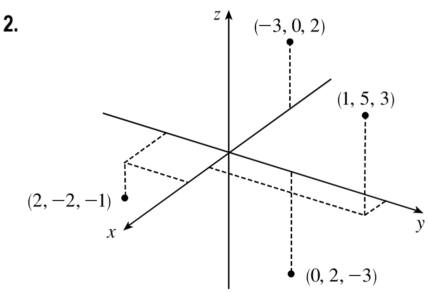
12 □ VECTORS AND THE GEOMETRY OF SPACE

12.1 Three-Dimensional Coordinate Systems

1. We start at the origin, which has coordinates $(0, 0, 0)$. First we move 4 units along the positive x-axis, affecting only the x-coordinate, bringing us to the point $(4, 0, 0)$. We then move 3 units straight downward, in the negative z-direction. Thus only the z-coordinate is affected, and we arrive at $(4, 0, -3)$.

2.

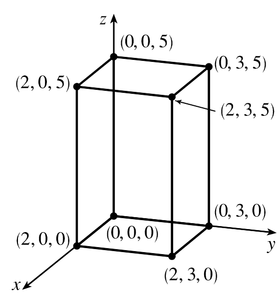


3. The distance from a point to the yz-plane is the absolute value of the x-coordinate of the point. $C(2, 4, 6)$ has the x-coordinate with the smallest absolute value, so C is the point closest to the yz-plane. $A(-4, 0, -1)$ must lie in the xz-plane since the distance from A to the xz-plane, given by the y-coordinate of A, is 0.

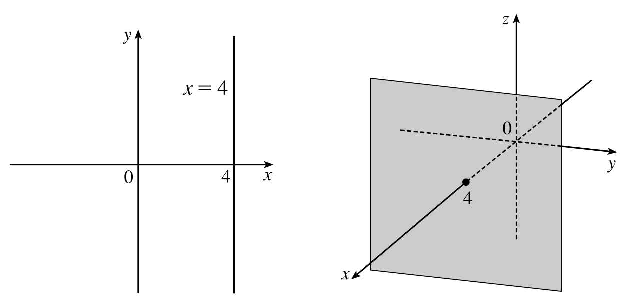
4. The projection of $(2, 3, 5)$ onto the xy-plane is $(2, 3, 0)$; onto the yz-plane, $(0, 3, 5)$; onto the xz-plane, $(2, 0, 5)$.

The length of the diagonal of the box is the distance between the origin and $(2, 3, 5)$, given by

$$\sqrt{(2-0)^2 + (3-0)^2 + (5-0)^2} = \sqrt{38} \approx 6.16$$



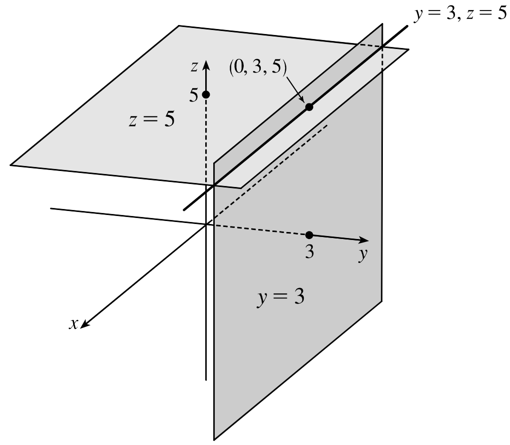
5. In $\mathbb{R}^2$, the equation $x = 4$ represents a line parallel to the y-axis and 4 units to the right of it. In $\mathbb{R}^3$, the equation $x = 4$ represents the set $\{(x, y, z) \mid x = 4\}$, the set of all points whose x-coordinate is 4. This is the vertical plane that is parallel to the yz-plane and 4 units in front of it.



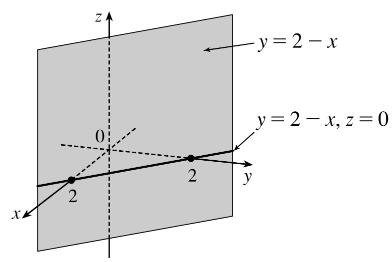
6. In $\mathbb{R}^3$, the equation $y = 3$ represents a vertical plane that is parallel to the xz-plane and 3 units to the right of it. The equation $z = 5$ represents a horizontal plane parallel to the xy-plane and 5 units above it. The pair of equations $y = 3$, $z = 5$ represents the set of points that are simultaneously on both planes, or in other words, the line of intersection of the planes $y = 3$, $z = 5$.

[continued]

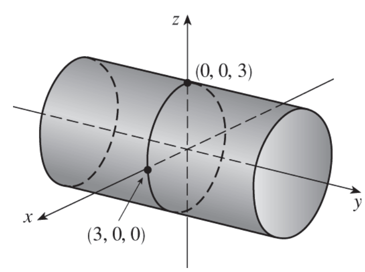
This line can also be described as the set $\{(x, 3, 5) \mid x \in \mathbb{R}\}$, which is the set of all points in $\mathbb{R}^3$ whose x-coordinate may vary but whose y- and z-coordinates are fixed at 3 and 5, respectively. Thus the line is parallel to the x-axis and intersects the yz-plane in the point $(0, 3, 5)$.



7. The equation $x + y = 2$ represents the set of all points in $\mathbb{R}^3$ whose x- and y-coordinates have a sum of 2, or equivalently where $y = 2 - x$. This is the set $\{(x, 2 - x, z) \mid x \in \mathbb{R}, z \in \mathbb{R}\}$ which is a vertical plane that intersects the xy-plane in the line $y = 2 - x$, $z = 0$.



8. The equation $x^2 + z^2 = 9$ has no restrictions on y, and the x- and z-coordinates satisfy the equation for a circle of radius 3 with center on the origin. Thus the surface $x^2 + z^2 = 9$ in $\mathbb{R}^3$ consists of all possible vertical circles (parallel to the xz-plane) $x^2 + z^2 = 9$, $y = k$, and is therefore a circular cylinder with radius 3 whose axis is the y-axis.



9. The distance between the points $P_1(3, 5, -2)$ and $P_2(-1, 1, -4)$ is

$$|P_1P_2| = \sqrt{(-1-3)^2 + (1-5)^2 + [-4-(-2)]^2} = \sqrt{16 + 16 + 4} = 6$$

10. The distance between the points $P_1(-6, -3, 0)$ and $P_2(2, 4, 5)$ is

$$|P_1P_2| = \sqrt{[2-(-6)]^2 + [4-(-3)]^2 + (5-0)^2} = \sqrt{64 + 49 + 25} = \sqrt{138}$$

11. We can find the lengths of the sides of the triangle by using the distance formula between pairs of vertices:

$$\begin{aligned}
|PQ| &= \sqrt{(7-3)^2 + [0-(-2)]^2 + [1-(-3)]^2} = \sqrt{16 + 4 + 16} = \sqrt{36} = 6 \\
|QR| &= \sqrt{(1-7)^2 + (2-0)^2 + (1-1)^2} = \sqrt{36 + 4 + 0} = \sqrt{40} = 2\sqrt{10} \\
|RP| &= \sqrt{(3-1)^2 + (-2-2)^2 + (-3-1)^2} = \sqrt{4 + 16 + 16} = \sqrt{36} = 6
\end{aligned}$$

The longest side is QR, but the Pythagorean Theorem is not satisfied: $|PQ|^2 + |RP|^2 \neq |QR|^2$. Thus PQR is not a right triangle. PQR is isosceles, as two sides have the same length.

12. Compute the lengths of the sides of the triangle by using the distance formula between pairs of vertices:

$$|PQ| = \sqrt{(4-2)^2 + [1-(-1)]^2 + (1-0)^2} = \sqrt{4+4+1} = \sqrt{9} = 3$$
$$|QR| = \sqrt{(4-4)^2 + (-5-1)^2 + (4-1)^2} = \sqrt{0+36+9} = \sqrt{45} = 3\sqrt{5}$$
$$|RP| = \sqrt{(2-4)^2 + [-1-(-5)]^2 + (0-4)^2} = \sqrt{4+16+16} = \sqrt{36} = 6$$

Since the Pythagorean Theorem is satisfied by $|PQ|^2 + |RP|^2 = |QR|^2$, PQR is a right triangle. PQR is not isosceles, as no two sides have the same length.

13. (a) First we find the distances between points:

$$|AB| = \sqrt{(3-2)^2 + (7-4)^2 + (-2-2)^2} = \sqrt{26}$$
$$|BC| = \sqrt{(1-3)^2 + (3-7)^2 + [3-(-2)]^2} = \sqrt{45}$$
$$|AC| = \sqrt{(1-2)^2 + (3-4)^2 + (3-2)^2} = \sqrt{3}$$

In order for the points to lie on a straight line, the sum of the two shortest distances must be equal to the longest distance. Since $\sqrt{26} + \sqrt{3} \neq \sqrt{45}$, the three points do not lie on a straight line.

(b) First we find the distances between points:

$$|DE| = \sqrt{(1-0)^2 + [-2-(-5)]^2 + (4-5)^2} = \sqrt{11}$$
$$|EF| = \sqrt{(3-1)^2 + [4-(-2)]^2 + (2-4)^2} = \sqrt{44} = 2\sqrt{11}$$
$$|DF| = \sqrt{(3-0)^2 + [4-(-5)]^2 + (2-5)^2} = \sqrt{99} = 3\sqrt{11}$$

Since $\sqrt{11} + 2\sqrt{11} = 3\sqrt{11}$, the three points lie on a straight line.

14. (a) The distance from a point to the xy-plane is the absolute value of the z-coordinate of the point. Thus, the distance from $(4, -2, 6)$ to the xy-plane is $|6| = 6$.

(b) Similarly, the distance to the yz-plane is the absolute value of the x-coordinate of the point: $|4| = 4$.

(c) The distance to the xz-plane is the absolute value of the y-coordinate of the point: $|-2| = 2$.

(d) The point on the x-axis closest to $(4, -2, 6)$ is the point $(4, 0, 0)$. (Approach the x-axis perpendicularly.) The distance from $(4, -2, 6)$ to the x-axis is the distance between these two points: $\sqrt{(4-4)^2 + (-2-0)^2 + (6-0)^2} = \sqrt{40} = 2\sqrt{10} \approx 6.32$.

(e) The point on the y-axis closest to $(4, -2, 6)$ is $(0, -2, 0)$. The distance between these points is $\sqrt{(4-0)^2 + [-2-(-2)]^2 + (6-0)^2} = \sqrt{52} = 2\sqrt{13} \approx 7.21$.

(f) The point on the z-axis closest to $(4, -2, 6)$ is $(0, 0, 6)$. The distance between these points is $\sqrt{(4-0)^2 + (-2-0)^2 + (6-6)^2} = \sqrt{20} = 2\sqrt{5} \approx 4.47$.

15. An equation of the sphere with center $(-3, 2, 5)$ and radius 4 is $[x-(-3)]^2 + (y-2)^2 + (z-5)^2 = 4^2$, or $(x+3)^2 + (y-2)^2 + (z-5)^2 = 16$. The intersection of this sphere with the yz-plane is the set of points on the sphere whose x-coordinate is 0. Putting $x = 0$ into the equation, we have $9 + (y-2)^2 + (z-5)^2 = 16$, $x = 0$ or $(y-2)^2 + (z-5)^2 = 7$, $x = 0$, which represents a circle in the yz-plane with center $(0, 2, 5)$ and radius $\sqrt{7}$.

16. An equation of the sphere with center $(2, -6, 4)$ and radius 5 is $(x-2)^2 + [y-(-6)]^2 + (z-4)^2 = 5^2$, or $(x-2)^2 + (y+6)^2 + (z-4)^2 = 25$. The intersection of this sphere with the xy-plane is the set of points on the sphere whose z-coordinate is 0. Putting $z = 0$ into the equation, we have $(x-2)^2 + (y+6)^2 = 9, z = 0$ which represents a circle in the xy-plane with center $(2, -6, 0)$ and radius 3. To find the intersection with the xz-plane, we set $y = 0$: $(x-2)^2 + (z-4)^2 = -11$. Since no points satisfy this equation, the sphere does not intersect the xz-plane. (Also note that the distance from the center of the sphere to the xz-plane is greater than the radius of the sphere.) To find the intersection with the yz-plane, we set $x = 0$: $(y+6)^2 + (z-4)^2 = 21, x = 0$, a circle in the yz-plane with center $(0, -6, 4)$ and radius $\sqrt{21}$.

17. The radius of the sphere is the distance between $(4, 3, -1)$ and $(3, 8, 1)$: $r = \sqrt{(3-4)^2 + (8-3)^2 + [1-(-1)]^2} = \sqrt{30}$. Thus, an equation of the sphere is $(x-3)^2 + (y-8)^2 + (z-1)^2 = 30$.

18. If the sphere passes through the origin, the radius of the sphere must be the distance from the origin to the point $(1, 2, 3)$: $r = \sqrt{(1-0)^2 + (2-0)^2 + (3-0)^2} = \sqrt{14}$. Then an equation of the sphere is $(x-1)^2 + (y-2)^2 + (z-3)^2 = 14$.

19. Completing squares in the equation $x^2 + y^2 + z^2 + 8x - 2z = 8$ gives $(x^2 + 8x + 16) + y^2 + (z^2 - 2z + 1) = 8 + 16 + 1 \quad \Rightarrow \quad (x+4)^2 + y^2 + (z-1)^2 = 25$, which we recognize as an equation of a sphere with center $(-4, 0, 1)$ and radius $\sqrt{25} = 5$.

20. Completing squares in the equation $x^2 - 6x + y^2 + 4y + z^2 + 10z = 0$ gives $(x^2 - 6x + 9) + (y^2 + 4y + 4) + (z^2 + 10z + 25) = 9 + 4 + 25 \quad \Rightarrow \quad (x-3)^2 + (y+2)^2 + (z+5)^2 = 38$, which we recognize as an equation of a sphere with center $(3, -2, -5)$ and radius $\sqrt{38}$.

21. Completing squares in the equation $2x^2 - 2x + 2y^2 + 4y + 2z^2 = -1$ gives $2\left(x^2 - x + \frac{1}{4}\right) + 2(y^2 + 2y + 1) + 2z^2 = -1 + \frac{1}{2} + 2 \quad \Rightarrow \quad 2\left(x - \frac{1}{2}\right)^2 + 2(y+1)^2 + 2z^2 = \frac{3}{2} \quad \Rightarrow$ $\left(x - \frac{1}{2}\right)^2 + (y+1)^2 + z^2 = \frac{3}{4}$, which we recognize as an equation of a sphere with center $\left(\frac{1}{2}, -1, 0\right)$ and radius $\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$.

22. Completing the squares in the equation $4x^2 - 16x + 4y^2 + 6y + 4z^2 = -12$ gives $4(x^2 - 4x + 4) + 4\left(y^2 + \frac{3}{2}y + \frac{9}{16}\right) + 4z^2 = -12 + 16 + \frac{9}{4} \quad \Rightarrow \quad 4(x-2)^2 + 4\left(y + \frac{3}{4}\right)^2 + 4z^2 = \frac{25}{4} \quad \Rightarrow$ $(x-2)^2 + \left(y + \frac{3}{4}\right)^2 + z^2 = \frac{25}{16}$, which we recognize as the equation of a sphere with center $\left(2, -\frac{3}{4}, 0\right)$ and radius $\sqrt{\frac{25}{16}} = \frac{5}{4}$.

23. If the midpoint of the line segment from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ is $Q = \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}, \dfrac{z_1 + z_2}{2}\right)$, then the distances $|P_1Q|$ and $|QP_2|$ are equal, and each is half of $|P_1P_2|$. We verify that this is the case:

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

[continued]

$$|P_1Q| = \sqrt{\left[\tfrac{1}{2}(x_1+x_2)-x_1\right]^2 + \left[\tfrac{1}{2}(y_1+y_2)-y_1\right]^2 + \left[\tfrac{1}{2}(z_1+z_2)-z_1\right]^2}$$
$$= \sqrt{\left(\tfrac{1}{2}x_2-\tfrac{1}{2}x_1\right)^2 + \left(\tfrac{1}{2}y_2-\tfrac{1}{2}y_1\right)^2 + \left(\tfrac{1}{2}z_2-\tfrac{1}{2}z_1\right)^2} = \sqrt{\left(\tfrac{1}{2}\right)^2\left[(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2\right]}$$
$$= \tfrac{1}{2}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2} = \tfrac{1}{2}\,|P_1P_2|$$

$$|QP_2| = \sqrt{\left[x_2-\tfrac{1}{2}(x_1+x_2)\right]^2 + \left[y_2-\tfrac{1}{2}(y_1+y_2)\right]^2 + \left[z_2-\tfrac{1}{2}(z_1+z_2)\right]^2}$$
$$= \sqrt{\left(\tfrac{1}{2}x_2-\tfrac{1}{2}x_1\right)^2 + \left(\tfrac{1}{2}y_2-\tfrac{1}{2}y_1\right)^2 + \left(\tfrac{1}{2}z_2-\tfrac{1}{2}z_1\right)^2} = \sqrt{\left(\tfrac{1}{2}\right)^2\left[(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2\right]}$$
$$= \tfrac{1}{2}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2} = \tfrac{1}{2}\,|P_1P_2|$$

So Q is indeed the midpoint of P_1P_2.

24. By Exercise 23(a), the midpoint of the diameter that has endpoints $(5,4,3)$ and $(1,6,-9)$ (and thus the center of the sphere) is $\left(\dfrac{5+1}{2}, \dfrac{4+6}{2}, \dfrac{3+(-9)}{2}\right) = (3,5,-3)$. The radius is half the diameter, so $r = \frac{1}{2}\sqrt{(1-5)^2+(6-4)^2+(-9-3)^2} = \frac{1}{2}\sqrt{164} = \sqrt{41}$. Therefore, an equation of the sphere is $(x-3)^2+(y-5)^2+(z+3)^2 = 41$.

25. (a) Since the sphere touches the xy-plane, its radius is the distance from its center, $(-1,4,5)$, to the xy-plane, which is 5. Therefore, an equation is $(x+1)^2+(y-4)^2+(z-5)^2 = 25$.

(b) Since the sphere touches the yz-plane, its radius is the distance from its center, $(-1,4,5)$, to the yz-plane, which is 1. Therefore, an equation is $(x+1)^2+(y-4)^2+(z-5)^2 = 1$.

(c) Since the sphere touches the xz-plane, its radius is the distance from its center, $(-1,4,5)$, to the xz-plane, which is 4. Therefore, an equation is $(x+1)^2+(y-4)^2+(z-5)^2 = 16$.

26. The shortest distance from the center, $(7,3,8)$, to any of the three coordinate planes is 3, which is the distance to the xz-plane. Therefore, an equation of the sphere is $(x-7)^2+(y-3)^2+(z-8)^2 = 9$.

27. The equation $z=-2$ represents a plane, parallel to the xy-plane and 2 units below it.

28. The equation $x=3$ represents a plane, parallel to the yz-plane and 3 units in front of it.

29. The inequality $y \geq 1$ represents a half-space consisting of all the points on or to the right of the plane $y=1$.

30. The inequality $x < 4$ represents a half-space consisting of all the points behind the plane $x=4$.

31. The inequality $-1 \leq x \leq 2$ represents all points on or between the vertical planes $x=-1$ and $x=2$.

32. The equation $z=y$ represents a plane, perpendicular to the yz-plane, and intersecting the yz-plane in the line $z=y$, $x=0$.

33. Because $z=-1$, all points in the region must lie in the horizontal plane $z=-1$. In addition, $x^2+y^2=4$, so the region consists of all points that lie on a circle with radius 2 and center on the z-axis that is contained in the plane $z=-1$.

34. Here $x^2 + y^2 = 4$ with no restrictions on z, so a point in the region must lie on a circle of radius 2, center on the z-axis, but it could be in any horizontal plane $z = k$ (parallel to the xy-plane). Thus the region consists of all possible circles $x^2 + y^2 = 4$, $z = k$ and is therefore a circular cylinder with radius 2 whose axis is the z-axis.

35. The inequality $y^2 + z^2 \leq 25$ is equivalent to $\sqrt{y^2 + z^2} \leq 5$, which describes the set of all points in $\mathbb{R}^3$ whose distance from the x-axis is at most 5. Thus, the inequality represents the region consisting of all points on or inside a circular cylinder of radius 5 with axis the x-axis.

36. The inequality $x^2 + z^2 \leq 25$ is equivalent to $\sqrt{x^2 + z^2} \leq 5$, which describes the set of all points in $\mathbb{R}^3$ whose distance from the y-axis is at most 5. Further, $0 \leq y \leq 2$ consists of the points on or between the planes $y = 0$ and $y = 2$. Thus, the inequalities represent the region consisting of all points on or inside a circular cylinder of radius 5 with axis the y-axis from $y = 0$ to $y = 2$.

37. The equation $x^2 + y^2 + z^2 = 4$ is equivalent to $\sqrt{x^2 + y^2 + z^2} = 2$, so the region consists of those points whose distance from the origin is 2. This is the set of all points on a sphere with radius 2 and center $(0, 0, 0)$.

38. The inequality $x^2 + y^2 + z^2 \leq 4$ is equivalent to $\sqrt{x^2 + y^2 + z^2} \leq 2$, so the region consists of those points whose distance from the origin is at most 2. This is the set of all points on or inside a sphere with radius 2 and center $(0, 0, 0)$.

39. The inequalities $1 \leq x^2 + y^2 + z^2 \leq 5$ are equivalent to $1 \leq \sqrt{x^2 + y^2 + z^2} \leq \sqrt{5}$, so the region consists of those points whose distance from the origin is at least 1 and at most $\sqrt{5}$. This is the set of all points on or between spheres with radii 1 and $\sqrt{5}$ and centers $(0, 0, 0)$.

40. The inequalities $1 \leq x^2 + y^2 \leq 5$ are equivalent to $1 \leq \sqrt{x^2 + y^2} \leq \sqrt{5}$, which represents the set of all points in $\mathbb{R}^3$ whose distance is at least 1 and at most $\sqrt{5}$ from the z-axis. Thus, the region consists of all points on or between a circular cylinder of radius 1 and a circular cylinder of radius $\sqrt{5}$ with axis the z-axis.

41. The inequalities $0 \leq x \leq 3, 0 \leq y \leq 3, 0 \leq z \leq 3$ represent the set of all points in $\mathbb{R}^3$ that lie on or between the planes $x = 3$, $y = 3$, $z = 3$ in the first octant. Thus, the region is a cube with dimensions $3 \times 3 \times 3$.

42. The inequality $x^2 + y^2 + z^2 > 2z \quad \Leftrightarrow \quad x^2 + y^2 + (z - 1)^2 > 1$ is equivalent to $\sqrt{x^2 + y^2 + (z - 1)^2} > 1$, so the region consists of those points whose distance from the point $(0, 0, 1)$ is greater than 1. This is the set of all points outside the sphere with radius 1 and center $(0, 0, 1)$.

43. This describes all points whose x-coordinate is between 0 and 5, that is, $0 < x < 5$.

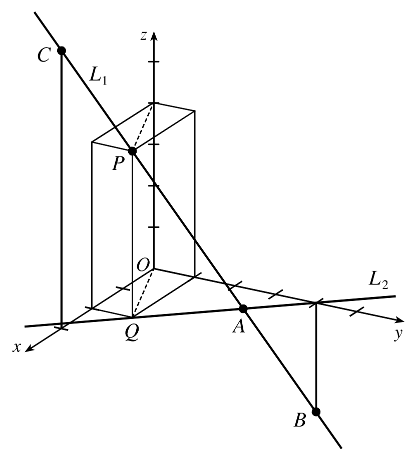
44. For any point on or above the disk in the xy-plane with center the origin and radius 2 we have $x^2 + y^2 \leq 4$. Also each point lies on or between the planes $z = 0$ and $z = 8$, so the region is described by $x^2 + y^2 \leq 4,\ 0 \leq z \leq 8$.

45. This describes a region all of whose points have a distance to the origin which is greater than r, but smaller than R. So inequalities describing the region are $r < \sqrt{x^2 + y^2 + z^2} < R$, or $r^2 < x^2 + y^2 + z^2 < R^2$.

46. The solid sphere itself is represented by $\sqrt{x^2+y^2+z^2} \le 2$. Since we want only the upper hemisphere, we restrict the z-coordinate to nonnegative values. Then inequalities describing the region are $\sqrt{x^2+y^2+z^2} \le 2$, $z \ge 0$, or $x^2+y^2+z^2 \le 4$, $z \ge 0$.

47. (a) To find the x- and y-coordinates of the point P, we project it onto L_2 and project the resulting point Q onto the x- and y-axes. To find the z-coordinate, we project P onto either the xz-plane or the yz-plane (using our knowledge of its x- or y-coordinate) and then project the resulting point onto the z-axis. (Or, we could draw a line parallel to QO from P to the z-axis.) The coordinates of P are $(2, 1, 4)$.

(b) A is the intersection of L_1 and L_2, B is directly below the y-intercept of L_2, and C is directly above the x-intercept of L_2.



48. Let $P = (x, y, z)$. Then $2\,|PB| = |PA| \;\Leftrightarrow\; 4\,|PB|^2 = |PA|^2 \;\Leftrightarrow$

$4\big[(x-6)^2 + (y-2)^2 + (z+2)^2\big] = (x+1)^2 + (y-5)^2 + (z-3)^2 \;\Leftrightarrow$

$4\big(x^2 - 12x + 36\big) - x^2 - 2x + 4\big(y^2 - 4y + 4\big) - y^2 + 10y + 4\big(z^2 + 4z + 4\big) - z^2 + 6z = 35 \;\Leftrightarrow$

$3x^2 - 50x + 3y^2 - 6y + 3z^2 + 22z = 35 - 144 - 16 - 16 \;\Leftrightarrow\; x^2 - \frac{50}{3}x + y^2 - 2y + z^2 + \frac{22}{3}z = -\frac{141}{3}$.

By completing the square three times we get $\left(x - \frac{25}{3}\right)^2 + (y-1)^2 + \left(z + \frac{11}{3}\right)^2 = \frac{-423\,+\,625\,+\,9\,+\,121}{9} = \frac{332}{9}$, which is an equation of a sphere with center $\left(\frac{25}{3}, 1, -\frac{11}{3}\right)$ and radius $\frac{\sqrt{332}}{3}$.

49. We need to find a set of points $\big\{P(x,y,z) \;\big|\; |AP| = |BP|\big\}$.

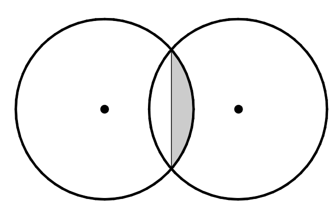
$\sqrt{(x+1)^2 + (y-5)^2 + (z-3)^2} = \sqrt{(x-6)^2 + (y-2)^2 + (z+2)^2} \;\Rightarrow$

$(x+1)^2 + (y-5)^2 + (z-3)^2 = (x-6)^2 + (y-2)^2 + (z+2)^2 \;\Rightarrow$

$x^2 + 2x + 1 + y^2 - 10y + 25 + z^2 - 6z + 9 = x^2 - 12x + 36 + y^2 - 4y + 4 + z^2 + 4z + 4 \;\Rightarrow\; 14x - 6y - 10z = 9$.

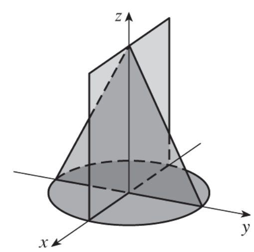
Thus, the set of points is a plane perpendicular to the line segment joining A and B (since this plane must contain the perpendicular bisector of the line segment AB).

50. Completing the square three times in the first equation gives $(x+2)^2 + (y-1)^2 + (z+2)^2 = 2^2$, a sphere with center $(-2, 1, -2)$ and radius 2. The second equation is that of a sphere with center $(0,0,0)$ and radius 2. The distance between the centers of the spheres is $\sqrt{(-2-0)^2 + (1-0)^2 + (-2-0)^2} = \sqrt{4+1+4} = 3$. Since the spheres have the same radius, the volume inside both spheres is symmetrical about the plane containing the circle of intersection of the spheres. The distance from this plane to the center of the circles is $\frac{3}{2}$. So the region inside both spheres consists of two caps of spheres of height $h = 2 - \frac{3}{2} = \frac{1}{2}$. From Exercise 5.2.61, the volume of a cap of a sphere is $V = \pi h^2\left(r - \frac{1}{3}h\right) = \pi\left(\frac{1}{2}\right)^2\left(2 - \frac{1}{3}\cdot\frac{1}{2}\right) = \frac{11\pi}{24}$. So the total volume is $2 \cdot \frac{11\pi}{24} = \frac{11\pi}{12}$.

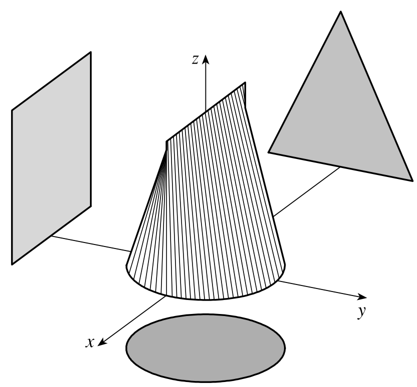


51. The sphere $x^2 + y^2 + z^2 = 4$ has center $(0, 0, 0)$ and radius 2. Completing squares in $x^2 - 4x + y^2 - 4y + z^2 - 4z = -11$ gives $(x^2 - 4x + 4) + (y^2 - 4y + 4) + (z^2 - 4z + 4) = -11 + 4 + 4 + 4 \quad \Rightarrow \quad (x-2)^2 + (y-2)^2 + (z-2)^2 = 1$, so this is the sphere with center $(2, 2, 2)$ and radius 1. The (shortest) distance between the spheres is measured along the line segment connecting their centers. The distance between $(0, 0, 0)$ and $(2, 2, 2)$ is $\sqrt{(2-0)^2 + (2-0)^2 + (2-0)^2} = \sqrt{12} = 2\sqrt{3}$, and subtracting the radius of each circle, the distance between the spheres is $2\sqrt{3} - 2 - 1 = 2\sqrt{3} - 3$.

52. There are many different solids that fit the given description. However, any possible solid must have a circular horizontal cross-section at its top or at its base. Here we illustrate a solid with a circular base in the xy-plane. (A circular cross-section at the top results in an inverted version of the solid described below.) The vertical cross-section through the center of the base that is parallel to the xz-plane must be a square, and the vertical cross-section parallel to the yz-plane (perpendicular to the square) through the center of the base must be a triangle with two vertices on the circle and the third vertex at the center of the top side of the square. (See the figure.)



The solid can include any additional points that do not extend beyond these three "silhouettes" when viewed from directions parallel to the coordinate axes. One possibility shown here is to draw the circular base and the vertical square first. Then draw a surface formed by line segments parallel to the yz-plane that connect the top of the square to the circle.



Problem 8 in the Problems Plus section at the end of the chapter illustrates another possible solid.

12.2 Vectors

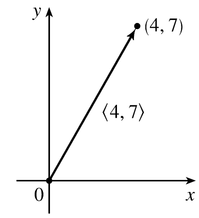
1. (a) The cost of a theater ticket is a scalar, because it has only magnitude.

(b) The current in a river is a vector, because it has both magnitude (the speed of the current) and direction at any given location.

(c) If we assume that the initial path is linear, the initial flight path from Houston to Dallas is a vector, because it has both magnitude (distance) and direction.

(d) The population of the world is a scalar, because it has only magnitude.

2. If the initial point of the vector $\langle 4, 7\rangle$ is placed at the origin, then $\langle 4, 7\rangle$ is the position vector of the point $(4, 7)$.



3. Vectors are equal when they share the same length and direction (but not necessarily location). Using the symmetry of the parallelogram as a guide, we see that $\overrightarrow{AB} = \overrightarrow{DC}$, $\overrightarrow{DA} = \overrightarrow{CB}$, $\overrightarrow{DE} = \overrightarrow{EB}$, and $\overrightarrow{EA} = \overrightarrow{CE}$.

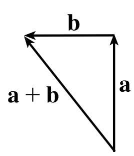
4. (a) The initial point of $\overrightarrow{BC}$ is positioned at the terminal point of $\overrightarrow{AB}$, so by the Triangle Law the sum $\overrightarrow{AB} + \overrightarrow{BC}$ is the vector with initial point A and terminal point C, namely $\overrightarrow{AC}$.

(b) By the Triangle Law, $\overrightarrow{CD} + \overrightarrow{DB}$ is the vector with initial point C and terminal point B, namely $\overrightarrow{CB}$.

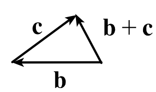
(c) First we consider $\overrightarrow{DB} - \overrightarrow{AB}$ as $\overrightarrow{DB} + \left(-\overrightarrow{AB}\right)$. Then since $-\overrightarrow{AB}$ has the same length as $\overrightarrow{AB}$ but points in the opposite direction, we have $-\overrightarrow{AB} = \overrightarrow{BA}$ and so $\overrightarrow{DB} - \overrightarrow{AB} = \overrightarrow{DB} + \overrightarrow{BA} = \overrightarrow{DA}$.

(d) We use the Triangle Law twice: $\overrightarrow{DC} + \overrightarrow{CA} + \overrightarrow{AB} = \left(\overrightarrow{DC} + \overrightarrow{CA}\right) + \overrightarrow{AB} = \overrightarrow{DA} + \overrightarrow{AB} = \overrightarrow{DB}$.

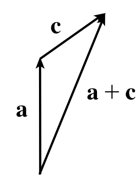
5. (a)



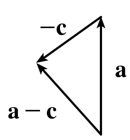
(b)



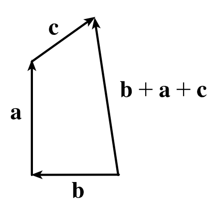
(c)



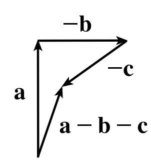
(d)



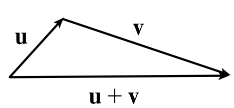
(e)



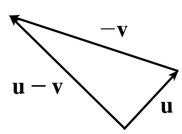
(f)



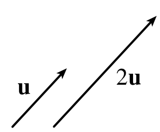
6. (a)



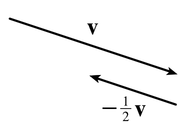
(b)



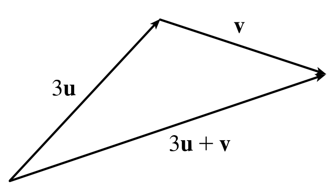
(c)



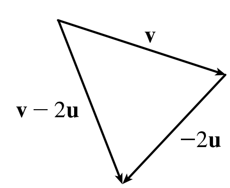
(d)



(e)

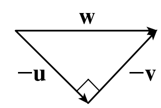


(f)



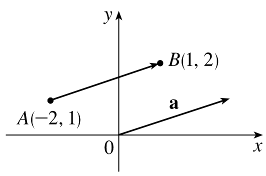
7. Because the tail of $\mathbf{d}$ is the midpoint of QR we have $\overrightarrow{QR} = 2\mathbf{d}$, and by the Triangle Law, $\mathbf{a} + 2\mathbf{d} = \mathbf{b} \;\Rightarrow\; 2\mathbf{d} = \mathbf{b} - \mathbf{a} \;\Rightarrow\; \mathbf{d} = \frac{1}{2}(\mathbf{b} - \mathbf{a}) = \frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{a}$. Again by the Triangle Law, we have $\mathbf{c} + \mathbf{d} = \mathbf{b}$ so $\mathbf{c} = \mathbf{b} - \mathbf{d} = \mathbf{b} - \left(\frac{1}{2}\mathbf{b} - \frac{1}{2}\mathbf{a}\right) = \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$.

8. We are given $\mathbf{u}+\mathbf{v}+\mathbf{w}=\mathbf{0}$, so $\mathbf{w}=(-\mathbf{u})+(-\mathbf{v})$. (See the figure.)

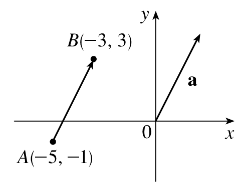


Vectors $-\mathbf{u}$, $-\mathbf{v}$, and $\mathbf{w}$ form a right triangle, so from the Pythagorean Theorem we have $|-\mathbf{u}|^2+|-\mathbf{v}|^2=|\mathbf{w}|^2$. But $|-\mathbf{u}|=|\mathbf{u}|=1$ and $|-\mathbf{v}|=|\mathbf{v}|=1$, so $|\mathbf{w}|=\sqrt{|-\mathbf{u}|^2+|-\mathbf{v}|^2}=\sqrt{2}$.

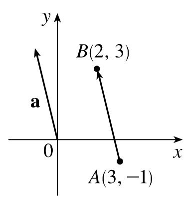
9. $\mathbf{a}=\langle 1-(-2),2-1\rangle=\langle 3,1\rangle$



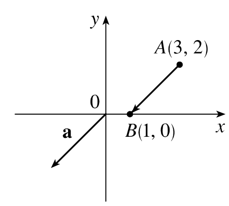
10. $\mathbf{a}=\langle -3-(-5),3-(-1)\rangle=\langle 2,4\rangle$



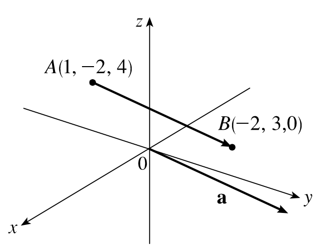
11. $\mathbf{a}=\langle 2-3,3-(-1)\rangle=\langle -1,4\rangle$



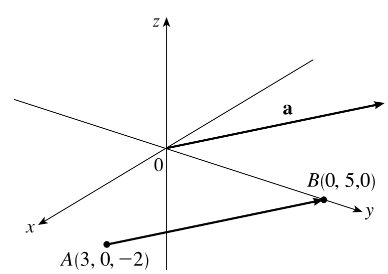
12. $\mathbf{a}=\langle 1-3,0-2\rangle=\langle -2,-2\rangle$



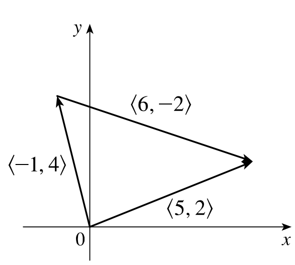
13. $\mathbf{a}=\langle -2-1,3-(-2),0-4\rangle=\langle -3,5,-4\rangle$



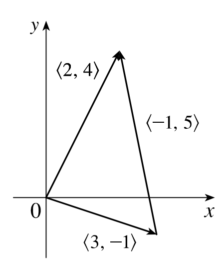
14. $\mathbf{a}=\langle 0-3,5-0,0-(-2)\rangle=\langle -3,5,2\rangle$



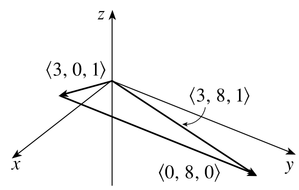
15. $\langle -1,4\rangle+\langle 6,-2\rangle=\langle -1+6,4+(-2)\rangle=\langle 5,2\rangle$



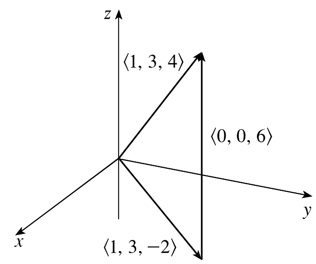
16. $\langle 3,-1\rangle+\langle -1,5\rangle=\langle 3+(-1),-1+5\rangle=\langle 2,4\rangle$



17. $\langle 3,0,1\rangle + \langle 0,8,0\rangle = \langle 3+0, 0+8, 1+0\rangle$
$= \langle 3,8,1\rangle$



18. $\langle 1,3,-2\rangle + \langle 0,0,6\rangle = \langle 1+0, 3+0, -2+6\rangle$
$= \langle 1,3,4\rangle$



19. $\mathbf{a}+\mathbf{b} = \langle -3,4\rangle + \langle 9,-1\rangle = \langle -3+9, 4+(-1)\rangle = \langle 6,3\rangle$

$4\,\mathbf{a}+2\,\mathbf{b} = 4\,\langle -3,4\rangle + 2\,\langle 9,-1\rangle = \langle -12,16\rangle + \langle 18,-2\rangle = \langle 6,14\rangle$

$|\mathbf{a}| = \sqrt{(-3)^2+4^2} = \sqrt{25} = 5$

$|\mathbf{a}-\mathbf{b}| = |\langle -3-9, 4-(-1)\rangle| = |\langle -12,5\rangle| = \sqrt{(-12)^2+5^2} = \sqrt{169} = 13$

20. $\mathbf{a}+\mathbf{b} = (5\,\mathbf{i}+3\,\mathbf{j}) + (-\mathbf{i}-2\,\mathbf{j}) = 4\,\mathbf{i}+\mathbf{j}$

$4\,\mathbf{a}+2\,\mathbf{b} = 4\,(5\,\mathbf{i}+3\,\mathbf{j}) + 2\,(-\mathbf{i}-2\,\mathbf{j}) = 20\,\mathbf{i}+12\,\mathbf{j}-2\,\mathbf{i}-4\,\mathbf{j} = 18\,\mathbf{i}+8\,\mathbf{j}$

$|\mathbf{a}| = \sqrt{5^2+3^2} = \sqrt{34}$

$|\mathbf{a}-\mathbf{b}| = |(5\,\mathbf{i}+3\,\mathbf{j}) - (-\mathbf{i}-2\,\mathbf{j})| = |6\,\mathbf{i}+5\,\mathbf{j}| = \sqrt{6^2+5^2} = \sqrt{61}$

21. $\mathbf{a}+\mathbf{b} = (4\,\mathbf{i}-3\,\mathbf{j}+2\,\mathbf{k}) + (2\,\mathbf{i}-4\,\mathbf{k}) = 6\,\mathbf{i}-3\,\mathbf{j}-2\mathbf{k}$

$4\,\mathbf{a}+2\,\mathbf{b} = 4\,(4\,\mathbf{i}-3\,\mathbf{j}+2\,\mathbf{k}) + 2\,(2\,\mathbf{i}-4\,\mathbf{k}) = 16\,\mathbf{i}-12\,\mathbf{j}+8\,\mathbf{k}+4\,\mathbf{i}-8\,\mathbf{k} = 20\,\mathbf{i}-12\,\mathbf{j}$

$|\mathbf{a}| = \sqrt{4^2+(-3)^2+2^2} = \sqrt{29}$

$|\mathbf{a}-\mathbf{b}| = |(4\,\mathbf{i}-3\,\mathbf{j}+2\,\mathbf{k}) - (2\,\mathbf{i}-4\,\mathbf{k})| = |2\,\mathbf{i}-3\,\mathbf{j}+6\,\mathbf{k}| = \sqrt{2^2+(-3)^2+6^2} = \sqrt{49} = 7$

22. $\mathbf{a}+\mathbf{b} = \langle 8,1,-4\rangle + \langle 5,-2,1\rangle = \langle 8+5, 1+(-2), -4+1\rangle = \langle 13,-1,-3\rangle$

$4\,\mathbf{a}+2\,\mathbf{b} = 4\,\langle 8,1,-4\rangle + 2\,\langle 5,-2,1\rangle = \langle 32,4,-16\rangle + \langle 10,-4,2\rangle = \langle 42,0,-14\rangle$

$|\mathbf{a}| = \sqrt{8^2+1^2+(-4)^2} = \sqrt{81} = 9$

$|\mathbf{a}-\mathbf{b}| = |\langle 8-5, 1-(-2), -4-1\rangle| = |\langle 3,3,-5\rangle| = \sqrt{3^2+3^2+(-5)^2} = \sqrt{43}$

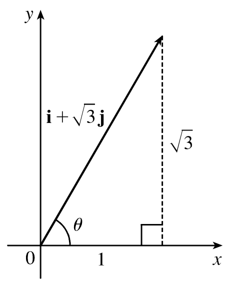
23. The vector $\langle 6,-2\rangle$ has length $|\langle 6,-2\rangle| = \sqrt{6^2+(-2)^2} = \sqrt{40} = 2\sqrt{10}$, so by Equation 4 the unit vector with the same direction is $\dfrac{1}{2\sqrt{10}}\,\langle 6,-2\rangle = \left\langle \dfrac{3}{\sqrt{10}}, -\dfrac{1}{\sqrt{10}}\right\rangle$.

24. The vector $-5\,\mathbf{i}+3\,\mathbf{j}-\mathbf{k}$ has length $|-5\,\mathbf{i}+3\,\mathbf{j}-\mathbf{k}| = \sqrt{(-5)^2+3^2+(-1)^2} = \sqrt{35}$, so by Equation 4 the unit vector with the same direction is $\dfrac{1}{\sqrt{35}}(-5\,\mathbf{i}+3\,\mathbf{j}-\mathbf{k}) = -\dfrac{5}{\sqrt{35}}\,\mathbf{i} + \dfrac{3}{\sqrt{35}}\,\mathbf{j} - \dfrac{1}{\sqrt{35}}\,\mathbf{k}$.

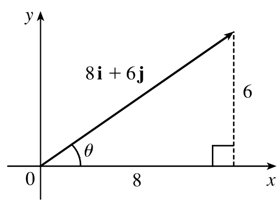
25. The vector $8\,\mathbf{i}-\mathbf{j}+4\,\mathbf{k}$ has length $|8\,\mathbf{i}-\mathbf{j}+4\,\mathbf{k}| = \sqrt{8^2+(-1)^2+4^2} = \sqrt{81} = 9$, so by Equation 4 the unit vector with the same direction is $\frac{1}{9}(8\,\mathbf{i}-\mathbf{j}+4\,\mathbf{k}) = \frac{8}{9}\,\mathbf{i} - \frac{1}{9}\,\mathbf{j} + \frac{4}{9}\,\mathbf{k}$.

