \cdot \frac{-x\sqrt{r^2 - x^2} - x^2 + r^2 - x^2}{\sqrt{r^2 - x^2}} \end{aligned}$$

$S'(x) = 0 \quad\Rightarrow\quad x\sqrt{r^2 - x^2} = r^2 - 2x^2 \quad (\star) \quad\Rightarrow\quad \left(x\sqrt{r^2 - x^2}\right)^2 = (r^2 - 2x^2)^2 \quad\Rightarrow$
$x^2(r^2 - x^2) = r^4 - 4r^2x^2 + 4x^4 \quad\Rightarrow\quad r^2x^2 - x^4 = r^4 - 4r^2x^2 + 4x^4 \quad\Rightarrow\quad 5x^4 - 5r^2x^2 + r^4 = 0$.

This is a quadratic equation in x^2. By the quadratic formula, $x^2 = \frac{5 \pm \sqrt{5}}{10}r^2$, but we reject the solution with the + sign since it doesn't satisfy $(\star)$. [The right side is negative and the left side is positive.] So $x = \sqrt{\frac{5 - \sqrt{5}}{10}}\,r$. Since $S(0) = S(r) = 0$, the maximum surface area occurs at the critical number and $x^2 = \frac{5-\sqrt{5}}{10}r^2 \quad\Rightarrow\quad y^2 = r^2 - \frac{5-\sqrt{5}}{10}r^2 = \frac{5+\sqrt{5}}{10}r^2 \quad\Rightarrow$ the surface area is

$$2\pi\left(\tfrac{5+\sqrt{5}}{10}\right)r^2 + 4\pi\sqrt{\tfrac{5-\sqrt{5}}{10}}\sqrt{\tfrac{5+\sqrt{5}}{10}}\,r^2 = \pi r^2\left[2\cdot\tfrac{5+\sqrt{5}}{10} + 4\tfrac{\sqrt{(5-\sqrt{5})(5+\sqrt{5})}}{10}\right] = \pi r^2\left[\tfrac{5+\sqrt{5}}{5} + \tfrac{2\sqrt{20}}{5}\right]$$
$$= \pi r^2\left[\tfrac{5+\sqrt{5}+2\cdot 2\sqrt{5}}{5}\right] = \pi r^2\left[\tfrac{5+5\sqrt{5}}{5}\right] = \pi r^2\left(1+\sqrt{5}\right).$$

40.



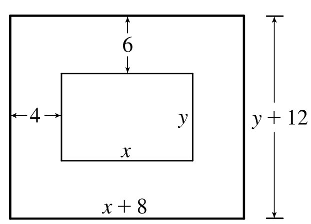
Perimeter $= 10 \quad\Rightarrow\quad 2y + x + \pi\left(\dfrac{x}{2}\right) = 10 \quad\Rightarrow$

$y = \dfrac{1}{2}\left(10 - x - \dfrac{\pi x}{2}\right) = 5 - \dfrac{x}{2} - \dfrac{\pi x}{4}$. The area is the area of the rectangle plus the area of the semicircle, or $xy + \frac{1}{2}\pi\left(\dfrac{x}{2}\right)^2$, so $A(x) = x\left(5 - \dfrac{x}{2} - \dfrac{\pi x}{4}\right) + \frac{1}{8}\pi x^2 = 5x - \frac{1}{2}x^2 - \frac{\pi}{8}x^2$.

$A'(x) = 5 - \left(1 + \frac{\pi}{4}\right)x = 0 \quad\Rightarrow\quad x = \dfrac{5}{1+\pi/4} = \dfrac{20}{4+\pi}$. $A''(x) = -\left(1 + \dfrac{\pi}{4}\right) < 0$, so this gives a maximum.

The dimensions are $x = \dfrac{20}{4+\pi}$ m and $y = 5 - \dfrac{10}{4+\pi} - \dfrac{5\pi}{4+\pi} = \dfrac{20 + 5\pi - 10 - 5\pi}{4+\pi} = \dfrac{10}{4+\pi}$ m, so the height of the rectangle is half the base.

41.



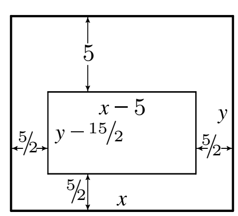
$xy = 384 \quad\Rightarrow\quad y = 384/x$. Total area is

$A(x) = (8+x)(12 + 384/x) = 12(40 + x + 256/x)$, so

$A'(x) = 12(1 - 256/x^2) = 0 \quad\Rightarrow\quad x = 16$. There is an absolute minimum when $x = 16$ since $A'(x) < 0$ for $0 < x < 16$ and $A'(x) > 0$ for $x > 16$.

When $x = 16$, $y = 384/16 = 24$, so the dimensions are 24 cm and 36 cm.

42.

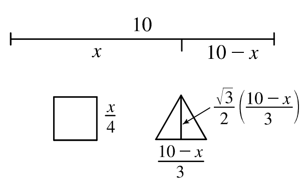


$xy = 900$, so $y = 900/x$. The printed area is

$(x-5)(y - {}^{15}\!/_2) = (x-5)(900/x - {}^{15}\!/_2) = {}^{1875}\!/_2 - {}^{15x}\!/_2 - {}^{4500}\!/_x = A(x)$.

$A'(x) = -{}^{15}\!/_2 + {}^{4500}\!/_{x^2} = 0$ when $x^2 = 600 \Rightarrow x = 10\sqrt{6}$. This gives an absolute maximum since $A'(x) > 0$ for $0 < x < 10\sqrt{6}$ and $A'(x) < 0$ for $x > 10\sqrt{6}$. When $x = 10\sqrt{6}$, $y = {}^{900}\!/_{10\sqrt{6}} = {}^{90}\!/_{\sqrt{6}}$, so the dimensions are $10\sqrt{6}$ cm. and ${}^{90}\!/_{\sqrt{6}}$ cm.

43.



Let x be the length of the wire used for the square. The total area is

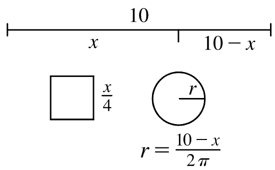
$$A(x) = \left(\frac{x}{4}\right)^2 + \frac{1}{2}\left(\frac{10-x}{3}\right)\frac{\sqrt{3}}{2}\left(\frac{10-x}{3}\right)$$
$$= \tfrac{1}{16}x^2 + \tfrac{\sqrt{3}}{36}(10-x)^2,\ 0 \le x \le 10$$

$A'(x) = \frac{1}{8}x - \frac{\sqrt{3}}{18}(10-x) = 0 \quad\Leftrightarrow\quad \frac{9}{72}x + \frac{4\sqrt{3}}{72}x - \frac{40\sqrt{3}}{72} = 0 \quad\Leftrightarrow\quad x = \frac{40\sqrt{3}}{9+4\sqrt{3}}$.

Now $A(0) = \left(\frac{\sqrt{3}}{36}\right)100 \approx 4.81$, $A(10) = \frac{100}{16} = 6.25$ and $A\left(\frac{40\sqrt{3}}{9+4\sqrt{3}}\right) \approx 2.72$, so

(a) The maximum area occurs when $x = 10$ m, and all the wire is used for the square.

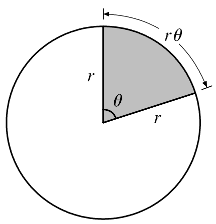
(b) The minimum area occurs when $x = \frac{40\sqrt{3}}{9+4\sqrt{3}} \approx 4.35$ m.

44. 

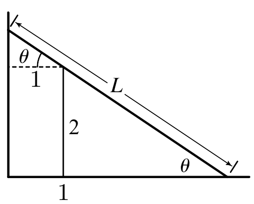
Total area is $A(x) = \left(\frac{x}{4}\right)^2 + \pi\left(\frac{10-x}{2\pi}\right)^2 = \frac{x^2}{16} + \frac{(10-x)^2}{4\pi},\ 0 \le x \le 10.$

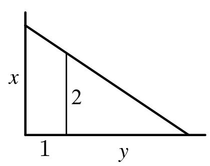
$A'(x) = \frac{x}{8} - \frac{10-x}{2\pi} = \left(\frac{1}{2\pi} + \frac{1}{8}\right)x - \frac{5}{\pi} = 0 \quad\Rightarrow\quad x = 40/(4+\pi).$

$A(0) = 25/\pi \approx 7.96$, $A(10) = 6.25$, and $A(40/(4+\pi)) \approx 3.5$, so the maximum occurs when $x = 0$ m and the minimum occurs when $x = 40/(4+\pi)$ m.

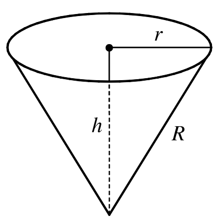
45. 

From the figure, the perimeter of the slice is $2r + r\theta = 60$, so $\theta = \dfrac{60-2r}{r}$. The area A of the slice is $A = \frac{1}{2}r^2\theta = \frac{1}{2}r^2\left(\dfrac{60-2r}{r}\right) = r(30-r) = 30r - r^2$ for $0 \le r \le 30$. $A'(r) = 30 - 2r$, so $A' = 0$ when $r = 15$. Since $A(0) = 0$, $A(30) = 0$, and $A(15) = 225\ \mathrm{cm}^2$, the largest piece comes from a pizza with radius $15\ \mathrm{cm}$ and diameter $30\ \mathrm{cm}$. Note that $\theta = 2$ radians $\approx 114.6°$, which is about 32% of the whole pizza.

46. 



$L = 2csc\theta + sec\theta,\ 0 < \theta < \frac{\pi}{2},\ \frac{dL}{d\theta} = -2csc\theta cot\theta + sec\theta tan\theta$. Let $\frac{dL}{d\theta} = 0$, then we can get $sec\theta tan\theta = 2csc\theta cot\theta \implies tan^3\theta = 2 \implies \theta = arctan2^{\frac{1}{3}}$. When $0 < \theta < arctan2^{\frac{1}{3}}$, $\frac{dL}{d\theta} < 0$. When $arctan2^{\frac{1}{3}} < \theta < \frac{\pi}{2}$, $\frac{dL}{d\theta} > 0$. Hence, L has an absolute minimum when $\theta = arctan2^{\frac{1}{3}}$, and the shortest ladder has length $L = 2csc(arctan2^{\frac{1}{3}}) + sec(arctan2^{\frac{1}{3}}) \approx 4.16$ m.

47. 

$h^2 + r^2 = R^2 \quad\Rightarrow\quad V = \frac{\pi}{3}r^2h = \frac{\pi}{3}(R^2 - h^2)h = \frac{\pi}{3}(R^2h - h^3).$

$V'(h) = \frac{\pi}{3}(R^2 - 3h^2) = 0$ when $h = \frac{1}{\sqrt{3}}R$. This gives an absolute maximum, since $V'(h) > 0$ for $0 < h < \frac{1}{\sqrt{3}}R$ and $V'(h) < 0$ for $h > \frac{1}{\sqrt{3}}R$. The maximum volume is $V\left(\frac{1}{\sqrt{3}}R\right) = \frac{\pi}{3}\left(\frac{1}{\sqrt{3}}R^3 - \frac{1}{3\sqrt{3}}R^3\right) = \frac{2}{9\sqrt{3}}\pi R^3.$

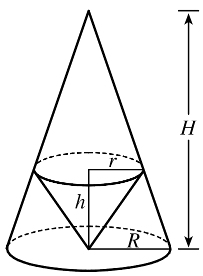
48. The volume and surface area of a cone with radius r and height h are given by $V = \frac{1}{3}\pi r^2 h$ and $S = \pi r\sqrt{r^2 + h^2}$.

We'll minimize $A = S^2$ subject to $V = 27$. $V = 27 \quad\Rightarrow\quad \frac{1}{3}\pi r^2 h = 27 \quad\Rightarrow\quad r^2 = \dfrac{81}{\pi h}$ **(1)**.

$A = \pi^2 r^2(r^2 + h^2) = \pi^2\left(\dfrac{81}{\pi h}\right)\left(\dfrac{81}{\pi h} + h^2\right) = \dfrac{81^2}{h^2} + 81\pi h$, so $A' = 0 \quad\Rightarrow\quad \dfrac{-2\cdot 81^2}{h^3} + 81\pi = 0 \quad\Rightarrow$

$81\pi = \dfrac{2\cdot 81^2}{h^3} \quad\Rightarrow\quad h^3 = \dfrac{162}{\pi} \quad\Rightarrow\quad h = \sqrt[3]{\dfrac{162}{\pi}} = 3\sqrt[3]{\dfrac{6}{\pi}} \approx 3.722$. From **(1)**, $r^2 = \dfrac{81}{\pi h} = \dfrac{81}{\pi\cdot 3\sqrt[3]{6/\pi}} = \dfrac{27}{\sqrt[3]{6\pi^2}} \quad\Rightarrow$

$r = \dfrac{3\sqrt{3}}{\sqrt[6]{6\pi^2}} \approx 2.632$. $A'' = 6\cdot 81^2/h^4 > 0$, so A and hence S has an absolute minimum at these values of r and h.

49. 

By similar triangles, $\dfrac{H}{R} = \dfrac{H-h}{r}$ **(1)**. The volume of the inner cone is $V = \frac{1}{3}\pi r^2 h$,

so we'll solve **(1)** for h. $\dfrac{Hr}{R} = H - h \quad\Rightarrow$

$h = H - \dfrac{Hr}{R} = \dfrac{HR - Hr}{R} = \dfrac{H}{R}(R - r)$ **(2)**.

Thus, $V(r) = \dfrac{\pi}{3} r^2 \cdot \dfrac{H}{R}(R - r) = \dfrac{\pi H}{3R}(Rr^2 - r^3) \quad\Rightarrow$

$V'(r) = \dfrac{\pi H}{3R}(2Rr - 3r^2) = \dfrac{\pi H}{3R} r(2R - 3r)$.

$V'(r) = 0 \quad\Rightarrow\quad r = 0$ or $2R = 3r \quad\Rightarrow\quad r = \frac{2}{3}R$ and from **(2)**, $h = \dfrac{H}{R}\left(R - \frac{2}{3}R\right) = \dfrac{H}{R}\left(\frac{1}{3}R\right) = \frac{1}{3}H$.

$V'(r)$ changes from positive to negative at $r = \frac{2}{3}R$, so the inner cone has a maximum volume of

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{2}{3}R\right)^2\left(\frac{1}{3}H\right) = \frac{4}{27} \cdot \frac{1}{3}\pi R^2 H$, which is approximately 15% of the volume of the larger cone.

50. We need to minimize F for $0 \le \theta < \pi/2$. $F(\theta) = \dfrac{\mu W}{\mu \sin\theta + \cos\theta} \quad\Rightarrow\quad F'(\theta) = \dfrac{-\mu W\,(\mu\cos\theta - \sin\theta)}{(\mu\sin\theta + \cos\theta)^2}$ [by the Reciprocal Rule]. $F'(\theta) > 0 \quad\Rightarrow\quad \mu\cos\theta - \sin\theta < 0 \quad\Rightarrow\quad \mu\cos\theta < \sin\theta \quad\Rightarrow\quad \mu < \tan\theta \quad\Rightarrow\quad \theta > \tan^{-1}\mu$.

So F is decreasing on $\left(0, \tan^{-1}\mu\right)$ and increasing on $\left(\tan^{-1}\mu, \frac{\pi}{2}\right)$. Thus, F attains its minimum value at $\theta = \tan^{-1}\mu$.

This maximum value is $F(\tan^{-1}\mu) = \dfrac{\mu W}{\sqrt{\mu^2 + 1}}$.

51. $P(R) = \dfrac{E^2 R}{(R+r)^2} \quad\Rightarrow$

$$P'(R) = \frac{(R+r)^2 \cdot E^2 - E^2 R \cdot 2(R+r)}{[(R+r)^2]^2} = \frac{(R^2 + 2Rr + r^2)E^2 - 2E^2R^2 - 2E^2Rr}{(R+r)^4}$$

$$= \frac{E^2r^2 - E^2R^2}{(R+r)^4} = \frac{E^2(r^2 - R^2)}{(R+r)^4} = \frac{E^2(r+R)(r-R)}{(R+r)^4} = \frac{E^2(r-R)}{(R+r)^3}$$

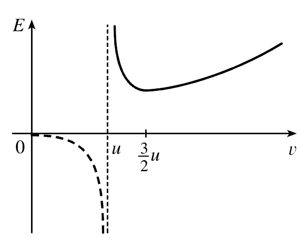
$P'(R) = 0 \quad\Rightarrow\quad R = r \quad\Rightarrow\quad P(r) = \dfrac{E^2 r}{(r+r)^2} = \dfrac{E^2 r}{4r^2} = \dfrac{E^2}{4r}$.

The expression for $P'(R)$ shows that $P'(R) > 0$ for $R < r$ and $P'(R) < 0$ for $R > r$. Thus, the maximum value of the power is $E^2/(4r)$, and this occurs when $R = r$.

52. (a) $E(v) = \dfrac{aLv^3}{v - u} \quad\Rightarrow\quad E'(v) = aL\dfrac{(v-u)3v^2 - v^3}{(v-u)^2} = 0$ when

$2v^3 = 3uv^2 \quad\Rightarrow\quad 2v = 3u \quad\Rightarrow\quad v = \frac{3}{2}u$.

The First Derivative Test shows that this value of v gives the minimum value of E.

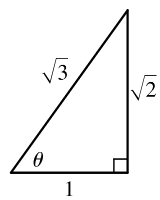
(b) 

53. $S = 6sh - \frac{3}{2}s^2\cot\theta + \left(\frac{3}{2}\sqrt{3}\,s^2\right)\csc\theta$

(a) $\dfrac{dS}{d\theta} = \frac{3}{2}s^2\csc^2\theta - \left(\frac{3}{2}\sqrt{3}\,s^2\right)\csc\theta\,\cot\theta$ or $\frac{3}{2}s^2\csc\theta\left(\csc\theta - \sqrt{3}\cot\theta\right)$.

(b) $\dfrac{dS}{d\theta} = 0$ when $\csc\theta - \sqrt{3}\cot\theta = 0 \quad\Rightarrow\quad \dfrac{1}{\sin\theta} - \sqrt{3}\,\dfrac{\cos\theta}{\sin\theta} = 0 \quad\Rightarrow\quad \cos\theta = \frac{1}{\sqrt{3}}$. The First Derivative Test shows that the minimum surface area occurs when $\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 55^\circ$.

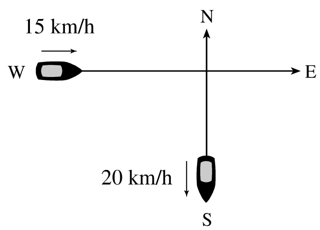
(c)



If $\cos\theta = \frac{1}{\sqrt{3}}$, then $\cot\theta = \frac{1}{\sqrt{2}}$ and $\csc\theta = \frac{\sqrt{3}}{\sqrt{2}}$, so the surface area is

$$S = 6sh - \tfrac{3}{2}s^2\tfrac{1}{\sqrt{2}} + 3s^2\tfrac{\sqrt{3}}{2}\tfrac{\sqrt{3}}{\sqrt{2}} = 6sh - \tfrac{3}{2\sqrt{2}}s^2 + \tfrac{9}{2\sqrt{2}}s^2$$
$$= 6sh + \tfrac{6}{2\sqrt{2}}s^2 = 6s\left(h + \tfrac{1}{2\sqrt{2}}s\right)$$

54.



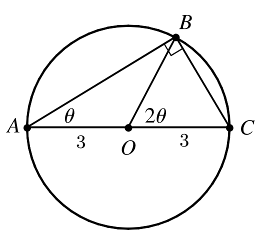
Let t be the time, in hours, after 2:00 PM. The position of the boat heading south at time t is $(0, -20t)$. The position of the boat heading east at time t is $(-15 + 15t, 0)$. If $D(t)$ is the distance between the boats at time t, we minimize $f(t) = [D(t)]^2 = 20^2t^2 + 15^2(t-1)^2$.

$f'(t) = 800t + 450(t-1) = 1250t - 450 = 0$ when $t = \frac{450}{1250} = 0.36$ h.

$0.36 \text{ h} \times \frac{60 \text{ min}}{\text{h}} = 21.6 \text{ min} = 21 \text{ min } 36 \text{ s}$. Since $f''(t) > 0$, this gives a minimum, so the boats are closest together at 2:21:36 PM.

55. Here $T(x) = \dfrac{\sqrt{x^2+25}}{6} + \dfrac{5-x}{8}$, $0 \le x \le 5$ $\Rightarrow$ $T'(x) = \dfrac{x}{6\sqrt{x^2+25}} - \dfrac{1}{8} = 0$ $\Leftrightarrow$ $8x = 6\sqrt{x^2+25}$ $\Leftrightarrow$ $16x^2 = 9(x^2+25)$ $\Leftrightarrow$ $x = \frac{15}{\sqrt{7}}$. But $\frac{15}{\sqrt{7}} > 5$, so T has no critical number. Since $T(0) \approx 1.46$ and $T(5) \approx 1.18$, she should row directly to B.

56.



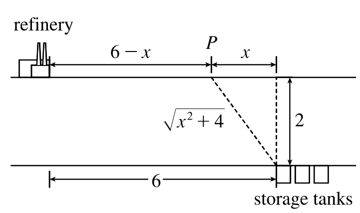
In isosceles triangle AOB, $\angle O = 180^\circ - \theta - \theta$, so $\angle BOC = 2\theta$. The distance rowed is $6\cos\theta$ while the distance walked is the length of arc $BC = 3(2\theta) = 6\theta$. The time taken is given by $T(\theta) = \dfrac{6\cos\theta}{3} + \dfrac{6\theta}{6} = 2\cos\theta + \theta$, $0 \le \theta \le \frac{\pi}{2}$.

$T'(\theta) = -2\sin\theta + 1 = 0$ $\Leftrightarrow$ $\sin\theta = \frac{1}{2}$ $\Rightarrow$ $\theta = \frac{\pi}{6}$.

Check the value of T at $\theta = \frac{\pi}{6}$ and at the endpoints of the domain of T; that is, $\theta = 0$ and $\theta = \frac{\pi}{2}$.

$T(0) = 2$, $T\left(\frac{\pi}{6}\right) = \sqrt{3} + \frac{\pi}{6} \approx 2.26$, and $T\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \approx 1.57$. Therefore, the minimum value of T is $\frac{\pi}{2}$ when $\theta = \frac{\pi}{2}$; that is, the woman should walk all the way. Note that $T''(\theta) = -2\cos\theta < 0$ for $0 \le \theta < \frac{\pi}{2}$, so $\theta = \frac{\pi}{6}$ gives a maximum time.

57. There are $(6-x)$ km over land and $\sqrt{x^2+4}$ km under the river.



We need to minimize the cost C (measured in \$100,000) of the pipeline.

$C(x) = (6-x)(4) + \left(\sqrt{x^2+4}\right)(8)$ $\Rightarrow$

$C'(x) = -4 + 8 \cdot \frac{1}{2}(x^2+4)^{-1/2}(2x) = -4 + \dfrac{8x}{\sqrt{x^2+4}}$.

$C'(x) = 0$ $\Rightarrow$ $4 = \dfrac{8x}{\sqrt{x^2+4}}$ $\Rightarrow$ $\sqrt{x^2+4} = 2x$ $\Rightarrow$ $x^2 + 4 = 4x^2$ $\Rightarrow$ $4 = 3x^2$ $\Rightarrow$ $x^2 = \frac{4}{3}$ $\Rightarrow$

$x = 2/\sqrt{3}$ $[0 \le x \le 6]$. Compare the costs for $x = 0$, $2/\sqrt{3}$, and 6. $C(0) = 24 + 16 = 40$,

$C(2/\sqrt{3}) = 24 - 8/\sqrt{3} + 32/\sqrt{3} = 24 + 24/\sqrt{3} \approx 37.9$, and $C(6) = 0 + 8\sqrt{40} \approx 50.6$. So the minimum cost is about \$3.79 million when P is $6 - 2/\sqrt{3} \approx 4.85$ km east of the refinery.

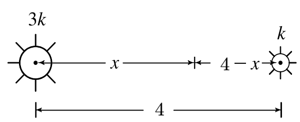
58. The distance from the refinery to P is now $\sqrt{(6-x)^2 + 1^2} = \sqrt{x^2 - 12x + 37}$.

Thus, $C(x) = 4\sqrt{x^2 - 12x + 37} + 8\sqrt{x^2 + 4} \Rightarrow$

$C'(x) = 4 \cdot \frac{1}{2}(x^2 - 12x + 37)^{-1/2}(2x - 12) + 8 \cdot \frac{1}{2}(x^2 + 4)^{-1/2}(2x) = \dfrac{4(x-6)}{\sqrt{x^2 - 12x + 37}} + \dfrac{8x}{\sqrt{x^2 + 4}}$.

$C'(x) = 0 \Rightarrow x \approx 1.12$ [from a graph of C' or a numerical rootfinder]. $C(0) \approx 40.3$, $C(1.12) \approx 38.3$, and $C(6) \approx 54.6$. So the minimum cost is slightly higher (than in the previous exercise) at about \$3.83 million when P is approximately 4.88 km from the point on the bank 1 km south of the refinery.

59.



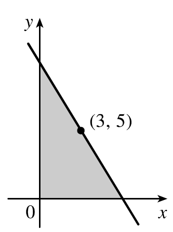
The total illumination is $I(x) = \dfrac{3k}{x^2} + \dfrac{k}{(4-x)^2}$, $0 < x < 10$. Then

$I'(x) = \dfrac{-6k}{x^3} + \dfrac{2k}{(4-x)^3} = 0 \Rightarrow 6k(4-x)^3 = 2kx^3 \Rightarrow$

$3(4-x)^3 = x^3 \Rightarrow \sqrt[3]{3}\,(4-x) = x \Rightarrow 4\sqrt[3]{3} - \sqrt[3]{3}\,x = x \Rightarrow 4\sqrt[3]{3} = x + \sqrt[3]{3}\,x \Rightarrow$

$4\sqrt[3]{3} = (1 + \sqrt[3]{3})x \Rightarrow x = \dfrac{4\sqrt[3]{3}}{1 + \sqrt[3]{3}} \approx 2.36$ m. This gives a minimum since $I''(x) > 0$ for $0 < x < 4$.

60.

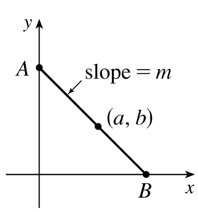


The line with slope m (where $m < 0$) through $(3, 5)$ has equation $y - 5 = m(x - 3)$ or $y = mx + (5 - 3m)$. The y-intercept is $5 - 3m$ and the x-intercept is $-5/m + 3$. So the triangle has area $A(m) = \frac{1}{2}(5 - 3m)(-5/m + 3) = 15 - 25/(2m) - \frac{9}{2}m$. Now

$A'(m) = \dfrac{25}{2m^2} - \dfrac{9}{2} = 0 \Leftrightarrow m^2 = \frac{25}{9} \Rightarrow m = -\frac{5}{3}$ (since $m < 0$).

$A''(m) = -\dfrac{25}{m^3} > 0$, so there is an absolute minimum when $m = -\frac{5}{3}$. Thus, an equation of the line is $y - 5 = -\frac{5}{3}(x - 3)$ or $y = -\frac{5}{3}x + 10$.

61.



Every line segment in the first quadrant passing through (a, b) with endpoints on the x- and y-axes satisfies an equation of the form $y - b = m(x - a)$, where $m < 0$. By setting $x = 0$ and then $y = 0$, we find its endpoints, $A(0, b - am)$ and $B\left(a - \frac{b}{m}, 0\right)$. The distance d from A to B is given by $d = \sqrt{\left[\left(a - \frac{b}{m}\right) - 0\right]^2 + [0 - (b - am)]^2}$.

It follows that the square of the length of the line segment, as a function of m, is given by

$S(m) = \left(a - \dfrac{b}{m}\right)^2 + (am - b)^2 = a^2 - \dfrac{2ab}{m} + \dfrac{b^2}{m^2} + a^2m^2 - 2abm + b^2$. Thus,

$$S'(m) = \frac{2ab}{m^2} - \frac{2b^2}{m^3} + 2a^2m - 2ab = \frac{2}{m^3}(abm - b^2 + a^2m^4 - abm^3)$$
$$= \frac{2}{m^3}[b(am - b) + am^3(am - b)] = \frac{2}{m^3}(am - b)(b + am^3)$$

Thus, $S'(m) = 0 \Leftrightarrow m = b/a$ or $m = -\sqrt[3]{\frac{b}{a}}$. Since $b/a > 0$ and $m < 0$, m must equal $-\sqrt[3]{\frac{b}{a}}$. Since $\dfrac{2}{m^3} < 0$, we see that $S'(m) < 0$ for $m < -\sqrt[3]{\frac{b}{a}}$ and $S'(m) > 0$ for $m > -\sqrt[3]{\frac{b}{a}}$. Thus, S has its absolute minimum value when $m = -\sqrt[3]{\frac{b}{a}}$.

That value is

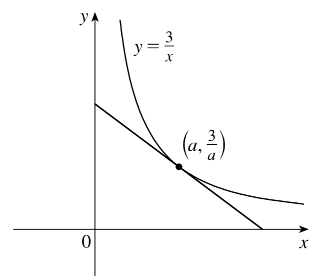
$$S\left(-\sqrt[3]{\tfrac{b}{a}}\right) = \left(a + b\sqrt[3]{\tfrac{a}{b}}\right)^2 + \left(-a\sqrt[3]{\tfrac{b}{a}} - b\right)^2 = \left(a + \sqrt[3]{ab^2}\right)^2 + \left(\sqrt[3]{a^2 b} + b\right)^2$$

$$= a^2 + 2a^{4/3}b^{2/3} + a^{2/3}b^{4/3} + a^{4/3}b^{2/3} + 2a^{2/3}b^{4/3} + b^2 = a^2 + 3a^{4/3}b^{2/3} + 3a^{2/3}b^{4/3} + b^2$$

The last expression is of the form $x^3 + 3x^2y + 3xy^2 + y^3 \quad [= (x+y)^3]$ with $x = a^{2/3}$ and $y = b^{2/3}$, so we can write it as $(a^{2/3} + b^{2/3})^3$ and the shortest such line segment has length $\sqrt{S} = (a^{2/3} + b^{2/3})^{3/2}$.

62. $y = 1 + 40x^3 - 3x^5 \quad \Rightarrow \quad y' = 120x^2 - 15x^4$, so the tangent line to the curve at $x = a$ has slope $m(a) = 120a^2 - 15a^4$. Now $m'(a) = 240a - 60a^3 = -60a(a^2 - 4) = -60a(a+2)(a-2)$, so $m'(a) > 0$ for $a < -2$, and $0 < a < 2$, and $m'(a) < 0$ for $-2 < a < 0$ and $a > 2$. Thus, m is increasing on $(-\infty, -2)$, decreasing on $(-2, 0)$, increasing on $(0, 2)$, and decreasing on $(2, \infty)$. Clearly, $m(a) \to -\infty$ as $a \to \pm\infty$, so the maximum value of $m(a)$ must be one of the two local maxima, $m(-2)$ or $m(2)$. But both $m(-2)$ and $m(2)$ equal $120 \cdot 2^2 - 15 \cdot 2^4 = 480 - 240 = 240$. So 240 is the largest slope, and it occurs at the points $(-2, -223)$ and $(2, 225)$. *Note:* $a = 0$ corresponds to a local *minimum* of m.

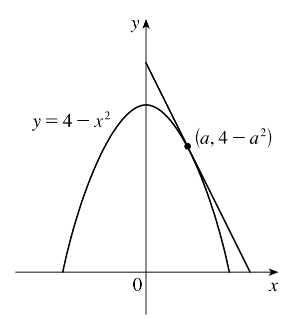
63. $y = \dfrac{3}{x} \quad \Rightarrow \quad y' = -\dfrac{3}{x^2}$, so an equation of the tangent line at the point $(a, \frac{3}{a})$ is $y - \dfrac{3}{a} = -\dfrac{3}{a^2}(x - a)$, or $y = -\dfrac{3}{a^2}x + \dfrac{6}{a}$. The y-intercept $\quad [x = 0] \quad$ is $6/a$. The x-intercept $\quad [y = 0] \quad$ is $2a$. The distance d of the line segment that has endpoints at the intercepts is $d = \sqrt{(2a - 0)^2 + (0 - 6/a)^2}$. Let $S = d^2$, so $S = 4a^2 + \dfrac{36}{a^2} \quad \Rightarrow$



$S' = 8a - \dfrac{72}{a^3}$. $S' = 0 \quad \Leftrightarrow \quad \dfrac{72}{a^3} = 8a \quad \Leftrightarrow \quad a^4 = 9 \quad \Leftrightarrow \quad a^2 = 3 \quad \Rightarrow \quad a = \sqrt{3}$.

$S'' = 8 + \dfrac{216}{a^4} > 0$, so there is an absolute minimum at $a = \sqrt{3}$. Thus, $S = 4(3) + \frac{36}{3} = 12 + 12 = 24$ and hence, $d = \sqrt{24} = 2\sqrt{6}$.

64. $y = 4 - x^2 \quad \Rightarrow \quad y' = -2x$, so an equation of the tangent line at $(a, 4 - a^2)$ is $y - (4 - a^2) = -2a(x - a)$, or $y = -2ax + a^2 + 4$. The y-intercept $\quad [x = 0]$ is $a^2 + 4$. The x-intercept $\quad [y = 0] \quad$ is $\dfrac{a^2 + 4}{2a}$. The area A of the triangle is



$$A = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2} \cdot \frac{a^2 + 4}{2a}(a^2 + 4) = \frac{1}{4}\frac{a^4 + 8a^2 + 16}{a} = \frac{1}{4}\left(a^3 + 8a + \frac{16}{a}\right).$$

$$A' = 0 \quad \Rightarrow \quad \frac{1}{4}\left(3a^2 + 8 - \frac{16}{a^2}\right) = 0 \quad \Rightarrow \quad 3a^4 + 8a^2 - 16 = 0 \quad \Rightarrow$$

$(3a^2 - 4)(a^2 + 4) = 0 \quad \Rightarrow \quad a^2 = \dfrac{4}{3} \quad \Rightarrow \quad a = \dfrac{2}{\sqrt{3}}$. $A'' = \dfrac{1}{4}\left(6a + \dfrac{32}{a^3}\right) > 0$, so there is an absolute minimum at $a = \dfrac{2}{\sqrt{3}}$. Thus, $A = \dfrac{1}{2} \cdot \dfrac{4/3 + 4}{2(2/\sqrt{3})}\left(\dfrac{4}{3} + 4\right) = \dfrac{1}{2} \cdot \dfrac{4\sqrt{3}}{3} \cdot \dfrac{16}{3} = \dfrac{32}{9}\sqrt{3}$.

65. (a) If $c(x) = \dfrac{C(x)}{x}$, then, by the Quotient Rule, we have $c'(x) = \dfrac{xC'(x) - C(x)}{x^2}$. Now $c'(x) = 0$ when

$xC'(x) - C(x) = 0$ and this gives $C'(x) = \dfrac{C(x)}{x} = c(x)$. Therefore, the marginal cost equals the average cost.

(b) (i) $C(x) = 16{,}000 + 200x + 4x^{3/2}$, $C(1000) = 16{,}000 + 200{,}000 + 40{,}000\sqrt{10} \approx 216{,}000 + 126{,}491$, so

$C(1000) \approx \$342{,}491$. $c(x) = C(x)/x = \dfrac{16{,}000}{x} + 200 + 4x^{1/2}$, $c(1000) \approx \$342.49/\text{unit}$. $C'(x) = 200 + 6x^{1/2}$,

$C'(1000) = 200 + 60\sqrt{10} \approx \$389.74/\text{unit}$.

(ii) We must have $C'(x) = c(x) \quad\Leftrightarrow\quad 200 + 6x^{1/2} = \dfrac{16{,}000}{x} + 200 + 4x^{1/2} \quad\Leftrightarrow\quad 2x^{3/2} = 16{,}000 \quad\Leftrightarrow$

$x = (8{,}000)^{2/3} = 400$ units. To check that this is a minimum, we calculate

$c'(x) = \dfrac{-16{,}000}{x^2} + \dfrac{2}{\sqrt{x}} = \dfrac{2}{x^2}\left(x^{3/2} - 8000\right)$. This is negative for $x < (8000)^{2/3} = 400$, zero at $x = 400$,

and positive for $x > 400$, so c is decreasing on $(0, 400)$ and increasing on $(400, \infty)$. Thus, c has an absolute minimum at $x = 400$. [*Note:* $c''(x)$ is *not* positive for all $x > 0$.]

(iii) The minimum average cost is $c(400) = 40 + 200 + 80 = \$320/\text{unit}$.

66. (a) The total profit is $P(x) = R(x) - C(x)$. In order to maximize profit we look for the critical numbers of P, that is, the numbers where the marginal profit is 0. But if $P'(x) = R'(x) - C'(x) = 0$, then $R'(x) = C'(x)$. Therefore, if the profit is a maximum, then the marginal revenue equals the marginal cost.

(b) $C(x) = 16{,}000 + 500x - 1.6x^2 + 0.004x^3$, $p(x) = 1700 - 7x$. Then $R(x) = xp(x) = 1700x - 7x^2$. If the profit is maximum, then $R'(x) = C'(x) \quad\Leftrightarrow\quad 1700 - 14x = 500 - 3.2x + 0.012x^2 \quad\Leftrightarrow\quad 0.012x^2 + 10.8x - 1200 = 0 \quad\Leftrightarrow$

$x^2 + 900x - 100{,}000 = 0 \quad\Leftrightarrow\quad (x + 1000)(x - 100) = 0 \quad\Leftrightarrow\quad x = 100$ (since $x > 0$). The profit is maximized if $P''(x) < 0$, but since $P''(x) = R''(x) - C''(x)$, we can just check the condition $R''(x) < C''(x)$. Now $R''(x) = -14 < -3.2 + 0.024x = C''(x)$ for $x > 0$, so there is a maximum at $x = 100$.

67. (a) We are given that the demand function p is linear and $p(27{,}000) = 10$, $p(33{,}000) = 8$, so the slope is

$\frac{10 - 8}{27{,}000 - 33{,}000} = -\frac{1}{3000}$ and an equation of the line is $y - 10 = \left(-\frac{1}{3000}\right)(x - 27{,}000) \quad\Rightarrow$

$y = p(x) = -\frac{1}{3000}x + 19 = 19 - (x/3000)$.

(b) The revenue is $R(x) = xp(x) = 19x - (x^2/3000) \quad\Rightarrow\quad R'(x) = 19 - (x/1500) = 0$ when $x = 28{,}500$. Since $R''(x) = -1/1500 < 0$, the maximum revenue occurs when $x = 28{,}500 \quad\Rightarrow\quad$ the price is $p(28{,}500) = \$9.50$.

68. (a) Let $p(x)$ be the demand function. Then $p(x)$ is linear and $y = p(x)$ passes through $(20, 10)$ and $(18, 11)$, so the slope is $-\frac{1}{2}$ and an equation of the line is $y - 10 = -\frac{1}{2}(x - 20) \quad\Leftrightarrow\quad y = -\frac{1}{2}x + 20$. Thus, the demand is $p(x) = -\frac{1}{2}x + 20$ and the revenue is $R(x) = xp(x) = -\frac{1}{2}x^2 + 20x$.

(b) The cost is $C(x) = 6x$, so the profit is $P(x) = R(x) - C(x) = -\frac{1}{2}x^2 + 14x$. Then $0 = P'(x) = -x + 14 \quad\Rightarrow$

$x = 14$. Since $P''(x) = -1 < 0$, the selling price for maximum profit is $p(14) = -\frac{1}{2}(14) + 20 = \$13$.

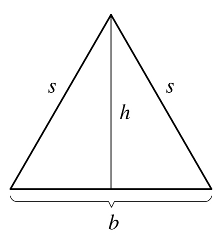
69. (a) As in Example 6, we see that the demand function p is linear. We are given that $p(1200) = 350$ and deduce that $p(1280) = 340$, since a \$10 reduction in price increases sales by 80 per week. The slope for p is $\dfrac{340 - 350}{1280 - 1200} = -\dfrac{1}{8}$, so an equation is $p - 350 = -\frac{1}{8}(x - 1200)$ or $p(x) = -\frac{1}{8}x + 500$, where $x \geq 1200$.

(b) $R(x) = x\,p(x) = -\frac{1}{8}x^2 + 500x$. $R'(x) = -\frac{1}{4}x + 500 = 0$ when $x = 4(500) = 2000$. $p(2000) = 250$, so the price should be set at \$250 to maximize revenue.

(c) $C(x) = 35{,}000 + 120x \;\Rightarrow\; P(x) = R(x) - C(x) = -\frac{1}{8}x^2 + 500x - 35{,}000 - 120x = -\frac{1}{8}x^2 + 380x - 35{,}000$. $P'(x) = -\frac{1}{4}x + 380 = 0$ when $x = 4(380) = 1520$. $p(1520) = 310$, so the price should be set at \$310 to maximize profit.

