# 13 □ VECTOR FUNCTIONS

## 13.1 Vector Functions and Space Curves

**1.** The component functions $\ln(t+1)$, $\dfrac{t}{\sqrt{9-t^2}}$, and $2^t$ are all defined when $t+1>0 \;\Rightarrow\; t>-1$ and $9-t^2>0 \;\Rightarrow$ $-3<t<3$, so the domain of $\mathbf{r}$ is $(-1,3)$.

**2.** The component functions $\cos t$, $\ln t$, and $\dfrac{1}{t-2}$ are all defined when $t>0$ and $t\neq 2$, so the domain of $\mathbf{r}$ is $(0,2)\cup(2,\infty)$.

**3.** $\lim\limits_{t\to 0} e^{-3t}=e^0=1$, $\lim\limits_{t\to 0}\dfrac{t^2}{\sin^2 t}=\lim\limits_{t\to 0}\dfrac{1}{\dfrac{\sin^2 t}{t^2}}=\dfrac{1}{\lim\limits_{t\to 0}\dfrac{\sin^2 t}{t^2}}=\dfrac{1}{\left(\lim\limits_{t\to 0}\dfrac{\sin t}{t}\right)^2}=\dfrac{1}{1^2}=1$,

and $\lim\limits_{t\to 0}\cos 2t=\cos 0=1$. Thus,

$$\lim_{t\to 0}\left(e^{-3t}\,\mathbf{i}+\frac{t^2}{\sin^2 t}\,\mathbf{j}+\cos 2t\,\mathbf{k}\right)=\left[\lim_{t\to 0}e^{-3t}\right]\mathbf{i}+\left[\lim_{t\to 0}\frac{t^2}{\sin^2 t}\right]\mathbf{j}+\left[\lim_{t\to 0}\cos 2t\right]\mathbf{k}=\mathbf{i}+\mathbf{j}+\mathbf{k}.$$

**4.** $\lim\limits_{t\to 1}\dfrac{t^2-t}{t-1}=\lim\limits_{t\to 1}\dfrac{t\,(t-1)}{t-1}=\lim\limits_{t\to 1}t=1$, $\lim\limits_{t\to 1}\sqrt{t+8}=3$, $\lim\limits_{t\to 1}\dfrac{\sin\pi t}{\ln t}=\lim\limits_{t\to 1}\dfrac{\pi\cos\pi t}{1/t}=-\pi$ [by l'Hospital's Rule].

Thus, the given limit equals $\mathbf{i}+3\,\mathbf{j}-\pi\,\mathbf{k}$.

**5.** $\lim\limits_{t\to\infty}\dfrac{1+t^2}{1-t^2}=\lim\limits_{t\to\infty}\dfrac{(1/t^2)+1}{(1/t^2)-1}=\dfrac{0+1}{0-1}=-1$, $\lim\limits_{t\to\infty}\tan^{-1}t=\frac{\pi}{2}$, $\lim\limits_{t\to\infty}\dfrac{1-e^{-2t}}{t}=\lim\limits_{t\to\infty}\dfrac{1}{t}-\dfrac{1}{te^{2t}}=0-0=0$. Thus,

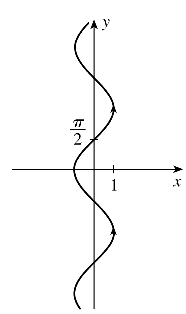
$\lim\limits_{t\to\infty}\left\langle\dfrac{1+t^2}{1-t^2},\tan^{-1}t,\dfrac{1-e^{-2t}}{t}\right\rangle=\left\langle -1,\frac{\pi}{2},0\right\rangle$.

**6.** $\lim\limits_{t\to\infty}te^{-t}=\lim\limits_{t\to\infty}\dfrac{t}{e^t}=\lim\limits_{t\to\infty}\dfrac{1}{e^t}=0$ [by l'Hospital's Rule], $\lim\limits_{t\to\infty}\dfrac{t^3+t}{2t^3-1}=\lim\limits_{t\to\infty}\dfrac{1+(1/t^2)}{2-(1/t^3)}=\dfrac{1+0}{2-0}=\dfrac{1}{2}$,

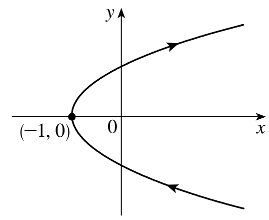
and $\lim\limits_{t\to\infty}t\sin\dfrac{1}{t}=\lim\limits_{t\to\infty}\dfrac{\sin(1/t)}{1/t}=\lim\limits_{t\to\infty}\dfrac{\cos(1/t)(-1/t^2)}{-1/t^2}=\lim\limits_{t\to\infty}\cos\dfrac{1}{t}=\cos 0=1$ [again by l'Hospital's Rule].

Thus, $\lim\limits_{t\to\infty}\left\langle te^{-t},\dfrac{t^3+t}{2t^3-1},t\sin\dfrac{1}{t}\right\rangle=\left\langle 0,\frac{1}{2},1\right\rangle$.

**7.** The corresponding parametric equations for this curve are $x=-\cos t$, $y=t$. We can make a table of values or we can eliminate the parameter: $t=y \;\Rightarrow\; x=-\cos y$, with $y\in\mathbb{R}$. By comparing different values of $t$, we find the direction in which $t$ increases as indicated in the graph.

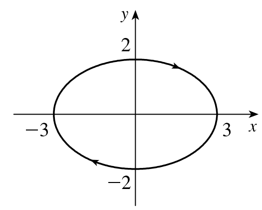


**8.** The corresponding parametric equations for this curve are $x = t^2 - 1,\ y = t$. We can make a table of values, or we can eliminate the parameter: $t = y \quad \Rightarrow \quad x = y^2 - 1$, with $y \in \mathbb{R}$. Thus, the curve is a parabola with vertex $(-1, 0)$ that opens to the right. By comparing different values of $t$, we find the direction in which $t$ increases as indicated in the graph.

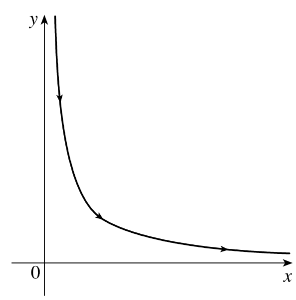



**9.** The corresponding parametric equations for this curve are $x = 3\sin t$, $y = 2\cos t$. We can make a table of values, or we can eliminate the parameter: $x = 3\sin t$, $y = 2\cos t \quad \Rightarrow \quad \dfrac{x}{3} = \sin t$, $\dfrac{y}{2} = \cos t \quad \Rightarrow$

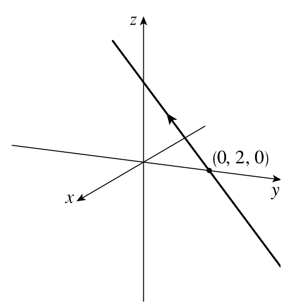
$$\frac{x^2}{9} + \frac{y^2}{4} = \sin^2 t + \cos^2 t = 1,$$ which we recognize as the equation of an ellipse with $x \in [-3, 3]$ and $y \in [-2, 2]$. By comparing different values of $t$, we find the direction in which $t$ increases as indicated in the graph.



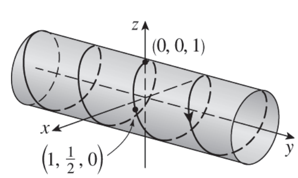

**10.** The corresponding parametric equations for this curve are $x = e^t$, $y = e^{-t}$. We can make a table of values, or we can eliminate the parameter: $y = e^{-t} = 1/e^t = 1/x$ with $x, y > 0$. By comparing different values of $t$, we find the direction in which $t$ increases as indicated in the graph.



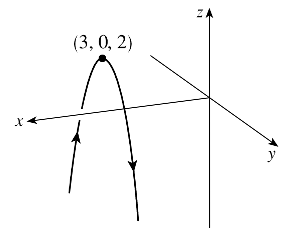

**11.** The corresponding parametric equations are $x = t,\ y = 2 - t,\ z = 2t$, which are parametric equations of a line through the point $(0, 2, 0)$ and with direction vector $\langle 1, -1, 2 \rangle$.



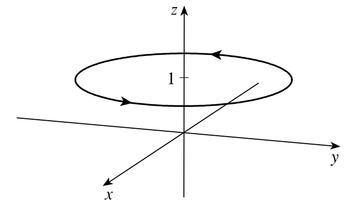

**12.** The corresponding parametric equations are $x = \sin \pi t,\ y = t,\ z = \cos \pi t$. Note that $x^2 + z^2 = \sin^2 \pi t + \cos^2 \pi t = 1$, so the curve lies on the circular cylinder $x^2 + z^2 = 1$. A point $(x, y, z)$ on the curve lies directly to the left or right of the point $(x, 0, z)$ which moves clockwise (when viewed from the left) along the circle $x^2 + z^2 = 1$ in the $xz$-plane as $t$ increases. Since $y = t$, the curve is a helix that spirals toward the right around the cylinder.



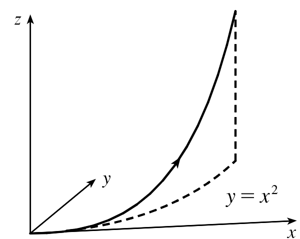
**13.** The corresponding parametric equations are $x = 3,\ y = t,\ z = 2 - t^2$. Eliminating the parameter in $y$ and $z$ gives $z = 2 - y^2$. Because $x = 3$, the curve is a parabola in the vertical plane $x = 3$ with vertex $(3, 0, 2)$.



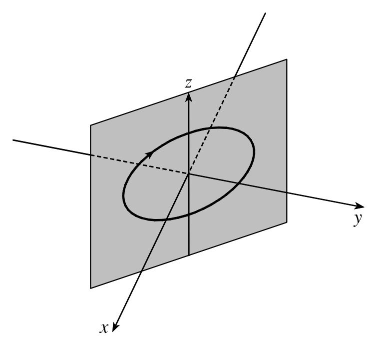

**14.** The corresponding parametric equations are $x = 2\cos t,\ y = 2\sin t$, $z = 1$. Eliminating the parameter in $x$ and $y$ gives $x^2 + y^2 = 4\cos^2 t + 4\sin^2 t = 4(\cos^2 t + \sin^2 t) = 4$. Since $z = 1$, the curve is a circle of radius 2 centered at $(0, 0, 1)$ in the horizontal plane $z = 1$.



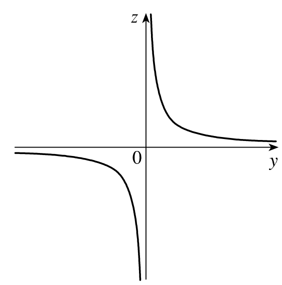

**15.** The parametric equations are $x = t^2$, $y = t^4$, $z = t^6$. These are positive for $t \neq 0$ and 0 when $t = 0$. So the curve lies entirely in the first octant. The projection of the graph onto the $xy$-plane is $y = x^2$, $y > 0$, a half parabola. The projection onto the $xz$-plane is $z = x^3$, $z > 0$, a half cubic, and the projection onto the $yz$-plane is $y^3 = z^2$.



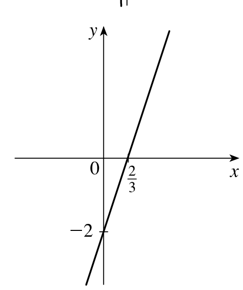

**16.** If $x = \cos t,\ y = -\cos t,\ z = \sin t$, then $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$, so the curve is contained in the intersection of circular cylinders along the $x$- and $y$-axes. Furthermore, $y = -x$, so the curve is an ellipse in the plane $y = -x$, centered at the origin.



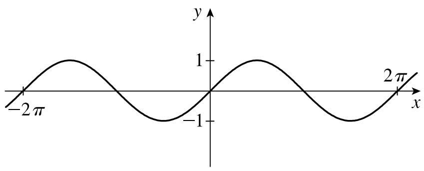

**17.** The projection of the curve defined by the vector function $\mathbf{r}(t) = \langle t^2, t^3, t^{-3} \rangle$ onto the $yz$-plane is given by $\mathbf{r}(t) = \langle 0, t^3, t^{-3} \rangle$ [we use 0 for the $x$-component], whose graph is the curve $z = 1/y$, $x = 0$, since $z = t^{-3} = 1/t^3$.



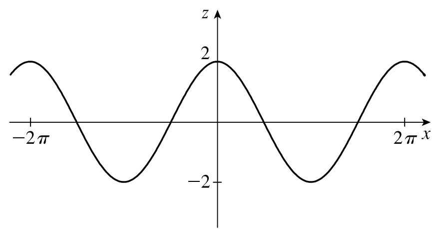

**18.** The projection of the curve defined by the vector function $\mathbf{r}(t) = \langle t + 1, 3t + 1, \cos(t/2) \rangle$ onto the $xy$-plane is given by $\mathbf{r}(t) = \langle t + 1, 3t + 1, 0 \rangle$ [we use 0 for the $z$-component], whose graph is the curve $y = 3x - 2$, $z = 0$, since $y = 3t + 1 = 3(x - 1) + 1 = 3x - 2$.



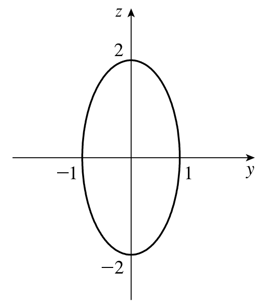
**19.** The projection of the curve onto the $xy$-plane is given by $\mathbf{r}(t) = \langle t, \sin t, 0 \rangle$ [we use 0 for the $z$-component] whose graph is the curve $y = \sin x$, $z = 0$. Similarly, the projection onto the $xz$-plane is $\mathbf{r}(t) = \langle t, 0, 2\cos t \rangle$, whose graph is the cosine wave $z = 2\cos x$, $y = 0$, and the projection onto the $yz$-plane is $\mathbf{r}(t) = \langle 0, \sin t, 2\cos t \rangle$ whose graph is the ellipse $y^2 + \frac{1}{4}z^2 = 1$, $x = 0$.



$xy$-plane

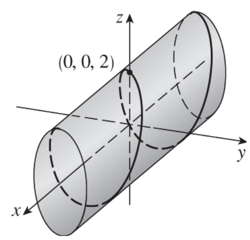


$xz$-plane

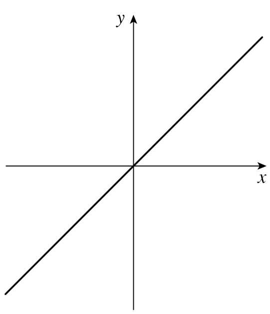


$yz$-plane

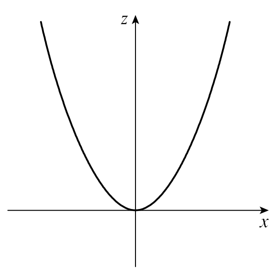
From the projection onto the $yz$-plane we see that the curve lies on an elliptical cylinder with axis the $x$-axis. The other two projections show that the curve oscillates both vertically and horizontally as we move in the $x$-direction, suggesting that the curve is an elliptical helix that spirals along the cylinder.



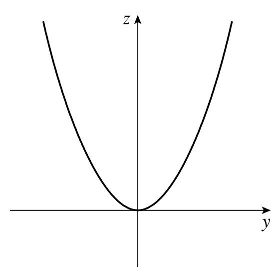

**20.** The projection of the curve onto the $xy$-plane is given by $\mathbf{r}(t) = \langle t, t, 0 \rangle$ whose graph is the line $y = x$, $z = 0$. The projection onto the $xz$-plane is $\mathbf{r}(t) = \langle t, 0, t^2 \rangle$ whose graph is the parabola $z = x^2$, $y = 0$. The projection onto the $yz$-plane is $\mathbf{r}(t) = \langle 0, t, t^2 \rangle$ whose graph is the parabola $z = y^2$, $x = 0$.



$xy$-plane

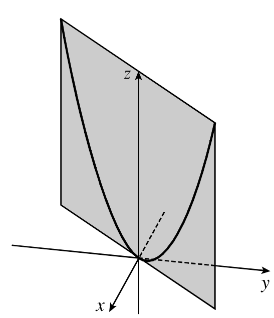


$xz$-plane



$yz$-plane

From the projection onto the $xy$-plane we see that the curve lies on the vertical plane $y = x$. The other two projections show that the curve is a parabola contained in this plane.



**21.** We take $\mathbf{r}_0 = \langle -2, 1, 0 \rangle$ and $\mathbf{r}_1 = \langle 5, 2, -3 \rangle$. Then, by Equation 12.5.4 we have a vector equation for the line segment:

$$\mathbf{r}(t) = (1-t)\langle -2, 1, 0 \rangle + t\langle 5, 2, -3 \rangle \quad \Rightarrow \quad \mathbf{r}(t) = \langle -2 + 7t, 1 + t, -3t \rangle, \quad 0 \le t \le 1$$

with corresponding parametric equations $x = -2 + 7t$, $y = 1 + t$, $z = -3t$, $0 \le t \le 1$.

**22.** We take $\mathbf{r}_0 = \langle 0, 0, 0 \rangle$ and $\mathbf{r}_1 = \langle -7, 4, 6 \rangle$. Then, by Equation 12.5.4 we have a vector equation for the line segment:

$$\mathbf{r}(t) = (1-t)\langle 0, 0, 0 \rangle + t\langle -7, 4, 6 \rangle \quad \Rightarrow \quad \mathbf{r}(t) = \langle -7t, 4t, 6t \rangle, \quad 0 \le t \le 1$$

with corresponding parametric equations $x = -7t$, $y = 4t$, $z = 6t$, $0 \le t \le 1$.

**23.** We take $\mathbf{r}_0 = \langle 3.5, -1.4, 2.1 \rangle$ and $\mathbf{r}_1 = \langle 1.8, 0.3, 2.1 \rangle$. Then, by Equation 12.5.4 we have a vector equation for the line segment:

$$\mathbf{r}(t) = (1-t)\langle 3.5, -1.4, 2.1 \rangle + t\langle 1.8, 0.3, 2.1 \rangle \quad \Rightarrow \quad \mathbf{r}(t) = \langle 3.5 - 1.7t, -1.4 + 1.7t, 2.1 \rangle, \quad 0 \le t \le 1$$

with corresponding parametric equations $x = 3.5 - 1.7t$, $y = -1.4 + 1.7t$, $z = 2.1$, $0 \le t \le 1$.

**24.** We take $\mathbf{r}_0 = \langle a, b, c \rangle$ and $\mathbf{r}_1 = \langle u, v, w \rangle$. Then, by Equation 12.5.4 we have a vector equation for the line segment:

$$\mathbf{r}(t) = (1-t)\,\mathbf{r}_0 + t\,\mathbf{r}_1 = (1-t)\langle a, b, c \rangle + t\langle u, v, w \rangle$$

$$\Rightarrow \qquad \mathbf{r}(t) = \langle a + (u-a)t, b + (v-b)t, c + (w-c)t \rangle, \quad 0 \le t \le 1$$

with corresponding parametric equations $x = a + (u-a)t,\ y = b + (v-b)t,\ z = c + (w-c)t,\ 0 \le t \le 1$.

**25.** $x = t\cos t,\ y = t,\ z = t\sin t,\ t \ge 0$. At any point $(x, y, z)$ on the curve, $x^2 + z^2 = t^2\cos^2 t + t^2\sin^2 t = t^2 = y^2$ so the curve lies on the circular cone $x^2 + z^2 = y^2$ with axis the $y$-axis. Also notice that $y \ge 0$; the graph is II.

**26.** $x = \cos t,\ y = \sin t,\ z = 1/(1+t^2)$. At any point on the curve we have $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, so the curve lies on the circular cylinder $x^2 + y^2 = 1$ with axis the $z$-axis. Notice that $0 < z \le 1$ and $z = 1$ only for $t = 0$. A point $(x, y, z)$ on the curve lies directly above the point $(x, y, 0)$, which moves counterclockwise around the unit circle in the $xy$-plane as $t$ increases, and $z \to 0$ as $t \to \pm\infty$. The graph must be VI.

**27.** $x = t,\ y = 1/(1+t^2),\ z = t^2$. At any point on the curve we have $z = x^2$, so the curve lies on a parabolic cylinder parallel to the $y$-axis. Notice that $0 < y \le 1$ and $z \ge 0$. Also the curve passes through $(0, 1, 0)$ when $t = 0$ and $y \to 0$, $z \to \infty$ as $t \to \pm\infty$, so the graph must be V.

**28.** $x = \cos t,\ y = \sin t,\ z = \cos 2t$. $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, so the curve lies on a circular cylinder with axis the $z$-axis. A point $(x, y, z)$ on the curve lies directly above or below $(x, y, 0)$, which moves around the unit circle in the $xy$-plane with period $2\pi$. At the same time, the $z$-value of the point $(x, y, z)$ oscillates with a period of $\pi$. So the curve repeats itself and the graph is I.

**29.** $x = \cos 8t,\ y = \sin 8t,\ z = e^{0.8t},\ t \ge 0$. $x^2 + y^2 = \cos^2 8t + \sin^2 8t = 1$, so the curve lies on a circular cylinder with axis the $z$-axis. A point $(x, y, z)$ on the curve lies directly above the point $(x, y, 0)$, which moves counterclockwise around the unit circle in the $xy$-plane as $t$ increases. The curve starts at $(1, 0, 1)$, when $t = 0$, and $z \to \infty$ (at an increasing rate) as $t \to \infty$, so the graph is IV.

**30.** $x = \cos^2 t,\ y = \sin^2 t,\ z = t.\quad x + y = \cos^2 t + \sin^2 t = 1$, so the curve lies in the vertical plane $x + y = 1$. $x$ and $y$ are periodic, both with period $\pi$, and $z$ increases as $t$ increases, so the graph is III.

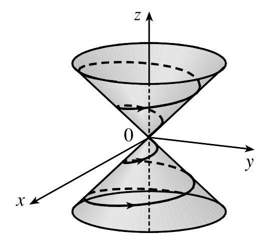
**31.** As $y = 4$ in the vector equation $\mathbf{r}(t) = \langle t, 4, t^2 \rangle$, the curve $z = x^2$ lies in the plane $y = 4$.

**32.** $\mathbf{r}(t) = \langle t, t^2, t \rangle$. Consider the projection of the curve in the $xz$-plane, $\mathbf{r}(t) = \langle t, 0, t \rangle$. This is the line $z = x$, $y = 0$. Thus, the curve is contained in the plane $z = x$.

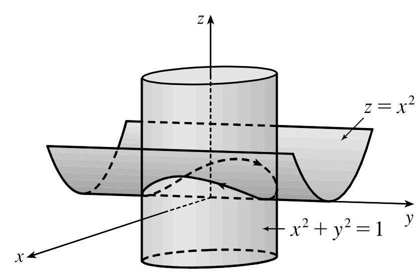
**33.** $\mathbf{r}(t) = \langle \sin t, \cos t, -\cos t \rangle$. Consider the projection of the curve in the $yz$-plane, $\mathbf{r}(t) = \langle 0, \cos t, -\cos t \rangle$. This is the line $z = -y$, $x = 0$. Thus, the curve is contained in the plane $z = -y$.

**34.** $\mathbf{r}(t) = \langle 2t, \sin t, t + 1 \rangle$. Consider the projection in the $xz$-plane, $\mathbf{r}(t) = \langle 2t, 0, t + 1 \rangle$. This is the line with parametric equations $x = 2t$, $z = t + 1$, $y = 0 \quad \Rightarrow \quad x = 2t = 2(z - 1) = 2z - 2, y = 0$. Thus, the curve is contained in the plane $x = 2z - 2$.

**35.** If $x = t\cos t,\ y = t\sin t,\ z = t$, then $x^2 + y^2 = t^2\cos^2 t + t^2 \sin^2 t = t^2 = z^2$, so the curve lies on the cone $z^2 = x^2 + y^2$. Since $z = t$, the curve is a spiral on this cone.




**36.** If $x = \sin t$, $y = \cos t$, $z = \sin^2 t$, then $x^2 = \sin^2 t = z$ and $x^2 + y^2 = \sin^2 t + \cos^2 t = 1$, so the curve is contained in the intersection of the parabolic cylinder $z = x^2$ with the circular cylinder $x^2 + y^2 = 1$. We get the complete intersection for $0 \le t \le 2\pi$.




**37.** Here $x = 2t,\ y = e^t,\ z = e^{2t}$. Then $t = x/2 \quad \Rightarrow \quad y = e^t = e^{x/2}$, so the curve lies on the cylinder $y = e^{x/2}$. Also $z = e^{2t} = e^x$, so the curve lies on the cylinder $z = e^x$. Since $z = e^{2t} = \left(e^t\right)^2 = y^2$, the curve also lies on the parabolic cylinder $z = y^2$.

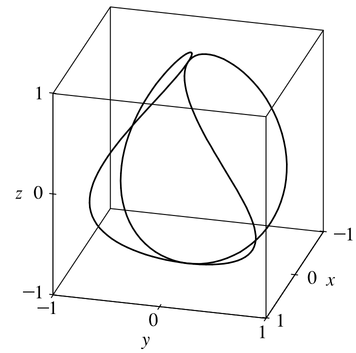
**38.** Here $x = t^2,\ y = \ln t,\ z = 1/t$. The domain of $\mathbf{r}$ is $(0, \infty)$, so $x = t^2 \quad \Rightarrow \quad t = \sqrt{x} \quad \Rightarrow \quad y = \ln\sqrt{x}$. Thus one surface containing the curve is the cylinder $y = \ln\sqrt{x}$ or $y = \ln x^{1/2} = \frac{1}{2}\ln x$. Also $z = 1/t = 1/\sqrt{x}$, so the curve also lies on the cylinder $z = 1/\sqrt{x}$ or $x = 1/z^2$, $z > 0$. Finally $z = 1/t \quad \Rightarrow \quad t = 1/z \quad \Rightarrow \quad y = \ln(1/z)$, so the curve also lies on the cylinder $y = \ln(1/z)$ or $y = \ln z^{-1} = -\ln z$. Note that the surface $y = \ln(xz)$ also contains the curve, since $\ln(xz) = \ln(t^2 \cdot 1/t) = \ln t = y$.

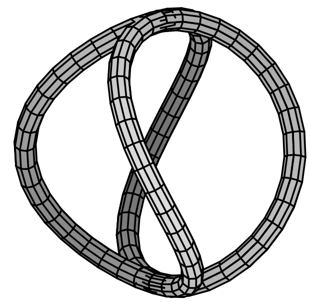
**39.** Parametric equations for the curve are $x = t,\ y = 0,\ z = 2t - t^2$. Substituting into the equation of the paraboloid gives $2t - t^2 = t^2 \quad \Rightarrow \quad 2t = 2t^2 \quad \Rightarrow \quad t = 0, 1$. Since $\mathbf{r}(0) = \mathbf{0}$ and $\mathbf{r}(1) = \mathbf{i} + \mathbf{k}$, the points of intersection are $(0, 0, 0)$ and $(1, 0, 1)$.

**40.** Parametric equations for the helix are $x = \sin t$, $y = \cos t$, $z = t$. Substituting into the equation of the sphere gives $\sin^2 t + \cos^2 t + t^2 = 5 \;\Rightarrow\; 1 + t^2 = 5 \;\Rightarrow\; t = \pm 2$. Since $\mathbf{r}(2) = \langle \sin 2, \cos 2, 2\rangle$ and $\mathbf{r}(-2) = \langle \sin(-2), \cos(-2), -2\rangle$, the points of intersection are $(\sin 2, \cos 2, 2) \approx (0.909, -0.416, 2)$ and $(\sin(-2), \cos(-2), -2) \approx (-0.909, -0.416, -2)$.

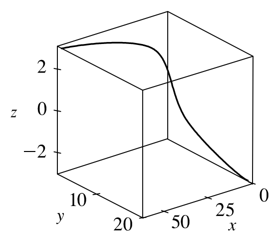
**41.** $\mathbf{r}(t) = \langle \cos t\, \sin 2t, \sin t\, \sin 2t, \cos 2t\rangle$.

We include both a regular plot and a plot showing a tube of radius 0.08 around the curve.

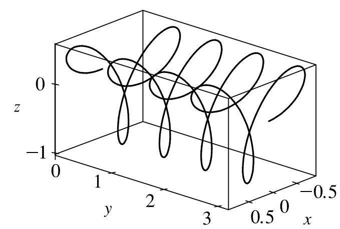


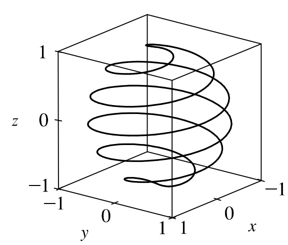
**42.** $\mathbf{r}(t) = \left\langle te^t, e^{-t}, t\right\rangle$



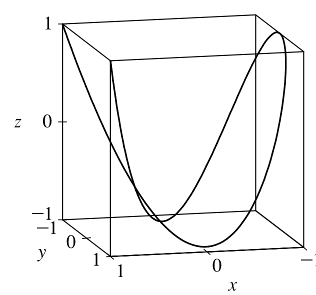

**43.** $\mathbf{r}(t) = \left\langle \sin 3t \cos t, \frac{1}{4}t, \sin 3t \sin t\right\rangle$



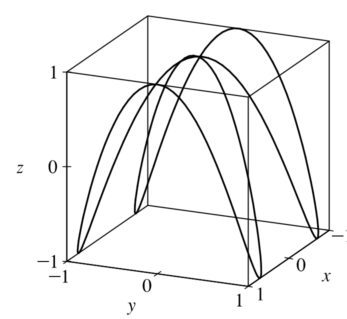

**44.** $\mathbf{r}(t) = \langle \cos(8\cos t)\sin t, \sin(8\cos t)\sin t, \cos t\rangle$




**45.** $\mathbf{r}(t) = \langle \cos 2t, \cos 3t, \cos 4t\rangle$

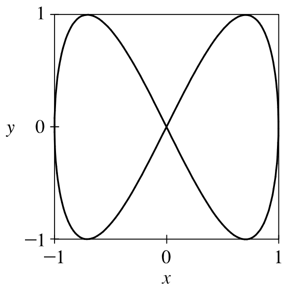



**46.** $x = \sin t$, $y = \sin 2t$, $z = \cos 4t$.

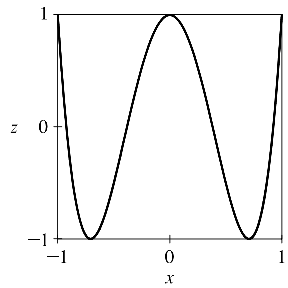



[continued]

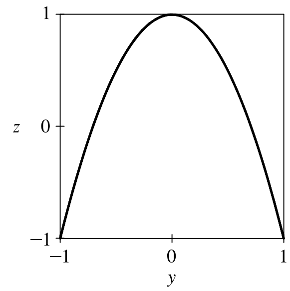
We graph the projections onto the coordinate planes.




$xy$-plane

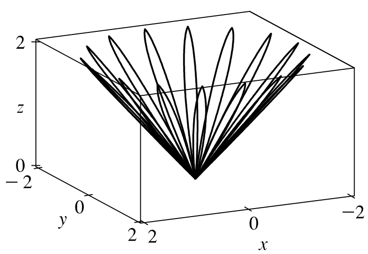



$xz$-plane




$yz$-plane

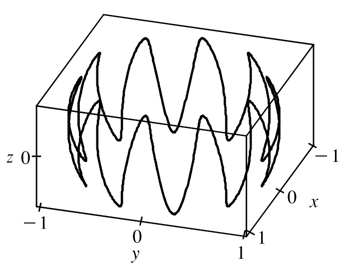
From the projection onto the $xy$-plane we see that from above the curve appears to be shaped like a "figure eight." The curve can be visualized as this shape wrapped around an almost parabolic cylindrical surface, the profile of which is visible in the projection onto the $yz$-plane.

**47.** 


$x = (1 + \cos 16t)\cos t$, $y = (1 + \cos 16t)\sin t$, $z = 1 + \cos 16t$. At any point on the graph,

$$\begin{aligned} x^2 + y^2 &= (1 + \cos 16t)^2 \cos^2 t + (1 + \cos 16t)^2 \sin^2 t \\ &= (1 + \cos 16t)^2 = z^2, \text{ so the graph lies on the cone } x^2 + y^2 = z^2. \end{aligned}$$

From the graph at left, we see that this curve looks like the projection of a leaved two-dimensional curve onto a cone.

**48.** 


$x = \sqrt{1 - 0.25\cos^2 10t}\,\cos t,\ \ y = \sqrt{1 - 0.25\cos^2 10t}\,\sin t,$

$z = 0.5\cos 10t$. At any point on the graph,

$$\begin{aligned} x^2 + y^2 + z^2 &= (1 - 0.25\cos^2 10t)\cos^2 t \\ &\quad + (1 - 0.25\cos^2 10t)\sin^2 t + 0.25\cos^2 t \\ &= 1 - 0.25\cos^2 10t + 0.25\cos^2 10t = 1, \end{aligned}$$

so the graph lies on the sphere $x^2 + y^2 + z^2 = 1$, and since $z = 0.5\cos 10t$ the graph resembles a trigonometric curve with ten peaks projected onto the sphere. We get the complete graph for $0 \le t \le 2\pi$.

**49.** If $t = -1$, then $x = 1,\ y = 4,\ z = 0$, so the curve passes through the point $(1, 4, 0)$. If $t = 3$, then $x = 9,\ y = -8,\ z = 28$, so the curve passes through the point $(9, -8, 28)$. For the point $(4, 7, -6)$ to be on the curve, we require $y = 1 - 3t = 7 \;\Rightarrow\; t = -2$. But then $z = 1 + (-2)^3 = -7 \ne -6$, so $(4, 7, -6)$ is not on the curve.

**50.** The projection of the curve $C$ of intersection onto the $xy$-plane is the circle $x^2 + y^2 = 4$, $z = 0$. Then we can write $x = 2\cos t,\ y = 2\sin t,\ 0 \le t \le 2\pi$. Since $C$ also lies on the surface $z = xy$, we have $z = xy = (2\cos t)(2\sin t) = 4\cos t\sin t$, or $2\sin(2t)$. Then parametric equations for $C$ are $x = 2\cos t,\ y = 2\sin t$, $z = 2\sin(2t),\ 0 \le t \le 2\pi$, and the corresponding vector function is $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + 2\sin(2t)\,\mathbf{k}$, $0 \le t \le 2\pi$.

**51.** Both equations are solved for $z$, so we can substitute to eliminate $z$: $\sqrt{x^2+y^2} = 1+y \quad\Rightarrow\quad x^2+y^2 = 1+2y+y^2 \quad\Rightarrow$ $x^2 = 1+2y \quad\Rightarrow\quad y = \frac{1}{2}(x^2-1)$. We can form parametric equations for the curve $C$ of intersection by choosing a parameter $x = t$, then $y = \frac{1}{2}(t^2-1)$ and $z = 1+y = 1+\frac{1}{2}(t^2-1) = \frac{1}{2}(t^2+1)$. Thus, a vector function representing $C$ is $\mathbf{r}(t) = t\,\mathbf{i} + \frac{1}{2}(t^2-1)\,\mathbf{j} + \frac{1}{2}(t^2+1)\,\mathbf{k}$.

**52.** The projection of the curve $C$ of intersection onto the $xy$-plane is the parabola $y = x^2$, $z = 0$. Then we can choose the parameter $x = t \quad\Rightarrow\quad y = t^2$. Since $C$ also lies on the surface $z = 4x^2 + y^2$, we have $z = 4x^2 + y^2 = 4t^2 + (t^2)^2$. Then parametric equations for $C$ are $x = t,\ y = t^2,\ z = 4t^2 + t^4$, and the corresponding vector function is $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + (4t^2+t^4)\,\mathbf{k}$.

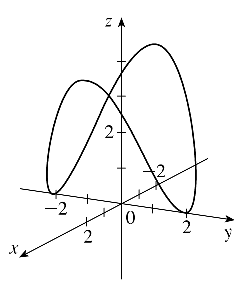
**53.** The projection of the curve $C$ of intersection onto the $xy$-plane is the circle $x^2+y^2 = 1$, $z = 0$, so we can write $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$. Since $C$ also lies on the surface $z = x^2 - y^2$, we have $z = x^2 - y^2 = \cos^2 t - \sin^2 t$ or $\cos 2t$. Thus parametric equations for $C$ are $x = \cos t$, $y = \sin t$, $z = \cos 2t$, $0 \le t \le 2\pi$, and the corresponding vector function is $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \cos 2t\,\mathbf{k}$, $0 \le t \le 2\pi$.

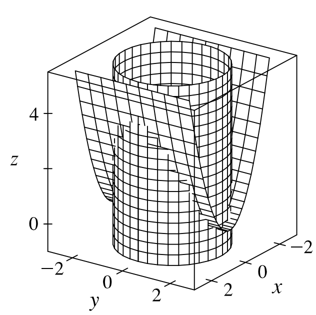
**54.** The projection of the curve $C$ of intersection onto the $xz$-plane is the circle $x^2+z^2 = 1$, $y = 0$, so we can write $x = \cos t$, $z = \sin t$, $0 \le t \le 2\pi$. $C$ also lies on the surface $x^2 + y^2 + 4z^2 = 4$, and since $y \ge 0$ we can write

$$y = \sqrt{4 - x^2 - 4z^2} = \sqrt{4 - \cos^2 t - 4\sin^2 t} = \sqrt{4 - \cos^2 t - 4(1-\cos^2 t)} = \sqrt{3\cos^2 t} = \sqrt{3}\,|\cos t\,|$$

Thus parametric equations for $C$ are $x = \cos t$, $y = \sqrt{3}\,|\cos t\,|$, $z = \sin t$, $0 \le t \le 2\pi$, and the corresponding vector function is $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sqrt{3}\,|\cos t\,|\,\mathbf{j} + \sin t\,\mathbf{k}$, $0 \le t \le 2\pi$.

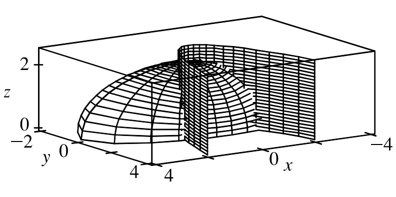
**55.**

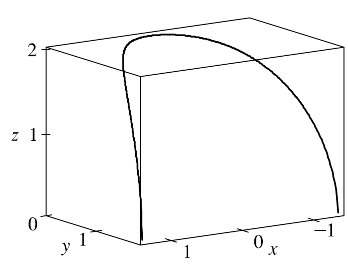



The projection of the curve $C$ of intersection onto the $xy$-plane is the circle $x^2+y^2 = 4$, $z = 0$. Then we can write $x = 2\cos t,\ y = 2\sin t,\ 0 \le t \le 2\pi$. Since $C$ also lies on the surface $z = x^2$, we have $z = x^2 = (2\cos t)^2 = 4\cos^2 t$. Then parametric equations for $C$ are $x = 2\cos t,\ y = 2\sin t,\ z = 4\cos^2 t,\ 0 \le t \le 2\pi$.

**56.**



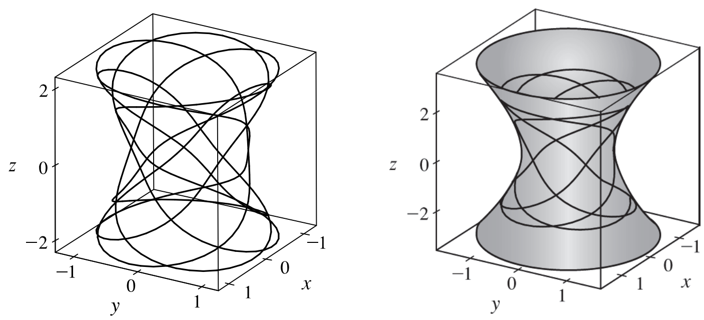

$x = t \quad\Rightarrow\quad y = t^2 \quad\Rightarrow\quad 4z^2 = 16 - x^2 - 4y^2 = 16 - t^2 - 4t^4 \quad\Rightarrow\quad z = \sqrt{4 - \left(\frac{1}{2}t\right)^2 - t^4}$.

Note that $z$ is positive because the intersection is with the top half of the ellipsoid. Hence the curve is given by $x = t,\ y = t^2,\ z = \sqrt{4 - \frac{1}{4}t^2 - t^4}$.

**57.** For the particles to collide, we require $\mathbf{r}_1(t) = \mathbf{r}_2(t) \quad\Leftrightarrow\quad \langle t^2, 7t-12, t^2\rangle = \langle 4t-3, t^2, 5t-6\rangle$. Equating components gives $t^2 = 4t-3$, $7t-12 = t^2$, and $t^2 = 5t-6$. From the first equation, $t^2-4t+3=0 \quad\Leftrightarrow\quad (t-3)(t-1)=0$ so $t=1$ or $t=3$. $t=1$ does not satisfy the other two equations, but $t=3$ does. The particles collide when $t=3$, at the point $(9,9,9)$.

**58.** The particles collide provided $\mathbf{r}_1(t) = \mathbf{r}_2(t) \quad\Leftrightarrow\quad \langle t, t^2, t^3\rangle = \langle 1+2t, 1+6t, 1+14t\rangle$. Equating components gives $t = 1+2t$, $t^2 = 1+6t$, and $t^3 = 1+14t$. The first equation gives $t=-1$, but this does not satisfy the other equations, so the particles do not collide. For the paths to intersect, we need to find a value for $t$ and a value for $s$ where $\mathbf{r}_1(t) = \mathbf{r}_2(s) \quad\Leftrightarrow\quad \langle t, t^2, t^3\rangle = \langle 1+2s, 1+6s, 1+14s\rangle$. Equating components, $t = 1+2s$, $t^2 = 1+6s$, and $t^3 = 1+14s$. Substituting the first equation into the second gives $(1+2s)^2 = 1+6s \quad\Rightarrow\quad 4s^2-2s=0 \quad\Rightarrow\quad 2s(2s-1)=0 \quad\Rightarrow\quad s=0$ or $s=\frac{1}{2}$. From the first equation, $s=0 \quad\Rightarrow\quad t=1$ and $s=\frac{1}{2} \quad\Rightarrow\quad t=2$. Checking, we see that both pairs of values satisfy the third equation. Thus the paths intersect twice, at the point $(1,1,1)$ when $s=0$ and $t=1$, and at $(2,4,8)$ when $s=\frac{1}{2}$ and $t=2$.

**59.** (a) We plot the parametric equations for $0 \le t \le 2\pi$ in the first figure. We get a better idea of the shape of the curve if we plot it simultaneously with the hyperboloid of one sheet from part (b), as shown in the second figure.




(b) Here $x = \frac{27}{26}\sin 8t - \frac{8}{39}\sin 18t$, $y = -\frac{27}{26}\cos 8t + \frac{8}{39}\cos 18t$, $z = \frac{144}{65}\sin 5t$.

For any point on the curve,

$$\begin{aligned}
x^2+y^2 &= \left(\tfrac{27}{26}\sin 8t - \tfrac{8}{39}\sin 18t\right)^2 + \left(-\tfrac{27}{26}\cos 8t + \tfrac{8}{39}\cos 18t\right)^2 \\
&= \tfrac{27^2}{26^2}\sin^2 8t - 2\cdot\tfrac{27\cdot 8}{26\cdot 39}\sin 8t\sin 18t + \tfrac{64}{39^2}\sin^2 18t \\
&\qquad + \tfrac{27^2}{26^2}\cos^2 8t - 2\cdot\tfrac{27\cdot 8}{26\cdot 39}\cos 8t\cos 18t + \tfrac{64}{39^2}\cos^2 18t \\
&= \tfrac{27^2}{26^2}\left(\sin^2 8t + \cos^2 8t\right) + \tfrac{64}{39^2}\left(\sin^2 18t + \cos^2 18t\right) - \tfrac{72}{169}\left(\sin 8t\sin 18t + \cos 8t\cos 18t\right) \\
&= \tfrac{27^2}{26^2} + \tfrac{64}{39^2} - \tfrac{72}{169}\cos\left(18t-8t\right) = \tfrac{27^2}{26^2} + \tfrac{64}{39^2} - \tfrac{72}{169}\cos 10t
\end{aligned}$$

using the trigonometric identities $\sin^2\theta + \cos^2\theta = 1$ and $\cos(x-y) = \cos x\cos y + \sin x\sin y$. Also $z^2 = \frac{144^2}{65^2}\sin^2 5t$, and the identity $\sin^2 x = \dfrac{1-\cos 2x}{2}$ gives $z^2 = \frac{144^2}{65^2}\cdot\frac{1}{2}\left[1-\cos(2\cdot 5t)\right] = \frac{144^2}{2\cdot 65^2} - \frac{144^2}{2\cdot 65^2}\cos 10t$.

[continued]

Then

$$\begin{aligned}144(x^2+y^2)-25z^2 &= 144\left(\tfrac{27^2}{26^2}+\tfrac{64}{39^2}-\tfrac{72}{169}\cos 10t\right)-25\left(\tfrac{144^2}{2\cdot 65^2}-\tfrac{144^2}{2\cdot 65^2}\cos 10t\right)\\ &= 144\left(\tfrac{27^2}{26^2}+\tfrac{64}{39^2}-\tfrac{25\cdot 144}{2\cdot 65^2}-\tfrac{72}{169}\cos 10t+\tfrac{25\cdot 144}{2\cdot 65^2}\cos 10t\right)\\ &= 144\left(\tfrac{27^2}{26^2}+\tfrac{64}{39^2}-\tfrac{72}{169}-\tfrac{72}{169}\cos 10t+\tfrac{72}{169}\cos 10t\right)=144\left(\tfrac{25}{36}\right)=100\end{aligned}$$

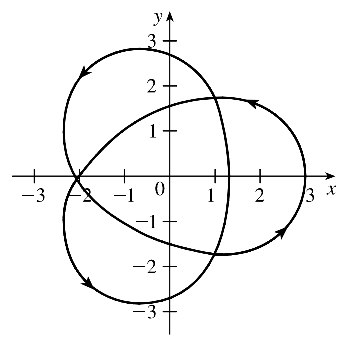
Thus the curve lies on the surface $144(x^2+y^2)-25z^2=100$ or $144x^2+144y^2-25z^2=100$, a hyperboloid of one sheet with axis the $z$-axis.

**60.** The projection of the curve onto the $xy$-plane is given by the parametric equations $x=(2+\cos 1.5t)\cos t$, $y=(2+\cos 1.5t)\sin t$. If we convert to polar coordinates, we have

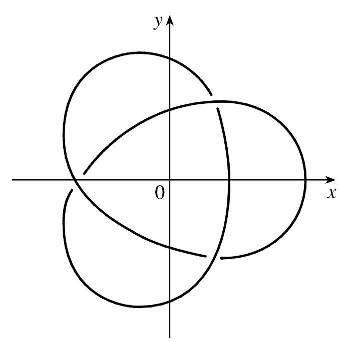
$$\begin{aligned}r^2=x^2+y^2 &= [(2+\cos 1.5t)\cos t]^2+[(2+\cos 1.5t)\sin t]^2\\ &= (2+\cos 1.5t)^2(\cos^2 t+\sin^2 t)=(2+\cos 1.5t)^2 \quad\Rightarrow\quad r=2+\cos 1.5t\end{aligned}$$

Also, $\tan\theta=\dfrac{y}{x}=\dfrac{(2+\cos 1.5t)\sin t}{(2+\cos 1.5t)\cos t}=\tan t \quad\Rightarrow\quad \theta=t.$

Thus, the polar equation of the curve is $r=2+\cos 1.5\theta$. At $\theta=0$, we have $r=3$, and $r$ decreases to 1 as $\theta$ increases to $\frac{2\pi}{3}$. For $\frac{2\pi}{3}\le\theta\le\frac{4\pi}{3}$, $r$ increases to 3; $r$ decreases to 1 again at $\theta=2\pi$, increases to 3 at $\theta=\frac{8\pi}{3}$, decreases to 1 at $\theta=\frac{10\pi}{3}$, and completes the closed curve by increasing to 3 at $\theta=4\pi$. We sketch an approximate graph as shown in the figure.