26. $|\langle 6,2,-3\rangle| = \sqrt{6^2+2^2+(-3)^2} = \sqrt{49} = 7$, so a unit vector in the direction of $\langle 6,2,-3\rangle$ is $\mathbf{u} = \frac{1}{7}\langle 6,2,-3\rangle$.

A vector in the same direction but with length 4 is $4\mathbf{u} = 4\cdot\frac{1}{7}\langle 6,2,-3\rangle = \left\langle \frac{24}{7},\frac{8}{7},-\frac{12}{7}\right\rangle$.

27. 

From the figure, we see that $\tan\theta = \dfrac{\sqrt{3}}{1} = \sqrt{3} \quad\Rightarrow\quad \theta = 60^\circ$.

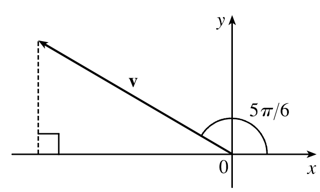
28. 

From the figure, we see that $\tan\theta = \frac{6}{8} = \frac{3}{4}$, so $\theta = \tan^{-1}\left(\frac{3}{4}\right) \approx 36.9^\circ$.

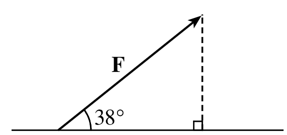
29. From the figure, we see that the x-component of $\mathbf{v}$ is

$v_1 = |\mathbf{v}|\cos(5\pi/6) = 4\left(-\frac{\sqrt{3}}{2}\right) = -2\sqrt{3}$ and the y-component is

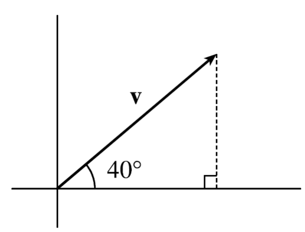
$v_2 = |\mathbf{v}|\sin(5\pi/6) = 4\left(\frac{1}{2}\right) = 2$. Thus, $\mathbf{v} = \left\langle -2\sqrt{3}, 2\right\rangle$.



30. From the figure, we see that the horizontal component of the force $\mathbf{F}$ is $|\mathbf{F}|\cos 38^\circ = 50\cos 38^\circ \approx 39.4$ N, and the vertical component is $|\mathbf{F}|\sin 38^\circ = 50\sin 38^\circ \approx 30.8$ N.



31. The velocity vector $\mathbf{v}$ makes an angle of 40° with the horizontal and has magnitude equal to the speed at which the football was thrown. From the figure, we see that the horizontal component of $\mathbf{v}$ is $|\mathbf{v}|\cos 40^\circ = 20\cos 40^\circ \approx 15.32$ m/s and the vertical component $\mathbf{v}$ is $|\mathbf{v}|\sin 40^\circ = 20\sin 40^\circ \approx 12.86$ m/s.



32. The given force vectors can be expressed in terms of their horizontal and vertical components as $20\cos 45^\circ\,\mathbf{i} + 20\sin 45^\circ\,\mathbf{j} = 10\sqrt{2}\,\mathbf{i} + 10\sqrt{2}\,\mathbf{j}$ and $16\cos 30^\circ\,\mathbf{i} - 16\sin 30^\circ\,\mathbf{j} = 8\sqrt{3}\,\mathbf{i} - 8\,\mathbf{j}$. The resultant force $\mathbf{F}$ is the sum of these two vectors: $\mathbf{F} = \left(10\sqrt{2}+8\sqrt{3}\right)\mathbf{i} + \left(10\sqrt{2}-8\right)\mathbf{j} \approx 28.00\,\mathbf{i} + 6.14\,\mathbf{j}$. Then we have $|\mathbf{F}| \approx \sqrt{(28.00)^2+(6.14)^2} \approx 28.7$ lb and, letting θ be the angle $\mathbf{F}$ makes with the positive x-axis,

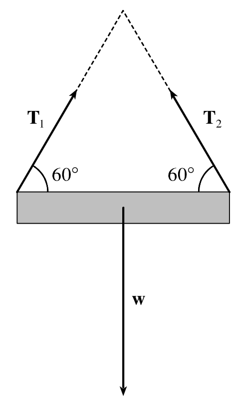
$$\tan\theta = \frac{10\sqrt{2}-8}{10\sqrt{2}+8\sqrt{3}} \quad\Rightarrow\quad \theta = \tan^{-1}\left(\frac{10\sqrt{2}-8}{10\sqrt{2}+8\sqrt{3}}\right) \approx 12.4^\circ.$$

33. The given force vectors can be expressed in terms of their horizontal and vertical components as $-300\,\mathbf{i}$ and $200\cos 60^\circ\,\mathbf{i} + 200\sin 60^\circ\,\mathbf{j} = 200\left(\frac{1}{2}\right)\,\mathbf{i} + 200\left(\frac{\sqrt{3}}{2}\right)\,\mathbf{j} = 100\,\mathbf{i} + 100\sqrt{3}\,\mathbf{j}$. The resultant force $\mathbf{F}$ is the sum of these two vectors: $\mathbf{F} = (-300+100)\,\mathbf{i} + \left(0 + 100\sqrt{3}\right)\mathbf{j} = -200\,\mathbf{i} + 100\sqrt{3}\,\mathbf{j}$. Then we have $|\mathbf{F}| \approx \sqrt{(-200)^2 + \left(100\sqrt{3}\right)^2} = \sqrt{70{,}000} = 100\sqrt{7} \approx 264.6$ N. Let θ be the angle $\mathbf{F}$ makes with the positive x-axis. Then $\tan\theta = \dfrac{100\sqrt{3}}{-200} = -\dfrac{\sqrt{3}}{2}$ and the terminal point of $\mathbf{F}$ lies in the second quadrant, so $\theta = \tan^{-1}\left(-\dfrac{\sqrt{3}}{2}\right) + 180^\circ \approx -40.9^\circ + 180^\circ = 139.1^\circ$.

34. Call the two tensile forces $\mathbf{T_1}$ and $\mathbf{T_2}$, corresponding to the left and right sides, respectively. Putting the forces into vertical and horizontal components gives us

$$\mathbf{T_1} = |\mathbf{T_1}|\cos 60^\circ\mathbf{i} + |\mathbf{T_1}|\sin 60^\circ\mathbf{j} \quad (1) \qquad \text{and}$$

$$\mathbf{T_2} = -|\mathbf{T_2}|\cos 60^\circ\mathbf{i} + |\mathbf{T_2}|\sin 60^\circ\mathbf{j} \quad (2)$$



The resultant of these forces, $\mathbf{T_1} + \mathbf{T_2}$, counterbalances the force of gravity acting on the beam [which is $-500g\mathbf{j} = -500(9.8)\mathbf{j} = -4900\mathbf{j}$], so $\mathbf{T_1} + \mathbf{T_2} = 4900\mathbf{j}$.

Thus, $\mathbf{T_1} + \mathbf{T_2} = (|\mathbf{T_1}|\cos 60^\circ - |\mathbf{T_2}|\cos 60^\circ)\mathbf{i} + (|\mathbf{T_1}|\sin 60^\circ + |\mathbf{T_2}|\sin 60^\circ)\mathbf{j} = 4900\mathbf{j}$. Setting components equal to each other gives us $|\mathbf{T_1}|\cos 60^\circ - |\mathbf{T_2}|\cos 60^\circ = 0 \Rightarrow |\mathbf{T_1}| = |\mathbf{T_2}|$ and $|\mathbf{T_1}|\sin 60^\circ + |\mathbf{T_2}|\sin 60^\circ = 4900$. Subbing in the first equation gives us

$$|\mathbf{T_1}|\sin 60^\circ + |\mathbf{T_1}|\sin 60^\circ = 4900 \Rightarrow 2|\mathbf{T_1}|\sin 60^\circ = 4900 \Rightarrow$$

$$2\left(\frac{\sqrt{3}}{2}\right)|\mathbf{T_1}| = 4900 \Rightarrow |\mathbf{T_1}| = |\mathbf{T_2}| = \frac{4900}{\sqrt{3}} \approx 2829 \text{ N}$$

Finally, from (1) and (2), $\mathbf{T_1} \approx 1414\mathbf{i} + 2450\mathbf{j}$ and $\mathbf{T_2} \approx -1414\mathbf{i} + 2450\mathbf{j}$.

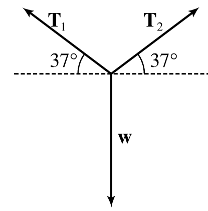
35. Call the two tension vectors $\mathbf{T}_2$ and $\mathbf{T}_3$, corresponding to the ropes of length 2 m and 3 m. In terms of vertical and horizontal components,

$$\mathbf{T}_2 = -|\mathbf{T}_2|\cos 50^\circ\mathbf{i} + |\mathbf{T}_2|\sin 50^\circ\mathbf{j} \quad \mathbf{(1)} \qquad \text{and} \qquad \mathbf{T}_3 = |\mathbf{T}_3|\cos 38^\circ\mathbf{i} + |\mathbf{T}_3|\sin 38^\circ\mathbf{j} \quad \mathbf{(2)}$$

The resultant of these forces, $\mathbf{T}_2 + \mathbf{T}_3$, counterbalances the weight of the hoist (which is $-350\,\mathbf{j}$), so $\mathbf{T}_2 + \mathbf{T}_3 = 350\,\mathbf{j} \Rightarrow (-|\mathbf{T}_2|\cos 50^\circ + |\mathbf{T}_3|\cos 38^\circ)\,\mathbf{i} + (|\mathbf{T}_2|\sin 50^\circ + |\mathbf{T}_3|\sin 38^\circ)\,\mathbf{j} = 350\,\mathbf{j}$. Equating components, we have $-|\mathbf{T}_2|\cos 50^\circ + |\mathbf{T}_3|\cos 38^\circ = 0 \Rightarrow |\mathbf{T}_2| = |\mathbf{T}_3|\dfrac{\cos 38^\circ}{\cos 50^\circ}$ and $|\mathbf{T}_2|\sin 50^\circ + |\mathbf{T}_3|\sin 38^\circ = 350$. Substituting the first equation into the second gives $|\mathbf{T}_3|\dfrac{\cos 38^\circ}{\cos 50^\circ}\sin 50^\circ + |\mathbf{T}_3|\sin 38^\circ = 350 \Rightarrow |\mathbf{T}_3|(\cos 38^\circ\tan 50^\circ + \sin 38^\circ) = 350$, so the magnitudes of the tensions are $|\mathbf{T}_3| = \dfrac{350}{\cos 38^\circ\tan 50^\circ + \sin 38^\circ} \approx 225.11$ N and $|\mathbf{T}_2| = |\mathbf{T}_3|\dfrac{\cos 38^\circ}{\cos 50^\circ} \approx 275.97$ N.

Finally, from **(1)** and **(2)**, the tension vectors are $\mathbf{T}_2 \approx -177.39\,\mathbf{i} + 211.41\,\mathbf{j}$ and $\mathbf{T}_3 \approx 177.39\,\mathbf{i} + 138.59\,\mathbf{j}$.

36. We can consider the weight of the chain to be concentrated at its midpoint. The forces acting on the chain then are the tension vectors $\mathbf{T}_1$, $\mathbf{T}_2$ in each end of the chain and the weight $\mathbf{w}$, as shown in the figure. We know $|\mathbf{T}_1| = |\mathbf{T}_2| = 25$ N so, in terms of vertical and horizontal components, we have

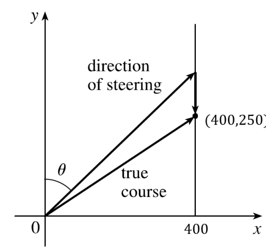


$$\mathbf{T}_1 = -25\cos 37°\,\mathbf{i} + 25\sin 37°\,\mathbf{j} \qquad \mathbf{T}_2 = 25\cos 37°\,\mathbf{i} + 25\sin 37°\,\mathbf{j}$$

The resultant vector $\mathbf{T}_1 + \mathbf{T}_2$ of the tensions counterbalances the weight $\mathbf{w}$, giving $\mathbf{T}_1 + \mathbf{T}_2 = -\mathbf{w}$. Since $\mathbf{w} = -|\mathbf{w}|\,\mathbf{j}$, we have $(-25\cos 37°\,\mathbf{i} + 25\sin 37°\,\mathbf{j}) + (25\cos 37°\,\mathbf{i} + 25\sin 37°\,\mathbf{j}) = |\mathbf{w}|\,\mathbf{j} \;\Rightarrow\; 50\sin 37°\,\mathbf{j} = |\mathbf{w}|\,\mathbf{j} \;\Rightarrow$ $|\mathbf{w}| = 50\sin 37° \approx 30.1$. So the weight is 30.1 N, and since $w = mg$, the mass is $\frac{30.1}{9.8} \approx 3.07$ kg.

37. Let $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ be the force vectors where $|\mathbf{v}_1| = 25$, $|\mathbf{v}_2| = 12$, and $|\mathbf{v}_3| = 4$. Set up coordinate axes so that the object is at the origin and $\mathbf{v}_1$, $\mathbf{v}_2$ lie in the xy-plane. We can position the vectors so that $\mathbf{v}_1 = 25\,\mathbf{i}$, $\mathbf{v}_2 = 12\cos 100°\,\mathbf{i} + 12\sin 100°\,\mathbf{j}$, and $\mathbf{v}_3 = 4\,\mathbf{k}$. The magnitude of a force that counterbalances the three given forces must match the magnitude of the resultant force. We have $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = (25 + 12\cos 100°)\,\mathbf{i} + 12\sin 100°\,\mathbf{j} + 4\,\mathbf{k}$, so the counterbalancing force must have magnitude $|\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3| = \sqrt{(25 + 12\cos 100°)^2 + (12\sin 100°)^2 + 4^2} \approx 26.1$ N.

38. (a) Set up coordinate axes so that the rower is at the origin, the channel is bordered by the y-axis and the line $x = 400$, and the current flows in the negative x-direction. The rower wants to reach the point $(400, 250)$. Let θ be the angle, measured from the positive x-axis, in the direction she should steer. (See the figure.)



In still water, the kayak has velocity $\mathbf{v}_b = \langle 2\sin\theta, 2\cos\theta\rangle$ and the velocity of the current is $\mathbf{v}_c = \langle 0, -0.5\rangle$, so the true path of the rower is determined by the velocity vector $\mathbf{v} = \mathbf{v}_b + \mathbf{v}_c = \langle 2\sin\theta, 2\cos\theta - 0.5\rangle$. Let t be the time (in seconds) after the rower departs; then the position of the kayak at time t is given by $t\mathbf{v}$ and she crosses the channel when $t\mathbf{v} = \langle 2\sin\theta, 2\cos\theta - 0.5\rangle\, t = \langle 400, 250\rangle$. Thus, $2(\sin\theta)t = 400 \Rightarrow t = \frac{400}{2\sin\theta}$ and $(2\cos\theta - 0.5)t = 250$. Substituting gives

$$(2\cos\theta - 0.5)\frac{400}{2\sin\theta} = 250 \Rightarrow 800\cos\theta - 200 = 500\sin\theta \Rightarrow 8\cos\theta - 2 = 5\sin\theta \quad (\mathbf{1})$$

Squaring both sides, we have

$$64\cos^2\theta - 32\cos\theta + 4 = 25\sin^2\theta = 25(1 - \cos^2\theta)$$

$$89\cos^2\theta - 32\cos\theta - 21 = 0$$

The quadratic formula gives

$$\cos\theta = \frac{32 \pm \sqrt{(-32)^2 - 4(89)(-21)}}{2(89)} = \frac{32 \pm \sqrt{8500}}{178} \approx 0.6977 \text{or} - 0.3381$$

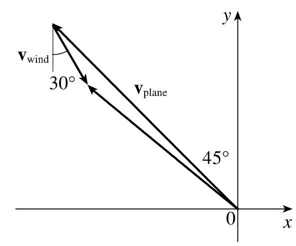
[continued]

The acute value for θ is approximately $\cos^{-1}(0.6977) \approx 45.8°$. Thus, the rower should steer in the direction that is $45.8°$ from the bank, toward upstream.

Alternate solution: We could solve $(\mathbf{1})$ graphically by plotting $y = 8\cos\theta - 2$ and $y = 5\sin\theta$ on a graphical device and finding the approximate intersection point (0.799, 3.582). Thus, $\theta \approx 0.799$ radians, or equivalently, $45.8°$.

(b) From part (a), we know the trim is completed when $t = \frac{400}{2\sin\theta}$. But $\theta \approx 45.8°$, so the time required is approximately $t = \frac{400}{2\sin 45.8°} \approx 279$ seconds, or 4.65 minutes.

39. In still air, the velocity vector of the plane is 290 km/h N45°E, which can be written as $\mathbf{v}_{plane} = 290(-\sin 45°\,\mathbf{i} + \cos 45°\,\mathbf{j})$. The wind is blowing at 55 km/h in the direction of S30°E, or equivalently, $\mathbf{v}_{wind} = 55(\sin 30°\,\mathbf{i} - \cos 30°\,\mathbf{j})$. The velocity of the plane relative to the ground is



$$\mathbf{v} = \mathbf{v}_{plane} + \mathbf{v}_{wind} = (55\cos 30° - 290\cos 45°)\mathbf{i} + (290\sin 45° - 55\sin 30°)\mathbf{j}$$

$$= \left(55\frac{1}{2} - 290\frac{1}{\sqrt{2}}\right)\mathbf{i} + \left(290\frac{1}{\sqrt{2}} - 55\frac{\sqrt{3}}{2}\right)\mathbf{j}$$

$$\approx -177.6\mathbf{i} + 157.4\mathbf{j}$$

The ground speed is $|\mathbf{v}| \approx \sqrt{(-177.6)^2 + (157.4)^2} \approx 237$ km/h. The angle that the velocity vector makes with the x-axis is $\theta \approx \tan^{-1}\left(\frac{157.4}{177.6}\right) \approx 41.5°$. Therefore, the true course of the plane is about $\text{N}(90 - 41.5°)$ W = N48.5° W.

40. With respect to the water's surface, the dog's velocity is the sum of the velocity of the ship with respect to the water and the velocity of the dog with respect to the ship. If we let north be the positive y direction and west be the negative x direction, we have $\mathbf{v} = \langle -32, 0\rangle + \langle 0, 4\rangle = \langle -32, 4\rangle$. Then, the speed of the dog is $|\mathbf{v}| = \sqrt{(-32)^2 + 4^2} \approx 32.2$ km/h. The vector $\mathbf{v}$ makes an angle of $\tan^{-1}\left(\dfrac{4}{-32}\right) \approx -7.1°$ and $-7.1° + 180° = 172.9°$. Therefore, the dog's direction is N $(172.9 - 90)°$ W or N 82.9° W.

41. The slope of the tangent line to the graph of $y = x^2$ at the point $(2, 4)$ is

$$\left.\frac{dy}{dx}\right|_{x=2} = \left.2x\right|_{x=2} = 4$$

Thus, a parallel vector is $\mathbf{i} + 4\,\mathbf{j}$, which has length $|\mathbf{i} + 4\,\mathbf{j}| = \sqrt{1^2 + 4^2} = \sqrt{17}$, and so unit vectors parallel to the tangent line are $\pm\frac{1}{\sqrt{17}}\,(\mathbf{i} + 4\,\mathbf{j})$.

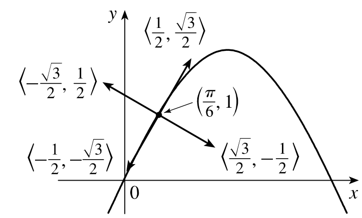
42. (a) The slope of the tangent line to the graph of $y = 2\sin x$ at the point $(\pi/6, 1)$ is

$$\left.\frac{dy}{dx}\right|_{x=\pi/6} = 2\cos x\Big|_{x=\pi/6} = 2\cdot\frac{\sqrt{3}}{2} = \sqrt{3}$$

Thus, a parallel vector is $\mathbf{i} + \sqrt{3}\,\mathbf{j}$, which has length $\left|\mathbf{i} + \sqrt{3}\,\mathbf{j}\right| = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = \sqrt{4} = 2$, and so unit vectors parallel to the tangent line are $\pm\frac{1}{2}\left(\mathbf{i} + \sqrt{3}\,\mathbf{j}\right)$.

(b) The slope of the tangent line is $\sqrt{3}$, so the slope of a line perpendicular to the tangent line is $-\frac{1}{\sqrt{3}}$ and a vector in this direction is $\sqrt{3}\,\mathbf{i} - \mathbf{j}$. Since $\left|\sqrt{3}\,\mathbf{i} - \mathbf{j}\right| = \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2} = 2$, unit vectors perpendicular to the tangent line are $\pm\frac{1}{2}\left(\sqrt{3}\,\mathbf{i} - \mathbf{j}\right)$.

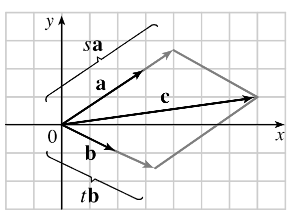
(c)



43. By the Triangle Law, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$. Then $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{AC} + \overrightarrow{CA}$, but $\overrightarrow{AC} + \overrightarrow{CA} = \overrightarrow{AC} + \left(-\overrightarrow{AC}\right) = \mathbf{0}$. So $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \mathbf{0}$.

44. $\overrightarrow{AC} = \frac{1}{3}\overrightarrow{AB}$ and $\overrightarrow{BC} = \frac{2}{3}\overrightarrow{BA}$. $\mathbf{c} = \overrightarrow{OA} + \overrightarrow{AC} = \mathbf{a} + \frac{1}{3}\overrightarrow{AB} \;\Rightarrow\; \overrightarrow{AB} = 3\,\mathbf{c} - 3\,\mathbf{a}$. $\mathbf{c} = \overrightarrow{OB} + \overrightarrow{BC} = \overrightarrow{OB} + \frac{2}{3}\overrightarrow{BA} \;\Rightarrow\; \overrightarrow{BA} = \frac{3}{2}\,\mathbf{c} - \frac{3}{2}\,\mathbf{b}$. $\overrightarrow{BA} = -\overrightarrow{AB}$, so $\frac{3}{2}\,\mathbf{c} - \frac{3}{2}\,\mathbf{b} = 3\,\mathbf{a} - 3\,\mathbf{c} \;\Leftrightarrow\; \mathbf{c} + 2\,\mathbf{c} = 2\,\mathbf{a} + \mathbf{b} \;\Leftrightarrow\; \mathbf{c} = \frac{2}{3}\,\mathbf{a} + \frac{1}{3}\,\mathbf{b}$.

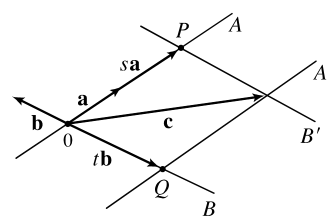
45. (a), (b)



(c) From the sketch, we estimate that $s \approx 1.3$ and $t \approx 1.6$.

(d) $\mathbf{c} = s\,\mathbf{a} + t\,\mathbf{b} \;\Leftrightarrow\; 7 = 3s + 2t$ and $1 = 2s - t$. Solving these equations gives $s = \frac{9}{7}$ and $t = \frac{11}{7}$.

46. Draw $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ emanating from the origin. Extend $\mathbf{a}$ and $\mathbf{b}$ to form lines A and B, and draw lines A' and B' parallel to these two lines through the terminal point of $\mathbf{c}$. Since $\mathbf{a}$ and $\mathbf{b}$ are not parallel, A and B' must meet (at P), and A' and B must also meet (at Q). Now we see that $\overrightarrow{OP} + \overrightarrow{OQ} = \mathbf{c}$, so if



$s = \dfrac{\left|\overrightarrow{OP}\right|}{|\mathbf{a}|}$ $\left(\text{or its negative, if } \mathbf{a} \text{ points in the direction opposite } \overrightarrow{OP}\right)$ and $t = \dfrac{\left|\overrightarrow{OQ}\right|}{|\mathbf{b}|}$ (or its negative, as in the diagram), then $\mathbf{c} = s\mathbf{a} + t\mathbf{b}$, as required.

Argument using components: Since $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ all lie in the same plane, we can consider them to be vectors in two dimensions. Let $\mathbf{a} = \langle a_1, a_2\rangle$, $\mathbf{b} = \langle b_1, b_2\rangle$, and $\mathbf{c} = \langle c_1, c_2\rangle$. We need $sa_1 + tb_1 = c_1$ and $sa_2 + tb_2 = c_2$. Multiplying the first equation by a_2 and the second by a_1 and subtracting, we get $t = \dfrac{c_2a_1 - c_1a_2}{b_2a_1 - b_1a_2}$. Similarly $s = \dfrac{b_2c_1 - b_1c_2}{b_2a_1 - b_1a_2}$. Since $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{0}$ and $\mathbf{a}$ is not a scalar multiple of $\mathbf{b}$, the denominator is not zero.

47. $|\mathbf{r} - \mathbf{r}_0|$ is the distance between the points (x, y, z) and (x_0, y_0, z_0), so the set of points is a sphere with radius 1 and center (x_0, y_0, z_0).

Alternate method: $|\mathbf{r} - \mathbf{r}_0| = 1 \quad \Leftrightarrow \quad \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} = 1 \quad \Leftrightarrow$

$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = 1$, which is the equation of a sphere with radius 1 and center (x_0, y_0, z_0).

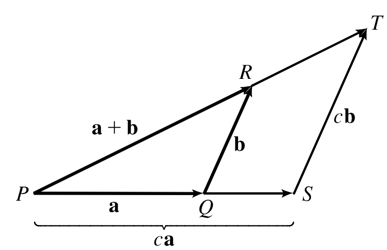
48. Let P_1 and P_2 be the points with position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively. Then $|\mathbf{r} - \mathbf{r}_1| + |\mathbf{r} - \mathbf{r}_2|$ is the sum of the distances from (x, y) to P_1 and P_2. Since this sum is constant, the set of points (x, y) represents an ellipse with foci P_1 and P_2. The condition $k > |\mathbf{r}_1 - \mathbf{r}_2|$ assures us that the ellipse is not degenerate.

49.
$$\begin{aligned}
\mathbf{a} + (\mathbf{b} + \mathbf{c}) &= \langle a_1, a_2 \rangle + (\langle b_1, b_2 \rangle + \langle c_1, c_2 \rangle) = \langle a_1, a_2 \rangle + \langle b_1 + c_1, b_2 + c_2 \rangle \\
&= \langle a_1 + b_1 + c_1, a_2 + b_2 + c_2 \rangle = \langle (a_1 + b_1) + c_1, (a_2 + b_2) + c_2 \rangle \\
&= \langle a_1 + b_1, a_2 + b_2 \rangle + \langle c_1, c_2 \rangle = (\langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle) + \langle c_1, c_2 \rangle \\
&= (\mathbf{a} + \mathbf{b}) + \mathbf{c}
\end{aligned}$$

50. *Algebraically:*
$$\begin{aligned}
c(\mathbf{a} + \mathbf{b}) &= c\left(\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle\right) = c\,\langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle \\
&= \langle c\,(a_1 + b_1)\,, c\,(a_2 + b_2)\,, c\,(a_3 + b_3) \rangle = \langle ca_1 + cb_1, ca_2 + cb_2, ca_3 + cb_3 \rangle \\
&= \langle ca_1, ca_2, ca_3 \rangle + \langle cb_1, cb_2, cb_3 \rangle = c\,\mathbf{a} + c\,\mathbf{b}
\end{aligned}$$

Geometrically:

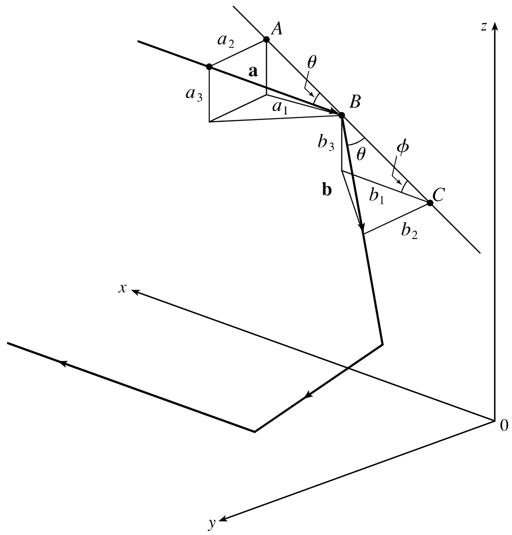
According to the Triangle Law, if $\mathbf{a} = \overrightarrow{PQ}$ and $\mathbf{b} = \overrightarrow{QR}$, then $\mathbf{a} + \mathbf{b} = \overrightarrow{PR}$. Construct triangle PST as shown so that $\overrightarrow{PS} = c\,\mathbf{a}$ and $\overrightarrow{ST} = c\,\mathbf{b}$. (We have drawn the case where $c > 1$.) By the Triangle Law, $\overrightarrow{PT} = c\,\mathbf{a} + c\,\mathbf{b}$. But triangle PQR and triangle PST are similar triangles because $c\,\mathbf{b}$ is parallel to $\mathbf{b}$. Therefore, $\overrightarrow{PR}$ and $\overrightarrow{PT}$ are parallel and, in fact, $\overrightarrow{PT} = c\overrightarrow{PR}$. Thus, $c\,\mathbf{a} + c\,\mathbf{b} = c(\mathbf{a} + \mathbf{b})$.



51. Consider triangle ABC, where D and E are the midpoints of AB and BC. We know that $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ **(1)** and $\overrightarrow{DB} + \overrightarrow{BE} = \overrightarrow{DE}$ **(2)**. However, $\overrightarrow{DB} = \frac{1}{2}\overrightarrow{AB}$, and $\overrightarrow{BE} = \frac{1}{2}\overrightarrow{BC}$. Substituting these expressions for $\overrightarrow{DB}$ and $\overrightarrow{BE}$ into **(2)** gives $\frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} = \overrightarrow{DE}$. Comparing this with **(1)** gives $\overrightarrow{DE} = \frac{1}{2}\overrightarrow{AC}$. Therefore $\overrightarrow{AC}$ and $\overrightarrow{DE}$ are parallel and $\left|\overrightarrow{DE}\right| = \frac{1}{2}\left|\overrightarrow{AC}\right|$.

52. The question states that the light ray strikes all three mirrors, so it is not parallel to any of them and $a_1 \neq 0$, $a_2 \neq 0$ and $a_3 \neq 0$. Let $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, as in the diagram. We can let $|\mathbf{b}| = |\mathbf{a}|$, since only its direction is important. Then $\dfrac{|b_2|}{|\mathbf{b}|} = \sin\theta = \dfrac{|a_2|}{|\mathbf{a}|} \quad \Rightarrow \quad |b_2| = |a_2|$.

[continued]



From the diagram $b_2\,\mathbf{j}$ and $a_2\,\mathbf{j}$ point in opposite directions, so $b_2 = -a_2$. $|AB| = |BC|$, so $|b_3| = \sin\phi\ |BC| = \sin\phi\ |AB| = |a_3|$, and $|b_1| = \cos\phi\ |BC| = \cos\phi\ |AB| = |a_1|$. $b_3\,\mathbf{k}$ and $a_3\,\mathbf{k}$ have the same direction, as do $b_1\,\mathbf{i}$ and $a_1\,\mathbf{i}$, so $\mathbf{b} = \langle a_1, -a_2, a_3\rangle$. When the ray hits the other mirrors, similar arguments show that these reflections will reverse the signs of the other two coordinates, so the final reflected ray will be $\langle -a_1, -a_2, -a_3\rangle = -\mathbf{a}$, which is parallel to $\mathbf{a}$.

DISCOVERY PROJECT The Shape of a Hanging Chain

1. As $s(x)$ is the length of the chain with uniform density ρ, the mass of the chain is given by $\rho s(x)$. Then the downward gravitational force is given by $\mathbf{w} = \langle 0, -g\rho s(x)\rangle$. Also, $\mathbf{T}_0 = \langle |\mathbf{T}_0|\cos 180^\circ, |\mathbf{T}_0|\sin 180^\circ\rangle = \langle -|\mathbf{T}_0|, 0\rangle$. As the system is in equilibrium, we have

$$\begin{aligned}\mathbf{T}_0 + \mathbf{T} + \mathbf{w} &= \mathbf{0}\\ \mathbf{T} &= -\mathbf{T}_0 - \mathbf{w}\\ &= -\langle -|\mathbf{T}_0|, 0\rangle - \langle 0, -g\rho s(x)\rangle\\ &= \langle |\mathbf{T}_0|, g\rho s(x)\rangle\end{aligned}$$

2. Note that the vector $\mathbf{T}$ is parallel to the tangent line to the curve at the point (x, y). Thus, the slope of the tangent line can be written as

$$\frac{dy}{dx} = \frac{g\rho s(x)}{|\mathbf{T}_0|} = \frac{s(x)}{|\mathbf{T}_0|/(g\rho)} = \frac{s(x)}{a} \qquad \text{where} \quad a = \frac{|\mathbf{T}_0|}{g\rho}$$

3. By Equation 8.1.6, $s'(x) = \int_0^x \sqrt{1 + \left(\frac{dy}{dt}\right)^2}$, so differentiating both sides of the equation from Problem 2 gives

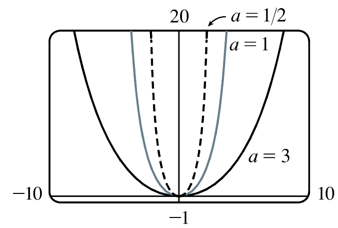
$\dfrac{d^2y}{dx^2} = \dfrac{1}{a}\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$. Making the substitution $z = \dfrac{dy}{dx}$, we have $\dfrac{dz}{dx} = \dfrac{1}{a}\sqrt{1+z^2} \quad\Rightarrow\quad \dfrac{dz}{\sqrt{1+z^2}} = \dfrac{dx}{a}$.

From Table 3.11.6 we know that an antiderivative of $1/\sqrt{1+x^2}$ is $\sinh^{-1} x$, so integrating both sides of the preceding equation gives $\sinh^{-1} z = \dfrac{x}{a} + C$. We are given that $y'(0) = 0 \quad\Rightarrow\quad z(0) = 0 \quad\Rightarrow\quad C = 0$, so $\sinh^{-1} z = \dfrac{x}{a} \quad\Rightarrow$ $z = \sinh\dfrac{x}{a}$.

[continued]

As $z = \dfrac{dy}{dx}$, $\dfrac{dy}{dx} = \sinh\dfrac{x}{a} \quad\Rightarrow\quad dy = \sinh\dfrac{x}{a}\,dx \quad\Rightarrow\quad \int dy = \int \sinh\dfrac{x}{a}\,dx \quad\Rightarrow\quad y = a\cosh\dfrac{x}{a} + C$. From the initial condition $y(0) = 0$, we have $0 = a\cosh 0 + C \quad\Rightarrow\quad 0 = a + C \quad\Rightarrow\quad -a = C$. Therefore, the equation of the curve is $y = a\cosh\dfrac{x}{a} - a$.

4. As the value of a increases, the graph of $y = a\cosh\dfrac{x}{a} - a$ is stretched horizontally.



12.3 The Dot Product

1. (a) $\mathbf{a}\cdot\mathbf{b}$ is a scalar, and the dot product is defined only for vectors, so $(\mathbf{a}\cdot\mathbf{b})\cdot\mathbf{c}$ has no meaning.

(b) $(\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}$ is a scalar multiple of a vector, so it does have meaning.

(c) Both $|\mathbf{a}|$ and $\mathbf{b}\cdot\mathbf{c}$ are scalars, so $|\mathbf{a}|\,(\mathbf{b}\cdot\mathbf{c})$ is an ordinary product of real numbers, and has meaning.

(d) Both $\mathbf{a}$ and $\mathbf{b}+\mathbf{c}$ are vectors, so the dot product $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})$ has meaning.

(e) $\mathbf{a}\cdot\mathbf{b}$ is a scalar, but $\mathbf{c}$ is a vector, and so the two quantities cannot be added and $\mathbf{a}\cdot\mathbf{b}+\mathbf{c}$ has no meaning.

(f) $|\mathbf{a}|$ is a scalar, and the dot product is defined only for vectors, so $|\mathbf{a}|\cdot(\mathbf{b}+\mathbf{c})$ has no meaning.

2. $\mathbf{a}\cdot\mathbf{b} = \langle 5, -2\rangle\cdot\langle 3, 4\rangle = (5)(3) + (-2)(4) = 15 - 8 = 7$

3. $\mathbf{a}\cdot\mathbf{b} = \langle 1.5, 0.4\rangle\cdot\langle -4, 6\rangle = (1.5)(-4) + (0.4)(6) = -6 + 2.4 = -3.6$

4. $\mathbf{a}\cdot\mathbf{b} = \langle 6, -2, 3\rangle\cdot\langle 2, 5, -1\rangle = (6)(2) + (-2)\,(5) + (3)(-1) = 12 - 10 - 3 = -1$

5. $\mathbf{a}\cdot\mathbf{b} = \left\langle 4, 1, \frac{1}{4}\right\rangle\cdot\langle 6, -3, -8\rangle = (4)(6) + (1)(-3) + \left(\frac{1}{4}\right)(-8) = 19$

6. $\mathbf{a}\cdot\mathbf{b} = \langle p, -p, 2p\rangle\cdot\langle 2q, q, -q\rangle = (p)(2q) + (-p)(q) + (2p)(-q) = 2pq - pq - 2pq = -pq$

7. $\mathbf{a}\cdot\mathbf{b} = (2\,\mathbf{i}+\mathbf{j})\cdot(\mathbf{i}-\mathbf{j}+\mathbf{k}) = (2)(1) + (1)(-1) + (0)(1) = 1$

8. $\mathbf{a}\cdot\mathbf{b} = (3\,\mathbf{i}+2\,\mathbf{j}-\mathbf{k})\cdot(4\,\mathbf{i}+5\,\mathbf{k}) = (3)(4) + (2)(0) + (-1)(5) = 7$

9. By Theorem 3, $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta = (7)(4)\cos 30^\circ = 28\left(\frac{\sqrt{3}}{2}\right) = 14\sqrt{3}$.

10. By Theorem 3, $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta = (80)(50)\cos\frac{3\pi}{4} = 4000\left(-\frac{\sqrt{2}}{2}\right) = -2000\sqrt{2}$.

11. $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are all unit vectors, so the triangle is an equilateral triangle. Thus the angle between $\mathbf{u}$ and $\mathbf{v}$ is 60° and $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos 60^\circ = (1)(1)\left(\frac{1}{2}\right) = \frac{1}{2}$. If $\mathbf{w}$ is moved so it has the same initial point as $\mathbf{u}$, we can see that the angle between them is 120° and we have $\mathbf{u}\cdot\mathbf{w} = |\mathbf{u}|\,|\mathbf{w}|\cos 120^\circ = (1)(1)\left(-\frac{1}{2}\right) = -\frac{1}{2}$.

12. $\mathbf{u}$ is a unit vector, so $\mathbf{w}$ is also a unit vector, and $|\mathbf{v}|$ can be determined by examining the right triangle formed by $\mathbf{u}$ and $\mathbf{v}$. Since the angle between $\mathbf{u}$ and $\mathbf{v}$ is $45°$, we have $|\mathbf{v}| = |\mathbf{u}|\cos 45° = \frac{\sqrt{2}}{2}$. Then $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos 45° = (1)\left(\frac{\sqrt{2}}{2}\right)\frac{\sqrt{2}}{2} = \frac{1}{2}$.

Since $\mathbf{u}$ and $\mathbf{w}$ are orthogonal, $\mathbf{u}\cdot\mathbf{w} = 0$.

13. (a) $\mathbf{i}\cdot\mathbf{j} = \langle 1,0,0\rangle\cdot\langle 0,1,0\rangle = (1)(0)+(0)(1)+(0)(0) = 0$. Similarly, $\mathbf{j}\cdot\mathbf{k} = (0)(0)+(1)(0)+(0)(1) = 0$ and $\mathbf{k}\cdot\mathbf{i} = (0)(1)+(0)(0)+(1)(0) = 0$.

Another method: Because $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are mutually perpendicular, the cosine factor in each dot product (see Theorem 3) is $\cos\frac{\pi}{2} = 0$.

(b) By Property 1 of the dot product, $\mathbf{i}\cdot\mathbf{i} = |\mathbf{i}|^2 = 1^2 = 1$ since $\mathbf{i}$ is a unit vector. Similarly, $\mathbf{j}\cdot\mathbf{j} = |\mathbf{j}|^2 = 1$ and $\mathbf{k}\cdot\mathbf{k} = |\mathbf{k}|^2 = 1$.

14. The dot product $\mathbf{A}\cdot\mathbf{P}$ is

$$\begin{aligned}\langle a,b,c\rangle\cdot\langle 4,2.5,1\rangle &= a(4)+b(2.5)+c(1)\\ &= (\text{number of hamburgers sold})(\text{price per hamburger})\\ &\qquad + (\text{number of hot dogs sold})(\text{price per hot dog})\\ &\qquad\qquad + (\text{number of bottles sold})(\text{price per bottle})\end{aligned}$$

so it is equal to the vendor's total revenue for that day.

15. $\mathbf{u} = \langle 5,1\rangle$, $\mathbf{v} = \langle 3,2\rangle \;\Rightarrow\; |\mathbf{u}| = \sqrt{5^2+1^2} = \sqrt{26}$, $|\mathbf{v}| = \sqrt{3^2+2^2} = \sqrt{13}$, and $\mathbf{u}\cdot\mathbf{v} = 5(3)+1(2) = 17$. From Corollary 6, we have $\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \dfrac{17}{\sqrt{26}\sqrt{13}} = \dfrac{17}{13\sqrt{2}}$ and the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\theta = \cos^{-1}\left(\dfrac{17}{13\sqrt{2}}\right) \approx 22°$.

16. $\mathbf{a} = \mathbf{i} - 3\,\mathbf{j}$, $\mathbf{b} = -3\,\mathbf{i} + 4\,\mathbf{j} \;\Rightarrow\; |\mathbf{a}| = \sqrt{1^2+(-3)^2} = \sqrt{10}$, $|\mathbf{b}| = \sqrt{(-3)^2+4^2} = 5$, and $\mathbf{a}\cdot\mathbf{b} = 1(-3)+(-3)(4) = -15$. From Corollary 6, we have $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \dfrac{-15}{5\sqrt{10}} = \dfrac{-3}{\sqrt{10}}$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\theta = \cos^{-1}\left(\dfrac{-3}{\sqrt{10}}\right) \approx 162°$.

17. $\mathbf{a} = \langle 1,-4,1\rangle$, $\mathbf{b} = \langle 0,2,-2\rangle \;\Rightarrow\; |\mathbf{a}| = \sqrt{1^2+(-4)^2+1^2} = \sqrt{18} = 3\sqrt{2}$, $|\mathbf{b}| = \sqrt{0^2+2^2+(-2)^2} = \sqrt{8} = 2\sqrt{2}$, and $\mathbf{a}\cdot\mathbf{b} = (1)(0)+(-4)(2)+(1)(-2) = -10$. From Corollary 6, we have $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{-10}{3\sqrt{2}\cdot 2\sqrt{2}} = -\dfrac{10}{12} = -\dfrac{5}{6}$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\theta = \cos^{-1}\left(-\frac{5}{6}\right) \approx 146°$.

18. $\mathbf{a} = \langle -1,3,4\rangle$, $\mathbf{b} = \langle 5,2,1\rangle \;\Rightarrow\; |\mathbf{a}| = \sqrt{(-1)^2+3^2+4^2} = \sqrt{26}$, $|\mathbf{b}| = \sqrt{5^2+2^2+1^2} = \sqrt{30}$, and $\mathbf{a}\cdot\mathbf{b} = (-1)(5)+(3)(2)+(4)(1) = 5$. From Corollary 6, we have $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{5}{\sqrt{26}\cdot\sqrt{30}} = \dfrac{5}{\sqrt{780}} = \dfrac{5}{2\sqrt{195}}$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\theta = \cos^{-1}\left(\dfrac{5}{2\sqrt{195}}\right) \approx 80°$.

19. $\mathbf{u} = \mathbf{i} - 4\,\mathbf{j} + \mathbf{k}$, $\mathbf{v} = -3\,\mathbf{i} + \mathbf{j} + 5\,\mathbf{k} \quad \Rightarrow \quad |\mathbf{u}| = \sqrt{1^2 + (-4)^2 + 1^2} = \sqrt{18} = 3\sqrt{2}$, $|\mathbf{v}| = \sqrt{(-3)^2 + 1^2 + 5^2} = \sqrt{35}$,

and $\mathbf{u} \cdot \mathbf{v} = 1(-3) + (-4)(1) + 1(5) = -2$. From Corollary 6, we have $\cos\theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \dfrac{-2}{3\sqrt{2}\sqrt{35}} = \dfrac{-2}{3\sqrt{70}}$ and the angle

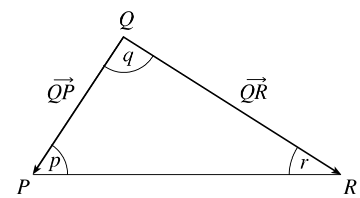
between $\mathbf{u}$ and $\mathbf{v}$ is $\theta = \cos^{-1}\left(\dfrac{-2}{3\sqrt{70}}\right) \approx 95^\circ$.