70. Let w denote the number of operating wells. Then the amount of daily oil production for each well is $240 - 8(w - 16) = 368 - 8w$, where $w \geq 16$. The total daily oil production P for all wells is given by $P(w) = w(368 - 8w) = 368w - 8w^2$. Now $P'(w) = 368 - 16w$ and $P'(w) = 0 \;\Leftrightarrow\; w = \frac{368}{16} = 23$. $P''(w) = -16 < 0$, so the daily production is maximized when the company adds $23 - 16 = 7$ wells.

71.



Here $s^2 = h^2 + b^2/4$, so $h^2 = s^2 - b^2/4$. The area is $A = \frac{1}{2}b\sqrt{s^2 - b^2/4}$. Let the perimeter be p, so $2s + b = p$ or $s = (p - b)/2 \;\Rightarrow$ $A(b) = \frac{1}{2}b\sqrt{(p - b)^2/4 - b^2/4} = b\sqrt{p^2 - 2pb}/4$. Now

$$A'(b) = \frac{\sqrt{p^2 - 2pb}}{4} - \frac{bp/4}{\sqrt{p^2 - 2pb}} = \frac{-3pb + p^2}{4\sqrt{p^2 - 2pb}}.$$

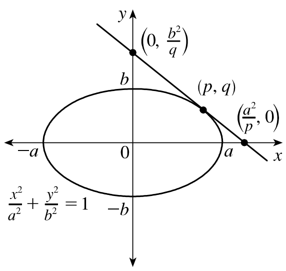
Therefore, $A'(b) = 0 \;\Rightarrow\; -3pb + p^2 = 0 \;\Rightarrow\; b = p/3$. Since $A'(b) > 0$ for $b < p/3$ and $A'(b) < 0$ for $b > p/3$, there is an absolute maximum when $b = p/3$. But then $2s + p/3 = p$, so $s = p/3 \;\Rightarrow\; s = b \;\Rightarrow$ the triangle is equilateral.

72. From Exercise 57, with K replacing 8 for the "under river" cost (measured in \$100,000), we see that $C'(x) = 0 \;\Leftrightarrow$ $4\sqrt{x^2 + 4} = Kx \;\Leftrightarrow\; 16x^2 + 64 = K^2x^2 \;\Leftrightarrow\; 64 = (K^2 - 16)x^2 \;\Leftrightarrow\; x = \dfrac{8}{\sqrt{K^2 - 16}}$. Also from Exercise 57, we have $C(x) = (6 - x)4 + \sqrt{x^2 + 4}\,K$. We now compare costs for using the minimum distance possible under the river $[x = 0]$ and using the critical number above. $C(0) = 24 + 2K$ and

$$\begin{aligned} C\left(\frac{8}{\sqrt{K^2 - 16}}\right) &= 24 - \frac{32}{\sqrt{K^2 - 16}} + \sqrt{\frac{64}{K^2 - 16} + 4}\,K = 24 - \frac{32}{\sqrt{K^2 - 16}} + \sqrt{\frac{4K^2}{K^2 - 16}}\,K \\ &= 24 - \frac{32}{\sqrt{K^2 - 16}} + \frac{2K^2}{\sqrt{K^2 - 16}} = 24 + \frac{2(K^2 - 16)}{\sqrt{K^2 - 16}} = 24 + 2\sqrt{K^2 - 16} \end{aligned}$$

Since $\sqrt{K^2 - 16} < K$, we see that $C\left(\dfrac{8}{\sqrt{K^2 - 16}}\right) < C(0)$ for any cost K, so the minimum distance possible for the "under river" portion of the pipeline should *never* be used.

73. (a)



Using implicit differentiation, $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1 \;\Rightarrow\; \dfrac{2x}{a^2}+\dfrac{2y\,y'}{b^2}=0 \;\Rightarrow$ $\dfrac{2y\,y'}{b^2}=-\dfrac{2x}{a^2} \;\Rightarrow\; y'=-\dfrac{b^2x}{a^2y}$. At (p,q), $y'=-\dfrac{b^2p}{a^2q}$, and an equation of the tangent line is $y-q=-\dfrac{b^2p}{a^2q}(x-p) \;\Leftrightarrow\; y=-\dfrac{b^2p}{a^2q}x+\dfrac{b^2p^2}{a^2q}+q \;\Leftrightarrow$ $y=-\dfrac{b^2p}{a^2q}x+\dfrac{b^2p^2+a^2q^2}{a^2q}$. The last term is the y-intercept, but not the term we want, namely b^2/q. Since (p,q) is on the ellipse, we know $\dfrac{p^2}{a^2}+\dfrac{q^2}{b^2}=1$. To use that relationship we must divide b^2p^2 in the y-intercept by a^2b^2, so divide all terms by a^2b^2. $\dfrac{(b^2p^2+a^2q^2)/a^2b^2}{(a^2q)/a^2b^2}=\dfrac{p^2/a^2+q^2/b^2}{q/b^2}=\dfrac{1}{q/b^2}=\dfrac{b^2}{q}$. So the tangent line has equation $y=-\dfrac{b^2p}{a^2q}x+\dfrac{b^2}{q}$. Let $y=0$ and solve for x to find that x-intercept: $\dfrac{b^2p}{a^2q}x=\dfrac{b^2}{q} \;\Leftrightarrow$ $x=\dfrac{b^2a^2q}{qb^2p}=\dfrac{a^2}{p}$.

(b) The portion of the tangent line cut off by the coordinate axes is the distance between the intercepts, $(a^2/p,0)$ and $(0,b^2/q)$: $\sqrt{\left(\dfrac{a^2}{p}\right)^2+\left(-\dfrac{b^2}{q}\right)^2}=\sqrt{\dfrac{a^4}{p^2}+\dfrac{b^4}{q^2}}$. To eliminate p or q, we turn to the relationship $\dfrac{p^2}{a^2}+\dfrac{q^2}{b^2}=1 \;\Leftrightarrow$ $\dfrac{q^2}{b^2}=1-\dfrac{p^2}{a^2} \;\Leftrightarrow\; q^2=b^2-\dfrac{b^2p^2}{a^2} \;\Leftrightarrow\; q^2=\dfrac{b^2(a^2-p^2)}{a^2}$. Now substitute for q^2 and use the square S of the distance. $S(p)=\dfrac{a^4}{p^2}+\dfrac{b^4a^2}{b^2(a^2-p^2)}=\dfrac{a^4}{p^2}+\dfrac{a^2b^2}{a^2-p^2}$ for $0<p<a$. Note that as $p\to 0$ or $p\to a$, $S(p)\to\infty$, so the minimum value of S must occur at a critical number. Now $S'(p)=-\dfrac{2a^4}{p^3}+\dfrac{2a^2b^2p}{(a^2-p^2)^2}$ and $S'(p)=0 \;\Leftrightarrow$

$$\frac{2a^4}{p^3}=\frac{2a^2b^2p}{(a^2-p^2)^2} \;\Leftrightarrow\; a^2(a^2-p^2)^2=b^2p^4 \;\Rightarrow\; a(a^2-p^2)=bp^2 \;\Leftrightarrow\; a^3=(a+b)p^2 \;\Leftrightarrow\; p^2=\frac{a^3}{a+b}.$$

Substitute for p^2 in $S(p)$:

$$\frac{a^4}{\dfrac{a^3}{a+b}}+\frac{a^2b^2}{a^2-\dfrac{a^3}{a+b}}=\frac{a^4(a+b)}{a^3}+\frac{a^2b^2(a+b)}{a^2(a+b)-a^3}=\frac{a(a+b)}{1}+\frac{a^2b^2(a+b)}{a^2b}$$
$$=a(a+b)+b(a+b)=(a+b)(a+b)=(a+b)^2$$

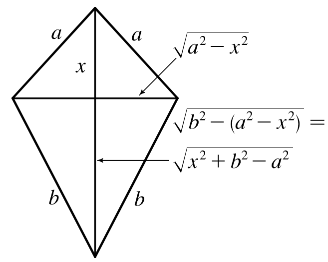
Taking the square root gives us the desired minimum length of $a+b$.

(c) The triangle formed by the tangent line and the coordinate axes has area $A=\dfrac{1}{2}\left(\dfrac{a^2}{p}\right)\left(\dfrac{b^2}{q}\right)$. As in part (b), we'll use the square of the area and substitute for q^2. $S=\dfrac{a^4b^4}{4p^2q^2}=\dfrac{a^4b^4a^2}{4p^2b^2(a^2-p^2)}=\dfrac{a^6b^2}{4p^2(a^2-p^2)}$. Minimizing S (and hence A) is equivalent to maximizing $p^2(a^2-p^2)$. Let $f(p)=p^2(a^2-p^2)=a^2p^2-p^4$ for $0<p<a$. As in part (b), the

minimum value of S must occur at a critical number. Now $f'(p) = 2a^2p - 4p^3 = 2p(a^2 - 2p^2)$. $f'(p) = 0 \Rightarrow$ $p^2 = a^2/2 \Rightarrow p = a/\sqrt{2}$ $[p > 0]$. Substitute for p^2 in $S(p)$: $\dfrac{a^6b^2}{4\left(\frac{a^2}{2}\right)\left(a^2 - \frac{a^2}{2}\right)} = \dfrac{a^6b^2}{a^4} = a^2b^2 = (ab)^2$.

Taking the square root gives us the desired minimum area of ab.

74. See the figure. The area is given by



$$A(x) = \tfrac{1}{2}\left(2\sqrt{a^2 - x^2}\right)x + \tfrac{1}{2}\left(2\sqrt{a^2 - x^2}\right)\left(\sqrt{x^2 + b^2 - a^2}\right)$$
$$= \sqrt{a^2 - x^2}\left(x + \sqrt{x^2 + b^2 - a^2}\right)$$

for $0 \le x \le a$. Now

$$A'(x) = \sqrt{a^2 - x^2}\left(1 + \frac{x}{\sqrt{x^2 + b^2 - a^2}}\right) + \left(x + \sqrt{x^2 + b^2 - a^2}\right)\frac{-x}{\sqrt{a^2 - x^2}}$$
$$= 0 \quad \Leftrightarrow$$
$$\frac{x}{\sqrt{a^2 - x^2}}\left(x + \sqrt{x^2 + b^2 - a^2}\right) = \sqrt{a^2 - x^2}\left(\frac{x + \sqrt{x^2 + b^2 - a^2}}{\sqrt{x^2 + b^2 - a^2}}\right).$$

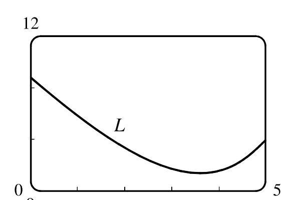
Except for the trivial case where $x = 0$, $a = b$ and $A(x) = 0$, we have $x + \sqrt{x^2 + b^2 - a^2} > 0$. Hence, cancelling this factor gives $\dfrac{x}{\sqrt{a^2 - x^2}} = \dfrac{\sqrt{a^2 - x^2}}{\sqrt{x^2 + b^2 - a^2}} \Rightarrow x\sqrt{x^2 + b^2 - a^2} = a^2 - x^2 \Rightarrow$ $x^2(x^2 + b^2 - a^2) = a^4 - 2a^2x^2 + x^4 \Rightarrow x^2(b^2 - a^2) = a^4 - 2a^2x^2 \Rightarrow x^2(b^2 + a^2) = a^4 \Rightarrow x = \dfrac{a^2}{\sqrt{a^2 + b^2}}$.

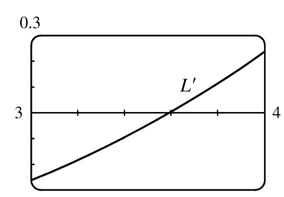
Now we must check the value of A at this point as well as at the endpoints of the domain to see which gives the maximum value. $A(0) = a\sqrt{b^2 - a^2}$, $A(a) = 0$ and

$$A\left(\frac{a^2}{\sqrt{a^2 + b^2}}\right) = \sqrt{a^2 - \left(\frac{a^2}{\sqrt{a^2 + b^2}}\right)^2}\left[\frac{a^2}{\sqrt{a^2 + b^2}} + \sqrt{\left(\frac{a^2}{\sqrt{a^2 + b^2}}\right)^2 + b^2 - a^2}\right]$$
$$= \frac{ab}{\sqrt{a^2 + b^2}}\left[\frac{a^2}{\sqrt{a^2 + b^2}} + \frac{b^2}{\sqrt{a^2 + b^2}}\right] = \frac{ab(a^2 + b^2)}{a^2 + b^2} = ab$$

Since $b \ge \sqrt{b^2 - a^2}$, $A\left(a^2/\sqrt{a^2 + b^2}\right) \ge A(0)$. So there is an absolute maximum when $x = \dfrac{a^2}{\sqrt{a^2 + b^2}}$. In this case the horizontal piece should be $\dfrac{2ab}{\sqrt{a^2 + b^2}}$ and the vertical piece should be $\dfrac{a^2 + b^2}{\sqrt{a^2 + b^2}} = \sqrt{a^2 + b^2}$.

75. Note that $|AD| = |AP| + |PD| \Rightarrow 5 = x + |PD| \Rightarrow |PD| = 5 - x$.





Using the Pythagorean Theorem for ΔPDB and ΔPDC gives us

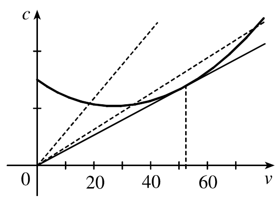
$$L(x) = |AP| + |BP| + |CP| = x + \sqrt{(5 - x)^2 + 2^2} + \sqrt{(5 - x)^2 + 3^2}$$
$$= x + \sqrt{x^2 - 10x + 29} + \sqrt{x^2 - 10x + 34} \quad \Rightarrow$$

$L'(x) = 1 + \dfrac{x - 5}{\sqrt{x^2 - 10x + 29}} + \dfrac{x - 5}{\sqrt{x^2 - 10x + 34}}$. From the graphs of L and L', it seems that the minimum value of L is about $L(3.59) = 9.35$ m.

76. We note that since c is the consumption in liters per hour, and v is the velocity in kilometers per hour, then

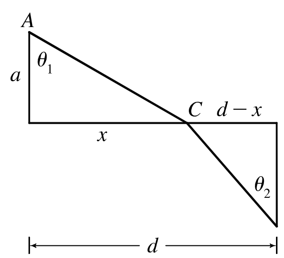
$\dfrac{c}{v} = \dfrac{\text{liters/hour}}{\text{kilometers/hour}} = \dfrac{\text{liters}}{\text{kilometers}}$ gives us the consumption in liters per kilometers, that is, the quantity G. To find the minimum,

we calculate $\dfrac{dG}{dv} = \dfrac{d}{dv}\left(\dfrac{c}{v}\right) = \dfrac{v\,\dfrac{dc}{dv} - c\,\dfrac{dv}{dv}}{v^2} = \dfrac{v\,\dfrac{dc}{dv} - c}{v^2}$.



This is 0 when $v\,\dfrac{dc}{dv} - c = 0 \quad\Leftrightarrow\quad \dfrac{dc}{dv} = \dfrac{c}{v}$. This implies that the tangent line of $c(v)$ passes through the origin, and this occurs when $v \approx 48$ km/h. Note that the slope of the secant line through the origin and a point $(v, c(v))$ on the graph is equal to $G(v)$, and it is intuitively clear that G is minimized in the case where the secant is in fact a tangent.

77.



The total time is

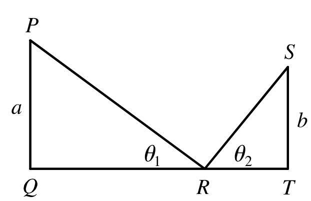
$$T(x) = (\text{time from } A \text{ to } C) + (\text{time from } C \text{ to } B)$$
$$= \frac{\sqrt{a^2+x^2}}{v_1} + \frac{\sqrt{b^2+(d-x)^2}}{v_2}, \quad 0 < x < d$$

$$T'(x) = \frac{x}{v_1\sqrt{a^2+x^2}} - \frac{d-x}{v_2\sqrt{b^2+(d-x)^2}} = \frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2}$$

The minimum occurs when $T'(x) = 0 \quad\Rightarrow\quad \dfrac{\sin\theta_1}{v_1} = \dfrac{\sin\theta_2}{v_2}$, or,

equivalently, $\dfrac{\sin\theta_1}{\sin\theta_2} = \dfrac{v_1}{v_2}$. [*Note:* $T''(x) > 0$]

78.



If $d = |QT|$, we minimize $f(\theta_1) = |PR| + |RS| = a\csc\theta_1 + b\csc\theta_2$.

Differentiating with respect to θ_1, and setting $\dfrac{df}{d\theta_1}$ equal to 0, we get

$$\frac{df}{d\theta_1} = 0 = -a\csc\theta_1\,\cot\theta_1 - b\csc\theta_2\,\cot\theta_2\,\frac{d\theta_2}{d\theta_1}.$$

So we need to find an expression for $\dfrac{d\theta_2}{d\theta_1}$. We can do this by observing that $|QT| = \text{constant} = a\cot\theta_1 + b\cot\theta_2$.

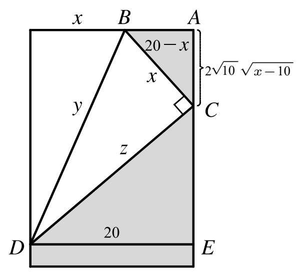
Differentiating this equation implicitly with respect to θ_1, we get $-a\csc^2\theta_1 - b\csc^2\theta_2\,\dfrac{d\theta_2}{d\theta_1} = 0 \quad\Rightarrow$

$\dfrac{d\theta_2}{d\theta_1} = -\dfrac{a\csc^2\theta_1}{b\csc^2\theta_2}$. We substitute this into the expression for $\dfrac{df}{d\theta_1}$ to get

$$-a\csc\theta_1\,\cot\theta_1 - b\csc\theta_2\,\cot\theta_2\left(-\frac{a\csc^2\theta_1}{b\csc^2\theta_2}\right) = 0 \quad\Leftrightarrow\quad -a\csc\theta_1\,\cot\theta_1 + a\frac{\csc^2\theta_1\,\cot\theta_2}{\csc\theta_2} = 0 \quad\Leftrightarrow$$

$\cot\theta_1\,\csc\theta_2 = \csc\theta_1\,\cot\theta_2 \quad\Leftrightarrow\quad \dfrac{\cot\theta_1}{\csc\theta_1} = \dfrac{\cot\theta_2}{\csc\theta_2} \quad\Leftrightarrow\quad \cos\theta_1 = \cos\theta_2$. Since θ_1 and θ_2 are both acute, we have $\theta_1 = \theta_2$.

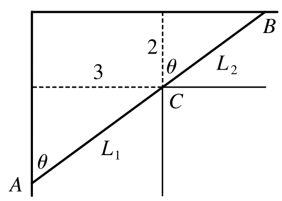
79.



$y^2 = x^2 + z^2$, but triangles CDE and BCA are similar, so $z/20 = x/2\sqrt{10}\sqrt{x-10} \Rightarrow z = \sqrt{10}\,x/\sqrt{x-10}$. Thus, we minimize $f(x) = y^2 = x^2 + 10x^2/(x-10) = x^3/(x-10)$, $10 < x \le 20$.

$f'(x) = \frac{(x-10)(3x^2)-x^3}{(x-10)^2} = \frac{x^2[3(x-10)-x]}{(x-10)^2} = \frac{2x^2(x-15)}{(x-10)^2} = 0$ when $x = 15$. $f'(x) < 0$ when $x < 15$, $f'(x) > 0$ when $x > 15$, so the minimum occurs when $x = 15$ cm.

80.

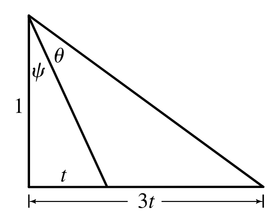


Paradoxically, we solve this maximum problem by solving a minimum problem. Let L be the length of the line ACB going from wall to wall touching the inner corner C. As $\theta \to 0$ or $\theta \to \frac{\pi}{2}$, we have $L \to \infty$ and there will be an angle that makes L a minimum. A pipe of this length will just fit around the corner.

From the diagram, $L = L_1 + L_2 = 3\csc\theta + 2\sec\theta \quad\Rightarrow\quad dL/d\theta = -3\csc\theta\,\cot\theta + 2\sec\theta\,\tan\theta = 0$ when $2\sec\theta\,\tan\theta = 3\csc\theta\,\cot\theta \quad\Leftrightarrow\quad \tan^3\theta = \frac{3}{2} = 1.5 \quad\Leftrightarrow\quad \tan\theta = \sqrt[3]{1.5}$. Then $\sec^2\theta = 1 + \left(\frac{3}{2}\right)^{2/3}$ and $\csc^2\theta = 1 + \left(\frac{3}{2}\right)^{-2/3}$, so the longest pipe has length $L = 3\left[1 + \left(\frac{3}{2}\right)^{-2/3}\right]^{1/2} + 2\left[1 + \left(\frac{3}{2}\right)^{2/3}\right]^{1/2} \approx 7.02\text{ m}$.

Or, use $\theta = \tan^{-1}\left(\sqrt[3]{1.5}\right) \approx 0.853 \quad\Rightarrow\quad L = 3\csc\theta + 2\sec\theta \approx 7.02\text{ m}$.

81.



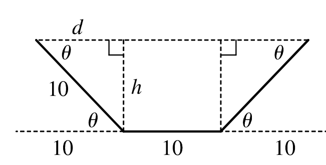
It suffices to maximize $\tan\theta$. Now

$$\frac{3t}{1} = \tan(\psi + \theta) = \frac{\tan\psi + \tan\theta}{1 - \tan\psi\,\tan\theta} = \frac{t + \tan\theta}{1 - t\tan\theta}. \text{ So}$$

$$3t(1 - t\tan\theta) = t + \tan\theta \quad\Rightarrow\quad 2t = (1 + 3t^2)\tan\theta \quad\Rightarrow\quad \tan\theta = \frac{2t}{1 + 3t^2}.$$

Let $f(t) = \tan\theta = \dfrac{2t}{1+3t^2} \quad\Rightarrow\quad f'(t) = \dfrac{2(1+3t^2) - 2t(6t)}{(1+3t^2)^2} = \dfrac{2(1-3t^2)}{(1+3t^2)^2} = 0 \quad\Leftrightarrow\quad 1 - 3t^2 = 0 \quad\Leftrightarrow$ $t = \frac{1}{\sqrt{3}}$ since $t \ge 0$. Now $f'(t) > 0$ for $0 \le t < \frac{1}{\sqrt{3}}$ and $f'(t) < 0$ for $t > \frac{1}{\sqrt{3}}$, so f has an absolute maximum when $t = \frac{1}{\sqrt{3}}$ and $\tan\theta = \dfrac{2(1/\sqrt{3})}{1 + 3(1/\sqrt{3})^2} = \dfrac{1}{\sqrt{3}} \quad\Rightarrow\quad \theta = \frac{\pi}{6}$. Substituting for t and θ in $3t = \tan(\psi + \theta)$ gives us $\sqrt{3} = \tan\left(\psi + \frac{\pi}{6}\right) \quad\Rightarrow\quad \psi = \frac{\pi}{6}$.

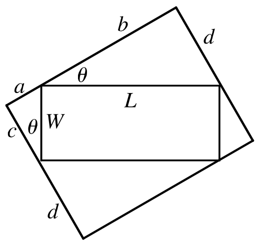
82. We maximize the cross-sectional area



$$\begin{aligned} A(\theta) &= 10h + 2\left(\tfrac{1}{2}dh\right) = 10h + dh = 10(10\sin\theta) + (10\cos\theta)(10\sin\theta) \\ &= 100(\sin\theta + \sin\theta\,\cos\theta),\ 0 \le \theta \le \tfrac{\pi}{2} \\ A'(\theta) &= 100(\cos\theta + \cos^2\theta - \sin^2\theta) = 100(\cos\theta + 2\cos^2\theta - 1) \\ &= 100(2\cos\theta - 1)(\cos\theta + 1) = 0 \text{ when } \cos\theta = \tfrac{1}{2} \quad\Leftrightarrow\quad \theta = \tfrac{\pi}{3} \qquad [\cos\theta \ne -1 \text{ since } 0 \le \theta \le \tfrac{\pi}{2}.] \end{aligned}$$

Now $A(0) = 0$, $A\left(\frac{\pi}{2}\right) = 100$ and $A\left(\frac{\pi}{3}\right) = 75\sqrt{3} \approx 129.9$, so the maximum occurs when $\theta = \frac{\pi}{3}$.

83.



In the small triangle with sides a and c and hypotenuse W, $\sin\theta = \dfrac{a}{W}$ and $\cos\theta = \dfrac{c}{W}$. In the triangle with sides b and d and hypotenuse L, $\sin\theta = \dfrac{d}{L}$ and $\cos\theta = \dfrac{b}{L}$. Thus, $a = W\sin\theta$, $c = W\cos\theta$, $d = L\sin\theta$, and $b = L\cos\theta$, so the area of the circumscribed rectangle is

$$\begin{aligned} A(\theta) &= (a+b)(c+d) = (W\sin\theta + L\cos\theta)(W\cos\theta + L\sin\theta) \\ &= W^2\sin\theta\,\cos\theta + WL\sin^2\theta + LW\cos^2\theta + L^2\sin\theta\,\cos\theta \\ &= LW\sin^2\theta + LW\cos^2\theta + (L^2+W^2)\sin\theta\,\cos\theta \\ &= LW(\sin^2\theta + \cos^2\theta) + (L^2+W^2)\cdot\tfrac{1}{2}\cdot 2\sin\theta\,\cos\theta = LW + \tfrac{1}{2}(L^2+W^2)\sin 2\theta,\ \ 0 \le \theta \le \tfrac{\pi}{2} \end{aligned}$$

This expression shows, without calculus, that the maximum value of $A(\theta)$ occurs when $\sin 2\theta = 1 \;\Leftrightarrow\; 2\theta = \frac{\pi}{2} \;\Rightarrow\; \theta = \frac{\pi}{4}$. So the maximum area is $A\left(\frac{\pi}{4}\right) = LW + \frac{1}{2}(L^2+W^2) = \frac{1}{2}(L^2 + 2LW + W^2) = \frac{1}{2}(L+W)^2$.

84. (a) Let D be the point such that $a = |AD|$. From the text figure, $\sin\theta = \dfrac{b}{|BC|} \;\Rightarrow\; |BC| = b\csc\theta$ and

$\cos\theta = \dfrac{|BD|}{|BC|} = \dfrac{a - |AB|}{|BC|} \;\Rightarrow\; |BC| = (a - |AB|)\sec\theta$. Eliminating $|BC|$ gives

$(a - |AB|)\sec\theta = b\csc\theta \;\Rightarrow\; b\cot\theta = a - |AB| \;\Rightarrow\; |AB| = a - b\cot\theta$. The total resistance is

$$R(\theta) = C\frac{|AB|}{r_1^4} + C\frac{|BC|}{r_2^4} = C\left(\frac{a - b\cot\theta}{r_1^4} + \frac{b\csc\theta}{r_2^4}\right).$$

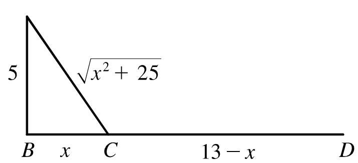
(b) $R'(\theta) = C\left(\dfrac{b\csc^2\theta}{r_1^4} - \dfrac{b\csc\theta\,\cot\theta}{r_2^4}\right) = bC\csc\theta\left(\dfrac{\csc\theta}{r_1^4} - \dfrac{\cot\theta}{r_2^4}\right)$.

$R'(\theta) = 0 \;\Leftrightarrow\; \dfrac{\csc\theta}{r_1^4} = \dfrac{\cot\theta}{r_2^4} \;\Leftrightarrow\; \dfrac{r_2^4}{r_1^4} = \dfrac{\cot\theta}{\csc\theta} = \cos\theta$.

$R'(\theta) > 0 \;\Leftrightarrow\; \dfrac{\csc\theta}{r_1^4} > \dfrac{\cot\theta}{r_2^4} \;\Rightarrow\; \cos\theta < \dfrac{r_2^4}{r_1^4}$ and $R'(\theta) < 0$ when $\cos\theta > \dfrac{r_2^4}{r_1^4}$, so there is an absolute minimum when $\cos\theta = r_2^4/r_1^4$.

(c) When $r_2 = \frac{2}{3}r_1$, we have $\cos\theta = \left(\frac{2}{3}\right)^4$, so $\theta = \cos^{-1}\left(\frac{2}{3}\right)^4 \approx 79^\circ$.

85. (a)



If k = energy/km over land, then energy/km over water $= 1.4k$. So the total energy is $E = 1.4k\sqrt{25+x^2} + k(13-x)$, $0 \le x \le 13$, and so $\dfrac{dE}{dx} = \dfrac{1.4kx}{(25+x^2)^{1/2}} - k$.

Set $\dfrac{dE}{dx} = 0$: $1.4kx = k(25+x^2)^{1/2} \;\Rightarrow\; 1.96x^2 = x^2 + 25 \;\Rightarrow\; 0.96x^2 = 25 \;\Rightarrow\; x = \frac{5}{\sqrt{0.96}} \approx 5.1$.

Testing against the value of E at the endpoints: $E(0) = 1.4k(5) + 13k = 20k$, $E(5.1) \approx 17.9k$, $E(13) \approx 19.5k$.

Thus, to minimize energy, the bird should fly to a point about 5.1 km from B.

(b) If W/L is large, the bird would fly to a point C that is closer to B than to D to minimize the energy used flying over water. If W/L is small, the bird would fly to a point C that is closer to D than to B to minimize the distance of the flight.

$E = W\sqrt{25+x^2} + L(13-x) \quad\Rightarrow\quad \dfrac{dE}{dx} = \dfrac{Wx}{\sqrt{25+x^2}} - L = 0$ when $\dfrac{W}{L} = \dfrac{\sqrt{25+x^2}}{x}$. By the same sort of argument as in part (a), this ratio will give the minimal expenditure of energy if the bird heads for the point x km from B.

(c) For flight direct to D, $x = 13$, so from part (b), $W/L = \frac{\sqrt{25+13^2}}{13} \approx 1.07$. There is no value of W/L for which the bird should fly directly to B. But note that $\lim\limits_{x\to 0^+}(W/L) = \infty$, so if the point at which E is a minimum is close to B, then W/L is large.

(d) Assuming that the birds instinctively choose the path that minimizes the energy expenditure, we can use the equation for $dE/dx = 0$ from part (a) with $1.4k = c$, $x = 4$, and $k = 1$: $c(4) = 1 \cdot (25+4^2)^{1/2} \quad\Rightarrow\quad c = \sqrt{41}/4 \approx 1.6$.

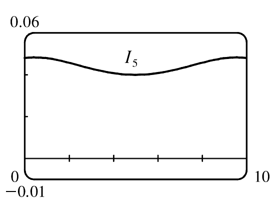
86. (a) $I(x) \propto \dfrac{\text{strength of source}}{(\text{distance from source})^2}$. Adding the intensities from the left and right lightbulbs,

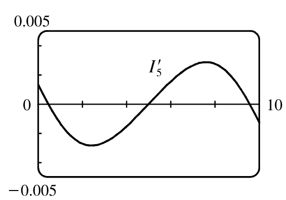
$$I(x) = \frac{k}{x^2+d^2} + \frac{k}{(10-x)^2+d^2} = \frac{k}{x^2+d^2} + \frac{k}{x^2-20x+100+d^2}.$$

(b) The magnitude of the constant k won't affect the location of the point of maximum intensity, so for convenience we take $k = 1$. $I'(x) = -\dfrac{2x}{(x^2+d^2)^2} - \dfrac{2(x-10)}{(x^2-20x+100+d^2)^2}$.

Substituting $d = 5$ into the equations for $I(x)$and $I'(x)$, we get

$$I_5(x) = \frac{1}{x^2+25} + \frac{1}{x^2-20x+125} \qquad \text{and} \qquad I_5'(x) = -\frac{2x}{(x^2+25)^2} - \frac{2(x-10)}{(x^2-20x+125)^2}$$

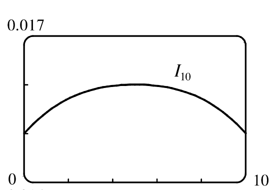


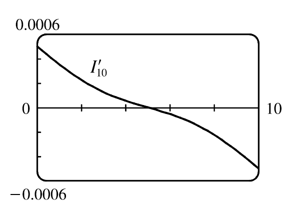


From the graphs, it appears that $I_5(x)$ has a minimum at $x = 5$ m.

(c) Substituting $d = 10$ into the equations for $I(x)$ and $I'(x)$ gives

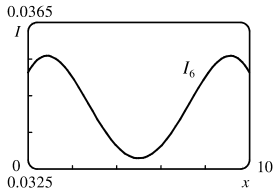
$$I_{10}(x) = \frac{1}{x^2+100} + \frac{1}{x^2-20x+200} \qquad \text{and} \qquad I_{10}'(x) = -\frac{2x}{(x^2+100)^2} - \frac{2(x-10)}{(x^2-20x+200)^2}$$

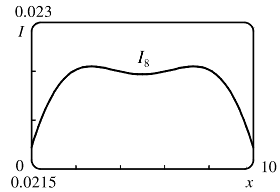


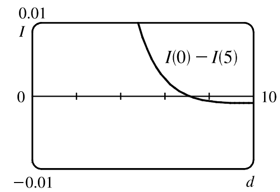


From the graphs, it seems that for $d = 10$, the intensity is minimized at the endpoints, that is, $x = 0$ and $x = 10$. The midpoint is now the most brightly lit point!

(d) From the first figures in parts (b) and (c), we see that the minimal illumination changes from the midpoint ($x = 5$ with $d = 5$) to the endpoints ($x = 0$ and $x = 10$ with $d = 10$).







So we try $d = 6$ (see the first figure) and we see that the minimum value still occurs at $x = 5$. Next, we let $d = 8$ (see the second figure) and we see that the minimum value occurs at the endpoints. It appears that for some value of d between 6 and 8, we must have minima at both the midpoint and the endpoints, that is, $I(5)$ must equal $I(0)$. To find this value of d, we solve $I(0) = I(5)$ (with $k = 1$):

$$\frac{1}{d^2} + \frac{1}{100 + d^2} = \frac{1}{25 + d^2} + \frac{1}{25 + d^2} = \frac{2}{25 + d^2} \quad \Rightarrow \quad (25 + d^2)(100 + d^2) + d^2(25 + d^2) = 2d^2(100 + d^2) \quad \Rightarrow$$

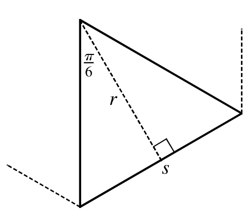
$$2500 + 125d^2 + d^4 + 25d^2 + d^4 = 200d^2 + 2d^4 \quad \Rightarrow \quad 2500 = 50d^2 \quad \Rightarrow \quad d^2 = 50 \quad \Rightarrow \quad d = 5\sqrt{2} \approx 7.071$$

[for $0 \le d \le 10$]. The third figure, a graph of $I(0) - I(5)$ with d independent, confirms that $I(0) - I(5) = 0$, that is, $I(0) = I(5)$, when $d = 5\sqrt{2}$. Thus, the point of minimal illumination changes abruptly from the midpoint to the endpoints when $d = 5\sqrt{2}$.

APPLIED PROJECT The Shape of a Can

1. In this case, the amount of metal used in the making of each top or bottom is $(2r)^2 = 4r^2$. So the quantity we want to minimize is $A = 2\pi rh + 2(4r^2)$. But $V = \pi r^2 h \quad \Leftrightarrow \quad h = V/\pi r^2$. Substituting this expression for h in A gives $A = 2V/r + 8r^2$. Differentiating A with respect to r, we get $dA/dr = -2V/r^2 + 16r = 0 \quad \Rightarrow$ $16r^3 = 2V = 2\pi r^2 h \quad \Leftrightarrow \quad \dfrac{h}{r} = \dfrac{8}{\pi} \approx 2.55$. This gives a minimum because $\dfrac{d^2A}{dr^2} = 16 + \dfrac{4V}{r^3} > 0$.

2.

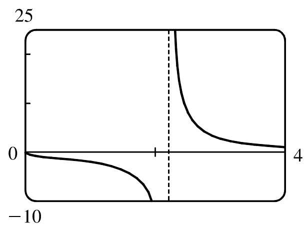


We need to find the area of metal used up by each end, that is, the area of each hexagon. We subdivide the hexagon into six congruent triangles, each sharing one side (s in the diagram) with the hexagon. We calculate the length of $s = 2r\tan\frac{\pi}{6} = \frac{2}{\sqrt{3}}r$, so the area of each triangle is $\frac{1}{2}sr = \frac{1}{\sqrt{3}}r^2$, and the total area of the hexagon is $6 \cdot \frac{1}{\sqrt{3}}r^2 = 2\sqrt{3}\,r^2$. So the quantity we want to minimize is $A = 2\pi rh + 2 \cdot 2\sqrt{3}\,r^2$. Substituting for h as in Problem 1 and differentiating, we get $\dfrac{dA}{dr} = -\dfrac{2V}{r^2} + 8\sqrt{3}\,r$.

Setting this equal to 0, we get $8\sqrt{3}\,r^3 = 2V = 2\pi r^2 h \quad \Rightarrow \quad \dfrac{h}{r} = \dfrac{4\sqrt{3}}{\pi} \approx 2.21$. Again this minimizes A because $\dfrac{d^2A}{dr^2} = 8\sqrt{3} + \dfrac{4V}{r^3} > 0$.

3. Let $C = 4\sqrt{3}\,r^2 + 2\pi rh + k\,(4\pi r + h) = 4\sqrt{3}\,r^2 + 2\pi r\left(\dfrac{V}{\pi r^2}\right) + k\left(4\pi r + \dfrac{V}{\pi r^2}\right)$. Then

$\dfrac{dC}{dr} = 8\sqrt{3}\,r - \dfrac{2V}{r^2} + 4k\pi - \dfrac{2kV}{\pi r^3}$. Setting this equal to 0, dividing by 2 and substituting $\dfrac{V}{r^2} = \pi h$ and $\dfrac{V}{\pi r^3} = \dfrac{h}{r}$ in the second and fourth terms respectively, we get $0 = 4\sqrt{3}\,r - \pi h + 2k\pi - \dfrac{kh}{r}$ $\Leftrightarrow$ $k\left(2\pi - \dfrac{h}{r}\right) = \pi h - 4\sqrt{3}\,r$ $\Rightarrow$ $\dfrac{k}{r}\,\dfrac{2\pi - h/r}{\pi h/r - 4\sqrt{3}} = 1$. We now multiply by $\dfrac{\sqrt[3]{V}}{k}$, noting that $\dfrac{\sqrt[3]{V}}{k}\,\dfrac{k}{r} = \sqrt[3]{\dfrac{V}{r^3}} = \sqrt[3]{\dfrac{\pi h}{r}}$, and get $\dfrac{\sqrt[3]{V}}{k} = \sqrt[3]{\dfrac{\pi h}{r}} \cdot \dfrac{2\pi - h/r}{\pi h/r - 4\sqrt{3}}$.

4. 

Let $\sqrt[3]{V}/k = T$ and $h/r = x$ so that $T(x) = \sqrt[3]{\pi x} \cdot \dfrac{2\pi - x}{\pi x - 4\sqrt{3}}$. We see from the graph of T that when the ratio $\sqrt[3]{V}/k$ is large; that is, either the volume of the can is large or the cost of joining (proportional to k) is small, the optimum value of h/r is about 2.21, but when $\sqrt[3]{V}/k$ is small, indicating small volume or expensive joining, the optimum value of h/r is larger. (The part of the graph for $\sqrt[3]{V}/k < 0$ has no physical meaning, but confirms the location of the asymptote.)

5. Our conclusion is usually true in practice. But there are exceptions, such as cans of tuna, which may have to do with the shape of a reasonable slice of tuna. And for a comfortable grip on a soda or beer can, the geometry of the human hand is a restriction on the radius. Other possible considerations are packaging, transportation and stocking constraints, aesthetic appeal and other marketing concerns. Also, there may be better models than ours which prescribe a differently shaped can in special circumstances.

APPLIED PROJECT Planes and Birds: Minimizing Energy

1. $P(v) = Av^3 + \dfrac{BL^2}{v}$ $\Rightarrow$ $P'(v) = 3Av^2 - \dfrac{BL^2}{v^2}$. $P'(v) = 0$ $\Leftrightarrow$ $3Av^2 = \dfrac{BL^2}{v^2}$ $\Leftrightarrow$ $v^4 = \dfrac{BL^2}{3A}$ $\Rightarrow$ $v = \sqrt[4]{\dfrac{BL^2}{3A}}$. $P''(v) = 6Av + \dfrac{2BL^2}{v^3} > 0$, so the speed that minimizes the required power is $v_P = \left(\dfrac{BL^2}{3A}\right)^{1/4}$.