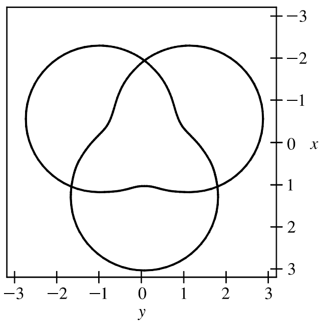



We can determine how the curve passes over itself by investigating the maximum and minimum values of $z$ for $0\le t\le 4\pi$. Since $z=\sin 1.5t$, $z$ is maximized where $\sin 1.5t=1 \quad\Rightarrow\quad 1.5t=\frac{\pi}{2},\frac{5\pi}{2}$, or $\frac{9\pi}{2} \quad\Rightarrow$ $t=\frac{\pi}{3},\frac{5\pi}{3}$, or $3\pi$. $z$ is minimized where $\sin 1.5t=-1 \quad\Rightarrow$ $1.5t=\frac{3\pi}{2},\frac{7\pi}{2}$, or $\frac{11\pi}{2} \quad\Rightarrow\quad t=\pi,\frac{7\pi}{3}$, or $\frac{11\pi}{3}$. Note that these are precisely the values for which $\cos 1.5t=0 \quad\Rightarrow\quad r=2$, and on the graph of the projection, these six points appear to be at the three self-intersections we see. Comparing the maximum and minimum values of $z$ at these intersections, we can determine where the curve passes over itself, as indicated in the figure.

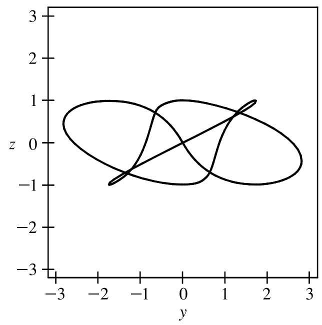



[continued]

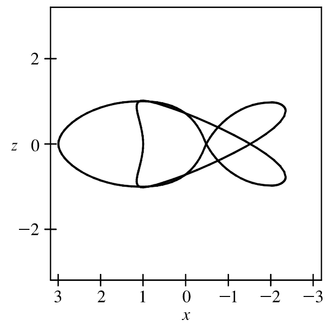
We show a computer-drawn graph of the curve from above, as well as views from the front and from the right side.




Top view

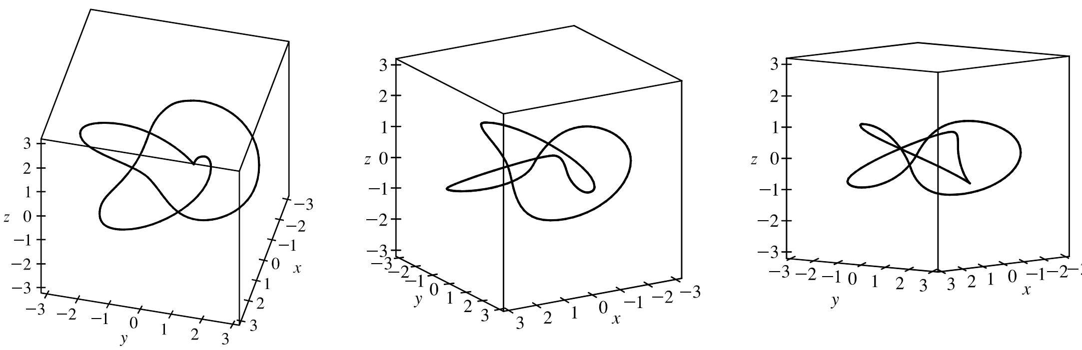


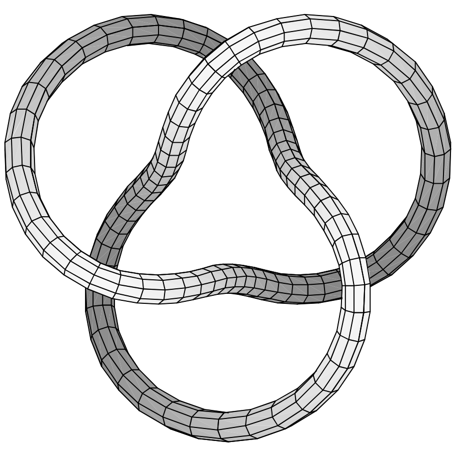

Front view

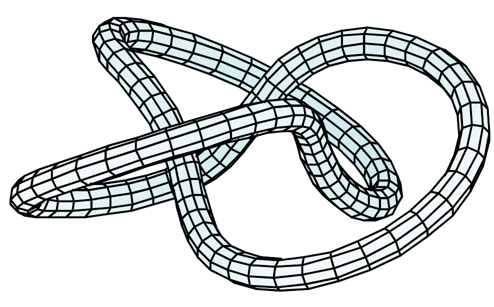



Side view

The top view graph shows a more accurate representation of the projection of the trefoil knot onto the $xy$-plane (the axes are rotated $90^\circ$). Notice the indentations the graph exhibits at the points corresponding to $r = 1$. Finally, we graph several additional viewpoints of the trefoil knot, along with two plots showing a tube of radius 0.2 around the curve.





**61.** Let $\mathbf{u}(t) = \langle u_1(t), u_2(t), u_3(t) \rangle$ and $\mathbf{v}(t) = \langle v_1(t), v_2(t), v_3(t) \rangle$. In each part of this problem the basic procedure is to use Equation 1 and then analyze the individual component functions using the limit properties we have already developed for real-valued functions.

(a) $\lim\limits_{t\to a} \mathbf{u}(t) + \lim\limits_{t\to a} \mathbf{v}(t) = \left\langle \lim\limits_{t\to a} u_1(t), \lim\limits_{t\to a} u_2(t), \lim\limits_{t\to a} u_3(t) \right\rangle + \left\langle \lim\limits_{t\to a} v_1(t), \lim\limits_{t\to a} v_2(t), \lim\limits_{t\to a} v_3(t) \right\rangle$ and the limits of these component functions must each exist since the vector functions both possess limits as $t \to a$. Then adding the two vectors and using the addition property of limits for real-valued functions, we have that

$$\begin{aligned}
\lim_{t\to a} \mathbf{u}(t) + \lim_{t\to a} \mathbf{v}(t) &= \left\langle \lim_{t\to a} u_1(t) + \lim_{t\to a} v_1(t), \lim_{t\to a} u_2(t) + \lim_{t\to a} v_2(t), \lim_{t\to a} u_3(t) + \lim_{t\to a} v_3(t) \right\rangle \\
&= \left\langle \lim_{t\to a} [u_1(t) + v_1(t)]\,, \lim_{t\to a} [u_2(t) + v_2(t)]\,, \lim_{t\to a} [u_3(t) + v_3(t)] \right\rangle \\
&= \lim_{t\to a} \langle u_1(t) + v_1(t), u_2(t) + v_2(t), u_3(t) + v_3(t) \rangle \qquad \text{[using (1) backward]} \\
&= \lim_{t\to a} [\mathbf{u}(t) + \mathbf{v}(t)]
\end{aligned}$$

(b) $$\begin{aligned}
\lim_{t\to a} c\mathbf{u}(t) &= \lim_{t\to a} \langle cu_1(t), cu_2(t), cu_3(t) \rangle = \left\langle \lim_{t\to a} cu_1(t), \lim_{t\to a} cu_2(t), \lim_{t\to a} cu_3(t) \right\rangle \\
&= \left\langle c \lim_{t\to a} u_1(t), c \lim_{t\to a} u_2(t), c \lim_{t\to a} u_3(t) \right\rangle = c \left\langle \lim_{t\to a} u_1(t), \lim_{t\to a} u_2(t), \lim_{t\to a} u_3(t) \right\rangle \\
&= c \lim_{t\to a} \langle u_1(t), u_2(t), u_3(t) \rangle = c \lim_{t\to a} \mathbf{u}(t)
\end{aligned}$$

(c) $$\begin{aligned}
\lim_{t\to a} \mathbf{u}(t) \cdot \lim_{t\to a} \mathbf{v}(t) &= \left\langle \lim_{t\to a} u_1(t), \lim_{t\to a} u_2(t), \lim_{t\to a} u_3(t) \right\rangle \cdot \left\langle \lim_{t\to a} v_1(t), \lim_{t\to a} v_2(t), \lim_{t\to a} v_3(t) \right\rangle \\
&= \left[\lim_{t\to a} u_1(t)\right] \left[\lim_{t\to a} v_1(t)\right] + \left[\lim_{t\to a} u_2(t)\right] \left[\lim_{t\to a} v_2(t)\right] + \left[\lim_{t\to a} u_3(t)\right] \left[\lim_{t\to a} v_3(t)\right] \\
&= \lim_{t\to a} u_1(t)v_1(t) + \lim_{t\to a} u_2(t)v_2(t) + \lim_{t\to a} u_3(t)v_3(t) \\
&= \lim_{t\to a} [u_1(t)v_1(t) + u_2(t)v_2(t) + u_3(t)v_3(t)] = \lim_{t\to a} [\mathbf{u}(t) \cdot \mathbf{v}(t)]
\end{aligned}$$

(d) $$\begin{aligned}
\lim_{t\to a} \mathbf{u}(t) \times \lim_{t\to a} \mathbf{v}(t) &= \left\langle \lim_{t\to a} u_1(t), \lim_{t\to a} u_2(t), \lim_{t\to a} u_3(t) \right\rangle \times \left\langle \lim_{t\to a} v_1(t), \lim_{t\to a} v_2(t), \lim_{t\to a} v_3(t) \right\rangle \\
&= \Big\langle \left[\lim_{t\to a} u_2(t)\right] \left[\lim_{t\to a} v_3(t)\right] - \left[\lim_{t\to a} u_3(t)\right] \left[\lim_{t\to a} v_2(t)\right], \\
&\qquad \left[\lim_{t\to a} u_3(t)\right] \left[\lim_{t\to a} v_1(t)\right] - \left[\lim_{t\to a} u_1(t)\right] \left[\lim_{t\to a} v_3(t)\right], \\
&\qquad\qquad \left[\lim_{t\to a} u_1(t)\right] \left[\lim_{t\to a} v_2(t)\right] - \left[\lim_{t\to a} u_2(t)\right] \left[\lim_{t\to a} v_1(t)\right] \Big\rangle \\
&= \Big\langle \lim_{t\to a} [u_2(t)v_3(t) - u_3(t)v_2(t)],\ \lim_{t\to a} [u_3(t)v_1(t) - u_1(t)v_3(t)], \\
&\qquad \lim_{t\to a} [u_1(t)v_2(t) - u_2(t)v_1(t)] \Big\rangle \\
&= \lim_{t\to a} \langle u_2(t)v_3(t) - u_3(t)v_2(t), u_3(t)\, v_1(t) - u_1(t)v_3(t), u_1(t)v_2(t) - u_2(t)v_1(t) \rangle \\
&= \lim_{t\to a} [\mathbf{u}(t) \times \mathbf{v}(t)]
\end{aligned}$$

**62.** Let $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$. If $\lim\limits_{t\to a} \mathbf{r}(t) = \mathbf{b}$, then $\lim\limits_{t\to a} \mathbf{r}(t)$ exists, so by (1),

$\mathbf{b} = \lim\limits_{t\to a} \mathbf{r}(t) = \left\langle \lim\limits_{t\to a} f(t), \lim\limits_{t\to a} g(t), \lim\limits_{t\to a} h(t) \right\rangle$. By the definition of equal vectors we have $\lim\limits_{t\to a} f(t) = b_1$, $\lim\limits_{t\to a} g(t) = b_2$

and $\lim_{t \to a} h(t) = b_3$. But these are limits of real-valued functions, so by the definition of limits, for every $\varepsilon > 0$ there exists $\delta_1 > 0$, $\delta_2 > 0$, $\delta_3 > 0$ so that if $0 < |t - a| < \delta_1$, then $|f(t) - b_1| < \varepsilon/3$; if $0 < |t - a| < \delta_2$, then $|g(t) - b_2| < \varepsilon/3$; and if $0 < |t - a| < \delta_3$, then $|h(t) - b_3| < \varepsilon/3$. Letting $\delta =$ minimum of $\{\delta_1, \delta_2, \delta_3\}$, then if $0 < |t - a| < \delta$ we have $|f(t) - b_1| + |g(t) - b_2| + |h(t) - b_3| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$. But

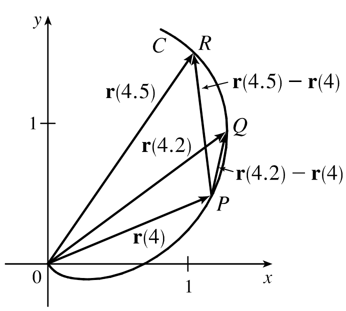
$$\begin{aligned} |\mathbf{r}(t) - \mathbf{b}| &= |\langle f(t) - b_1, g(t) - b_2, h(t) - b_3 \rangle| = \sqrt{(f(t) - b_1)^2 + (g(t) - b_2)^2 + (h(t) - b_3)^2} \\ &\leq \sqrt{[f(t) - b_1]^2} + \sqrt{[g(t) - b_2]^2} + \sqrt{[h(t) - b_3]^2} = |f(t) - b_1| + |g(t) - b_2| + |h(t) - b_3| \end{aligned}$$

Thus, for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < |t - a| < \delta$, then $|\mathbf{r}(t) - \mathbf{b}| \leq |f(t) - b_1| + |g(t) - b_2| + |h(t) - b_3| < \varepsilon$.

Conversely, suppose for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < |t - a| < \delta$, then $|\mathbf{r}(t) - \mathbf{b}| < \varepsilon \;\Leftrightarrow\; |\langle f(t) - b_1, g(t) - b_2, h(t) - b_3 \rangle| < \varepsilon \;\Leftrightarrow\; \sqrt{[f(t) - b_1]^2 + [g(t) - b_2]^2 + [h(t) - b_3]^2} < \varepsilon \;\Leftrightarrow\; [f(t) - b_1]^2 + [g(t) - b_2]^2 + [h(t) - b_3]^2 < \varepsilon^2$. But each term on the left side of the last inequality is positive, so if $0 < |t - a| < \delta$, then $[f(t) - b_1]^2 < \varepsilon^2$, $[g(t) - b_2]^2 < \varepsilon^2$ and $[h(t) - b_3]^2 < \varepsilon^2$ or, taking the square root of both sides in each of the above, $|f(t) - b_1| < \varepsilon$, $|g(t) - b_2| < \varepsilon$ and $|h(t) - b_3| < \varepsilon$. And by definition of limits of real-valued functions we have $\lim_{t \to a} f(t) = b_1$, $\lim_{t \to a} g(t) = b_2$, and $\lim_{t \to a} h(t) = b_3$. But by (1), $\lim_{t \to a} \mathbf{r}(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle$, so $\lim_{t \to a} \mathbf{r}(t) = \langle b_1, b_2, b_3 \rangle = \mathbf{b}$.

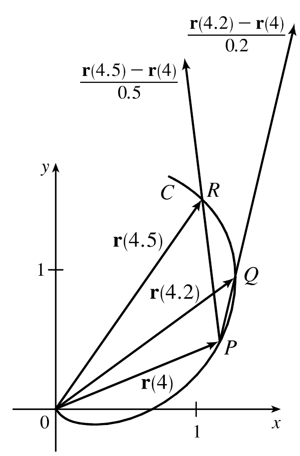
## 13.2 Derivatives and Integrals of Vector Functions

**1.** (a)



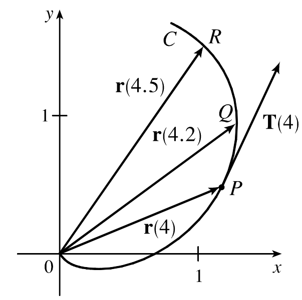

(b) $\dfrac{\mathbf{r}(4.5) - \mathbf{r}(4)}{0.5} = 2[\mathbf{r}(4.5) - \mathbf{r}(4)]$, so we draw a vector in the same direction but with twice the length of the vector $\mathbf{r}(4.5) - \mathbf{r}(4)$.

$\dfrac{\mathbf{r}(4.2) - \mathbf{r}(4)}{0.2} = 5[\mathbf{r}(4.2) - \mathbf{r}(4)]$, so we draw a vector in the same direction but with 5 times the length of the vector $\mathbf{r}(4.2) - \mathbf{r}(4)$.

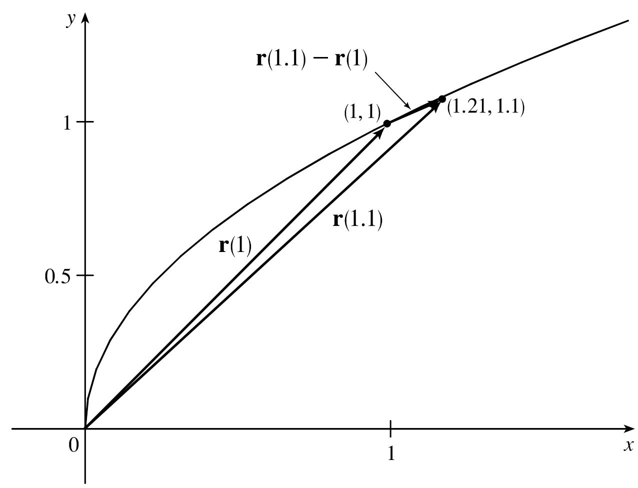



(c) By Definition 1, $\mathbf{r}'(4) = \lim_{h \to 0} \dfrac{\mathbf{r}(4 + h) - \mathbf{r}(4)}{h}$. $\mathbf{T}(4) = \dfrac{\mathbf{r}'(4)}{|\mathbf{r}'(4)|}$.

(d) $\mathbf{T}(4)$ is a unit vector in the same direction as $\mathbf{r}'(4)$, that is, parallel to the tangent line to the curve at $\mathbf{r}(4)$ with length 1.

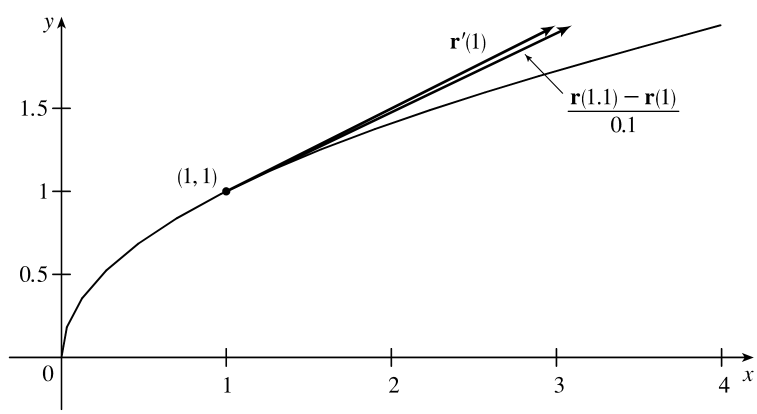



**2.** (a) The curve can be represented by the parametric equations $x = t^2$, $y = t$, $0 \le t \le 2$. Eliminating the parameter, we have $x = y^2$, $0 \le y \le 2$, a portion of which we graph here, along with the vectors $\mathbf{r}(1)$, $\mathbf{r}(1.1)$, and $\mathbf{r}(1.1) - \mathbf{r}(1)$.




(b) Since $\mathbf{r}(t) = \langle t^2, t \rangle$, we differentiate components, giving $\mathbf{r}'(t) = \langle 2t, 1 \rangle$, so $\mathbf{r}'(1) = \langle 2, 1 \rangle$.

$$\frac{\mathbf{r}(1.1) - \mathbf{r}(1)}{0.1} = \frac{\langle 1.21, 1.1 \rangle - \langle 1, 1 \rangle}{0.1} = 10 \langle 0.21, 0.1 \rangle = \langle 2.1, 1 \rangle.$$

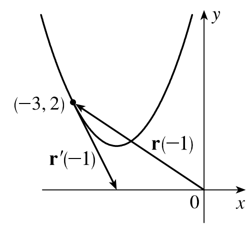



As we can see from the graph, these vectors are very close in length and direction. $\mathbf{r}'(1)$ is defined to be

$\lim_{h\to 0} \dfrac{\mathbf{r}(1+h)-\mathbf{r}(1)}{h}$, and we recognize $\dfrac{\mathbf{r}(1.1)-\mathbf{r}(1)}{0.1}$ as the expression after the limit sign with $h=0.1$. Since $h$ is close to 0, we would expect $\dfrac{\mathbf{r}(1.1)-\mathbf{r}(1)}{0.1}$ to be a vector close to $\mathbf{r}'(1)$.

**3.** $\mathbf{r}(t)=\langle t-2, t^2+1\rangle$, $\mathbf{r}(-1)=\langle -3,2\rangle$. Since $(x+2)^2=t^2=y-1 \;\Rightarrow\; y=(x+2)^2+1$, the curve is a parabola.

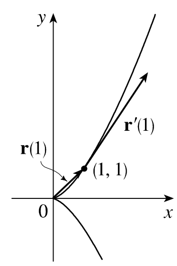
(a), (c)




(b) $\mathbf{r}'(t)=\langle 1,2t\rangle$, $\mathbf{r}'(-1)=\langle 1,-2\rangle$

**4.** $\mathbf{r}(t)=\langle t^2,t^3\rangle$, $\mathbf{r}(1)=\langle 1,1\rangle$. Since $x=t^2=(t^3)^{2/3}=y^{2/3}$, the curve is the graph of $x=y^{2/3}$.

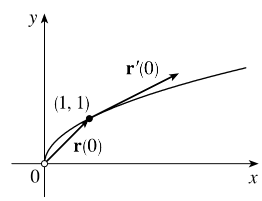
(a), (c)




(b) $\mathbf{r}'(t)=\langle 2t,3t^2\rangle$, $\mathbf{r}'(1)=\langle 2,3\rangle$

**5.** $\mathbf{r}(t)=e^{2t}\,\mathbf{i}+e^t\,\mathbf{j}$, $\mathbf{r}(0)=\mathbf{i}+\mathbf{j}$. Since $x=e^{2t}=(e^t)^2=y^2$, the curve is part of a parabola. Note that here $x>0$, $y>0$.

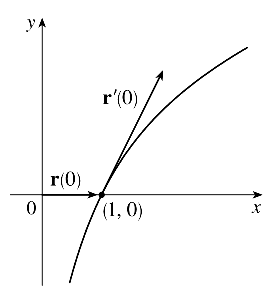
(a), (c)




(b) $\mathbf{r}'(t)=2e^{2t}\,\mathbf{i}+e^t\,\mathbf{j}$, $\mathbf{r}'(0)=2\,\mathbf{i}+\mathbf{j}$

**6.** $\mathbf{r}(t)=e^t\,\mathbf{i}+2t\,\mathbf{j}$, $\mathbf{r}(0)=\mathbf{i}$. Since $x=e^t \;\Leftrightarrow\; t=\ln x$ and $y=2t=2\ln x$, the curve is the graph of $y=2\ln x$.

(a), (c)

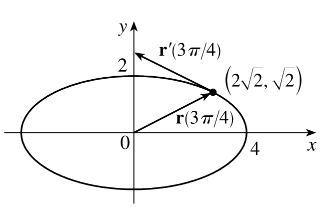



(b) $\mathbf{r}'(t)=e^t\,\mathbf{i}+2\,\mathbf{j}$, $\mathbf{r}'(0)=\mathbf{i}+2\,\mathbf{j}$

**7.** $\mathbf{r}(t)=4\sin t\,\mathbf{i}-2\cos t\,\mathbf{j}$, $\mathbf{r}(3\pi/4)=4(\sqrt{2}/2)\,\mathbf{i}-2(-\sqrt{2}/2)\,\mathbf{j}=2\sqrt{2}\,\mathbf{i}+\sqrt{2}\,\mathbf{j}$.

Here $(x/4)^2+(y/2)^2=\sin^2 t+\cos^2 t=1$, so the curve is the ellipse $\dfrac{x^2}{16}+\dfrac{y^2}{4}=1$.

(a), (c)

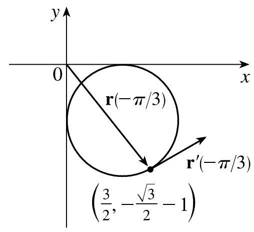



(b) $\mathbf{r}'(t)=4\cos t\,\mathbf{i}+2\sin t\,\mathbf{j}$, $\mathbf{r}'(3\pi/4)=-2\sqrt{2}\,\mathbf{i}+\sqrt{2}\,\mathbf{j}$.

**8.** $\mathbf{r}(t) = (\cos t + 1)\,\mathbf{i} + (\sin t - 1)\,\mathbf{j}$, $\mathbf{r}(-\pi/3) = \left(\frac{1}{2} + 1\right)\mathbf{i} + \left(-\frac{\sqrt{3}}{2} - 1\right)\mathbf{j} = \frac{3}{2}\,\mathbf{i} + \left(-\frac{\sqrt{3}}{2} - 1\right)\mathbf{j} \approx 1.5\,\mathbf{i} - 1.87\,\mathbf{j}$.

Here $(x-1)^2 + (y+1)^2 = \cos^2 t + \sin^2 t = 1$, so the curve is a circle of radius 1 with center $(1, -1)$.

(a), (c)




(b) $\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$,

$$\mathbf{r}'(-\pi/3) = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\,\mathbf{j} \approx 0.87\,\mathbf{i} + 0.5\,\mathbf{j}$$

**9.** $\mathbf{r}(t) = \left\langle \sqrt{t-2}, 3,\ 1/t^2 \right\rangle \quad \Rightarrow$

$$\mathbf{r}'(t) = \left\langle \frac{d}{dt}\left[\sqrt{t-2}\right], \frac{d}{dt}[3], \frac{d}{dt}\left[1/t^2\right] \right\rangle = \left\langle \tfrac{1}{2}(t-2)^{-1/2}, 0, -2t^{-3} \right\rangle = \left\langle \frac{1}{2\sqrt{t-2}}, 0, -\frac{2}{t^3} \right\rangle$$

**10.** $\mathbf{r}(t) = \left\langle e^{-t}, t - t^3, \ln t \right\rangle \quad \Rightarrow \quad \mathbf{r}'(t) = \left\langle -e^{-t}, 1 - 3t^2, 1/t \right\rangle$

**11.** $\mathbf{r}(t) = t^2\ \mathbf{i} + \cos\left(t^2\right)\ \mathbf{j} + \sin^2 t\ \mathbf{k} \quad \Rightarrow$

$$\mathbf{r}'(t) = 2t\,\mathbf{i} + \left[-\sin(t^2)\cdot 2t\right]\mathbf{j} + (2\sin t\cdot\cos t)\,\mathbf{k} = 2t\,\mathbf{i} - 2t\sin(t^2)\,\mathbf{j} + 2\sin t\cos t\,\mathbf{k}$$

**12.** $\mathbf{r}(t) = \dfrac{1}{1+t}\,\mathbf{i} + \dfrac{t}{1+t}\,\mathbf{j} + \dfrac{t^2}{1+t}\,\mathbf{k} \quad \Rightarrow$

$$\mathbf{r}'(t) = \frac{0 - 1(1)}{(1+t)^2}\,\mathbf{i} + \frac{(1+t)\cdot 1 - t(1)}{(1+t)^2}\,\mathbf{j} + \frac{(1+t)\cdot 2t - t^2(1)}{(1+t)^2}\,\mathbf{k} = -\frac{1}{(1+t)^2}\,\mathbf{i} + \frac{1}{(1+t)^2}\,\mathbf{j} + \frac{t^2 + 2t}{(1+t)^2}\,\mathbf{k}$$

**13.** $\mathbf{r}(t) = t\sin t\,\mathbf{i} + e^t\cos t\,\mathbf{j} + \sin t\cos t\,\mathbf{k} \quad \Rightarrow$

$$\begin{aligned}\mathbf{r}'(t) &= [t\cdot\cos t + (\sin t)\cdot 1]\,\mathbf{i} + \left[e^t(-\sin t) + (\cos t)e^t\right]\mathbf{j} + [(\sin t)(-\sin t) + (\cos t)(\cos t)]\,\mathbf{k} \\ &= (t\cos t + \sin t)\,\mathbf{i} + e^t(\cos t - \sin t)\,\mathbf{j} + \left(\cos^2 t - \sin^2 t\right)\mathbf{k}\end{aligned}$$

**14.** $\mathbf{r}(t) = \sin^2 at\,\mathbf{i} + te^{bt}\,\mathbf{j} + \cos^2 ct\,\mathbf{k} \quad \Rightarrow$

$$\begin{aligned}\mathbf{r}'(t) &= [2(\sin at)\cdot(\cos at)(a)]\,\mathbf{i} + \left[t\cdot e^{bt}(b) + e^{bt}\cdot 1\right]\mathbf{j} + [2(\cos ct)\cdot(-\sin ct)(c)]\,\mathbf{k} \\ &= 2a\sin at\cos at\,\mathbf{i} + e^{bt}(bt+1)\,\mathbf{j} - 2c\sin ct\cos ct\,\mathbf{k}\end{aligned}$$

**15.** $\mathbf{r}(t) = \mathbf{a} + t\,\mathbf{b} + t^2\,\mathbf{c} \quad \Rightarrow \quad \mathbf{r}'(t) = \mathbf{0} + \mathbf{b} + 2t\,\mathbf{c} = \mathbf{b} + 2t\,\mathbf{c}$ by Formulas 1 and 3 of Theorem 3.

**16.** To find $\mathbf{r}'(t)$, by Formula 5 of Theorem 3, we first expand $\mathbf{r}(t) = t\,\mathbf{a}\times(\mathbf{b} + t\,\mathbf{c}) = t(\mathbf{a}\times\mathbf{b}) + t^2(\mathbf{a}\times\mathbf{c})$, so $\mathbf{r}'(t) = \mathbf{a}\times\mathbf{b} + 2t(\mathbf{a}\times\mathbf{c})$.

**17.** $\mathbf{r}(t) = \left\langle t^2 - 2t, 1 + 3t, \frac{1}{3}t^3 + \frac{1}{2}t^2 \right\rangle \quad \Rightarrow \quad \mathbf{r}'(t) = \left\langle 2t - 2, 3, t^2 + t \right\rangle \quad \Rightarrow \quad \mathbf{r}'(2) = \langle 2, 3, 6\rangle$.

So $|\mathbf{r}'(2)| = \sqrt{2^2 + 3^2 + 6^2} = \sqrt{49} = 7$ and $\mathbf{T}(2) = \dfrac{\mathbf{r}'(2)}{|\mathbf{r}'(2)|} = \frac{1}{7}\langle 2, 3, 6\rangle = \left\langle \frac{2}{7}, \frac{3}{7}, \frac{6}{7} \right\rangle$.

**18.** $\mathbf{r}(t) = \left\langle \tan^{-1} t, 2e^{2t}, 8te^t \right\rangle \quad \Rightarrow \quad \mathbf{r}'(t) = \left\langle 1/(1+t^2), 4e^{2t}, 8te^t + 8e^t \right\rangle \quad \Rightarrow \quad \mathbf{r}'(0) = \langle 1, 4, 8\rangle$.

So $|\mathbf{r}'(0)| = \sqrt{1^2 + 4^2 + 8^2} = \sqrt{81} = 9$ and $\mathbf{T}(0) = \dfrac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \frac{1}{9}\langle 1, 4, 8\rangle = \left\langle \frac{1}{9}, \frac{4}{9}, \frac{8}{9} \right\rangle$.

**19.** $\mathbf{r}(t) = \cos t\,\mathbf{i} + 3t\,\mathbf{j} + 2\sin 2t\,\mathbf{k} \Rightarrow \mathbf{r}'(t) = -\sin t\,\mathbf{i} + 3\,\mathbf{j} + 4\cos 2t\,\mathbf{k} \Rightarrow \mathbf{r}'(0) = 3\,\mathbf{j} + 4\,\mathbf{k}$. So

$|\mathbf{r}'(0)| = \sqrt{0^2 + 3^2 + 4^2} = \sqrt{25} = 5$ and $\mathbf{T}(0) = \dfrac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \frac{1}{5}(3\,\mathbf{j} + 4\,\mathbf{k}) = \frac{3}{5}\,\mathbf{j} + \frac{4}{5}\,\mathbf{k}$.

**20.** $\mathbf{r}(t) = \sin^2 t\,\mathbf{i} + \cos^2 t\,\mathbf{j} + \tan^2 t\,\mathbf{k} \Rightarrow \mathbf{r}'(t) = 2\sin t\cos t\,\mathbf{i} - 2\cos t\sin t\,\mathbf{j} + 2\tan t\sec^2 t\,\mathbf{k} \Rightarrow$

$\mathbf{r}'\left(\frac{\pi}{4}\right) = 2 \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2}\,\mathbf{i} - 2 \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2}\,\mathbf{j} + 2 \cdot 1 \cdot (\sqrt{2})^2\,\mathbf{k} = \mathbf{i} - \mathbf{j} + 4\,\mathbf{k}$. So $\left|\mathbf{r}'\left(\frac{\pi}{4}\right)\right| = \sqrt{1^2 + 1^2 + 4^2} = \sqrt{18} = 3\sqrt{2}$

and $\mathbf{T}\left(\frac{\pi}{4}\right) = \dfrac{\mathbf{r}'\left(\frac{\pi}{4}\right)}{\left|\mathbf{r}'\left(\frac{\pi}{4}\right)\right|} = \dfrac{1}{3\sqrt{2}}(\mathbf{i} - \mathbf{j} + 4\,\mathbf{k}) = \dfrac{1}{3\sqrt{2}}\,\mathbf{i} - \dfrac{1}{3\sqrt{2}}\,\mathbf{j} + \dfrac{4}{3\sqrt{2}}\,\mathbf{k}$.

**21.** The point $(2, -2, 4)$ corresponds to $t = 1$ [note that $4/t = 4$]. Then

$$\mathbf{r}(t) = \left\langle t^3 + 1, 3t - 5, 4/t \right\rangle \Rightarrow \mathbf{r}'(t) = \left\langle 3t^2, 3, -4/t^2 \right\rangle \Rightarrow \mathbf{r}'(1) = \langle 3, 3, -4 \rangle$$

So
$$|\mathbf{r}'(1)| = \sqrt{3^2 + 3^2 + (-4)^2} = \sqrt{34}$$

and
$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{|\mathbf{r}'(1)|} = \frac{1}{\sqrt{34}}\langle 3, 3, -4 \rangle = \left\langle \frac{3}{\sqrt{34}}, \frac{3}{\sqrt{34}}, -\frac{4}{\sqrt{34}} \right\rangle$$

**22.** The point $(0, 0, 1)$ corresponds to $t = 0$ [note that $5t = 0$]. Then

$$\mathbf{r}(t) = \sin t\,\mathbf{i} + 5t\,\mathbf{j} + \cos t\,\mathbf{k} \Rightarrow \mathbf{r}'(t) = \cos t\,\mathbf{i} + 5\,\mathbf{j} - \sin t\,\mathbf{k} \Rightarrow \mathbf{r}'(0) = \mathbf{i} + 5\,\mathbf{j}$$

So
$$|\mathbf{r}'(0)| = \sqrt{1^2 + 5^2 + 0^2} = \sqrt{26}$$

and
$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \frac{1}{\sqrt{26}}(\mathbf{i} + 5\mathbf{j}) = \frac{1}{\sqrt{26}}\,\mathbf{i} + \frac{5}{\sqrt{26}}\,\mathbf{j}$$

**23.** $\mathbf{r}(t) = \left\langle t^4, t, t^2 \right\rangle \Rightarrow \mathbf{r}'(t) = \left\langle 4t^3, 1, 2t \right\rangle$. Then $\mathbf{r}'(1) = \langle 4, 1, 2 \rangle$, $|\mathbf{r}'(1)| = \sqrt{4^2 + 1^2 + 2^2} = \sqrt{21}$, and

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{|\mathbf{r}'(1)|} = \frac{1}{\sqrt{21}}\langle 4, 1, 2 \rangle = \left\langle \frac{4}{\sqrt{21}}, \frac{1}{\sqrt{21}}, \frac{2}{\sqrt{21}} \right\rangle$$

$\mathbf{r}''(t) = \left\langle 12t^2, 0, 2 \right\rangle$, so

$$\begin{aligned} \mathbf{r}'(t) \times \mathbf{r}''(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4t^3 & 1 & 2t \\ 12t^2 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2t \\ 0 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 4t^3 & 2t \\ 12t^2 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 4t^3 & 1 \\ 12t^2 & 0 \end{vmatrix}\mathbf{k} \\ &= (2 - 0)\,\mathbf{i} - (8t^3 - 24t^3)\,\mathbf{j} + (0 - 12t^2)\,\mathbf{k} = \left\langle 2, 16t^3, -12t^2 \right\rangle \end{aligned}$$

**24.** $\mathbf{r}(t) = \left\langle e^{2t}, e^{-3t}, t \right\rangle \Rightarrow \mathbf{r}'(t) = \left\langle 2e^{2t}, -3e^{-3t}, 1 \right\rangle$. Then $\mathbf{r}'(0) = \langle 2, -3, 1 \rangle$, $|\mathbf{r}'(0)| = \sqrt{2^2 + (-3)^2 + 1^2} = \sqrt{14}$, and

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \frac{1}{\sqrt{14}}\langle 2, -3, 1 \rangle = \left\langle \frac{2}{\sqrt{14}}, -\frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}} \right\rangle$$

$\mathbf{r}''(t) = \left\langle 4e^{2t}, 9e^{-3t}, 0 \right\rangle \Rightarrow \mathbf{r}''(0) = \langle 4, 9, 0 \rangle$. Then

$$\begin{aligned} \mathbf{r}'(0) \times \mathbf{r}''(0) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 1 \\ 4 & 9 & 0 \end{vmatrix} = \begin{vmatrix} -3 & 1 \\ 9 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 1 \\ 4 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & -3 \\ 4 & 9 \end{vmatrix}\mathbf{k} \\ &= (0 - 9)\,\mathbf{i} - (0 - 4)\,\mathbf{j} + [18 - (-12)]\,\mathbf{k} = \langle -9, 4, 30 \rangle \end{aligned}$$

**25.** The vector equation for the curve is $\mathbf{r}(t) = \left\langle t^2 + 1, 4\sqrt{t}, e^{t^2 - t} \right\rangle$, so $\mathbf{r}'(t) = \left\langle 2t, 2/\sqrt{t}, (2t-1)e^{t^2-t} \right\rangle$. The point $(2, 4, 1)$ corresponds to $t = 1$, so the tangent vector there is $\mathbf{r}'(1) = \langle 2, 2, 1 \rangle$. Thus, the tangent line goes through the point $(2, 4, 1)$ and is parallel to the vector $\langle 2, 2, 1 \rangle$. Parametric equations are $x = 2 + 2t$, $y = 4 + 2t$, $z = 1 + t$.

**26.** The vector equation for the curve is $\mathbf{r}(t) = \langle \ln(t+1), t\cos 2t, 2^t \rangle$, so $\mathbf{r}'(t) = \langle 1/(t+1), \cos 2t - 2t\sin 2t, 2^t \ln 2 \rangle$. The point $(0, 0, 1)$ corresponds to $t = 0$, so the tangent vector there is $\mathbf{r}'(0) = \langle 1, 1, \ln 2 \rangle$. Thus, the tangent line goes through the point $(0, 0, 1)$ and is parallel to the vector $\langle 1, 1, \ln 2 \rangle$. Parametric equations are $x = 0 + 1 \cdot t = t$, $y = 0 + 1 \cdot t = t$, $z = 1 + (\ln 2)t$.

**27.** The vector equation for the curve is $\mathbf{r}(t) = \langle e^{-t}\cos t, e^{-t}\sin t, e^{-t} \rangle$, so

$$\begin{aligned}\mathbf{r}'(t) &= \langle e^{-t}(-\sin t) + (\cos t)(-e^{-t}), e^{-t}\cos t + (\sin t)(-e^{-t}), (-e^{-t}) \rangle \\ &= \langle -e^{-t}(\cos t + \sin t), e^{-t}(\cos t - \sin t), -e^{-t} \rangle\end{aligned}$$

The point $(1, 0, 1)$ corresponds to $t = 0$, so the tangent vector there is $\mathbf{r}'(0) = \langle -e^0(\cos 0 + \sin 0), e^0(\cos 0 - \sin 0), -e^0 \rangle = \langle -1, 1, -1 \rangle$. Thus, the tangent line is parallel to the vector $\langle -1, 1, -1 \rangle$ and parametric equations are $x = 1 + (-1)t = 1 - t$, $y = 0 + 1 \cdot t = t$, $z = 1 + (-1)t = 1 - t$.

**28.** The vector equation for the curve is $\mathbf{r}(t) = \langle \sqrt{t^2+3}, \ln(t^2+3), t \rangle$, so $\mathbf{r}'(t) = \langle t/\sqrt{t^2+3}, 2t/(t^2+3), 1 \rangle$. At $(2, \ln 4, 1)$, $t = 1$ and $\mathbf{r}'(1) = \langle \frac{1}{2}, \frac{1}{2}, 1 \rangle$. Thus, parametric equations of the tangent line are $x = 2 + \frac{1}{2}t$, $y = \ln 4 + \frac{1}{2}t$, $z = 1 + t$.

**29.** First we parametrize the curve $C$ of intersection. The projection of $C$ onto the $xy$-plane is contained in the circle $x^2 + y^2 = 25$, $z = 0$, so we can write $x = 5\cos t$, $y = 5\sin t$. $C$ also lies on the cylinder $y^2 + z^2 = 20$, and $z \geq 0$ near the point $(3, 4, 2)$, so we can write $z = \sqrt{20 - y^2} = \sqrt{20 - 25\sin^2 t}$. A vector equation then for $C$ is

$\mathbf{r}(t) = \left\langle 5\cos t, 5\sin t, \sqrt{20 - 25\sin^2 t} \right\rangle \Rightarrow \mathbf{r}'(t) = \left\langle -5\sin t, 5\cos t, \frac{1}{2}(20 - 25\sin^2 t)^{-1/2}(-50\sin t\cos t) \right\rangle$.

The point $(3, 4, 2)$ corresponds to $t = \cos^{-1}\left(\frac{3}{5}\right)$, so the tangent vector there is

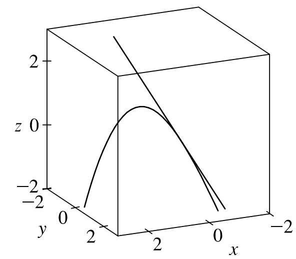
$\mathbf{r}'\left(\cos^{-1}\left(\frac{3}{5}\right)\right) = \left\langle -5\left(\frac{4}{5}\right), 5\left(\frac{3}{5}\right), \frac{1}{2}\left(20 - 25\left(\frac{4}{5}\right)^2\right)^{-1/2}\left(-50\left(\frac{4}{5}\right)\left(\frac{3}{5}\right)\right) \right\rangle = \langle -4, 3, -6 \rangle$.

The tangent line is parallel to this vector and passes through $(3, 4, 2)$, so a vector equation for the line is $\mathbf{r}(t) = (3 - 4t)\mathbf{i} + (4 + 3t)\mathbf{j} + (2 - 6t)\mathbf{k}$.

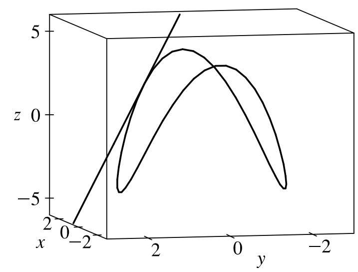
**30.** $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, e^t \rangle \Rightarrow \mathbf{r}'(t) = \langle -2\sin t, 2\cos t, e^t \rangle$. The tangent line to the curve is parallel to the plane when the curve's tangent vector is orthogonal to the plane's normal vector. Thus we require $\langle -2\sin t, 2\cos t, e^t \rangle \cdot \langle \sqrt{3}, 1, 0 \rangle = 0 \Rightarrow$ $-2\sqrt{3}\sin t + 2\cos t + 0 = 0 \Rightarrow \tan t = \frac{1}{\sqrt{3}} \Rightarrow t = \frac{\pi}{6}$ [since $0 \leq t \leq \pi$].

$\mathbf{r}\left(\frac{\pi}{6}\right) = \left\langle \sqrt{3}, 1, e^{\pi/6} \right\rangle$, so the point is $(\sqrt{3}, 1, e^{\pi/6})$.

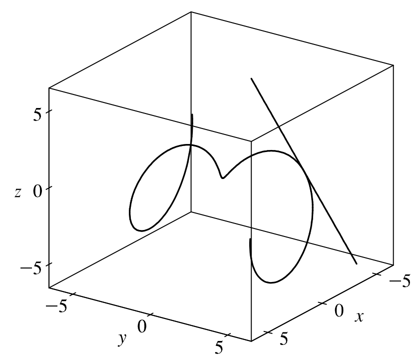
**31.** $\mathbf{r}(t) = \langle t, e^{-t}, 2t - t^2 \rangle \;\Rightarrow\; \mathbf{r}'(t) = \langle 1, -e^{-t}, 2 - 2t \rangle$. At $(0, 1, 0)$, $t = 0$ and $\mathbf{r}'(0) = \langle 1, -1, 2 \rangle$. Thus, parametric equations of the tangent line are $x = t$, $y = 1 - t$, $z = 2t$.



**32.** $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, 4\cos 2t \rangle$, $\mathbf{r}'(t) = \langle -2\sin t, 2\cos t, -8\sin 2t \rangle$. At $(\sqrt{3}, 1, 2)$, $t = \frac{\pi}{6}$ and $\mathbf{r}'(\frac{\pi}{6}) = \langle -1, \sqrt{3}, -4\sqrt{3} \rangle$. Thus, parametric equations of the tangent line are $x = \sqrt{3} - t$, $y = 1 + \sqrt{3}\,t$, $z = 2 - 4\sqrt{3}\,t$.



**33.** $\mathbf{r}(t) = \langle t\cos t, t, t\sin t \rangle \;\Rightarrow\; \mathbf{r}'(t) = \langle \cos t - t\sin t, 1, t\cos t + \sin t \rangle$. At $(-\pi, \pi, 0)$, $t = \pi$ and $\mathbf{r}'(\pi) = \langle -1, 1, -\pi \rangle$. Thus, parametric equations of the tangent line are $x = -\pi - t$, $y = \pi + t$, $z = -\pi t$.



**34.** (a) The tangent line to the curve $\mathbf{r}(t) = \langle \sin \pi t, 2\sin \pi t, \cos \pi t \rangle$ at $t = 0$ is the line through the point with position vector $\mathbf{r}(0) = \langle \sin 0, 2\sin 0, \cos 0 \rangle = \langle 0, 0, 1 \rangle$, and in the direction of the tangent vector, $\mathbf{r}'(0) = \langle \pi\cos 0, 2\pi\cos 0, -\pi\sin 0 \rangle = \langle \pi, 2\pi, 0 \rangle$. So an equation of the line is $\langle x, y, z \rangle = \mathbf{r}(0) + u\,\mathbf{r}'(0) = \langle 0 + \pi u, 0 + 2\pi u, 1 \rangle = \langle \pi u, 2\pi u, 1 \rangle$.

$\mathbf{r}\left(\frac{1}{2}\right) = \left\langle \sin\frac{\pi}{2}, 2\sin\frac{\pi}{2}, \cos\frac{\pi}{2} \right\rangle = \langle 1, 2, 0 \rangle$,

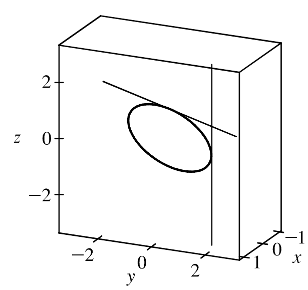
$\mathbf{r}'\left(\frac{1}{2}\right) = \left\langle \pi\cos\frac{\pi}{2}, 2\pi\cos\frac{\pi}{2}, -\pi\sin\frac{\pi}{2} \right\rangle = \langle 0, 0, -\pi \rangle$.