20. $\mathbf{a} = 8\,\mathbf{i} - \mathbf{j} + 4\,\mathbf{k}$, $\mathbf{b} = 4\,\mathbf{j} + 2\,\mathbf{k} \quad \Rightarrow \quad |\mathbf{a}| = \sqrt{8^2 + (-1)^2 + 4^2} = \sqrt{81} = 9$, $|\mathbf{b}| = \sqrt{0^2 + 4^2 + 2^2} = \sqrt{20} = 2\sqrt{5}$, and

$\mathbf{a} \cdot \mathbf{b} = (8)(0) + (-1)(4) + (4)(2) = 4$. From Corollary 6, we have $\cos\theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{4}{9 \cdot 2\sqrt{5}} = \dfrac{2}{9\sqrt{5}}$ and the angle

between $\mathbf{a}$ and $\mathbf{b}$ is $\theta = \cos^{-1}\left(\dfrac{2}{9\sqrt{5}}\right) \approx 84^\circ$.

21. Let p, q, and r be the angles at vertices P, Q, and R respectively. Then p is the angle between vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, q is the angle between vectors $\overrightarrow{QP}$ and $\overrightarrow{QR}$, and r is the angle between vectors $\overrightarrow{RP}$ and $\overrightarrow{RQ}$.



Thus $\cos p = \dfrac{\overrightarrow{PQ} \cdot \overrightarrow{PR}}{\left|\overrightarrow{PQ}\right| \left|\overrightarrow{PR}\right|} = \dfrac{\langle -2, 3 \rangle \cdot \langle 1, 4 \rangle}{\sqrt{(-2)^2 + 3^2}\,\sqrt{1^2 + 4^2}} = \dfrac{-2 + 12}{\sqrt{13}\,\sqrt{17}} = \dfrac{10}{\sqrt{221}}$ and $p = \cos^{-1}\left(\dfrac{10}{\sqrt{221}}\right) \approx 48^\circ$. Similarly,

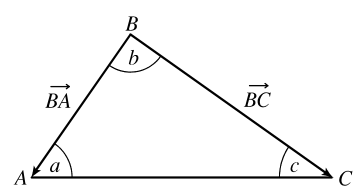
$\cos q = \dfrac{\overrightarrow{QP} \cdot \overrightarrow{QR}}{\left|\overrightarrow{QP}\right| \left|\overrightarrow{QR}\right|} = \dfrac{\langle 2, -3 \rangle \cdot \langle 3, 1 \rangle}{\sqrt{4 + 9}\,\sqrt{9 + 1}} = \dfrac{6 - 3}{\sqrt{13}\,\sqrt{10}} = \dfrac{3}{\sqrt{130}}$ so $q = \cos^{-1}\left(\dfrac{3}{\sqrt{130}}\right) \approx 75^\circ$ and

$r \approx 180^\circ - (48^\circ + 75^\circ) = 57^\circ$.

Alternate solution: Apply the Law of Cosines three times as follows:

$$\cos p = \frac{\left|\overrightarrow{QR}\right|^2 - \left|\overrightarrow{PQ}\right|^2 - \left|\overrightarrow{PR}\right|^2}{2\left|\overrightarrow{PQ}\right|\,\left|\overrightarrow{PR}\right|} \qquad \cos q = \frac{\left|\overrightarrow{PR}\right|^2 - \left|\overrightarrow{PQ}\right|^2 - \left|\overrightarrow{QR}\right|^2}{2\left|\overrightarrow{PQ}\right|\,\left|\overrightarrow{QR}\right|} \qquad \cos r = \frac{\left|\overrightarrow{PQ}\right|^2 - \left|\overrightarrow{PR}\right|^2 - \left|\overrightarrow{QR}\right|^2}{2\left|\overrightarrow{PR}\right|\,\left|\overrightarrow{QR}\right|}$$

22. Let a, b, and c be the angles at vertices A, B, and C. Then a is the angle between vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$, b is the angle between vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$, and c is the angle between vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$.



Thus $\cos a = \dfrac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\left|\overrightarrow{AB}\right| \left|\overrightarrow{AC}\right|} = \dfrac{\langle 2, -2, 1 \rangle \cdot \langle 0, 3, 4 \rangle}{\sqrt{2^2 + (-2)^2 + 1^2}\,\sqrt{0^2 + 3^2 + 4^2}} = \dfrac{0 - 6 + 4}{3 \cdot 5} = -\dfrac{2}{15}$ and $a = \cos^{-1}\left(-\frac{2}{15}\right) \approx 98^\circ$.

Similarly, $\cos b = \dfrac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{\left|\overrightarrow{BA}\right| \left|\overrightarrow{BC}\right|} = \dfrac{\langle -2, 2, -1 \rangle \cdot \langle -2, 5, 3 \rangle}{\sqrt{4 + 4 + 1}\,\sqrt{4 + 25 + 9}} = \dfrac{4 + 10 - 3}{3 \cdot \sqrt{38}} = \dfrac{11}{3\sqrt{38}}$ so $b = \cos^{-1}\left(\dfrac{11}{3\sqrt{38}}\right) \approx 54^\circ$ and

$c \approx 180^\circ - (98^\circ + 54^\circ) = 28^\circ$.

[continued]

Alternate solution: Apply the Law of Cosines three times as follows:

$$\cos a = \frac{\left|\overrightarrow{BC}\right|^2 - \left|\overrightarrow{AB}\right|^2 - \left|\overrightarrow{AC}\right|^2}{2\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|} \qquad \cos b = \frac{\left|\overrightarrow{AC}\right|^2 - \left|\overrightarrow{AB}\right|^2 - \left|\overrightarrow{BC}\right|^2}{2\left|\overrightarrow{AB}\right|\left|\overrightarrow{BC}\right|} \qquad \cos c = \frac{\left|\overrightarrow{AB}\right|^2 - \left|\overrightarrow{AC}\right|^2 - \left|\overrightarrow{BC}\right|^2}{2\left|\overrightarrow{AC}\right|\left|\overrightarrow{BC}\right|}$$

23. (a) $\mathbf{a}\cdot\mathbf{b} = (9)(-2) + (3)(6) = 0$, so $\mathbf{a}$ and $\mathbf{b}$ are orthogonal (and not parallel).

(b) $\mathbf{a}\cdot\mathbf{b} = (4)(3) + (5)(-1) + (-2)(5) = -3 \neq 0$, so $\mathbf{a}$ and $\mathbf{b}$ are not orthogonal. Also, since $\mathbf{a}$ is not a scalar multiple of $\mathbf{b}$, $\mathbf{a}$ and $\mathbf{b}$ are not parallel.

(c) $\mathbf{a}\cdot\mathbf{b} = (-8)(6) + (12)(-9) + (4)(-3) = -168 \neq 0$, so $\mathbf{a}$ and $\mathbf{b}$ are not orthogonal. Because $\mathbf{a} = -\frac{4}{3}\,\mathbf{b}$, $\mathbf{a}$ and $\mathbf{b}$ are parallel.

(d) $\mathbf{a}\cdot\mathbf{b} = (3)(5) + (-1)(9) + (3)(-2) = 0$, so $\mathbf{a}$ and $\mathbf{b}$ are orthogonal (and not parallel).

24. (a) $\mathbf{u}\cdot\mathbf{v} = (-5)(3) + (4)(4) + (-2)(-1) = 3 \neq 0$, so $\mathbf{u}$ and $\mathbf{v}$ are not orthogonal. Also, $\mathbf{u}$ is not a scalar multiple of $\mathbf{v}$, so $\mathbf{u}$ and $\mathbf{v}$ are not parallel.

(b) $\mathbf{u}\cdot\mathbf{v} = (9)(-6) + (-6)(4) + (3)(-2) = -84 \neq 0$, so $\mathbf{u}$ and $\mathbf{v}$ are not orthogonal. Because $\mathbf{u} = -\frac{3}{2}\,\mathbf{v}$, $\mathbf{u}$ and $\mathbf{v}$ are parallel.

(c) $\mathbf{u}\cdot\mathbf{v} = (c)(c) + (c)(0) + (c)(-c) = c^2 + 0 - c^2 = 0$, so $\mathbf{u}$ and $\mathbf{v}$ are orthogonal (and not parallel). (Note that if $c = 0$ then $\mathbf{u} = \mathbf{v} = \mathbf{0}$, and the zero vector is considered orthogonal to all vectors. Although in this case $\mathbf{u}$ and $\mathbf{v}$ are identical, they are not considered parallel, as only nonzero vectors can be parallel.)

25. $\overrightarrow{QP} = \langle -1, -3, 2\rangle$, $\overrightarrow{QR} = \langle 4, -2, -1\rangle$, and $\overrightarrow{QP}\cdot\overrightarrow{QR} = -4 + 6 - 2 = 0$. Thus $\overrightarrow{QP}$ and $\overrightarrow{QR}$ are orthogonal, so the angle of the triangle at vertex Q is a right angle.

26. By Theorem 3, vectors $\langle 2, 1, -1\rangle$ and $\langle 1, x, 0\rangle$ meet at an angle of 45° when

$\langle 2, 1, -1\rangle \cdot \langle 1, x, 0\rangle = \sqrt{4+1+1}\,\sqrt{1+x^2+0}\,\cos 45^\circ$ or $2 + x - 0 = \sqrt{6}\,\sqrt{1+x^2}\cdot\frac{\sqrt{2}}{2} \quad\Leftrightarrow\quad 2 + x = \sqrt{3}\sqrt{1+x^2}$.

Squaring both sides gives $4 + 4x + x^2 = 3 + 3x^2 \quad\Leftrightarrow\quad 2x^2 - 4x - 1 = 0$. By the quadratic formula,

$x = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(-1)}}{2(2)} = \dfrac{4 \pm \sqrt{24}}{4} = \dfrac{4 \pm 2\sqrt{6}}{4} = 1 \pm \dfrac{\sqrt{6}}{2}$. (You can verify that both values are valid.)

27. Let $\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$ be a vector orthogonal to both $\mathbf{i}+\mathbf{j}$ and $\mathbf{i}+\mathbf{k}$. Then $\mathbf{a}\cdot(\mathbf{i}+\mathbf{j}) = 0 \quad\Leftrightarrow\quad a_1 + a_2 = 0$ and $\mathbf{a}\cdot(\mathbf{i}+\mathbf{k}) = 0 \quad\Leftrightarrow\quad a_1 + a_3 = 0$, so $a_1 = -a_2 = -a_3$. Furthermore $\mathbf{a}$ is to be a unit vector, so $1 = a_1^2 + a_2^2 + a_3^2 = 3a_1^2$ implies $a_1 = \pm\frac{1}{\sqrt{3}}$. Thus $\mathbf{a} = \frac{1}{\sqrt{3}}\,\mathbf{i} - \frac{1}{\sqrt{3}}\,\mathbf{j} - \frac{1}{\sqrt{3}}\,\mathbf{k}$ and $\mathbf{a} = -\frac{1}{\sqrt{3}}\,\mathbf{i} + \frac{1}{\sqrt{3}}\,\mathbf{j} + \frac{1}{\sqrt{3}}\,\mathbf{k}$ are two such unit vectors.

28. Let $\mathbf{u} = \langle a, b\rangle$ be a unit vector. By Theorem 3 we need $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}|\;|\mathbf{v}|\cos 60^\circ \quad\Leftrightarrow\quad 3a + 4b = (1)(5)\frac{1}{2} \quad\Leftrightarrow$ $b = \frac{5}{8} - \frac{3}{4}a$. Since $\mathbf{u}$ is a unit vector, $|\mathbf{u}| = \sqrt{a^2+b^2} = 1 \quad\Leftrightarrow\quad a^2 + b^2 = 1 \quad\Leftrightarrow\quad a^2 + \left(\frac{5}{8} - \frac{3}{4}a\right)^2 = 1 \quad\Leftrightarrow$

$\frac{25}{16}a^2 - \frac{15}{16}a + \frac{25}{64} = 1 \quad \Leftrightarrow \quad 100a^2 - 60a - 39 = 0$. By the quadratic formula,

$a = \dfrac{-(-60) \pm \sqrt{(-60)^2 - 4(100)(-39)}}{2(100)} = \dfrac{60 \pm \sqrt{19{,}200}}{200} = \dfrac{3 \pm 4\sqrt{3}}{10}$. If $a = \dfrac{3 + 4\sqrt{3}}{10}$, then

$b = \dfrac{5}{8} - \dfrac{3}{4}\left(\dfrac{3 + 4\sqrt{3}}{10}\right) = \dfrac{4 - 3\sqrt{3}}{10}$, and if $a = \dfrac{3 - 4\sqrt{3}}{10}$, then $b = \dfrac{5}{8} - \dfrac{3}{4}\left(\dfrac{3 - 4\sqrt{3}}{10}\right) = \dfrac{4 + 3\sqrt{3}}{10}$. Thus the two

unit vectors are $\left\langle \dfrac{3 + 4\sqrt{3}}{10}, \dfrac{4 - 3\sqrt{3}}{10} \right\rangle \approx \langle 0.9928, -0.1196 \rangle$ and $\left\langle \dfrac{3 - 4\sqrt{3}}{10}, \dfrac{4 + 3\sqrt{3}}{10} \right\rangle \approx \langle -0.3928, 0.9196 \rangle$.

29. The line $y = 4 - 3x \quad \Leftrightarrow \quad y = -3x + 4$ has slope -3, so a vector parallel to the line is $\mathbf{a} = \langle 1, -3 \rangle$. The line $y = 3x + 2$ has slope 3, so a vector parallel to the line is $\mathbf{b} = \langle 1, 3 \rangle$. The angle between the lines is the same as the angle θ between the vectors. Here we have $\mathbf{a} \cdot \mathbf{b} = 1(1) + 3(-3) = -8$, $|\mathbf{a}| = \sqrt{1^2 + (-3)^2} = \sqrt{10}$, and $|\mathbf{b}| = \sqrt{1^2 + 3^2} = \sqrt{10}$. Then $\cos\theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \dfrac{-8}{\sqrt{10}\sqrt{10}} = -\dfrac{4}{5}$ and $\theta = \cos^{-1}\left(-\dfrac{4}{5}\right) \approx 143.1^\circ$. Therefore, the acute angle between the two lines is approximately $180^\circ - 143.1^\circ = 36.9^\circ$.

30. The line $5x - y = 8 \quad \Leftrightarrow \quad y = 5x - 8$ has slope 5, so a vector parallel to the line is $\mathbf{a} = \langle 1, 5 \rangle$. The line $x + 3y = 15 \quad \Leftrightarrow \quad y = -\frac{1}{3}x + 5$ has slope $-\frac{1}{3}$, so a vector parallel to the line is $\mathbf{b} = \langle 3, -1 \rangle$. The angle between the lines is the same as the angle θ between the vectors. Here we have $\mathbf{a} \cdot \mathbf{b} = 1(3) + 5(-1) = -2$, $|\mathbf{a}| = \sqrt{1^2 + 5^2} = \sqrt{26}$, and $|\mathbf{b}| = \sqrt{3^2 + (-1)^2} = \sqrt{10}$. Then $\cos\theta = \dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \dfrac{-2}{\sqrt{26}\sqrt{10}} = -\dfrac{1}{\sqrt{65}}$ and $\theta = \cos^{-1}\left(-\dfrac{1}{\sqrt{65}}\right) \approx 97.1^\circ$. Therefore, the acute angle between the two lines is approximately $180^\circ - 97.1^\circ = 82.9^\circ$.

31. The curves $y = x^2$ and $y = x^3$ meet when $x^2 = x^3 \quad \Leftrightarrow \quad x^3 - x^2 = 0 \quad \Leftrightarrow \quad x^2(x - 1) = 0 \quad \Leftrightarrow \quad x = 0, x = 1$. We have $\dfrac{d}{dx}x^2 = 2x$ and $\dfrac{d}{dx}x^3 = 3x^2$, so the tangent lines of both curves have slope 0 at $x = 0$. Thus the angle between the curves is 0° at the point $(0, 0)$. For $x = 1$, $\left.\dfrac{d}{dx}\,x^2\right|_{x=1} = 2$ and $\left.\dfrac{d}{dx}\,x^3\right|_{x=1} = 3$ so the tangent lines at the point $(1, 1)$ have slopes 2 and 3. Vectors parallel to the tangent lines are $\langle 1, 2 \rangle$ and $\langle 1, 3 \rangle$, and the angle θ between them is given by

$$\cos\theta = \frac{\langle 1, 2 \rangle \cdot \langle 1, 3 \rangle}{|\langle 1, 2 \rangle|\,|\langle 1, 3 \rangle|} = \frac{1 + 6}{\sqrt{5}\,\sqrt{10}} = \frac{7}{5\sqrt{2}}$$

Thus $\theta = \cos^{-1}\left(\dfrac{7}{5\sqrt{2}}\right) \approx 8.1^\circ$.

32. The curves $y = \sin x$ and $y = \cos x$ meet when $\sin x = \cos x \quad \Leftrightarrow \quad \tan x = 1 \quad \Leftrightarrow \quad x = \pi/4 \;\; [0 \le x \le \pi/2]$. Thus the point of intersection is $\left(\pi/4, \sqrt{2}/2\right)$. We have $\left.\dfrac{d}{dx}\,\sin x\right|_{x=\pi/4} = \left.\cos x\right|_{x=\pi/4} = \dfrac{\sqrt{2}}{2}$ and $\left.\dfrac{d}{dx}\,\cos x\right|_{x=\pi/4} = \left.-\sin x\right|_{x=\pi/4} = -\dfrac{\sqrt{2}}{2}$, so the tangent lines at that point have slopes $\dfrac{\sqrt{2}}{2}$ and $-\dfrac{\sqrt{2}}{2}$. Vectors parallel to the tangent lines are $\left\langle 1, \frac{\sqrt{2}}{2} \right\rangle$ and $\left\langle 1, -\frac{\sqrt{2}}{2} \right\rangle$, and the angle θ between them is given by

$$\cos\theta = \frac{\langle 1, \sqrt{2}/2\rangle \cdot \langle 1, -\sqrt{2}/2\rangle}{|\langle 1, \sqrt{2}/2\rangle|\;|\langle 1, -\sqrt{2}/2\rangle|} = \frac{1-\frac{1}{2}}{\sqrt{\frac{3}{2}}\sqrt{\frac{3}{2}}} = \frac{1/2}{3/2} = \frac{1}{3}$$

Thus $\theta = \cos^{-1}\left(\frac{1}{3}\right) \approx 70.5°$.

33. $|\langle 4,1,8\rangle| = \sqrt{4^2+1^2+8^2} = \sqrt{81} = 9$. Using Equations 8 and 9, we have $\cos\alpha = \frac{4}{9}$, $\cos\beta = \frac{1}{9}$, and $\cos\gamma = \frac{8}{9}$. The direction angles are given by $\alpha = \cos^{-1}\left(\frac{4}{9}\right) \approx 63.6°$, $\beta = \cos^{-1}\left(\frac{1}{9}\right) \approx 83.6°$, and $\gamma = \cos^{-1}\left(\frac{8}{9}\right) \approx 27.3°$.

34. $|\langle -6,2,9\rangle| = \sqrt{(-6)^2+2^2+9^2} = \sqrt{121} = 11$. Using Equations 8 and 9, we have $\cos\alpha = -\frac{6}{11}$, $\cos\beta = \frac{2}{11}$, and $\cos\gamma = \frac{9}{11}$. The direction angles are given by $\alpha = \cos^{-1}\left(-\frac{6}{11}\right) \approx 123.1°$, $\beta = \cos^{-1}\left(\frac{2}{11}\right) \approx 79.5°$, and $\gamma = \cos^{-1}\left(\frac{9}{11}\right) \approx 35.1°$.

35. $|3\mathbf{i} - \mathbf{j} - 2\mathbf{k}| = \sqrt{3^2+(-1)^2+(-2)^2} = \sqrt{14}$. Using Equations 8 and 9, we have $\cos\alpha = \dfrac{3}{\sqrt{14}}$, $\cos\beta = -\dfrac{1}{\sqrt{14}}$, and $\cos\gamma = -\dfrac{2}{\sqrt{14}}$. The direction angles are given by $\alpha = \cos^{-1}\left(\dfrac{3}{\sqrt{14}}\right) \approx 36.7°$, $\beta = \cos^{-1}\left(-\dfrac{1}{\sqrt{14}}\right) \approx 105.5°$, and $\gamma = \cos^{-1}\left(-\dfrac{2}{\sqrt{14}}\right) \approx 122.3°$.

36. $|-0.7\mathbf{i} + 1.2\mathbf{j} - 0.8\mathbf{k}| = \sqrt{(-0.7)^2+(1.2)^2+(-0.8)^2} = \sqrt{2.57}$. Using Equations 8 and 9, we have $\cos\alpha = -\dfrac{0.7}{\sqrt{2.57}}$, $\cos\beta = \dfrac{1.2}{\sqrt{2.57}}$, and $\cos\gamma = -\dfrac{0.8}{\sqrt{2.57}}$. The direction angles are given by $\alpha = \cos^{-1}\left(-\dfrac{0.7}{\sqrt{2.57}}\right) \approx 115.9°$, $\beta = \cos^{-1}\left(\dfrac{1.2}{\sqrt{2.57}}\right) \approx 41.5°$, and $\gamma = \cos^{-1}\left(-\dfrac{0.8}{\sqrt{2.57}}\right) \approx 119.9°$.

37. $|\langle c,c,c\rangle| = \sqrt{c^2+c^2+c^2} = \sqrt{3c^2} = c\sqrt{3}$. Using Equations 8 and 9, we have $\cos\alpha = \cos\beta = \cos\gamma = \dfrac{c}{c\sqrt{3}} = \dfrac{1}{\sqrt{3}}$. The direction angles are given by $\alpha = \beta = \gamma = \cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right) \approx 54.7°$.

38. Since $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, $\cos^2\gamma = 1 - \cos^2\alpha - \cos^2\beta = 1 - \cos^2\left(\frac{\pi}{4}\right) - \cos^2\left(\frac{\pi}{3}\right) = 1 - \left(\frac{\sqrt{2}}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = \frac{1}{4}$. Thus $\cos\gamma = \pm\frac{1}{2}$ and $\gamma = \frac{\pi}{3}$ or $\gamma = \frac{2\pi}{3}$.

39. $|\mathbf{a}| = \sqrt{(-5)^2+12^2} = \sqrt{169} = 13$. The scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\operatorname{comp}_{\mathbf{a}}\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \dfrac{-5\cdot 4 + 12\cdot 6}{13} = 4$ and the vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\operatorname{proj}_{\mathbf{a}}\mathbf{b} = \left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\right)\dfrac{\mathbf{a}}{|\mathbf{a}|} = 4\cdot\frac{1}{13}\langle -5,12\rangle = \left\langle -\frac{20}{13}, \frac{48}{13}\right\rangle$.

40. $|\mathbf{a}| = \sqrt{1^2+4^2} = \sqrt{17}$. The scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\operatorname{comp}_{\mathbf{a}}\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \dfrac{1\cdot 2 + 4\cdot 3}{\sqrt{17}} = \dfrac{14}{\sqrt{17}}$ and the vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\operatorname{proj}_{\mathbf{a}}\mathbf{b} = \left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\right)\dfrac{\mathbf{a}}{|\mathbf{a}|} = \frac{14}{\sqrt{17}}\cdot\frac{1}{\sqrt{17}}\langle 1,4\rangle = \left\langle \frac{14}{17}, \frac{56}{17}\right\rangle$.

41. $|\mathbf{a}| = \sqrt{4^2 + 7^2 + (-4)^2} = \sqrt{81} = 9$ so the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is

$\text{comp}_{\mathbf{a}}\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \dfrac{(4)(3) + (7)(-1) + (-4)(1)}{9} = \dfrac{1}{9}$. The vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is

$\text{proj}_{\mathbf{a}}\mathbf{b} = \left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\right)\dfrac{\mathbf{a}}{|\mathbf{a}|} = \frac{1}{9}\cdot\frac{1}{9}\langle 4, 7, -4\rangle = \frac{1}{81}\langle 4, 7, -4\rangle = \left\langle \frac{4}{81}, \frac{7}{81}, -\frac{4}{81}\right\rangle$.

42. $|\mathbf{a}| = \sqrt{1 + 16 + 64} = \sqrt{81} = 9$ so the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\text{comp}_{\mathbf{a}}\,\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \frac{1}{9}(-12 + 4 + 16) = \frac{8}{9}$, while the vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\text{proj}_{\mathbf{a}}\,\mathbf{b} = \left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\right)\dfrac{\mathbf{a}}{|\mathbf{a}|} = \frac{8}{9}\cdot\frac{1}{9}\langle -1, 4, 8\rangle = \frac{8}{81}\langle -1, 4, 8\rangle = \left\langle -\frac{8}{81}, \frac{32}{81}, \frac{64}{81}\right\rangle$.

43. $|\mathbf{a}| = \sqrt{9 + 9 + 1} = \sqrt{19}$ so the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\text{comp}_{\mathbf{a}}\,\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \dfrac{6 - 12 - 1}{\sqrt{19}} = -\dfrac{7}{\sqrt{19}}$ while the vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\text{proj}_{\mathbf{a}}\,\mathbf{b} = -\dfrac{7}{\sqrt{19}}\dfrac{\mathbf{a}}{|\mathbf{a}|} = -\frac{7}{\sqrt{19}}\cdot\frac{1}{\sqrt{19}}(3\,\mathbf{i} - 3\,\mathbf{j} + \mathbf{k}) = -\frac{7}{19}(3\,\mathbf{i} - 3\,\mathbf{j} + \mathbf{k}) = -\frac{21}{19}\,\mathbf{i} + \frac{21}{19}\,\mathbf{j} - \frac{7}{19}\,\mathbf{k}$.

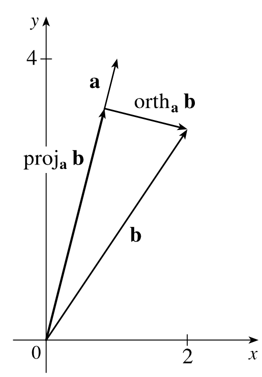
44. $|\mathbf{a}| = \sqrt{1 + 4 + 9} = \sqrt{14}$ so the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\text{comp}_{\mathbf{a}}\,\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \dfrac{5 + 0 - 3}{\sqrt{14}} = \dfrac{2}{\sqrt{14}}$ while the vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\text{proj}_{\mathbf{a}}\,\mathbf{b} = \dfrac{2}{\sqrt{14}}\dfrac{\mathbf{a}}{|\mathbf{a}|} = \frac{2}{\sqrt{14}}\cdot\frac{1}{\sqrt{14}}(\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}) = \frac{1}{7}(\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}) = \frac{1}{7}\,\mathbf{i} + \frac{2}{7}\,\mathbf{j} + \frac{3}{7}\,\mathbf{k}$.

45. $(\text{orth}_{\mathbf{a}}\,\mathbf{b})\cdot\mathbf{a} = (\mathbf{b} - \text{proj}_{\mathbf{a}}\,\mathbf{b})\cdot\mathbf{a} = \mathbf{b}\cdot\mathbf{a} - (\text{proj}_{\mathbf{a}}\,\mathbf{b})\cdot\mathbf{a} = \mathbf{b}\cdot\mathbf{a} - \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\,\mathbf{a}\cdot\mathbf{a} = \mathbf{b}\cdot\mathbf{a} - \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\,|\mathbf{a}|^2 = \mathbf{b}\cdot\mathbf{a} - \mathbf{a}\cdot\mathbf{b} = 0$.

So they are orthogonal by (7).

46. Using the formula in Exercise 45 and the result of Exercise 40, we have

$\text{orth}_{\mathbf{a}}\,\mathbf{b} = \mathbf{b} - \text{proj}_{\mathbf{a}}\,\mathbf{b} = \langle 2, 3\rangle - \left\langle \frac{14}{17}, \frac{56}{17}\right\rangle = \left\langle \frac{20}{17}, -\frac{5}{17}\right\rangle$.



47. $\text{comp}_{\mathbf{a}}\,\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = 2 \iff \mathbf{a}\cdot\mathbf{b} = 2\,|\mathbf{a}| = 2\sqrt{10}$. If $\mathbf{b} = \langle b_1, b_2, b_3\rangle$, then we need $3b_1 + 0b_2 - 1b_3 = 2\sqrt{10}$.

One possible solution is obtained by taking $b_1 = 0$, $b_2 = 0$, $b_3 = -2\sqrt{10}$. In general, $\mathbf{b} = \left\langle s, t, 3s - 2\sqrt{10}\right\rangle$, $s, t \in \mathbb{R}$.

48. (a) $\text{comp}_{\mathbf{a}}\,\mathbf{b} = \text{comp}_{\mathbf{b}}\,\mathbf{a} \iff \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \dfrac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{b}|} \iff \dfrac{1}{|\mathbf{a}|} = \dfrac{1}{|\mathbf{b}|}$ or $\mathbf{a}\cdot\mathbf{b} = 0 \iff |\mathbf{b}| = |\mathbf{a}|$ or $\mathbf{a}\cdot\mathbf{b} = 0$.

That is, if $\mathbf{a}$ and $\mathbf{b}$ are orthogonal or if they have the same length.

(b) $\text{proj}_{\mathbf{a}}\,\mathbf{b} = \text{proj}_{\mathbf{b}}\,\mathbf{a} \iff \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\,\mathbf{a} = \dfrac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{b}|^2}\,\mathbf{b} \iff \mathbf{a}\cdot\mathbf{b} = 0$ or $\dfrac{\mathbf{a}}{|\mathbf{a}|^2} = \dfrac{\mathbf{b}}{|\mathbf{b}|^2}$.

But $\dfrac{\mathbf{a}}{|\mathbf{a}|^2} = \dfrac{\mathbf{b}}{|\mathbf{b}|^2} \Rightarrow \dfrac{|\mathbf{a}|}{|\mathbf{a}|^2} = \dfrac{|\mathbf{b}|}{|\mathbf{b}|^2} \Rightarrow |\mathbf{a}| = |\mathbf{b}|$. Substituting this into the previous equation gives $\mathbf{a} = \mathbf{b}$.

So $\text{proj}_{\mathbf{a}}\,\mathbf{b} = \text{proj}_{\mathbf{b}}\,\mathbf{a} \iff \mathbf{a}$ and $\mathbf{b}$ are orthogonal, or they are equal.

49. The displacement vector is $\mathbf{D} = (6-0)\,\mathbf{i} + (12-10)\,\mathbf{j} + (20-8)\,\mathbf{k} = 6\,\mathbf{i} + 2\,\mathbf{j} + 12\,\mathbf{k}$ so, by Equation 12, the work done is

$W = \mathbf{F}\cdot\mathbf{D} = (8\,\mathbf{i} - 6\,\mathbf{j} + 9\,\mathbf{k})\cdot(6\,\mathbf{i} + 2\,\mathbf{j} + 12\,\mathbf{k}) = 48 - 12 + 108 = 144$ joules.

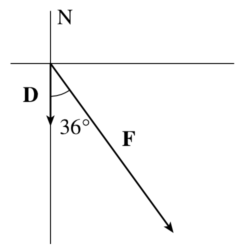
50. Here $|\mathbf{D}| = 1000$ m, $|\mathbf{F}| = 1500$ N, and $\theta = 30^\circ$. Thus

$W = \mathbf{F}\cdot\mathbf{D} = |\mathbf{F}|\,|\mathbf{D}|\cos\theta = (1500)(1000)\left(\frac{\sqrt{3}}{2}\right) = 750{,}000\sqrt{3}$ joules.

51. Here $|\mathbf{D}| = 80$ m, $|\mathbf{F}| = 30$ N, and $\theta = 40^\circ$. Thus,

$W = \mathbf{F}\cdot\mathbf{D} = |\mathbf{F}|\,|\mathbf{D}|\cos\theta = (30)(80)\cos 40^\circ \approx 1838.5$ J.

52. $W = \mathbf{F}\cdot\mathbf{D} = |\mathbf{F}|\,|\mathbf{D}|\cos\theta = (2000)(40)\cos 36 \approx 64{,}721$ J.



53. First note that $\mathbf{n} = \langle a, b\rangle$ is perpendicular to the line, because if $Q_1 = (a_1, b_1)$ and $Q_2 = (a_2, b_2)$ lie on the line, then

$\mathbf{n}\cdot\overrightarrow{Q_1Q_2} = aa_2 - aa_1 + bb_2 - bb_1 = 0$, since $aa_2 + bb_2 = -c = aa_1 + bb_1$ from the equation of the line.

Let $P_2 = (x_2, y_2)$ lie on the line. Then the distance from P_1 to the line is the absolute value of the scalar projection

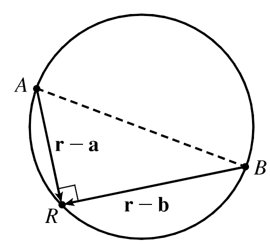
of $\overrightarrow{P_1P_2}$ onto $\mathbf{n}$. $\quad \operatorname{comp}_{\mathbf{n}}\left(\overrightarrow{P_1P_2}\right) = \dfrac{|\mathbf{n}\cdot\langle x_2 - x_1, y_2 - y_1\rangle|}{|\mathbf{n}|} = \dfrac{|ax_2 - ax_1 + by_2 - by_1|}{\sqrt{a^2+b^2}} = \dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$

since $ax_2 + by_2 = -c$. The required distance is $\dfrac{|(3)(-2) + (-4)(3) + 5|}{\sqrt{3^2 + (-4)^2}} = \dfrac{13}{5}$.

54. $(\mathbf{r}-\mathbf{a})\cdot(\mathbf{r}-\mathbf{b}) = 0$ implies that the vectors $\mathbf{r}-\mathbf{a}$ and $\mathbf{r}-\mathbf{b}$ are orthogonal. From the diagram (in which A, B and R are the terminal points of the vectors), we see that this implies that R lies on a sphere whose diameter is the line from A to B. The center of this circle is the midpoint of AB, that is,

$\frac{1}{2}(\mathbf{a}+\mathbf{b}) = \left\langle \frac{1}{2}(a_1+b_1), \frac{1}{2}(a_2+b_2), \frac{1}{2}(a_3+b_3)\right\rangle$, and its radius is

$\frac{1}{2}\,|\mathbf{a}-\mathbf{b}| = \frac{1}{2}\sqrt{(a_1-b_1)^2 + (a_2-b_2)^2 + (a_3-b_3)^2}$.



Or: Expand the given equation, substitute $\mathbf{r}\cdot\mathbf{r} = x^2 + y^2 + z^2$ and complete the squares.

55. For convenience, consider the unit cube positioned so that its back left corner is at the origin, and its edges lie along the coordinate axes. The diagonal of the cube that begins at the origin and ends at $(1,1,1)$ has vector representation $\langle 1,1,1\rangle$. The angle θ between this vector and the vector of the edge which also begins at the origin and runs along the x-axis [that is,

$\langle 1,0,0\rangle$] is given by $\cos\theta = \dfrac{\langle 1,1,1\rangle\cdot\langle 1,0,0\rangle}{|\langle 1,1,1\rangle|\,|\langle 1,0,0\rangle|} = \dfrac{1}{\sqrt{3}} \quad\Rightarrow\quad \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 54.7^\circ$.

56. Consider a cube with sides of unit length, wholly within the first octant and with edges along each of the three coordinate axes. $\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $\mathbf{i}+\mathbf{j}$ are vector representations of a diagonal of the cube and a diagonal of one of its faces. If θ is the angle between these diagonals, then $\cos\theta = \dfrac{(\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot(\mathbf{i}+\mathbf{j})}{|\mathbf{i}+\mathbf{j}+\mathbf{k}|\,|\mathbf{i}+\mathbf{j}|} = \dfrac{1+1}{\sqrt{3}\sqrt{2}} = \sqrt{\dfrac{2}{3}} \quad\Rightarrow\quad \theta = \cos^{-1}\sqrt{\tfrac{2}{3}} \approx 35.3^\circ$.

57. Consider the H—C—H combination consisting of the sole carbon atom and the two hydrogen atoms that are at $(1,0,0)$ and $(0,1,0)$ (or any H—C—H combination, for that matter). Vector representations of the line segments emanating from the carbon atom and extending to these two hydrogen atoms are $\left\langle 1-\frac{1}{2}, 0-\frac{1}{2}, 0-\frac{1}{2}\right\rangle = \left\langle \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\rangle$ and $\left\langle 0-\frac{1}{2}, 1-\frac{1}{2}, 0-\frac{1}{2}\right\rangle = \left\langle -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\rangle$. The bond angle, θ, is therefore given by

$$\cos\theta = \frac{\left\langle \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\rangle \cdot \left\langle -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\rangle}{\left|\left\langle \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right\rangle\right| \left|\left\langle -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\rangle\right|} = \frac{-\frac{1}{4}-\frac{1}{4}+\frac{1}{4}}{\sqrt{\frac{3}{4}}\sqrt{\frac{3}{4}}} = -\frac{1}{3} \quad\Rightarrow\quad \theta = \cos^{-1}\left(-\tfrac{1}{3}\right) \approx 109.5^\circ.$$

58. Let α be the angle between $\mathbf{a}$ and $\mathbf{c}$ and β be the angle between $\mathbf{c}$ and $\mathbf{b}$. We need to show that $\alpha = \beta$. Now

$\cos\alpha = \dfrac{\mathbf{a}\cdot\mathbf{c}}{|\mathbf{a}|\,|\mathbf{c}|} = \dfrac{\mathbf{a}\cdot|\mathbf{a}|\,\mathbf{b}+\mathbf{a}\cdot|\mathbf{b}|\,\mathbf{a}}{|\mathbf{a}|\,|\mathbf{c}|} = \dfrac{|\mathbf{a}|\,\mathbf{a}\cdot\mathbf{b}+|\mathbf{a}|^2\,|\mathbf{b}|}{|\mathbf{a}|\,|\mathbf{c}|} = \dfrac{\mathbf{a}\cdot\mathbf{b}+|\mathbf{a}|\,|\mathbf{b}|}{|\mathbf{c}|}$. Similarly,

$\cos\beta = \dfrac{\mathbf{b}\cdot\mathbf{c}}{|\mathbf{b}|\,|\mathbf{c}|} = \dfrac{|\mathbf{a}|\,|\mathbf{b}|+\mathbf{b}\cdot\mathbf{a}}{|\mathbf{c}|}$. Thus $\cos\alpha = \cos\beta$. However $0^\circ \le \alpha \le 180^\circ$ and $0^\circ \le \beta \le 180^\circ$, so $\alpha = \beta$ and $\mathbf{c}$ bisects the angle between $\mathbf{a}$ and $\mathbf{b}$.

59. Let $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$.

Property 2: $\quad \mathbf{a}\cdot\mathbf{b} = \langle a_1, a_2, a_3\rangle \cdot \langle b_1, b_2, b_3\rangle = a_1b_1 + a_2b_2 + a_3b_3$

$$= b_1a_1 + b_2a_2 + b_3a_3 = \langle b_1, b_2, b_3\rangle \cdot \langle a_1, a_2, a_3\rangle = \mathbf{b}\cdot\mathbf{a}$$

Property 4: $\quad (c\,\mathbf{a})\cdot\mathbf{b} = \langle ca_1, ca_2, ca_3\rangle \cdot \langle b_1, b_2, b_3\rangle = (ca_1)b_1 + (ca_2)b_2 + (ca_3)b_3$

$$= c\,(a_1b_1 + a_2b_2 + a_3b_3) = c\,(\mathbf{a}\cdot\mathbf{b}) = a_1(cb_1) + a_2(cb_2) + a_3(cb_3)$$

$$= \langle a_1, a_2, a_3\rangle \cdot \langle cb_1, cb_2, cb_3\rangle = \mathbf{a}\cdot(c\,\mathbf{b})$$

Property 5: $\quad \mathbf{0}\cdot\mathbf{a} = \langle 0,0,0\rangle \cdot \langle a_1, a_2, a_3\rangle = (0)(a_1) + (0)(a_2) + (0)(a_3) = 0$

60. Let the figure be called quadrilateral $ABCD$. The diagonals can be represented by $\overrightarrow{AC}$ and $\overrightarrow{BD}$. $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$ and $\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{BC} - \overrightarrow{DC} = \overrightarrow{BC} - \overrightarrow{AB}$ (Since opposite sides of the object are of the same length and parallel, $\overrightarrow{AB} = \overrightarrow{DC}$.) Thus,

$$\overrightarrow{AC}\cdot\overrightarrow{BD} = \left(\overrightarrow{AB} + \overrightarrow{BC}\right)\cdot\left(\overrightarrow{BC} - \overrightarrow{AB}\right) = \overrightarrow{AB}\cdot\left(\overrightarrow{BC} - \overrightarrow{AB}\right) + \overrightarrow{BC}\cdot\left(\overrightarrow{BC} - \overrightarrow{AB}\right)$$

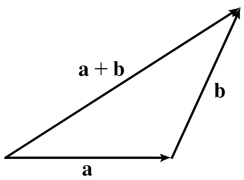
$$= \overrightarrow{AB}\cdot\overrightarrow{BC} - \left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{BC}\right|^2 - \overrightarrow{AB}\cdot\overrightarrow{BC} = \left|\overrightarrow{BC}\right|^2 - \left|\overrightarrow{AB}\right|^2$$

But $\left|\overrightarrow{AB}\right|^2 = \left|\overrightarrow{BC}\right|^2$ because all sides of the quadrilateral are equal in length. Therefore, $\overrightarrow{AC}\cdot\overrightarrow{BD} = 0$, and since both of these vectors are nonzero, this tells us that the diagonals of the quadrilateral are perpendicular.

61. $|\mathbf{a}\cdot\mathbf{b}| = \big|\,|\mathbf{a}|\,|\mathbf{b}|\cos\theta\big| = |\mathbf{a}|\,|\mathbf{b}|\,|\cos\theta|$. Since $|\cos\theta| \le 1$, $|\mathbf{a}\cdot\mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\,|\cos\,\theta| \le |\mathbf{a}|\,|\mathbf{b}|$.

Note: We have equality in the case of $\cos\theta = \pm 1$, so $\theta = 0$ or $\theta = \pi$, thus equality when $\mathbf{a}$ and $\mathbf{b}$ are parallel.

62. (a)

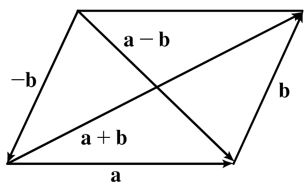


The Triangle Inequality states that the length of the longest side of a triangle is less than or equal to the sum of the lengths of the two shortest sides.

(b) $$\begin{aligned} |\mathbf{a}+\mathbf{b}|^2 &= (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) = (\mathbf{a}\cdot\mathbf{a}) + 2(\mathbf{a}\cdot\mathbf{b}) + (\mathbf{b}\cdot\mathbf{b}) = |\mathbf{a}|^2 + 2(\mathbf{a}\cdot\mathbf{b}) + |\mathbf{b}|^2 \\ &\le |\mathbf{a}|^2 + 2\,|\mathbf{a}|\,|\mathbf{b}| + |\mathbf{b}|^2 \qquad \text{[by the Cauchy-Schwartz Inequality]} \\ &= (|\mathbf{a}| + |\mathbf{b}|)^2 \end{aligned}$$

Thus, taking the square root of both sides, $|\mathbf{a}+\mathbf{b}| \le |\mathbf{a}| + |\mathbf{b}|$.

63. (a)



The Parallelogram Law states that the sum of the squares of the lengths of the diagonals of a parallelogram equals the sum of the squares of its (four) sides.

(b) $|\mathbf{a}+\mathbf{b}|^2 = (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) = |\mathbf{a}|^2 + 2(\mathbf{a}\cdot\mathbf{b}) + |\mathbf{b}|^2$ and $|\mathbf{a}-\mathbf{b}|^2 = (\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = |\mathbf{a}|^2 - 2(\mathbf{a}\cdot\mathbf{b}) + |\mathbf{b}|^2$.

Adding these two equations gives $|\mathbf{a}+\mathbf{b}|^2 + |\mathbf{a}-\mathbf{b}|^2 = 2\,|\mathbf{a}|^2 + 2\,|\mathbf{b}|^2$.

64. If the vectors $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ are orthogonal, then $(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = 0$. But

$$\begin{aligned} (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) &= (\mathbf{u}+\mathbf{v})\cdot\mathbf{u} - (\mathbf{u}+\mathbf{v})\cdot\mathbf{v} && \text{by Property 3 of the dot product} \\ &= \mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{v} && \text{by Property 3} \\ &= |\mathbf{u}|^2 + \mathbf{u}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{v} - |\mathbf{v}|^2 && \text{by Properties 1 and 2} \\ &= |\mathbf{u}|^2 - |\mathbf{v}|^2 \end{aligned}$$

Thus $|\mathbf{u}|^2 - |\mathbf{v}|^2 = 0 \;\Rightarrow\; |\mathbf{u}|^2 = |\mathbf{v}|^2 \;\Rightarrow\; |\mathbf{u}| = |\mathbf{v}|$ [since $|\mathbf{u}|, |\mathbf{v}| \ge 0$].

