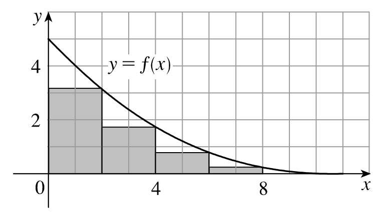
4 □ INTEGRALS

4.1 The Area and Distance Problems

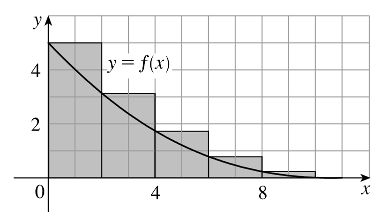
1. (a) Since f is *decreasing*, we can obtain a *lower* estimate by using *right* endpoints. We are instructed to use five rectangles, so $n = 5$.

$$R_5 = \sum_{i=1}^{5} f(x_i)\,\Delta x \qquad \left[\Delta x = \frac{b-a}{n} = \frac{10-0}{5} = 2\right]$$
$$= f(x_1)\cdot 2 + f(x_2)\cdot 2 + f(x_3)\cdot 2 + f(x_4)\cdot 2 + f(x_5)\cdot 2$$
$$= 2[f(2) + f(4) + f(6) + f(8) + f(10)]$$
$$\approx 2(3.2 + 1.8 + 0.8 + 0.2 + 0)$$
$$= 2(6) = 12$$

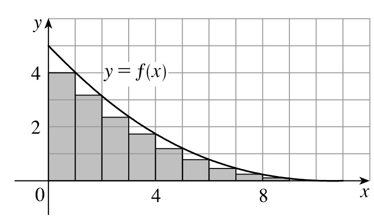


Since f is *decreasing*, we can obtain an *upper* estimate by using *left* endpoints.

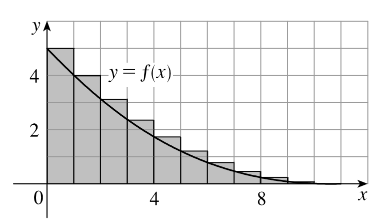
$$L_5 = \sum_{i=1}^{5} f(x_{i-1})\,\Delta x$$
$$= f(x_0)\cdot 2 + f(x_1)\cdot 2 + f(x_2)\cdot 2 + f(x_3)\cdot 2 + f(x_4)\cdot 2$$
$$= 2[f(0) + f(2) + f(4) + f(6) + f(8)]$$
$$\approx 2(5 + 3.2 + 1.8 + 0.8 + 0.2)$$
$$= 2(11) = 22$$



(b) $$R_{10} = \sum_{i=1}^{10} f(x_i)\,\Delta x \qquad \left[\Delta x = \tfrac{10-0}{10} = 1\right]$$
$$= 1[f(x_1) + f(x_2) + \cdots + f(x_{10})]$$
$$= f(1) + f(2) + \cdots + f(10)$$
$$\approx 4 + 3.2 + 2.5 + 1.8 + 1.3 + 0.8 + 0.5 + 0.2 + 0.1 + 0$$
$$= 14.4$$



$$L_{10} = \sum_{i=1}^{10} f(x_{i-1})\,\Delta x$$
$$= f(0) + f(1) + \cdots + f(9)$$
$$= R_{10} + 1\cdot f(0) - 1\cdot f(10) \qquad \left[\begin{array}{c}\text{add leftmost upper rectangle,}\\ \text{subtract rightmost lower rectangle}\end{array}\right]$$
$$= 14.4 + 5 - 0$$
$$= 19.4$$



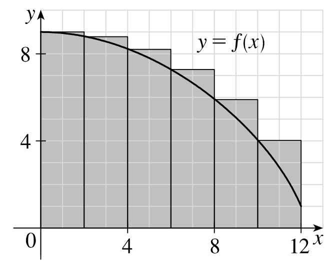
2. (a) (i) $L_6 = \sum\limits_{i=1}^{6} f(x_{i-1})\,\Delta x \qquad [\Delta x = \frac{12-0}{6} = 2]$

$= 2[f(x_0) + f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)]$

$= 2[f(0) + f(2) + f(4) + f(6) + f(8) + f(10)]$

$\approx 2(9 + 8.8 + 8.2 + 7.3 + 5.9 + 4.1)$

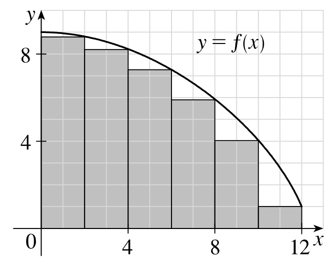
$= 2(43.3) = 86.6$



(ii) $R_6 = L_6 + 2 \cdot f(12) - 2 \cdot f(0)$

$\approx 86.6 + 2(1) - 2(9) = 70.6$

[Add area of rightmost lower rectangle and subtract area of leftmost upper rectangle.]

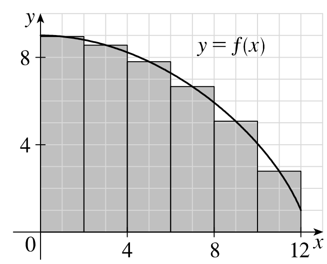


(iii) $M_6 = \sum\limits_{i=1}^{6} f(\overline{x}_i)\,\Delta x$

$= 2[f(1) + f(3) + f(5) + f(7) + f(9) + f(11)]$

$\approx 2(8.9 + 8.5 + 7.8 + 6.6 + 5.1 + 2.8)$

$= 2(39.7) = 79.4$



(b) Since f is *decreasing*, we obtain an *overestimate* by using *left* endpoints; that is, L_6.

(c) Since f is *decreasing*, we obtain an *underestimate* by using *right* endpoints; that is, R_6.

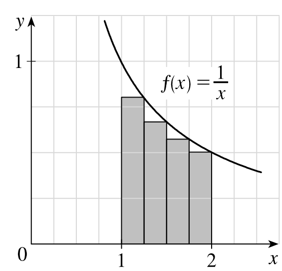
(d) M_6 gives the best estimate, since the area of each rectangle appears to be closer to the true area than the overestimates and underestimates in L_6 and R_6.

3. (a) $R_4 = \sum\limits_{i=1}^{4} f(x_i)\,\Delta x \qquad \left[\Delta x = \dfrac{2-1}{4} = \dfrac{1}{4}\right] = \left[\sum\limits_{i=1}^{4} f(x_i)\right] \Delta x$

$= [f(x_1) + f(x_2) + f(x_3) + f(x_4)]\,\Delta x$

$= \left[\dfrac{1}{5/4} + \dfrac{1}{6/4} + \dfrac{1}{7/4} + \dfrac{1}{8/4}\right] \dfrac{1}{4} = \left[\frac{4}{5} + \frac{2}{3} + \frac{4}{7} + \frac{1}{2}\right] \frac{1}{4} \approx 0.6345$

Since f is *decreasing* on $[1, 2]$, an *underestimate* is obtained by using the *right* endpoint approximation, R_4.

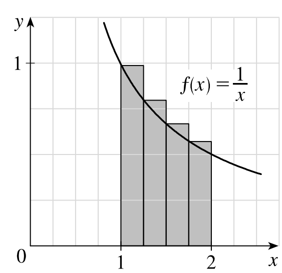


(b) $L_4 = \sum\limits_{i=1}^{4} f(x_{i-1})\,\Delta x = \left[\sum\limits_{i=1}^{4} f(x_{i-1})\right] \Delta x$

$= [f(x_0) + f(x_1) + f(x_2) + f(x_3)]\,\Delta x$

$= \left[\dfrac{1}{1} + \dfrac{1}{5/4} + \dfrac{1}{6/4} + \dfrac{1}{7/4}\right] \dfrac{1}{4} = \left[1 + \frac{4}{5} + \frac{2}{3} + \frac{4}{7}\right] \frac{1}{4} \approx 0.7595$

L_4 is an overestimate. Alternatively, we could just add the area of the leftmost upper rectangle and subtract the area of the rightmost lower rectangle; that is, $L_4 = R_4 + f(1) \cdot \frac{1}{4} - f(2) \cdot \frac{1}{4}$.



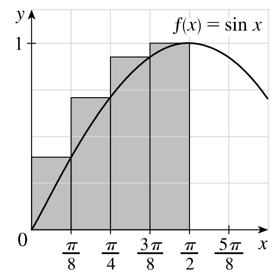
4. (a) $R_4 = \sum_{i=1}^{4} f(x_i)\,\Delta x \qquad \left[\Delta x = \dfrac{\pi/2 - 0}{4} = \dfrac{\pi}{8}\right] = \left[\sum_{i=1}^{4} f(x_i)\right]\Delta x$

$= [f(x_1) + f(x_2) + f(x_3) + f(x_4)]\,\Delta x$

$= \left[\sin\frac{\pi}{8} + \sin\frac{2\pi}{8} + \sin\frac{3\pi}{8} + \sin\frac{4\pi}{8}\right]\frac{\pi}{8}$

≈ 1.1835

Since f is *increasing* on $\left[0, \frac{\pi}{2}\right]$, R_4 is an *overestimate*.



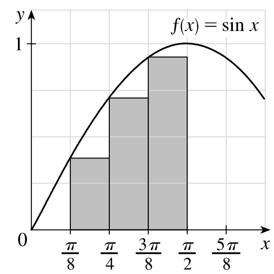
(b) $L_4 = \sum_{i=1}^{4} f(x_{i-1})\,\Delta x = \left[\sum_{i=1}^{4} f(x_{i-1})\right]\Delta x$

$= [f(x_0) + f(x_1) + f(x_2) + f(x_3)]\,\Delta x$

$= \left[\sin 0 + \sin\frac{\pi}{8} + \sin\frac{2\pi}{8} + \sin\frac{3\pi}{8}\right]\frac{\pi}{8}$

≈ 0.7908

Since f is *increasing* on $\left[0, \frac{\pi}{2}\right]$, L_4 is an *underestimate*.



5. (a) $f(x) = 1 + x^2$ and $\Delta x = \dfrac{2-(-1)}{3} = 1 \quad\Rightarrow$

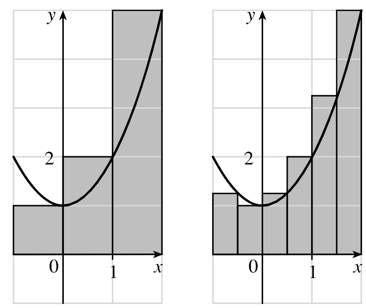
$R_3 = 1\cdot f(0) + 1\cdot f(1) + 1\cdot f(2) = 1\cdot 1 + 1\cdot 2 + 1\cdot 5 = 8.$

$\Delta x = \dfrac{2-(-1)}{6} = 0.5 \quad\Rightarrow$

$R_6 = 0.5[f(-0.5) + f(0) + f(0.5) + f(1) + f(1.5) + f(2)]$

$= 0.5(1.25 + 1 + 1.25 + 2 + 3.25 + 5)$

$= 0.5(13.75) = 6.875$

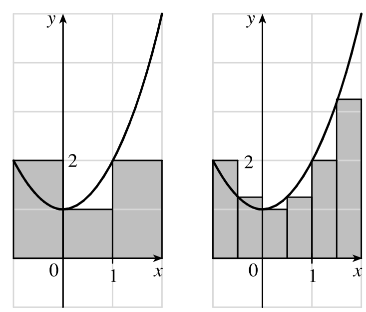


(b) $L_3 = 1\cdot f(-1) + 1\cdot f(0) + 1\cdot f(1) = 1\cdot 2 + 1\cdot 1 + 1\cdot 2 = 5$

$L_6 = 0.5[f(-1) + f(-0.5) + f(0) + f(0.5) + f(1) + f(1.5)]$

$= 0.5(2 + 1.25 + 1 + 1.25 + 2 + 3.25)$

$= 0.5(10.75) = 5.375$



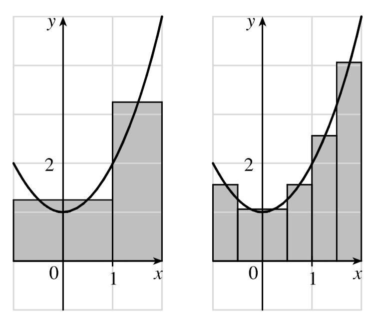
(c) $M_3 = 1\cdot f(-0.5) + 1\cdot f(0.5) + 1\cdot f(1.5)$

$= 1\cdot 1.25 + 1\cdot 1.25 + 1\cdot 3.25 = 5.75$

$M_6 = 0.5[f(-0.75) + f(-0.25) + f(0.25)$
$\quad + f(0.75) + f(1.25) + f(1.75)]$

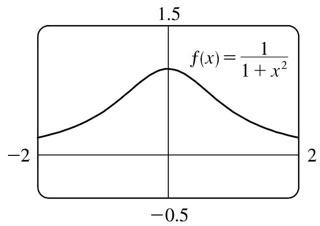
$= 0.5(1.5625 + 1.0625 + 1.0625 + 1.5625 + 2.5625 + 4.0625)$

$= 0.5(11.875) = 5.9375$

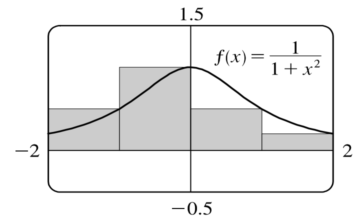


(d) M_6 appears to be the best estimate.

6. (a)



(b) $f(x) = 1/(1+x^2)$ and $\Delta x = \frac{2-(-2)}{4} = 1 \Rightarrow$



(i) $R_4 = \sum_{i=1}^{4} f(x_i)\,\Delta x$

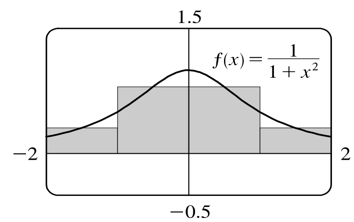
$= f(-1)\cdot 1 + f(0)\cdot 1 + f(1)\cdot 1 + f(2)\cdot 1$

$= \frac{1}{2} + 1 + \frac{1}{2} + \frac{1}{5} = \frac{11}{5} = 2.2$

(ii) $M_4 = \sum_{i=1}^{4} f(\overline{x}_i)\,\Delta x \qquad \left[\overline{x}_i = \frac{1}{2}(x_{i-1}+x_i)\right]$

$= f(-1.5)\cdot 1 + f(-0.5)\cdot 1 + f(0.5)\cdot 1 + f(1.5)\cdot 1$

$= \frac{4}{13} + \frac{4}{5} + \frac{4}{5} + \frac{4}{13} = \frac{144}{65} \approx 2.2154$



(c) $n = 8$, so $\Delta x = \frac{2-(-2)}{8} = \frac{1}{2}$.

$$R_8 = \tfrac{1}{2}[f(-1.5) + f(-1) + f(-0.5) + f(0) + f(0.5) + f(1) + f(1.5) + f(2)]$$

$$= \tfrac{1}{2}\left[\tfrac{4}{13} + \tfrac{1}{2} + \tfrac{4}{5} + 1 + \tfrac{4}{5} + \tfrac{1}{2} + \tfrac{4}{13} + \tfrac{1}{5}\right] = \tfrac{287}{130} \approx 2.2077$$

$$M_8 = \tfrac{1}{2}[f(-1.75) + f(-1.25) + f(-0.75) + f(-0.25) + f(0.25) + f(0.75) + f(1.25) + f(1.75)]$$

$$= \tfrac{1}{2}\left[2\left(\tfrac{16}{65} + \tfrac{16}{41} + \tfrac{16}{25} + \tfrac{16}{17}\right)\right] \approx 2.2176$$

7. $f(x) = 6 - x^2,\ -2 \le x \le 2,\ \Delta x = \dfrac{2-(-2)}{n} = \dfrac{4}{n}.$

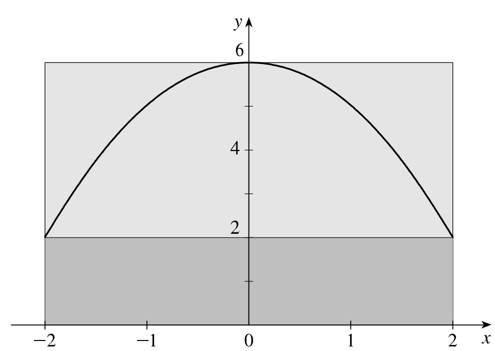
$\boldsymbol{n = 2}$**:**

The maximum values of f on both subintervals occur at $x = 0$, so

upper sum $= f(0)\cdot 2 + f(0)\cdot 2 = 6\cdot 2 + 6\cdot 2 = 24.$

The minimum values of f on the subintervals occur at $x = -2$ and $x = 2$, so

lower sum $= f(-2)\cdot 2 + f(2)\cdot 2 = 2\cdot 2 + 2\cdot 2 = 8.$

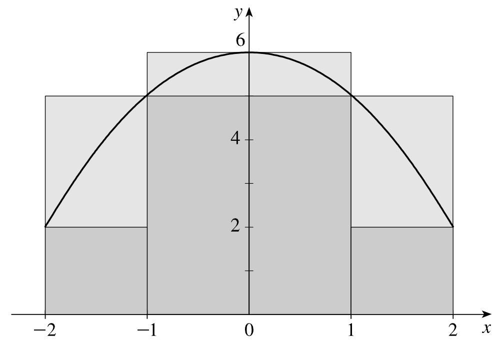


[continued]

$\boldsymbol{n = 4}$:

$$\text{upper sum} = [f(-1) + f(0) + f(0) + f(1)](1) = [5 + 6 + 6 + 5](1) = 22$$

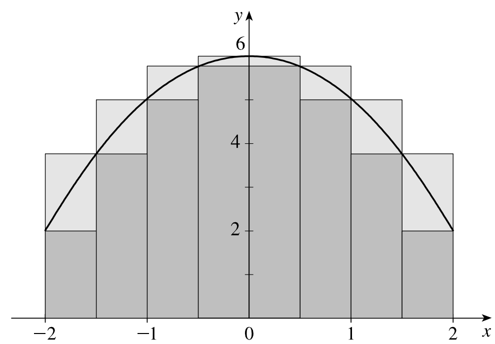
$$\text{lower sum} = [f(-2) + f(-1) + f(1) + f(2)](1) = [2 + 5 + 5 + 2](1) = 14$$



$\boldsymbol{n = 8}$:

$$\text{upper sum} = [f(-1.5) + f(-1) + f(-0.5) + f(0) + f(0) + f(0.5) + f(1) + f(1.5)](0.5) = 20.5$$

$$\text{lower sum} = [f(-2) + f(-1.5) + f(-1) + f(-0.5) + f(0.5) + f(1) + f(1.5) + f(2)](0.5) = 16.5$$

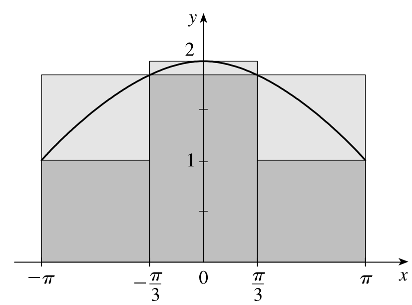


8. $f(x) = 1 + \cos(x/2),\ -\pi \le x \le \pi,\ \Delta x = \dfrac{2\pi}{n}$.

$\boldsymbol{n = 3}$:

$$\text{upper sum} = \left[f\left(-\tfrac{\pi}{3}\right) + f(0) + f\left(\tfrac{\pi}{3}\right)\right]\left(\tfrac{2\pi}{3}\right) \approx 12.005$$

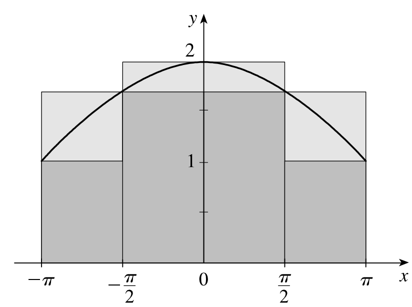
$$\text{lower sum} = \left[f(-\pi) + f\left(\pm\tfrac{\pi}{3}\right) + f(\pi)\right]\left(\tfrac{2\pi}{3}\right) \approx 8.097$$



$\boldsymbol{n = 4}$:

$$\text{upper sum} = \left[f\left(-\tfrac{\pi}{2}\right) + f(0) + f(0) + f\left(\tfrac{\pi}{2}\right)\right]\left(\tfrac{\pi}{2}\right) \approx 11.646$$

$$\text{lower sum} = \left[f(-\pi) + f\left(-\tfrac{\pi}{2}\right) + f\left(\tfrac{\pi}{2}\right) + f(\pi)\right]\left(\tfrac{\pi}{2}\right) \approx 8.505$$



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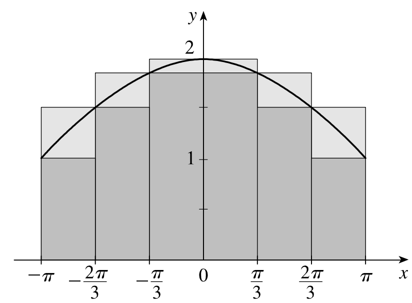
$\boldsymbol{n = 6}$**:**

$$\begin{aligned}\text{upper sum} &= \left[f\left(-\tfrac{2\pi}{3}\right) + f\left(-\tfrac{\pi}{3}\right) + f(0) \right.\\ &\quad \left. + f(0) + f\left(\tfrac{\pi}{3}\right) + f\left(\tfrac{2\pi}{3}\right)\right]\left(\tfrac{\pi}{3}\right)\\ &\approx 11.239\end{aligned}$$

$$\begin{aligned}\text{lower sum} &= \left[f(-\pi) + f\left(-\tfrac{2\pi}{3}\right) + f\left(-\tfrac{\pi}{3}\right) \right.\\ &\quad \left. + f\left(\tfrac{\pi}{3}\right) + f\left(\tfrac{2\pi}{3}\right) + f(\pi)\right]\left(\tfrac{\pi}{3}\right)\\ &\approx 9.144\end{aligned}$$



9. Since v is an increasing function, L_6 will give us a lower estimate and R_6 will give us an upper estimate.

$$L_6 = 0 \cdot 0.5 + 1.9 \cdot 0.5 + 3.3 \cdot 0.5 + 4.5 \cdot 0.5 + 5.5 \cdot 0.5 + 5.9 \cdot 0.5 = 21.1 \cdot 0.5 = 10.55\,\text{m}$$

$$R_6 = 1.9 \cdot 0.5 + 3.3 \cdot 0.5 + 4.5 \cdot 0.5 + 5.5 \cdot 0.5 + 5.9 \cdot 0.5 + 6.2 \cdot 0.5 = 27.3 \cdot 0.5 = 13.65\,\text{m}$$

10. (a) The velocities are given with units mi/h, so we must convert the 10-second intervals to hours:

$$10 \text{ seconds} = \frac{10 \text{ seconds}}{3600 \text{ seconds/h}} = \frac{1}{360}\text{ h}$$

$$\begin{aligned}\text{distance} \approx L_6 &= (294.4 \text{ km/h})\left(\frac{1}{360}\text{ h}\right) + (270.4 \text{ km/h})\left(\frac{1}{360}\text{ h}\right) + (171.6 \text{ km/h})\left(\frac{1}{360}\text{ h}\right) + (160.6 \text{ km/h})\left(\frac{1}{360}\text{ h}\right)\\ &\quad + (200.4 \text{ km/h})\left(\frac{1}{360}\text{ h}\right) + (283.4 \text{ km/h})\left(\frac{1}{360}\text{ h}\right)\\ &= \frac{1380.8}{360} \approx 3.836 \text{ km}\end{aligned}$$

(b) Distance $\approx R_6 = \left(\frac{1}{360}\right)(270.4 + 171.6 + 160.6 + 200.4 + 283.4 + 282.6) = \dfrac{1369}{360} \approx 3.803$ km

(c) The velocity is neither increasing nor decreasing on the given interval, so the estimates in parts (a) and (b) are neither upper nor lower estimates.

11. Lower estimate for oil leakage: $R_5 = (7.6 + 6.8 + 6.2 + 5.7 + 5.3)(2) = (31.6)(2) = 63.2$ L.

Upper estimate for oil leakage: $L_5 = (8.7 + 7.6 + 6.8 + 6.2 + 5.7)(2) = (35)(2) = 70$ L.

12. We can find an upper estimate by using the final velocity for each time interval. Thus, the distance d traveled after 62 seconds can be approximated by

$$d = \sum_{i=1}^{6} v(t_i)\,\Delta t_i = (56 \text{ m/s})(10 \text{ s}) + 97(5) + 136(5) + 226(12) + 404(27) + 440(3) = 16{,}665 \text{ m}$$

13. For a decreasing function, using left endpoints gives us an overestimate and using right endpoints results in an underestimate. We will use M_6 to get an estimate. $\Delta t = 1$, so

$$M_6 = 1[v(0.5) + v(1.5) + v(2.5) + v(3.5) + v(4.5) + v(5.5)] \approx 12.5 + 10 + 7.5 + 5 + 2.5 + 1.25 = 38.75\text{ m}$$

For a very rough check on the above calculation, we can draw a line from $(0, 17.5)$ to $(6, 0)$ and calculate the area of the triangle: $\frac{1}{2} \times 17.5 \times 6 = 52.5$. This is clearly an overestimate, so our midpoint estimate of 38.75 is reasonable.

14. For an increasing function, using left endpoints gives us an underestimate and using right endpoints results in an overestimate. We will use M_6 to get an estimate. $\Delta t = \frac{30-0}{6} = 5\text{ s} = \frac{5}{3600}\text{ h} = \frac{1}{720}\text{ h}$.

$$\begin{aligned} M_6 &= \tfrac{1}{720}[v(2.5) + v(7.5) + v(12.5) + v(17.5) + v(22.5) + v(27.5)] \\ &= \tfrac{1}{720}(31.25 + 66 + 88 + 103.5 + 113.75 + 119.25) = \tfrac{1}{720}(521.75) \approx 0.725\text{ km} \end{aligned}$$

For a very rough check on the above calculation, we can draw a line from $(0, 0)$ to $(30, 120)$ and calculate the area of the triangle: $\frac{1}{2}(30)(120) = 1800$. Divide by 3600 to get 0.5, which is clearly an underestimate, making our midpoint estimate of 0.725 seem reasonable. Of course, answers will vary due to different readings of the graph.

15. $f(t) = -t(t-21)(t+1)$ and $\Delta t = \frac{12-0}{6} = 2$

$$\begin{aligned} M_6 &= 2 \cdot f(1) + 2 \cdot f(3) + 2 \cdot f(5) + 2 \cdot f(7) + 2 \cdot f(9) + 2 \cdot f(11) \\ &= 2 \cdot 40 + 2 \cdot 216 + 2 \cdot 480 + 2 \cdot 784 + 2 \cdot 1080 + 2 \cdot 1320 \\ &= 7840\text{ (infected cells/mL)} \cdot \text{days} \end{aligned}$$

Thus, the total amount of infection needed to develop symptoms of measles is about 7840 infected cells per mL of blood plasma.

16. $f(x) = \dfrac{2x}{x^2+1}$, $1 \le x \le 3$. $\Delta x = (3-1)/n = 2/n$ and $x_i = 1 + i\Delta x = 1 + 2i/n$.

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2(1+2i/n)}{(1+2i/n)^2+1} \cdot \frac{2}{n}.$$

17. $f(x) = 2 + \sin^2 x$, $0 \le x \le \pi$. $\Delta x = (\pi - 0)/n = \pi/n$ and $x_i = 0 + i\,\Delta x = \pi i/n$.

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \left[2 + \sin^2(\pi i/n)\right] \cdot \frac{\pi}{n}.$$

18. $f(x) = x^2 + \sqrt{1+2x}$, $4 \le x \le 7$. $\Delta x = (7-4)/n = 3/n$ and $x_i = 4 + i\,\Delta x = 4 + 3i/n$.

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \left[(4+3i/n)^2 + \sqrt{1+2(4+3i/n)}\right] \cdot \frac{3}{n}.$$

19. $f(x) = x\sqrt{x^3+8}$, $1 \le x \le 5$. $\Delta x = (5-1)/n = 4/n$ and $x_i = 1 + i\,\Delta x = 1 + 4i/n$.

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \left[(1+4i/n)\sqrt{(1+4i/n)^3+8}\right] \cdot \frac{4}{n}.$$

20. $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left(\frac{i}{n}\right)^5$ can be interpreted as the area of the region lying under the graph of $y = x^5$ on the interval $[0, 1]$, since for $y = x^5$ on $[0, 1]$ with $\Delta x = \dfrac{1-0}{n} = \dfrac{1}{n}$, $x_i = 0 + i\,\Delta x = \dfrac{i}{n}$, and $x_i^* = x_i$, the expression for area is $A = \displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^5 \cdot \frac{1}{n}$. Note that this answer is not unique. For example, we could also use $y = (x-1)^5$ on $[1, 2]$.

21. $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\frac{1}{1+(2i/n)}$ can be interpreted as the area of the region lying under the graph of $y=\frac{1}{1+x}$ on the interval $[0,2]$, since for $y=\frac{1}{1+x}$ on $[0,2]$ with $\Delta x=\frac{2-0}{n}=\frac{2}{n}$, $x_i=0+i\,\Delta x=\frac{2i}{n}$, and $x_i^*=x_i$, the expression for area is $A=\lim_{n\to\infty}\sum_{i=1}^{n}f(x_i^*)\,\Delta x=\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{1}{1+(2i/n)}\right)\cdot\frac{2}{n}$. Note that this answer is not unique. We could use $y=\frac{1}{x}$ on $[1,3]$ or, in general, $y=\frac{1}{x-n}$ on $[n+1,n+3]$, where n is any real number. The given answer results from the general case with $n=-1$.

22. $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}\sqrt{1+\frac{3i}{n}}$ can be interpreted as the area of the region lying under the graph of $y=\sqrt{1+x}$ on the interval $[0,3]$, since for $y=\sqrt{1+x}$ on $[0,3]$ with $\Delta x=\frac{3-0}{n}=\frac{3}{n}$, $x_i=0+i\,\Delta x=\frac{3i}{n}$, and $x_i^*=x_i$, the expression for the area is $A=\lim_{n\to\infty}\sum_{i=1}^{n}f(x_i^*)\,\Delta x=\lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{1+\frac{3i}{n}}\,\frac{3}{n}$. Note that this answer is not unique. We could use $y=\sqrt{x}$ on $[1,4]$ or, in general, $y=\sqrt{x-n}$ on $[n+1,n+4]$, where n is any real number.

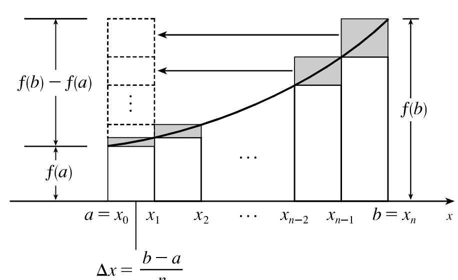
23. $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\pi}{4n}\tan\frac{i\pi}{4n}$ can be interpreted as the area of the region lying under the graph of $y=\tan x$ on the interval $\left[0,\frac{\pi}{4}\right]$, since for $y=\tan x$ on $\left[0,\frac{\pi}{4}\right]$ with $\Delta x=\frac{\pi/4-0}{n}=\frac{\pi}{4n}$, $x_i=0+i\,\Delta x=\frac{i\pi}{4n}$, and $x_i^*=x_i$, the expression for the area is $A=\lim_{n\to\infty}\sum_{i=1}^{n}f(x_i^*)\,\Delta x=\lim_{n\to\infty}\sum_{i=1}^{n}\tan\left(\frac{i\pi}{4n}\right)\frac{\pi}{4n}$. Note that this answer is not unique, since the expression for the area is the same for the function $y=\tan(x-k\pi)$ on the interval $\left[k\pi,k\pi+\frac{\pi}{4}\right]$, where k is any integer.

24. (a) $\Delta x=\frac{1-0}{n}=\frac{1}{n}$ and $x_i=0+i\,\Delta x=\frac{i}{n}$. $\quad A=\lim_{n\to\infty}R_n=\lim_{n\to\infty}\sum_{i=1}^{n}f(x_i)\,\Delta x=\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^3\cdot\frac{1}{n}$.

(b) $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^3}{n^3}\cdot\frac{1}{n}=\lim_{n\to\infty}\frac{1}{n^4}\sum_{i=1}^{n}i^3=\lim_{n\to\infty}\frac{1}{n^4}\left[\frac{n(n+1)}{2}\right]^2=\lim_{n\to\infty}\frac{(n+1)^2}{4n^2}=\frac{1}{4}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2=\frac{1}{4}$

25. (a) Since f is an increasing function, L_n is an underestimate of A [lower sum] and R_n is an overestimate of A [upper sum]. Thus, A, L_n, and R_n are related by the inequality $L_n<A<R_n$.

(b)
$$\begin{aligned}R_n&=f(x_1)\Delta x+f(x_2)\Delta x+\cdots+f(x_n)\Delta x\\L_n&=f(x_0)\Delta x+f(x_1)\Delta x+\cdots+f(x_{n-1})\Delta x\\R_n-L_n&=f(x_n)\Delta x-f(x_0)\Delta x\\&=\Delta x[f(x_n)-f(x_0)]\\&=\frac{b-a}{n}[f(b)-f(a)]\end{aligned}$$



[continued]

In the diagram, $R_n - L_n$ is the sum of the areas of the shaded rectangles. By sliding the shaded rectangles to the left so that they stack on top of the leftmost shaded rectangle, we form a rectangle of height $f(b) - f(a)$ and width $\dfrac{b-a}{n}$.

(c) $A > L_n$, so $R_n - A < R_n - L_n$; that is, $R_n - A < \dfrac{b-a}{n}[f(b) - f(a)]$.

26. From Exercise 25, we have $R_n - A < \dfrac{b-a}{n}[f(b) - f(a)] = \dfrac{3-1}{n}[f(3) - f(1)] = \dfrac{2}{n}(e^3 - e)$. Solving $\dfrac{2}{n}(e^3 - e) < 0.0001$ for n gives us $2(e^3 - e) < 0.0001n \Rightarrow n > \dfrac{2(e^3 - e)}{0.0001} \quad \Rightarrow \quad n > 347{,}345.1$. Thus, a value of n that assures us that $R_n - A < 0.0001$ is $n = 347{,}346$. [This is not the *least* value of n.]

27. Here is one possible algorithm (ordered sequence of operations) for calculating the sums:

1 Let SUM = 0, X_MIN = 0, X_MAX = 1, N = 10 (depending on which sum we are calculating), DELTA_X = (X_MAX - X_MIN)/N, and RIGHT_ENDPOINT = X_MIN + DELTA_X.

2 Repeat steps 2a, 2b in sequence until RIGHT_ENDPOINT > X_MAX.

2a Add (RIGHT_ENDPOINT)^4 to SUM.

2b Add DELTA_X to RIGHT_ENDPOINT.

At the end of this procedure, (DELTA_X)·(SUM) is equal to the answer we are looking for. We find that

$R_{10} = \dfrac{1}{10}\displaystyle\sum_{i=1}^{10}\left(\frac{i}{10}\right)^4 \approx 0.2533$, $R_{30} = \dfrac{1}{30}\displaystyle\sum_{i=1}^{30}\left(\frac{i}{30}\right)^4 \approx 0.2170$, $R_{50} = \dfrac{1}{50}\displaystyle\sum_{i=1}^{50}\left(\frac{i}{50}\right)^4 \approx 0.2101$, and

$R_{100} = \dfrac{1}{100}\displaystyle\sum_{i=1}^{100}\left(\frac{i}{100}\right)^4 \approx 0.2050$. It appears that the exact area is 0.2. The following display shows the program SUMRIGHT and its output from a TI-83/4 Plus calculator. To generalize the program, we have input (rather than assign) values for Xmin, Xmax, and N. Also, the function, x^4, is assigned to Y_1, enabling us to evaluate any right sum merely by changing Y_1 and running the program.

```
PROGRAM:SUMRIGHT
:0→S
:Prompt Xmin
:Prompt Xmax
:Prompt N
:(Xmax-Xmin)/N→D
:Xmin+D→R
:For(I,1,N)
:S+Y₁(R)→S
:R+D→R
:End
:D*S→Z
:Disp Z
```

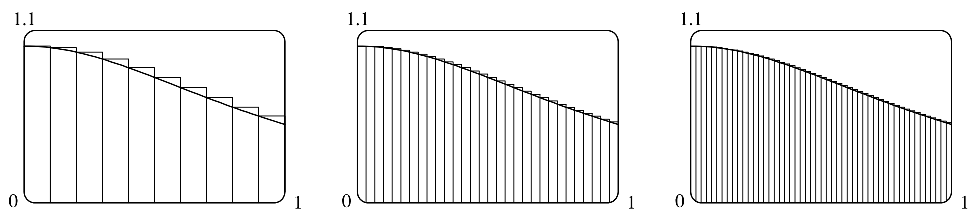
```
prgmSUMRIGHT
Xmin=?0
Xmax=?1
N=?10
              .25333
                Done
```

28. We can use the algorithm from Exercise 27 with X_MIN = 0, X_MAX = $\pi/2$, and cos(RIGHT_ENDPOINT) instead of (RIGHT_ENDPOINT)^4 in step 2a. We find that $R_{10} = \dfrac{\pi/2}{10}\displaystyle\sum_{i=1}^{10}\cos\left(\frac{i\pi}{20}\right) \approx 0.9194$, $R_{30} = \dfrac{\pi/2}{30}\displaystyle\sum_{i=1}^{30}\cos\left(\frac{i\pi}{60}\right) \approx 0.9736$, and $R_{50} = \dfrac{\pi/2}{50}\displaystyle\sum_{i=1}^{50}\cos\left(\frac{i\pi}{100}\right) \approx 0.9842$, and $R_{100} = \dfrac{\pi/2}{100}\displaystyle\sum_{i=1}^{100}\cos\left(\frac{i\pi}{200}\right) \approx 0.9921$. It appears that the exact area is 1.

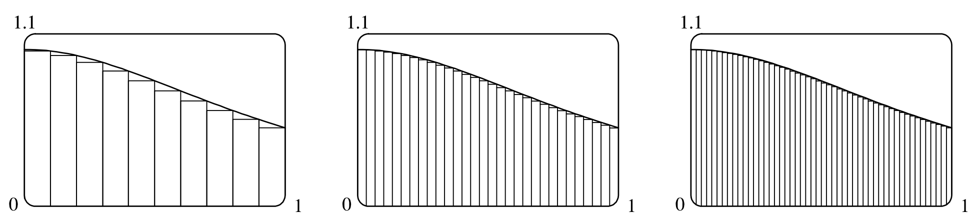
29. In Maple, we have to perform a number of steps before getting a numerical answer. After loading the student package [command: `with(student);`] we use the command `left_sum:=leftsum(1/(x^2+1),x=0..1,10` [or `30`, or `50`]`);` which gives us the expression in summation notation. To get a numerical approximation to the sum, we use `evalf(left_sum);`. Mathematica does not have a special command for these sums, so we must type them in manually. For example, the first left sum is given by `(1/10)*Sum[1/(((i-1)/10)^2+1)],{i,1,10}]`, and we use the `N` command on the resulting output to get a numerical approximation.

(a) With $f(x) = \dfrac{1}{x^2+1}$, $0 \le x \le 1$, the left sums are of the form $L_n = \dfrac{1}{n}\sum\limits_{i=1}^{n} \dfrac{1}{\left(\frac{i-1}{n}\right)^2+1}$. Specifically, $L_{10} \approx 0.8100$, $L_{30} \approx 0.7937$, and $L_{50} \approx 0.7904$. The right sums are of the form $R_n = \dfrac{1}{n}\sum\limits_{i=1}^{n} \dfrac{1}{\left(\frac{i}{n}\right)^2+1}$. Specifically, $R_{10} \approx 0.7600$, $R_{30} \approx 0.7770$, and $R_{50} \approx 0.7804$.

(b) In Maple, we use the `leftbox` (with the same arguments as `left_sum`) and `rightbox` commands to generate the graphs.



left endpoints, $n = 10$ left endpoints, $n = 30$ left endpoints, $n = 50$



right endpoints, $n = 10$ right endpoints, $n = 30$ right endpoints, $n = 50$

(c) We know that since $y = 1/(x^2+1)$ is a decreasing function on $(0, 1)$, all of the left sums are larger than the actual area, and all of the right sums are smaller than the actual area. Since the left sum with $n = 50$ is about $0.7904 < 0.791$ and the right sum with $n = 50$ is about $0.7804 > 0.780$, we conclude that $0.780 < R_{50} <$ exact area $< L_{50} < 0.791$, so the exact area is between 0.780 and 0.791.

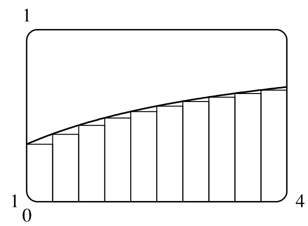
30. (a) With $f(x) = x/(x+2)$, $1 \le x \le 4$, and $x_i = 1 + 3i/n$, the left sums are of the form

$L_n = \sum\limits_{i=1}^{n} f(x_{i-1})\,\Delta x = \dfrac{3}{n}\sum\limits_{i=1}^{n} \dfrac{1+3(i-1)/n}{3+3(i-1)/n}$. In particular, $L_{10} \approx 1.5625$, $L_{30} \approx 1.5969$, and $L_{50} \approx 1.6037$.