So the equation of the second line is

$\langle x, y, z \rangle = \mathbf{r}\left(\frac{1}{2}\right) + v\mathbf{r}'\left(\frac{1}{2}\right) = \langle 1, 2, 0 \rangle + v\,\langle 0, 0, -\pi \rangle = \langle 1, 2, -\pi v \rangle$.

The lines intersect where $\langle \pi u, 2\pi u, 1 \rangle = \langle 1, 2, -\pi v \rangle$, so the point of intersection is $(1, 2, 1)$.

(b)



**35.** The angle of intersection of the two curves is the angle between the two tangent vectors to the curves at the point of intersection. Since $\mathbf{r}_1'(t) = \langle 1, 2t, 3t^2 \rangle$ and $t = 0$ at $(0, 0, 0)$, $\mathbf{r}_1'(0) = \langle 1, 0, 0 \rangle$ is a tangent vector to $\mathbf{r}_1$ at $(0, 0, 0)$. Similarly, $\mathbf{r}_2'(t) = \langle \cos t, 2\cos 2t, 1 \rangle$ and since $\mathbf{r}_2(0) = \langle 0, 0, 0 \rangle$, $\mathbf{r}_2'(0) = \langle 1, 2, 1 \rangle$ is a tangent vector to $\mathbf{r}_2$ at $(0, 0, 0)$. If $\theta$ is the angle between these two tangent vectors, then $\cos\theta = \frac{1}{\sqrt{1}\sqrt{6}}\langle 1, 0, 0 \rangle \cdot \langle 1, 2, 1 \rangle = \frac{1}{\sqrt{6}}$ and $\theta = \cos^{-1}\left(\frac{1}{\sqrt{6}}\right) \approx 66^\circ$.

**36.** To find the point of intersection, we must find the values of $t$ and $s$ which satisfy the following three equations simultaneously: $t = 3 - s$, $1 - t = s - 2$, $3 + t^2 = s^2$. Solving the last two equations gives $t = 1$, $s = 2$ (check these in the first equation). Thus the point of intersection is $(1, 0, 4)$. To find the angle $\theta$ of intersection, we proceed as in Exercise 35. The tangent vectors to the respective curves at $(1, 0, 4)$ are $\mathbf{r}_1'(1) = \langle 1, -1, 2\rangle$ and $\mathbf{r}_2'(2) = \langle -1, 1, 4\rangle$. So

$\cos\theta = \frac{1}{\sqrt{6}\sqrt{18}}(-1 - 1 + 8) = \frac{6}{6\sqrt{3}} = \frac{1}{\sqrt{3}}$ and $\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 55°$.

*Note:* In Exercise 35, the curves intersect when the value of both parameters is zero. However, as seen in this exercise, it is not necessary for the parameters to be of equal value at the point of intersection.

**37.** 
$$\begin{aligned}\int_0^2 (t\,\mathbf{i} - t^3\,\mathbf{j} + 3t^5\,\mathbf{k})\,dt &= \left(\int_0^2 t\,dt\right)\mathbf{i} - \left(\int_0^2 t^3\,dt\right)\mathbf{j} + \left(\int_0^2 3t^5\,dt\right)\mathbf{k}\\ &= \left[\tfrac{1}{2}t^2\right]_0^2\,\mathbf{i} - \left[\tfrac{1}{4}t^4\right]_0^2\,\mathbf{j} + \left[\tfrac{1}{2}t^6\right]_0^2\,\mathbf{k}\\ &= \tfrac{1}{2}(4-0)\,\mathbf{i} - \tfrac{1}{4}(16-0)\,\mathbf{j} + \tfrac{1}{2}(64-0)\,\mathbf{k} = 2\,\mathbf{i} - 4\,\mathbf{j} + 32\,\mathbf{k}\end{aligned}$$

**38.** 
$$\begin{aligned}\int_1^4 \left(2t^{3/2}\,\mathbf{i} + (t+1)\sqrt{t}\,\mathbf{k}\right)dt &= \left(\int_1^4 2t^{3/2}\,dt\right)\mathbf{i} + \left[\int_1^4 (t^{3/2} + t^{1/2})\,dt\right]\mathbf{k}\\ &= \left[\tfrac{4}{5}t^{5/2}\right]_1^4\,\mathbf{i} + \left[\tfrac{2}{5}t^{5/2} + \tfrac{2}{3}t^{3/2}\right]_1^4\,\mathbf{k}\\ &= \tfrac{4}{5}(4^{5/2} - 1)\,\mathbf{i} + \left(\tfrac{2}{5}(4)^{5/2} + \tfrac{2}{3}(4)^{3/2} - \tfrac{2}{5} - \tfrac{2}{3}\right)\mathbf{k}\\ &= \tfrac{4}{5}(31)\,\mathbf{i} + \left(\tfrac{2}{5}(32) + \tfrac{2}{3}(8) - \tfrac{2}{5} - \tfrac{2}{3}\right)\mathbf{k} = \tfrac{124}{5}\,\mathbf{i} + \tfrac{256}{15}\,\mathbf{k}\end{aligned}$$

**39.** 
$$\begin{aligned}\int_0^1 \left(\frac{1}{t+1}\,\mathbf{i} + \frac{1}{t^2+1}\,\mathbf{j} + \frac{t}{t^2+1}\,\mathbf{k}\right)dt &= \left(\int_0^1 \frac{1}{t+1}\,dt\right)\mathbf{i} + \left(\int_0^1 \frac{1}{t^2+1}\,dt\right)\mathbf{j} + \left(\int_0^1 \frac{t}{t^2+1}\,dt\right)\mathbf{k}\\ &= \left[\ln|t+1|\right]_0^1\,\mathbf{i} + \left[\tan^{-1} t\right]_0^1\,\mathbf{j} + \left[\tfrac{1}{2}\ln(t^2+1)\right]_0^1\,\mathbf{k}\\ &= (\ln 2 - \ln 1)\,\mathbf{i} + \left(\tfrac{\pi}{4} - 0\right)\mathbf{j} + \tfrac{1}{2}(\ln 2 - \ln 1)\,\mathbf{k} = \ln 2\,\mathbf{i} + \tfrac{\pi}{4}\,\mathbf{j} + \tfrac{1}{2}\ln 2\,\mathbf{k}\end{aligned}$$

**40.** $\int_0^{\pi/4} (\sec t\tan t\,\mathbf{i} + t\cos 2t\,\mathbf{j} + \sin^2 2t\cos 2t\,\mathbf{k})\,dt$

$$= \left(\int_0^{\pi/4} \sec t\tan t\,dt\right)\mathbf{i} + \left(\int_0^{\pi/4} t\cos 2t\,dt\right)\mathbf{j} + \left(\int_0^{\pi/4} \sin^2 2t\cos 2t\,dt\right)\mathbf{k}$$

$$= \left[\sec t\right]_0^{\pi/4}\,\mathbf{i} + \left(\left[\tfrac{1}{2}t\sin 2t\right]_0^{\pi/4} - \int_0^{\pi/4} \tfrac{1}{2}\sin 2t\,dt\right)\mathbf{j} + \left[\tfrac{1}{6}\sin^3 2t\right]_0^{\pi/4}\,\mathbf{k}$$

[For the $y$-component, integrate by parts with $u = t$, $dv = \cos 2t\,dt$.]

$$= \left(\sec\tfrac{\pi}{4} - \sec 0\right)\mathbf{i} + \left(\tfrac{\pi}{8}\sin\tfrac{\pi}{2} - 0 - \left[-\tfrac{1}{4}\cos 2t\right]_0^{\pi/4}\right)\mathbf{j} + \tfrac{1}{6}\left(\sin^3\tfrac{\pi}{2} - \sin^3 0\right)\mathbf{k}$$

$$= (\sqrt{2} - 1)\,\mathbf{i} + \left(\tfrac{\pi}{8} + \tfrac{1}{4}\cos\tfrac{\pi}{2} - \tfrac{1}{4}\cos 0\right)\mathbf{j} + \tfrac{1}{6}(1 - 0)\,\mathbf{k} = (\sqrt{2} - 1)\,\mathbf{i} + \left(\tfrac{\pi}{8} - \tfrac{1}{4}\right)\mathbf{j} + \tfrac{1}{6}\,\mathbf{k}$$

**41.** 
$$\begin{aligned}\int \left(\frac{1}{1+t^2}\,\mathbf{i} + te^{t^2}\,\mathbf{j} + \sqrt{t}\,\mathbf{k}\right)dt &= \left(\int \frac{1}{1+t^2}\,dt\right)\mathbf{i} + \left(\int te^{t^2}\,dt\right)\mathbf{j} + \left(\int \sqrt{t}\,dt\right)\mathbf{k}\\ &= \tan^{-1} t\,\mathbf{i} + \tfrac{1}{2}e^{t^2}\,\mathbf{j} + \tfrac{2}{3}t^{3/2}\,\mathbf{k} + \mathbf{C}\end{aligned}$$

where $\mathbf{C}$ is a vector constant of integration.

**42.** $\displaystyle\int\left(t\cos t^2\,\mathbf{i}+\frac{1}{t}\,\mathbf{j}+\sec^2 t\,\mathbf{k}\right)dt=\left(\int t\cos t^2\,dt\right)\mathbf{i}+\left(\int\frac{1}{t}\,dt\right)\mathbf{j}+\left(\int\sec^2 t\,dt\right)\mathbf{k}$

$$=\tfrac{1}{2}\sin t^2\,\mathbf{i}+\ln|t|\,\mathbf{j}+\tan t\,\mathbf{k}+\mathbf{C}$$

where $\mathbf{C}$ is a vector constant of integration.

**43.** $\mathbf{r}'(t)=2t\,\mathbf{i}+3t^2\,\mathbf{j}+\sqrt{t}\,\mathbf{k}\quad\Rightarrow\quad\mathbf{r}(t)=t^2\,\mathbf{i}+t^3\,\mathbf{j}+\frac{2}{3}t^{3/2}\,\mathbf{k}+\mathbf{C}$, where $\mathbf{C}$ is a constant vector.

But $\mathbf{i}+\mathbf{j}=\mathbf{r}\,(1)=\mathbf{i}+\mathbf{j}+\frac{2}{3}\mathbf{k}+\mathbf{C}$. Thus, $\mathbf{C}=-\frac{2}{3}\mathbf{k}$ and $\mathbf{r}(t)=t^2\,\mathbf{i}+t^3\,\mathbf{j}+\left(\frac{2}{3}t^{3/2}-\frac{2}{3}\right)\mathbf{k}$.

**44.** $\mathbf{r}'(t)=t\,\mathbf{i}+e^t\,\mathbf{j}+te^t\,\mathbf{k}\quad\Rightarrow\quad\mathbf{r}(t)=\frac{1}{2}t^2\,\mathbf{i}+e^t\,\mathbf{j}+\left(te^t-e^t\right)\,\mathbf{k}+\mathbf{C}$. But $\mathbf{i}+\mathbf{j}+\mathbf{k}=\mathbf{r}\,(0)=\mathbf{j}-\mathbf{k}+\mathbf{C}$.

Thus, $\mathbf{C}=\mathbf{i}+2\,\mathbf{k}$ and $\mathbf{r}(t)=\left(\frac{1}{2}t^2+1\right)\mathbf{i}+e^t\,\mathbf{j}+(te^t-e^t+2)\,\mathbf{k}$.

For Exercises 45–48, let $\mathbf{u}(t)=\langle u_1(t),u_2(t),u_3(t)\rangle$ and $\mathbf{v}(t)=\langle v_1(t),v_2(t),v_3(t)\rangle$. In each of these exercises, the procedure is to apply Theorem 2 so that the corresponding properties of derivatives of real-valued functions can be used.

**45.**
$$\begin{aligned}\frac{d}{dt}\,[\mathbf{u}(t)+\mathbf{v}(t)]&=\frac{d}{dt}\,\langle u_1(t)+v_1(t),u_2(t)+v_2(t),u_3(t)+v_3(t)\rangle\\&=\left\langle\frac{d}{dt}\,[u_1(t)+v_1\,(t)]\,,\frac{d}{dt}\,[u_2(t)+v_2(t)]\,,\frac{d}{dt}\,[u_3(t)+v_3(t)]\right\rangle\\&=\langle u_1'(t)+v_1'(t),u_2'(t)+v_2'(t),u_3'(t)+v_3'(t)\rangle\\&=\langle u_1'(t),u_2'\,(t)\,,u_3'(t)\rangle+\langle v_1'(t),v_2'(t),v_3'(t)\rangle=\mathbf{u}'(t)+\mathbf{v}'(t)\end{aligned}$$

**46.**
$$\begin{aligned}\frac{d}{dt}\,[f(t)\,\mathbf{u}(t)]&=\frac{d}{dt}\,\langle f(t)u_1(t),f(t)u_2(t),f(t)u_3(t)\rangle\\&=\left\langle\frac{d}{dt}\,[f(t)u_1(t)]\,,\frac{d}{dt}\,[f(t)u_2(t)]\,,\frac{d}{dt}\,[f(t)u_3(t)]\right\rangle\\&=\langle f'(t)u_1(t)+f(t)u_1'(t),f'(t)u_2(t)+f(t)u_2'(t),f'(t)u_3(t)+f(t)u_3'(t)\rangle\\&=f'(t)\,\langle u_1(t),u_2(t),u_3(t)\rangle+f(t)\,\langle u_1'(t),u_2'(t),u_3'(t)\rangle=f'(t)\,\mathbf{u}(t)+f(t)\,\mathbf{u}'(t)\end{aligned}$$

**47.**
$$\begin{aligned}\frac{d}{dt}\,[\mathbf{u}(t)\times\mathbf{v}(t)]\;&=\frac{d}{dt}\,\langle u_2(t)v_3(t)-u_3(t)v_2(t),u_3(t)v_1(t)-u_1(t)v_3(t),u_1(t)v_2(t)-u_2(t)v_1(t)\rangle\\&=\langle u_2'v_3(t)+u_2(t)v_3'(t)-u_3'(t)v_2(t)-u_3(t)v_2'(t),\\&\qquad u_3'(t)v_1(t)+u_3(t)v_1'\,(t)-u_1'(t)v_3(t)-u_1(t)v_3'(t),\\&\qquad\quad u_1'(t)v_2(t)+u_1(t)v_2'(t)-u_2'(t)v_1(t)-u_2(t)v_1'(t)\rangle\\&=\langle u_2'(t)v_3(t)-u_3'(t)v_2\,(t)\,,u_3'(t)v_1(t)-u_1'(t)v_3(t),u_1'(t)v_2(t)-u_2'(t)v_1(t)\rangle\\&\qquad+\langle u_2(t)v_3'(t)-u_3(t)v_2'(t),u_3(t)v_1'\,(t)-u_1(t)v_3'(t),u_1(t)v_2'(t)-u_2(t)v_1'(t)\rangle\\&=\mathbf{u}'(t)\times\mathbf{v}(t)+\mathbf{u}(t)\times\mathbf{v}'(t)\end{aligned}$$

*Alternate solution:* Let $\mathbf{r}(t)=\mathbf{u}(t)\times\mathbf{v}(t)$. Then

$$\begin{aligned}\mathbf{r}(t+h)-\mathbf{r}(t)&=[\mathbf{u}(t+h)\times\mathbf{v}(t+h)]-[\mathbf{u}(t)\times\mathbf{v}(t)]\\&=[\mathbf{u}(t+h)\times\mathbf{v}(t+h)]-[\mathbf{u}(t)\times\mathbf{v}(t)]+[\mathbf{u}(t+h)\times\mathbf{v}(t)]-[\mathbf{u}(t+h)\times\mathbf{v}(t)]\\&=\mathbf{u}(t+h)\times[\mathbf{v}(t+h)-\mathbf{v}(t)]+[\mathbf{u}(t+h)-\mathbf{u}(t)]\times\mathbf{v}(t)\end{aligned}$$

[continued]

(Be careful of the order of the cross product.) Dividing through by $h$ and taking the limit as $h \to 0$ we have

$$\mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{u}(t+h)\times[\mathbf{v}(t+h)-\mathbf{v}(t)]}{h} + \lim_{h\to 0} \frac{[\mathbf{u}(t+h)-\mathbf{u}(t)]\times\mathbf{v}(t)}{h} = \mathbf{u}(t)\times\mathbf{v}'(t) + \mathbf{u}'(t)\times\mathbf{v}(t)$$

by Exercise 13.1.61(a) and Definition 1.

**48.** $\dfrac{d}{dt}[\mathbf{u}(f(t))] = \dfrac{d}{dt}\langle u_1(f(t)), u_2(f(t)), u_3(f(t))\rangle = \left\langle \dfrac{d}{dt}[u_1(f(t))], \dfrac{d}{dt}[u_2(f(t))], \dfrac{d}{dt}[u_3(f(t))]\right\rangle$

$$= \langle f'(t)u_1'(f(t)), f'(t)u_2'(f(t)), f'(t)u_3'(f(t))\rangle = f'(t)\,\mathbf{u}'(t)$$

**49.** $\dfrac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$ [by Formula 4 of Theorem 3]

$$\begin{aligned}
&= \langle \cos t, -\sin t, 1\rangle\cdot\langle t, \cos t, \sin t\rangle + \langle \sin t, \cos t, t\rangle\cdot\langle 1, -\sin t, \cos t\rangle\\
&= t\cos t - \cos t\,\sin t + \sin t + \sin t - \cos t\,\sin t + t\cos t\\
&= 2t\cos t + 2\sin t - 2\cos t\,\sin t
\end{aligned}$$

**50.** $\dfrac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$ [by Formula 5 of Theorem 3]

$$\begin{aligned}
&= \langle \cos t, -\sin t, 1\rangle\times\langle t, \cos t, \sin t\rangle + \langle \sin t, \cos t, t\rangle\times\langle 1, -\sin t, \cos t\rangle\\
&= \left\langle -\sin^2 t - \cos t, t - \cos t\,\sin t, \cos^2 t + t\sin t\right\rangle\\
&\qquad + \left\langle \cos^2 t + t\sin t, t - \cos t\,\sin t, -\sin^2 t - \cos t\right\rangle\\
&= \left\langle \cos^2 t - \sin^2 t - \cos t + t\sin t, 2t - 2\cos t\,\sin t, \cos^2 t - \sin^2 t - \cos t + t\sin t\right\rangle
\end{aligned}$$

**51.** By Formula 4 of Theorem 3, $f'(t) = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$, and $\mathbf{v}'(t) = \left\langle 1, 2t, 3t^2\right\rangle$, so

$f'(2) = \mathbf{u}'(2)\cdot\mathbf{v}(2) + \mathbf{u}(2)\cdot\mathbf{v}'(2) = \langle 3, 0, 4\rangle\cdot\langle 2, 4, 8\rangle + \langle 1, 2, -1\rangle\cdot\langle 1, 4, 12\rangle = 6 + 0 + 32 + 1 + 8 - 12 = 35.$

**52.** By Formula 5 of Theorem 3, $\mathbf{r}'(t) = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$, so

$$\begin{aligned}
\mathbf{r}'(2) &= \mathbf{u}'(2)\times\mathbf{v}(2) + \mathbf{u}(2)\times\mathbf{v}'(2) = \langle 3, 0, 4\rangle\times\langle 2, 4, 8\rangle + \langle 1, 2, -1\rangle\times\langle 1, 4, 12\rangle\\
&= \langle -16, -16, 12\rangle + \langle 28, -13, 2\rangle = \langle 12, -29, 14\rangle
\end{aligned}$$

**53.** $\mathbf{r}(t) = \mathbf{a}\cos\omega t + \mathbf{b}\sin\omega t \quad\Rightarrow\quad \mathbf{r}'(t) = -\mathbf{a}\,\omega\sin\omega t + \mathbf{b}\,\omega\cos\omega t$ by Formulas 1 and 3 of Theorem 3. Then

$$\begin{aligned}
\mathbf{r}(t)\times\mathbf{r}'(t) &= (\mathbf{a}\cos\omega t + \mathbf{b}\sin\omega t)\times(-\mathbf{a}\,\omega\sin\omega t + \mathbf{b}\,\omega\cos\omega t)\\
&= (\mathbf{a}\cos\omega t + \mathbf{b}\sin\omega t)\times(-\mathbf{a}\,\omega\sin\omega t) + (\mathbf{a}\cos\omega t + \mathbf{b}\sin\omega t)\times(\mathbf{b}\,\omega\cos\omega t)\\
&\qquad \text{[by Property 3 of Theorem 12.4.11]}\\
&= \mathbf{a}\cos\omega t\times(-\mathbf{a}\,\omega\sin\omega t) + \mathbf{b}\sin\omega t\times(-\mathbf{a}\,\omega\sin\omega t) + \mathbf{a}\cos\omega t\times\mathbf{b}\,\omega\cos\omega t + \mathbf{b}\sin\omega t\times\mathbf{b}\,\omega\cos\omega t\\
&\qquad \text{[by Property 4]}\\
&= (\cos\omega t)(-\omega\sin\omega t)(\mathbf{a}\times\mathbf{a}) + (\sin\omega t)(-\omega\sin\omega t)(\mathbf{b}\times\mathbf{a}) + (\cos\omega t)(\omega\cos\omega t)(\mathbf{a}\times\mathbf{b})\\
&\qquad + (\sin\omega t)(\omega\cos\omega t)(\mathbf{b}\times\mathbf{b}) \qquad \text{[by Property 2]}\\
&= \mathbf{0} + \left(\omega\sin^2\omega t\right)(\mathbf{a}\times\mathbf{b}) + \left(\omega\cos^2\omega t\right)(\mathbf{a}\times\mathbf{b}) + \mathbf{0} \qquad \text{[by Property 1 and Example 12.4.2]}\\
&= \omega\left(\sin^2\omega t + \cos^2\omega t\right)(\mathbf{a}\times\mathbf{b}) = \omega\,(\mathbf{a}\times\mathbf{b}) = \omega\,\mathbf{a}\times\mathbf{b} \qquad \text{[by Property 2]}
\end{aligned}$$

**54.** From Exercise 53, $\mathbf{r}'(t) = -\mathbf{a}\,\omega\sin\omega t + \mathbf{b}\,\omega\cos\omega t \quad \Rightarrow \quad \mathbf{r}''(t) = -\mathbf{a}\,\omega^2\cos\omega t - \mathbf{b}\,\omega^2\sin\omega t$. Then

$$\begin{aligned}\mathbf{r}''(t) + \omega^2\mathbf{r}(t) &= \left(-\mathbf{a}\,\omega^2\cos\omega t - \mathbf{b}\,\omega^2\sin\omega t\right) + \omega^2\left(\mathbf{a}\cos\omega t + \mathbf{b}\sin\omega t\right)\\ &= -\mathbf{a}\,\omega^2\cos\omega t - \mathbf{b}\,\omega^2\sin\omega t + \mathbf{a}\,\omega^2\cos\omega t + \mathbf{b}\,\omega^2\sin\omega t = \mathbf{0}\end{aligned}$$

**55.** $\dfrac{d}{dt}\left[\mathbf{r}(t)\times\mathbf{r}'(t)\right] = \mathbf{r}'(t)\times\mathbf{r}'(t) + \mathbf{r}(t)\times\mathbf{r}''(t)$ by Formula 5 of Theorem 3. But $\mathbf{r}'(t)\times\mathbf{r}'(t) = \mathbf{0}$ (by Example 12.4.2).

Thus, $\dfrac{d}{dt}\left[\mathbf{r}(t)\times\mathbf{r}'(t)\right] = \mathbf{r}(t)\times\mathbf{r}''(t)$.

**56.**
$$\begin{aligned}\frac{d}{dt}\left(\mathbf{u}(t)\cdot[\mathbf{v}(t)\times\mathbf{w}(t)]\right) &= \mathbf{u}'(t)\cdot[\mathbf{v}(t)\times\mathbf{w}(t)] + \mathbf{u}(t)\cdot\frac{d}{dt}\left[\mathbf{v}(t)\times\mathbf{w}(t)\right]\\ &= \mathbf{u}'(t)\cdot[\mathbf{v}(t)\times\mathbf{w}(t)] + \mathbf{u}(t)\cdot[\mathbf{v}'(t)\times\mathbf{w}(t) + \mathbf{v}(t)\times\mathbf{w}'(t)]\\ &= \mathbf{u}'(t)\cdot[\mathbf{v}(t)\times\mathbf{w}(t)] + \mathbf{u}(t)\cdot[\mathbf{v}'(t)\times\mathbf{w}(t)] + \mathbf{u}(t)\cdot[\mathbf{v}(t)\times\mathbf{w}'(t)]\\ &= \mathbf{u}'(t)\cdot[\mathbf{v}(t)\times\mathbf{w}(t)] - \mathbf{v}'(t)\cdot[\mathbf{u}(t)\times\mathbf{w}(t)] + \mathbf{w}'(t)\cdot[\mathbf{u}(t)\times\mathbf{v}(t)]\end{aligned}$$

**57.** $\dfrac{d}{dt}\,|\mathbf{r}(t)| = \dfrac{d}{dt}\,[\mathbf{r}(t)\cdot\mathbf{r}(t)]^{1/2} = \frac{1}{2}[\mathbf{r}(t)\cdot\mathbf{r}(t)]^{-1/2}\,[2\mathbf{r}(t)\cdot\mathbf{r}'(t)] = \dfrac{1}{|\mathbf{r}(t)|}\,\mathbf{r}(t)\cdot\mathbf{r}'(t)$

**58.** Since $\mathbf{r}(t)\cdot\mathbf{r}'(t) = 0$, by Formula 4 of Theorem 3 we have

$\dfrac{d}{dt}\,|\mathbf{r}(t)|^2 = \dfrac{d}{dt}\,[\mathbf{r}(t)\cdot\mathbf{r}(t)] = \mathbf{r}'(t)\cdot\mathbf{r}(t) + \mathbf{r}(t)\cdot\mathbf{r}'(t) = 2\,[\mathbf{r}(t)\cdot\mathbf{r}'(t)] = 0$. This is true for all $t$, thus $|\mathbf{r}(t)|^2$, and so $|\mathbf{r}(t)|$ is a constant, and hence the curve lies on a sphere with center the origin.

**59.** Since $\mathbf{u}(t) = \mathbf{r}(t)\cdot[\mathbf{r}'(t)\times\mathbf{r}''(t)]$,

$$\begin{aligned}\mathbf{u}'(t) &= \mathbf{r}'(t)\cdot[\mathbf{r}'(t)\times\mathbf{r}''(t)] + \mathbf{r}(t)\cdot\frac{d}{dt}\left[\mathbf{r}'(t)\times\mathbf{r}''(t)\right]\\ &= 0 + \mathbf{r}(t)\cdot[\mathbf{r}''(t)\times\mathbf{r}''(t) + \mathbf{r}'(t)\times\mathbf{r}'''(t)] && [\text{since } \mathbf{r}'(t)\perp\mathbf{r}'(t)\times\mathbf{r}''(t)]\\ &= \mathbf{r}(t)\cdot[\mathbf{r}'(t)\times\mathbf{r}'''(t)] && [\text{since } \mathbf{r}''(t)\times\mathbf{r}''(t) = \mathbf{0}]\end{aligned}$$

**60.** The tangent vector $\mathbf{r}'(t)$ is defined as $\lim\limits_{h\to 0}\dfrac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$. Here we assume that this limit exists and $\mathbf{r}'(t)\neq\mathbf{0}$; then we know that this vector lies on the tangent line to the curve. As in Figure 1, let points $P$ and $Q$ have position vectors $\mathbf{r}(t)$ and $\mathbf{r}(t+h)$. The vector $\mathbf{r}(t+h)-\mathbf{r}(t)$ points from $P$ to $Q$, so $\mathbf{r}(t+h)-\mathbf{r}(t) = \overrightarrow{PQ}$. If $h>0$ then $t<t+h$, so $Q$ lies "ahead" of $P$ on the curve. If $h$ is sufficiently small (we can take $h$ to be as small as we like since $h\to 0$), then $\overrightarrow{PQ}$ approximates the curve from $P$ to $Q$ and hence points approximately in the direction of the curve as $t$ increases. Since $h$ is positive, $\dfrac{1}{h}\,\overrightarrow{PQ} = \dfrac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$ points in the same direction. If $h<0$, then $t>t+h$ so $Q$ lies "behind" $P$ on the curve. For $h$ sufficiently small, $\overrightarrow{PQ}$ approximates the curve but points in the direction of decreasing $t$. However, $h$ is negative, so $\dfrac{1}{h}\,\overrightarrow{PQ} = \dfrac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$ points in the opposite direction, that is, in the direction of increasing $t$. In both cases, the difference quotient $\dfrac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$ points in the direction of increasing $t$. The tangent vector $\mathbf{r}'(t)$ is the limit of this difference quotient, so it must also point in the direction of increasing $t$.

## 13.3 Arc Length and Curvature

**1.** (a) $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle = \langle 3-t, 2t, 4t+1\rangle \;\Rightarrow\; \mathbf{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle = \langle -1, 2, 4\rangle$. Then

$$L = \int_1^3 \sqrt{(-1)^2 + 2^2 + 4^2}\, dt = \int_1^3 \sqrt{21}\, dt = \Big[t\sqrt{21}\Big]_1^3 = 3\sqrt{21} - \sqrt{21} = 2\sqrt{21}$$

(b) $\mathbf{r}(1) = \langle 2, 2, 5\rangle \;\Rightarrow\; P_1 = (2, 2, 5)$; $\mathbf{r}(3) = \langle 0, 6, 13\rangle \;\Rightarrow\; P_2 = (0, 6, 13)$. Then

$$|P_2P_1| = \sqrt{(0-2)^2 + (6-2)^2 + (13-5)^2} = \sqrt{84} = 2\sqrt{21}$$

**2.** (a) $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k} = (t+2)\,\mathbf{i} - t\,\mathbf{j} + (3t-5)\,\mathbf{k} \;\Rightarrow\; \mathbf{r}'(t) = \mathbf{i} - \mathbf{j} + 3\,\mathbf{k}$. Then

$$L = \int_{-1}^2 \sqrt{1^2 + (-1)^2 + 3^2}\, dt = \int_{-1}^2 \sqrt{11}\, dt = \Big[t\sqrt{11}\Big]_{-1}^2 = 2\sqrt{11} + \sqrt{11} = 3\sqrt{11}$$

(b) $\mathbf{r}(-1) = \mathbf{i} + \mathbf{j} - 8\,\mathbf{k} \;\Rightarrow\; P_1 = (1, 1, -8)$; $\mathbf{r}(2) = 4\,\mathbf{i} - 2\,\mathbf{j} + \mathbf{k} \;\Rightarrow\; P_2 = (4, -2, 1)$. Then

$$|P_2P_1| = \sqrt{(4-1)^2 + (-2-1)^2 + [1-(-8)]^2} = \sqrt{99} = 3\sqrt{11}$$

**3.** $\mathbf{r}(t) = \langle t, 3\cos t, 3\sin t\rangle \;\Rightarrow\; \mathbf{r}'(t) = \langle 1, -3\sin t, 3\cos t\rangle \;\Rightarrow$

$|\mathbf{r}'(t)| = \sqrt{1^2 + (-3\sin t)^2 + (3\cos t)^2} = \sqrt{1 + 9(\sin^2 t + \cos^2 t)} = \sqrt{10}$.

Then using Formula 3, we have $L = \int_{-5}^5 |\mathbf{r}'(t)|\, dt = \int_{-5}^5 \sqrt{10}\, dt = \sqrt{10}\, t\Big]_{-5}^5 = 10\sqrt{10}$.

**4.** $\mathbf{r}(t) = \left\langle 2t, t^2, \frac{1}{3}t^3\right\rangle \;\Rightarrow\; \mathbf{r}'(t) = \left\langle 2, 2t, t^2\right\rangle \;\Rightarrow$

$|\mathbf{r}'(t)| = \sqrt{2^2 + (2t)^2 + (t^2)^2} = \sqrt{4 + 4t^2 + t^4} = \sqrt{(2+t^2)^2} = 2 + t^2$ for $0 \le t \le 1$. Then using Formula 3, we have $L = \int_0^1 |\mathbf{r}'(t)|\, dt = \int_0^1 (2+t^2)\, dt = 2t + \frac{1}{3}t^3\Big]_0^1 = \frac{7}{3}$.

**5.** $\mathbf{r}(t) = \sqrt{2}\, t\,\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k} \;\Rightarrow\; \mathbf{r}'(t) = \sqrt{2}\,\mathbf{i} + e^t\mathbf{j} - e^{-t}\mathbf{k} \;\Rightarrow$

$|\mathbf{r}'(t)| = \sqrt{\left(\sqrt{2}\right)^2 + (e^t)^2 + (-e^{-t})^2} = \sqrt{2 + e^{2t} + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}$ [since $e^t + e^{-t} > 0$].

Then $L = \int_0^1 |\mathbf{r}'(t)|\, dt = \int_0^1 (e^t + e^{-t})\, dt = \left[e^t - e^{-t}\right]_0^1 = e - e^{-1}$.

**6.** $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \ln\cos t\,\mathbf{k} \;\Rightarrow\; \mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \dfrac{-\sin t}{\cos t}\,\mathbf{k} = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} - \tan t\,\mathbf{k}$,

$|\mathbf{r}'(t)| = \sqrt{(-\sin t)^2 + \cos^2 t + (-\tan t)^2} = \sqrt{1 + \tan^2 t} = \sqrt{\sec^2 t} = |\sec t|$. Since $\sec t > 0$ for $0 \le t \le \pi/4$, here we can say $|\mathbf{r}'(t)| = \sec t$. Then

$$\begin{aligned} L &= \int_0^{\pi/4} \sec t\, dt = \Big[\ln|\sec t + \tan t|\Big]_0^{\pi/4} = \ln\left|\sec\tfrac{\pi}{4} + \tan\tfrac{\pi}{4}\right| - \ln|\sec 0 + \tan 0| \\ &= \ln\left|\sqrt{2} + 1\right| - \ln|1 + 0| = \ln(\sqrt{2} + 1) \end{aligned}$$

**7.** $\mathbf{r}(t) = \mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k} \;\Rightarrow\; \mathbf{r}'(t) = 2t\,\mathbf{j} + 3t^2\,\mathbf{k} \;\Rightarrow\; |\mathbf{r}'(t)| = \sqrt{4t^2 + 9t^4} = t\sqrt{4 + 9t^2}$ [since $t \ge 0$].

Then $L = \int_0^1 |\mathbf{r}'(t)|\, dt = \int_0^1 t\sqrt{4 + 9t^2}\, dt = \frac{1}{18} \cdot \frac{2}{3}(4 + 9t^2)^{3/2}\Big]_0^1 = \frac{1}{27}(13^{3/2} - 4^{3/2}) = \frac{1}{27}(13^{3/2} - 8)$.

**8.** $\mathbf{r}(t) = t^2\,\mathbf{i} + 9t\,\mathbf{j} + 4t^{3/2}\,\mathbf{k} \quad\Rightarrow\quad \mathbf{r}'(t) = 2t\,\mathbf{i} + 9\,\mathbf{j} + 6\sqrt{t}\,\mathbf{k} \quad\Rightarrow$

$|\mathbf{r}'(t)| = \sqrt{4t^2 + 81 + 36t} = \sqrt{(2t+9)^2} = |2t+9| = 2t+9$ [since $2t + 9 \geq 0$ for $1 \leq t \leq 4$]. Then

$L = \int_1^4 |\mathbf{r}'(t)|\,dt = \int_1^4 (2t+9)\,dt = \Big[t^2 + 9t\Big]_1^4 = 52 - 10 = 42.$

**9.** $\mathbf{r}(t) = \langle t^2, t^3, t^4\rangle \quad\Rightarrow\quad \mathbf{r}'(t) = \langle 2t, 3t^2, 4t^3\rangle \quad\Rightarrow\quad |\mathbf{r}'(t)| = \sqrt{(2t)^2 + (3t^2)^2 + (4t^3)^2} = \sqrt{4t^2 + 9t^4 + 16t^6}$, so

$L = \int_0^2 |\mathbf{r}'(t)|\,dt = \int_0^2 \sqrt{4t^2 + 9t^4 + 16t^6}\,dt \approx 18.6833.$

**10.** $\mathbf{r}(t) = \langle t, e^{-t}, te^{-t}\rangle \quad\Rightarrow\quad \mathbf{r}'(t) = \langle 1, -e^{-t}, (1-t)e^{-t}\rangle \quad\Rightarrow$

$|\mathbf{r}'(t)| = \sqrt{1^2 + (-e^{-t})^2 + [(1-t)e^{-t}]^2} = \sqrt{1 + e^{-2t} + (1-t)^2 e^{-2t}} = \sqrt{1 + (2 - 2t + t^2)e^{-2t}}$, so

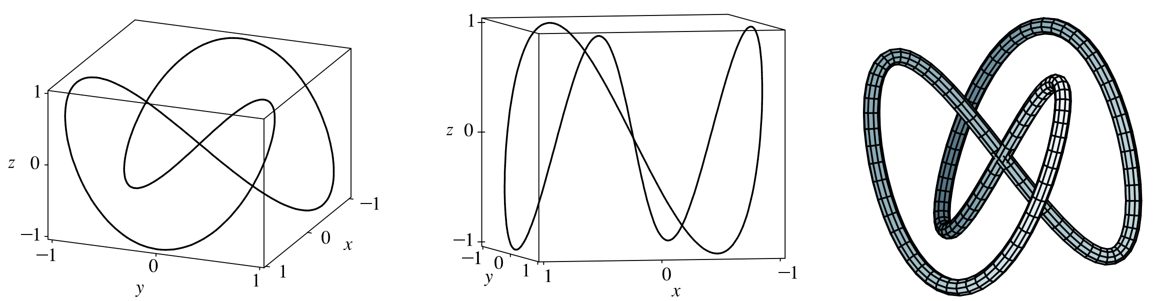
$L = \int_1^3 |\mathbf{r}'(t)|\,dt = \int_1^3 \sqrt{1 + (2 + 2t + t^2)e^{-2t}}\,dt \approx 2.0454.$

**11.** $\mathbf{r}(t) = \langle \cos \pi t, 2t, \sin 2\pi t\rangle \quad\Rightarrow\quad \mathbf{r}'(t) = \langle -\pi \sin \pi t, 2, 2\pi \cos 2\pi t\rangle \quad\Rightarrow\quad |\mathbf{r}'(t)| = \sqrt{\pi^2 \sin^2 \pi t + 4 + 4\pi^2 \cos^2 2\pi t}.$

The point $(1, 0, 0)$ corresponds to $t = 0$ and $(1, 4, 0)$ corresponds to $t = 2$, so the length is

$L = \int_0^2 |\mathbf{r}'(t)|\,dt = \int_0^2 \sqrt{\pi^2 \sin^2 \pi t + 4 + 4\pi^2 \cos^2 2\pi t}\,dt \approx 10.3311.$

**12.** We plot two different views of the curve with parametric equations $x = \sin t$, $y = \sin 2t$, $z = \sin 3t$. To help visualize the curve, we also include a plot showing a tube of radius 0.07 around the curve.




The complete curve is given by the parameter interval $[0, 2\pi]$ and we have $\mathbf{r}'(t) = \langle \cos t, 2\cos 2t, 3\cos 3t\rangle \quad\Rightarrow$

$|\mathbf{r}'(t)| = \sqrt{\cos^2 t + 4\cos^2 2t + 9\cos^2 3t}$, so $L = \int_0^{2\pi} |\mathbf{r}'(t)|\,dt = \int_0^{2\pi} \sqrt{\cos^2 t + 4\cos^2 2t + 9\cos^2 3t}\,dt \approx 16.0264.$

**13.** The projection of the curve $C$ onto the $xy$-plane is the curve $x^2 = 2y$ or $y = \frac{1}{2}x^2$, $z = 0$. Then we can choose the parameter $x = t \quad\Rightarrow\quad y = \frac{1}{2}t^2$. Since $C$ also lies on the surface $3z = xy$, we have $z = \frac{1}{3}xy = \frac{1}{3}(t)(\frac{1}{2}t^2) = \frac{1}{6}t^3$. Then parametric equations for $C$ are $x = t$, $y = \frac{1}{2}t^2$, $z = \frac{1}{6}t^3$ and the corresponding vector equation is $\mathbf{r}(t) = \langle t, \frac{1}{2}t^2, \frac{1}{6}t^3\rangle$. The origin corresponds to $t = 0$ and the point $(6, 18, 36)$ corresponds to $t = 6$, so

$$\begin{aligned} L &= \int_0^6 |\mathbf{r}'(t)|\,dt = \int_0^6 \left|\langle 1, t, \tfrac{1}{2}t^2\rangle\right|\,dt = \int_0^6 \sqrt{1^2 + t^2 + \left(\tfrac{1}{2}t^2\right)^2}\,dt = \int_0^6 \sqrt{1 + t^2 + \tfrac{1}{4}t^4}\,dt \\ &= \int_0^6 \sqrt{(1 + \tfrac{1}{2}t^2)^2}\,dt = \int_0^6 (1 + \tfrac{1}{2}t^2)\,dt = \left[t + \tfrac{1}{6}t^3\right]_0^6 = 6 + 36 = 42 \end{aligned}$$

**14.** Let $C$ be the curve of intersection. The projection of $C$ onto the $xy$-plane is the ellipse $4x^2 + y^2 = 4$ or $x^2 + y^2/4 = 1$, $z = 0$. Then we can write $x = \cos t$, $y = 2\sin t$, $0 \leq t \leq 2\pi$. Since $C$ also lies on the plane $x + y + z = 2$, we have

$z = 2 - x - y = 2 - \cos t - 2\sin t$. Then parametric equations for $C$ are $x = \cos t$, $y = 2\sin t$, $z = 2 - \cos t - 2\sin t$, $0 \le t \le 2\pi$, and the corresponding vector equation is $\mathbf{r}(t) = \langle \cos t, 2\sin t, 2 - \cos t - 2\sin t\rangle$. Differentiating gives $\mathbf{r}'(t) = \langle -\sin t, 2\cos t, \sin t - 2\cos t\rangle \quad \Rightarrow$

$|\mathbf{r}'(t)| = \sqrt{(-\sin t)^2 + (2\cos t)^2 + (\sin t - 2\cos t)^2} = \sqrt{2\sin^2 t + 8\cos^2 t - 4\sin t\,\cos t}$. The length of $C$ is

$L = \int_0^{2\pi} |\mathbf{r}'(t)|\,dt = \int_0^{2\pi} \sqrt{2\sin^2 t + 8\cos^2 t - 4\sin t\,\cos t}\,dt \approx 13.5191$.

**15.** (a) $\mathbf{r}(t) = (5 - t)\,\mathbf{i} + (4t - 3)\,\mathbf{j} + 3t\,\mathbf{k} \quad \Rightarrow \quad \mathbf{r}'(t) = -\mathbf{i} + 4\,\mathbf{j} + 3\,\mathbf{k}$ and $\frac{ds}{dt} = |\mathbf{r}'(t)| = \sqrt{1 + 16 + 9} = \sqrt{26}$. The point $P\,(4, 1, 3)$ corresponds to $t = 1$, so the arc length function from $P$ is

$s(t) = \int_1^t |\mathbf{r}'(u)|\,du = \int_1^t \sqrt{26}\,du = \sqrt{26}\,u\big|_1^t = \sqrt{26}\,(t - 1)$. Since $s = \sqrt{26}\,(t - 1)$, we have $t = \dfrac{s}{\sqrt{26}} + 1$.

Substituting for $t$ in the original equation, the reparametrization of the curve with respect to arc length is

$$\begin{aligned}\mathbf{r}(t(s)) &= \left[5 - \left(\frac{s}{\sqrt{26}} + 1\right)\right]\mathbf{i} + \left[4\left(\frac{s}{\sqrt{26}} + 1\right) - 3\right]\mathbf{j} + 3\left(\frac{s}{\sqrt{26}} + 1\right)\mathbf{k} \\ &= \left(4 - \frac{s}{\sqrt{26}}\right)\mathbf{i} + \left(\frac{4s}{\sqrt{26}} + 1\right)\mathbf{j} + \left(\frac{3s}{\sqrt{26}} + 3\right)\mathbf{k}\end{aligned}$$

(b) The point 4 units along the curve from $P$ has position vector

$\mathbf{r}(t(4)) = \left(4 - \dfrac{4}{\sqrt{26}}\right)\mathbf{i} + \left(\dfrac{4(4)}{\sqrt{26}} + 1\right)\mathbf{j} + \left(\dfrac{3(4)}{\sqrt{26}} + 3\right)\mathbf{k}$, so the point is $\left(4 - \dfrac{4}{\sqrt{26}}, \dfrac{16}{\sqrt{26}} + 1, \dfrac{12}{\sqrt{26}} + 3\right)$.

**16.** (a) $\mathbf{r}(t) = e^t \sin t\,\mathbf{i} + e^t \cos t\,\mathbf{j} + \sqrt{2}\,e^t\,\mathbf{k} \quad \Rightarrow \quad \mathbf{r}'(t) = e^t\,(\cos t + \sin t)\,\mathbf{i} + e^t\,(\cos t - \sin t)\,\mathbf{j} + \sqrt{2}\,e^t\,\mathbf{k}$ and

$$\begin{aligned}\frac{ds}{dt} &= |\mathbf{r}'(t)| = \sqrt{e^{2t}(\cos t + \sin t)^2 + e^{2t}(\cos t - \sin t)^2 + 2e^{2t}} \\ &= \sqrt{e^{2t}\left[2(\cos^2 t + \sin^2 t) + 2\cos t\sin t - 2\cos t\sin t + 2\right]} = \sqrt{4e^{2t}} = 2e^t\end{aligned}$$

The point $P\left(0, 1, \sqrt{2}\,\right)$ corresponds to $t = 0$, so the arc length function from $P$ is

$s(t) = \int_0^t |\mathbf{r}'(u)|\,du = \int_0^t 2e^u\,du = 2e^u\big|_0^t = 2(e^t - 1)$. Since $s = 2(e^t - 1)$, we have $e^t = \dfrac{s}{2} + 1 \quad \Leftrightarrow$

$t = \ln\left(\frac{1}{2}s + 1\right)$. Substituting for $t$ in the original equation, the reparametrization of the curve with respect to arc length is

$\mathbf{r}(t(s)) = \left(\frac{1}{2}s + 1\right)\sin\!\left(\ln\left(\frac{1}{2}s + 1\right)\right)\mathbf{i} + \left(\frac{1}{2}s + 1\right)\cos\!\left(\ln\left(\frac{1}{2}s + 1\right)\right)\mathbf{j} + \left(\frac{\sqrt{2}}{2}s + \sqrt{2}\right)\mathbf{k}$.

(b) The point 4 units along the curve from $P$ has position vector

$\mathbf{r}(t(4)) = \left(\frac{1}{2}(4) + 1\right)\sin\!\left(\ln\left(\frac{1}{2}(4) + 1\right)\right)\mathbf{i} + \left(\frac{1}{2}(4) + 1\right)\cos\!\left(\ln\left(\frac{1}{2}(4) + 1\right)\right)\mathbf{j} + \left(\frac{\sqrt{2}}{2}(4) + \sqrt{2}\right)\mathbf{k}$, so the point is $\left(3\sin(\ln 3), 3\cos(\ln 3), 3\sqrt{2}\,\right)$.

**17.** Here $\mathbf{r}(t) = \langle 3\sin t, 4t, 3\cos t\rangle$, so $\mathbf{r}'(t) = \langle 3\cos t, 4, -3\sin t\rangle$ and $|\mathbf{r}'(t)| = \sqrt{9\cos^2 t + 16 + 9\sin^2 t} = \sqrt{25} = 5$. The point $(0, 0, 3)$ corresponds to $t = 0$, so the arc length function beginning at $(0, 0, 3)$ and measuring in the positive direction is given by $s(t) = \int_0^t |\mathbf{r}'(u)|\,du = \int_0^t 5\,du = 5t$. $\;s(t) = 5 \;\Rightarrow\; 5t = 5 \;\Rightarrow\; t = 1$, thus your location after moving 5 units along the curve is $(3\sin 1, 4, 3\cos 1)$.