65. $$\begin{aligned} \operatorname{proj}_{\mathbf{a}}\mathbf{b}\cdot\operatorname{proj}_{\mathbf{b}}\mathbf{a} &= \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}\cdot\frac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{b}|^2}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\cdot\frac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{b}|^2}\,(\mathbf{a}\cdot\mathbf{b}) && \text{by Property 4 of the dot product} \\ &= \frac{(\mathbf{a}\cdot\mathbf{b})^2}{|\mathbf{a}|^2\,|\mathbf{b}|^2}\,(\mathbf{a}\cdot\mathbf{b}) = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}\right)^2(\mathbf{a}\cdot\mathbf{b}) && \text{by Property 2} \\ &= (\cos\theta)^2\,(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\mathbf{b})\cos^2\theta && \text{by Corollary 6} \end{aligned}$$

66. (a) Suppose that $\mathbf{u}$ and $\mathbf{v}$ are nonzero orthogonal vectors. Then

$$\begin{aligned} |\mathbf{u}+\mathbf{v}|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) \\ &= \mathbf{u}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} \\ &= |\mathbf{u}|^2 + 0 + 0 + |\mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 \end{aligned}$$

(b) Suppose that $|\mathbf{u}+\mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2$. From part (a), we know that $|\mathbf{u}+\mathbf{v}|^2 = |\mathbf{u}|^2 + 2(\mathbf{u}\cdot\mathbf{v}) + |\mathbf{v}|^2$. Thus, $2(\mathbf{u}\cdot\mathbf{v}) = 0$, which implies that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

12.4 The Cross Product

1. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 0 \\ 1 & 0 & 5 \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ 0 & 5 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 0 \\ 1 & 5 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix}\mathbf{k}$

$= (15-0)\,\mathbf{i} - (10-0)\,\mathbf{j} + (0-3)\,\mathbf{k} = 15\,\mathbf{i} - 10\,\mathbf{j} - 3\,\mathbf{k}$

Now $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} = \langle 15,-10,-3\rangle\cdot\langle 2,3,0\rangle = 30-30+0 = 0$ and

$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} = \langle 15,-10,-3\rangle\cdot\langle 1,0,5\rangle = 15+0-15 = 0$, so $\mathbf{a}\times\mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

2. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 3 & -2 \\ 2 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 3 & -2 \\ -1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 4 & -2 \\ 2 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 4 & 3 \\ 2 & -1 \end{vmatrix}\mathbf{k}$

$= (3-2)\,\mathbf{i} - [4-(-4)]\,\mathbf{j} + (-4-6)\,\mathbf{k} = \mathbf{i} - 8\,\mathbf{j} - 10\,\mathbf{k}$

Now $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} = \langle 1,-8,-10\rangle\cdot\langle 4,3,-2\rangle = 4-24+20 = 0$ and

$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} = \langle 1,-8,-10\rangle\cdot\langle 2,-1,1\rangle = 2+8-10 = 0$, so $\mathbf{a}\times\mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

3. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & -4 \\ -1 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 2 & -4 \\ 3 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & -4 \\ -1 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & 2 \\ -1 & 3 \end{vmatrix}\mathbf{k}$

$= [2-(-12)]\,\mathbf{i} - (0-4)\,\mathbf{j} + [0-(-2)]\,\mathbf{k} = 14\,\mathbf{i} + 4\,\mathbf{j} + 2\,\mathbf{k}$

Since $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} = (14\,\mathbf{i}+4\,\mathbf{j}+2\,\mathbf{k})\cdot(2\,\mathbf{j}-4\,\mathbf{k}) = 0+8-8 = 0$, $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{a}$.

Since $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} = (14\,\mathbf{i}+4\,\mathbf{j}+2\,\mathbf{k})\cdot(-\mathbf{i}+3\,\mathbf{j}+\mathbf{k}) = -14+12+2 = 0$, $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{b}$.

4. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 3 & -3 \\ 3 & -3 & 3 \end{vmatrix} = \begin{vmatrix} 3 & -3 \\ -3 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & -3 \\ 3 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 3 \\ 3 & -3 \end{vmatrix}\mathbf{k}$

$= (9-9)\,\mathbf{i} - [9-(-9)]\,\mathbf{j} + (-9-9)\,\mathbf{k} = -18\,\mathbf{j} - 18\,\mathbf{k}$

Since $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} = (-18\,\mathbf{j}-18\,\mathbf{k})\cdot(3\,\mathbf{i}+3\,\mathbf{j}-3\,\mathbf{k}) = 0-54+54 = 0$, $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{a}$.

Since $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} = (-18\,\mathbf{j}-18\,\mathbf{k})\cdot(3\,\mathbf{i}-3\,\mathbf{j}+3\,\mathbf{k}) = 0+54-54 = 0$, $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{b}$.

5. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ 1 & 2 & -3 \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & \frac{1}{4} \\ 2 & -3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} \frac{1}{2} & \frac{1}{4} \\ 1 & -3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} \frac{1}{2} & \frac{1}{3} \\ 1 & 2 \end{vmatrix}\mathbf{k}$

$= \left(-1-\frac{1}{2}\right)\mathbf{i} - \left(-\frac{3}{2}-\frac{1}{4}\right)\mathbf{j} + \left(1-\frac{1}{3}\right)\mathbf{k} = -\frac{3}{2}\,\mathbf{i} + \frac{7}{4}\,\mathbf{j} + \frac{2}{3}\,\mathbf{k}$

Since $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} = \left(-\frac{3}{2}\,\mathbf{i}+\frac{7}{4}\,\mathbf{j}+\frac{2}{3}\,\mathbf{k}\right)\cdot\left(\frac{1}{2}\,\mathbf{i}+\frac{1}{3}\,\mathbf{j}+\frac{1}{4}\,\mathbf{k}\right) = -\frac{3}{4}+\frac{7}{12}+\frac{1}{6} = 0$, $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{a}$.

Since $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} = \left(-\frac{3}{2}\,\mathbf{i}+\frac{7}{4}\,\mathbf{j}+\frac{2}{3}\,\mathbf{k}\right)\cdot(\mathbf{i}+2\,\mathbf{j}-3\,\mathbf{k}) = -\frac{3}{2}+\frac{7}{2}-2 = 0$, $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{b}$.

6. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & \cos t & \sin t \\ 1 & -\sin t & \cos t \end{vmatrix} = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix}\mathbf{i} - \begin{vmatrix} t & \sin t \\ 1 & \cos t \end{vmatrix}\mathbf{j} + \begin{vmatrix} t & \cos t \\ 1 & -\sin t \end{vmatrix}\mathbf{k}$

$= [\cos^2 t - (-\sin^2 t)]\,\mathbf{i} - (t\cos t - \sin t)\,\mathbf{j} + (-t\sin t - \cos t)\,\mathbf{k} = \mathbf{i} + (\sin t - t\cos t)\,\mathbf{j} + (-t\sin t - \cos t)\,\mathbf{k}$

Since
$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} &= [\,\mathbf{i} + (\sin t - t\cos t)\,\mathbf{j} + (-t\sin t - \cos t)\,\mathbf{k}\,]\cdot(t\,\mathbf{i} + \cos t\,\mathbf{j} + \sin t\,\mathbf{k}) \\ &= t + \sin t\cos t - t\cos^2 t - t\sin^2 t - \sin t\cos t \\ &= t - t\left(\cos^2 t + \sin^2 t\right) = 0\end{aligned}$$

$\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{a}$.

Since
$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} &= [\,\mathbf{i} + (\sin t - t\cos t)\,\mathbf{j} + (-t\sin t - \cos t)\,\mathbf{k}\,]\cdot(\mathbf{i} - \sin t\,\mathbf{j} + \cos t\,\mathbf{k}) \\ &= 1 - \sin^2 t + t\sin t\cos t - t\sin t\cos t - \cos^2 t \\ &= 1 - \left(\sin^2 t + \cos^2 t\right) = 0\end{aligned}$$

$\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{b}$.

7. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t^3 & t^2 & t \\ t & 2t & 3t \end{vmatrix} = \begin{vmatrix} t^2 & t \\ 2t & 3t \end{vmatrix}\mathbf{i} - \begin{vmatrix} t^3 & t \\ t & 3t \end{vmatrix}\mathbf{j} + \begin{vmatrix} t^3 & t^2 \\ t & 2t \end{vmatrix}\mathbf{k}$

$= (3t^3 - 2t^2)\mathbf{i} - (3t^4 - t^2)\mathbf{j} + (2t^4 - t^3)\mathbf{k}$

Since
$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} &= \left\langle 3t^3 - 2t^2, t^2 - 3t^4, 2t^4 - t^3\right\rangle\cdot\left\langle t^3, t^2, t\right\rangle \\ &= 3t^6 - 2t^5 + t^4 - 3t^6 + 2t^5 - t^4 = 0\end{aligned}$$

$\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{a}$.

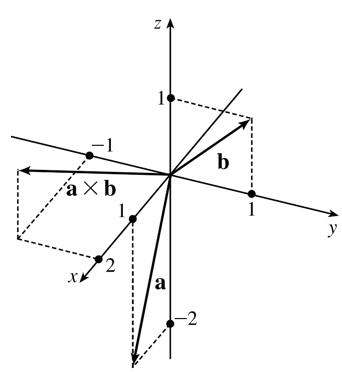
Since
$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} &= \left\langle 3t^3 - 2t^2, t^2 - 3t^4, 2t^4 - t^3\right\rangle\cdot\langle t, 2t, 3t\rangle \\ &= 3t^4 - 2t^3 + 2t^3 - 6t^5 + 6t^5 - 3t^4 = 0\end{aligned}$$

$\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{b}$.

8. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{vmatrix}$

$= \begin{vmatrix} 0 & -2 \\ 1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\mathbf{k}$

$= 2\,\mathbf{i} - \mathbf{j} + \mathbf{k}$



9. According to the discussion following Example 4, $\mathbf{i}\times\mathbf{j} = \mathbf{k}$, so $(\mathbf{i}\times\mathbf{j})\times\mathbf{k} = \mathbf{k}\times\mathbf{k} = \mathbf{0}$ [by Example 2].

10. $\mathbf{k}\times(\mathbf{i}-2\mathbf{j}) = \mathbf{k}\times\mathbf{i}+\mathbf{k}\times(-2\mathbf{j})$ by Property 3 of the cross product

$= \mathbf{k}\times\mathbf{i}+(-2)(\mathbf{k}\times\mathbf{j})$ by Property 2

$= \mathbf{j}+(-2)(-\mathbf{i}) = 2\,\mathbf{i}+\mathbf{j}$ by the discussion following Example 4

11. $(\mathbf{j}-\mathbf{k})\times(\mathbf{k}-\mathbf{i}) = (\mathbf{j}-\mathbf{k})\times\mathbf{k}+(\mathbf{j}-\mathbf{k})\times(-\mathbf{i})$ by Property 3 of the cross product

$= \mathbf{j}\times\mathbf{k}+(-\mathbf{k})\times\mathbf{k}+\mathbf{j}\times(-\mathbf{i})+(-\mathbf{k})\times(-\mathbf{i})$ by Property 4

$= (\mathbf{j}\times\mathbf{k})+(-1)(\mathbf{k}\times\mathbf{k})+(-1)(\mathbf{j}\times\mathbf{i})+(-1)^2(\mathbf{k}\times\mathbf{i})$ by Property 2

$= \mathbf{i}+(-1)\,\mathbf{0}+(-1)(-\mathbf{k})+\mathbf{j} = \mathbf{i}+\mathbf{j}+\mathbf{k}$ by Example 2 and the discussion following Example 4

12. $(\mathbf{i}+\mathbf{j})\times(\mathbf{i}-\mathbf{j}) = (\mathbf{i}+\mathbf{j})\times\mathbf{i}+(\mathbf{i}+\mathbf{j})\times(-\mathbf{j})$ by Property 3 of the cross product

$= \mathbf{i}\times\mathbf{i}+\mathbf{j}\times\mathbf{i}+\mathbf{i}\times(-\mathbf{j})+\mathbf{j}\times(-\mathbf{j})$ by Property 4

$= (\mathbf{i}\times\mathbf{i})+(\mathbf{j}\times\mathbf{i})+(-1)(\mathbf{i}\times\mathbf{j})+(-1)(\mathbf{j}\times\mathbf{j})$ by Property 2

$= \mathbf{0}+(-\mathbf{k})+(-1)\,\mathbf{k}+(-1)\,\mathbf{0} = -2\,\mathbf{k}$ by Example 2 and the discussion following Example 4

13. (a) Since $\mathbf{b}\times\mathbf{c}$ is a vector, the dot product $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ is meaningful and is a scalar.

(b) $\mathbf{b}\cdot\mathbf{c}$ is a scalar, so $\mathbf{a}\times(\mathbf{b}\cdot\mathbf{c})$ is meaningless, as the cross product is defined only for two *vectors*.

(c) Since $\mathbf{b}\times\mathbf{c}$ is a vector, the cross product $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ is meaningful and results in another vector.

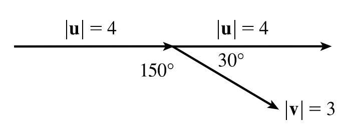
(d) $\mathbf{b}\cdot\mathbf{c}$ is a scalar, so the dot product $\mathbf{a}\cdot(\mathbf{b}\cdot\mathbf{c})$ is meaningless, as the dot product is defined only for two vectors.

(e) Since $(\mathbf{a}\cdot\mathbf{b})$ and $(\mathbf{c}\cdot\mathbf{d})$ are both scalars, the cross product $(\mathbf{a}\cdot\mathbf{b})\times(\mathbf{c}\cdot\mathbf{d})$ is meaningless.

(f) $\mathbf{a}\times\mathbf{b}$ and $\mathbf{c}\times\mathbf{d}$ are both vectors, so the dot product $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})$ is meaningful and is a scalar.

14. Using Theorem 9, we have $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta = (10)(8)\sin 60^\circ = 80\cdot\frac{\sqrt{3}}{2} = 40\sqrt{3}$. By the right-hand rule, $\mathbf{u}\times\mathbf{v}$ is directed into the page.

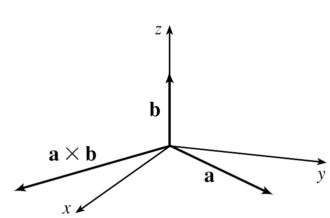
15. If we sketch $\mathbf{u}$ and $\mathbf{v}$ starting from the same initial point, we see that the angle between them is 30°.



Using Theorem 9, we have $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta = (4)(3)\sin 30^\circ = 12\cdot\frac{1}{2} = 6$. By the right-hand rule, $\mathbf{u}\times\mathbf{v}$ is directed into the page.

16. (a) $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\sin\theta = 3\cdot 2\cdot\sin\frac{\pi}{2} = 6$

(b) $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{k}$, so it lies in the xy-plane, and its z-coordinate is 0. By the right-hand rule, its y-component is negative and its x-component is positive.



17. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 4 & 2 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 3 \\ 2 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix}\mathbf{k} = (-1-6)\,\mathbf{i} - (2-12)\,\mathbf{j} + [4-(-4)]\,\mathbf{k} = -7\,\mathbf{i} + 10\,\mathbf{j} + 8\,\mathbf{k}$

$\mathbf{b}\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 2 & 1 \\ 2 & -1 & 3 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 4 & 1 \\ 2 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 4 & 2 \\ 2 & -1 \end{vmatrix}\mathbf{k} = [6-(-1)]\,\mathbf{i} - (12-2)\,\mathbf{j} + (-4-4)\,\mathbf{k} = 7\,\mathbf{i} - 10\,\mathbf{j} - 8\,\mathbf{k}$

Notice $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$ here, as we know is always true by Property 1 of the cross product.

18. $\mathbf{b}\times\mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 0 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & -1 \\ 0 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}\mathbf{k} = 4\,\mathbf{i} - 6\,\mathbf{j} + 2\,\mathbf{k}$ so

$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 4 & -6 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ -6 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 1 \\ 4 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 0 \\ 4 & -6 \end{vmatrix}\mathbf{k} = 6\,\mathbf{i} + 2\,\mathbf{j} - 6\,\mathbf{k}.$

$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 2 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}\mathbf{k} = -\,\mathbf{i} + 3\,\mathbf{j} + \mathbf{k}$ so

$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & 1 \\ 0 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & 1 \\ 0 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & 3 \\ 0 & 1 \end{vmatrix}\mathbf{k} = 8\,\mathbf{i} + 3\,\mathbf{j} - \mathbf{k}.$

Thus $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) \neq (\mathbf{a}\times\mathbf{b})\times\mathbf{c}$.

19. By Theorem 8, the cross product of two vectors is orthogonal to both vectors. So we calculate

$$\langle 3,2,1\rangle \times \langle -1,1,0\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 1 \\ -1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 1 \\ -1 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 2 \\ -1 & 1 \end{vmatrix}\mathbf{k} = -\mathbf{i} - \mathbf{j} + 5\,\mathbf{k}.$$

So two unit vectors orthogonal to both given vectors are $\pm\dfrac{\langle -1,-1,5\rangle}{\sqrt{1+1+25}} = \pm\dfrac{\langle -1,-1,5\rangle}{3\sqrt{3}}$, that is, $\left\langle -\frac{1}{3\sqrt{3}}, -\frac{1}{3\sqrt{3}}, \frac{5}{3\sqrt{3}} \right\rangle$

and $\left\langle \frac{1}{3\sqrt{3}}, \frac{1}{3\sqrt{3}}, -\frac{5}{3\sqrt{3}} \right\rangle$.

20. By Theorem 8, the cross product of two vectors is orthogonal to both vectors. So we calculate

$$(\mathbf{j}-\mathbf{k})\times(\mathbf{i}+\mathbf{j}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}\mathbf{k} = \mathbf{i} - \mathbf{j} - \mathbf{k}$$

Thus two unit vectors orthogonal to both given vectors are $\pm\frac{1}{\sqrt{3}}(\mathbf{i}-\mathbf{j}-\mathbf{k})$, that is, $\frac{1}{\sqrt{3}}\,\mathbf{i} - \frac{1}{\sqrt{3}}\,\mathbf{j} - \frac{1}{\sqrt{3}}\,\mathbf{k}$ and

$-\frac{1}{\sqrt{3}}\,\mathbf{i} + \frac{1}{\sqrt{3}}\,\mathbf{j} + \frac{1}{\sqrt{3}}\,\mathbf{k}$.

21. Let $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$. Then

$$\mathbf{0} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 0 \\ a_1 & a_2 & a_3 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ a_2 & a_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0 & 0 \\ a_1 & a_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0 & 0 \\ a_1 & a_2 \end{vmatrix} \mathbf{k} = \mathbf{0},$$

$$\mathbf{a} \times \mathbf{0} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ 0 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ 0 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ 0 & 0 \end{vmatrix} \mathbf{k} = \mathbf{0}.$$

22. Let $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$.

$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} &= \left\langle \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right\rangle \cdot \langle b_1, b_2, b_3 \rangle = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} b_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} b_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} b_3 \\ &= (a_2b_3b_1 - a_3b_2b_1) - (a_1b_3b_2 - a_3b_1b_2) + (a_1b_2b_3 - a_2b_1b_3) = 0\end{aligned}$$

23. $$\begin{aligned}\mathbf{a} \times \mathbf{b} &= \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle \\ &= \langle (-1)(b_2a_3 - b_3a_2), (-1)(b_3a_1 - b_1a_3), (-1)(b_1a_2 - b_2a_1) \rangle \\ &= -\langle b_2a_3 - b_3a_2, b_3a_1 - b_1a_3, b_1a_2 - b_2a_1 \rangle = -\mathbf{b} \times \mathbf{a}\end{aligned}$$

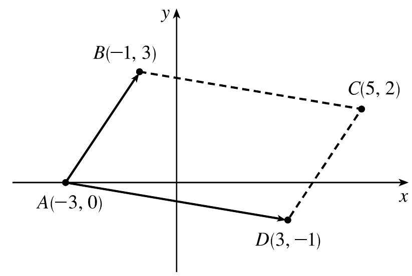
24. $c\mathbf{a} = \langle ca_1, ca_2, ca_3 \rangle$, so

$$\begin{aligned}(c\,\mathbf{a}) \times \mathbf{b} &= \langle ca_2b_3 - ca_3b_2, ca_3b_1 - ca_1b_3, ca_1b_2 - ca_2b_1 \rangle \\ &= c\langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle = c(\mathbf{a} \times \mathbf{b}) \\ &= \langle ca_2b_3 - ca_3b_2, ca_3b_1 - ca_1b_3, ca_1b_2 - ca_2b_1 \rangle \\ &= \langle a_2(cb_3) - a_3(cb_2), a_3(cb_1) - a_1(cb_3), a_1(cb_2) - a_2(cb_1) \rangle \\ &= \mathbf{a} \times (c\,\mathbf{b})\end{aligned}$$

25. $$\begin{aligned}\mathbf{a} \times (\mathbf{b} + \mathbf{c}) &= \mathbf{a} \times \langle b_1 + c_1, b_2 + c_2, b_3 + c_3 \rangle \\ &= \langle a_2(b_3 + c_3) - a_3(b_2 + c_2), a_3(b_1 + c_1) - a_1(b_3 + c_3), a_1(b_2 + c_2) - a_2(b_1 + c_1) \rangle \\ &= \langle a_2b_3 + a_2c_3 - a_3b_2 - a_3c_2, a_3b_1 + a_3c_1 - a_1b_3 - a_1c_3, a_1b_2 + a_1c_2 - a_2b_1 - a_2c_1 \rangle \\ &= \langle (a_2b_3 - a_3b_2) + (a_2c_3 - a_3c_2), (a_3b_1 - a_1b_3) + (a_3c_1 - a_1c_3), (a_1b_2 - a_2b_1) + (a_1c_2 - a_2c_1) \rangle \\ &= \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle + \langle a_2c_3 - a_3c_2, a_3c_1 - a_1c_3, a_1c_2 - a_2c_1 \rangle \\ &= (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})\end{aligned}$$

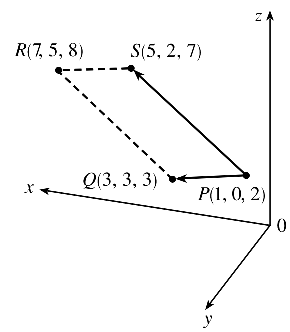
26. $$\begin{aligned}(\mathbf{a} + \mathbf{b}) \times \mathbf{c} &= -\mathbf{c} \times (\mathbf{a} + \mathbf{b}) && \text{by Property 1 of the cross product} \\ &= -(\mathbf{c} \times \mathbf{a} + \mathbf{c} \times \mathbf{b}) && \text{by Property 3} \\ &= -(-\mathbf{a} \times \mathbf{c} + (-\mathbf{b} \times \mathbf{c})) && \text{by Property 1} \\ &= \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c} && \text{by Property 2}\end{aligned}$$

27. By plotting the vertices, we can see that the parallelogram is determined by the vectors $\overrightarrow{AB} = \langle 2, 3 \rangle$ and $\overrightarrow{AD} = \langle 6, -1 \rangle$. We know that the area of the parallelogram determined by two vectors is equal to the length of the cross product of these vectors. In order to compute the cross product, we consider the vector $\overrightarrow{AB}$ as the three-dimensional vector $\langle 2, 3, 0 \rangle$ (and similarly for $\overrightarrow{AD}$), and then the area of parallelogram $ABCD$ is



$$\left|\overrightarrow{AB} \times \overrightarrow{AD}\right| = \left\| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 0 \\ 6 & -1 & 0 \end{matrix} \right\| = |(0-0)\,\mathbf{i} - (0-0)\,\mathbf{j} + (-2-18)\,\mathbf{k}| = |-20\,\mathbf{k}| = 20$$

28. By plotting the vertices, we can see that the parallelogram is determined by the vectors $\overrightarrow{PQ} = \langle 2, 3, 1 \rangle$ and $\overrightarrow{PS} = \langle 4, 2, 5 \rangle$. Thus the area of parallelogram $PQRS$ is



$$\left|\overrightarrow{PQ} \times \overrightarrow{PS}\right| = \left\| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 1 \\ 4 & 2 & 5 \end{matrix} \right\| = |(15-2)\,\mathbf{i} - (10-4)\,\mathbf{j} + (4-12)\,\mathbf{k}|$$

$$= |13\,\mathbf{i} - 6\,\mathbf{j} - 8\,\mathbf{k}| = \sqrt{169 + 36 + 64} = \sqrt{269} \approx 16.40$$

29. (a) Because the plane through P, Q, and R contains the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, a vector orthogonal to these vectors (such as their cross product) is also orthogonal to the plane. $\overrightarrow{PQ} = \langle 2, 1, 3 \rangle$ and $\overrightarrow{PR} = \langle 5, 4, 2 \rangle$, so

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \langle (1)(2) - (3)(4), (3)(5) - (2)(2), (2)(4) - (1)(5) \rangle = \langle -10, 11, 3 \rangle$$

Therefore, $\langle -10, 11, 3 \rangle$ (or any nonzero scalar multiple) is orthogonal to the plane through P, Q, and R.

(b) The area of the triangle determined by P, Q, and R is equal to half the area of the parallelogram determined by the three points. Using part (a), the area of the parallelogram is

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = |\langle -10, 11, 3 \rangle| = \sqrt{(-10)^2 + 11^2 + 3^2} = \sqrt{230}$$

So the area of triangle PQR is $\frac{1}{2}\sqrt{230}$.

30. (a) Because the plane through P, Q, and R contains the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, a vector orthogonal to these vectors (such as their cross product) is also orthogonal to the plane. $\overrightarrow{PQ} = \langle 3, 3, -6 \rangle$ and $\overrightarrow{PR} = \langle 2, 3, 1 \rangle$, so

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \langle (3)(1) - (-6)(3), (-6)(2) - (3)(1), (3)(3) - (3)(2) \rangle = \langle 21, -15, 3 \rangle$$

Therefore, $\langle 21, -15, 3 \rangle$ (or any nonzero scalar multiple) is orthogonal to the plane through P, Q, and R.

(b) The area of the triangle determined by P, Q, and R is equal to half the area of the parallelogram determined by the three points. Using part (a), the area of the parallelogram is

$$|\overrightarrow{PQ}\times\overrightarrow{PR}| = |\langle 21,-15,3\rangle| = \sqrt{21^2+(-15)^2+3^2} = \sqrt{675} = 15\sqrt{3}$$

So the area of triangle PQR is $\frac{15}{2}\sqrt{3}$.

31. (a) Because the plane through P, Q, and R contains the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, a vector orthogonal to these vectors (such as their cross product) is also orthogonal to the plane. $\overrightarrow{PQ} = \langle -4,3,3\rangle$ and $\overrightarrow{PR} = \langle -3,-2,2\rangle$, so

$$\overrightarrow{PQ}\times\overrightarrow{PR} = \langle (3)(2)-(3)(-2),(3)(-3)-(-4)(2),(-4)(-2)-(3)(-3)\rangle = \langle 12,-1,17\rangle$$

Therefore, $\langle 12,-1,17\rangle$ (or any nonzero scalar multiple) is orthogonal to the plane through P, Q, and R.

(b) The area of the triangle determined by P, Q, and R is equal to half the area of the parallelogram determined by the three points. Using part (a), the area of the parallelogram is

$$|\overrightarrow{PQ}\times\overrightarrow{PR}| = |\langle 12,-1,17\rangle| = \sqrt{12^2+(-1)^2+17^2} = \sqrt{434}$$

So the area of triangle PQR is $\frac{1}{2}\sqrt{434}$.

32. (a) $\overrightarrow{PQ} = \langle -3,1,-2\rangle$ and $\overrightarrow{PR} = \langle 1,4,-7\rangle$, so a vector orthogonal to the plane through P, Q, and R is $\overrightarrow{PQ}\times\overrightarrow{PR} = \langle (1)(-7)-(-2)(4),(-2)(1)-(-3)(-7),(-3)(4)-(1)(1)\rangle = \langle 1,-23,-13\rangle$ (or any nonzero scalar multiple).

(b) The area of the parallelogram determined by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is $\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right| = |\langle 1,-23,-13\rangle| = \sqrt{1+529+169} = \sqrt{699}$, so the area of triangle PQR is $\frac{1}{2}\sqrt{699}$.

33. By Equation 14, the volume of the parallelepiped determined by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is the magnitude of their scalar triple product, which is $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ 2 & 1 & 4 \end{vmatrix} = 1\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} - 2\begin{vmatrix} -1 & 2 \\ 2 & 4 \end{vmatrix} + 3\begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix} = 1(4-2)-2(-4-4)+3(-1-2) = 9.$

Thus the volume of the parallelepiped is 9 cubic units.

34. $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} + 0\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = 0+1+0 = 1.$

So the volume of the parallelepiped determined by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is 1 cubic unit.

35. $\mathbf{a} = \overrightarrow{PQ} = \langle 4,2,2\rangle$, $\mathbf{b} = \overrightarrow{PR} = \langle 3,3,-1\rangle$, and $\mathbf{c} = \overrightarrow{PS} = \langle 5,5,1\rangle$.

$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} 4 & 2 & 2 \\ 3 & 3 & -1 \\ 5 & 5 & 1 \end{vmatrix} = 4\begin{vmatrix} 3 & -1 \\ 5 & 1 \end{vmatrix} - 2\begin{vmatrix} 3 & -1 \\ 5 & 1 \end{vmatrix} + 2\begin{vmatrix} 3 & 3 \\ 5 & 5 \end{vmatrix} = 32-16+0 = 16,$

so the volume of the parallelepiped is 16 cubic units.

36. $\mathbf{a} = \overrightarrow{PQ} = \langle -4, 2, 4\rangle$, $\mathbf{b} = \overrightarrow{PR} = \langle 2, 1, -2\rangle$ and $\mathbf{c} = \overrightarrow{PS} = \langle -3, 4, 1\rangle$.

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} -4 & 2 & 4 \\ 2 & 1 & -2 \\ -3 & 4 & 1 \end{vmatrix} = -4\begin{vmatrix} 1 & -2 \\ 4 & 1 \end{vmatrix} - 2\begin{vmatrix} 2 & -2 \\ -3 & 1 \end{vmatrix} + 4\begin{vmatrix} 2 & 1 \\ -3 & 4 \end{vmatrix} = -36 + 8 + 44 = 16,$$

so the volume of the parallelepiped is 16 cubic units.

37. $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} 1 & 5 & -2 \\ 3 & -1 & 0 \\ 5 & 9 & -4 \end{vmatrix} = 1\begin{vmatrix} -1 & 0 \\ 9 & -4 \end{vmatrix} - 5\begin{vmatrix} 3 & 0 \\ 5 & -4 \end{vmatrix} + (-2)\begin{vmatrix} 3 & -1 \\ 5 & 9 \end{vmatrix} = 4 + 60 - 64 = 0$, which says that the volume of the parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is 0, and thus these three vectors are coplanar.

38. $\mathbf{u} = \overrightarrow{AB} = \langle 2, -4, 4\rangle$, $\mathbf{v} = \overrightarrow{AC} = \langle 4, -1, -2\rangle$ and $\mathbf{w} = \overrightarrow{AD} = \langle 2, 3, -6\rangle$.

$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} 2 & -4 & 4 \\ 4 & -1 & -2 \\ 2 & 3 & -6 \end{vmatrix} = 2\begin{vmatrix} -1 & -2 \\ 3 & -6 \end{vmatrix} - (-4)\begin{vmatrix} 4 & -2 \\ 2 & -6 \end{vmatrix} + 4\begin{vmatrix} 4 & -1 \\ 2 & 3 \end{vmatrix} = 24 - 80 + 56 = 0$, so the volume of the parallelepiped determined by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ is 0, which says these vectors lie in the same plane. Therefore, their initial and terminal points A, B, C and D also lie in the same plane.

39. Using the notation of the text, $|\mathbf{r}| = 0.18$ m, $|\mathbf{F}| = 60$ N, and the angle between $\mathbf{r}$ and $\mathbf{F}$ is $\theta = 70^\circ + 10^\circ = 80^\circ$. (Move $\mathbf{F}$ so that both vectors start from the same point.) Then the magnitude of the torque is $|\boldsymbol{\tau}| = |\mathbf{r}\times\mathbf{F}| = |\mathbf{r}|\,|\mathbf{F}|\sin\theta = (0.18)(60)\sin 80^\circ = 10.8\sin 80^\circ \approx 10.6$ N·m.

40. (a) Let $\mathbf{r}$ be the positional vector from point P to the handle. Then $|\mathbf{r}| = \sqrt{30^2 + 60^2} = 10\sqrt{45}$ cm $= \frac{\sqrt{45}}{10}$ m. Moving $\mathbf{F}$ so that both vectors start from the same point makes an angle between $\mathbf{r}$ and $\mathbf{F}$ of $\theta = \tan^{-1}\frac{60}{30} \approx 63.4^\circ$. Therefore,

$$|\tau| = |\mathbf{r}||\mathbf{F}|\sin\theta = (\frac{\sqrt{45}}{10})(90)\sin 63.4^\circ \approx 54 \text{ N·m}$$

(b) Let $\mathbf{r}_Q$ be the positional vector from point P to the point Q. Then $|\mathbf{r}_Q| = \sqrt{20^2 + 20^2} = 20\sqrt{2}$ cm $= \frac{\sqrt{2}}{5}$ m and the angle that the force vector makes with $|\mathbf{r}_Q|$ is $\theta = 45^\circ$. Therefore, $|\tau| = |\mathbf{r}_Q||\mathbf{F}|\sin\theta = (\frac{\sqrt{2}}{5})(90)\sin 45^\circ = 18$ N·m.

41. Using the notation of the text, $\mathbf{r} = \langle 0, 0.3, 0\rangle$ (measuring in meters) and $\mathbf{F}$ has direction $\langle 0, 3, -4\rangle$. The angle θ between them can be determined by $\cos\theta = \dfrac{\langle 0, 0.3, 0\rangle\cdot\langle 0, 3, -4\rangle}{|\langle 0, 0.3, 0\rangle|\,|\langle 0, 3, -4\rangle|} \quad\Rightarrow\quad \cos\theta = \dfrac{0.9}{(0.3)(5)} \quad\Rightarrow\quad \cos\theta = 0.6 \quad\Rightarrow$

$\theta = \cos^{-1}(0.6) \approx 53.1^\circ$. Then $|\boldsymbol{\tau}| = |\mathbf{r}|\,|\mathbf{F}|\sin\theta \quad\Rightarrow\quad 100 \approx 0.3\,|\mathbf{F}|\sin 53.1^\circ \quad\Rightarrow\quad |\mathbf{F}| \approx \dfrac{100}{0.3\sin 53.1^\circ} \approx 417$ N.

42. Since $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\sin\theta$, $0 \le \theta \le \pi$, $|\mathbf{u}\times\mathbf{v}|$ achieves its maximum value for $\sin\theta = 1 \quad\Rightarrow\quad \theta = \frac{\pi}{2}$, in which case $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| = 3\,|5\,\mathbf{j}| = 15$. The minimum value is zero, which occurs when $\sin\theta = 0 \quad\Rightarrow\quad \theta = 0$ or π, so when $\mathbf{u}$, $\mathbf{v}$ are parallel. Thus, when $\mathbf{u}$ points in the same direction as $\mathbf{v}$, so $\mathbf{u} = 3\,\mathbf{j}$, $|\mathbf{u}\times\mathbf{v}| = 0$. As $\mathbf{u}$ rotates counterclockwise,

$\mathbf{u}\times\mathbf{v}$ is directed in the negative z-direction (by the right-hand rule) and the length increases until $\theta=\frac{\pi}{2}$, in which case $\mathbf{u}=-3\,\mathbf{i}$ and $|\mathbf{u}\times\mathbf{v}|=|\mathbf{u}|\,|\mathbf{v}|=|-3\,\mathbf{i}|\,|5\,\mathbf{j}|=15$. As $\mathbf{u}$ rotates to the negative y-axis, $\mathbf{u}\times\mathbf{v}$ remains pointed in the negative z-direction and the length of $\mathbf{u}\times\mathbf{v}$ decreases to 0, after which the direction of $\mathbf{u}\times\mathbf{v}$ reverses to point in the positive z-direction and $|\mathbf{u}\times\mathbf{v}|$ increases. When $\mathbf{u}=3\,\mathbf{i}$ (so $\theta=\frac{\pi}{2}$), $|\mathbf{u}\times\mathbf{v}|$ again reaches its maximum of 15, after which $|\mathbf{u}\times\mathbf{v}|$ decreases to 0 as $\mathbf{u}$ rotates to the positive y-axis.

43. From Theorem 9 we have $|\mathbf{a}\times\mathbf{b}|=|\mathbf{a}|\,|\mathbf{b}|\sin\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, and from Theorem 12.3.3 we have $\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}|\,|\mathbf{b}|\cos\theta \quad\Rightarrow\quad |\mathbf{a}|\,|\mathbf{b}|=\dfrac{\mathbf{a}\cdot\mathbf{b}}{\cos\theta}$. Substituting the second equation into the first gives $|\mathbf{a}\times\mathbf{b}|=\dfrac{\mathbf{a}\cdot\mathbf{b}}{\cos\theta}\sin\theta$, so $\dfrac{|\mathbf{a}\times\mathbf{b}|}{\mathbf{a}\cdot\mathbf{b}}=\tan\theta$. Here $|\mathbf{a}\times\mathbf{b}|=|\langle 1,2,2\rangle|=\sqrt{1+4+4}=3$, so $\tan\theta=\dfrac{|\mathbf{a}\times\mathbf{b}|}{\mathbf{a}\cdot\mathbf{b}}=\dfrac{3}{\sqrt{3}}=\sqrt{3} \quad\Rightarrow\quad \theta=60^\circ$.

44. (a) Let $\mathbf{v}=\langle v_1,v_2,v_3\rangle$. Then

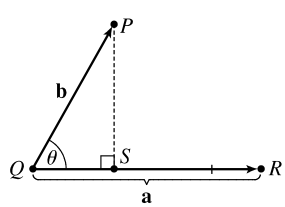
$$\langle 1,2,1\rangle\times\mathbf{v}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&2&1\\v_1&v_2&v_3\end{vmatrix}=\begin{vmatrix}2&1\\v_2&v_3\end{vmatrix}\mathbf{i}-\begin{vmatrix}1&1\\v_1&v_3\end{vmatrix}\mathbf{j}+\begin{vmatrix}1&2\\v_1&v_2\end{vmatrix}\mathbf{k}=(2v_3-v_2)\,\mathbf{i}-(v_3-v_1)\,\mathbf{j}+(v_2-2v_1)\,\mathbf{k}.$$

If $\langle 1,2,1\rangle\times\mathbf{v}=\langle 3,1,-5\rangle$ then $\langle 2v_3-v_2,v_1-v_3,v_2-2v_1\rangle=\langle 3,1,-5\rangle \quad\Leftrightarrow\quad 2v_3-v_2=3$ **(1)**, $v_1-v_3=1$ **(2)**, and $v_2-2v_1=-5$ **(3)**. From **(3)** we have $v_2=2v_1-5$ and from **(2)** we have $v_3=v_1-1$; substitution into **(1)** gives $2\,(v_1-1)-(2v_1-5)=3 \quad\Rightarrow\quad 3=3$, so this is a dependent system. If we let $v_1=a$ then $v_2=2a-5$ and $v_3=a-1$, so $\mathbf{v}$ is any vector of the form $\langle a,2a-5,a-1\rangle$.

(b) If $\langle 1,2,1\rangle\times\mathbf{v}=\langle 3,1,5\rangle$ then $2v_3-v_2=3$ **(1)**, $v_1-v_3=1$ **(2)**, and $v_2-2v_1=5$ **(3)**. From **(3)** we have $v_2=2v_1+5$ and from **(2)** we have $v_3=v_1-1$; substitution into **(1)** gives $2\,(v_1-1)-(2v_1+5)=3 \quad\Rightarrow\quad -7=3$, so this is an inconsistent system and has no solution.

Alternatively, if we use matrices to solve the system we could show that the determinant is 0 (and hence the system has no solution).

45. (a)



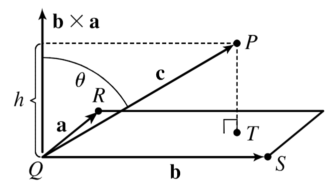
The distance between a point and a line is the length of the perpendicular from the point to the line, here $\left|\overrightarrow{PS}\right|=d$. But referring to triangle PQS, $d=\left|\overrightarrow{PS}\right|=\left|\overrightarrow{QP}\right|\sin\theta=|\mathbf{b}|\sin\theta$. But θ is the angle between $\overrightarrow{QP}=\mathbf{b}$ and $\overrightarrow{QR}=\mathbf{a}$. Thus by Theorem 9, $\sin\theta=\dfrac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}|\,|\mathbf{b}|}$ and so $d=|\mathbf{b}|\sin\theta=\dfrac{|\mathbf{b}|\,|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}|\,|\mathbf{b}|}=\dfrac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}|}$.

(b) $\mathbf{a}=\overrightarrow{QR}=\langle -1,-2,-1\rangle$ and $\mathbf{b}=\overrightarrow{QP}=\langle 1,-5,-7\rangle$. Then

$\mathbf{a}\times\mathbf{b}=\langle(-2)(-7)-(-1)(-5),(-1)(1)-(-1)(-7),(-1)(-5)-(-2)(1)\rangle=\langle 9,-8,7\rangle$.

Thus the distance is $d=\dfrac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}|}=\frac{1}{\sqrt{6}}\sqrt{81+64+49}=\sqrt{\frac{194}{6}}=\sqrt{\frac{97}{3}}$.

46. (a) The distance between a point and a plane is the length of the perpendicular from the point to the plane, here $\left|\overrightarrow{TP}\right| = d$. But $\overrightarrow{TP}$ is parallel to $\mathbf{b} \times \mathbf{a}$ (because $\mathbf{b} \times \mathbf{a}$ is perpendicular to $\mathbf{b}$ and $\mathbf{a}$) and $d = \left|\overrightarrow{TP}\right| =$ the absolute value of the scalar projection of $\mathbf{c}$ along $\mathbf{b} \times \mathbf{a}$, which is $|\mathbf{c}|\,|\cos\theta|$. (Notice that this is the same setup as the development of the volume of a parallelepiped with $h = |\mathbf{c}|\,|\cos\theta|$). Thus $d = |\mathbf{c}|\,|\cos\theta| = h = V/A$ where $A = |\mathbf{a} \times \mathbf{b}|$, the area of the base. So finally $d = \dfrac{V}{A} = \dfrac{|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|}{|\mathbf{a} \times \mathbf{b}|}$.



(b) $\mathbf{a} = \overrightarrow{QR} = \langle -1, 2, 0\rangle$, $\mathbf{b} = \overrightarrow{QS} = \langle -1, 0, 3\rangle$ and $\mathbf{c} = \overrightarrow{QP} = \langle 1, 1, 4\rangle$. Then

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} -1 & 2 & 0 \\ -1 & 0 & 3 \\ 1 & 1 & 4 \end{vmatrix} = (-1)\begin{vmatrix} 0 & 3 \\ 1 & 4 \end{vmatrix} - 2\begin{vmatrix} -1 & 3 \\ 1 & 4 \end{vmatrix} + 0 = 17$$

and $$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & 0 \\ -1 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & 2 \\ -1 & 0 \end{vmatrix}\mathbf{k} = 6\,\mathbf{i} + 3\,\mathbf{j} + 2\,\mathbf{k}$$

Thus $d = \dfrac{|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|}{|\mathbf{a} \times \mathbf{b}|} = \dfrac{17}{\sqrt{36 + 9 + 4}} = \dfrac{17}{7}$.

47. From Theorem 9 we have $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}| \sin\theta$ so

$$\begin{aligned} |\mathbf{a} \times \mathbf{b}|^2 &= |\mathbf{a}|^2\,|\mathbf{b}|^2 \sin^2\theta = |\mathbf{a}|^2\,|\mathbf{b}|^2\left(1 - \cos^2\theta\right) \\ &= |\mathbf{a}|^2\,|\mathbf{b}|^2 - (|\mathbf{a}|\,|\mathbf{b}|\cos\theta)^2 = |\mathbf{a}|^2\,|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 \end{aligned}$$

by Theorem 12.3.3.

48. If $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$ then $\mathbf{b} = -(\mathbf{a} + \mathbf{c})$, so

$\mathbf{a} \times \mathbf{b} = \mathbf{a} \times [-(\mathbf{a} + \mathbf{c})] = -[\mathbf{a} \times (\mathbf{a} + \mathbf{c})]$	by Property 2 of the cross product (with $c = -1$)
$= -[(\mathbf{a} \times \mathbf{a}) + (\mathbf{a} \times \mathbf{c})]$	by Property 3
$= -[\mathbf{0} + (\mathbf{a} \times \mathbf{c})] = -\mathbf{a} \times \mathbf{c}$	by Example 2
$= \mathbf{c} \times \mathbf{a}$	by Property 1

Similarly, $\mathbf{a} = -(\mathbf{b} + \mathbf{c})$ so

$$\begin{aligned} \mathbf{c} \times \mathbf{a} &= \mathbf{c} \times [-(\mathbf{b} + \mathbf{c})] = -[\mathbf{c} \times (\mathbf{b} + \mathbf{c})] \\ &= -[(\mathbf{c} \times \mathbf{b}) + (\mathbf{c} \times \mathbf{c})] = -[(\mathbf{c} \times \mathbf{b}) + \mathbf{0}] \\ &= -\mathbf{c} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} \end{aligned}$$

Thus $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a}$.