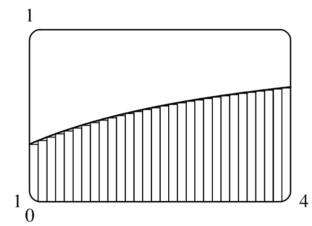
[continued]

The right sums are of the form $R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x = \frac{3}{n} \sum_{i=1}^{n} \frac{1+3i/n}{3+3i/n}$. In particular, $R_{10} \approx 1.6625$, $R_{30} \approx 1.6302$, and $R_{50} \approx 1.6237$.

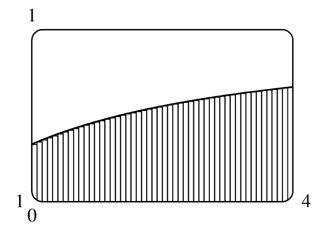
(b) In Maple, we use the `leftbox` and `rightbox` commands (with the same arguments as `leftsum` and `rightsum` above) to generate the graphs.



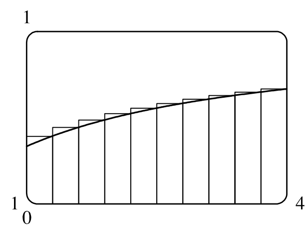
left endpoints, $n = 10$



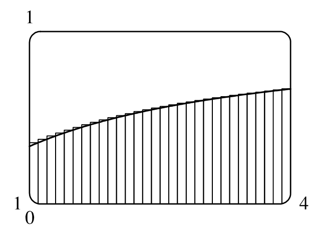
left endpoints, $n = 30$



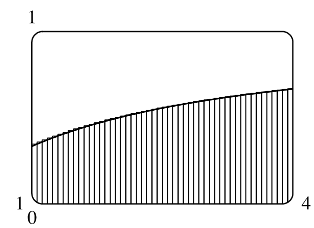
left endpoints, $n = 50$



right endpoints, $n = 10$



right endpoints, $n = 30$



right endpoints, $n = 50$

(c) $f'(x) = \dfrac{(x+2)\cdot 1 - x\cdot 1}{(x+2)^2} = \dfrac{2}{(x+2)^2} > 0$, so f is an increasing function. Thus, the left sums are underestimates of the area A and the right sums are overestimates. The results in part (a) show that $1.603 < L_{50} < A < R_{50} < 1.624$.

31. (a) $y = f(x) = x^5$. $\Delta x = \dfrac{2-0}{n} = \dfrac{2}{n}$ and $x_i = 0 + i\,\Delta x = \dfrac{2i}{n}$.

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2i}{n}\right)^5 \cdot \frac{2}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{32i^5}{n^5}\cdot\frac{2}{n} = \lim_{n\to\infty} \frac{64}{n^6} \sum_{i=1}^{n} i^5.$$

(b) $\displaystyle\sum_{i=1}^{n} i^5 \overset{\text{CAS}}{=} \frac{n^2(n+1)^2\left(2n^2+2n-1\right)}{12}$

(c)
$$\begin{aligned}\lim_{n\to\infty} \frac{64}{n^6}\cdot\frac{n^2(n+1)^2\left(2n^2+2n-1\right)}{12} &= \frac{64}{12}\lim_{n\to\infty} \frac{\left(n^2+2n+1\right)\left(2n^2+2n-1\right)}{n^2\cdot n^2} \\ &= \frac{16}{3}\lim_{n\to\infty}\left(1+\frac{2}{n}+\frac{1}{n^2}\right)\left(2+\frac{2}{n}-\frac{1}{n^2}\right) = \frac{16}{3}\cdot 1\cdot 2 = \frac{32}{3}\end{aligned}$$

32. (a) $y = f(x) = x^4 + 5x^2 + x,\ 2 \le x \le 7 \ \Rightarrow\ \Delta x = \dfrac{7-2}{n} = \dfrac{5}{n}$, $x_i = 2 + i\,\Delta x = 2 + \dfrac{5i}{n} \ \Rightarrow$

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{5}{n}\sum_{i=1}^{n}\left[\left(2+\frac{5i}{n}\right)^4 + 5\left(2+\frac{5i}{n}\right)^2 + \left(2+\frac{5i}{n}\right)\right]$$

(b) $R_n \overset{\text{CAS}}{=} \dfrac{5}{n}\cdot\dfrac{4723n^4 + 7845n^3 + 3475n^2 - 125}{6n^3}$

(c) $A = \displaystyle\lim_{n\to\infty} R_n \overset{\text{CAS}}{=} \frac{23{,}615}{6} = 3935.8\overline{3}$

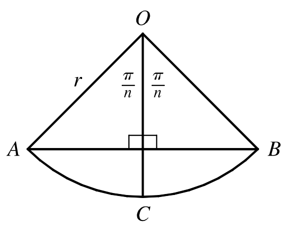
33. $y = f(x) = \cos x$. $\Delta x = \dfrac{b-0}{n} = \dfrac{b}{n}$ and $x_i = 0 + i\,\Delta x = \dfrac{bi}{n}$.

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \cos\left(\frac{bi}{n}\right)\cdot\frac{b}{n}$$

$$\overset{\text{CAS}}{=} \lim_{n\to\infty}\left[\frac{b\sin\left(b\left(\frac{1}{2n}+1\right)\right)}{2n\sin\left(\frac{b}{2n}\right)} - \frac{b}{2n}\right] \overset{\text{CAS}}{=} \sin b$$

If $b = \frac{\pi}{2}$, then $A = \sin\frac{\pi}{2} = 1$.

34. (a)



The diagram shows one of the n congruent triangles, $\triangle AOB$, with central angle $2\pi/n$. O is the center of the circle and AB is one of the sides of the polygon. Radius OC is drawn so as to bisect $\angle AOB$. It follows that OC intersects AB at right angles and bisects AB. Thus, $\triangle AOB$ is divided into two right triangles with legs of length $\frac{1}{2}(AB) = r\sin(\pi/n)$ and $r\cos(\pi/n)$. $\triangle AOB$ has area $2\cdot\frac{1}{2}[r\sin(\pi/n)][r\cos(\pi/n)] = r^2\sin(\pi/n)\,\cos(\pi/n) = \frac{1}{2}r^2\sin(2\pi/n)$, so $A_n = n\cdot\text{area}(\triangle AOB) = \frac{1}{2}nr^2\sin(2\pi/n)$.

(b) To use Equation 2.4.5, $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta} = 1$, we need to have the same expression in the denominator as we have in the argument of the sine function—in this case, $2\pi/n$.

$\lim\limits_{n\to\infty} A_n = \lim\limits_{n\to\infty}\frac{1}{2}nr^2\sin(2\pi/n) = \lim\limits_{n\to\infty}\frac{1}{2}nr^2\dfrac{\sin(2\pi/n)}{2\pi/n}\cdot\dfrac{2\pi}{n} = \lim\limits_{n\to\infty}\dfrac{\sin(2\pi/n)}{2\pi/n}\pi r^2$. Let $\theta = \dfrac{2\pi}{n}$.

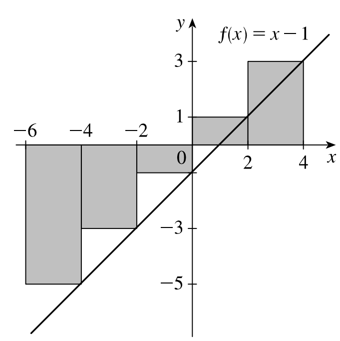
Then as $n\to\infty$, $\theta\to 0$, so $\lim\limits_{n\to\infty} A_n = \lim\limits_{n\to\infty}\dfrac{\sin(2\pi/n)}{2\pi/n}\pi r^2 = \lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}\pi r^2 = (1)\,\pi r^2 = \pi r^2$.

4.2 The Definite Integral

1. $f(x) = x - 1$, $-6 \le x \le 4$. $\Delta x = \dfrac{b-a}{n} = \dfrac{4-(-6)}{5} = 2$.

Since we are using right endpoints, $x_i^* = x_i$.

$$\begin{aligned} R_5 &= \sum_{i=1}^{5} f(x_i)\,\Delta x \\ &= (\Delta x)[f(x_1)+f(x_2)+f(x_3)+f(x_4)+f(x_5)+f(x_6)] \\ &= 2[f(-4)+f(-2)+f(0)+f(2)+f(4)] \\ &= 2[-5+(-3)+(-1)+1+3] \\ &= 2(-5) = -10 \end{aligned}$$

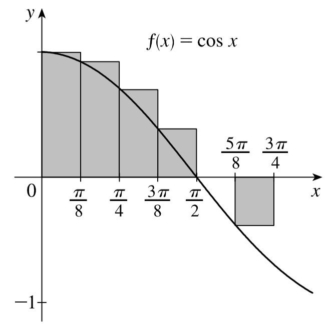


The Riemann sum represents the sum of the areas of the two rectangles above the x-axis minus the sum of the areas of the three rectangles below the x-axis; that is, the *net area* of the rectangles with respect to the x-axis.

2. $f(x) = \cos x$, $0 \le x \le \frac{3\pi}{4}$. $\Delta x = \dfrac{b-a}{n} = \dfrac{3\pi/4 - 0}{6} = \dfrac{\pi}{8}$.

Since we are using left endpoints, $x_i^* = x_{i-1}$.

$$\begin{aligned} L_6 &= \sum_{i=1}^{6} f(x_{i-1})\,\Delta x \\ &= (\Delta x)[f(x_0) + f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)] \\ &= \tfrac{\pi}{8}\left[f(0) + f\left(\tfrac{\pi}{8}\right) + f\left(\tfrac{2\pi}{8}\right) + f\left(\tfrac{3\pi}{8}\right) + f\left(\tfrac{4\pi}{8}\right) + f\left(\tfrac{5\pi}{8}\right)\right] \\ &\approx 1.033186 \end{aligned}$$

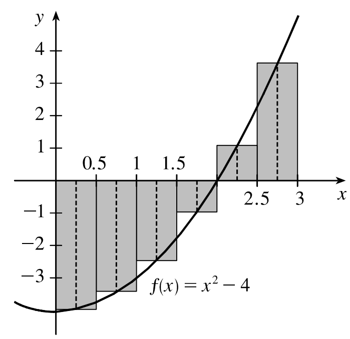


The Riemann sum represents the sum of the areas of the four rectangles above the x-axis minus the area of the rectangle below the x-axis; that is, the *net area* of the rectangles with respect to the x-axis. A sixth rectangle is degenerate, with height 0, and has no area.

3. $f(x) = x^2 - 4$, $0 \le x \le 3$. $\Delta x = \dfrac{b-a}{n} = \dfrac{3-0}{6} = \dfrac{1}{2}$.

Since we are using midpoints, $x_i^* = \overline{x}_i = \frac{1}{2}(x_{i-1} + x_i)$.

$$\begin{aligned} M_6 &= \sum_{i=1}^{6} f(\overline{x}_i)\,\Delta x \\ &= (\Delta x)[f(\overline{x}_1) + f(\overline{x}_2) + f(\overline{x}_3) + f(\overline{x}_4) + f(\overline{x}_5) + f(\overline{x}_6)] \\ &= \tfrac{1}{2}\left[f\left(\tfrac{1}{4}\right) + f\left(\tfrac{3}{4}\right) + f\left(\tfrac{5}{4}\right) + f\left(\tfrac{7}{4}\right) + f\left(\tfrac{9}{4}\right) + f\left(\tfrac{11}{4}\right)\right] \\ &= \tfrac{1}{2}\left(-\tfrac{63}{16} - \tfrac{55}{16} - \tfrac{39}{16} - \tfrac{15}{16} + \tfrac{17}{16} + \tfrac{57}{16}\right) = \tfrac{1}{2}\left(-\tfrac{98}{16}\right) = -\tfrac{49}{16} \end{aligned}$$

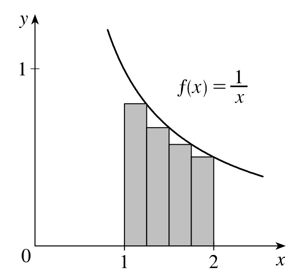


The Riemann sum represents the sum of the areas of the two rectangles above the x-axis minus the sum of the areas of the four rectangles below the x-axis; that is, the *net area* of the rectangles with respect to the x-axis.

4. (a) $f(x) = \dfrac{1}{x}$, $1 \le x \le 2$. $\Delta x = \dfrac{b-a}{n} = \dfrac{2-1}{4} = \dfrac{1}{4}$.

Since we are using right endpoints, $x_i^* = x_i$.

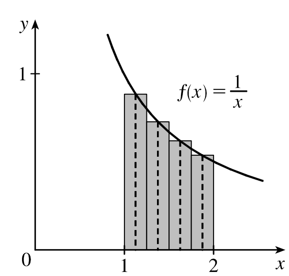
$$\begin{aligned} R_4 &= \sum_{i=1}^{4} f(x_i)\,\Delta x \\ &= (\Delta x)[f(x_1) + f(x_2) + f(x_3) + f(x_4)] \\ &= \tfrac{1}{4}\left[f\left(\tfrac{5}{4}\right) + f\left(\tfrac{6}{4}\right) + f\left(\tfrac{7}{4}\right) + f\left(\tfrac{8}{4}\right)\right] \\ &= \tfrac{1}{4}\left[\tfrac{4}{5} + \tfrac{2}{3} + \tfrac{4}{7} + \tfrac{1}{2}\right] \\ &\approx 0.634524 \end{aligned}$$



The Riemann sum represents the sum of the areas of the four rectangles.

(b) Since we are using midpoints, $x_i^* = \overline{x}_i = \frac{1}{2}(x_{i-1} + x_i)$.

$$\begin{aligned} M_4 &= \sum_{i=1}^{4} f(\overline{x}_i)\,\Delta x \\ &= (\Delta x)[f(\overline{x}_1) + f(\overline{x}_2) + f(\overline{x}_3) + f(\overline{x}_4)] \\ &= \tfrac{1}{4}\left[f\left(\tfrac{9}{8}\right) + f\left(\tfrac{11}{8}\right) + f\left(\tfrac{13}{8}\right) + f\left(\tfrac{15}{8}\right)\right] \\ &= \tfrac{1}{4}\left(\tfrac{8}{9} + \tfrac{8}{11} + \tfrac{8}{13} + \tfrac{8}{15}\right) \approx 0.691220 \end{aligned}$$



The Riemann sum represents the sum of the areas of the four rectangles.

5. (a) $\int_0^{10} f(x)\,dx \approx R_5 = [f(2)+f(4)+f(6)+f(8)+f(10)]\,\Delta x$

$$= [-1+1+3+(-1)+0](2) = 2(2) = 4$$

(b) $\int_0^{10} f(x)\,dx \approx L_5 = [f(0)+f(2)+f(4)+f(6)+f(8)]\,\Delta x$

$$= [-1+(-1)+1+3+(-1)](2) = 1(2) = 2$$

(c) $\int_0^{10} f(x)\,dx \approx M_5 = [f(1)+f(3)+f(5)+f(7)+f(9)]\,\Delta x$

$$= [2+0+2+1+(-2)](2) = 3(2) = 6$$

6. (a) $\int_{-2}^{4} g(x)\,dx \approx R_6 = [g(-1)+g(0)+g(1)+g(2)+g(3)+g(4)]\,\Delta x$

$$= \left[\tfrac{3}{2}+0+\left(-\tfrac{3}{2}\right)+\tfrac{1}{2}+(-1)+\tfrac{1}{2}\right](1) = 0(1) = 0$$

(b) $\int_{-2}^{4} g(x)\,dx \approx L_6 = [g(-2)+g(-1)+g(0)+g(1)+g(2)+g(3)]\,\Delta x$

$$= \left[0+\tfrac{3}{2}+0+\left(-\tfrac{3}{2}\right)+\tfrac{1}{2}+(-1)\right](1) = -\tfrac{1}{2}(1) = -\tfrac{1}{2}$$

(c) $\int_{-2}^{4} g(x)\,dx \approx M_6 = \left[g\left(-\tfrac{3}{2}\right)+g\left(-\tfrac{1}{2}\right)+g\left(\tfrac{1}{2}\right)+g\left(\tfrac{3}{2}\right)+g\left(\tfrac{5}{2}\right)+g\left(\tfrac{7}{2}\right)\right]\Delta x$

$$= \left[1+1+(-1)+\left(-\tfrac{1}{2}\right)+0+\left(-\tfrac{1}{2}\right)\right](1) = 0(1) = 0$$

7. Since f is increasing, $L_5 \le \int_{10}^{30} f(x)\,dx \le R_5$.

Lower estimate $= L_5 = \sum\limits_{i=1}^{5} f(x_{i-1})\Delta x = 4[f(10)+f(14)+f(18)+f(22)+f(26)]$

$$= 4[-12+(-6)+(-2)+1+3] = 4(-16) = -64$$

Upper estimate $= R_5 = \sum\limits_{i=1}^{5} f(x)\Delta x = 4[f(14)+f(18)+f(22)+f(26)+f(30)]$

$$= 4[-6+(-2)+1+3+8] = 4(4) = 16$$

8. (a) Using the right endpoints to approximate $\int_3^9 f(x)\,dx$, we have

$$\sum_{i=1}^{3} f(x_i)\,\Delta x = 2[f(5)+f(7)+f(9)] = 2\,(-0.6+0.9+1.8) = 4.2.$$

Since f is *increasing*, using *right* endpoints gives an *overestimate*.

(b) Using the left endpoints to approximate $\int_3^9 f(x)\,dx$, we have

$$\sum_{i=1}^{3} f(x_{i-1})\,\Delta x = 2[f(3)+f(5)+f(7)] = 2\,(-3.4-0.6+0.9) = -6.2.$$

Since f is *increasing*, using *left* endpoints gives an *underestimate*.

(c) Using the midpoint of each interval to approximate $\int_3^9 f(x)\,dx$, we have

$$\sum_{i=1}^{3} f(\overline{x}_i)\,\Delta x = 2[f(4)+f(6)+f(8)] = 2\,(-2.1+0.3+1.4) = -0.8.$$

We cannot say anything about the midpoint estimate compared to the exact value of the integral.

9. $\Delta x = (8-0)/4 = 2$, so the endpoints are 0, 2, 4, 6, and 8, and the midpoints are 1, 3, 5, and 7. The Midpoint Rule gives $\displaystyle\int_0^8 x^2\,dx \approx \sum_{i=1}^{4} f(\overline{x}_i)\,\Delta x = 2(1^2+3^2+5^2+7^2) = 2(84) = 168.$

10. $\Delta x = (2-0)/4 = \frac{1}{2}$, so the endpoints are 0, $\frac{1}{2}$, 1, $\frac{3}{2}$, and 2, and the midpoints are $\frac{1}{4}$, $\frac{3}{4}$, $\frac{5}{4}$, and $\frac{7}{4}$. The Midpoint Rule gives

$$\int_0^2 (8x+3)\,dx \approx \sum_{i=1}^{4} f(\overline{x}_i)\,\Delta x = \left[\left(8\cdot\tfrac{1}{4}+3\right)+\left(8\cdot\tfrac{3}{4}+3\right)+\left(8\cdot\tfrac{5}{4}+3\right)+\left(8\cdot\tfrac{7}{4}+3\right)\right]\left(\tfrac{1}{2}\right) = (44)\left(\tfrac{1}{2}\right) = 22.$$

11. $\Delta x = (8-0)/4 = 2$, so the endpoints are 0, 2, 4, 6, and 8, and the midpoints are 1, 3, 5, and 7. The Midpoint Rule gives

$$\int_0^8 \sin\sqrt{x}\,dx \approx \sum_{i=1}^{4} f(\bar{x}_i)\,\Delta x = 2\left(\sin\sqrt{1}+\sin\sqrt{3}+\sin\sqrt{5}+\sin\sqrt{7}\right) \approx 2(3.0910) = 6.1820.$$

12. $\Delta x = (1-0)/5 = \frac{1}{5}$, so the endpoints are 0, $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$, and 1, and the midpoints are $\frac{1}{10}$, $\frac{3}{10}$, $\frac{5}{10}$, $\frac{7}{10}$, and $\frac{9}{10}$. The Midpoint Rule gives

$$\int_0^1 \sqrt{x^3+1}\,dx \approx \sum_{i=1}^{5} f(\overline{x}_i)\,\Delta x = \tfrac{1}{5}\left(\sqrt{\left(\tfrac{1}{10}\right)^3+1}+\sqrt{\left(\tfrac{3}{10}\right)^3+1}+\sqrt{\left(\tfrac{5}{10}\right)^3+1}+\sqrt{\left(\tfrac{7}{10}\right)^3+1}+\sqrt{\left(\tfrac{9}{10}\right)^3+1}\right)$$
$$\approx 1.1097$$

13. $\Delta x = (3-1)/5 = \frac{2}{5}$, so the endpoints are 1, $\frac{7}{5}$, $\frac{9}{5}$, $\frac{11}{5}$, $\frac{13}{5}$, and 3, and the midpoints are $\frac{6}{5}$, $\frac{8}{5}$, 2, $\frac{12}{5}$, and $\frac{14}{5}$. The Midpoint Rule gives

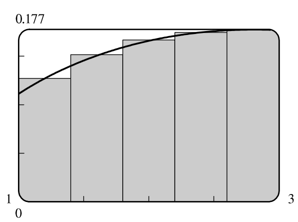
$$\int_1^3 \frac{x}{x^2+8}\,dx \approx \sum_{i=1}^{5} f(\overline{x}_i)\,\Delta x = \left(\frac{6/5}{(6/5)^2+8}+\frac{8/5}{(8/5)^2+8}+\frac{2}{2^2+8}+\frac{12/5}{(12/5)^2+8}+\frac{14/5}{(14/5)^2+8}\right)\left(\frac{2}{5}\right)$$
$$\approx 0.3186$$

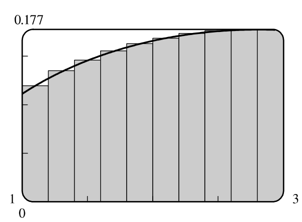
14. $\Delta x = (\pi-0)/4 = \frac{\pi}{4}$, so the endpoints are $\frac{\pi}{4}$, $\frac{2\pi}{4}$, $\frac{3\pi}{4}$, and $\frac{4\pi}{4}$, and the midpoints are $\frac{\pi}{8}$, $\frac{3\pi}{8}$, $\frac{5\pi}{8}$, and $\frac{7\pi}{8}$. The Midpoint Rule gives

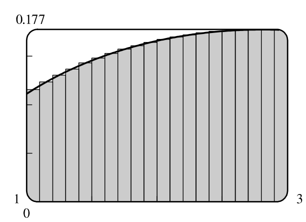
$$\int_0^{\pi} x\sin^2 x\,dx \approx \sum_{i=1}^{5} f(\overline{x}_i)\,\Delta x = \frac{\pi}{4}\left(\frac{\pi}{8}\sin^2\frac{\pi}{8}+\frac{3\pi}{8}\sin^2\frac{3\pi}{8}+\frac{5\pi}{8}\sin^2\frac{5\pi}{8}+\frac{7\pi}{8}\sin^2\frac{7\pi}{8}\right) \approx 2.4674$$

15. Using Mathematica and the Riemann Sum notebook from MathWorld, we obtain the following for $f(x) = x/(x^2+8)$:

$M_5 \approx 0.318595$ $\qquad$ $M_{10} \approx 0.318144$ $\qquad$ $M_{20} \approx 0.318032$







16. For $f(x) = x/(x+1)$ on $[0,2]$, we calculate $L_{100} \approx 0.89469$ and $R_{100} \approx 0.90802$. Since f is increasing on $[0,2]$, L_{100} is an underestimate of $\displaystyle\int_0^2 \frac{x}{x+1}\,dx$ and R_{100} is an overestimate. Thus, $0.8946 < \displaystyle\int_0^2 \frac{x}{x+1}\,dx < 0.9081$.

17. We'll create the table of values to approximate $\int_0^{\pi} \sin x\,dx$ by using the program in the solution to Exercise 4.1.27 with $Y_1 = \sin x$, Xmin $= 0$, Xmax $= \pi$, and $n = 5$, 10, 50, and 100.

The values of R_n appear to be approaching 2.

n	R_n
5	1.933766
10	1.983524
50	1.999342
100	1.999836

18. $\int_0^2 \sqrt{1+x^4}\,dx$ with $n = 5$, 10, 50, and 100.

n	L_n	R_n
5	3.080614	4.329856
10	3.354110	3.978731
50	3.591540	3.716464
100	3.622383	3.684845

The value of the integral lies between 3.622 and 3.685. Note that $f(x) = \sqrt{1+x^4}$ is increasing on $(0, 2)$. We cannot make a similar statement for $\int_{-1}^2 \sqrt{1+x^4}\,dx$ since f is decreasing on $(-1, 0)$.

19. On $[0, \pi]$, $\lim\limits_{n\to\infty} \sum\limits_{i=1}^n \dfrac{\sin x_i}{1+x_i}\,\Delta x = \displaystyle\int_0^\pi \frac{\sin x}{1+x}\,dx.$

20. On $[2, 5]$, $\lim\limits_{n\to\infty} \sum\limits_{i=1}^n x_i\sqrt{1+x_i^3}\,\Delta x = \int_2^5 x\sqrt{1+x^3}\,dx.$

21. On $[2, 7]$, $\lim\limits_{n\to\infty} \sum\limits_{i=1}^n [5(x_i^*)^3 - 4x_i^*]\,\Delta x = \int_2^7 (5x^3 - 4x)\,dx.$

22. On $[1, 3]$, $\lim\limits_{n\to\infty} \sum\limits_{i=1}^n \dfrac{x_i^*}{(x_i^*)^2+4}\,\Delta x = \displaystyle\int_1^3 \frac{x}{x^2+4}\,dx.$

23. For $\displaystyle\int_0^4 (x - x^2)\,dx$, $\Delta x = \dfrac{4-0}{n} = \dfrac{4}{n}$, and $x_i = 0 + i\,\Delta x = \dfrac{4i}{n}$. Then

$$\int_0^4 (x-x^2)\,dx = \lim_{n\to\infty}\sum_{i=1}^n f\left(\frac{4i}{n}\right)\frac{4}{n} = \lim_{n\to\infty}\sum_{i=1}^n \left[\left(\frac{4i}{n}\right) - \left(\frac{4i}{n}\right)^2\right]\frac{4}{n} = \lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^n\left[\frac{4i}{n} - \frac{16i^2}{n^2}\right] = \lim_{n\to\infty} R_n.$$

$$\begin{aligned}
\lim_{n\to\infty}\frac{4}{n}\sum_{i=1}^n\left[\frac{4i}{n} - \frac{16i^2}{n^2}\right] &= \lim_{n\to\infty}\frac{4}{n}\left[\frac{4}{n}\sum_{i=1}^n i - \frac{16}{n^2}\sum_{i=1}^n i^2\right] = \lim_{n\to\infty}\left[\frac{16}{n^2}\frac{n(n+1)}{2} - \frac{64}{n^3}\frac{n(n+1)(2n+1)}{6}\right] \\
&= \lim_{n\to\infty}\left[\frac{8}{n}(n+1) - \frac{32}{3n^2}(n+1)(2n+1)\right] \\
&= \lim_{n\to\infty}\left[8\left(1+\frac{1}{n}\right) - \frac{32}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right] = 8(1) - \frac{32}{3}(1)(2) = -\frac{40}{3}
\end{aligned}$$

24. For $\displaystyle\int_1^3 (x^3 + 5x^2)\,dx$, $\Delta x = \dfrac{3-1}{n} = \dfrac{2}{n}$, and $x_i = 1 + i\,\Delta x = 1 + \dfrac{2i}{n}$. Then

$$\begin{aligned}
\int_1^3 (x^3+5x^2)\,dx &= \lim_{n\to\infty}\sum_{i=1}^n f\left(1+\frac{2i}{n}\right)\frac{2}{n} = \lim_{n\to\infty}\sum_{i=1}^n\left[\left(1+\frac{2i}{n}\right)^3 + 5\left(1+\frac{2i}{n}\right)^2\right]\frac{2}{n} \\
&= \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^n\left[\left(1+\frac{6i}{n}+\frac{12i^2}{n^2}+\frac{8i^3}{n^3}\right) + \left(5+\frac{20i}{n}+\frac{20i^2}{n^2}\right)\right] \\
&= \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^n\left[6+\frac{26i}{n}+\frac{32i^2}{n^2}+\frac{8i^3}{n^3}\right] = \lim_{n\to\infty} R_n
\end{aligned}$$

[continued]

$$\lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}\left[6+\frac{26i}{n}+\frac{32i^2}{n^2}+\frac{8i^3}{n^3}\right]=\lim_{n\to\infty}\frac{2}{n}\left[\sum_{i=1}^{n}6+\frac{26}{n}\sum_{i=1}^{n}i+\frac{32}{n^2}\sum_{i=1}^{n}i^2+\frac{8}{n^3}\sum_{i=1}^{n}i^3\right]$$

$$=\lim_{n\to\infty}\left\{\frac{2}{n}\cdot n(6)+\frac{52}{n^2}\frac{n(n+1)}{2}+\frac{64}{n^3}\frac{n(n+1)(2n+1)}{6}+\frac{16}{n^4}\left[\frac{n(n+1)}{2}\right]^2\right\}$$

$$=\lim_{n\to\infty}\left[2(6)+\frac{26}{n}(n+1)+\frac{32}{3n^2}(n+1)(2n+1)+\frac{4}{n^2}(n+1)^2\right]$$

$$=\lim_{n\to\infty}\left[12+26\left(1+\frac{1}{n}\right)+\frac{32}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+4\left(1+\frac{1}{n}\right)^2\right]$$

$$=12+26(1)+\frac{32}{3}(1)(2)+4(1)^2=\frac{190}{3}$$

25. $f(x)=\sqrt{4+x^2}$, $a=1$, $b=3$, and $\Delta x=\dfrac{3-1}{n}=\dfrac{2}{n}$. Using Theorem 4, we get $x_i^*=x_i=1+i\,\Delta x=1+\dfrac{2i}{n}$, so

$$\int_1^3\sqrt{4+x^2}\,dx=\lim_{n\to\infty}R_n=\lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{4+\left(1+\frac{2i}{n}\right)^2}\cdot\frac{2}{n}.$$

26. $f(x)=x^2+\dfrac{1}{x}$, $a=2$, $b=5$, and $\Delta x=\dfrac{5-2}{n}=\dfrac{3}{n}$. Using Theorem 4, we get $x_i^*=x_i=2+i\,\Delta x=2+\dfrac{3i}{n}$, so

$$\int_2^5\left(x^2+\frac{1}{x}\right)dx=\lim_{n\to\infty}R_n=\lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(2+\frac{3i}{n}\right)^2+\frac{1}{2+\dfrac{3i}{n}}\right]\cdot\frac{3}{n}.$$

27. Note that $\Delta x=\dfrac{2-0}{n}=\dfrac{2}{n}$ and $x_i=0+i\,\Delta x=\dfrac{2i}{n}$.

$$\int_0^2 3x\,dx=\lim_{n\to\infty}\sum_{i=1}^{n}f(x_i)\,\Delta x=\lim_{n\to\infty}\sum_{i=1}^{n}f\left(\frac{2i}{n}\right)\frac{2}{n}=\lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}3\left(\frac{2i}{n}\right)=\lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}\frac{6i}{n}$$

$$=\lim_{n\to\infty}\frac{2}{n}\left(\frac{6}{n}\right)\sum_{i=1}^{n}i=\lim_{n\to\infty}\frac{2}{n}\left(\frac{6}{n}\right)\left[\frac{n(n+1)}{2}\right]=\lim_{n\to\infty}6\left(\frac{n+1}{n}\right)$$

$$=\lim_{n\to\infty}6\left(1+\frac{1}{n}\right)=6(1)=6$$

28. Note that $\Delta x=\dfrac{3-0}{n}=\dfrac{3}{n}$ and $x_i=0+i\,\Delta x=\dfrac{3i}{n}$.

$$\int_0^3 x^2\,dx=\lim_{n\to\infty}\sum_{i=1}^{n}f(x_i)\,\Delta x=\lim_{n\to\infty}\sum_{i=1}^{n}f\left(\frac{3i}{n}\right)\frac{3}{n}=\lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left(\frac{3i}{n}\right)^2=\lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\frac{9i^2}{n^2}$$

$$=\lim_{n\to\infty}\frac{3}{n}\left(\frac{9}{n^2}\right)\sum_{i=1}^{n}i^2=\lim_{n\to\infty}\frac{3}{n}\left(\frac{9}{n^2}\right)\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$=\frac{9}{2}\lim_{n\to\infty}\frac{(n+1)(2n+1)}{n^2}=\frac{9}{2}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

$$=\frac{9}{2}(1)(2)=9$$

29. Note that $\Delta x = \dfrac{3-0}{n} = \dfrac{3}{n}$ and $x_i = 0 + i\,\Delta x = \dfrac{3i}{n}$.

$$\int_0^3 (5x+2)\,dx = \lim_{n\to\infty} \sum_{i=1}^n f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^n \left[5\left(\frac{3i}{n}\right)+2\right]\frac{3}{n} = \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^n \left(\frac{15i}{n}+2\right)$$

$$= \lim_{n\to\infty} \frac{3}{n}\left(\frac{15}{n}\sum_{i=1}^n i + \sum_{i=1}^n 2\right) = \lim_{n\to\infty}\left[\frac{45}{n^2}\frac{n(n+1)}{2} + \frac{3}{n}\cdot n(2)\right]$$

$$= \lim_{n\to\infty}\left[\frac{45}{2n}(n+1)+3(2)\right] = \lim_{n\to\infty}\left[\frac{45}{2}\left(1+\frac{1}{n}\right)+6\right] = \frac{45}{2}(1)+6 = \frac{57}{2}$$

30. Note that $\Delta x = \dfrac{4-0}{n} = \dfrac{4}{n}$ and $x_i = 0 + i\,\Delta x = \dfrac{4i}{n}$.

$$\int_0^4 (6-x^2)\,dx = \lim_{n\to\infty} \sum_{i=1}^n f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^n f\left(\frac{4i}{n}\right)\frac{4}{n} = \lim_{n\to\infty} \frac{4}{n}\sum_{i=1}^n \left[6-\left(\frac{4i}{n}\right)^2\right]$$

$$= \lim_{n\to\infty} \frac{4}{n}\sum_{i=1}^n \left(6-\frac{16i^2}{n^2}\right) = \lim_{n\to\infty}\left(\frac{4}{n}\sum_{i=1}^n 6 - \frac{64}{n^3}\sum_{i=1}^n i^2\right)$$

$$= \lim_{n\to\infty}\left[\frac{4}{n}\cdot n(6) - \frac{64}{n^3}\frac{n(n+1)(2n+1)}{6}\right] = \lim_{n\to\infty}\left[4(6) - \frac{32}{3n^2}(n+1)(2n+1)\right]$$

$$= \lim_{n\to\infty}\left[24 - \frac{32}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right] = 24 - \frac{32}{3}(1)(2) = \frac{8}{3}$$

31. Note that $\Delta x = \dfrac{5-1}{n} = \dfrac{4}{n}$ and $x_i = 1 + i\,\Delta x = 1 + \dfrac{4i}{n}$.

$$\int_1^5 (3x^2+7x)\,dx = \lim_{n\to\infty} \sum_{i=1}^n f(x_i)\,\Delta x = \lim_{n\to\infty} \sum_{i=1}^n f\left(1+\frac{4i}{n}\right)\frac{4}{n}$$

$$= \lim_{n\to\infty} \frac{4}{n}\sum_{i=1}^n \left[3\left(1+\frac{4i}{n}\right)^2 + 7\left(1+\frac{4i}{n}\right)\right]$$

$$= \lim_{n\to\infty} \frac{4}{n}\sum_{i=1}^n \left[3\left(1+\frac{8i}{n}+\frac{16i^2}{n^2}\right) + 7\left(1+\frac{4i}{n}\right)\right]$$

$$= \lim_{n\to\infty} \frac{4}{n}\sum_{i=1}^n \left[3+\frac{24i}{n}+\frac{48i^2}{n^2}+7+\frac{28i}{n}\right] = \lim_{n\to\infty} \frac{4}{n}\sum_{i=1}^n \left[10+\frac{52i}{n}+\frac{48i^2}{n^2}\right]$$

$$= \lim_{n\to\infty} \frac{4}{n}\left[\sum_{i=1}^n 10 + \frac{52}{n}\sum_{i=1}^n i + \frac{48}{n^2}\sum_{i=1}^n i^2\right]$$

$$= \lim_{n\to\infty}\left[\frac{4}{n}\cdot n(10) + \frac{208}{n^2}\frac{n(n+1)}{2} + \frac{192}{n^3}\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty}\left[4\,(10) + \frac{104}{n}(n+1) + \frac{32}{n^2}(n+1)(2n+1)\right]$$

$$= \lim_{n\to\infty}\left[40 + 104\left(1+\frac{1}{n}\right) + 32\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right]$$

$$= 40 + 104(1) + 32(1)(2) = 208$$

32. Note that $\Delta x = \dfrac{2-(-1)}{n} = \dfrac{3}{n}$ and $x_i = -1 + i\,\Delta x = -1 + \dfrac{3i}{n}$.

$$\begin{aligned}
\int_{-1}^{2}(4x^2+x+2)\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(-1+\frac{3i}{n}\right)\frac{3}{n} \\
&= \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left[4\left(-1+\frac{3i}{n}\right)^2 + \left(-1+\frac{3i}{n}\right)+2\right] \\
&= \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left[4\left(1-\frac{6i}{n}+\frac{9i^2}{n^2}\right) + \left(-1+\frac{3i}{n}\right)+2\right] \\
&= \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left[5-\frac{21i}{n}+\frac{36i^2}{n^2}\right] = \lim_{n\to\infty}\frac{3}{n}\left[\sum_{i=1}^{n}5-\frac{21}{n}\sum_{i=1}^{n}i+\frac{36}{n^2}\sum_{i=1}^{n}i^2\right] \\
&= \lim_{n\to\infty}\left[\frac{3}{n}\cdot n(5)-\frac{63}{n^2}\frac{n(n+1)}{2}+\frac{108}{n^3}\frac{n(n+1)(2n+1)}{6}\right] \\
&= \lim_{n\to\infty}\left[3(5)-\frac{63}{2n}(n+1)+\frac{18}{n^2}(n+1)(2n+1)\right] \\
&= \lim_{n\to\infty}\left[15-\frac{63}{2}\left(1+\frac{1}{n}\right)+18\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right] = 15-\frac{63}{2}(1)+18(1)(2) = \frac{39}{2}
\end{aligned}$$

33. Note that $\Delta x = \dfrac{1-0}{n} = \dfrac{1}{n}$ and $x_i = 0 + i\,\Delta x = \dfrac{i}{n}$.

$$\begin{aligned}
\int_{0}^{1}(x^3-3x^2)\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(\frac{i}{n}\right)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n}\left[\left(\frac{i}{n}\right)^3-3\left(\frac{i}{n}\right)^2\right]\frac{1}{n} \\
&= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left[\frac{i^3}{n^3}-\frac{3i^2}{n^2}\right] = \lim_{n\to\infty}\frac{1}{n}\left[\frac{1}{n^3}\sum_{i=1}^{n}i^3-\frac{3}{n^2}\sum_{i=1}^{n}i^2\right] \\
&= \lim_{n\to\infty}\left\{\frac{1}{n^4}\left[\frac{n(n+1)}{2}\right]^2-\frac{3}{n^3}\frac{n(n+1)(2n+1)}{6}\right\} = \lim_{n\to\infty}\left[\frac{1}{4}\frac{n+1}{n}\frac{n+1}{n}-\frac{1}{2}\frac{n+1}{n}\frac{2n+1}{n}\right] \\
&= \lim_{n\to\infty}\left[\frac{1}{4}\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)-\frac{1}{2}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right] = \frac{1}{4}(1)(1)-\frac{1}{2}(1)(2) = -\frac{3}{4}
\end{aligned}$$

34. Note that $\Delta x = \dfrac{2-0}{n} = \dfrac{2}{n}$ and $x_i = 0 + i\,\Delta x = \dfrac{2i}{n}$.

$$\begin{aligned}
\int_{0}^{2}(2x-x^3)\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\frac{2}{n} = \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}\left[2\left(\frac{2i}{n}\right)-\left(\frac{2i}{n}\right)^3\right] \\
&= \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}\left[\frac{4i}{n}-\frac{8i^3}{n^3}\right] = \lim_{n\to\infty}\frac{2}{n}\left[\frac{4}{n}\sum_{i=1}^{n}i-\frac{8}{n^3}\sum_{i=1}^{n}i^3\right] \\
&= \lim_{n\to\infty}\left\{\frac{8}{n^2}\frac{n(n+1)}{2}-\frac{16}{n^4}\left[\frac{n(n+1)}{2}\right]^2\right\} = \lim_{n\to\infty}\left[4\frac{n+1}{n}-4\frac{(n+1)^2}{n^2}\right] \\
&= \lim_{n\to\infty}\left[4\left(1+\frac{1}{n}\right)-4\frac{n+1}{n}\frac{n+1}{n}\right] = \lim_{n\to\infty}\left[4\left(1+\frac{1}{n}\right)-4\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)\right] \\
&= 4(1)-4(1)(1) = 0
\end{aligned}$$

35. (a) Think of $\int_0^2 f(x)\,dx$ as the area of a trapezoid with bases 1 and 3 and height 2. The area of a trapezoid is $A = \frac{1}{2}(b+B)h$, so $\int_0^2 f(x)\,dx = \frac{1}{2}(1+3)2 = 4$.