**18.** $\mathbf{r}(t) = \left(\dfrac{2}{t^2+1} - 1\right)\mathbf{i} + \dfrac{2t}{t^2+1}\,\mathbf{j} \quad \Rightarrow \quad \mathbf{r}'(t) = \dfrac{-4t}{(t^2+1)^2}\,\mathbf{i} + \dfrac{-2t^2+2}{(t^2+1)^2}\,\mathbf{j}$,

$$\frac{ds}{dt} = |\mathbf{r}'(t)| = \sqrt{\left[\frac{-4t}{(t^2+1)^2}\right]^2 + \left[\frac{-2t^2+2}{(t^2+1)^2}\right]^2} = \sqrt{\frac{4t^4+8t^2+4}{(t^2+1)^4}} = \sqrt{\frac{4(t^2+1)^2}{(t^2+1)^4}} = \sqrt{\frac{4}{(t^2+1)^2}} = \frac{2}{t^2+1}.$$

Since the initial point $(1,0)$ corresponds to $t=0$, the arc length function is

$s(t) = \displaystyle\int_0^t |\mathbf{r}'(u)|\,du = \int_0^t \frac{2}{u^2+1}\,du = 2\arctan t$. Then $\arctan t = \frac{1}{2}s \quad \Rightarrow \quad t = \tan\frac{1}{2}s$. Substituting, we have

$$\mathbf{r}(t(s)) = \left[\frac{2}{\tan^2\left(\frac{1}{2}s\right)+1} - 1\right]\mathbf{i} + \frac{2\tan\left(\frac{1}{2}s\right)}{\tan^2\left(\frac{1}{2}s\right)+1}\,\mathbf{j} = \frac{1-\tan^2\left(\frac{1}{2}s\right)}{1+\tan^2\left(\frac{1}{2}s\right)}\,\mathbf{i} + \frac{2\tan\left(\frac{1}{2}s\right)}{\sec^2\left(\frac{1}{2}s\right)}\,\mathbf{j}$$

$$= \frac{1-\tan^2\left(\frac{1}{2}s\right)}{\sec^2\left(\frac{1}{2}s\right)}\,\mathbf{i} + 2\tan\left(\tfrac{1}{2}s\right)\cos^2\left(\tfrac{1}{2}s\right)\mathbf{j} = \left[\cos^2\left(\tfrac{1}{2}s\right) - \sin^2\left(\tfrac{1}{2}s\right)\right]\mathbf{i} + 2\sin\left(\tfrac{1}{2}s\right)\cos\left(\tfrac{1}{2}s\right)\mathbf{j} = \cos s\,\mathbf{i} + \sin s\,\mathbf{j}$$

With this parametrization, we recognize the function as representing the unit circle. Note here that the curve approaches, but does not include, the point $(-1,0)$, since $\cos s = -1$ for $s = \pi + 2k\pi$ ($k$ an integer) but then $t = \tan\left(\frac{1}{2}s\right)$ is undefined.

**19.** (a) $\mathbf{r}(t) = \left\langle t^2, \sin t - t\cos t, \cos t + t\sin t\right\rangle \quad \Rightarrow$

$\mathbf{r}'(t) = \langle 2t, \cos t + t\sin t - \cos t,\ -\sin t + t\cos t + \sin t\rangle = \langle 2t, t\sin t, t\cos t\rangle \quad \Rightarrow$

$|\mathbf{r}'(t)| = \sqrt{4t^2 + t^2\sin^2 t + t^2\cos^2 t} = \sqrt{4t^2 + t^2(\cos^2 t + \sin^2 t)} = \sqrt{5t^2} = \sqrt{5}\,t$ [since $t > 0$]. Then

$\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{5}\,t}\langle 2t, t\sin t, t\cos t\rangle = \dfrac{1}{\sqrt{5}}\langle 2, \sin t, \cos t\rangle$. $\quad \mathbf{T}'(t) = \dfrac{1}{\sqrt{5}}\langle 0, \cos t, -\sin t\rangle \quad \Rightarrow$

$|\mathbf{T}'(t)| = \frac{1}{\sqrt{5}}\sqrt{0 + \cos^2 t + \sin^2 t} = \dfrac{1}{\sqrt{5}}$. Thus, $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \dfrac{1/\sqrt{5}}{1/\sqrt{5}}\langle 0, \cos t, -\sin t\rangle = \langle 0, \cos t, -\sin t\rangle$.

(b) By Formula 9, the curvature is $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{1/\sqrt{5}}{\sqrt{5}\,t} = \dfrac{1}{5t}$.

**20.** (a) $\mathbf{r}(t) = \langle 5\sin t, t, 5\cos t\rangle \quad \Rightarrow \quad \mathbf{r}'(t) = \langle 5\cos t, 1, -5\sin t\rangle \quad \Rightarrow \quad |\mathbf{r}'(t)| = \sqrt{25\cos^2 t + 1 + 25\sin^2 t} = \sqrt{26}$.

Then $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{26}}\langle 5\cos t, 1, -5\sin t\rangle$. $\quad \mathbf{T}'(t) = \dfrac{1}{\sqrt{26}}\langle -5\sin t, 0, -5\cos t\rangle \quad \Rightarrow$

$|\mathbf{T}'(t)| = \dfrac{1}{\sqrt{26}}\sqrt{25\sin^2 t + 0^2 + 25\cos^2 t} = \dfrac{5}{\sqrt{26}}$. Thus,

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \dfrac{\sqrt{26}}{5}\cdot\dfrac{1}{\sqrt{26}}\langle -5\sin t, 0, -5\cos t\rangle = \langle -\sin t, 0, -\cos t\rangle$.

(b) By Formula 9, the curvature is $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{5/\sqrt{26}}{\sqrt{26}} = \dfrac{5}{26}$.

**21.** (a) $\mathbf{r}(t) = \left\langle t, t^2, 4\right\rangle \quad \Rightarrow \quad \mathbf{r}'(t) = \langle 1, 2t, 0\rangle \quad \Rightarrow \quad |\mathbf{r}'(t)| = \sqrt{1 + 4t^2 + 0}$. Then $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{1+4t^2}}\langle 1, 2t, 0\rangle$.

$$\mathbf{T}'(t) = \frac{1}{\sqrt{1+4t^2}}\langle 0, 2, 0\rangle - \frac{4t}{(1+4t^2)^{3/2}}\langle 1, 2t, 0\rangle \qquad \text{[by Formula 3 of Theorem 13.2.3]}$$

$$= \frac{1}{(1+4t^2)^{3/2}}\left[(1+4t^2)\langle 0, 2, 0\rangle - 4t\langle 1, 2t, 0\rangle\right] = \frac{1}{(1+4t^2)^{3/2}}\langle -4t, 2, 0\rangle$$

[continued]

$$|\mathbf{T}'(t)| = \frac{1}{(1+4t^2)^{3/2}}\sqrt{16t^2+4} = \frac{2}{1+4t^2}. \text{ Thus,}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{1+4t^2}{2}\cdot\frac{1}{(1+4t^2)^{3/2}}\langle -4t, 2, 0\rangle = \frac{1}{\sqrt{1+4t^2}}\langle -2t, 1, 0\rangle.$$

(b) By Formula 9, the curvature is $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{2/(1+4t^2)}{\sqrt{1+4t^2}} = \dfrac{2}{(1+4t^2)^{3/2}}$.

**22.** (a) $\mathbf{r}(t) = \left\langle t, t, \frac{1}{2}t^2\right\rangle \;\Rightarrow\; \mathbf{r}'(t) = \langle 1, 1, t\rangle \;\Rightarrow\; |\mathbf{r}'(t)| = \sqrt{1+1+t^2} = \sqrt{2+t^2}$.

Then $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{2+t^2}}\langle 1, 1, t\rangle$.

$$\begin{aligned}\mathbf{T}'(t) &= \frac{1}{\sqrt{2+t^2}}\langle 0,0,1\rangle - \frac{t}{(2+t^2)^{3/2}}\langle 1,1,t\rangle \qquad \text{[by Formula 3 of Theorem 13.2.3]}\\ &= \frac{1}{(2+t^2)^{3/2}}\left[(2+t^2)\langle 0,0,1\rangle - t\langle 1,1,t\rangle\right] = \frac{1}{(2+t^2)^{3/2}}\langle -t,-t,2\rangle\end{aligned}$$

$$|\mathbf{T}'(t)| = \frac{1}{(2+t^2)^{3/2}}\sqrt{4+2t^2} = \frac{\sqrt{2}}{2+t^2}. \text{ Thus,}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{2+t^2}{\sqrt{2}}\cdot\frac{1}{(2+t^2)^{3/2}}\langle -t,-t,2\rangle = \frac{1}{\sqrt{4+2t^2}}\langle -t,-t,2\rangle.$$

(b) By Formula 9, the curvature is $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{\sqrt{2}/(2+t^2)}{\sqrt{2+t^2}} = \dfrac{\sqrt{2}}{(2+t^2)^{3/2}}$.

**23.** (a) $\mathbf{r}(t) = \left\langle t, \frac{1}{2}t^2, t^2\right\rangle \;\Rightarrow\; \mathbf{r}'(t) = \langle 1, t, 2t\rangle \;\Rightarrow\; |\mathbf{r}'(t)| = \sqrt{1+t^2+4t^2} = \sqrt{1+5t^2}$.

Then $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{1+5t^2}}\langle 1, t, 2t\rangle$.

$$\begin{aligned}\mathbf{T}'(t) &= \frac{-5t}{(1+5t^2)^{3/2}}\langle 1,t,2t\rangle + \frac{1}{\sqrt{1+5t^2}}\langle 0,1,2\rangle \qquad \text{[by Formula 3 of Theorem 13.2.3]}\\ &= \frac{1}{(1+5t^2)^{3/2}}\left(\left\langle -5t,-5t^2,-10t^2\right\rangle + \left\langle 0, 1+5t^2, 2+10t^2\right\rangle\right) = \frac{1}{(1+5t^2)^{3/2}}\langle -5t,1,2\rangle\end{aligned}$$

$$|\mathbf{T}'(t)| = \frac{1}{(1+5t^2)^{3/2}}\sqrt{25t^2+1+4} = \frac{1}{(1+5t^2)^{3/2}}\sqrt{25t^2+5} = \frac{\sqrt{5}\sqrt{5t^2+1}}{(1+5t^2)^{3/2}} = \frac{\sqrt{5}}{1+5t^2}.$$

Thus, $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \dfrac{1+5t^2}{\sqrt{5}}\cdot\dfrac{1}{(1+5t^2)^{3/2}}\langle -5t,1,2\rangle = \dfrac{1}{\sqrt{5+25t^2}}\langle -5t,1,2\rangle$.

(b) By Formula 9, the curvature is $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{\sqrt{5}/(1+5t^2)}{\sqrt{1+5t^2}} = \dfrac{\sqrt{5}}{(1+5t^2)^{3/2}}$.

**24.** (a) $\mathbf{r}(t) = \left\langle \sqrt{2}\,t, e^t, e^{-t}\right\rangle \;\Rightarrow\; \mathbf{r}'(t) = \left\langle \sqrt{2}, e^t, -e^{-t}\right\rangle \;\Rightarrow\; |\mathbf{r}'(t)| = \sqrt{2+e^{2t}+e^{-2t}} = \sqrt{(e^t+e^{-t})^2} = e^t+e^{-t}$.

Then $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{e^t+e^{-t}}\left\langle \sqrt{2}, e^t, -e^{-t}\right\rangle = \dfrac{1}{e^{2t}+1}\left\langle \sqrt{2}\,e^t, e^{2t}, -1\right\rangle \quad \left[\text{after multiplying by } \dfrac{e^t}{e^t}\right]$.

[continued]

$$\mathbf{T}'(t) = \frac{1}{e^{2t}+1}\left\langle \sqrt{2}\,e^t, 2e^{2t}, 0\right\rangle - \frac{2e^{2t}}{\left(e^{2t}+1\right)^2}\left\langle \sqrt{2}\,e^t, e^{2t}, -1\right\rangle \qquad \text{[by Formula 3 of Theorem 13.2.3]}$$

$$= \frac{1}{(e^{2t}+1)^2}\left[(e^{2t}+1)\left\langle \sqrt{2}\,e^t, 2e^{2t}, 0\right\rangle - 2e^{2t}\left\langle \sqrt{2}\,e^t, e^{2t}, -1\right\rangle\right] = \frac{1}{(e^{2t}+1)^2}\left\langle \sqrt{2}\,e^t\left(1-e^{2t}\right), 2e^{2t}, 2e^{2t}\right\rangle$$

$$|\mathbf{T}'(t)| = \frac{1}{(e^{2t}+1)^2}\sqrt{2e^{2t}(1-2e^{2t}+e^{4t})+4e^{4t}+4e^{4t}} = \frac{1}{\left(e^{2t}+1\right)^2}\sqrt{2e^{2t}(1+2e^{2t}+e^{4t})}$$

$$= \frac{1}{(e^{2t}+1)^2}\sqrt{2e^{2t}\left(1+e^{2t}\right)^2} = \frac{\sqrt{2}\,e^t(1+e^{2t})}{(e^{2t}+1)^2} = \frac{\sqrt{2}\,e^t}{e^{2t}+1}$$

Thus, $$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{e^{2t}+1}{\sqrt{2}\,e^t}\frac{1}{(e^{2t}+1)^2}\left\langle \sqrt{2}\,e^t(1-e^{2t}), 2e^{2t}, 2e^{2t}\right\rangle$$

$$= \frac{1}{\sqrt{2}\,e^t(e^{2t}+1)}\left\langle \sqrt{2}\,e^t(1-e^{2t}), 2e^{2t}, 2e^{2t}\right\rangle = \frac{1}{e^{2t}+1}\left\langle 1-e^{2t}, \sqrt{2}\,e^t, \sqrt{2}\,e^t\right\rangle$$

(b) By Formula 9, the curvature is

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{\sqrt{2}\,e^t}{e^{2t}+1}\cdot\frac{1}{e^t+e^{-t}} = \frac{\sqrt{2}\,e^t}{e^{3t}+2e^t+e^{-t}} = \frac{\sqrt{2}\,e^{2t}}{e^{4t}+2e^{2t}+1} = \frac{\sqrt{2}\,e^{2t}}{(e^{2t}+1)^2}.$$

**25.** $\mathbf{r}(t) = t^3\,\mathbf{j} + t^2\,\mathbf{k} \;\Rightarrow\; \mathbf{r}'(t) = 3t^2\,\mathbf{j} + 2t\,\mathbf{k}, \quad \mathbf{r}''(t) = 6t\,\mathbf{j} + 2\,\mathbf{k}, \quad |\mathbf{r}'(t)| = \sqrt{0^2+(3t^2)^2+(2t)^2} = \sqrt{9t^4+4t^2},$

$\mathbf{r}'(t)\times\mathbf{r}''(t) = -6t^2\,\mathbf{i}, \quad |\mathbf{r}'(t)\times\mathbf{r}''(t)| = 6t^2$. Then $\kappa(t) = \dfrac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \dfrac{6t^2}{\left(\sqrt{9t^4+4t^2}\,\right)^3} = \dfrac{6t^2}{(9t^4+4t^2)^{3/2}}.$

**26.** $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + e^t\,\mathbf{k} \;\Rightarrow\; \mathbf{r}'(t) = \mathbf{i} + 2t\,\mathbf{j} + e^t\,\mathbf{k}, \quad \mathbf{r}''(t) = 2\,\mathbf{j} + e^t\,\mathbf{k},$

$|\mathbf{r}'(t)| = \sqrt{1^2+(2t)^2+(e^t)^2} = \sqrt{1+4t^2+e^{2t}}, \quad \mathbf{r}'(t)\times\mathbf{r}''(t) = (2t-2)e^t\,\mathbf{i} - e^t\,\mathbf{j} + 2\,\mathbf{k},$

$|\mathbf{r}'(t)\times\mathbf{r}''(t)| = \sqrt{[(2t-2)e^t]^2+(-e^t)^2+2^2} = \sqrt{(2t-2)^2e^{2t}+e^{2t}+4} = \sqrt{(4t^2-8t+5)e^{2t}+4}.$

Then $\kappa(t) = \dfrac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \dfrac{\sqrt{(4t^2-8t+5)e^{2t}+4}}{\left(\sqrt{1+4t^2+e^{2t}}\,\right)^3} = \dfrac{\sqrt{(4t^2-8t+5)e^{2t}+4}}{(1+4t^2+e^{2t})^{3/2}}.$

**27.** $\mathbf{r}(t) = \sqrt{6}\,t^2\,\mathbf{i} + 2t\,\mathbf{j} + 2t^3\,\mathbf{k} \;\Rightarrow\; \mathbf{r}'(t) = 2\sqrt{6}\,t\,\mathbf{i} + 2\,\mathbf{j} + 6t^2\,\mathbf{k}, \quad \mathbf{r}''(t) = 2\sqrt{6}\,\mathbf{i} + 12t\,\mathbf{k},$

$|\mathbf{r}'(t)| = \sqrt{24t^2+4+36t^4} = \sqrt{4(9t^4+6t^2+1)} = \sqrt{4(3t^2+1)^2} = 2(3t^2+1),$

$\mathbf{r}'(t)\times\mathbf{r}''(t) = 24t\,\mathbf{i} - 12\sqrt{6}\,t^2\,\mathbf{j} - 4\sqrt{6}\,\mathbf{k},$

$|\mathbf{r}'(t)\times\mathbf{r}''(t)| = \sqrt{576t^2+864t^4+96} = \sqrt{96(9t^4+6t^2+1)} = \sqrt{96(3t^2+1)^2} = 4\sqrt{6}\,(3t^2+1).$

Then $\kappa(t) = \dfrac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \dfrac{4\sqrt{6}\,(3t^2+1)}{8(3t^2+1)^3} = \dfrac{\sqrt{6}}{2(3t^2+1)^2}.$

**28.** $\mathbf{r}(t) = \left\langle t^2, \ln t, t\ln t\right\rangle \;\Rightarrow\; \mathbf{r}'(t) = \langle 2t, 1/t, 1+\ln t\rangle, \quad \mathbf{r}''(t) = \left\langle 2, -1/t^2, 1/t\right\rangle.$ The point $(1,0,0)$ corresponds to $t=1$, and $\mathbf{r}'(1) = \langle 2,1,1\rangle, \quad |\mathbf{r}'(1)| = \sqrt{2^2+1^2+1^2} = \sqrt{6}, \quad \mathbf{r}''(1) = \langle 2,-1,1\rangle, \quad \mathbf{r}'(1)\times\mathbf{r}''(1) = \langle 2,0,-4\rangle,$

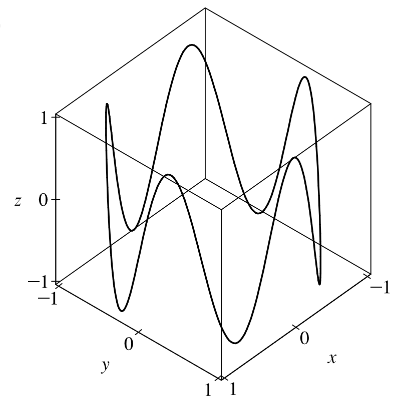
$|\mathbf{r}'(1)\times\mathbf{r}''(1)| = \sqrt{2^2+0^2+(-4)^2} = \sqrt{20} = 2\sqrt{5}$. Then $\kappa(1) = \dfrac{|\mathbf{r}'(1)\times\mathbf{r}''(1)|}{|\mathbf{r}'(1)|^3} = \dfrac{2\sqrt{5}}{\left(\sqrt{6}\right)^3} = \dfrac{2\sqrt{5}}{6\sqrt{6}}$ or $\dfrac{\sqrt{30}}{18}.$

**29.** $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle \Rightarrow \mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle$. The point $(1, 1, 1)$ corresponds to $t = 1$, and $\mathbf{r}'(1) = \langle 1, 2, 3 \rangle \Rightarrow$

$|\mathbf{r}'(1)| = \sqrt{1 + 4 + 9} = \sqrt{14}$. $\mathbf{r}''(t) = \langle 0, 2, 6t \rangle \Rightarrow \mathbf{r}''(1) = \langle 0, 2, 6 \rangle$. $\mathbf{r}'(1) \times \mathbf{r}''(1) = \langle 6, -6, 2 \rangle$, so

$|\mathbf{r}'(1) \times \mathbf{r}''(1)| = \sqrt{36 + 36 + 4} = \sqrt{76}$. Then $\kappa(1) = \dfrac{|\mathbf{r}'(1) \times \mathbf{r}''(1)|}{|\mathbf{r}'(1)|^3} = \dfrac{\sqrt{76}}{\sqrt{14}^3} = \dfrac{1}{7}\sqrt{\dfrac{19}{14}}$.

**30.**




Note that we get the complete curve for $0 \le t < 2\pi$.

$\mathbf{r}(t) = \langle \cos t, \sin t, \sin 5t \rangle \Rightarrow \mathbf{r}'(t) = \langle -\sin t, \cos t, 5\cos 5t \rangle$,

$\mathbf{r}''(t) = \langle -\cos t, -\sin t, -25\sin 5t \rangle$. The point $(1, 0, 0)$ corresponds to $t = 0$, and $\mathbf{r}'(0) = \langle 0, 1, 5 \rangle \Rightarrow$

$|\mathbf{r}'(0)| = \sqrt{0^2 + 1^2 + 5^2} = \sqrt{26}$, $\mathbf{r}''(0) = \langle -1, 0, 0 \rangle$,

$\mathbf{r}'(0) \times \mathbf{r}''(0) = \langle 0, -5, 1 \rangle \Rightarrow$

$|\mathbf{r}'(0) \times \mathbf{r}''(0)| = \sqrt{0^2 + (-5)^2 + 1^2} = \sqrt{26}$. The curvature at the point $(1, 0, 0)$ is $\kappa(0) = \dfrac{|\mathbf{r}'(0) \times \mathbf{r}''(0)|}{|\mathbf{r}'(0)|^3} = \dfrac{\sqrt{26}}{\left(\sqrt{26}\right)^3} = \dfrac{1}{26}$.

**31.** $f(x) = x^4 \Rightarrow f'(x) = 4x^3 \Rightarrow f''(x) = 12x^2$. By Formula 11, the curvature is

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}} = \frac{|12x^2|}{[1 + (4x^3)^2]^{3/2}} = \frac{12x^2}{(1 + 16x^6)^{3/2}}.$$

**32.** $f(x) = \tan x \Rightarrow f'(x) = \sec^2 x \Rightarrow f''(x) = 2\sec x \cdot \sec x \tan x = 2\sec^2 x \tan x$. By Formula 11, the curvature is

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}} = \frac{|2\sec^2 x \tan x|}{[1 + (\sec^2 x)^2]^{3/2}} = \frac{2\sec^2 x\,|\tan x|}{(1 + \sec^4 x)^{3/2}}.$$

**33.** $f(x) = xe^x \Rightarrow f'(x) = xe^x + e^x \Rightarrow f''(x) = xe^x + 2e^x$. By Formula 11, the curvature is

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}} = \frac{|xe^x + 2e^x|}{[1 + (xe^x + e^x)^2]^{3/2}} = \frac{e^x\,|x + 2|}{[1 + (xe^x + e^x)^2]^{3/2}}.$$

**34.** $y = \ln x \Rightarrow y' = \dfrac{1}{x} \Rightarrow y'' = -\dfrac{1}{x^2}$. By Formula 11, the curvature is

$$\kappa(x) = \frac{|y''(x)|}{\left[1 + (y'(x))^2\right]^{3/2}} = \left|\frac{-1}{x^2}\right| \frac{1}{(1 + 1/x^2)^{3/2}} = \frac{1}{x^2}\frac{(x^2)^{3/2}}{(x^2+1)^{3/2}} = \frac{|x|}{(x^2+1)^{3/2}} = \frac{x}{(x^2+1)^{3/2}} \quad [\text{since } x > 0].$$

To find the maximum curvature, we first find the critical numbers of $\kappa(x)$:

$$\kappa'(x) = \frac{(x^2+1)^{3/2} - x\left(\frac{3}{2}\right)(x^2+1)^{1/2}(2x)}{[(x^2+1)^{3/2}]^2} = \frac{(x^2+1)^{1/2}[(x^2+1) - 3x^2]}{(x^2+1)^3} = \frac{1 - 2x^2}{(x^2+1)^{5/2}};$$

$\kappa'(x) = 0 \Rightarrow 1 - 2x^2 = 0$, so the only critical number in the domain is $x = \frac{1}{\sqrt{2}}$. Since $\kappa'(x) > 0$ for $0 < x < \frac{1}{\sqrt{2}}$ and $\kappa'(x) < 0$ for $x > \frac{1}{\sqrt{2}}$, $\kappa(x)$ attains its maximum at $x = \frac{1}{\sqrt{2}}$. Thus, the maximum curvature occurs at $\left(\frac{1}{\sqrt{2}}, \ln\frac{1}{\sqrt{2}}\right)$.

Since $\displaystyle\lim_{x\to\infty} \frac{x}{(x^2+1)^{3/2}} = 0$, $\kappa(x)$ approaches 0 as $x \to \infty$.

**35.** Since $y = y' = y'' = e^x$, the curvature is $\kappa(x) = \dfrac{|y''(x)|}{[1+(y'(x))^2]^{3/2}} = \dfrac{e^x}{(1+e^{2x})^{3/2}} = e^x(1+e^{2x})^{-3/2}$.

To find the maximum curvature, we first find the critical numbers of $\kappa(x)$:

$$\kappa'(x) = e^x(1+e^{2x})^{-3/2} + e^x\left(-\tfrac{3}{2}\right)(1+e^{2x})^{-5/2}(2e^{2x}) = e^x\frac{1+e^{2x}-3e^{2x}}{(1+e^{2x})^{5/2}} = e^x\frac{1-2e^{2x}}{(1+e^{2x})^{5/2}}.$$

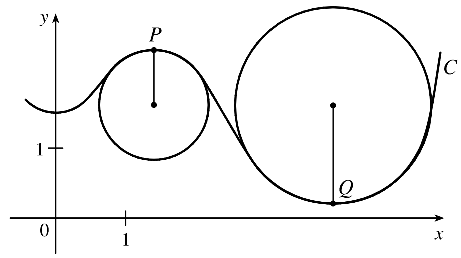
$\kappa'(x) = 0$ when $1 - 2e^{2x} = 0$, so $e^{2x} = \frac{1}{2}$ or $x = -\frac{1}{2}\ln 2$. And since $1 - 2e^{2x} > 0$ for $x < -\frac{1}{2}\ln 2$ and $1 - 2e^{2x} < 0$ for $x > -\frac{1}{2}\ln 2$, the maximum curvature is attained at the point $\left(-\frac{1}{2}\ln 2, e^{(-\ln 2)/2}\right) = \left(-\frac{1}{2}\ln 2, \frac{1}{\sqrt{2}}\right)$.

Since $\lim\limits_{x\to\infty} e^x(1+e^{2x})^{-3/2} = 0$, $\kappa(x)$ approaches 0 as $x \to \infty$.

**36.** We can take the parabola as having its vertex at the origin and opening upward, so the equation is $f(x) = ax^2$, $a > 0$. Then by Formula 11, $\kappa(x) = \dfrac{|f''(x)|}{[1+(f'(x))^2]^{3/2}} = \dfrac{|2a|}{[1+(2ax)^2]^{3/2}} = \dfrac{2a}{(1+4a^2x^2)^{3/2}}$, thus $\kappa(0) = 2a$. We want $\kappa(0) = 4$, so $a = 2$ and the equation is $y = 2x^2$.

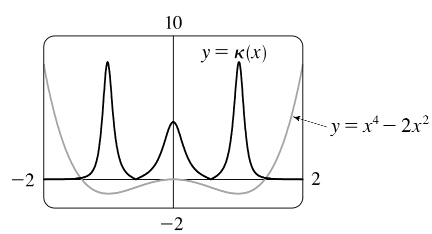
**37.** (a) $C$ appears to be changing direction more quickly at $P$ than $Q$, so we would expect the curvature to be greater at $P$.

(b) First we sketch approximate osculating circles at $P$ and $Q$. Using the axes scale as a guide, we measure the radius of the osculating circle at $P$ to be approximately 0.8 units, thus $\rho = \dfrac{1}{\kappa} \Rightarrow \kappa = \dfrac{1}{\rho} \approx \dfrac{1}{0.8} \approx 1.3$. Similarly, we estimate the radius of the osculating circle at $Q$ to be 1.4 units, so $\kappa = \dfrac{1}{\rho} \approx \dfrac{1}{1.4} \approx 0.7$.




**38.** $y = x^4 - 2x^2 \Rightarrow y' = 4x^3 - 4x \Rightarrow y'' = 12x^2 - 4$.

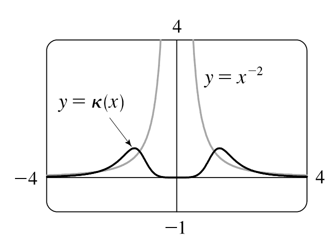
$\kappa(x) = \dfrac{|y''|}{\left[1+(y')^2\right]^{3/2}} = \dfrac{\left|12x^2-4\right|}{\left[1+(4x^3-4x)^2\right]^{3/2}}$. The graph of the curvature here is what we would expect. The graph of $y = x^4 - 2x^2$ appears to be bending most sharply at the origin and near $x = \pm 1$.




**39.** $y = x^{-2} \Rightarrow y' = -2x^{-3} \Rightarrow y'' = 6x^{-4}$.

$\kappa(x) = \dfrac{|y''|}{\left[1+(y')^2\right]^{3/2}} = \dfrac{\left|6x^{-4}\right|}{\left[1+(-2x^{-3})^2\right]^{3/2}} = \dfrac{6}{x^4\,(1+4x^{-6})^{3/2}}$.

The appearance of the two humps in this graph is perhaps a little surprising, but it is explained by the fact that $y = x^{-2}$ increases asymptotically at the origin from both directions, and so its graph has very little bend there. [Note that $\kappa(0)$ is undefined.]

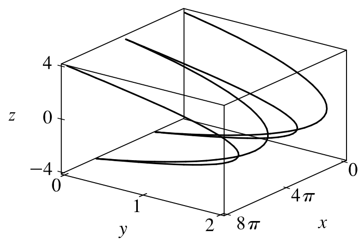


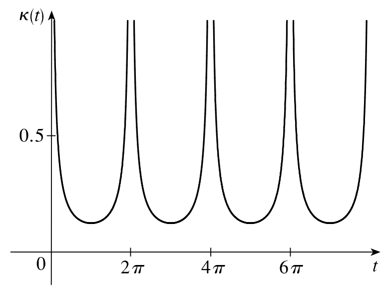
**40.** $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t, 4\cos(t/2)\rangle \quad \Rightarrow \quad \mathbf{r}'(t) = \langle 1 - \cos t, \sin t, -2\sin(t/2)\rangle \quad \Rightarrow$

$\mathbf{r}''(t) = \langle \sin t, \cos t, -\cos(t/2)\rangle$. Using a CAS, $\mathbf{r}'(t) \times \mathbf{r}''(t) = \left\langle -2\sin^3(t/2), -\sin(t/2)\sin t, \cos t - 1\right\rangle$,

$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{3 - 4\cos t + \cos 2t}$ or $2\sqrt{2}\sin^2(t/2)$, and $|\mathbf{r}'(t)| = 2\sqrt{1 - \cos t}$ or $2\sqrt{2}\,|\sin(t/2)|$.

(To compute cross products in Maple, use the `VectorCalculus` or `LinearAlgebra` package and the `CrossProduct(a,b)` command. Here loading the `RealDomain` package will give simpler results. In Mathematica, use `Cross[a,b]`.) Then $\kappa(t) = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \dfrac{\sqrt{3 - 4\cos t + \cos 2t}}{8\,(1 - \cos t)^{3/2}}$ or $\dfrac{1}{4\sqrt{2 - 2\cos t}}$ or $\dfrac{1}{8\,|\sin(t/2)|}$. We plot the space curve and its curvature function for $0 \le t \le 8\pi$ below.



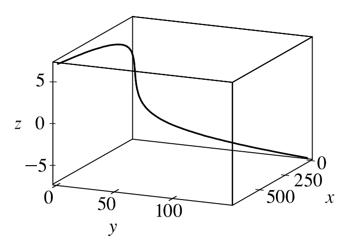

The asymptotes in the graph of $\kappa(t)$ correspond to the sharp cusps we see in the graph of $\mathbf{r}(t)$. The space curve bends most sharply as it approaches these cusps (mostly in the $x$-direction) and bends most gradually between these, near its intersections with the $xy$-plane, where $t = \pi + 2n\pi$ ($n$ an integer). (The bending we see in the $z$-direction on the curve near these points is deceiving; most of the curvature occurs in the $x$-direction.) The curvature graph has local minima at these values of $t$.

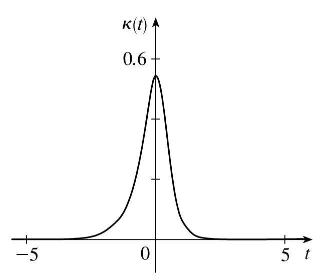
**41.** $\mathbf{r}(t) = \left\langle te^t, e^{-t}, \sqrt{2}t\right\rangle \quad \Rightarrow \quad \mathbf{r}'(t) = \left\langle (t+1)e^t, -e^{-t}, \sqrt{2}\right\rangle \quad \Rightarrow \quad \mathbf{r}''(t) = \left\langle (t+2)e^t, e^{-t}, 0\right\rangle$.

Then $\mathbf{r}'(t) \times \mathbf{r}''(t) = \left\langle -\sqrt{2}e^{-t}, \sqrt{2}(t+2)e^t, 2t+3\right\rangle$, $\quad |\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{2e^{-2t} + 2(t+2)^2e^{2t} + (2t+3)^2}$,

$|\mathbf{r}'(t)| = \sqrt{(t+1)^2e^{2t} + e^{-2t} + 2}$, and $\kappa(t) = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \dfrac{\sqrt{2e^{-2t} + 2(t+2)^2e^{2t} + (2t+3)^2}}{\left[(t+1)^2e^{2t} + e^{-2t} + 2\right]^{3/2}}$.

We plot the space curve and its curvature function for $-5 \le t \le 5$ below.



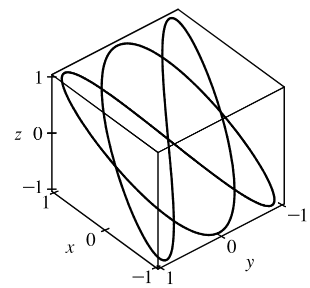




From the graph of $\kappa(t)$ we see that curvature is maximized for $t = 0$, so the curve bends most sharply at the point $(0, 1, 0)$. The curve bends more gradually as we move away from this point, becoming almost linear. This is reflected in the curvature graph, where $\kappa(t)$ becomes nearly 0 as $|t|$ increases.

**42.** Notice that the curve $a$ is highest for the same $x$-values at which curve $b$ is turning more sharply, and $a$ is 0 or near 0 where $b$ is nearly straight. So, $a$ must be the graph of $y = \kappa(x)$, and $b$ is the graph of $y = f(x)$.

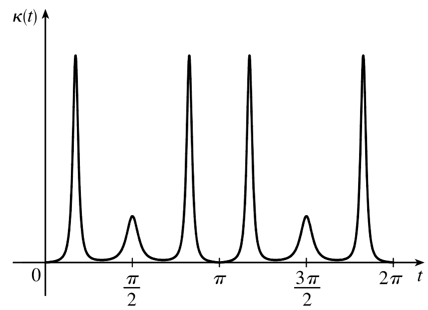
**43.** Notice that the curve $b$ has two inflection points at which the graph appears almost straight. We would expect the curvature to be 0 or nearly 0 at these values, but the curve $a$ isn't near 0 there. Thus, $a$ must be the graph of $y = f(x)$ rather than the graph of curvature, and $b$ is the graph of $y = \kappa(x)$.

**44.** (a) The complete curve for $\mathbf{r}(t) = \langle \sin 3t, \sin 2t, \sin 3t \rangle$ is given by $0 \leq t \leq 2\pi$. Curvature appears to have a local (or absolute) maximum at 6 points. (Look at points where the curve appears to turn more sharply.)



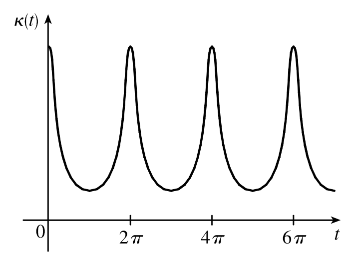

(b) Using a CAS, we find (after simplifying)

$\kappa(t) = \dfrac{3\sqrt{2}\sqrt{(5\sin t + \sin 5t)^2}}{(9\cos 6t + 2\cos 4t + 11)^{3/2}}$. (To compute cross products in Maple, use the `VectorCalculus` or `LinearAlgebra` package and the `CrossProduct(a,b)` command; in Mathematica, use `Cross[a,b]`.) The graph shows 6 local (or absolute) maximum points for $0 \leq t \leq 2\pi$, as observed in part (a).




**45.** $\mathbf{r}(t) = \left\langle t - \frac{3}{2}\sin t, 1 - \frac{3}{2}\cos t, t \right\rangle$. Using a CAS, we find (after simplifying) $\kappa(t) = \dfrac{6\sqrt{4\cos^2 t - 12\cos t + 13}}{(17 - 12\cos t)^{3/2}}$.

(To compute cross products in Maple, use the `VectorCalculus` or `LinearAlgebra` package and the `CrossProduct(a,b)` command; in Mathematica, use `Cross[a,b]`.) Curvature is largest at integer multiples of $2\pi$.




**46.** Here $\mathbf{r}(t) = \langle f(t), g(t) \rangle$, $\mathbf{r}'(t) = \langle f'(t), g'(t) \rangle$, $\mathbf{r}''(t) = \langle f''(t), g''(t) \rangle$,

$|\mathbf{r}'(t)|^3 = \left[\sqrt{(f'(t))^2 + (g'(t))^2}\right]^3 = [(f'(t))^2 + (g'(t))^2]^{3/2} = (\dot{x}^2 + \dot{y}^2)^{3/2}$, and

$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = |\langle 0, 0, f'(t)\,g''(t) - f''(t)\,g'(t) \rangle| = \left[(\dot{x}\ddot{y} - \ddot{x}\dot{y})^2\right]^{1/2} = |\dot{x}\ddot{y} - \dot{y}\ddot{x}|$. Thus, by Theorem 10,

$\kappa(t) = \dfrac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{[\dot{x}^2 + \dot{y}^2]^{3/2}}$.

**47.** $x = t^2 \;\Rightarrow\; \dot{x} = 2t \;\Rightarrow\; \ddot{x} = 2$, $\quad y = t^3 \;\Rightarrow\; \dot{y} = 3t^2 \;\Rightarrow\; \ddot{y} = 6t$.

Then $\kappa(t) = \dfrac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{[\dot{x}^2 + \dot{y}^2]^{3/2}} = \dfrac{\left|(2t)(6t) - (3t^2)(2)\right|}{[(2t)^2 + (3t^2)^2]^{3/2}} = \dfrac{\left|12t^2 - 6t^2\right|}{(4t^2 + 9t^4)^{3/2}} = \dfrac{6t^2}{(4t^2 + 9t^4)^{3/2}}$.

**48.** $x = a\cos\omega t \quad\Rightarrow\quad \dot{x} = -a\omega\sin\omega t \quad\Rightarrow\quad \ddot{x} = -a\omega^2\cos\omega t$,

$y = b\sin\omega t \quad\Rightarrow\quad \dot{y} = b\omega\cos\omega t \quad\Rightarrow\quad \ddot{y} = -b\omega^2\sin\omega t$. Then

$$\kappa(t) = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{[\dot{x}^2 + \dot{y}^2]^{3/2}} = \frac{\left|(-a\omega\sin\omega t)(-b\omega^2\sin\omega t) - (b\omega\cos\omega t)(-a\omega^2\cos\omega t)\right|}{[(-a\omega\sin\omega t)^2 + (b\omega\cos\omega t)^2]^{3/2}}$$

$$= \frac{\left|ab\omega^3\sin^2\omega t + ab\omega^3\cos^2\omega t\right|}{(a^2\omega^2\sin^2\omega t + b^2\omega^2\cos^2\omega t)^{3/2}} = \frac{\left|ab\omega^3\right|}{(a^2\omega^2\sin^2\omega t + b^2\omega^2\cos^2\omega t)^{3/2}}$$

**49.** $x = e^t\cos t \quad\Rightarrow\quad \dot{x} = e^t(\cos t - \sin t) \quad\Rightarrow\quad \ddot{x} = e^t(-\sin t - \cos t) + e^t(\cos t - \sin t) = -2e^t\sin t$,

$y = e^t\sin t \quad\Rightarrow\quad \dot{y} = e^t(\cos t + \sin t) \quad\Rightarrow\quad \ddot{y} = e^t(-\sin t + \cos t) + e^t(\cos t + \sin t) = 2e^t\cos t$. Then

$$\kappa(t) = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{[\dot{x}^2 + \dot{y}^2]^{3/2}} = \frac{\left|e^t(\cos t - \sin t)(2e^t\cos t) - e^t(\cos t + \sin t)(-2e^t\sin t)\right|}{\left([e^t(\cos t - \sin t)]^2 + [e^t(\cos t + \sin t)]^2\right)^{3/2}}$$

$$= \frac{\left|2e^{2t}(\cos^2 t - \sin t\cos t + \sin t\cos t + \sin^2 t)\right|}{\left[e^{2t}(\cos^2 t - 2\cos t\sin t + \sin^2 t + \cos^2 t + 2\cos t\sin t + \sin^2 t)\right]^{3/2}} = \frac{\left|2e^{2t}(1)\right|}{\left[e^{2t}(1+1)\right]^{3/2}} = \frac{2e^{2t}}{e^{3t}(2)^{3/2}} = \frac{1}{\sqrt{2}\,e^t}$$

**50.** $f(x) = e^{cx}$, $\quad f'(x) = ce^{cx}$, $\quad f''(x) = c^2e^{cx}$. Using Formula 11, we have

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}} = \frac{\left|c^2e^{cx}\right|}{[1 + (ce^{cx})^2]^{3/2}} = \frac{c^2e^{cx}}{(1 + c^2e^{2cx})^{3/2}}$$ so the curvature at $x = 0$ is

$\kappa(0) = \dfrac{c^2}{(1 + c^2)^{3/2}}$. To determine the maximum value for $\kappa(0)$, let $f(c) = \dfrac{c^2}{(1 + c^2)^{3/2}}$. Then

$f'(c) = \dfrac{2c\cdot(1 + c^2)^{3/2} - c^2\cdot\frac{3}{2}(1 + c^2)^{1/2}(2c)}{[(1 + c^2)^{3/2}]^2} = \dfrac{(1 + c^2)^{1/2}\left[2c(1 + c^2) - 3c^3\right]}{(1 + c^2)^3} = \dfrac{(2c - c^3)}{(1 + c^2)^{5/2}}$. We have a critical number when $2c - c^3 = 0 \quad\Rightarrow\quad c(2 - c^2) = 0 \quad\Rightarrow\quad c = 0$ or $c = \pm\sqrt{2}$. $\quad f'(c)$ is positive for $c < -\sqrt{2}$, $0 < c < \sqrt{2}$ and negative elsewhere, so $f$ achieves its maximum value when $c = \sqrt{2}$ or $-\sqrt{2}$. In either case, $\kappa(0) = \dfrac{2}{3^{3/2}}$, so the members of the family with the largest value of $\kappa(0)$ are $f(x) = e^{\sqrt{2}x}$ and $f(x) = e^{-\sqrt{2}x}$.

**51.** $\mathbf{r}(t) = \left\langle t^2, \frac{2}{3}t^3, t\right\rangle$. $\left(1, \frac{2}{3}, 1\right)$ corresponds to $t = 1$.

$\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{\left\langle 2t, 2t^2, 1\right\rangle}{\sqrt{4t^2 + 4t^4 + 1}} = \dfrac{\left\langle 2t, 2t^2, 1\right\rangle}{2t^2 + 1}$, so $\mathbf{T}(1) = \left\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right\rangle$.

$\mathbf{T}'(t) = -4t(2t^2 + 1)^{-2}\left\langle 2t, 2t^2, 1\right\rangle + (2t^2 + 1)^{-1}\left\langle 2, 4t, 0\right\rangle \qquad$ [by Formula 3 of Theorem 13.2.3]

$\quad = (2t^2 + 1)^{-2}\left\langle -8t^2 + 4t^2 + 2, -8t^3 + 8t^3 + 4t, -4t\right\rangle = 2(2t^2 + 1)^{-2}\left\langle 1 - 2t^2, 2t, -2t\right\rangle$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{2(2t^2 + 1)^{-2}\left\langle 1 - 2t^2, 2t, -2t\right\rangle}{2(2t^2 + 1)^{-2}\sqrt{(1 - 2t^2)^2 + (2t)^2 + (-2t)^2}} = \frac{\left\langle 1 - 2t^2, 2t, -2t\right\rangle}{\sqrt{1 - 4t^2 + 4t^4 + 8t^2}} = \frac{\left\langle 1 - 2t^2, 2t, -2t\right\rangle}{1 + 2t^2}$$

$\mathbf{N}(1) = \left\langle -\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right\rangle$ and $\mathbf{B}(1) = \mathbf{T}(1)\times\mathbf{N}(1) = \left\langle -\frac{4}{9} - \frac{2}{9}, -\left(-\frac{4}{9} + \frac{1}{9}\right), \frac{4}{9} + \frac{2}{9}\right\rangle = \left\langle -\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right\rangle$.

**52.** $(1, 0, 0)$ corresponds to $t = 0$. $\quad \mathbf{r}(t) = \langle\cos t, \sin t, \ln\cos t\rangle$, and in Exercise 6, we found that $\mathbf{r}'(t) = \langle -\sin t, \cos t, -\tan t\rangle$ and $|\mathbf{r}'(t)| = |\sec t|$. Here we can assume $-\frac{\pi}{2} < t < \frac{\pi}{2}$ and then $\sec t > 0 \quad\Rightarrow\quad |\mathbf{r}'(t)| = \sec t$.

[continued]

$\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{\langle -\sin t, \cos t, -\tan t\rangle}{\sec t} = \left\langle -\sin t\cos t, \cos^2 t, -\sin t\right\rangle$ and $\mathbf{T}(0) = \langle 0, 1, 0\rangle$.

$\mathbf{T}'(t) = \langle -[(\sin t)(-\sin t) + (\cos t)(\cos t)], 2(\cos t)(-\sin t), -\cos t\rangle = \left\langle \sin^2 t - \cos^2 t, -2\sin t\cos t, -\cos t\right\rangle$, so

$$\mathbf{N}(0) = \frac{\mathbf{T}'(0)}{|\mathbf{T}'(0)|} = \frac{\langle -1, 0, -1\rangle}{\sqrt{1+0+1}} = \frac{1}{\sqrt{2}}\langle -1, 0, -1\rangle = \left\langle -\tfrac{1}{\sqrt{2}}, 0, -\tfrac{1}{\sqrt{2}}\right\rangle.$$

Finally, $\mathbf{B}(0) = \mathbf{T}(0) \times \mathbf{N}(0) = \langle 0, 1, 0\rangle \times \left\langle -\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right\rangle = \left\langle -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right\rangle$.

**53.** $\mathbf{r}(t) = \langle \sin 2t, -\cos 2t, 4t\rangle \Rightarrow \mathbf{r}'(t) = \langle 2\cos 2t, 2\sin 2t, 4\rangle$. The point $(0, 1, 2\pi)$ corresponds to $t = \pi/2$, and the normal plane there has normal vector $\mathbf{r}'(\pi/2) = \langle -2, 0, 4\rangle$. An equation for the normal plane is

$-2(x-0) + 0(y-1) + 4(z-2\pi) = 0$ or $-2x + 4z = 8\pi$ or $x - 2z = -4\pi$.