49.
$$\begin{aligned}
(\mathbf{a}-\mathbf{b})\times(\mathbf{a}+\mathbf{b}) &= (\mathbf{a}-\mathbf{b})\times\mathbf{a}+(\mathbf{a}-\mathbf{b})\times\mathbf{b} && \text{by Property 3 of the cross product}\\
&= \mathbf{a}\times\mathbf{a}+(-\mathbf{b})\times\mathbf{a}+\mathbf{a}\times\mathbf{b}+(-\mathbf{b})\times\mathbf{b} && \text{by Property 4}\\
&= (\mathbf{a}\times\mathbf{a})-(\mathbf{b}\times\mathbf{a})+(\mathbf{a}\times\mathbf{b})-(\mathbf{b}\times\mathbf{b}) && \text{by Property 2 (with } c=-1)\\
&= \mathbf{0}-(\mathbf{b}\times\mathbf{a})+(\mathbf{a}\times\mathbf{b})-\mathbf{0} && \text{by Example 2}\\
&= (\mathbf{a}\times\mathbf{b})+(\mathbf{a}\times\mathbf{b}) && \text{by Property 1}\\
&= 2(\mathbf{a}\times\mathbf{b})
\end{aligned}$$

50. Let $\mathbf{a}=\langle a_1,a_2,a_3\rangle$, $\mathbf{b}=\langle b_1,b_2,b_3\rangle$ and $\mathbf{c}=\langle c_1,c_2,c_3\rangle$, so $\mathbf{b}\times\mathbf{c}=\langle b_2c_3-b_3c_2, b_3c_1-b_1c_3, b_1c_2-b_2c_1\rangle$ and

$$\begin{aligned}
\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) &= \langle a_2(b_1c_2-b_2c_1)-a_3(b_3c_1-b_1c_3), a_3(b_2c_3-b_3c_2)-a_1(b_1c_2-b_2c_1),\\
&\qquad a_1(b_3c_1-b_1c_3)-a_2(b_2c_3-b_3c_2)\rangle\\
&= \langle a_2b_1c_2-a_2b_2c_1-a_3b_3c_1+a_3b_1c_3, a_3b_2c_3-a_3b_3c_2-a_1b_1c_2+a_1b_2c_1,\\
&\qquad a_1b_3c_1-a_1b_1c_3-a_2b_2c_3+a_2b_3c_2\rangle\\
&= \langle (a_2c_2+a_3c_3)b_1-(a_2b_2+a_3b_3)c_1, (a_1c_1+a_3c_3)b_2-(a_1b_1+a_3b_3)c_2,\\
&\qquad (a_1c_1+a_2c_2)b_3-(a_1b_1+a_2b_2)c_3\rangle\\
(\star)\quad &= \langle (a_2c_2+a_3c_3)b_1-(a_2b_2+a_3b_3)c_1+a_1b_1c_1-a_1b_1c_1,\\
&\qquad (a_1c_1+a_3c_3)b_2-(a_1b_1+a_3b_3)c_2+a_2b_2c_2-a_2b_2c_2,\\
&\qquad (a_1c_1+a_2c_2)b_3-(a_1b_1+a_2b_2)c_3+a_3b_3c_3-a_3b_3c_3\rangle\\
&= \langle (a_1c_1+a_2c_2+a_3c_3)b_1-(a_1b_1+a_2b_2+a_3b_3)c_1,\\
&\qquad (a_1c_1+a_2c_2+a_3c_3)b_2-(a_1b_1+a_2b_2+a_3b_3)c_2,\\
&\qquad (a_1c_1+a_2c_2+a_3c_3)b_3-(a_1b_1+a_2b_2+a_3b_3)c_3\rangle\\
&= (a_1c_1+a_2c_2+a_3c_3)\langle b_1,b_2,b_3\rangle-(a_1b_1+a_2b_2+a_3b_3)\langle c_1,c_2,c_3\rangle\\
&= (\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}
\end{aligned}$$

($\star$) Here we look ahead to see what terms are still needed to arrive at the desired equation. By adding and subtracting the same terms, we don't change the value of the component.

51. $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})+\mathbf{b}\times(\mathbf{c}\times\mathbf{a})+\mathbf{c}\times(\mathbf{a}\times\mathbf{b})$

$$\begin{aligned}
&= [(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}]+[(\mathbf{b}\cdot\mathbf{a})\mathbf{c}-(\mathbf{b}\cdot\mathbf{c})\mathbf{a}]+[(\mathbf{c}\cdot\mathbf{b})\mathbf{a}-(\mathbf{c}\cdot\mathbf{a})\mathbf{b}] && \text{by Exercise 50}\\
&= (\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}+(\mathbf{a}\cdot\mathbf{b})\mathbf{c}-(\mathbf{b}\cdot\mathbf{c})\mathbf{a}+(\mathbf{b}\cdot\mathbf{c})\mathbf{a}-(\mathbf{a}\cdot\mathbf{c})\mathbf{b}=\mathbf{0}
\end{aligned}$$

52. Let $\mathbf{c}\times\mathbf{d}=\mathbf{v}$. Then

$$\begin{aligned}
(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) &= (\mathbf{a}\times\mathbf{b})\cdot\mathbf{v}=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{v}) && \text{by Property 5 of the cross product}\\
&= \mathbf{a}\cdot[\mathbf{b}\times(\mathbf{c}\times\mathbf{d})]=\mathbf{a}\cdot[(\mathbf{b}\cdot\mathbf{d})\mathbf{c}-(\mathbf{b}\cdot\mathbf{c})\mathbf{d}] && \text{by Exercise 50}\\
&= (\mathbf{b}\cdot\mathbf{d})(\mathbf{a}\cdot\mathbf{c})-(\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{d}) && \text{by Properties 3 and 4 of the dot product}\\
&= \begin{vmatrix}\mathbf{a}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c}\\ \mathbf{a}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{d}\end{vmatrix}
\end{aligned}$$

53. (a) No. If $\mathbf{a}\cdot\mathbf{b}=\mathbf{a}\cdot\mathbf{c}$, then $\mathbf{a}\cdot(\mathbf{b}-\mathbf{c})=0$, so $\mathbf{a}$ is perpendicular to $\mathbf{b}-\mathbf{c}$, which can happen if $\mathbf{b}\neq\mathbf{c}$. For example, let $\mathbf{a}=\langle 1,1,1\rangle$, $\mathbf{b}=\langle 1,0,0\rangle$ and $\mathbf{c}=\langle 0,1,0\rangle$.

(b) No. If $\mathbf{a}\times\mathbf{b}=\mathbf{a}\times\mathbf{c}$ then $\mathbf{a}\times(\mathbf{b}-\mathbf{c})=\mathbf{0}$, which implies that $\mathbf{a}$ is parallel to $\mathbf{b}-\mathbf{c}$, which of course can happen if $\mathbf{b}\neq\mathbf{c}$.

(c) Yes. Since $\mathbf{a}\cdot\mathbf{c}=\mathbf{a}\cdot\mathbf{b}$, $\mathbf{a}$ is perpendicular to $\mathbf{b}-\mathbf{c}$, by part (a). From part (b), $\mathbf{a}$ is also parallel to $\mathbf{b}-\mathbf{c}$. Thus since $\mathbf{a}\neq\mathbf{0}$ but is both parallel and perpendicular to $\mathbf{b}-\mathbf{c}$, we have $\mathbf{b}-\mathbf{c}=\mathbf{0}$, so $\mathbf{b}=\mathbf{c}$.

54. (a) $\mathbf{k}_i$ is perpendicular to $\mathbf{v}_j$ if $i\neq j$ by the definition of $\mathbf{k}_i$ and Theorem 8.

(b) $\mathbf{k}_1\cdot\mathbf{v}_1=\dfrac{\mathbf{v}_2\times\mathbf{v}_3}{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)}\cdot\mathbf{v}_1=\dfrac{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)}{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)}=1$

$$\mathbf{k}_2\cdot\mathbf{v}_2=\frac{\mathbf{v}_3\times\mathbf{v}_1}{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)}\cdot\mathbf{v}_2=\frac{\mathbf{v}_2\cdot(\mathbf{v}_3\times\mathbf{v}_1)}{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)}=\frac{(\mathbf{v}_2\times\mathbf{v}_3)\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)}=1\qquad\text{[by Property 5 of the cross product]}$$

$$\mathbf{k}_3\cdot\mathbf{v}_3=\frac{(\mathbf{v}_1\times\mathbf{v}_2)\cdot\mathbf{v}_3}{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)}=\frac{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)}{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)}=1\qquad\text{[by Property 5]}$$

(c) $$\mathbf{k}_1\cdot(\mathbf{k}_2\times\mathbf{k}_3)=\mathbf{k}_1\cdot\left(\frac{\mathbf{v}_3\times\mathbf{v}_1}{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)}\times\frac{\mathbf{v}_1\times\mathbf{v}_2}{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)}\right)=\frac{\mathbf{k}_1}{[\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)]^2}\cdot[(\mathbf{v}_3\times\mathbf{v}_1)\times(\mathbf{v}_1\times\mathbf{v}_2)]$$
$$=\frac{\mathbf{k}_1}{[\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)]^2}\cdot\left([(\mathbf{v}_3\times\mathbf{v}_1)\cdot\mathbf{v}_2]\,\mathbf{v}_1-[(\mathbf{v}_3\times\mathbf{v}_1)\cdot\mathbf{v}_1]\,\mathbf{v}_2\right)\qquad\text{[by Exercise 50]}$$

But $(\mathbf{v}_3\times\mathbf{v}_1)\cdot\mathbf{v}_1=0$ since $\mathbf{v}_3\times\mathbf{v}_1$ is orthogonal to $\mathbf{v}_1$, and $(\mathbf{v}_3\times\mathbf{v}_1)\cdot\mathbf{v}_2=\mathbf{v}_2\cdot(\mathbf{v}_3\times\mathbf{v}_1)=(\mathbf{v}_2\times\mathbf{v}_3)\cdot\mathbf{v}_1=\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)$. Thus,

$$\mathbf{k}_1\cdot(\mathbf{k}_2\times\mathbf{k}_3)=\frac{\mathbf{k}_1}{[\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)]^2}\cdot[\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)]\,\mathbf{v}_1=\frac{\mathbf{k}_1\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)}=\frac{1}{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)}\qquad\text{[by part (b)]}$$

DISCOVERY PROJECT The Geometry of a Tetrahedron

1. Set up a coordinate system so that vertex S is at the origin, $R=(0,y_1,0)$, $Q=(x_2,y_2,0)$, $P=(x_3,y_3,z_3)$.

Then $\overrightarrow{SR}=\langle 0,y_1,0\rangle$, $\overrightarrow{SQ}=\langle x_2,y_2,0\rangle$, $\overrightarrow{SP}=\langle x_3,y_3,z_3\rangle$, $\overrightarrow{QR}=\langle -x_2,y_1-y_2,0\rangle$, and $\overrightarrow{QP}=\langle x_3-x_2,y_3-y_2,z_3\rangle$.

Let

$$\mathbf{v}_S=\overrightarrow{QR}\times\overrightarrow{QP}=(y_1z_3-y_2z_3)\,\mathbf{i}+x_2z_3\,\mathbf{j}+(-x_2y_3-x_3y_1+x_3y_2+x_2y_1)\,\mathbf{k}$$

Then $\mathbf{v}_S$ is an outward normal to the face opposite vertex S. Similarly,

$\mathbf{v}_R=\overrightarrow{SQ}\times\overrightarrow{SP}=y_2z_3\,\mathbf{i}-x_2z_3\,\mathbf{j}+(x_2y_3-x_3y_2)\,\mathbf{k}$, $\mathbf{v}_Q=\overrightarrow{SP}\times\overrightarrow{SR}=-y_1z_3\,\mathbf{i}+x_3y_1\,\mathbf{k}$, and

$\mathbf{v}_P=\overrightarrow{SR}\times\overrightarrow{SQ}=-x_2y_1\,\mathbf{k}\quad\Rightarrow\quad\mathbf{v}_S+\mathbf{v}_R+\mathbf{v}_Q+\mathbf{v}_P=\mathbf{0}$. Now

$$\begin{aligned}|\mathbf{v}_S|&=\text{ area of the parallelogram determined by }\overrightarrow{QR}\text{ and }\overrightarrow{QP}\\&=2\,(\text{area of triangle }RQP)=2|\mathbf{v}_1|\end{aligned}$$

So $\mathbf{v}_S=2\mathbf{v}_1$, and similarly $\mathbf{v}_R=2\mathbf{v}_2$, $\mathbf{v}_Q=2\mathbf{v}_3$, $\mathbf{v}_P=2\mathbf{v}_4$. Thus $\mathbf{v}_1+\mathbf{v}_2+\mathbf{v}_3+\mathbf{v}_4=\mathbf{0}$.

2. (a) Let $S = (x_0, y_0, z_0)$, $R = (x_1, y_1, z_1)$, $Q = (x_2, y_2, z_2)$, $P = (x_3, y_3, z_3)$ be the four vertices. Then

$$\begin{aligned}\text{Volume} &= \tfrac{1}{3}(\text{distance from } S \text{ to plane } RQP) \times (\text{area of triangle } RQP) \\ &= \tfrac{1}{3}\frac{\left|\mathbf{N}\cdot\overrightarrow{SR}\right|}{|\mathbf{N}|}\cdot\tfrac{1}{2}\left|\overrightarrow{RQ}\times\overrightarrow{RP}\right|\end{aligned}$$

where $\mathbf{N}$ is a vector which is normal to the face RQP. Thus $\mathbf{N} = \overrightarrow{RQ}\times\overrightarrow{RP}$. Therefore

$$V = \left|\tfrac{1}{6}\left(\overrightarrow{RQ}\times\overrightarrow{RP}\right)\cdot\overrightarrow{SR}\right| = \frac{1}{6}\left|\begin{vmatrix} x_0 - x & y_0 - y_1 & z_0 - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix}\right|$$

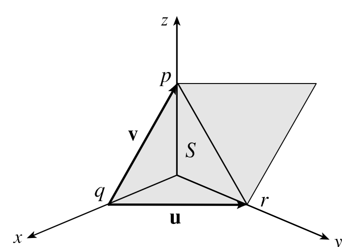
(b) Using the formula from part (a), $V = \dfrac{1}{6}\left|\begin{vmatrix} 1-1 & 1-2 & 1-3 \\ 1-1 & 1-2 & 2-3 \\ 3-1 & -1-2 & 2-3 \end{vmatrix}\right| = \dfrac{1}{6}\left|2(1-2)\right| = \dfrac{1}{3}.$

3. We define a vector $\mathbf{v}_1$ to have length equal to the area of the face opposite vertex P, so we can say $|\mathbf{v}_1| = A$, and direction perpendicular to the face and pointing outward, as in Problem 1. Similarly, we define $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ so that $|\mathbf{v}_2| = B$, $|\mathbf{v}_3| = C$, and $|\mathbf{v}_4| = D$ and with the analogous directions. From Problem 1, we know $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0} \;\Rightarrow\; \mathbf{v}_4 = -(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) \;\Rightarrow\; |\mathbf{v}_4| = |-(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3)| = |\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3| \;\Rightarrow\; |\mathbf{v}_4|^2 = |\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3|^2 \;\Rightarrow$

$$\begin{aligned}\mathbf{v}_4\cdot\mathbf{v}_4 &= (\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3)\cdot(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) \\ &= \mathbf{v}_1\cdot\mathbf{v}_1 + \mathbf{v}_1\cdot\mathbf{v}_2 + \mathbf{v}_1\cdot\mathbf{v}_3 + \mathbf{v}_2\cdot\mathbf{v}_1 + \mathbf{v}_2\cdot\mathbf{v}_2 + \mathbf{v}_2\cdot\mathbf{v}_3 + \mathbf{v}_3\cdot\mathbf{v}_1 + \mathbf{v}_3\cdot\mathbf{v}_2 + \mathbf{v}_3\cdot\mathbf{v}_3\end{aligned}$$

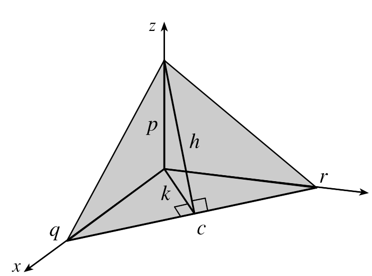
Since the vertex S is trirectangular, we know the three faces meeting at S are mutually perpendicular, so the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are also mutually perpendicular. Therefore, $\mathbf{v}_i\cdot\mathbf{v}_j = 0$ for $i \neq j$ and $i, j \in \{1, 2, 3\}$. Thus we have $\mathbf{v}_4\cdot\mathbf{v}_4 = \mathbf{v}_1\cdot\mathbf{v}_1 + \mathbf{v}_2\cdot\mathbf{v}_2 + \mathbf{v}_3\cdot\mathbf{v}_3 \;\Rightarrow\; |\mathbf{v}_4|^2 = |\mathbf{v}_1|^2 + |\mathbf{v}_2|^2 + |\mathbf{v}_3|^2 \;\Rightarrow\; D^2 = A^2 + B^2 + C^2$.

Another method: We introduce a coordinate system, as shown. Recall that the area of the parallelogram spanned by two vectors is equal to the length of their cross product, so since $\mathbf{u}\times\mathbf{v} = \langle -q, r, 0\rangle \times \langle -q, 0, p\rangle = \langle pr, pq, qr\rangle$, we have $|\mathbf{u}\times\mathbf{v}| = \sqrt{(pr)^2 + (pq)^2 + (qr)^2}$, and therefore



$$\begin{aligned}D^2 &= \left(\tfrac{1}{2}|\mathbf{u}\times\mathbf{v}|\right)^2 = \tfrac{1}{4}[(pr)^2 + (pq)^2 + (qr)^2] \\ &= \left(\tfrac{1}{2}pr\right)^2 + \left(\tfrac{1}{2}pq\right)^2 + \left(\tfrac{1}{2}qr\right)^2 = A^2 + B^2 + C^2.\end{aligned}$$

A third method: We draw a line from S perpendicular to QR, as shown. Now $D = \frac{1}{2}ch$, so $D^2 = \frac{1}{4}c^2h^2$. Substituting $h^2 = p^2 + k^2$, we get $D^2 = \frac{1}{4}c^2\left(p^2 + k^2\right) = \frac{1}{4}c^2p^2 + \frac{1}{4}c^2k^2$. But $C = \frac{1}{2}ck$, so $D^2 = \frac{1}{4}c^2p^2 + C^2$. Now substituting $c^2 = q^2 + r^2$ gives $D^2 = \frac{1}{4}p^2q^2 + \frac{1}{4}q^2r^2 + C^2 = A^2 + B^2 + C^2$.



12.5 Equations of Lines and Planes

1. (a) True; each of the first two lines has a direction vector parallel to the direction vector of the third line, so these vectors are each scalar multiples of the third direction vector. Then the first two direction vectors are also scalar multiples of each other, so these vectors, and hence the two lines, are parallel.

(b) False; for example, the x- and y-axes are both perpendicular to the z-axis, yet the x- and y-axes are not parallel.

(c) True; each of the first two planes has a normal vector parallel to the normal vector of the third plane, so these two normal vectors are parallel to each other and the planes are parallel.

(d) False; for example, the xy- and yz-planes are not parallel, yet they are both perpendicular to the xz-plane.

(e) False; the x- and y-axes are not parallel, yet they are both parallel to the plane $z = 1$.

(f) True; if each line is perpendicular to a plane, then the lines' direction vectors are both parallel to a normal vector for the plane. Thus, the direction vectors are parallel to each other and the lines are parallel.

(g) False; the planes $y = 1$ and $z = 1$ are not parallel, yet they are both parallel to the x-axis.

(h) True; if each plane is perpendicular to a line, then any normal vector for each plane is parallel to a direction vector for the line. Thus, the normal vectors are parallel to each other and the planes are parallel.

(i) True; see Figure 9 and the accompanying discussion.

(j) False; they can be skew, as in Example 3.

(k) True. Consider any normal vector for the plane and any direction vector for the line. If the normal vector is perpendicular to the direction vector, the line and plane are parallel. Otherwise, the vectors meet at an angle θ, $0^\circ \leq \theta < 90^\circ$, and the line will intersect the plane at an angle $90^\circ - \theta$.

2. For this line, we have $\mathbf{r}_0 = 4\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 6\mathbf{k}$, so a vector equation is $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v} = (4\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) + t(2\mathbf{i} - \mathbf{j} + 6\mathbf{k}) = (4 + 2t)\,\mathbf{i} + (2 - t)\,\mathbf{j} + (-3 + 6t)\,\mathbf{k}$, and parametric equations are $x = 4 + 2t$, $y = 2 - t$, $z = -3 + 6t$.

3. For this line, we have $\mathbf{r}_0 = -\mathbf{i} + 8\mathbf{j} + 7\mathbf{k}$ and $\mathbf{v} = \frac{1}{2}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{1}{4}\mathbf{k}$, so a vector equation is $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v} = (-\mathbf{i} + 8\mathbf{j} + 7\mathbf{k}) + t\left(\frac{1}{2}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{1}{4}\mathbf{k}\right) = \left(-1 + \frac{1}{2}t\right)\mathbf{i} + \left(8 + \frac{1}{3}t\right)\mathbf{j} + \left(7 + \frac{1}{4}t\right)\mathbf{k}$, and parametric equations are $x = -1 + \frac{1}{2}t$, $y = 8 + \frac{1}{3}t$, $z = 7 + \frac{1}{4}t$.

4. The direction vector for this line is the same as the given line, $\mathbf{v} = -3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$**.** Here $\mathbf{r}_0 = 6\mathbf{i} - 2\mathbf{k}$, so a vector equation is $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v} = (6\,\mathbf{i} - 2\,\mathbf{k}) + t(-3\,\mathbf{i} + 4\,\mathbf{j} + 5\,\mathbf{k}) = (6 - 3t)\,\mathbf{i} + 4t\,\mathbf{j} + (-2 + 5t)\,\mathbf{k}$, and parametric equations are $x = 6 - 3t$, $y = 4t$, $z = -2 + 5t$.

5. A line perpendicular to the given plane has the same direction as a normal vector to the plane, such as $\mathbf{n} = \langle 3, -2, 2 \rangle$. So $\mathbf{r}_0 = 5\mathbf{i} + 7\mathbf{j} + \mathbf{k}$ and we can take $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$. Then a vector equation is

$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v} = (5\,\mathbf{i} + 7\,\mathbf{j} + \mathbf{k}) + t(3\,\mathbf{i} - 2\,\mathbf{j} + 2\,\mathbf{k}) = (5 + 3t)\,\mathbf{i} + (7 - 2t)\,\mathbf{j} + (1 + 2t)\,\mathbf{k}$, and parametric equations are $x = 5 + 3t$, $y = 7 - 2t$, $z = 1 + 2t$.

6. The vector $\mathbf{v} = \langle 1 - (-5), 6 - 2, -2 - 5 \rangle = \langle 6, 4, -7 \rangle$ is parallel to the line. Letting $P_0 = (-5, 2, 5)$, parametric equations are $x = -5 + 6t$, $y = 2 + 4t$, $z = 5 - 7t$ and symmetric equations are $\dfrac{x+5}{6} = \dfrac{y-2}{4} = \dfrac{z-5}{-7}$.

7. The vector $\mathbf{v} = \langle 8 - 0, -1 - 0, 3 - 0 \rangle = \langle 8, -1, 3 \rangle$ is parallel to the line. Letting $P_0 = (0, 0, 0)$, parametric equations are $x = 8t$, $y = -t$, $z = 3t$ and symmetric equations are $\dfrac{x}{8} = \dfrac{y}{-1} = \dfrac{z}{3}$ or $\dfrac{x}{8} = -y = \dfrac{z}{3}$.

8. The vector $\mathbf{v} = \langle 1.3 - 0.4, 0.8 - (-0.2), -2.3 - 1.1 \rangle = \langle 0.9, 1, -3.4 \rangle$ is parallel to the line. Letting $P_0 = (0.4, -0.2, 1.1)$, parametric equations are $x = 0.4 + 0.9t$, $y = -0.2 + t$, $z = 1.1 - 3.4t$ and symmetric equations are $\dfrac{x - 0.4}{0.9} = \dfrac{y + 0.2}{1} = \dfrac{z - 1.1}{-3.4}$ or $\dfrac{x - 0.4}{0.9} = y + 0.2 = \dfrac{z - 1.1}{-3.4}$.

9. The vector $\mathbf{v} = \langle -7 - 12, 9 - 9, 11 - (-13) \rangle = \langle -19, 0, 24 \rangle$ is parallel to the line. Letting $P_0 = (12, 9, -13)$, parametric equations are $x = 12 - 19t$, $y = 9$, $z = -13 + 24t$ and symmetric equations are $\dfrac{x - 12}{-19} = \dfrac{z + 13}{24}$, $y = 9$. Notice here that the direction number $b = 0$, so rather than writing $\dfrac{y-9}{0}$ in the symmetric equation, we must write $y = 9$ separately.

10. $\mathbf{v} = (\mathbf{i} + \mathbf{j}) \times (\mathbf{j} + \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ is the direction of the line perpendicular to both $\mathbf{i} + \mathbf{j}$ and $\mathbf{j} + \mathbf{k}$.

With $P_0 = (2, 1, 0)$, parametric equations are $x = 2 + t$, $y = 1 - t$, $z = t$ and symmetric equations are $x - 2 = \dfrac{y - 1}{-1} = z$ or $x - 2 = 1 - y = z$.

11. The given line $\dfrac{x}{2} = \dfrac{y}{3} = \dfrac{z+1}{1}$ has direction $\mathbf{v} = \langle 2, 3, 1 \rangle$. Taking $(-6, 2, 3)$ as P_0, parametric equations are $x = -6 + 2t$, $y = 2 + 3t$, $z = 3 + t$ and symmetric equations are $\dfrac{x+6}{2} = \dfrac{y-2}{3} = z - 3$.

12. Setting $z = 0$ we see that $(1, 0, 0)$ satisfies the equations of both planes, so they do in fact have a line of intersection. The line is perpendicular to the normal vectors of both planes, so a direction vector for the line is $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 1, 2, 3 \rangle \times \langle 1, -1, 1 \rangle = \langle 5, 2, -3 \rangle$. Taking the point $(1, 0, 0)$ as P_0, parametric equations are $x = 1 + 5t$, $y = 2t$, $z = -3t$, and symmetric equations are $\dfrac{x-1}{5} = \dfrac{y}{2} = \dfrac{z}{-3}$.

13. Direction vectors of the lines are $\mathbf{v}_1 = \langle -2 - (-4), 0 - (-6), -3 - 1 \rangle = \langle 2, 6, -4 \rangle$ and $\mathbf{v}_2 = \langle 5 - 10, 3 - 18, 14 - 4 \rangle = \langle -5, -15, 10 \rangle$. Since $\mathbf{v}_2 = -\frac{5}{2}\mathbf{v}_1$, the direction vectors, and thus the lines, are parallel.

14. Direction vectors of the lines are $\mathbf{v}_1 = \langle 1 - (-2), 1 - 4, 1 - 0 \rangle = \langle 3, -3, 1 \rangle$ and $\mathbf{v}_2 = \langle 3 - 2, -1 - 3, -8 - 4 \rangle = \langle 1, -4, -12 \rangle$. Since $\mathbf{v}_1 \cdot \mathbf{v}_2 = 3 + 12 - 12 \neq 0$, the direction vectors, and thus the lines, are not perpendicular.

15. (a) The line passes through the point $(1, -5, 6)$ and a direction vector for the line is $\langle -1, 2, -3 \rangle$, so symmetric equations for the line are $\dfrac{x-1}{-1} = \dfrac{y+5}{2} = \dfrac{z-6}{-3}$.

(b) The line intersects the xy-plane when $z = 0$, so we need $\dfrac{x-1}{-1} = \dfrac{y+5}{2} = \dfrac{0-6}{-3}$ or $\dfrac{x-1}{-1} = 2 \Rightarrow x = -1$, $\dfrac{y+5}{2} = 2 \Rightarrow y = -1$. Thus the point of intersection with the xy-plane is $(-1, -1, 0)$. Similarly for the yz-plane, we need $x = 0 \Rightarrow 1 = \dfrac{y+5}{2} = \dfrac{z-6}{-3} \Rightarrow y = -3$, $z = 3$. Thus the line intersects the yz-plane at $(0, -3, 3)$. For the xz-plane, we need $y = 0 \Rightarrow \dfrac{x-1}{-1} = \dfrac{5}{2} = \dfrac{z-6}{-3} \Rightarrow x = -\frac{3}{2}$, $z = -\frac{3}{2}$. So the line intersects the xz-plane at $\left(-\frac{3}{2}, 0, -\frac{3}{2}\right)$.

16. (a) A vector normal to the plane $x - y + 3z = 7$ is $\mathbf{n} = \langle 1, -1, 3 \rangle$, and since the line is to be perpendicular to the plane, $\mathbf{n}$ is also a direction vector for the line. Thus parametric equations of the line are $x = 2 + t$, $y = 4 - t$, $z = 6 + 3t$.

(b) On the xy-plane, $z = 0$. So $z = 6 + 3t = 0 \Rightarrow t = -2$ in the parametric equations of the line, and therefore $x = 0$ and $y = 6$, giving the point of intersection $(0, 6, 0)$. For the yz-plane, $x = 0$ so we get the same point of interesection: $(0, 6, 0)$. For the xz-plane, $y = 0$ which implies $t = 4$, so $x = 6$ and $z = 18$ and the point of intersection is $(6, 0, 18)$.

17. From Equation 4, the line segment from $\mathbf{r}_0 = 6\,\mathbf{i} - \mathbf{j} + 9\,\mathbf{k}$ to $\mathbf{r}_1 = 7\,\mathbf{i} + 6\,\mathbf{j}$ has vector equation

$$\begin{aligned} \mathbf{r}(t) &= (1-t)\,\mathbf{r}_0 + t\,\mathbf{r}_1 = (1-t)(6\,\mathbf{i} - \mathbf{j} + 9\,\mathbf{k}) + t(7\,\mathbf{i} + 6\,\mathbf{j}) \\ &= (6\,\mathbf{i} - \mathbf{j} + 9\,\mathbf{k}) - t(6\,\mathbf{i} - \mathbf{j} + 9\,\mathbf{k}) + t(7\,\mathbf{i} + 6\,\mathbf{j}) \\ &= (6\,\mathbf{i} - \mathbf{j} + 9\,\mathbf{k}) + t(\,\mathbf{i} + 7\,\mathbf{j} - 9\,\mathbf{k}), \quad 0 \le t \le 1. \end{aligned}$$

18. From Equation 4, the line segment from $\mathbf{r}_0 = -2\,\mathbf{i} + 18\,\mathbf{j} + 31\,\mathbf{k}$ to $\mathbf{r}_1 = 11\,\mathbf{i} - 4\,\mathbf{j} + 48\,\mathbf{k}$ has vector equation

$$\begin{aligned} \mathbf{r}(t) &= (1-t)\,\mathbf{r}_0 + t\,\mathbf{r}_1 = (1-t)(-2\,\mathbf{i} + 18\,\mathbf{j} + 31\,\mathbf{k}) + t(11\,\mathbf{i} - 4\,\mathbf{j} + 48\,\mathbf{k}) \\ &= (-2\,\mathbf{i} + 18\,\mathbf{j} + 31\,\mathbf{k}) + t(13\,\mathbf{i} - 22\,\mathbf{j} + 17\,\mathbf{k}), \quad 0 \le t \le 1. \end{aligned}$$

The corresponding parametric equations are $x = -2 + 13t$, $y = 18 - 22t$, $z = 31 + 17t$, $0 \le t \le 1$.

19. Since the direction vectors $\langle 2, -1, 3 \rangle$ and $\langle 4, -2, 5 \rangle$ are not scalar multiples of each other, the lines aren't parallel. For the lines to intersect, we must be able to find one value of t and one value of s that produce the same point from the respective parametric equations. Thus we need to satisfy the following three equations: $3 + 2t = 1 + 4s$, $4 - t = 3 - 2s$, $1 + 3t = 4 + 5s$. Solving the last two equations we get $t = 1$, $s = 0$ and checking, we see that these values don't satisfy the first equation. Thus the lines aren't parallel and don't intersect, so they must be skew lines.

20. Since the direction vectors are $\mathbf{v}_1 = \langle -12, 9, -3 \rangle$ and $\mathbf{v}_2 = \langle 8, -6, 2 \rangle$, we have $\mathbf{v}_1 = -\frac{3}{2}\mathbf{v}_2$ so the lines are parallel.

21. Since the direction vectors $\langle 1, -2, -3 \rangle$ and $\langle 1, 3, -7 \rangle$ aren't scalar multiples of each other, the lines aren't parallel. Parametric equations of the lines are L_1: $x = 2 + t$, $y = 3 - 2t$, $z = 1 - 3t$ and L_2: $x = 3 + s$, $y = -4 + 3s$, $z = 2 - 7s$. Thus, for the lines to intersect, the three equations $2 + t = 3 + s$, $3 - 2t = -4 + 3s$, and $1 - 3t = 2 - 7s$ must be satisfied simultaneously. Solving the first two equations gives $t = 2$, $s = 1$ and checking, we see that these values do satisfy the third equation, so the lines intersect when $t = 2$ and $s = 1$, that is, at the point $(4, -1, -5)$.

22. The direction vectors $\langle 1, -1, 3\rangle$ and $\langle 2, -2, 7\rangle$ are not parallel, so neither are the lines. Parametric equations for the lines are L_1: $x = t$, $y = 1 - t$, $z = 2 + 3t$ and L_2: $x = 2 + 2s$, $y = 3 - 2s$, $z = 7s$. Thus, for the lines to interesect, the three equations $t = 2 + 2s$, $1 - t = 3 - 2s$, and $2 + 3t = 7s$ must be satisfied simultaneously. Solving the last two equations gives $t = -10$, $s = -4$ and checking, we see that these values don't satisfy the first equation. Thus, the lines aren't parallel and don't intersect, so they must be skew.

23. $5\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$ is a normal vector to the plane. $(3, 2, 1)$ is a point on the plane. Setting $a = 5$, $b = 4$, $c = 6$ and $x_0 = 3$, $y_0 = 2$, $z_0 = 1$ in Equation 7 gives $5(x - 3) + 4(y - 2) + 6(z - 1) = 0$, or $5x + 4y + 6z = 29$, as an equation of the plane.

24. $\langle 6, 1, -1\rangle$ is a normal vector to the plane. $(-3, 4, 2)$ is a point on the plane. Setting $a = 6$, $b = 1$, $c = -1$ and $x_0 = -3$, $y_0 = 4$, $z_0 = 2$ in Equation 7 gives $6(x + 3) + 1(y - 4) - (z - 2) = 0$, or $6x + y - z = -16$, as an equation of the plane.

25. Since the plane is perpendicular to the vector $-\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$, we can take $\langle -1, 2, 3\rangle$ as a normal vector to the plane. $(5, -2, 4)$ is a point on the plane. Setting $a = -1$, $b = 2$, $c = 3$ and $x_0 = 5$, $y_0 = -2$, $z_0 = 4$ in Equation 7 gives $-(x - 5) + 2(y + 2) + 3(z - 4) = 0$, or $-x + 2y + 3z = 3$, as an equation of the plane.

26. Since the line is perpendicular to the plane, its direction vector, $\langle -8, -7, 2\rangle$, is a normal vector to the plane. $(0, 0, 0)$ is a point on the plane. Setting $a = -8$, $b = -7$, $c = 2$ and $x_0 = 0$, $y_0 = 0$, $z_0 = 0$ in Equation 7 gives $-8(x - 0) - 7(y - 0) + 2(z - 0) = 0$, or $-8x - 7y + 2z = 0$, as an equation of the plane.

27. Since the line is perpendicular to the plane, its direction vector, $\langle 4, -1, 5\rangle$, is a normal vector to the plane. $(1, 3, -1)$ is a point on the plane. Setting $a = 4$, $b = -1$, $c = 5$ and $x_0 = 1$, $y_0 = 3$, $z_0 = -1$ in Equation 7 gives $4(x - 1) - 1(y - 3) + 5(z + 1) = 0$, or $4x - y + 5z = -4$, as an equation of the plane.

28. Since the two planes are parallel, they will have the same normal vectors. The plane is $z = 2x - 3y \;\Leftrightarrow\; 2x - 3y - z = 0$, so we can take $\mathbf{n} = \langle 2, -3, -1\rangle$, and an equation of the plane is $2(x - 9) - 3(y + 4) - 1(z + 5) = 0$, or $2x - 3y - z = 35$.

29. Since the two planes are parallel, they will have the same normal vectors. The plane is $2x - y + 3z = 1$, so we can take $\mathbf{n} = \langle 2, -1, 3\rangle$, and an equation of the plane is $2(x - 2.1) - 1(y - 1.7) + 3(z + 0.9) = 0$, or $2x - y + 3z = -0.2$, or $10x - 5y + 15z = -1$.

30. First, a normal vector for the plane $5x + 2y + z = 1$ is $\mathbf{n} = \langle 5, 2, 1\rangle$. A direction vector for the line is $\mathbf{v} = \langle 1, -1, -3\rangle$, and since $\mathbf{n} \cdot \mathbf{v} = 0$, we know the line is perpendicular to $\mathbf{n}$ and hence parallel to the plane. Thus, there is a parallel plane which contains the line. By putting $t = 0$, we know that the point $(1, 2, 4)$ is on the line and hence the new plane. We can use the same normal vector $\mathbf{n} = \langle 5, 2, 1\rangle$, so an equation of the plane is $5(x - 1) + 2(y - 2) + 1(z - 4) = 0$ or $5x + 2y + z = 13$.

31. The vector from $(0, 1, 1)$ to $(1, 0, 1)$, namely $\mathbf{a} = \langle 1 - 0, 0 - 1, 1 - 1\rangle = \langle 1, -1, 0\rangle$, and the vector from $(0, 1, 1)$ to $(1, 1, 0)$, $\mathbf{b} = \langle 1 - 0, 1 - 1, 0 - 1\rangle = \langle 1, 0, -1\rangle$, both lie in the plane, so $\mathbf{a} \times \mathbf{b}$ is a normal vector to the plane. Thus, we can take

$\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle (-1)((-1) - (0)(0), (0)(1) - (1)(-1), (1)(0) - (-1)(1) \rangle = \langle 1, 1, 1 \rangle$. If P_0 is the point $(0, 1, 1)$, an equation of the plane is $1(x - 0) + 1(y - 1) + 1(z - 1) = 0$ or $x + y + z = 2$.

32. Here the vectors $\mathbf{a} = \langle 3, -2, 1 \rangle$ and $\mathbf{b} = \langle 1, 1, 1 \rangle$ lie in the plane, so $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle (-2)(1) - (1)(1), (1)(1) - (3)(1), (3)(1) - (-2)(1) \rangle = \langle -3, -2, 5 \rangle$ is a normal vector to the plane. We can take the origin as P_0, so an equation of the plane is $-3(x - 0) - 2(y - 0) + 5(z - 0) = 0$ or $-3x - 2y + 5z = 0$ or $3x + 2y - 5z = 0$.

33. Here the vectors $\mathbf{a} = \langle 3 - 2, -8 - 1, 6 - 2 \rangle = \langle 1, -9, 4 \rangle$ and $\mathbf{b} = \langle -2 - 2, -3 - 1, 1 - 2 \rangle = \langle -4, -4, -1 \rangle$ lie in the plane, so a normal vector to the plane is $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle 9 + 16, -16 + 1, -4 - 36 \rangle = \langle 25, -15, -40 \rangle$ and an equation of the plane is $25(x - 2) - 15(y - 1) - 40(z - 2) = 0$ or $25x - 15y - 40z = -45$ or $5x - 3y - 8z = -9$.

34. The vectors $\mathbf{a} = \langle -2 - 3, -2 - 0, 3 - (-1) \rangle = \langle -5, -2, 4 \rangle$ and $\mathbf{b} = \langle 7 - 3, 1 - 0, -4 - (-1) \rangle = \langle 4, 1, -3 \rangle$ lie in the plane, so a normal vector to the plane is $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle 6 - 4, 16 - 15, -5 + 8 \rangle = \langle 2, 1, 3 \rangle$ and an equation of the plane is $2(x - 3) + 1(y - 0) + 3[z - (-1)] = 0$ or $2x + y + 3z = 3$.

35. If we first find two nonparallel vectors in the plane, their cross product will be a normal vector to the plane. Since the given line lies in the plane, its direction vector $\mathbf{a} = \langle -1, 2, -3 \rangle$ is one vector in the plane. We can verify that the given point $(3, 5, -1)$ does not lie on this line, so to find another nonparallel vector $\mathbf{b}$ which lies in the plane, we can pick any point on the line and find a vector connecting the points. If we put $t = 0$, we see that $(4, -1, 0)$ is on the line, so $\mathbf{b} = \langle 4 - 3, -1 - 5, 0 - (-1) \rangle = \langle 1, -6, 1 \rangle$ and $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle 2 - 18, -3 + 1, 6 - 2 \rangle = \langle -16, -2, 4 \rangle$. Thus, an equation of the plane is $-16(x - 3) - 2(y - 5) + 4[z - (-1)] = 0$ or $-16x - 2y + 4z = -62$ or $8x + y - 2z = 31$.

36. Since the line $\dfrac{x}{3} = \dfrac{y + 4}{1} = \dfrac{z}{2}$ lies in the plane, its direction vector $\mathbf{a} = \langle 3, 1, 2 \rangle$ is parallel to the plane. The point $(0, -4, 0)$ is on the line (put $t = 0$ in the corresponding parametric equations), and we can verify that the given point $(6, -1, 3)$ in the plane is not on the line. The vector connecting these two points, $\mathbf{b} = \langle 6, 3, 3 \rangle$, is therefore parallel to the plane, but not parallel to $\mathbf{a}$. Then $\mathbf{a} \times \mathbf{b} = \langle 3 - 6, 12 - 9, 9 - 6 \rangle = \langle -3, 3, 3 \rangle$ is a normal vector to the plane, and an equation of the plane is $-3(x - 0) + 3[y - (-4)] + 3(z - 0) = 0$ or $-3x + 3y + 3z = -12$ or $x - y - z = 4$.

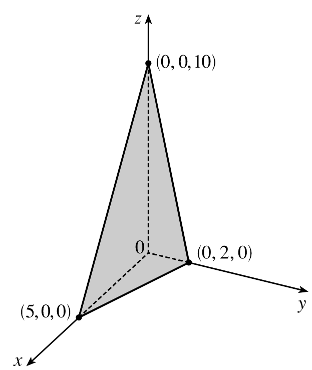
37. Normal vectors for the given planes are $\mathbf{n}_1 = \langle 1, 2, 3 \rangle$ and $\mathbf{n}_2 = \langle 2, -1, 1 \rangle$. A direction vector, then, for the line of intersection is $\mathbf{a} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 2 + 3, 6 - 1, -1 - 4 \rangle = \langle 5, 5, -5 \rangle$, and $\mathbf{a}$ is parallel to the desired plane. Another vector parallel to the plane is the vector connecting any point on the line of intersection to the given point $(3, 1, 4)$ in the plane. Setting $z = 0$, the equations of the planes reduce to $x + 2y = 1$ and $2x - y = -3$ with simultaneous solution $x = -1$ and $y = 1$. So a point on the line is $(-1, 1, 0)$ and another vector parallel to the plane is $\mathbf{b} = \langle 3 - (-1), 1 - 1, 4 - 0 \rangle = \langle 4, 0, 4 \rangle$. Then a normal vector to the plane is $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle 20 - 0, -20 - 20, 0 - 20 \rangle = \langle 20, -40, -20 \rangle$. Equivalently, we can take $\langle 1, -2, -1 \rangle$ as a normal vector, and an equation of the plane is $1(x - 3) - 2(y - 1) - 1(z - 4) = 0$ or $x - 2y - z = -3$.

38. The points $(0,-2,5)$ and $(-1,3,1)$ lie in the desired plane, so the vector $\mathbf{v}_1 = \langle -1,5,-4\rangle$ connecting them is parallel to the plane. The desired plane is perpendicular to the plane $2z = 5x + 4y$ or $5x + 4y - 2z = 0$ and for perpendicular planes, a normal vector for one plane is parallel to the other plane, so $\mathbf{v}_2 = \langle 5,4,-2\rangle$ is also parallel to the desired plane.
A normal vector to the desired plane is $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle -10+16, -20-2, -4-25\rangle = \langle 6,-22,-29\rangle$.
Taking $(x_0,y_0,z_0) = (0,-2,5)$, the equation we are looking for is $6(x-0) - 22(y+2) - 29(z-5) = 0$ or $6x - 22y - 29z = -101$.

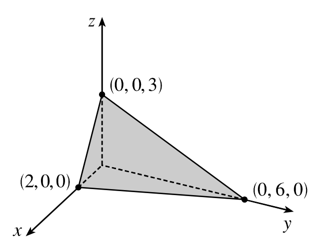
39. If a plane is perpendicular to two other planes, its normal vector is perpendicular to the normal vectors of the other two planes. Thus $\langle 2,1,-2\rangle \times \langle 1,0,3\rangle = \langle 3-0,-2-6,0-1\rangle = \langle 3,-8,-1\rangle$ is a normal vector to the desired plane. The point $(1,5,1)$ lies on the plane, so an equation is $3(x-1) - 8(y-5) - (z-1) = 0$ or $3x - 8y - z = -38$.

40. $\mathbf{n}_1 = \langle 1,0,-1\rangle$ and $\mathbf{n}_2 = \langle 0,1,2\rangle$. Setting $z = 0$, it is easy to see that $(1,3,0)$ is a point on the line of intersection of $x - z = 1$ and $y + 2z = 3$. The direction of this line is $\mathbf{v}_1 = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 1,-2,1\rangle$. A second vector parallel to the desired plane is $\mathbf{v}_2 = \langle 1,1,-2\rangle$, since it is perpendicular to $x + y - 2z = 1$. Therefore, a normal of the plane in question is $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle 4-1, 1+2, 1+2\rangle = \langle 3,3,3\rangle$, or we can use $\langle 1,1,1\rangle$. Taking $(x_0,y_0,z_0) = (1,3,0)$, the equation we are looking for is $(x-1) + (y-3) + z = 0 \quad \Leftrightarrow \quad x + y + z = 4$.