2. $E(v) = \dfrac{P(v)}{v} = Av^2 + \dfrac{BL^2}{v^2}$ $\Rightarrow$ $E'(v) = 2Av - \dfrac{2BL^2}{v^3}$. $E'(v) = 0$ $\Leftrightarrow$ $2Av = \dfrac{2BL^2}{v^3}$ $\Leftrightarrow$ $v^4 = \dfrac{BL^2}{A}$ $\Rightarrow$ $v = \sqrt[4]{\dfrac{BL^2}{A}}$. $E''(v) = 2A + \dfrac{6BL^2}{v^4} > 0$, so the speed that minimizes the energy needed to propel the plane is $v_E = \left(\dfrac{BL^2}{A}\right)^{1/4}$.

3. $\dfrac{v_E}{v_P} = \dfrac{\left(\dfrac{BL^2}{A}\right)^{1/4}}{\left(\dfrac{BL^2}{3A}\right)^{1/4}} = \left(\dfrac{\dfrac{BL^2}{A}}{\dfrac{BL^2}{3A}}\right)^{1/4} = 3^{1/4} \approx 1.316$. Thus, $v_E \approx 1.316\, v_P$, so the speed for minimum energy is about 31.6% greater (faster) than the speed for minimum power.

4. Since x is the fraction of flying time spent in flapping mode, $1 - x$ is the fraction of time spent in folded mode. The average power $\overline{P}$ is the weighted average of P_{flap} and P_{fold}, so

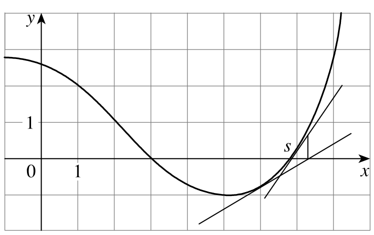
$$\begin{aligned}\overline{P} &= xP_{\text{flap}} + (1-x)P_{\text{fold}} = x\left[(A_b + A_w)\,v^3 + \frac{B(mg/x)^2}{v}\right] + (1-x)A_bv^3 \\ &= xA_bv^3 + xA_wv^3 + x\frac{Bm^2g^2}{x^2v} + A_bv^3 - xA_bv^3 = A_bv^3 + xA_wv^3 + \frac{Bm^2g^2}{xv}\end{aligned}$$

5. $\overline{P}(x) = A_bv^3 + xA_wv^3 + \dfrac{Bm^2g^2}{xv} \;\Rightarrow\; \overline{P}'(x) = A_wv^3 - \dfrac{Bm^2g^2}{x^2v}$. $\overline{P}'(x) = 0 \;\Leftrightarrow\; A_wv^3 = \dfrac{Bm^2g^2}{x^2v} \;\Leftrightarrow$ $x^2 = \dfrac{Bm^2g^2}{A_wv^4} \;\Rightarrow\; x = \dfrac{mg}{v^2}\sqrt{\dfrac{B}{A_w}}$. Since $\overline{P}''(x) = \dfrac{2Bm^2g^2}{x^3v} > 0$, this critical number, call it $x_{\overline{P}}$, gives an absolute minimum for the average power. If the bird flies slowly, then v is smaller and $x_{\overline{P}}$ increases, and the bird spends a larger fraction of its flying time flapping. If the bird flies faster and faster, then v is larger and $x_{\overline{P}}$ decreases, and the bird spends a smaller fraction of its flying time flapping, while still minimizing average power.

6. $\overline{E}(x) = \dfrac{\overline{P}(x)}{v} \;\Rightarrow\; \overline{E}'(x) = \dfrac{1}{v}\overline{P}'(x)$, so $\overline{E}'(x) = 0 \;\Leftrightarrow\; \overline{P}'(x) = 0$. The value of x that minimizes $\overline{E}$ is the same value of x that minimizes $\overline{P}$, namely $x_{\overline{P}} = \dfrac{mg}{v^2}\sqrt{\dfrac{B}{A_w}}$.

3.8 Newton's Method

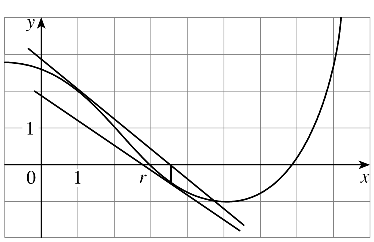
1. (a)



The tangent line at $x_1 = 6$ intersects the x-axis at $x \approx 7.3$, so $x_2 = 7.3$. The tangent line at $x = 7.3$ intersects the x-axis at $x \approx 6.8$, so $x_3 \approx 6.8$.

(b) $x_1 = 8$ would be a better first approximation because the tangent line at $x = 8$ intersects the x-axis closer to s than does the first approximation $x_1 = 6$.

2.



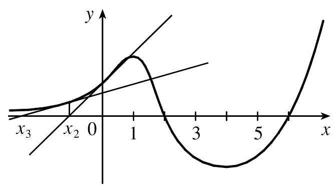
The tangent line at $x_1 = 1$ intersects the x-axis at $x \approx 3.5$, so $x_2 = 3.5$. The tangent line at $x = 3.5$ intersects the x-axis at $x \approx 2.8$, so $x_3 = 2.8$.

3. Since the tangent line $y = 9 - 2x$ is tangent to the curve $y = f(x)$ at the point $(2, 5)$, we have $x_1 = 2$, $f(x_1) = 5$, and $f'(x_1) = -2$ [the slope of the tangent line]. Thus, by Equation 2,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{5}{-2} = \frac{9}{2}$$

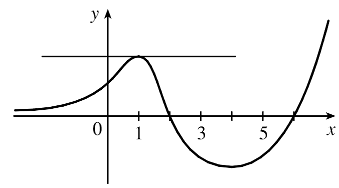
Note that geometrically $\frac{9}{2}$ represents the x-intercept of the tangent line $y = 9 - 2x$.

4. (a)



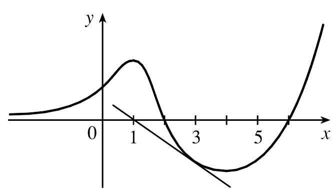
If $x_1 = 0$, then x_2 is negative, and x_3 is even more negative. The sequence of approximations does not converge, that is, Newton's method fails.

(b)



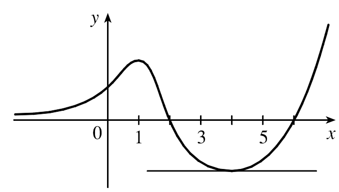
If $x_1 = 1$, the tangent line is horizontal and Newton's method fails.

(c)



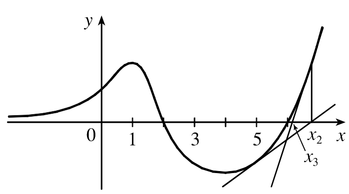
If $x_1 = 3$, then $x_2 = 1$ and we have the same situation as in part (b). Newton's method fails again.

(d)



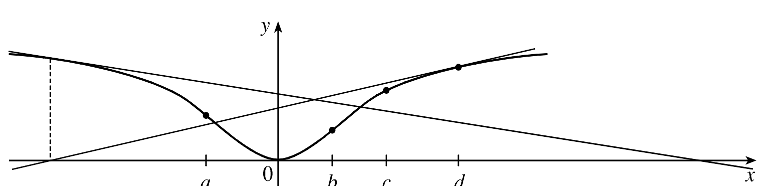
If $x_1 = 4$, the tangent line is horizontal and Newton's method fails.

(e)



If $x_1 = 5$, then x_2 is greater than 6, x_3 gets closer to 6, and the sequence of approximations converges to 6. Newton's method succeeds!

5. The initial approximations $x_1 = a, b$, and c will work, resulting in a second approximation closer to the origin, and lead to the solution of the equation $f(x) = 0$, namely, $x = 0$. The initial approximation $x_1 = d$ will not work because it will result in successive approximations farther and farther from the origin.



6. $f(x) = 2x^3 - 3x^2 + 2 \quad \Rightarrow \quad f'(x) = 6x^2 - 6x$, so $x_{n+1} = x_n - \dfrac{2x_n^3 - 3x_n^2 + 2}{6x_n^2 - 6x_n}$. Now $x_1 = -1 \quad \Rightarrow$

$$x_2 = -1 - \frac{2(-1)^3 - 3(-1)^2 + 2}{6(-1)^2 - 6(-1)} = -1 - \frac{-3}{12} = -\frac{3}{4} \quad \Rightarrow$$

$$x_3 = -\frac{3}{4} - \frac{2\left(-\frac{3}{4}\right)^3 - 3\left(-\frac{3}{4}\right)^2 + 2}{6\left(-\frac{3}{4}\right)^2 - 6\left(-\frac{3}{4}\right)} = -\frac{3}{4} - \frac{-17/32}{63/8} = -\frac{43}{63} \approx -0.6825.$$

7. $f(x) = \frac{2}{x} - x^2 + 1 \quad \Rightarrow \quad f'(x) = -\frac{2}{x^2} - 2x$, so $x_{n+1} = x_n - \frac{2/x_n - x_n^2 + 1}{-2/x_n^2 - 2x_n}$. Now $x_1 = 2 \quad \Rightarrow$

$x_2 = 2 - \frac{1 - 4 + 1}{-1/2 - 4} = 2 - \frac{-2}{-9/2} = \frac{14}{9} \quad \Rightarrow \quad x_3 = \frac{14}{9} - \frac{2/(14/9) - (14/9)^2 + 1}{-2(14/9)^2 - 2(14/9)} \approx 1.5215.$

8. Solving $x^5 = x^2 + 1$ is the same as solving $f(x) = x^5 - x^2 - 1 = 0$. $f'(x) = 5x^4 - 2x$, so $x_{n+1} = x_n - \frac{x_n^5 - x_n^2 - 1}{5x_n^4 - 2x_n}$.

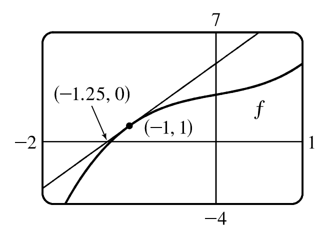
Now, $x_1 = 1 \quad \Rightarrow \quad x_2 = 1 - \frac{1^5 - 1^2 - 1}{5(1)^4 - 2(1)} = 1 - \frac{-1}{3} = \frac{4}{3} \quad \Rightarrow \quad x_3 = \frac{4}{3} - \frac{(4/3)^5 - (4/3)^2 - 1}{5(4/3)^4 - 2(4/3)} \approx 1.2240.$

9. $f(x) = x^3 + x + 3 \quad \Rightarrow \quad f'(x) = 3x^2 + 1$, so $x_{n+1} = x_n - \frac{x_n^3 + x_n + 3}{3x_n^2 + 1}$.

Now $x_1 = -1 \quad \Rightarrow$

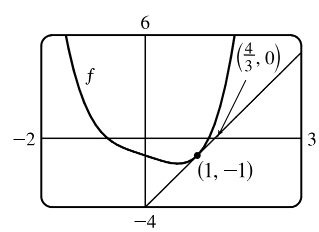
$x_2 = -1 - \frac{(-1)^3 + (-1) + 3}{3(-1)^2 + 1} = -1 - \frac{-1 - 1 + 3}{3 + 1} = -1 - \frac{1}{4} = -1.25.$

Newton's method follows the tangent line at $(-1, 1)$ down to its intersection with the x-axis at $(-1.25, 0)$, giving the second approximation $x_2 = -1.25$.



10. $f(x) = x^4 - x - 1 \quad \Rightarrow \quad f'(x) = 4x^3 - 1$, so $x_{n+1} = x_n - \frac{x_n^4 - x_n - 1}{4x_n^3 - 1}$.

Now $x_1 = 1 \quad \Rightarrow \quad x_2 = 1 - \frac{1^4 - 1 - 1}{4 \cdot 1^3 - 1} = 1 - \frac{-1}{3} = \frac{4}{3}$. Newton's method follows the tangent line at $(1, -1)$ up to its intersection with the x-axis at $\left(\frac{4}{3}, 0\right)$, giving the second approximation $x_2 = \frac{4}{3}$.



11. To approximate $x = \sqrt[4]{75}$ (so that $x^4 = 75$), we can take $f(x) = x^4 - 75$. So $f'(x) = 4x^3$, and thus,

$x_{n+1} = x_n - \frac{x_n^4 - 75}{4x_n^3}$. Since $\sqrt[4]{81} = 3$ and 81 is reasonably close to 75, we'll use $x_1 = 3$. We need to find approximations until they agree to eight decimal places. $x_1 = 3 \quad \Rightarrow \quad x_2 = 2.9\overline{4}$, $x_3 \approx 2.94283228$, $x_4 \approx 2.94283096 \approx x_5$. So $\sqrt[4]{75} \approx 2.94283096$, to eight decimal places.

To use Newton's method on a calculator, assign f to Y_1 and f' to Y_2. Then store x_1 in X and enter $\mathrm{X} - \mathrm{Y}_1/\mathrm{Y}_2 \to \mathrm{X}$ to get x_2 and further approximations (repeatedly press ENTER).

12. $f(x) = x^8 - 500 \quad \Rightarrow \quad f'(x) = 8x^7$, so $x_{n+1} = x_n - \frac{x_n^8 - 500}{8x_n^7}$. Since $\sqrt[8]{256} = 2$ and 256 is reasonably close to 500, we'll use $x_1 = 2$. We need to find approximations until they agree to eight decimal places. $x_1 = 2 \quad \Rightarrow \quad x_2 \approx 2.23828125$, $x_3 \approx 2.18055972$, $x_4 \approx 2.17461675$, $x_5 \approx 2.17455928 \approx x_6$. So $\sqrt[8]{500} \approx 2.17455928$, to eight decimal places.

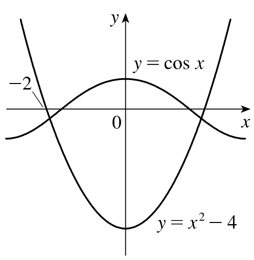
13. (a) Let $f(x) = 3x^4 - 8x^3 + 2$. The polynomial f is continuous on $[2, 3]$, $f(2) = -14 < 0$, and $f(3) = 29 > 0$, so by the Intermediate Value Theorem, there is a number c in $(2, 3)$ such that $f(c) = 0$. In other words, the equation $3x^4 - 8x^3 + 2 = 0$ has a solution in $[2, 3]$.

(b) $f'(x) = 12x^3 - 24x^2 \quad \Rightarrow \quad x_{n+1} = x_n - \dfrac{3x_n^4 - 8x_n^3 + 2}{12x_n^3 - 24x_n^2}$. Taking $x_1 = 2.5$, we get $x_2 = 2.655$, $x_3 \approx 2.630725$, $x_4 \approx 2.630021$, $x_5 \approx 2.630020 \approx x_6$. To six decimal places, the solution is 2.630020. Note that taking $x_1 = 2$ is not allowed since $f'(2) = 0$.

14. (a) Let $f(x) = -2x^5 + 9x^4 - 7x^3 - 11x$. The polynomial f is continuous on $[3, 4]$, $f(3) = 21 > 0$, and $f(4) = -236 < 0$, so by the Intermediate Value Theorem, there is a number c in $(3, 4)$ such that $f(c) = 0$. In other words, the equation $-2x^5 + 9x^4 - 7x^3 - 11x = 0$ has a solution in $[3, 4]$.

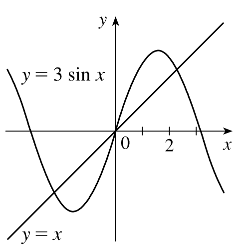
(b) $f'(x) = -10x^4 + 36x^3 - 21x^2 - 11$. $\quad x_{n+1} = x_n - \dfrac{-2x_n^5 + 9x_n^4 - 7x_n^3 - 11x_n}{-10x_n^4 + 36x_n^3 - 21x_n^2 - 11}$. Taking $x_1 = 3.5$, we get $x_2 \approx 3.329174$, $x_3 = 3.278706$, $x_4 \approx 3.274501$, and $x_5 \approx 3.274473 \approx x_6$. To six decimal places, the solution is 3.274473.

15.



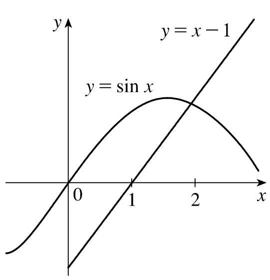
From the graph, we see that the negative solution of $\cos x = x^2 - 4$ is near $x = -2$. Solving $\cos x = x^2 - 4$ is the same as solving $f(x) = \cos x - x^2 + 4 = 0$. $f'(x) = -\sin x - 2x$, so $x_{n+1} = x_n - \dfrac{\cos x_n - x_n^2 + 4}{-\sin x_n - 2x_n}$. $\quad x_1 = -2 \quad \Rightarrow$ $x_2 \approx -1.915233$, $x_3 \approx -1.914021 \approx x_4$. Thus, the negative solution is -1.914021, to six decimal places.

16.



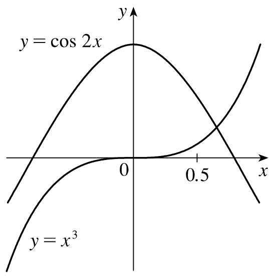
$3\sin x = x$, so $f(x) = 3\sin x - x \quad \Rightarrow \quad f'(x) = 3\cos x - 1 \quad \Rightarrow$ $x_{n+1} = x_n - \dfrac{3\sin x_n - x_n}{3\cos x_n - 1}$. From the figure, the positive root of $3\sin x = x$ is near 2. $\quad x_1 = 2 \quad \Rightarrow \quad x_2 \approx 2.323732$, $x_3 \approx 2.279595$, $x_4 \approx 2.278863 \approx x_5$. So the positive root is 2.278863, to six decimal places.

17.



From the graph, we see that there appears to be a point of intersection near $x = 2$. Solving $\sin x = x - 1$ is the same as solving $f(x) = \sin x - x + 1 = 0$. $f'(x) = \cos x - 1$, so $x_{n+1} = x_n - \dfrac{\sin x_n - x_n + 1}{\cos x_n - 1}$. $\quad x_1 = 2 \quad \Rightarrow$ $x_2 \approx 1.935951$, $x_3 \approx 1.934564$, $x_4 \approx 1.934563 \approx x_5$. Thus, the solution is 1.934563, to six decimal places.

18.

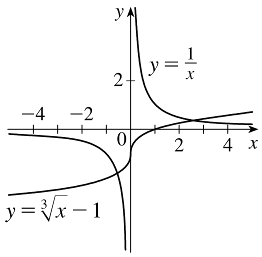


From the graph, we see that there appears to be a point of intersection near $x = 0.5$. Solving $\cos 2x = x^3$ is the same as solving $f(x) = \cos 2x - x^3 = 0$.

$f'(x) = -2\sin 2x - 3x^2$, so $x_{n+1} = x_n - \dfrac{\cos 2x_n - x_n^3}{-2\sin 2x_n - 3x_n^2}$. $x_1 = 0.5 \Rightarrow$

$x_2 \approx 0.670700$, $x_3 \approx 0.648160$, $x_4 \approx 0.647766$, $x_5 \approx 0.647765 \approx x_6$. Thus, the solution 0.647765, to six decimal places.

19.



From the graph, we see that there appear to be points of intersection near $x = -0.5$ and $x = 2.5$. Solving $\dfrac{1}{x} = \sqrt[3]{x} - 1$ is the same as solving

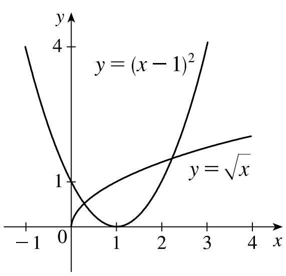
$f(x) = \dfrac{1}{x} - \sqrt[3]{x} + 1 = 0$. $f'(x) = -\dfrac{1}{x^2} - \dfrac{1}{3x^{2/3}}$, so

$$x_{n+1} = x_n - \frac{1/x_n - \sqrt[3]{x_n} + 1}{-1/x_n^2 - 1/(3x_n^{2/3})}.$$

$x_1 = -0.5$	$x_1 = 2.5$
$x_2 \approx -0.545549$	$x_2 \approx 2.625502$
$x_3 \approx -0.549672$	$x_3 \approx 2.629654$
$x_4 \approx -0.549700 \approx x_5$	$x_4 \approx 2.629658 \approx x_5$

To six decimal places, the roots of the equation are -0.549700 and 2.629658.

20.



From the graph, we see that there appear to be points of intersection near $x = 0.3$ and $x = 2.2$. Solving $(x-1)^2 = \sqrt{x}$ is the same as solving

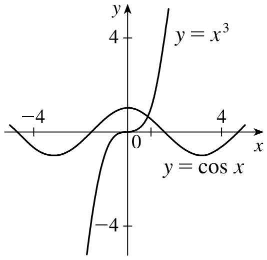
$f(x) = (x-1)^2 - \sqrt{x} = 0$. $f'(x) = 2(x-1) - \dfrac{1}{2\sqrt{x}}$, so

$$x_{n+1} = x_n - \frac{(x_n - 1)^2 - \sqrt{x_n}}{2(x_n - 1) - 1/(2\sqrt{x_n})}.$$

$x_1 = 0.3$	$x_1 = 2.2$
$x_2 \approx 0.275043$	$x_2 \approx 2.220961$
$x_3 \approx 0.275508 \approx x_4$	$x_3 \approx 2.220744 \approx x_4$

To six decimal places, the roots of the equation are 0.275508 and 2.220744.

21.



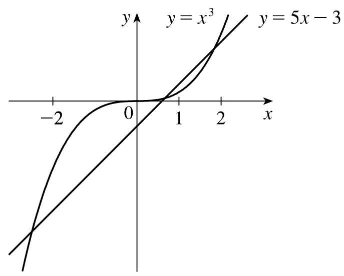
From the graph, we see that there appears to be a point of intersection near $x = 1$. Solving $x^3 = \cos x$ is the same as solving $f(x) = x^3 - \cos x = 0$.

$f'(x) = 3x^2 + \sin x$, so $x_{n+1} = x_n - \dfrac{x_n^3 - \cos x_n}{3x_n^2 + \sin x_n}$.

$x_1 = 1 \Rightarrow x_2 \approx 0.880333$, $x_3 \approx 0.865684$, $x_4 = 0.865474 \approx x_5$.

To six decimal places, the only solution is 0.865474.

22.



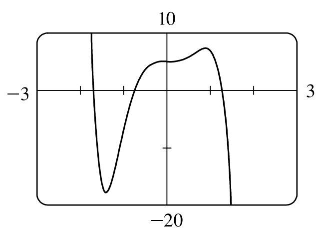
From the figure, we see that there appear to be points of intersection near $x = -2$, $x = 1$, and $x = 2$. Solving $x^3 = 5x - 3$ is the same as solving $f(x) = x^3 - 5x + 3 = 0$. $f'(x) = 3x^2 - 5$, so

$$x_{n+1} = x_n - \frac{x_n^3 - 5x_n + 3}{3x_n^2 - 5}.$$

$x_1 = -2$	$x_1 = 1$	$x_1 = 2$
$x_2 \approx -2.714286$	$x_2 = 0.5$	$x_2 \approx 1.857143$
$x_3 \approx -2.513979$	$x_3 \approx 0.647059$	$x_3 \approx 1.834787$
$x_4 \approx -2.491151$	$x_4 \approx 0.656573$	$x_4 \approx 1.834244$
$x_5 \approx -2.490864 \approx x_6$	$x_5 \approx 0.656620 \approx x_6$	$x_5 \approx 1.834243 \approx x_6$

To six decimal places, the solutions of the equation are -2.490864, 0.656620, and 1.834243.

23.



$f(x) = -2x^7 - 5x^4 + 9x^3 + 5 \quad\Rightarrow\quad f'(x) = -14x^6 - 20x^3 + 27x^2 \quad\Rightarrow$

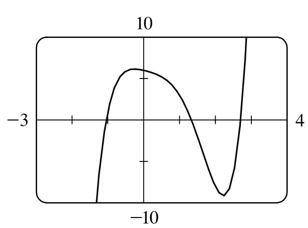
$$x_{n+1} = x_n - \frac{-2x_n^7 - 5x_n^4 + 9x_n^3 + 5}{-14x_n^6 - 20x_n^3 + 27x_n^2}.$$

From the graph of f, there appear to be solutions near -1.7, -0.7, and 1.3.

$x_1 = -1.7$	$x_1 = -0.7$	$x_1 = 1.3$
$x_2 = -1.693255$	$x_2 \approx -0.74756345$	$x_2 = 1.268776$
$x_3 \approx -1.69312035$	$x_3 \approx -0.74467752$	$x_3 \approx 1.26589387$
$x_4 \approx -1.69312029 \approx x_5$	$x_4 \approx -0.74466668 \approx x_5$	$x_4 \approx 1.26587094 \approx x_5$

To eight decimal places, the solutions of the equation are -1.69312029, -0.74466668, and 1.26587094.

24.



$f(x) = x^5 - 3x^4 + x^3 - x^2 - x + 6 \quad\Rightarrow$

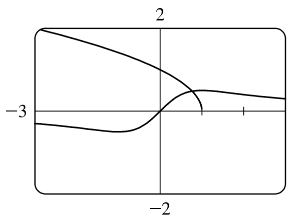
$f'(x) = 5x^4 - 12x^3 + 3x^2 - 2x - 1 \quad\Rightarrow$

$$x_{n+1} = x_n - \frac{x_n^5 - 3x_n^4 + x_n^3 - x_n^2 - x_n + 6}{5x_n^4 - 12x_n^3 + 3x_n^2 - 2x_n - 1}.$$

From the graph of f, there appear to be solutions near -1, 1.3, and 2.7.

$x_1 = -1$	$x_1 = 1.3$	$x_1 = 2.7$
$x_2 \approx -1.04761905$	$x_2 \approx 1.33313045$	$x_2 \approx 2.70556135$
$x_3 \approx -1.04451724$	$x_3 \approx 1.33258330$	$x_3 \approx 2.70551210$
$x_4 \approx -1.04450307 \approx x_5$	$x_4 \approx 1.33258316 \approx x_5$	$x_4 \approx 2.70551209 \approx x_5$

To eight decimal places, the solutions of the equation are -1.04450307, 1.33258316, and 2.70551209.

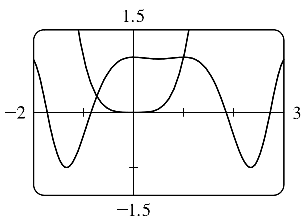
25. 

Solving $\dfrac{x}{x^2+1} = \sqrt{1-x}$ is the same as solving

$f(x) = \dfrac{x}{x^2+1} - \sqrt{1-x} = 0$. $f'(x) = \dfrac{1-x^2}{(x^2+1)^2} + \dfrac{1}{2\sqrt{1-x}} \Rightarrow$

$$x_{n+1} = x_n - \frac{\dfrac{x_n}{x_n^2+1} - \sqrt{1-x_n}}{\dfrac{1-x_n^2}{(x_n^2+1)^2} + \dfrac{1}{2\sqrt{1-x_n}}}.$$

From the graph, we see that the curves intersect at about 0.8. $x_1 = 0.8 \Rightarrow x_2 \approx 0.76757581$, $x_3 \approx 0.76682610$, $x_4 \approx 0.76682579 \approx x_5$. To eight decimal places, the solution of the equation is 0.76682579.

26. 

Solving $\cos(x^2 - x) = x^4$ is the same as solving

$f(x) = \cos(x^2 - x) - x^4 = 0$. $f'(x) = -(2x-1)\sin(x^2-x) - 4x^3 \Rightarrow$

$x_{n+1} = x_n - \dfrac{\cos(x_n^2 - x_n) - x_n^4}{-(2x_n - 1)\sin(x_n^2 - x_n) - 4x_n^3}$. From the equations $y = \cos(x^2 - x)$ and $y = x^4$ and the graph, we deduce that one solution of the equation $\cos(x^2 - x) = x^4$ is $x = 1$. We also see that the graphs intersect at approximately $x = -0.7$. $x_1 = -0.7 \Rightarrow x_2 \approx -0.73654354$, $x_3 \approx -0.73486274$, $x_4 \approx -0.73485910 \approx x_5$. To eight decimal places, one solution of the equation is -0.73485910; the other solution is 1.

27. (a) $f(x) = x^2 - a \Rightarrow f'(x) = 2x$, so Newton's method gives

$$x_{n+1} = x_n - \frac{x_n^2 - a}{2x_n} = x_n - \frac{1}{2}x_n + \frac{a}{2x_n} = \frac{1}{2}x_n + \frac{a}{2x_n} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right).$$

(b) Using (a) with $a = 1000$ and $x_1 = \sqrt{900} = 30$, we get $x_2 \approx 31.666667$, $x_3 \approx 31.622807$, and $x_4 \approx 31.622777 \approx x_5$.

So $\sqrt{1000} \approx 31.622777$.

28. (a) $f(x) = \dfrac{1}{x} - a \Rightarrow f'(x) = -\dfrac{1}{x^2}$, so $x_{n+1} = x_n - \dfrac{1/x_n - a}{-1/x_n^2} = x_n + x_n - ax_n^2 = 2x_n - ax_n^2$.

(b) Using (a) with $a = 1.6894$ and $x_1 = \frac{1}{2} = 0.5$, we get $x_2 = 0.5754$, $x_3 \approx 0.588485$, and $x_4 \approx 0.588789 \approx x_5$.

So $1/1.6984 \approx 0.588789$.

29. $f(x) = x^3 - 3x + 6 \Rightarrow f'(x) = 3x^2 - 3$. If $x_1 = 1$, then $f'(x_1) = 0$ and the tangent line used for approximating x_2 is horizontal. Attempting to find x_2 results in trying to divide by zero.

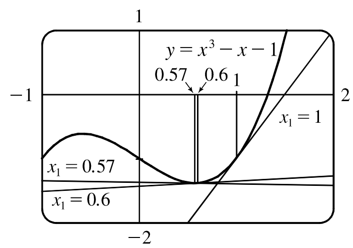
30. $x^3 - x = 1 \Leftrightarrow x^3 - x - 1 = 0$. $f(x) = x^3 - x - 1 \Rightarrow f'(x) = 3x^2 - 1$, so $x_{n+1} = x_n - \dfrac{x_n^3 - x_n - 1}{3x_n^2 - 1}$.

(a) $x_1 = 1$, $x_2 = 1.5$, $x_3 \approx 1.347826$, $x_4 \approx 1.325200$, $x_5 \approx 1.324718 \approx x_6$

(b) $x_1 = 0.6$, $x_2 = 17.9$, $x_3 \approx 11.946802$, $x_4 \approx 7.985520$, $x_5 \approx 5.356909$, $x_6 \approx 3.624996$, $x_7 \approx 2.505589$, $x_8 \approx 1.820129$, $x_9 \approx 1.461044$, $x_{10} \approx 1.339323$, $x_{11} \approx 1.324913$, $x_{12} \approx 1.324718 \approx x_{13}$

(c) $x_1 = 0.57$, $x_2 \approx -54.165455$, $x_3 \approx -36.114293$, $x_4 \approx -24.082094$, $x_5 \approx -16.063387$, $x_6 \approx -10.721483$, $x_7 \approx -7.165534$, $x_8 \approx -4.801704$, $x_9 \approx -3.233425$, $x_{10} \approx -2.193674$, $x_{11} \approx -1.496867$, $x_{12} \approx -0.997546$, $x_{13} \approx -0.496305$, $x_{14} \approx -2.894162$, $x_{15} \approx -1.967962$, $x_{16} \approx -1.341355$, $x_{17} \approx -0.870187$, $x_{18} \approx -0.249949$, $x_{19} \approx -1.192219$, $x_{20} \approx -0.731952$, $x_{21} \approx 0.355213$, $x_{22} \approx -1.753322$, $x_{23} \approx -1.189420$, $x_{24} \approx -0.729123$, $x_{25} \approx 0.377844$, $x_{26} \approx -1.937872$, $x_{27} \approx -1.320350$, $x_{28} \approx -0.851919$, $x_{29} \approx -0.200959$, $x_{30} \approx -1.119386$, $x_{31} \approx -0.654291$, $x_{32} \approx 1.547010$, $x_{33} \approx 1.360051$, $x_{34} \approx 1.325828$, $x_{35} \approx 1.324719$, $x_{36} \approx 1.324718 \approx x_{37}$.

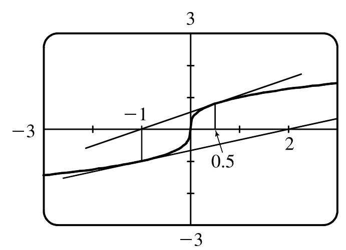
(d)



From the figure, we see that the tangent line corresponding to $x_1 = 1$ results in a sequence of approximations that converges quite quickly ($x_5 \approx x_6$). The tangent line corresponding to $x_1 = 0.6$ is close to being horizontal, so x_2 is quite far from the solution. But the sequence still converges — just a little more slowly ($x_{12} \approx x_{13}$). Lastly, the tangent line corresponding to $x_1 = 0.57$ is very nearly horizontal, x_2 is farther away from the solution, and the sequence takes more iterations to converge ($x_{36} \approx x_{37}$).

31. For $f(x) = x^{1/3}$, $f'(x) = \frac{1}{3}x^{-2/3}$ and

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^{1/3}}{\frac{1}{3}x_n^{-2/3}} = x_n - 3x_n = -2x_n.$$

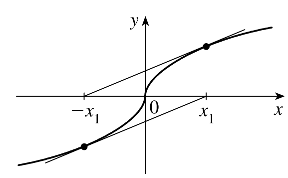


Therefore, each successive approximation becomes twice as large as the previous one in absolute value, so the sequence of approximations fails to converge to the solution, which is 0. In the figure, we have $x_1 = 0.5$, $x_2 = -2(0.5) = -1$, and $x_3 = -2(-1) = 2$.

32. According to Newton's Method, for $x_n > 0$,

$$x_{n+1} = x_n - \frac{\sqrt{x_n}}{1/\left(2\sqrt{x_n}\right)} = x_n - 2x_n = -x_n \text{ and for } x_n < 0,$$

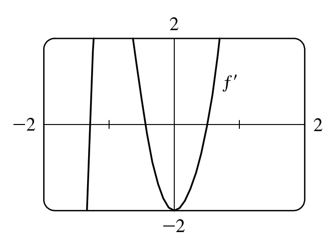
$$x_{n+1} = x_n - \frac{-\sqrt{-x_n}}{1/\left(2\sqrt{-x_n}\right)} = x_n - [-2(-x_n)] = -x_n. \text{ So we can see that}$$



after choosing any value x_1 the subsequent values will alternate between $-x_1$ and x_1 and never approach the solution.

33. (a) $f(x) = x^6 - x^4 + 3x^3 - 2x \Rightarrow f'(x) = 6x^5 - 4x^3 + 9x^2 - 2 \Rightarrow$ $f''(x) = 30x^4 - 12x^2 + 18x$. To find the critical numbers of f, we'll find the zeros of f'. From the graph of f', it appears there are zeros at approximately $x = -1.3$, -0.4, and 0.5. Try $x_1 = -1.3 \Rightarrow$

$$x_2 = x_1 - \frac{f'(x_1)}{f''(x_1)} \approx -1.293344 \Rightarrow x_3 \approx -1.293227 \approx x_4.$$

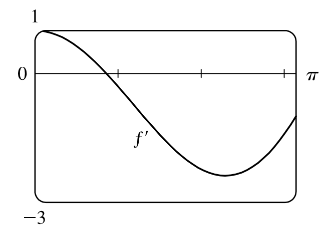


[continued]

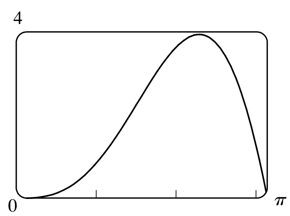
Now try $x_1 = -0.4 \quad\Rightarrow\quad x_2 \approx -0.443755 \quad\Rightarrow\quad x_3 \approx -0.441735 \quad\Rightarrow\quad x_4 \approx -0.441731 \approx x_5$. Finally try $x_1 = 0.5 \quad\Rightarrow\quad x_2 \approx 0.507937 \quad\Rightarrow\quad x_3 \approx 0.507854 \approx x_4$. Therefore, $x = -1.293227, -0.441731$, and 0.507854 are all the critical numbers correct to six decimal places.

(b) There are two critical numbers where f' changes from negative to positive, so f changes from decreasing to increasing. $f(-1.293227) \approx -2.0212$ and $f(0.507854) \approx -0.6721$, so -2.0212 is the absolute minimum value of f correct to four decimal places.

34. $f(x) = x\cos x \quad\Rightarrow\quad f'(x) = \cos x - x\sin x$. $f'(x)$ exists for all x, so to find the maximum of f, we can examine the zeros of f'. From the graph of f', we see that a good choice for x_1 is $x_1 = 0.9$. Use $g(x) = \cos x - x\sin x$ and $g'(x) = -2\sin x - x\cos x$ to obtain $x_2 \approx 0.860781$, $x_3 \approx 0.860334 \approx x_4$. Now we have $f(0) = 0$, $f(\pi) = -\pi$, and $f(0.860334) \approx 0.561096$, so 0.561096 is the absolute maximum value of f correct to six decimal places.



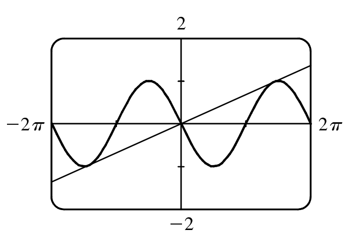
35.



$$y = x^2\sin x \quad\Rightarrow\quad y' = x^2\cos x + (\sin x)(2x) \quad\Rightarrow$$
$$\begin{aligned} y'' &= x^2(-\sin x) + (\cos x)(2x) + (\sin x)(2) + 2x\cos x \\ &= -x^2\sin x + 4x\cos x + 2\sin x \quad\Rightarrow \end{aligned}$$
$$\begin{aligned} y''' &= -x^2\cos x + (\sin x)(-2x) + 4x(-\sin x) + (\cos x)(4) + 2\cos x \\ &= -x^2\cos x - 6x\sin x + 6\cos x. \end{aligned}$$

From the graph of $y = x^2\sin x$, we see that $x = 1.5$ is a reasonable guess for the x-coordinate of the inflection point. Using Newton's method with $g(x) = y''$ and $g'(x) = y'''$, we get $x_1 = 1.5 \quad\Rightarrow\quad x_2 \approx 1.520092$, $x_3 \approx 1.519855 \approx x_4$. The inflection point is about $(1.519855, 2.306964)$.

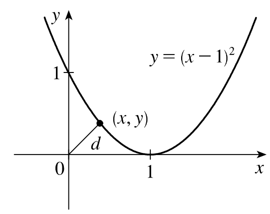
36.



$f(x) = -\sin x \quad\Rightarrow\quad f'(x) = -\cos x$. At $x = a$, the slope of the tangent line is $f'(a) = -\cos a$. The line through the origin and $(a, f(a))$ is $y = \dfrac{-\sin a - 0}{a - 0}x$. If this line is to be tangent to f at $x = a$, then its slope must equal $f'(a)$. Thus, $\dfrac{-\sin a}{a} = -\cos a \quad\Rightarrow\quad \tan a = a$.

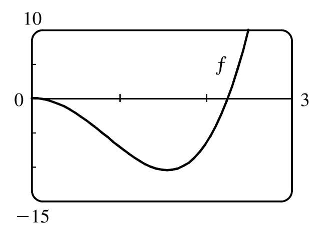
To solve this equation using Newton's method, let $g(x) = \tan x - x$, $g'(x) = \sec^2 x - 1$, and $x_{n+1} = x_n - \dfrac{\tan x_n - x_n}{\sec^2 x_n - 1}$ with $x_1 = 4.5$ (estimated from the figure). $x_2 \approx 4.493614$, $\quad x_3 \approx 4.493410$, $\quad x_4 \approx 4.493409 \approx x_5$. Thus, the slope of the line that has the largest slope is $f'(x_5) \approx 0.217234$.

37. We need to minimize the distance from $(0,0)$ to an arbitrary point (x,y) on the curve $y=(x-1)^2$. $d=\sqrt{x^2+y^2}$ $\Rightarrow$ $d(x)=\sqrt{x^2+[(x-1)^2]^2}=\sqrt{x^2+(x-1)^4}$. When $d'=0$, d will be minimized and equivalently, $s=d^2$ will be minimized, so we will use Newton's method with $f=s'$ and $f'=s''$.



$f(x)=2x+4(x-1)^3$ $\Rightarrow$ $f'(x)=2+12(x-1)^2$, so $x_{n+1}=x_n-\dfrac{2x_n+4(x_n-1)^3}{2+12(x_n-1)^2}$. Try $x_1=0.5$ $\Rightarrow$ $x_2=0.4$, $x_3\approx 0.410127$, $x_4\approx 0.410245\approx x_5$. Now $d(0.410245)\approx 0.537841$ is the minimum distance and the point on the parabola is $(0.410245, 0.347810)$, correct to six decimal places.