(b) $$\begin{aligned} \int_0^5 f(x)\,dx &= \underbrace{\int_0^2 f(x)\,dx}_{\text{trapezoid}} + \underbrace{\int_2^3 f(x)\,dx}_{\text{rectangle}} + \underbrace{\int_3^5 f(x)\,dx}_{\text{triangle}} \\ &= \tfrac{1}{2}(1+3)2 + 3\cdot 1 + \tfrac{1}{2}\cdot 2\cdot 3 = 4+3+3 = 10 \end{aligned}$$

(c) $\int_5^7 f(x)\,dx$ is the negative of the area of the triangle with base 2 and height 3. $\int_5^7 f(x)\,dx = -\frac{1}{2}\cdot 2\cdot 3 = -3$.

(d) $\int_3^7 f(x)\,dx = \int_3^5 f(x)\,dx + \int_5^7 f(x)\,dx$. $\int_3^5 f(x)\,dx$ is the area of the triangle with base 2 and height 3. $\int_3^5 f(x)\,dx = \frac{1}{2}\cdot 2\cdot 3 = 3$. From part (c), $\int_5^7 f(x)\,dx = -3$. Thus, $\int_3^7 f(x)\,dx = 3 + (-3) = 0$.

Or: Since $\int_3^5 f(x)\,dx$ is the same figure as in part (c), but with opposite sign, it has value 3. Thus, $\int_3^7 f(x)\,dx = 3 + (-3) = 0$.

(e) $\int_3^7 |f(x)|\,dx = \int_3^5 |f(x)|\,dx + \int_5^7 |f(x)|\,dx = \int_3^5 f(x)\,dx + \int_5^7 [-f(x)]\,dx$. From part (d), $\int_3^5 f(x)\,dx = 3$. From part (c), $\int_5^7 f(x)\,dx = -3$, so $\int_5^7 [-f(x)]\,dx = -(-3) = 3$. Thus, $\int_3^7 |f(x)|\,dx = 3+3 = 6$.

(f) $\int_2^0 f(x)\,dx = -\int_0^2 f(x)\,dx$. From part (a), $\int_0^2 f(x)\,dx = 4$, so $\int_2^0 f(x)\,dx = -4$.

36. (a) $\int_0^2 g(x)\,dx = \frac{1}{2}\cdot 4\cdot 2 = 4$ [area of a triangle]

(b) $\int_2^6 g(x)\,dx = -\frac{1}{2}\pi(2)^2 = -2\pi$ [negative of the area of a semicircle]

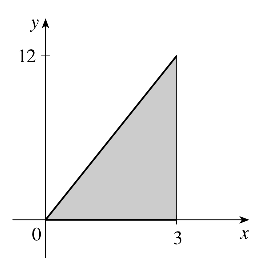
(c) $\int_6^7 g(x)\,dx = \frac{1}{2}\cdot 1\cdot 1 = \frac{1}{2}$ [area of a triangle]

$\int_0^7 g(x)\,dx = \int_0^2 g(x)\,dx + \int_2^6 g(x)\,dx + \int_6^7 g(x)\,dx = 4 - 2\pi + \frac{1}{2} = 4.5 - 2\pi$

37. (a) Note that $\Delta x = \dfrac{3-0}{n} = \dfrac{3}{n}$ and $x_i = 0 + i\,\Delta x = \dfrac{3i}{n}$.

$$\begin{aligned} \int_0^3 4x\,dx &= \lim_{n\to\infty}\sum_{i=1}^n f(x_i)\,\Delta x = \lim_{n\to\infty}\sum_{i=1}^n f\left(\frac{3i}{n}\right)\frac{3}{n} = \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^n 4\left(\frac{3i}{n}\right) \\ &= \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^n \frac{12i}{n} = \lim_{n\to\infty}\frac{3}{n}\left(\frac{12}{n}\right)\sum_{i=1}^n i = \lim_{n\to\infty}\frac{36}{n^2}\left[\frac{n(n+1)}{2}\right] \\ &= \lim_{n\to\infty} 18\left(\frac{n+1}{n}\right) = \lim_{n\to\infty} 18\left(1+\frac{1}{n}\right) = 18(1) = 18 \end{aligned}$$

(b) $\int_0^3 4x\,dx$ can be interpreted as the area of the shaded triangle; that is, $\frac{1}{2}(3)(12) = 18$.



38. (a) Note that $\Delta x = \dfrac{4-(-1)}{n} = \dfrac{5}{n}$ and $x_i = -1 + i\,\Delta x = -1 + \dfrac{5i}{n}$.

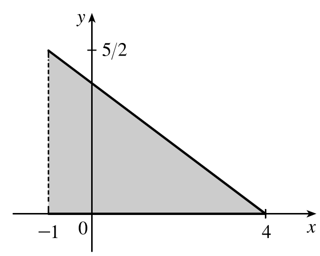
$$\int_{-1}^{4}\left(2-\frac{1}{2}x\right)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} f\left(-1+\frac{5i}{n}\right)\frac{5}{n} = \lim_{n\to\infty}\frac{5}{n}\sum_{i=1}^{n}\left[2-\frac{1}{2}\left(-1+\frac{5i}{n}\right)\right]$$

$$= \lim_{n\to\infty}\frac{5}{n}\sum_{i=1}^{n}\left[2+\frac{1}{2}-\frac{5i}{2n}\right] = \lim_{n\to\infty}\frac{5}{n}\sum_{i=1}^{n}\left[\frac{5}{2}-\frac{5i}{2n}\right] = \lim_{n\to\infty}\left[\frac{5}{n}\sum_{i=1}^{n}\frac{5}{2}-\frac{25}{2n^2}\sum_{i=1}^{n} i\right]$$

$$= \lim_{n\to\infty}\left[\frac{5}{n}\cdot n\left(\frac{5}{2}\right)-\frac{25}{2n^2}\,\frac{n(n+1)}{2}\right] = \lim_{n\to\infty}\left[\frac{25}{2}-\frac{25}{4n}(n+1)\right]$$

$$= \lim_{n\to\infty}\left[\frac{25}{2}-\frac{25}{4}\left(1+\frac{1}{n}\right)\right] = \frac{25}{2}-\frac{25}{4}(1) = \frac{25}{4}$$

(b) $\displaystyle\int_{-1}^{4}\left(2-\frac{1}{2}x\right)dx$ can be interpreted as the area of the shaded triangle; that is, $\dfrac{1}{2}(5)\left(\dfrac{5}{2}\right) = \dfrac{25}{4}$.



39. (a) $\Delta x = (8-0)/8 = 1$ and $x_i^* = x_i = 0 + 1i = i$.

$\int_0^8 (3-2x)\,dx$

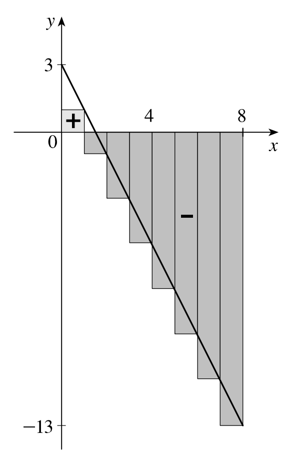
$$\approx \sum_{i=1}^{8} f(x_i^*)\,\Delta x$$

$$= 1\{[3-2(1)] + [3-2(2)] + \cdots + [3-2(8)]\}$$

$$= 1[1+(-1)+(-3)+(-5)+(-7)+(-9)+(-11)+(-13)]$$

$$= -48$$

(b)



(c) Note that $\Delta x = \dfrac{8-0}{n} = \dfrac{8}{n}$ and $x_i = 0 + i\,\Delta x = \dfrac{8i}{n}$.

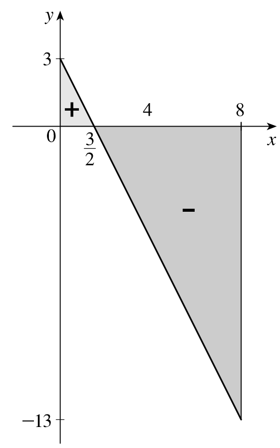
$$\int_0^8 (3-2x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n}\left[3-2\left(\frac{8i}{n}\right)\right]\left(\frac{8}{n}\right) = \lim_{n\to\infty}\frac{8}{n}\sum_{i=1}^{n}\left[3-\frac{16i}{n}\right]$$

$$= \lim_{n\to\infty}\left[\frac{8}{n}\sum_{i=1}^{n}3-\frac{128}{n^2}\sum_{i=1}^{n} i\right] = \lim_{n\to\infty}\left[\frac{8}{n}\cdot n(3)-\frac{128}{n^2}\,\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty}\left[8(3)-\frac{64}{n}(n+1)\right] = \lim_{n\to\infty}\left[24-64\left(1+\frac{1}{n}\right)\right]$$

$$= 24 - 64(1) = -40$$

(d) $\int_0^8 (3-2x)\,dx = A_1 - A_2$, where A_1 is the area marked $+$ and A_2 is the area marked $-$.



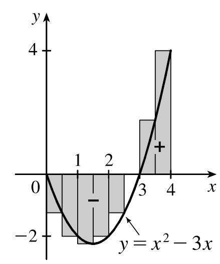
40. (a) $\Delta x = (4-0)/8 = 0.5$ and $x_i^* = x_i = 0.5i$.

$$\int_0^4 (x^2-3x)\,dx \approx \sum_{i=1}^{8} f(x_i^*)\,\Delta x$$

$$= 0.5\{[0.5^2 - 3(0.5)] + [1.0^2 - 3(1.0)] + \cdots + [3.5^2 - 3(3.5)] + [4.0^2 - 3(4.0)]\}$$

$$= \tfrac{1}{2}\left(-\tfrac{5}{4} - 2 - \tfrac{9}{4} - 2 - \tfrac{5}{4} + 0 + \tfrac{7}{4} + 4\right) = -1.5$$

(b)



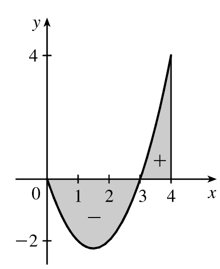
(c) $$\int_0^4 (x^2-3x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(\frac{4i}{n}\right)^2 - 3\left(\frac{4i}{n}\right)\right]\left(\frac{4}{n}\right)$$

$$= \lim_{n\to\infty} \frac{4}{n}\left[\frac{16}{n^2}\sum_{i=1}^{n} i^2 - \frac{12}{n}\sum_{i=1}^{n} i\right] = \lim_{n\to\infty}\left[\frac{64}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{48}{n^2}\cdot\frac{n(n+1)}{2}\right]$$

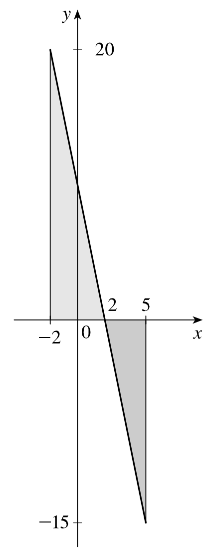
$$= \lim_{n\to\infty}\left[\frac{32}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) - 24\left(1+\frac{1}{n}\right)\right]$$

$$= \tfrac{32}{3}\cdot 2 - 24 = -\tfrac{8}{3}$$

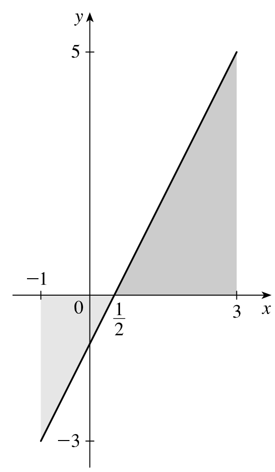
(d) $\int_0^4 (x^2-3x)\,dx = A_1 - A_2$, where A_1 is the area marked $+$ and A_2 is the area marked $-$.



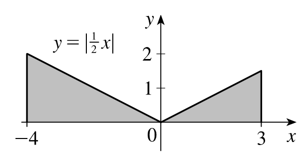
41. $\int_{-2}^{5}(10-5x)\,dx$ can be interpreted as the difference of the areas of the two shaded triangles; that is, $\frac{1}{2}(4)(20)-\frac{1}{2}(3)(15)=40-\frac{45}{2}=\frac{35}{2}$.



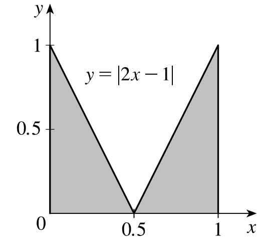
42. $\int_{-1}^{3}(2x-1)\,dx$ can be interpreted as the difference of the areas of the two shaded triangles; that is, $-\frac{1}{2}\left(\frac{3}{2}\right)(3)+\frac{1}{2}\left(\frac{5}{2}\right)(5)=-\frac{9}{4}+\frac{25}{4}=\frac{16}{4}=4$.



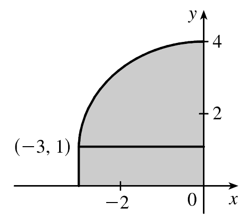
43. $\int_{-4}^{3}\left|\frac{1}{2}x\right|\,dx$ can be interpreted as the sum of the areas of the two shaded triangles; that is, $\frac{1}{2}(4)(2)+\frac{1}{2}(3)\left(\frac{3}{2}\right)=4+\frac{9}{4}=\frac{25}{4}$.



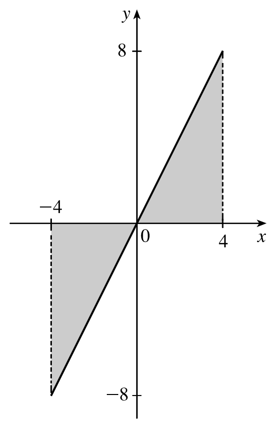
44. $\int_{0}^{1}|2x-1|\,dx$ can be interpreted as the sum of the areas of the two shaded triangles; that is, $2\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)(1)=\frac{1}{2}$.

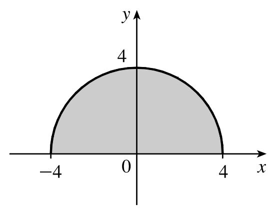


45. $\int_{-3}^{0}\left(1+\sqrt{9-x^2}\right)dx$ can be interpreted as the area under the graph of $f(x)=1+\sqrt{9-x^2}$ between $x=-3$ and $x=0$. This is equal to one-quarter the area of the circle with radius 3, plus the area of the rectangle, so $\int_{-3}^{0}\left(1+\sqrt{9-x^2}\right)dx=\frac{1}{4}\pi\cdot 3^2+1\cdot 3=3+\frac{9}{4}\pi$.



46. $\int_{-4}^{4}\left(2x-\sqrt{16-x^2}\right)dx=\int_{-4}^{4}2x\,dx-\int_{-4}^{4}\sqrt{16-x^2}\,dx$. By symmetry, the value of the first integral is 0 since the shaded area above the x-axis equals the shaded area below the x-axis. The second integral can be interpreted as one-half the area of a circle with radius 4; that is, $\frac{1}{2}\pi(4)^2=8\pi$. Thus, the value of the original integral is $0-8\pi=-8\pi$.





47. $$\int_a^b x\,dx=\lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^{n}\left[a+\frac{b-a}{n}i\right]=\lim_{n\to\infty}\left[\frac{a(b-a)}{n}\sum_{i=1}^{n}1+\frac{(b-a)^2}{n^2}\sum_{i=1}^{n}i\right]$$

$$=\lim_{n\to\infty}\left[\frac{a(b-a)}{n}n+\frac{(b-a)^2}{n^2}\cdot\frac{n(n+1)}{2}\right]=a\,(b-a)+\lim_{n\to\infty}\frac{(b-a)^2}{2}\left(1+\frac{1}{n}\right)$$

$$=a(b-a)+\tfrac{1}{2}(b-a)^2=(b-a)\left(a+\tfrac{1}{2}b-\tfrac{1}{2}a\right)=(b-a)\tfrac{1}{2}(b+a)=\tfrac{1}{2}\left(b^2-a^2\right)$$

48. $$\int_a^b x^2\,dx=\lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^{n}\left[a+\frac{b-a}{n}i\right]^2=\lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^{n}\left[a^2+2a\frac{b-a}{n}i+\frac{(b-a)^2}{n^2}i^2\right]$$

$$=\lim_{n\to\infty}\left[\frac{(b-a)^3}{n^3}\sum_{i=1}^{n}i^2+\frac{2a\,(b-a)^2}{n^2}\sum_{i=1}^{n}i+\frac{a^2\,(b-a)}{n}\sum_{i=1}^{n}1\right]$$

$$=\lim_{n\to\infty}\left[\frac{(b-a)^3}{n^3}\frac{n\,(n+1)\,(2n+1)}{6}+\frac{2a\,(b-a)^2}{n^2}\frac{n\,(n+1)}{2}+\frac{a^2\,(b-a)}{n}n\right]$$

$$=\lim_{n\to\infty}\left[\frac{(b-a)^3}{6}\cdot 1\cdot\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+a\,(b-a)^2\cdot 1\cdot\left(1+\frac{1}{n}\right)+a^2\,(b-a)\right]$$

$$=\frac{(b-a)^3}{3}+a\,(b-a)^2+a^2\,(b-a)=\frac{b^3-3ab^2+3a^2b-a^3}{3}+ab^2-2a^2b+a^3+a^2b-a^3$$

$$=\frac{b^3}{3}-\frac{a^3}{3}-ab^2+a^2b+ab^2-a^2b=\frac{b^3-a^3}{3}$$

49. $\Delta x = (\pi - 0)/n = \pi/n$ and $x_i^* = x_i = \pi i/n$.

$$\int_0^{\pi} \sin 5x\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}(\sin 5x_i)\left(\frac{\pi}{n}\right) = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\sin\frac{5\pi i}{n}\right)\frac{\pi}{n} \overset{\text{CAS}}{=} \pi\lim_{n\to\infty}\frac{1}{n}\cot\left(\frac{5\pi}{2n}\right) \overset{\text{CAS}}{=} \pi\left(\frac{2}{5\pi}\right) = \frac{2}{5}$$

50. $\Delta x = (10 - 2)/n = 8/n$ and $x_i^* = x_i = 2 + 8i/n$.

$$\begin{aligned}\int_2^{10} x^6\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n}\left(2+\frac{8i}{n}\right)^6\left(\frac{8}{n}\right) = 8\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(2+\frac{8i}{n}\right)^6 \\ &\overset{\text{CAS}}{=} 8\lim_{n\to\infty}\frac{1}{n}\cdot\frac{64\big(58{,}593n^6 + 164{,}052n^5 + 131{,}208n^4 - 27{,}776n^2 + 2048\big)}{21n^5} \\ &\overset{\text{CAS}}{=} 8\left(\frac{1{,}249{,}984}{7}\right) = \frac{9{,}999{,}872}{7} \approx 1{,}428{,}553.1\end{aligned}$$

51. $\int_1^1 \sqrt{1+x^4}\,dx = 0$ since the limits of integration are equal.

52.
$$\begin{aligned}\int_\pi^0 \sin^4\theta\,d\theta &= -\int_0^\pi \sin^4\theta\,d\theta && \text{[because we reversed the limits of integration]}\\ &= -\int_0^\pi \sin^4 x\,dx && \text{[we can use any letter without changing the value of the integral]}\\ &= -\tfrac{3}{8}\pi && \text{[given value]}\end{aligned}$$

53. $\int_0^1 (5 - 6x^2)\,dx = \int_0^1 5\,dx - 6\int_0^1 x^2\,dx = 5(1-0) - 6\left(\frac{1}{3}\right) = 5 - 2 = 3$

54. $\int_2^5 (1 + 3x^4)\,dx = \int_2^5 1\,dx + \int_2^5 3x^4\,dx = 1(5-2) + 3\int_2^5 x^4\,dx = 1(3) + 3(618.6) = 1858.8$

55.
$$\begin{aligned}\int_1^4 (2x^2 - 3x + 1)\,dx &= 2\int_1^4 x^2\,dx - 3\int_1^4 x\,dx + \int_1^4 1\,dx\\ &= 2\cdot\tfrac{1}{3}(4^3 - 1^3) - 3\cdot\tfrac{1}{2}(4^2 - 1^2) + 1(4-1) = \tfrac{45}{2} = 22.5\end{aligned}$$

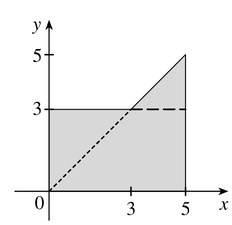
56.
$$\begin{aligned}\int_0^{\pi/2}(2\cos x - 5x)\,dx &= \int_0^{\pi/2} 2\cos x\,dx - \int_0^{\pi/2} 5x\,dx = 2\int_0^{\pi/2}\cos x\,dx - 5\int_0^{\pi/2} x\,dx\\ &= 2(1) - 5\frac{(\pi/2)^2 - 0^2}{2} = 2 - \frac{5\pi^2}{8}\end{aligned}$$

57.
$$\begin{aligned}\int_{-2}^2 f(x)\,dx + \int_2^5 f(x)\,dx - \int_{-2}^{-1} f(x)\,dx &= \int_{-2}^5 f(x)\,dx + \int_{-1}^{-2} f(x)\,dx && \text{[by Property 5 and reversing limits]}\\ &= \int_{-1}^5 f(x)\,dx && \text{[Property 5]}\end{aligned}$$

58. $\int_2^4 f(x)\,dx + \int_4^8 f(x)\,dx = \int_2^8 f(x)\,dx$, so $\int_4^8 f(x)\,dx = \int_2^8 f(x)\,dx - \int_2^4 f(x)\,dx = 7.3 - 5.9 = 1.4$.

59. $\int_0^9 [2f(x) + 3g(x)]\,dx = 2\int_0^9 f(x)\,dx + 3\int_0^9 g(x)\,dx = 2(37) + 3(16) = 122$

60. If $f(x) = \begin{cases} 3 & \text{for } x < 3 \\ x & \text{for } x \geq 3 \end{cases}$, then $\int_0^5 f(x)\,dx$ can be interpreted as the area of the shaded region, which consists of a 5-by-3 rectangle surmounted by an isosceles right triangle whose legs have length 2. Thus, $\int_0^5 f(x)\,dx = 5(3) + \frac{1}{2}(2)(2) = 17$.



61. $\int_0^3 f(x)\,dx$ is clearly less than -1 and has the smallest value. The slope of the tangent line of f at $x=1$, $f'(1)$, has a value between -1 and 0, so it has the next smallest value. The largest value is $\int_3^8 f(x)\,dx$, followed by $\int_4^8 f(x)\,dx$, which has a value about 1 unit less than $\int_3^8 f(x)\,dx$. Still positive, but with a smaller value than $\int_4^8 f(x)\,dx$, is $\int_0^8 f(x)\,dx$. Ordering these quantities from smallest to largest gives us

$$\int_0^3 f(x)\,dx < f'(1) < \int_0^8 f(x)\,dx < \int_4^8 f(x)\,dx < \int_3^8 f(x)\,dx \text{ or } \mathrm{B} < \mathrm{E} < \mathrm{A} < \mathrm{D} < \mathrm{C}$$

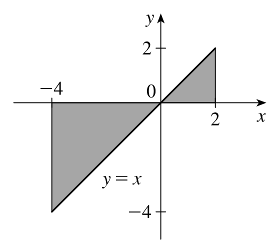
62. $F(0) = \int_2^0 f(t)\,dt = -\int_0^2 f(t)\,dt$, so $F(0)$ is negative, and similarly, so is $F(1)$. $F(3)$ and $F(4)$ are negative since they represent negatives of areas below the x-axis. Since $F(2) = \int_2^2 f(t)\,dt = 0$ is the only nonnegative value, choice C is the largest.

63. $I = \int_{-4}^2 [f(x) + 2x + 5]\,dx = \int_{-4}^2 f(x)\,dx + 2\int_{-4}^2 x\,dx + \int_{-4}^2 5\,dx = I_1 + 2I_2 + I_3$

$I_1 = -3$ [area below x-axis] $+ 3 - 3 = -3$

$I_2 = -\frac{1}{2}(4)(4)$ [area of triangle, see figure] $+ \frac{1}{2}(2)(2)$

$= -8 + 2 = -6$



$I_3 = 5[2 - (-4)] = 5(6) = 30$

Thus, $I = -3 + 2(-6) + 30 = 15$.

64. Using Integral Comparison Property 8, $m \le f(x) \le M \;\Rightarrow\; m(2-0) \le \int_0^2 f(x)\,dx \le M(2-0) \;\Rightarrow$ $2m \le \int_0^2 f(x)\,dx \le 2M$.

65. $x^2 - 4x + 4 = (x-2)^2 \ge 0$ on $[0, 4]$, so $\int_0^4 (x^2 - 4x + 4)\,dx \ge 0$ [Property 6].

66. $x^2 \le x$ on $[0,1]$, so $\sqrt{1+x^2} \le \sqrt{1+x}$ on $[0,1]$. Hence, $\int_0^1 \sqrt{1+x^2}\,dx \le \int_0^1 \sqrt{1+x}\,dx$ [Property 7].

67. If $-1 \le x \le 1$, then $0 \le x^2 \le 1$ and $1 \le 1 + x^2 \le 2$, so $1 \le \sqrt{1+x^2} \le \sqrt{2}$ and $1[1-(-1)] \le \int_{-1}^1 \sqrt{1+x^2}\,dx \le \sqrt{2}\,[1-(-1)]$ [Property 8]; that is, $2 \le \int_{-1}^1 \sqrt{1+x^2}\,dx \le 2\sqrt{2}$.

68. If $\dfrac{\pi}{6} \le x \le \dfrac{\pi}{3}$, then $\dfrac{1}{2} \le \sin x \le \dfrac{\sqrt{3}}{2}$ $\left(\sin x \text{ is increasing on } \left[\frac{\pi}{6}, \frac{\pi}{3}\right]\right)$, so

$\dfrac{1}{2}\left(\dfrac{\pi}{3} - \dfrac{\pi}{6}\right) \le \displaystyle\int_{\pi/6}^{\pi/3} \sin x\,dx \le \dfrac{\sqrt{3}}{2}\left(\dfrac{\pi}{3} - \dfrac{\pi}{6}\right)$ [Property 8]; that is, $\dfrac{\pi}{12} \le \displaystyle\int_{\pi/6}^{\pi/3} \sin x\,dx \le \dfrac{\sqrt{3}\,\pi}{12}$.

69. If $0 \le x \le 1$, then $0 \le x^3 \le 1$, so $0(1-0) \le \int_0^1 x^3\,dx \le 1(1-0)$ [Property 8]; that is, $0 \le \int_0^1 x^3\,dx \le 1$.

70. If $0 \le x \le 3$, then $4 \le x + 4 \le 7$ and $\dfrac{1}{7} \le \dfrac{1}{x+4} \le \dfrac{1}{4}$, so $\dfrac{1}{7}(3-0) \le \displaystyle\int_0^3 \dfrac{1}{x+4}\,dx \le \dfrac{1}{4}(3-0)$ [Property 8]; that is,

$\dfrac{3}{7} \le \displaystyle\int_0^3 \dfrac{1}{x+4}\,dx \le \dfrac{3}{4}$.

71. If $\frac{\pi}{4} \le x \le \frac{\pi}{3}$, then $1 \le \tan x \le \sqrt{3}$, so $1\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \le \int_{\pi/4}^{\pi/3} \tan x\,dx \le \sqrt{3}\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$ or $\frac{\pi}{12} \le \int_{\pi/4}^{\pi/3} \tan x\,dx \le \frac{\pi}{12}\sqrt{3}$.

72. Let $f(x) = x^3 - 3x + 3$ for $0 \le x \le 2$. Then $f'(x) = 3x^2 - 3 = 3(x+1)(x-1)$, so f is decreasing on $(0,1)$ and increasing on $(1,2)$. f has the absolute minimum value $f(1) = 1$. Since $f(0) = 3$ and $f(2) = 5$, the absolute maximum value of f is $f(2) = 5$. Thus, $1 \le x^3 - 3x + 3 \le 5$ for x in $[0,2]$. It follows from Property 8 that $1 \cdot (2-0) \le \int_0^2 \left(x^3 - 3x + 3\right) dx \le 5 \cdot (2-0)$; that is, $2 \le \int_0^2 \left(x^3 - 3x + 3\right) dx \le 10$.

73. For $-1 \le x \le 1$, $0 \le x^4 \le 1$ and $1 \le \sqrt{1+x^4} \le \sqrt{2}$, so $1[1-(-1)] \le \int_{-1}^1 \sqrt{1+x^4}\,dx \le \sqrt{2}\,[1-(-1)]$ or $2 \le \int_{-1}^1 \sqrt{1+x^4}\,dx \le 2\sqrt{2}$.

74. Let $f(x) = x - 2\sin x$ for $\pi \le x \le 2\pi$. Then $f'(x) = 1 - 2\cos x$ and $f'(x) = 0 \;\Rightarrow\; \cos x = \frac{1}{2} \;\Rightarrow\; x = \frac{5\pi}{3}$. f has the absolute maximum value $f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} - 2\sin\frac{5\pi}{3} = \frac{5\pi}{3} + \sqrt{3} \approx 6.97$ since $f(\pi) = \pi$ and $f(2\pi) = 2\pi$ are both smaller than 6.97. Thus, $\pi \le f(x) \le \frac{5\pi}{3} + \sqrt{3} \;\Rightarrow\; \pi(2\pi - \pi) \le \int_\pi^{2\pi} f(x)\,dx \le \left(\frac{5\pi}{3} + \sqrt{3}\right)(2\pi - \pi)$; that is, $\pi^2 \le \int_\pi^{2\pi} (x - 2\sin x)\,dx \le \frac{5}{3}\pi^2 + \sqrt{3}\,\pi$.

75. $\sqrt{x^4+1} \ge \sqrt{x^4} = x^2$, so $\int_1^3 \sqrt{x^4+1}\,dx \ge \int_1^3 x^2\,dx = \frac{1}{3}\left(3^3 - 1^3\right) = \frac{26}{3}$.

76. $0 \le \sin x \le 1$ for $0 \le x \le \frac{\pi}{2}$, so $x\sin x \le x \;\Rightarrow\; \int_0^{\pi/2} x\sin x\,dx \le \int_0^{\pi/2} x\,dx = \frac{1}{2}\left[\left(\frac{\pi}{2}\right)^2 - 0^2\right] = \frac{\pi^2}{8}$.

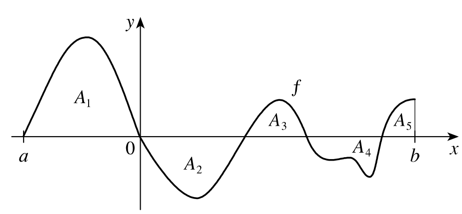
77. $1/x < \sqrt{x} < x$ for $1 < x \le 2$ and $\sqrt{x}$ is an increasing function, so $\sqrt{1/x} < \sqrt{\sqrt{x}} < \sqrt{x}$, and hence $\int_1^2 \sqrt{1/x}\,dx < \int_1^2 \sqrt{\sqrt{x}}\,dx < \int_1^2 \sqrt{x}\,dx$. Thus, $\int_1^2 \sqrt{x}\,dx$ has the largest value.

78. $x^2 < \sqrt{x}$ for $0 < x \le 0.5$ and cosine is a decreasing function on $[0, 0.5]$, so $\cos(x^2) > \cos\sqrt{x}$, and hence, $\int_0^{0.5} \cos(x^2)\,dx > \int_0^{0.5} \cos\sqrt{x}\,dx$. Thus, $\int_0^{0.5} \cos(x^2)\,dx$ is larger.

79. Using right endpoints as in the proof of Property 2, we calculate

$$\int_a^b cf(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^n cf(x_i)\,\Delta x = \lim_{n\to\infty} c\sum_{i=1}^n f(x_i)\,\Delta x = c\lim_{n\to\infty} \sum_{i=1}^n f(x_i)\,\Delta x = c\int_a^b f(x)\,dx.$$

80. (a) See the figure.



$$\left|\int_a^b f(x)\,dx\right| = |A_1 - A_2 + A_3 - A_4 + A_5|$$
$$\le |A_1 + A_2 + A_3 + A_4 + A_5|$$
$$= \int_a^b |f(x)|\,dx$$

(b) Since $-|f(x)| \le f(x) \le |f(x)|$, it follows from Property 7 that

$$-\int_a^b |f(x)|\,dx \le \int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx \quad\Rightarrow\quad \left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$$

Note that the definite integral is a real number, and so the following property applies: $-a \le b \le a \;\Rightarrow\; |b| \le a$ for all real numbers b and nonnegative numbers a.

(c) $\left|\int_a^b f(x)\sin 2x\,dx\right| \le \int_a^b |f(x)\sin 2x|\;dx$ [by part (b)] $= \int_a^b |f(x)|\,|\sin 2x|\,dx \le \int_a^b |f(x)|\;dx$ by Property 7 since

$|\sin 2x| \le 1 \quad\Rightarrow\quad |f(x)|\,|\sin 2x| \le |f(x)|$.

81. Suppose that f is integrable on $[0,1]$, that is, $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} f(x_i^*)\,\Delta x$ exists for any choice of x_i^* in $[x_{i-1}, x_i]$. Let n denote a positive integer and divide the interval $[0,1]$ into n equal subintervals $\left[0,\dfrac{1}{n}\right], \left[\dfrac{1}{n},\dfrac{2}{n}\right], \ldots, \left[\dfrac{n-1}{n},1\right]$. If we choose x_i^* to be a rational number in the ith subinterval, then we obtain the Riemann sum $\sum\limits_{i=1}^{n} f(x_i^*)\cdot\dfrac{1}{n} = 0$, so

$\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n} f(x_i^*)\cdot\dfrac{1}{n} = \lim\limits_{n\to\infty} 0 = 0$. Now suppose we choose x_i^* to be an irrational number. Then we get

$\sum\limits_{i=1}^{n} f(x_i^*)\cdot\dfrac{1}{n} = \sum\limits_{i=1}^{n} 1\cdot\dfrac{1}{n} = n\cdot\dfrac{1}{n} = 1$ for each n, so $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n} f(x_i^*)\cdot\dfrac{1}{n} = \lim\limits_{n\to\infty} 1 = 1$. Since the value of

$\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n} f(x_i^*)\,\Delta x$ depends on the choice of the sample points x_i^*, the limit does not exist, and f is not integrable on $[0,1]$.

82. Partition the interval $[0,1]$ into n equal subintervals and choose $x_1^* = \dfrac{1}{n^2}$. Then with $f(x) = \dfrac{1}{x}$,

$\sum\limits_{i=1}^{n} f(x_i^*)\,\Delta x \ge f(x_1^*)\Delta x = \dfrac{1}{1/n^2}\cdot\dfrac{1}{n} = n$. Thus, $\sum\limits_{i=1}^{n} f(x_i^*)\,\Delta x$ can be made arbitrarily large and hence, f is not integrable on $[0,1]$.

83. $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\dfrac{i^4}{n^5} = \lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\dfrac{i^4}{n^4}\cdot\dfrac{1}{n} = \lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\left(\dfrac{i}{n}\right)^4\dfrac{1}{n}$. At this point, we need to recognize the limit as being of the form

$\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n} f(x_i)\,\Delta x$, where $\Delta x = (1-0)/n = 1/n$, $x_i = 0 + i\,\Delta x = i/n$, and $f(x) = x^4$. Thus, the definite integral is $\int_0^1 x^4\,dx$.

84. $\lim\limits_{n\to\infty}\dfrac{1}{n}\sum\limits_{i=1}^{n}\dfrac{1}{1+(i/n)^2} = \lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\dfrac{1}{1+(i/n)^2}\cdot\dfrac{1}{n} = \lim\limits_{n\to\infty}\sum\limits_{i=1}^{n} f(x_i)\,\Delta x$, where $\Delta x = (1-0)/n = 1/n$,

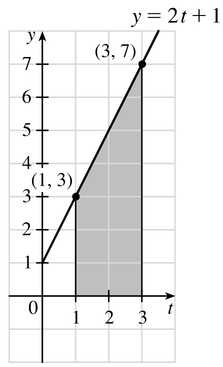
$x_i = 0 + i\,\Delta x = i/n$, and $f(x) = \dfrac{1}{1+x^2}$. Thus, the definite integral is $\displaystyle\int_0^1 \frac{dx}{1+x^2}$.

85. Choose $x_i = 1 + \dfrac{i}{n}$ and $x_i^* = \sqrt{x_{i-1}x_i} = \sqrt{\left(1+\dfrac{i-1}{n}\right)\left(1+\dfrac{i}{n}\right)}$. Then

$$\begin{aligned}
\int_1^2 x^{-2}\,dx &= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\left(1+\frac{i-1}{n}\right)\left(1+\frac{i}{n}\right)} = \lim_{n\to\infty} n\sum_{i=1}^{n}\frac{1}{(n+i-1)(n+i)} \\
&= \lim_{n\to\infty} n\sum_{i=1}^{n}\left(\frac{1}{n+i-1}-\frac{1}{n+i}\right) \quad [\text{by the hint}] \quad = \lim_{n\to\infty} n\left(\sum_{i=0}^{n-1}\frac{1}{n+i} - \sum_{i=1}^{n}\frac{1}{n+i}\right) \\
&= \lim_{n\to\infty} n\left(\left[\frac{1}{n}+\frac{1}{n+1}+\cdots+\frac{1}{2n-1}\right] - \left[\frac{1}{n+1}+\cdots+\frac{1}{2n-1}+\frac{1}{2n}\right]\right) \\
&= \lim_{n\to\infty} n\left(\frac{1}{n}-\frac{1}{2n}\right) = \lim_{n\to\infty}\left(1-\tfrac{1}{2}\right) = \tfrac{1}{2}
\end{aligned}$$

DISCOVERY PROJECT Area Functions

1. (a)

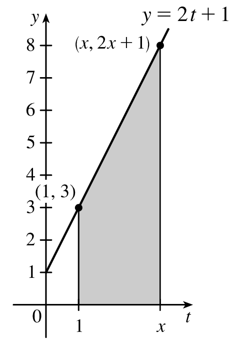


Area of trapezoid $= \frac{1}{2}(b_1 + b_2)h = \frac{1}{2}(3+7)2$
$= 10$ square units

Or:

Area of rectangle + area of triangle
$= b_r h_r + \frac{1}{2} b_t h_t = (2)(3) + \frac{1}{2}(2)(4) = 10$ square units

(b)

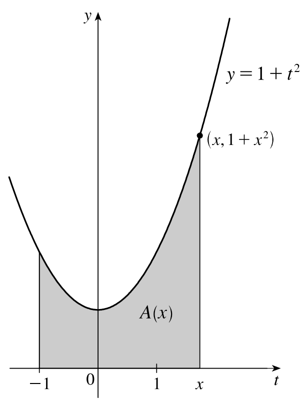


As in part (a),

$$A(x) = \tfrac{1}{2}[3 + (2x+1)](x-1) = \tfrac{1}{2}(2x+4)(x-1)$$
$$= (x+2)(x-1) = x^2 + x - 2 \text{ square units}$$

(c) $A'(x) = 2x + 1$. This is the y-coordinate of the point $(x, 2x+1)$ on the given line.

2. (a)

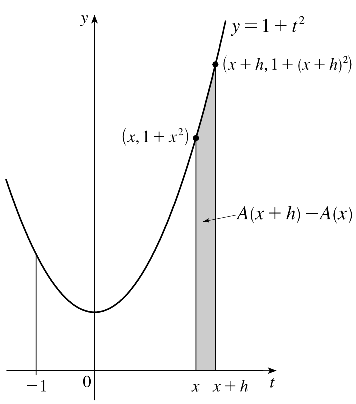


(b) $A(x) = \int_{-1}^{x} \left(1 + t^2\right) dt = \int_{-1}^{x} 1\, dt + \int_{-1}^{x} t^2\, dt$ [Property 2]

$$= 1[x - (-1)] + \frac{x^3 - (-1)^3}{3} \quad \begin{bmatrix}\text{Property 1 and} \\ \text{Exercise 4.2.28}\end{bmatrix}$$
$$= x + 1 + \tfrac{1}{3}x^3 + \tfrac{1}{3}$$
$$= \tfrac{1}{3}x^3 + x + \tfrac{4}{3}$$

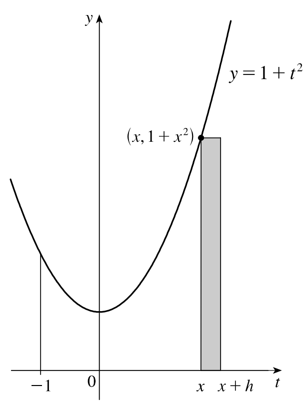
(c) $A'(x) = x^2 + 1$. This is the y-coordinate of the point $\left(x, 1 + x^2\right)$ on the given curve.

(d)



$A(x+h) - A(x)$ is the area under the curve $y = 1 + t^2$ from $t = x$ to $t = x + h$.