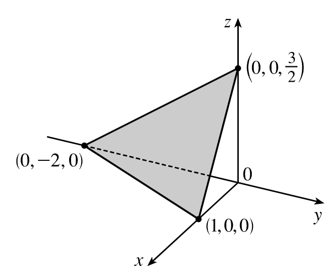
41. To find the x-intercept we set $y = z = 0$ in the equation $2x + 5y + z = 10$ and obtain $2x = 10 \quad \Rightarrow \quad x = 5$ so the x-intercept is $(5,0,0)$. When $x = z = 0$ we get $5y = 10 \quad \Rightarrow \quad y = 2$, so the y-intercept is $(0,2,0)$. Setting $x = y = 0$ gives $z = 10$, so the z-intercept is $(0,0,10)$ and we graph the portion of the plane that lies in the first octant.



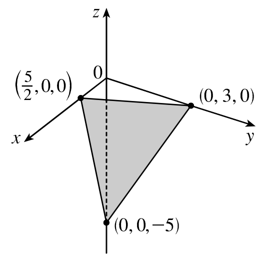
42. To find the x-intercept we set $y = z = 0$ in the equation $3x + y + 2z = 6$ and obtain $3x = 6 \quad \Rightarrow \quad x = 2$ so the x-intercept is $(2,0,0)$. When $x = z = 0$ we get $y = 6$ so the y-intercept is $(0,6,0)$. Setting $x = y = 0$ gives $2z = 6 \quad \Rightarrow \quad z = 3$, so the z-intercept is $(0,0,3)$. The figure shows the portion of the plane that lies in the first octant.



43. Setting $y = z = 0$ in the equation $6x - 3y + 4z = 6$ gives $6x = 6 \quad \Rightarrow$ $x = 1$, when $x = z = 0$, we have $-3y = 6 \;\Rightarrow\; y = -2$, and $x = y = 0$ implies $4z = 6 \quad \Rightarrow \quad z = \frac{3}{2}$, so the intercepts are $(1,0,0)$, $(0,-2,0)$, and $\left(0,0,\frac{3}{2}\right)$. The figure shows the portion of the plane cut off by the coordinate planes.



44. Setting $y = z = 0$ in the equation $6x + 5y - 3z = 15$ gives $6x = 15 \Rightarrow x = \frac{5}{2}$, when $x = z = 0$, we have $5y = 15 \Rightarrow y = 3$, and $x = y = 0$ implies $-3z = 15 \Rightarrow z = -5$, so the intercepts are $(\frac{5}{2}, 0, 0)$, $(0, 3, 0)$, and $(0, 0, -5)$. The figure shows the portion of the plane cut off by the coordinate planes.



45. Substitute the parametric equations of the line into the equation of the plane: $x + 2y - z = 7 \Rightarrow$ $(2 - 2t) + 2(3t) - (1 + t) = 7 \Rightarrow 3t + 1 = 7 \Rightarrow t = 2$. Therefore, the point of intersection of the line and the plane is given by $x = 2 - 2(2) = -2$, $y = 3(2) = 6$, and $z = 1 + 2 = 3$, that is, the point $(-2, 6, 3)$.

46. Substitute the parametric equations of the line into the equation of the plane: $3(t - 1) - (1 + 2t) + 2(3 - t) = 5 \Rightarrow$ $-t + 2 = 5 \Rightarrow t = -3$. Therefore, the point of intersection of the line and the plane is given by $x = -3 - 1 = -4$, $y = 1 + 2(-3) = -5$, and $z = 3 - (-3) = 6$, that is, the point $(-4, -5, 6)$.

47. Parametric equations for the line are $x = \frac{1}{5}t$, $y = 2t$, $z = t - 2$ and substitution into the equation of the plane gives $10\left(\frac{1}{5}t\right) - 7(2t) + 3(t - 2) + 24 = 0 \Rightarrow -9t + 18 = 0 \Rightarrow t = 2$. Thus $x = \frac{1}{5}(2) = \frac{2}{5}$, $y = 2(2) = 4$, $z = 2 - 2 = 0$ and the point of intersection is $\left(\frac{2}{5}, 4, 0\right)$.

48. A direction vector for the line through $(-3, 1, 0)$ and $(-1, 5, 6)$ is $\mathbf{v} = \langle 2, 4, 6\rangle$ and, taking $P_0 = (-3, 1, 0)$, parametric equations for the line are $x = -3 + 2t$, $y = 1 + 4t$, $z = 6t$. Substitution of the parametric equations into the equation of the plane gives $2(-3 + 2t) + (1 + 4t) - (6t) = -2 \Rightarrow 2t - 5 = -2 \Rightarrow t = \frac{3}{2}$. Then $x = -3 + 2\left(\frac{3}{2}\right) = 0$, $y = 1 + 4\left(\frac{3}{2}\right) = 7$, and $z = 6\left(\frac{3}{2}\right) = 9$, and the point of intersection is $(0, 7, 9)$.

49. Setting $x = 0$, we see that $(0, 1, 0)$ satisfies the equations of both planes, so that they do in fact have a line of intersection. $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 1, 1, 1\rangle \times \langle 1, 0, 1\rangle = \langle 1, 0, -1\rangle$ is the direction of this line. Therefore, direction numbers of the intersecting line are $1, 0, -1$.

50. The angle between the two planes is the same as the angle between their normal vectors. The normal vectors of the two planes are $\langle 1, 1, 1\rangle$ and $\langle 1, 2, 3\rangle$. The cosine of the angle θ between these two planes is

$$\cos\theta = \frac{\langle 1, 1, 1\rangle \cdot \langle 1, 2, 3\rangle}{|\langle 1, 1, 1\rangle|\,|\langle 1, 2, 3\rangle|} = \frac{1 + 2 + 3}{\sqrt{1 + 1 + 1}\,\sqrt{1 + 4 + 9}} = \frac{6}{\sqrt{42}} = \sqrt{\frac{6}{7}}.$$

51. Normal vectors for the planes are $\mathbf{n}_1 = \langle 1, 4, -3\rangle$ and $\mathbf{n}_2 = \langle -3, 6, 7\rangle$. The normals aren't parallel (they are not scalar multiples of each other), so neither are the planes. But $\mathbf{n}_1 \cdot \mathbf{n}_2 = -3 + 24 - 21 = 0$, so the normals, and thus the planes, are perpendicular.

52. Normal vectors for the planes are $\mathbf{n}_1 = \langle 9, -3, 6\rangle$ and $\mathbf{n}_2 = \langle 6, -2, 4\rangle$ (the plane's equation is $6x - 2y + 4z = 0$). Since $\mathbf{n}_1 = \frac{3}{2}\mathbf{n}_2$, the normals, and thus the planes, are parallel.

53. Normal vectors for the planes are $\mathbf{n}_1 = \langle 1, 2, -1 \rangle$ and $\mathbf{n}_2 = \langle 2, -2, 1 \rangle$. The normals are not parallel (they are not scalar multiples of each other), so neither are the planes. Furthermore, $\mathbf{n}_1 \cdot \mathbf{n}_2 = 2 - 4 - 1 = -3 \neq 0$, so the planes aren't perpendicular. The angle between the planes is the same as the angle between the normals, given by

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|} = \frac{-3}{\sqrt{6}\,\sqrt{9}} = -\frac{1}{\sqrt{6}} \quad \Rightarrow \quad \theta = \cos^{-1}\left(-\frac{1}{\sqrt{6}}\right) \approx 114.1^\circ.$$

54. Normal vectors for the planes are $\mathbf{n}_1 = \langle 1, -1, 3 \rangle$ and $\mathbf{n}_2 = \langle 3, 1, -1 \rangle$. The normals are not parallel, so neither are the planes. Since $\mathbf{n}_1 \cdot \mathbf{n}_2 = 3 - 1 - 3 = -1 \neq 0$, the planes aren't perpendicular. The angle between the planes is given by

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|} = \frac{-1}{\sqrt{11}\,\sqrt{11}} = -\frac{1}{11} \quad \Rightarrow \quad \theta = \cos^{-1}\left(-\tfrac{1}{11}\right) \approx 95.2^\circ.$$

55. The planes are $2x - 3y - z = 0$ and $4x - 6y - 2z = 3$ with normal vectors $\mathbf{n}_1 = \langle 2, -3, -1 \rangle$ and $\mathbf{n}_2 = \langle 4, -6, -2 \rangle$. Since $\mathbf{n}_2 = 2\mathbf{n}_1$, the normals, and thus the planes, are parallel.

56. The normals are $\mathbf{n}_1 = \langle 5, 2, 3 \rangle$ and $\mathbf{n}_2 = \langle 4, -1, -6 \rangle$, which are not scalar multiples of each other, so the planes aren't parallel. Since $\mathbf{n}_1 \cdot \mathbf{n}_2 = 20 - 2 - 18 = 0$, the normals, and thus the planes, are perpendicular.

57. (a) To find a point on the line of intersection, set one of the variables equal to a constant, say $z = 0$. (This will fail if the line of intersection does not cross the xy-plane; in that case, try setting x or y equal to 0.) The equations of the two planes reduce to $x + y = 1$ and $x + 2y = 1$. Solving these two equations gives $x = 1$, $y = 0$. Thus a point on the line is $(1, 0, 0)$.

A vector $\mathbf{v}$ in the direction of this intersecting line is perpendicular to the normal vectors of both planes, so we can take $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 1, 1, 1 \rangle \times \langle 1, 2, 2 \rangle = \langle 2 - 2, 1 - 2, 2 - 1 \rangle = \langle 0, -1, 1 \rangle$. By Equations 2, parametric equations for the line are $x = 1$, $y = -t$, $z = t$.

(b) The angle between the planes satisfies $\cos\theta = \dfrac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|} = \dfrac{1 + 2 + 2}{\sqrt{3}\,\sqrt{9}} = \dfrac{5}{3\sqrt{3}}$. Therefore $\theta = \cos^{-1}\left(\dfrac{5}{3\sqrt{3}}\right) \approx 15.8^\circ$.

58. (a) If we set $z = 0$ then the equations of the planes reduce to $3x - 2y = 1$ and $2x + y = 3$ and solving these two equations gives $x = 1$, $y = 1$. Thus a point on the line of intersection is $(1, 1, 0)$. A vector $\mathbf{v}$ in the direction of this intersecting line is perpendicular to the normal vectors of both planes, so let $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 3, -2, 1 \rangle \times \langle 2, 1, -3 \rangle = \langle 5, 11, 7 \rangle$. By Equations 2, parametric equations for the line are $x = 1 + 5t$, $y = 1 + 11t$, $z = 7t$.

(b) $\cos\theta = \dfrac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|} = \dfrac{6 - 2 - 3}{\sqrt{14}\,\sqrt{14}} = \dfrac{1}{14} \quad \Rightarrow \quad \theta = \cos^{-1}\left(\tfrac{1}{14}\right) \approx 85.9^\circ$.

59. Setting $z = 0$, the equations of the two planes become $5x - 2y = 1$ and $4x + y = 6$. Solving these two equations gives $x = 1$, $y = 2$ so a point on the line of intersection is $(1, 2, 0)$. A vector $\mathbf{v}$ in the direction of this intersecting line is perpendicular to the normal vectors of both planes. So we can use $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 5, -2, -2 \rangle \times \langle 4, 1, 1 \rangle = \langle 0, -13, 13 \rangle$ or equivalently we can take $\mathbf{v} = \langle 0, -1, 1 \rangle$, and symmetric equations for the line are $x = 1$, $\dfrac{y - 2}{-1} = \dfrac{z}{1}$ or $x = 1$, $y - 2 = -z$.

60. If we set $z = 0$ then the equations of the planes reduce to $2x - y - 5 = 0$ and $4x + 3y - 5 = 0$ and solving these two equations gives $x = 2$, $y = -1$. Thus a point on the line of intersection is $(2, -1, 0)$. A vector $\mathbf{v}$ in the direction of this intersecting line is perpendicular to the normal vectors of both planes, so take $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 2, -1, -1 \rangle \times \langle 4, 3, -1 \rangle = \langle 4, -2, 10 \rangle$ or equivalently we can take $\mathbf{v} = \langle 2, -1, 5 \rangle$. Symmetric equations for the line are $\dfrac{x-2}{2} = \dfrac{y+1}{-1} = \dfrac{z}{5}$.

61. The distance from a point (x, y, z) to $(1, 0, -2)$ is $d_1 = \sqrt{(x-1)^2 + y^2 + (z+2)^2}$ and the distance from (x, y, z) to $(3, 4, 0)$ is $d_2 = \sqrt{(x-3)^2 + (y-4)^2 + z^2}$. The plane consists of all points (x, y, z) where $d_1 = d_2 \;\Rightarrow\; d_1^2 = d_2^2 \;\Leftrightarrow$
$(x-1)^2 + y^2 + (z+2)^2 = (x-3)^2 + (y-4)^2 + z^2 \;\Leftrightarrow$
$x^2 - 2x + y^2 + z^2 + 4z + 5 = x^2 - 6x + y^2 - 8y + z^2 + 25 \;\Leftrightarrow\; 4x + 8y + 4z = 20$ so an equation for the plane is $4x + 8y + 4z = 20$ or equivalently $x + 2y + z = 5$.

Alternatively, you can argue that the segment joining points $(1, 0, -2)$ and $(3, 4, 0)$ is perpendicular to the plane and the plane includes the midpoint of the segment.

62. The distance from a point (x, y, z) to $(2, 5, 5)$ is $d_1 = \sqrt{(x-2)^2 + (y-5)^2 + (z-5)^2}$ and the distance from (x, y, z) to $(-6, 3, 1)$ is $d_2 = \sqrt{(x+6)^2 + (y-3)^2 + (z-1)^2}$. The plane consists of all points (x, y, z) where $d_1 = d_2 \;\Rightarrow$
$d_1^2 = d_2^2 \;\Leftrightarrow\; (x-2)^2 + (y-5)^2 + (z-5)^2 = (x+6)^2 + (y-3)^2 + (z-1)^2 \;\Leftrightarrow$
$x^2 - 4x + y^2 - 10y + z^2 - 10z + 54 = x^2 + 12x + y^2 - 6y + z^2 - 2z + 46 \;\Leftrightarrow\; 16x + 4y + 8z = 8$ so an equation for the plane is $16x + 4y + 8z = 8$ or equivalently $4x + y + 2z = 2$.

63. The plane contains the points $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$. Thus the vectors $\mathbf{a} = \langle -a, b, 0 \rangle$ and $\mathbf{b} = \langle -a, 0, c \rangle$ lie in the plane, and $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle bc - 0, 0 + ac, 0 + ab \rangle = \langle bc, ac, ab \rangle$ is a normal vector to the plane. The equation of the plane is therefore $bcx + acy + abz = abc + 0 + 0$ or $bcx + acy + abz = abc$. Notice that if $a \neq 0$, $b \neq 0$ and $c \neq 0$ then we can rewrite the equation as $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$. This is a good equation to remember!

64. (a) For the lines to intersect, we must be able to find one value of t and one value of s satisfying the three equations $1 + t = 2 - s$, $1 - t = s$ and $2t = 2$. From the third we get $t = 1$, and putting this in the second gives $s = 0$. These values of s and t do satisfy the first equation, so the lines intersect at the point $P_0 = (1 + 1, 1 - 1, 2(1)) = (2, 0, 2)$.

(b) The direction vectors of the lines are $\langle 1, -1, 2 \rangle$ and $\langle -1, 1, 0 \rangle$, so a normal vector for the plane is $\langle -1, 1, 0 \rangle \times \langle 1, -1, 2 \rangle = \langle 2, 2, 0 \rangle$ and it contains the point $(2, 0, 2)$. Then an equation of the plane is $2(x-2) + 2(y-0) + 0(z-2) = 0 \;\Leftrightarrow\; x + y = 2$.

65. Two vectors which are perpendicular to the required line are the normal of the given plane, $\langle 1, 1, 1 \rangle$, and a direction vector for the given line, $\langle 1, -1, 2 \rangle$. So a direction vector for the required line is $\langle 1, 1, 1 \rangle \times \langle 1, -1, 2 \rangle = \langle 3, -1, -2 \rangle$. Thus L is given by $\langle x, y, z \rangle = \langle 0, 1, 2 \rangle + t\langle 3, -1, -2 \rangle$, or in parametric form, $x = 3t$, $y = 1 - t$, $z = 2 - 2t$.

66. Let L be the given line. Then $(1,1,0)$ is the point on L corresponding to $t=0$. L is in the direction of $\mathbf{a}=\langle 1,-1,2\rangle$ and $\mathbf{b}=\langle -1,0,2\rangle$ is the vector joining $(1,1,0)$ and $(0,1,2)$. Then

$\mathbf{b}-\operatorname{proj}_{\mathbf{a}}\mathbf{b}=\langle -1,0,2\rangle-\dfrac{\langle 1,-1,2\rangle\cdot\langle -1,0,2\rangle}{1^2+(-1)^2+2^2}\langle 1,-1,2\rangle=\langle -1,0,2\rangle-\frac{1}{2}\langle 1,-1,2\rangle=\left\langle -\frac{3}{2},\frac{1}{2},1\right\rangle$ is a direction vector for the required line. Thus $2\left\langle -\frac{3}{2},\frac{1}{2},1\right\rangle=\langle -3,1,2\rangle$ is also a direction vector, and the line has parametric equations $x=-3t$, $y=1+t$, $z=2+2t$. (Notice that this is the same line as in Exercise 65.)

67. Let P_i have normal vector $\mathbf{n}_i$. Then $\mathbf{n}_1=\langle 3,6,-3\rangle$, $\mathbf{n}_2=\langle 4,-12,8\rangle$, $\mathbf{n}_3=\langle 3,-9,6\rangle$, $\mathbf{n}_4=\langle 1,2,-1\rangle$. Now $\mathbf{n}_1=3\mathbf{n}_4$, so $\mathbf{n}_1$ and $\mathbf{n}_4$ are parallel, and hence P_1 and P_4 are parallel; similarly P_2 and P_3 are parallel because $\mathbf{n}_2=\frac{4}{3}\mathbf{n}_3$. However, $\mathbf{n}_1$ and $\mathbf{n}_2$ are not parallel (so not all four planes are parallel). Notice that the point $(2,0,0)$ lies on both P_1 and P_4, so these two planes are identical. The point $\left(\frac{5}{4},0,0\right)$ lies on P_2 but not on P_3, so these are different planes.

68. Let L_i have direction vector $\mathbf{v}_i$. Rewrite the symmetric equations for L_3 as $\dfrac{x-1}{1/2}=\dfrac{y-1}{-1/4}=\dfrac{z+1}{1}$; then $\mathbf{v}_1=\langle 6,-3,12\rangle$, $\mathbf{v}_2=\langle 2,1,4\rangle$, $\mathbf{v}_3=\left\langle \frac{1}{2},-\frac{1}{4},1\right\rangle$, and $\mathbf{v}_4=\langle 4,2,8\rangle$. $\mathbf{v}_1=12\mathbf{v}_3$, so L_1 and L_3 are parallel. $\mathbf{v}_4=2\mathbf{v}_2$, so L_2 and L_4 are parallel. (Note that L_1 and L_2 are not parallel.) L_1 contains the point $(1,1,5)$, but this point does not lie on L_3, so they're not identical. $(3,1,5)$ lies on L_4 and also on L_2 (for $t=1$), so L_2 and L_4 are the same line.

69. Let $Q=(1,3,4)$ and $R=(2,1,1)$, points on the line corresponding to $t=0$ and $t=1$. Let $P=(4,1,-2)$. Then $\mathbf{a}=\overrightarrow{QR}=\langle 1,-2,-3\rangle$, $\mathbf{b}=\overrightarrow{QP}=\langle 3,-2,-6\rangle$. The distance is

$$d=\frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}|}=\frac{|\langle 1,-2,-3\rangle\times\langle 3,-2,-6\rangle|}{|\langle 1,-2,-3\rangle|}=\frac{|\langle 6,-3,4\rangle|}{|\langle 1,-2,-3\rangle|}=\frac{\sqrt{6^2+(-3)^2+4^2}}{\sqrt{1^2+(-2)^2+(-3)^2}}=\frac{\sqrt{61}}{\sqrt{14}}=\sqrt{\frac{61}{14}}.$$

70. Let $Q=(0,6,3)$ and $R=(2,4,4)$, points on the line corresponding to $t=0$ and $t=1$. Let $P=(0,1,3)$. Then $\mathbf{a}=\overrightarrow{QR}=\langle 2,-2,1\rangle$ and $\mathbf{b}=\overrightarrow{QP}=\langle 0,-5,0\rangle$. The distance is

$$d=\frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}|}=\frac{|\langle 2,-2,1\rangle\times\langle 0,-5,0\rangle|}{|\langle 2,-2,1\rangle|}=\frac{|\langle 5,0,-10\rangle|}{|\langle 2,-2,1\rangle|}=\frac{\sqrt{5^2+0^2+(-10)^2}}{\sqrt{2^2+(-2)^2+1^2}}=\frac{\sqrt{125}}{\sqrt{9}}=\frac{5\sqrt{5}}{3}.$$

71. By Equation 9, the distance is $D=\dfrac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}=\dfrac{|3(1)+2(-2)+6(4)-5|}{\sqrt{3^2+2^2+6^2}}=\dfrac{|18|}{\sqrt{49}}=\dfrac{18}{7}$.

72. By Equation 9, the distance is $D=\dfrac{|1(-6)-2(3)-4(5)-8|}{\sqrt{1^2+(-2)^2+(-4)^2}}=\dfrac{|-40|}{\sqrt{21}}=\dfrac{40}{\sqrt{21}}$.

73. Put $y=z=0$ in the equation of the first plane to get the point $(2,0,0)$ on the plane. Because the planes are parallel, the distance D between them is the distance from $(2,0,0)$ to the second plane. By Equation 9,

$$D=\frac{|4(2)-6(0)+2(0)-3|}{\sqrt{4^2+(-6)^2+(2)^2}}=\frac{5}{\sqrt{56}}=\frac{5}{2\sqrt{14}}\text{ or }\frac{5\sqrt{14}}{28}.$$

74. Put $x = y = 0$ in the equation of the first plane to get the point $(0, 0, 0)$ on the plane. Because the planes are parallel the distance D between them is the distance from $(0, 0, 0)$ to the second plane $3x - 6y + 9z - 1 = 0$. By Equation 9,

$$D = \frac{|3(0) - 6(0) + 9(0) - 1|}{\sqrt{3^2 + (-6)^2 + 9^2}} = \frac{1}{\sqrt{126}} = \frac{1}{3\sqrt{14}}.$$

75. The distance between two parallel planes is the same as the distance between a point on one of the planes and the other plane. Let $P_0 = (x_0, y_0, z_0)$ be a point on the plane given by $ax + by + cz + d_1 = 0$. Then $ax_0 + by_0 + cz_0 + d_1 = 0$ and the distance between P_0 and the plane given by $ax + by + cz + d_2 = 0$ is, from Equation 9,

$$D = \frac{|ax_0 + by_0 + cz_0 + d_2|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|-d_1 + d_2|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}.$$

76. The planes must have parallel normal vectors, so if $ax + by + cz + d = 0$ is such a plane, then for some $t \neq 0$, $\langle a, b, c\rangle = t\langle 1, 2, -2\rangle = \langle t, 2t, -2t\rangle$. So this plane is given by the equation $x + 2y - 2z + k = 0$, where $k = d/t$. By Exercise 75, the distance between the planes is $2 = \dfrac{|1 - k|}{\sqrt{1^2 + 2^2 + (-2)^2}} \quad\Leftrightarrow\quad 6 = |1 - k| \quad\Leftrightarrow\quad k = 7$ or -5. So the desired planes have equations $x + 2y - 2z = 7$ and $x + 2y - 2z = -5$.

77. L_1: $x = y = z \quad\Rightarrow\quad x = y$ **(1)**. L_2: $x + 1 = y/2 = z/3 \quad\Rightarrow\quad x + 1 = y/2$ **(2)**. The solution of **(1)** and **(2)** is $x = y = -2$. However, when $x = -2$, $x = z \quad\Rightarrow\quad z = -2$, but $x + 1 = z/3 \quad\Rightarrow\quad z = -3$, a contradiction. Hence the lines do not intersect. For L_1, $\mathbf{v}_1 = \langle 1, 1, 1\rangle$, and for L_2, $\mathbf{v}_2 = \langle 1, 2, 3\rangle$, so the lines are not parallel. Thus the lines are skew lines. If two lines are skew, they can be viewed as lying in two parallel planes and so the distance between the skew lines would be the same as the distance between these parallel planes. The common normal vector to the planes must be perpendicular to both $\langle 1, 1, 1\rangle$ and $\langle 1, 2, 3\rangle$, the direction vectors of the two lines. So set $\mathbf{n} = \langle 1, 1, 1\rangle \times \langle 1, 2, 3\rangle = \langle 3 - 2, -3 + 1, 2 - 1\rangle = \langle 1, -2, 1\rangle$. From above, we know that $(-2, -2, -2)$ and $(-2, -2, -3)$ are points of L_1 and L_2, respectively. So in the notation of Equation 8, $1(-2) - 2(-2) + 1(-2) + d_1 = 0 \quad\Rightarrow\quad d_1 = 0$ and $1(-2) - 2(-2) + 1(-3) + d_2 = 0 \quad\Rightarrow\quad d_2 = 1$.

By Exercise 75, the distance between these two skew lines is $D = \dfrac{|0 - 1|}{\sqrt{1 + 4 + 1}} = \dfrac{1}{\sqrt{6}}$.

Alternate solution (without reference to planes): A vector which is perpendicular to both of the lines is $\mathbf{n} = \langle 1, 1, 1\rangle \times \langle 1, 2, 3\rangle = \langle 1, -2, 1\rangle$. Pick any point on each of the lines, say $(-2, -2, -2)$ and $(-2, -2, -3)$, and form the vector $\mathbf{b} = \langle 0, 0, 1\rangle$ connecting the two points. The distance between the two skew lines is the absolute value of the scalar projection of $\mathbf{b}$ along $\mathbf{n}$, that is, $D = \dfrac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|} = \dfrac{|1 \cdot 0 - 2 \cdot 0 + 1 \cdot 1|}{\sqrt{1 + 4 + 1}} = \dfrac{1}{\sqrt{6}}$.

78. First notice that if two lines are skew, they can be viewed as lying in two parallel planes and so the distance between the skew lines would be the same as the distance between these parallel planes. The common normal vector to the planes must be perpendicular to both $\mathbf{v}_1 = \langle 1, 6, 2\rangle$ and $\mathbf{v}_2 = \langle 2, 15, 6\rangle$, the direction vectors of the two lines, respectively. Thus, set

$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle 36 - 30, 4 - 6, 15 - 12 \rangle = \langle 6, -2, 3 \rangle$. Setting $t = 0$ and $s = 0$ gives the points $(1, 1, 0)$ and $(1, 5, -2)$. So in the notation of Equation 8, $6 - 2 + 0 + d_1 = 0 \quad \Rightarrow \quad d_1 = -4$ and $6 - 10 - 6 + d_2 = 0 \quad \Rightarrow \quad d_2 = 10$.

Then by Exercise 75, the distance between the two skew lines is given by $D = \dfrac{|-4 - 10|}{\sqrt{36 + 4 + 9}} = \dfrac{14}{7} = 2$.

Alternate solution (without reference to planes): We already know that the direction vectors of the two lines are $\mathbf{v}_1 = \langle 1, 6, 2 \rangle$ and $\mathbf{v}_2 = \langle 2, 15, 6 \rangle$. Then $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle 6, -2, 3 \rangle$ is perpendicular to both lines. Pick any point on each of the lines, say $(1, 1, 0)$ and $(1, 5, -2)$, and form the vector $\mathbf{b} = \langle 0, 4, -2 \rangle$ connecting the two points. Then the distance between the two skew lines is the absolute value of the scalar projection of $\mathbf{b}$ along $\mathbf{n}$, that is,

$$D = \frac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|} = \frac{1}{\sqrt{36 + 4 + 9}} \, |0 - 8 - 6| = \frac{14}{7} = 2.$$

79. A direction vector for L_1 is $\mathbf{v}_1 = \langle 2, 0, -1 \rangle$ and a direction vector for L_2 is $\mathbf{v}_2 = \langle 3, 2, 2 \rangle$. These vectors are not parallel so neither are the lines. Parametric equations for the lines are L_1: $x = 2t$, $y = 0$, $z = -t$, and L_2: $x = 1 + 3s$, $y = -1 + 2s$, $z = 1 + 2s$. No values of t and s satisfy these equations simultaneously, so the lines don't intersect and hence are skew. We can view the lines as lying in two parallel planes; a common normal vector to the planes is $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle 2, -7, 4 \rangle$. Line L_1 passes through the origin, so $(0, 0, 0)$ lies on one of the planes, and $(1, -1, 1)$ is a point on L_2 and therefore on the other plane. Equations of the planes then are $2x - 7y + 4z = 0$ and $2x - 7y + 4z - 13 = 0$, and by Exercise 75, the distance between the two skew lines is $D = \dfrac{|0 - (-13)|}{\sqrt{4 + 49 + 16}} = \dfrac{13}{\sqrt{69}}$.

Alternate solution (without reference to planes): Direction vectors of the two lines are $\mathbf{v}_1 = \langle 2, 0, -1 \rangle$ and $\mathbf{v}_2 = \langle 3, 2, 2 \rangle$. Then $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle 2, -7, 4 \rangle$ is perpendicular to both lines. Pick any point on each of the lines, say $(0, 0, 0)$ and $(1, -1, 1)$, and form the vector $\mathbf{b} = \langle 1, -1, 1 \rangle$ connecting the two points. Then the distance between the two skew lines is the absolute value of the scalar projection of $\mathbf{b}$ along $\mathbf{n}$, that is, $D = \dfrac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|} = \dfrac{|2 + 7 + 4|}{\sqrt{4 + 49 + 16}} = \dfrac{13}{\sqrt{69}}$.

80. A direction vector for the line L_1 is $\mathbf{v}_1 = \langle 1, 2, 2 \rangle$. A normal vector for the plane P_1 is $\mathbf{n}_1 = \langle 1, -1, 2 \rangle$. The vector from the point $(0, 0, 1)$ to $(3, 2, -1)$, $\langle 3, 2, -2 \rangle$, is parallel to the plane P_2, as is the vector from $(0, 0, 1)$ to $(1, 2, 1)$, namely $\langle 1, 2, 0 \rangle$. Thus a normal vector for P_2 is $\langle 3, 2, -2 \rangle \times \langle 1, 2, 0 \rangle = \langle 4, -2, 4 \rangle$, or we can use $\mathbf{n}_2 = \langle 2, -1, 2 \rangle$, and a direction vector for the line L_2 of intersection of these planes is $\mathbf{v}_2 = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 1, -1, 2 \rangle \times \langle 2, -1, 2 \rangle = \langle 0, 2, 1 \rangle$. Notice that the point $(3, 2, -1)$ lies on both planes, so it also lies on L_2. The lines are skew, so we can view them as lying in two parallel planes; a common normal vector to the planes is $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle -2, -1, 2 \rangle$. Line L_1 passes through the point $(1, 2, 6)$, so $(1, 2, 6)$ lies on one of the planes, and $(3, 2, -1)$ is a point on L_2 and therefore on the other plane. Equations of the planes then are $-2x - y + 2z - 8 = 0$ and $-2x - y + 2z + 10 = 0$, and by Exercise 75, the distance between the lines is

$$D = \frac{|-8 - 10|}{\sqrt{4 + 1 + 4}} = \frac{18}{3} = 6.$$

[continued]

Alternatively, direction vectors for the lines are $\mathbf{v}_1 = \langle 1, 2, 2\rangle$ and $\mathbf{v}_2 = \langle 0, 2, 1\rangle$, so $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle -2, -1, 2\rangle$ is perpendicular to both lines. Pick any point on each of the lines, say $(1, 2, 6)$ and $(3, 2, -1)$, and form the vector $\mathbf{b} = \langle 2, 0, -7\rangle$ connecting the two points. Then the distance between the two skew lines is the absolute value of the scalar projection of $\mathbf{b}$ along $\mathbf{n}$, that is, $D = \dfrac{|\mathbf{n}\cdot\mathbf{b}|}{|\mathbf{n}|} = \dfrac{|-4+0-14|}{\sqrt{4+1+4}} = \dfrac{18}{3} = 6$.

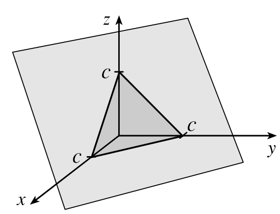
81. (a) A direction vector from tank A to tank B is $\langle 765 - 325, 675 - 810, 599 - 561\rangle = \langle 440, -135, 38\rangle$. Taking tank A's position $(325, 810, 561)$ as the initial point, parametric equations for the line of sight are $x = 325 + 440t$, $y = 810 - 135t$, $z = 561 + 38t$ for $0 \le t \le 1$.

(b) We divide the line of sight into 5 equal segments, corresponding to $\Delta t = 0.2$, and compute the elevation from the z-component of the parametric equations in part (a):

t	$z = 561 + 38t$	terrain elevation
0	561.0	
0.2	568.6	549
0.4	576.2	566
0.6	583.8	586
0.8	591.4	589
1.0	599.0	

Since the terrain is higher than the line of sight when $t = 0.6$, the tanks can't see each other.

82. (a) The planes $x + y + z = c$ have normal vector $\langle 1, 1, 1\rangle$, so they are all parallel. Their x-, y-, and z-intercepts are all c. When $c > 0$ their intersection with the first octant is an equilateral triangle and when $c < 0$ their intersection with the octant diagonally opposite the first is an equilateral triangle.



(b) The planes $x + y + cz = 1$ have x-intercept 1, y-intercept 1, and z-intercept $1/c$. The plane with $c = 0$ is parallel to the z-axis. As c gets larger, the planes get closer to the xy-plane.

(c) The planes $y\cos\theta + z\sin\theta = 1$ have normal vectors $\langle 0, \cos\theta, \sin\theta\rangle$, which are perpendicular to the x-axis, and so the planes are parallel to the x-axis. We look at their intersection with the yz-plane. These are lines that are perpendicular to $\langle\cos\theta, \sin\theta\rangle$ and pass through $(\cos\theta, \sin\theta)$, since $\cos^2\theta + \sin^2\theta = 1$. So these are the tangent lines to the unit circle. Thus the family consists of all planes tangent to the circular cylinder with radius 1 and axis the x-axis.

83. If $a \ne 0$, then $ax + by + cz + d = 0 \quad\Rightarrow\quad a(x + d/a) + b(y - 0) + c(z - 0) = 0$ which by (7) is the scalar equation of the plane through the point $(-d/a, 0, 0)$ with normal vector $\langle a, b, c\rangle$. Similarly, if $b \ne 0$ (or if $c \ne 0$) the equation of the plane can be rewritten as $a(x - 0) + b(y + d/b) + c(z - 0) = 0$ [or as $a(x - 0) + b(y - 0) + c(z + d/c) = 0$] which by (7) is the scalar equation of a plane through the point $(0, -d/b, 0)$ [or the point $(0, 0, -d/c)$] with normal vector $\langle a, b, c\rangle$.

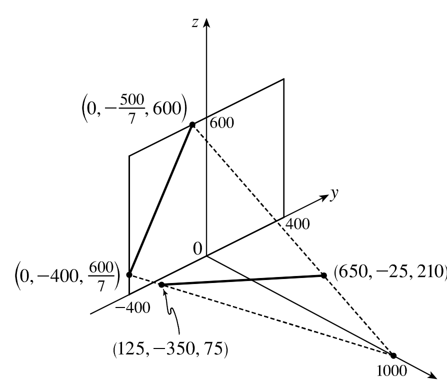
DISCOVERY PROJECT Putting 3D in Perspective

1. If we view the screen from the camera's location, the vertical clipping plane on the left passes through the points $(1000, 0, 0)$, $(0, -400, 0)$, and $(0, -400, 600)$. A vector from the first point to the second is $\mathbf{v}_1 = \langle -1000, -400, 0 \rangle$ and a vector from the first point to the third is $\mathbf{v}_2 = \langle -1000, -400, 600 \rangle$. A normal vector for the clipping plane is $\mathbf{v}_1 \times \mathbf{v}_2 = -240{,}000\,\mathbf{i} + 600{,}000\,\mathbf{j}$ or $-2\,\mathbf{i} + 5\,\mathbf{j}$, and an equation for the plane is $-2(x - 1000) + 5(y - 0) + 0(z - 0) = 0 \quad \Rightarrow \quad 2x - 5y = 2000$. By symmetry, the vertical clipping plane on the right is given by $2x + 5y = 2000$. The lower clipping plane is $z = 0$. The upper clipping plane passes through the points $(1000, 0, 0)$, $(0, -400, 600)$, and $(0, 400, 600)$. Vectors from the first point to the second and third points are $\mathbf{v}_1 = \langle -1000, -400, 600 \rangle$ and $\mathbf{v}_2 = \langle -1000, 400, 600 \rangle$, and a normal vector for the plane is $\mathbf{v}_1 \times \mathbf{v}_2 = -480{,}000\,\mathbf{i} - 800{,}000\,\mathbf{k}$ or $3\,\mathbf{i} + 5\,\mathbf{k}$. An equation for the plane is $3(x - 1000) + 0(y - 0) + 5(z - 0) = 0 \quad \Rightarrow \quad 3x + 5z = 3000$.

A direction vector for the line L is $\mathbf{v} = \langle 630, 390, 162 \rangle$ and taking $P_0 = (230, -285, 102)$, parametric equations are $x = 230 + 630t$, $y = -285 + 390t$, $z = 102 + 162t$. L intersects the left clipping plane when $2(230 + 630t) - 5(-285 + 390t) = 2000 \quad \Rightarrow \quad t = -\frac{1}{6}$. The corresponding point is $(125, -350, 75)$. L intersects the right clipping plane when $2(230 + 630t) + 5(-285 + 390t) = 2000 \quad \Rightarrow \quad t = \frac{593}{642}$. The corresponding point is approximately $(811.9, 75.2, 251.6)$, but this point is not contained within the viewing volume. L intersects the upper clipping plane when $3(230 + 630t) + 5(102 + 162t) = 3000 \quad \Rightarrow \quad t = \frac{2}{3}$, corresponding to the point $(650, -25, 210)$, and L intersects the lower clipping plane when $z = 0 \quad \Rightarrow \quad 102 + 162t = 0 \quad \Rightarrow \quad t = -\frac{17}{27}$. The corresponding point is approximately $(-166.7, -530.6, 0)$, which is not contained within the viewing volume. Thus L should be clipped at the points $(125, -350, 75)$ and $(650, -25, 210)$.

2. A sight line from the camera at $(1000, 0, 0)$ to the left endpoint $(125, -350, 75)$ of the clipped line has direction $\mathbf{v} = \langle -875, -350, 75 \rangle$. Parametric equations are $x = 1000 - 875t$, $y = -350t$, $z = 75t$. This line intersects the screen when $x = 0 \quad \Rightarrow \quad 1000 - 875t = 0 \quad \Rightarrow \quad t = \frac{8}{7}$, corresponding to the point $\left(0, -400, \frac{600}{7}\right)$. Similarly, a sight line from the camera to the right endpoint $(650, -25, 210)$ of the clipped line has direction $\langle -350, -25, 210 \rangle$ and parametric equations are $x = 1000 - 350t$, $y = -25t$, $z = 210t$. $\quad x = 0 \quad \Rightarrow \quad 1000 - 350t = 0 \quad \Rightarrow \quad t = \frac{20}{7}$, corresponding to the point $\left(0, -\frac{500}{7}, 600\right)$. Thus the projection of the clipped line is the line segment between the points $\left(0, -400, \frac{600}{7}\right)$ and $\left(0, -\frac{500}{7}, 600\right)$.

3. From Equation 12.5.4, equations for the four sides of the screen are $\mathbf{r}_1(t) = (1 - t)\langle 0, -400, 0 \rangle + t\,\langle 0, -400, 600 \rangle$,
$\mathbf{r}_2(t) = (1 - t)\langle 0, -400, 600 \rangle + t\,\langle 0, 400, 600 \rangle$,
$\mathbf{r}_3(t) = (1 - t)\langle 0, 400, 0 \rangle + t\,\langle 0, 400, 600 \rangle$, and
$\mathbf{r}_4(t) = (1 - t)\langle 0, -400, 0 \rangle + t\,\langle 0, 400, 0 \rangle$. The clipped line segment connects the points $(125, -350, 75)$ and $(650, -25, 210)$, so an equation for the segment is
$\mathbf{r}_5(t) = (1 - t)\langle 125, -350, 75 \rangle + t\,\langle 650, -25, 210 \rangle$.
The projection of the clipped segment connects the points



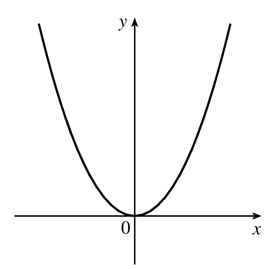
$\left(0, -400, \frac{600}{7}\right)$ and $\left(0, -\frac{500}{7}, 600\right)$, so an equation is $\mathbf{r}_6(t) = (1-t)\left\langle 0, -400, \frac{600}{7}\right\rangle + t\left\langle 0, -\frac{500}{7}, 600\right\rangle$.

The sight line on the left connects the points $(1000, 0, 0)$ and $\left(0, -400, \frac{600}{7}\right)$, so an equation is $\mathbf{r}_7(t) = (1-t)\langle 1000, 0, 0\rangle + t\left\langle 0, -400, \frac{600}{7}\right\rangle$. The other sight line connects $(1000, 0, 0)$ to $\left(0, -\frac{500}{7}, 600\right)$, so an equation is $\mathbf{r}_8(t) = (1-t)\langle 1000, 0, 0\rangle + t\left\langle 0, -\frac{500}{7}, 600\right\rangle$.

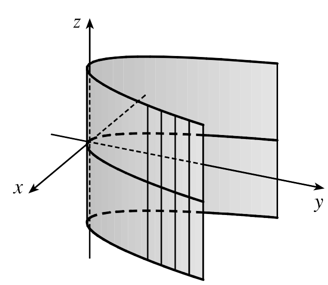
4. The vector from $(621, -147, 206)$ to $(563, 31, 242)$, $\mathbf{v}_1 = \langle -58, 178, 36\rangle$, lies in the plane of the rectangle, as does the vector from $(621, -147, 206)$ to $(657, -111, 86)$, $\mathbf{v}_2 = \langle 36, 36, -120\rangle$. A normal vector for the plane is $\mathbf{v}_1 \times \mathbf{v}_2 = \langle -1888, -142, -708\rangle$ or $\langle 8, 2, 3\rangle$, and an equation of the plane is $8x + 2y + 3z = 5292$. The line L intersects this plane when $8(230 + 630t) + 2(-285 + 390t) + 3(102 + 162t) = 5292 \quad \Rightarrow \quad t = \frac{1858}{3153} \approx 0.589$. The corresponding point is approximately $(601.25, -55.18, 197.46)$. Starting at this point, a portion of the line is hidden behind the rectangle. The line becomes visible again at the left edge of the rectangle, specifically the edge between the points $(621, -147, 206)$ and $(657, -111, 86)$. (This is most easily determined by graphing the rectangle and the line.) A plane through these two points and the camera's location, $(1000, 0, 0)$, will clip the line at the point it becomes visible. Two vectors in this plane are $\mathbf{v}_1 = \langle -379, -147, 206\rangle$ and $\mathbf{v}_2 = \langle -343, -111, 86\rangle$. A normal vector for the plane is $\mathbf{v}_1 \times \mathbf{v}_2 = \langle 10{,}224, -38{,}064, -8352\rangle$ and an equation of the plane is $213x - 793y - 174z = 213{,}000$. L intersects this plane when $213(230 + 630t) - 793(-285 + 390t) - 174(102 + 162t) = 213{,}000 \quad \Rightarrow \quad t = \frac{44{,}247}{203{,}268} \approx 0.2177$. The corresponding point is approximately $(367.14, -200.11, 137.26)$. Thus the portion of L that should be removed is the segment between the points $(601.25, -55.18, 197.46)$ and $(367.14, -200.11, 137.26)$.

12.6 Cylinders and Quadric Surfaces

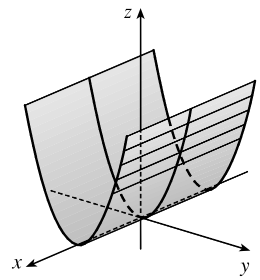
1. (a) In $\mathbb{R}^2$, the equation $y = x^2$ represents a parabola.



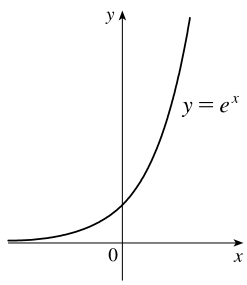
(b) In $\mathbb{R}^3$, the equation $y = x^2$ doesn't involve z, so any horizontal plane with equation $z = k$ intersects the graph in a curve with equation $y = x^2$. Thus, the surface is a parabolic cylinder, made up of infinitely many shifted copies of the same parabola. The rulings are parallel to the z-axis.



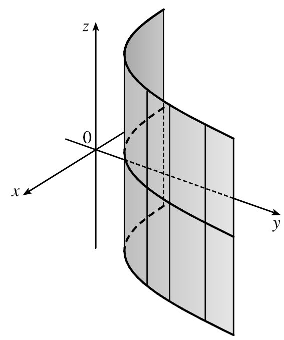
(c) In $\mathbb{R}^3$, the equation $z = y^2$ also represents a parabolic cylinder. Since x doesn't appear, the graph is formed by moving the parabola $z = y^2$ in the direction of the x-axis. Thus, the rulings of the cylinder are parallel to the x-axis.