38. Let the radius of the circle be r. Using $s=r\theta$, we have $5=r\theta$ and so $r=5/\theta$. From the Law of Cosines we get $4^2=r^2+r^2-2\cdot r\cdot r\cdot\cos\theta$ $\Leftrightarrow$ $16=2r^2(1-\cos\theta)=2(5/\theta)^2(1-\cos\theta)$. Multiplying by θ^2 gives $16\theta^2=50(1-\cos\theta)$, so we take $f(\theta)=16\theta^2+50\cos\theta-50$ and $f'(\theta)=32\theta-50\sin\theta$. The formula for Newton's method is $\theta_{n+1}=\theta_n-\dfrac{16\theta_n^2+50\cos\theta_n-50}{32\theta_n-50\sin\theta_n}$. From the graph of f, we can use $\theta_1=2.2$, giving us $\theta_2\approx 2.2662$, $\theta_3\approx 2.2622\approx\theta_4$. So correct to four decimal places, the angle is 2.2622 radians $\approx 130^\circ$.



39. In this case, $A=18{,}000$, $R=375$, and $n=5(12)=60$. So the formula $A=\dfrac{R}{i}[1-(1+i)^{-n}]$ becomes

$18{,}000=\dfrac{375}{x}[1-(1+x)^{-60}]$ $\Leftrightarrow$ $48x=1-(1+x)^{-60}$ [multiply each term by $(1+x)^{60}$] $\Leftrightarrow$ $48x(1+x)^{60}-(1+x)^{60}+1=0$. Let the LHS be called $f(x)$, so that

$$\begin{aligned}f'(x)&=48x(60)(1+x)^{59}+48(1+x)^{60}-60(1+x)^{59}\\&=12(1+x)^{59}[4x(60)+4(1+x)-5]=12(1+x)^{59}(244x-1)\end{aligned}$$

$x_{n+1}=x_n-\dfrac{48x_n(1+x_n)^{60}-(1+x_n)^{60}+1}{12(1+x_n)^{59}(244x_n-1)}$. An interest rate of 1% per month seems like a reasonable estimate for $x=i$. So let $x_1=1\%=0.01$, and we get $x_2\approx 0.0082202$, $x_3\approx 0.0076802$, $x_4\approx 0.0076291$, $x_5\approx 0.0076286\approx x_6$. Thus, the dealer is charging a monthly interest rate of 0.76286% (or 9.55% per year, compounded monthly).

40. (a) $p(x)=x^5-(2+r)x^4+(1+2r)x^3-(1-r)x^2+2(1-r)x+r-1$ $\Rightarrow$

$p'(x)=5x^4-4(2+r)x^3+3(1+2r)x^2-2(1-r)x+2(1-r)$. So we use

$$x_{n+1}=x_n-\frac{x_n^5-(2+r)x_n^4+(1+2r)x_n^3-(1-r)x_n^2+2(1-r)x_n+r-1}{5x_n^4-4(2+r)x_n^3+3(1+2r)x_n^2-2(1-r)x_n+2(1-r)}.$$

We substitute in the value $r\approx 3.04042\times 10^{-6}$ in order to evaluate the approximations numerically. The libration point L_1 is slightly less than 1 AU from the sun, so we take $x_1=0.95$ as our first approximation, and get $x_2\approx 0.96682$,

$x_3 \approx 0.97770$, $x_4 \approx 0.98451$, $x_5 \approx 0.98830$, $x_6 \approx 0.98976$, $x_7 \approx 0.98998$, $x_8 \approx 0.98999 \approx x_9$. So, to five decimal places, L_1 is located 0.98999 AU from the sun (or 0.01001 AU from the earth).

(b) In this case we use Newton's method with the function

$p(x) - 2rx^2 = x^5 - (2+r)x^4 + (1+2r)x^3 - (1+r)x^2 + 2(1-r)x + r - 1 \quad \Rightarrow$

$\left[p(x) - 2rx^2\right]' = 5x^4 - 4(2+r)x^3 + 3(1+2r)x^2 - 2(1+r)x + 2(1-r)$. So

$x_{n+1} = x_n - \dfrac{x_n^5 - (2+r)x_n^4 + (1+2r)x_n^3 - (1+r)x_n^2 + 2(1-r)x_n + r - 1}{5x_n^4 - 4(2+r)x_n^3 + 3(1+2r)x_n^2 - 2(1+r)x_n + 2(1-r)}$. Again, we substitute

$r \approx 3.04042 \times 10^{-6}$. L_2 is slightly more than 1 AU from the sun and, judging from the result of part (a), probably less than 0.02 AU from earth. So we take $x_1 = 1.02$ and get $x_2 \approx 1.01422$, $x_3 \approx 1.01118$, $x_4 \approx 1.01018$, $x_5 \approx 1.01008 \approx x_6$. So, to five decimal places, L_2 is located 1.01008 AU from the sun (or 0.01008 AU from the earth).

3.9 Antiderivatives

1. (a) $f(x) = 6 \quad \Rightarrow \quad F(x) = 6x$ is an antiderivative.

(b) $g(t) = 3t^2 \quad \Rightarrow \quad G(t) = 3\,\dfrac{t^{2+1}}{2+1} = t^3$ is an antiderivative.

2. (a) $f(x) = 2x = 2x^1 \quad \Rightarrow \quad F(x) = 2\,\dfrac{x^{1+1}}{1+1} = x^2$ is an antiderivative.

(b) $g(x) = -1/x^2 = -x^{-2} \quad \Rightarrow \quad G(x) = -\dfrac{x^{-2+1}}{-2+1} = x^{-1} = 1/x$ is an antiderivative.

3. (a) $h(q) = \cos q \quad \Rightarrow \quad H(q) = \sin q$ is an antiderivative.

(b) $f(x) = \sec x \tan x \quad \Rightarrow \quad F(x) = \sec x$ is an antiderivative.

4. (a) $g(t) = \sin t \quad \Rightarrow \quad G(t) = -\cos t$ is an antiderivative.

(b) $r(\theta) = \sec^2\theta \quad \Rightarrow \quad R(\theta) = \tan\theta$ is an antiderivative.

5. $f(x) = 4x + 7 = 4x^1 + 7 \quad \Rightarrow \quad F(x) = 4\dfrac{x^{1+1}}{1+1} + 7x + C = 2x^2 + 7x + C$

Check: $F'(x) = 2(2x) + 7 + 0 = 4x + 7 = f(x)$

6. $f(x) = x^2 - 3x + 2 \quad \Rightarrow \quad F(x) = \dfrac{x^3}{3} - 3\dfrac{x^2}{2} + 2x + C = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x + C$

Check: $F'(x) = \frac{1}{3}(3x^2) - \frac{3}{2}(2x) + 2 + 0 = x^2 - 3x + 2 = f(x)$

7. $f(x) = 2x^3 - \frac{2}{3}x^2 + 5x \quad \Rightarrow \quad F(x) = 2\,\dfrac{x^{3+1}}{3+1} - \dfrac{2}{3}\,\dfrac{x^{2+1}}{2+1} + 5\,\dfrac{x^{1+1}}{1+1} = \frac{1}{2}x^4 - \frac{2}{9}x^3 + \frac{5}{2}x^2 + C$

Check: $F'(x) = \frac{1}{2}(4x^3) - \frac{2}{9}(3x^2) + \frac{5}{2}(2x) + 0 = 2x^3 - \frac{2}{3}x^2 + 5x = f(x)$

8. $f(x) = 6x^5 - 8x^4 - 9x^2 \quad \Rightarrow \quad F(x) = 6\,\dfrac{x^6}{6} - 8\,\dfrac{x^5}{5} - 9\,\dfrac{x^3}{3} + C = x^6 - \frac{8}{5}x^5 - 3x^3 + C$

9. $f(x) = x(12x+8) = 12x^2 + 8x \quad \Rightarrow \quad F(x) = 12\,\dfrac{x^3}{3} + 8\,\dfrac{x^2}{2} + C = 4x^3 + 4x^2 + C$

10. $f(x) = (x-5)^2 = x^2 - 10x + 25 \quad \Rightarrow \quad F(x) = \dfrac{x^3}{3} - 10\dfrac{x^2}{2} + 25x + C = \frac{1}{3}x^3 - 5x^2 + 25x + C$

11. $g(x) = 4x^{-2/3} - 2x^{5/3} \quad \Rightarrow \quad G(x) = 4(3x^{1/3}) - 2\left(\frac{3}{8}x^{8/3}\right) + C = 12x^{1/3} - \frac{3}{4}x^{8/3} + C$

12. $h(z) = 3z^{0.8} + z^{-2.5} \quad \Rightarrow \quad H(z) = 3\,\dfrac{z^{1.8}}{1.8} + \dfrac{z^{-1.5}}{-1.5} = \dfrac{5}{3}\,z^{1.8} - \dfrac{2}{3}\,z^{-1.5} + C$

13. $f(x) = 3\sqrt{x} - 2\sqrt[3]{x} = 3x^{1/2} - 2x^{1/3} \quad \Rightarrow \quad F(x) = 3\left(\frac{2}{3}x^{3/2}\right) - 2\left(\frac{3}{4}x^{4/3}\right) + C = 2x^{3/2} - \frac{3}{2}x^{4/3} + C$

14. $g(x) = \sqrt{x}\,(2 - x + 6x^2) = 2x^{1/2} - x^{3/2} + 6x^{5/2} \quad \Rightarrow$

$$G(x) = 2\,\frac{x^{3/2}}{3/2} - \frac{x^{5/2}}{5/2} + 6\,\frac{x^{7/2}}{7/2} + C = \frac{4}{3}\,x^{3/2} - \frac{2}{5}\,x^{5/2} + \frac{12}{7}\,x^{7/2} + C$$

15. $f(t) = \dfrac{2t - 4 + 3\sqrt{t}}{\sqrt{t}} = 2t^{1/2} - 4t^{-1/2} + 3 \quad \Rightarrow \quad F(t) = 2\,\dfrac{t^{3/2}}{3/2} - 4\,\dfrac{t^{1/2}}{1/2} + 3t + C = \dfrac{4}{3}\,t^{3/2} - 8t^{1/2} + 3t + C$

16. $f(x) = \sqrt[4]{5} + \sqrt[4]{x} = \sqrt[4]{5} + x^{1/4} \quad \Rightarrow \quad F(x) = \sqrt[4]{5}\,x + \dfrac{x^{5/4}}{5/4} + C = \sqrt[4]{5}\,x + \dfrac{4}{5}\,x^{5/4} + C$

17. $f(x) = \dfrac{10}{x^9} = 10x^{-9}$ has domain $(-\infty, 0) \cup (0, \infty)$, so $F(x) = \begin{cases} \dfrac{10x^{-8}}{-8} + C_1 = -\dfrac{5}{4x^8} + C_1 & \text{if } x < 0 \\ -\dfrac{5}{4x^8} + C_2 & \text{if } x > 0 \end{cases}$

See Example 1(c) for a similar problem.

18. $g(x) = \dfrac{5 - 4x^3 + 2x^6}{x^6} = 5x^{-6} - 4x^{-3} + 2$ has domain $(-\infty, 0) \cup (0, \infty)$, so

$$G(x) = \begin{cases} 5\,\dfrac{x^{-5}}{-5} - 4\,\dfrac{x^{-2}}{-2} + 2x + C_1 = -\dfrac{1}{x^5} + \dfrac{2}{x^2} + 2x + C_1 & \text{if } x < 0 \\ -\dfrac{1}{x^5} + \dfrac{2}{x^2} + 2x + C_2 & \text{if } x > 0 \end{cases}$$

19. $f(\theta) = 2\sin\theta - 3\sec\theta\tan\theta \quad \Rightarrow \quad F(\theta) = -2\cos\theta - 3\sec\theta + C_n$ on the interval $\left(n\pi - \frac{\pi}{2}, n\pi + \frac{\pi}{2}\right)$.

20. $f(t) = 3\cos t - 4\sin t \quad \Rightarrow \quad F(t) = 3(\sin t) - 4(-\cos t) + C = 3\sin t + 4\cos t + C$

21. $h(\theta) = 2\sin\theta - \sec^2\theta \quad \Rightarrow \quad H(\theta) = -2\cos\theta - \tan\theta + C_n$ on the interval $\left(n\pi - \frac{\pi}{2}, n\pi + \frac{\pi}{2}\right)$, n an integer.

22. $h(x) = \sec^2 x + \pi\cos x \quad \Rightarrow \quad H(x) = \tan x + \pi\sin x + C_n$ on the interval $\left(n\pi - \frac{\pi}{2}, n\pi + \frac{\pi}{2}\right)$.

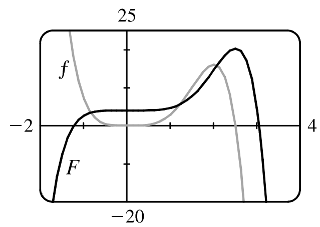
23. $g(v) = \sqrt[3]{v^2} - 2\sec^2 v \quad \Rightarrow \quad G(v) = \frac{3}{5}v^{5/3} - 2\tan v + C_n$ on $(n\pi - \frac{\pi}{2}, n\pi + \frac{\pi}{2})$, n an integer.

24. $f(x) = 1 + 2\sin x + 3/\sqrt{x} = 1 + 2\sin x + 3x^{-1/2} \quad \Rightarrow \quad F(x) = x - 2\cos x + 3\,\dfrac{x^{1/2}}{1/2} + C = x - 2\cos x + 6\sqrt{x} + C$

25. $f(x) = 5x^4 - 2x^5 \quad \Rightarrow \quad F(x) = 5 \cdot \dfrac{x^5}{5} - 2 \cdot \dfrac{x^6}{6} + C = x^5 - \frac{1}{3}x^6 + C.$

$F(0) = 4 \quad \Rightarrow \quad 0^5 - \frac{1}{3} \cdot 0^6 + C = 4 \quad \Rightarrow \quad C = 4$, so $F(x) = x^5 - \frac{1}{3}x^6 + 4$.

The graph confirms our answer since $f(x) = 0$ when F has a local maximum, f is positive when F is increasing, and f is negative when F is decreasing.

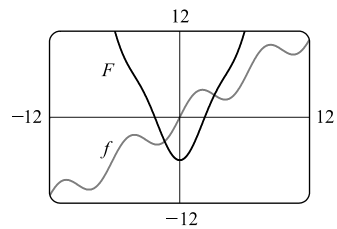


26. $f(x) = x + 2\sin x \quad \Rightarrow \quad F(x) = \frac{1}{2}x^2 - 2\cos x + C.$

$F(0) = -6 \quad \Rightarrow \quad 0 - 2 + C = -6 \quad \Rightarrow \quad C = -4$, so

$F(x) = \frac{1}{2}x^2 - 2\cos x - 4.$

The graph confirms our answer since $f(x) = 0$ when F has a local minimum, f is positive when F is increasing, and f is negative when F is decreasing.



27. $f''(x) = 24x \quad \Rightarrow \quad f'(x) = 24\left(\dfrac{x^2}{2}\right) + C = 12x^2 + C \quad \Rightarrow \quad f(x) = 12\left(\dfrac{x^3}{3}\right) + Cx + D = 4x^3 + Cx + D$

28. $f''(t) = t^2 - 4 \quad \Rightarrow \quad f'(t) = \dfrac{t^3}{3} - 4t + C \quad \Rightarrow \quad f(t) = \dfrac{1}{3}\left(\dfrac{t^4}{4}\right) - 4\left(\dfrac{t^2}{2}\right) + Ct + D = \dfrac{1}{12}t^4 - 2t^2 + Ct + D$

29. $f''(x) = 4x^3 + 24x - 1 \quad \Rightarrow \quad f'(x) = 4\left(\dfrac{x^4}{4}\right) + 24\left(\dfrac{x^2}{2}\right) - x + C = x^4 + 12x^2 - x + C \quad \Rightarrow$

$f(x) = \dfrac{x^5}{5} + 12\left(\dfrac{x^3}{3}\right) - \dfrac{x^2}{2} + Cx + D = \dfrac{1}{5}x^5 + 4x^3 - \dfrac{1}{2}x^2 + Cx + D$

30. $f''(x) = 6x - x^4 + 3x^5 \quad \Rightarrow \quad f'(x) = 6\left(\dfrac{x^2}{2}\right) - \dfrac{x^5}{5} + 3\left(\dfrac{x^6}{6}\right) + C = 3x^2 - \dfrac{1}{5}x^5 + \dfrac{1}{2}x^6 + C \quad \Rightarrow$

$f(x) = 3\left(\dfrac{x^3}{3}\right) - \dfrac{1}{5}\left(\dfrac{x^6}{6}\right) + \dfrac{1}{2}\left(\dfrac{x^7}{7}\right) + Cx + D = x^3 - \dfrac{1}{30}x^6 + \dfrac{1}{14}x^7 + Cx + D$

31. $f''(x) = 4 - \sqrt[3]{x} \quad \Rightarrow \quad f'(x) = 4x - \frac{3}{4}x^{4/3} + C \quad \Rightarrow \quad f(x) = 4 \cdot \frac{1}{2}x^2 - \frac{3}{4} \cdot \frac{3}{7}x^{7/3} + Cx + D = 2x^2 - \frac{9}{28}x^{7/3} + Cx + D$

32. $f''(x) = x^{2/3} + x^{-2/3}$ has domain $(-\infty, 0) \cup (0, \infty)$, so

$$f'(x) = \begin{cases} \frac{3}{5}x^{5/3} + 3x^{1/3} + C_1 & \text{if } x < 0 \\ \frac{3}{5}x^{5/3} + 3x^{1/3} + C_2 & \text{if } x > 0 \end{cases} \quad \text{and} \quad f(x) = \begin{cases} \frac{9}{40}x^{8/3} + \frac{9}{4}x^{4/3} + C_1x + D_1 & \text{if } x < 0 \\ \frac{9}{40}x^{8/3} + \frac{9}{4}x^{4/3} + C_2x + D_2 & \text{if } x > 0 \end{cases}$$

33. $f'''(t) = 12 + \sin t \quad \Rightarrow \quad f''(t) = 12t - \cos t + C_1 \quad \Rightarrow \quad f'(t) = 6t^2 - \sin t + C_1t + D \quad \Rightarrow$

$f(t) = 2t^3 + \cos t + Ct^2 + Dt + E$, where $C = \frac{1}{2}C_1$.

34. $f'''(t) = \sqrt{t} - 2\cos t = t^{1/2} - 2\cos t \quad \Rightarrow \quad f''(t) = \frac{2}{3}t^{3/2} - 2\sin t + C_1 \quad \Rightarrow \quad f'(t) = \frac{4}{15}t^{5/2} + 2\cos t + C_1t + D \quad \Rightarrow$

$f(t) = \frac{8}{105}t^{7/2} + 2\sin t + Ct^2 + Dt + E$, where $C = \frac{1}{2}C_1$.

35. $f'(x) = 5x^4 - 3x^2 + 4 \quad\Rightarrow\quad f(x) = x^5 - x^3 + 4x + C.\quad f(-1) = -1 + 1 - 4 + C$ and $f(-1) = 2 \quad\Rightarrow$
$-4 + C = 2 \quad\Rightarrow\quad C = 6$, so $f(x) = x^5 - x^3 + 4x + 6$.

36. $f'(x) = \sqrt{x} - 2 \quad\Rightarrow\quad f(x) = \frac{2}{3}x^{3/2} - 2x + C$.
$f(9) = \frac{2}{3}(9)^{3/2} - 2(9) + C = 18 - 18 + C = C$ and $f(9) = 4 \quad\Rightarrow\quad C = 4$, so $f(x) = \frac{2}{3}x^{3/2} - 2x + 4$.

37. $f'(x) = 5x^{2/3} \quad\Rightarrow\quad f(x) = 5\left(\frac{3}{5}x^{5/3}\right) + C = 3x^{5/3} + C$.
$f(8) = 3 \cdot 32 + C$ and $f(8) = 21 \quad\Rightarrow\quad 96 + C = 21 \quad\Rightarrow\quad C = -75$, so $f(x) = 3x^{5/3} - 75$.

38. $f'(t) = t + \dfrac{1}{t^3},\ t > 0 \quad\Rightarrow\quad f(t) = \dfrac{1}{2}t^2 - \dfrac{1}{2t^2} + C.\quad f(1) = \dfrac{1}{2} - \dfrac{1}{2} + C$ and $f(1) = 6 \quad\Rightarrow\quad C = 6$, so
$f(t) = \dfrac{1}{2}t^2 - \dfrac{1}{2t^2} + 6$.

39. $f'(t) = \sec t\,(\sec t + \tan t) = \sec^2 t + \sec t\,\tan t,\ -\frac{\pi}{2} < t < \frac{\pi}{2} \quad\Rightarrow\quad f(t) = \tan t + \sec t + C.\quad f\left(\frac{\pi}{4}\right) = 1 + \sqrt{2} + C$
and $f\left(\frac{\pi}{4}\right) = -1 \quad\Rightarrow\quad 1 + \sqrt{2} + C = -1 \quad\Rightarrow\quad C = -2 - \sqrt{2}$, so $f(t) = \tan t + \sec t - 2 - \sqrt{2}$.
Note: The fact that f is defined and continuous on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ means that we have only one constant of integration.

40. $f'(x) = \dfrac{x+1}{\sqrt{x}} = x^{1/2} + x^{-1/2} \quad\Rightarrow\quad f(x) = \frac{2}{3}x^{3/2} + 2x^{1/2} + C$.
$f(1) = \frac{2}{3} + 2 + C = \frac{8}{3} + C$ and $f(1) = 5 \quad\Rightarrow\quad C = 5 - \frac{8}{3} = \frac{7}{3}$, so $f(x) = \frac{2}{3}x^{3/2} + 2\sqrt{x} + \frac{7}{3}$.

41. $f''(x) = -2 + 12x - 12x^2 \quad\Rightarrow\quad f'(x) = -2x + 6x^2 - 4x^3 + C.\ f'(0) = C$ and $f'(0) = 12 \quad\Rightarrow\quad C = 12$, so
$f'(x) = -2x + 6x^2 - 4x^3 + 12$ and hence, $f(x) = -x^2 + 2x^3 - x^4 + 12x + D.\ f(0) = D$ and $f(0) = 4 \quad\Rightarrow\quad D = 4$,
so $f(x) = -x^2 + 2x^3 - x^4 + 12x + 4$.

42. $f''(x) = 8x^3 + 5 \quad\Rightarrow\quad f'(x) = 2x^4 + 5x + C.\ f'(1) = 2 + 5 + C$ and $f'(1) = 8 \quad\Rightarrow\quad C = 1$, so
$f'(x) = 2x^4 + 5x + 1.\ f(x) = \frac{2}{5}x^5 + \frac{5}{2}x^2 + x + D.\ f(1) = \frac{2}{5} + \frac{5}{2} + 1 + D = D + \frac{39}{10}$ and $f(1) = 0 \quad\Rightarrow\quad D = -\frac{39}{10}$,
so $f(x) = \frac{2}{5}x^5 + \frac{5}{2}x^2 + x - \frac{39}{10}$.

43. $f''(\theta) = \sin\theta + \cos\theta \quad\Rightarrow\quad f'(\theta) = -\cos\theta + \sin\theta + C.\ f'(0) = -1 + C$ and $f'(0) = 4 \quad\Rightarrow\quad C = 5$, so
$f'(\theta) = -\cos\theta + \sin\theta + 5$ and hence, $f(\theta) = -\sin\theta - \cos\theta + 5\theta + D.\ f(0) = -1 + D$ and $f(0) = 3 \quad\Rightarrow\quad D = 4$,
so $f(\theta) = -\sin\theta - \cos\theta + 5\theta + 4$.

44. $f''(t) = 4 - 6/t^4 \quad\Rightarrow\quad f'(t) = 4t - 6\left(-\frac{1}{3}\right)t^{-3} + C = 4t + 2/t^3 + C$ for $t > 0$. Now $f'(2) = 9 \quad\Rightarrow$
$8 + \frac{1}{4} + C = 9 \quad\Rightarrow\quad C = \frac{3}{4}$, so $f'(t) = 4t + 2/t^3 + \frac{3}{4}.\ f(t) = 2t^2 - 1/t^2 + \frac{3}{4}t + D$ and $f(1) = 6 \quad\Rightarrow$
$2 - 1 + \frac{3}{4} + D = 6 \quad\Rightarrow\quad D = \frac{17}{4}$, so $f(t) = 2t^2 - 1/t^2 + \frac{3}{4}t + \frac{17}{4}$.

45. $f''(x) = 4 + 6x + 24x^2 \quad\Rightarrow\quad f'(x) = 4x + 3x^2 + 8x^3 + C \quad\Rightarrow\quad f(x) = 2x^2 + x^3 + 2x^4 + Cx + D.\quad f(0) = D$ and
$f(0) = 3 \quad\Rightarrow\quad D = 3$, so $f(x) = 2x^2 + x^3 + 2x^4 + Cx + 3.\quad f(1) = 8 + C$ and $f(1) = 10 \quad\Rightarrow\quad C = 2$,
so $f(x) = 2x^2 + x^3 + 2x^4 + 2x + 3$.

46. $f''(x) = 20x^3 + 12x^2 + 4 \quad\Rightarrow\quad f'(x) = 5x^4 + 4x^3 + 4x + C \quad\Rightarrow\quad f(x) = x^5 + x^4 + 2x^2 + Cx + D$.

$f(0) = D$ and $f(0) = 8 \quad\Rightarrow\quad D = 8$. $f(1) = 1 + 1 + 2 + C + 8 = C + 12$ and $f(1) = 5 \quad\Rightarrow\quad C = -7$, so

$f(x) = x^5 + x^4 + 2x^2 - 7x + 8$.

47. $f''(t) = \sqrt[3]{t} - \cos t = t^{1/3} - \cos t \quad\Rightarrow\quad f'(t) = \frac{3}{4}t^{4/3} - \sin t + C \quad\Rightarrow\quad f(t) = \frac{9}{28}t^{7/3} + \cos t + Ct + D$.

$f(0) = 0 + 1 + 0 + D$ and $f(0) = 2 \quad\Rightarrow\quad D = 1$, so $f(t) = \frac{9}{28}t^{7/3} + \cos t + Ct + 1$. $f(1) = \frac{9}{28} + \cos 1 + C + 1$ and

$f(1) = 2 \quad\Rightarrow\quad C = 2 - \frac{9}{28} - \cos 1 - 1 = \frac{19}{28} - \cos 1$, so $f(t) = \frac{9}{28}t^{7/3} + \cos t + \left(\frac{19}{28} - \cos 1\right)t + 1$.

48. $f'''(x) = \cos x \quad\Rightarrow\quad f''(x) = \sin x + C$. $f''(0) = C$ and $f''(0) = 3 \quad\Rightarrow\quad C = 3$. $f''(x) = \sin x + 3 \quad\Rightarrow$

$f'(x) = -\cos x + 3x + D$. $f'(0) = -1 + D$ and $f'(0) = 2 \quad\Rightarrow\quad D = 3$. $f'(x) = -\cos x + 3x + 3 \quad\Rightarrow$

$f(x) = -\sin x + \frac{3}{2}x^2 + 3x + E$. $f(0) = E$ and $f(0) = 1 \quad\Rightarrow\quad E = 1$. Thus, $f(x) = -\sin x + \frac{3}{2}x^2 + 3x + 1$.

49. "The slope of its tangent line at $(x, f(x))$ is $3 - 4x$" means that $f'(x) = 3 - 4x$, so $f(x) = 3x - 2x^2 + C$.

"The graph of f passes through the point $(2, 5)$" means that $f(2) = 5$, but $f(2) = 3(2) - 2(2)^2 + C$, so $5 = 6 - 8 + C \quad\Rightarrow$

$C = 7$. Thus, $f(x) = 3x - 2x^2 + 7$ and $f(1) = 3 - 2 + 7 = 8$.

50. $f'(x) = x^3 \quad\Rightarrow\quad f(x) = \frac{1}{4}x^4 + C$. $x + y = 0 \quad\Rightarrow\quad y = -x \quad\Rightarrow\quad m = -1$. Now $m = f'(x) \quad\Rightarrow\quad -1 = x^3 \quad\Rightarrow$

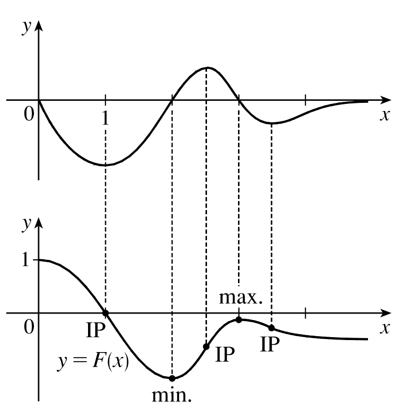
$x = -1 \quad\Rightarrow\quad y = 1$ (from the equation of the tangent line), so $(-1, 1)$ is a point on the graph of f. From f,

$1 = \frac{1}{4}(-1)^4 + C \quad\Rightarrow\quad C = \frac{3}{4}$. Therefore, the function is $f(x) = \frac{1}{4}x^4 + \frac{3}{4}$.

51. b is the antiderivative of f. For small x, f is negative, so the graph of its antiderivative must be decreasing. But both a and c are increasing for small x, so only b can be f's antiderivative. Also, f is positive where b is increasing, which supports our conclusion.

52. We know right away that c cannot be f's antiderivative, since the slope of c is not zero at the x-value where $f = 0$. Now f is positive when a is increasing and negative when a is decreasing, so a is the antiderivative of f.

53.



The graph of F must start at $(0, 1)$. Where the given graph, $y = f(x)$, has a local minimum or maximum, the graph of F will have an inflection point.

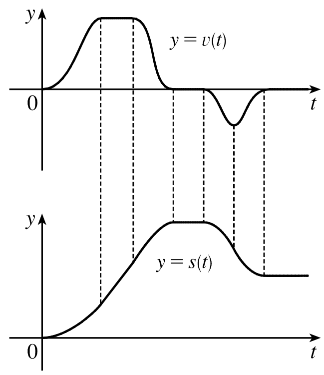
Where f is negative (positive), F is decreasing (increasing).

Where f changes from negative to positive, F will have a minimum.

Where f changes from positive to negative, F will have a maximum.

Where f is decreasing (increasing), F is concave downward (upward).

54.

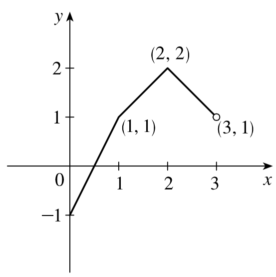


Where v is positive (negative), s is increasing (decreasing).

Where v is increasing (decreasing), s is concave upward (downward).

Where v is horizontal (a steady velocity), s is linear.

55.



$$f'(x) = \begin{cases} 2 & \text{if } 0 \le x < 1 \\ 1 & \text{if } 1 < x < 2 \\ -1 & \text{if } 2 < x < 3 \end{cases} \quad \Rightarrow \quad f(x) = \begin{cases} 2x + C & \text{if } 0 \le x < 1 \\ x + D & \text{if } 1 < x < 2 \\ -x + E & \text{if } 2 < x < 3 \end{cases}$$

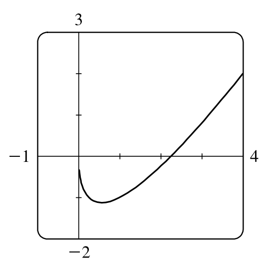
$f(0) = -1 \quad \Rightarrow \quad 2(0) + C = -1 \quad \Rightarrow \quad C = -1$. Starting at the point $(0, -1)$ and moving to the right on a line with slope 2 gets us to the point $(1, 1)$.

The slope for $1 < x < 2$ is 1, so we get to the point $(2, 2)$. Here we have used the fact that f is continuous. We can include the point $x = 1$ on either the first or the second part of f. The line connecting $(1, 1)$ to $(2, 2)$ is $y = x$, so $D = 0$. The slope for $2 < x < 3$ is -1, so we get to $(3, 1)$. $\quad f(2) = 2 \quad \Rightarrow \quad -2 + E = 2 \quad \Rightarrow \quad E = 4$. Thus,

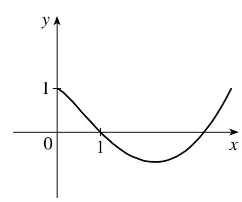
$$f(x) = \begin{cases} 2x - 1 & \text{if } 0 \le x \le 1 \\ x & \text{if } 1 < x < 2 \\ -x + 4 & \text{if } 2 \le x < 3 \end{cases}$$

Note that $f'(x)$ does not exist at $x = 1$, 2, or 3.

56. (a)



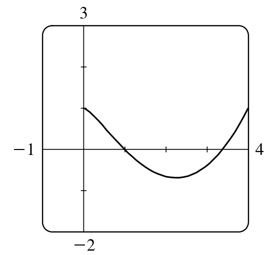
(b) Since $F(0) = 1$, we can start our graph at $(0, 1)$. f has a minimum at about $x = 0.5$, so its derivative is zero there. f is decreasing on $(0, 0.5)$, so its derivative is negative and hence, F is CD on $(0, 0.5)$ and has an IP at $x \approx 0.5$. On $(0.5, 2.2)$, f is negative and increasing (f' is positive), so F is decreasing and CU. On $(2.2, \infty)$, f is positive and increasing, so F is increasing and CU.



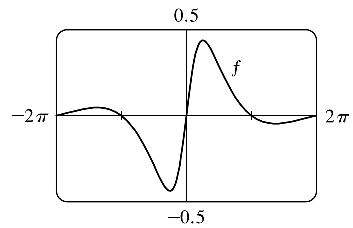
(c) $f(x) = 2x - 3\sqrt{x} \quad \Rightarrow \quad F(x) = x^2 - 3 \cdot \frac{2}{3}x^{3/2} + C.$

$F(0) = C$ and $F(0) = 1 \quad \Rightarrow \quad C = 1$, so $F(x) = x^2 - 2x^{3/2} + 1.$

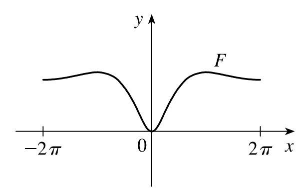
(d)



57. $f(x) = \dfrac{\sin x}{1 + x^2}, \ -2\pi \le x \le 2\pi$

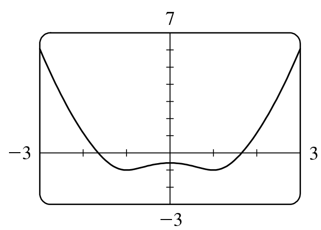


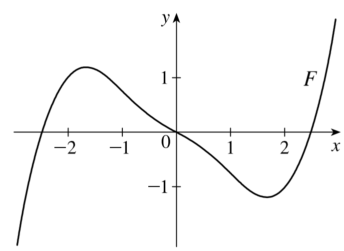
Note that the graph of f is one of an odd function, so the graph of F will be one of an even function.



58. $f(x) = \sqrt{x^4 - 2x^2 + 2} - 2, \ -3 \le x \le 3$

Note that the graph of f is one of an even function, so the graph of F will be one of an odd function.





59. $v(t) = s'(t) = 2\cos t + 4\sin t \quad \Rightarrow \quad s(t) = 2\sin t - 4\cos t + C. \quad s(0) = -4 + C$ and $s(0) = 3 \quad \Rightarrow \quad C = 7$, so $s(t) = 2\sin t - 4\cos t + 7.$

60. $v(t) = s'(t) = t^2 - 3\sqrt{t} = t^2 - 3t^{1/2} \quad \Rightarrow \quad s(t) = \frac{1}{3}t^3 - 2t^{3/2} + C. \quad s(4) = \frac{64}{3} - 16 + C$ and $s(4) = 8 \quad \Rightarrow$ $C = 8 - \frac{64}{3} + 16 = \frac{8}{3}$, so $s(t) = \frac{1}{3}t^3 - 2t^{3/2} + \frac{8}{3}.$

61. $a(t) = v'(t) = 2t + 1 \quad \Rightarrow \quad v(t) = t^2 + t + C. \quad v(0) = C$ and $v(0) = -2 \quad \Rightarrow \quad C = -2$, so $v(t) = t^2 + t - 2$ and $s(t) = \frac{1}{3}t^3 + \frac{1}{2}t^2 - 2t + D. \quad s(0) = D$ and $s(0) = 3 \quad \Rightarrow \quad D = 3$, so $s(t) = \frac{1}{3}t^3 + \frac{1}{2}t^2 - 2t + 3.$

62. $a(t) = v'(t) = 3\cos t - 2\sin t \quad \Rightarrow \quad v(t) = 3\sin t + 2\cos t + C. \quad v(0) = 2 + C$ and $v(0) = 4 \quad \Rightarrow \quad C = 2$, so $v(t) = 3\sin t + 2\cos t + 2$ and $s(t) = -3\cos t + 2\sin t + 2t + D. \quad s(0) = -3 + D$ and $s(0) = 0 \quad \Rightarrow \quad D = 3$, so $s(t) = -3\cos t + 2\sin t + 2t + 3.$

63. $a(t) = v'(t) = \sin t - \cos t \quad \Rightarrow \quad v(t) = s'(t) = -\cos t - \sin t + C \quad \Rightarrow \quad s(t) = -\sin t + \cos t + Ct + D.$

$s(0) = 1 + D = 0$ and $s(\pi) = -1 + C\pi + D = 6 \quad \Rightarrow \quad D = -1$ and $C = \dfrac{8}{\pi}$. Thus, $s(t) = -\sin t + \cos t + \dfrac{8}{\pi}t - 1.$

64. $a(t) = t^2 - 4t + 6 \quad \Rightarrow \quad v(t) = \frac{1}{3}t^3 - 2t^2 + 6t + C \quad \Rightarrow \quad s(t) = \frac{1}{12}t^4 - \frac{2}{3}t^3 + 3t^2 + Ct + D$. $s(0) = D$ and $s(0) = 0 \quad \Rightarrow \quad D = 0$. $s(1) = \frac{29}{12} + C$ and $s(1) = 20 \quad \Rightarrow \quad C = \frac{211}{12}$. Thus, $s(t) = \frac{1}{12}t^4 - \frac{2}{3}t^3 + 3t^2 + \frac{211}{12}t$.

65. (a) We first observe that since the stone is dropped 450 m above the ground, $v(0) = 0$ and $s(0) = 450$.

$v'(t) = a(t) = -9.8 \quad \Rightarrow \quad v(t) = -9.8t + C$. Now $v(0) = 0 \quad \Rightarrow \quad C = 0$, so $v(t) = -9.8t \quad \Rightarrow$

$s(t) = -4.9t^2 + D$. Last, $s(0) = 450 \quad \Rightarrow \quad D = 450 \quad \Rightarrow \quad s(t) = 450 - 4.9t^2$.

(b) The stone reaches the ground when $s(t) = 0$. $450 - 4.9t^2 = 0 \quad \Rightarrow \quad t^2 = 450/4.9 \quad \Rightarrow \quad t_1 = \sqrt{450/4.9} \approx 9.58$ s.

(c) The velocity with which the stone strikes the ground is $v(t_1) = -9.8\sqrt{450/4.9} \approx -93.9$ m/s.

(d) This is just reworking parts (a) and (b) with $v(0) = -5$. Using $v(t) = -9.8t + C$, $v(0) = -5 \quad \Rightarrow \quad 0 + C = -5 \quad \Rightarrow$

$v(t) = -9.8t - 5$. So $s(t) = -4.9t^2 - 5t + D$ and $s(0) = 450 \quad \Rightarrow \quad D = 450 \quad \Rightarrow \quad s(t) = -4.9t^2 - 5t + 450$.

Solving $s(t) = 0$ by using the quadratic formula gives us $t = \left(5 \pm \sqrt{8845}\right)/(-9.8) \quad \Rightarrow \quad t_1 \approx 9.09$ s.

66. $v'(t) = a(t) = a \quad \Rightarrow \quad v(t) = at + C$ and $v_0 = v(0) = C \quad \Rightarrow \quad v(t) = at + v_0 \quad \Rightarrow$

$s(t) = \frac{1}{2}at^2 + v_0t + D \quad \Rightarrow \quad s_0 = s(0) = D \quad \Rightarrow \quad s(t) = \frac{1}{2}at^2 + v_0t + s_0$

67. By Exercise 66 with $a = -9.8$, $s(t) = -4.9t^2 + v_0t + s_0$ and $v(t) = s'(t) = -9.8t + v_0$. So

$[v(t)]^2 = (-9.8t + v_0)^2 = (9.8)^2 t^2 - 19.6v_0t + v_0^2 = v_0^2 + 96.04t^2 - 19.6v_0t = v_0^2 - 19.6\left(-4.9t^2 + v_0t\right)$.

But $-4.9t^2 + v_0t$ is just $s(t)$ without the s_0 term; that is, $s(t) - s_0$. Thus, $[v(t)]^2 = v_0^2 - 19.6\,[s(t) - s_0]$.

68. For the first ball, $s_1(t) = -4.9t^2 + 15t + 130$. For the second ball, $a_2(t) = \frac{dv}{dt} = -9.8 \implies v_2(t) = -9.8t + C$. Since $v_2(1) = -9.8 \times 1 + C = 8 \implies C = 17.8$; therefore, $v_2(t) = -9.8t + 17.8$. So $s_2(t) = -4.9t^2 + 17.8t + D$. Since $s_2(1) = -4.9 \times 1 + 17.8 \times 1 + D = 130 \implies D = 117.1$; therefore $s_2(t) = -4.9t^2 + 17.8t + 117.1$. The balls pass each other when $s_1(t) = s_2(t)$, so $-4.9t^2 + 15t + 130 = -4.9t^2 + 17.8t + 117.1 \implies 2.8t = 12.9 \implies t \approx 4.6$ s.