(e)



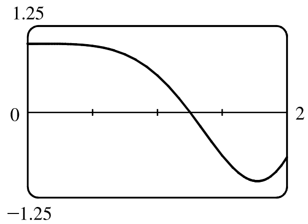
An approximating rectangle is shown in the figure.

It has height $1 + x^2$, width h, and area $h(1 + x^2)$, so

$$A(x+h) - A(x) \approx h(1 + x^2) \quad \Rightarrow \quad \frac{A(x+h) - A(x)}{h} \approx 1 + x^2.$$

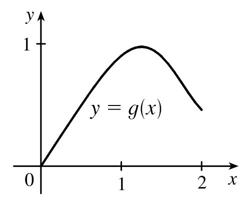
(f) Part (e) says that the average rate of change of A is approximately $1 + x^2$. As h approaches 0, the quotient approaches the instantaneous rate of change—namely, $A'(x)$. So the result of part (c), $A'(x) = x^2 + 1$, is geometrically plausible.

3. (a) $f(x) = \cos(x^2)$

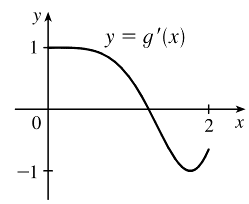


(b) $g(x)$ starts to decrease at that value of x where $\cos(t^2)$ changes from positive to negative; that is, at about $x = 1.25$.

(c) $g(x) = \int_0^x \cos(t^2)\,dt$. Using an integration command, we find that $g(0) = 0$, $g(0.2) \approx 0.200$, $g(0.4) \approx 0.399$, $g(0.6) \approx 0.592$, $g(0.8) \approx 0.768$, $g(1.0) \approx 0.905$, $g(1.2) \approx 0.974$, $g(1.4) \approx 0.950$, $g(1.6) \approx 0.826$, $g(1.8) \approx 0.635$, and $g(2.0) \approx 0.461$.



(d) We sketch the graph of g' using the method of Example 1 in Section 2.2. The graphs of $g'(x)$ and $f(x)$ look alike, so we guess that $g'(x) = f(x)$.



4. In Problems 1 and 2, we showed that if $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$, for the functions $f(t) = 2t + 1$ and $f(t) = 1 + t^2$. In Problem 3 we guessed that the same is true for $f(t) = \cos(t^2)$, based on visual evidence. So we conjecture that $g'(x) = f(x)$ for any continuous function f. This turns out to be true and is proved in Section 4.3 (the Fundamental Theorem of Calculus).

4.3 The Fundamental Theorem of Calculus

1. One process undoes what the other one does. The precise version of this statement is given by the Fundamental Theorem of Calculus. See the statement of this theorem and the paragraph that follows it.

2. (a) $g(x) = \int_0^x f(t)\,dt$, so $g(0) = \int_0^0 f(t)\,dt = 0$.

$g(1) = \int_0^1 f(t)\,dt = \frac{1}{2} \cdot 1 \cdot 1 \quad [\text{area of triangle}] \quad = \frac{1}{2}$.

$$\begin{aligned} g(2) &= \int_0^2 f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt \quad [\text{below the } t\text{-axis}] \\ &= \tfrac{1}{2} - \tfrac{1}{2} \cdot 1 \cdot 1 = 0. \end{aligned}$$

$g(3) = g(2) + \int_2^3 f(t)\,dt = 0 - \frac{1}{2} \cdot 1 \cdot 1 = -\frac{1}{2}$.

$g(4) = g(3) + \int_3^4 f(t)\,dt = -\frac{1}{2} + \frac{1}{2} \cdot 1 \cdot 1 = 0$.

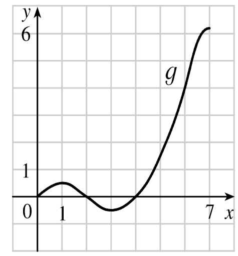
$g(5) = g(4) + \int_4^5 f(t)\,dt = 0 + 1.5 = 1.5$.

$g(6) = g(5) + \int_5^6 f(t)\,dt = 1.5 + 2.5 = 4$.

(b) $g(7) = g(6) + \int_6^7 f(t)\,dt \approx 4 + 2.2$ [estimate from the graph] $= 6.2$.

(c) The answers from part (a) and part (b) indicate that g has a minimum at $x = 3$ and a maximum at $x = 7$. This makes sense from the graph of f since we are subtracting area on $1 < x < 3$ and adding area on $3 < x < 7$.

(d)



3. (a) $g(x) = \int_0^x f(t)\,dt$.

$g(0) = \int_0^0 f(t)\,dt = 0$

$g(1) = \int_0^1 f(t)\,dt = 1 \cdot 2 = 2$ [rectangle],

$g(2) = \int_0^2 f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt = g(1) + \int_1^2 f(t)\,dt$
$= 2 + 1 \cdot 2 + \frac{1}{2} \cdot 1 \cdot 2 = 5$ [rectangle plus triangle],

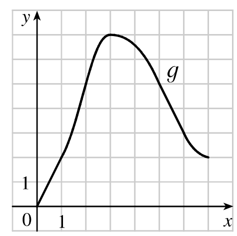
$g(3) = \int_0^3 f(t)\,dt = g(2) + \int_2^3 f(t)\,dt = 5 + \frac{1}{2} \cdot 1 \cdot 4 = 7$,

$g(6) = g(3) + \int_3^6 f(t)\,dt$ [the integral is negative since f lies under the t-axis]
$= 7 + \left[-\left(\frac{1}{2} \cdot 2 \cdot 2 + 1 \cdot 2\right)\right] = 7 - 4 = 3$

(b) g is increasing on $(0, 3)$ because as x increases from 0 to 3, we keep adding more area.

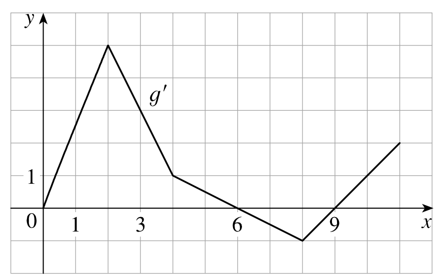
(c) g has a maximum value when we start subtracting area; that is, at $x = 3$.

(d)



4. (a) $g(x) = \int_0^x f(t)\,dt \quad \Rightarrow \quad g'(x) = f(x)$ by FTC1.

Thus, the graph of g' is the graph of f.



(b) $g(3) = \int_0^3 f(t)\,dt = \frac{1}{2} \cdot 2 \cdot 5 + \frac{1}{2}(5 + 3) \cdot 1 = 9$ [triangle plus trapezoid]

$g'(3) = f(3) = 3$ by FTC1.

From part (a), $g''(3) = f'(3)$, which is the slope of the given graph at $x = 3$.

Thus, $g''(3) = \dfrac{1 - 5}{4 - 2} = \dfrac{-4}{2} = -2$ [slope of line between $(2, 5)$ and $(4, 1)$].

(c) g has a local maximum at $x = 6$ because $f = g'$ changes from positive to negative there, so g changes from increasing to decreasing there. (We start subtracting area at $x = 6$).

(d) g has a local minimum at $x = 9$ because $f = g'$ changes from negative to positive there, so g changes from decreasing to increasing there. (We start adding area at $x = 9$.)

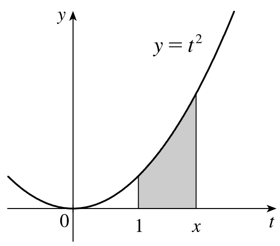
5. (a) $g(x) = \int_0^x f(t)\,dt = 3x$ [area of a rectangle with base x and height 3]

(b) $g(x) = 3x \quad \Rightarrow \quad g'(x) = 3 = f(x)$, so g is an antiderivative of f. Since $f(t) = 3$, 3 is the integrand in part (a), and 3 is the integrand evaluated at upper limit of integration x (since 3 is constant), verifying FTC1.

6. (a) $g(x) = \int_0^x f(t)\,dt = \frac{1}{2} \cdot x \cdot 3x = \frac{3}{2}x^2$ [area of a triangle with base x and height $3x$]

(b) $g(x) = \frac{3}{2}x^2 \quad \Rightarrow \quad g'(x) = \frac{3}{2} \cdot 2x = 3x = f(x)$, so g is an antiderivative of f. Since $f(t) = 3t$, $3t$ is the integrand in part (a), and $3x$ is the integrand evaluated at upper limit of integration x, verifying FTC1.

7.

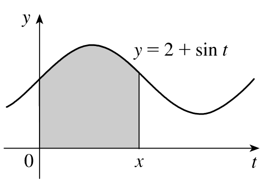


(a) By FTC1 with $f(t) = t^2$ and $a = 1$, $g(x) = \int_1^x t^2\,dt \quad \Rightarrow$

$g'(x) = f(x) = x^2$.

(b) Using FTC2, $g(x) = \int_1^x t^2\,dt = \left[\frac{1}{3}t^3\right]_1^x = \frac{1}{3}x^3 - \frac{1}{3} \quad \Rightarrow \quad g'(x) = x^2$.

8.



(a) By FTC1 with $f(t) = 2 + \sin t$ and $a = 0$, $g(x) = \int_0^x (2 + \sin t)\,dt \quad \Rightarrow$

$g'(x) = f(x) = 2 + \sin x$.

(b) Using FTC2,

$$g(x) = \int_0^x (2 + \sin t)\,dt = [2t - \cos t]_0^x = (2x - \cos x) - (0 - 1)$$
$$= 2x - \cos x + 1 \quad \Rightarrow$$
$$g'(x) = 2 - (-\sin x) + 0 = 2 + \sin x$$

9. $f(t) = \sqrt{t + t^3}$ and $g(x) = \int_0^x \sqrt{t + t^3}\,dt$, so by FTC1, $g'(x) = f(x) = \sqrt{x + x^3}$.

10. $f(t) = \cos(t^2)$ and $g(x) = \int_1^x \cos(t^2)\,dt$, so by FTC1, $g'(x) = f(x) = \cos(x^2)$.

11. $f(t) = \sin(1 + t^3)$ and $g(w) = \int_0^w \sin(1 + t^3)\,dt$, so by FTC1, $g'(w) = f(w) = \sin(1 + w^3)$.

12. $f(t) = \dfrac{\sqrt{t}}{t+1}$ and $h(u) = \displaystyle\int_0^u \frac{\sqrt{t}}{t+1}\,dt$, so by FTC1, $h'(u) = f(u) = \dfrac{\sqrt{u}}{u+1}$.

13. $F(x) = \displaystyle\int_x^0 \sqrt{1 + \sec t}\,dt = -\int_0^x \sqrt{1 + \sec t}\,dt \overset{\text{FTC1}}{\Rightarrow} F'(x) = -\frac{d}{dx}\int_0^x \sqrt{1 + \sec t}\,dt = -\sqrt{1 + \sec x}$

14. $R(y) = \displaystyle\int_y^2 t^3 \sin t\,dt = -\int_2^y t^3 \sin t\,dt \quad \Rightarrow \quad R'(y) = -\frac{d}{dy}\int_2^y t^3 \sin t\,dt = -y^3 \sin y$

15. Let $u = \dfrac{1}{x}$. Then $\dfrac{du}{dx} = -\dfrac{1}{x^2}$. Also, $\dfrac{dh}{dx} = \dfrac{dh}{du}\dfrac{du}{dx}$, so

$$h'(x) = \frac{d}{dx}\int_2^{1/x} \sin^4 t\,dt = \frac{d}{du}\int_2^u \sin^4 t\,dt \cdot \frac{du}{dx} = \sin^4 u\,\frac{du}{dx} = \frac{-\sin^4(1/x)}{x^2}.$$

16. Let $u = \sqrt{x}$. Then $\dfrac{du}{dx} = \dfrac{1}{2\sqrt{x}}$. Also, $\dfrac{dh}{dx} = \dfrac{dh}{du}\dfrac{du}{dx}$, so by FTC1,

$$h'(x) = \frac{d}{dx}\int_1^{\sqrt{x}} \frac{z^2}{z^4+1}\,dz = \frac{d}{du}\int_1^u \frac{z^2}{z^4+1}\,dz \cdot \frac{du}{dx} = \frac{u^2}{u^4+1}\,\frac{du}{dx} = \frac{x}{x^2+1}\,\frac{1}{2\sqrt{x}} = \frac{\sqrt{x}}{2(x^2+1)}.$$

17. Let $u = 3x+2$. Then $\dfrac{du}{dx} = 3$. Also, $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$, so by FTC1,

$$y' = \frac{d}{dx}\int_1^{3x+2} \frac{t}{1+t^3}\,dt = \frac{d}{du}\int_1^u \frac{t}{1+t^3}\,dt \cdot \frac{du}{dx} = \frac{u}{1+u^3}\,\frac{du}{dx} = \frac{3x+2}{1+(3x+2)^3}\cdot 3 = \frac{3(3x+2)}{1+(3x+2)^3}$$

18. Let $u = x^4$. Then $\dfrac{du}{dx} = 4x^3$. Also, $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$, so

$$y' = \frac{d}{dx}\int_0^{x^4} \cos^2\theta\,d\theta = \frac{d}{du}\int_0^u \cos^2\theta\,d\theta \cdot \frac{du}{dx} = \cos^2 u\,\frac{du}{dx} = \cos^2(x^4)\cdot 4x^3.$$

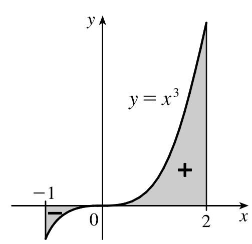
19. Let $u = \sqrt{x}$. Then $\dfrac{du}{dx} = \dfrac{1}{2\sqrt{x}}$. Also, $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$, so by FTC1,

$$y' = \frac{d}{dx}\int_{\sqrt{x}}^{\pi/4} \theta\tan\theta\,d\theta = -\frac{d}{du}\int_{\pi/4}^{\sqrt{x}} \theta\tan\theta\,d\theta \cdot \frac{du}{dx} = -u\tan u\,\frac{du}{dx} = -\sqrt{x}\tan\sqrt{x}\cdot\frac{1}{2\sqrt{x}} = -\frac{1}{2}\tan\sqrt{x}$$

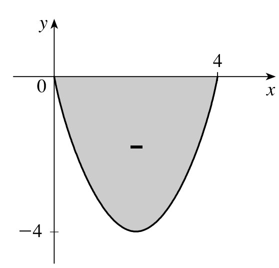
20. $y = \displaystyle\int_{1/x}^4 \sqrt{1+\frac{1}{t}}\,dt = -\int_4^{1/x}\sqrt{1+\frac{1}{t}}\,dt$. Let $u = \dfrac{1}{x}$. Then $\dfrac{du}{dx} = -\dfrac{1}{x^2}$. Also, $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$, so by FTC1,

$$\begin{aligned} y' &= \frac{d}{dx}\left[-\int_4^{1/x}\sqrt{1+\frac{1}{t}}\,dt\right] = -\frac{d}{du}\int_4^u \sqrt{1+\frac{1}{t}}\,dt\cdot\frac{du}{dx} \\ &= -\sqrt{1+\frac{1}{u}}\cdot\left(-\frac{1}{x^2}\right) = -\sqrt{1+\frac{1}{1/x}}\cdot\left(-\frac{1}{x^2}\right) = \frac{1}{x^2}\sqrt{1+x} \end{aligned}$$

21. $\int_{-1}^2 x^3\,dx = \left[\frac14 x^4\right]_{-1}^2 = 4 - \frac14 = \frac{15}{4} = 3.75$



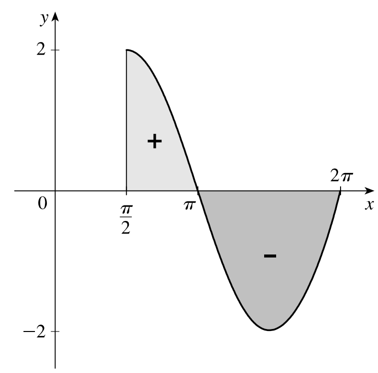
22. $\int_0^4 (x^2-4x)\,dx = \left[\frac13 x^3 - 2x^2\right]_0^4$
$= \left(\frac{64}{3} - 32\right) - 0 = -\frac{32}{3}$



23. $\int_{\pi/2}^{2\pi}(2\sin x)\,dx = \Big[-2\cos x\Big]_{\pi/2}^{2\pi}$

$= (-2\cos 2\pi) - \left(-2\cos\frac{\pi}{2}\right)$

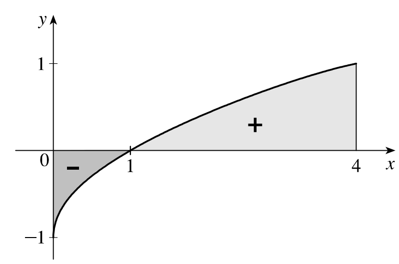
$= -2(1) + 2(0) = -2$



24. $\int_0^4 (\sqrt{x} - 1)\,dx = \Big[\frac{2}{3}x^{3/2} - x\Big]_0^4$

$= \big[\frac{2}{3}(8) - 4\big] - 0$

$= \frac{4}{3}$



25. $\int_1^3 (x^2 + 2x - 4)\,dx = \big[\frac{1}{3}x^3 + x^2 - 4x\big]_1^3 = (9 + 9 - 12) - \left(\frac{1}{3} + 1 - 4\right) = 6 + \frac{8}{3} = \frac{26}{3}$

26. $\int_{-1}^1 x^{100}\,dx = \big[\frac{1}{101}x^{101}\big]_{-1}^1 = \frac{1}{101} - \left(-\frac{1}{101}\right) = \frac{2}{101}$

27. $\int_0^2 \left(\frac{4}{5}t^3 - \frac{3}{4}t^2 + \frac{2}{5}t\right)dt = \big[\frac{1}{5}t^4 - \frac{1}{4}t^3 + \frac{1}{5}t^2\big]_0^2 = \left(\frac{16}{5} - 2 + \frac{4}{5}\right) - 0 = 2$

28. $\int_0^1 (1 - 8v^3 + 16v^7)\,dv = \big[v - 2v^4 + 2v^8\big]_0^1 = (1 - 2 + 2) - 0 = 1$

29. $\displaystyle\int_1^9 \sqrt{x}\,dx = \int_1^9 x^{1/2}\,dx = \left[\frac{x^{3/2}}{3/2}\right]_1^9 = \frac{2}{3}\Big[x^{3/2}\Big]_1^9 = \frac{2}{3}(9^{3/2} - 1^{3/2}) = \frac{2}{3}(27 - 1) = \frac{52}{3}$

30. $\displaystyle\int_1^8 x^{-2/3}\,dx = \left[\frac{x^{1/3}}{1/3}\right]_1^8 = 3\Big[x^{1/3}\Big]_1^8 = 3(8^{1/3} - 1^{1/3}) = 3(2 - 1) = 3$

31. $\displaystyle\int_0^4 (t^2 + t^{3/2})\,dt = \left[\frac{1}{3}t^3 + \frac{2}{5}t^{5/2}\right]_0^4 = \left(\frac{64}{3} + \frac{64}{5}\right) - 0 = \frac{512}{15}$

32. $\displaystyle\int_1^3 \left(\frac{1}{z^2} + \frac{1}{z^3}\right)dz = \left[-\frac{1}{z} - \frac{1}{2z^2}\right]_1^3 = \left(-\frac{1}{3} - \frac{1}{18}\right) - \left(-1 - \frac{1}{2}\right) = \frac{10}{9}$

33. $\int_{\pi/2}^0 \cos\theta\,d\theta = \Big[\sin\theta\Big]_{\pi/2}^0 = \sin 0 - \sin\dfrac{\pi}{2} = 0 - 1 = -1$

34. $\displaystyle\int_{-5}^5 \pi\,dx = \big[\pi x\big]_{-5}^5 = 5\pi - (-5\pi) = 10\pi$

35. $\displaystyle\int_0^1 (u + 2)(u - 3)\,du = \int_0^1 (u^2 - u - 6)\,du = \big[\tfrac{1}{3}u^3 - \tfrac{1}{2}u^2 - 6u\big]_0^1 = \left(\tfrac{1}{3} - \tfrac{1}{2} - 6\right) - 0 = -\tfrac{37}{6}$

36. $\displaystyle\int_0^4 (4-t)\sqrt{t}\,dt = \int_0^4 (4-t)t^{1/2}\,dt = \int_0^4 (4t^{1/2} - t^{3/2})\,dt = \left[\tfrac{8}{3}t^{3/2} - \tfrac{2}{5}t^{5/2}\right]_0^4 = \tfrac{8}{3}(8) - \tfrac{2}{5}(32) = \tfrac{320-192}{15} = \tfrac{128}{15}$

37. $\displaystyle\int_1^4 \frac{2+x^2}{\sqrt{x}}\,dx = \int_1^4 \left(\frac{2}{\sqrt{x}} + \frac{x^2}{\sqrt{x}}\right)dx = \int_1^4 (2x^{-1/2} + x^{3/2})\,dx$

$\displaystyle= \left[4x^{1/2} + \tfrac{2}{5}x^{5/2}\right]_1^4 = \left[4(2) + \tfrac{2}{5}(32)\right] - \left(4 + \tfrac{2}{5}\right) = 8 + \tfrac{64}{5} - 4 - \tfrac{2}{5} = \tfrac{82}{5}$

38. $\int_{-1}^2 (3u-2)(u+1)\,du = \int_{-1}^2 (3u^2+u-2)\,du = \left[u^3 + \tfrac{1}{2}u^2 - 2u\right]_{-1}^2 = (8+2-4) - \left(-1+\tfrac{1}{2}+2\right) = 6 - \tfrac{3}{2} = \tfrac{9}{2}$

39. $\displaystyle\int_1^2 \frac{s^4+1}{s^2}\,ds = \int_1^2 (s^2 + s^{-2})\,ds = \left[\frac{1}{3}s^3 - \frac{1}{s}\right]_1^2 = \left(\frac{8}{3} - \frac{1}{2}\right) - \left(\frac{1}{3} - 1\right) = \frac{7}{3} + \frac{1}{2} = \frac{17}{6}$

40. $\int_5^5 \sqrt{t^2 + \sin t}\,dt = F(5) - F(5)$ [where F is any antiderivative of $\sqrt{t^2 - 5t}$] $= 0$

41. $\displaystyle\int_0^{\pi/3} \sec\theta\,\tan\theta\,d\theta = \Big[\sec\theta\Big]_0^{\pi/3} = \sec\frac{\pi}{3} - \sec 0 = 2 - 1 = 1$

42. $\displaystyle\int_{\pi/4}^{\pi/3} \csc^2\theta\,d\theta = \Big[-\cot\theta\Big]_{\pi/4}^{\pi/3} = \left(-\cot\frac{\pi}{3}\right) - \left(-\cot\frac{\pi}{4}\right) = -\frac{1}{\sqrt{3}} - (-1) = 1 - \frac{1}{\sqrt{3}}$

43. $\int_0^1 (1+r)^3\,dr = \int_0^1 (1 + 3r + 3r^2 + r^3)\,dr = \left[r + \tfrac{3}{2}r^2 + r^3 + \tfrac{1}{4}r^4\right]_0^1 = \left(1 + \tfrac{3}{2} + 1 + \tfrac{1}{4}\right) - 0 = \tfrac{15}{4}$

44. $\displaystyle\int_1^{18} \sqrt{\frac{3}{z}}\,dz = \int_1^{18} \sqrt{3}z^{-1/2}\,dz = \sqrt{3}\left[2z^{1/2}\right]_1^{18} = 2\sqrt{3}(18^{1/2} - 1^{1/2}) = 2\sqrt{3}(3\sqrt{2} - 1)$

45. If $f(x) = \begin{cases} \sin x & \text{if } 0 \le x < \pi/2 \\ \cos x & \text{if } \pi/2 \le x \le \pi \end{cases}$ then

$\int_0^\pi f(x)\,dx = \int_0^{\pi/2} \sin x\,dx + \int_{\pi/2}^\pi \cos x\,dx = \big[-\cos x\big]_0^{\pi/2} + \big[\sin x\big]_{\pi/2}^\pi = -\cos\tfrac{\pi}{2} + \cos 0 + \sin\pi - \sin\tfrac{\pi}{2}$

$= -0 + 1 + 0 - 1 = 0$

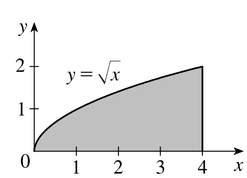
Note that f is integrable by Theorem 3 in Section 4.2.

46. If $f(x) = \begin{cases} 2 & \text{if } -2 \le x \le 0 \\ 4 - x^2 & \text{if } 0 < x \le 2 \end{cases}$ then

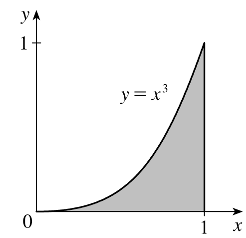
$\int_{-2}^2 f(x)\,dx = \int_{-2}^0 2\,dx + \int_0^2 (4 - x^2)\,dx = \big[2x\big]_{-2}^0 + \left[4x - \tfrac{1}{3}x^3\right]_0^2 = [0 - (-4)] + \left(\tfrac{16}{3} - 0\right) = \tfrac{28}{3}$

Note that f is integrable by Theorem 3 in Section 4.2.

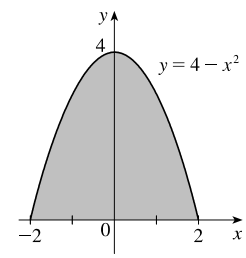
47. Area $\displaystyle= \int_0^4 \sqrt{x}\,dx = \int_0^4 x^{1/2}\,dx = \left[\tfrac{2}{3}x^{3/2}\right]_0^4 = \tfrac{2}{3}(8) - 0 = \tfrac{16}{3}$



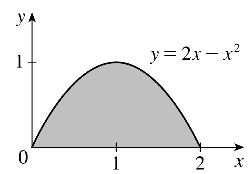
48. Area $= \int_0^1 x^3\,dx = \left[\frac{1}{4}x^4\right]_0^1 = \frac{1}{4} - 0 = \frac{1}{4}$



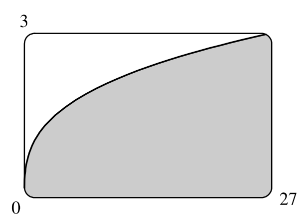
49. Area $= \int_{-2}^2 (4 - x^2)\,dx = \left[4x - \frac{1}{3}x^3\right]_{-2}^2 = \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right) = \frac{32}{3}$



50. Area $= \int_0^2 (2x - x^2)\,dx = \left[x^2 - \frac{1}{3}x^3\right]_0^2 = \left(4 - \frac{8}{3}\right) - 0 = \frac{4}{3}$

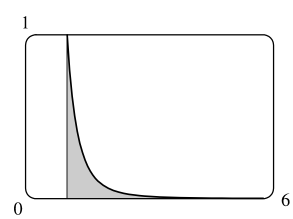


51. From the graph, it appears that the area is about 60. The actual area is $\int_0^{27} x^{1/3}dx = \left[\frac{3}{4}x^{4/3}\right]_0^{27} = \frac{3}{4} \cdot 81 - 0 = \frac{243}{4} = 60.75$. This is $\frac{3}{4}$ of the area of the viewing rectangle.

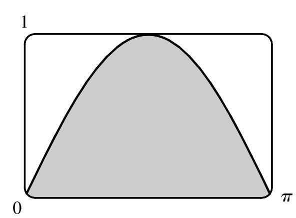


52. From the graph, it appears that the area is about $\frac{1}{3}$. The actual area is

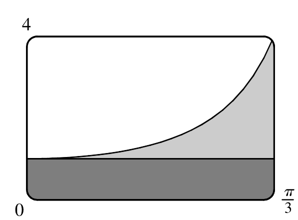
$$\int_1^6 x^{-4}\,dx = \left[\frac{x^{-3}}{-3}\right]_1^6 = \left[\frac{-1}{3x^3}\right]_1^6 = -\frac{1}{3 \cdot 216} + \frac{1}{3} = \frac{215}{648} \approx 0.3318.$$



53. It appears that the area under the graph is about $\frac{2}{3}$ of the area of the viewing rectangle, or about $\frac{2}{3}\pi \approx 2.1$. The actual area is $\int_0^\pi \sin x\,dx = [-\cos x]_0^\pi = (-\cos\pi) - (-\cos 0) = -(-1) + 1 = 2.$



54. Splitting up the region as shown, we estimate that the area under the graph is $\frac{\pi}{3} + \frac{1}{4}\left(3 \cdot \frac{\pi}{3}\right) \approx 1.8$. The actual area is $\int_0^{\pi/3} \sec^2 x\,dx = \left[\tan x\right]_0^{\pi/3} = \sqrt{3} - 0 = \sqrt{3} \approx 1.73.$



55. $f(x) = x^{-4}$ is not continuous on the interval $[-2, 1]$, so FTC2 cannot be applied. In fact, f has an infinite discontinuity at $x = 0$, so $\int_{-2}^{1} x^{-4}\, dx$ does not exist.

56. $f(x) = \dfrac{4}{x^3}$ is not continuous on the interval $[-1, 2]$, so FTC2 cannot be applied. In fact, f has an infinite discontinuity at $x = 0$, so $\displaystyle\int_{-1}^{2} \frac{4}{x^3}\, dx$ does not exist.

57. $f(\theta) = \sec\theta\, \tan\theta$ is not continuous on the interval $[\pi/3, \pi]$, so FTC2 cannot be applied. In fact, f has an infinite discontinuity at $x = \pi/2$, so $\int_{\pi/3}^{\pi} \sec\theta\, \tan\theta\, d\theta$ does not exist.

58. $f(x) = \sec^2 x$ is not continuous on the interval $[0, \pi]$, so FTC2 cannot be applied. In fact, f has an infinite discontinuity at $x = \pi/2$, so $\int_0^{\pi} \sec^2 x\, dx$ does not exist.

59. $$g(x) = \int_{2x}^{3x} \frac{u^2-1}{u^2+1}\, du = \int_{2x}^{0} \frac{u^2-1}{u^2+1}\, du + \int_{0}^{3x} \frac{u^2-1}{u^2+1}\, du = -\int_{0}^{2x} \frac{u^2-1}{u^2+1}\, du + \int_{0}^{3x} \frac{u^2-1}{u^2+1}\, du \quad\Rightarrow$$

$$g'(x) = -\frac{(2x)^2-1}{(2x)^2+1}\cdot\frac{d}{dx}(2x) + \frac{(3x)^2-1}{(3x)^2+1}\cdot\frac{d}{dx}(3x) = -2\cdot\frac{4x^2-1}{4x^2+1} + 3\cdot\frac{9x^2-1}{9x^2+1}$$

60. $$g(x) = \int_{1-2x}^{1+2x} t\sin t\, dt = \int_{1-2x}^{0} t\sin t\, dt + \int_{0}^{1+2x} t\sin t\, dt = -\int_{0}^{1-2x} t\sin t\, dt + \int_{0}^{1+2x} t\sin t\, dt \quad\Rightarrow$$

$$\begin{aligned} g'(x) &= -(1-2x)\sin(1-2x)\cdot\frac{d}{dx}(1-2x) + (1+2x)\sin(1+2x)\cdot\frac{d}{dx}(1+2x) \\ &= 2(1-2x)\sin(1-2x) + 2(1+2x)\sin(1+2x) \end{aligned}$$

61. $$h(x) = \int_{\sqrt{x}}^{x^3} \cos(t^2)\, dt = \int_{\sqrt{x}}^{0} \cos(t^2)\, dt + \int_{0}^{x^3} \cos(t^2)\, dt = -\int_{0}^{\sqrt{x}} \cos(t^2)\, dt + \int_{0}^{x^3} \cos(t^2)\, dt \quad\Rightarrow$$

$$h'(x) = -\cos\!\left((\sqrt{x}\,)^2\right)\cdot\frac{d}{dx}(\sqrt{x}\,) + [\cos(x^3)^2]\cdot\frac{d}{dx}(x^3) = -\frac{1}{2\sqrt{x}}\cos x + 3x^2\cos(x^6)$$

62. $$g(x) = \int_{\tan x}^{x^2} \frac{1}{\sqrt{2+t^4}}\, dt = \int_{\tan x}^{1} \frac{dt}{\sqrt{2+t^4}} + \int_{1}^{x^2} \frac{dt}{\sqrt{2+t^4}} = -\int_{1}^{\tan x} \frac{dt}{\sqrt{2+t^4}} + \int_{1}^{x^2} \frac{dt}{\sqrt{2+t^4}} \quad\Rightarrow$$

$$g'(x) = \frac{-1}{\sqrt{2+\tan^4 x}}\,\frac{d}{dx}(\tan x) + \frac{1}{\sqrt{2+x^8}}\,\frac{d}{dx}(x^2) = -\frac{\sec^2 x}{\sqrt{2+\tan^4 x}} + \frac{2x}{\sqrt{2+x^8}}$$

63. $F(x) = \displaystyle\int_{\pi}^{x} \frac{\cos t}{t}\, dt \quad\Rightarrow\quad F'(x) = \frac{\cos x}{x}$, so the slope at $x = \pi$ is $\dfrac{\cos\pi}{\pi} = -\dfrac{1}{\pi}$. The y-coordinate of the point on F at $x = \pi$ is $F(\pi) = \displaystyle\int_{\pi}^{\pi} \frac{\cos t}{t}\, dt = 0$ since the limits are equal. An equation of the tangent line is $y - 0 = -\dfrac{1}{\pi}(x - \pi)$, or $y = -\dfrac{1}{\pi}x + 1$.

64. $f(x) = \int_0^x (1-t^2)\cos^2 t\, dt$ is increasing when $f'(x) = (1-x^2)\cos^2 x$ is positive. Since $\cos^2 x \geq 0$, $f'(x) > 0 \quad\Rightarrow$ $1 - x^2 > 0 \quad\Leftrightarrow\quad |x| < 1$, so f is increasing on $(-1, 1)$.

Note: The zeros of $\cos x$ do not affect the intervals of increase; that is, if $f'(x) = (9 - x^2)\cos^2 x$, then f is increasing on $(-3, 3)$, even though $f'(x) = 0$ when $x = \pm\frac{\pi}{2}$.

65. $y = \int_0^x \frac{t^2}{t^2+t+2}\,dt \quad \Rightarrow \quad y' = \frac{x^2}{x^2+x+2} \quad \Rightarrow$

$$y'' = \frac{(x^2+x+2)(2x) - x^2(2x+1)}{(x^2+x+2)^2} = \frac{2x^3+2x^2+4x-2x^3-x^2}{(x^2+x+2)^2} = \frac{x^2+4x}{(x^2+x+2)^2} = \frac{x(x+4)}{(x^2+x+2)^2}.$$

The curve y is concave downward when $y'' < 0$; that is, on the interval $(-4, 0)$.

66. If $F(x) = \int_1^x f(t)\,dt$, then by FTC1, $F'(x) = f(x)$, and also, $F''(x) = f'(x)$. F is concave downward where F'' is negative; that is, where f' is negative. The given graph shows that f is decreasing ($f' < 0$) on the interval $(-1, 1)$.

67. By FTC2, $\int_1^4 f'(x)\,dx = f(4) - f(1)$, so $17 = f(4) - 12 \quad \Rightarrow \quad f(4) = 17 + 12 = 29$.

68. $g(y) = \int_3^y f(x)\,dx \quad \Rightarrow \quad g'(y) = f(y)$. Since $f(x) = \int_0^{\sin x} \sqrt{1+t^2}\,dt$, $g''(y) = f'(y) = \sqrt{1+\sin^2 y} \cdot \cos y$,

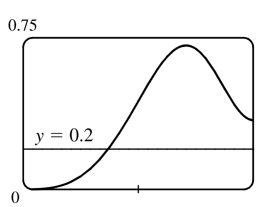
so $g''(\frac{\pi}{6}) = \sqrt{1+\sin^2(\frac{\pi}{6})} \cdot \cos\frac{\pi}{6} = \sqrt{1+(\frac{1}{2})^2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{5}}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{15}}{4}$.

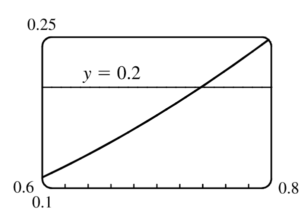
69. (a) The Fresnel function $S(x) = \int_0^x \sin\left(\frac{\pi}{2}t^2\right)dt$ has local maximum values where $0 = S'(x) = \sin\left(\frac{\pi}{2}t^2\right)$ and S' changes from positive to negative. For $x > 0$, this happens when $\frac{\pi}{2}x^2 = (2n-1)\pi$ [odd multiples of π] $\Leftrightarrow$ $x^2 = 2(2n-1) \quad \Leftrightarrow \quad x = \sqrt{4n-2}$, n any positive integer. For $x < 0$, S' changes from positive to negative where $\frac{\pi}{2}x^2 = 2n\pi$ [even multiples of π] $\Leftrightarrow \quad x^2 = 4n \quad \Leftrightarrow \quad x = -2\sqrt{n}$. S' does not change sign at $x = 0$.

(b) S is concave upward on those intervals where $S''(x) > 0$. Differentiating our expression for $S'(x)$, we get $S''(x) = \cos\left(\frac{\pi}{2}x^2\right)\left(2\frac{\pi}{2}x\right) = \pi x\cos\left(\frac{\pi}{2}x^2\right)$. For $x > 0$, $S''(x) > 0$ where $\cos(\frac{\pi}{2}x^2) > 0 \quad \Leftrightarrow \quad 0 < \frac{\pi}{2}x^2 < \frac{\pi}{2}$ or $\left(2n-\frac{1}{2}\right)\pi < \frac{\pi}{2}x^2 < \left(2n+\frac{1}{2}\right)\pi$, n any integer $\Leftrightarrow \quad 0 < x < 1$ or $\sqrt{4n-1} < x < \sqrt{4n+1}$, n any positive integer.

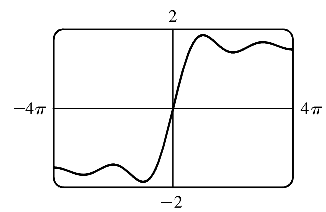
For $x < 0$, $S''(x) > 0$ where $\cos(\frac{\pi}{2}x^2) < 0 \quad \Leftrightarrow \quad \left(2n-\frac{3}{2}\right)\pi < \frac{\pi}{2}x^2 < \left(2n-\frac{1}{2}\right)\pi$, n any integer $\Leftrightarrow$ $4n-3 < x^2 < 4n-1 \quad \Leftrightarrow \quad \sqrt{4n-3} < |x| < \sqrt{4n-1} \quad \Rightarrow \quad \sqrt{4n-3} < -x < \sqrt{4n-1} \quad \Rightarrow$ $-\sqrt{4n-3} > x > -\sqrt{4n-1}$, so the intervals of upward concavity for $x < 0$ are $\left(-\sqrt{4n-1}, -\sqrt{4n-3}\right)$, n any positive integer. To summarize: S is concave upward on the intervals $(0,1)$, $\left(-\sqrt{3},-1\right)$, $\left(\sqrt{3},\sqrt{5}\right)$, $\left(-\sqrt{7},-\sqrt{5}\right)$, $\left(\sqrt{7},3\right), \ldots$.

(c) In Maple, we use `plot({int(sin(Pi*t^2/2),t=0..x),0.2},x=0..2);`. Note that Maple recognizes the Fresnel function, calling it `FresnelS(x)`. In Mathematica, we use `Plot[{Integrate[Sin[Pi*t^2/2],{t,0,x}],0.2},{x,0,2}]`. In Derive, we load the utility file `FRESNEL` and plot `FRESNEL_SIN(x)`. From the graphs, we see that $\int_0^x \sin\left(\frac{\pi}{2}t^2\right)dt = 0.2$ at $x \approx 0.74$.





70. (a) In Maple, we should start by setting `si:=int(sin(t)/t,t=0..x);`. In Mathematica, the command is `si=Integrate[Sin[t]/t,{t,0,x}]`. Note that both systems recognize this function; Maple calls it `Si(x)` and Mathematica calls it `SinIntegral[x]`. In Maple, the command to generate the graph is `plot(si,x=-4*Pi..4*Pi);`. In Mathematica, it is `Plot[si,{x,-4*Pi,4*Pi}]`. In Derive, we load the utility file `EXP_INT` and plot `SI(x)`.



(b) $\mathrm{Si}(x)$ has local maximum values where $\mathrm{Si}'(x)$ changes from positive to negative, passing through 0. From the Fundamental Theorem we know that $\mathrm{Si}'(x) = \dfrac{d}{dx}\displaystyle\int_0^x \frac{\sin t}{t}\,dt = \frac{\sin x}{x}$, so we must have $\sin x = 0$ for a maximum, and for $x > 0$ we must have $x = (2n-1)\pi$, n any positive integer, for Si' to be changing from positive to negative at x. For $x < 0$, we must have $x = 2n\pi$, n any positive integer, for a maximum, since the denominator of $\mathrm{Si}'(x)$ is negative for $x < 0$. Thus, the local maxima occur at $x = \pi, -2\pi, 3\pi, -4\pi, 5\pi, -6\pi, \ldots$.