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\langle 2\cos 2t, 2\sin 2t, 4\rangle}{\sqrt{4\cos^2 2t + 4\sin^2 2t + 16}} = \frac{1}{2\sqrt{5}}\langle 2\cos 2t, 2\sin 2t, 4\rangle = \frac{1}{\sqrt{5}}\langle \cos 2t, \sin 2t, 2\rangle \quad\Rightarrow$$

$\mathbf{T}'(t) = \frac{1}{\sqrt{5}}\langle -2\sin 2t, 2\cos 2t, 0\rangle \Rightarrow |\mathbf{T}'(t)| = \frac{1}{\sqrt{5}}\sqrt{4\sin^2 2t + 4\cos^2 2t} = \frac{2}{\sqrt{5}}$, and

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \langle -\sin 2t, \cos 2t, 0\rangle$. Then $\mathbf{T}(\pi/2) = \frac{1}{\sqrt{5}}\langle -1, 0, 2\rangle$, $\mathbf{N}(\pi/2) = \langle 0, -1, 0\rangle$, and

$\mathbf{B}(\pi/2) = \mathbf{T}(\pi/2) \times \mathbf{N}(\pi/2) = \frac{1}{\sqrt{5}}\langle 2, 0, 1\rangle$. Since $\mathbf{B}(\pi/2)$ is normal to the osculating plane, so is $\langle 2, 0, 1\rangle$, and an equation of the plane is $2(x-0) + 0(y-1) + 1(z-2\pi) = 0$ or $2x + z = 2\pi$.

**54.** $\mathbf{r}(t) = \left\langle \ln t, 2t, t^2\right\rangle \Rightarrow \mathbf{r}'(t) = \langle 1/t, 2, 2t\rangle$. The point $(0, 2, 1)$ corresponds to $t = 1$, and the normal plane there has normal vector $\mathbf{r}'(1) = \langle 1, 2, 2\rangle$. An equation for the normal plane is $1(x-0) + 2(y-2) + 2(z-1) = 0$ or $x + 2y + 2z = 6$.

$|\mathbf{r}'(t)| = \sqrt{1/t^2 + 4 + 4t^2} = \sqrt{[(1/t) + 2t]^2} = (1/t) + 2t$ [since $t > 0$] and then

$\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{\langle 1/t, 2, 2t\rangle}{(1/t) + 2t} = \dfrac{1}{1+2t^2}\left\langle 1, 2t, 2t^2\right\rangle$ $\left[\text{after multiplying by } \dfrac{t}{t}\right]$. By Formula 3 of Theorem 13.2.3,

$$\begin{aligned}\mathbf{T}'(t) &= -\frac{4t}{(1+2t^2)^2}\left\langle 1, 2t, 2t^2\right\rangle + \frac{1}{1+2t^2}\langle 0, 2, 4t\rangle \\ &= \frac{1}{(1+2t^2)^2}\left\langle -4t, -8t^2 + 2(1+2t^2), -8t^3 + 4t(1+2t^2)\right\rangle = \frac{1}{(1+2t^2)^2}\left\langle -4t, 2-4t^2, 4t\right\rangle\end{aligned}$$

Then

$$\begin{aligned}|\mathbf{T}'(t)| &= \frac{1}{(1+2t^2)^2}\sqrt{16t^2 + (2-4t^2)^2 + 16t^2} = \frac{1}{(1+2t^2)^2}\sqrt{16t^2 + 4 + 16t^4} \\ &= \frac{1}{(1+2t^2)^2}\cdot 2\sqrt{(1+2t^2)^2} = \frac{2}{1+2t^2}\end{aligned}$$

and $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \dfrac{1}{2(1+2t^2)}\left\langle -4t, 2-4t^2, 4t\right\rangle = \dfrac{1}{1+2t^2}\left\langle -2t, 1-2t^2, 2t\right\rangle$.

Thus $\mathbf{T}(1) = \frac{1}{3}\langle 1, 2, 2\rangle$, $\mathbf{N}(1) = \frac{1}{3}\langle -2, -1, 2\rangle$, and $\mathbf{B}(1) = \mathbf{T}(1) \times \mathbf{N}(1) = \frac{1}{9}\langle 6, -6, 3\rangle$ is normal to the osculating plane.

[continued]

We can take the parallel vector $\langle 2, -2, 1\rangle$ as a normal vector for the plane, so an equation is

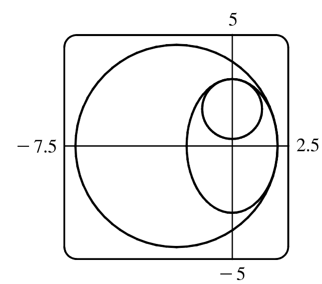
$2(x-0) - 2(y-2) + 1(z-1) = 0$ or $2x - 2y + z = -3$.

*Note:* Since $\mathbf{r}'(1)$ is parallel to $\mathbf{T}(1)$ and $\mathbf{T}'(1)$ is parallel to $\mathbf{N}(1)$, we could have taken $\mathbf{r}'(1) \times \mathbf{T}'(1)$ as a normal vector for the plane.

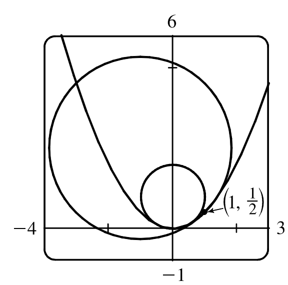
**55.** The ellipse is given by the parametric equations $x = 2\cos t$, $y = 3\sin t$, so using the result from Exercise 46,

$$\kappa(t) = \frac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{[\dot{x}^2 + \dot{y}^2]^{3/2}} = \frac{|(-2\sin t)(-3\sin t) - (3\cos t)(-2\cos t)|}{(4\sin^2 t + 9\cos^2 t)^{3/2}} = \frac{6}{(4\sin^2 t + 9\cos^2 t)^{3/2}}.$$

At $(2, 0)$, $t = 0$. Now $\kappa(0) = \frac{6}{27} = \frac{2}{9}$, so the radius of the osculating circle is $1/\kappa(0) = \frac{9}{2}$ and its center is $\left(-\frac{5}{2}, 0\right)$. Its equation is therefore $\left(x + \frac{5}{2}\right)^2 + y^2 = \frac{81}{4}$. At $(0, 3)$, $t = \frac{\pi}{2}$, and $\kappa\left(\frac{\pi}{2}\right) = \frac{6}{8} = \frac{3}{4}$. So the radius of the osculating circle is $\frac{4}{3}$ and its center is $\left(0, \frac{5}{3}\right)$. Hence its equation is $x^2 + \left(y - \frac{5}{3}\right)^2 = \frac{16}{9}$.




**56.** $y = \frac{1}{2}x^2 \quad\Rightarrow\quad y' = x$ and $y'' = 1$, so Formula 11 gives $\kappa(x) = \dfrac{1}{(1 + x^2)^{3/2}}$. So the curvature at $(0, 0)$ is $\kappa(0) = 1$ and the osculating circle has radius 1 and center $(0, 1)$, and hence equation $x^2 + (y-1)^2 = 1$. The curvature at $\left(1, \frac{1}{2}\right)$ is $\kappa(1) = \dfrac{1}{(1 + 1^2)^{3/2}} = \dfrac{1}{2\sqrt{2}}$. The tangent line to the parabola at $\left(1, \frac{1}{2}\right)$ has slope 1, so the normal line has slope $-1$. Thus the center of the osculating circle lies in the direction of the unit vector $\left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$. The circle has radius $2\sqrt{2}$, so its center has position vector $\left\langle 1, \frac{1}{2}\right\rangle + 2\sqrt{2}\left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \left\langle -1, \frac{5}{2}\right\rangle$. So the equation of the circle is $(x+1)^2 + \left(y - \frac{5}{2}\right)^2 = 8$.




**57.** Here $\mathbf{r}(t) = \left\langle t^3, 3t, t^4 \right\rangle$, and $\mathbf{r}'(t) = \left\langle 3t^2, 3, 4t^3 \right\rangle$ is normal to the normal plane for any $t$. The given plane has normal vector $\langle 6, 6, -8\rangle$, and the planes are parallel when their normal vectors are parallel. Thus we need to find a value for $t$ where $\left\langle 3t^2, 3, 4t^3 \right\rangle = k\,\langle 6, 6, -8\rangle$ for some $k \neq 0$. From the $y$-component we see that $k = \frac{1}{2}$, and $\left\langle 3t^2, 3, 4t^3 \right\rangle = \frac{1}{2}\langle 6, 6, -8\rangle = \langle 3, 3, -4\rangle$ for $t = -1$. Thus, the planes are parallel at the point $(-1, -3, 1)$.

**58.** To find the osculating plane, we first calculate the unit tangent and normal vectors.

In Maple, we use the `VectorCalculus` package and set `r:=<t^3,3*t,t^4>;`. After differentiating, the `Normalize` command converts the tangent vector to the unit tangent vector: `T:=Normalize(diff(r,t));`. After simplifying, we find that $\mathbf{T}(t) = \dfrac{\left\langle 3t^2, 3, 4t^3 \right\rangle}{\sqrt{16t^6 + 9t^4 + 9}}$. We use a similar procedure to compute the unit normal vector, `N:=Normalize(diff(T,t));`. After simplifying, we have $\mathbf{N}(t) = \dfrac{\left\langle -t(8t^6 - 9), -3t^3(3 + 8t^2), 6t^2(t^4 + 3)\right\rangle}{\sqrt{t^2(4t^6 + 36t^2 + 9)(16t^6 + 9t^4 + 9)}}$. Then we use the command `B:=CrossProduct(T,N);`. After simplification, we find that $\mathbf{B}(t) = \dfrac{\left\langle 6t^2, -2t^4, -3t\right\rangle}{\sqrt{t^2(4t^6 + 36t^2 + 9)}}$.

In Mathematica, we define the vector function `r={t^3,3*t,t^4}` and use the command `Dt` to differentiate. We find $\mathbf{T}(t)$ by dividing the result by its magnitude, computed using the `Norm` command. (You may wish to include the option `Element[t,Reals]` to obtain simpler expressions.) $\mathbf{N}(t)$ is found similarly, and we use `Cross[T,N]` to find $\mathbf{B}(t)$.

Now $\mathbf{B}(t)$ is parallel to $\left\langle 6t^2, -2t^4, -3t\right\rangle$, so if $\mathbf{B}(t)$ is parallel to $\langle 1, 1, 1\rangle$ for some $t \neq 0$ [since $\mathbf{B}(0) = \mathbf{0}$], then $\left\langle 6t^2, -2t^4, -3t\right\rangle = k\langle 1, 1, 1\rangle$ for some value of $k$. But then $6t^2 = -2t^4 = -3t$ which has no solution for $t \neq 0$. So there is no such osculating plane.

**59.** First we parametrize the curve of intersection. We can choose $y = t$; then $x = y^2 = t^2$ and $z = x^2 = t^4$, and the curve is given by $\mathbf{r}(t) = \left\langle t^2, t, t^4\right\rangle$. $\mathbf{r}'(t) = \left\langle 2t, 1, 4t^3\right\rangle$ and the point $(1, 1, 1)$ corresponds to $t = 1$, so $\mathbf{r}'(1) = \langle 2, 1, 4\rangle$ is a normal vector for the normal plane. Thus, an equation of the normal plane is

$2(x - 1) + 1(y - 1) + 4(z - 1) = 0$ or $2x + y + 4z = 7$. $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{4t^2 + 1 + 16t^6}}\left\langle 2t, 1, 4t^3\right\rangle$ and

$\mathbf{T}'(t) = -\frac{1}{2}(4t^2 + 1 + 16t^6)^{-3/2}(8t + 96t^5)\left\langle 2t, 1, 4t^3\right\rangle + (4t^2 + 1 + 16t^6)^{-1/2}\left\langle 2, 0, 12t^2\right\rangle$. A normal vector for the osculating plane is $\mathbf{B}(1) = \mathbf{T}(1) \times \mathbf{N}(1)$, but $\mathbf{r}'(1) = \langle 2, 1, 4\rangle$ is parallel to $\mathbf{T}(1)$ and

$\mathbf{T}'(1) = -\frac{1}{2}(21)^{-3/2}(104)\langle 2, 1, 4\rangle + (21)^{-1/2}\langle 2, 0, 12\rangle = \frac{2}{21\sqrt{21}}\langle -31, -26, 22\rangle$ is parallel to $\mathbf{N}(1)$ as is $\langle -31, -26, 22\rangle$, so $\langle 2, 1, 4\rangle \times \langle -31, -26, 22\rangle = \langle 126, -168, -21\rangle$ is normal to the osculating plane. Thus, an equation for the osculating plane is $126(x - 1) - 168(y - 1) - 21(z - 1) = 0$ or $6x - 8y - z = -3$.

**60.** $\mathbf{r}(t) = \left\langle t + 2, 1 - t, \frac{1}{2}t^2\right\rangle \Rightarrow \mathbf{r}'(t) = \langle 1, -1, t\rangle$, $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{2 + t^2}}\langle 1, -1, t\rangle$,

$$\begin{aligned}\mathbf{T}'(t) &= -\tfrac{1}{2}(2 + t^2)^{-3/2}(2t)\langle 1, -1, t\rangle + (2 + t^2)^{-1/2}\langle 0, 0, 1\rangle \\ &= -(2 + t^2)^{-3/2}\left[t\langle 1, -1, t\rangle - (2 + t^2)\langle 0, 0, 1\rangle\right] = \frac{-1}{(2 + t^2)^{3/2}}\langle t, -t, -2\rangle\end{aligned}$$

A normal vector for the osculating plane is $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$, but $\mathbf{r}'(t) = \langle 1, -1, t\rangle$ is parallel to $\mathbf{T}(t)$ and $\langle t, -t, -2\rangle$ is parallel to $\mathbf{T}'(t)$ and hence parallel to $\mathbf{N}(t)$, so $\langle 1, -1, t\rangle \times \langle t, -t, -2\rangle = \left\langle t^2 + 2, t^2 + 2, 0\right\rangle$ is normal to the osculating plane for any $t$. All such vectors are parallel to $\langle 1, 1, 0\rangle$, so at any point $\left(t + 2, 1 - t, \frac{1}{2}t^2\right)$ on the curve, an equation for the osculating plane is $1[x - (t + 2)] + 1[y - (1 - t)] + 0\left(z - \frac{1}{2}t^2\right) = 0$ or $x + y = 3$. Because the osculating plane at every point on the curve is the same, we can conclude that the curve itself lies in that same plane. In fact, we can easily verify that the parametric equations of the curve satisfy $x + y = 3$.

**61.** $\mathbf{r}(t) = \langle e^t \cos t, e^t \sin t, e^t \rangle \quad \Rightarrow \quad \mathbf{r}'(t) = \langle e^t(\cos t - \sin t), e^t(\cos t + \sin t), e^t \rangle$ so

$$|\mathbf{r}'(t)| = \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\cos t + \sin t)^2 + e^{2t}}$$
$$= \sqrt{e^{2t}\left[2(\cos^2 t + \sin^2 t) - 2\cos t \sin t + 2\cos t \sin t + 1\right]} = \sqrt{3e^{2t}} = \sqrt{3}\, e^t$$

and $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{3}\, e^t} \langle e^t(\cos t - \sin t), e^t(\cos t + \sin t), e^t \rangle = \dfrac{1}{\sqrt{3}} \langle \cos t - \sin t, \cos t + \sin t, 1 \rangle$. The vector $\mathbf{k} = \langle 0, 0, 1 \rangle$ is parallel to the $z$-axis, so for any $t$, the angle $\alpha$ between $\mathbf{T}(t)$ and the $z$-axis is given by

$$\cos\alpha = \frac{\mathbf{T}(t) \cdot \mathbf{k}}{|\mathbf{T}(t)|\ |\mathbf{k}|} = \frac{\frac{1}{\sqrt{3}} \langle \cos t - \sin t, \cos t + \sin t, 1 \rangle \cdot \langle 0, 0, 1 \rangle}{\frac{1}{\sqrt{3}} \sqrt{(\cos t - \sin t)^2 + (\cos t + \sin t)^2 + 1}\ \sqrt{1}} = \frac{1}{\sqrt{2(\cos^2 t + \sin^2 t) + 1}} = \frac{1}{\sqrt{3}}.$$ Thus, the angle is constant; specifically, $\alpha = \cos^{-1}(1/\sqrt{3}) \approx 54.7^\circ$.

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{(1/\sqrt{3}\,) \langle -\sin t - \cos t, -\sin t + \cos t, 0 \rangle}{(1/\sqrt{3}\,)\sqrt{2\left(\sin^2 t + \cos^2 t\right)}} = \frac{1}{\sqrt{2}} \langle -\sin t - \cos t, -\sin t + \cos t, 0 \rangle,$$ and the angle $\beta$ made with the $z$-axis is given by $\cos\beta = \dfrac{\mathbf{N}(t) \cdot \mathbf{k}}{|\mathbf{N}(t)|\ |\mathbf{k}|} = 0$, so $\beta = 90^\circ$.

$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \frac{1}{\sqrt{6}} \langle \sin t - \cos t, -\sin t - \cos t, 2 \rangle$ and the angle $\gamma$ made with the $z$-axis is given by

$$\cos\gamma = \frac{\mathbf{B}(t) \cdot \mathbf{k}}{|\mathbf{B}(t)|\ |\mathbf{k}|} = \frac{\frac{1}{\sqrt{6}} \langle \sin t - \cos t, -\sin t - \cos t, 2 \rangle \cdot \langle 0, 0, 1 \rangle}{\frac{1}{\sqrt{6}} \sqrt{(\sin t - \cos t)^2 + (-\sin t - \cos t)^2 + 4}\,\sqrt{1}} = \frac{2}{\sqrt{6}}$$ or equivalently $\dfrac{\sqrt{6}}{3}$. Again, the angle is constant, specifically, $\gamma = \cos^{-1}(2/\sqrt{6}) \approx 35.3^\circ$.

**62.** If vectors $\mathbf{T}$ and $\mathbf{B}$ lie in the rectifying plane, then $\mathbf{N}$ is a normal vector for the plane, as it is orthogonal to both $\mathbf{T}$ and $\mathbf{B}$. The point $\left(\sqrt{2}/2, \sqrt{2}/2, 1\right)$ corresponds to $t = \pi/4$, so we can take $\mathbf{T}'(\pi/4)$ as a normal vector for the plane [since it is parallel to $\mathbf{N}(\pi/4)$]. $\quad \mathbf{r}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \tan t\,\mathbf{k} \quad \Rightarrow \quad \mathbf{r}'(t) = \cos t\,\mathbf{i} - \sin t\,\mathbf{j} + \sec^2 t\,\mathbf{k}$ and

$|\mathbf{r}'(t)| = \sqrt{\cos^2 t + \sin^2 t + \sec^4 t} = \sqrt{1 + \sec^4 t}$. Then $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{1 + \sec^4 t}} \left(\cos t\,\mathbf{i} - \sin t\,\mathbf{j} + \sec^2 t\,\mathbf{k}\right)$.

By Formula 3 of Theorem 13.2.3,

$$\mathbf{T}'(t) = -\frac{2\sec^4 t \tan t}{(1 + \sec^4 t)^{3/2}} \left(\cos t\,\mathbf{i} - \sin t\,\mathbf{j} + \sec^2 t\,\mathbf{k}\right) + \frac{1}{\sqrt{1 + \sec^4 t}} \left(-\sin t\,\mathbf{i} - \cos t\,\mathbf{j} + 2\sec^2 t \tan t\,\mathbf{k}\right)$$ and

$$\begin{aligned}
\mathbf{T}'(\pi/4) &= -\frac{2(\sqrt{2}\,)^4(1)}{[1 + (\sqrt{2}\,)^4]^{3/2}} \left(\frac{\sqrt{2}}{2}\,\mathbf{i} - \frac{\sqrt{2}}{2}\,\mathbf{j} + (\sqrt{2}\,)^2\,\mathbf{k}\right) + \frac{1}{\sqrt{1 + (\sqrt{2}\,)^4}} \left(-\frac{\sqrt{2}}{2}\,\mathbf{i} - \frac{\sqrt{2}}{2}\,\mathbf{j} + 2(\sqrt{2}\,)^2(1)\,\mathbf{k}\right) \\
&= -\frac{8}{5\sqrt{5}} \left(\frac{\sqrt{2}}{2}\,\mathbf{i} - \frac{\sqrt{2}}{2}\,\mathbf{j} + 2\,\mathbf{k}\right) + \frac{1}{\sqrt{5}} \left(-\frac{\sqrt{2}}{2}\,\mathbf{i} - \frac{\sqrt{2}}{2}\,\mathbf{j} + 4\,\mathbf{k}\right) = -\frac{13\sqrt{2}}{10\sqrt{5}}\,\mathbf{i} + \frac{3\sqrt{2}}{10\sqrt{5}}\,\mathbf{j} + \frac{4}{5\sqrt{5}}\,\mathbf{k}
\end{aligned}$$

We can take the parallel vector $-13\sqrt{2}\,\mathbf{i} + 3\sqrt{2}\,\mathbf{j} + 8\,\mathbf{k}$ as a normal for the plane, so an equation for the plane is

$-13\sqrt{2}\left(x - \frac{\sqrt{2}}{2}\right) + 3\sqrt{2}\left(y - \frac{\sqrt{2}}{2}\right) + 8\,(z - 1) = 0$ or $-13\sqrt{2}\,x + 3\sqrt{2}\,y + 8z = -2$ or $13x - 3y - 4\sqrt{2}\,z = \sqrt{2}$.

**63.** $\kappa = \left|\dfrac{d\mathbf{T}}{ds}\right| = \left|\dfrac{d\mathbf{T}/dt}{ds/dt}\right| = \dfrac{|d\mathbf{T}/dt|}{ds/dt}$ and $\mathbf{N} = \dfrac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$, so $\kappa\mathbf{N} = \dfrac{\left|\dfrac{d\mathbf{T}}{dt}\right| \dfrac{d\mathbf{T}}{dt}}{\left|\dfrac{d\mathbf{T}}{dt}\right| \dfrac{ds}{dt}} = \dfrac{d\mathbf{T}/dt}{ds/dt} = \dfrac{d\mathbf{T}}{ds}$ by the Chain Rule.

**64.** For a plane curve, $\mathbf{T} = |\mathbf{T}|\cos\phi\,\mathbf{i} + |\mathbf{T}|\sin\phi\,\mathbf{j} = \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}$. Then

$\dfrac{d\mathbf{T}}{ds} = \left(\dfrac{d\mathbf{T}}{d\phi}\right)\left(\dfrac{d\phi}{ds}\right) = (-\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j})\left(\dfrac{d\phi}{ds}\right)$ and $\left|\dfrac{d\mathbf{T}}{ds}\right| = |-\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}|\left|\dfrac{d\phi}{ds}\right| = \left|\dfrac{d\phi}{ds}\right|$. Hence for a plane curve, the curvature is $\kappa = |d\phi/ds|$.

**65.** (a) $|\mathbf{B}| = 1 \;\Rightarrow\; \mathbf{B}\cdot\mathbf{B} = 1 \;\Rightarrow\; \dfrac{d}{ds}(\mathbf{B}\cdot\mathbf{B}) = 0 \;\Rightarrow\; 2\,\dfrac{d\mathbf{B}}{ds}\cdot\mathbf{B} = 0 \;\Rightarrow\; \dfrac{d\mathbf{B}}{ds} \perp \mathbf{B}$.

This shows that $d\mathbf{B}/ds$ is perpendicular to $\mathbf{B}$. Alternatively, note that this is a direct result of Theorem 13.2.4.

(b) $\mathbf{B} = \mathbf{T}\times\mathbf{N} \;\Rightarrow$

$$\begin{aligned}
\frac{d\mathbf{B}}{ds} &= \frac{d}{ds}(\mathbf{T}\times\mathbf{N}) = \frac{d}{dt}(\mathbf{T}\times\mathbf{N})\,\frac{1}{ds/dt} = \frac{d}{dt}(\mathbf{T}\times\mathbf{N})\,\frac{1}{|\mathbf{r}'(t)|} && \text{[by Formula 7]}\\
&= [(\mathbf{T}'\times\mathbf{N}) + (\mathbf{T}\times\mathbf{N}')]\,\frac{1}{|\mathbf{r}'(t)|} && \text{[by Formula 5 of Theorem 13.2.3]}\\
&= \left[\left(\mathbf{T}'\times\frac{\mathbf{T}'}{|\mathbf{T}'|}\right) + (\mathbf{T}\times\mathbf{N}')\right]\frac{1}{|\mathbf{r}'(t)|}\\
&= [\mathbf{0} + (\mathbf{T}\times\mathbf{N}')]\,\frac{1}{|\mathbf{r}'(t)|} = \frac{\mathbf{T}\times\mathbf{N}'}{|\mathbf{r}'(t)|} && [\mathbf{a}\times c\mathbf{a} = \mathbf{0}]\\
\Rightarrow\; \frac{d\mathbf{B}}{ds} &\perp \mathbf{T} && \text{[by Theorem 12.4.8]}
\end{aligned}$$

(c) $\mathbf{B} = \mathbf{T}\times\mathbf{N} \;\Rightarrow\; \mathbf{B}\perp\mathbf{T}$ and $\mathbf{B}\perp\mathbf{N}$. Since $\mathbf{T}\perp\mathbf{N}$, $\mathbf{B}$, $\mathbf{T}$, and $\mathbf{N}$ form an orthogonal set of vectors in the three-dimensional space $\mathbb{R}^3$. From parts (a) and (b), $d\mathbf{B}/ds$ is perpendicular to both $\mathbf{B}$ and $\mathbf{T}$, so $d\mathbf{B}/ds$ is parallel to $\mathbf{N}$.

**66.** We need to find $\mathbf{T}(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$ in terms of $t$. $\mathbf{r}(t) = \langle\sin t, 3t, \cos t\rangle \;\Rightarrow\; \mathbf{r}'(t) = \langle\cos t, 3, -\sin t\rangle \;\Rightarrow$

$|\mathbf{r}'(t)| = \sqrt{\cos^2 t + 3^2 + \sin^2 t} = \sqrt{10}$. Then $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{10}}\langle\cos t, 3, -\sin t\rangle \;\Rightarrow$

$\mathbf{T}'(t) = \dfrac{1}{\sqrt{10}}\langle-\sin t, 0, -\cos t\rangle$, and $|\mathbf{T}'(t)| = \dfrac{1}{\sqrt{10}}\sqrt{\sin^2 t + 0^2 + \cos^2 t} = \dfrac{1}{\sqrt{10}}$.

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \dfrac{\sqrt{10}}{1}\cdot\dfrac{1}{\sqrt{10}}\langle-\sin t, 0, -\cos t\rangle = \langle-\sin t, 0, -\cos t\rangle$.

Then

$$\begin{aligned}
\mathbf{B}(t) = \mathbf{T}(t)\times\mathbf{N}(t) &= \frac{1}{\sqrt{10}}\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \cos t & 3 & -\sin t\\ -\sin t & 0 & -\cos t\end{vmatrix}\\
&= \frac{1}{\sqrt{10}}\left[-3\cos t\,\mathbf{i} - (-\cos^2 t - \sin^2 t)\,\mathbf{j} + 3\sin t\,\mathbf{k}\right] = \frac{1}{\sqrt{10}}\langle-3\cos t, 1, 3\sin t\rangle
\end{aligned}$$

$$\mathbf{B}'(t) = \frac{1}{\sqrt{10}}\langle 3\sin t, 0, 3\cos t\rangle \;\Rightarrow\; \mathbf{B}'(\pi/2) = \frac{1}{\sqrt{10}}\langle 3, 0, 0\rangle \quad\text{and}\quad \mathbf{N}(\pi/2) = \langle-1, 0, 0\rangle$$

Thus, the torsion is $\tau(t) = -\dfrac{\mathbf{B}'(\pi/2)\cdot\mathbf{N}(\pi/2)}{|\mathbf{r}'(\pi/2)|} = -\dfrac{-3/\sqrt{10}}{\sqrt{10}} = \dfrac{3}{10}$.

**67.** We need to find $\mathbf{T}(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$ in terms of $t$. $\mathbf{r}(t) = \left\langle \frac{1}{2}t^2, 2t, t \right\rangle \Rightarrow \mathbf{r}'(t) = \langle t, 2, 1 \rangle \Rightarrow$

$|\mathbf{r}'(t)| = \sqrt{t^2 + 2^2 + 1^2} = \sqrt{t^2 + 5}$. Then $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{t^2 + 5}} \langle t, 2, 1 \rangle \Rightarrow$

$$\begin{aligned}\mathbf{T}'(t) &= \frac{1}{\sqrt{t^2+5}} \langle 1, 0, 0 \rangle - \frac{t}{(t^2+5)^{3/2}} \langle t, 2, 1 \rangle = \frac{1}{(t^2+5)^{3/2}} \left[ (t^2+5) \langle 1, 0, 0 \rangle - t \langle t, 2, 1 \rangle \right] \\ &= \frac{1}{(t^2+5)^{3/2}} \langle 5, -2t, -t \rangle, \text{ and}\end{aligned}$$

$$|\mathbf{T}'(t)| = \frac{1}{(t^2+5)^{3/2}} \sqrt{25 + 5t^2} = \frac{\sqrt{5}}{t^2+5}.$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{t^2+5}{\sqrt{5}} \cdot \frac{1}{(t^2+5)^{3/2}} \langle 5, -2t, -t \rangle = \frac{1}{\sqrt{5}\sqrt{t^2+5}} \langle 5, -2t, -t \rangle.$$

Then $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \dfrac{1}{\sqrt{t^2+5}} \cdot \dfrac{1}{\sqrt{5}\sqrt{t^2+5}} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & 2 & 1 \\ 5 & -2t & -t \end{vmatrix}$

$$= \frac{1}{\sqrt{5}(t^2+5)} \left[ (-2t + 2t)\mathbf{i} - (-t^2 - 5)\mathbf{j} + (-2t^2 - 10)\mathbf{k} \right] = \left\langle 0, \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \right\rangle$$

Since $\mathbf{B}(t)$ is constant, $\mathbf{B}'(t) = \mathbf{0}$, and $\mathbf{B}'(1) \cdot \mathbf{N}(1) = 0$. Thus, the torsion is $\tau(1) = -\dfrac{\mathbf{B}'(1) \cdot \mathbf{N}(1)}{|\mathbf{r}'(1)|} = -\dfrac{0}{\sqrt{6}} = 0$.

**68.** $\mathbf{r} = \left\langle t, \frac{1}{2}t^2, \frac{1}{3}t^3 \right\rangle \Rightarrow \mathbf{r}' = \left\langle 1, t, t^2 \right\rangle \Rightarrow \mathbf{r}'' = \langle 0, 1, 2t \rangle \Rightarrow \mathbf{r}''' = \langle 0, 0, 2 \rangle$.

$\mathbf{r}' \times \mathbf{r}'' = \left\langle t^2, -2t, 1 \right\rangle$. By Theorem 15, the torsion is $\tau = \dfrac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2} = \dfrac{\left\langle t^2, -2t, 1 \right\rangle \cdot \langle 0, 0, 2 \rangle}{\left( \sqrt{(t^2)^2 + (-2t)^2 + 1^2} \right)^2} = \dfrac{2}{t^4 + 4t^2 + 1}$.

The torsion at $t = 0$ is $\tau(0) = \frac{2}{1} = 2$.

**69.** $\mathbf{r} = \left\langle e^t, e^{-t}, t \right\rangle \Rightarrow \mathbf{r}' = \left\langle e^t, -e^{-t}, 1 \right\rangle \Rightarrow \mathbf{r}'' = \left\langle e^t, e^{-t}, 0 \right\rangle \Rightarrow \mathbf{r}''' = \left\langle e^t, -e^{-t}, 0 \right\rangle$.

$\mathbf{r}' \times \mathbf{r}'' = \left\langle -e^{-t}, e^t, 2 \right\rangle$. By Theorem 15, the torsion is

$$\tau = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2} = \frac{\left\langle -e^{-t}, e^t, 2 \right\rangle \cdot \left\langle e^t, -e^{-t}, 0 \right\rangle}{\left( \sqrt{(-e^{-t})^2 + (e^t)^2 + 2^2} \right)^2} = \frac{-1 - 1 + 0}{e^{-2t} + e^{2t} + 4} = \frac{-2}{e^{2t} + e^{-2t} + 4}$$

The torsion at $t = 0$ is $\tau(0) = \frac{-2}{1+1+4} = -\frac{1}{3}$.

**70.** $\mathbf{r} = \langle \cos t, \sin t, \sin t \rangle \Rightarrow \mathbf{r}' = \langle -\sin t, \cos t, \cos t \rangle \Rightarrow \mathbf{r}'' = \langle -\cos t, -\sin t, -\sin t \rangle \Rightarrow$

$\mathbf{r}''' = \langle \sin t, -\cos t, -\cos t \rangle$. $\mathbf{r}' \times \mathbf{r}'' = \langle 0, -1, 1 \rangle$. By Theorem 15, the torsion is

$$\tau = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2} = \frac{\langle 0, -1, 1 \rangle \cdot \langle \sin t, -\cos t, -\cos t \rangle}{\left( \sqrt{0^2 + (-1)^2 + 1^2} \right)^2} = \frac{0 + \cos t - \cos t}{1 + 1} = 0$$

The torsion at $t = 0$, or any value of $t$, is 0.

**71.** $\mathbf{N} = \mathbf{B} \times \mathbf{T} \quad \Rightarrow$

$$\frac{d\mathbf{N}}{ds} = \frac{d}{ds}(\mathbf{B} \times \mathbf{T}) = \frac{d\mathbf{B}}{ds} \times \mathbf{T} + \mathbf{B} \times \frac{d\mathbf{T}}{ds} \qquad \text{[by Formula 5 of Theorem 13.2.3]}$$

$$= -\tau\mathbf{N} \times \mathbf{T} + \mathbf{B} \times \kappa\mathbf{N} \qquad \text{[by Formulas 3 and 1]}$$

$$= -\tau(\mathbf{N} \times \mathbf{T}) + \kappa(\mathbf{B} \times \mathbf{N}) \qquad \text{[by Property 2 of Theorem 12.4.11]}$$

But $\mathbf{B} \times \mathbf{N} = \mathbf{B} \times (\mathbf{B} \times \mathbf{T}) = (\mathbf{B} \cdot \mathbf{T})\,\mathbf{B} - (\mathbf{B} \cdot \mathbf{B})\,\mathbf{T}$ [by Property 6 of Theorem 12.4.11]

$$= 0 - \mathbf{T} = -\mathbf{T} \quad \Rightarrow$$

$d\mathbf{N}/ds = \tau(\mathbf{T} \times \mathbf{N}) - \kappa\mathbf{T} = -\kappa\,\mathbf{T} + \tau\,\mathbf{B}$.

**72.** (a) $\mathbf{r}' = s'\,\mathbf{T} \quad \Rightarrow \quad \mathbf{r}'' = s''\,\mathbf{T} + s'\,\mathbf{T}' = s''\,\mathbf{T} + s'\,\dfrac{d\mathbf{T}}{ds}s' = s''\,\mathbf{T} + \kappa(s')^2\,\mathbf{N}$ by the first Frenet-Serret formula.

(b) Using part (a), we have

$$\mathbf{r}' \times \mathbf{r}'' = (s'\,\mathbf{T}) \times [s''\,\mathbf{T} + \kappa(s')^2\,\mathbf{N}]$$

$$= [(s'\,\mathbf{T}) \times (s''\,\mathbf{T})] + \left[(s'\mathbf{T}) \times (\kappa(s')^2\,\mathbf{N})\right] \qquad \text{[by Property 3 of Theorem 12.4.11 ]}$$

$$= (s's'')(\mathbf{T} \times \mathbf{T}) + \kappa(s')^3(\mathbf{T} \times \mathbf{N}) = \mathbf{0} + \kappa(s')^3\,\mathbf{B} = \kappa(s')^3\,\mathbf{B}$$

(c) Using part (a), we have

$$\mathbf{r}''' = [s''\,\mathbf{T} + \kappa(s')^2\,\mathbf{N}]' = s'''\,\mathbf{T} + s''\,\mathbf{T}' + \kappa'(s')^2\,\mathbf{N} + 2\kappa s's''\,\mathbf{N} + \kappa(s')^2\,\mathbf{N}'$$

$$= s'''\,\mathbf{T} + s''\frac{d\mathbf{T}}{ds}\,s' + \kappa'(s')^2\,\mathbf{N} + 2\kappa s's''\,\mathbf{N} + \kappa(s')^2\,\frac{d\mathbf{N}}{ds}\,s'$$

$$= s'''\,\mathbf{T} + s''s'\kappa\,\mathbf{N} + \kappa'(s')^2\,\mathbf{N} + 2\kappa s's''\,\mathbf{N} + \kappa(s')^3(-\kappa\,\mathbf{T} + \tau\,\mathbf{B}) \qquad \text{[by Formulas 1 and 2]}$$

$$= [s''' - \kappa^2(s')^3]\,\mathbf{T} + [3\kappa s's'' + \kappa'(s')^2]\,\mathbf{N} + \kappa\tau(s')^3\,\mathbf{B}$$

(d) Using parts (b) and (c) and the facts that $\mathbf{B} \cdot \mathbf{T} = 0$, $\mathbf{B} \cdot \mathbf{N} = 0$, and $\mathbf{B} \cdot \mathbf{B} = 1$, we get

$$\frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2} = \frac{\kappa(s')^3\,\mathbf{B} \cdot \left\{[s''' - \kappa^2(s')^3]\,\mathbf{T} + [3\kappa s's'' + \kappa'(s')^2]\,\mathbf{N} + \kappa\tau(s')^3\,\mathbf{B}\right\}}{|\kappa(s')^3\,\mathbf{B}|^2} = \frac{\kappa(s')^3\kappa\tau(s')^3}{[\kappa(s')^3]^2} = \tau.$$

**73.** First we find the quantities required to compute $\kappa$:

$$\mathbf{r}'(t) = \langle -a\sin t, a\cos t, b\rangle \quad \Rightarrow \quad \mathbf{r}''(t) = \langle -a\cos t, -a\sin t, 0\rangle \quad \Rightarrow \quad \mathbf{r}'''(t) = \langle a\sin t, -a\cos t, 0\rangle$$

$$|\mathbf{r}'(t)| = \sqrt{(-a\sin t)^2 + (a\cos t)^2 + b^2} = \sqrt{a^2 + b^2}$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \end{vmatrix} = ab\sin t\,\mathbf{i} - ab\cos t\,\mathbf{j} + a^2\,\mathbf{k}$$

$$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{(ab\sin t)^2 + (-ab\cos t)^2 + (a^2)^2} = \sqrt{a^2b^2 + a^4} = a\sqrt{a^2 + b^2}$$

$$(\mathbf{r}'(t) \times \mathbf{r}''(t)) \cdot \mathbf{r}'''(t) = (ab\sin t)(a\sin t) + (-ab\cos t)(-a\cos t) + (a^2)(0) = a^2 b$$

[continued]

By Theorem 10, $\kappa(t) = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \dfrac{a\sqrt{a^2+b^2}}{\left(\sqrt{a^2+b^2}\right)^3} = \dfrac{a}{a^2+b^2}$, which is a constant.

By Theorem 15, the torsion is

$$\tau = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2} = \frac{\langle ab\sin t, -ab\cos t, a^2 \rangle \cdot \langle a\sin t, -a\cos t, 0\rangle}{\left(\sqrt{a^2b^2+a^4}\right)^2} = \frac{a^2 b}{a^2(a^2+b^2)} = \frac{b}{a^2+b^2},$$ which is also a constant.

**74.** $\mathbf{r} = \langle \sinh t, \cosh t, t\rangle \quad\Rightarrow\quad \mathbf{r}' = \langle \cosh t, \sinh t, 1\rangle$, $\mathbf{r}'' = \langle \sinh t, \cosh t, 0\rangle$, $\mathbf{r}''' = \langle \cosh t, \sinh t, 0\rangle \quad\Rightarrow$

$\mathbf{r}' \times \mathbf{r}'' = \left\langle -\cosh t, \sinh t, \cosh^2 t - \sinh^2 t\right\rangle = \langle -\cosh t, \sinh t, 1\rangle \quad\Rightarrow$

$$\kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3} = \frac{|\langle -\cosh t, \sinh t, 1\rangle|}{|\langle \cosh t, \sinh t, 1\rangle|^3} = \frac{\sqrt{\cosh^2 t + \sinh^2 t + 1}}{\left(\cosh^2 t + \sinh^2 t + 1\right)^{3/2}} = \frac{1}{\cosh^2 t + \sinh^2 t + 1} = \frac{1}{2\cosh^2 t},$$

$$\tau = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2} = \frac{\langle -\cosh t, \sinh t, 1\rangle \cdot \langle \cosh t, \sinh t, 0\rangle}{\cosh^2 t + \sinh^2 t + 1} = \frac{-\cosh^2 t + \sinh^2 t}{2\cosh^2 t} = \frac{-1}{2\cosh^2 t}$$

So at the point $(0, 1, 0)$, $t = 0$, and $\kappa = \frac{1}{2}$ and $\tau = -\frac{1}{2}$.

**75.** (a) At any point $P$ on $C$ where $\kappa(t) \neq 0$, the circle of curvature of $C$ at $P$ has center a distance $1/\kappa(t)$ from $P$ in the direction of the unit normal vector $\mathbf{N}(t)$. The position vector of $P$ is $\mathbf{r}(t)$, so we get a position vector for the center of curvature by adding $\dfrac{1}{\kappa(t)}\mathbf{N}(t)$ to $\mathbf{r}(t)$: $\mathbf{r}_e(t) = \mathbf{r}(t) + \dfrac{1}{\kappa(t)}\mathbf{N}(t)$, $\kappa(t) \neq 0$.

(b) $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k} \quad\Rightarrow\quad \mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k} \quad\Rightarrow\quad |\mathbf{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1^2} = \sqrt{2}$.

Then $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = -\dfrac{\sin t}{\sqrt{2}}\,\mathbf{i} + \dfrac{\cos t}{\sqrt{2}}\,\mathbf{j} + \dfrac{1}{\sqrt{2}}\,\mathbf{k} \quad\Rightarrow\quad \mathbf{T}'(t) = -\dfrac{\cos t}{\sqrt{2}}\,\mathbf{i} - \dfrac{\sin t}{\sqrt{2}}\,\mathbf{j} \quad\Rightarrow\quad |\mathbf{T}'(t)| = \dfrac{1}{\sqrt{2}}$.

So $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j}$ and $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{1/\sqrt{2}}{\sqrt{2}} = \dfrac{1}{2}$. Thus,

$$\mathbf{r}_e(t) = \mathbf{r}(t) + \frac{1}{\kappa(t)}\mathbf{N}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k} + 2(-\cos t\,\mathbf{i} - \sin t\,\mathbf{j}) = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j} + t\,\mathbf{k}.$$

(c) The parabola $y = x^2$ can be parameterized $x = t$, $y = t^2$, which gives the corresponding vector equation

$\mathbf{r}(t) = \left\langle t, t^2\right\rangle \quad\Rightarrow\quad \mathbf{r}'(t) = \langle 1, 2t\rangle \quad\Rightarrow\quad |\mathbf{r}'(t)| = \sqrt{1+4t^2}$. Then $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{1+4t^2}}\langle 1, 2t\rangle \quad\Rightarrow$

$$\begin{aligned}
\mathbf{T}'(t) &= \frac{1}{\sqrt{1+4t^2}}\langle 0, 2\rangle - \frac{4t}{(1+4t^2)^{3/2}}\langle 1, 2t\rangle = \frac{1}{(1+4t^2)^{3/2}}\left[\left(1+4t^2\right)\langle 0,2\rangle - 4t\,\langle 1, 2t\rangle\right] \\
&= \frac{1}{(1+4t^2)^{3/2}}\langle -4t, 2\rangle \quad\Rightarrow
\end{aligned}$$

$$|\mathbf{T}'(t)| = \frac{1}{(1+4t^2)^{3/2}}\sqrt{16t^2+4} = \frac{2}{1+4t^2}.$$

So $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \dfrac{1+4t^2}{2} \cdot \dfrac{1}{(1+4t^2)^{3/2}}\langle -4t, 2\rangle = \dfrac{1}{\sqrt{1+4t^2}}\langle -2t, 1\rangle$ and

$\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{2/(1+4t^2)}{\sqrt{1+4t^2}} = \dfrac{2}{(1+4t^2)^{3/2}}$. Thus,

$$\mathbf{r}_e(t) = \mathbf{r}(t) + \frac{1}{\kappa(t)}\mathbf{N}(t) = \left\langle t, t^2 \right\rangle + \frac{(1+4t^2)^{3/2}}{2} \cdot \frac{1}{\sqrt{1+4t^2}} \left\langle -2t, 1 \right\rangle$$
$$= \left\langle t, t^2 \right\rangle + (1+4t^2)\left\langle -t, \tfrac{1}{2} \right\rangle = \left\langle -4t^3, \tfrac{1}{2} + 3t^2 \right\rangle$$

To obtain a function form of the answer, note that $x = -4t^3 \quad \Rightarrow \quad t = (-x/4)^{1/3}$, so

$y_e = \frac{1}{2} + 3t^2 = \frac{1}{2} + 3[(-x/4)^{1/3}]^2 = \frac{1}{2} + 3(x/4)^{2/3}$.

**76.** (a) If $C$ is planar, then it lies in a plane that we can express in the form $ax + by + cz = d$, where $a$, $b$, $c$, and $d$ are not all zero. (See Equation 12.5.8.) For any $t$, the point $(x(t), y(t), z(t))$ lies on the curve and hence on the plane, so the equation $ax(t) + by(t) + cz(t) = d$ must be satisfied.

Conversely, if there exist scalars $a$, $b$, $c$, and $d$, not all zero, such that $ax(t) + by(t) + cz(t) = d$ for all $t$, then each point $(x(t), y(t), z(t))$ on $C$ satisfies the equation $ax + by + cz = d$. By Exercise 12.5.83, this equation represents a plane, and hence $C$ is planar.

(b) By part (a) there exist scalars $a$, $b$, $c$, and $d$, not all zero, such that for all $t$, $ax(t) + by(t) + cz(t) = d \quad \Rightarrow$ $\langle a, b, c \rangle \cdot \mathbf{r}(t) = d$. In addition, for all $t$, $ax'(t) + by'(t) + cz'(t) = 0 \quad \Rightarrow \quad \langle a, b, c \rangle \cdot \mathbf{r}'(t) = 0 \quad \Rightarrow \quad \mathbf{r}'$ is perpendicular to the normal vector $\mathbf{n} = \langle a, b, c \rangle$ of the plane containing $C \quad \Rightarrow \quad \mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$ is also perpendicular to $\mathbf{n}$.

Similarly, $\langle a, b, c \rangle \cdot \mathbf{r}''(t) = 0 \quad \Rightarrow \quad \mathbf{r}''$ is perpendicular to $\mathbf{n} \quad \Rightarrow$

$\mathbf{T}'(t) \cdot \mathbf{n} = \dfrac{d}{dt}\left(\dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}\right) \cdot \mathbf{n} = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \cdot \mathbf{n} + \dfrac{d}{dt}\left(\dfrac{1}{|\mathbf{r}'(t)|}\right)\mathbf{r}'(t) \cdot \mathbf{n} = 0 + 0 = 0$. So then $\mathbf{N} \cdot \mathbf{n} = \dfrac{\mathbf{T}'(t) \cdot \mathbf{n}}{|\mathbf{T}'(t)|} = 0$, and hence $\mathbf{N}$ is perpendicular to $\mathbf{n}$. So, since $\mathbf{n} = \langle a, b, c \rangle$ is perpendicular to both $\mathbf{T}$ and $\mathbf{N}$, it follows that $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ is parallel to $\mathbf{n}$ and hence is normal to the plane containing $C$.