69. Using Exercise 66 with $a = -9.8$, $v_0 = 0$, and $s_0 = h$ (the height of the cliff), we know that the height at time t is $s(t) = -4.9 + h$. $v(t) = s'(t) = -9.8t$ and $v(t) = -40 \Rightarrow -9.8t = -40 \Rightarrow t = 4.08$, so $0 = s(4.08) = -4.9(4.08)^2 + h \Rightarrow h = -4.9(4.08)^2 = 82$ m.

70. (a) $EIy'' = mg(L - x) + \frac{1}{2}\rho g(L - x)^2 \quad \Rightarrow \quad EIy' = -\frac{1}{2}mg(L - x)^2 - \frac{1}{6}\rho g(L - x)^3 + C \quad \Rightarrow$

$EIy = \frac{1}{6}mg(L - x)^3 + \frac{1}{24}\rho g(L - x)^4 + Cx + D$. Since the left end of the board is fixed, we must have $y = y' = 0$ when $x = 0$. Thus, $0 = -\frac{1}{2}mgL^2 - \frac{1}{6}\rho gL^3 + C$ and $0 = \frac{1}{6}mgL^3 + \frac{1}{24}\rho gL^4 + D$. It follows that

$EIy = \frac{1}{6}mg(L - x)^3 + \frac{1}{24}\rho g(L - x)^4 + \left(\frac{1}{2}mgL^2 + \frac{1}{6}\rho gL^3\right)x - \left(\frac{1}{6}mgL^3 + \frac{1}{24}\rho gL^4\right)$ and

$$f(x) = y = \frac{1}{EI}\left[\tfrac{1}{6}mg(L - x)^3 + \tfrac{1}{24}\rho g(L - x)^4 + \left(\tfrac{1}{2}mgL^2 + \tfrac{1}{6}\rho gL^3\right)x - \left(\tfrac{1}{6}mgL^3 + \tfrac{1}{24}\rho gL^4\right)\right]$$

(b) $f(L) < 0$, so the end of the board is a *distance* approximately $-f(L)$ below the horizontal. From our result in (a), we calculate

$$-f(L) = \frac{-1}{EI}\left[\tfrac{1}{2}mgL^3 + \tfrac{1}{6}\rho gL^4 - \tfrac{1}{6}mgL^3 - \tfrac{1}{24}\rho gL^4\right] = \frac{-1}{EI}\left(\tfrac{1}{3}mgL^3 + \tfrac{1}{8}\rho gL^4\right) = -\frac{gL^3}{EI}\left(\frac{m}{3} + \frac{\rho L}{8}\right)$$

Note: This is positive because g is negative.

71. Marginal cost $= 1.92 - 0.002x = C'(x) \quad\Rightarrow\quad C(x) = 1.92x - 0.001x^2 + K$. But $C(1) = 1.92 - 0.001 + K = 562 \quad\Rightarrow$ $K = 560.081$. Therefore, $C(x) = 1.92x - 0.001x^2 + 560.081 \quad\Rightarrow\quad C(100) = 742.081$, so the cost of producing 100 items is \$742.08.

72. Let the mass, measured from one end, be $m(x)$. Then $m(0) = 0$ and $\rho = \dfrac{dm}{dx} = x^{-1/2} \quad\Rightarrow\quad m(x) = 2x^{1/2} + C$ and $m(0) = C = 0$, so $m(x) = 2\sqrt{x}$. Thus, the mass of the 100-centimeter rod is $m(100) = 2\sqrt{100} = 20$ g.

73. Taking the upward direction to be positive we have that for $0 \le t \le 10$ (using the subscript 1 to refer to $0 \le t \le 10$),

$a_1(t) = -(9 - 0.9t) = v_1'(t) \quad\Rightarrow\quad v_1(t) = -9t + 0.45t^2 + v_0$, but $v_1(0) = v_0 = -10 \quad\Rightarrow$

$v_1(t) = -9t + 0.45t^2 - 10 = s_1'(t) \quad\Rightarrow\quad s_1(t) = -\frac{9}{2}t^2 + 0.15t^3 - 10t + s_0$. But $s_1(0) = 500 = s_0 \quad\Rightarrow$

$s_1(t) = -\frac{9}{2}t^2 + 0.15t^3 - 10t + 500$. $s_1(10) = -450 + 150 - 100 + 500 = 100$, so it takes

more than 10 seconds for the raindrop to fall. Now for $t > 10$, $a(t) = 0 = v'(t) \quad\Rightarrow$

$v(t) = \text{constant} = v_1(10) = -9(10) + 0.45(10)^2 - 10 = -55 \quad\Rightarrow\quad v(t) = -55$.

At 55 m/s, it will take $100/55 \approx 1.8$ s to fall the last 100 m. Hence, the total time is $10 + \frac{100}{55} = \frac{130}{11} \approx 11.8$ s.

74. $v'(t) = a(t) = -7$. The initial velocity is $80\text{ km/h} = \frac{80\cdot 1000}{3600} = \frac{200}{9}\text{ m/s}$, so $v(t) = -7t + \frac{200}{9}$.

The car stops when $v(t) = 0 \Leftrightarrow t = \frac{200}{9.7} = \frac{200}{63}$. Since $s(t) = -\frac{7t^2}{2} + \frac{200}{9}t$, the distance covered is $s\left(\frac{200}{63}\right) = -\frac{7}{2}\left(\frac{200}{63}\right)^2 + \frac{200}{9}\left(\frac{200}{63}\right) = \frac{20{,}000}{567} \approx 35.27$ m.

75. $a(t) = k$, the initial velocity is $50\text{ km/h} = 50\cdot\frac{1000}{3600} = \frac{125}{9}\text{ m/s}$ and the final velocity (after 5 seconds) is $80\text{ km/h} = 80\cdot\frac{1000}{3600} = \frac{200}{9}\text{ m/s}$. So $v(t) = kt + C$ and $v(0) = {}^{125}\!/_9 \Rightarrow C = {}^{125}\!/_9$. Thus, $v(t) = kt + {}^{125}\!/_9 \Rightarrow v(5) = 5k + \frac{125}{9}$. But $v(5) = \frac{200}{9}$, so $5k + \frac{125}{9} = \frac{200}{9} \Rightarrow 5k = \frac{75}{9} \Rightarrow k = \frac{5}{3} \approx 1.67\text{ m/s}^2$.

76. $a(t) = -5 \Rightarrow v(t) = -5t + v_0$ where v_0 where is the car's speed (in m/s) when the brakes were applied. The car stops when $-5t + v_0 = 0 \Leftrightarrow t = {}^1\!/_5 v_0$. Now $s(t) = \frac{1}{2}(-5)t^2 + v_0 t = -{}^5\!/_2 + v_0 t$. The car travels 60 m in the time that it takes to stop, so $s({}^1\!/_5 v_0) = 60 \Rightarrow 60 - \frac{5}{2}\left(\frac{1}{5}v_0\right)^2 + v_0\left(\frac{1}{5}v_0\right) = \frac{1}{10}v_0^2 \Rightarrow v_0^2 = 10.60 = 600 \Rightarrow 24\text{ m/s}\ [86.4\text{ km/h}]$.

77. Let the acceleration be $a(t) = k\text{ km/h}^2$. We have $v(0) = 100\text{ km/h}$ and we can take the initial position $s(0)$ to be 0.

We want the time t_f for which $v(t) = 0$ to satisfy $s(t) < 0.08$ km. In general, $v'(t) = a(t) = k$, so $v(t) = kt + C$, where $C = v(0) = 100$. Now $s'(t) = v(t) = kt + 100$, so $s(t) = \frac{1}{2}kt^2 + 100t + D$, where $D = s(0) = 0$.

Thus, $s(t) = \frac{1}{2}kt^2 + 100t$. Since $v(t_f) = 0$, we have $kt_f + 100 = 0$ or $t_f = -100/k$, so

$s(t_f) = \dfrac{1}{2}k\left(-\dfrac{100}{k}\right)^2 + 100\left(-\dfrac{100}{k}\right) = 10{,}000\left(\dfrac{1}{2k} - \dfrac{1}{k}\right) = -\dfrac{5{,}000}{k}$. The condition $s(t_f)$ must satisfy is

$-\dfrac{5{,}000}{k} < 0.08 \quad \Rightarrow \quad -\dfrac{5{,}000}{0.08} > k \quad [k \text{ is negative}] \quad \Rightarrow \quad k < -62{,}500 \text{ km/h}^2$, or equivalently,

$k < -\frac{3125}{648} \approx -4.82 \text{ m/s}^2$.

78. (a) When $0 < t \le 3$, $a(t) = 18t \implies v(t) = 9t^2 + c$, but $v(0) = 0 \implies c = 0$, so $v(t) = 9t^2$, therefore $s(t) = 3t^3 + d$, but $s(0) = 0 \implies d = 0$, hence $s(t) = 3t^3$ and we could get $v(3) = 81$ s, $(3) = 81$.

When $3 < t \le 17$. $a(t) = -9.8 \implies v(t) = -9.8(t-3) + c$, but $v(3) = c = 81$, so $v(t) = -9.8(t-3) + 81$, therefore $s(t) = -4.9(t-3)^2 + 81(t-3) + d$, but $s(3) = d = 81$, hence $s(t) = -4.9(t-3)^2 + 81(t-3) + 81$ and we could get $v(17) = -56.2$ s, $(17) = 254.6$.

When $17 < t \le 22$, $\frac{\Delta v}{\Delta t} = \frac{-5.5-(-56.2)}{22-17} = \frac{50.7}{5} = 10.14 \implies v(t) = 10.14(t-17) + c$, but $v(17) = -56.2$, so $v(t) = 10.14(t-17) - 56.2$, therefore $s(t) = 5.07(t-17)^2 - 56.2(t-17) + d$, but $s(17) = 254.6$, hence $s(t) = 5.07(t-17)^2 - 56.2(t-17) + 254.6$ and we could get $v(22) = -5.5$ s, $(22) = 100.35$.

When $t > 22$, $v(t) = -5.5$, so $s(t) = -5.5(t-22) + c$, but $s(22) = 100.35$, so $s(t) = -5.5(t-22) + 100.35$.

In a summary,

$$v(t) = \begin{cases} 9t^2 & 0 < t \le 3 \\ -9.8(t-3) + 81 & 3 < t \le 17 \\ 10.14(t-17) - 56.2 & 17 < t \le 22 \\ -5.5 & t > 22 \end{cases}$$

$$s(t) = \begin{cases} 3t^3 & 0 < t \le 3 \\ -4.9(t-3)^2 + 81(t-3) + 81 & 3 < t \le 17 \\ 5.07(t-17)^2 - 56.2(t-17) + 254.6 & 17 < t \le 22 \\ -5.5(t-22) + 100.35 & t > 22 \end{cases}$$

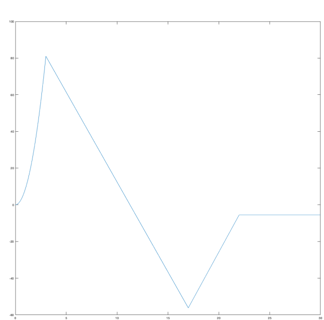


Figure 1: $v(t)$

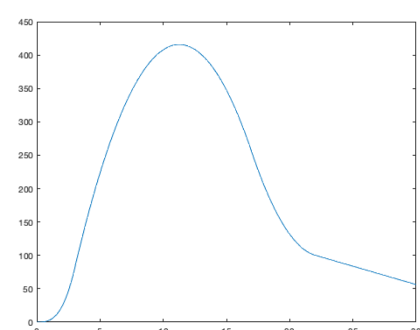


Figure 2: $s(t)$

(b) To find the maximum height, set $v(t)$ on $3 < t \leq 17$ equal to 0 such that $-9.8(t-3)+81=0 \implies t_1 \approx 11.27$ s.

(c) To find the time to land, set $s(t) = -5.5(t-22)+100.35 = 0$, then $t - 22 = \frac{100.35}{5.5} \implies t = 22 + \frac{100.35}{5.5} \approx 40.25$ s.

79. (a) First note that 145 km/h $= \frac{725}{18}$ m/s and $a(t) = 1.2$ m/s $\implies v(t) = 1.2t + c$, but $v(0) = 0 \implies c = 0$, so $v(t) = 1.2t$. Now $1.2t = \frac{725}{18} \implies t \approx 33.56$ s, Therefore $s(t) = 0.6t^2 + d$, since $s(0) = 0 \implies d = 0$, so $s(t) = 0.6t^2$, therefore $s(33.56) = 0.6 \times 33.56^2 \approx 675.76$ m and we know 20 minutes $= 20 \times 60 = 1200$ s, so for $33.56 < t < 1233.56$, we have $v(t) = \frac{725}{18} \implies s(1233.56) = \frac{725}{18}(1233.56 - 33.56) + 675.76 \approx 49{,}009.09$ m ≈ 49 km.

(b) As in part (a), the train accelerates for 33.56 s and travels 675.76 m while doing so. Similarly, it decelerates for 33.56 s and travels 675.76 m at the end of this trip. During the remaining $1200 - 2 \times 33.56 = 1132.88$ s it travels at $\frac{725}{18}$ m/s, so the distance traveled is $\frac{725}{18} \times 1132.88 \approx 45{,}629.89$ m. Thus, the total distance is $675.76 \times 2 + 45{,}629.89 = 46{,}981.41$ m ≈ 47 km/h.

(c) 72 km $=$ 72000 m. Subtract $2(675.76)$ to take care of the speeding up and slowing down, and we have 70,648.48 m at $\frac{725}{18}$ m/s for a trip of $70{,}648.48/(\frac{725}{18}) \approx 1754$ s at 145 km/h. The total time is $1754 + 2(33.56) = 1821.12$ s ≈ 30.352 min.

(d) $37.5 \times 60 = 2250$ s. $2250 - 2 \times 33.56 = 2182.88$ s at maximum speed. $2182.88(\frac{725}{18}) + 2(675.76) = 89{,}273.07556$ m $\approx$ 89 km.

3 Review

TRUE-FALSE QUIZ

1. False. For example, take $f(x) = x^3$, then $f'(x) = 3x^2$ and $f'(0) = 0$, but $f(0) = 0$ is not a maximum or minimum; $(0,0)$ is an inflection point.

2. False. For example, $f(x) = |x|$ has an absolute minimum at 0, but $f'(0)$ does not exist.

3. False. For example, $f(x) = x$ is continuous on $(0,1)$ but attains neither a maximum nor a minimum value on $(0,1)$. Don't confuse this with f being continuous on the *closed* interval $[a,b]$, which would make the statement true.

4. True. By the Mean Value Theorem, $f'(c) = \dfrac{f(1) - f(-1)}{1-(-1)} = \dfrac{0}{2} = 0$. Note that $|c| < 1 \quad \Leftrightarrow \quad c \in (-1,1)$.

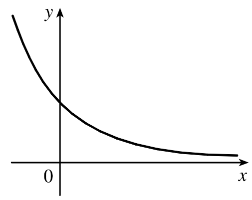
5. True. This is an example of part (b) of the Increasing/Decreasing Test.

6. False. For example, the curve $y = f(x) = 1$ has no inflection points but $f''(c) = 0$ for all c.

7. False. $f'(x) = g'(x) \quad \Rightarrow \quad f(x) = g(x) + C$. For example, if $f(x) = x + 2$ and $g(x) = x + 1$, then $f'(x) = g'(x) = 1$, but $f(x) \neq g(x)$.

8. False. Assume there is a function f such that $f(1) = -2$ and $f(3) = 0$. Then by the Mean Value Theorem there exists a number $c \in (1, 3)$ such that $f'(c) = \dfrac{f(3) - f(1)}{3 - 1} = \dfrac{0 - (-2)}{2} = 1$. But $f'(x) > 1$ for all x, a contradiction.

9. True. The graph of one such function is sketched.



10. False. At any point $(a, f(a))$, we know that $f'(a) < 0$. So since the tangent line at $(a, f(a))$ is not horizontal, it must cross the x-axis—at $x = b$, say. But since $f''(x) > 0$ for all x, the graph of f must lie above all of its tangents; in particular, $f(b) > 0$. But this is a contradiction, since we are given that $f(x) < 0$ for all x.

11. True. Let $x_1 < x_2$ where $x_1, x_2 \in I$. Then $f(x_1) < f(x_2)$ and $g(x_1) < g(x_2)$ [since f and g are increasing on I], so $(f + g)(x_1) = f(x_1) + g(x_1) < f(x_2) + g(x_2) = (f + g)(x_2)$.

12. False. $f(x) = x$ and $g(x) = 2x$ are both increasing on $(0, 1)$, but $f(x) - g(x) = -x$ is not increasing on $(0, 1)$.

13. False. Take $f(x) = x$ and $g(x) = x - 1$. Then both f and g are increasing on $(0, 1)$. But $f(x)\, g(x) = x(x - 1)$ is not increasing on $(0, 1)$.

14. True. Let $x_1 < x_2$ where $x_1, x_2 \in I$. Then $0 < f(x_1) < f(x_2)$ and $0 < g(x_1) < g(x_2)$ [since f and g are both positive and increasing]. Hence, $f(x_1)\, g(x_1) < f(x_2)\, g(x_1) < f(x_2)\, g(x_2)$. So fg is increasing on I.

15. True. Let $x_1, x_2 \in I$ and $x_1 < x_2$. Then $f(x_1) < f(x_2)$ [f is increasing] $\Rightarrow$ $\dfrac{1}{f(x_1)} > \dfrac{1}{f(x_2)}$ [f is positive] $\Rightarrow$ $g(x_1) > g(x_2)$ $\Rightarrow$ $g(x) = 1/f(x)$ is decreasing on I.

16. False. If f is even, then $f(x) = f(-x)$. Using the Chain Rule to differentiate this equation, we get $f'(x) = f'(-x)\dfrac{d}{dx}(-x) = -f'(-x)$. Thus, $f'(-x) = -f'(x)$, so f' is odd.

17. True. If f is periodic, then there is a number p such that $f(x + p) = f(p)$ for all x. Differentiating gives $f'(x) = f'(x + p) \cdot (x + p)' = f'(x + p) \cdot 1 = f'(x + p)$, so f' is periodic.

18. False. The most general antiderivative of $f(x) = x^{-2}$ is $F(x) = -1/x + C_1$ for $x < 0$ and $F(x) = -1/x + C_2$ for $x > 0$ [see Example 3.9.1(c)].

19. True. By the Mean Value Theorem, there exists a number c in $(0, 1)$ such that $f(1) - f(0) = f'(c)(1 - 0) = f'(c)$. Since $f'(c)$ is nonzero, $f(1) - f(0) \neq 0$, so $f(1) \neq f(0)$.

20. False. Consider $f(x) = \sin x$ for $x \geq 0$. $\lim\limits_{x \to \infty} f(x) \neq \pm\infty$ and f has no horizontal asymptote.

EXERCISES

1. $f(x) = x^3 - 9x^2 + 24x - 2$, $[0, 5]$. $f'(x) = 3x^2 - 18x + 24 = 3(x^2 - 6x + 8) = 3(x-2)(x-4)$. $f'(x) = 0 \Leftrightarrow x = 2$ or $x = 4$. $f'(x) > 0$ for $0 < x < 2$, $f'(x) < 0$ for $2 < x < 4$, and $f'(x) > 0$ for $4 < x < 5$, so $f(2) = 18$ is a local maximum value and $f(4) = 14$ is a local minimum value. Checking the endpoints, we find $f(0) = -2$ and $f(5) = 18$. Thus, $f(0) = -2$ is the absolute minimum value and $f(2) = f(5) = 18$ is the absolute maximum value.

2. $f(x) = x\sqrt{1-x}$, $[-1, 1]$. $f'(x) = x \cdot \frac{1}{2}(1-x)^{-1/2}(-1) + (1-x)^{1/2}(1) = (1-x)^{-1/2}\left[-\frac{1}{2}x + (1-x)\right] = \dfrac{1 - \frac{3}{2}x}{\sqrt{1-x}}$. $f'(x) = 0 \Rightarrow x = \frac{2}{3}$. $f'(x)$ does not exist $\Leftrightarrow x = 1$. $f'(x) > 0$ for $-1 < x < \frac{2}{3}$ and $f'(x) < 0$ for $\frac{2}{3} < x < 1$, so $f\left(\frac{2}{3}\right) = \frac{2}{3}\sqrt{\frac{1}{3}} = \frac{2}{9}\sqrt{3}$ $[\approx 0.38]$ is a local maximum value. Checking the endpoints, we find $f(-1) = -\sqrt{2}$ and $f(1) = 0$. Thus, $f(-1) = -\sqrt{2}$ is the absolute minimum value and $f\left(\frac{2}{3}\right) = \frac{2}{9}\sqrt{3}$ is the absolute maximum value.

3. $f(x) = \dfrac{3x-4}{x^2+1}$, $[-2, 2]$. $f'(x) = \dfrac{(x^2+1)(3) - (3x-4)(2x)}{(x^2+1)^2} = \dfrac{-(3x^2 - 8x - 3)}{(x^2+1)^2} = \dfrac{-(3x+1)(x-3)}{(x^2+1)^2}$. $f'(x) = 0 \Rightarrow x = -\frac{1}{3}$ or $x = 3$, but 3 is not in the interval. $f'(x) > 0$ for $-\frac{1}{3} < x < 2$ and $f'(x) < 0$ for $-2 < x < -\frac{1}{3}$, so $f\left(-\frac{1}{3}\right) = \frac{-5}{10/9} = -\frac{9}{2}$ is a local minimum value. Checking the endpoints, we find $f(-2) = -2$ and $f(2) = \frac{2}{5}$. Thus, $f\left(-\frac{1}{3}\right) = -\frac{9}{2}$ is the absolute minimum value and $f(2) = \frac{2}{5}$ is the absolute maximum value.

4. $f(x) = \sqrt{x^2 + x + 1}$, $[-2, 1]$. $f'(x) = \frac{1}{2}(x^2 + x + 1)^{-1/2}(2x+1) = \dfrac{2x+1}{2\sqrt{x^2+x+1}}$. $f'(x) = 0 \Rightarrow x = -\dfrac{1}{2}$. $f'(x) > 0$ for $-\frac{1}{2} < x < 1$ and $f'(x) < 0$ for $-2 < x < -\frac{1}{2}$, so $f\left(-\frac{1}{2}\right) = \sqrt{3}/2$ is a local minimum value. Checking the endpoints, we find $f(-2) = f(1) = \sqrt{3}$. Thus, $f\left(-\frac{1}{2}\right) = \sqrt{3}/2$ is the absolute minimum value and $f(-2) = f(1) = \sqrt{3}$ is the absolute maximum value.

5. $f(x) = x + 2\cos x$, $[-\pi, \pi]$. $f'(x) = 1 - 2\sin x$. $f'(x) = 0 \Rightarrow \sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$. $f'(x) > 0$ for $\left(-\pi, \frac{\pi}{6}\right)$ and $\left(\frac{5\pi}{6}, \pi\right)$, and $f'(x) < 0$ for $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$, so $f\left(\frac{\pi}{6}\right) = \frac{\pi}{6} + \sqrt{3} \approx 2.26$ is a local maximum value and $f\left(\frac{5\pi}{6}\right) = \frac{5\pi}{6} - \sqrt{3} \approx 0.89$ is a local minimum value. Checking the endpoints, we find $f(-\pi) = -\pi - 2 \approx -5.14$ and $f(\pi) = \pi - 2 \approx 1.14$. Thus, $f(-\pi) = -\pi - 2$ is the absolute minimum value and $f\left(\frac{\pi}{6}\right) = \frac{\pi}{6} + \sqrt{3}$ is the absolute maximum value.

6. $f(x) = \sin x + \cos^2 x$, $[0, \pi]$. $f'(x) = \cos x - 2\cos x \sin x = \cos x\,(1 - 2\sin x)$, so $f'(x) = 0$ for x in $(0, \pi)$ $\Leftrightarrow$ $\cos x = 0$ or $\sin x = \frac{1}{2} \Leftrightarrow x = \frac{\pi}{6}, \frac{\pi}{2}$, or $\frac{5\pi}{6}$. $f'(x) = \cos x - \sin 2x \Rightarrow f''(x) = -\sin x - 2\cos 2x$, so $f''\left(\frac{\pi}{6}\right) = -\frac{1}{2} - 2\left(\frac{1}{2}\right) = -\frac{3}{2}$, $f''\left(\frac{\pi}{2}\right) = -1 - 2(-1) = 1$, and $f''\left(\frac{5\pi}{6}\right) = -\frac{1}{2} - 2\left(\frac{1}{2}\right) = -\frac{3}{2}$. Thus, $f\left(\frac{\pi}{6}\right) = \frac{5}{4}$ and $f\left(\frac{5\pi}{6}\right) = \frac{5}{4}$ are local maxima and $f\left(\frac{\pi}{2}\right) = 1$ is a local minimum. $f(0) = 1$ and $f(\pi) = 1$, so f has its absolute minimum value of 1 at 0, $\frac{\pi}{2}$, and π. f attains its absolute maximum value of $\frac{5}{4}$ at $\frac{\pi}{6}$ and $\frac{5\pi}{6}$.

7. $\displaystyle \lim_{x\to\infty} \frac{3x^4+x-5}{6x^4-2x^2+1} = \lim_{x\to\infty} \frac{3+\dfrac{1}{x^3}-\dfrac{5}{x^4}}{6-\dfrac{2}{x^2}+\dfrac{1}{x^4}} = \frac{3+0+0}{6-0+0} = \frac{1}{2}$

8. $\displaystyle \lim_{t\to\infty} \frac{t^3-t+2}{(2t-1)(t^2+t+1)} = \lim_{t\to\infty} \frac{1-1/t^2+2/t^3}{(2-1/t)(1+1/t+1/t^2)} = \frac{1}{2\cdot 1} = \frac{1}{2}$

9. $\displaystyle \lim_{x\to-\infty} \frac{\sqrt{4x^2+1}}{3x-1} = \lim_{x\to-\infty} \frac{\sqrt{4x^2+1}/\sqrt{x^2}}{(3x-1)/\sqrt{x^2}} = \lim_{x\to-\infty} \frac{\sqrt{4+1/x^2}}{-3+1/x}$ [since $-x = |x| = \sqrt{x^2}$ for $x<0$]

$\displaystyle = \frac{2}{-3+0} = -\frac{2}{3}$

10. $\displaystyle \lim_{x\to-\infty} (x^2+x^3) = \lim_{x\to-\infty} x^2(1+x) = -\infty$ since $x^2 \to \infty$ and $1+x \to -\infty$ as $x \to -\infty$.

11. $\displaystyle \lim_{x\to\infty} \left(\sqrt{4x^2+3x}-2x\right) = \lim_{x\to\infty} \frac{\sqrt{4x^2+3x}-2x}{1} \cdot \frac{\sqrt{4x^2+3x}+2x}{\sqrt{4x^2+3x}+2x} = \lim_{x\to\infty} \frac{(4x^2+3x)-4x^2}{\sqrt{4x^2+3x}+2x}$

$\displaystyle = \lim_{x\to\infty} \frac{3x}{\sqrt{4x^2+3x}+2x} = \lim_{x\to\infty} \frac{3x/\sqrt{x^2}}{\left(\sqrt{4x^2+3x}+2x\right)/\sqrt{x^2}}$

$\displaystyle = \lim_{x\to\infty} \frac{3}{\sqrt{4+3/x}+2}$ [since $x = |x| = \sqrt{x^2}$ for $x>0$]

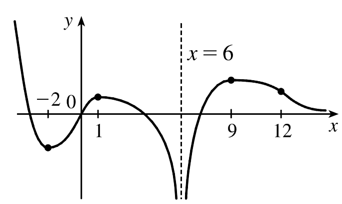
$\displaystyle = \frac{3}{2+2} = \frac{3}{4}$

12. $0 \le \sin^4 x \le 1$, so $0 \le \dfrac{\sin^4 x}{\sqrt{x}} \le \dfrac{1}{\sqrt{x}}$. Since $\displaystyle\lim_{x\to\infty} 0 = 0$ and $\displaystyle\lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$, $\displaystyle\lim_{x\to\infty} \frac{\sin^4 x}{\sqrt{x}} = 0$ by the Squeeze Theorem.

13. $f(0)=0$, $f'(-2)=f'(1)=f'(9)=0$, $\displaystyle\lim_{x\to\infty} f(x)=0$, $\displaystyle\lim_{x\to 6} f(x)=-\infty$,

$f'(x)<0$ on $(-\infty,-2)$, $(1,6)$, and $(9,\infty)$, $f'(x)>0$ on $(-2,1)$ and $(6,9)$,

$f''(x)>0$ on $(-\infty,0)$ and $(12,\infty)$, $f''(x)<0$ on $(0,6)$ and $(6,12)$



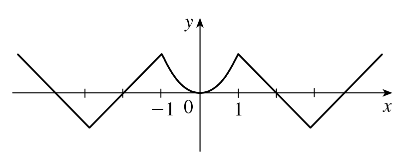
14. For $0<x<1$, $f'(x)=2x$, so $f(x)=x^2+C$. Since $f(0)=0$,

$f(x)=x^2$ on $[0,1]$. For $1<x<3$, $f'(x)=-1$, so $f(x)=-x+D$.

$1=f(1)=-1+D \quad\Rightarrow\quad D=2$, so $f(x)=2-x$. For $x>3$, $f'(x)=1$,

so $f(x)=x+E$. $-1=f(3)=3+E \quad\Rightarrow\quad E=-4$, so $f(x)=x-4$.

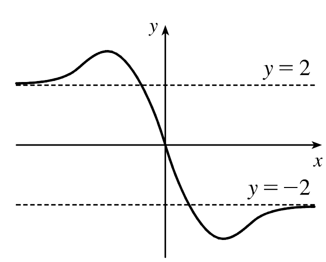
Since f is even, its graph is symmetric about the y-axis.



15. f is odd, $f'(x)<0$ for $0<x<2$, $\quad f'(x)>0$ for $x>2$,

$f''(x)>0$ for $0<x<3$, $\quad f''(x)<0$ for $x>3$,

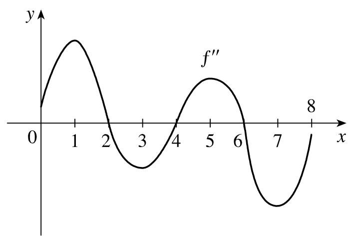
$\lim_{x\to\infty} f(x) = -2$



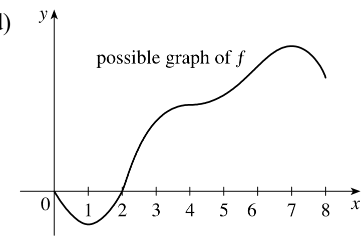
16. (a) Since $f'(x) > 0$ on $(1, 4)$ and $(4, 7)$, f is increasing on these intervals. Since $f'(x) < 0$ on $(0, 1)$ and $(7, 8)$, f is decreasing on these intervals.

(b) Since $f'(7) = 0$ and f' changes from positive to negative there, f changes from increasing to decreasing and has a local maximum at $x = 7$. Since $f'(x) = 0$ at $x = 1$ and f' changes from negative to positive there, f changes from decreasing to increasing and has a local minimum at $x = 1$.

(c)



(d)

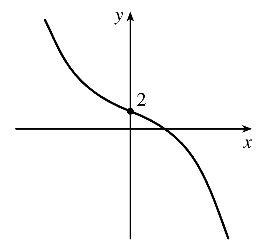


17. $y = f(x) = 2 - 2x - x^3$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 2$. The x-intercept (approximately 0.770917) can be found using Newton's Method. **C.** No symmetry **D.** No asymptote

E. $f'(x) = -2 - 3x^2 = -(3x^2 + 2) < 0$, so f is decreasing on $\mathbb{R}$.

F. No extreme values **G.** $f''(x) = -6x < 0$ on $(0, \infty)$ and $f''(x) > 0$ on $(-\infty, 0)$, so f is CD on $(0, \infty)$ and CU on $(-\infty, 0)$. There is an IP at $(0, 2)$.

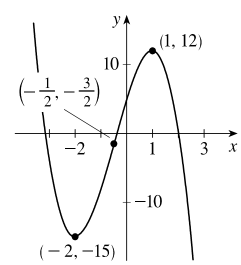
H.



18. $y = f(x) = -2x^3 - 3x^2 + 12x + 5$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 5$; x-intercept: $f(x) = 0 \Leftrightarrow x \approx -3.15, -0.39, 2.04$ **C.** No symmetry **D.** No asymptote

E. $f'(x) = -6x^2 - 6x + 12 = -6(x^2 + x - 2) = -6(x + 2)(x - 1)$. $f'(x) > 0$ for $-2 < x < 1$, so f is increasing on $(-2, 1)$ and decreasing on $(-\infty, -2)$ and $(1, \infty)$. **F.** Local minimum value $f(-2) = -15$, local maximum value $f(1) = 12$ **G.** $f''(x) = -12x - 6 = -12\left(x + \frac{1}{2}\right)$. $f''(x) > 0$ for $x < -\frac{1}{2}$, so f is CU on $\left(-\infty, -\frac{1}{2}\right)$ and CD on $\left(-\frac{1}{2}, \infty\right)$. There is an IP at $\left(-\frac{1}{2}, -\frac{3}{2}\right)$.

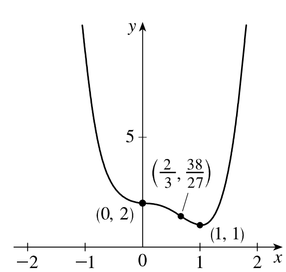
H.



19. $y = f(x) = 3x^4 - 4x^3 + 2$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 2$; no x-intercept **C.** No symmetry **D.** No asymptote

E. $f'(x) = 12x^3 - 12x^2 = 12x^2(x - 1)$. $f'(x) > 0$ for $x > 1$, so f is increasing on $(1, \infty)$ and decreasing on $(-\infty, 1)$. **F.** $f'(x)$ does not change sign at $x = 0$, so there is no local extremum there. $f(1) = 1$ is a local minimum value. **G.** $f''(x) = 36x^2 - 24x = 12x(3x - 2)$. $f''(x) < 0$ for $0 < x < \frac{2}{3}$, so f is CD on $\left(0, \frac{2}{3}\right)$ and f is CU on $(-\infty, 0)$ and $\left(\frac{2}{3}, \infty\right)$. There are inflection points at $(0, 2)$ and $\left(\frac{2}{3}, \frac{38}{27}\right)$.

H.



20. $y = f(x) = \dfrac{x}{1-x^2}$ **A.** $D = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$ **B.** y-intercept: $f(0) = 0$; x-intercept: 0

C. $f(-x) = -f(x)$, so f is odd and the graph is symmetric about the origin. **D.** $\lim\limits_{x\to\pm\infty} \dfrac{x}{1-x^2} = 0$, so $y = 0$ is a HA.

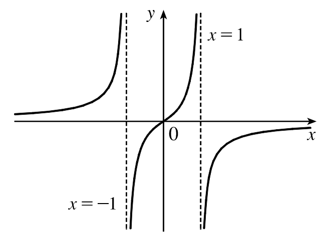
$\lim\limits_{x\to-1^-} \dfrac{x}{1-x^2} = \infty$ and $\lim\limits_{x\to-1^+} \dfrac{x}{1-x^2} = -\infty$, so $x = -1$ is a VA. Similarly, $\lim\limits_{x\to 1^-} \dfrac{x}{1-x^2} = \infty$ and

$\lim\limits_{x\to 1^+} \dfrac{x}{1-x^2} = -\infty$, so $x = 1$ is a VA. **E.** $f'(x) = \dfrac{(1-x^2)(1) - x(-2x)}{(1-x^2)^2} = \dfrac{1+x^2}{(1-x^2)^2} > 0$ for $x \neq \pm 1$, so f is

increasing on $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$. **F.** No extreme values

G. $$f''(x) = \frac{(1-x^2)^2(2x) - (1+x^2)2(1-x^2)(-2x)}{[(1-x^2)^2]^2}$$
$$= \frac{2x(1-x^2)[(1-x^2) + 2(1+x^2)]}{(1-x^2)^4} = \frac{2x(3+x^2)}{(1-x^2)^3}$$

H.



$f''(x) > 0$ for $x < -1$ and $0 < x < 1$, and $f''(x) < 0$ for $-1 < x < 0$ and $x > 1$, so f is CU on $(-\infty, -1)$ and $(0, 1)$, and f is CD on $(-1, 0)$ and $(1, \infty)$. $(0, 0)$ is an IP.

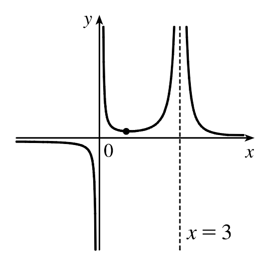
21. $y = f(x) = \dfrac{1}{x(x-3)^2}$ **A.** $D = \{x \mid x \neq 0, 3\} = (-\infty, 0) \cup (0, 3) \cup (3, \infty)$ **B.** No intercepts. **C.** No symmetry.

D. $\lim\limits_{x\to\pm\infty} \dfrac{1}{x(x-3)^2} = 0$, so $y = 0$ is a HA. $\lim\limits_{x\to 0^+} \dfrac{1}{x(x-3)^2} = \infty$, $\lim\limits_{x\to 0^-} \dfrac{1}{x(x-3)^2} = -\infty$, $\lim\limits_{x\to 3} \dfrac{1}{x(x-3)^2} = \infty$,

so $x = 0$ and $x = 3$ are VA. **E.** $f'(x) = -\dfrac{(x-3)^2 + 2x(x-3)}{x^2(x-3)^4} = \dfrac{3(1-x)}{x^2(x-3)^3} \Rightarrow f'(x) > 0 \Leftrightarrow 1 < x < 3$,

so f is increasing on $(1, 3)$ and decreasing on $(-\infty, 0)$, $(0, 1)$, and $(3, \infty)$.

H.



F. Local minimum value $f(1) = \frac{1}{4}$ **G.** $f''(x) = \dfrac{6(2x^2 - 4x + 3)}{x^3(x-3)^4}$.

Note that $2x^2 - 4x + 3 > 0$ for all x since it has negative discriminant.

So $f''(x) > 0 \Leftrightarrow x > 0 \Rightarrow f$ is CU on $(0, 3)$ and $(3, \infty)$ and CD on $(-\infty, 0)$. No IP

22. $y = f(x) = \dfrac{1}{x^2} - \dfrac{1}{(x-2)^2}$ **A.** $D = \{x \mid x \neq 0, 2\}$ **B.** y-intercept: none; x-intercept: $f(x) = 0 \Rightarrow$

$\dfrac{1}{x^2} = \dfrac{1}{(x-2)^2} \Leftrightarrow (x-2)^2 = x^2 \Leftrightarrow x^2 - 4x + 4 = x^2 \Leftrightarrow 4x = 4 \Leftrightarrow x = 1$ **C.** No symmetry

D. $\lim\limits_{x\to 0} f(x) = \infty$ and $\lim\limits_{x\to 2} f(x) = -\infty$, so $x = 0$ and $x = 2$ are VA; $\lim\limits_{x\to\pm\infty} f(x) = 0$, so $y = 0$ is a HA

E. $f'(x) = -\dfrac{2}{x^3} + \dfrac{2}{(x-2)^3} > 0 \Rightarrow \dfrac{-(x-2)^3 + x^3}{x^3(x-2)^3} > 0 \Leftrightarrow \dfrac{-x^3 + 6x^2 - 12x + 8 + x^3}{x^3(x-2)^3} > 0 \Leftrightarrow$

$\dfrac{2(3x^2 - 6x + 4)}{x^3(x-2)^3} > 0$. The numerator is positive (the discriminant of the quadratic is negative), so $f'(x) > 0$ if $x < 0$ or $x > 2$, and hence, f is increasing on $(-\infty, 0)$ and $(2, \infty)$ and decreasing on $(0, 2)$.

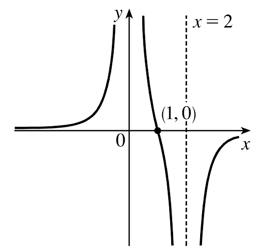
[continued]

F. No extreme values **G.** $f''(x) = \dfrac{6}{x^4} - \dfrac{6}{(x-2)^4} > 0 \;\Rightarrow$

$\dfrac{(x-2)^4 - x^4}{x^4(x-2)^4} > 0 \;\Leftrightarrow\; \dfrac{x^4 - 8x^3 + 24x^2 - 32x + 16 - x^4}{x^4(x-2)^4} > 0 \;\Leftrightarrow$

$\dfrac{-8(x^3 - 3x^2 + 4x - 2)}{x^4(x-2)^4} > 0 \;\Leftrightarrow\; \dfrac{-8(x-1)(x^2 - 2x + 2)}{x^4(x-2)^4} > 0$. So f'' is positive for $x < 1$ $[x \neq 0]$ and negative for $x > 1$ $[x \neq 2]$. Thus, f is CU on $(-\infty, 0)$ and $(0, 1)$ and f is CD on $(1, 2)$ and $(2, \infty)$. IP at $(1, 0)$

H.