(c) To find the first inflection point, we solve $\mathrm{Si}''(x) = \dfrac{\cos x}{x} - \dfrac{\sin x}{x^2} = 0$. We can see from the graph that the first inflection point lies somewhere between $x = 3$ and $x = 5$. Using a rootfinder gives the value $x \approx 4.4934$. To find the y-coordinate of the inflection point, we evaluate $\mathrm{Si}(4.4934) \approx 1.6556$. So the coordinates of the first inflection point to the right of the origin are about $(4.4934, 1.6556)$. Alternatively, we could graph $S''(x)$ and estimate the first positive x-value at which it changes sign.

(d) It seems from the graph that the function has horizontal asymptotes at $y \approx 1.5$, with $\lim\limits_{x\to\pm\infty} \mathrm{Si}(x) \approx \pm 1.5$ respectively. Using the limit command, we get $\lim\limits_{x\to\infty} \mathrm{Si}(x) = \frac{\pi}{2}$. Since $\mathrm{Si}(x)$ is an odd function, $\lim\limits_{x\to-\infty} \mathrm{Si}(x) = -\frac{\pi}{2}$. So $\mathrm{Si}(x)$ has the horizontal asymptotes $y = \pm\frac{\pi}{2}$.

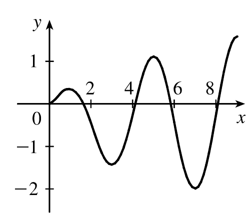
(e) We use the `fsolve` command in Maple (or `FindRoot` in Mathematica) to find that the solution is $x \approx 1.1$. Or, as in Exercise 69(c), we graph $y = \mathrm{Si}(x)$ and $y = 1$ on the same screen to see where they intersect.

71. (a) By FTC1, $g'(x) = f(x)$. So $g'(x) = f(x) = 0$ at $x = 1, 3, 5, 7$, and 9. g has local maxima at $x = 1$ and 5 (since $f = g'$ changes from positive to negative there) and local minima at $x = 3$ and 7. There is no local maximum or minimum at $x = 9$, since f is not defined for $x > 9$.

(b) We can see from the graph that $\left|\int_0^1 f\,dt\right| < \left|\int_1^3 f\,dt\right| < \left|\int_3^5 f\,dt\right| < \left|\int_5^7 f\,dt\right| < \left|\int_7^9 f\,dt\right|$. So $g(1) = \left|\int_0^1 f\,dt\right|$, $g(5) = \int_0^5 f\,dt = g(1) - \left|\int_1^3 f\,dt\right| + \left|\int_3^5 f\,dt\right|$, and $g(9) = \int_0^9 f\,dt = g(5) - \left|\int_5^7 f\,dt\right| + \left|\int_7^9 f\,dt\right|$. Thus, $g(1) < g(5) < g(9)$, and so the absolute maximum of $g(x)$ occurs at $x = 9$.

(c) g is concave downward on those intervals where $g'' < 0$. But $g'(x) = f(x)$, so $g''(x) = f'(x)$, which is negative on (approximately) $\left(\frac{1}{2}, 2\right)$, $(4, 6)$ and $(8, 9)$. So g is concave downward on these intervals.

(d)

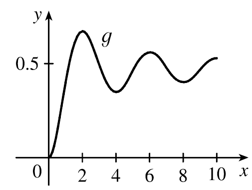


72. (a) By FTC1, $g'(x) = f(x)$. So $g'(x) = f(x) = 0$ at $x = 2$, 4, 6, 8, and 10. g has local maxima at $x = 2$ and 6 (since $f = g'$ changes from positive to negative there) and local minima at $x = 4$ and 8. There is no local maximum or minimum at $x = 10$, since f is not defined for $x > 10$.

(b) We can see from the graph that $\left|\int_0^2 f\,dt\right| > \left|\int_2^4 f\,dt\right| > \left|\int_4^6 f\,dt\right| > \left|\int_6^8 f\,dt\right| > \left|\int_8^{10} f\,dt\right|$. So $g(2) = \left|\int_0^2 f\,dt\right|$, $g(6) = \int_0^6 f\,dt = g(2) - \left|\int_2^4 f\,dt\right| + \left|\int_4^6 f\,dt\right|$, and $g(10) = \int_0^{10} f\,dt = g(6) - \left|\int_6^8 f\,dt\right| + \left|\int_8^{10} f\,dt\right|$. Thus, $g(2) > g(6) > g(10)$, and so the absolute maximum of $g(x)$ occurs at $x = 2$.

(c) g is concave downward on those intervals where $g'' < 0$. But $g'(x) = f(x)$, so $g''(x) = f'(x)$, which is negative on $(1, 3)$, $(5, 7)$ and $(9, 10)$. So g is concave downward on these intervals.

(d)



73. $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \left(\dfrac{i^4}{n^5} + \dfrac{i}{n^2}\right) = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \left(\dfrac{i^4}{n^4} + \dfrac{i}{n}\right)\dfrac{1}{n} = \lim\limits_{n\to\infty} \dfrac{1-0}{n} \sum\limits_{i=1}^{n} \left[\left(\dfrac{i}{n}\right)^4 + \dfrac{i}{n}\right] = \displaystyle\int_0^1 (x^4 + x)\,dx$

$= \left[\frac{1}{5}x^5 + \frac{1}{2}x^2\right]_0^1 = \left(\frac{1}{5} + \frac{1}{2}\right) - 0 = \frac{7}{10}$

74. $\lim\limits_{n\to\infty} \dfrac{1}{n}\left(\sqrt{\dfrac{1}{n}} + \sqrt{\dfrac{2}{n}} + \cdots + \sqrt{\dfrac{n}{n}}\right) = \lim\limits_{n\to\infty} \dfrac{1-0}{n} \sum\limits_{i=1}^{n} \sqrt{\dfrac{i}{n}} = \displaystyle\int_0^1 \sqrt{x}\,dx = \left[\dfrac{2x^{3/2}}{3}\right]_0^1 = \dfrac{2}{3} - 0 = \dfrac{2}{3}$

75. Suppose $h < 0$. Since f is continuous on $[x + h, x]$, the Extreme Value Theorem says that there are numbers u and v in $[x + h, x]$ such that $f(u) = m$ and $f(v) = M$, where m and M are the absolute minimum and maximum values of f on $[x + h, x]$. By Integral Property 4.2.8, $m(-h) \le \int_{x+h}^{x} f(t)\,dt \le M(-h)$; that is, $f(u)(-h) \le -\int_x^{x+h} f(t)\,dt \le f(v)(-h)$.

Since $-h > 0$, we can divide this inequality by $-h$: $f(u) \le \dfrac{1}{h}\displaystyle\int_x^{x+h} f(t)\,dt \le f(v)$. By Equation 2,

$\dfrac{g(x+h) - g(x)}{h} = \dfrac{1}{h}\displaystyle\int_x^{x+h} f(t)\,dt$ for $h \ne 0$, and hence $f(u) \le \dfrac{g(x+h) - g(x)}{h} \le f(v)$, which is Equation 3 in the case where $h < 0$.

76. $\dfrac{d}{dx}\displaystyle\int_{g(x)}^{h(x)} f(t)\,dt = \dfrac{d}{dx}\left[\int_{g(x)}^{0} f(t)\,dt + \int_0^{h(x)} f(t)\,dt\right] = \dfrac{d}{dx}\int_0^{h(x)} f(t)\,dt - \dfrac{d}{dx}\int_0^{g(x)} f(t)\,dt.$

Using the Chain Rule in conjunction with FTC1 twice gives us $f(h(x))\,h'(x) - f(g(x))\,g'(x)$.

77. (a) Let $f(x) = \sqrt{x} \;\Rightarrow\; f'(x) = 1/(2\sqrt{x}\,) > 0$ for $x > 0 \;\Rightarrow\; f$ is increasing on $(0, \infty)$. If $x \ge 0$, then $x^3 \ge 0$, so $1 + x^3 \ge 1$ and since f is increasing, this means that $f(1 + x^3) \ge f(1) \;\Rightarrow\; \sqrt{1 + x^3} \ge 1$ for $x \ge 0$. Next let $g(t) = t^2 - t \;\Rightarrow\; g'(t) = 2t - 1 \;\Rightarrow\; g'(t) > 0$ when $t \ge 1$. Thus, g is increasing on $(1, \infty)$. And since $g(1) = 0$, $g(t) \ge 0$ when $t \ge 1$. Now let $t = \sqrt{1 + x^3}$, where $x \ge 0$. $\sqrt{1 + x^3} \ge 1$ (from above) $\;\Rightarrow\; t \ge 1 \;\Rightarrow\; g(t) \ge 0 \;\Rightarrow$ $(1 + x^3) - \sqrt{1 + x^3} \ge 0$ for $x \ge 0$. Therefore, $1 \le \sqrt{1 + x^3} \le 1 + x^3$ for $x \ge 0$.

(b) From part (a) and Integral Property 4.2.7: $\int_0^1 1\,dx \le \int_0^1 \sqrt{1+x^3}\,dx \le \int_0^1 (1+x^3)\,dx \quad\Leftrightarrow$

$\big[x\big]_0^1 \le \int_0^1 \sqrt{1+x^3}\,dx \le \big[x+\frac14 x^4\big]_0^1 \quad\Leftrightarrow\quad 1 \le \int_0^1 \sqrt{1+x^3}\,dx \le 1+\frac14 = 1.25.$

78. (a) For $0 \le x \le 1$, we have $x^2 \le x$. Since $f(x) = \cos x$ is a decreasing function on $[0,1]$, $\cos(x^2) \ge \cos x$.

(b) $\pi/6 < 1$, so by part (a), $\cos(x^2) \ge \cos x$ on $[0, \pi/6]$. Thus,

$\int_0^{\pi/6} \cos(x^2)\,dx \ge \int_0^{\pi/6} \cos x\,dx = \big[\sin x\big]_0^{\pi/6} = \sin(\pi/6) - \sin 0 = \frac12 - 0 = \frac12.$

79. $0 < \dfrac{x^2}{x^4+x^2+1} < \dfrac{x^2}{x^4} = \dfrac{1}{x^2}$ on $[5,10]$, so

$$0 \le \int_5^{10} \frac{x^2}{x^4+x^2+1}\,dx < \int_5^{10} \frac{1}{x^2}\,dx = \left[-\frac1x\right]_5^{10} = -\frac{1}{10} - \left(-\frac15\right) = \frac{1}{10} = 0.1.$$

80. (a) If $x < 0$, then $g(x) = \int_0^x f(t)\,dt = \int_0^x 0\,dt = 0$.

If $0 \le x \le 1$, then $g(x) = \int_0^x f(t)\,dt = \int_0^x t\,dt = \big[\frac12 t^2\big]_0^x = \frac12 x^2$.

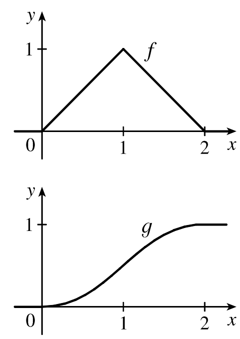
If $1 < x \le 2$, then

$$g(x) = \int_0^x f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^x f(t)\,dt = g(1) + \int_1^x (2-t)\,dt$$
$$= \tfrac12(1)^2 + \big[2t - \tfrac12 t^2\big]_1^x = \tfrac12 + \big(2x - \tfrac12 x^2\big) - \big(2 - \tfrac12\big) = 2x - \tfrac12 x^2 - 1.$$

If $x > 2$, then $g(x) = \int_0^x f(t)\,dt = g(2) + \int_2^x 0\,dt = 1 + 0 = 1$. So

$$g(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac12 x^2 & \text{if } 0 \le x \le 1 \\ 2x - \frac12 x^2 - 1 & \text{if } 1 < x \le 2 \\ 1 & \text{if } x > 2 \end{cases}$$

(b)



(c) f is not differentiable at its corners at $x = 0$, 1, and 2. f is differentiable on $(-\infty, 0)$, $(0,1)$, $(1,2)$ and $(2,\infty)$.

g is differentiable on $(-\infty, \infty)$.

81. Using FTC1, we differentiate both sides of $6 + \displaystyle\int_a^x \frac{f(t)}{t^2}\,dt = 2\sqrt{x}$ to get $\dfrac{f(x)}{x^2} = 2\dfrac{1}{2\sqrt{x}} \quad\Rightarrow\quad f(x) = x^{3/2}$.

To find a, we substitute $x = a$ in the original equation to obtain $6 + \displaystyle\int_a^a \frac{f(t)}{t^2}\,dt = 2\sqrt{a} \quad\Rightarrow\quad 6 + 0 = 2\sqrt{a} \quad\Rightarrow$

$3 = \sqrt{a} \quad\Rightarrow\quad a = 9.$

82. Note that $h'(u)$ is an antiderivative of $h''(u)$, so by FTC2, $\int_1^2 h''(u)\,du = \int_1^2 (h')'(u)\,du = h'(2) - h'(1) = 5 - 2 = 3$.

The other information is unnecessary.

83. (a) Let $F(t) = \int_0^t f(s)\,ds$. Then, by FTC1, $F'(t) = f(t) =$ rate of depreciation, so $F(t)$ represents the loss in value over the interval $[0,t]$.

(b) $C(t) = \dfrac1t\left[A + \displaystyle\int_0^t f(s)\,ds\right] = \dfrac{A + F(t)}{t}$ represents the average expenditure per unit of t during the interval $[0,t]$, assuming that there has been only one overhaul during that time period. The company wants to minimize average expenditure.

(c) $C(t) = \dfrac{1}{t}\left[A + \displaystyle\int_0^t f(s)\,ds\right]$. Using FTC1, we have $C'(t) = -\dfrac{1}{t^2}\left[A + \displaystyle\int_0^t f(s)\,ds\right] + \dfrac{1}{t}f(t)$.

$$C'(t) = 0 \quad\Rightarrow\quad t\,f(t) = A + \int_0^t f(s)\,ds \quad\Rightarrow\quad f(t) = \frac{1}{t}\left[A + \int_0^t f(s)\,ds\right] = C(t).$$

84. $\displaystyle\int_0^3 (2\sin x - e^x)\,dx = \Big[-2\cos x - e^x\Big]_0^3 = (-2\cos 3 - e^3) - (-2 - 1) = 3 - 2\cos 3 - e^3$

85. $\displaystyle\int_1^9 \frac{1}{2x}\,dx = \frac{1}{2}\int_1^9 \frac{1}{x}\,dx = \tfrac{1}{2}\Big[\ln|x|\Big]_1^9 = \tfrac{1}{2}(\ln 9 - \ln 1) = \tfrac{1}{2}\ln 9 - 0 = \ln 9^{1/2} = \ln 3$

86. $\displaystyle\int_0^1 10^x\,dx = \left[\frac{10^x}{\ln 10}\right]_0^1 = \frac{10}{\ln 10} - \frac{1}{\ln 10} = \frac{9}{\ln 10}$

87. $\displaystyle\int_{1/2}^{1/\sqrt{2}} \frac{4}{\sqrt{1-x^2}}\,dx = \Big[4\arcsin x\Big]_{1/2}^{1/\sqrt{2}} = 4\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = 4\left(\frac{\pi}{12}\right) = \frac{\pi}{3}$

88. $\displaystyle\int_0^1 \frac{4}{t^2+1}\,dt = 4\int_0^1 \frac{1}{1+t^2}\,dt = 4\big[\tan^{-1} t\big]_0^1 = 4\big(\tan^{-1} 1 - \tan^{-1} 0\big) = 4\big(\tfrac{\pi}{4} - 0\big) = \pi$

89. $\int_{-1}^1 e^{u+1}\,du = \big[e^{u+1}\big]_{-1}^1 = e^2 - e^0 = e^2 - 1$ [or start with $e^{u+1} = e^u e^1$]

90. $\displaystyle\int_1^3 \frac{y^3 - 2y^2 - y}{y^2}\,dy = \int_1^3 \left(y - 2 - \frac{1}{y}\right)dy = \big[\tfrac{1}{2}y^2 - 2y - \ln|y|\big]_1^3 = \big(\tfrac{9}{2} - 6 - \ln 3\big) - \big(\tfrac{1}{2} - 2 - 0\big) = -\ln 3$

4.4 Indefinite Integrals and the Net Change Theorem

1. $\displaystyle\frac{d}{dx}\left(\frac{1}{2}x + \frac{1}{4}\sin 2x + C\right) = \frac{1}{2} + \frac{1}{4}\cos 2x \cdot 2 + 0 = \frac{1}{2} + \frac{1}{2}\cos 2x$

$$= \tfrac{1}{2} + \tfrac{1}{2}\big(2\cos^2 x - 1\big) = \tfrac{1}{2} + \cos^2 x - \tfrac{1}{2} = \cos^2 x$$

2. $\displaystyle\frac{d}{dx}(\tan x - x + C) = \sec^2 x - 1 + 0 = \tan^2 x$

3. $\displaystyle\frac{d}{dx}\left[-\frac{\sqrt{1+x^2}}{x} + C\right] = \frac{d}{dx}\left[-\frac{(1+x^2)^{1/2}}{x} + C\right] = -\frac{x \cdot \frac{1}{2}(1+x^2)^{-1/2}(2x) - (1+x^2)^{1/2} \cdot 1}{(x)^2} + 0$

$$= -\frac{(1+x^2)^{-1/2}\left[x^2 - (1+x^2)\right]}{x^2} = -\frac{-1}{(1+x^2)^{1/2}x^2} = \frac{1}{x^2\sqrt{1+x^2}}$$

4. $\displaystyle\frac{d}{dx}\left[\frac{2}{15b^2}(3bx - 2a)(a+bx)^{3/2} + C\right] = \frac{2}{15b^2}\left[(3bx - 2a)\tfrac{3}{2}(a+bx)^{1/2}(b) + (a+bx)^{3/2}(3b) + 0\right]$

$$= \frac{2}{15b^2}(3b)(a+bx)^{1/2}\left[(3bx - 2a)\tfrac{1}{2} + (a+bx)\right]$$

$$= \frac{2}{5b}(a+bx)^{1/2}\left(\frac{5}{2}bx\right) = x\sqrt{a+bx}$$

5. $\int(3x^2 + 4x + 1)\,dx = 3 \cdot \tfrac{1}{3}x^3 + 4 \cdot \tfrac{1}{2}x^2 + x + C = x^3 + 2x^2 + x + C$

6. $\int(5 + 2\sqrt{x}\,)\,dx = \int(5 + 2x^{1/2})\,dx = 5x + 2 \cdot \tfrac{2}{3}x^{3/2} + C = 5x + \tfrac{4}{3}x^{3/2} + C$

7. $\int (x + \cos x)\, dx = \frac{1}{2}x^2 + \sin x + C$

8. $\displaystyle\int \left(\sqrt[3]{x} + \frac{1}{\sqrt[3]{x}} \right) dx = \int (x^{1/3} + x^{-1/3})\, dx = \frac{3}{4}x^{4/3} + \frac{3}{2}x^{2/3} + C$

9. $\displaystyle\int (x^{1.3} + 7x^{2.5})\, dx = \frac{1}{2.3}x^{2.3} + \frac{7}{3.5}x^{3.5} + C = \frac{1}{2.3}x^{2.3} + 2x^{3.5} + C$

10. $\int \sqrt[4]{x^5}\, dx = \int x^{5/4}\, dx = \frac{4}{9}x^{9/4} + C$

11. $\int \left(5 + \frac{2}{3}x^2 + \frac{3}{4}x^3\right) dx = 5x + \frac{2}{3} \cdot \frac{1}{3}x^3 + \frac{3}{4} \cdot \frac{1}{4}x^4 + C = 5x + \frac{2}{9}x^3 + \frac{3}{16}x^4 + C$

12. $\int (u^6 - 2u^5 - u^3 + \frac{2}{7})\, du = \frac{1}{7}u^7 - 2 \cdot \frac{1}{6}u^6 - \frac{1}{4}u^4 + \frac{2}{7}u + C = \frac{1}{7}u^7 - \frac{1}{3}u^6 - \frac{1}{4}u^4 + \frac{2}{7}u + C$

13. $\displaystyle\int (u + 4)(2u + 1)\, du = \int (2u^2 + 9u + 4)\, du = 2\frac{u^3}{3} + 9\frac{u^2}{2} + 4u + C = \frac{2}{3}u^3 + \frac{9}{2}u^2 + 4u + C$

14. $\int \sqrt{t}\,(t^2 + 3t + 2)\, dt = \int t^{1/2}(t^2 + 3t + 2)\, dt = \int (t^{5/2} + 3t^{3/2} + 2t^{1/2})\, dt$

$= \frac{2}{7}t^{7/2} + 3 \cdot \frac{2}{5}t^{5/2} + 2 \cdot \frac{2}{3}t^{3/2} + C = \frac{2}{7}t^{7/2} + \frac{6}{5}t^{5/2} + \frac{4}{3}t^{3/2} + C$

15. $\displaystyle\int \frac{1 + \sqrt{x} + x}{\sqrt{x}}\, dx = \int \left(\frac{1}{\sqrt{x}} + 1 + \sqrt{x} \right) dx = \int (x^{-1/2} + 1 + x^{1/2})\, dx$

$= 2x^{1/2} + x + \frac{2}{3}x^{3/2} + C = 2\sqrt{x} + x + \frac{2}{3}x^{3/2} + C$

16. $\displaystyle\int \left(u^2 + 1 + \frac{1}{u^2} \right) du = \int (u^2 + 1 + u^{-2})\, du = \frac{u^3}{3} + u + \frac{u^{-1}}{-1} + C = \frac{1}{3}u^3 + u - \frac{1}{u} + C$

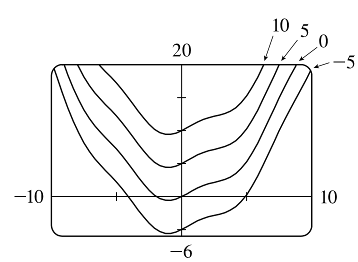
17. $\int (2 + \tan^2\theta)\, d\theta = \int [2 + (\sec^2\theta - 1)]\, d\theta = \int (1 + \sec^2\theta)\, d\theta = \theta + \tan\theta + C$

18. $\int \sec t\,(\sec t + \tan t)\, dt = \int (\sec^2 t + \sec t\, \tan t)\, dt = \tan t + \sec t + C$

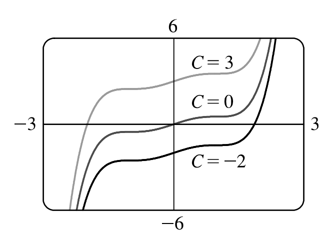
19. $\int 3\csc^2 t\, dt = -3\cot t + C$

20. $\displaystyle\int \frac{\sin 2x}{\sin x}\, dx = \int \frac{2\sin x\, \cos x}{\sin x}\, dx = \int 2\cos x\, dx = 2\sin x + C$

21. $\int \left(\cos x + \frac{1}{2}x\right) dx = \sin x + \frac{1}{4}x^2 + C$. The members of the family in the figure correspond to $C = -5$, 0, 5, and 10.



22. $\int (1 - x^2)^2\, dx = \int (1 - 2x^2 + x^4)\, dx = x - \frac{2}{3}x^3 + \frac{1}{5}x^5 + C$



23. $\int_{-2}^{3}(x^2-3)\,dx=\left[\frac{1}{3}x^3-3x\right]_{-2}^{3}=(9-9)-\left(-\frac{8}{3}+6\right)=\frac{8}{3}-\frac{18}{3}=-\frac{10}{3}$

24. $\int_{1}^{2}(4x^3-3x^2+2x)\,dx=\left[x^4-x^3+x^2\right]_{1}^{2}=(16-8+4)-(1-1+1)=12-1=11$

25. $\displaystyle\int_{1}^{4}(8t^3-6t^{-2})\,dt=\left[2t^4+\frac{6}{t}\right]_{1}^{4}=\left(2\cdot4^4+\frac{6}{4}\right)-\left(2\cdot1^4+\frac{6}{1}\right)=\left(512+\frac{3}{2}\right)-(2+6)=\frac{1011}{2}=505.5$

26. $\int_{0}^{8}\left(\frac{1}{8}+\frac{1}{2}w+\frac{1}{3}w^{1/3}\right)dw=\left[\frac{1}{8}w+\frac{1}{4}w^2+\frac{1}{4}w^{4/3}\right]_{0}^{8}=(1+16+4)-0=21$

27. $\int_{0}^{2}(2x-3)(4x^2+1)\,dx=\int_{0}^{2}(8x^3-12x^2+2x-3)\,dx=\left[2x^4-4x^3+x^2-3x\right]_{0}^{2}=(32-32+4-6)-0=-2$

28. $\displaystyle\int_{1}^{2}\left(\frac{1}{x^2}-\frac{4}{x^3}\right)dx=\int_{1}^{2}(x^{-2}-4x^{-3})\,dx=\left[\frac{x^{-1}}{-1}-\frac{4x^{-2}}{-2}\right]_{1}^{2}=\left[-\frac{1}{x}+\frac{2}{x^2}\right]_{1}^{2}=\left(-\frac{1}{2}+\frac{1}{2}\right)-(-1+2)=-1$

29. $\int_{0}^{\pi}(4\sin\theta-3\cos\theta)\,d\theta=\left[-4\cos\theta-3\sin\theta\right]_{0}^{\pi}=(4-0)-(-4-0)=8$

30. $\int_{-1}^{1}t(1-t)^2\,dt=\int_{-1}^{1}t(1-2t+t^2)\,dt=\int_{-1}^{1}(t-2t^2+t^3)\,dt=\left[\frac{1}{2}t^2-\frac{2}{3}t^3+\frac{1}{4}t^4\right]_{-1}^{1}$

$=\left(\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right)-\left(\frac{1}{2}+\frac{2}{3}+\frac{1}{4}\right)=-\frac{4}{3}$

31. $\displaystyle\int_{1}^{4}\left(\frac{4+6u}{\sqrt{u}}\right)du=\int_{1}^{4}\left(\frac{4}{\sqrt{u}}+\frac{6u}{\sqrt{u}}\right)du=\int_{1}^{4}(4u^{-1/2}+6u^{1/2})\,du=\left[8u^{1/2}+4u^{3/2}\right]_{1}^{4}=(16+32)-(8+4)=36$

32. $\displaystyle\int_{1}^{2}\left(2-\frac{1}{p^2}\right)^2dp=\int_{1}^{2}\left(4-\frac{4}{p^2}+\frac{1}{p^4}\right)dp=\int_{1}^{2}(4-4p^{-2}+p^{-4})\,dp=\left[4p+4p^{-1}-\frac{1}{3}p^{-3}\right]_{1}^{2}$

$$=\left(8+2-\frac{1}{24}\right)-\left(4+4-\frac{1}{3}\right)=2-\frac{1}{24}+\frac{1}{3}=\frac{48-1+8}{24}=\frac{55}{24}$$

33. $\displaystyle\int_{\pi/6}^{\pi/3}(4\sec^2y)\,dy=\left[4\tan y\right]_{\pi/6}^{\pi/3}=4\tan\frac{\pi}{3}-4\tan\frac{\pi}{6}=4\cdot\sqrt{3}-4\cdot\frac{1}{\sqrt{3}}=4\sqrt{3}-\frac{4\sqrt{3}}{3}=\frac{8\sqrt{3}}{3}$ or $\displaystyle\frac{8}{\sqrt{3}}$

34. $\displaystyle\int_{0}^{\pi/2}\left(\sqrt{t}-3\cos t\right)dt=\int_{0}^{\pi/2}\left(t^{1/2}-3\cos t\right)dt=\left[\frac{2}{3}t^{3/2}-3\sin t\right]_{0}^{\pi/2}=\left[\frac{2}{3}\left(\frac{\pi}{2}\right)^{3/2}-3\sin\frac{\pi}{2}\right]-0$

$$=\frac{2}{3}\left(\frac{\pi}{2}\right)^{3/2}-3\cdot1=\frac{2}{3}\left(\frac{\pi}{2}\right)^{3/2}-3$$

35. $\int_{0}^{1}x\left(\sqrt[3]{x}+\sqrt[4]{x}\right)dx=\int_{0}^{1}(x^{4/3}+x^{5/4})\,dx=\left[\frac{3}{7}x^{7/3}+\frac{4}{9}x^{9/4}\right]_{0}^{1}=\left(\frac{3}{7}+\frac{4}{9}\right)-0=\frac{55}{63}$

36. $\displaystyle\int_{1}^{8}\left(\frac{2}{\sqrt[3]{w}}-\sqrt[3]{w}\right)dw=\int_{1}^{8}(2w^{-1/3}-w^{1/3})\,dw=\left[3w^{2/3}-\frac{3}{4}w^{4/3}\right]_{1}^{8}=(12-12)-\left(3-\frac{3}{4}\right)=-\frac{9}{4}$

37. $\int_{1}^{4}\sqrt{5/x}\,dx=\sqrt{5}\int_{1}^{4}x^{-1/2}\,dx=\sqrt{5}\left[2\sqrt{x}\right]_{1}^{4}=\sqrt{5}\,(2\cdot2-2\cdot1)=2\sqrt{5}$

38. $\displaystyle\int_{\pi/6}^{\pi/2}\csc t\,\cot t\,dt=\left[-\csc t\right]_{\pi/6}^{\pi/2}=\left(-\csc\frac{\pi}{2}\right)-\left(-\csc\frac{\pi}{6}\right)=-1-(-2)=1$

39. $\displaystyle\int_0^{\pi/4} \frac{1+\cos^2\theta}{\cos^2\theta}\,d\theta = \int_0^{\pi/4}\left(\frac{1}{\cos^2\theta}+\frac{\cos^2\theta}{\cos^2\theta}\right)d\theta = \int_0^{\pi/4}(\sec^2\theta+1)\,d\theta$

$$= \big[\tan\theta+\theta\big]_0^{\pi/4} = \left(\tan\tfrac{\pi}{4}+\tfrac{\pi}{4}\right)-(0+0) = 1+\tfrac{\pi}{4}$$

40. $\displaystyle\int_0^{\pi/3}\frac{\sin\theta+\sin\theta\,\tan^2\theta}{\sec^2\theta}\,d\theta = \int_0^{\pi/3}\frac{\sin\theta\,(1+\tan^2\theta)}{\sec^2\theta}\,d\theta = \int_0^{\pi/3}\frac{\sin\theta\,\sec^2\theta}{\sec^2\theta}\,d\theta = \int_0^{\pi/3}\sin\theta\,d\theta$

$$= \big[-\cos\theta\big]_0^{\pi/3} = -\tfrac{1}{2}-(-1) = \tfrac{1}{2}$$

41. $\displaystyle\int_0^{64}\sqrt{u}\,(u-\sqrt[3]{u})\,du = \int_0^{64}(u^{3/2}-u^{5/6})\,du = \left[\frac{2}{5}u^{5/2}-\frac{6}{11}u^{11/6}\right]_0^{64} = \left(\frac{65{,}536}{5}-\frac{12{,}288}{11}\right)-0 = \frac{659{,}456}{55}$

42. $\int_0^1(1+x^2)^3\,dx = \int_0^1(1+3x^2+3x^4+x^6)\,dx = \left[x+x^3+\frac{3}{5}x^5+\frac{1}{7}x^7\right]_0^1 = \left(1+1+\frac{3}{5}+\frac{1}{7}\right)-0 = \frac{96}{35}$

43. $|x-3| = \begin{cases} x-3 & \text{if } x-3\ge 0 \\ -(x-3) & \text{if } x-3<0\end{cases} = \begin{cases} x-3 & \text{if } x\ge 3 \\ 3-x & \text{if } x<3\end{cases}$

Thus, $\int_2^5|x-3|\,dx = \int_2^3(3-x)\,dx+\int_3^5(x-3)\,dx = \left[3x-\frac{1}{2}x^2\right]_2^3+\left[\frac{1}{2}x^2-3x\right]_3^5$

$$= \left(9-\tfrac{9}{2}\right)-(6-2)+\left(\tfrac{25}{2}-15\right)-\left(\tfrac{9}{2}-9\right) = \tfrac{5}{2}$$

44. $|2x-1| = \begin{cases} 2x-1 & \text{if } 2x-1\ge 0 \\ -(2x-1) & \text{if } 2x-1<0\end{cases} = \begin{cases} 2x-1 & \text{if } x\ge \frac{1}{2} \\ 1-2x & \text{if } x<\frac{1}{2}\end{cases}$

Thus, $\int_0^2|2x-1|\,dx = \int_0^{1/2}(1-2x)\,dx+\int_{1/2}^2(2x-1)\,dx = \left[x-x^2\right]_0^{1/2}+\left[x^2-x\right]_{1/2}^2$

$$= \left(\tfrac{1}{2}-\tfrac{1}{4}\right)-0+(4-2)-\left(\tfrac{1}{4}-\tfrac{1}{2}\right) = \tfrac{1}{4}+2-\left(-\tfrac{1}{4}\right) = \tfrac{5}{2}$$

45. $\int_{-1}^2(x-2\,|x|)\,dx = \int_{-1}^0[x-2(-x)]\,dx+\int_0^2[x-2(x)]\,dx = \int_{-1}^0 3x\,dx+\int_0^2(-x)\,dx = 3\left[\frac{1}{2}x^2\right]_{-1}^0-\left[\frac{1}{2}x^2\right]_0^2$

$$= 3\left(0-\tfrac{1}{2}\right)-(2-0) = -\tfrac{7}{2} = -3.5$$

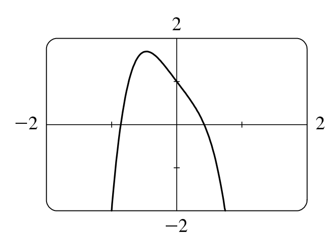
46. $\int_0^{3\pi/2}|\sin x|\,dx = \int_0^\pi\sin x\,dx+\int_\pi^{3\pi/2}(-\sin x)\,dx = \big[-\cos x\big]_0^\pi+\big[\cos x\big]_\pi^{3\pi/2}$

$$= [1-(-1)]+[0-(-1)] = 2+1 = 3$$

47. The graph shows that $y=1-2x-5x^4$ has x-intercepts at $x=a\approx-0.86$ and at $x=b\approx0.42$. So the area of the region that lies under the curve and above the x-axis is

$\int_a^b(1-2x-5x^4)\,dx = \left[x-x^2-x^5\right]_a^b$

$$= (b-b^2-b^5)-(a-a^2-a^5)\approx 1.36$$

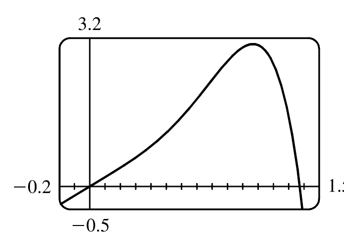


48. The graph shows that $y=2x+3x^4-2x^6$ has x-intercepts at $x=0$ and at $x=a\approx1.37$. So the area of the region that lies under the curve and above the x-axis is

$\int_0^a\left(2x+3x^4-2x^6\right)dx = \left[x^2+\frac{3}{5}x^5-\frac{2}{7}x^7\right]_0^a$

$$= \left(a^2+\tfrac{3}{5}a^5-\tfrac{2}{7}a^7\right)-0$$

$$\approx 2.18$$



49. $A = \int_0^2 (2y - y^2)\,dy = \left[y^2 - \frac{1}{3}y^3\right]_0^2 = \left(4 - \frac{8}{3}\right) - 0 = \frac{4}{3}$

50. $y = \sqrt[4]{x} \quad\Rightarrow\quad x = y^4$, so $A = \int_0^1 y^4\,dy = \left[\frac{1}{5}y^5\right]_0^1 = \frac{1}{5}$.

51. If $w'(t)$ is the rate of change of weight in kilograms per year, then $w(t)$ represents the weight in kilograms of the child at age t. We know from the Net Change Theorem that $\int_5^{10} w'(t)\,dt = w(10) - w(5)$, so the integral represents the increase in the child's weight (in kilograms) between the ages of 5 and 10.

52. $\int_a^b I(t)\,dt = \int_a^b Q'(t)\,dt = Q(b) - Q(a)$ by the Net Change Theorem, so it represents the change in the charge Q from time $t = a$ to $t = b$.

53. Since $r(t)$ is the rate at which oil leaks, we can write $r(t) = -V'(t)$, where $V(t)$ is the volume of oil at time t. [Note that the minus sign is needed because V is decreasing, so $V'(t)$ is negative, but $r(t)$ is positive.] Thus, by the Net Change Theorem, $\int_0^{120} r(t)\,dt = -\int_0^{120} V'(t)\,dt = -\left[V(120) - V(0)\right] = V(0) - V(120)$, which is the number of liters of oil that leaked from the tank in the first two hours (120 minutes).

54. By the Net Change Theorem, $\int_0^{15} n'(t)\,dt = n(15) - n(0) = n(15) - 100$ represents the increase in the bee population in 15 weeks. So $100 + \int_0^{15} n'(t)\,dt = n(15)$ represents the total bee population after 15 weeks.

55. By the Net Change Theorem, $\int_{1000}^{5000} R'(x)\,dx = R(5000) - R(1000)$, so it represents the increase in revenue when production is increased from 1000 units to 5000 units.

56. The slope of the trail is the rate of change of the elevation E, so $f(x) = E'(x)$. By the Net Change Theorem, $\int_3^5 f(x)\,dx = \int_3^5 E'(x)\,dx = E(5) - E(3)$ is the change in the elevation E between $x = 3$ kilometers and $x = 5$ kilometers from the start of the trail.

57. The function h gives the rate of change in the total number of heartbeats H, so $h(t) = H'(t)$. By the Net Change Theorem, $\int_0^{30} h(t)\,dt = H(30) - H(0) = H(30) - 0 = H(30)$ represents the total number of heartbeats during the first 30 minutes of an exercise session.

58. The units for $a(x)$ are kilograms per meter and the units for x are meters, so the units for da/dx are kilograms per meter per meter, denoted $(kg/m)/m$. The unit of measurement for $\int_2^8 a(x)$ is the product of kilograms per meter and meters, that is, kilograms.

59. In general, the unit of measurement for $\int_a^b f(x)\,dx$ is the product of the unit for $f(x)$ and the unit for x. Since $f(x)$ is measured in newtons and x is measured in meters, the units for $\int_0^{100} f(x)\,dx$ are newton-meters (or joules). (A newton-meter is abbreviated N·m.)

60. (a) At $t = 2$, the distance from the charging station (the displacement) is given by $\int_0^2 v(t)\,dt = \frac{1}{2} \cdot 2 \cdot 5$ [triangle] $= 5$ m.

At $t = 4$, the displacement is $\int_0^4 v(t)\,dt = 5 + \int_2^4 v(t)\,dt = 5 + 2 \cdot 2$ [square] $+ \frac{1}{2} \cdot 1 \cdot 3$ [triangle] $= \frac{21}{2}$ m.

At $t = 6$, the displacement is $\frac{21}{2} + \int_4^6 v(t)\,dt = \frac{21}{2} + \frac{1}{2} \cdot 2 \cdot 2$ [triangle] $= \frac{25}{2}$ m.

At $t = 8$, the displacement is $\frac{25}{2} + \int_6^8 v(t)\,dt = \frac{25}{2} + 0 = \frac{25}{2}$ m.

At $t = 10$, the displacement is $\frac{25}{2} + \int_8^{10} v(t)\,dt = \frac{25}{2} - \frac{1}{2} \cdot 2 \cdot 4$ [triangle] $= \frac{17}{2}$ m.

At $t = 12$, the displacement is $\frac{17}{2} + \int_{10}^{12} v(t)\,dt = \frac{17}{2} - \frac{1}{2} \cdot 2 \cdot 4$ [triangle] $= \frac{9}{2}$ m.

(b) The vehicle is farthest from the charging station when the absolute value of the displacement is greatest. The displacement of this autonomous vehicle is never negative, so we look for the largest values of the displacement integral (or, equivalently, the places where the area above the t-axis is the greatest). This occurs from $t = 6$ to $t = 8$, so the vehicle is farthest from the charging station then.

(c) The total distance traveled is given by $\int_0^{12} |v(t)|\, dt = \int_0^8 v(t)\, dt + \int_8^{12} [-v(t)]\, dt = \frac{25}{2} + \frac{1}{2} \cdot 4 \cdot 4 = \frac{41}{2}$ m.