(c) By part (b), $\mathbf{B}$ is normal to the plane containing $C$ and so $\mathbf{B}(t) = \dfrac{\langle a, b, c \rangle}{\sqrt{a^2 + b^2 + c^2}}$ for all $t$, that is, $\mathbf{B}$ is a constant vector.

Therefore, $\mathbf{B}'(t) = \mathbf{0}$ and $\tau(t) = -\dfrac{\mathbf{B}'(t) \cdot \mathbf{N}(t)}{|\mathbf{r}'(t)|} = -\dfrac{0}{|\mathbf{r}'(t)|} = 0$ for all $t$.

(d) The projection of the curve $\mathbf{r}(t) = \left\langle t, 2t, t^2 \right\rangle$ in the $xy$-plane is the curve $\mathbf{r}(t) = \langle t, 2t, 0 \rangle$, which is the line with corresponding parametric equations $x = t,\ y = 2t,\ z = 0$. Therefore, the equation of the plane containing $\mathbf{r}(t)$ is $y = 2x$, or $2x - y = 0$, with normal vector $\mathbf{n} = \langle 2, -1, 0 \rangle$. By part (b), $\mathbf{B} = \frac{1}{\sqrt{5}} \langle 2, -1, 0 \rangle$.

**77.** For one helix, the vector equation is $\mathbf{r}(t) = \langle 10 \cos t, 10 \sin t, 34t/(2\pi) \rangle$ (measuring in angstroms), because the radius of each helix is 10 angstroms, and $z$ increases by 34 angstroms for each increase of $2\pi$ in $t$. Using the arc length formula, letting $t$ go from 0 to $2.9 \times 10^8 \times 2\pi$, we find the approximate length of each helix to be

$$L = \int_0^{2.9 \times 10^8 \times 2\pi} |\mathbf{r}'(t)|\, dt = \int_0^{2.9 \times 10^8 \times 2\pi} \sqrt{(-10 \sin t)^2 + (10 \cos t)^2 + \left(\tfrac{34}{2\pi}\right)^2}\, dt = \sqrt{100 + \left(\tfrac{34}{2\pi}\right)^2}\, t \Big]_0^{2.9 \times 10^8 \times 2\pi}$$
$$= 2.9 \times 10^8 \times 2\pi \sqrt{100 + \left(\tfrac{34}{2\pi}\right)^2} \approx 2.07 \times 10^{10} \text{ Å} \text{ — more than two meters!}$$

**78.** (a) For the function $F(x) = \begin{cases} 0 & \text{if } x < 0 \\ P(x) & \text{if } 0 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$ to be continuous, we must have $P(0) = 0$ and $P(1) = 1$.

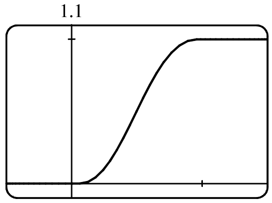
For $F'$ to be continuous, we must have $P'(0) = P'(1) = 0$. The curvature of the curve $y = F(x)$ at the point $(x, F(x))$ is $\kappa(x) = \dfrac{|F''(x)|}{\left(1 + [F'(x)]^2\right)^{3/2}}$. For $\kappa(x)$ to be continuous, we must have $P''(0) = P''(1) = 0$.

Write $P(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$. Then $P'(x) = 5ax^4 + 4bx^3 + 3cx^2 + 2dx + e$ and $P''(x) = 20ax^3 + 12bx^2 + 6cx + 2d$. Our six conditions are:

$$P(0) = 0 \quad \Rightarrow \quad f = 0 \quad \textbf{(1)} \qquad\qquad P(1) = 1 \quad \Rightarrow \quad a + b + c + d + e + f = 1 \quad \textbf{(2)}$$
$$P'(0) = 0 \quad \Rightarrow \quad e = 0 \quad \textbf{(3)} \qquad\qquad P'(1) = 0 \quad \Rightarrow \quad 5a + 4b + 3c + 2d + e = 0 \quad \textbf{(4)}$$
$$P''(0) = 0 \quad \Rightarrow \quad d = 0 \quad \textbf{(5)} \qquad\qquad P''(1) = 0 \quad \Rightarrow \quad 20a + 12b + 6c + 2d = 0 \quad \textbf{(6)}$$

From **(1)**, **(3)**, and **(5)**, we have $d = e = f = 0$. Thus **(2)**, **(4)** and **(6)** become **(7)** $a + b + c = 1$, **(8)** $5a + 4b + 3c = 0$, and **(9)** $10a + 6b + 3c = 0$. Subtracting **(8)** from **(9)** gives **(10)** $5a + 2b = 0$. Multiplying **(7)** by 3 and subtracting from **(8)** gives **(11)** $2a + b = -3$. Multiplying **(11)** by 2 and subtracting from **(10)** gives $a = 6$. By **(10)**, $b = -15$. By **(7)**, $c = 10$. Thus, $P(x) = 6x^5 - 15x^4 + 10x^3$.

(b)




## 13.4 Motion in Space: Velocity and Acceleration

**1.** (a) If $\mathbf{r}(t) = x(t)\,\mathbf{i} + y\,(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ is the position vector of the particle at time *t*, then the average velocity over the time interval $[0, 1]$ is

$$\mathbf{v}_{\text{ave}} = \frac{\mathbf{r}(1) - \mathbf{r}(0)}{1 - 0} = \frac{(4.5\,\mathbf{i} + 6.0\,\mathbf{j} + 3.0\,\mathbf{k}) - (2.7\,\mathbf{i} + 9.8\,\mathbf{j} + 3.7\,\mathbf{k})}{1} = 1.8\,\mathbf{i} - 3.8\,\mathbf{j} - 0.7\,\mathbf{k}$$

Similarly, over the other intervals we have

$$[0.5, 1]: \quad \mathbf{v}_{\text{ave}} = \frac{\mathbf{r}(1) - \mathbf{r}(0.5)}{1 - 0.5} = \frac{(4.5\,\mathbf{i} + 6.0\,\mathbf{j} + 3.0\,\mathbf{k}) - (3.5\,\mathbf{i} + 7.2\,\mathbf{j} + 3.3\,\mathbf{k})}{0.5} = 2.0\,\mathbf{i} - 2.4\,\mathbf{j} - 0.6\,\mathbf{k}$$

$$[1, 2]: \quad \mathbf{v}_{\text{ave}} = \frac{\mathbf{r}(2) - \mathbf{r}(1)}{2 - 1} = \frac{(7.3\,\mathbf{i} + 7.8\,\mathbf{j} + 2.7\,\mathbf{k}) - (4.5\,\mathbf{i} + 6.0\,\mathbf{j} + 3.0\,\mathbf{k})}{1} = 2.8\,\mathbf{i} + 1.8\,\mathbf{j} - 0.3\,\mathbf{k}$$

$$[1, 1.5]: \quad \mathbf{v}_{\text{ave}} = \frac{\mathbf{r}(1.5) - \mathbf{r}(1)}{1.5 - 1} = \frac{(5.9\,\mathbf{i} + 6.4\,\mathbf{j} + 2.8\,\mathbf{k}) - (4.5\,\mathbf{i} + 6.0\,\mathbf{j} + 3.0\,\mathbf{k})}{0.5} = 2.8\,\mathbf{i} + 0.8\,\mathbf{j} - 0.4\,\mathbf{k}$$

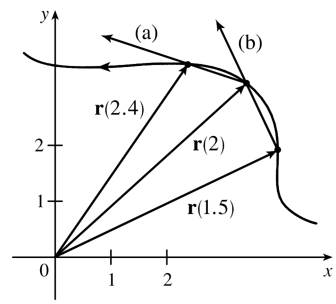
(b) We can estimate the velocity at $t = 1$ by averaging the average velocities over the time intervals $[0.5, 1]$ and $[1, 1.5]$:

$\mathbf{v}(1) \approx \frac{1}{2}[(2\,\mathbf{i} - 2.4\,\mathbf{j} - 0.6\,\mathbf{k}) + (2.8\,\mathbf{i} + 0.8\,\mathbf{j} - 0.4\,\mathbf{k})] = 2.4\,\mathbf{i} - 0.8\,\mathbf{j} - 0.5\,\mathbf{k}$. Then the speed is

$|\mathbf{v}(1)| \approx \sqrt{(2.4)^2 + (-0.8)^2 + (-0.5)^2} \approx 2.58$.

**2.** (a) The average velocity over $2 \le t \le 2.4$ is

$\dfrac{\mathbf{r}(2.4) - \mathbf{r}(2)}{2.4 - 2} = 2.5\,[\mathbf{r}(2.4) - \mathbf{r}(2)]$, so we sketch a vector in the same direction but 2.5 times the length of $[\mathbf{r}(2.4) - \mathbf{r}(2)]$.

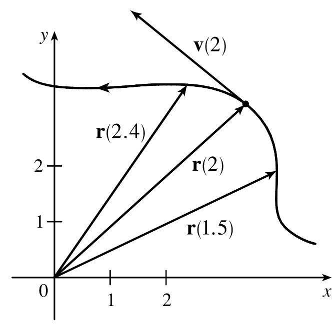



(b) The average velocity over $1.5 \le t \le 2$ is

$\dfrac{\mathbf{r}(2) - \mathbf{r}(1.5)}{2 - 1.5} = 2[\mathbf{r}(2) - \mathbf{r}(1.5)]$, so we sketch a vector in the same direction but twice the length of $[\mathbf{r}(2) - \mathbf{r}(1.5)]$.

(c) Using Equation 2, we have $\mathbf{v}(2) = \lim\limits_{h\to 0} \dfrac{\mathbf{r}(2+h) - \mathbf{r}(2)}{h}$.

(d) $\mathbf{v}(2)$ is tangent to the curve at $\mathbf{r}(2)$ and points in the direction of increasing $t$. Its length is the speed of the particle at $t = 2$. We can estimate the speed by averaging the lengths of the vectors found in parts (a) and (b) which represent the average speed over $2 \le t \le 2.4$ and $1.5 \le t \le 2$ respectively. Using the axes scale as a guide, we estimate the vectors to have lengths 2.8 and 2.7. Thus, we estimate the speed at $t = 2$ to be $|\mathbf{v}(2)| \approx \frac{1}{2}(2.8 + 2.7) = 2.75$ and we draw the velocity vector $\mathbf{v}(2)$ with this length.



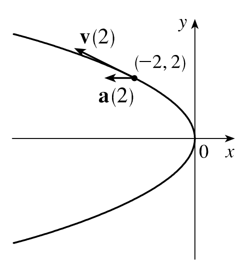

**3.** $\mathbf{r}(t) = \left\langle -\frac{1}{2}t^2, t \right\rangle \Rightarrow$ At $t = 2$:

$\mathbf{v}(t) = \mathbf{r}'(t) = \langle -t, 1 \rangle$ $\quad$ $\mathbf{v}(2) = \langle -2, 1 \rangle$

$\mathbf{a}(t) = \mathbf{r}''(t) = \langle -1, 0 \rangle$ $\quad$ $\mathbf{a}(2) = \langle -1, 0 \rangle$

$|\mathbf{v}(t)| = \sqrt{t^2 + 1}$

Notice that $x = -\frac{1}{2}y^2$, so the path is a parabola.



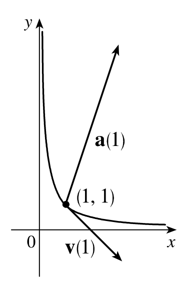

**4.** $\mathbf{r}(t) = \left\langle t^2, 1/t^2 \right\rangle \Rightarrow$ At $t = 1$:

$\mathbf{v}(t) = \mathbf{r}'(t) = \left\langle 2t, -2/t^3 \right\rangle$ $\quad$ $\mathbf{v}(1) = \langle 2, -2 \rangle$

$\mathbf{a}(t) = \mathbf{r}''(t) = \left\langle 2, 6/t^4 \right\rangle$ $\quad$ $\mathbf{a}(1) = \langle 2, 6 \rangle$

$|\mathbf{v}(t)| = \sqrt{4t^2 + 4/t^6} = 2\sqrt{t^2 + 1/t^6}$

Notice that $y = 1/x$ and $x > 0$, so the path is part of the hyperbola $y = 1/x$.



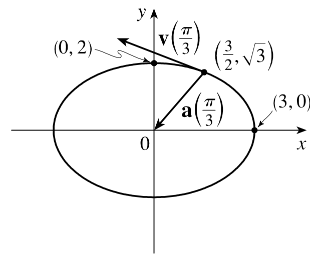
**5.** $\mathbf{r}(t) = 3\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} \quad \Rightarrow$ At $t = \pi/3$:

$\mathbf{v}(t) = -3\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$ $\qquad \mathbf{v}\left(\frac{\pi}{3}\right) = -\frac{3\sqrt{3}}{2}\,\mathbf{i} + \mathbf{j}$

$\mathbf{a}(t) = -3\cos t\,\mathbf{i} - 2\sin t\,\mathbf{j}$ $\qquad \mathbf{a}\left(\frac{\pi}{3}\right) = -\frac{3}{2}\,\mathbf{i} - \sqrt{3}\,\mathbf{j}$

$$|\mathbf{v}(t)| = \sqrt{9\sin^2 t + 4\cos^2 t} = \sqrt{5\sin^2 t + 4\sin^2 t + 4\cos^2 t} = \sqrt{4 + 5\sin^2 t}$$

Notice that $x^2/9 + y^2/4 = \sin^2 t + \cos^2 t = 1$, so the path is an ellipse.



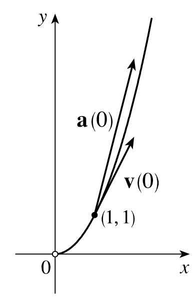

**6.** $\mathbf{r}(t) = e^t\,\mathbf{i} + e^{2t}\,\mathbf{j} \quad \Rightarrow$ At $t = 0$:

$\mathbf{v}(t) = e^t\,\mathbf{i} + 2e^{2t}\,\mathbf{j}$ $\qquad \mathbf{v}(0) = \mathbf{i} + 2\,\mathbf{j}$

$\mathbf{a}(t) = e^t\,\mathbf{i} + 4e^{2t}\,\mathbf{j}$ $\qquad \mathbf{a}(0) = \mathbf{i} + 4\,\mathbf{j}$

$$|\mathbf{v}(t)| = \sqrt{e^{2t} + 4e^{4t}} = e^t\sqrt{1 + 4e^{2t}}$$

Notice that $y = e^{2t} = \left(e^t\right)^2 = x^2$, so the particle travels along a parabola, but $x = e^t$, so $x > 0$.



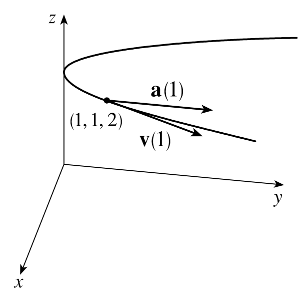

**7.** $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + 2\,\mathbf{k} \quad \Rightarrow$ At $t = 1$:

$\mathbf{v}(t) = \mathbf{i} + 2t\,\mathbf{j}$ $\qquad \mathbf{v}(1) = \mathbf{i} + 2\,\mathbf{j}$

$\mathbf{a}(t) = 2\,\mathbf{j}$ $\qquad \mathbf{a}(1) = 2\,\mathbf{j}$

$$|\mathbf{v}(t)| = \sqrt{1 + 4t^2}$$

Here $x = t$, $y = t^2 \quad \Rightarrow \quad y = x^2$ and $z = 2$, so the path of the particle is a parabola in the plane $z = 2$.



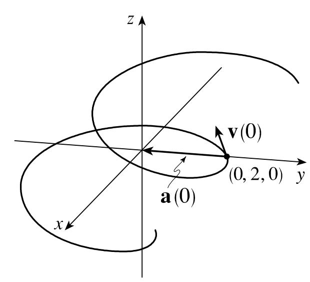

**8.** $\mathbf{r}(t) = t\,\mathbf{i} + 2\cos t\,\mathbf{j} + \sin t\,\mathbf{k} \quad \Rightarrow$ At $t = 0$:

$\mathbf{v}(t) = \mathbf{i} - 2\sin t\,\mathbf{j} + \cos t\,\mathbf{k}$ $\qquad \mathbf{v}(0) = \mathbf{i} + \mathbf{k}$

$\mathbf{a}(t) = -2\cos t\,\mathbf{j} - \sin t\,\mathbf{k}$ $\qquad \mathbf{a}(0) = -2\,\mathbf{j}$

$$|\mathbf{v}(t)| = \sqrt{1 + 4\sin^2 t + \cos^2 t} = \sqrt{2 + 3\sin^2 t}$$

Since $y^2/4 + z^2 = 1$, $x = t$, the path of the particle is an elliptical helix about the $x$-axis.




**9.** $\mathbf{r}(t) = \left\langle t^2 + t, t^2 - t, t^3 \right\rangle \quad \Rightarrow \quad \mathbf{v}(t) = \mathbf{r}'(t) = \left\langle 2t + 1, 2t - 1, 3t^2 \right\rangle, \quad \mathbf{a}(t) = \mathbf{v}'(t) = \langle 2, 2, 6t \rangle,$

$$|\mathbf{v}(t)| = \sqrt{(2t+1)^2 + (2t-1)^2 + (3t^2)^2} = \sqrt{9t^4 + 8t^2 + 2}.$$

**10.** $\mathbf{r}(t) = \langle 2\cos t, 3t, 2\sin t \rangle \quad \Rightarrow \quad \mathbf{v}(t) = \mathbf{r}'(t) = \langle -2\sin t, 3, 2\cos t \rangle, \quad \mathbf{a}(t) = \mathbf{v}'(t) = \langle -2\cos t, 0, -2\sin t \rangle,$

$$|\mathbf{v}(t)| = \sqrt{4\sin^2 t + 9 + 4\cos^2 t} = \sqrt{13}.$$

**11.** $\mathbf{r}(t) = \sqrt{2}\,t\,\mathbf{i} + e^t\,\mathbf{j} + e^{-t}\,\mathbf{k} \quad \Rightarrow \quad \mathbf{v}(t) = \mathbf{r}'(t) = \sqrt{2}\,\mathbf{i} + e^t\,\mathbf{j} - e^{-t}\,\mathbf{k}, \quad \mathbf{a}(t) = \mathbf{v}'(t) = e^t\,\mathbf{j} + e^{-t}\,\mathbf{k},$

$|\mathbf{v}(t)| = \sqrt{2 + e^{2t} + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}.$

**12.** $\mathbf{r}(t) = t^2\,\mathbf{i} + 2t\,\mathbf{j} + \ln t\,\mathbf{k} \quad \Rightarrow \quad \mathbf{v}(t) = \mathbf{r}'(t) = 2t\,\mathbf{i} + 2\,\mathbf{j} + (1/t)\,\mathbf{k}, \quad \mathbf{a}(t) = \mathbf{v}'(t) = 2\,\mathbf{i} - (1/t^2)\,\mathbf{k},$

$|\mathbf{v}(t)| = \sqrt{4t^2 + 4 + (1/t^2)} = \sqrt{[2t + (1/t)]^2} = |2t + (1/t)|.$

**13.** $\mathbf{r}(t) = e^t(\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}) = e^t\cos t\,\mathbf{i} + e^t\sin t\,\mathbf{j} + te^t\,\mathbf{k} \quad \Rightarrow$

$$\begin{aligned}\mathbf{v}(t) = \mathbf{r}'(t) &= [e^t(-\sin t) + (\cos t)e^t]\,\mathbf{i} + [e^t\cos t + (\sin t)e^t]\,\mathbf{j} + (te^t + e^t)\,\mathbf{k} \\ &= e^t[(\cos t - \sin t)\,\mathbf{i} + (\sin t + \cos t)\,\mathbf{j} + (t+1)\,\mathbf{k}]\end{aligned}$$

$$\begin{aligned}\mathbf{a}(t) = \mathbf{v}'(t) &= [e^t(-\sin t - \cos t) + (\cos t - \sin t)e^t]\,\mathbf{i} + [e^t(\cos t - \sin t) + (\sin t + \cos t)e^t]\,\mathbf{j} \\ &\qquad + [e^t \cdot 1 + (t+1)e^t]\,\mathbf{k} \\ &= e^t[-2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} + (t+2)\,\mathbf{k}]\end{aligned}$$

$$\begin{aligned}|\mathbf{v}(t)| &= \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\sin t + \cos t)^2 + e^{2t}(t+1)^2} \\ &= \sqrt{e^{2t}}\sqrt{\cos^2 t + \sin^2 t - 2\cos t\sin t + \sin^2 t + \cos^2 t + 2\sin t\cos t + t^2 + 2t + 1} \\ &= e^t\sqrt{t^2 + 2t + 3}\end{aligned}$$

**14.** $\mathbf{r}(t) = \langle t^2, \sin t - t\cos t, \cos t + t\sin t\rangle \quad \Rightarrow$

$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 2t, \cos t - (-t\sin t + \cos t), -\sin t + t\cos t + \sin t\rangle = \langle 2t, t\sin t, t\cos t\rangle,$

$\mathbf{a}(t) = \mathbf{v}'(t) = \langle 2, t\cos t + \sin t, -t\sin t + \cos t\rangle,$

$|\mathbf{v}(t)| = \sqrt{4t^2 + t^2\sin^2 t + t^2\cos^2 t} = \sqrt{4t^2 + t^2} = \sqrt{5t^2} = \sqrt{5}\,t \quad$ [since $t \geq 0$].

**15.** $\mathbf{a}(t) = 2\,\mathbf{i} + 2t\,\mathbf{k} \quad \Rightarrow \quad \mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int (2\,\mathbf{i} + 2t\,\mathbf{k})\,dt = 2t\,\mathbf{i} + t^2\,\mathbf{k} + \mathbf{C}$. Then $\mathbf{v}(0) = \mathbf{C}$ but we were given that $\mathbf{v}(0) = 3\,\mathbf{i} - \mathbf{j}$, so $\mathbf{C} = 3\,\mathbf{i} - \mathbf{j}$ and $\mathbf{v}(t) = 2t\,\mathbf{i} + t^2\,\mathbf{k} + 3\,\mathbf{i} - \mathbf{j} = (2t+3)\,\mathbf{i} - \mathbf{j} + t^2\,\mathbf{k}$.

$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int \left[(2t+3)\,\mathbf{i} - \mathbf{j} + t^2\,\mathbf{k}\right] dt = (t^2 + 3t)\,\mathbf{i} - t\,\mathbf{j} + \frac{1}{3}t^3\,\mathbf{k} + \mathbf{D}$. Here $\mathbf{r}(0) = \mathbf{D}$ and we were given that $\mathbf{r}(0) = \mathbf{j} + \mathbf{k}$, so $\mathbf{D} = \mathbf{j} + \mathbf{k}$ and $\mathbf{r}(t) = (t^2 + 3t)\,\mathbf{i} + (1 - t)\,\mathbf{j} + \left(\frac{1}{3}t^3 + 1\right)\mathbf{k}$.

**16.** $\mathbf{a}(t) = \sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} + 6t\,\mathbf{k} \quad \Rightarrow \quad \mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int (\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} + 6t\,\mathbf{k})\,dt = -\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + 3t^2\,\mathbf{k} + \mathbf{C}$.

Then $\mathbf{v}(0) = -\mathbf{i} + \mathbf{C}$ but we were given that $\mathbf{v}(0) = -\mathbf{k}$, so $-\mathbf{i} + \mathbf{C} = -\mathbf{k} \quad \Rightarrow \quad \mathbf{C} = \mathbf{i} - \mathbf{k}$ and $\mathbf{v}(t) = (1 - \cos t)\,\mathbf{i} + 2\sin t\,\mathbf{j} + (3t^2 - 1)\,\mathbf{k}$.

$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int \left[(1 - \cos t)\,\mathbf{i} + 2\sin t\,\mathbf{j} + (3t^2 - 1)\,\mathbf{k}\right] dt = (t - \sin t)\,\mathbf{i} - 2\cos t\,\mathbf{j} + (t^3 - t)\,\mathbf{k} + \mathbf{D}$. Here $\mathbf{r}(0) = -2\,\mathbf{j} + \mathbf{D}$ and we were given that $\mathbf{r}(0) = \mathbf{j} - 4\,\mathbf{k}$, so $\mathbf{D} = 3\,\mathbf{j} - 4\,\mathbf{k}$ and $\mathbf{r}(t) = (t - \sin t)\,\mathbf{i} + (3 - 2\cos t)\,\mathbf{j} + (t^3 - t - 4)\,\mathbf{k}$.

**17.** (a) $\mathbf{a}(t) = 2t\,\mathbf{i} + \sin t\,\mathbf{j} + \cos 2t\,\mathbf{k} \quad \Rightarrow$

$\mathbf{v}(t) = \int (2t\,\mathbf{i} + \sin t\,\mathbf{j} + \cos 2t\,\mathbf{k})\,dt = t^2\,\mathbf{i} - \cos t\,\mathbf{j} + \frac{1}{2}\sin 2t\,\mathbf{k} + \mathbf{C}$

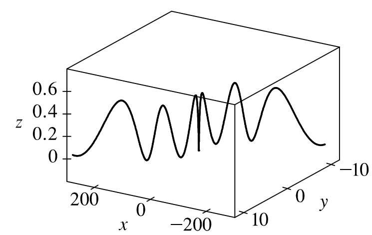
and $\mathbf{i} = \mathbf{v}(0) = -\mathbf{j} + \mathbf{C}$, so $\mathbf{C} = \mathbf{i} + \mathbf{j}$

and $\mathbf{v}(t) = \left(t^2 + 1\right)\mathbf{i} + (1 - \cos t)\,\mathbf{j} + \frac{1}{2}\sin 2t\,\mathbf{k}$.

$\mathbf{r}(t) = \int [\left(t^2+1\right)\,\mathbf{i} + (1-\cos t)\,\mathbf{j} + \frac{1}{2}\sin 2t\,\mathbf{k}]dt$

$\quad = \left(\frac{1}{3}t^3 + t\right)\,\mathbf{i} + (t - \sin t)\,\mathbf{j} - \frac{1}{4}\cos 2t\,\mathbf{k} + \mathbf{D}$

(b)




But $\mathbf{j} = \mathbf{r}(0) = -\frac{1}{4}\,\mathbf{k} + \mathbf{D}$, so $\mathbf{D} = \mathbf{j} + \frac{1}{4}\,\mathbf{k}$ and $\mathbf{r}(t) = \left(\frac{1}{3}t^3 + t\right)\mathbf{i} + (t - \sin t + 1)\,\mathbf{j} + \left(\frac{1}{4} - \frac{1}{4}\cos 2t\right)\mathbf{k}$.

**18.** (a) $\mathbf{a}(t) = t\,\mathbf{i} + e^t\,\mathbf{j} + e^{-t}\,\mathbf{k} \quad \Rightarrow$

$\mathbf{v}(t) = \int \left(t\,\mathbf{i} + e^t\,\mathbf{j} + e^{-t}\,\mathbf{k}\right)dt = \frac{1}{2}t^2\,\mathbf{i} + e^t\,\mathbf{j} - e^{-t}\,\mathbf{k} + \mathbf{C}$

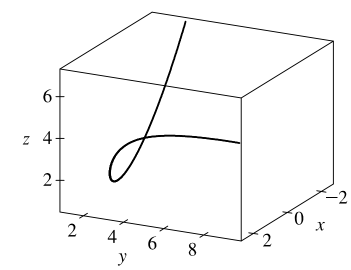
and $\mathbf{k} = \mathbf{v}(0) = \mathbf{j} - \mathbf{k} + \mathbf{C}$, so $\mathbf{C} = -\mathbf{j} + 2\,\mathbf{k}$

and $\mathbf{v}(t) = \frac{1}{2}t^2\,\mathbf{i} + \left(e^t - 1\right)\,\mathbf{j} + \left(2 - e^{-t}\right)\,\mathbf{k}$.

$\mathbf{r}(t) = \int \left[\frac{1}{2}t^2\,\mathbf{i} + (e^t - 1)\,\mathbf{j} + (2 - e^{-t})\,\mathbf{k}\right]dt$

$\quad = \frac{1}{6}t^3\,\mathbf{i} + (e^t - t)\,\mathbf{j} + (e^{-t} + 2t)\,\mathbf{k} + \mathbf{D}$

(b)




But $\mathbf{j} + \mathbf{k} = \mathbf{r}(0) = \mathbf{j} + \mathbf{k} + \mathbf{D}$, so $\mathbf{D} = \mathbf{0}$ and $\mathbf{r}(t) = \frac{1}{6}t^3\,\mathbf{i} + (e^t - t)\,\mathbf{j} + (e^{-t} + 2t)\,\mathbf{k}$.

**19.** $\mathbf{r}(t) = \left\langle t^2, 5t, t^2 - 16t\right\rangle \quad \Rightarrow \quad \mathbf{v}(t) = \langle 2t, 5, 2t - 16\rangle$, $|\mathbf{v}(t)| = \sqrt{4t^2 + 25 + 4t^2 - 64t + 256} = \sqrt{8t^2 - 64t + 281}$

and $\dfrac{d}{dt}\,|\mathbf{v}(t)| = \frac{1}{2}(8t^2 - 64t + 281)^{-1/2}(16t - 64)$. This is zero if and only if the numerator is zero, that is,

$16t - 64 = 0$ or $t = 4$. Since $\dfrac{d}{dt}\,|\mathbf{v}(t)| < 0$ for $t < 4$ and $\dfrac{d}{dt}\,|\mathbf{v}(t)| > 0$ for $t > 4$, the minimum speed of $\sqrt{153}$ is attained

at $t = 4$ units of time.

**20.** Since $\mathbf{r}(t) = t^3\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$, $\mathbf{a}(t) = \mathbf{r}''(t) = 6t\,\mathbf{i} + 2\,\mathbf{j} + 6t\,\mathbf{k}$. By Newton's Second Law,

$\mathbf{F}(t) = m\,\mathbf{a}(t) = 6mt\,\mathbf{i} + 2m\,\mathbf{j} + 6mt\,\mathbf{k}$ is the required force.

**21.** $|\mathbf{F}(t)| = 20$ N in the direction of the positive $z$-axis, so $\mathbf{F}(t) = 20\,\mathbf{k}$. Also $m = 4$ kg, $\mathbf{r}(0) = \mathbf{0}$ and $\mathbf{v}(0) = \mathbf{i} - \mathbf{j}$.

Since $20\mathbf{k} = \mathbf{F}(t) = 4\,\mathbf{a}(t)$, $\mathbf{a}(t) = 5\,\mathbf{k}$. Then $\mathbf{v}(t) = 5t\,\mathbf{k} + \mathbf{c}_1$ where $\mathbf{c}_1 = \mathbf{i} - \mathbf{j}$ so $\mathbf{v}(t) = \mathbf{i} - \mathbf{j} + 5t\,\mathbf{k}$ and the

speed is $|\mathbf{v}(t)| = \sqrt{1 + 1 + 25t^2} = \sqrt{25t^2 + 2}$. Also $\mathbf{r}(t) = t\,\mathbf{i} - t\,\mathbf{j} + \frac{5}{2}t^2\,\mathbf{k} + \mathbf{c}_2$ and $\mathbf{0} = \mathbf{r}(0)$, so $\mathbf{c}_2 = \mathbf{0}$

and $\mathbf{r}(t) = t\,\mathbf{i} - t\,\mathbf{j} + \frac{5}{2}t^2\,\mathbf{k}$.

**22.** The argument here is the same as that in the proof of Theorem 13.2.4 with $\mathbf{r}(t)$ replaced by $\mathbf{v}(t)$ and $\mathbf{r}'(t)$ replaced by $\mathbf{a}(t)$.

**23.** $|\mathbf{v}(0)| = 200$ m/s and, since the angle of elevation is $60^\circ$, a unit vector in the direction of the velocity is

$(\cos 60^\circ)\mathbf{i} + (\sin 60^\circ)\mathbf{j} = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\,\mathbf{j}$. Thus $\mathbf{v}(0) = 200\left(\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\,\mathbf{j}\right) = 100\,\mathbf{i} + 100\,\sqrt{3}\,\mathbf{j}$ and if we set up the axes so that the

projectile starts at the origin, then $\mathbf{r}(0) = \mathbf{0}$. Ignoring air resistance, the only force is that due to gravity, so

$\mathbf{F}(t) = m\mathbf{a}(t) = -mg\,\mathbf{j}$ where $g \approx 9.8$ m/s$^2$. Thus $\mathbf{a}(t) = -9.8\,\mathbf{j}$ and, integrating, we have $\mathbf{v}(t) = -9.8t\,\mathbf{j} + \mathbf{C}$. But

$100\,\mathbf{i} + 100\sqrt{3}\,\mathbf{j} = \mathbf{v}(0) = \mathbf{C}$, so $\mathbf{v}(t) = 100\,\mathbf{i} + \left(100\sqrt{3} - 9.8t\right)\mathbf{j}$ and then (integrating again) $\mathbf{r}(t) = 100\,t\,\mathbf{i} + \left(100\sqrt{3}\,t - 4.9t^2\right)\mathbf{j} + \mathbf{D}$ where $\mathbf{0} = \mathbf{r}(0) = \mathbf{D}$. Thus, the position function of the projectile is $\mathbf{r}(t) = 100\,t\,\mathbf{i} + \left(100\sqrt{3}\,t - 4.9t^2\right)\mathbf{j}$.

(a) Parametric equations for the projectile are $x(t) = 100t$, $y(t) = 100\sqrt{3}\,t - 4.9t^2$. The projectile reaches the ground when $y(t) = 0$ (and $t > 0$) $\Rightarrow$ $100\sqrt{3}\,t - 4.9t^2 = t\left(100\sqrt{3} - 4.9t\right) = 0$ $\Rightarrow$ $t = \frac{100\sqrt{3}}{4.9} \approx 35.3$ s. So the range is $x\left(\frac{100\sqrt{3}}{4.9}\right) = 100\left(\frac{100\sqrt{3}}{4.9}\right) \approx 3535$ m.

(b) The maximum height is reached when $y(t)$ has a critical number (or equivalently, when the vertical component of velocity is 0): $y'(t) = 0$ $\Rightarrow$ $100\sqrt{3} - 9.8t = 0$ $\Rightarrow$ $t = \frac{100\sqrt{3}}{9.8} \approx 17.7$ s. Thus, the maximum height is $y\left(\frac{100\sqrt{3}}{9.8}\right) = 100\sqrt{3}\left(\frac{100\sqrt{3}}{9.8}\right) - 4.9\left(\frac{100\sqrt{3}}{9.8}\right)^2 \approx 1531$ m.

(c) From part (a), impact occurs at $t = \frac{100\sqrt{3}}{4.9}$ s. Thus, the velocity at impact is $\mathbf{v}\left(\frac{100\sqrt{3}}{4.9}\right) = 100\,\mathbf{i} + \left[100\sqrt{3} - 9.8\left(\frac{100\sqrt{3}}{4.9}\right)\right]\mathbf{j} = 100\,\mathbf{i} - 100\sqrt{3}\,\mathbf{j}$ and the speed is $\left|\mathbf{v}\left(\frac{100\sqrt{3}}{4.9}\right)\right| = \sqrt{10{,}000 + 30{,}000} = 200\ \text{m/s}$.

**24.** As in Exercise 23, $\mathbf{v}(t) = 100\,\mathbf{i} + \left(100\sqrt{3} - 9.8t\right)\mathbf{j}$ and $\mathbf{r}(t) = 100\,t\,\mathbf{i} + \left(100\sqrt{3}\,t - 4.9t^2\right)\mathbf{j} + \mathbf{D}$. But $\mathbf{r}(0) = 100\,\mathbf{j}$, so $\mathbf{D} = 100\,\mathbf{j}$ and $\mathbf{r}(t) = 100\,t\,\mathbf{i} + \left(100 + 100\sqrt{3}\,t - 4.9t^2\right)\mathbf{j}$.

(a) $y = 0$ $\Rightarrow$ $100 + 100\sqrt{3}\,t - 4.9t^2 = 0$ or $4.9t^2 - 100\sqrt{3}\,t - 100 = 0$. From the quadratic formula, we have

$$t = \frac{100\sqrt{3} \pm \sqrt{(-100\sqrt{3}\,)^2 - 4(4.9)(-100)}}{2(4.9)} = \frac{100\sqrt{3} \pm \sqrt{31{,}960}}{9.8}.$$

Taking the positive $t$-value gives $t = \frac{100\sqrt{3} + \sqrt{31{,}960}}{9.8} \approx 35.9$ s. Thus the range is $x = 100 \cdot \frac{100\sqrt{3} + \sqrt{31{,}960}}{9.8} \approx 3592$ m.

(b) The maximum height is attained when $\dfrac{dy}{dt} = 0$ $\Rightarrow$ $100\sqrt{3} - 9.8t = 0$ $\Rightarrow$ $t = \frac{100\sqrt{3}}{9.8} \approx 17.7$ s and the maximum height is $100 + 100\sqrt{3}\left(\frac{100\sqrt{3}}{9.8}\right) - 4.9\left(\frac{100\sqrt{3}}{9.8}\right)^2 \approx 1631$ m.

*Alternate solution:* Because the projectile is fired in the same direction and with the same velocity as in Exercise 23, but from a point 100 m higher, the maximum height reached is 100 m higher than that found in Exercise 23, that is, $1531\text{ m} + 100\text{ m} = 1631$ m.

(c) From part (a), impact occurs at $t = \frac{100\sqrt{3} + \sqrt{31{,}960}}{9.8}$ s. Thus, the velocity at impact is $\mathbf{v}\left(\frac{100\sqrt{3} + \sqrt{31{,}960}}{9.8}\right) = 100\,\mathbf{i} + \left[100\sqrt{3} - 9.8\left(\frac{100\sqrt{3} + \sqrt{31{,}960}}{9.8}\right)\right]\mathbf{j} = 100\,\mathbf{i} - \sqrt{31{,}960}\,\mathbf{j}$ and the speed is $|\mathbf{v}| = \sqrt{10{,}000 + 31{,}960} = \sqrt{41{,}960} \approx 205\ \text{m/s}$.

**25.** As in Example 5, $\mathbf{r}(t) = (v_0 \cos 45°)t\,\mathbf{i} + \left[(v_0 \sin 45°)t - \frac{1}{2}gt^2\right]\mathbf{j} = \frac{1}{2}\left[v_0\sqrt{2}\,t\,\mathbf{i} + \left(v_0\sqrt{2}\,t - gt^2\right)\mathbf{j}\right]$. The ball lands when $y = 0$ (and $t > 0$) $\Rightarrow \quad t = \dfrac{v_0\sqrt{2}}{g}$ s. Now since it lands 90 m away, $90 = x = \frac{1}{2}v_0\sqrt{2}\,\dfrac{v_0\sqrt{2}}{g}$ or $v_0^2 = 90g$ and the initial velocity is $v_0 = \sqrt{90g} \approx 30$ m/s.

**26.** Let $\alpha$ be the angle of elevation. Here $v_0 = 400$ m/s and from Example 5, the horizontal distance traveled by the projectile is $d = \dfrac{v_0^2 \sin 2\alpha}{g}$. We want $\dfrac{400^2 \sin 2\alpha}{g} = 3000 \quad\Rightarrow\quad \sin 2\alpha = \dfrac{3000g}{400^2} \approx 0.1838 \quad\Rightarrow\quad 2\alpha \approx \sin^{-1}(0.1838) \approx 10.6°$ or $2\alpha \approx 180° - 10.6° = 169.4°$. Thus, two angles of elevation are $\alpha \approx 5.3°$ and $\alpha \approx 84.7°$.

**27.** As in Example 5, $\mathbf{r}(t) = (v_0 \cos 36°)t\,\mathbf{i} + \left[(v_0 \sin 36°)t - \frac{1}{2}gt^2\right]\mathbf{j}$ and then $\mathbf{v}(t) = \mathbf{r}'(t) = (v_0 \cos 36°)\,\mathbf{i} + [(v_0 \sin 36°) - gt]\,\mathbf{j}$. The shell reaches its maximum height when the vertical component of velocity is zero, so $(v_0 \sin 36°) - gt = 0 \quad\Rightarrow\quad t = \dfrac{v_0 \sin 36°}{g}$. The vertical height of the shell at that time is 500 m, so

$$(v_0 \sin 36°)\left(\frac{v_0 \sin 36°}{g}\right) - \frac{1}{2}g\left(\frac{v_0 \sin 36°}{g}\right)^2 = 500 \quad\Rightarrow\quad \frac{v_0^2 \sin^2 36°}{2g} = 500 \quad\Rightarrow$$

$$v_0 = \sqrt{\frac{1000g}{\sin^2 36°}} \approx \frac{\sqrt{1000(9.8)}}{\sin 36°} \approx 168 \text{ m/s}.$$

**28.** Here, $v_0 = 35$ m/s, the angle of elevation is $\alpha = 50°$, and if we place the origin at home plate, then $\mathbf{r}(0) = 1\mathbf{j}$. As in Example 5, we have $\mathbf{r}(t) = -\frac{1}{2}gt^2\mathbf{j} + t\mathbf{v}_0 + \mathbf{D}$, where $\mathbf{D} = \mathbf{r}(0) = 1\mathbf{j}$ and $\mathbf{v}_0 = v_0 \cos\alpha\mathbf{i} + v_o \sin\alpha\mathbf{j}$, so $\mathbf{r}(t) = (v_0 \cos\alpha)t\mathbf{i} + [(v_0 \sin\alpha - \frac{1}{2}gt^2 + 1]\mathbf{j}$. Thus, parametric equations for the trajectory of the ball are $x = (v_0 \cos\alpha)t$, $y = (v_0 \sin\alpha)t - \frac{1}{2}gt^2 + 1$. The ball reaches the fence when $x = 120 \Rightarrow (v_0 \cos\alpha)t = 120 \Rightarrow t = \frac{120}{v_0 \cos\alpha} = \frac{120}{35 \cos 50°}$ $\approx 5.33$ s. At this time, the height of the ball is $y = (v_0 \sin\alpha)t - \frac{1}{2}gt^2 + 1 \approx (35 \sin 50°)(5.33) - \frac{1}{2}(9.8)(5.33)^2 + 1$ $\approx 4.7$ m. Since the fence is 4 m high, the ball clears the fence.

**29.** Place the catapult at the origin and assume the catapult is 100 meters from the city, so the city lies between $(100, 0)$ and $(600, 0)$. The initial speed is $v_0 = 80$ m/s and let $\theta$ be the angle the catapult is set at. As in Example 5, the trajectory of the catapulted rock is given by $\mathbf{r}\,(t) = (80\cos\theta)t\,\mathbf{i} + \left[(80\sin\theta)t - 4.9t^2\right]\mathbf{j}$. The top of the near city wall is at $(100, 15)$, which the rock will hit when $(80\cos\theta)\,t = 100 \quad\Rightarrow\quad t = \dfrac{5}{4\cos\theta}$ and $(80\sin\theta)t - 4.9t^2 = 15 \quad\Rightarrow$

$80\sin\theta \cdot \dfrac{5}{4\cos\theta} - 4.9\left(\dfrac{5}{4\cos\theta}\right)^2 = 15 \quad\Rightarrow\quad 100\tan\theta - 7.65625\sec^2\theta = 15$. Replacing $\sec^2\theta$ with $\tan^2\theta + 1$ gives $7.65625\tan^2\theta - 100\tan\theta + 22.65625 = 0$. Using the quadratic formula, we have $\tan\theta \approx 0.230635, 12.8306 \quad\Rightarrow$ $\theta \approx 13.0°, 85.5°$. So for $13.0° < \theta < 85.5°$, the rock will land beyond the near city wall. The base of the far wall is located at $(600, 0)$ which the rock hits if $(80\cos\theta)t = 600 \quad\Rightarrow\quad t = \dfrac{15}{2\cos\theta}$ and $(80\sin\theta)t - 4.9t^2 = 0 \quad\Rightarrow$

$80\sin\theta\cdot\dfrac{15}{2\cos\theta}-4.9\left(\dfrac{15}{2\cos\theta}\right)^2=0 \quad\Rightarrow\quad 600\tan\theta-275.625\sec^2\theta=0 \quad\Rightarrow$

$275.625\tan^2\theta-600\tan\theta+275.625=0$. Solutions are $\tan\theta\approx 0.658678, 1.51819 \quad\Rightarrow\quad \theta\approx 33.4^\circ, 56.6^\circ$. Thus, the rock lands beyond the enclosed city ground for $33.4^\circ<\theta<56.6^\circ$, and the angles that allow the rock to land on city ground are $13.0^\circ<\theta<33.4^\circ, 56.6^\circ<\theta<85.5^\circ$. If you consider that the rock can hit the far wall and bounce back into the city, we calculate the angles that cause the rock to hit the top of the wall at $(600,15)$: $(80\cos\theta)t=600 \quad\Rightarrow\quad t=\dfrac{15}{2\cos\theta}$ and $(80\sin\theta)t-4.9t^2=15 \quad\Rightarrow\quad 600\tan\theta-275.625\sec^2\theta=15 \quad\Rightarrow\quad 275.625\tan^2\theta-600\tan\theta+290.625=0$. Solutions are $\tan\theta\approx 0.727506, 1.44936 \quad\Rightarrow\quad \theta\approx 36.0^\circ, 55.4^\circ$, so the catapult should be set with angle $\theta$ where $13.0^\circ<\theta<36.0^\circ, 55.4^\circ<\theta<85.5^\circ$.

**30.** If we place the projectile at the origin then, as in Example 5, $\mathbf{r}(t)=(v_0\cos\alpha)t\,\mathbf{i}+\left[(v_0\sin\alpha)t-\frac{1}{2}gt^2\right]\mathbf{j}$ and $\mathbf{v}(t)=(v_0\cos\alpha)\,\mathbf{i}+[(v_0\sin\alpha)-gt]\,\mathbf{j}$. The maximum height is reached when the vertical component of velocity is zero, so $(v_0\sin\alpha)-gt=0 \quad\Rightarrow\quad t=\dfrac{v_0\sin\alpha}{g}$, and the corresponding height is the vertical component of the position function:

$$(v_0\sin\alpha)t-\tfrac{1}{2}gt^2=(v_0\sin\alpha)\left(\frac{v_0\sin\alpha}{g}\right)-\tfrac{1}{2}g\left(\frac{v_0\sin\alpha}{g}\right)^2=\frac{1}{2g}v_0^2\sin^2\alpha$$

Half that time is $t=\dfrac{v_0\sin\alpha}{2g}$, when the height of the projectile is

$$\begin{aligned}(v_0\sin\alpha)t-\tfrac{1}{2}gt^2&=(v_0\sin\alpha)\left(\frac{v_0\sin\alpha}{2g}\right)-\tfrac{1}{2}g\left(\frac{v_0\sin\alpha}{2g}\right)^2\\&=\frac{1}{2g}v_0^2\sin^2\alpha-\frac{1}{8g}v_0^2\sin^2\alpha=\frac{3}{8g}v_0^2\sin^2\alpha=\frac{3}{4}\left(\frac{1}{2g}v_0^2\sin^2\alpha\right)\end{aligned}$$

or three-quarters of the maximum height.