23. $y = f(x) = \dfrac{(x-1)^3}{x^2} = \dfrac{x^3 - 3x^2 + 3x - 1}{x^2} = x - 3 + \dfrac{3x - 1}{x^2}$ **A.** $D = \{x \mid x \neq 0\} = (-\infty, 0) \cup (0, \infty)$

B. y-intercept: none; x-intercept: $f(x) = 0 \;\Leftrightarrow\; x = 1$ **C.** No symmetry **D.** $\lim\limits_{x \to 0^-} \dfrac{(x-1)^3}{x^2} = -\infty$ and

$\lim\limits_{x \to 0^+} f(x) = -\infty$, so $x = 0$ is a VA. $f(x) - (x - 3) = \dfrac{3x - 1}{x^2} \to 0$ as $x \to \pm\infty$, so $y = x - 3$ is a SA.

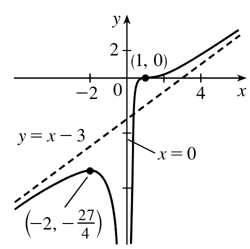
E. $f'(x) = \dfrac{x^2 \cdot 3(x-1)^2 - (x-1)^3(2x)}{(x^2)^2} = \dfrac{x(x-1)^2[3x - 2(x-1)]}{x^4} = \dfrac{(x-1)^2(x+2)}{x^3}$. $f'(x) < 0$ for $-2 < x < 0$, so f is increasing on $(-\infty, -2)$, decreasing on $(-2, 0)$, and increasing on $(0, \infty)$.

F. Local maximum value $f(-2) = -\frac{27}{4}$ **G.** $f(x) = x - 3 + \dfrac{3}{x} - \dfrac{1}{x^2} \;\Rightarrow$

$f'(x) = 1 - \dfrac{3}{x^2} + \dfrac{2}{x^3} \;\Rightarrow\; f''(x) = \dfrac{6}{x^3} - \dfrac{6}{x^4} = \dfrac{6x - 6}{x^4} = \dfrac{6(x-1)}{x^4}$.

$f''(x) > 0$ for $x > 1$, so f is CD on $(-\infty, 0)$ and $(0, 1)$, and f is CU on $(1, \infty)$. There is an inflection point at $(1, 0)$.

H.



24. $y = f(x) = \sqrt{1-x} + \sqrt{1+x}$ **A.** $1 - x \geq 0$ and $1 + x \geq 0 \;\Rightarrow\; x \leq 1$ and $x \geq -1$, so $D = [-1, 1]$.

B. y-intercept: $f(0) = 1 + 1 = 2$; no x-intercept because $f(x) > 0$ for all x.

C. $f(-x) = f(x)$, so the curve is symmetric about the y-axis. **D.** No asymptote

E. $f'(x) = \frac{1}{2}(1-x)^{-1/2}(-1) + \frac{1}{2}(1+x)^{-1/2} = \dfrac{-1}{2\sqrt{1-x}} + \dfrac{1}{2\sqrt{1+x}} = \dfrac{-\sqrt{1+x} + \sqrt{1-x}}{2\sqrt{1-x}\sqrt{1+x}} > 0 \;\Rightarrow$

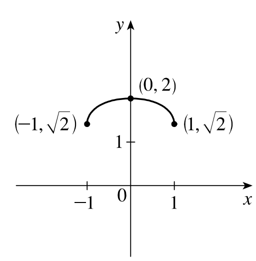
$-\sqrt{1+x} + \sqrt{1-x} > 0 \;\Rightarrow\; \sqrt{1-x} > \sqrt{1+x} \;\Rightarrow\; 1 - x > 1 + x \;\Rightarrow\; -2x > 0 \;\Rightarrow\; x < 0$, so $f'(x) > 0$ for $-1 < x < 0$ and $f'(x) < 0$ for $0 < x < 1$. Thus, f is increasing on $(-1, 0)$ and decreasing on $(0, 1)$. **F.** Local maximum value $f(0) = 2$

G. $f''(x) = -\frac{1}{2}\left(-\frac{1}{2}\right)(1-x)^{-3/2}(-1) + \frac{1}{2}\left(-\frac{1}{2}\right)(1+x)^{-3/2}$

$$= \frac{-1}{4(1-x)^{3/2}} + \frac{-1}{4(1+x)^{3/2}} < 0$$

for all x in the domain, so f is CD on $(-1, 1)$. No IP

H.



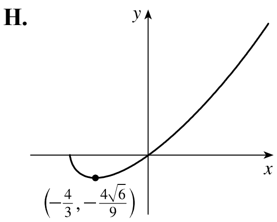
25. $y = f(x) = x\sqrt{2+x}$ **A.** $D = [-2, \infty)$ **B.** y-intercept: $f(0) = 0$; x-intercepts: -2 and 0 **C.** No symmetry

D. No asymptote **E.** $f'(x) = \dfrac{x}{2\sqrt{2+x}} + \sqrt{2+x} = \dfrac{1}{2\sqrt{2+x}}[x + 2(2+x)] = \dfrac{3x+4}{2\sqrt{2+x}} = 0$ when $x = -\frac{4}{3}$, so f is decreasing on $\left(-2, -\frac{4}{3}\right)$ and increasing on $\left(-\frac{4}{3}, \infty\right)$. **F.** Local minimum value $f\left(-\frac{4}{3}\right) = -\frac{4}{3}\sqrt{\frac{2}{3}} = -\frac{4\sqrt{6}}{9} \approx -1.09$, no local maximum

G. $$f''(x) = \frac{2\sqrt{2+x}\cdot 3 - (3x+4)\dfrac{1}{\sqrt{2+x}}}{4(2+x)} = \frac{6(2+x)-(3x+4)}{4(2+x)^{3/2}} = \frac{3x+8}{4(2+x)^{3/2}}$$

$f''(x) > 0$ for $x > -2$, so f is CU on $(-2, \infty)$. No IP

H.



26. $y = f(x) = x^{2/3}(x-3)^2$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \Leftrightarrow x = 0, 3$

C. No symmetry **D.** No asymptote

E. $f'(x) = x^{2/3}\cdot 2(x-3) + (x-3)^2\cdot\frac{2}{3}x^{-1/3} = \frac{2}{3}x^{-1/3}(x-3)[3x + (x-3)] = \frac{2}{3}x^{-1/3}(x-3)(4x-3)$.

$f'(x) > 0 \Leftrightarrow 0 < x < \frac{3}{4}$ or $x > 3$, so f is decreasing on $(-\infty, 0)$, increasing on $\left(0, \frac{3}{4}\right)$, decreasing on $\left(\frac{3}{4}, 3\right)$, and increasing on $(3, \infty)$. **F.** Local minimum value $f(0) = f(3) = 0$; local maximum value

$f\left(\frac{3}{4}\right) = \left(\frac{3}{4}\right)^{2/3}\left(-\frac{9}{4}\right)^2 = \frac{81}{16}\sqrt[3]{\frac{9}{16}} = \frac{81}{32}\sqrt[3]{\frac{9}{2}}$ $[\approx 4.18]$

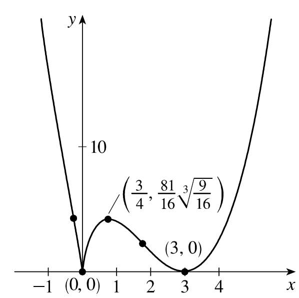
G. $f'(x) = \left(\frac{2}{3}x^{-1/3}\right)(4x^2 - 15x + 9) \Rightarrow$

$$\begin{aligned} f''(x) &= \left(\tfrac{2}{3}x^{-1/3}\right)(8x-15) + (4x^2-15x+9)\left(-\tfrac{2}{9}x^{-4/3}\right) \\ &= \tfrac{2}{9}x^{-4/3}[3x(8x-15) - (4x^2-15x+9)] \\ &= \tfrac{2}{9}x^{-4/3}(20x^2 - 30x - 9) \end{aligned}$$

$f''(x) = 0 \Leftrightarrow x \approx -0.26$ or 1.76. $f''(x)$ does not exist at $x = 0$.

f is CU on $(-\infty, -0.26)$, CD on $(-0.26, 0)$, CD on $(0, 1.76)$, and CU on $(1.76, \infty)$. There are inflection points at $(-0.26, 4.28)$ and $(1.76, 2.25)$.

H.



27. $y = f(x) = \sin^2 x - 2\cos x$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = -2$ **C.** $f(-x) = f(x)$, so f is symmetric with respect to the y-axis. f has period 2π. **D.** No asymptote **E.** $y' = 2\sin x\cos x + 2\sin x = 2\sin x\,(\cos x + 1)$. $y' = 0 \Leftrightarrow \sin x = 0$ or $\cos x = -1 \Leftrightarrow x = n\pi$ or $x = (2n+1)\pi$. $y' > 0$ when $\sin x > 0$, since $\cos x + 1 \geq 0$ for all x. Therefore, $y' > 0$ [and so f is increasing] on $(2n\pi, (2n+1)\pi)$; $y' < 0$ [and so f is decreasing] on $((2n-1)\pi, 2n\pi)$.

F. Local maximum values are $f((2n+1)\pi) = 2$; local minimum values are $f(2n\pi) = -2$.

G. $y' = \sin 2x + 2\sin x \Rightarrow y'' = 2\cos 2x + 2\cos x = 2(2\cos^2 x - 1) + 2\cos x = 4\cos^2 x + 2\cos x - 2$

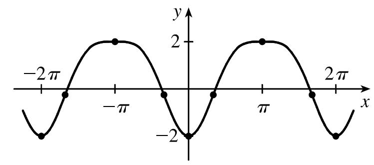
$$= 2(2\cos^2 x + \cos x - 1) = 2(2\cos x - 1)(\cos x + 1)$$

[continued]

$y'' = 0 \;\Leftrightarrow\; \cos x = \frac{1}{2}$ or $-1 \;\Leftrightarrow\; x = 2n\pi \pm \frac{\pi}{3}$ or $x = (2n+1)\pi$.

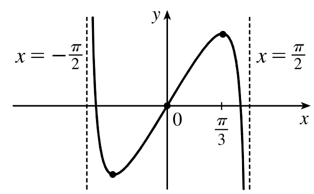
$y'' > 0$ [and so f is CU] on $\left(2n\pi - \frac{\pi}{3}, 2n\pi + \frac{\pi}{3}\right)$; $y'' \le 0$ [and so f is CD] on $\left(2n\pi + \frac{\pi}{3}, 2n\pi + \frac{5\pi}{3}\right)$. There are inflection points at $\left(2n\pi \pm \frac{\pi}{3}, -\frac{1}{4}\right)$.

H.



28. $y = f(x) = 4x - \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$ **A.** $D = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. **B.** y-intercept $= f(0) = 0$ **C.** $f(-x) = -f(x)$, so the curve is symmetric about $(0, 0)$. **D.** $\lim\limits_{x \to \pi/2^-} (4x - \tan x) = -\infty$, $\lim\limits_{x \to -\pi/2^+} (4x - \tan x) = \infty$, so $x = \frac{\pi}{2}$ and $x = -\frac{\pi}{2}$ are VA. **E.** $f'(x) = 4 - \sec^2 x > 0 \;\Leftrightarrow\; \sec x < 2 \;\Leftrightarrow\; \cos x > \frac{1}{2} \;\Leftrightarrow\; -\frac{\pi}{3} < x < \frac{\pi}{3}$, so f is increasing on $\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$ and decreasing on $\left(-\frac{\pi}{2}, -\frac{\pi}{3}\right)$ and $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$. **F.** $f\left(\frac{\pi}{3}\right) = \frac{4\pi}{3} - \sqrt{3}$ is a local maximum value, $f\left(-\frac{\pi}{3}\right) = \sqrt{3} - \frac{4\pi}{3}$ is a local minimum value.
G. $f''(x) = -2\sec^2 x \tan x > 0 \;\Leftrightarrow\; \tan x < 0 \;\Leftrightarrow\; -\frac{\pi}{2} < x < 0$, so f is CU on $\left(-\frac{\pi}{2}, 0\right)$ and CD on $\left(0, \frac{\pi}{2}\right)$. IP at $(0, 0)$

H.



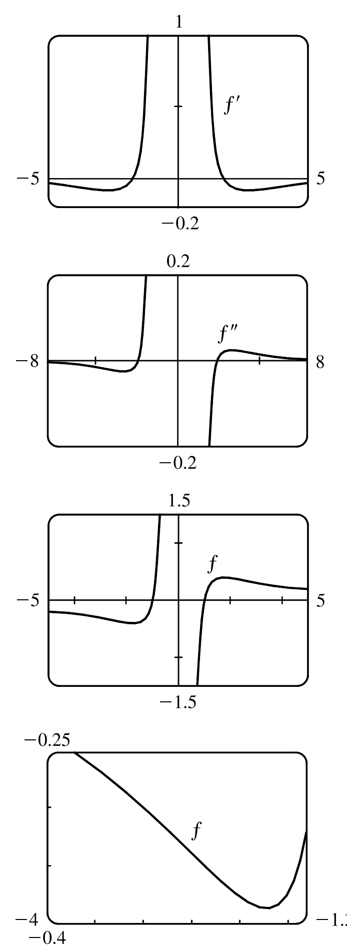
29. $f(x) = \dfrac{x^2 - 1}{x^3} \;\Rightarrow\; f'(x) = \dfrac{x^3(2x) - (x^2 - 1)3x^2}{x^6} = \dfrac{3 - x^2}{x^4} \;\Rightarrow$

$f''(x) = \dfrac{x^4(-2x) - (3 - x^2)4x^3}{x^8} = \dfrac{2x^2 - 12}{x^5}$

Estimates: From the graphs of f' and f'', it appears that f is increasing on $(-1.73, 0)$ and $(0, 1.73)$ and decreasing on $(-\infty, -1.73)$ and $(1.73, \infty)$; f has a local maximum of about $f(1.73) = 0.38$ and a local minimum of about $f(-1.7) = -0.38$; f is CU on $(-2.45, 0)$ and $(2.45, \infty)$, and CD on $(-\infty, -2.45)$ and $(0, 2.45)$; and f has inflection points at about $(-2.45, -0.34)$ and $(2.45, 0.34)$.

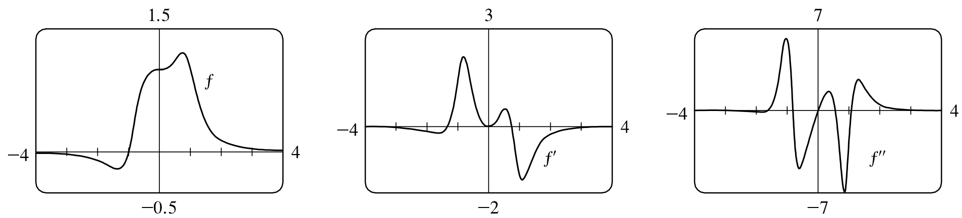
Exact: Now $f'(x) = \dfrac{3 - x^2}{x^4}$ is positive for $0 < x^2 < 3$, that is, f is increasing on $\left(-\sqrt{3}, 0\right)$ and $\left(0, \sqrt{3}\right)$; and $f'(x)$ is negative (and so f is decreasing) on $\left(-\infty, -\sqrt{3}\right)$ and $\left(\sqrt{3}, \infty\right)$. $f'(x) = 0$ when $x = \pm\sqrt{3}$.

f' goes from positive to negative at $x = \sqrt{3}$, so f has a local maximum of $f\left(\sqrt{3}\right) = \frac{\left(\sqrt{3}\right)^2 - 1}{\left(\sqrt{3}\right)^3} = \frac{2\sqrt{3}}{9}$; and since f is odd, we know that maxima on the interval $(0, \infty)$ correspond to minima on $(-\infty, 0)$, so f has a local minimum of $f\left(-\sqrt{3}\right) = -\frac{2\sqrt{3}}{9}$. Also, $f''(x) = \dfrac{2x^2 - 12}{x^5}$ is positive (so f is CU) on $\left(-\sqrt{6}, 0\right)$ and $\left(\sqrt{6}, \infty\right)$, and negative (so f is CD) on $\left(-\infty, -\sqrt{6}\right)$ and $\left(0, \sqrt{6}\right)$. There are IP at $\left(\sqrt{6}, \frac{5\sqrt{6}}{36}\right)$ and $\left(-\sqrt{6}, -\frac{5\sqrt{6}}{36}\right)$.

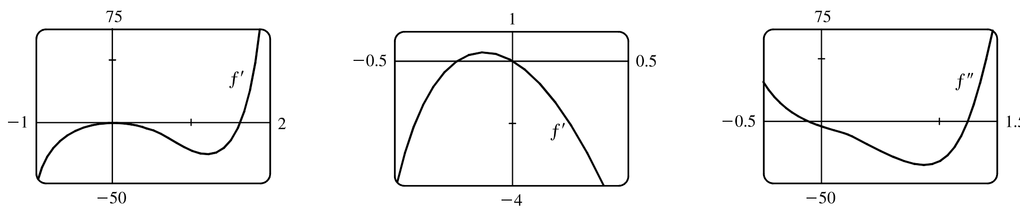


30. $f(x) = \dfrac{x^3+1}{x^6+1} \quad\Rightarrow\quad f'(x) = -\dfrac{3x^2(x^6+2x^3-1)}{(x^6+1)^2} \quad\Rightarrow\quad f''(x) = \dfrac{6x(2x^{12}+7x^9-9x^6-5x^3+1)}{(x^6+1)^3}$.

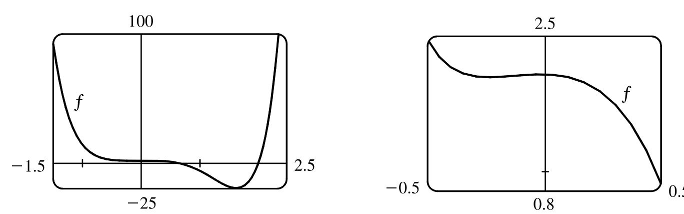
$f(x)=0 \;\Leftrightarrow\; x=-1$. $f'(x)=0 \;\Leftrightarrow\; x=0$ or $x\approx -1.34, 0.75$. $f''(x)=0 \;\Leftrightarrow\; x=0$ or $x\approx -1.64, -0.82, 0.54, 1.09$. From the graphs of f and f', it appears that f is decreasing on $(-\infty,-1.34)$, increasing on $(-1.34, 0.75)$, and decreasing on $(0.75,\infty)$. f has a local minimum value of $f(-1.34)\approx -0.21$ and a local maximum value of $f(0.75)\approx 1.21$. From the graphs of f and f'', it appears that f is CD on $(-\infty,-1.64)$, CU on $(-1.64,-0.82)$, CD on $(-0.82,0)$, CU on $(0,0.54)$, CD on $(0.54,1.09)$ and CU on $(1.09,\infty)$. There are inflection points at about $(-1.64,-0.17)$, $(-0.82,0.34)$, $(0.54,1.13)$, $(1.09,0.86)$, and at $(0,1)$.



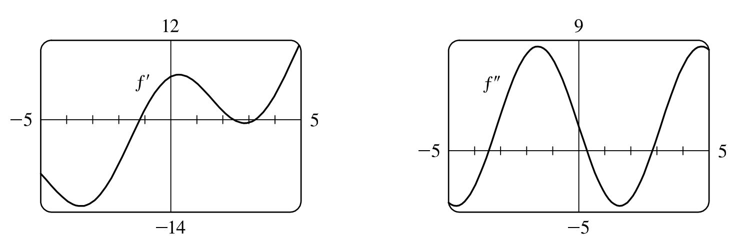
31. $f(x)=3x^6-5x^5+x^4-5x^3-2x^2+2 \quad\Rightarrow\quad f'(x)=18x^5-25x^4+4x^3-15x^2-4x \quad\Rightarrow$
$f''(x)=90x^4-100x^3+12x^2-30x-4$



From the graphs of f' and f'', it appears that f is increasing on $(-0.23,0)$ and $(1.62,\infty)$ and decreasing on $(-\infty,-0.23)$ and $(0,1.62)$; f has a local maximum of $f(0)=2$ and local minima of about $f(-0.23)=1.96$ and $f(1.62)=-19.2$; f is CU on $(-\infty,-0.12)$ and $(1.24,\infty)$ and CD on $(-0.12,1.24)$; and f has inflection points at about $(-0.12,1.98)$ and $(1.24,-12.1)$.

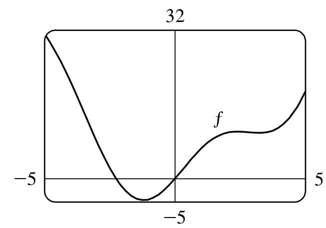


32. $f(x)=x^2+6.5\sin x,\ -5\le x\le 5 \quad\Rightarrow\quad f'(x)=2x+6.5\cos x \quad\Rightarrow\quad f''(x)=2-6.5\sin x.\quad f(x)=0 \;\Leftrightarrow$
$x\approx -2.25$ and $x=0$; $\;f'(x)=0 \;\Leftrightarrow\; x\approx -1.19, 2.40, 3.24$; $\;f''(x)=0 \;\Leftrightarrow\; x\approx -3.45, 0.31, 2.83$.



[continued]

From the graphs of f' and f'', it appears that f is decreasing on $(-5, -1.19)$ and $(2.40, 3.24)$ and increasing on $(-1.19, 2.40)$ and $(3.24, 5)$; f has a local maximum of about $f(2.40) = 10.15$ and local minima of about $f(-1.19) = -4.62$ and $f(3.24) = 9.86$; f is CU on $(-3.45, 0.31)$ and $(2.83, 5)$ and CD on $(-5, -3.45)$ and $(0.31, 2.83)$; and f has inflection points at about $(-3.45, 13.93)$, $(0.31, 2.10)$, and $(2.83, 10.00)$.



33. Let $f(x) = 3x + 2\cos x + 5$. Then $f(0) = 7 > 0$ and $f(-\pi) = -3\pi - 2 + 5 = -3\pi + 3 = -3(\pi - 1) < 0$, and since f is continuous on $\mathbb{R}$ (hence on $[-\pi, 0]$), the Intermediate Value Theorem assures us that there is at least one solution of f in $[-\pi, 0]$. Now $f'(x) = 3 - 2\sin x > 0$ implies that f is increasing on $\mathbb{R}$, so there is exactly one solution of f, and hence, exactly one real solution of the equation $3x + 2\cos x + 5 = 0$.

34. By the Mean Value Theorem, $f'(c) = \dfrac{f(4) - f(0)}{4 - 0} \quad\Leftrightarrow\quad 4f'(c) = f(4) - 1$ for some c with $0 < c < 4$. Since $2 \le f'(c) \le 5$, we have $4(2) \le 4f'(c) \le 4(5) \quad\Leftrightarrow\quad 4(2) \le f(4) - 1 \le 4(5) \quad\Leftrightarrow\quad 8 \le f(4) - 1 \le 20 \quad\Leftrightarrow$ $9 \le f(4) \le 21$.

35. Since f is continuous on $[32, 33]$ and differentiable on $(32, 33)$, then by the Mean Value Theorem there exists a number c in $(32, 33)$ such that $f'(c) = \frac{1}{5}c^{-4/5} = \dfrac{\sqrt[5]{33} - \sqrt[5]{32}}{33 - 32} = \sqrt[5]{33} - 2$, but $\frac{1}{5}c^{-4/5} > 0 \quad\Rightarrow\quad \sqrt[5]{33} - 2 > 0 \quad\Rightarrow\quad \sqrt[5]{33} > 2$. Also f' is decreasing, so that $f'(c) < f'(32) = \frac{1}{5}(32)^{-4/5} = 0.0125 \quad\Rightarrow\quad 0.0125 > f'(c) = \sqrt[5]{33} - 2 \quad\Rightarrow\quad \sqrt[5]{33} < 2.0125$. Therefore, $2 < \sqrt[5]{33} < 2.0125$.

36. Since the point $(1, 3)$ is on the curve $y = ax^3 + bx^2$, we have $3 = a(1)^3 + b(1)^2 \quad\Rightarrow\quad 3 = a + b$ **(1)**.

$y' = 3ax^2 + 2bx \quad\Rightarrow\quad y'' = 6ax + 2b$. $\;y'' = 0$ [for inflection points] $\quad\Leftrightarrow\quad x = \dfrac{-2b}{6a} = -\dfrac{b}{3a}$. Since we want $x = 1$, $1 = -\dfrac{b}{3a} \quad\Rightarrow\quad b = -3a$. Combining with **(1)** gives us $3 = a - 3a \quad\Leftrightarrow\quad 3 = -2a \quad\Leftrightarrow\quad a = -\frac{3}{2}$. Hence, $b = -3\left(-\frac{3}{2}\right) = \frac{9}{2}$ and the curve is $y = -\frac{3}{2}x^3 + \frac{9}{2}x^2$.

37. (a) $g(x) = f(x^2) \quad\Rightarrow\quad g'(x) = 2xf'(x^2)$ by the Chain Rule. Since $f'(x) > 0$ for all $x \ne 0$, we must have $f'(x^2) > 0$ for $x \ne 0$, so $g'(x) = 0 \quad\Leftrightarrow\quad x = 0$. Now $g'(x)$ changes sign (from negative to positive) at $x = 0$, since one of its factors, $f'(x^2)$, is positive for all x, and its other factor, $2x$, changes from negative to positive at this point, so by the First Derivative Test, f has a local and absolute minimum at $x = 0$.

(b) $g'(x) = 2xf'(x^2) \quad\Rightarrow\quad g''(x) = 2[xf''(x^2)(2x) + f'(x^2)] = 4x^2f''(x^2) + 2f'(x^2)$ by the Product Rule and the Chain Rule. But $x^2 > 0$ for all $x \ne 0$, $f''(x^2) > 0$ [since f is CU for $x > 0$], and $f'(x^2) > 0$ for all $x \ne 0$, so since all of its factors are positive, $g''(x) > 0$ for $x \ne 0$. Whether $g''(0)$ is positive or 0 doesn't matter [since the sign of g'' does not change there]; g is concave upward on $\mathbb{R}$.

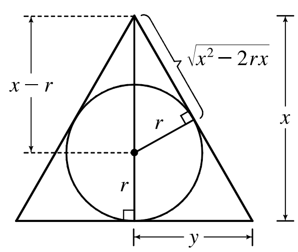
38. Call the two integers x and y. Then $x + 4y = 1000$, so $x = 1000 - 4y$. Their product is $P = xy = (1000 - 4y)y$, so our problem is to maximize the function $P(y) = 1000y - 4y^2$, where $0 < y < 250$ and y is an integer. $P'(y) = 1000 - 8y$, so $P'(y) = 0 \quad \Leftrightarrow \quad y = 125$. $\quad P''(y) = -8 < 0$, so $P(125) = 62{,}500$ is an absolute maximum. Since the optimal y turned out to be an integer, we have found the desired pair of numbers, namely $x = 1000 - 4(125) = 500$ and $y = 125$.

39. If $B = 0$, the line is vertical and the distance from $x = -\dfrac{C}{A}$ to (x_1, y_1) is $\left|x_1 + \dfrac{C}{A}\right| = \dfrac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$, so assume $B \neq 0$. The square of the distance from (x_1, y_1) to the line is $f(x) = (x - x_1)^2 + (y - y_1)^2$ where $Ax + By + C = 0$, so we minimize $f(x) = (x - x_1)^2 + \left(-\dfrac{A}{B}x - \dfrac{C}{B} - y_1\right)^2 \quad \Rightarrow \quad f'(x) = 2\,(x - x_1) + 2\left(-\dfrac{A}{B}x - \dfrac{C}{B} - y_1\right)\left(-\dfrac{A}{B}\right)$.

$f'(x) = 0 \quad \Rightarrow \quad x = \dfrac{B^2x_1 - ABy_1 - AC}{A^2 + B^2}$ and this gives a minimum since $f''(x) = 2\left(1 + \dfrac{A^2}{B^2}\right) > 0$. Substituting this value of x into $f(x)$ and simplifying gives $f(x) = \dfrac{(Ax_1 + By_1 + C)^2}{A^2 + B^2}$, so the minimum distance is $\sqrt{f(x)} = \dfrac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$.

40. On the hyperbola $xy = 8$, if $d(x)$ is the distance from the point $(x, y) = (x, 8/x)$ to the point $(3, 0)$, then $[d(x)]^2 = (x - 3)^2 + 64/x^2 = f(x)$. $f'(x) = 2(x - 3) - 128/x^3 = 0 \quad \Rightarrow \quad x^4 - 3x^3 - 64 = 0 \quad \Rightarrow$ $(x - 4)\left(x^3 + x^2 + 4x + 16\right) = 0 \quad \Rightarrow \quad x = 4$ since the solution must have $x > 0$. Then $y = \frac{8}{4} = 2$, so the point is $(4, 2)$.

41.



By similar triangles, $\dfrac{y}{x} = \dfrac{r}{\sqrt{x^2 - 2rx}}$, so the area of the triangle is

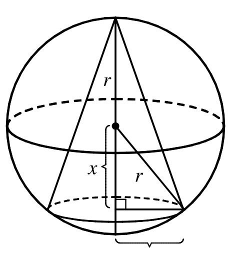
$$A(x) = \tfrac{1}{2}(2y)x = xy = \frac{rx^2}{\sqrt{x^2 - 2rx}} \quad \Rightarrow$$

$$A'(x) = \frac{2rx\sqrt{x^2 - 2rx} - rx^2(x - r)/\sqrt{x^2 - 2rx}}{x^2 - 2rx} = \frac{rx^2\,(x - 3r)}{(x^2 - 2rx)^{3/2}} = 0$$

when $x = 3r$.

$A'(x) < 0$ when $2r < x < 3r$, $A'(x) > 0$ when $x > 3r$. So $x = 3r$ gives a minimum and $A(3r) = \dfrac{r(9r^2)}{\sqrt{3}\,r} = 3\sqrt{3}\,r^2$.

42.



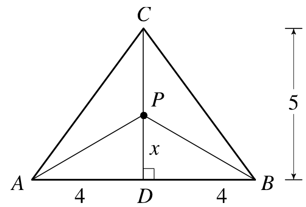
The volume of the cone is $V = \frac{1}{3}\pi y^2(r + x) = \frac{1}{3}\pi(r^2 - x^2)(r + x), \; -r \le x \le r$.

$$V'(x) = \tfrac{\pi}{3}[(r^2 - x^2)\,(1) + (r + x)(-2x)] = \tfrac{\pi}{3}[(r + x)(r - x - 2x)]$$
$$= \tfrac{\pi}{3}(r + x)(r - 3x) = 0 \text{ when } x = -r \text{ or } x = r/3.$$

Now $V(r) = 0 = V(-r)$, so the maximum occurs at $x = r/3$ and the volume is

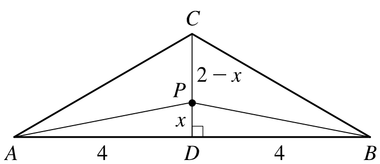
$$V\left(\frac{r}{3}\right) = \frac{\pi}{3}\left(r^2 - \frac{r^2}{9}\right)\left(\frac{4r}{3}\right) = \frac{32\pi r^3}{81}.$$

43.



We minimize $L(x) = |PA| + |PB| + |PC| = 2\sqrt{x^2+16} + (5-x)$, $0 \le x \le 5$. $L'(x) = 2x/\sqrt{x^2+16} - 1 = 0 \quad\Leftrightarrow\quad 2x = \sqrt{x^2+16} \quad\Leftrightarrow\quad 4x^2 = x^2+16 \quad\Leftrightarrow\quad x = \frac{4}{\sqrt{3}}$. $L(0) = 13$, $L\left(\frac{4}{\sqrt{3}}\right) \approx 11.9$, $L(5) \approx 12.8$, so the minimum occurs when $x = \frac{4}{\sqrt{3}} \approx 2.3$.

44.

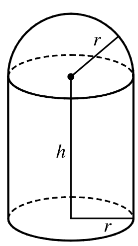


If $|CD| = 2$, the last part of $L(x)$ changes from $(5-x)$ to $(2-x)$ with $0 \le x \le 2$. But we still get $L'(x) = 0 \quad\Leftrightarrow\quad x = \frac{4}{\sqrt{3}}$, which isn't in the interval $[0,2]$. Now $L(0) = 10$ and $L(2) = 2\sqrt{20} = 4\sqrt{5} \approx 8.9$. The minimum occurs when $P = C$.

45. $v = K\sqrt{\dfrac{L}{C} + \dfrac{C}{L}} \quad\Rightarrow\quad \dfrac{dv}{dL} = \dfrac{K}{2\sqrt{(L/C)+(C/L)}}\left(\dfrac{1}{C} - \dfrac{C}{L^2}\right) = 0 \quad\Leftrightarrow\quad \dfrac{1}{C} = \dfrac{C}{L^2} \quad\Leftrightarrow\quad L^2 = C^2 \quad\Leftrightarrow\quad L = C.$

This gives the minimum velocity since $v' < 0$ for $0 < L < C$ and $v' > 0$ for $L > C$.

46.



We minimize the surface area $S = \pi r^2 + 2\pi r h + \frac{1}{2}(4\pi r^2) = 3\pi r^2 + 2\pi r h$.

Solving $V = \pi r^2 h + \frac{2}{3}\pi r^3$ for h, we get $h = \dfrac{V - \frac{2}{3}\pi r^3}{\pi r^2} = \dfrac{V}{\pi r^2} - \frac{2}{3}r$, so

$S(r) = 3\pi r^2 + 2\pi r\left[\dfrac{V}{\pi r^2} - \frac{2}{3}r\right] = \frac{5}{3}\pi r^2 + \dfrac{2V}{r}$.

$S'(r) = -\dfrac{2V}{r^2} + \frac{10}{3}\pi r = \dfrac{\frac{10}{3}\pi r^3 - 2V}{r^2} = 0 \quad\Leftrightarrow\quad \frac{10}{3}\pi r^3 = 2V \quad\Leftrightarrow\quad r^3 = \dfrac{3V}{5\pi} \quad\Leftrightarrow\quad r = \sqrt[3]{\dfrac{3V}{5\pi}}$.

This gives an absolute minimum since $S'(r) < 0$ for $0 < r < \sqrt[3]{\dfrac{3V}{5\pi}}$ and $S'(r) > 0$ for $r > \sqrt[3]{\dfrac{3V}{5\pi}}$. Thus,

$$h = \frac{V - \frac{2}{3}\pi \cdot \dfrac{3V}{5\pi}}{\pi\sqrt[3]{\dfrac{(3V)^2}{(5\pi)^2}}} = \frac{\left(V - \frac{2}{5}V\right)\sqrt[3]{(5\pi)^2}}{\pi\sqrt[3]{(3V)^2}} = \frac{3V\sqrt[3]{(5\pi)^2}}{5\pi\sqrt[3]{(3V)^2}} = \sqrt[3]{\frac{3V}{5\pi}} = r$$

47. Let x denote the number of \$1 decreases in ticket price. Then the ticket price is $\$12 - \$1(x)$, and the average attendance is $11{,}000 + 1000(x)$. Now the revenue per game is

$$\begin{aligned} R(x) &= (\text{price per person}) \times (\text{number of people per game}) \\ &= (12-x)(11{,}000 + 1000x) = -1000x^2 + 1000x + 132{,}000 \end{aligned}$$

for $0 \le x \le 4$ [since the seating capacity is 15,000] $\quad\Rightarrow\quad R'(x) = -2000x + 1000 = 0 \quad\Leftrightarrow\quad x = 0.5$. This is a maximum since $R''(x) = -2000 < 0$ for all x. Now we must check the value of $R(x) = (12-x)(11{,}000 + 1000x)$ at $x = 0.5$ and at the endpoints of the domain to see which value of x gives the maximum value of R.

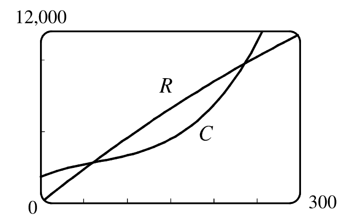
$R(0) = (12)(11{,}000) = 132{,}000$, $R(0.5) = (11.5)(11{,}500) = 132{,}250$, and $R(4) = (8)(15{,}000) = 120{,}000$. Thus, the maximum revenue of \$132,250 per game occurs when the average attendance is 11,500 and the ticket price is \$11.50.

48. (a) $C(x) = 1800 + 25x - 0.2x^2 + 0.001x^3$ and

$R(x) = xp(x) = 48.2x - 0.03x^2$.

The profit is maximized when $C'(x) = R'(x)$.

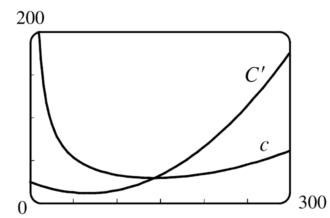
From the figure, we estimate that the tangents are parallel when $x \approx 160$.



(b) $C'(x) = 25 - 0.4x + 0.003x^2$ and $R'(x) = 48.2 - 0.06x$. $C'(x) = R'(x) \quad \Rightarrow \quad 0.003x^2 - 0.34x - 23.2 = 0 \quad \Rightarrow$ $x_1 \approx 161.3$ $(x > 0)$. $R''(x) = -0.06$ and $C''(x) = -0.4 + 0.006x$, so $R''(x_1) = -0.06 < C''(x_1) \approx 0.57 \quad \Rightarrow$ profit is maximized by producing 161 units.

(c) $c(x) = \dfrac{C(x)}{x} = \dfrac{1800}{x} + 25 - 0.2x + 0.001x^2$ is the average cost. Since the average cost is minimized when the marginal cost equals the average cost, we graph $c(x)$ and $C'(x)$ and estimate the point of intersection.

From the figure, $C'(x) = c(x) \quad \Leftrightarrow \quad x \approx 144$.



49. $f(x) = x^5 - x^4 + 3x^2 - 3x - 2 \quad \Rightarrow \quad f'(x) = 5x^4 - 4x^3 + 6x - 3$, so $x_{n+1} = x_n - \dfrac{x_n^5 - x_n^4 + 3x_n^2 - 3x_n - 2}{5x_n^4 - 4x_n^3 + 6x_n - 3}$.

Now $x_1 = 1 \quad \Rightarrow \quad x_2 = 1.5 \quad \Rightarrow \quad x_3 \approx 1.343860 \quad \Rightarrow \quad x_4 \approx 1.300320 \quad \Rightarrow \quad x_5 \approx 1.297396 \quad \Rightarrow$ $x_6 \approx 1.297383 \approx x_7$, so the solution in $[1, 2]$ is 1.297383, to six decimal places.

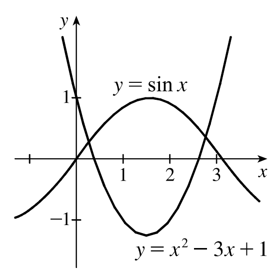
50. Graphing $y = \sin x$ and $y = x^2 - 3x + 1$ shows that there are two solutions, one about 0.3 and the other about 2.8.

$f(x) = \sin x - x^2 + 3x - 1 \quad \Rightarrow \quad f'(x) = \cos x - 2x + 3 \quad \Rightarrow$

$x_{n+1} = x_n - \dfrac{\sin x_n - x_n^2 + 3x_n - 1}{\cos x_n - 2x_n + 3}$. Now $x_1 = 0.3 \quad \Rightarrow$

$x_2 \approx 0.268552 \quad \Rightarrow \quad x_3 \approx 0.268881 \approx x_4$ and $x_1 = 2.8 \quad \Rightarrow$

$x_2 \approx 2.770354 \quad \Rightarrow \quad x_3 \approx 2.770058 \approx x_4$, so to six decimal places, the solutions are 0.268881 and 2.770058.

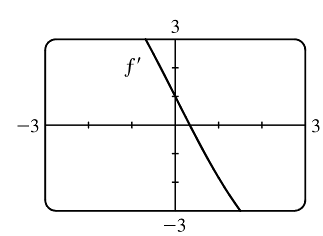


51. $f(t) = \cos t + t - t^2 \quad \Rightarrow \quad f'(t) = -\sin t + 1 - 2t$. $f'(t)$ exists for all t, so to find the maximum of f, we can examine the zeros of f'.

From the graph of f', we see that a good choice for t_1 is $t_1 = 0.3$.

Use $g(t) = -\sin t + 1 - 2t$ and $g'(t) = -\cos t - 2$ to obtain

$t_2 \approx 0.33535293$, $t_3 \approx 0.33541803 \approx t_4$. Since $f''(t) = -\cos t - 2 < 0$ for all t, $f(0.33541803) \approx 1.16718557$ is the absolute maximum.