61. (a) Displacement $= \int_0^3 (3t - 5)\, dt = \left[\frac{3}{2}t^2 - 5t\right]_0^3 = \frac{27}{2} - 15 = -\frac{3}{2}$ m

(b) Distance traveled $= \int_0^3 |3t - 5|\, dt = \int_0^{5/3} (5 - 3t)\, dt + \int_{5/3}^3 (3t - 5)\, dt$

$$= \left[5t - \tfrac{3}{2}t^2\right]_0^{5/3} + \left[\tfrac{3}{2}t^2 - 5t\right]_{5/3}^3 = \tfrac{25}{3} - \tfrac{3}{2} \cdot \tfrac{25}{9} + \tfrac{27}{2} - 15 - \left(\tfrac{3}{2} \cdot \tfrac{25}{9} - \tfrac{25}{3}\right) = \tfrac{41}{6} \text{ m}$$

62. (a) Displacement $= \int_2^4 (t^2 - 2t - 3)\, dt = \left[\frac{1}{3}t^3 - t^2 - 3t\right]_2^4 = \left(\frac{64}{3} - 16 - 12\right) - \left(\frac{8}{3} - 4 - 6\right) = \frac{2}{3}$ m

(b) $v(t) = t^2 - 2t - 3 = (t + 1)(t - 3)$, so $v(t) < 0$ for $-1 < t < 3$, but on the interval $[2, 4]$, $v(t) < 0$ for $2 \le t < 3$.

$$\begin{aligned} \text{Distance traveled} &= \int_2^4 \left|t^2 - 2t - 3\right|\, dt = \int_2^3 -(t^2 - 2t - 3)\, dt + \int_3^4 (t^2 - 2t - 3)\, dt \\ &= \left[-\tfrac{1}{3}t^3 + t^2 + 3t\right]_2^3 + \left[\tfrac{1}{3}t^3 - t^2 - 3t\right]_3^4 \\ &= (-9 + 9 + 9) - \left(-\tfrac{8}{3} + 4 + 6\right) + \left(\tfrac{64}{3} - 16 - 12\right) - (9 - 9 - 9) = 4 \text{ m} \end{aligned}$$

63. (a) $v'(t) = a(t) = t + 4 \quad \Rightarrow \quad v(t) = \frac{1}{2}t^2 + 4t + C \quad \Rightarrow \quad v(0) = C = 5 \quad \Rightarrow \quad v(t) = \frac{1}{2}t^2 + 4t + 5$ m/s

(b) Distance traveled $= \int_0^{10} |v(t)|\, dt = \int_0^{10} \left|\frac{1}{2}t^2 + 4t + 5\right|\, dt = \int_0^{10} \left(\frac{1}{2}t^2 + 4t + 5\right)\, dt = \left[\frac{1}{6}t^3 + 2t^2 + 5t\right]_0^{10}$

$= \frac{500}{3} + 200 + 50 = 416\frac{2}{3}$ m

64. (a) $v'(t) = a(t) = 2t + 3 \quad \Rightarrow \quad v(t) = t^2 + 3t + C \quad \Rightarrow \quad v(0) = C = -4 \quad \Rightarrow \quad v(t) = t^2 + 3t - 4$

(b) Distance traveled $= \int_0^3 \left|t^2 + 3t - 4\right|\, dt = \int_0^3 |(t + 4)(t - 1)|\, dt = \int_0^1 \left(-t^2 - 3t + 4\right)\, dt + \int_1^3 \left(t^2 + 3t - 4\right)\, dt$

$$\begin{aligned} &= \left[-\tfrac{1}{3}t^3 - \tfrac{3}{2}t^2 + 4t\right]_0^1 + \left[\tfrac{1}{3}t^3 + \tfrac{3}{2}t^2 - 4t\right]_1^3 \\ &= \left(-\tfrac{1}{3} - \tfrac{3}{2} + 4\right) + \left(9 + \tfrac{27}{2} - 12\right) - \left(\tfrac{1}{3} + \tfrac{3}{2} - 4\right) = \tfrac{89}{6} \text{ m} \end{aligned}$$

65. Since $m'(x) = \rho(x)$, $m = \int_0^4 \rho(x)\, dx = \int_0^4 \left(9 + 2\sqrt{x}\right)\, dx = \left[9x + \frac{4}{3}x^{3/2}\right]_0^4 = 36 + \frac{32}{3} - 0 = \frac{140}{3} = 46\frac{2}{3}$ kg.

66. By the Net Change Theorem, the amount of water that flows from the tank during the first 10 minutes is

$\int_0^{10} r(t)\, dt = \int_0^{10} (200 - 4t)\, dt = \left[200t - 2t^2\right]_0^{10} = (2000 - 200) - 0 = 1800$ liters.

67. Let s be the position of the car. We know from Equation 2 that $s(100) - s(0) = \int_0^{100} v(t)\, dt$. We use the Midpoint Rule for $0 \le t \le 100$ with $n = 5$. Note that the length of each of the five time intervals is 20 seconds $= \frac{20}{3600}$ hour $= \frac{1}{180}$ hour.

So the distance traveled is

$\int_0^{100} v(t)\, dt \approx \frac{1}{180}[v(10) + v(30) + v(50) + v(70) + v(90)] = \frac{1}{180}(61 + 93 + 82 + 85 + 76) = \frac{397}{180} \approx 2.2$ kilometers.

68. (a) By the Net Change Theorem, the total amount spewed into the atmosphere is $Q(6) - Q(0) = \int_0^6 r(t)\,dt = Q(6)$ since $Q(0) = 0$. The rate $r(t)$ is positive, so Q is an increasing function. Thus, an upper estimate for $Q(6)$ is R_6 and a lower estimate for $Q(6)$ is L_6. $\Delta t = \dfrac{b-a}{n} = \dfrac{6-0}{6} = 1$.

$R_6 = \sum\limits_{i=1}^{6} r(t_i)\,\Delta t = 10 + 24 + 36 + 46 + 54 + 60 = 230$ tonnes.

$L_6 = \sum\limits_{i=1}^{6} r(t_{i-1})\,\Delta t = R_6 + r(0) - r(6) = 230 + 2 - 60 = 172$ tonnes.

(b) $\Delta t = \dfrac{b-a}{n} = \dfrac{6-0}{3} = 2$. $Q(6) \approx M_3 = 2[r(1) + r(3) + r(5)] = 2(10 + 36 + 54) = 2(100) = 200$ tonnes.

69. Use the midpoint of each of four 2-day intervals. Let $t = 0$ correspond to July 18 and note that the inflow rate, $r(t)$, is in m³/s.

$\text{Amount of water} = \int_0^8 r(t)\,dt \approx [r(1) + r(3) + r(5) + r(7)]\,\dfrac{8-0}{4} \approx [181 + 120 + 108 + 74](2) = 966.$

Now multiply by the number of seconds in a day, $24 \cdot 60^2$, to get $83{,}462{,}400 \text{ m}^3$.

70. By the Net Change Theorem, the amount of water after four days is

$$\begin{aligned} 25{,}000 + \int_0^4 r(t)\,dt \approx 25{,}000 + M_4 &= 25{,}000 + \tfrac{4-0}{4}\,[r(0.5) + r(1.5) + r(2.5) + r(3.5)] \\ &\approx 25{,}000 + [1500 + 1770 + 740 + (-690)] = 28{,}320 \text{ liters} \end{aligned}$$

71. To use the Midpoint Rule, we'll use the midpoint of each of three 2-second intervals.

$v(6) - v(0) = \int_0^6 a(t)\,dt \approx [a(1) + a(3) + a(5)]\,\dfrac{6-0}{3} \approx (0.2 + 2.5 + 2.3)(2) = 10 \text{ m/s}$

72. Let $M(t)$ denote the number of megabits transmitted at time t (in hours) [note that $D(t)$ is measured in megabits/second]. By the Net Change Theorem and the Midpoint Rule,

$$\begin{aligned} M(8) - M(0) &= \int_0^8 3600D(t)\,dt \approx 3600 \cdot \tfrac{8-0}{4}[D(1) + D(3) + D(5) + D(7)] \\ &\approx 7200(0.32 + 0.50 + 0.56 + 0.83) = 7200(2.21) = 15{,}912 \text{ megabits} \end{aligned}$$

73. Power is the rate of change of energy with respect to time; that is, $P(t) = E'(t)$. By the Net Change Theorem and the Midpoint Rule, the electric energy used on that day is

$$\begin{aligned} E(24) - E(0) &= \int_0^{24} P(t)\,dt \approx \frac{24-0}{12}[P(1) + P(3) + P(5) + \cdots + P(21) + P(23)] \\ &\approx 2(11 + 10.5 + 11 + 14 + 15.1 + 15.5 + 15.1 + 15 + 15 + 16.1 + 15 + 13) \\ &= 2(166.3) = 332.6 \text{ gigawatt-hours} \end{aligned}$$

74. (a) From Exercise 3.1.70(a), $v(t) = 0.0004t^3 - 0.0349t^2 + 7.6025t - 6.5897$

(b) $h(125) - h(0) = \int_0^{125} v(t)\,dt = [0.0001t^4 - 0.0116t^3 + 3.80125t^2 - 6.5897t]\big|_0^{125} \approx 60{,}328.6 \text{ m}$

75. $\int (\sin x + \sinh x)\,dx = -\cos x + \cosh x + C$

76. $\displaystyle\int_{-10}^{10} \frac{2e^x}{\sinh x + \cosh x}\,dx = \int_{-10}^{10} \frac{2e^x}{\dfrac{e^x - e^{-x}}{2} + \dfrac{e^x + e^{-x}}{2}}\,dx = \int_{-10}^{10} \frac{2e^x}{e^x}\,dx = \int_{-10}^{10} 2\,dx = \big[2x\big]_{-10}^{10}$

$= 20 - (-20) = 40$

77. $\displaystyle\int \left(x^2 + 1 + \frac{1}{x^2+1}\right) dx = \frac{x^3}{3} + x + \tan^{-1}x + C$

78. $\displaystyle\int_1^2 \frac{(x-1)^3}{x^2}\,dx = \int_1^2 \frac{x^3 - 3x^2 + 3x - 1}{x^2}\,dx = \int_1^2 \left(x - 3 + \frac{3}{x} - \frac{1}{x^2}\right) dx = \left[\tfrac{1}{2}x^2 - 3x + 3\ln|x| + \frac{1}{x}\right]_1^2$

$= \left(2 - 6 + 3\ln 2 + \frac{1}{2}\right) - \left(\frac{1}{2} - 3 + 0 + 1\right) = 3\ln 2 - 2$

79. $\displaystyle\int_0^{1/\sqrt{3}} \frac{t^2 - 1}{t^4 - 1}\,dt = \int_0^{1/\sqrt{3}} \frac{t^2 - 1}{(t^2+1)(t^2-1)}\,dt = \int_0^{1/\sqrt{3}} \frac{1}{t^2+1}\,dt = \big[\arctan t\big]_0^{1/\sqrt{3}} = \arctan\!\left(1/\sqrt{3}\right) - \arctan 0$

$= \frac{\pi}{6} - 0 = \frac{\pi}{6}$

80. $B = 3A \;\Rightarrow\; \int_0^b e^x\,dx = 3\int_0^a e^x\,dx \;\Rightarrow\; [e^x]_0^b = 3\,[e^x]_0^a \;\Rightarrow\; e^b - 1 = 3(e^a - 1) \;\Rightarrow\; e^b = 3e^a - 2 \;\Rightarrow$

$b = \ln(3e^a - 2)$

4.5 The Substitution Rule

1. Let $u = 2x$. Then $du = 2\,dx$ and $dx = \frac{1}{2}\,du$, so $\int \cos 2x\,dx = \int \cos u\,\left(\frac{1}{2}\,du\right) = \frac{1}{2}\sin u + C = \frac{1}{2}\sin 2x + C$.

2. Let $u = 2x^2 + 3$. Then $du = 4x\,dx$ and $x\,dx = \frac{1}{4}\,du$, so

$$\int x(2x^2+3)^4\,dx = \int u^4\left(\frac{1}{4}\,du\right) = \frac{1}{4}\frac{u^5}{5} + C = \frac{1}{20}(2x^2+3)^5 + C.$$

3. Let $u = x^3 + 1$. Then $du = 3x^2\,dx$ and $x^2\,dx = \frac{1}{3}\,du$, so

$$\int x^2\sqrt{x^3+1}\,dx = \int \sqrt{u}\,\left(\tfrac{1}{3}\,du\right) = \frac{1}{3}\frac{u^{3/2}}{3/2} + C = \frac{1}{3}\cdot\frac{2}{3}u^{3/2} + C = \tfrac{2}{9}(x^3+1)^{3/2} + C.$$

4. Let $u = \sin\theta$. Then $du = \cos\theta\,d\theta$, so $\int \sin^2\theta\,\cos\theta\,d\theta = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}\sin^3\theta + C$.

5. Let $u = x^4 - 5$. Then $du = 4x^3\,dx$ and $x^3\,dx = \frac{1}{4}\,du$, so

$$\int \frac{x^3}{(x^4-5)^2}\,dx = \int \frac{1}{u^2}\left(\frac{1}{4}\,du\right) = \frac{1}{4}\int u^{-2}\,du = \frac{1}{4}\frac{u^{-1}}{-1} + C = -\frac{1}{4u} + C = -\frac{1}{4\,(x^4-5)} + C.$$

6. Let $u = 1 + \dfrac{1}{x}$. Then $du = -\dfrac{1}{x^2}\,dx$ and $\dfrac{1}{x^2}\,dx = -du$, so

$$\int \frac{1}{x^2}\sqrt{1 + \frac{1}{x}}\,dx = \int \sqrt{u}\,(-du) = -\frac{2}{3}u^{3/2} + C = -\frac{2}{3}\left(1 + \frac{1}{x}\right)^{3/2} + C.$$

7. Let $u = \sqrt{t}$. Then $du = \dfrac{1}{2\sqrt{t}}\,dt$ and $\dfrac{1}{\sqrt{t}}\,dt = 2\,du$, so $\displaystyle\int \frac{\cos\sqrt{t}}{\sqrt{t}}\,dt = \int \cos u\,(2\,du) = 2\sin u + C = 2\sin\sqrt{t} + C$.

8. Let $u = z - 1$. Then $z = u + 1$ and $du = dz$, so

$$\int z\sqrt{z-1}\,dz = \int (u+1)\sqrt{u}\,du = \int (u^{3/2} + u^{1/2})\,du = \tfrac{2}{5}u^{5/2} + \tfrac{2}{3}u^{3/2} + C$$
$$= \tfrac{2}{5}(z-1)^{5/2} + \tfrac{2}{3}(z-1)^{3/2} + C$$

9. Let $u = 1 - x^2$. Then $du = -2x\,dx$ and $x\,dx = -\frac{1}{2}\,du$, so

$\int x\sqrt{1-x^2}\,dx = \int \sqrt{u}\,\left(-\frac{1}{2}\,du\right) = -\frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C = -\frac{1}{3}(1-x^2)^{3/2} + C.$

10. Let $u = 5 - 3x$. Then $du = -3\,dx$ and $dx = -\frac{1}{3}\,du$, so

$\int (5-3x)^{10}\,dx = \int u^{10}\left(-\frac{1}{3}\,du\right) = -\frac{1}{3}\cdot\frac{1}{11}u^{11} + C = -\frac{1}{33}(5-3x)^{11} + C.$

11. Let $u = x^3$. Then $du = 3x^2\,dx$ and $x^2\,dx = \frac{1}{3}\,du$, so $\int x^2\cos(x^3)\,dx = \int \cos u\left(\frac{1}{3}\,du\right) = \frac{1}{3}\sin u + C = \frac{1}{3}\sin(x^3) + C.$

12. Let $u = 1 + \cos t$. Then $du = -\sin t\,dt$ and $\sin t\,dt = -du$, so

$\int \sin t\,\sqrt{1+\cos t}\,dt = \int \sqrt{u}\,(-du) = -\frac{2}{3}u^{3/2} + C = -\frac{2}{3}(1+\cos t)^{3/2} + C.$

13. Let $u = \dfrac{\pi}{3}t$. Then $du = \dfrac{\pi}{3}\,dt$ and $dt = \dfrac{3}{\pi}\,du$, so

$$\int \sin\left(\frac{\pi t}{3}\right)dt = \int \sin u\left(\frac{3}{\pi}\,du\right) = \frac{3}{\pi}\cdot(-\cos u) + C = -\frac{3}{\pi}\cos\left(\frac{\pi}{3}t\right) + C.$$

14. Let $u = 2\theta$. Then $du = 2\,d\theta$ and $d\theta = \frac{1}{2}\,du$, so $\int \sec^2 2\theta\,d\theta = \int \sec^2 u\left(\frac{1}{2}\,du\right) = \frac{1}{2}\tan u + C = \frac{1}{2}\tan 2\theta + C.$

15. Let $u = 3t$. Then $du = 3\,dt$ and $dt = \frac{1}{3}\,du$, so $\int \sec 3t\,\tan 3t\,dt = \int \sec u\,\tan u\left(\frac{1}{3}\,du\right) = \frac{1}{3}\sec u + C = \frac{1}{3}\sec 3t + C.$

16. Let $u = 4 - y^3$. Then $du = -3y^2\,dy$ and $y^2\,dy = -\frac{1}{3}\,du$, so

$\int y^2(4-y^3)^{2/3}\,dy = \int u^{2/3}\left(-\frac{1}{3}\,du\right) = -\frac{1}{3}\cdot\frac{3}{5}u^{5/3} + C = -\frac{1}{5}(4-y^3)^{5/3} + C.$

17. Let $u = 1 + 5t$. Then $du = 5\,dt$ and $dt = \frac{1}{5}\,du$, so

$\int \cos(1+5t)\,dt = \int \cos u\left(\frac{1}{5}\,du\right) = \frac{1}{5}\sin u + C = \frac{1}{5}\sin(1+5t) + C.$

18. Let $u = 1/x$. Then $du = -\dfrac{1}{x^2}\,dx$ and $\dfrac{1}{x^2}\,dx = -du$, so $\displaystyle\int \frac{\sin(1/x)}{x^2}\,dx = \int \sin u\,(-du) = \cos u + C = \cos(1/x) + C.$

19. Let $u = \cos\theta$. Then $du = -\sin\theta\,d\theta$ and $\sin\theta\,d\theta = -du$, so

$\int \cos^3\theta\,\sin\theta\,d\theta = \int u^3(-du) = -\frac{1}{4}u^4 + C = -\frac{1}{4}\cos^4\theta + C.$

20. Let $u = \cos x$. Then $du = -\sin x\,dx$ and $-du = \sin x\,dx$, so

$\int \sin x\,\sin(\cos x)\,dx = \int \sin u\,(-du) = (-\cos u)(-1) + C = \cos(\cos x) + C.$

21. Let $u = x^2 + \dfrac{2}{x}$. Then $du = \left(2x - \dfrac{2}{x^2}\right)dx = 2\left(x - \dfrac{1}{x^2}\right)dx$ and $\left(x - \dfrac{1}{x^2}\right)dx = \dfrac{1}{2}\,du$, so

$$\int \left(x - \frac{1}{x^2}\right)\left(x^2 + \frac{2}{x}\right)^5 dx = \int u^5\left(\frac{1}{2}\,du\right) = \frac{1}{2}\cdot\frac{1}{6}u^6 + C = \frac{1}{12}\left(x^2 + \frac{2}{x}\right)^6 + C.$$

22. Let $u = x + 2$. Then $du = dx$ and $x = u - 2$, so

$\int x\sqrt{x+2}\,dx = \int (u-2)\sqrt{u}\,du = \int (u^{3/2} - 2u^{1/2})\,du = \frac{2}{5}u^{5/2} - 2\cdot\frac{2}{3}u^{3/2} + C = \frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + C.$

23. Let $u = 3ax + bx^3$. Then $du = (3a + 3bx^2)\,dx = 3(a + bx^2)\,dx$, so

$$\int \frac{a+bx^2}{\sqrt{3ax+bx^3}}\,dx = \int \frac{\frac{1}{3}\,du}{u^{1/2}} = \frac{1}{3}\int u^{-1/2}\,du = \tfrac{1}{3}\cdot 2u^{1/2} + C = \tfrac{2}{3}\sqrt{3ax+bx^3} + C.$$

24. Let $u = \tan x$. Then $du = \sec^2 x\,dx$, so

$$\int \frac{\sec^2 x}{\tan^2 x}\,dx = \int \frac{1}{u^2}\,du = \int u^{-2}\,du = -1u^{-1} + C = -\frac{1}{\tan x} + C = -\cot x + C.$$

Or: $\displaystyle\int \frac{\sec^2 x}{\tan^2 x}\,dx = \int \left(\frac{1}{\cos^2 x}\cdot\frac{\cos^2 x}{\sin^2 x}\right)dx = \int \csc^2 x\,dx = -\cot x + C$

25. Let $u = 1 + z^3$. Then $du = 3z^2\,dz$ and $z^2\,dz = \frac{1}{3}\,du$, so

$$\int \frac{z^2}{\sqrt[3]{1+z^3}}\,dz = \int u^{-1/3}\left(\tfrac{1}{3}\,du\right) = \tfrac{1}{3}\cdot\tfrac{3}{2}u^{2/3} + C = \tfrac{1}{2}(1+z^3)^{2/3} + C.$$

26. Let $u = 1 + \tan t$. Then $du = \sec^2 t\,dt$, so

$$\int \frac{dt}{\cos^2 t\,\sqrt{1+\tan t}} = \int \frac{\sec^2 t\,dt}{\sqrt{1+\tan t}} = \int \frac{du}{\sqrt{u}} = \int u^{-1/2}\,du = \frac{u^{1/2}}{1/2} + C = 2\sqrt{1+\tan t} + C.$$

27. Let $u = \cot x$. Then $du = -\csc^2 x\,dx$ and $\csc^2 x\,dx = -du$, so

$$\int \sqrt{\cot x}\,\csc^2 x\,dx = \int \sqrt{u}\,(-du) = -\frac{u^{3/2}}{3/2} + C = -\tfrac{2}{3}(\cot x)^{3/2} + C.$$

28. Let $u = \dfrac{\pi}{x}$. Then $du = -\dfrac{\pi}{x^2}\,dx$ and $\dfrac{1}{x^2}\,dx = -\dfrac{1}{\pi}\,du$, so

$$\int \frac{\cos(\pi/x)}{x^2}\,dx = \int \cos u\left(-\frac{1}{\pi}\,du\right) = -\frac{1}{\pi}\sin u + C = -\frac{1}{\pi}\sin\frac{\pi}{x} + C$$

29. Let $u = \sec x$. Then $du = \sec x\,\tan x\,dx$, so

$\int \sec^3 x\,\tan x\,dx = \int \sec^2 x\,(\sec x\,\tan x)\,dx = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}\sec^3 x + C.$

30. Let $u = 2 + x$. Then $du = dx$, $x = u - 2$, and $x^2 = (u-2)^2$, so

$$\begin{aligned}\int x^2\sqrt{2+x}\,dx &= \int (u-2)^2\sqrt{u}\,du = \int (u^2 - 4u + 4)u^{1/2}\,du = \int (u^{5/2} - 4u^{3/2} + 4u^{1/2})\,du \\ &= \tfrac{2}{7}u^{7/2} - \tfrac{8}{5}u^{5/2} + \tfrac{8}{3}u^{3/2} + C = \tfrac{2}{7}(2+x)^{7/2} - \tfrac{8}{5}(2+x)^{5/2} + \tfrac{8}{3}(2+x)^{3/2} + C\end{aligned}$$

31. Let $u = 2x + 5$. Then $du = 2\,dx$ and $x = \frac{1}{2}(u-5)$, so

$$\begin{aligned}\int x(2x+5)^8\,dx &= \int \tfrac{1}{2}(u-5)u^8\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{4}\int (u^9 - 5u^8)\,du \\ &= \tfrac{1}{4}\left(\tfrac{1}{10}u^{10} - \tfrac{5}{9}u^9\right) + C = \tfrac{1}{40}(2x+5)^{10} - \tfrac{5}{36}(2x+5)^9 + C\end{aligned}$$

32. Let $u = x^2 + 1$ [so $x^2 = u - 1$]. Then $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$, so

$$\int x^3\sqrt{x^2+1}\,dx = \int x^2\sqrt{x^2+1}\,x\,dx = \int (u-1)\sqrt{u}\,\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\int (u^{3/2} - u^{1/2})\,du$$
$$= \tfrac{1}{2}\left(\tfrac{2}{5}u^{5/2} - \tfrac{2}{3}u^{3/2}\right) + C = \tfrac{1}{5}(x^2+1)^{5/2} - \tfrac{1}{3}(x^2+1)^{3/2} + C$$

Or: Let $u = \sqrt{x^2+1}$. Then $u^2 = x^2 + 1 \;\Rightarrow\; 2u\,du = 2x\,dx \;\Rightarrow\; u\,du = x\,dx$, so

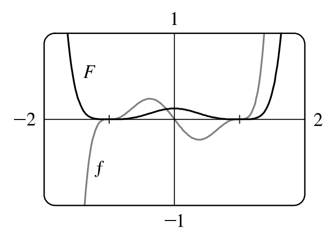
$$\int x^3\sqrt{x^2+1}\,dx = \int x^2\sqrt{x^2+1}\,x\,dx = \int (u^2-1)\,u\cdot u\,du = \int (u^4 - u^2)\,du$$
$$= \tfrac{1}{5}u^5 - \tfrac{1}{3}u^3 + C = \tfrac{1}{5}(x^2+1)^{5/2} - \tfrac{1}{3}(x^2+1)^{3/2} + C$$

Note: This answer can be written as $\frac{1}{15}\sqrt{x^2+1}(3x^4 + x^2 - 2) + C$.

33. $f(x) = x(x^2-1)^3$. $\quad u = x^2 - 1 \;\Rightarrow\; du = 2x\,dx$, so

$$\int x(x^2-1)^3\,dx = \int u^3\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{8}u^4 + C = \tfrac{1}{8}(x^2-1)^4 + C$$

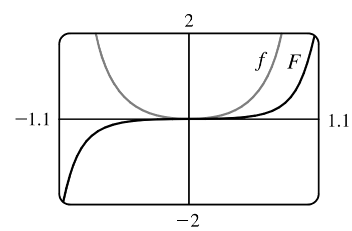
Where f is positive (negative), F is increasing (decreasing). Where f changes from negative to positive (positive to negative), F has a local minimum (maximum).



34. $f(\theta) = \tan^2\theta\,\sec^2\theta$. $u = \tan\theta \;\Rightarrow\; du = \sec^2\theta\,d\theta$, so

$$\int \tan^2\theta\,\sec^2\theta\,d\theta = \int u^2\,du = \tfrac{1}{3}u^3 + C = \tfrac{1}{3}\tan^3\theta + C$$

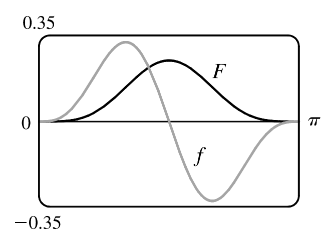
Note that f is positive and F is increasing. At $x = 0$, $f = 0$ and F has a horizontal tangent.



35. $f(x) = \sin^3 x\,\cos x$. $u = \sin x \;\Rightarrow\; du = \cos x\,dx$, so

$$\int \sin^3 x\,\cos x\,dx = \int u^3\,du = \tfrac{1}{4}u^4 + C = \tfrac{1}{4}\sin^4 x + C$$

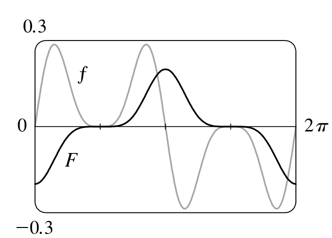
Note that at $x = \frac{\pi}{2}$, f changes from positive to negative and F has a local maximum. Also, both f and F are periodic with period π, so at $x = 0$ and at $x = \pi$, f changes from negative to positive and F has local minima.



36. $f(x) = \sin x\cos^4 x$. $u = \cos x \;\Rightarrow\; du = -\sin x\,dx$, so

$$\int \sin x\cos^4 x\,dx = \int u^4\,(-du) = -\tfrac{1}{5}u^5 + C = -\tfrac{1}{5}\cos^5 x + C$$

Note that at $x = \pi$, f changes from positive to negative and F has a local maximum. Also, both f and F are periodic with period 2π, so at $x = 0$ and at $x = 2\pi$, f changes from negative to positive and F has a local minimum.



37. Let $u = \frac{\pi}{2}t$, so $du = \frac{\pi}{2}\,dt$. When $t = 0$, $u = 0$; when $t = 1$, $u = \frac{\pi}{2}$. Thus,

$$\int_0^1 \cos(\pi t/2)\,dt = \int_0^{\pi/2} \cos u\,\left(\tfrac{2}{\pi}\,du\right) = \tfrac{2}{\pi}\left[\sin u\right]_0^{\pi/2} = \tfrac{2}{\pi}\left(\sin\tfrac{\pi}{2} - \sin 0\right) = \tfrac{2}{\pi}(1-0) = \tfrac{2}{\pi}$$

38. Let $u = 3t - 1$, so $du = 3\,dt$. When $t = 0$, $u = -1$; when $t = 1$, $u = 2$. Thus,

$$\int_0^1 (3t-1)^{50}\,dt = \int_{-1}^2 u^{50}\left(\tfrac{1}{3}\,du\right) = \tfrac{1}{3}\left[\tfrac{1}{51}u^{51}\right]_{-1}^2 = \tfrac{1}{153}\left[2^{51} - (-1)^{51}\right] = \tfrac{1}{153}(2^{51}+1)$$

39. Let $u = 1 + 7x$, so $du = 7\,dx$. When $x = 0$, $u = 1$; when $x = 1$, $u = 8$. Thus,

$$\int_0^1 \sqrt[3]{1+7x}\,dx = \int_1^8 u^{1/3}\left(\tfrac{1}{7}\,du\right) = \tfrac{1}{7}\left[\tfrac{3}{4}u^{4/3}\right]_1^8 = \tfrac{3}{28}(8^{4/3} - 1^{4/3}) = \tfrac{3}{28}(16-1) = \tfrac{45}{28}$$

40. Let $u = \frac{1}{2}t$, so $du = \frac{1}{2}\,dt$. When $t = \frac{\pi}{3}$, $u = \frac{\pi}{6}$; when $t = \frac{2\pi}{3}$, $u = \frac{\pi}{3}$. Thus,

$$\int_{\pi/3}^{2\pi/3} \csc^2\left(\tfrac{1}{2}t\right)dt = \int_{\pi/6}^{\pi/3} \csc^2 u\,(2\,du) = 2\left[-\cot u\right]_{\pi/6}^{\pi/3} = -2\left(\cot\frac{\pi}{3} - \cot\frac{\pi}{6}\right)$$

$$= -2\left(\frac{1}{\sqrt{3}} - \sqrt{3}\right) = -2\left(\tfrac{1}{3}\sqrt{3} - \sqrt{3}\right) = \tfrac{4}{3}\sqrt{3}$$

41. Let $u = \cos t$, so $du = -\sin t\,dt$. When $t = 0$, $u = 1$; when $t = \frac{\pi}{6}$, $u = \sqrt{3}/2$. Thus,

$$\int_0^{\pi/6} \frac{\sin t}{\cos^2 t}\,dt = \int_1^{\sqrt{3}/2} \frac{1}{u^2}\,(-du) = \left[\frac{1}{u}\right]_1^{\sqrt{3}/2} = \frac{2}{\sqrt{3}} - 1.$$

42. Let $u = 2 + \sqrt{x}$. Then $du = \dfrac{1}{2\sqrt{x}}\,dx$ and $\dfrac{1}{\sqrt{x}}\,dx = 2\,du$. When $x = 1$, $u = 3$; when $x = 4$, $u = 4$. Thus,

$$\int_1^4 \frac{\sqrt{2+\sqrt{x}}}{\sqrt{x}}\,dx = \int_3^4 \sqrt{u}\,(2\,du) = \left[\frac{4}{3}u^{3/2}\right]_3^4 = \frac{4}{3}(4^{3/2} - 3^{3/2}) = \frac{4}{3}\left(8 - 3\sqrt{3}\right) \text{ or } \frac{32}{3} - 4\sqrt{3}.$$

43. $\int_{-\pi/4}^{\pi/4} (x^3 + x^4\tan x)\,dx = 0$ by Theorem 6(b), since $f(x) = x^3 + x^4\tan x$ is an odd function.

44. Let $u = \sin x$, so $du = \cos x\,dx$. When $x = 0$, $u = 0$; when $x = \frac{\pi}{2}$, $u = 1$. Thus,

$$\int_0^{\pi/2} \cos x\,\sin(\sin x)\,dx = \int_0^1 \sin u\,du = \left[-\cos u\right]_0^1 = -(\cos 1 - 1) = 1 - \cos 1.$$

45. Let $u = 1 + 2x$, so $du = 2\,dx$. When $x = 0$, $u = 1$; when $x = 13$, $u = 27$. Thus,

$$\int_0^{13} \frac{dx}{\sqrt[3]{(1+2x)^2}} = \int_1^{27} u^{-2/3}\left(\tfrac{1}{2}\,du\right) = \left[\tfrac{1}{2}\cdot 3u^{1/3}\right]_1^{27} = \tfrac{3}{2}(3-1) = 3.$$

46. Assume $a > 0$. Let $u = a^2 - x^2$, so $du = -2x\,dx$. When $x = 0$, $u = a^2$; when $x = a$, $u = 0$. Thus,

$$\int_0^a x\sqrt{a^2 - x^2}\,dx = \int_{a^2}^0 u^{1/2}\left(-\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\int_0^{a^2} u^{1/2}\,du = \tfrac{1}{2}\cdot\left[\tfrac{2}{3}u^{3/2}\right]_0^{a^2} = \tfrac{1}{3}a^3.$$

47. Let $u = x^2 + a^2$, so $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$. When $x = 0$, $u = a^2$; when $x = a$, $u = 2a^2$. Thus,

$$\int_0^a x\sqrt{x^2 + a^2}\,dx = \int_{a^2}^{2a^2} u^{1/2}\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\left[\tfrac{2}{3}u^{3/2}\right]_{a^2}^{2a^2} = \left[\tfrac{1}{3}u^{3/2}\right]_{a^2}^{2a^2} = \tfrac{1}{3}\left[(2a^2)^{3/2} - (a^2)^{3/2}\right] = \tfrac{1}{3}\left(2\sqrt{2} - 1\right)a^3$$

48. $\int_{-\pi/3}^{\pi/3} x^4\sin x\,dx = 0$ by Theorem 6(b), since $f(x) = x^4\sin x$ is an odd function.

49. Let $u = x - 1$, so $u + 1 = x$ and $du = dx$. When $x = 1$, $u = 0$; when $x = 2$, $u = 1$. Thus,

$$\int_1^2 x\sqrt{x-1}\,dx = \int_0^1 (u+1)\sqrt{u}\,du = \int_0^1 (u^{3/2} + u^{1/2})\,du = \left[\tfrac{2}{5}u^{5/2} + \tfrac{2}{3}u^{3/2}\right]_0^1 = \tfrac{2}{5} + \tfrac{2}{3} = \tfrac{16}{15}.$$

50. Let $u = 1 + 2x$, so $x = \frac{1}{2}(u-1)$ and $du = 2\,dx$. When $x = 0$, $u = 1$; when $x = 4$, $u = 9$. Thus,

$$\int_0^4 \frac{x}{\sqrt{1+2x}}\,dx = \int_1^9 \frac{\frac{1}{2}(u-1)}{\sqrt{u}}\frac{du}{2} = \tfrac{1}{4}\int_1^9 (u^{1/2} - u^{-1/2})\,du = \tfrac{1}{4}\left[\tfrac{2}{3}u^{3/2} - 2u^{1/2}\right]_1^9 = \tfrac{1}{4}\cdot\tfrac{2}{3}\left[u^{3/2} - 3u^{1/2}\right]_1^9$$
$$= \tfrac{1}{6}[(27-9) - (1-3)] = \tfrac{20}{6} = \tfrac{10}{3}$$

51. Let $u = x^{-2}$, so $du = -2x^{-3}\,dx$. When $x = \frac{1}{2}$, $u = 4$; when $x = 1$, $u = 1$. Thus,

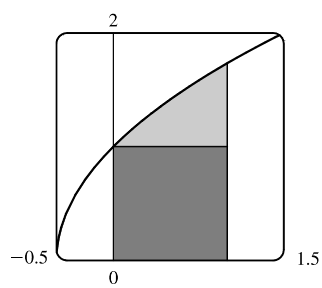
$$\int_{1/2}^1 \frac{\cos(x^{-2})}{x^3}\,dx = \int_4^1 \cos u\left(\frac{du}{-2}\right) = \frac{1}{2}\int_1^4 \cos u\,du = \frac{1}{2}\big[\sin u\big]_1^4 = \frac{1}{2}(\sin 4 - \sin 1).$$

52. Let $u = 1 + \sqrt{x}$, so $du = \dfrac{1}{2\sqrt{x}}\,dx \;\Rightarrow\; 2\sqrt{x}\,du = dx \;\Rightarrow\; 2(u-1)\,du = dx$. When $x = 0$, $u = 1$; when $x = 1$, $u = 2$. Thus,

$$\int_0^1 \frac{dx}{(1+\sqrt{x}\,)^4} = \int_1^2 \frac{1}{u^4}\cdot[2(u-1)\,du] = 2\int_1^2\left(\frac{1}{u^3} - \frac{1}{u^4}\right)du = 2\left[-\frac{1}{2u^2} + \frac{1}{3u^3}\right]_1^2$$
$$= 2\left[\left(-\tfrac{1}{8} + \tfrac{1}{24}\right) - \left(-\tfrac{1}{2} + \tfrac{1}{3}\right)\right] = 2\left(\tfrac{1}{12}\right) = \tfrac{1}{6}$$

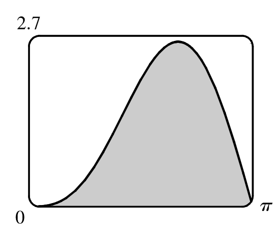
53. From the graph, it appears that the area under the curve is about $1 + \left(\text{a little more than } \frac{1}{2}\cdot 1\cdot 0.7\right)$, or about 1.4. The exact area is given by $A = \int_0^1 \sqrt{2x+1}\,dx$. Let $u = 2x + 1$, so $du = 2\,dx$. The limits change to $2\cdot 0 + 1 = 1$ and $2\cdot 1 + 1 = 3$, and

$$A = \int_1^3 \sqrt{u}\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\left[\tfrac{2}{3}u^{3/2}\right]_1^3 = \tfrac{1}{3}\left(3\sqrt{3} - 1\right) = \sqrt{3} - \tfrac{1}{3} \approx 1.399.$$



54. From the graph, it appears that the area under the curve is almost $\frac{1}{2}\cdot\pi\cdot 2.6$, or about 4. The exact area is given by

$$A = \int_0^\pi (2\sin x - \sin 2x)\,dx = -2\left[\cos x\right]_0^\pi - \int_0^\pi \sin 2x\,dx$$
$$= -2(-1-1) - 0 = 4$$



Note: $\int_0^\pi \sin 2x\,dx = 0$ since it is clear from the graph of $y = \sin 2x$ that $\int_{\pi/2}^\pi \sin 2x\,dx = -\int_0^{\pi/2} \sin 2x\,dx$.

55. First write the integral as a sum of two integrals:

$I = \int_{-2}^2 (x+3)\sqrt{4-x^2}\,dx = I_1 + I_2 = \int_{-2}^2 x\sqrt{4-x^2}\,dx + \int_{-2}^2 3\sqrt{4-x^2}\,dx$. $I_1 = 0$ by Theorem 6(b), since $f(x) = x\sqrt{4-x^2}$ is an odd function and we are integrating from $x = -2$ to $x = 2$. We interpret I_2 as three times the area of a semicircle with radius 2, so $I = 0 + 3\cdot\frac{1}{2}\left(\pi\cdot 2^2\right) = 6\pi$.

56. Let $u = x^2$. Then $du = 2x\,dx$ and the limits are unchanged ($0^2 = 0$ and $1^2 = 1$), so

$I = \int_0^1 x\sqrt{1 - x^4}\,dx = \frac{1}{2}\int_0^1 \sqrt{1 - u^2}\,du$. But this integral can be interpreted as the area of a quarter-circle with radius 1.

So $I = \frac{1}{2} \cdot \frac{1}{4}\left(\pi \cdot 1^2\right) = \frac{1}{8}\pi$.

57. The volume of inhaled air in the lungs at time t is

$$V(t) = \int_0^t f(u)\,du = \int_0^t \frac{1}{2}\sin\left(\frac{2\pi}{5}u\right)du = \int_0^{2\pi t/5} \frac{1}{2}\sin v\left(\frac{5}{2\pi}\,dv\right) \qquad \left[\text{substitute } v = \tfrac{2\pi}{5}u,\ dv = \tfrac{2\pi}{5}\,du\right]$$

$$= \frac{5}{4\pi}\left[-\cos v\right]_0^{2\pi t/5} = \frac{5}{4\pi}\left[-\cos\left(\frac{2\pi}{5}t\right) + 1\right] = \frac{5}{4\pi}\left[1 - \cos\left(\frac{2\pi}{5}t\right)\right] \text{ liters}$$

58. Let $u = \dfrac{\pi t}{12}$. Then $du = \dfrac{\pi}{12}\,dt$ and

$$\int_0^{24} R(t)\,dt = \int_0^{24}\left[85 - 0.18\cos\left(\frac{\pi t}{12}\right)\right]dt = \int_0^{2\pi}(85 - 0.18\cos u)\left(\frac{12}{\pi}\,du\right) = \frac{12}{\pi}\left[85u - 0.18\sin u\right]_0^{2\pi}$$

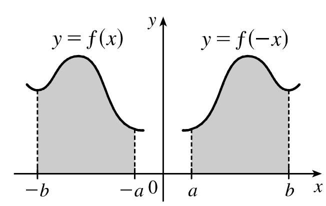
$$= \frac{12}{\pi}\left[(85 \cdot 2\pi - 0) - (0 - 0)\right] = 2040 \text{ kcal}$$

59. Let $u = 2x$. Then $du = 2\,dx$, so $\int_0^2 f(2x)\,dx = \int_0^4 f(u)\left(\frac{1}{2}\,du\right) = \frac{1}{2}\int_0^4 f(u)\,du = \frac{1}{2}(10) = 5$.