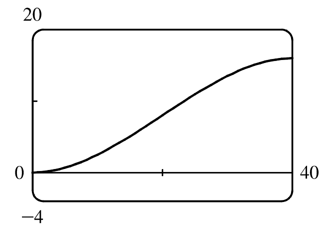
**31.** Here $a(t)=-4\mathbf{j}-32\mathbf{k}$ so $v(t)=-4t\mathbf{j}-32t\mathbf{k}+v_0=-4t\mathbf{j}-32t\mathbf{k}+50\mathbf{i}+80\mathbf{k}=50\mathbf{i}-4t\mathbf{j}+(80-32t)\mathbf{k}$ and $r(t)=50t\mathbf{i}-2t^2\mathbf{j}+(80t-16t^2)\mathbf{k}$ (note that $r_0=0$). The ball lands when the $z$-component of $r(t)$ is zero and $t>0$: $80t-16t^2=16t(5-t)=0 \implies t=5$. The position of the ball then is $r(5)=50(5)\mathbf{i}-2(5)^2\mathbf{j}+[80(5)-16(5)^2]\mathbf{k}=250\mathbf{i}-50\mathbf{j}$, or equivalently, the point $(250,-50,0)$. This is a distance of $\sqrt{250^2+(-50)^2+0^2}=\sqrt{65{,}000}\approx 255$ m from the origin at an angle of $tan^{-1}(\frac{50}{250})\approx 11.3^\circ$ from the eastern direction toward the south. The speed of the ball is $|v(5)|=|50\mathbf{i}-20\mathbf{j}-80\mathbf{k}|=\sqrt{50^2+(-20)^2+(-80)^2}=\sqrt{9300}\approx 96.4$ m/s.

**32.** Place the ball at the origin and consider $\mathbf{j}$ to be pointing in the northward direction with $\mathbf{i}$ pointing east and $\mathbf{k}$ pointing upward. Force = mass $\times$ acceleration $\Rightarrow$ acceleration = force/mass, so the wind applies a constant acceleration of $4\text{ N}/0.8\text{ kg}=5\text{ m/s}^2$ in the easterly direction. Combined with the acceleration due to gravity, the acceleration acting on the ball is $\mathbf{a}(t)=5\,\mathbf{i}-9.8\,\mathbf{k}$. Then $\mathbf{v}(t)=\int\mathbf{a}(t)\,dt=5t\,\mathbf{i}-9.8t\,\mathbf{k}+\mathbf{C}$ where $\mathbf{C}$ is a constant vector.

We know $\mathbf{v}(0)=\mathbf{C}=-30\cos 30^\circ\,\mathbf{j}+30\sin 30^\circ\,\mathbf{k}=-15\sqrt{3}\,\mathbf{j}+15\,\mathbf{k} \quad\Rightarrow\quad \mathbf{C}=-15\sqrt{3}\,\mathbf{j}+15\,\mathbf{k}$ and

$\mathbf{v}(t) = 5t\,\mathbf{i} - 15\sqrt{3}\,\mathbf{j} + (15 - 9.8t)\,\mathbf{k}$. $\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = 2.5t^2\,\mathbf{i} - 15\sqrt{3}\,t\,\mathbf{j} + \left(15t - 4.9t^2\right)\mathbf{k} + \mathbf{D}$ but $\mathbf{r}(0) = \mathbf{D} = \mathbf{0}$ so $\mathbf{r}(t) = 2.5t^2\,\mathbf{i} - 15\sqrt{3}\,t\,\mathbf{j} + \left(15t - 4.9t^2\right)\mathbf{k}$. The ball lands when $15t - 4.9t^2 = 0 \Rightarrow t = 0, t = 15/4.9 \approx 3.0612$ s, so the ball lands at approximately $\mathbf{r}(3.0612) \approx 23.43\,\mathbf{i} - 79.53\,\mathbf{j}$ which is 82.9 m away in the direction S $16.4°$E. Its speed is approximately $|\mathbf{v}(3.0612)| \approx \left|15.306\,\mathbf{i} - 15\sqrt{3}\,\mathbf{j} - 15\,\mathbf{k}\right| \approx 33.68$ m/s.

**33.** (a) After $t$ seconds, the boat will be $5t$ meters west of point $A$. The velocity of the water at that location is $\frac{3}{400}(5t)(40 - 5t)\,\mathbf{j}$. The velocity of the boat in still water is $5\,\mathbf{i}$, so the resultant velocity of the boat is $\mathbf{v}(t) = 5\,\mathbf{i} + \frac{3}{400}(5t)(40 - 5t)\,\mathbf{j} = 5\mathbf{i} + \left(\frac{3}{2}t - \frac{3}{16}t^2\right)\mathbf{j}$. Integrating, we obtain $\mathbf{r}(t) = 5t\,\mathbf{i} + \left(\frac{3}{4}t^2 - \frac{1}{16}t^3\right)\mathbf{j} + \mathbf{C}$. If we place the origin at $A$ (and consider $\mathbf{j}$ to coincide with the northern direction) then $\mathbf{r}(0) = \mathbf{0} \Rightarrow \mathbf{C} = \mathbf{0}$ and we have $\mathbf{r}(t) = 5t\,\mathbf{i} + \left(\frac{3}{4}t^2 - \frac{1}{16}t^3\right)\mathbf{j}$. The boat reaches the east bank when $5t = 40$, that is, when $t = 8$ s, and it is located at $\mathbf{r}(8) = 5(8)\mathbf{i} + \left(\frac{3}{4}(8)^2 - \frac{1}{16}(8)^3\right)\mathbf{j} = 40\,\mathbf{i} + 16\,\mathbf{j}$. Thus the boat is 16 m downstream.




(b) Let $\alpha$ be the angle north of east that the boat heads. Then the velocity of the boat in still water is given by $5(\cos\alpha)\,\mathbf{i} + 5(\sin\alpha)\,\mathbf{j}$. At $t$ seconds, the boat is $5(\cos\alpha)t$ meters from the west bank, at which point the velocity of the water is $\frac{3}{400}[5(\cos\alpha)t][40 - 5(\cos\alpha)t]\,\mathbf{j}$. The resultant velocity of the boat is given by

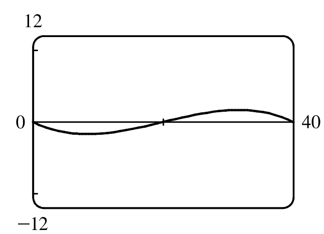
$$\mathbf{v}(t) = 5(\cos\alpha)\,\mathbf{i} + \left[5\sin\alpha + \tfrac{3}{400}(5t\cos\alpha)(40 - 5t\cos\alpha)\right]\mathbf{j} = (5\cos\alpha)\,\mathbf{i} + \left(5\sin\alpha + \tfrac{3}{2}t\cos\alpha - \tfrac{3}{16}t^2\cos^2\alpha\right)\mathbf{j}.$$

Integrating, $\mathbf{r}(t) = (5t\cos\alpha)\,\mathbf{i} + \left(5t\sin\alpha + \frac{3}{4}t^2\cos\alpha - \frac{1}{16}t^3\cos^2\alpha\right)\mathbf{j}$ (where we have again placed the origin at $A$). The boat will reach the east bank when $5t\cos\alpha = 40 \Rightarrow t = \dfrac{40}{5\cos\alpha} = \dfrac{8}{\cos\alpha}$.

In order to land at point $B(40, 0)$ we need $5t\sin\alpha + \frac{3}{4}t^2\cos\alpha - \frac{1}{16}t^3\cos^2\alpha = 0 \Rightarrow$

$$5\left(\frac{8}{\cos\alpha}\right)\sin\alpha + \frac{3}{4}\left(\frac{8}{\cos\alpha}\right)^2\cos\alpha - \frac{1}{16}\left(\frac{8}{\cos\alpha}\right)^3\cos^2\alpha = 0 \Rightarrow \frac{1}{\cos\alpha}(40\sin\alpha + 48 - 32) = 0 \Rightarrow$$

$40\sin\alpha + 16 = 0 \Rightarrow \sin\alpha = -\frac{2}{5}$. Thus $\alpha = \sin^{-1}\left(-\frac{2}{5}\right) \approx -23.6°$, so the boat should head $23.6°$ south of east (upstream). The path does seem realistic. The boat initially heads upstream to counteract the effect of the current. Near the center of the river, the current is stronger and the boat is pushed downstream. When the boat nears the eastern bank, the current is slower and the boat is able to progress upstream to arrive at point $B$.




**34.** As in Exercise 33(b), let $\alpha$ be the angle north of east that the boat heads, so the velocity of the boat in still water is given by $5(\cos\alpha)\,\mathbf{i} + 5(\sin\alpha)\,\mathbf{j}$. At $t$ seconds, the boat is $5(\cos\alpha)t$ meters from the west bank, at which point the velocity of the water is $3\sin(\pi x/40)\,\mathbf{j} = 3\sin[\pi \cdot 5(\cos\alpha)t/40]\,\mathbf{j} = 3\sin\left(\frac{\pi}{8}t\cos\alpha\right)\mathbf{j}$. The resultant velocity of the boat then is given by $\mathbf{v}(t) = 5(\cos\alpha)\,\mathbf{i} + \left[5\sin\alpha + 3\sin\left(\frac{\pi}{8}t\cos\alpha\right)\right]\mathbf{j}$. Integrating,

$$\mathbf{r}\,(t) = (5t\cos\alpha)\,\mathbf{i} + \left[5t\sin\alpha - \frac{24}{\pi\cos\alpha}\cos\left(\tfrac{\pi}{8}t\cos\alpha\right)\right]\mathbf{j} + \mathbf{C}.$$

If we place the origin at $A$, then $\mathbf{r}(0) = \mathbf{0} \;\Rightarrow\; -\dfrac{24}{\pi\cos\alpha}\mathbf{j} + \mathbf{C} = \mathbf{0} \;\Rightarrow\; \mathbf{C} = \dfrac{24}{\pi\cos\alpha}\mathbf{j}$ and

$\mathbf{r}(t) = (5t\cos\alpha)\,\mathbf{i} + \left[5t\sin\alpha - \dfrac{24}{\pi\cos\alpha}\cos\left(\tfrac{\pi}{8}t\cos\alpha\right) + \dfrac{24}{\pi\cos\alpha}\right]\mathbf{j}$. The boat will reach the east bank when

$5t\cos\alpha = 40 \;\Rightarrow\; t = \dfrac{8}{\cos\alpha}$. In order to land at point $B(40, 0)$ we need

$$5t\sin\alpha - \frac{24}{\pi\cos\alpha}\cos\left(\tfrac{\pi}{8}t\cos\alpha\right) + \frac{24}{\pi\cos\alpha} = 0 \;\Rightarrow$$

$$5\left(\frac{8}{\cos\alpha}\right)\sin\alpha - \frac{24}{\pi\cos\alpha}\cos\left[\frac{\pi}{8}\left(\frac{8}{\cos\alpha}\right)\cos\alpha\right] + \frac{24}{\pi\cos\alpha} = 0 \;\Rightarrow\; \frac{1}{\cos\alpha}\left(40\sin\alpha - \frac{24}{\pi}\cos\pi + \frac{24}{\pi}\right) = 0 \;\Rightarrow$$

$40\sin\alpha + \dfrac{48}{\pi} = 0 \;\Rightarrow\; \sin\alpha = -\dfrac{6}{5\pi}$. Thus $\alpha = \sin^{-1}\left(-\dfrac{6}{5\pi}\right) \approx -22.5^\circ$, so the boat should head $22.5^\circ$ south of east.

**35.** If $\mathbf{r}'(t) = \mathbf{c} \times \mathbf{r}(t)$, then $\mathbf{r}'(t)$ is perpendicular to both $\mathbf{c}$ and $\mathbf{r}(t)$. Remember that $\mathbf{r}'(t)$ points in the direction of motion, so if $\mathbf{r}'(t)$ is always perpendicular to $\mathbf{c}$, the path of the particle must lie in a plane perpendicular to $\mathbf{c}$. But $\mathbf{r}'(t)$ is also perpendicular to the position vector $\mathbf{r}(t)$ which confines the path to a sphere centered at the origin. Considering both restrictions, the path must be contained in a circle that lies in a plane perpendicular to $\mathbf{c}$, and the circle is centered on a line through the origin in the direction of $\mathbf{c}$.

**36.** (a) From Equation 7 we have $\mathbf{a} = v'\mathbf{T} + \kappa v^2\mathbf{N}$. If a particle moves along a straight line, then $\kappa = 0$ [see Section 13.3], so the acceleration vector becomes $\mathbf{a} = v'\mathbf{T}$. Because the acceleration vector is a scalar multiple of the unit tangent vector, it is parallel to the tangent vector.

(b) If the speed of the particle is constant, then $v' = 0$ and Equation 7 gives $\mathbf{a} = \kappa v^2\mathbf{N}$. Thus the acceleration vector is parallel to the unit normal vector (which is perpendicular to the tangent vector and points in the direction that the curve is turning).

**37.** $\mathbf{r}(t) = (t^2+1)\,\mathbf{i} + t^3\,\mathbf{j} \;\Rightarrow\; \mathbf{r}'(t) = 2t\,\mathbf{i} + 3t^2\,\mathbf{j}$,

$|\mathbf{r}'(t)| = \sqrt{(2t)^2 + (3t^2)^2} = \sqrt{4t^2 + 9t^4} = t\sqrt{4 + 9t^2}$ [since $t \geq 0$], $\quad \mathbf{r}''(t) = 2\,\mathbf{i} + 6t\,\mathbf{j}, \quad \mathbf{r}'(t) \times \mathbf{r}''(t) = 6t^2\,\mathbf{k}$.

Then Equation 9 gives $a_T = \dfrac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \dfrac{(2t)(2) + (3t^2)(6t)}{t\sqrt{4+9t^2}} = \dfrac{4t + 18t^3}{t\sqrt{4+9t^2}} = \dfrac{4 + 18t^2}{\sqrt{4+9t^2}}$

$$\left[\begin{aligned} &\text{or by Equation 8,}\quad a_T = v' = \frac{d}{dt}\left[t\sqrt{4+9t^2}\,\right] = t\cdot\tfrac{1}{2}\left(4+9t^2\right)^{-1/2}(18t) + \left(4+9t^2\right)^{1/2}\cdot 1 \\ &\qquad = \left(4+9t^2\right)^{-1/2}\left(9t^2 + 4 + 9t^2\right) = (4+18t^2)/\sqrt{4+9t^2} \end{aligned}\right]$$

and Equation 10 gives $a_N = \dfrac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \dfrac{6t^2}{t\sqrt{4+9t^2}} = \dfrac{6t}{\sqrt{4+9t^2}}$.

**38.** $\mathbf{r}(t) = 2t^2\,\mathbf{i} + \left(\frac{2}{3}t^3 - 2t\right)\mathbf{j} \quad \Rightarrow \quad \mathbf{r}'(t) = 4t\,\mathbf{i} + (2t^2 - 2)\,\mathbf{j}$,

$|\mathbf{r}'(t)| = \sqrt{16t^2 + (2t^2 - 2)^2} = \sqrt{4t^4 + 8t^2 + 4} = \sqrt{4\,(t^2+1)^2} = 2(t^2+1)$,

$\mathbf{r}''(t) = 4\,\mathbf{i} + 4t\,\mathbf{j}$, $\quad \mathbf{r}'(t) \times \mathbf{r}''(t) = (8t^2 + 8)\,\mathbf{k}$. Then Equation 9 gives

$$a_T = \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \frac{(4t)(4) + (2t^2-2)(4t)}{2(t^2+1)} = \frac{8t(t^2+1)}{2(t^2+1)} = 4t \quad \left[\text{or by Equation 8, } a_T = v' = \frac{d}{dt}\left[2(t^2+1)\right] = 4t\right]$$

and Equation 10 gives $a_N = \dfrac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \dfrac{8(t^2+1)}{2(t^2+1)} = 4$.

**39.** $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k} \quad \Rightarrow \quad \mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k}$, $\quad |\mathbf{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$,

$\mathbf{r}''(t) = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j}$, $\quad \mathbf{r}'(t)\times\mathbf{r}''(t) = \sin t\,\mathbf{i} - \cos t\,\mathbf{j} + \mathbf{k}$.

Then $a_T = \dfrac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \dfrac{\sin t\,\cos t - \sin t\,\cos t}{\sqrt{2}} = 0$ and $a_N = \dfrac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \dfrac{\sqrt{\sin^2 t + \cos^2 t + 1}}{\sqrt{2}} = \dfrac{\sqrt{2}}{\sqrt{2}} = 1$.

**40.** $\mathbf{r}(t) = t\,\mathbf{i} + 2e^t\,\mathbf{j} + e^{2t}\,\mathbf{k} \quad \Rightarrow \quad \mathbf{r}'(t) = \mathbf{i} + 2e^t\,\mathbf{j} + 2e^{2t}\,\mathbf{k}$, $\quad |\mathbf{r}'(t)| = \sqrt{1 + 4e^{2t} + 4e^{4t}} = \sqrt{(1+2e^{2t})^2} = 1 + 2e^{2t}$,

$\mathbf{r}''(t) = 2e^t\,\mathbf{j} + 4e^{2t}\,\mathbf{k}$, $\quad \mathbf{r}'(t)\times\mathbf{r}''(t) = 4e^{3t}\,\mathbf{i} - 4e^{2t}\,\mathbf{j} + 2e^t\,\mathbf{k}$,

$|\mathbf{r}'(t)\times\mathbf{r}''(t)| = \sqrt{16e^{6t} + 16e^{4t} + 4e^{2t}} = \sqrt{4e^{2t}(2e^{2t}+1)^2} = 2e^t(2e^{2t}+1)$. Then

$$a_T = \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \frac{4e^{2t} + 8e^{4t}}{1 + 2e^{2t}} = \frac{4e^{2t}(1 + 2e^{2t})}{1 + 2e^{2t}} = 4e^{2t} \text{ and } a_N = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \frac{2e^t(2e^{2t}+1)}{1+2e^{2t}} = 2e^t.$$

**41.** $\mathbf{r}(t) = \ln t\,\mathbf{i} + (t^2 + 3t)\,\mathbf{j} + 4\sqrt{t}\,\mathbf{k} \quad \Rightarrow \quad \mathbf{r}'(t) = (1/t)\,\mathbf{i} + (2t+3)\,\mathbf{j} + (2/\sqrt{t}\,)\,\mathbf{k} \quad \Rightarrow$

$\mathbf{r}''(t) = (-1/t^2)\,\mathbf{i} + 2\,\mathbf{j} - (1/t^{3/2})\,\mathbf{k}$. The point $(0,4,4)$ corresponds to $t = 1$, where

$\mathbf{r}'(1) = \mathbf{i} + 5\,\mathbf{j} + 2\,\mathbf{k}$, $\quad \mathbf{r}''(1) = -\mathbf{i} + 2\,\mathbf{j} - \mathbf{k}$, and $\mathbf{r}'(1)\times\mathbf{r}''(1) = -9\,\mathbf{i} - \mathbf{j} + 7\,\mathbf{k}$. Thus at the point $(0,4,4)$,

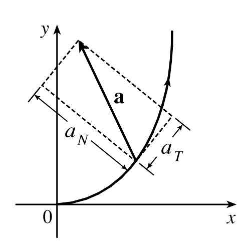
$$a_T = \frac{\mathbf{r}'(1)\cdot\mathbf{r}''(1)}{|\mathbf{r}'(1)|} = \frac{-1 + 10 - 2}{\sqrt{1 + 25 + 4}} = \frac{7}{\sqrt{30}} \text{ and } a_N = \frac{|\mathbf{r}'(1)\times\mathbf{r}''(1)|}{|\mathbf{r}'(1)|} = \frac{\sqrt{81 + 1 + 49}}{\sqrt{30}} = \sqrt{\frac{131}{30}}.$$

**42.** $\mathbf{r}(t) = t^{-1}\,\mathbf{i} + t^{-2}\,\mathbf{j} + t^{-3}\,\mathbf{k} \quad \Rightarrow \quad \mathbf{r}'(t) = -t^{-2}\,\mathbf{i} - 2t^{-3}\,\mathbf{j} - 3t^{-4}\,\mathbf{k} \quad \Rightarrow \quad \mathbf{r}''(t) = 2t^{-3}\,\mathbf{i} + 6t^{-4}\,\mathbf{j} + 12t^{-5}\,\mathbf{k}$. The point $(1,1,1)$ corresponds to $t = 1$, where $\mathbf{r}'(1) = -\mathbf{i} - 2\,\mathbf{j} - 3\,\mathbf{k}$, $\quad \mathbf{r}''(1) = 2\,\mathbf{i} + 6\,\mathbf{j} + 12\,\mathbf{k}$, and

$\mathbf{r}'(1)\times\mathbf{r}''(1) = -6\,\mathbf{i} + 6\,\mathbf{j} - 2\,\mathbf{k}$. Thus at the point $(1,1,1)$, $a_T = \dfrac{\mathbf{r}'(1)\cdot\mathbf{r}''(1)}{|\mathbf{r}'(1)|} = \dfrac{-2 - 12 - 36}{\sqrt{1+4+9}} = -\dfrac{50}{\sqrt{14}}$ and

$$a_N = \frac{|\mathbf{r}'(1)\times\mathbf{r}''(1)|}{|\mathbf{r}'(1)|} = \frac{\sqrt{36+36+4}}{\sqrt{14}} = \sqrt{\frac{76}{14}} = \sqrt{\frac{38}{7}}.$$

**43.** The tangential component of $\mathbf{a}$ is the length of the projection of $\mathbf{a}$ onto $\mathbf{T}$, so we sketch the scalar projection of $\mathbf{a}$ in the tangential direction to the curve and estimate its length to be 4.5 (using the fact that $\mathbf{a}$ has length 10 as a guide). Similarly, the normal component of $\mathbf{a}$ is the length of the projection of $\mathbf{a}$ onto $\mathbf{N}$, so we sketch the scalar projection of $\mathbf{a}$ in the normal direction to the curve and estimate its length to be 9.0. Thus $a_T \approx 4.5\ \text{cm/s}^2$ and $a_N \approx 9.0\ \text{cm/s}^2$.



**44.** $\mathbf{L}(t) = m\,\mathbf{r}(t) \times \mathbf{v}(t) \quad \Rightarrow$

$$\mathbf{L}'(t) = m[\mathbf{r}'(t) \times \mathbf{v}(t) + \mathbf{r}\,(t) \times \mathbf{v}'(t)] \qquad \text{[by Formula 5 of Theorem 13.2.3]}$$
$$= m[\mathbf{v}(t) \times \mathbf{v}(t) + \mathbf{r}(t) \times \mathbf{v}'(t)] = m[\mathbf{0} + \mathbf{r}(t) \times \mathbf{a}\,(t)] = \boldsymbol{\tau}(t)$$

So if the torque is always $\mathbf{0}$, then $\mathbf{L}'(t) = \mathbf{0}$ for all $t$, and so $\mathbf{L}(t)$ is constant.

**45.** If the engines are turned off at time $t$, then the spacecraft will continue to travel in the direction of $\mathbf{v}(t)$, so we need a $t$ such that for some scalar $s > 0$, $\mathbf{r}(t) + s\,\mathbf{v}(t) = \langle 6, 4, 9\rangle$. $\mathbf{v}(t) = \mathbf{r}'(t) = \mathbf{i} + \dfrac{1}{t}\,\mathbf{j} + \dfrac{8t}{(t^2+1)^2}\,\mathbf{k} \quad \Rightarrow$

$$\mathbf{r}(t) + s\,\mathbf{v}(t) = \left\langle 3 + t + s, 2 + \ln t + \frac{s}{t}, 7 - \frac{4}{t^2+1} + \frac{8st}{(t^2+1)^2}\right\rangle \quad \Rightarrow \quad 3 + t + s = 6 \quad \Rightarrow \quad s = 3 - t,$$

so $7 - \dfrac{4}{t^2+1} + \dfrac{8(3-t)t}{(t^2+1)^2} = 9 \quad \Leftrightarrow \quad \dfrac{24t - 12t^2 - 4}{(t^2+1)^2} = 2 \quad \Leftrightarrow \quad t^4 + 8t^2 - 12t + 3 = 0.$

It is easily seen that $t = 1$ is a root of this polynomial. Also $2 + \ln 1 + \dfrac{3-1}{1} = 4$, so $t = 1$ is the desired solution.

**46.** (a) $m\,\dfrac{d\mathbf{v}}{dt} = \dfrac{dm}{dt}\,\mathbf{v}_e \quad \Leftrightarrow \quad \dfrac{d\mathbf{v}}{dt} = \dfrac{1}{m}\,\dfrac{dm}{dt}\,\mathbf{v}_e$. Integrating both sides of this equation with respect to $t$ gives

$$\int_0^t \frac{d\mathbf{v}}{du}\,du = \mathbf{v}_e \int_0^t \frac{1}{m}\,\frac{dm}{du}\,du \quad \Rightarrow \quad \int_{\mathbf{v}(0)}^{\mathbf{v}(t)} d\mathbf{v} = \mathbf{v}_e \int_{m(0)}^{m(t)} \frac{dm}{m} \quad \text{[Substitution Rule]} \quad \Rightarrow$$

$$\mathbf{v}(t) - \mathbf{v}(0) = \mathbf{v}_e \ln \frac{m(t)}{m(0)} \quad \Rightarrow \quad \mathbf{v}(t) = \mathbf{v}(0) - \mathbf{v}_e \ln \frac{m(0)}{m(t)}.$$

(b) $|\mathbf{v}(t)| = 2\,|\mathbf{v}_e|$, and $|\mathbf{v}(0)| = 0$. Therefore, by part (a), $2\,|\mathbf{v}_e| = \left| -\ln\!\left(\dfrac{m(0)}{m(t)}\right)\mathbf{v}_e \right| \quad \Rightarrow$

$$2\,|\mathbf{v}_e| = \ln\!\left(\frac{m(0)}{m\,(t)}\right)|\mathbf{v}_e|. \quad \left[\textit{Note: } m(0) > m(t) \text{ so that } \ln\!\left(\frac{m(0)}{m(t)}\right) > 0\right] \quad \Rightarrow \quad m(t) = e^{-2}m(0).$$

Thus $\dfrac{m(0) - e^{-2}m(0)}{m(0)} = 1 - e^{-2}$ is the fraction of the initial mass that is burned as fuel.

## APPLIED PROJECT Kepler's Laws

**1.** With $\mathbf{r} = (r\cos\theta)\,\mathbf{i} + (r\sin\theta)\,\mathbf{j}$ and $\mathbf{h} = \alpha\,\mathbf{k}$ where $\alpha > 0$,

(a) $$\mathbf{h} = \mathbf{r} \times \mathbf{r}' = [(r\cos\theta)\,\mathbf{i} + (r\sin\theta)\,\mathbf{j}] \times \left[\left(r'\cos\theta - r\sin\theta\,\frac{d\theta}{dt}\right)\mathbf{i} + \left(r'\sin\theta + r\cos\theta\,\frac{d\theta}{dt}\right)\mathbf{j}\right]$$
$$= \left[rr'\cos\theta\sin\theta + r^2\cos^2\theta\,\frac{d\theta}{dt} - rr'\cos\theta\sin\theta + r^2\sin^2\theta\,\frac{d\theta}{dt}\right]\mathbf{k} = r^2\,\frac{d\theta}{dt}\,\mathbf{k}$$

(b) Since $\mathbf{h} = \alpha\,\mathbf{k}$, $\alpha > 0$, $\alpha = |\mathbf{h}|$. But by part (a), $\alpha = |\mathbf{h}| = r^2\,(d\theta/dt)$.

(c) $A(t) = \frac{1}{2}\int_{\theta_0}^{\theta} |\mathbf{r}|^2\,d\theta = \frac{1}{2}\int_{t_0}^{t} r^2\,(d\theta/dt)\,dt$ in polar coordinates. Thus, by the Fundamental Theorem of Calculus,

$$\frac{dA}{dt} = \frac{r^2}{2}\,\frac{d\theta}{dt}.$$

(d) $\dfrac{dA}{dt} = \dfrac{r^2}{2}\dfrac{d\theta}{dt} = \dfrac{h}{2} =$ constant since $\mathbf{h}$ is a constant vector and $h = |\mathbf{h}|$.

**2.** (a) Since $dA/dt = \frac{1}{2}h$, a constant, $A(t) = \frac{1}{2}ht + c_1$. But $A(0) = 0$, so $A(t) = \frac{1}{2}ht$. But $A(T) =$ area of the ellipse $= \pi ab$ and $A(T) = \frac{1}{2}hT$, so $T = 2\pi ab/h$.

(b) $h^2/(GM) = ed$ where $e$ is the eccentricity of the ellipse. But $a = ed/(1 - e^2)$ or $ed = a(1 - e^2)$ and $1 - e^2 = b^2/a^2$. Hence $h^2/(GM) = ed = b^2/a$.

(c) $T^2 = \dfrac{4\pi a^2 b^2}{h^2} = 4\pi^2 a^2 b^2 \dfrac{a}{GMb^2} = \dfrac{4\pi^2}{GM}a^3$.

**3.** From Problem 2, $T^2 = \dfrac{4\pi^2}{GM}a^3$. $T \approx 365.25$ days $\times\, 24 \cdot 60^2\, \dfrac{\text{seconds}}{\text{day}} \approx 3.1558 \times 10^7$ seconds. Therefore,

$a^3 = \dfrac{GMT^2}{4\pi^2} \approx \dfrac{(6.67 \times 10^{-11})(1.99 \times 10^{30})(3.1558 \times 10^7)^2}{4\pi^2} \approx 3.348 \times 10^{33}\text{ m}^3 \quad\Rightarrow\quad a \approx 1.496 \times 10^{11}$ m. Thus, the length of the major axis of the earth's orbit (that is, $2a$) is approximately $2.99 \times 10^{11}\text{ m} = 2.99 \times 10^8$ km.

**4.** We can adapt the equation $T^2 = \dfrac{4\pi^2}{GM}a^3$ from Problem 2(c) with the earth at the center of the system, so $T$ is the period of the satellite's orbit about the earth, $M$ is the mass of the earth, and $a$ is the length of the semimajor axis of the satellite's orbit (measured from the earth's center). Since we want the satellite to remain fixed above a particular point on the earth's equator, $T$ must coincide with the period of the earth's own rotation, so $T = 24\text{ h} = 86{,}400$ s. The mass of the earth is $M = 5.98 \times 10^{24}$ kg, so $a = \left(\dfrac{T^2GM}{4\pi^2}\right)^{1/3} \approx \left[\dfrac{(86{,}400)^2(6.67 \times 10^{-11})(5.98 \times 10^{24})}{4\pi^2}\right]^{1/3} \approx 4.23 \times 10^7$ m. If we assume a circular orbit, the radius of the orbit is $a$, and since the radius of the earth is $6.37 \times 10^6$ m, the required altitude above the earth's surface for the satellite is $4.23 \times 10^7 - 6.37 \times 10^6 \approx 3.59 \times 10^7$ m, or 35,900 km.

## 13 Review

### TRUE-FALSE QUIZ

**1.** True. If we reparametrize the curve by replacing $u = t^3$, we have $\mathbf{r}(u) = u\,\mathbf{i} + 2u\,\mathbf{j} + 3u\,\mathbf{k}$, which is a line through the origin with direction vector $\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$.

**2.** True. Parametric equations for the curve are $x = 0$, $y = t^2$, $z = 4t$, and since $t = z/4$, we have $y = t^2 = (z/4)^2$ or $y = \frac{1}{16}z^2$, $x = 0$. This is an equation of a parabola in the $yz$-plane.

**3.** False. The vector function $\mathbf{r}(t) = \langle 2t, 3 - t, 0\rangle$ represents a line, but the line does not pass through the origin; the $x$-component is 0 only for $t = 0$, which corresponds to the point $(0, 3, 0)$, not $(0, 0, 0)$.

**4.** True. See Theorem 13.2.2.

**5.** False. By Formula 5 of Theorem 13.2.3, $\dfrac{d}{dt}\left[\mathbf{u}(t) \times \mathbf{v}(t)\right] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$.

**6.** False. For example, let $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$. Then $|\mathbf{r}(t)| = \sqrt{\cos^2 t + \sin^2 t} = 1 \quad \Rightarrow \quad \dfrac{d}{dt}\,|\mathbf{r}(t)| = 0$, but

$|\mathbf{r}'(t)| = |\langle -\sin t, \cos t \rangle| = \sqrt{(-\sin t)^2 + \cos^2 t} = 1$.

**7.** False. $\kappa$ is the magnitude of the rate of change of the unit tangent vector $\mathbf{T}$ with respect to arc length $s$, not with respect to $t$.

**8.** False. The binormal vector, by the definition given in Section 13.3, is $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = -\left[\mathbf{N}(t) \times \mathbf{T}(t)\right]$.

**9.** True. At an inflection point where $f$ is twice continuously differentiable we must have $f''(x) = 0$, and by Equation 13.3.11, the curvature is 0 there.

**10.** True. From Equation 13.3.9 , $\kappa(t) = 0 \quad \Leftrightarrow \quad |\mathbf{T}'(t)| = 0 \quad \Leftrightarrow \quad \mathbf{T}'(t) = \mathbf{0}$ for all $t$. But then $\mathbf{T}(t) = \mathbf{C}$, a constant vector, which is true only for a straight line.

**11.** False. If $\mathbf{r}(t)$ is the position of a moving particle at time $t$ and $|\mathbf{r}(t)| = 1$ then the particle lies on the unit circle or the unit sphere, but this does not mean that the speed $|\mathbf{r}'(t)|$ must be constant. As a counterexample, let $\mathbf{r}(t) = \left\langle t, \sqrt{1-t^2} \right\rangle$, then $\mathbf{r}'(t) = \left\langle 1, -t/\sqrt{1-t^2} \right\rangle$ and $|\mathbf{r}(t)| = \sqrt{t^2 + 1 - t^2} = 1$ but $|\mathbf{r}'(t)| = \sqrt{1 + t^2/(1-t^2)} = 1/\sqrt{1-t^2}$ which is not constant.

**12.** True. See Theorem 13.2.4.

**13.** True. See the discussion preceding Example 8 in Section 13.3.

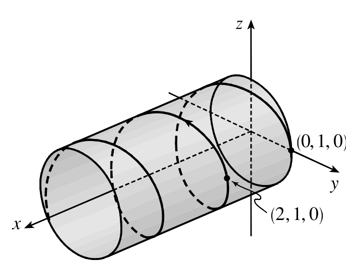
**14.** False. For example, $\mathbf{r}_1(t) = \langle t, t \rangle$ and $\mathbf{r}_2(t) = \langle 2t, 2t \rangle$ both represent the same plane curve (the line $y = x$), but the tangent vector $\mathbf{r}_1'(t) = \langle 1, 1 \rangle$ for all $t$, while $\mathbf{r}_2'(t) = \langle 2, 2 \rangle$. In fact, different parametrizations give parallel tangent vectors at a point, but their magnitudes may differ.

**15.** True. The projection in the $xz$-plane is given by $\mathbf{r}(t) = \langle \cos 2t, 0, \sin 2t \rangle$. Since $x^2 + z^2 = (\cos 2t)^2 + (\sin 2t)^2 = 1$, the projection onto the $xz$-plane is a circle with radius 1.

**16.** True. Note that the direction vector for both lines is $\mathbf{d} = \langle 1, 2, 1 \rangle$, so the lines are parallel. Further, the point $(0, 0, 1)$ is contained on both lines for $t = 0$ and $t = 1$, respectively. The lines are parallel and have a point in common; therefore, they are the same line.

## EXERCISES

**1.** (a) The corresponding parametric equations for the curve are $x = t$, $y = \cos \pi t$, $z = \sin \pi t$. Since $y^2 + z^2 = 1$, the curve is contained in a circular cylinder with axis the $x$-axis. Since $x = t$, the curve is a helix.




(b) $\mathbf{r}(t) = t\,\mathbf{i} + \cos \pi t\,\mathbf{j} + \sin \pi t\,\mathbf{k} \quad \Rightarrow$

$\mathbf{r}'(t) = \mathbf{i} - \pi \sin \pi t\,\mathbf{j} + \pi \cos \pi t\,\mathbf{k} \quad \Rightarrow$

$\mathbf{r}''(t) = -\pi^2 \cos \pi t\,\mathbf{j} - \pi^2 \sin \pi t\,\mathbf{k}$

**2.** (a) The expressions $\sqrt{2-t}$, $(e^t-1)/t$, and $\ln(t+1)$ are all defined when $2-t \geq 0 \Rightarrow t \leq 2, t \neq 0$, and $t+1>0 \Rightarrow t>-1$. Thus the domain of $\mathbf{r}$ is $(-1,0) \cup (0,2]$.

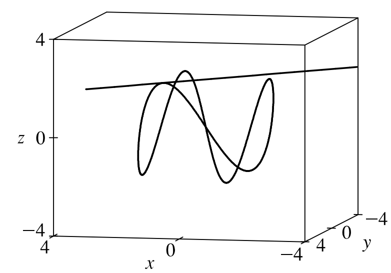
(b) $\displaystyle \lim_{t\to 0} \mathbf{r}(t) = \left\langle \lim_{t\to 0} \sqrt{2-t}, \lim_{t\to 0} \frac{e^t-1}{t}, \lim_{t\to 0} \ln(t+1) \right\rangle = \left\langle \sqrt{2-0}, \lim_{t\to 0} \frac{e^t}{1}, \ln(0+1) \right\rangle$

$= \langle \sqrt{2}, 1, 0 \rangle$ [using l'Hospital's Rule in the $y$-component]

(c) $\displaystyle \mathbf{r}'(t) = \left\langle \frac{d}{dt}\sqrt{2-t}, \frac{d}{dt}\frac{e^t-1}{t}, \frac{d}{dt}\ln(t+1) \right\rangle = \left\langle -\frac{1}{2\sqrt{2-t}}, \frac{te^t-e^t+1}{t^2}, \frac{1}{t+1} \right\rangle$

**3.** The projection of the curve $C$ of intersection onto the $xy$-plane is the circle $x^2+y^2=16, z=0$. So we can write $x=4\cos t,\ y=4\sin t,\ 0 \leq t \leq 2\pi$. From the equation of the plane, we have $z=5-x=5-4\cos t$, so parametric equations for $C$ are $x=4\cos t,\ y=4\sin t,\ z=5-4\cos t, 0 \leq t \leq 2\pi$, and the corresponding vector function is $\mathbf{r}(t) = 4\cos t\,\mathbf{i} + 4\sin t\,\mathbf{j} + (5-4\cos t)\,\mathbf{k}, 0 \leq t \leq 2\pi$.

**4.** The curve is given by $\mathbf{r}(t) = \langle 2\sin t, 2\sin 2t, 2\sin 3t \rangle$, so $\mathbf{r}'(t) = \langle 2\cos t, 4\cos 2t, 6\cos 3t \rangle$. The point $(1, \sqrt{3}, 2)$ corresponds to $t=\frac{\pi}{6}$ (or $\frac{\pi}{6}+2k\pi$, $k$ an integer), so the tangent vector there is $\mathbf{r}'(\frac{\pi}{6}) = \langle \sqrt{3}, 2, 0 \rangle$. Then the tangent line has direction vector $\langle \sqrt{3}, 2, 0 \rangle$ and includes the point $(1, \sqrt{3}, 2)$, so parametric equations are $x=1+\sqrt{3}\,t, y=\sqrt{3}+2t, z=2$.




**5.** $\displaystyle \int_0^1 (t^2\,\mathbf{i} + t\cos \pi t\,\mathbf{j} + \sin \pi t\,\mathbf{k})\,dt = \left(\int_0^1 t^2\,dt\right)\mathbf{i} + \left(\int_0^1 t\cos \pi t\,dt\right)\mathbf{j} + \left(\int_0^1 \sin \pi t\,dt\right)\mathbf{k}$

$\displaystyle = \left[\tfrac{1}{3}t^3\right]_0^1 \mathbf{i} + \left(\left[\tfrac{t}{\pi}\sin \pi t\right]_0^1 - \int_0^1 \tfrac{1}{\pi}\sin \pi t\,dt\right)\mathbf{j} + \left[-\tfrac{1}{\pi}\cos \pi t\right]_0^1 \mathbf{k}$

$\displaystyle = \tfrac{1}{3}\,\mathbf{i} + \left[\tfrac{1}{\pi^2}\cos \pi t\right]_0^1 \mathbf{j} + \tfrac{2}{\pi}\,\mathbf{k} = \tfrac{1}{3}\,\mathbf{i} - \tfrac{2}{\pi^2}\,\mathbf{j} + \tfrac{2}{\pi}\,\mathbf{k}$

where we integrated by parts in the $y$-component.

**6.** (a) $C$ intersects the $xz$-plane where $y=0 \Rightarrow 2t-1=0 \Rightarrow t=\frac{1}{2}$, so the point is $\left(2-\left(\frac{1}{2}\right)^3, 0, \ln\frac{1}{2}\right) = \left(\frac{15}{8}, 0, -\ln 2\right)$.

(b) The curve is given by $\mathbf{r}(t) = \langle 2-t^3, 2t-1, \ln t \rangle$, so $\mathbf{r}'(t) = \langle -3t^2, 2, 1/t \rangle$. The point $(1,1,0)$ corresponds to $t=1$, so the tangent vector there is $\mathbf{r}'(1) = \langle -3, 2, 1 \rangle$. Then the tangent line has direction vector $\langle -3, 2, 1 \rangle$ and includes the point $(1,1,0)$, so parametric equations are $x=1-3t, y=1+2t, z=t$.

(c) The normal plane has normal vector $\mathbf{r}'(1) = \langle -3, 2, 1 \rangle$ and equation $-3(x-1)+2(y-1)+z=0$ or $3x-2y-z=1$.

**7.** $\mathbf{r}(t) = \langle t^2, t^3, t^4 \rangle \Rightarrow \mathbf{r}'(t) = \langle 2t, 3t^2, 4t^3 \rangle \Rightarrow |\mathbf{r}'(t)| = \sqrt{4t^2+9t^4+16t^6}$ and $L = \int_0^3 |\mathbf{r}'(t)|\,dt = \int_0^3 \sqrt{4t^2+9t^4+16t^6}\,dt$. Using Simpson's Rule with $f(t) = \sqrt{4t^2+9t^4+16t^6}$ and $n=6$, we have $\Delta t = \frac{3-0}{6} = \frac{1}{2}$ and

$$\begin{aligned} L &\approx \tfrac{\Delta t}{3}\left[f(0) + 4f\left(\tfrac{1}{2}\right) + 2f(1) + 4f\left(\tfrac{3}{2}\right) + 2f(2) + 4f\left(\tfrac{5}{2}\right) + f(3)\right] \\ &= \tfrac{1}{6}\Bigg[\sqrt{0+0+0} + 4\cdot\sqrt{4\left(\tfrac{1}{2}\right)^2 + 9\left(\tfrac{1}{2}\right)^4 + 16\left(\tfrac{1}{2}\right)^6} + 2\cdot\sqrt{4(1)^2 + 9(1)^4 + 16(1)^6} \\ &\qquad + 4\cdot\sqrt{4\left(\tfrac{3}{2}\right)^2 + 9\left(\tfrac{3}{2}\right)^4 + 16\left(\tfrac{3}{2}\right)^6} + 2\cdot\sqrt{4(2)^2 + 9(2)^4 + 16(2)^6} \\ &\qquad + 4\cdot\sqrt{4\left(\tfrac{5}{2}\right)^2 + 9\left(\tfrac{5}{2}\right)^4 + 16\left(\tfrac{5}{2}\right)^6} + \sqrt{4(3)^2 + 9(3)^4 + 16(3)^6}\,\Bigg] \\ &\approx 86.631 \end{aligned}$$

**8.** $\mathbf{r}(t) = \left\langle 2t^{3/2}, \cos 2t, \sin 2t\right\rangle \Rightarrow \mathbf{r}'(t) = \left\langle 3t^{1/2}, -2\sin 2t, 2\cos 2t\right\rangle \Rightarrow$

$|\mathbf{r}'(t)| = \sqrt{9t + 4(\sin^2 2t + \cos^2 2t)} = \sqrt{9t+4}$.

Thus, $L = \int_0^1 \sqrt{9t+4}\,dt = \int_4^{13} \frac{1}{9}u^{1/2}\,du = \frac{1}{9}\left[\frac{2}{3}u^{3/2}\right]_4^{13} = \frac{2}{27}(13^{3/2} - 8)$.

**9.** $\mathbf{r}_1(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$ and $\mathbf{r}_2(t) = (1+t)\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$. The angle $\theta$ of intersection of the two curves is the angle between their respective tangents at the point of intersection. For both curves, the point $(1,0,0)$ corresponds to $t = 0$.

$\mathbf{r}_1'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{r}_1'(0) = \mathbf{j} + \mathbf{k}$, and $\mathbf{r}_2'(t) = \mathbf{i} + 2t\,\mathbf{j} + 3t^2\,\mathbf{k} \Rightarrow \mathbf{r}_2'(0) = \mathbf{i}$.

$\mathbf{r}_1'(0)\cdot\mathbf{r}_2'(0) = (\mathbf{j} + \mathbf{k})\cdot\mathbf{i} = 0$. Therefore, the curves intersect in a right angle, that is, $\theta = 90^\circ$.

**10.** $\mathbf{r}(t) = e^t\,\mathbf{i} + e^t\sin t\,\mathbf{j} + e^t\cos t\,\mathbf{k}$. The parametric value corresponding to the point $(1,0,1)$ is $t = 0$.

$\mathbf{r}'(t) = e^t\,\mathbf{i} + e^t(\cos t + \sin t)\,\mathbf{j} + e^t(\cos t - \sin t)\,\mathbf{k} \Rightarrow |\mathbf{r}'(t)| = e^t\sqrt{1^2 + (\cos t + \sin t)^2 + (\cos t - \sin t)^2} = \sqrt{3}\,e^t$

and $s(t) = \int_0^t e^u\sqrt{3}\,du = \sqrt{3}(e^t - 1) \Rightarrow 1 + \frac{1}{\sqrt{3}}s = e^t \Rightarrow t = \ln\left(1 + \frac{1}{\sqrt{3}}s\right)$.

Therefore, $\mathbf{r}(t(s)) = \left(1 + \frac{1}{\sqrt{3}}s\right)\mathbf{i} + \left(1 + \frac{1}{\sqrt{3}}s\right)\sin\ln\left(1 + \frac{1}{\sqrt{3}}s\right)\mathbf{j} + \left(1 + \frac{1}{\sqrt{3}}s\right)\cos\ln\left(1 + \frac{1}{\sqrt{3}}s\right)\mathbf{k}$.

**11.** (a) $\mathbf{r}(t) = \left\langle \sin^3 t, \cos^3 t, \sin^2 t\right\rangle \Rightarrow \mathbf{r}'(t) = \left\langle 3\sin^2 t\cos t, -3\cos^2 t\sin t, 2\sin t\cos t\right\rangle$,

$$\begin{aligned} |\mathbf{r}'(t)| &= \sqrt{9\sin^4 t\cos^2 t + 9\cos^4 t\sin^2 t + 4\sin^2 t\cos^2 t} \\ &= \sqrt{\sin^2 t\cos^2 t\left(9\sin^2 t + 9\cos^2 t + 4\right)} = \sqrt{13}\,\sin t\cos t \quad [\text{since } 0 \le t \le \pi/2 \;\Rightarrow\; \sin t, \cos t \ge 0] \end{aligned}$$

Then $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{13}\,\sin t\cos t}\left\langle 3\sin^2 t\cos t, -3\cos^2 t\sin t, 2\sin t\cos t\right\rangle = \frac{1}{\sqrt{13}}\left\langle 3\sin t, -3\cos t, 2\right\rangle$.

(b) $\mathbf{T}'(t) = \frac{1}{\sqrt{13}}\left\langle 3\cos t, 3\sin t, 0\right\rangle$, $|\mathbf{T}'(t)| = \frac{1}{\sqrt{13}}\sqrt{9\cos^2 t + 9\sin^2 t + 0} = \frac{3}{\sqrt{13}}$, and

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{1}{3}\left\langle 3\cos t, 3\sin t, 0\right\rangle = \left\langle \cos t, \sin t, 0\right\rangle$.