52. $y = f(x) = x \sin x, \;\; 0 \le x \le 2\pi$. **A.** $D = [0, 2\pi]$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \quad \Leftrightarrow \quad x = 0$ or $\sin x = 0 \quad \Leftrightarrow \quad x = 0, \pi$, or 2π. **C.** There is no symmetry on D, but if f is defined for all real numbers x, then f is an even function. **D.** No asymptote **E.** $f'(x) = x\cos x + \sin x$. To find critical numbers in $(0, 2\pi)$, we graph f' and see that there are two critical numbers, about 2 and 4.9. To find them more precisely, we use Newton's method, setting

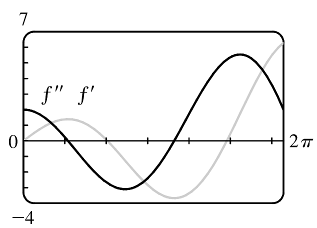
$g(x) = f'(x) = x\cos x + \sin x$, so that $g'(x) = f''(x) = 2\cos x - x\sin x$ and $x_{n+1} = x_n - \dfrac{x_n \cos x_n + \sin x_n}{2\cos x_n - x_n \sin x_n}$.

$x_1 = 2 \;\Rightarrow\; x_2 \approx 2.029048$, $x_3 \approx 2.028758 \approx x_4$ and $x_1 = 4.9 \;\Rightarrow\; x_2 \approx 4.913214$, $x_3 \approx 4.913180 \approx x_4$, so the critical numbers, to six decimal places, are $r_1 = 2.028758$ and $r_2 = 4.913180$. By checking sample values of f' in $(0, r_1)$, (r_1, r_2), and $(r_2, 2\pi)$, we see that f is increasing on $(0, r_1)$, decreasing on (r_1, r_2), and increasing on $(r_2, 2\pi)$. **F.** Local maximum value $f(r_1) \approx 1.819706$, local minimum value $f(r_2) \approx -4.814470$. **G.** $f''(x) = 2\cos x - x\sin x$. To find points where $f''(x) = 0$, we graph f'' and find that $f''(x) = 0$ at about 1 and 3.6. To find the values more precisely, we use Newton's method. Set $h(x) = f''(x) = 2\cos x - x\sin x$. Then $h'(x) = -3\sin x - x\cos x$, so

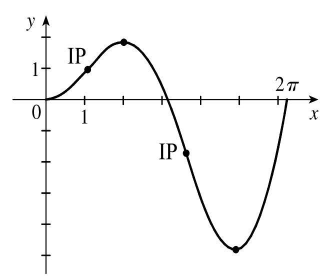
$x_{n+1} = x_n - \dfrac{2\cos x_n - x_n \sin x_n}{-3\sin x_n - x_n \cos x_n}$. $x_1 = 1 \;\Rightarrow\; x_2 \approx 1.078028$, $x_3 \approx 1.076874 \approx x_4$ and $x_1 = 3.6 \;\Rightarrow$

$x_2 \approx 3.643996$, $x_3 \approx 3.643597 \approx x_4$, so the zeros of f'', to six decimal places, are $r_3 = 1.076874$ and $r_4 = 3.643597$.

By checking sample values of f'' in $(0, r_3)$, (r_3, r_4), and $(r_4, 2\pi)$, we see that f is CU on $(0, r_3)$, CD on (r_3, r_4), and CU on $(r_4, 2\pi)$. f has inflection points at $(r_3, f(r_3) \approx 0.948166)$ and $(r_4, f(r_4) \approx -1.753240)$.



H.



53. $f(x) = 4\sqrt{x} - 6x^2 + 3 = 4x^{1/2} - 6x^2 + 3 \;\Rightarrow\; F(x) = 4\left(\frac{2}{3}x^{3/2}\right) - 6\left(\frac{1}{3}x^3\right) + 3x + C = \frac{8}{3}x^{3/2} - 2x^3 + 3x + C$

54. $g(x) = \cos x + 2\sec^2 x \;\Rightarrow\; G(x) = \sin x + 2\tan x + C_n$ on the interval $\left(n\pi - \frac{\pi}{2}, n\pi + \frac{\pi}{2}\right)$.

55. $h(t) = t^{-3} + 5\sin t \;\Rightarrow\; H(t) = \begin{cases} -\frac{1}{2}t^{-2} - 5\cos t + C_1 & \text{if } t < 0 \\ -\frac{1}{2}t^{-2} - 5\cos t + C_2 & \text{if } t > 0 \end{cases}$.

See Example 3.9.1(c) for a similar problem.

56. $f(x) = \dfrac{3x^5 - 4x^2 + 1}{x^2} = 3x^3 - 4 + \dfrac{1}{x^2} \;\Rightarrow$

$$F(x) = \begin{cases} \frac{3}{4}x^4 - 4x - \dfrac{1}{x} + C_1 & \text{if } x < 0 \\ \frac{3}{4}x^4 - 4x - \dfrac{1}{x} + C_2 & \text{if } x > 0 \end{cases}$$

57. $f'(t) = 2t - 3\sin t \;\Rightarrow\; f(t) = t^2 + 3\cos t + C$.

$f(0) = 3 + C$ and $f(0) = 5 \;\Rightarrow\; C = 2$, so $f(t) = t^2 + 3\cos t + 2$.

58. $f'(u) = \dfrac{u^2 + \sqrt{u}}{u} = u + u^{-1/2} \quad\Rightarrow\quad f(u) = \frac{1}{2}u^2 + 2u^{1/2} + C.$

$f(1) = \frac{1}{2} + 2 + C$ and $f(1) = 3 \quad\Rightarrow\quad C = \frac{1}{2}$, so $f(u) = \frac{1}{2}u^2 + 2\sqrt{u} + \frac{1}{2}$.

59. $f''(x) = 1 - 6x + 48x^2 \quad\Rightarrow\quad f'(x) = x - 3x^2 + 16x^3 + C.\quad f'(0) = C$ and $f'(0) = 2 \quad\Rightarrow\quad C = 2$, so

$f'(x) = x - 3x^2 + 16x^3 + 2$ and hence, $f(x) = \frac{1}{2}x^2 - x^3 + 4x^4 + 2x + D$.

$f(0) = D$ and $f(0) = 1 \quad\Rightarrow\quad D = 1$, so $f(x) = \frac{1}{2}x^2 - x^3 + 4x^4 + 2x + 1$.

60. $f''(x) = 5x^3 + 6x^2 + 2 \quad\Rightarrow\quad f'(x) = \frac{5}{4}x^4 + 2x^3 + 2x + C \quad\Rightarrow\quad f(x) = \frac{1}{4}x^5 + \frac{1}{2}x^4 + x^2 + Cx + D$. Now $f(0) = D$ and $f(0) = 3$, so $D = 3$. Also, $f(1) = \frac{1}{4} + \frac{1}{2} + 1 + C + 3 = C + \frac{19}{4}$ and $f(1) = -2$, so $C + \frac{19}{4} = -2 \quad\Rightarrow\quad C = -\frac{27}{4}$.

Thus, $f(x) = \frac{1}{4}x^5 + \frac{1}{2}x^4 + x^2 - \frac{27}{4}x + 3$.

61. $v(t) = s'(t) = 2t - \sin t \quad\Rightarrow\quad s(t) = t^2 + \cos t + C.$

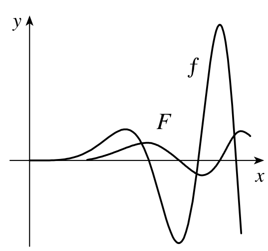
$s(0) = 0 + 1 + C = C + 1$ and $s(0) = 3 \quad\Rightarrow\quad C + 1 = 3 \quad\Rightarrow\quad C = 2$, so $s(t) = t^2 + \cos t + 2$.

62. $a(t) = v'(t) = \sin t + 3\cos t \quad\Rightarrow\quad v(t) = -\cos t + 3\sin t + C.$

$v(0) = -1 + 0 + C$ and $v(0) = 2 \quad\Rightarrow\quad C = 3$, so $v(t) = -\cos t + 3\sin t + 3$ and $s(t) = -\sin t - 3\cos t + 3t + D$.

$s(0) = -3 + D$ and $s(0) = 0 \quad\Rightarrow\quad D = 3$, and $s(t) = -\sin t - 3\cos t + 3t + 3$.

63. $f(x) = x^2\sin(x^2),\ 0 \le x \le \pi$



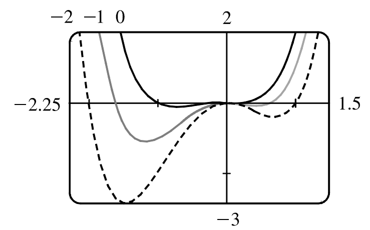
64. $f(x) = x^4 + x^3 + cx^2 \quad\Rightarrow\quad f'(x) = 4x^3 + 3x^2 + 2cx$. This is 0 when $x\left(4x^2 + 3x + 2c\right) = 0 \quad\Leftrightarrow\quad x = 0$ or $4x^2 + 3x + 2c = 0$. Using the quadratic formula, we find that the solutions of this last equation are $x = \dfrac{-3 \pm \sqrt{9 - 32c}}{8}$.

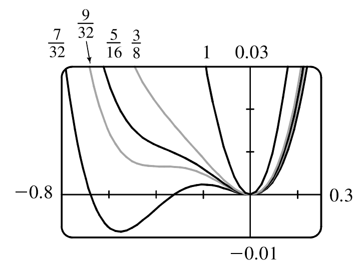
Now if $9 - 32c < 0 \quad\Leftrightarrow\quad c > \frac{9}{32}$, then $(0, 0)$ is the only critical point, a minimum. If $c = \frac{9}{32}$, then there are two critical points (a minimum at $x = 0$, and a horizontal tangent with no maximum or minimum at $x = -\frac{3}{8}$) and if $c < \frac{9}{32}$, then there are three critical points except when $c = 0$, in which case the solution with the $+$ sign coincides with the critical point at $x = 0$.

For $0 < c < \frac{9}{32}$, there is a minimum at $x = -\dfrac{3}{8} - \dfrac{\sqrt{9 - 32c}}{8}$, a maximum at $x = -\dfrac{3}{8} + \dfrac{\sqrt{9 - 32c}}{8}$, and a minimum at $x = 0$. For $c = 0$, there is a minimum at $x = -\frac{3}{4}$ and a horizontal tangent with no extremum at $x = 0$, and for $c < 0$, there is a maximum at $x = 0$, and there are minima at $x = -\dfrac{3}{8} \pm \dfrac{\sqrt{9 - 32c}}{8}$. Now we calculate $f''(x) = 12x^2 + 6x + 2c$.

The solutions of this equation are $x = \dfrac{-6 \pm \sqrt{36 - 4 \cdot 12 \cdot 2c}}{24}$. So if $36 - 96c \le 0 \quad\Leftrightarrow\quad c \ge \frac{3}{8}$, then there is no inflection point. If $c < \frac{3}{8}$, then there are two inflection points at $x = -\dfrac{1}{4} \pm \dfrac{\sqrt{9-24c}}{12}$.

Value of c	No. of CP	No. of IP
$c < 0$	3	2
$c = 0$	2	2
$0 < c < \frac{9}{32}$	3	2
$c = \frac{9}{32}$	2	2
$\frac{9}{32} < c < \frac{3}{8}$	1	2
$c \geq \frac{3}{8}$	1	0

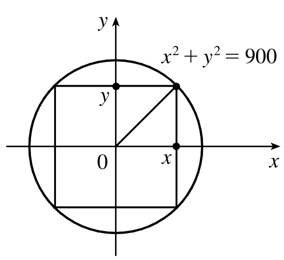




65. Choosing the positive direction to be upward, we have $a(t) = -9.8 \quad\Rightarrow\quad v(t) = -9.8t + v_0$, but $v(0) = 0 = v_0 \quad\Rightarrow$ $v(t) = -9.8t = s'(t) \quad\Rightarrow\quad s(t) = -4.9t^2 + s_0$, but $s(0) = s_0 = 500 \quad\Rightarrow\quad s(t) = -4.9t^2 + 500$. When $s = 0$, $-4.9t^2 + 500 = 0 \quad\Rightarrow\quad t_1 = \sqrt{\frac{500}{4.9}} \approx 10.1 \quad\Rightarrow\quad v(t_1) = -9.8\sqrt{\frac{500}{4.9}} \approx -98.995\ \mathrm{m/s}$. Since the canister has been designed to withstand an impact velocity of $100\ \mathrm{m/s}$, the canister will *not burst*.

66. Let $s_A(t)$ and $s_B(t)$ be the position functions for cars A and B and let $f(t) = s_A(t) - s_B(t)$. Since A passed B twice, there must be three values of t such that $f(t) = 0$. Then by three applications of Rolle's Theorem (see Exercise 3.2.26), there is a number c such that $f''(c) = 0$. So $s_A''(c) = s_B''(c)$; that is, A and B had equal accelerations at $t = c$. We assume that f is continuous on $[0, T]$ and twice differentiable on $(0, T)$, where T is the total time of the race.

67. (a)



The cross-sectional area of the rectangular beam is

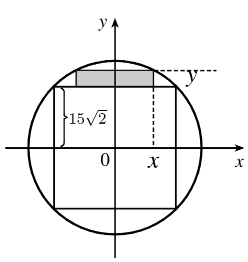
$A = 2x \cdot 2y = 4xy = 4x\sqrt{900 - x^2},\ 0 \leq x \leq 30$, so

$$\begin{aligned}\frac{dA}{dx} &= 4x\left(\tfrac{1}{2}\right)(900 - x^2)^{-1/2}(-2x) + (900 - x^2)^{1/2} \cdot 4 \\ &= \tfrac{-4x^2}{(900-x^2)^{1/2}} + 4(900 - x^2)^{1/2} = \tfrac{4[-x^2 = (900 - x^2)]}{(900-x^2)^{1/2}}.\end{aligned}$$

$\frac{dA}{dx} = 0$ when $-x^2 + (900 - x^2) = 0 \Rightarrow x^2 = 450 \Rightarrow x = 15\sqrt{2} \Rightarrow y = \sqrt{900 - (15\sqrt{2})^2} = 15\sqrt{2}$.

Since $A(0) = A(30) = 0$, the rectangle of maximum area is a square.

(b)



The cross-sectional area of each rectangular plank (shaded in the figure) is

$A = 2x(y - 15\sqrt{2}) = 2x[\sqrt{900 - x^2} - 15\sqrt{2}],\ 0 \leq x \leq 15\sqrt{2}$, so

$$\begin{aligned}\frac{dA}{dx} &= 2\left(\sqrt{900 - x^2}\right) - 15\sqrt{2}) + 2x\left(\tfrac{1}{2}\right)(900 - x^2)^{-1/2}(-2x) \\ &= 2(900 - x^2)^{1/2} - 30\sqrt{2} - \tfrac{2x^2}{(900-x^2)^{1/2}}\end{aligned}$$

Set $\frac{dA}{dx} = 0:\quad (900 - x^2) - 15\sqrt{2}(900 - x^2)^{1/2} - x^2 = 0 \Rightarrow 900 - 2x^2 = 15\sqrt{2}(900 - x^2)^{1/2} \Rightarrow$

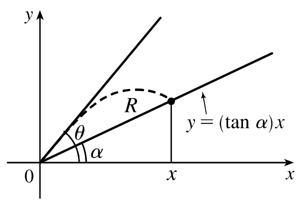
$4x^4 - 3600x^2 + 810{,}000 = 450(900 - x^2) \Rightarrow 4x^4 - 3150x^2 + 405{,}000 = 0 \Rightarrow$

$2x^4 - 1575x^2 + 202{,}500 = 0 \Rightarrow x^2 = \frac{1575 \pm \sqrt{860{,}625}}{4} \Rightarrow x \approx 25.01$ or 12.72. But $25.01 > 15\sqrt{2}$,

so $x_1 \approx 12.72 \Rightarrow y - 15\sqrt{2} = \sqrt{900 - x_1^2} - 15\sqrt{2} \approx 5.96$. Each plank should have dimensions about 12.72 m by 5.96 m.

(c) From the figure in part (a), the width is $2x$ and the depth is $2y$, so the strength is
$S = k(2x)(2y)^2 = 8kxy^2 = 8kx(900 - x^2) = 7200kx - 8kx^3,\ 0 \le x \le 30.$
$dS/dx = 7200k - 24kx^2 = 0$ when $24kx^2 = 7200 \Rightarrow x^2 = 300 \Rightarrow x = 10\sqrt{3} \Rightarrow$
$y = \sqrt{600} = 10\sqrt{6} = \sqrt{2}x$. Since $S(0) = S(30) = 0$, the maximum strength occurs when $x = 10\sqrt{3}$.
The dimensions should be $20\sqrt{3} \approx 34.64$ centimeters by $20\sqrt{6} \approx 48.99$ centimeters.

68. (a)



$y = (\tan\theta)x - \dfrac{g}{2v^2\cos^2\theta}x^2$. The parabola intersects the line when

$(\tan\alpha)x = (\tan\theta)x - \dfrac{g}{2v^2\cos^2\theta}x^2 \quad \Rightarrow$

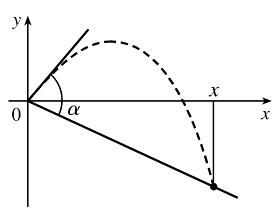
$x = \dfrac{(\tan\theta - \tan\alpha)2v^2\cos^2\theta}{g} \quad \Rightarrow$

$$R(\theta) = \frac{x}{\cos\alpha} = \left(\frac{\sin\theta}{\cos\theta} - \frac{\sin\alpha}{\cos\alpha}\right)\frac{2v^2\cos^2\theta}{g\cos\alpha} = \left(\frac{\sin\theta}{\cos\theta} - \frac{\sin\alpha}{\cos\alpha}\right)(\cos\theta\ \cos\alpha)\frac{2v^2\cos\theta}{g\cos^2\alpha}$$
$$= (\sin\theta\ \cos\alpha - \sin\alpha\ \cos\theta)\frac{2v^2\cos\theta}{g\cos^2\alpha} = \sin(\theta - \alpha)\frac{2v^2\cos\theta}{g\cos^2\alpha}$$

(b) $$R'(\theta) = \frac{2v^2}{g\cos^2\alpha}[\cos\theta \cdot \cos(\theta - \alpha) + \sin(\theta - \alpha)(-\sin\theta)] = \frac{2v^2}{g\cos^2\alpha}\cos[\theta + (\theta - \alpha)]$$
$$= \frac{2v^2}{g\cos^2\alpha}\cos(2\theta - \alpha) = 0$$

when $\cos(2\theta - \alpha) = 0 \quad \Rightarrow \quad 2\theta - \alpha = \frac{\pi}{2} \quad \Rightarrow \quad \theta = \dfrac{\pi/2 + \alpha}{2} = \dfrac{\pi}{4} + \dfrac{\alpha}{2}$. The First Derivative Test shows that this gives a maximum value for $R(\theta)$. [This could be done without calculus by applying the formula for $\sin x\ \cos y$ to $R(\theta)$.]

(c)



Replacing α by $-\alpha$ in part (a), we get $R(\theta) = \dfrac{2v^2\cos\theta\ \sin(\theta + \alpha)}{g\cos^2\alpha}$.

Proceeding as in part (b), or simply by replacing α by $-\alpha$ in the result of part (b), we see that $R(\theta)$ is maximized when $\theta = \dfrac{\pi}{4} - \dfrac{\alpha}{2}$.

69. Let c be the diameter of the semicircle. Then, from the given figure and the Law of Cosines, $c^2 = a^2 + a^2 - 2a \cdot a \cdot \cos\theta$.
The radius of the semicircle is $\frac{1}{2}c$, or $\frac{1}{2}\sqrt{2a^2 - 2a^2\cos\theta}$. The area of the figure is given by

$$A(\theta) = \text{area of triangle} + \text{area of circle}$$
$$= \tfrac{1}{2}a \cdot a \cdot \sin\theta + \tfrac{1}{2}\pi\left(\tfrac{1}{2}\sqrt{2a^2 - 2a^2\cos\theta}\right)^2 = \tfrac{1}{2}a^2\sin\theta + \tfrac{1}{8}\pi(2a^2 - 2a^2\cos\theta)$$

$A'(\theta) = \frac{1}{2}a^2\cos\theta + \frac{1}{4}\pi a^2\sin\theta = 0 \quad \Rightarrow \quad \frac{1}{4}\pi a^2\sin\theta = -\frac{1}{2}a^2\cos\theta \quad \Rightarrow \quad \tan\theta = -\dfrac{2}{\pi} \quad \Rightarrow$

$\theta = \tan^{-1}\left(-\frac{2}{\pi}\right) + n\pi$ (n an integer). We let $n = 1$ so that $0 \le \theta \le \pi$, giving

$\theta = \tan^{-1}\left(-\frac{2}{\pi}\right) + \pi = \tan^{-1}\left(-\frac{2}{\pi}\right) + 180^\circ \approx 147.5^\circ$.

$A''(\theta) = -\frac{1}{2}a^2 \sin\theta + \frac{1}{4}\pi a^2 \cos\theta < 0$ [θ. is a second-quadrant angle, so $\sin\theta > 0$ and $\cos\theta < 0$], so this value of θ gives a maximum.

70. (a) $V'(t)$ is the rate of change of the volume of the water with respect to time. $H'(t)$ is the rate of change of the height of the water with respect to time. Since the volume and the height are increasing, $V'(t)$ and $H'(t)$ are positive.

(b) $V'(t)$ is constant, so $V''(t)$ is zero (the slope of a constant function is 0).

(c) At first, the height H of the water increases quickly because the tank is narrow. But as the sphere widens, the rate of increase of the height slows down, reaching a minimum at $t = t_2$. Thus, the height is increasing at a decreasing rate on $(0, t_2)$, so its graph is concave downward and $H''(t_1) < 0$. As the sphere narrows for $t > t_2$, the rate of increase of the height begins to increase, and the graph of H is concave upward. Therefore, $H''(t_2) = 0$ and $H''(t_3) > 0$.

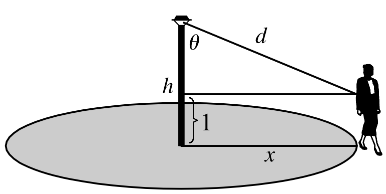
71. (a) $I = \dfrac{k\cos\theta}{d^2} = \dfrac{k(h/d)}{d^2} = k\dfrac{h}{d^3} = k\dfrac{h}{\left(\sqrt{20^2 + h^2}\right)^3} = k\dfrac{h}{(400 + h^2)^{3/2}} \quad\Rightarrow$

$$\frac{dI}{dh} = k\frac{(400 + h^2)^{3/2} - h\frac{3}{2}(400 + h^2)^{1/2}\cdot 2h}{[(400 + h^2)^{3/2}]^2} = \frac{k(400 + h^2)^{1/2}(400 + h^2 - 3h^2)}{(400 + h^2)^3}$$

$$= \frac{k(400 - 2h)^2}{(400 + h^2)^{5/2}} \qquad [k \text{ is the constant of proportionality}]$$

Set $dI/dh = 0$: $400 - 2h^2 = 0 \;\Rightarrow\; h^2 = 200 \;\Rightarrow\; h = \sqrt{200} = 10\sqrt{2}$. By the First Derivative Test, I has a local maximum at $h = 10\sqrt{2} \approx 14$ m.

(b)



$\dfrac{dx}{dt} = 1$ m/s

$I = \dfrac{k\cos\theta}{d^2} = \dfrac{k[(h-1)/d]}{d^2} = \dfrac{k(h-1)}{d^3}$

$= \dfrac{k(h-1)}{[(h-1)^2 + x^2]^{3/2}} = k(h-1)\left[(h-1)^2 + x^2\right]^{-3/2}$ so

$$\frac{dI}{dt} = \frac{dI}{dx}\cdot\frac{dx}{dt} = k(h-1)\left(-\tfrac{3}{2}\right)\left[(h-1)^2 + x^2\right]^{-5/2}\cdot 2x\cdot\frac{dx}{dt}$$

$$= k(h-1)(-3x)\left[(h-1)^2 + x^2\right]^{-5/2} = \frac{-3xk(h-1)}{[(h-1)^2 + x^2]^{5/2}}$$

When $x = 20$,

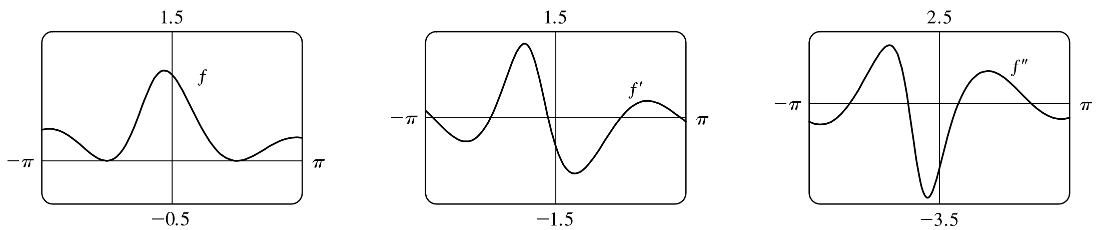
$$\left.\frac{dI}{dt}\right|_{x=20} = -\frac{60k(h-1)}{[(h-1)^2 + 400]^{5/2}}$$

72. $f(x) = \dfrac{\cos^2 x}{\sqrt{x^2 + x + 1}}$, $-\pi \leq x \leq \pi \quad \Rightarrow \quad f'(x) = -\dfrac{\cos x\,[(2x+1)\cos x + 4(x^2+x+1)\sin x]}{2(x^2+x+1)^{3/2}} \quad \Rightarrow$

$$f''(x) = -\frac{(8x^4 + 16x^3 + 16x^2 + 8x + 9)\cos^2 x - 8(x^2+x+1)(2x+1)\sin x\,\cos x - 8(x^2+x+1)^2\sin^2 x}{4(x^2+x+1)^{5/2}}$$

$f(x) = 0 \quad \Leftrightarrow \quad x = \pm\frac{\pi}{2}$; $\quad f'(x) = 0 \quad \Leftrightarrow \quad x \approx -2.96, -1.57, -0.18, 1.57, 3.01$;

$f''(x) = 0 \quad \Leftrightarrow \quad x \approx -2.16, -0.75, 0.46$, and 2.21.



The x-coordinates of the maximum points are the values at which f' changes from positive to negative, that is, -2.96, -0.18, and 3.01. The x-coordinates of the minimum points are the values at which f' changes from negative to positive, that is, -1.57 and 1.57. The x-coordinates of the inflection points are the values at which f'' changes sign, that is, -2.16, -0.75, 0.46, and 2.21.

PROBLEMS PLUS

1. Let $f(x) = \sin x - \cos x$ on $[0, 2\pi]$ since f has period 2π. $f'(x) = \cos x + \sin x = 0 \quad \Leftrightarrow \quad \cos x = -\sin x \quad \Leftrightarrow$ $\tan x = -1 \quad \Leftrightarrow \quad x = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$. Evaluating f at its critical numbers and endpoints, we get $f(0) = -1$, $f\left(\frac{3\pi}{4}\right) = \sqrt{2}$, $f\left(\frac{7\pi}{4}\right) = -\sqrt{2}$, and $f(2\pi) = -1$. So f has absolute maximum value $\sqrt{2}$ and absolute minimum value $-\sqrt{2}$. Thus, $-\sqrt{2} \leq \sin x - \cos x \leq \sqrt{2} \quad \Rightarrow \quad |\sin x - \cos x| \leq \sqrt{2}$.

2. $x^2y^2\left(4 - x^2\right)\left(4 - y^2\right) = x^2\left(4 - x^2\right)y^2\left(4 - y^2\right) = f(x)f(y)$, where $f(t) = t^2\left(4 - t^2\right)$. We will show that $0 \leq f(t) \leq 4$ for $|t| \leq 2$, which gives $0 \leq f(x)f(y) \leq 16$ for $|x| \leq 2$ and $|y| \leq 2$.

$f(t) = 4t^2 - t^4 \quad \Rightarrow \quad f'(t) = 8t - 4t^3 = 4t\left(2 - t^2\right) = 0 \quad \Rightarrow \quad t = 0$ or $\pm\sqrt{2}$.

$f(0) = 0$, $f\left(\pm\sqrt{2}\right) = 2(4 - 2) = 4$, and $f(2) = 0$. So 0 is the absolute minimum value of $f(t)$ on $[-2, 2]$ and 4 is the absolute maximum value of $f(t)$ on $[-2, 2]$. We conclude that $0 \leq f(t) \leq 4$ for $|t| \leq 2$ and hence, $0 \leq f(x)f(y) \leq 4^2$ or $0 \leq x^2\left(4 - x^2\right)y^2\left(4 - y^2\right) \leq 16$.

3. $y = \dfrac{\sin x}{x} \quad \Rightarrow \quad y' = \dfrac{x\cos x - \sin x}{x^2} \quad \Rightarrow \quad y'' = \dfrac{-x^2\sin x - 2x\cos x + 2\sin x}{x^3}$. If (x, y) is an inflection point, then $y'' = 0 \quad \Rightarrow \quad (2 - x^2)\sin x = 2x\cos x \quad \Rightarrow \quad (2 - x^2)^2\sin^2 x = 4x^2\cos^2 x \quad \Rightarrow$

$(2 - x^2)^2\sin^2 x = 4x^2(1 - \sin^2 x) \quad \Rightarrow \quad (4 - 4x^2 + x^4)\sin^2 x = 4x^2 - 4x^2\sin^2 x \quad \Rightarrow$

$(4 + x^4)\sin^2 x = 4x^2 \quad \Rightarrow \quad (x^4 + 4)\dfrac{\sin^2 x}{x^2} = 4 \quad \Rightarrow \quad y^2(x^4 + 4) = 4$ since $y = \dfrac{\sin x}{x}$.

4. Let $P\left(a, 1 - a^2\right)$ be the point of contact. The equation of the tangent line at P is $y - \left(1 - a^2\right) = (-2a)(x - a) \quad \Rightarrow$ $y - 1 + a^2 = -2ax + 2a^2 \quad \Rightarrow \quad y = -2ax + a^2 + 1$. To find the x-intercept, put $y = 0$: $2ax = a^2 + 1 \quad \Rightarrow$ $x = \dfrac{a^2 + 1}{2a}$. To find the y-intercept, put $x = 0$: $y = a^2 + 1$. Therefore, the area of the triangle is

$\dfrac{1}{2}\left(\dfrac{a^2 + 1}{2a}\right)\left(a^2 + 1\right) = \dfrac{\left(a^2 + 1\right)^2}{4a}$. Therefore, we minimize the function $A(a) = \dfrac{\left(a^2 + 1\right)^2}{4a}$, $a > 0$.

$$A'(a) = \frac{(4a)2\left(a^2 + 1\right)(2a) - \left(a^2 + 1\right)^2(4)}{16a^2} = \frac{\left(a^2 + 1\right)\left[4a^2 - \left(a^2 + 1\right)\right]}{4a^2} = \frac{\left(a^2 + 1\right)\left(3a^2 - 1\right)}{4a^2}.$$

$A'(a) = 0$ when $3a^2 - 1 = 0 \quad \Rightarrow \quad a = \frac{1}{\sqrt{3}}$. $A'(a) < 0$ for $a < \frac{1}{\sqrt{3}}$, $A'(a) > 0$ for $a > \frac{1}{\sqrt{3}}$. So by the First Derivative Test, there is an absolute minimum when $a = \frac{1}{\sqrt{3}}$. The required point is $\left(\frac{1}{\sqrt{3}}, \frac{2}{3}\right)$ and the corresponding minimum area is $A\left(\frac{1}{\sqrt{3}}\right) = \frac{4\sqrt{3}}{9}$.

5. Differentiating $x^2 + xy + y^2 = 12$ implicitly with respect to x gives $2x + y + x\dfrac{dy}{dx} + 2y\dfrac{dy}{dx} = 0$, so $\dfrac{dy}{dx} = -\dfrac{2x+y}{x+2y}$.

At a highest or lowest point, $\dfrac{dy}{dx} = 0 \quad \Leftrightarrow \quad y = -2x$. Substituting $-2x$ for y in the original equation gives

$x^2 + x(-2x) + (-2x)^2 = 12$, so $3x^2 = 12$ and $x = \pm 2$. If $x = 2$, then $y = -2x = -4$, and if $x = -2$ then $y = 4$. Thus, the highest and lowest points are $(-2, 4)$ and $(2, -4)$.

6.
$$f(x) = \frac{1}{1+|x|} + \frac{1}{1+|x-2|}$$
$$= \begin{cases} \dfrac{1}{1-x} + \dfrac{1}{1-(x-2)} & \text{if } x < 0 \\ \dfrac{1}{1+x} + \dfrac{1}{1-(x-2)} & \text{if } 0 \le x < 2 \\ \dfrac{1}{1+x} + \dfrac{1}{1+(x-2)} & \text{if } x \ge 2 \end{cases} \quad \Rightarrow \quad f'(x) = \begin{cases} \dfrac{1}{(1-x)^2} + \dfrac{1}{(3-x)^2} & \text{if } x < 0 \\ \dfrac{-1}{(1+x)^2} + \dfrac{1}{(3-x)^2} & \text{if } 0 < x < 2 \\ \dfrac{-1}{(1+x)^2} - \dfrac{1}{(x-1)^2} & \text{if } x > 2 \end{cases}$$

We see that $f'(x) > 0$ for $x < 0$ and $f'(x) < 0$ for $x > 2$. For $0 < x < 2$, we have

$f'(x) = \dfrac{1}{(3-x)^2} - \dfrac{1}{(x+1)^2} = \dfrac{(x^2+2x+1)-(x^2-6x+9)}{(3-x)^2(x+1)^2} = \dfrac{8(x-1)}{(3-x)^2(x+1)^2}$, so $f'(x) < 0$ for $0 < x < 1$,

$f'(1) = 0$ and $f'(x) > 0$ for $1 < x < 2$. We have shown that $f'(x) > 0$ for $x < 0$; $f'(x) < 0$ for $0 < x < 1$; $f'(x) > 0$ for $1 < x < 2$; and $f'(x) < 0$ for $x > 2$. Therefore, by the First Derivative Test, the local maxima of f are at $x = 0$ and $x = 2$, where f takes the value $\frac{4}{3}$. Therefore, $\frac{4}{3}$ is the absolute maximum value of f.

7. Since

$$\frac{f(x+n) - f(x)}{n} = f'(x) \qquad \textbf{(1)}$$

holds for all real numbers x and all positive integers n, we have

$$\frac{f(x+n) - f(x)}{n} = \frac{f(x+2n) - f(x)}{2n}$$

for every real number x. It follows that

$$f(x+2n) - 2f(x+n) = -f(x) \qquad \textbf{(2)}$$

Now, again from **(1)**, we can write

$$nf'(x) = f(x+n) - f(x)$$

The right-hand side of this equation is differentiable by hypothesis, so the left-hand side is also differentiable. Differentiating and then using **(1)** again — twice this time — we get

$$\begin{aligned} nf''(x) &= f'(x+n) - f'(x) \\ &= \frac{f(x+2n) - f(x+n)}{n} - \frac{f(x+n) - f(x)}{n} = \frac{f(x+2n) - 2f(x+n) + f(x)}{n} \end{aligned}$$

[continued]

Rearranging this last equation and simplifying using **(2)**, we get

$$n^2 f''(x) = f(x+2n) - 2f(x+n) + f(x) = -f(x) + f(x) = 0$$

Thus, $f''(x) = 0$ for all x, so f is a linear function.

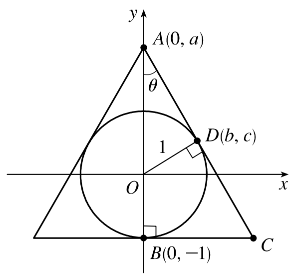
8. If $f''(x) > 0$ for all x, then f' is increasing on $(-\infty, \infty)$, so $f'(0)$ must be greater than $f'(-1)$. But $f'(0) = 0 < \frac{1}{2} = f'(-1)$, so such a function cannot exist.

9. (a) $y = x^2 \;\Rightarrow\; y' = 2x$, so the slope of the tangent line at $P(a, a^2)$ is $2a$ and the slope of the normal line is $-\dfrac{1}{2a}$ for $a \neq 0$. An equation of the normal line is $y - a^2 = -\dfrac{1}{2a}(x - a)$. Substitute x^2 for y to find the x-coordinates of the two points of intersection of the parabola and the normal line. $x^2 - a^2 = -\dfrac{x}{2a} + \dfrac{1}{2} \;\Leftrightarrow\; x^2 + \left(\dfrac{1}{2a}\right)x - \dfrac{1}{2} - a^2 = 0$. We know that a is a root of this quadratic equation, so $x - a$ is a factor, and we have $(x-a)\left(x + \dfrac{1}{2a} + a\right) = 0$, and hence, $x = -a - \dfrac{1}{2a}$ is the x-coordinate of the point Q. We want to minimize the y-coordinate of Q, which is $\left(-a - \dfrac{1}{2a}\right)^2 = a^2 + 1 + \dfrac{1}{4a^2} = y(a)$. Now $y'(a) = 2a - \dfrac{1}{2a^3} = \dfrac{4a^4 - 1}{2a^3} = \dfrac{(2a^2+1)(2a^2-1)}{2a^3} = 0 \;\Rightarrow$ $a = \dfrac{1}{\sqrt{2}}$ for $a > 0$. Since $y''(a) = 2 + \dfrac{3}{2a^4} > 0$, we see that $a = \dfrac{1}{\sqrt{2}}$ gives us the minimum value of the y-coordinate of Q.

(b) The square S of the distance from $P(a, a^2)$ to $Q\left(-a - \dfrac{1}{2a}, \left(-a - \dfrac{1}{2a}\right)^2\right)$ is given by

$$\begin{aligned} S &= \left(-a - \frac{1}{2a} - a\right)^2 + \left[\left(-a - \frac{1}{2a}\right)^2 - a^2\right]^2 = \left(-2a - \frac{1}{2a}\right)^2 + \left[\left(a^2 + 1 + \frac{1}{4a^2}\right) - a^2\right]^2 \\ &= \left(4a^2 + 2 + \frac{1}{4a^2}\right) + \left(1 + \frac{1}{4a^2}\right)^2 = \left(4a^2 + 2 + \frac{1}{4a^2}\right) + 1 + \frac{2}{4a^2} + \frac{1}{16a^4} \\ &= 4a^2 + 3 + \frac{3}{4a^2} + \frac{1}{16a^4} \end{aligned}$$

$S' = 8a - \dfrac{6}{4a^3} - \dfrac{4}{16a^5} = 8a - \dfrac{3}{2a^3} - \dfrac{1}{4a^5} = \dfrac{32a^6 - 6a^2 - 1}{4a^5} = \dfrac{(2a^2-1)(4a^2+1)^2}{4a^5}$. The only real positive zero of the equation $S' = 0$ is $a = \dfrac{1}{\sqrt{2}}$. Since $S'' = 8 + \dfrac{9}{2a^4} + \dfrac{5}{4a^6} > 0$, $a = \dfrac{1}{\sqrt{2}}$ corresponds to the shortest possible length of the line segment PQ.

10. 

$\overline{AC}$ is tangent to the unit circle at D. To find the slope of $\overline{AC}$ at D, use implicit differentiation. $x^2 + y^2 = 1 \;\Rightarrow\; 2x + 2y\,y' = 0 \;\Rightarrow\; y\,y' = -x \;\Rightarrow\; y' = -\dfrac{x}{y}$.

Thus, the tangent line at $D(b, c)$ has equation $y = -\dfrac{b}{c}\,x + a$. At D, $x = b$ and $y = c$, so $c = -\dfrac{b}{c}(b) + a \;\Rightarrow\; a = c + \dfrac{b^2}{c} = \dfrac{c^2 + b^2}{c} = \dfrac{1}{c}$, and hence $c = \dfrac{1}{a}$.

Since $b^2 + c^2 = 1$, $b = \sqrt{1 - c^2} = \sqrt{1 - 1/a^2} = \sqrt{\dfrac{a^2 - 1}{a^2}} = \dfrac{\sqrt{a^2 - 1}}{a}$, and now we have both b and c in terms of a. At C, $y = -1$, so $-1 = -\dfrac{b}{c}\,x + a \;\Rightarrow\; \dfrac{b}{c}\,x = a + 1 \;\Rightarrow$
$x = \dfrac{c}{b}(a + 1) = \dfrac{1/a}{\sqrt{a^2 - 1}/a}(a + 1) = \dfrac{a + 1}{\sqrt{(a + 1)(a - 1)}} = \sqrt{\dfrac{a + 1}{a - 1}}$, and C has coordinates $\left(\sqrt{\dfrac{a + 1}{a - 1}}, -1\right)$. Let S be the square of the distance from A to C. Then $S(a) = \left(0 - \sqrt{\dfrac{a + 1}{a - 1}}\right)^2 + (a + 1)^2 = \dfrac{a + 1}{a - 1} + (a + 1)^2 \;\Rightarrow$

$$\begin{aligned} S'(a) &= \frac{(a - 1)(1) - (a + 1)(1)}{(a - 1)^2} + 2(a + 1) = \frac{-2 + 2(a + 1)(a - 1)^2}{(a - 1)^2} \\ &= \frac{-2 + 2(a^3 - a^2 - a + 1)}{(a - 1)^2} = \frac{2a^3 - 2a^2 - 2a}{(a - 1)^2} = \frac{2a(a^2 - a - 1)}{(a - 1)^2} \end{aligned}$$

Using the quadratic formula, we find that the solutions of $a^2 - a - 1 = 0$ are $a = \dfrac{1 \pm \sqrt{5}}{2}$, so $a_1 = \dfrac{1 + \sqrt{5}}{2}$ (the "golden mean") since $a > 0$. For $1 < a < a_1$, $S'(a) < 0$, and for $a > a_1$, $S'(a) > 0$, so a_1 minimizes S.