60. Let $u = x^2$. Then $du = 2x\,dx$, so $\int_0^3 x f(x^2)\,dx = \int_0^9 f(u)\left(\frac{1}{2}\,du\right) = \frac{1}{2}\int_0^9 f(u)\,du = \frac{1}{2}(4) = 2$.

61. Let $u = -x$. Then $du = -dx$, so

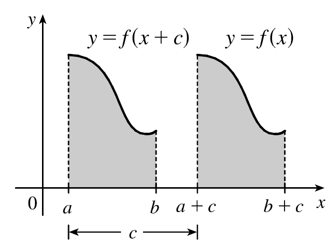
$$\int_a^b f(-x)\,dx = \int_{-a}^{-b} f(u)(-du) = \int_{-b}^{-a} f(u)\,du = \int_{-b}^{-a} f(x)\,dx$$



From the diagram, we see that the equality follows from the fact that we are reflecting the graph of f, and the limits of integration, about the y-axis.

62. Let $u = x + c$. Then $du = dx$, so

$$\int_a^b f(x + c)\,dx = \int_{a+c}^{b+c} f(u)\,du = \int_{a+c}^{b+c} f(x)\,dx$$



From the diagram, we see that the equality follows from the fact that we are translating the graph of f, and the limits of integration, by a distance c.

63. Let $u = 1 - x$. Then $x = 1 - u$ and $dx = -du$, so

$\int_0^1 x^a(1 - x)^b\,dx = \int_1^0 (1 - u)^a\,u^b(-du) = \int_0^1 u^b(1 - u)^a\,du = \int_0^1 x^b(1 - x)^a\,dx$.

64. Let $u = \pi - x$. Then $du = -dx$. When $x = \pi$, $u = 0$ and when $x = 0$, $u = \pi$. So

$$\int_0^\pi x f(\sin x)\,dx = -\int_\pi^0 (\pi - u)\,f(\sin(\pi - u))\,du = \int_0^\pi (\pi - u)\,f(\sin u)\,du$$

$$= \pi\int_0^\pi f(\sin u)\,du - \int_0^\pi u\,f(\sin u)\,du = \pi\int_0^\pi f(\sin x)\,dx - \int_0^\pi x\,f(\sin x)\,dx \quad \Rightarrow$$

$2\int_0^\pi x\,f(\sin x)\,dx = \pi\int_0^\pi f(\sin x)\,dx \quad \Rightarrow \quad \int_0^\pi x\,f(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx$.

65. $\int_0^{\pi/2} f(\cos x)\,dx = \int_0^{\pi/2} f\left[\sin\left(\frac{\pi}{2} - x\right)\right] dx \qquad \left[u = \frac{\pi}{2} - x,\, du = -dx\right]$

$$= \int_{\pi/2}^0 f(\sin u)(-du) = \int_0^{\pi/2} f(\sin u)\,du = \int_0^{\pi/2} f(\sin x)\,dx$$

Continuity of f is needed in order to apply the substitution rule for definite integrals.

66. In Exercise 65, take $f(x) = x^2$, so $\int_0^{\pi/2} \cos^2 x\,dx = \int_0^{\pi/2} \sin^2 x\,dx$. Now

$\int_0^{\pi/2} \cos^2 x\,dx + \int_0^{\pi/2} \sin^2 x\,dx = \int_0^{\pi/2} (\cos^2 x + \sin^2 x)\,dx = \int_0^{\pi/2} 1\,dx = \left[x\right]_0^{\pi/2} = \frac{\pi}{2}$,

so $2\int_0^{\pi/2} \cos^2 x\,dx = \frac{\pi}{2} \;\Rightarrow\; \int_0^{\pi/2} \cos^2 x\,dx = \frac{\pi}{4} \quad \left[= \int_0^{\pi/2} \sin^2 x\,dx\right]$.

67. Let $u = 4x + 7$. Then $du = 4\,dx$ and $dx = \dfrac{1}{4}\,du$, so $\displaystyle\int \frac{dx}{4x+7} = \int \frac{1}{u}\left(\frac{1}{4}\,du\right) = \frac{1}{4}\ln|u| + C = \frac{1}{4}\ln|4x+7| + C$.

68. Let $u = -5r$. Then $du = -5\,dr$ and $dr = -\frac{1}{5}\,du$, so $\int e^{-5r}\,dr = \int e^u\left(-\frac{1}{5}\,du\right) = -\frac{1}{5}e^u + C = -\frac{1}{5}e^{-5r} + C$.

69. Let $u = \ln x$. Then $du = \dfrac{dx}{x}$, so $\displaystyle\int \frac{(\ln x)^2}{x}\,dx = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}(\ln x)^3 + C$.

70. Let $u = ax + b$. Then $du = a\,dx$ and $dx = (1/a)\,du$, so

$$\int \frac{dx}{ax+b} = \int \frac{(1/a)\,du}{u} = \frac{1}{a}\int \frac{1}{u}\,du = \frac{1}{a}\ln|u| + C = \frac{1}{a}\ln|ax+b| + C.$$

71. Let $u = 2 + 3e^r$. Then $du = 3e^r\,dr$ and $e^r\,dr = \frac{1}{3}\,du$, so

$$\int e^r(2+3e^r)^{3/2}\,dr = \int u^{3/2}\left(\tfrac{1}{3}\,du\right) = \tfrac{1}{3}\cdot\tfrac{2}{5}u^{5/2} + C = \tfrac{2}{15}(2+3e^r)^{5/2} + C.$$

72. Let $u = \cos t$. Then $du = -\sin t\,dt$ and $\sin t\,dt = -du$, so $\int e^{\cos t}\sin t\,dt = \int e^u\,(-du) = -e^u + C = -e^{\cos t} + C$.

73. Let $u = \arctan x$. Then $du = \dfrac{1}{x^2+1}\,dx$, so $\displaystyle\int \frac{(\arctan x)^2}{x^2+1}\,dx = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}(\arctan x)^3 + C$.

74. Let $u = 3t^2 + 6t - 5$. Then $du = (6t+6)\,dt = 6(t+1)\,dt$ and $(t+1)\,dt = \frac{1}{6}\,du$, so

$$\int \frac{t+1}{3t^2+6t-5}\,dt = \int \frac{1}{u}\left(\frac{1}{6}\,du\right) = \frac{1}{6}\ln|u| + C = \frac{1}{6}\ln\left|3t^2+6t-5\right| + C.$$

75. Let $u = 1 + x^2$. Then $du = 2x\,dx$, so

$$\int \frac{1+x}{1+x^2}\,dx = \int \frac{1}{1+x^2}\,dx + \int \frac{x}{1+x^2}\,dx = \tan^{-1}x + \int \frac{\frac{1}{2}\,du}{u} = \tan^{-1}x + \tfrac{1}{2}\ln|u| + C$$
$$= \tan^{-1}x + \tfrac{1}{2}\ln\left|1+x^2\right| + C = \tan^{-1}x + \tfrac{1}{2}\ln\left(1+x^2\right) + C \quad [\text{since } 1+x^2 > 0].$$

76. Let $u = \ln x$. Then $du = (1/x)\,dx$, so $\displaystyle\int \frac{\sin(\ln x)}{x}\,dx = \int \sin u\,du = -\cos u + C = -\cos(\ln x) + C$.

77. $\displaystyle\int \frac{\sin 2x}{1+\cos^2 x}\,dx = 2\int \frac{\sin x\cos x}{1+\cos^2 x}\,dx = 2I$. Let $u = \cos x$. Then $du = -\sin x\,dx$, so

$$2I = -2\int \frac{u\,du}{1+u^2} = -2\cdot\tfrac{1}{2}\ln(1+u^2) + C = -\ln(1+u^2) + C = -\ln(1+\cos^2 x) + C.$$

Or: Let $u = 1 + \cos^2 x$.

78. Let $u = \cos x$. Then $du = -\sin x\,dx$ and $\sin x\,dx = -du$, so

$$\int \frac{\sin x}{1+\cos^2 x}\,dx = \int \frac{-du}{1+u^2} = -\tan^{-1}u + C = -\tan^{-1}(\cos x) + C.$$

79. $\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx$. Let $u = \sin x$. Then $du = \cos x\,dx$, so $\int \cot x\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln|\sin x| + C$.

80. Let $u = x^2$. Then $du = 2x\,dx$, so $\int \frac{x}{1+x^4}\,dx = \int \frac{\frac{1}{2}\,du}{1+u^2} = \frac{1}{2}\tan^{-1}u + C = \frac{1}{2}\tan^{-1}(x^2) + C$.

81. Let $u = \ln x$, so $du = \frac{dx}{x}$. When $x = e$, $u = 1$; when $x = e^4$; $u = 4$. Thus,

$$\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}} = \int_1^4 u^{-1/2}\,du = 2\left[u^{1/2}\right]_1^4 = 2(2-1) = 2.$$

82. Let $u = e^x$. Then $du = e^x\,dx$. When $x = 0$, $u = 1$; when $x = 1$, $u = e$. Thus,

$$\int_0^1 \frac{e^x}{1+e^{2x}}\,dx = \int_1^e \frac{1}{1+u^2}\,du = \left[\arctan u\right]_1^e = \arctan e - \arctan 1 = \arctan e - \frac{\pi}{4}.$$

83. Let $u = e^z + z$, so $du = (e^z + 1)\,dz$. When $z = 0$, $u = 1$; when $z = 1$, $u = e + 1$. Thus,

$$\int_0^1 \frac{e^z+1}{e^z+z}\,dz = \int_1^{e+1} \frac{1}{u}\,du = \left[\ln|u|\right]_1^{e+1} = \ln|e+1| - \ln|1| = \ln(e+1).$$

84. Let $u = (x-1)^2$, so $du = 2(x-1)\,dx$. When $x = 0$, $u = 1$; when $x = 2$, $u = 1$. Thus,

$$\int_0^2 (x-1)e^{(x-1)^2}\,dx = \int_1^1 e^u\left(\tfrac{1}{2}\,du\right) = 0 \text{ since the limits are equal.}$$

85. $\frac{x\sin x}{1+\cos^2 x} = x\cdot\frac{\sin x}{2-\sin^2 x} = x\,f(\sin x)$, where $f(t) = \frac{t}{2-t^2}$. By Exercise 64,

$$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx = \int_0^\pi x\,f(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+\cos^2 x}\,dx$$

Let $u = \cos x$. Then $du = -\sin x\,dx$. When $x = \pi$, $u = -1$ and when $x = 0$, $u = 1$. So

$$\begin{aligned}\frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+\cos^2 x}\,dx &= -\frac{\pi}{2}\int_1^{-1} \frac{du}{1+u^2} = \frac{\pi}{2}\int_{-1}^1 \frac{du}{1+u^2} = \frac{\pi}{2}\left[\tan^{-1}u\right]_{-1}^1 \\ &= \frac{\pi}{2}[\tan^{-1}1 - \tan^{-1}(-1)] = \frac{\pi}{2}\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \frac{\pi^2}{4}\end{aligned}$$

4 Review

TRUE-FALSE QUIZ

1. True by Property 2 of the Integral in Section 4.2.

2. False. Try $a = 0$, $b = 2$, $f(x) = g(x) = 1$ as a counterexample.

3. True by Property 3 of the Integral in Section 4.2.

4. False. You can't take a variable outside the integral sign. For example, using $f(x) = 1$ on $[0, 1]$,

$\int_0^1 x\, f(x)\, dx = \int_0^1 x\, dx = \left[\frac{1}{2}x^2\right]_0^1 = \frac{1}{2}$ (a constant) while $x \int_0^1 1\, dx = x\, [x]_0^1 = x \cdot 1 = x$ (a variable).

5. False. For example, let $f(x) = x^2$. Then $\int_0^1 \sqrt{x^2}\, dx = \int_0^1 x\, dx = \frac{1}{2}$, but $\sqrt{\int_0^1 x^2\, dx} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}}$.

6. True. The definite integral is a number; it does not depend on the choice of variable name. See Note 2 of Section 4.2.

7. True by the Net Change Theorem.

8. False. For example, let $v(t) = 1 - t$. On $0 \le t \le 2$, the distance traveled is

$\int_0^2 |v(t)|\, dt = \int_0^1 (1 - t)\, dt + \int_1^2 (t - 1)\, dt = \frac{1}{2} + \frac{1}{2} = 1$, but $\int_0^2 v(t)\, dt = \int_0^2 (1 - t)\, dt = 0$.

The given integral represents net displacement.

9. False. $\int_a^b f'(x)\, [f(x)]^4\, dx$ is a definite integral and, thus, is a number; $\frac{1}{5}[f(x)]^5 + C$ is a family of functions.

The statement would be true without the limits of integration.

10. False. For example, let $a = 0$, $b = 1$, $f(x) = 3$, $g(x) = x$. $f(x) > g(x)$ for each x in $(0, 1)$, but $f'(x) = 0 < 1 = g'(x)$ for $x \in (0, 1)$.

11. True by Comparison Property 7 of the Integral in Section 4.2.

12. True. $\int_{-5}^{5} \left(ax^2 + bx + c\right) dx = \int_{-5}^{5} \left(ax^2 + c\right) dx + \int_{-5}^{5} bx\, dx$

$$= 2\int_0^5 \left(ax^2 + c\right) dx + 0 \quad \text{[because } ax^2 + c \text{ is even and } bx \text{ is odd]}$$

13. False. For example, the function $y = |x|$ is continuous on $\mathbb{R}$, but has no derivative at $x = 0$.

14. True by FTC1.

15. True. By Property 5 in Section 4.2, $\displaystyle\int_\pi^{3\pi} \frac{\sin x}{x}\, dx = \int_\pi^{2\pi} \frac{\sin x}{x}\, dx + \int_{2\pi}^{3\pi} \frac{\sin x}{x}\, dx \quad \Rightarrow$

$$\int_\pi^{2\pi} \frac{\sin x}{x}\, dx = \int_\pi^{3\pi} \frac{\sin x}{x}\, dx - \int_{2\pi}^{3\pi} \frac{\sin x}{x}\, dx \quad \Rightarrow \quad \int_\pi^{2\pi} \frac{\sin x}{x}\, dx = \int_\pi^{3\pi} \frac{\sin x}{x}\, dx + \int_{3\pi}^{2\pi} \frac{\sin x}{x}\, dx$$

[by reversing limits].

16. False. For example, $\int_0^1 \left(x - \frac{1}{2}\right) dx = \left[\frac{1}{2}x^2 - \frac{1}{2}x\right]_0^1 = \left(\frac{1}{2} - \frac{1}{2}\right) - (0 - 0) = 0$, but $f(x) = x - \frac{1}{2} \ne 0$.

17. False. $\int_a^b f(x)\, dx$ is a constant, so $\dfrac{d}{dx}\left(\int_a^b f(x)\, dx\right) = 0$, not $f(x)$ [unless $f(x) = 0$]. Compare the given statement carefully with FTC1, in which the upper limit in the integral is x.

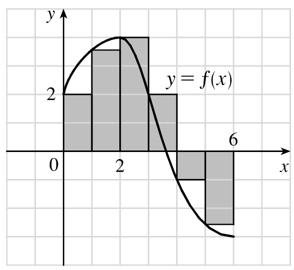
18. False. See the paragraph before Figure 4 in Section 4.2, and notice that $y = x - x^3 < 0$ for $1 < x \le 2$.

19. False. The function $f(x) = 1/x^4$ is not bounded on the interval $[-2, 1]$. It has an infinite discontinuity at $x = 0$, so it is not integrable on the interval. (If the integral were to exist, a positive value would be expected, by Comparison Property 4.2.6 of Integrals.)

20. False. For example, if $f(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{if } -1 \le x < 0 \end{cases}$ then f has a jump discontinuity at 0, but $\int_{-1}^{1} f(x)\,dx$ exists and is equal to 1.

EXERCISES

1. (a)



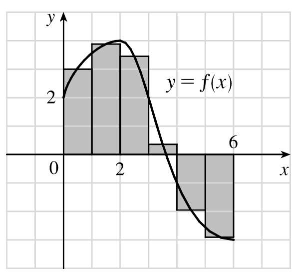
$$L_6 = \sum_{i=1}^{6} f(x_{i-1})\,\Delta x \quad [\Delta x = \tfrac{6-0}{6} = 1]$$

$$= f(x_0)\cdot 1 + f(x_1)\cdot 1 + f(x_2)\cdot 1 + f(x_3)\cdot 1 + f(x_4)\cdot 1 + f(x_5)\cdot 1$$

$$\approx 2 + 3.5 + 4 + 2 + (-1) + (-2.5) = 8$$

The Riemann sum represents the sum of the areas of the four rectangles above the x-axis minus the sum of the areas of the two rectangles below the x-axis.

(b)



$$M_6 = \sum_{i=1}^{6} f(\overline{x}_i)\,\Delta x \quad [\Delta x = \tfrac{6-0}{6} = 1]$$

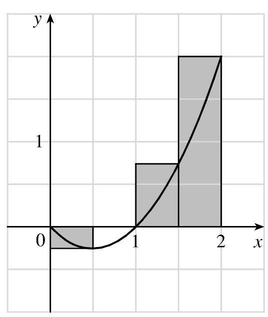
$$= f(\overline{x}_1)\cdot 1 + f(\overline{x}_2)\cdot 1 + f(\overline{x}_3)\cdot 1 + f(\overline{x}_4)\cdot 1 + f(\overline{x}_5)\cdot 1 + f(\overline{x}_6)\cdot 1$$

$$= f(0.5) + f(1.5) + f(2.5) + f(3.5) + f(4.5) + f(5.5)$$

$$\approx 3 + 3.9 + 3.4 + 0.3 + (-2) + (-2.9) = 5.7$$

The Riemann sum represents the sum of the areas of the four rectangles above the x-axis minus the sum of the areas of the two rectangles below the x-axis.

2. (a)



$f(x) = x^2 - x$ and $\Delta x = \frac{2-0}{4} = 0.5 \quad \Rightarrow$

$$R_4 = 0.5f(0.5) + 0.5f(1) + 0.5f(1.5) + 0.5f(2)$$

$$= 0.5(-0.25 + 0 + 0.75 + 2) = 1.25$$

The Riemann sum represents the sum of the areas of the two rectangles above the x-axis minus the area of the rectangle below the x-axis. (The second rectangle vanishes.)

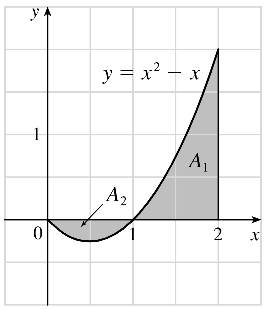
(b) $\displaystyle\int_0^2 (x^2 - x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x \qquad [\Delta x = 2/n \text{ and } x_i = 2i/n]$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{4i^2}{n^2} - \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \lim_{n\to\infty} \frac{2}{n}\left[\frac{4}{n^2}\sum_{i=1}^{n} i^2 - \frac{2}{n}\sum_{i=1}^{n} i\right]$$

$$= \lim_{n\to\infty} \left[\frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{4}{n^2}\cdot\frac{n(n+1)}{2}\right] = \lim_{n\to\infty}\left[\frac{4}{3}\cdot\frac{n+1}{n}\cdot\frac{2n+1}{n} - 2\cdot\frac{n+1}{n}\right]$$

$$= \lim_{n\to\infty}\left[\frac{4}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) - 2\left(1+\frac{1}{n}\right)\right] = \tfrac{4}{3}\cdot 1\cdot 2 - 2\cdot 1 = \tfrac{2}{3}$$

(c) $\int_0^2 (x^2 - x)\,dx = \left[\frac{1}{3}x^3 - \frac{1}{2}x^2\right]_0^2 = \left(\frac{8}{3} - 2\right) = \frac{2}{3}$

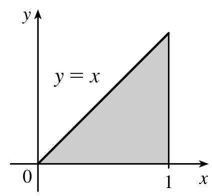
(d) 

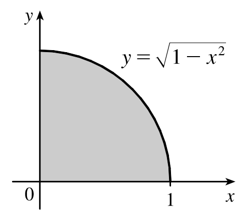
$\int_0^2 (x^2 - x)\,dx = A_1 - A_2$, where A_1 and A_2 are the areas shown in the diagram.

3. $\int_0^1 \left(x + \sqrt{1 - x^2}\right)dx = \int_0^1 x\,dx + \int_0^1 \sqrt{1 - x^2}\,dx = I_1 + I_2$.

I_1 can be interpreted as the area of the triangle shown in the figure and I_2 can be interpreted as the area of the quarter-circle.

Area $= \frac{1}{2}(1)(1) + \frac{1}{4}(\pi)(1)^2 = \frac{1}{2} + \frac{\pi}{4}$.





4. On $[0, \pi]$, $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \sin x_i\,\Delta x = \int_0^\pi \sin x\,dx = [-\cos x]_0^\pi = -(-1) - (-1) = 2$.

5. $\int_0^6 f(x)\,dx = \int_0^4 f(x)\,dx + \int_4^6 f(x)\,dx \quad\Rightarrow\quad 10 = 7 + \int_4^6 f(x)\,dx \quad\Rightarrow\quad \int_4^6 f(x)\,dx = 10 - 7 = 3$

6. (a) $\int_1^5 (x + 2x^5)\,dx = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} f(x_i)\,\Delta x \qquad \left[\Delta x = \dfrac{5-1}{n} = \dfrac{4}{n},\ x_i = 1 + \dfrac{4i}{n}\right]$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(1 + \frac{4i}{n}\right) + 2\left(1 + \frac{4i}{n}\right)^5\right] \cdot \frac{4}{n} \overset{\text{CAS}}{=} \lim_{n\to\infty} \frac{1305n^4 + 3126n^3 + 2080n^2 - 256}{n^3} \cdot \frac{4}{n}$$

$= 5220$

(b) $\int_1^5 (x + 2x^5)\,dx = \left[\frac{1}{2}x^2 + \frac{2}{6}x^6\right]_1^5 = \left(\frac{25}{2} + \frac{15{,}625}{3}\right) - \left(\frac{1}{2} + \frac{1}{3}\right) = 12 + 5208 = 5220$

7. First note that either a or b must be the graph of $\int_0^x f(t)\,dt$, since $\int_0^0 f(t)\,dt = 0$, and $c(0) \neq 0$. Now notice that $b > 0$ when c is increasing, and that $c > 0$ when a is increasing. It follows that c is the graph of $f(x)$, b is the graph of $f'(x)$, and a is the graph of $\int_0^x f(t)\,dt$.

8. (a) By FTC2, we have $\displaystyle\int_0^{\pi/2} \frac{d}{dx}\left(\sin\frac{x}{2}\cos\frac{x}{3}\right)dx = \left[\sin\frac{x}{2}\cos\frac{x}{3}\right]_0^{\pi/2} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - 0 \cdot 1 = \frac{\sqrt{6}}{4}$.

(b) $\displaystyle\frac{d}{dx}\int_0^{\pi/2} \sin\frac{x}{2}\cos\frac{x}{3}\,dx = 0$, since the definite integral is a constant.

(c) $\displaystyle\frac{d}{dx}\int_x^{\pi/2} \sin\frac{t}{2}\cos\frac{t}{3}\,dt = \frac{d}{dx}\left(-\int_{\pi/2}^{x} \sin\frac{t}{2}\cos\frac{t}{3}\,dt\right) = -\frac{d}{dx}\int_{\pi/2}^{x} \sin\frac{t}{2}\cos\frac{t}{2}\,dt = -\sin\frac{x}{2}\cos\frac{x}{3}$, by FTC1.

9. $g(4) = \int_0^4 f(t)\,dt = \int_0^1 f(t)\,dt + \int_1^2 f(t)\,dt + \int_2^3 f(t)\,dt + \int_3^4 f(t)\,dt$

$$= -\tfrac{1}{2}\cdot 1\cdot 2 \quad \begin{bmatrix}\text{area of triangle,}\\ \textit{below } t\text{-axis}\end{bmatrix} \quad + \tfrac{1}{2}\cdot 1\cdot 2 + 1\cdot 2 + \tfrac{1}{2}\cdot 1\cdot 2 = 3$$

By FTC1, $g'(x) = f(x)$, so $g'(4) = f(4) = 0$.

10. $g(x) = \int_0^x f(t)\,dt \quad \Rightarrow \quad g'(x) = f(x) \quad$ [by FTC1] $\quad \Rightarrow \quad g''(x) = f'(x)$, so $g''(4) = f'(4) = -2$, which is the slope of the line segment at $x = 4$.

11. $\int_{-1}^0 (x^2 + 5x)\,dx = \left[\frac{1}{3}x^3 + \frac{5}{2}x^2\right]_{-1}^0 = 0 - \left(-\frac{1}{3} + \frac{5}{2}\right) = -\frac{13}{6}$

12. $\int_0^T (x^4 - 8x + 7)\,dx = \left[\frac{1}{5}x^5 - 4x^2 + 7x\right]_0^T = \left(\frac{1}{5}T^5 - 4T^2 + 7T\right) - 0 = \frac{1}{5}T^5 - 4T^2 + 7T$

13. $\int_0^1 (1 - x^9)\,dx = \left[x - \frac{1}{10}x^{10}\right]_0^1 = \left(1 - \frac{1}{10}\right) - 0 = \frac{9}{10}$

14. Let $u = 1 - x$, so $du = -dx$ and $dx = -du$. When $x = 0$, $u = 1$; when $x = 1$, $u = 0$. Thus,

$\int_0^1 (1 - x)^9\,dx = \int_1^0 u^9(-du) = \int_0^1 u^9\,du = \frac{1}{10}\left[u^{10}\right]_0^1 = \frac{1}{10}(1 - 0) = \frac{1}{10}$.

15. $\displaystyle\int_1^9 \frac{\sqrt{u} - 2u^2}{u}\,du = \int_1^9 (u^{-1/2} - 2u)\,du = \left[2u^{1/2} - u^2\right]_1^9 = (6 - 81) - (2 - 1) = -76$

16. $\int_0^1 \left(\sqrt[4]{u} + 1\right)^2\,du = \int_0^1 (u^{1/2} + 2u^{1/4} + 1)\,du = \left[\frac{2}{3}u^{3/2} + \frac{8}{5}u^{5/4} + u\right]_0^1 = \left(\frac{2}{3} + \frac{8}{5} + 1\right) - 0 = \frac{49}{15}$

17. Let $u = y^2 + 1$, so $du = 2y\,dy$ and $y\,dy = \frac{1}{2}\,du$. When $y = 0$, $u = 1$; when $y = 1$, $u = 2$. Thus,

$\int_0^1 y(y^2 + 1)^5\,dy = \int_1^2 u^5\left(\frac{1}{2}\,du\right) = \frac{1}{2}\left[\frac{1}{6}u^6\right]_1^2 = \frac{1}{12}(64 - 1) = \frac{63}{12} = \frac{21}{4}$.

18. Let $u = 1 + y^3$, so $du = 3y^2\,dy$ and $y^2\,dy = \frac{1}{3}\,du$. When $y = 0$, $u = 1$; when $y = 2$, $u = 9$. Thus,

$\int_0^2 y^2\sqrt{1 + y^3}\,dy = \int_1^9 u^{1/2}\left(\frac{1}{3}\,du\right) = \frac{1}{3}\left[\frac{2}{3}u^{3/2}\right]_1^9 = \frac{2}{9}(27 - 1) = \frac{52}{9}$.

19. $\displaystyle\int_1^5 \frac{dt}{(t-4)^2}$ does not exist because the function $f(t) = \dfrac{1}{(t-4)^2}$ has an infinite discontinuity at $t = 4$; that is, f is discontinuous on the interval $[1, 5]$.

20. Let $u = 3\pi t$, so $du = 3\pi\,dt$. When $t = 0$, $u = 1$; when $t = 1$, $u = 3\pi$. Thus,

$\displaystyle\int_0^1 \sin(3\pi t)\,dt = \int_0^{3\pi} \sin u\left(\frac{1}{3\pi}\,du\right) = \frac{1}{3\pi}\left[-\cos u\right]_0^{3\pi} = -\frac{1}{3\pi}(-1 - 1) = \frac{2}{3\pi}$.

21. Let $u = v^3$, so $du = 3v^2\,dv$. When $v = 0$, $u = 0$; when $v = 1$, $u = 1$. Thus,

$\int_0^1 v^2\cos(v^3)\,dv = \int_0^1 \cos u\left(\frac{1}{3}\,du\right) = \frac{1}{3}\left[\sin u\right]_0^1 = \frac{1}{3}(\sin 1 - 0) = \frac{1}{3}\sin 1$.

22. $\displaystyle\int_{-1}^1 \frac{\sin x}{1 + x^2}\,dx = 0$ by Theorem 4.5.6(b), since $f(x) = \dfrac{\sin x}{1 + x^2}$ is an odd function.

23. $\displaystyle\int_{-\pi/4}^{\pi/4} \frac{t^4 \tan t}{2+\cos t}\,dt = 0$ by Theorem 4.5.6, since $f(t) = \dfrac{t^4 \tan t}{2+\cos t}$ is an odd function.

24. Let $u = x^2 + 4x$. Then $du = (2x+4)\,dx = 2(x+2)\,dx$, so

$$\int \frac{x+2}{\sqrt{x^2+4x}}\,dx = \int u^{-1/2}\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\cdot 2u^{1/2} + C = \sqrt{u} + C = \sqrt{x^2+4x} + C.$$

25. Let $u = \sin \pi t$. Then $du = \pi \,\cos \pi t\,dt$, so $\int \sin \pi t\,\cos \pi t\,dt = \int u\left(\frac{1}{\pi}\,du\right) = \frac{1}{\pi}\cdot\frac{1}{2}u^2 + C = \frac{1}{2\pi}(\sin \pi t)^2 + C$.

26. Let $u = \cos x$. Then $du = -\sin x\,dx$, so $\int \sin x\,\cos(\cos x)\,dx = -\int \cos u\,du = -\sin u + C = -\sin(\cos x) + C$.

27. Let $u = 2\theta$. Then $du = 2\,d\theta$, so

$$\textstyle\int_0^{\pi/8} \sec 2\theta\,\tan 2\theta\,d\theta = \int_0^{\pi/4} \sec u\,\tan u\left(\frac{1}{2}\,du\right) = \frac{1}{2}\big[\sec u\big]_0^{\pi/4} = \frac{1}{2}\left(\sec\frac{\pi}{4} - \sec 0\right) = \frac{1}{2}\left(\sqrt{2}-1\right) = \frac{1}{2}\sqrt{2} - \frac{1}{2}.$$

28. Let $u = 1 + \tan t$, so $du = \sec^2 t\,dt$. When $t = 0$, $u = 1$; when $t = \frac{\pi}{4}$, $u = 2$. Thus,

$$\textstyle\int_0^{\pi/4} (1+\tan t)^3 \sec^2 t\,dt = \int_1^2 u^3\,du = \left[\frac{1}{4}u^4\right]_1^2 = \frac{1}{4}\left(2^4 - 1^4\right) = \frac{1}{4}(16-1) = \frac{15}{4}.$$

29. Let $u = 1 - x$. Then $x = 1 - u$, $du = -dx$, and $dx = -du$, so

$$\begin{aligned}\textstyle\int x(1-x)^{2/3}\,dx &= \textstyle\int (1-u)\cdot u^{2/3}\,(-du) = \int \left(u^{5/3} - u^{2/3}\right)du = \frac{3}{8}u^{8/3} - \frac{3}{5}u^{5/3} + C \\ &= \tfrac{3}{8}(1-x)^{8/3} - \tfrac{3}{5}(1-x)^{5/3} + C\end{aligned}$$

30. Let $u = \tan\theta$. Then $du = \sec^2\theta\,d\theta$, so $\int \sec^2\theta\,\tan^3\theta\,d\theta = \int u^3\,du = \frac{1}{4}u^4 + C = \frac{1}{4}\tan^4\theta + C$.

31. Since $x^2 - 4 < 0$ for $0 \le x < 2$ and $x^2 - 4 > 0$ for $2 < x \le 3$, we have $\left|x^2-4\right| = -(x^2-4) = 4 - x^2$ for $0 \le x < 2$ and $\left|x^2-4\right| = x^2 - 4$ for $2 < x \le 3$. Thus,

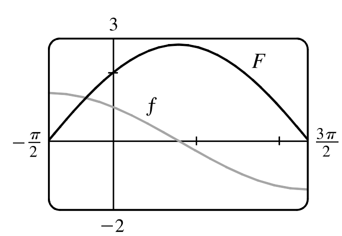
$$\begin{aligned}\int_0^3 \left|x^2-4\right|dx &= \int_0^2 (4-x^2)\,dx + \int_2^3 (x^2-4)\,dx = \left[4x - \frac{x^3}{3}\right]_0^2 + \left[\frac{x^3}{3} - 4x\right]_2^3 \\ &= \left(8 - \tfrac{8}{3}\right) - 0 + (9-12) - \left(\tfrac{8}{3} - 8\right) = \tfrac{16}{3} - 3 + \tfrac{16}{3} = \tfrac{32}{3} - \tfrac{9}{3} = \tfrac{23}{3}\end{aligned}$$

32. Since $\sqrt{x} - 1 < 0$ for $0 \le x < 1$ and $\sqrt{x} - 1 > 0$ for $1 < x \le 4$, we have $\left|\sqrt{x}-1\right| = -\left(\sqrt{x}-1\right) = 1 - \sqrt{x}$ for $0 \le x < 1$ and $\left|\sqrt{x}-1\right| = \sqrt{x} - 1$ for $1 < x \le 4$. Thus,

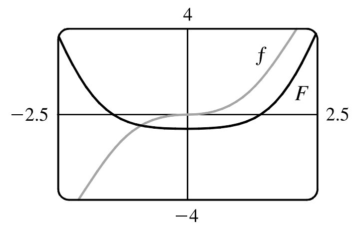
$$\begin{aligned}\int_0^4 \left|\sqrt{x}-1\right|dx &= \int_0^1 \left(1-\sqrt{x}\right)dx + \int_1^4 \left(\sqrt{x}-1\right)dx = \left[x - \tfrac{2}{3}x^{3/2}\right]_0^1 + \left[\tfrac{2}{3}x^{3/2} - x\right]_1^4 \\ &= \left(1 - \tfrac{2}{3}\right) - 0 + \left(\tfrac{16}{3} - 4\right) - \left(\tfrac{2}{3} - 1\right) = \tfrac{1}{3} + \tfrac{16}{3} - 4 + \tfrac{1}{3} = 6 - 4 = 2\end{aligned}$$

33. Let $u = 1 + \sin x$. Then $du = \cos x\,dx$, so

$$\int \frac{\cos x\,dx}{\sqrt{1+\sin x}} = \textstyle\int u^{-1/2}\,du = 2u^{1/2} + C = 2\sqrt{1+\sin x} + C.$$

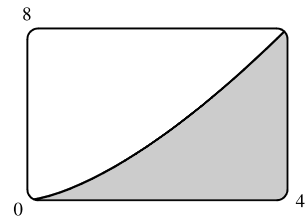


34. Let $u = x^2 + 1$. Then $x^2 = u - 1$ and $x\,dx = \frac{1}{2}\,du$, so



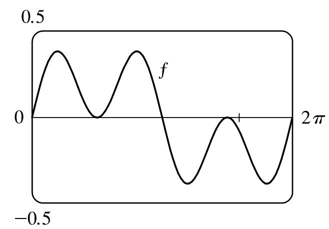
$$\int \frac{x^3}{\sqrt{x^2+1}}\,dx = \int \frac{(u-1)}{\sqrt{u}}\left(\tfrac{1}{2}\,du\right) = \frac{1}{2}\int (u^{1/2} - u^{-1/2})\,du$$
$$= \tfrac{1}{2}\left(\tfrac{2}{3}u^{3/2} - 2u^{1/2}\right) + C = \tfrac{1}{3}(x^2+1)^{3/2} - (x^2+1)^{1/2} + C$$
$$= \tfrac{1}{3}(x^2+1)^{1/2}\left[(x^2+1) - 3\right] + C = \tfrac{1}{3}\sqrt{x^2+1}\,(x^2-2) + C$$

35. From the graph, it appears that the area under the curve $y = x\sqrt{x}$ between $x = 0$ and $x = 4$ is somewhat less than half the area of an 8×4 rectangle, so perhaps about 13 or 14. To find the exact value, we evaluate



$\int_0^4 x\sqrt{x}\,dx = \int_0^4 x^{3/2}\,dx = \left[\frac{2}{5}x^{5/2}\right]_0^4 = \frac{2}{5}(4)^{5/2} = \frac{64}{5} = 12.8.$

36. From the graph, it seems as though $\int_0^{2\pi} \cos^2 x\,\sin x\,dx$ is equal to 0. To evaluate the integral, let $u = \cos x \;\Rightarrow\; du = -\sin x\,dx$. Thus, $I = \int_1^1 u^2\,(-du) = 0$.



37. Area $= \int_0^4 (x^2+5)\,dx = \left[\frac{1}{3}x^3 + 5x\right]_0^4 = \left(\frac{64}{3} + 20\right) - 0 = \frac{124}{3}$

38. Area $= \int_0^{\pi/2} \sin x\,dx = \left[-\cos x\right]_0^{\pi/2} = -\cos\dfrac{\pi}{2} - (-\cos 0) = 0 + 1 = 1.$

39. (a) $\int_1^5 f(x)\,dx = \int_1^3 f(x)\,dx + \int_3^4 f(x)\,dx + \int_4^5 f(x)\,dx = 3 - 2 + 1 = 2$

(b) $\int_1^5 |f(x)|\,dx = \int_1^3 f(x)\,dx + \int_3^4 [-f(x)]\,dx + \int_4^5 f(x)\,dx = 3 + 2 + 1 = 6$

40. (a) $\int_1^4 f(x)\,dx + \int_3^5 f(x)\,dx = \int_1^3 f(x)\,dx + \int_3^4 f(x)\,dx + \int_3^4 f(x)\,dx + \int_4^5 f(x)\,dx = 3 - 2 - 2 + 1 = 0$

(b) $\int_1^3 2f(x)\,dx + \int_3^5 6f(x)\,dx = 2\left[\int_1^3 f(x)\,dx\right] + 6\left[\int_3^5 f(x)\,dx\right]$
$$= 2\left[\int_1^3 f(x)\,dx\right] + 6\left[\int_3^4 f(x)\,dx + \int_4^5 f(x)\,dx\right]$$
$$= 2(3) + 6(-2+1) = 6 + 6(-1) = 0$$

41. $F(x) = \displaystyle\int_0^x \frac{t^2}{1+t^3}\,dt \;\Rightarrow\; F'(x) = \frac{d}{dx}\int_0^x \frac{t^2}{1+t^3}\,dt = \frac{x^2}{1+x^3}$

42. $F(x) = \displaystyle\int_x^1 \sqrt{t + \sin t}\,dt = -\int_1^x \sqrt{t+\sin t}\,dt \;\Rightarrow\; F'(x) = -\frac{d}{dx}\int_1^x \sqrt{t+\sin t}\,dt = -\sqrt{x+\sin x}$

43. Let $u = x^4$. Then $\dfrac{du}{dx} = 4x^3$. Also, $\dfrac{dg}{dx} = \dfrac{dg}{du}\dfrac{du}{dx}$, so

$$g'(x) = \frac{d}{dx}\int_0^{x^4} \cos(t^2)\,dt = \frac{d}{du}\int_0^u \cos(t^2)\,dt \cdot \frac{du}{dx} = \cos(u^2)\,\frac{du}{dx} = 4x^3\cos(x^8).$$

44. Let $u = \sin x$. Then $\dfrac{du}{dx} = \cos x$. Also, $\dfrac{dg}{dx} = \dfrac{dg}{du}\dfrac{du}{dx}$, so

$$g'(x) = \frac{d}{dx}\int_1^{\sin x} \frac{1-t^2}{1+t^4}\,dt = \frac{d}{du}\int_1^u \frac{1-t^2}{1+t^4}\,dt \cdot \frac{du}{dx} = \frac{1-u^2}{1+u^4}\cdot\frac{du}{dx} = \frac{1-\sin^2 x}{1+\sin^4 x}\cdot\cos x = \frac{\cos^3 x}{1+\sin^4 x}$$

45. $y = \displaystyle\int_{\sqrt{x}}^{x} \frac{\cos\theta}{\theta}\,d\theta = \int_1^x \frac{\cos\theta}{\theta}\,d\theta + \int_{\sqrt{x}}^{1} \frac{\cos\theta}{\theta}\,d\theta = \int_1^x \frac{\cos\theta}{\theta}\,d\theta - \int_1^{\sqrt{x}} \frac{\cos\theta}{\theta}\,d\theta \quad\Rightarrow$

$$y' = \frac{\cos x}{x} - \frac{\cos\sqrt{x}}{\sqrt{x}}\,\frac{1}{2\sqrt{x}} = \frac{2\cos x - \cos\sqrt{x}}{2x}$$

46. $y = \int_{2x}^{3x+1}\sin\left(t^4\right)dt = \int_{2x}^{0}\sin\left(t^4\right)dt + \int_0^{3x+1}\sin\left(t^4\right)dt = \int_0^{3x+1}\sin\left(t^4\right)dt - \int_0^{2x}\sin\left(t^4\right)dt \quad\Rightarrow$

$$y' = \sin\left[(3x+1)^4\right]\cdot\frac{d}{dx}(3x+1) - \sin\left[(2x)^4\right]\cdot\frac{d}{dx}(2x) = 3\sin\left[(3x+1)^4\right] - 2\sin\left[(2x)^4\right]$$

47. If $1 \le x \le 3$, then $\sqrt{1^2+3} \le \sqrt{x^2+3} \le \sqrt{3^2+3} \quad\Rightarrow\quad 2 \le \sqrt{x^2+3} \le 2\sqrt{3}$, so

$2(3-1) \le \int_1^3 \sqrt{x^2+3}\,dx \le 2\sqrt{3}(3-1)$; that is, $4 \le \int_1^3 \sqrt{x^2+3}\,dx \le 4\sqrt{3}$.