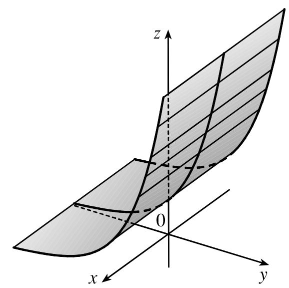
2. (a)



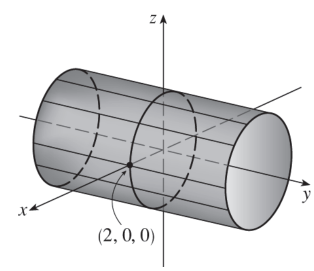
(b) Since the equation $y = e^x$ doesn't involve z, horizontal traces are copies of the curve $y = e^x$. The rulings are parallel to the z-axis.



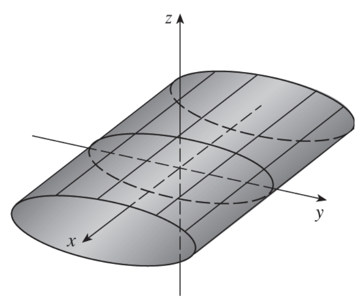
(c) The equation $z = e^y$ doesn't involve x, so vertical traces in $x = k$ (parallel to the yz-plane) are copies of the curve $z = e^y$. The rulings are parallel to the x-axis.



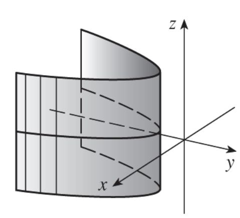
3. Since y is missing from the equation, the vertical traces $x^2 + z^2 = 4$, $y = k$ are copies of the same circle in the plane $y = k$. Thus, the surface $x^2 + z^2 = 4$ is a circular cylinder of radius 2 with rulings parallel to the y-axis.



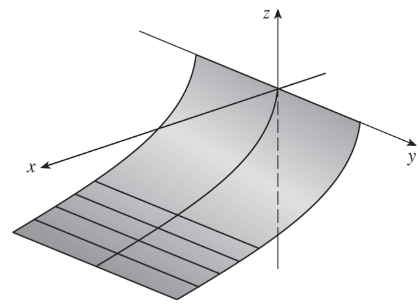
4. Since x is missing from the equation, the vertical traces $y^2 + 9z^2 = 9$, $x = k$ are copies of the same ellipse in the plane $x = k$. Thus, the surface $y^2 + 9z^2 = 9$ is an elliptic cylinder with rulings parallel to the x-axis.



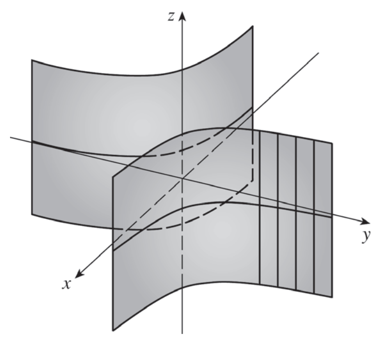
5. Since z is missing from the equation, the horizontal traces $x^2 + y + 1 = 0 \quad \Rightarrow \quad y = -x^2 - 1$, $z = k$ are copies of the same parabola in the plane $z = k$. Thus, the surface $x^2 + y + 1 = 0$ is a parabolic cylinder with rulings parallel to the z-axis.



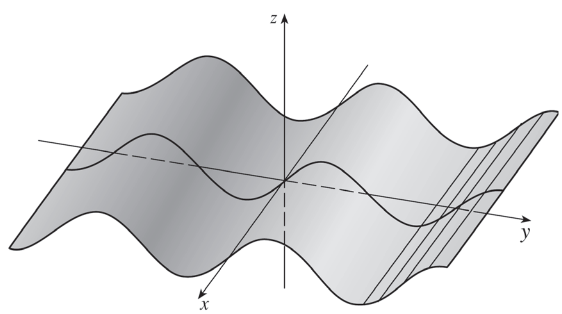
6. Since y is missing from the equation, the vertical traces $z = -\sqrt{x}$, $y = k$ are copies of the curve $z = -\sqrt{x}$ with rulings parallel to the y-axis.



7. Since z is missing, each horizontal trace $xy = 1$, $z = k$, is a copy of the same hyperbola in the plane $z = k$. Thus the surface $xy = 1$ is a hyperbolic cylinder with rulings parallel to the z-axis.



8. Since x is missing, each vertical trace $z = \sin y$, $x = k$, is a copy of a sine curve in the plane $x = k$. Thus the surface $z = \sin y$ is a cylindrical surface with rulings parallel to the x-axis.

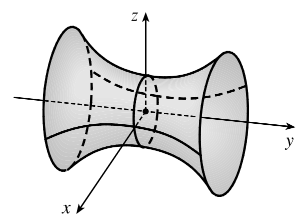


9. The trace in the xz-plane appears to be $z = \cos x$. The traces in the planes on the positive y-axis and negative y-axis are copies of the same graph. Therefore, an equation of the graph could be $z = \cos x$.

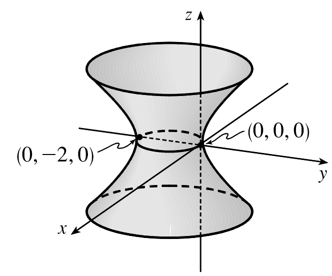
10. The trace in the yz-plane appears to be $z = y^3$. The traces in the planes on the positive x-axis and negative x-axis are copies of the same graph. Therefore, an equation of the graph could be $z = y^3$.

11. (a) The traces of $x^2 + y^2 - z^2 = 1$ in $x = k$ are $y^2 - z^2 = 1 - k^2$, a family of hyperbolas. (Note that the hyperbolas are oriented differently for $-1 < k < 1$ than for $k < -1$ or $k > 1$.) The traces in $y = k$ are $x^2 - z^2 = 1 - k^2$, a similar family of hyperbolas. The traces in $z = k$ are $x^2 + y^2 = 1 + k^2$, a family of circles. For $k = 0$, the trace in the xy-plane, the circle is of radius 1. As $|k|$ increases, so does the radius of the circle. This behavior, combined with the hyperbolic vertical traces, gives the graph of the hyperboloid of one sheet in Table 1.

(b) If we change the equation $x^2 + y^2 - z^2 = 1$ to $x^2 - y^2 + z^2 = 1$, the shape of the surface is unchanged, but the hyperboloid is rotated so that its axis is the y-axis. Traces in $y = k$ are circles, while traces in $x = k$ and $z = k$ are hyperbolas.

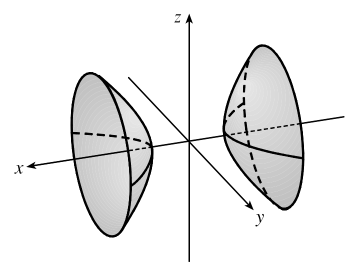


(c) Completing the square in y for $x^2 + y^2 + 2y - z^2 = 0$ gives $x^2 + (y+1)^2 - z^2 = 1$. The surface is a hyperboloid identical to the one in part (a) but shifted one unit in the negative y-direction.

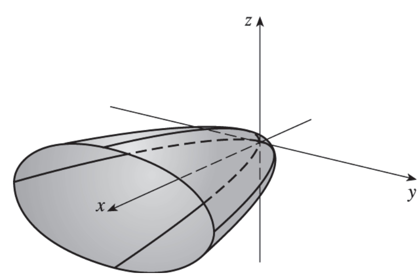


12. (a) The traces of $-x^2 - y^2 + z^2 = 1$ in $x = k$ are $-y^2 + z^2 = 1 + k^2$, a family of hyperbolas, as are the traces in $y = k$, $-x^2 + z^2 = 1 + k^2$. The traces in $z = k$ are $x^2 + y^2 = k^2 - 1$, a family of circles for $|k| > 1$. As $|k|$ increases, the radii of the circles increase; the traces are empty for $|k| < 1$. This behavior, combined with the vertical traces, gives the graph of the hyperboloid of two sheets in Table 1.

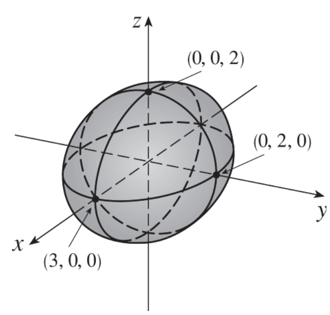
(b) If the equation in part (a) is changed to $x^2 - y^2 - z^2 = 1$, the graph has the same shape as the hyperboloid in part (a) but is rotated so that its axis is the x-axis. Traces in $x = k$, $|k| > 1$, are circles, while traces in $y = k$ and $z = k$ are hyperbolas.



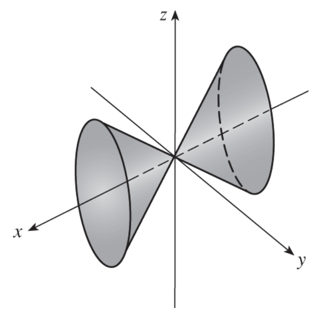
13. For $x = y^2 + 4z^2$, the traces in $x = k$ are $y^2 + 4z^2 = k$. When $k > 0$ we have a family of ellipses. When $k = 0$ we have just a point at the origin, and the trace is empty for $k < 0$. The traces in $y = k$ are $x = 4z^2 + k^2$, a family of parabolas opening in the positive x-direction. Similarly, the traces in $z = k$ are $x = y^2 + 4k^2$, a family of parabolas opening in the positive x-direction. We recognize the graph as an elliptic paraboloid with axis the x-axis and vertex the origin.



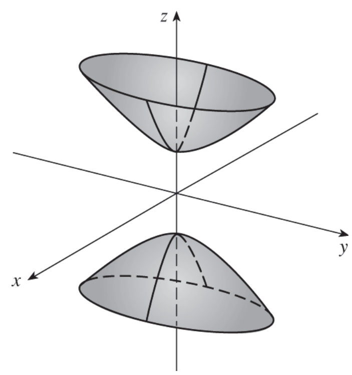
14. $4x^2 + 9y^2 + 9z^2 = 36$. The traces in $x = k$ are $9y^2 + 9z^2 = 36 - 4k^2 \quad \Leftrightarrow$ $y^2 + z^2 = 4 - \frac{4}{9}k^2$, a family of circles for $|k| < 3$. (The traces are a single point for $|k| = 3$ and are empty for $|k| > 3$.) The traces in $y = k$ are $4x^2 + 9z^2 = 36 - 9k^2$, a family of ellipses for $|k| < 2$. Similarly, the traces in $z = k$ are the ellipses $4x^2 + 9y^2 = 36 - 9k^2$, $|k| < 2$. The graph is an ellipsoid centered at the origin with intercepts $x = \pm 3$, $y = \pm 2$, $z = \pm 2$.



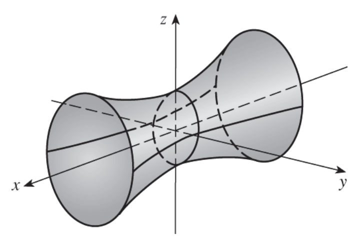
15. $x^2 = 4y^2 + z^2$. The traces in $x = k$ are the ellipses $4y^2 + z^2 = k^2$. The traces in $y = k$ are $x^2 - z^2 = 4k^2$, hyperbolas for $k \neq 0$ and two intersecting lines if $k = 0$. Similarly, the traces in $z = k$ are $x^2 - 4y^2 = k^2$, hyperbolas for $k \neq 0$ and two intersecting lines if $k = 0$. We recognize the graph as an elliptic cone with axis the x-axis and vertex the origin.



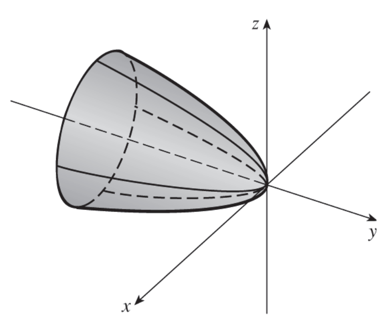
16. $z^2 - 4x^2 - y^2 = 4$. The traces in $x = k$ are the hyperbolas $z^2 - y^2 = 4 + 4k^2$, and the traces in $y = k$ are the hyperbolas $z^2 - 4x^2 = 4 + k^2$. The traces in $z = k$ are $4x^2 + y^2 = k^2 - 4$, a family of ellipses for $|k| > 2$. (The traces are a single point for $|k| = 2$ and are empty for $|k| < 2$.) The surface is a hyperboloid of two sheets with axis the z-axis.



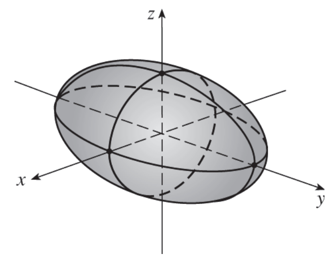
17. $9y^2 + 4z^2 = x^2 + 36$. The traces in $x = k$ are $9y^2 + 4z^2 = k^2 + 36$, a family of ellipses. The traces in $y = k$ are $4z^2 - x^2 = 9(4 - k^2)$, a family of hyperbolas for $|k| \neq 2$ and two intersecting lines when $|k| = 2$. (Note that the hyperbolas are oriented differently for $|k| < 2$ than for $|k| > 2$.) The traces in $z = k$ are $9y^2 - x^2 = 4(9 - k^2)$, a family of hyperbolas when $|k| \neq 3$ (oriented differently for $|k| < 3$ than for $|k| > 3$) and two intersecting lines when $|k| = 3$. We recognize the graph as a hyperboloid of one sheet with axis the x-axis.



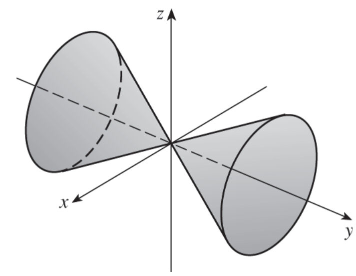
18. $3x^2 + y + 3z^2 = 0$. The traces in $x = k$ are the parabolas $y = -3z^2 - 3k^2$ which open to the left (in the negative y-direction). Traces in $y = k$ are $3x^2 + 3z^2 = -k \quad \Leftrightarrow \quad x^2 + z^2 = -\dfrac{k}{3}$, a family of circles for $k < 0$. (Traces are empty for $k > 0$ and a single point for $k = 0$.) Traces in $z = k$ are the parabolas $y = -3x^2 - 3k^2$ which open in the negative y-direction. The graph is a circular paraboloid with axis the y-axis, opening in the negative y-direction, and vertex the origin.



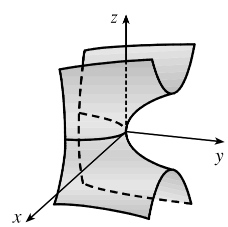
19. $\dfrac{x^2}{9}+\dfrac{y^2}{25}+\dfrac{z^2}{4}=1$. The traces in $x=k$ are $\dfrac{y^2}{25}+\dfrac{z^2}{4}=1-\dfrac{k^2}{9}$, a family of ellipses for $|k|<3$. (The traces are a single point for $|k|=3$ and are empty for $|k|>3$.) The traces in $y=k$ are the ellipses $\dfrac{x^2}{9}+\dfrac{z^2}{4}=1-\dfrac{k^2}{25}$, $|k|<5$, and the traces in $z=k$ are the ellipses $\dfrac{x^2}{9}+\dfrac{y^2}{25}=1-\dfrac{k^2}{4}$, $|k|<2$. The surface is an ellipsoid centered at the origin with intercepts $x=\pm3$, $y=\pm5$, $z=\pm2$.



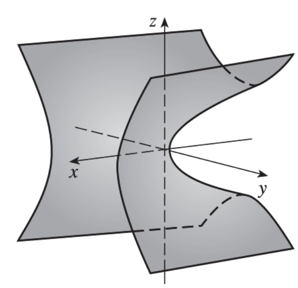
20. $3x^2-y^2+3z^2=0$. The traces in $x=k$ are $y^2-3z^2=3k^2$, a family of hyperbolas for $k\neq0$ and two intersecting lines if $k=0$. Traces in $y=k$ are the circles $3x^2+3z^2=k^2 \quad\Leftrightarrow\quad x^2+z^2=\frac{1}{3}k^2$. The traces in $z=k$ are $y^2-3x^2=3k^2$, hyperbolas for $k\neq0$ and two intersecting lines if $k=0$. We recognize the surface as a circular cone with axis the y-axis and vertex the origin.



21. $y=z^2-x^2$. The traces in $x=k$ are the parabolas $y=z^2-k^2$, opening in the positive y-direction. The traces in $y=k$ are $k=z^2-x^2$, two intersecting lines when $k=0$ and a family of hyperbolas for $k\neq0$ (note that the hyperbolas are oriented differently for $k>0$ than for $k<0$). The traces in $z=k$ are the parabolas $y=k^2-x^2$ which open in the negative y-direction. Thus the surface is a hyperbolic paraboloid centered at $(0,0,0)$.



22. $x=y^2-z^2$. The traces in $x=k$ are $y^2-z^2=k$, two intersecting lines when $k=0$ and a family of hyperbolas for $k\neq0$ (oriented differently for $k>0$ than for $k<0$). The traces in $y=k$ are the parabolas $x=-z^2+k^2$, opening in the negative x-direction, and the traces in $z=k$ are the parabolas $x=y^2-k^2$ which open in the positive x-direction. The graph is a hyperbolic paraboloid centered at $(0,0,0)$.



23. This is the equation of an ellipsoid: $x^2+4y^2+9z^2=x^2+\dfrac{y^2}{(1/2)^2}+\dfrac{z^2}{(1/3)^2}=1$, with x-intercepts ±1, y-intercepts $\pm\frac{1}{2}$ and z-intercepts $\pm\frac{1}{3}$. So the major axis is the x-axis and the only possible graph is VII.

24. This is the equation of an ellipsoid: $9x^2+4y^2+z^2=\dfrac{x^2}{(1/3)^2}+\dfrac{y^2}{(1/2)^2}+z^2=1$, with x-intercepts $\pm\frac{1}{3}$, y-intercepts $\pm\frac{1}{2}$ and z-intercepts ±1. So the major axis is the z-axis and the only possible graph is IV.

25. $x^2 - y^2 + z^2 = 1$ is the equation of a hyperboloid of one sheet, with $a = b = c = 1$. Since the coefficient of y^2 is negative, the axis of the hyperboloid is the y-axis. Hence, the correct graph is II.

26. $-x^2 + y^2 - z^2 = 1$ is the equation of a hyperboloid of two sheets, with $a = b = c = 1$. This surface does not intersect the xz-plane at all, so the axis of the hyperboloid is the y-axis. Hence, the correct graph is III.

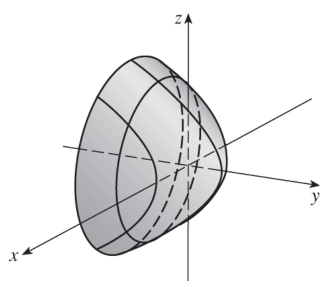
27. There are no real values of x and z that satisfy this equation, $y = 2x^2 + z^2$, for $y < 0$, so this surface does not extend to the left of the xz-plane. The surface intersects the plane $y = k > 0$ in an ellipse. Notice that y occurs to the first power whereas x and z occur to the second power. So the surface is an elliptic paraboloid with axis the y-axis. Its graph is VI.

28. $y^2 = x^2 + 2z^2$ is the equation of a cone with axis the y-axis. Its graph is I.

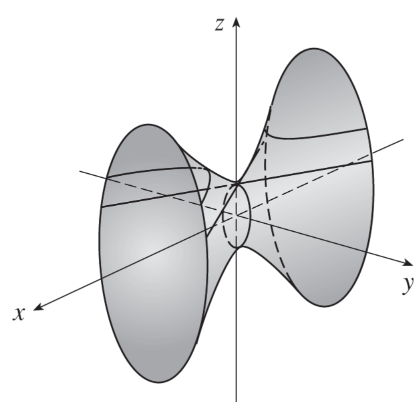
29. $x^2 + 2z^2 = 1$ is the equation of a cylinder because the variable y is missing from the equation. The intersection of the surface and the xz-plane is an ellipse. Its graph is VIII.

30. $y = x^2 - z^2$ is the equation of a hyperbolic paraboloid. The trace in the xy-plane is the parabola $y = x^2$. So the correct graph is V.

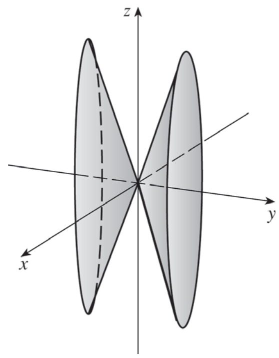
31. Vertical traces parallel to the xz-plane are circles centered at the origin whose radii increase as y decreases. (The trace in $y = 1$ is just a single point and the graph suggests that traces in $y = k$ are empty for $k > 1$.) The traces in vertical planes parallel to the yz-plane are parabolas opening to the left that shift to the left as $|x|$ increases. One surface that fits this description is a circular paraboloid, opening to the left, with vertex $(0, 1, 0)$.



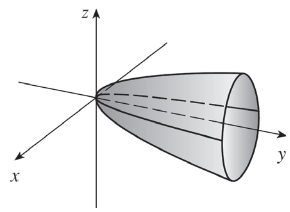
32. The vertical traces parallel to the yz-plane are ellipses that are smallest in the yz-plane and increase in size as $|x|$ increases. One surface that fits this description is a hyperboloid of one sheet with axis the x-axis. The horizontal traces in $z = k$ (hyperbolas and intersecting lines) also fit this surface, as shown in the figure.



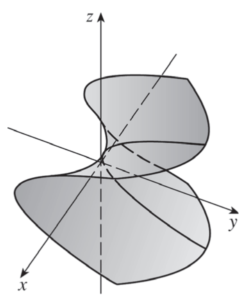
33. $y^2 = x^2 + \frac{1}{9}z^2$ or $y^2 = x^2 + \dfrac{z^2}{9}$ represents an elliptic cone with vertex $(0, 0, 0)$ and axis the y-axis.



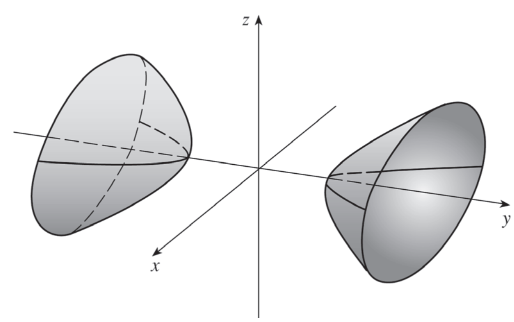
34. $4x^2 - y + 2z^2 = 0$ or $y = \dfrac{x^2}{1/4} + \dfrac{z^2}{1/2}$ or $\dfrac{y}{4} = x^2 + \dfrac{z^2}{2}$ represents an elliptic paraboloid with vertex $(0, 0, 0)$ and axis the y-axis.



35. $x^2 + 2y - 2z^2 = 0$ or $2y = 2z^2 - x^2$ or $y = z^2 - \dfrac{x^2}{2}$ represents a hyperbolic paraboloid with center $(0, 0, 0)$.



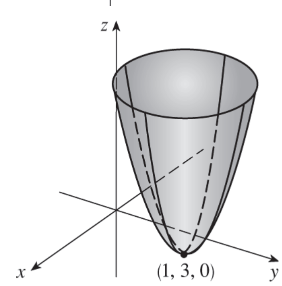
36. $y^2 = x^2 + 4z^2 + 4$ or $-x^2 + y^2 - 4z^2 = 4$ or $-\dfrac{x^2}{4} + \dfrac{y^2}{4} - z^2 = 1$ represents a hyperboloid of two sheets with axis the y-axis.



37. Completing squares in x and y gives

$\left(x^2 - 2x + 1\right) + \left(y^2 - 6y + 9\right) - z = 0 \quad \Leftrightarrow$

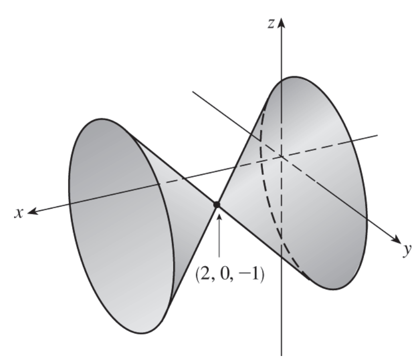
$(x-1)^2 + (y-3)^2 - z = 0$ or $z = (x-1)^2 + (y-3)^2$, a circular paraboloid opening upward with vertex $(1, 3, 0)$ and axis the vertical line $x = 1$, $y = 3$.



38. Completing squares in x and z gives

$\left(x^2 - 4x + 4\right) - y^2 - \left(z^2 + 2z + 1\right) + 3 = 0 + 4 - 1 \quad \Leftrightarrow$

$(x-2)^2 - y^2 - (z+1)^2 = 0$ or $(x-2)^2 = y^2 + (z+1)^2$, a circular cone with vertex $(2, 0, -1)$ and axis the horizontal line $y = 0$, $z = -1$.



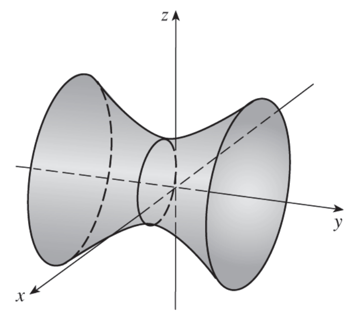
39. Completing squares in x and z gives

$\left(x^2-4x+4\right)-y^2+\left(z^2-2z+1\right)=0+4+1 \quad \Leftrightarrow$

$(x-2)^2-y^2+(z-1)^2=5$ or $\dfrac{(x-2)^2}{5}-\dfrac{y^2}{5}+\dfrac{(z-1)^2}{5}=1$, a

hyperboloid of one sheet with center $(2,0,1)$ and axis the horizontal line

$x=2$, $z=1$.



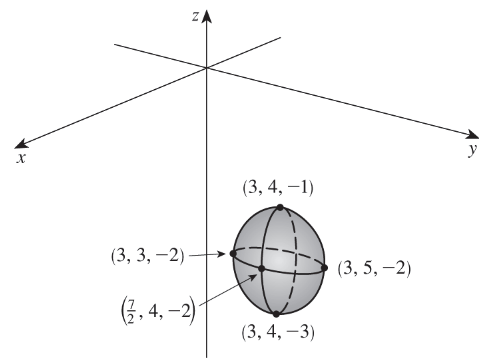
40. Completing squares in all three variables gives

$4(x^2-6x+9)+(y^2-8y+16)+(z^2+4z+4)=-55+36+16+4 \quad \Leftrightarrow$

$4\,(x-3)^2+(y-4)^2+(z+2)^2=1$ or

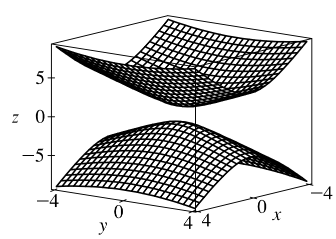
$\dfrac{(x-3)^2}{1/4}+(y-4)^2+(z+2)^2=1$, an ellipsoid with

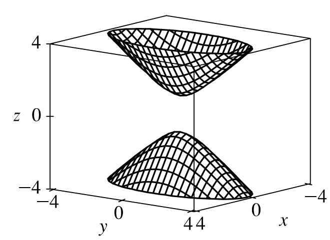
center $(3,4,-2)$.



41. Solving the equation for z we get $z=\pm\sqrt{1+4x^2+y^2}$, so we plot separately $z=\sqrt{1+4x^2+y^2}$ and

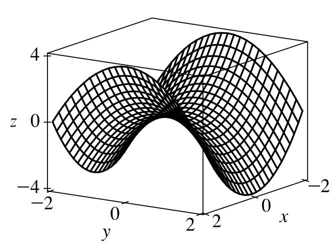
$z=-\sqrt{1+4x^2+y^2}$.



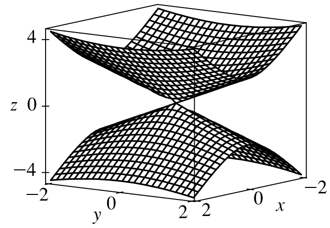


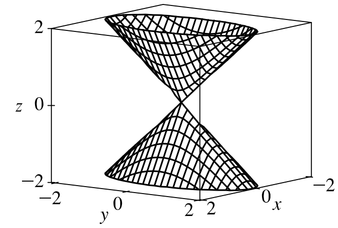
To restrict the z-range as in the second graph, we can use the option `view=-4..4` in Maple's `plot3d` command, or `PlotRange->{-4,4}` in Mathematica's `Plot3D` command.

42. We plot the surface $z=x^2-y^2$.

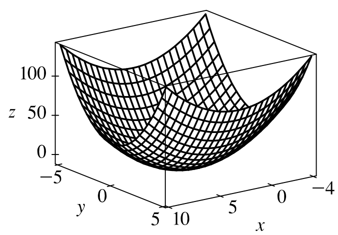


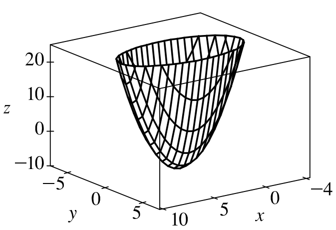
43. Solving the equation for z we get $z = \pm\sqrt{4x^2 + y^2}$, so we plot separately $z = \sqrt{4x^2 + y^2}$ and $z = -\sqrt{4x^2 + y^2}$.



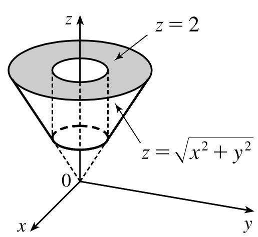


44. We plot the surface $z = x^2 - 6x + 4y^2$.

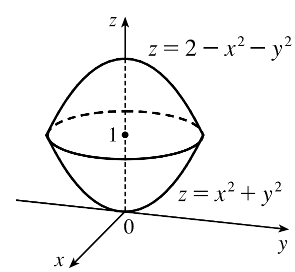




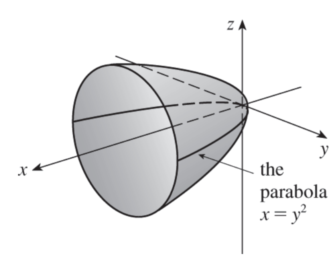
45.



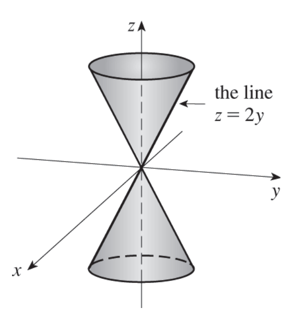
46.



47. The curve $y = \sqrt{x}$ is equivalent to $x = y^2$, $y \geq 0$. Rotating the curve about the x-axis creates a circular paraboloid with vertex at the origin, axis the x-axis, opening in the positive x-direction. The trace in the xy-plane is $x = y^2$, $z = 0$, and the trace in the xz-plane is a parabola of the same shape: $x = z^2$, $y = 0$. An equation for the surface is $x = y^2 + z^2$.



48. Rotating the line $z = 2y$ about the z-axis creates a (right) circular cone with vertex at the origin and axis the z-axis. Traces in $z = k$ ($k \neq 0$) are circles with center $(0, 0, k)$ and radius $y = z/2 = k/2$, so an equation for the trace is $x^2 + y^2 = (k/2)^2$, $z = k$. Thus, an equation for the surface is $x^2 + y^2 = (z/2)^2$ or $4x^2 + 4y^2 = z^2$.



49. Let $P = (x, y, z)$ be an arbitrary point equidistant from $(-1, 0, 0)$ and the plane $x = 1$. Then the distance from P to $(-1, 0, 0)$ is $\sqrt{(x+1)^2 + y^2 + z^2}$ and the distance from P to the plane $x = 1$ is $|x - 1| / \sqrt{1^2} = |x - 1|$ (by Equation 12.5.9). So $|x - 1| = \sqrt{(x+1)^2 + y^2 + z^2} \iff (x-1)^2 = (x+1)^2 + y^2 + z^2 \iff$ $x^2 - 2x + 1 = x^2 + 2x + 1 + y^2 + z^2 \iff -4x = y^2 + z^2$. Thus, the collection of all such points P is a circular paraboloid with vertex at the origin, axis the x-axis, which opens in the negative x-direction.

50. Let $P = (x, y, z)$ be an arbitrary point whose distance from the x-axis is twice its distance from the yz-plane. The distance from P to the x-axis is $\sqrt{(x-x)^2 + y^2 + z^2} = \sqrt{y^2 + z^2}$ and the distance from P to the yz-plane ($x = 0$) is $|x|/1 = |x|$. Thus $\sqrt{y^2 + z^2} = 2|x| \iff y^2 + z^2 = 4x^2 \iff x^2 = (y^2/2^2) + (z^2/2^2)$. So the surface is a right circular cone with vertex the origin and axis the x-axis.

51. (a) An equation for an ellipsoid centered at the origin with intercepts $x = \pm a$, $y = \pm b$, and $z = \pm c$ is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.

Here the poles of the model intersect the z-axis at $z = \pm 6356.523$ and the equator intersects the x- and y-axes at $x = \pm 6378.137$, $y = \pm 6378.137$, so an equation is

$$\frac{x^2}{(6378.137)^2} + \frac{y^2}{(6378.137)^2} + \frac{z^2}{(6356.523)^2} = 1$$

(b) Traces in $z = k$ are the circles $\dfrac{x^2}{(6378.137)^2} + \dfrac{y^2}{(6378.137)^2} = 1 - \dfrac{k^2}{(6356.523)^2} \iff$

$x^2 + y^2 = (6378.137)^2 - \left(\dfrac{6378.137}{6356.523}\right)^2 k^2$.

(c) To identify the traces in $y = mx$ we substitute $y = mx$ into the equation of the ellipsoid:

$$\begin{aligned}\frac{x^2}{(6378.137)^2} + \frac{(mx)^2}{(6378.137)^2} + \frac{z^2}{(6356.523)^2} &= 1 \\ \frac{(1+m^2)x^2}{(6378.137)^2} + \frac{z^2}{(6356.523)^2} &= 1 \\ \frac{x^2}{(6378.137)^2/(1+m^2)} + \frac{z^2}{(6356.523)^2} &= 1\end{aligned}$$

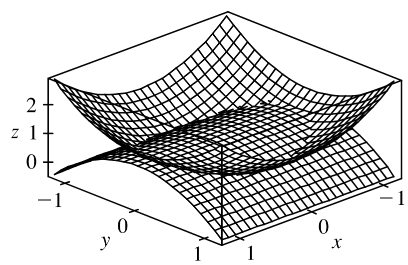
As expected, this is a family of ellipses.

52. If we position the hyperboloid on coordinate axes so that it is centered at the origin with axis the z-axis then its equation is given by $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$. Horizontal traces in $z = k$ are $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 + \dfrac{k^2}{c^2}$, a family of ellipses, but we know that the traces are circles so we must have $a = b$. The trace in $z = 0$ is $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2} = 1 \iff x^2 + y^2 = a^2$ and since the minimum radius of 100 m occurs there, we must have $a = 100$. The base of the tower is the trace in $z = -500$ given by $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2} = 1 + \dfrac{(-500)^2}{c^2}$ but $a = 100$ so the trace is $x^2 + y^2 = 100^2 + 50{,}000^2 \dfrac{1}{c^2}$. We know the base is a circle of radius 140, so we must have $100^2 + 50{,}000^2 \frac{1}{c^2} = 140^2 \quad \Rightarrow \quad c^2 = \frac{50{,}000^2}{140^2 - 100^2} = \frac{781{,}250}{3}$ and an equation for the tower is $\frac{x^2}{100^2} + \frac{y^2}{100^2} - \frac{z^2}{(781{,}250)/3} = 1$ or $\frac{x^2}{10{,}000} + \frac{y^2}{10{,}000} - \frac{3z^2}{781{,}250} = 1$, $-500 \leq z \leq 500$.

53. If (a, b, c) satisfies $z = y^2 - x^2$, then $c = b^2 - a^2$. L_1: $x = a + t$, $y = b + t$, $z = c + 2(b - a)t$, L_2: $x = a + t$, $y = b - t$, $z = c - 2(b + a)t$. Substitute the parametric equations of L_1 into the equation of the hyperbolic paraboloid in order to find the points of intersection: $z = y^2 - x^2 \quad \Rightarrow$
$c + 2(b - a)t = (b + t)^2 - (a + t)^2 = b^2 - a^2 + 2(b - a)t \quad \Rightarrow \quad c = b^2 - a^2$. As this is true for all values of t, L_1 lies on $z = y^2 - x^2$. Performing similar operations with L_2 gives: $z = y^2 - x^2 \quad \Rightarrow$
$c - 2(b + a)t = (b - t)^2 - (a + t)^2 = b^2 - a^2 - 2(b + a)t \quad \Rightarrow \quad c = b^2 - a^2$. This tells us that all of L_2 also lies on $z = y^2 - x^2$.

54. Any point on the curve of intersection must satisfy both $2x^2 + 4y^2 - 2z^2 + 6x = 2$ and $2x^2 + 4y^2 - 2z^2 - 5y = 0$. Subtracting, we get $6x + 5y = 2$, which is linear and therefore the equation of a plane. Thus the curve of intersection lies in this plane.

55.



The curve of intersection looks like a bent ellipse. The projection of this curve onto the xy-plane is the set of points $(x, y, 0)$ which satisfy $x^2 + y^2 = 1 - y^2 \quad \Leftrightarrow \quad x^2 + 2y^2 = 1 \quad \Leftrightarrow$
$x^2 + \frac{y^2}{\left(1/\sqrt{2}\right)^2} = 1$. This is an equation of an ellipse.

12 Review

TRUE-FALSE QUIZ

1. This is false, as the dot product of two vectors is a scalar, not a vector.

2. False. For example, if $\mathbf{u} = \mathbf{i}$ and $\mathbf{v} = -\mathbf{i}$ then $|\mathbf{u} + \mathbf{v}| = |\mathbf{0}| = 0$ but $|\mathbf{u}| + |\mathbf{v}| = 1 + 1 = 2$.

3. False. For example, if $\mathbf{u} = \mathbf{i}$ and $\mathbf{v} = \mathbf{j}$ then $|\mathbf{u} \cdot \mathbf{v}| = |0| = 0$ but $|\mathbf{u}|\,|\mathbf{v}| = 1 \cdot 1 = 1$. In fact, by Theorem 12.3.3, $|\mathbf{u} \cdot \mathbf{v}| = \big||\mathbf{u}|\,|\mathbf{v}| \cos\theta\big|$.

4. False. For example, $|\mathbf{i} \times \mathbf{i}| = |\mathbf{0}| = 0$ (see Example 12.4.2) but $|\mathbf{i}|\,|\mathbf{i}| = 1 \cdot 1 = 1$. In fact, by Theorem 12.4.9, $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| \sin\theta$.

5. True, by Theorem 12.3.2, property 2.

6. False. Property 1 of Theorem 12.4.11 says that $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$.

7. True. If θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then by Theorem 12.4.9, $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\sin\theta = |\mathbf{v}|\,|\mathbf{u}|\sin\theta = |\mathbf{v}\times\mathbf{u}|$.

(Or, by Theorem 12.4.11, $|\mathbf{u}\times\mathbf{v}| = |-\mathbf{v}\times\mathbf{u}| = |-1|\,|\mathbf{v}\times\mathbf{u}| = |\mathbf{v}\times\mathbf{u}|$.)

8. This is true by Theorem 12.3.2, property 4.

9. Theorem 12.4.11, property 2 tells us that this is true.

10. This is true by Theorem 12.4.11, property 4.

11. This is true by Theorem 12.4.11, property 5.

12. In general, this assertion is false; a counterexample is $\mathbf{i}\times(\mathbf{i}\times\mathbf{j}) \neq (\mathbf{i}\times\mathbf{i})\times\mathbf{j}$. (See the paragraph preceding Theorem 12.4.11.)

13. This is true because $\mathbf{u}\times\mathbf{v}$ is orthogonal to $\mathbf{u}$ (see Theorem 12.4.8), and the dot product of two orthogonal vectors is 0.

14.
$$\begin{aligned}(\mathbf{u}+\mathbf{v})\times\mathbf{v} &= \mathbf{u}\times\mathbf{v}+\mathbf{v}\times\mathbf{v} && \text{[by Theorem 12.4.11, property 4]}\\ &= \mathbf{u}\times\mathbf{v}+\mathbf{0} && \text{[by Example 12.4.2]}\\ &= \mathbf{u}\times\mathbf{v}\text{, so this is true.}\end{aligned}$$

15. This is false. A normal vector to the plane is $\mathbf{n} = \langle 6,-2,4\rangle$. Because $\langle 3,-1,2\rangle = \frac{1}{2}\mathbf{n}$, the vector is parallel to $\mathbf{n}$ and hence perpendicular to the plane.

16. This is false, because according to Equation 12.5.8, $ax+by+cz+d=0$ is the general equation of a plane.

17. This is false. In $\mathbb{R}^2$, $x^2+y^2=1$ represents a circle, but $\{(x,y,z)\mid x^2+y^2=1\}$ represents a *three-dimensional surface*, namely, a circular cylinder with axis the z-axis.

18. This is false. In $\mathbb{R}^3$ the graph of $y=x^2$ is a parabolic cylinder (see Example 12.6.1). A paraboloid has an equation such as $z=x^2+y^2$.

19. False. For example, $\mathbf{i}\cdot\mathbf{j}=\mathbf{0}$ but $\mathbf{i}\neq\mathbf{0}$ and $\mathbf{j}\neq\mathbf{0}$.

20. This is false. By Corollary 12.4.10, $\mathbf{u}\times\mathbf{v}=\mathbf{0}$ for any nonzero parallel vectors $\mathbf{u}$, $\mathbf{v}$. For instance, $\mathbf{i}\times\mathbf{i}=\mathbf{0}$.

21. This is true. If $\mathbf{u}$ and $\mathbf{v}$ are both nonzero, then by (7) in Section 12.3, $\mathbf{u}\cdot\mathbf{v}=0$ implies that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal. But $\mathbf{u}\times\mathbf{v}=\mathbf{0}$ implies that $\mathbf{u}$ and $\mathbf{v}$ are parallel (see Corollary 12.4.10). Two nonzero vectors can't be both parallel and orthogonal, so at least one of $\mathbf{u}$, $\mathbf{v}$ must be $\mathbf{0}$.

22. This is true. We know $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}|\;|\mathbf{v}|\;\cos\theta$ where $|\mathbf{u}|\geq 0$, $|\mathbf{v}|\geq 0$, and $|\cos\theta|\leq 1$, so $|\mathbf{u}\cdot\mathbf{v}| = |\mathbf{u}|\;|\mathbf{v}|\;|\cos\theta| \leq |\mathbf{u}|\;|\mathbf{v}|$.

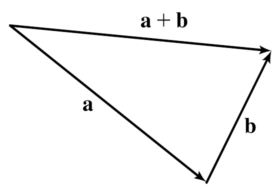
EXERCISES

1. (a) The radius of the sphere is the distance between the points $(-1, 2, 1)$ and $(6, -2, 3)$, namely, $\sqrt{[6-(-1)]^2 + (-2-2)^2 + (3-1)^2} = \sqrt{69}$. By the formula for an equation of a sphere (following Example 12.1.4), an equation of the sphere with center $(-1, 2, 1)$ and radius $\sqrt{69}$ is $(x+1)^2 + (y-2)^2 + (z-1)^2 = 69$.

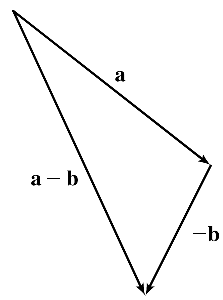
(b) The intersection of this sphere with the yz-plane is the set of points on the sphere whose x-coordinate is 0. Putting $x = 0$ into the equation, we have $(y-2)^2 + (z-1)^2 = 68, x = 0$ which represents a circle in the yz-plane with center $(0, 2, 1)$ and radius $\sqrt{68}$.

(c) Completing squares gives $(x-4)^2 + (y+1)^2 + (z+3)^2 = -1 + 16 + 1 + 9 = 25$. Thus the sphere is centered at $(4, -1, -3)$ and has radius 5.

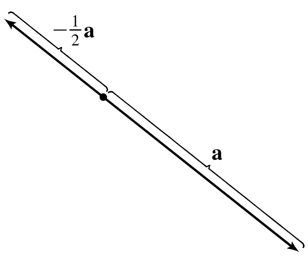
2. (a)



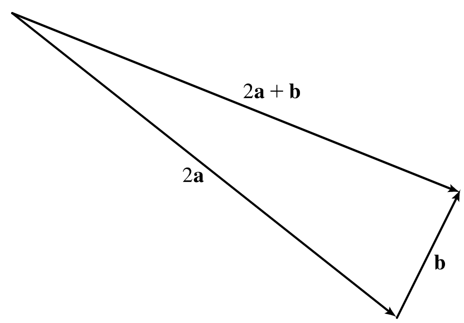
(b)



(c)



(d)



3. $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos 45^\circ = (2)(3)\frac{\sqrt{2}}{2} = 3\sqrt{2}$. $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| \sin 45^\circ = (2)(3)\frac{\sqrt{2}}{2} = 3\sqrt{2}$.

By the right-hand rule, $\mathbf{u} \times \mathbf{v}$ is directed out of the page.