Note: The minimum length of the equal sides is $\sqrt{S(a_1)} = \cdots = \sqrt{\dfrac{11 + 5\sqrt{5}}{2}} \approx 3.33$ and the corresponding length of the third side is $2\sqrt{\dfrac{a_1 + 1}{a_1 - 1}} = \cdots = 2\sqrt{2 + \sqrt{5}} \approx 4.12$, so the triangle is *not* equilateral.

Another method: In ΔABC, $\cos\theta = \dfrac{a + 1}{\overline{AC}}$, so $\overline{AC} = \dfrac{a + 1}{\cos\theta}$. In ΔADO, $\sin\theta = \dfrac{1}{a}$, so

$\cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - 1/a^2} = \dfrac{1}{a}\sqrt{a^2 - 1}$. Thus $\overline{AC} = \dfrac{a + 1}{(1/a)\sqrt{a^2 - 1}} = \dfrac{a(a + 1)}{\sqrt{a^2 - 1}} = f(a)$. Now find the minimum of f.

11. $A = (x_1, x_1^2)$ and $B = (x_2, x_2^2)$, where x_1 and x_2 are the solutions of the quadratic equation $x^2 = mx + b$. Let $P = (x, x^2)$ and set $A_1 = (x_1, 0)$, $B_1 = (x_2, 0)$, and $P_1 = (x, 0)$. Let $f(x)$ denote the area of triangle PAB. Then $f(x)$ can be expressed in terms of the areas of three trapezoids as follows:

$$\begin{aligned} f(x) &= \text{area}\,(A_1ABB_1) - \text{area}\,(A_1APP_1) - \text{area}\,(B_1BPP_1) \\ &= \tfrac{1}{2}(x_1^2 + x_2^2)(x_2 - x_1) - \tfrac{1}{2}(x_1^2 + x^2)(x - x_1) - \tfrac{1}{2}(x^2 + x_2^2)(x_2 - x) \end{aligned}$$

[continued]

After expanding and canceling terms, we get

$f(x) = \frac{1}{2}\left(x_2x_1^2 - x_1x_2^2 - xx_1^2 + x_1x^2 - x_2x^2 + xx_2^2\right) = \frac{1}{2}\left[x_1^2(x_2 - x) + x_2^2(x - x_1) + x^2(x_1 - x_2)\right]$

$f'(x) = \frac{1}{2}\left[-x_1^2 + x_2^2 + 2x(x_1 - x_2)\right]$. $f''(x) = \frac{1}{2}[2(x_1 - x_2)] = x_1 - x_2 < 0$ since $x_2 > x_1$.

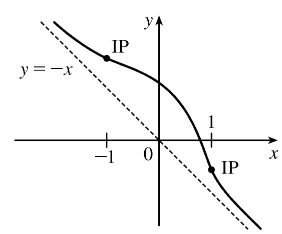
$f'(x) = 0 \quad \Rightarrow \quad 2x(x_1 - x_2) = x_1^2 - x_2^2 \quad \Rightarrow \quad x_P = \frac{1}{2}(x_1 + x_2)$.

$$\begin{aligned} f(x_P) &= \tfrac{1}{2}\left(x_1^2\left[\tfrac{1}{2}(x_2 - x_1)\right] + x_2^2\left[\tfrac{1}{2}(x_2 - x_1)\right] + \tfrac{1}{4}(x_1 + x_2)^2(x_1 - x_2)\right) \\ &= \tfrac{1}{2}\left[\tfrac{1}{2}(x_2 - x_1)\left(x_1^2 + x_2^2\right) - \tfrac{1}{4}(x_2 - x_1)(x_1 + x_2)^2\right] = \tfrac{1}{8}(x_2 - x_1)\left[2\left(x_1^2 + x_2^2\right) - \left(x_1^2 + 2x_1x_2 + x_2^2\right)\right] \\ &= \tfrac{1}{8}(x_2 - x_1)\left(x_1^2 - 2x_1x_2 + x_2^2\right) = \tfrac{1}{8}(x_2 - x_1)(x_1 - x_2)^2 = \tfrac{1}{8}(x_2 - x_1)(x_2 - x_1)^2 = \tfrac{1}{8}(x_2 - x_1)^3 \end{aligned}$$

To put this in terms of m and b, we solve the system $y = x_1^2$ and $y = mx_1 + b$, giving us $x_1^2 - mx_1 - b = 0 \quad \Rightarrow$ $x_1 = \frac{1}{2}\left(m - \sqrt{m^2 + 4b}\right)$. Similarly, $x_2 = \frac{1}{2}\left(m + \sqrt{m^2 + 4b}\right)$. The area is then $\frac{1}{8}(x_2 - x_1)^3 = \frac{1}{8}\left(\sqrt{m^2 + 4b}\right)^3$, and is attained at the point $P\left(x_P, x_P^2\right) = P\left(\frac{1}{2}m, \frac{1}{4}m^2\right)$.

Note: Another way to get an expression for $f(x)$ is to use the formula for an area of a triangle in terms of the coordinates of the vertices: $f(x) = \frac{1}{2}\left[\left(x_2x_1^2 - x_1x_2^2\right) + \left(x_1x^2 - xx_1^2\right) + \left(xx_2^2 - x_2x^2\right)\right]$.

12. If $f'(x) < 0$ for all x, $f''(x) > 0$ for $|x| > 1$, $f''(x) < 0$ for $|x| < 1$, and $\lim\limits_{x \to \pm\infty} [f(x) + x] = 0$, then f is decreasing everywhere, concave up on $(-\infty, -1)$ and $(1, \infty)$, concave down on $(-1, 1)$, and approaches the line $y = -x$ as $x \to \pm\infty$. An example of such a graph is sketched.



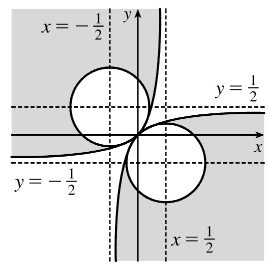
13. $f(x) = \left(a^2 + a - 6\right)\cos 2x + (a - 2)x + \cos 1 \quad \Rightarrow \quad f'(x) = -\left(a^2 + a - 6\right)\sin 2x\,(2) + (a - 2)$. The derivative exists for all x, so the only possible critical points will occur where $f'(x) = 0 \quad \Leftrightarrow \quad 2(a - 2)(a + 3)\sin 2x = a - 2 \quad \Leftrightarrow$ either $a = 2$ or $2(a + 3)\sin 2x = 1$, with the latter implying that $\sin 2x = \dfrac{1}{2(a + 3)}$. Since the range of $\sin 2x$ is $[-1, 1]$, this equation has no solution whenever either $\dfrac{1}{2(a + 3)} < -1$ or $\dfrac{1}{2(a + 3)} > 1$. Solving these inequalities, we get $-\frac{7}{2} < a < -\frac{5}{2}$.

14. To sketch the region $\left\{(x, y) \mid 2xy \le |x - y| \le x^2 + y^2\right\}$, we consider two cases.

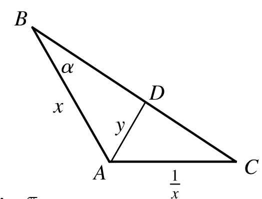
Case 1: $x \ge y$ This is the case in which (x, y) lies on or below the line $y = x$. The double inequality becomes $2xy \le x - y \le x^2 + y^2$. The right-hand inequality holds if and only if $x^2 - x + y^2 + y \ge 0 \quad \Leftrightarrow$ $\left(x - \frac{1}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 \ge \frac{1}{2} \quad \Leftrightarrow \quad (x, y)$ lies on or outside the circle with radius $\frac{1}{\sqrt{2}}$ centered at $\left(\frac{1}{2}, -\frac{1}{2}\right)$.

The left-hand inequality holds if and only if $2xy - x + y \le 0 \quad \Leftrightarrow \quad xy - \frac{1}{2}x + \frac{1}{2}y \le 0 \quad \Leftrightarrow$ $\left(x + \frac{1}{2}\right)\left(y - \frac{1}{2}\right) \le -\frac{1}{4} \quad \Leftrightarrow \quad (x, y)$ lies on or below the hyperbola $\left(x + \frac{1}{2}\right)\left(y - \frac{1}{2}\right) = -\frac{1}{4}$, which passes through the origin and approaches the lines $y = \frac{1}{2}$ and $x = -\frac{1}{2}$ asymptotically.

Case 2: $y \geq x$ This is the case in which (x, y) lies on or above the line $y = x$. The double inequality becomes $2xy \leq y - x \leq x^2 + y^2$. The right-hand inequality holds if and only if $x^2 + x + y^2 - y \geq 0 \Leftrightarrow$ $\left(x+\frac{1}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2 \geq \frac{1}{2} \Leftrightarrow (x, y)$ lies on or outside the circle of radius $\frac{1}{\sqrt{2}}$ centered at $\left(-\frac{1}{2}, \frac{1}{2}\right)$. The left-hand inequality holds if and only if $2xy + x - y \leq 0 \Leftrightarrow xy + \frac{1}{2}x - \frac{1}{2}y \leq 0 \Leftrightarrow \left(x-\frac{1}{2}\right)\left(y+\frac{1}{2}\right) \leq -\frac{1}{4} \Leftrightarrow (x, y)$ lies on or above the left-hand branch of the hyperbola $\left(x-\frac{1}{2}\right)\left(y+\frac{1}{2}\right) = -\frac{1}{4}$, which passes through the origin and approaches the lines $y = -\frac{1}{2}$ and $x = \frac{1}{2}$ asymptotically. Therefore, the region of interest consists of the points on or above the left branch of the hyperbola $\left(x-\frac{1}{2}\right)\left(y+\frac{1}{2}\right) = -\frac{1}{4}$ that are on or outside the circle $\left(x+\frac{1}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2 = \frac{1}{2}$, together with the points on or below the right branch of the hyperbola $\left(x+\frac{1}{2}\right)\left(y-\frac{1}{2}\right) = -\frac{1}{4}$ that are on or outside the circle $\left(x-\frac{1}{2}\right)^2 + \left(y+\frac{1}{2}\right)^2 = \frac{1}{2}$. Note that the inequalities are unchanged when x and y are interchanged, so the region is symmetric about the line $y = x$. So we need only have analyzed case 1 and then reflected that region about the line $y = x$, instead of considering case 2.



15. (a) Let $y = |AD|$, $x = |AB|$, and $1/x = |AC|$, so that $|AB| \cdot |AC| = 1$. We compute the area $\mathcal{A}$ of $\triangle ABC$ in two ways.



First, $\mathcal{A} = \frac{1}{2}|AB|\,|AC| \sin\frac{2\pi}{3} = \frac{1}{2} \cdot 1 \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}$.

Second,

$$\begin{aligned} \mathcal{A} &= (\text{area of } \triangle ABD) + (\text{area of } \triangle ACD) = \tfrac{1}{2}|AB|\,|AD| \sin\tfrac{\pi}{3} + \tfrac{1}{2}|AD|\,|AC| \sin\tfrac{\pi}{3} \\ &= \tfrac{1}{2}xy\,\tfrac{\sqrt{3}}{2} + \tfrac{1}{2}y(1/x)\,\tfrac{\sqrt{3}}{2} = \tfrac{\sqrt{3}}{4}y(x+1/x) \end{aligned}$$

Equating the two expressions for the area, we get $\frac{\sqrt{3}}{4}y\left(x+\dfrac{1}{x}\right) = \frac{\sqrt{3}}{4} \Leftrightarrow y = \dfrac{1}{x+1/x} = \dfrac{x}{x^2+1}$, $x > 0$.

Another method: Use the Law of Sines on the triangles ABD and ABC. In $\triangle ABD$, we have $\angle A + \angle B + \angle D = 180° \Leftrightarrow 60° + \alpha + \angle D = 180° \Leftrightarrow \angle D = 120° - \alpha$. Thus,

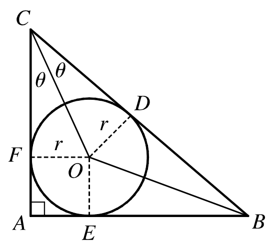
$\dfrac{x}{y} = \dfrac{\sin(120° - \alpha)}{\sin\alpha} = \dfrac{\sin 120° \cos\alpha - \cos 120° \sin\alpha}{\sin\alpha} = \dfrac{\frac{\sqrt{3}}{2}\cos\alpha + \frac{1}{2}\sin\alpha}{\sin\alpha} \Rightarrow \dfrac{x}{y} = \frac{\sqrt{3}}{2}\cot\alpha + \frac{1}{2}$, and by a similar argument with $\triangle ABC$, $\frac{\sqrt{3}}{2}\cot\alpha = x^2 + \frac{1}{2}$. Eliminating $\cot\alpha$ gives $\dfrac{x}{y} = \left(x^2 + \frac{1}{2}\right) + \frac{1}{2} \Rightarrow$

$y = \dfrac{x}{x^2+1}$, $x > 0$.

(b) We differentiate our expression for y with respect to x to find the maximum:

$\dfrac{dy}{dx} = \dfrac{(x^2+1) - x(2x)}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2} = 0$ when $x = 1$. This indicates a maximum by the First Derivative Test, since $y'(x) > 0$ for $0 < x < 1$ and $y'(x) < 0$ for $x > 1$, so the maximum value of y is $y(1) = \frac{1}{2}$.

16. (a)



From geometry, two tangents to a circle from a given point have the same length, so $|CF| = |CD|$, $|AE| = |AF|$, and $|BD| = |BE|$. Thus,

$$\begin{aligned}
&\tfrac{1}{2}(|BC| + |AC| - |AB|) \\
&\quad = \tfrac{1}{2}[(|BD| + |DC|) + (|AF| + |FC|) - (|AE| + |EB|)] \\
&\quad = \tfrac{1}{2}\Big[\Big(\underline{|BD|} + |CD|\Big) + \Big(\underline{|AF|} + |CD|\Big) - \Big(\underline{|AF|} + \underline{|BD|}\Big)\Big] \\
&\quad = \tfrac{1}{2}[2\,|CD|] = |CD|
\end{aligned}$$

(b) Using the result from part (a) and the fact that $a = |BC|$, we have $\tan\theta = \dfrac{r}{|CD|} \quad \Rightarrow$

$$\frac{r}{\tan\theta} = |CD| = \tfrac{1}{2}\left(|AC| + |BC| - |AB|\right) = \tfrac{1}{2}(a\cos 2\theta + a - a\sin 2\theta) \quad \Leftrightarrow$$

$$\begin{aligned}
r &= \tfrac{1}{2}a\tan\theta\,(2\cos^2\theta - 1 + 1 - 2\sin\theta\,\cos\theta) \\
&= \tfrac{1}{2}a\,(2\sin\theta\,\cos\theta - 2\sin^2\theta) && \text{[in terms of } \theta\text{]} \\
&= \tfrac{1}{2}a\,(\sin 2\theta + \cos 2\theta - 1) && \text{[in terms of } 2\theta\text{]}
\end{aligned}$$

(c) We differentiate r with respect to θ and set $dr/d\theta$ equal to 0 to find the maximum values:

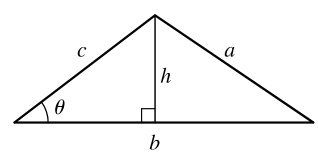
$dr/d\theta = \frac{1}{2}a\,(2\cos 2\theta - 2\sin 2\theta) = a(\cos 2\theta - \sin 2\theta)$. Since $0 < \theta < \frac{\pi}{4}$, $dr/d\theta = 0 \quad \Leftrightarrow \quad \cos 2\theta = \sin 2\theta \quad \Leftrightarrow$

$1 = \tan 2\theta \quad \Leftrightarrow \quad 2\theta = \frac{\pi}{4} \quad \Leftrightarrow \quad \theta = \frac{\pi}{8}$. This gives a maximum by the First Derivative Test, since

$dr/d\theta > 0$ for $0 < \theta < \frac{\pi}{8}$, and $dr/d\theta < 0$ for $\frac{\pi}{8} < \theta < \frac{\pi}{4}$. The maximum value is

$r\left(\frac{\pi}{8}\right) = \frac{1}{2}a\left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4} - 1\right) = \frac{1}{2}\left(\sqrt{2} - 1\right)a \approx 0.207a.$

17. (a) $A = \frac{1}{2}bh$ with $\sin\theta = h/c$, so $A = \frac{1}{2}bc\sin\theta$. But A is a constant, so differentiating this equation with respect to t, we get



$$\frac{dA}{dt} = 0 = \frac{1}{2}\left[bc\cos\theta\,\frac{d\theta}{dt} + b\,\frac{dc}{dt}\sin\theta + \frac{db}{dt}\,c\sin\theta\right] \quad \Rightarrow$$

$$bc\cos\theta\,\frac{d\theta}{dt} = -\sin\theta\left[b\,\frac{dc}{dt} + c\,\frac{db}{dt}\right] \quad \Rightarrow \quad \frac{d\theta}{dt} = -\tan\theta\left[\frac{1}{c}\frac{dc}{dt} + \frac{1}{b}\frac{db}{dt}\right].$$

(b) We use the Law of Cosines to get the length of side a in terms of those of b and c, and then we differentiate implicitly with respect to t: $a^2 = b^2 + c^2 - 2bc\cos\theta \quad \Rightarrow$

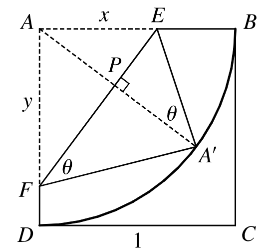
$$2a\,\frac{da}{dt} = 2b\,\frac{db}{dt} + 2c\,\frac{dc}{dt} - 2\left[bc(-\sin\theta)\,\frac{d\theta}{dt} + b\,\frac{dc}{dt}\,\cos\theta + \frac{db}{dt}\,c\cos\theta\right] \quad \Rightarrow$$

$\dfrac{da}{dt} = \dfrac{1}{a}\left(b\,\dfrac{db}{dt} + c\,\dfrac{dc}{dt} + bc\sin\theta\,\dfrac{d\theta}{dt} - b\,\dfrac{dc}{dt}\,\cos\theta - c\,\dfrac{db}{dt}\,\cos\theta\right)$. Now we substitute our value of a from the Law of Cosines and the value of $d\theta/dt$ from part (a), and simplify (primes signify differentiation by t):

$$\begin{aligned}
\frac{da}{dt} &= \frac{bb' + cc' + bc\sin\theta\,[-\tan\theta(c'/c + b'/b)] - (bc' + cb')(\cos\theta)}{\sqrt{b^2 + c^2 - 2bc\,\cos\theta}} \\
&= \frac{bb' + cc' - [\sin^2\theta\,(bc' + cb') + \cos^2\theta\,(bc' + cb')]/\cos\theta}{\sqrt{b^2 + c^2 - 2bc\cos\theta}} = \frac{bb' + cc' - (bc' + cb')\,\sec\theta}{\sqrt{b^2 + c^2 - 2bc\cos\theta}}
\end{aligned}$$

or $$\frac{da}{dt} = \frac{b\,\dfrac{db}{dt} + c\,\dfrac{dc}{dt} - \left(b\,\dfrac{dc}{dt} + c\,\dfrac{db}{dt}\right)\sec\theta}{\sqrt{b^2 + c^2 - 2bc\cos\theta}}.$$

18. Let $x = |AE|$, $y = |AF|$ as shown. The area $\mathcal{A}$ of the ΔAEF is $\mathcal{A} = \frac{1}{2}xy$. We need to find a relationship between x and y, so that we can take the derivative $d\mathcal{A}/dx$ and then find the maximum and minimum areas. Now let A' be the point on which A ends up after the fold has been performed, and let P be the intersection of AA' and EF. Note that AA' is perpendicular to EF since we are reflecting A through the line EF to get to A', and that $|AP| = |PA'|$ for the same reason.



But $|AA'| = 1$, since AA' is a radius of the circle. Since $|AP| + |PA'| = |AA'|$, we have $|AP| = \frac{1}{2}$. Another way to express the area of the triangle is $\mathcal{A} = \frac{1}{2}|EF|\,|AP| = \frac{1}{2}\sqrt{x^2+y^2}\left(\frac{1}{2}\right) = \frac{1}{4}\sqrt{x^2+y^2}$. Equating the two expressions for $\mathcal{A}$, we get $\frac{1}{2}xy = \frac{1}{4}\sqrt{x^2+y^2} \quad\Rightarrow\quad 4x^2y^2 = x^2+y^2 \quad\Rightarrow\quad y^2(4x^2-1) = x^2 \quad\Rightarrow\quad y = x/\sqrt{4x^2-1}$.

(Note that we could also have derived this result from the similarity of $\triangle A'PE$ and $\triangle A'FE$; that is,

$\dfrac{|A'P|}{|PE|} = \dfrac{|A'F|}{|A'E|} \quad\Rightarrow\quad \dfrac{\frac{1}{2}}{\sqrt{x^2-\left(\frac{1}{2}\right)^2}} = \dfrac{y}{x} \quad\Rightarrow\quad y = \dfrac{\frac{1}{2}x}{\sqrt{4x^2-1}/2} = \dfrac{x}{\sqrt{4x^2-1}}$.) Now we can substitute for y and

calculate $\dfrac{d\mathcal{A}}{dx}$: $\mathcal{A} = \dfrac{1}{2}\dfrac{x^2}{\sqrt{4x^2-1}} \quad\Rightarrow\quad \dfrac{d\mathcal{A}}{dx} = \dfrac{1}{2}\left[\dfrac{\sqrt{4x^2-1}\,(2x) - x^2\left(\frac{1}{2}\right)(4x^2-1)^{-1/2}(8x)}{4x^2-1}\right]$. This is 0 when

$2x\sqrt{4x^2-1} - 4x^3(4x^2-1)^{-1/2} = 0 \quad\Leftrightarrow\quad 2x(4x^2-1)^{-1/2}\left[(4x^2-1)-2x^2\right] = 0 \quad\Rightarrow\quad (4x^2-1)-2x^2 = 0$

$(x > 0) \quad\Leftrightarrow\quad 2x^2 = 1 \quad\Rightarrow\quad x = \frac{1}{\sqrt{2}}$. So this is one possible value for an extremum. We must also test the endpoints of the interval over which x ranges. The largest value that x can attain is 1, and the smallest value of x occurs when $y = 1 \quad\Leftrightarrow$ $1 = x/\sqrt{4x^2-1} \quad\Leftrightarrow\quad x^2 = 4x^2-1 \quad\Leftrightarrow\quad 3x^2 = 1 \quad\Leftrightarrow\quad x = \frac{1}{\sqrt{3}}$. This will give the same value of $\mathcal{A}$ as will $x = 1$, since the geometric situation is the same (reflected through the line $y = x$). We calculate

$\mathcal{A}\left(\frac{1}{\sqrt{2}}\right) = \dfrac{1}{2}\dfrac{(1/\sqrt{2})^2}{\sqrt{4(1/\sqrt{2})^2-1}} = \dfrac{1}{4} = 0.25$, and $\mathcal{A}(1) = \dfrac{1}{2}\dfrac{1^2}{\sqrt{4(1)^2-1}} = \dfrac{1}{2\sqrt{3}} \approx 0.29$. So the maximum area is

$\mathcal{A}(1) = \mathcal{A}\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{2\sqrt{3}}$ and the minimum area is $\mathcal{A}\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{4}$.

Another method: Use the angle θ (see diagram above) as a variable:

$\mathcal{A} = \frac{1}{2}xy = \frac{1}{2}\left(\frac{1}{2}\sec\theta\right)\left(\frac{1}{2}\csc\theta\right) = \dfrac{1}{8\sin\theta\,\cos\theta} = \dfrac{1}{4\sin 2\theta}$. $\mathcal{A}$ is minimized when $\sin 2\theta$ is maximal, that is, when $\sin 2\theta = 1 \quad\Rightarrow\quad 2\theta = \frac{\pi}{2} \quad\Rightarrow\quad \theta = \frac{\pi}{4}$. Also note that $A'E = x = \frac{1}{2}\sec\theta \le 1 \quad\Rightarrow\quad \sec\theta \le 2 \quad\Rightarrow$ $\cos\theta \ge \frac{1}{2} \quad\Rightarrow\quad \theta \le \frac{\pi}{3}$, and similarly, $A'F = y = \frac{1}{2}\csc\theta \le 1 \quad\Rightarrow\quad \csc\theta \le 2 \quad\Rightarrow\quad \sin\theta \le \frac{1}{2} \quad\Rightarrow\quad \theta \ge \frac{\pi}{6}$.

As above, we find that $\mathcal{A}$ is maximized at these endpoints: $\mathcal{A}\left(\frac{\pi}{6}\right) = \dfrac{1}{4\sin\frac{\pi}{3}} = \dfrac{1}{2\sqrt{3}} = \dfrac{1}{4\sin\frac{2\pi}{3}} = \mathcal{A}\left(\frac{\pi}{3}\right)$;

and minimized at $\theta = \frac{\pi}{4}$: $\mathcal{A}\left(\frac{\pi}{4}\right) = \dfrac{1}{4\sin\frac{\pi}{2}} = \dfrac{1}{4}$.

19. (a) Distance = rate × time, so time = distance/rate. $T_1 = \dfrac{D}{c_1}$, $T_2 = \dfrac{2\,|PR|}{c_1} + \dfrac{|RS|}{c_2} = \dfrac{2h\sec\theta}{c_1} + \dfrac{D-2h\tan\theta}{c_2}$,

$$T_3 = \frac{2\sqrt{h^2+D^2/4}}{c_1} = \frac{\sqrt{4h^2+D^2}}{c_1}.$$

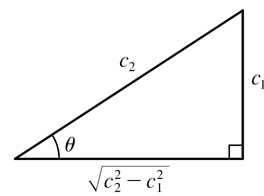
(b) $\dfrac{dT_2}{d\theta} = \dfrac{2h}{c_1} \cdot \sec\theta\,\tan\theta - \dfrac{2h}{c_2}\sec^2\theta = 0$ when $2h\sec\theta\left(\dfrac{1}{c_1}\tan\theta - \dfrac{1}{c_2}\sec\theta\right) = 0 \;\Rightarrow$

$\dfrac{1}{c_1}\dfrac{\sin\theta}{\cos\theta} - \dfrac{1}{c_2}\dfrac{1}{\cos\theta} = 0 \;\Rightarrow\; \dfrac{\sin\theta}{c_1\cos\theta} = \dfrac{1}{c_2\cos\theta} \;\Rightarrow\; \sin\theta = \dfrac{c_1}{c_2}$. The First Derivative Test shows that this gives a minimum.

(c) Using part (a) with $D = 1$ and $T_1 = 0.26$, we have $T_1 = \dfrac{D}{c_1} \;\Rightarrow\; c_1 = \frac{1}{0.26} \approx 3.85$ km/s. $T_3 = \dfrac{\sqrt{4h^2 + D^2}}{c_1} \;\Rightarrow$

$4h^2 + D^2 = T_3^2 c_1^2 \;\Rightarrow\; h = \frac{1}{2}\sqrt{T_3^2 c_1^2 - D^2} = \frac{1}{2}\sqrt{(0.34)^2(1/0.26)^2 - 1^2} \approx 0.42$ km. To find c_2, we use $\sin\theta = \dfrac{c_1}{c_2}$

from part (b) and $T_2 = \dfrac{2h\sec\theta}{c_1} + \dfrac{D - 2h\tan\theta}{c_2}$ from part (a). From the figure,



$\sin\theta = \dfrac{c_1}{c_2} \;\Rightarrow\; \sec\theta = \dfrac{c_2}{\sqrt{c_2^2 - c_1^2}}$ and $\tan\theta = \dfrac{c_1}{\sqrt{c_2^2 - c_1^2}}$, so

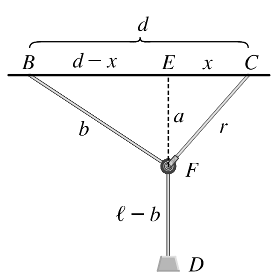
$T_2 = \dfrac{2hc_2}{c_1\sqrt{c_2^2 - c_1^2}} + \dfrac{D\sqrt{c_2^2 - c_1^2} - 2hc_1}{c_2\sqrt{c_2^2 - c_1^2}}$. Using the values for T_2 [given as 0.32],

h, c_1, and D, we can graph $\mathrm{Y}_1 = T_2$ and $\mathrm{Y}_2 = \dfrac{2hc_2}{c_1\sqrt{c_2^2 - c_1^2}} + \dfrac{D\sqrt{c_2^2 - c_1^2} - 2hc_1}{c_2\sqrt{c_2^2 - c_1^2}}$ and find their intersection points. Doing so gives us $c_2 \approx 4.10$ and 7.66, but if $c_2 = 4.10$, then $\theta = \sin^{-1}(c_1/c_2) \approx 69.6^\circ$, which implies that point S is to the left of point R in the diagram. So $c_2 = 7.66$ km/s.

20. A straight line intersects the curve $y = f(x) = x^4 + cx^3 + 12x^2 - 5x + 2$ in four distinct points if and only if the graph of f has two inflection points. $f'(x) = 4x^3 + 3cx^2 + 24x - 5$ and $f''(x) = 12x^2 + 6cx + 24$.

$f''(x) = 0 \;\Leftrightarrow\; x = \dfrac{-6c \pm \sqrt{(6c)^2 - 4(12)(24)}}{2(12)}$. There are two distinct roots for $f''(x) = 0$ (and hence two inflection points) if and only if the discriminant is positive; that is, $36c^2 - 1152 > 0 \;\Leftrightarrow\; c^2 > 32 \;\Leftrightarrow\; |c| > \sqrt{32}$. Thus, the desired values of c are $c < -4\sqrt{2}$ or $c > 4\sqrt{2}$.

21.



Let $a = |EF|$ and $b = |BF|$ as shown in the figure.

Since $\ell = |BF| + |FD|$, $|FD| = \ell - b$. Now

$$\begin{aligned} |ED| &= |EF| + |FD| = a + \ell - b \\ &\sqrt{r^2 - x^2} + \ell - \sqrt{(d - x)^2 + a^2} \\ &= \sqrt{r^2 - x^2} + \ell - \sqrt{(d - x)^2 + \left(\sqrt{r^2 - x^2}\right)^2} \\ &= \sqrt{r^2 - x^2} + \ell - \sqrt{d^2 - 2dx + x^2 + r^2 - x^2} \end{aligned}$$

Let $f(x) = \sqrt{r^2 - x^2} + \ell - \sqrt{d^2 + r^2 - 2dx}$.

$f'(x) = \frac{1}{2}\left(r^2 - x^2\right)^{-1/2}(-2x) - \frac{1}{2}\left(d^2 + r^2 - 2dx\right)^{-1/2}(-2d) = \dfrac{-x}{\sqrt{r^2 - x^2}} + \dfrac{d}{\sqrt{d^2 + r^2 - 2dx}}$.

[continued]

$f'(x) = 0 \quad \Rightarrow \quad \dfrac{x}{\sqrt{r^2 - x^2}} = \dfrac{d}{\sqrt{d^2 + r^2 - 2dx}} \quad \Rightarrow \quad \dfrac{x^2}{r^2 - x^2} = \dfrac{d^2}{d^2 + r^2 - 2dx} \quad \Rightarrow$

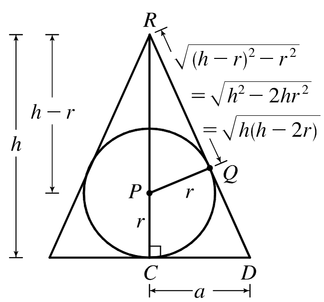
$d^2x^2 + r^2x^2 - 2dx^3 = d^2r^2 - d^2x^2 \quad \Rightarrow \quad 0 = 2dx^3 - 2d^2x^2 - r^2x^2 + d^2r^2 \quad \Rightarrow$

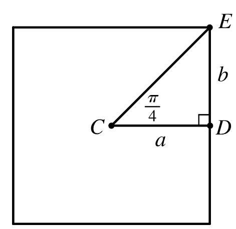
$0 = 2dx^2(x - d) - r^2(x^2 - d^2) \quad \Rightarrow \quad 0 = 2dx^2(x - d) - r^2(x + d)(x - d) \quad \Rightarrow \quad 0 = (x - d)\left[2dx^2 - r^2(x + d)\right]$

But $d > r > x$, so $x \neq d$. Thus, we solve $2dx^2 - r^2x - dr^2 = 0$ for x:

$x = \dfrac{-(-r^2) \pm \sqrt{(-r^2)^2 - 4(2d)(-dr^2)}}{2(2d)} = \dfrac{r^2 \pm \sqrt{r^4 + 8d^2r^2}}{4d}$. Because $\sqrt{r^4 + 8d^2r^2} > r^2$, the "negative" can be discarded. Thus, $x = \dfrac{r^2 + \sqrt{r^2}\,\sqrt{r^2 + 8d^2}}{4d} = \dfrac{r^2 + r\sqrt{r^2 + 8d^2}}{4d} \quad [r > 0] \quad = \dfrac{r}{4d}\left(r + \sqrt{r^2 + 8d^2}\right)$. The maximum value of $|ED|$ occurs at this value of x.

22.





Let $a = \overline{CD}$ denote the distance from the center C of the base to the midpoint D of a side of the base.

Since ΔPQR is similar to ΔDCR, $\dfrac{a}{h} = \dfrac{r}{\sqrt{h(h - 2r)}} \quad \Rightarrow \quad a = \dfrac{rh}{\sqrt{h(h - 2r)}} = r\dfrac{\sqrt{h}}{\sqrt{h - 2r}}$.

Let b denote one-half the length of a side of the base. The area A of the base is

$A = 8(\text{area of } \Delta CDE) = 8\left(\frac{1}{2}ab\right) = 4a\left(a \tan \frac{\pi}{4}\right) = 4a^2$.

The volume of the pyramid is $V = \frac{1}{3}Ah = \frac{1}{3}\left(4a^2\right)h = \dfrac{4}{3}\left(r\dfrac{\sqrt{h}}{\sqrt{h - 2r}}\right)^2 h = \frac{4}{3}r^2\dfrac{h^2}{h - 2r}$, with domain $h > 2r$.

Now $\dfrac{dV}{dh} = \dfrac{4}{3}r^2 \cdot \dfrac{(h - 2r)(2h) - h^2(1)}{(h - 2r)^2} = \dfrac{4}{3}r^2\dfrac{h^2 - 4hr}{(h - 2r)^2} = \dfrac{4}{3}r^2\dfrac{h(h - 4r)}{(h - 2r)^2}$

and

$$\begin{aligned}
\frac{d^2V}{dh^2} &= \frac{4}{3}r^2 \cdot \frac{(h - 2r)^2(2h - 4r) - \left(h^2 - 4hr\right)(2)(h - 2r)(1)}{\left[(h - 2r)^2\right]^2} \\
&= \frac{4}{3}r^2 \cdot \frac{2(h - 2r)\left[\left(h^2 - 4hr + 4r^2\right) - \left(h^2 - 4hr\right)\right]}{(h - 2r)^2} \\
&= \frac{8}{3}r^2 \cdot \frac{4r^2}{(h - 2r)^3} = \frac{32}{3}r^4 \cdot \frac{1}{(h - 2r)^3}.
\end{aligned}$$

[continued]

The first derivative is equal to zero for $h = 4r$ and the second derivative is positive for $h > 2r$, so the volume of the pyramid is minimized when $h = 4r$.

To extend our solution to a regular n-gon, we make the following changes:

(1) the number of sides of the base is n

(2) the number of triangles in the base is $2n$

(3) $\angle DCE = \dfrac{\pi}{n}$

(4) $b = a\tan\dfrac{\pi}{n}$

We then obtain the following results: $A = na^2\tan\dfrac{\pi}{n}$, $V = \dfrac{nr^2}{3}\cdot\tan\left(\dfrac{\pi}{n}\right)\cdot\dfrac{h^2}{h-2r}$, $\dfrac{dV}{dh} = \dfrac{nr^2}{3}\cdot\tan\left(\dfrac{\pi}{n}\right)\cdot\dfrac{h(h-4r)}{(h-2r)^2}$,

and $\dfrac{d^2V}{dh^2} = \dfrac{8nr^4}{3}\cdot\tan\left(\dfrac{\pi}{n}\right)\cdot\dfrac{1}{(h-2r)^3}$. Notice that the answer, $h = 4r$, is independent of the number of sides of the base of the polygon!

23. $V = \frac{4}{3}\pi r^3 \Rightarrow \dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt}$. But $\dfrac{dV}{dt}$ is proportional to the surface area, so $\dfrac{dV}{dt} = k\cdot 4\pi r^2$ for some constant k.

Therefore, $4\pi r^2\dfrac{dr}{dt} = k\cdot 4\pi r^2 \Leftrightarrow \dfrac{dr}{dt} = k =$ constant. An antiderivative of k with respect to t is kt, so $r = kt + C$.

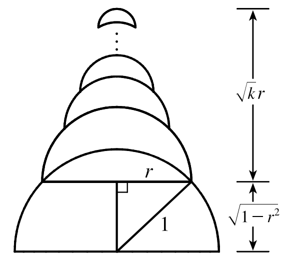
When $t = 0$, the radius r must equal the original radius r_0, so $C = r_0$, and $r = kt + r_0$. To find k we use the fact that when $t = 3$, $r = 3k + r_0$ and $V = \frac{1}{2}V_0 \Rightarrow \frac{4}{3}\pi(3k + r_0)^3 = \frac{1}{2}\cdot\frac{4}{3}\pi r_0^3 \Rightarrow (3k + r_0)^3 = \frac{1}{2}r_0^3 \Rightarrow$

$3k + r_0 = \frac{1}{\sqrt[3]{2}}r_0 \Rightarrow k = \frac{1}{3}r_0\left(\frac{1}{\sqrt[3]{2}} - 1\right)$. Since $r = kt + r_0$, $r = \frac{1}{3}r_0\left(\frac{1}{\sqrt[3]{2}} - 1\right)t + r_0$. When the snowball has melted completely we have $r = 0 \Rightarrow \frac{1}{3}r_0\left(\frac{1}{\sqrt[3]{2}} - 1\right)t + r_0 = 0$ which gives $t = \dfrac{3\sqrt[3]{2}}{\sqrt[3]{2} - 1}$. Hence, it takes

$\dfrac{3\sqrt[3]{2}}{\sqrt[3]{2} - 1} - 3 = \dfrac{3}{\sqrt[3]{2} - 1} \approx 11$ h 33 min longer.

24. By ignoring the bottom hemisphere of the initial spherical bubble, we can rephrase the problem as follows: Prove that the maximum height of a stack of n hemispherical bubbles is $\sqrt{n}$ if the radius of the bottom hemisphere is 1. We proceed by induction. The case $n = 1$ is obvious since $\sqrt{1}$ is the height of the first hemisphere. Suppose the assertion is true for $n = k$ and let's suppose we have $k + 1$ hemispherical bubbles forming a stack of maximum height. Suppose the second hemisphere (counting from the bottom) has radius r. Then by our induction hypothesis (scaled to the setting of a bottom hemisphere of radius r), the height of the stack formed by the top k bubbles is $\sqrt{k}\,r$. (If it were shorter, then the total stack of $k + 1$ bubbles wouldn't have maximum height.)

[continued]



The height of the whole stack is $H(r) = \sqrt{k}\,r + \sqrt{1-r^2}$. (See the figure.) We want to choose r so as to maximize $H(r)$. Note that $0 < r < 1$.

We calculate $H'(r) = \sqrt{k} - \dfrac{r}{\sqrt{1-r^2}}$ and $H''(r) = \dfrac{-1}{(1-r^2)^{3/2}}$.

$$H'(r) = 0 \quad\Leftrightarrow\quad r^2 = k(1-r^2) \quad\Leftrightarrow\quad (k+1)r^2 = k \quad\Leftrightarrow\quad r = \sqrt{\frac{k}{k+1}}.$$

This is the only critical number in $(0,1)$ and it represents a local maximum (hence an absolute maximum) since $H''(r) < 0$ on $(0,1)$. When $r = \sqrt{\dfrac{k}{k+1}}$,

$H(r) = \sqrt{k}\dfrac{\sqrt{k}}{\sqrt{k+1}} + \sqrt{1 - \dfrac{k}{k+1}} = \dfrac{k}{\sqrt{k+1}} + \dfrac{1}{\sqrt{k+1}} = \sqrt{k+1}$. Thus, the assertion is true for $n = k+1$ when it is true for $n = k$. By induction, it is true for all positive integers n.

Note: In general, a maximally tall stack of n hemispherical bubbles consists of bubbles with radii

$$1, \sqrt{\frac{n-1}{n}}, \sqrt{\frac{n-2}{n}}, \ldots, \sqrt{\frac{2}{n}}, \sqrt{\frac{1}{n}}.$$