(c) $\mathbf{B}(t) = \mathbf{T}(t)\times\mathbf{N}(t) = \frac{1}{\sqrt{13}}\left\langle 3\sin t, -3\cos t, 2\right\rangle \times \left\langle \cos t, \sin t, 0\right\rangle = \frac{1}{\sqrt{13}}\left\langle -2\sin t, 2\cos t, 3\right\rangle$

(d) $\kappa(t)=\dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}=\dfrac{3/\sqrt{13}}{\sqrt{13}\,\sin t\cos t}=\dfrac{3}{13\,\sin t\cos t}$ or $\frac{3}{13}\csc t\sec t$

(e) $\tau(t)=-\dfrac{\mathbf{B}'(t)\cdot\mathbf{N}(t)}{|\mathbf{r}'(t)|}=-\dfrac{\frac{1}{\sqrt{13}}\langle -2\cos t,-2\sin t,0\rangle\cdot\langle\cos t,\sin t,0\rangle}{\sqrt{13}\,\sin t\cos t}=-\dfrac{-2\cos^2 t-2\sin^2 t+0}{13\sin t\cos t}$

$=\dfrac{2}{13\,\sin t\cos t}$ or $\frac{2}{13}\csc t\sec t$

**12.** See the instructions for Exercises 13.3.46–49. $x=3\cos t,\ y=4\sin t\ \Rightarrow\ \dot{x}=-3\sin t,\ \dot{y}=4\cos t\ \Rightarrow$ $\ddot{x}=-3\cos t,\ \ddot{y}=-4\sin t$.

$$\kappa(t)=\frac{|\dot{x}\ddot{y}-\dot{y}\ddot{x}|}{[\dot{x}^2+\dot{y}^2]^{3/2}}=\frac{|(-3\sin t)(-4\sin t)-(4\cos t)(-3\cos t)|}{(9\sin^2 t+16\cos^2 t)^{3/2}}=\frac{12}{(9\sin^2 t+16\,\cos^2 t)^{3/2}}.$$

At $(3,0)$, $t=0$ and $\kappa(0)=12/(16)^{3/2}=\frac{12}{64}=\frac{3}{16}$. At $(0,4)$, $t=\frac{\pi}{2}$ and $\kappa\left(\frac{\pi}{2}\right)=12/9^{3/2}=\frac{12}{27}=\frac{4}{9}$.

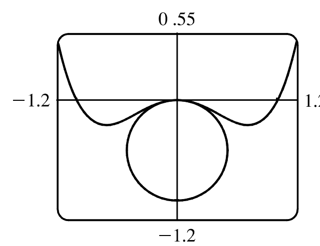
**13.** $y=x^4,\ y'=4x^3,\ y''=12x^2$, and $\kappa(x)=\dfrac{|y''|}{[1+(y')^2]^{3/2}}=\dfrac{\left|12x^2\right|}{(1+16x^6)^{3/2}}$, so $\kappa(1)=\dfrac{12}{17^{3/2}}$.

**14.** $y=x^4-x^2,\ y'=4x^3-2x,\ y''=12x^2-2$, and

$$\kappa(x)=\frac{|y''|}{[1+(y')^2]^{3/2}}=\frac{\left|12x^2-2\right|}{[1+(4x^3-2x)^2]^{3/2}}\ \Rightarrow\ \kappa(0)=2.$$

So the osculating circle has radius $\frac{1}{2}$ and center $\left(0,-\frac{1}{2}\right)$.

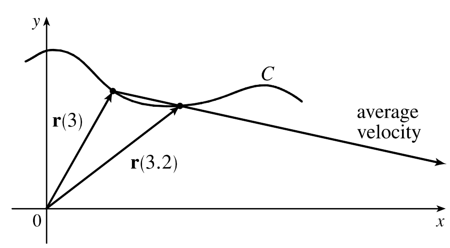
Thus, its equation is $x^2+\left(y+\frac{1}{2}\right)^2=\frac{1}{4}$.




**15.** $\mathbf{r}(t)=\langle\sin 2t,t,\cos 2t\rangle\ \Rightarrow\ \mathbf{r}'(t)=\langle 2\cos 2t,1,-2\sin 2t\rangle\ \Rightarrow\ \mathbf{T}(t)=\frac{1}{\sqrt{5}}\langle 2\cos 2t,1,-2\sin 2t\rangle\ \Rightarrow$ $\mathbf{T}'(t)=\frac{1}{\sqrt{5}}\langle -4\sin 2t,0,-4\cos 2t\rangle\ \Rightarrow\ \mathbf{N}(t)=\langle -\sin 2t,0,-\cos 2t\rangle$. So $\mathbf{N}=\mathbf{N}(\pi)=\langle 0,0,-1\rangle$ and $\mathbf{B}=\mathbf{T}\times\mathbf{N}=\frac{1}{\sqrt{5}}\langle -1,2,0\rangle$. So a normal to the osculating plane is $\langle -1,2,0\rangle$ and an equation is $-1(x-0)+2(y-\pi)+0(z-1)=0$ or $x-2y+2\pi=0$.

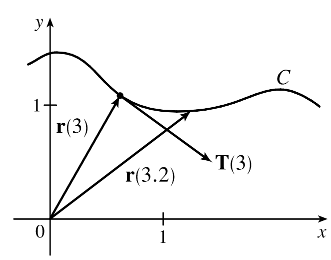
**16.** (a) The average velocity over $[3,3.2]$ is given by

$\dfrac{\mathbf{r}(3.2)-\mathbf{r}(3)}{3.2-3}=5[\mathbf{r}(3.2)-\mathbf{r}(3)]$, so we draw a vector with the same direction but 5 times the length of the vector $[\mathbf{r}(3.2)-\mathbf{r}(3)]$.




(b) $\mathbf{v}(3)=\mathbf{r}'(3)=\lim\limits_{h\to 0}\dfrac{\mathbf{r}(3+h)-\mathbf{r}(3)}{h}$

(c) $\mathbf{T}(3)=\dfrac{\mathbf{r}'(3)}{|\mathbf{r}'(3)|}$, a unit vector in the same direction as $\mathbf{r}'(3)$, that is, parallel to the tangent line to the curve at $\mathbf{r}(3)$, pointing in the direction corresponding to increasing $t$, and with length 1.



**17.** $\mathbf{r}(t) = t\ln t\,\mathbf{i} + t\,\mathbf{j} + e^{-t}\,\mathbf{k}$, $\quad \mathbf{v}(t) = \mathbf{r}'(t) = (1 + \ln t)\,\mathbf{i} + \mathbf{j} - e^{-t}\,\mathbf{k}$,

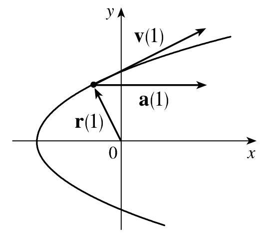
$|\mathbf{v}\,(t)| = \sqrt{(1+\ln t)^2 + 1^2 + (-e^{-t})^2} = \sqrt{2 + 2\ln t + (\ln t)^2 + e^{-2t}}$, $\quad \mathbf{a}(t) = \mathbf{v}'(t) = \frac{1}{t}\,\mathbf{i} + e^{-t}\,\mathbf{k}$

**18.** $\mathbf{r}(t) = (2t^2 - 3)\,\mathbf{i} + 2t\,\mathbf{j} \quad \Rightarrow \quad \mathbf{v}(t) = \mathbf{r}'(t) = 4t\,\mathbf{i} + 2\,\mathbf{j}$,

speed $= |\mathbf{v}(t)| = \sqrt{16t^2 + 4} = 2\sqrt{4t^2 + 1}$, and $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) = 4\,\mathbf{i}$.

At $t = 1$ we have $\mathbf{r}(1) = -\mathbf{i} + 2\,\mathbf{j}$, $\mathbf{v}(1) = 4\,\mathbf{i} + 2\,\mathbf{j}$, $\mathbf{a}(1) = 4\,\mathbf{i}$.

Notice that $y/2 = t \quad \Rightarrow \quad x = 2\,(y/2)^2 - 3 = \frac{1}{2}y^2 - 3$, so the path is a parabola.




**19.** $\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int (6t\,\mathbf{i} + 12t^2\,\mathbf{j} - 6t\,\mathbf{k})\,dt = 3t^2\,\mathbf{i} + 4t^3\,\mathbf{j} - 3t^2\,\mathbf{k} + \mathbf{C}$, but $\mathbf{i} - \mathbf{j} + 3\,\mathbf{k} = \mathbf{v}(0) = \mathbf{0} + \mathbf{C}$,

so $\mathbf{C} = \mathbf{i} - \mathbf{j} + 3\,\mathbf{k}$ and $\mathbf{v}(t) = (3t^2 + 1)\,\mathbf{i} + (4t^3 - 1)\,\mathbf{j} + (3 - 3t^2)\,\mathbf{k}$.

$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = (t^3 + t)\,\mathbf{i} + (t^4 - t)\,\mathbf{j} + (3t - t^3)\,\mathbf{k} + \mathbf{D}$.

But $\mathbf{r}(0) = \mathbf{0}$, so $\mathbf{D} = \mathbf{0}$ and $\mathbf{r}(t) = (t^3 + t)\,\mathbf{i} + (t^4 - t)\,\mathbf{j} + (3t - t^3)\,\mathbf{k}$.

**20.** We set up the axes so that the shot leaves the athlete's hand 2 m above the origin. Then we are given $r(0) = 2\mathbf{j}$, $|v(0)| = 13$ m/s, and $v(0)$ has direction given by a $45°$ angle of elevation. Then a unit vector in the direction of $v(0)$ is $\frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j}) \implies v(0) = \frac{13}{\sqrt{2}}(\mathbf{i}+\mathbf{j})$. Assuming air resistance is negligible, the only external force is due to gravity, so as in Example 13.4.5, we have $a = -g\mathbf{j}$, where $g = 9.8$ m/s. Since $v\ (t) = a(t)$, we integrate, giving $v(t) = -gt\mathbf{j} + C$, where $C = v(0) = \frac{13}{\sqrt{2}}(\mathbf{i}+\mathbf{j}) \implies v(t) = \frac{13}{\sqrt{2}}\mathbf{i} + (\frac{13}{\sqrt{2}} - gt)\mathbf{j}$. Since $r\,'(t) = v(t)$, we integrate again, so $r(t) = \frac{13}{\sqrt{2}}t\mathbf{i} + (\frac{13}{\sqrt{2}}t - \frac{1}{2}gt^2\mathbf{j}) + \mathbf{D}$. But $\mathbf{D} = r(0) = 2\mathbf{j} \implies r(t) = \frac{13}{\sqrt{2}}t\mathbf{i} + (\frac{13}{\sqrt{2}}t - \frac{1}{2}gt^2 + 2)\mathbf{j}$

(a) At 2 seconds, the shot is at

$$r(2) = \frac{13}{\sqrt{2}}(2)\mathbf{i} + (\frac{13}{\sqrt{2}}(2) - \frac{1}{2}g(2)^2 + 2)\mathbf{j} \approx 18.38\mathbf{i} + 0.78\mathbf{j} \qquad (1)$$

so the shot is about 0.78 m above the ground, at a horizontal distance of 18.38 m from the athlete.

(b) The shot reaches its maximum height when the vertical component of velocity is 0: $\frac{13}{\sqrt{2}} - gt = 0 \implies t = \frac{13}{\sqrt{2}g} \approx 0.94$ s. Then

$$r(0.94) = \frac{13}{\sqrt{2}}(0.94)\mathbf{i} + (\frac{13}{\sqrt{2}}(0.94) - \frac{1}{2}g(0.94)^2 + 2)\mathbf{j} \approx 8.64\mathbf{i} + 6.31\mathbf{j} \qquad (2)$$

so the maximum height is approximately 6.31 m.

(c) The shot hits the ground when the vertical component of $r(t)$ is 0, so $\frac{13}{\sqrt{2}}t - \frac{1}{2}gt^2 + 2 = 0 \implies 4.9t^2 - \frac{13}{\sqrt{2}}t - 2 = 0 \implies$ 2.07 s, so

$$r(2.07) = \frac{13}{\sqrt{2}}(2.07)\mathbf{i} + (\frac{13}{\sqrt{2}}(2.07) - \frac{1}{2}g(2.07)^2 + 2)\mathbf{j} \approx 19.03\mathbf{i} + 0.03\mathbf{j} \qquad (3)$$

thus, the shot lands approximately 19.03 m from the athlete.

**21.** Example 13.4.5 showed that the maximum horizontal range is achieved with an angle of elevation of $45°$. In this case, however, the projectile would hit the top of the tunnel using that angle. The horizontal range will be maximized with the largest angle of elevation that keeps the projectile within a height of 30 m. From Example 13.4.5, we know that the position function of the projectile is $\mathbf{r}(t) = (v_0\cos\alpha)t\,\mathbf{i} + \left[(v_0\sin\alpha)t - \frac{1}{2}gt^2\right]\mathbf{j}$ and the velocity is

$\mathbf{v}(t) = \mathbf{r}'(t) = (v_0 \cos\alpha)\,\mathbf{i} + [(v_0 \sin\alpha) - gt]\,\mathbf{j}$. The projectile achieves its maximum height when the vertical component of velocity is zero, so $(v_0 \sin\alpha) - gt = 0 \quad\Rightarrow\quad t = \dfrac{v_0 \sin\alpha}{g}$. We want the vertical height of the projectile at that time to be 30 m: $\quad (v_0 \sin\alpha)\left(\dfrac{v_0 \sin\alpha}{g}\right) - \tfrac{1}{2}g\left(\dfrac{v_0 \sin\alpha}{g}\right)^2 = 30 \quad\Rightarrow$

$$\left(\frac{v_0^2 \sin^2\alpha}{g}\right) - \frac{1}{2}\left(\frac{v_0^2 \sin^2\alpha}{g}\right) = 30 \quad\Rightarrow\quad \frac{v_0^2 \sin^2\alpha}{2g} = 30 \quad\Rightarrow\quad \sin^2\alpha = \frac{30(2g)}{v_0^2} = \frac{60(9.8)}{40^2} = 0.3675 \quad\Rightarrow$$

$\sin\alpha = \sqrt{0.3675}$. Thus, the desired angle of elevation is $\alpha = \sin^{-1}\sqrt{0.3675} \approx 37.3°$.

From the same example, the horizontal distance traveled is $d = \dfrac{v_0^2 \sin 2\alpha}{g} \approx \dfrac{40^2\,\sin(74.6°)}{9.8} \approx 157.4$ m.

**22.** $\mathbf{r}(t) = t\,\mathbf{i} + 2t\,\mathbf{j} + t^2\,\mathbf{k}, \quad \mathbf{r}'(t) = \mathbf{i} + 2\,\mathbf{j} + 2t\,\mathbf{k}, \quad \mathbf{r}''(t) = 2\,\mathbf{k}, \quad |\mathbf{r}'(t)| = \sqrt{1+4+4t^2} = \sqrt{4t^2+5}$.

Then $a_T = \dfrac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \dfrac{4t}{\sqrt{4t^2+5}}$ and $a_N = \dfrac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \dfrac{|4\,\mathbf{i} - 2\,\mathbf{j}|}{\sqrt{4t^2+5}} = \dfrac{2\sqrt{5}}{\sqrt{4t^2+5}}$.

**23.** (a) Instead of proceeding directly, we use Formula 3 of Theorem 13.2.3: $\mathbf{r}(t) = t\,\mathbf{R}(t) \quad\Rightarrow$

$\mathbf{v} = \mathbf{r}'(t) = \mathbf{R}(t) + t\,\mathbf{R}'(t) = \cos\omega t\,\mathbf{i} + \sin\omega t\,\mathbf{j} + t\,\mathbf{v}_d$.

(b) Using the same method as in part (a) and starting with $\mathbf{v} = \mathbf{R}(t) + t\,\mathbf{R}'(t)$, we have

$\mathbf{a} = \mathbf{v}' = \mathbf{R}'(t) + \mathbf{R}'(t) + t\,\mathbf{R}''(t) = 2\,\mathbf{R}'(t) + t\,\mathbf{R}''(t) = 2\,\mathbf{v}_d + t\,\mathbf{a}_d$.

(c) Here we have $\mathbf{r}(t) = e^{-t}\cos\omega t\,\mathbf{i} + e^{-t}\sin\omega t\,\mathbf{j} = e^{-t}\,\mathbf{R}(t)$. So, as in parts (a) and (b),

$\mathbf{v} = \mathbf{r}'(t) = e^{-t}\,\mathbf{R}'(t) - e^{-t}\,\mathbf{R}(t) = e^{-t}[\mathbf{R}'(t) - \mathbf{R}(t)] \quad\Rightarrow$

$$\begin{aligned} \mathbf{a} = \mathbf{v}' &= e^{-t}[\mathbf{R}''(t) - \mathbf{R}'(t)] - e^{-t}[\mathbf{R}'(t) - \mathbf{R}(t)] = e^{-t}[\mathbf{R}''(t) - 2\,\mathbf{R}'(t) + \mathbf{R}(t)] \\ &= e^{-t}\,\mathbf{a}_d - 2e^{-t}\,\mathbf{v}_d + e^{-t}\,\mathbf{R} \end{aligned}$$

Thus, the Coriolis acceleration (the sum of the "extra" terms not involving $\mathbf{a}_d$) is $-2e^{-t}\,\mathbf{v}_d + e^{-t}\,\mathbf{R}$.

**24.** (a) $F(x) = \begin{cases} 1 & \text{if } x \le 0 \\ \sqrt{1-x^2} & \text{if } 0 < x < \frac{1}{\sqrt{2}} \\ \sqrt{2} - x & \text{if } x \ge \frac{1}{\sqrt{2}} \end{cases} \quad\Rightarrow\quad F'(x) = \begin{cases} 0 & \text{if } x < 0 \\ -x/\sqrt{1-x^2} & \text{if } 0 < x < \frac{1}{\sqrt{2}} \\ -1 & \text{if } x > \frac{1}{\sqrt{2}} \end{cases} \quad\Rightarrow$

$$F''(x) = \begin{cases} 0 & \text{if } x < 0 \\ -1/(1-x^2)^{3/2} & \text{if } 0 < x < \frac{1}{\sqrt{2}} \\ 0 & \text{if } x > \frac{1}{\sqrt{2}} \end{cases}$$

since $\dfrac{d}{dx}[-x(1-x^2)^{-1/2}] = -(1-x^2)^{-1/2} - x^2(1-x^2)^{-3/2} = -(1-x^2)^{-3/2}$.

Now $\lim\limits_{x\to 0^+}\sqrt{1-x^2} = 1 = F(0)$ and $\lim\limits_{x\to(1/\sqrt{2})^-}\sqrt{1-x^2} = \frac{1}{\sqrt{2}} = F\left(\frac{1}{\sqrt{2}}\right)$, so $F$ is continuous. Also, since

$\lim_{x \to 0^+} F'(x) = 0 = \lim_{x \to 0^-} F'(x)$ and $\lim_{x \to (1/\sqrt{2})^-} F'(x) = -1 = \lim_{x \to (1/\sqrt{2})^+} F'(x)$, $F'$ is continuous. But

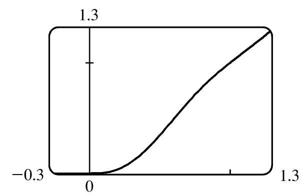
$\lim_{x \to 0^+} F''(x) = -1 \neq 0 = \lim_{x \to 0^-} F''(x)$, so $F''$ is not continuous at $x = 0$. (The same is true at $x = \frac{1}{\sqrt{2}}$.)

So $F$ does not have continuous curvature.

(b) Set $P(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$. The continuity conditions on $P$ are $P(0) = 0$, $P(1) = 1$, $P'(0) = 0$ and $P'(1) = 1$. Also, the curvature must be continuous. For $x \leq 0$ and $x \geq 1$, $\kappa(x) = 0$; elsewhere

$\kappa(x) = \dfrac{|P''(x)|}{(1 + [P'(x)]^2)^{3/2}}$, so we need $P''(0) = 0$ and $P''(1) = 0$.

The conditions $P(0) = P'(0) = P''(0) = 0$ imply that $d = e = f = 0$. The other conditions imply that $a + b + c = 1$, $5a + 4b + 3c = 1$, and $10a + 6b + 3c = 0$. From these, we find that $a = 3$, $b = -8$, and $c = 6$. Therefore $P(x) = 3x^5 - 8x^4 + 6x^3$. Since there was no solution with $a = 0$, this could not have been done with a polynomial of degree 4.



# PROBLEMS PLUS

**1.** (a) $\mathbf{r}(t) = R\cos\omega t\,\mathbf{i} + R\sin\omega t\,\mathbf{j} \quad\Rightarrow\quad \mathbf{v} = \mathbf{r}'(t) = -\omega R\sin\omega t\,\mathbf{i} + \omega R\cos\omega t\,\mathbf{j}$, so $\mathbf{r} = R(\cos\omega t\,\mathbf{i} + \sin\omega t\,\mathbf{j})$ and $\mathbf{v} = \omega R(-\sin\omega t\,\mathbf{i} + \cos\omega t\,\mathbf{j})$. $\mathbf{v}\cdot\mathbf{r} = \omega R^2(-\cos\omega t\sin\omega t + \sin\omega t\cos\omega t) = 0$, so $\mathbf{v}\perp\mathbf{r}$. Since $\mathbf{r}$ points along a radius of the circle, and $\mathbf{v}\perp\mathbf{r}$, $\mathbf{v}$ is tangent to the circle. Because it is a velocity vector, $\mathbf{v}$ points in the direction of motion.

(b) In (a), we wrote $\mathbf{v}$ in the form $\omega R\,\mathbf{u}$, where $\mathbf{u}$ is the unit vector $-\sin\omega t\,\mathbf{i} + \cos\omega t\,\mathbf{j}$. Clearly $|\mathbf{v}| = \omega R\,|\mathbf{u}| = \omega R$. At speed $\omega R$, the particle completes one revolution, a distance $2\pi R$, in time $T = \dfrac{2\pi R}{\omega R} = \dfrac{2\pi}{\omega}$.

(c) $\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = -\omega^2 R\cos\omega t\,\mathbf{i} - \omega^2 R\sin\omega t\,\mathbf{j} = -\omega^2 R(\cos\omega t\,\mathbf{i} + \sin\omega t\,\mathbf{j})$, so $\mathbf{a} = -\omega^2\mathbf{r}$. This shows that $\mathbf{a}$ is proportional to $\mathbf{r}$ and points in the opposite direction (toward the origin). Also, $|\mathbf{a}| = \omega^2\,|\mathbf{r}| = \omega^2 R$.

(d) By Newton's Second Law (see Section 13.4), $\mathbf{F} = m\mathbf{a}$, so $|\mathbf{F}| = m\,|\mathbf{a}| = mR\omega^2 = \dfrac{m\,(\omega R)^2}{R} = \dfrac{m\,|\mathbf{v}|^2}{R}$.

**2.** (a) Dividing the equation $|\mathbf{F}|sin\theta = \frac{mv_R^2}{R}$ by the equation $|\mathbf{F}|cos\theta = mg$, we obtain $tan\theta = \frac{v_R^2}{Rg}$, so $v_R^2 = Rgtan\theta$.

(b) $R = 120$ m and $\theta = 12^\circ$, so $v_R = \sqrt{Rgtan\theta} \approx \sqrt{120\cdot 9.8\cdot tan12^\circ} \approx 15.81$ m/s.

(c) We want to choose a new radius $R_1$ for which the new rated speed is $\frac{3}{2}$ of the old one:

$$\sqrt{R_1 gtan12^\circ} = \frac{3}{2}\sqrt{Rgtan12^\circ}$$

Squaring, we get $R_1 gtan12^\circ = \frac{9}{4}Rgtan12^\circ$, so $R_1 = \frac{9}{4}R = \frac{9}{4}(120) = 270$ m.

**3.** (a) The projectile reaches maximum height when $0 = \dfrac{dy}{dt} = \dfrac{d}{dt}\left[(v_0\sin\alpha)t - \frac{1}{2}gt^2\right] = v_0\sin\alpha - gt$; that is, when $t = \dfrac{v_0\sin\alpha}{g}$ and $y = (v_0\sin\alpha)\left(\dfrac{v_0\sin\alpha}{g}\right) - \dfrac{1}{2}g\left(\dfrac{v_0\sin\alpha}{g}\right)^2 = \dfrac{v_0^2\sin^2\alpha}{2g}$. This is the maximum height attained when the projectile is fired with an angle of elevation $\alpha$. This maximum height is largest when $\alpha = 90^\circ$. In that case, $\sin\alpha = 1$ and the maximum height is $\dfrac{v_0^2}{2g}$.

(b) Let $R = v_0^2/g$. We are asked to consider the parabola $x^2 + 2Ry - R^2 = 0$ which can be rewritten as $y = -\dfrac{1}{2R}x^2 + \dfrac{R}{2}$. The points on or inside this parabola are those for which $-R \le x \le R$ and $0 \le y \le \dfrac{-1}{2R}x^2 + \dfrac{R}{2}$. When the projectile is fired at angle of elevation $\alpha$, the points $(x, y)$ along its path satisfy the relations $x = (v_0\cos\alpha)\,t$ and $y = (v_0\sin\alpha)t - \frac{1}{2}gt^2$, where $0 \le t \le (2v_0\sin\alpha)/g$ (as in Example 13.4.5). Thus,
$|x| \le \left|v_0\cos\alpha\left(\dfrac{2v_0\sin\alpha}{g}\right)\right| = \left|\dfrac{v_0^2}{g}\sin 2\alpha\right| \le \left|\dfrac{v_0^2}{g}\right| = |R|$. This shows that $-R \le x \le R$.

For $t$ in the specified range, we also have $y = t\big(v_0\sin\alpha - \frac{1}{2}gt\big) = \frac{1}{2}gt\left(\dfrac{2v_0\sin\alpha}{g} - t\right) \ge 0$ and

$y = (v_0 \sin\alpha)\,\dfrac{x}{v_0\cos\alpha} - \dfrac{g}{2}\left(\dfrac{x}{v_0\cos\alpha}\right)^2 = (\tan\alpha)\,x - \dfrac{g}{2v_0^2\cos^2\alpha}x^2 = -\dfrac{1}{2R\cos^2\alpha}\,x^2 + (\tan\alpha)\,x$. Thus,

$$\begin{aligned} y - \left(\frac{-1}{2R}x^2 + \frac{R}{2}\right) &= \frac{-1}{2R\cos^2\alpha}\,x^2 + \frac{1}{2R}x^2 + (\tan\alpha)\,x - \frac{R}{2} \\ &= \frac{x^2}{2R}\left(1 - \frac{1}{\cos^2\alpha}\right) + (\tan\alpha)\,x - \frac{R}{2} = \frac{x^2(1-\sec^2\alpha) + 2R\,(\tan\alpha)\,x - R^2}{2R} \\ &= \frac{-(\tan^2\alpha)\,x^2 + 2R\,(\tan\alpha)\,x - R^2}{2R} = \frac{-\left[(\tan\alpha)\,x - R\right]^2}{2R} \le 0 \end{aligned}$$

We have shown that every target that can be hit by the projectile lies on or inside the parabola $y = -\dfrac{1}{2R}\,x^2 + \dfrac{R}{2}$.

Now let $(a,b)$ be any point on or inside the parabola $y = -\dfrac{1}{2R}\,x^2 + \dfrac{R}{2}$. Then $-R \le a \le R$ and $0 \le b \le -\dfrac{1}{2R}\,a^2 + \dfrac{R}{2}$.

We seek an angle $\alpha$ such that $(a,b)$ lies in the path of the projectile; that is, we wish to find an angle $\alpha$ such that

$b = -\dfrac{1}{2R\cos^2\alpha}\,a^2 + (\tan\alpha)\,a$ or equivalently $b = \dfrac{-1}{2R}\,(\tan^2\alpha + 1)a^2 + (\tan\alpha)\,a$. Rearranging this equation, we get

$\dfrac{a^2}{2R}\tan^2\alpha - a\tan\alpha + \left(\dfrac{a^2}{2R} + b\right) = 0$ or $a^2(\tan\alpha)^2 - 2aR(\tan\alpha) + (a^2 + 2bR) = 0$ $(\star)$. This quadratic equation

for $\tan\alpha$ has real solutions exactly when the discriminant is nonnegative. Now $B^2 - 4AC \ge 0 \quad\Leftrightarrow$

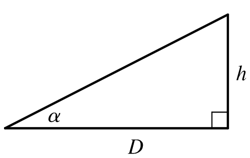
$(-2aR)^2 - 4a^2(a^2 + 2bR) \ge 0 \quad\Leftrightarrow\quad 4a^2(R^2 - a^2 - 2bR) \ge 0 \quad\Leftrightarrow\quad -a^2 - 2bR + R^2 \ge 0 \quad\Leftrightarrow$

$b \le \dfrac{1}{2R}\,(R^2 - a^2) \quad\Leftrightarrow\quad b \le \dfrac{-1}{2R}\,a^2 + \dfrac{R}{2}$. This condition is satisfied since $(a,b)$ is on or inside the parabola

$y = -\dfrac{1}{2R}\,x^2 + \dfrac{R}{2}$. It follows that $(a,b)$ lies in the path of the projectile when $\tan\alpha$ satisfies $(\star)$, that is, when

$\tan\alpha = \dfrac{2aR \pm \sqrt{4a^2(R^2 - a^2 - 2bR)}}{2a^2} = \dfrac{R \pm \sqrt{R^2 - 2bR - a^2}}{a}$.

(c)




If the gun is pointed at a target with height $h$ at a distance $D$ downrange, then $\tan\alpha = h/D$. When the projectile reaches a distance $D$ downrange (remember we are assuming that it doesn't hit the ground first), we have $D = x = (v_0\cos\alpha)t$, so $t = \dfrac{D}{v_0\cos\alpha}$ and $y = (v_0\sin\alpha)t - \frac{1}{2}gt^2 = D\tan\alpha - \dfrac{gD^2}{2v_0^2\cos^2\alpha}$.

Meanwhile, the target, whose $x$-coordinate is also $D$, has fallen from height $h$ to height

$h - \frac{1}{2}gt^2 = D\tan\alpha - \dfrac{gD^2}{2v_0^2\cos^2\alpha}$. Thus, the projectile hits the target.

**4.** (a) As in Problem 3, $\mathbf{r}(t) = (v_0\cos\alpha)t\,\mathbf{i} + \left[(v_0\sin\alpha)t - \frac{1}{2}gt^2\right]\mathbf{j}$, so $x = (v_0\cos\alpha)t$ and $y = (v_0\sin\alpha)t - \frac{1}{2}gt^2$. The difference here is that the projectile travels until it reaches a point where $x > 0$ and $y = -(\tan\theta)x$. (Here $0 \le \theta \le \frac{\pi}{2}$.)

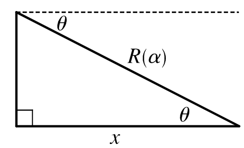
From the parametric equations, we obtain $t = \dfrac{x}{v_0\cos\alpha}$ and $y = \dfrac{(v_0\sin\alpha)x}{v_0\cos\alpha} - \dfrac{gx^2}{2v_0^2\cos^2\alpha} = (\tan\alpha)x - \dfrac{gx^2}{2v_0^2\cos^2\alpha}$.

Thus, the projectile hits the inclined plane at the point where $(\tan\alpha)x - \dfrac{gx^2}{2v_0^2\cos^2\alpha} = -(\tan\theta)x$. Since

$\dfrac{gx^2}{2v_0^2\cos^2\alpha} = (\tan\alpha + \tan\theta)x$ and $x > 0$, we must have $\dfrac{gx}{2v_0^2\cos^2\alpha} = \tan\alpha + \tan\theta$. It follows that

$x = \dfrac{2v_0^2 \cos^2 \alpha}{g}(\tan\alpha + \tan\theta)$ and $t = \dfrac{x}{v_0 \cos\alpha} = \dfrac{2v_0 \cos\alpha}{g}(\tan\alpha + \tan\theta)$. This means that the parametric equations are defined for $t$ in the interval $\left[0, \dfrac{2v_0 \cos\alpha}{g}(\tan\alpha + \tan\theta)\right]$.

(b) The downhill range (that is, the distance to the projectile's landing point as measured along the inclined plane) is $R(\alpha) = x\sec\theta$, where $x$ is the coordinate of the landing point calculated in part (a). Thus,




$$R(\alpha) = \frac{2v_0^2 \cos^2\alpha}{g}(\tan\alpha + \tan\theta)\sec\theta = \frac{2v_0^2}{g}\left(\frac{\sin\alpha\,\cos\alpha}{\cos\theta} + \frac{\cos^2\alpha\,\sin\theta}{\cos^2\theta}\right)$$
$$= \frac{2v_0^2\cos\alpha}{g\cos^2\theta}(\sin\alpha\,\cos\theta + \cos\alpha\,\sin\theta) = \frac{2v_0^2\cos\alpha\,\sin(\alpha+\theta)}{g\cos^2\theta}$$

$R(\alpha)$ is maximized when

$$0 = R'(\alpha) = \frac{2v_0^2}{g\cos^2\theta}[-\sin\alpha\,\sin(\alpha+\theta) + \cos\alpha\,\cos(\alpha+\theta)]$$
$$= \frac{2v_0^2}{g\cos^2\theta}\cos[(\alpha+\theta)+\alpha] = \frac{2v_0^2\cos(2\alpha+\theta)}{g\cos^2\theta}$$

This condition implies that $\cos(2\alpha+\theta) = 0 \;\Rightarrow\; 2\alpha+\theta = \frac{\pi}{2} \;\Rightarrow\; \alpha = \frac{1}{2}\left(\frac{\pi}{2} - \theta\right)$.

(c) The solution is similar to the solutions to parts (a) and (b). This time the projectile travels until it reaches a point where $x > 0$ and $y = (\tan\theta)x$. Since $\tan\theta = -\tan(-\theta)$, we obtain the solution from the previous one by replacing $\theta$ with $-\theta$. The desired angle is $\alpha = \frac{1}{2}\left(\frac{\pi}{2} + \theta\right)$.

(d) As observed in part (c), firing the projectile up an inclined plane with angle of inclination $\theta$ involves the same equations as in parts (a) and (b) but with $\theta$ replaced by $-\theta$. So if $R$ is the distance up an inclined plane, we know from part (b) that $R = \dfrac{2v_0^2\cos\alpha\sin(\alpha-\theta)}{g\cos^2(-\theta)} \;\Rightarrow\; v_0^2 = \dfrac{Rg\cos^2\theta}{2\cos\alpha\sin(\alpha-\theta)}$. $v_0^2$ is minimized (and hence $v_0$ is minimized) with respect to $\alpha$ when

$$0 = \frac{d}{d\alpha}(v_0^2) = \frac{Rg\cos^2\theta}{2}\cdot\frac{-(\cos\alpha\cos(\alpha-\theta) - \sin\alpha\sin(\alpha-\theta))}{[\cos\alpha\sin(\alpha-\theta)]^2}$$
$$= \frac{-Rg\cos^2\theta}{2}\cdot\frac{\cos[\alpha+(\alpha-\theta)]}{[\cos\alpha\sin(\alpha-\theta)]^2} = \frac{-Rg\cos^2\theta}{2}\cdot\frac{\cos(2\alpha-\theta)}{[\cos\alpha\sin(\alpha-\theta)]^2}$$

Since $\theta < \alpha < \frac{\pi}{2}$, this implies $\cos(2\alpha-\theta) = 0 \;\Leftrightarrow\; 2\alpha-\theta = \frac{\pi}{2} \;\Rightarrow\; \alpha = \frac{1}{2}\left(\frac{\pi}{2}+\theta\right)$. Thus the initial speed, and hence the energy required, is minimized for $\alpha = \frac{1}{2}\left(\frac{\pi}{2}+\theta\right)$.

**5.** (a) $a = -g\mathbf{j} \Longrightarrow v = v_0 - gt\mathbf{j} = \frac{1}{2}\mathbf{i} - gt\mathbf{j} \Longrightarrow s = s_0 + \frac{1}{2}t\mathbf{i} - \frac{1}{2}gt^2\mathbf{j} = 1.2\mathbf{j} + \frac{1}{2}t\mathbf{i} - \frac{1}{2}gt^2\mathbf{j} = \frac{1}{2}t\mathbf{i} + (1.2 - \frac{1}{2}gt^2)\mathbf{j}$. Therefore, $y = 0$ when $t = \sqrt{2.4/g}$ seconds. At that instant, the ball is $\frac{1}{2}\cdot t \approx 0.247$ m to the right of the table top. Its coordinates (relative to an origin on the floor directly under the table's edge) are (0.247, 0). At impact, the velocity is $v = \frac{1}{2}\mathbf{i} - \sqrt{2.4g}\,\mathbf{j} \Longrightarrow |v| = \sqrt{(\frac{1}{2})^2 + 2.4g} \approx 4.88$ m/s.

(b) The slope of the curve when $t = \sqrt{\frac{2.4}{g}}$ is $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-gt}{2} = \frac{-g\sqrt{2.4/g}}{2} = \frac{-\sqrt{2.4g}}{2}$. Thus, $cot\theta = \frac{\sqrt{2.4g}}{2}$ and $\theta \approx 22.5^\circ$.

(c) From (a), $|v| = \sqrt{0.25 + 2.4g}$. So the ball rebounds with speed $0.8\sqrt{0.25 + 2.4g} \approx 3.9$ m/s at angle of inclination $90^\circ - \theta \approx 67.5^\circ$. By Example 13.4.5, the horizontal distance traveled between bounces is $d = \frac{v_0^2 sin 2\alpha}{g}$, where $v_0 \approx 3.9$ m/s and $\alpha \approx 67.5^\circ$. Therefore, $d \approx 1.097$ m. So the ball strikes the floor at about $\frac{1}{2}\sqrt{\frac{2.4}{g}} + 1.097 \approx 1.34$ m.

**6.** By the Fundamental Theorem of Calculus, $\mathbf{r}'(t) = \left\langle \sin\left(\frac{1}{2}\pi t^2\right), \cos\left(\frac{1}{2}\pi t^2\right)\right\rangle$, $|\mathbf{r}'(t)| = 1$ and so $\mathbf{T}(t) = \mathbf{r}'(t)$.

Thus, $\mathbf{T}'(t) = \pi t\left\langle \cos\left(\frac{1}{2}\pi t^2\right), -\sin\left(\frac{1}{2}\pi t^2\right)\right\rangle$ and the curvature is $\kappa = |\mathbf{T}'(t)| = \sqrt{(\pi t)^2(1)} = \pi\,|t|$.

**7.** The trajectory of the projectile is given by $\mathbf{r}(t) = (v\cos\alpha)t\,\mathbf{i} + \left[(v\sin\alpha)t - \frac{1}{2}gt^2\right]\mathbf{j}$, so

$\mathbf{v}(t) = \mathbf{r}'(t) = v\cos\alpha\,\mathbf{i} + (v\sin\alpha - gt)\,\mathbf{j}$ and

$$|\mathbf{v}(t)| = \sqrt{(v\cos\alpha)^2 + (v\sin\alpha - gt)^2} = \sqrt{v^2 - (2vg\sin\alpha)\,t + g^2t^2} = \sqrt{g^2\left(t^2 - \frac{2v}{g}(\sin\alpha)\,t + \frac{v^2}{g^2}\right)}$$

$$= g\sqrt{\left(t - \frac{v}{g}\sin\alpha\right)^2 + \frac{v^2}{g^2} - \frac{v^2}{g^2}\sin^2\alpha} = g\sqrt{\left(t - \frac{v}{g}\sin\alpha\right)^2 + \frac{v^2}{g^2}\cos^2\alpha}$$

The projectile hits the ground when $(v\sin\alpha)t - \frac{1}{2}gt^2 = 0 \quad\Rightarrow\quad t = \frac{2v}{g}\sin\alpha$, so the distance traveled by the projectile is

$$L(\alpha) = \int_0^{(2v/g)\sin\alpha} |\mathbf{v}(t)|\;dt = \int_0^{(2v/g)\sin\alpha} g\sqrt{\left(t - \frac{v}{g}\sin\alpha\right)^2 + \frac{v^2}{g^2}\cos^2\alpha\,dt}$$

$$= g\left[\frac{t - (v/g)\sin\alpha}{2}\sqrt{\left(t - \frac{v}{g}\sin\alpha\right)^2 + \left(\frac{v}{g}\cos\alpha\right)^2}\right.$$
$$\left. + \frac{[(v/g)\cos\alpha]^2}{2}\ln\left(t - \frac{v}{g}\sin\alpha + \sqrt{\left(t - \frac{v}{g}\sin\alpha\right)^2 + \left(\frac{v}{g}\cos\alpha\right)^2}\right)\right]_0^{(2v/g)\sin\alpha}$$

[using Formula 21 in the Table of Integrals]

$$= \frac{g}{2}\left[\frac{v}{g}\sin\alpha\sqrt{\left(\frac{v}{g}\sin\alpha\right)^2 + \left(\frac{v}{g}\cos\alpha\right)^2} + \left(\frac{v}{g}\cos\alpha\right)^2\ln\left(\frac{v}{g}\sin\alpha + \sqrt{\left(\frac{v}{g}\sin\alpha\right)^2 + \left(\frac{v}{g}\cos\alpha\right)^2}\right)\right.$$
$$\left. + \frac{v}{g}\sin\alpha\sqrt{\left(\frac{v}{g}\sin\alpha\right)^2 + \left(\frac{v}{g}\cos\alpha\right)^2} - \left(\frac{v}{g}\cos\alpha\right)^2\ln\left(-\frac{v}{g}\sin\alpha + \sqrt{\left(\frac{v}{g}\sin\alpha\right)^2 + \left(\frac{v}{g}\cos\alpha\right)^2}\right)\right]$$

$$= \frac{g}{2}\left[\frac{v}{g}\sin\alpha\cdot\frac{v}{g} + \frac{v^2}{g^2}\cos^2\alpha\ln\left(\frac{v}{g}\sin\alpha + \frac{v}{g}\right) + \frac{v}{g}\sin\alpha\cdot\frac{v}{g} - \frac{v^2}{g^2}\cos^2\alpha\ln\left(-\frac{v}{g}\sin\alpha + \frac{v}{g}\right)\right]$$

$$= \frac{v^2}{g}\sin\alpha + \frac{v^2}{2g}\cos^2\alpha\ln\left(\frac{(v/g)\sin\alpha + v/g}{-(v/g)\sin\alpha + v/g}\right) = \frac{v^2}{g}\sin\alpha + \frac{v^2}{2g}\cos^2\alpha\ln\left(\frac{1 + \sin\alpha}{1 - \sin\alpha}\right)$$

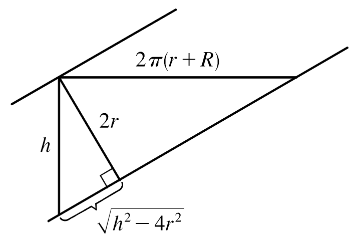
We want to maximize $L(\alpha)$ for $0 \le \alpha \le \pi/2$.

$$L'(\alpha) = \frac{v^2}{g}\cos\alpha + \frac{v^2}{2g}\left[\cos^2\alpha \cdot \frac{1-\sin\alpha}{1+\sin\alpha}\cdot\frac{2\cos\alpha}{(1-\sin\alpha)^2} - 2\cos\alpha\,\sin\alpha\,\ln\!\left(\frac{1+\sin\alpha}{1-\sin\alpha}\right)\right]$$
$$= \frac{v^2}{g}\cos\alpha + \frac{v^2}{2g}\left[\cos^2\alpha\cdot\frac{2}{\cos\alpha} - 2\cos\alpha\,\sin\alpha\,\ln\!\left(\frac{1+\sin\alpha}{1-\sin\alpha}\right)\right]$$
$$= \frac{v^2}{g}\cos\alpha + \frac{v^2}{g}\cos\alpha\left[1 - \sin\alpha\,\ln\!\left(\frac{1+\sin\alpha}{1-\sin\alpha}\right)\right] = \frac{v^2}{g}\cos\alpha\left[2 - \sin\alpha\,\ln\!\left(\frac{1+\sin\alpha}{1-\sin\alpha}\right)\right]$$

$L(\alpha)$ has critical points for $0 < \alpha < \pi/2$ when $L'(\alpha) = 0 \;\Rightarrow\; 2 - \sin\alpha\ln\!\left(\frac{1\,+\,\sin\alpha}{1\,-\,\sin\alpha}\right) = 0$ [since $\cos\alpha \neq 0$].

Solving by graphing (or using a CAS) gives $\alpha \approx 0.9855$. Compare values at the critical point and the endpoints: $L(0) = 0$, $L(\pi/2) = v^2/g$, and $L(0.9855) \approx 1.20v^2/g$. Thus, the distance traveled by the projectile is maximized for $\alpha \approx 0.9855$ or $\approx 56^\circ$.

**8.** As the cable is wrapped around the spool, think of the top or bottom of the cable forming a helix of radius $R + r$. Let $h$ be the vertical distance between coils. Then, from similar triangles,




$$\frac{2r}{\sqrt{h^2-4r^2}} = \frac{2\pi(r+R)}{h} \quad\Rightarrow\quad h^2r^2 = \pi^2(r+R)^2(h^2-4r^2) \quad\Rightarrow$$
$$h = \frac{2\pi r(r+R)}{\sqrt{\pi^2(r+R)^2 - r^2}}.$$

If we parametrize the helix by $x(t) = (R+r)\cos t$, $y(t) = (R+r)\sin t$, then we must have $z(t) = [h/(2\pi)]t$. The length of one complete cycle is

$$\ell = \int_0^{2\pi}\sqrt{[x'(t)]^2+[y'(t)]^2+[z'(t)]^2}\,dt = \int_0^{2\pi}\sqrt{(R+r)^2+\left(\frac{h}{2\pi}\right)^2}\,dt = 2\pi\sqrt{(R+r)^2+\left(\frac{h}{2\pi}\right)^2}$$
$$= 2\pi\sqrt{(R+r)^2+\frac{r^2(R+r)^2}{\pi^2(R+r)^2-r^2}} = 2\pi(R+r)\sqrt{1+\frac{r^2}{\pi^2(R+r)^2-r^2}} = \frac{2\pi^2(R+r)^2}{\sqrt{\pi^2(R+r)^2-r^2}}$$

The number of complete cycles is $[\![L/\ell]\!]$, and so the shortest length along the spool is

$$h\left[\!\!\left[\frac{L}{\ell}\right]\!\!\right] = \frac{2\pi r(R+r)}{\sqrt{\pi^2(R+r)^2-r^2}}\left[\!\!\left[\frac{L\sqrt{\pi^2(R+r)^2-r^2}}{2\pi^2(R+r)^2}\right]\!\!\right]$$

**9.** We can write the vector equation as $\mathbf{r}(t) = \mathbf{a}t^2 + \mathbf{b}t + \mathbf{c}$ where $\mathbf{a} = \langle a_1, a_2, a_3\rangle$, $\mathbf{b} = \langle b_1, b_2, b_3\rangle$, and $\mathbf{c} = \langle c_1, c_2, c_3\rangle$. Then $\mathbf{r}'(t) = 2t\,\mathbf{a} + \mathbf{b}$, which says that each tangent vector is the sum of a scalar multiple of $\mathbf{a}$ and the vector $\mathbf{b}$. Thus, the tangent vectors are all parallel to the plane determined by $\mathbf{a}$ and $\mathbf{b}$ so the curve must be parallel to this plane. [Here we assume that $\mathbf{a}$ and $\mathbf{b}$ are nonparallel. Otherwise, the tangent vectors are all parallel and the curve lies along a single line.] A normal vector for the plane is $\mathbf{a}\times\mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle$. The point $(c_1, c_2, c_3)$ lies on the plane (when $t = 0$), so an equation of the plane is

$$(a_2b_3 - a_3b_2)(x - c_1) + (a_3b_1 - a_1b_3)(y - c_2) + (a_1b_2 - a_2b_1)(z - c_3) = 0$$

or

$$(a_2b_3 - a_3b_2)x + (a_3b_1 - a_1b_3)y + (a_1b_2 - a_2b_1)z = a_2b_3c_1 - a_3b_2c_1 + a_3b_1c_2 - a_1b_3c_2 + a_1b_2c_3 - a_2b_1c_3$$