4. (a) $2\,\mathbf{a} + 3\,\mathbf{b} = 2\,\mathbf{i} + 2\,\mathbf{j} - 4\,\mathbf{k} + 9\,\mathbf{i} - 6\,\mathbf{j} + 3\,\mathbf{k} = 11\,\mathbf{i} - 4\,\mathbf{j} - \mathbf{k}$

(b) $|\mathbf{b}| = \sqrt{9+4+1} = \sqrt{14}$

(c) $\mathbf{a} \cdot \mathbf{b} = (1)(3) + (1)(-2) + (-2)(1) = -1$

(d) $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -2 \\ 3 & -2 & 1 \end{vmatrix} = (1-4)\,\mathbf{i} - (1+6)\,\mathbf{j} + (-2-3)\,\mathbf{k} = -3\,\mathbf{i} - 7\,\mathbf{j} - 5\,\mathbf{k}$

(e) $\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ 0 & 1 & -5 \end{vmatrix} = 9\,\mathbf{i} + 15\,\mathbf{j} + 3\,\mathbf{k}$, $\quad |\mathbf{b} \times \mathbf{c}| = 3\sqrt{9+25+1} = 3\sqrt{35}$

(f) $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} 1 & 1 & -2 \\ 3 & -2 & 1 \\ 0 & 1 & -5 \end{vmatrix} = \begin{vmatrix} -2 & 1 \\ 1 & -5 \end{vmatrix} - \begin{vmatrix} 3 & 1 \\ 0 & -5 \end{vmatrix} - 2\begin{vmatrix} 3 & -2 \\ 0 & 1 \end{vmatrix} = 9 + 15 - 6 = 18$

(g) $\mathbf{c}\times\mathbf{c} = \mathbf{0}$ for any $\mathbf{c}$.

(h) From part (e),

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{a}\times(9\,\mathbf{i} + 15\,\mathbf{j} + 3\,\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -2 \\ 9 & 15 & 3 \end{vmatrix}$$

$$= (3+30)\,\mathbf{i} - (3+18)\,\mathbf{j} + (15-9)\,\mathbf{k} = 33\,\mathbf{i} - 21\,\mathbf{j} + 6\,\mathbf{k}$$

(i) The scalar projection is $\operatorname{comp}_{\mathbf{a}}\mathbf{b} = |\mathbf{b}|\cos\theta = \mathbf{a}\cdot\mathbf{b}/\,|\mathbf{a}| = -\frac{1}{\sqrt{6}}$.

(j) The vector projection is $\operatorname{proj}_{\mathbf{a}}\mathbf{b} = -\dfrac{1}{\sqrt{6}}\left(\dfrac{\mathbf{a}}{|\mathbf{a}|}\right) = -\frac{1}{6}(\mathbf{i} + \mathbf{j} - 2\,\mathbf{k})$.

(k) $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{-1}{\sqrt{6}\,\sqrt{14}} = \dfrac{-1}{2\,\sqrt{21}}$ and $\theta = \cos^{-1}\left(\dfrac{-1}{2\,\sqrt{21}}\right) \approx 96^\circ$.

5. For the two vectors to be orthogonal, we need $\langle 3, 2, x\rangle \cdot \langle 2x, 4, x\rangle = 0 \quad\Leftrightarrow\quad (3)(2x) + (2)(4) + (x)(x) = 0 \quad\Leftrightarrow$

$x^2 + 6x + 8 = 0 \quad\Leftrightarrow\quad (x+2)(x+4) = 0 \quad\Leftrightarrow\quad x = -2$ or $x = -4$.

6. We know that the cross product of two vectors is orthogonal to both given vectors. So we calculate

$(\mathbf{j} + 2\,\mathbf{k}) \times (\mathbf{i} - 2\,\mathbf{j} + 3\,\mathbf{k}) = [3 - (-4)]\,\mathbf{i} - (0-2)\,\mathbf{j} + (0-1)\,\mathbf{k} = 7\,\mathbf{i} + 2\,\mathbf{j} - \mathbf{k}$.

Then two unit vectors orthogonal to both given vectors are $\pm\dfrac{7\,\mathbf{i} + 2\,\mathbf{j} - \mathbf{k}}{\sqrt{7^2 + 2^2 + (-1)^2}} = \pm\dfrac{1}{3\,\sqrt{6}}\,(7\,\mathbf{i} + 2\,\mathbf{j} - \mathbf{k})$,

that is, $\frac{7}{3\sqrt{6}}\,\mathbf{i} + \frac{2}{3\sqrt{6}}\,\mathbf{j} - \frac{1}{3\sqrt{6}}\,\mathbf{k}$ and $-\frac{7}{3\sqrt{6}}\,\mathbf{i} - \frac{2}{3\sqrt{6}}\,\mathbf{j} + \frac{1}{3\sqrt{6}}\,\mathbf{k}$.

7. (a) $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = 2$

(b) $\mathbf{u}\cdot(\mathbf{w}\times\mathbf{v}) = \mathbf{u}\cdot[-(\mathbf{v}\times\mathbf{w})] = -\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = -2$

(c) $\mathbf{v}\cdot(\mathbf{u}\times\mathbf{w}) = (\mathbf{v}\times\mathbf{u})\cdot\mathbf{w} = -(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = -2$

(d) $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{v} = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{v}) = \mathbf{u}\cdot\mathbf{0} = 0$

8. $(\mathbf{a}\times\mathbf{b})\cdot[(\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a})] = (\mathbf{a}\times\mathbf{b})\cdot([(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}]\,\mathbf{c} - [(\mathbf{b}\times\mathbf{c})\cdot\mathbf{c}]\,\mathbf{a})$

[by Property 6 of the cross product]

$$= (\mathbf{a}\times\mathbf{b})\cdot[(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}]\,\mathbf{c} = [\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})]\,(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$$

$$= [\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})]\,[\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})] = [\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})]^2$$

9. For simplicity, consider a unit cube positioned with its back left corner at the origin. Vector representations of the diagonals joining the points $(0,0,0)$ to $(1,1,1)$ and $(1,0,0)$ to $(0,1,1)$ are $\langle 1,1,1\rangle$ and $\langle -1,1,1\rangle$. Let θ be the angle between these two vectors. $\langle 1,1,1\rangle\cdot\langle -1,1,1\rangle = -1+1+1 = 1 = |\langle 1,1,1\rangle|\,|\langle -1,1,1\rangle|\cos\theta = 3\cos\theta \quad\Rightarrow\quad \cos\theta = \frac{1}{3} \quad\Rightarrow$

$\theta = \cos^{-1}\left(\frac{1}{3}\right) \approx 71^\circ$.

10. $\overrightarrow{AB} = \langle 1, 3, -1 \rangle$, $\overrightarrow{AC} = \langle -2, 1, 3 \rangle$ and $\overrightarrow{AD} = \langle -1, 3, 1 \rangle$. By Equation 12.4.13,

$$\overrightarrow{AB} \cdot \left(\overrightarrow{AC} \times \overrightarrow{AD}\right) = \begin{vmatrix} 1 & 3 & -1 \\ -2 & 1 & 3 \\ -1 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} - 3 \begin{vmatrix} -2 & 3 \\ -1 & 1 \end{vmatrix} - \begin{vmatrix} -2 & 1 \\ -1 & 3 \end{vmatrix} = -8 - 3 + 5 = -6.$$

The volume is $\left|\overrightarrow{AB} \cdot \left(\overrightarrow{AC} \times \overrightarrow{AD}\right)\right| = 6$ cubic units.

11. $\overrightarrow{AB} = \langle 1, 0, -1 \rangle$, $\overrightarrow{AC} = \langle 0, 4, 3 \rangle$, so

(a) a vector perpendicular to the plane is $\overrightarrow{AB} \times \overrightarrow{AC} = \langle 0 + 4, -(3 + 0), 4 - 0 \rangle = \langle 4, -3, 4 \rangle$.

(b) $\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\sqrt{16 + 9 + 16} = \frac{\sqrt{41}}{2}$.

12. $\mathbf{D} = (5 - 1)\,\mathbf{i} + (3 - 0)\,\mathbf{j} + (8 - 2)\,\mathbf{k} = 4\,\mathbf{i} + 3\,\mathbf{j} + 6\,\mathbf{k}$.
$W = \mathbf{F} \cdot \mathbf{D} = (3\,\mathbf{i} + 5\,\mathbf{j} + 10\,\mathbf{k}) \cdot (4\,\mathbf{i} + 3\,\mathbf{j} + 6\,\mathbf{k}) = (3)(4) + (5)(3) + (10)(6) = 12 + 15 + 60 = 87$ J

13. Let F_1 be the magnitude of the force directed 20° away from the direction of shore, and let F_2 be the magnitude of the other force. Separating these forces into components parallel to the direction of the resultant force and perpendicular to it gives $F_1 \cos 20^\circ + F_2 \cos 30^\circ = 255$ **(1)**, and $F_1 \sin 20^\circ - F_2 \sin 30^\circ = 0 \Rightarrow F_1 = F_2 \dfrac{\sin 30^\circ}{\sin 20^\circ}$ **(2)**. Substituting **(2)** into **(1)** gives $F_2(\sin 30^\circ \cot 20^\circ + \cos 30^\circ) = 255 \Rightarrow F_2 \approx 114$ N. Substituting this into **(2)** gives $F_1 \approx 166$ N.

14. $|\boldsymbol{\tau}| = |\mathbf{r}|\,|\mathbf{F}| \sin\theta = (0.40)(50)\sin(90^\circ - 30^\circ) \approx 17.3$ N·m.

15. The line has direction $\mathbf{v} = \langle -3, 2, 3 \rangle$. Letting $P_0 = (4, -1, 2)$, parametric equations are
$x = 4 - 3t,\; y = -1 + 2t,\; z = 2 + 3t$.

16. The line $\frac{1}{3}(x - 4) = \frac{1}{2}y = z + 2$, or $\dfrac{x - 4}{3} = \dfrac{y}{2} = \dfrac{z + 2}{1}$, has direction vector $\mathbf{v} = \langle 3, 2, 1 \rangle$ (or a nonzero scalar multiple). So parametric equations for the line through $(1, 0, -1)$ are $x = 1 + 3t,\; y = 2t,\; z = -1 + t$.

17. A direction vector for the line is a normal vector for the plane, $\mathbf{n} = \langle 2, -1, 5 \rangle$, and parametric equations for the line are
$x = -2 + 2t,\; y = 2 - t,\; z = 4 + 5t$.

18. Since the two planes are parallel, they will have the same normal vectors. Then we can take $\mathbf{n} = \langle 1, 4, -3 \rangle$ and an equation of the plane is $1(x - 2) + 4(y - 1) - 3(z - 0) = 0$ or $x + 4y - 3z = 6$.

19. Here the vectors $\mathbf{a} = \langle 4 - 3, 0 - (-1), 2 - 1 \rangle = \langle 1, 1, 1 \rangle$ and $\mathbf{b} = \langle 6 - 3, 3 - (-1), 1 - 1 \rangle = \langle 3, 4, 0 \rangle$ lie in the plane, so $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle -4, 3, 1 \rangle$ is a normal vector to the plane and an equation of the plane is
$-4(x - 3) + 3(y - (-1)) + 1(z - 1) = 0$ or $-4x + 3y + z = -14$.

20. If we first find two nonparallel vectors in the plane, their cross product will be a normal vector to the plane. Since the given line lies in the plane, its direction vector $\mathbf{a} = \langle 2, -1, 3 \rangle$ is one vector in the plane. We can verify that the given point $(1, 2, -2)$ does not lie on this line. The point $(0, 3, 1)$ is on the line (obtained by putting $t = 0$) and hence in the plane, so the vector $\mathbf{b} = \langle 0 - 1, 3 - 2, 1 - (-2) \rangle = \langle -1, 1, 3 \rangle$ lies in the plane, and a normal vector is $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle -6, -9, 1 \rangle$. Thus an equation of the plane is $-6(x - 1) - 9(y - 2) + (z + 2) = 0$ or $6x + 9y - z = 26$.

21. Substitution of the parametric equations into the equation of the plane gives $2x - y + z = 2(2-t) - (1+3t) + 4t = 2 \quad \Rightarrow$ $-t + 3 = 2 \quad \Rightarrow \quad t = 1$. When $t = 1$, the parametric equations give $x = 2 - 1 = 1$, $y = 1 + 3 = 4$ and $z = 4$. Therefore, the point of intersection is $(1, 4, 4)$.

22. Use the formula proven in Exercise 12.4.45(a). In the notation used in that exercise, $\mathbf{a}$ is just the direction of the line; that is, $\mathbf{a} = \langle 1, -1, 2 \rangle$. A point on the line is $(1, 2, -1)$ (setting $t = 0$), and therefore $\mathbf{b} = \langle 1 - 0, 2 - 0, -1 - 0 \rangle = \langle 1, 2, -1 \rangle$.

Hence $d = \dfrac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|} = \dfrac{|\langle 1, -1, 2 \rangle \times \langle 1, 2, -1 \rangle|}{\sqrt{1+1+4}} = \dfrac{|\langle -3, 3, 3 \rangle|}{\sqrt{6}} = \sqrt{\dfrac{27}{6}} = \dfrac{3}{\sqrt{2}}$.

23. Since the direction vectors $\langle 2, 3, 4 \rangle$ and $\langle 6, -1, 2 \rangle$ aren't parallel, neither are the lines. For the lines to intersect, the three equations $1 + 2t = -1 + 6s$, $2 + 3t = 3 - s$, $3 + 4t = -5 + 2s$ must be satisfied simultaneously. Solving the first two equations gives $t = \frac{1}{5}$, $s = \frac{2}{5}$ and checking we see these values don't satisfy the third equation. Thus the lines aren't parallel and they don't intersect, so they must be skew.

24. (a) The normal vectors are $\langle 1, 1, -1 \rangle$ and $\langle 2, -3, 4 \rangle$. Since these vectors aren't parallel, neither are the planes parallel. Also $\langle 1, 1, -1 \rangle \cdot \langle 2, -3, 4 \rangle = 2 - 3 - 4 = -5 \neq 0$ so the normal vectors, and thus the planes, are not perpendicular.

(b) $\cos\theta = \dfrac{\langle 1, 1, -1 \rangle \cdot \langle 2, -3, 4 \rangle}{\sqrt{3}\sqrt{29}} = -\dfrac{5}{\sqrt{87}}$ and $\theta = \cos^{-1}\left(-\frac{5}{\sqrt{87}}\right) \approx 122^\circ$ [or we can say $\approx 58^\circ$].

25. $\mathbf{n}_1 = \langle 1, 0, -1 \rangle$ and $\mathbf{n}_2 = \langle 0, 1, 2 \rangle$. Setting $z = 0$, it is easy to see that $(1, 3, 0)$ is a point on the line of intersection of $x - z = 1$ and $y + 2z = 3$. The direction of this line is $\mathbf{v}_1 = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 1, -2, 1 \rangle$. A second vector parallel to the desired plane is $\mathbf{v}_2 = \langle 1, 1, -2 \rangle$, since it is perpendicular to $x + y - 2z = 1$. Therefore, the normal of the plane in question is $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle 4 - 1, 1 + 2, 1 + 2 \rangle = 3\,\langle 1, 1, 1 \rangle$. Taking $(x_0, y_0, z_0) = (1, 3, 0)$, the equation we are looking for is $(x - 1) + (y - 3) + z = 0 \quad \Leftrightarrow \quad x + y + z = 4$.

26. (a) The vectors $\overrightarrow{AB} = \langle -1 - 2, -1 - 1, 10 - 1 \rangle = \langle -3, -2, 9 \rangle$ and $\overrightarrow{AC} = \langle 1 - 2, 3 - 1, -4 - 1 \rangle = \langle -1, 2, -5 \rangle$ lie in the plane, so $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \langle -3, -2, 9 \rangle \times \langle -1, 2, -5 \rangle = \langle -8, -24, -8 \rangle$ or equivalently $\langle 1, 3, 1 \rangle$ is a normal vector to the plane. The point $A(2, 1, 1)$ lies on the plane so an equation of the plane is $1(x - 2) + 3(y - 1) + 1(z - 1) = 0$ or $x + 3y + z = 6$.

(b) The line is perpendicular to the plane so it is parallel to a normal vector for the plane, namely $\langle 1, 3, 1 \rangle$. If the line passes through $B(-1, -1, 10)$ then symmetric equations are $\dfrac{x - (-1)}{1} = \dfrac{y - (-1)}{3} = \dfrac{z - 10}{1}$ or $x + 1 = \dfrac{y + 1}{3} = z - 10$.

(c) Normal vectors for the two planes are $\mathbf{n}_1 = \langle 1, 3, 1 \rangle$ and $\mathbf{n}_2 = \langle 2, -4, -3 \rangle$. The angle θ between the planes is given by

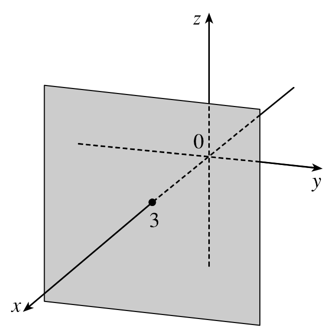
$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\ |\mathbf{n}_2|} = \frac{\langle 1, 3, 1 \rangle \cdot \langle 2, -4, -3 \rangle}{\sqrt{1^2 + 3^2 + 1^2}\,\sqrt{2^2 + (-4)^2 + (-3)^2}} = \frac{2 - 12 - 3}{\sqrt{11}\,\sqrt{29}} = -\frac{13}{\sqrt{319}}$$

Thus $\theta = \cos^{-1}\left(-\dfrac{13}{\sqrt{319}}\right) \approx 137^\circ$ or $180^\circ - 137^\circ = 43^\circ$.

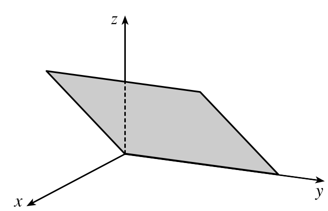
(d) From part (c), the point $(2,0,4)$ lies on the second plane, but notice that the point also satisfies the equation of the first plane, so the point lies on the line of intersection of the planes. A vector $\mathbf{v}$ in the direction of this intersecting line is perpendicular to the normal vectors of both planes, so take $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 1,3,1\rangle \times \langle 2,-4,-3\rangle = \langle -5,5,-10\rangle$ or equivalently we can take $\mathbf{v} = \langle 1,-1,2\rangle$. Parametric equations for the line are $x = 2+t$, $y=-t$, $z = 4+2t$.

27. By Exercise 12.5.75, $D = \dfrac{|-2-(-24)|}{\sqrt{3^2+1^2+(-4)^2}} = \dfrac{22}{\sqrt{26}}$.

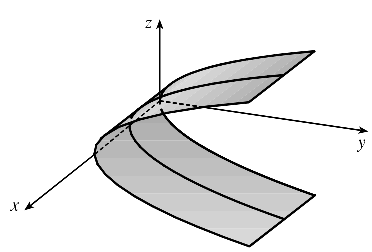
28. The equation $x = 3$ represents a plane parallel to the yz-plane and 3 units in front of it.



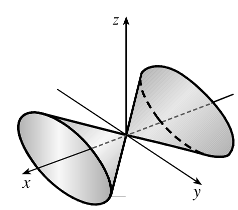
29. The equation $x = z$ represents a plane perpendicular to the xz-plane and intersecting the xz-plane in the line $x = z$, $y = 0$.



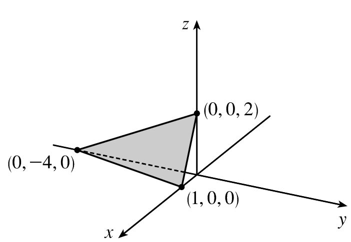
30. The equation $y = z^2$ represents a parabolic cylinder whose trace in the xz-plane is the x-axis and which opens to the right.



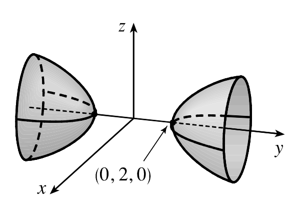
31. The equation $x^2 = y^2 + 4z^2$ represents a (right elliptical) cone with vertex at the origin and axis the x-axis.



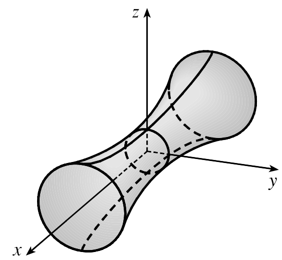
32. $4x - y + 2z = 4$ is a plane with intercepts $(1,0,0)$, $(0,-4,0)$, and $(0,0,2)$.



33. An equivalent equation is $-x^2 + \dfrac{y^2}{4} - z^2 = 1$, a hyperboloid of two sheets with axis the y-axis. For $|y| > 2$, traces parallel to the xz-plane are circles.



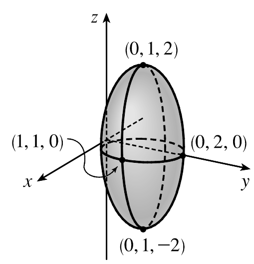
34. An equivalent equation is $-x^2 + y^2 + z^2 = 1$, a hyperboloid of one sheet with axis the x-axis.



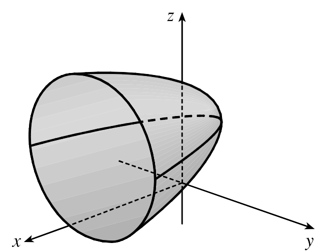
35. Completing the square in y gives

$4x^2 + 4(y-1)^2 + z^2 = 4$ or $x^2 + (y-1)^2 + \dfrac{z^2}{4} = 1$,

an ellipsoid centered at $(0, 1, 0)$.



36. Completing the square for $x = y^2 + z^2 - 2y - 4z + 5$ in y and z gives $x = (y-1)^2 + (z-2)^2$, which is a circular paraboloid with vertex $(0, 1, 2)$ and axis the horizontal line $y = 1$, $z = 2$.



37. $4x^2 + y^2 = 16 \quad \Leftrightarrow \quad \dfrac{x^2}{4} + \dfrac{y^2}{16} = 1$. The equation of the ellipsoid is $\dfrac{x^2}{4} + \dfrac{y^2}{16} + \dfrac{z^2}{c^2} = 1$, since the horizontal trace in the plane $z = 0$ must be the original ellipse. The traces of the ellipsoid in the yz-plane must be circles since the surface is obtained by rotation about the x-axis. Therefore, $c^2 = 16$ and the equation of the ellipsoid is $\dfrac{x^2}{4} + \dfrac{y^2}{16} + \dfrac{z^2}{16} = 1 \quad \Leftrightarrow$

$4x^2 + y^2 + z^2 = 16$.

38. The distance from a point $P(x, y, z)$ to the plane $y = 1$ is $|y - 1|$, so the given condition becomes

$|y-1| = 2\sqrt{(x-0)^2 + (y+1)^2 + (z-0)^2} \quad \Rightarrow \quad |y-1| = 2\sqrt{x^2 + (y+1)^2 + z^2} \quad \Rightarrow$

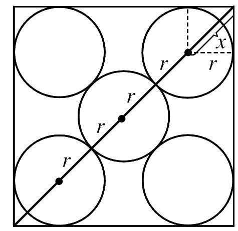
$(y-1)^2 = 4x^2 + 4(y+1)^2 + 4z^2 \quad \Leftrightarrow \quad -3 = 4x^2 + (3y^2 + 10y) + 4z^2 \quad \Leftrightarrow$

$\frac{16}{3} = 4x^2 + 3\left(y + \frac{5}{3}\right)^2 + 4z^2 \quad \Rightarrow \quad \frac{3}{4}x^2 + \frac{9}{16}\left(y + \frac{5}{3}\right)^2 + \frac{3}{4}z^2 = 1$.

This is the equation of an ellipsoid whose center is $\left(0, -\frac{5}{3}, 0\right)$.

☐ PROBLEMS PLUS

1. Since three-dimensional situations are often difficult to visualize and work with, let us first try to find an analogous problem in two dimensions. The analogue of a cube is a square and the analogue of a sphere is a circle. Thus a similar problem in two dimensions is the following: if five circles with the same radius r are contained in a square of side 1 m so that the circles touch each other and four of the circles touch two sides of the square, find r.



The diagonal of the square is $\sqrt{2}$. The diagonal is also $4r + 2x$. But x is the diagonal of a smaller square of side r. Therefore,

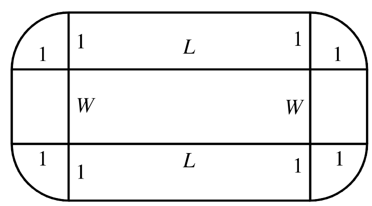
$x = \sqrt{2}\,r \quad \Rightarrow \quad \sqrt{2} = 4r + 2x = 4r + 2\sqrt{2}\,r = \left(4 + 2\sqrt{2}\right)r \quad \Rightarrow \quad r = \frac{\sqrt{2}}{4 + 2\sqrt{2}}$.

Let's use these ideas to solve the original three-dimensional problem. The diagonal of the cube is $\sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$. The diagonal of the cube is also $4r + 2x$ where x is the diagonal of a smaller cube with edge r. Therefore

$x = \sqrt{r^2 + r^2 + r^2} = \sqrt{3}\,r \quad \Rightarrow \quad \sqrt{3} = 4r + 2x = 4r + 2\sqrt{3}\,r = \left(4 + 2\sqrt{3}\right)r$. Thus $r = \dfrac{\sqrt{3}}{4 + 2\sqrt{3}} = \dfrac{2\sqrt{3} - 3}{2}$.

The radius of each ball is $\left(\sqrt{3} - \frac{3}{2}\right)$ m.

2. Try an analogous problem in two dimensions. Consider a rectangle with length L and width W and find the area of S in terms of L and W. Since S contains B, it has area



$$\begin{aligned} A(S) = LW &+ \text{the area of two } L \times 1 \text{ rectangles} \\ &+ \text{the area of two } 1 \times W \text{ rectangles} \\ &+ \text{the area of four quarter-circles of radius 1} \end{aligned}$$

as seen in the diagram. So $A(S) = LW + 2L + 2W + \pi \cdot 1^2$.

Now in three dimensions, the volume of S is

$$\begin{aligned} &LWH + 2(L \times W \times 1) + 2(1 \times W \times H) + 2(L \times 1 \times H) \\ &\quad + \text{the volume of 4 quarter-cylinders with radius 1 and height } W \\ &\quad + \text{the volume of 4 quarter-cylinders with radius 1 and height } L \\ &\quad + \text{the volume of 4 quarter-cylinders with radius 1 and height } H \\ &\quad + \text{the volume of 8 eighths of a sphere of radius 1} \end{aligned}$$

So

$$\begin{aligned} V(S) &= LWH + 2LW + 2WH + 2LH + \pi \cdot 1^2 \cdot W + \pi \cdot 1^2 \cdot L + \pi \cdot 1^2 \cdot H + \tfrac{4}{3}\pi \cdot 1^3 \\ &= LWH + 2(LW + WH + LH) + \pi(L + W + H) + \tfrac{4}{3}\pi. \end{aligned}$$

3. (a) We find the line of intersection L as in Example 12.5.7(b). Observe that the point $(-1, c, c)$ lies on both planes. Now since L lies in both planes, it is perpendicular to both of the normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$, and thus parallel to their cross product

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ c & 1 & 1 \\ 1 & -c & c \end{vmatrix} = \langle 2c, -c^2 + 1, -c^2 - 1 \rangle$$

So symmetric equations of L can be written as $\dfrac{x+1}{-2c} = \dfrac{y-c}{c^2-1} = \dfrac{z-c}{c^2+1}$, provided that $c \neq 0, \pm 1$.

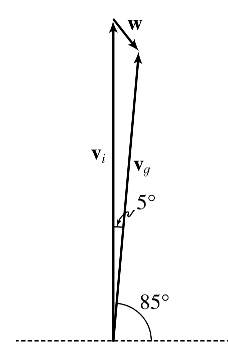
If $c = 0$, then the two planes are given by $y + z = 0$ and $x = -1$, so symmetric equations of L are $x = -1$, $y = -z$. If $c = -1$, then the two planes are given by $-x + y + z = -1$ and $x + y + z = -1$, and they intersect in the line $x = 0$, $y = -z - 1$. If $c = 1$, then the two planes are given by $x + y + z = 1$ and $x - y + z = 1$, and they intersect in the line $y = 0$, $x = 1 - z$.

(b) If we set $z = t$ in the symmetric equations and solve for x and y separately, we get $x + 1 = \dfrac{(t-c)(-2c)}{c^2+1}$,

$y - c = \dfrac{(t-c)(c^2-1)}{c^2+1} \quad \Rightarrow \quad x = \dfrac{-2ct + (c^2-1)}{c^2+1}$, $y = \dfrac{(c^2-1)t + 2c}{c^2+1}$. Eliminating c from these equations, we have $x^2 + y^2 = t^2 + 1$. So the curve traced out by L in the plane $z = t$ is a circle with center at $(0, 0, t)$ and radius $\sqrt{t^2+1}$.

(c) The area of a horizontal cross-section of the solid is $A(z) = \pi(z^2 + 1)$, so $V = \int_0^1 A(z)dz = \pi\left[\frac{1}{3}z^3 + z\right]_0^1 = \frac{4\pi}{3}$.

4. (a) We consider velocity vectors for the plane and the wind. Let $\mathbf{v}_i$ be the initial, intended velocity for the plane and $\mathbf{v}_g$ the actual velocity relative to the ground. If $\mathbf{w}$ is the velocity of the wind, $\mathbf{v}_g$ is the resultant, that is, the vector sum $\mathbf{v}_i + \mathbf{w}$ as shown in the figure. We know $\mathbf{v}_i = 180\,\mathbf{j}$, and since the plane actually flew 80 km in $\frac{1}{2}$ hour, $|\mathbf{v}_g| = 160$. Thus, $\mathbf{v}_g = (160\cos 85°)\,\mathbf{i} + (160\sin 85°)\,\mathbf{j} \approx 13.9\,\mathbf{i} + 159.4\,\mathbf{j}$. Finally, $\mathbf{v}_i + \mathbf{w} = \mathbf{v}_g$, so $\mathbf{w} = \mathbf{v}_g - \mathbf{v}_i \approx 13.9\,\mathbf{i} - 20.6\,\mathbf{j}$. Thus, the wind velocity is about $13.9\,\mathbf{i} - 20.6\,\mathbf{j}$, and the wind speed is $|\mathbf{w}| \approx \sqrt{(13.9)^2 + (-20.6)^2} \approx 24.9$ km/h.



(b) Let $\mathbf{v}$ be the velocity the pilot should have taken. The pilot would need to head a little west of north to compensate for the wind, so let θ be the angle $\mathbf{v}$ makes with the north direction. Then we can write $\mathbf{v} = \langle 180\cos(\theta + 90°), 180\sin(\theta + 90°)\rangle$. With the effect of the wind, the actual velocity (with respect to the ground) will be $\mathbf{v} + \mathbf{w}$, which we want to be due north. Equating horizontal components of the vectors, we must have $180\cos(\theta + 90°) + 160\cos 85° = 0 \quad \Rightarrow \quad \cos(\theta + 90°) = -\frac{160}{180}\cos 85° \approx -0.0775$, so $\theta \approx \cos^{-1}(-0.0775) - 90° \approx 4.4°$. Thus, the pilot should have headed about $4.4°$ west of north.

5. $\mathbf{v}_3 = \text{proj}_{\mathbf{v}_1}\mathbf{v}_2 = \dfrac{\mathbf{v}_1 \cdot \mathbf{v}_2}{|\mathbf{v}_1|^2}\,\mathbf{v}_1 = \dfrac{5}{2^2}\,\mathbf{v}_1$ so $|\mathbf{v}_3| = \dfrac{5}{2^2}\,|\mathbf{v}_1| = \dfrac{5}{2}$,

$$\mathbf{v}_4 = \text{proj}_{\mathbf{v}_2}\mathbf{v}_3 = \frac{\mathbf{v}_2 \cdot \mathbf{v}_3}{|\mathbf{v}_2|^2}\,\mathbf{v}_2 = \frac{\mathbf{v}_2 \cdot \frac{5}{2^2}\,\mathbf{v}_1}{|\mathbf{v}_2|^2}\,\mathbf{v}_2 = \frac{5}{2^2 \cdot 3^2}\,(\mathbf{v}_1 \cdot \mathbf{v}_2)\,\mathbf{v}_2 = \frac{5^2}{2^2 \cdot 3^2}\,\mathbf{v}_2 \quad\Rightarrow\quad |\mathbf{v}_4| = \frac{5^2}{2^2 \cdot 3^2}\,|\mathbf{v}_2| = \frac{5^2}{2^2 \cdot 3},$$

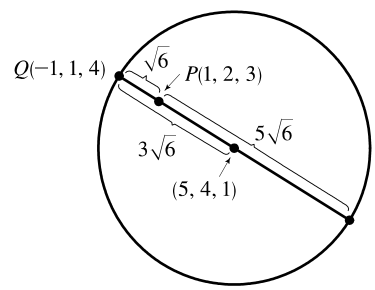
$$\mathbf{v}_5 = \text{proj}_{\mathbf{v}_3}\mathbf{v}_4 = \frac{\mathbf{v}_3 \cdot \mathbf{v}_4}{|\mathbf{v}_3|^2}\,\mathbf{v}_3 = \frac{\frac{5}{2^2}\,\mathbf{v}_1 \cdot \frac{5^2}{2^2\,3^2}\,\mathbf{v}_2}{\left(\frac{5}{2}\right)^2}\left(\frac{5}{2^2}\,\mathbf{v}_1\right) = \frac{5^2}{2^4 \cdot 3^2}\,(\mathbf{v}_1 \cdot \mathbf{v}_2)\;\mathbf{v}_1 = \frac{5^3}{2^4 \cdot 3^2}\,\mathbf{v}_1 \quad\Rightarrow$$

$|\mathbf{v}_5| = \dfrac{5^3}{2^4 \cdot 3^2}\,|\mathbf{v}_1| = \dfrac{5^3}{2^3 \cdot 3^2}$. Similarly, $|\mathbf{v}_6| = \dfrac{5^4}{2^4 \cdot 3^3}$, $|\mathbf{v}_7| = \dfrac{5^5}{2^5 \cdot 3^4}$, and in general, $|\mathbf{v}_n| = \dfrac{5^{n-2}}{2^{n-2} \cdot 3^{n-3}} = 3\left(\frac{5}{6}\right)^{n-2}$.

Thus,

$$\begin{aligned}\sum_{n=1}^{\infty} |\mathbf{v}_n| &= |\mathbf{v}_1| + |\mathbf{v}_2| + \sum_{n=3}^{\infty} 3\left(\tfrac{5}{6}\right)^{n-2} = 2 + 3 + \sum_{n=1}^{\infty} 3\left(\tfrac{5}{6}\right)^{n} \\ &= 5 + \sum_{n=1}^{\infty} \tfrac{5}{2}\left(\tfrac{5}{6}\right)^{n-1} = 5 + \frac{\frac{5}{2}}{1 - \frac{5}{6}} \quad \text{[sum of a geometric series]} \quad = 5 + 15 = 20\end{aligned}$$

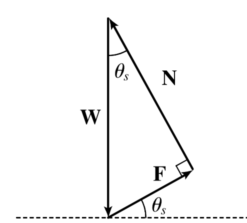
6. Completing squares in the inequality $x^2 + y^2 + z^2 < 136 + 2(x + 2y + 3z)$ gives $(x-1)^2 + (y-2)^2 + (z-3)^2 < 150$ which describes the set of all points (x, y, z) whose distance from the point $P\,(1, 2, 3)$ is less than $\sqrt{150} = 5\sqrt{6}$. The distance from P to $Q\,(-1, 1, 4)$ is $\sqrt{4 + 1 + 1} = \sqrt{6}$, so the largest possible sphere that passes through Q and satisfies the stated conditions extends $5\sqrt{6}$ units in the opposite direction, giving a diameter of $6\sqrt{6}$. (See the figure.)



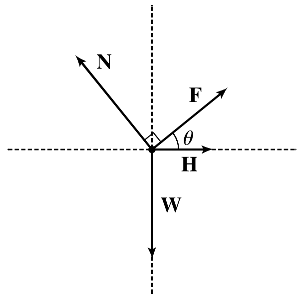
Thus the radius of the desired sphere is $3\sqrt{6}$, and its center is $3\sqrt{6}$ units from Q in the direction of P. A unit vector in this direction is $\mathbf{u} = \frac{1}{\sqrt{6}}\,\langle 2, 1, -1\rangle$, so starting at $Q(-1, 1, 4)$ and following the vector $3\sqrt{6}\,\mathbf{u} = \langle 6, 3, -3\rangle$ we arrive at the center of the sphere, $(5, 4, 1)$. An equation of the sphere then is $(x-5)^2 + (y-4)^2 + (z-1)^2 = \left(3\sqrt{6}\right)^2$ or $(x-5)^2 + (y-4)^2 + (z-1)^2 = 54$.

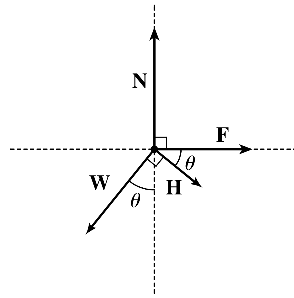
7. (a) When $\theta = \theta_s$, the block is not moving, so the sum of the forces on the block must be $\mathbf{0}$, thus $\mathbf{N} + \mathbf{F} + \mathbf{W} = \mathbf{0}$. This relationship is illustrated geometrically in the figure. Since the vectors form a right triangle, we have

$$\tan(\theta_s) = \frac{|\mathbf{F}|}{|\mathbf{N}|} = \frac{\mu_s n}{n} = \mu_s.$$



(b) We place the block at the origin and sketch the force vectors acting on the block, including the additional horizontal force $\mathbf{H}$, with initial points at the origin. We then rotate this system so that $\mathbf{F}$ lies along the positive x-axis and the inclined plane is parallel to the x-axis. (See the following figure.)





$|\mathbf{F}|$ is maximal, so $|\mathbf{F}| = \mu_s n$ for $\theta > \theta_s$. Then the vectors, in terms of components parallel and perpendicular to the inclined plane, are

$$\mathbf{N} = n\,\mathbf{j} \qquad\qquad \mathbf{F} = (\mu_s n)\,\mathbf{i}$$

$$\mathbf{W} = (-mg\sin\theta)\,\mathbf{i} + (-mg\cos\theta)\,\mathbf{j} \qquad\qquad \mathbf{H} = (h_{\min}\cos\theta)\,\mathbf{i} + (-h_{\min}\sin\theta)\,\mathbf{j}$$

Equating components, we have

$$\mu_s n - mg\sin\theta + h_{\min}\cos\theta = 0 \quad\Rightarrow\quad h_{\min}\cos\theta + \mu_s n = mg\sin\theta \qquad \textbf{(1)}$$

$$n - mg\cos\theta - h_{\min}\sin\theta = 0 \quad\Rightarrow\quad h_{\min}\sin\theta + mg\cos\theta = n \qquad \textbf{(2)}$$

(c) Since **(2)** is solved for n, we substitute into **(1)**:

$$h_{\min}\cos\theta + \mu_s(h_{\min}\sin\theta + mg\cos\theta) = mg\sin\theta \quad\Rightarrow$$

$$h_{\min}\cos\theta + h_{\min}\mu_s\sin\theta = mg\sin\theta - mg\mu_s\cos\theta \quad\Rightarrow$$

$$h_{\min} = mg\left(\frac{\sin\theta - \mu_s\cos\theta}{\cos\theta + \mu_s\sin\theta}\right) = mg\left(\frac{\tan\theta - \mu_s}{1 + \mu_s\tan\theta}\right)$$

From part (a) we know $\mu_s = \tan\theta_s$, so this becomes $h_{\min} = mg\left(\dfrac{\tan\theta - \tan\theta_s}{1 + \tan\theta_s\tan\theta}\right)$ and using a trigonometric identity, this is $mg\tan(\theta - \theta_s)$ as desired.

Note for $\theta = \theta_s$, $h_{\min} = mg\tan 0 = 0$, which makes sense since the block is at rest for θ_s, thus no additional force $\mathbf{H}$ is necessary to prevent it from moving. As θ increases, the factor $\tan(\theta - \theta_s)$, and hence the value of $h_{\min}$, increases slowly for small values of $\theta - \theta_s$ but much more rapidly as $\theta - \theta_s$ becomes significant. This seems reasonable, as the steeper the inclined plane, the less the horizontal components of the various forces affect the movement of the block, so we would need a much larger magnitude of horizontal force to keep the block motionless. If we allow $\theta \to 90°$, corresponding to the inclined plane being placed vertically, the value of $h_{\min}$ is quite large; this is to be expected, as it takes a great amount of horizontal force to keep an object from moving vertically. In fact, without friction (so $\theta_s = 0$), we would have $\theta \to 90° \quad\Rightarrow\quad h_{\min} \to \infty$, and it would be impossible to keep the block from slipping.

(d) Since $h_{\max}$ is the largest value of h that keeps the block from slipping, the force of friction is keeping the block from moving *up* the inclined plane; thus, $\mathbf{F}$ is directed *down* the plane. Our system of forces is similar to that in part (b), then, except that we have $\mathbf{F} = -(\mu_s n)\,\mathbf{i}$. (Note that $|\mathbf{F}|$ is again maximal.) Following our procedure in parts (b) and (c), we equate components:

$$-\mu_s n - mg\sin\theta + h_{\max}\cos\theta = 0 \quad\Rightarrow\quad h_{\max}\cos\theta - \mu_s n = mg\sin\theta$$

$$n - mg\cos\theta - h_{\max}\sin\theta = 0 \quad\Rightarrow\quad h_{\max}\sin\theta + mg\cos\theta = n$$

Then substituting,

$$h_{\max}\cos\theta - \mu_s(h_{\max}\sin\theta + mg\cos\theta) = mg\sin\theta \quad\Rightarrow$$

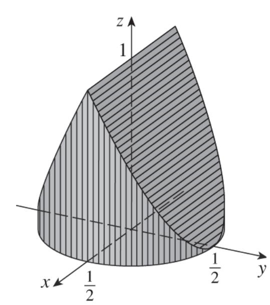
$$h_{\max}\cos\theta - h_{\max}\mu_s\sin\theta = mg\sin\theta + mg\mu_s\cos\theta \quad\Rightarrow$$

$$h_{\max} = mg\left(\frac{\sin\theta + \mu_s\cos\theta}{\cos\theta - \mu_s\sin\theta}\right) = mg\left(\frac{\tan\theta + \mu_s}{1 - \mu_s\tan\theta}\right)$$

$$= mg\left(\frac{\tan\theta + \tan\theta_s}{1 - \tan\theta_s\tan\theta}\right) = mg\tan(\theta + \theta_s)$$

We would expect $h_{\max}$ to increase as θ increases, with similar behavior as we established for $h_{\min}$, but with $h_{\max}$ values always larger than $h_{\min}$. We can see that this is the case if we graph $h_{\max}$ as a function of θ, as the curve is the graph of $h_{\min}$ translated $2\theta_s$ to the left, so the equation does seem reasonable. Notice that the equation predicts $h_{\max} \to \infty$ as $\theta \to (90^\circ - \theta_s)$. In fact, as $h_{\max}$ increases, the normal force increases as well. When $(90^\circ - \theta_s) \le \theta \le 90^\circ$, the horizontal force is completely counteracted by the sum of the normal and frictional forces, so no part of the horizontal force contributes to moving the block up the plane no matter how large its magnitude.

8. (a) The largest possible solid is achieved by starting with a circular cylinder of diameter 1 with axis the z-axis and with a height of 1. This is the largest solid that creates a square shadow with side length 1 in the y-direction and a circular disk shadow in the z-direction. For convenience, we place the base of the cylinder on the xy-plane so that it intersects the x- and y-axes at $\pm\frac{1}{2}$. We then remove as little as possible from the solid that leaves an isosceles triangle shadow in the x-direction. This is achieved by cutting the solid with planes parallel to the x-axis that intersect the z-axis at 1 and the y-axis at $\pm\frac{1}{2}$ (see the figure).



To compute the volume of this solid, we take vertical slices parallel to the xz-plane. The equation of the base of the solid is $x^2 + y^2 = \frac{1}{4}$, so a cross-section y units to the right of the origin is a rectangle with base $2\sqrt{\frac{1}{4} - y^2}$. For $0 \le y \le \frac{1}{2}$, the solid is cut off on top by the plane $z = 1 - 2y$, so the height of the rectangular cross-section is $1 - 2y$ and the cross-sectional area is $A(y) = 2\sqrt{\frac{1}{4} - y^2}\,(1 - 2y)$. The volume of the right half of the solid is

$$\int_0^{1/2} 2\sqrt{\tfrac{1}{4} - y^2}\,(1 - 2y)\,dy = 2\int_0^{1/2}\sqrt{\tfrac{1}{4} - y^2}\,dy - 4\int_0^{1/2} y\sqrt{\tfrac{1}{4} - y^2}\,dy$$

$$= 2\left[\tfrac{1}{4}\text{ area of a circle of radius }\tfrac{1}{2}\right] - 4\left[-\tfrac{1}{3}\left(\tfrac{1}{4} - y^2\right)^{3/2}\right]_0^{1/2}$$

$$= 2\left[\tfrac{1}{4}\cdot\pi\left(\tfrac{1}{2}\right)^2\right] + \tfrac{4}{3}\left[0 - \left(\tfrac{1}{4}\right)^{3/2}\right] = \tfrac{\pi}{8} - \tfrac{1}{6}$$

Thus the volume of the solid is $2\left(\frac{\pi}{8} - \frac{1}{6}\right) = \frac{\pi}{4} - \frac{1}{3} \approx 0.45$.

(b) There is not a smallest volume. We can remove portions of the solid from part (a) to create smaller and smaller volumes as long as we leave the "skeleton" shown in the figure intact (which has no volume at all and is not a solid). Thus, we can create solids with arbitrarily small volume.