48. If $2 \le x \le 4$, then $2^3+2 \le x^3+2 \le 4^3+2 \quad\Rightarrow\quad 10 \le x^3+2 \le 66$ and $\dfrac{1}{66} \le \dfrac{1}{x^3+2} \le \dfrac{1}{10}$, so

$\dfrac{1}{66}(4-2) \le \displaystyle\int_2^4 \frac{1}{x^3+2}\,dx \le \frac{1}{10}(4-2)$; that is, $\dfrac{1}{33} \le \displaystyle\int_2^4 \frac{1}{x^3+2}\,dx \le \frac{1}{5}$.

49. $0 \le x \le 1 \quad\Rightarrow\quad 0 \le \cos x \le 1 \quad\Rightarrow\quad x^2\cos x \le x^2 \quad\Rightarrow\quad \int_0^1 x^2\cos x\,dx \le \int_0^1 x^2\,dx = \frac{1}{3}\left[x^3\right]_0^1 = \frac{1}{3}$ [Property 4.2.7].

50. From a graph we see that $\dfrac{\sin x}{x}$ is decreasing on the interval $\left[\dfrac{\pi}{4}, \dfrac{\pi}{2}\right]$. Therefore, the largest value of $\dfrac{\sin x}{x}$ on $\left[\dfrac{\pi}{4}, \dfrac{\pi}{2}\right]$ is

$\dfrac{\sin(\pi/4)}{\pi/4} = \dfrac{\sqrt{2}/2}{\pi/4} = \dfrac{2\sqrt{2}}{\pi}$. (Alternatively we could apply the Closed Interval Method to find the maximum value.) By

Property 4.2.8 with $M = \dfrac{2\sqrt{2}}{\pi}$, we have $\displaystyle\int_{\pi/4}^{\pi/2} \frac{\sin x}{x}\,dx \le \frac{2\sqrt{2}}{\pi}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$.

51. $\Delta x = (3-0)/6 = \frac{1}{2}$, so the endpoints are $0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}$, and 3, and the midpoints are $\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}, \frac{9}{4}$, and $\frac{11}{4}$.

The Midpoint Rule gives

$$\int_0^3 \sin(x^3)\,dx \approx \sum_{i=1}^{6} f(\overline{x}_i)\,\Delta x = \tfrac{1}{2}\left[\sin\left(\tfrac{1}{4}\right)^3 + \sin\left(\tfrac{3}{4}\right)^3 + \sin\left(\tfrac{5}{4}\right)^3 + \sin\left(\tfrac{7}{4}\right)^3 + \sin\left(\tfrac{9}{4}\right)^3 + \sin\left(\tfrac{11}{4}\right)^3\right] \approx 0.2810.$$

52. (a) Displacement $= \int_0^5 \left(t^2 - t\right)dt = \left[\frac{1}{3}t^3 - \frac{1}{2}t^2\right]_0^5 = \frac{125}{3} - \frac{25}{2} = \frac{175}{6} = 29.1\overline{6}$ meters

(b) Distance traveled $= \int_0^5 \left|t^2 - t\right|dt = \int_0^5 |t(t-1)|\,dt = \int_0^1 \left(t - t^2\right)dt + \int_1^5 \left(t^2 - t\right)dt$

$= \left[\frac{1}{2}t^2 - \frac{1}{3}t^3\right]_0^1 + \left[\frac{1}{3}t^3 - \frac{1}{2}t^2\right]_1^5 = \frac{1}{2} - \frac{1}{3} - 0 + \left(\frac{125}{3} - \frac{25}{2}\right) - \left(\frac{1}{3} - \frac{1}{2}\right) = \frac{177}{6} = 29.5$ meters

53. Note that $r(t) = b'(t)$, where $b(t) =$ the number of barrels of oil consumed up to time t. So, by the Net Change Theorem,

$\int_{15}^{20} r(t)\,dt = b(20) - b(15)$ represents the number of barrels of oil consumed from Jan. 1, 2015, through Jan. 1, 2020.

54. Distance covered $= \int_0^{5.0} v(t)\,dt \approx M_5 = \frac{5.0-0}{5}[v(0.5) + v(1.5) + v(2.5) + v(3.5) + v(4.5)]$

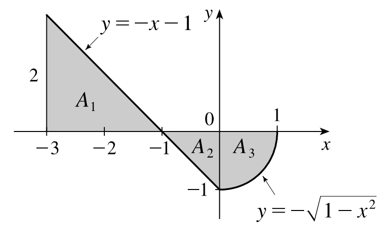
$= 1(4.67 + 8.86 + 10.22 + 10.67 + 10.81) = 45.23$ m

55. We use the Midpoint Rule with $n = 6$ and $\Delta t = \frac{24-0}{6} = 4$. The increase in the bee population was

$$\begin{aligned}\int_0^{24} r(t)\,dt \approx M_6 &= 4[r(2)+r(6)+r(10)+r(14)+r(18)+r(22)]\\ &\approx 4[50+1000+7000+8550+1350+150] = 4(18{,}100) = 72{,}400\end{aligned}$$

56. $A_1 = \frac{1}{2}bh = \frac{1}{2}(2)(2) = 2$, $A_2 = \frac{1}{2}bh = \frac{1}{2}(1)(1) = \frac{1}{2}$, and since $y = -\sqrt{1-x^2}$ for $0 \le x \le 1$ represents a quarter-circle with radius 1, $A_3 = \frac{1}{4}\pi r^2 = \frac{1}{4}\pi(1)^2 = \frac{\pi}{4}$. So

$$\int_{-3}^{1} f(x)\,dx = A_1 - A_2 - A_3 = 2 - \tfrac{1}{2} - \tfrac{\pi}{4} = \tfrac{1}{4}(6-\pi)$$



57. Let $u = 2\sin\theta$. Then $du = 2\cos\theta\,d\theta$ and when $\theta = 0$, $u = 0$; when $\theta = \frac{\pi}{2}$, $u = 2$. Thus,

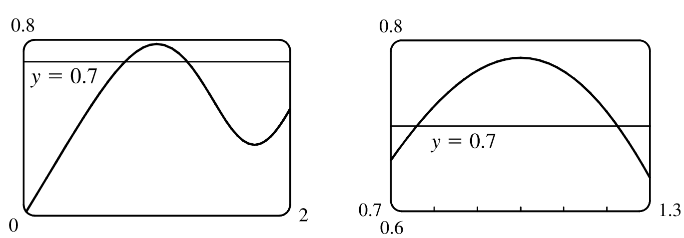
$\int_0^{\pi/2} f(2\sin\theta)\cos\theta\,d\theta = \int_0^2 f(u)\left(\frac{1}{2}\,du\right) = \frac{1}{2}\int_0^2 f(u)\,du = \frac{1}{2}\int_0^2 f(x)\,dx = \frac{1}{2}(6) = 3.$

58. (a) C is increasing on those intervals where C' is positive. By the Fundamental Theorem of Calculus,

$C'(x) = \dfrac{d}{dx}\left[\int_0^x \cos\left(\frac{\pi}{2}t^2\right)dt\right] = \cos\left(\frac{\pi}{2}x^2\right)$. This is positive when $\frac{\pi}{2}x^2$ is in the interval $\left(\left(2n-\frac{1}{2}\right)\pi, \left(2n+\frac{1}{2}\right)\pi\right)$, n any integer. This implies that $\left(2n-\frac{1}{2}\right)\pi < \frac{\pi}{2}x^2 < \left(2n+\frac{1}{2}\right)\pi \quad\Leftrightarrow\quad 0 \le |x| < 1$ or $\sqrt{4n-1} < |x| < \sqrt{4n+1}$, n any positive integer. So C is increasing on the intervals $(-1,1)$, $\left(\sqrt{3},\sqrt{5}\right)$, $\left(-\sqrt{5},-\sqrt{3}\right)$, $\left(\sqrt{7},3\right)$, $\left(-3,-\sqrt{7}\right)$, ...

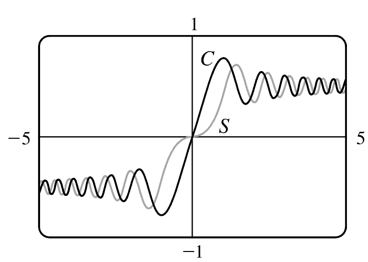
(b) C is concave upward on those intervals where $C'' > 0$. We differentiate C' to find C'': $C'(x) = \cos\left(\frac{\pi}{2}x^2\right) \Rightarrow$ $C''(x) = -\sin\left(\frac{\pi}{2}x^2\right)\left(\frac{\pi}{2}\cdot 2x\right) = -\pi x\sin\left(\frac{\pi}{2}x^2\right)$. For $x > 0$, this is positive where $(2n-1)\pi < \frac{\pi}{2}x^2 < 2n\pi$, n any positive integer $\Leftrightarrow$ $\sqrt{2(2n-1)} < x < 2\sqrt{n}$, n any positive integer. Since there is a factor of $-x$ in C'', the intervals of upward concavity for $x < 0$ are $\left(-\sqrt{2(2n+1)}, -2\sqrt{n}\right)$, n any nonnegative integer. That is, C is concave upward on $\left(-\sqrt{2},0\right)$, $\left(\sqrt{2},2\right)$, $\left(-\sqrt{6},-2\right)$, $\left(\sqrt{6},2\sqrt{2}\right)$, ...

(c)



From the graphs, we can determine that $\int_0^x \cos\left(\frac{\pi}{2}t^2\right)dt = 0.7$ at $x \approx 0.76$ and $x \approx 1.22$.

(d)



The graphs of $S(x)$ and $C(x)$ have similar shapes, except that S's flattens out near the origin, while C's does not. Note that for $x > 0$, C is increasing where S is concave up, and C is decreasing where S is concave down. Similarly, S is increasing where C is concave down, and S is decreasing where C is concave up. For $x < 0$, these relationships are reversed; that is, C is increasing where S is concave down, and S is increasing where C is concave up. See Example 4.3.3 and Exercise 4.3.69 for a discussion of $S(x)$.

59. $\displaystyle\int_0^x f(t)\,dt = x\sin x + \int_0^x \frac{f(t)}{1+t^2}\,dt \quad\Rightarrow\quad f(x) = x\cos x + \sin x + \frac{f(x)}{1+x^2}$ [by differentiation] $\Rightarrow$

$$f(x)\left(1 - \frac{1}{1+x^2}\right) = x\cos x + \sin x \quad\Rightarrow\quad f(x)\left(\frac{x^2}{1+x^2}\right) = x\cos x + \sin x \quad\Rightarrow\quad f(x) = \frac{1+x^2}{x^2}\,(x\cos x + \sin x)$$

60. $2\int_a^x f(t)\,dt = 2\sin x - 1 \quad\Rightarrow\quad \int_a^x f(t)\,dt = \sin x - \frac{1}{2}$. Differentiating both sides using FTC1 gives $f(x) = \cos x$.

We put $x = a$ into the last equation to get $0 = \sin a - \frac{1}{2}$, so $a = \frac{\pi}{6}$ satisfies the given equation.

61. Let $u = f(x)$ and $du = f'(x)\,dx$. So $2\int_a^b f(x)f'(x)\,dx = 2\int_{f(a)}^{f(b)} u\,du = \left[u^2\right]_{f(a)}^{f(b)} = [f(b)]^2 - [f(a)]^2$.

62. Let $F(x) = \displaystyle\int_2^x \sqrt{1+t^3}\,dt$. Then $F'(2) = \displaystyle\lim_{h\to 0} \frac{F(2+h) - F(2)}{h} = \lim_{h\to 0} \frac{1}{h}\int_2^{2+h} \sqrt{1+t^3}\,dt$, and $F'(x) = \sqrt{1+x^3}$, so

$$\lim_{h\to 0} \frac{1}{h}\int_2^{2+h} \sqrt{1+t^3}\,dt = F'(2) = \sqrt{1+2^3} = \sqrt{9} = 3.$$

63. Let $u = 1 - x$. Then $du = -dx$, so $\int_0^1 f(1-x)\,dx = \int_1^0 f(u)(-du) = \int_0^1 f(u)\,du = \int_0^1 f(x)\,dx$.

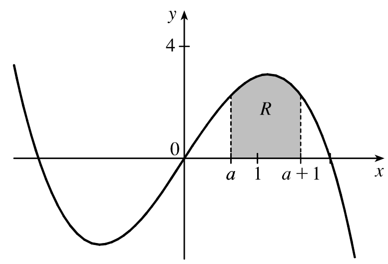
64. $\displaystyle\lim_{n\to\infty} \frac{1}{n}\left[\left(\frac{1}{n}\right)^9 + \left(\frac{2}{n}\right)^9 + \left(\frac{3}{n}\right)^9 + \cdots + \left(\frac{n}{n}\right)^9\right] = \lim_{n\to\infty} \frac{1-0}{n}\sum_{i=1}^n \left(\frac{i}{n}\right)^9 = \int_0^1 x^9\,dx = \left[\frac{x^{10}}{10}\right]_0^1 = \frac{1}{10}$

The limit is based on Riemann sums using right endpoints and subintervals of equal length.

PROBLEMS PLUS

1. Differentiating both sides of the equation $x \sin \pi x = \int_0^{x^2} f(t)\,dt$ (using FTC1 and the Chain Rule for the right side) gives $\sin \pi x + \pi x \cos \pi x = 2x f(x^2)$. Letting $x = 2$ so that $f(x^2) = f(4)$, we obtain $\sin 2\pi + 2\pi \cos 2\pi = 4f(4)$, so $f(4) = \frac{1}{4}(0 + 2\pi \cdot 1) = \frac{\pi}{2}$.

2. From the figure, it is clear that the value of a must be between 0 and 1 to obtain the maximum value of the area R.



$$R(a) = \int_a^{a+1} (4x - x^3)\,dx$$
$$= \int_0^{a+1} (4x - x^3)\,dx - \int_0^a (4x - x^3)\,dx$$

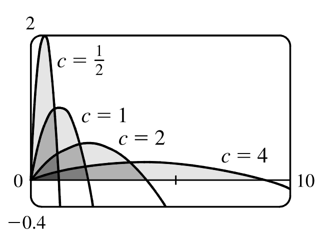
By FTC1,

$$R'(a) = 4(a+1) - (a+1)^3 - (4a - a^3)$$
$$= 4a + 4 - (a^3 + 3a^2 + 3a + 1) - 4a + a^3$$
$$= -3a^2 - 3a + 3$$

$R'(a) = 0 \quad \Leftrightarrow \quad a^2 + a - 1 = 0 \quad \Rightarrow \quad a = \dfrac{-1 + \sqrt{5}}{2} \approx 0.618$ for $a > 0$. To find the maximum value of R, show that $R(a) = \left[2x^2 - \frac{1}{4}x^4\right]_a^{a+1} = -a^3 - \frac{3}{2}a^2 + 3a + \frac{7}{4}$ and then $R\left(\frac{1}{2}\left(-1 + \sqrt{5}\right)\right) = \frac{5}{4}\sqrt{5} \approx 2.795$.

3. Differentiating the given equation, $\int_0^x f(t)\,dt = [f(x)]^2$, using FTC1 gives $f(x) = 2f(x)\,f'(x) \quad \Rightarrow$ $f(x)[2f'(x) - 1] = 0$, so $f(x) = 0$ or $f'(x) = \frac{1}{2}$. Since $f(x)$ is never 0, we must have $f'(x) = \frac{1}{2}$ and $f'(x) = \frac{1}{2} \quad \Rightarrow$ $f(x) = \frac{1}{2}x + C$. To find C, we substitute into the given equation to get $\int_0^x \left(\frac{1}{2}t + C\right) dt = \left(\frac{1}{2}x + C\right)^2 \quad \Leftrightarrow$ $\frac{1}{4}x^2 + Cx = \frac{1}{4}x^2 + Cx + C^2$. It follows that $C^2 = 0$, so $C = 0$, and $f(x) = \frac{1}{2}x$.

4. (a)

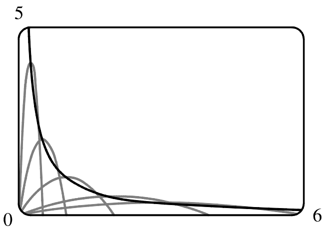


From the graph of $f(x) = \dfrac{2cx - x^2}{c^3}$, it appears that the areas are equal; that is, the area enclosed is independent of c.

(b) We first find the x-intercepts of the curve, to determine the limits of integration: $y = 0 \quad \Leftrightarrow \quad 2cx - x^2 = 0 \quad \Leftrightarrow \quad x = 0$ or $x = 2c$. Now we integrate the function between these limits to find the enclosed area:

$$A = \int_0^{2c} \frac{2cx - x^2}{c^3}\,dx = \frac{1}{c^3}\left[cx^2 - \tfrac{1}{3}x^3\right]_0^{2c} = \frac{1}{c^3}\left[c(2c)^2 - \tfrac{1}{3}(2c)^3\right] = \frac{1}{c^3}\left[4c^3 - \tfrac{8}{3}c^3\right] = \tfrac{4}{3}, \text{ a constant.}$$

(c)



The vertices of the family of parabolas seem to determine a branch of a hyperbola.

(d) For a particular c, the vertex is the point where the maximum occurs. We have seen that the x-intercepts are 0 and $2c$, so by symmetry, the maximum occurs at $x = c$, and its value is $\dfrac{2c(c) - c^2}{c^3} = \dfrac{1}{c}$. So we are interested in the curve consisting of all points of the form $\left(c, \dfrac{1}{c}\right)$, $c > 0$. This is the part of the hyperbola $y = 1/x$ lying in the first quadrant.

5. $f(x) = \displaystyle\int_0^{g(x)} \frac{1}{\sqrt{1+t^3}}\,dt$, where $g(x) = \displaystyle\int_0^{\cos x} [1 + \sin(t^2)]\,dt$. Using FTC1 and the Chain Rule (twice) we have

$f'(x) = \dfrac{1}{\sqrt{1+[g(x)]^3}}\,g'(x) = \dfrac{1}{\sqrt{1+[g(x)]^3}}\,[1+\sin(\cos^2 x)](-\sin x)$. Now $g\left(\frac{\pi}{2}\right) = \displaystyle\int_0^0 [1+\sin(t^2)]\,dt = 0$, so

$f'\left(\frac{\pi}{2}\right) = \dfrac{1}{\sqrt{1+0}}\,(1+\sin 0)(-1) = 1 \cdot 1 \cdot (-1) = -1.$

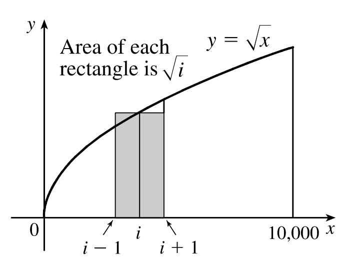
6. If $f(x) = \int_0^x x^2 \sin(t^2)\,dt = x^2 \int_0^x \sin(t^2)\,dt$, then $f'(x) = x^2 \sin(x^2) + 2x \int_0^x \sin(t^2)\,dt$, by the Product Rule and FTC1.

7. $f(x) = 2 + x - x^2 = (-x+2)(x+1) = 0 \quad \Leftrightarrow \quad x = 2$ or $x = -1$. $\quad f(x) \geq 0$ for $x \in [-1, 2]$ and $f(x) < 0$ everywhere else. The integral $\int_a^b (2 + x - x^2)\,dx$ has a maximum on the interval where the integrand is positive, which is $[-1, 2]$. So $a = -1$, $b = 2$. (Any larger interval gives a smaller integral since $f(x) < 0$ outside $[-1, 2]$. Any smaller interval also gives a smaller integral since $f(x) \geq 0$ in $[-1, 2]$.)

8. This sum can be interpreted as a Riemann sum, with the right endpoints of the subintervals as sample points and with $a = 0$, $b = 10{,}000$, and $f(x) = \sqrt{x}$. So we approximate

$$\sum_{i=1}^{10{,}000} \sqrt{i} \approx \lim_{n\to\infty} \frac{10{,}000}{n} \sum_{i=1}^{n} \sqrt{\frac{10{,}000i}{n}} = \int_0^{10{,}000} \sqrt{x}\,dx = \left[\tfrac{2}{3}x^{3/2}\right]_0^{10{,}000} = \tfrac{2}{3}(1{,}000{,}000) \approx 666{,}667.$$

Alternate method: We can use graphical methods as follows:



From the figure we see that $\int_{i-1}^{i} \sqrt{x}\,dx < \sqrt{i} < \int_i^{i+1} \sqrt{x}\,dx$, so

$\int_0^{10{,}000} \sqrt{x}\,dx < \displaystyle\sum_{i=1}^{10{,}000} \sqrt{i} < \int_1^{10{,}001} \sqrt{x}\,dx$. Since

$\int \sqrt{x}\,dx = \frac{2}{3}x^{3/2} + C$, we get $\int_0^{10{,}000} \sqrt{x}\,dx = 666{,}666.\overline{6}$ and

$\int_1^{10{,}001} \sqrt{x}\,dx = \frac{2}{3}[(10{,}001)^{3/2} - 1] \approx 666{,}766.$

Hence, $666{,}666.\overline{6} < \displaystyle\sum_{i=1}^{10{,}000} \sqrt{i} < 666{,}766$. We can estimate the sum by averaging these bounds:

$$\sum_{i=1}^{10{,}000} \approx \frac{666{,}666.\overline{6} + 666{,}766}{2} \approx 666{,}716.$$ The actual value is about 666,716.46.

9. (a) We can split the integral $\int_0^n [\![x]\!]\,dx$ into the sum $\sum\limits_{i=1}^{n}\left[\int_{i-1}^{i} [\![x]\!]\,dx\right]$. But on each of the intervals $[i-1, i)$ of integration, $[\![x]\!]$ is a constant function, namely $i-1$. So the ith integral in the sum is equal to $(i-1)[i-(i-1)] = (i-1)$. So the original integral is equal to $\sum\limits_{i=1}^{n}(i-1) = \sum\limits_{i=1}^{n-1} i = \dfrac{(n-1)n}{2}$.

(b) We can write $\int_a^b [\![x]\!]\,dx = \int_0^b [\![x]\!]\,dx - \int_0^a [\![x]\!]\,dx$.

Now $\int_0^b [\![x]\!]\,dx = \int_0^{[\![b]\!]} [\![x]\!]\,dx + \int_{[\![b]\!]}^b [\![x]\!]\,dx$. The first of these integrals is equal to $\frac{1}{2}([\![b]\!]-1)\,[\![b]\!]$, by part (a), and since $[\![x]\!] = [\![b]\!]$ on $[[\![b]\!]\,, b]$, the second integral is just $[\![b]\!]\,(b - [\![b]\!])$. So

$\int_0^b [\![x]\!]\,dx = \frac{1}{2}([\![b]\!]-1)\,[\![b]\!] + [\![b]\!]\,(b-[\![b]\!]) = \frac{1}{2}\,[\![b]\!]\,(2b-[\![b]\!]-1)$ and similarly $\int_0^a [\![x]\!]\,dx = \frac{1}{2}\,[\![a]\!]\,(2a-[\![a]\!]-1)$.

Therefore, $\int_a^b [\![x]\!]\,dx = \frac{1}{2}\,[\![b]\!]\,(2b-[\![b]\!]-1) - \frac{1}{2}\,[\![a]\!]\,(2a-[\![a]\!]-1)$.

10. By FTC1, $\dfrac{d}{dx}\displaystyle\int_0^x\left(\int_1^{\sin t}\sqrt{1+u^4}\,du\right)dt = \int_1^{\sin x}\sqrt{1+u^4}\,du$. Again using FTC1,

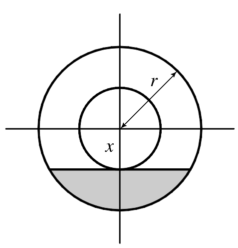
$$\frac{d^2}{dx^2}\int_0^x\left(\int_1^{\sin t}\sqrt{1+u^4}\,du\right)dt = \frac{d}{dx}\int_1^{\sin x}\sqrt{1+u^4}\,du = \sqrt{1+\sin^4 x}\,\cos x.$$

11. Let $Q(x) = \displaystyle\int_0^x P(t)\,dt = \left[at + \frac{b}{2}t^2 + \frac{c}{3}t^3 + \frac{d}{4}t^4\right]_0^x = ax + \frac{b}{2}x^2 + \frac{c}{3}x^3 + \frac{d}{4}x^4$. Then $Q(0) = 0$, and $Q(1) = 0$ by the given condition, $a + \dfrac{b}{2} + \dfrac{c}{3} + \dfrac{d}{4} = 0$. Also, $Q'(x) = P(x) = a + bx + cx^2 + dx^3$ by FTC1. By Rolle's Theorem, applied to Q on $[0, 1]$, there is a number r in $(0, 1)$ such that $Q'(r) = 0$, that is, such that $P(r) = 0$. Thus, the equation $P(x) = 0$ has a solution between 0 and 1.

More generally, if $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ and if $a_0 + \dfrac{a_1}{2} + \dfrac{a_2}{3} + \cdots + \dfrac{a_n}{n+1} = 0$, then the equation $P(x) = 0$ has a solution between 0 and 1. The proof is the same as before:

Let $Q(x) = \displaystyle\int_0^x P(t)\,dt = a_0x + \frac{a_1}{2}\,x^2 + \frac{a_2}{3}\,x^3 + \cdots + \frac{a_n}{n+1}\,x^n$. Then $Q(0) = Q(1) = 0$ and $Q'(x) = P(x)$. By Rolle's Theorem applied to Q on $[0, 1]$, there is a number r in $(0, 1)$ such that $Q'(r) = 0$, that is, such that $P(r) = 0$.

12.



Let x be the distance between the center of the disk and the surface of the liquid. The wetted circular region has area $\pi r^2 - \pi x^2$ while the unexposed wetted region (shaded in the diagram) has area $2\int_x^r \sqrt{r^2-t^2}\,dt$, so the exposed wetted region has area $A(x) = \pi r^2 - \pi x^2 - 2\int_x^r \sqrt{r^2-t^2}\,dt$, $0 \le x \le r$. By FTC1, we have $A'(x) = -2\pi x + 2\sqrt{r^2-x^2}$.

Now $A'(x) > 0 \;\Rightarrow\; -2\pi x + 2\sqrt{r^2-x^2} > 0 \;\Rightarrow\; \sqrt{r^2-x^2} > \pi x \;\Rightarrow\; r^2 - x^2 > \pi^2x^2 \;\Rightarrow\; r^2 > \pi^2x^2 + x^2 \;\Rightarrow\; r^2 > x^2(\pi^2+1) \;\Rightarrow\; x^2 < \dfrac{r^2}{\pi^2+1} \;\Rightarrow\; x < \dfrac{r}{\sqrt{\pi^2+1}}$, and we'll call this value x^*.

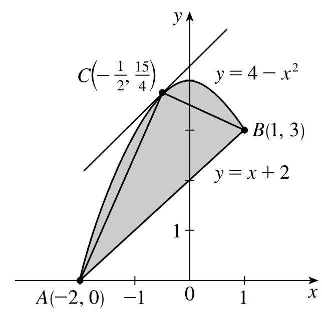
Since $A'(x) > 0$ for $0 < x < x^*$ and $A'(x) < 0$ for $x^* < x < r$, we have an absolute maximum when $x = x^*$.

13. Note that $\frac{d}{dx}\left(\int_0^x \left[\int_0^u f(t)\,dt\right] du\right) = \int_0^x f(t)\,dt$ by FTC1, while

$$\frac{d}{dx}\left[\int_0^x f(u)(x-u)\,du\right] = \frac{d}{dx}\left[x\int_0^x f(u)\,du\right] - \frac{d}{dx}\left[\int_0^x f(u)u\,du\right]$$
$$= \int_0^x f(u)\,du + xf(x) - f(x)x = \int_0^x f(u)\,du$$

Hence, $\int_0^x f(u)(x-u)\,du = \int_0^x \left[\int_0^u f(t)\,dt\right] du + C$. Setting $x = 0$ gives $C = 0$.

14. The parabola $y = 4 - x^2$ and the line $y = x + 2$ intersect when



$4 - x^2 = x + 2 \quad\Leftrightarrow\quad x^2 + x - 2 = 0 \quad\Leftrightarrow\quad (x+2)(x-1) = 0 \quad\Leftrightarrow$ $x = -2$ or 1. So the point A is $(-2, 0)$ and B is $(1, 3)$. The slope of the line $y = x + 2$ is 1 and the slope of the parabola $y = 4 - x^2$ at x-coordinate x is $-2x$. These slopes are equal when $x = -\frac{1}{2}$, so the point C is $\left(-\frac{1}{2}, \frac{15}{4}\right)$.

The area $\mathcal{A}_1$ of the parabolic segment is the area under the parabola from $x = -2$ to $x = 1$, minus the area under the line $y = x + 2$ from -2 to 1. Thus,

$$\mathcal{A}_1 = \int_{-2}^1 (4 - x^2)\,dx - \int_{-2}^1 (x+2)\,dx = \left[4x - \tfrac{1}{3}x^3\right]_{-2}^1 - \left[\tfrac{1}{2}x^2 + 2x\right]_{-2}^1$$
$$= \left[\left(4 - \tfrac{1}{3}\right) - \left(-8 + \tfrac{8}{3}\right)\right] - \left[\left(\tfrac{1}{2} + 2\right) - (2 - 4)\right] = 9 - \tfrac{9}{2} = \tfrac{9}{2}.$$

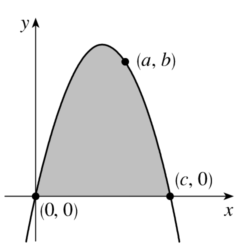
The area $\mathcal{A}_2$ of the inscribed triangle is the area under the line segment AC plus the area under the line segment CB minus the area under the line segment AB. The line through A and C has slope $\dfrac{15/4 - 0}{-1/2 + 2} = \dfrac{5}{2}$ and equation $y - 0 = \frac{5}{2}(x + 2)$, or $y = \frac{5}{2}x + 5$. The line through C and B has slope $\dfrac{3 - 15/4}{1 + 1/2} = -\dfrac{1}{2}$ and equation $y - 3 = -\frac{1}{2}(x - 1)$, or $y = -\frac{1}{2}x + \frac{7}{2}$.

Thus,

$$\mathcal{A}_2 = \int_{-2}^{-1/2} \left(\tfrac{5}{2}x + 5\right) dx + \int_{-1/2}^1 \left(-\tfrac{1}{2}x + \tfrac{7}{2}\right) dx - \int_{-2}^1 (x+2)\,dx = \left[\tfrac{5}{4}x^2 + 5x\right]_{-2}^{-1/2} + \left[-\tfrac{1}{4}x^2 + \tfrac{7}{2}x\right]_{-1/2}^1 - \tfrac{9}{2}$$
$$= \left[\left(\tfrac{5}{16} - \tfrac{5}{2}\right) - (5 - 10)\right] + \left[\left(-\tfrac{1}{4} + \tfrac{7}{2}\right) - \left(-\tfrac{1}{16} - \tfrac{7}{4}\right)\right] - \tfrac{9}{2} = \tfrac{45}{16} + \tfrac{81}{16} - \tfrac{72}{16} = \tfrac{54}{16} = \tfrac{27}{8}$$

Archimedes' result states that $\mathcal{A}_1 = \frac{4}{3}\mathcal{A}_2$, which is verified in this case since $\frac{4}{3} \cdot \frac{27}{8} = \frac{9}{2}$.

15.



Let c be the nonzero x-intercept so that the parabola has equation $f(x) = kx(x - c)$, or $y = kx^2 - ckx$, where $k < 0$. The area A under the parabola is

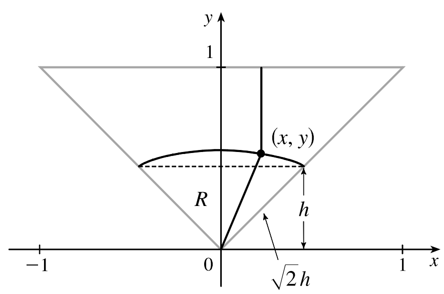
$$A = \int_0^c kx(x-c)\,dx = k\int_0^c (x^2 - cx)\,dx = k\left[\tfrac{1}{3}x^3 - \tfrac{1}{2}cx^2\right]_0^c$$
$$= k\left(\tfrac{1}{3}c^3 - \tfrac{1}{2}c^3\right) = -\tfrac{1}{6}kc^3$$

The point (a, b) is on the parabola, so $f(a) = b \quad\Rightarrow\quad b = ka(a - c) \quad\Rightarrow$ $k = \dfrac{b}{a(a-c)}$. Substituting for k in A gives $A(c) = -\dfrac{b}{6a} \cdot \dfrac{c^3}{a - c} \quad\Rightarrow$

$$A'(c) = -\frac{b}{6a} \cdot \frac{(a-c)(3c^2) - c^3(-1)}{(a-c)^2} = -\frac{b}{6a} \cdot \frac{c^2[3(a-c)+c]}{(a-c)^2} = -\frac{bc^2(3a-2c)}{6a(a-c)^2}$$

Now $A' = 0 \quad \Rightarrow \quad c = \frac{3}{2}a$. Since $A'(c) < 0$ for $a < c < \frac{3}{2}a$ and $A'(c) > 0$ for $c > \frac{3}{2}a$, so A has an absolute minimum when $c = \frac{3}{2}a$. Substituting for c in k gives us $k = \dfrac{b}{a\left(a - \frac{3}{2}a\right)} = -\dfrac{2b}{a^2}$, so $f(x) = -\dfrac{2b}{a^2}\,x\left(x - \frac{3}{2}a\right)$, or $f(x) = -\dfrac{2b}{a^2}\,x^2 + \dfrac{3b}{a}\,x$. Note that the vertex of the parabola is $\left(\frac{3}{4}a, \frac{9}{8}b\right)$ and the minimal area under the parabola is $A\left(\frac{3}{2}a\right) = \frac{9}{8}ab$.

16.



We restrict our attention to the triangle shown. A point in this triangle is closer to the side shown than to any other side, so if we find the area of the region R consisting of all points in the triangle that are closer to the center than to that side, we can multiply this area by 4 to find the total area. We find the equation of the set of points which are equidistant from the center and the side: the distance of the point (x, y) from the side is $1 - y$, and its distance from the center is $\sqrt{x^2 + y^2}$.

So the distances are equal if $\sqrt{x^2+y^2} = 1 - y \quad \Leftrightarrow \quad x^2 + y^2 = 1 - 2y + y^2 \quad \Leftrightarrow \quad y = \frac{1}{2}(1 - x^2)$. Note that the area we are interested in is equal to the area of a triangle plus a crescent-shaped area. To find these areas, we have to find the y-coordinate h of the horizontal line separating them. From the diagram, $1 - h = \sqrt{2}\,h \quad \Leftrightarrow \quad h = \frac{1}{1+\sqrt{2}} = \sqrt{2} - 1$. We calculate the areas in terms of h, and substitute afterward.

The area of the triangle is $\frac{1}{2}(2h)(h) = h^2$, and the area of the crescent-shaped section is

$$\int_{-h}^{h} \left[\tfrac{1}{2}(1-x^2) - h\right] dx = 2\int_0^h \left(\tfrac{1}{2} - h - \tfrac{1}{2}x^2\right) dx = 2\left[\left(\tfrac{1}{2} - h\right)x - \tfrac{1}{6}x^3\right]_0^h = h - 2h^2 - \tfrac{1}{3}h^3.$$

So the area of the whole region is

$$4\left[\left(h - 2h^2 - \tfrac{1}{3}h^3\right) + h^2\right] = 4h\left(1 - h - \tfrac{1}{3}h^2\right) = 4\left(\sqrt{2}-1\right)\left[1 - \left(\sqrt{2}-1\right) - \tfrac{1}{3}\left(\sqrt{2}-1\right)^2\right]$$

$$= 4\left(\sqrt{2}-1\right)\left(1 - \tfrac{1}{3}\sqrt{2}\right) = \tfrac{4}{3}\left(4\sqrt{2} - 5\right)$$

17. $$\lim_{n\to\infty}\left(\frac{1}{\sqrt{n}\,\sqrt{n+1}} + \frac{1}{\sqrt{n}\,\sqrt{n+2}} + \cdots + \frac{1}{\sqrt{n}\,\sqrt{n+n}}\right)$$

$$= \lim_{n\to\infty} \frac{1}{n}\left(\sqrt{\frac{n}{n+1}} + \sqrt{\frac{n}{n+2}} + \cdots + \sqrt{\frac{n}{n+n}}\right)$$

$$= \lim_{n\to\infty} \frac{1}{n}\left(\frac{1}{\sqrt{1+1/n}} + \frac{1}{\sqrt{1+2/n}} + \cdots + \frac{1}{\sqrt{1+1}}\right)$$

$$= \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{i}{n}\right) \qquad \left[\text{where } f(x) = \frac{1}{\sqrt{1+x}}\right]$$

$$= \int_0^1 \frac{1}{\sqrt{1+x}}\,dx = \left[2\sqrt{1+x}\,\right]_0^1 = 2\left(\sqrt{2}-1\right)$$

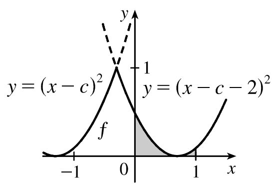
18. Note that the graphs of $(x-c)^2$ and $[(x-c)-2]^2$ intersect when $|x-c| = |x-c-2| \quad \Leftrightarrow$

$c - x = x - c - 2 \quad \Leftrightarrow \quad x = c+1$. The integration will proceed differently depending on the value of c.

Case 1: $-2 \le c < -1$

In this case, $f_c(x) = (x-c-2)^2$ for $x \in [0,1]$, so

$$g(c) = \int_0^1 (x-c-2)^2\,dx = \tfrac{1}{3}\left[(x-c-2)^3\right]_0^1 = \tfrac{1}{3}\left[(-c-1)^3 - (-c-2)^3\right]$$
$$= \tfrac{1}{3}(3c^2 + 9c + 7) = c^2 + 3c + \tfrac{7}{3} = \left(c + \tfrac{3}{2}\right)^2 + \tfrac{1}{12}$$



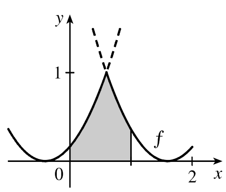
This is a parabola; its maximum value for $-2 \le c < -1$ is $g\,(-2) = \frac{1}{3}$, and its minimum value is $g\left(-\frac{3}{2}\right) = \frac{1}{12}$.

Case 2: $-1 \le c < 0$

In this case, $f_c(x) = \begin{cases} (x-c)^2 & \text{if } 0 \le x \le c+1 \\ (x-c-2)^2 & \text{if } c+1 < x \le 1 \end{cases}$

Therefore,

$$g(c) = \int_0^1 f_c(x)\,dx = \int_0^{c+1} (x-c)^2\,dx + \int_{c+1}^1 (x-c-2)^2\,dx$$
$$= \tfrac{1}{3}\left[(x-c)^3\right]_0^{c+1} + \tfrac{1}{3}\left[(x-c-2)^3\right]_{c+1}^1 = \tfrac{1}{3}\left[1 + c^3 + (-c-1)^3 - (-1)\right]$$
$$= -c^2 - c + \tfrac{1}{3} = -\left(c + \tfrac{1}{2}\right)^2 + \tfrac{7}{12}$$

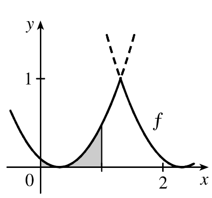


Again, this is a parabola, whose maximum value for $-1 \le c < 0$ is $g\left(-\frac{1}{2}\right) = \frac{7}{12}$, and whose minimum value on this c-interval is $g(-1) = \frac{1}{3}$.

Case 3: $0 \le c \le 2$

In this case, $f_c(x) = (x-c)^2$ for $x \in [0,1]$, so

$$g(c) = \int_0^1 (x-c)^2\,dx = \tfrac{1}{3}\left[(x-c)^3\right]_0^1 = \tfrac{1}{3}\left[(1-c)^3 - (-c)^3\right]$$
$$= c^2 - c + \tfrac{1}{3} = \left(c - \tfrac{1}{2}\right)^2 + \tfrac{1}{12}$$



This parabola has a maximum value of $g(2) = \frac{7}{3}$ and a minimum value of $g\left(\frac{1}{2}\right) = \frac{1}{12}$.

We conclude that $g(c)$ has an absolute maximum value of $g(2) = \frac{7}{3}$, and absolute minimum values of $g\left(-\frac{3}{2}\right) = g\left(\frac{1}{2}\right) = \frac{1}{12}$.