8 □ FURTHER APPLICATIONS OF INTEGRATION

8.1 Arc Length

1. $y = 3 - 2x \quad \Rightarrow \quad L = \int_{-1}^{3} \sqrt{1 + (dy/dx)^2}\, dx = \int_{-1}^{3} \sqrt{1 + (-2)^2}\, dx = \sqrt{5}\,\big[\,x\,\big]_{-1}^{3} = \sqrt{5}\,[3 - (-1)] = 4\sqrt{5}.$

The arc length can be calculated using the distance formula, since the curve is a line segment, so

$L = [\text{distance from } (-1, 5) \text{ to } (3, -3)] = \sqrt{[3 - (-1)]^2 + (-3 - 5)^2} = \sqrt{80} = 4\sqrt{5}.$

2. Using the arc length formula with $y = \sqrt{4 - x^2} \quad \Rightarrow \quad \dfrac{dy}{dx} = -\dfrac{x}{\sqrt{4 - x^2}}$, we get

$$\begin{aligned} L &= \int_0^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^2 \sqrt{1 + \frac{x^2}{4 - x^2}}\, dx = \int_0^2 \frac{\sqrt{4}}{\sqrt{4 - x^2}}\, dx = 2\int_0^2 \frac{dx}{\sqrt{4 - x^2}} \\ &= 2\left[\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2 = 2(\sin^{-1} 1 - \sin^{-1} 0) = 2\left(\frac{\pi}{2} - 0\right) = \pi \end{aligned}$$

The curve is one-quarter of a circle with radius 2, so the length of the arc is $\frac{1}{4}(2\pi \cdot 2) = \pi$.

3. $y = x^3 \quad \Rightarrow \quad dy/dx = 3x^2 \quad \Rightarrow \quad 1 + (dy/dx)^2 = 1 + (3x^2)^2$. So $L = \int_0^2 \sqrt{1 + 9x^4}\, dx$.

4. $y = e^x \quad \Rightarrow \quad dy/dx = e^x \quad \Rightarrow \quad 1 + (dy/dx)^2 = 1 + (e^x)^2 = 1 + e^{2x}$. So $L = \int_1^3 \sqrt{1 + e^{2x}}\, dx$.

5. $y = x - \ln x \quad \Rightarrow \quad dy/dx = 1 - 1/x \quad \Rightarrow \quad 1 + (dy/dx)^2 = 1 + (1 - 1/x)^2$. So $L = \int_1^4 \sqrt{1 + (1 - 1/x)^2}\, dx$.

6. $x = y^2 + y \quad \Rightarrow \quad dx/dy = 2y + 1 \quad \Rightarrow \quad 1 + (dx/dy)^2 = 1 + (2y + 1)^2$. So $L = \int_0^3 \sqrt{1 + (2y + 1)^2}\, dy$.

7. $x = \sin y \quad \Rightarrow \quad dx/dy = \cos y \quad \Rightarrow \quad 1 + (dx/dy)^2 = 1 + \cos^2 y$. So $L = \int_0^{\pi/2} \sqrt{1 + \cos^2 y}\, dy$.

8. $y^2 = \ln x \quad \Leftrightarrow \quad x = e^{y^2} \quad \Rightarrow \quad dx/dy = 2ye^{y^2} \quad \Rightarrow \quad 1 + (dx/dy)^2 = 1 + 4y^2 e^{2y^2}$. So $L = \int_{-1}^{1} \sqrt{1 + 4y^2 e^{2y^2}}\, dy$.

9. $y = \frac{2}{3}x^{3/2} \quad \Rightarrow \quad dy/dx = x^{1/2} \quad \Rightarrow \quad 1 + (dy/dx)^2 = 1 + x$. So

$L = \int_0^2 \sqrt{1 + x}\, dx = \int_0^2 (1 + x)^{1/2}\, dx = \left[\frac{2}{3}(1 + x)^{3/2}\right]_0^2 = \frac{2}{3}(3^{3/2} - 1^{3/2}) = \frac{2}{3}\big(3\sqrt{3} - 1\big) = 2\sqrt{3} - \frac{2}{3}.$

10. $y = (x + 4)^{3/2} \quad \Rightarrow \quad dy/dx = \frac{3}{2}(x + 4)^{1/2} \quad \Rightarrow \quad 1 + (dy/dx)^2 = 1 + \frac{9}{4}(x + 4)$. So

$L = \int_0^4 \sqrt{1 + \frac{9}{4}(x + 4)}\, dx = \int_0^4 \left(10 + \frac{9}{4}x\right)^{1/2} dx = \left[\frac{8}{27}\left(10 + \frac{9}{4}x\right)^{3/2}\right]_0^4 = \frac{8}{27}(19^{3/2} - 10^{3/2}).$

11. $y = \frac{2}{3}(1 + x^2)^{3/2} \quad \Rightarrow \quad dy/dx = 2x(1 + x^2)^{1/2} \quad \Rightarrow \quad 1 + (dy/dx)^2 = 1 + 4x^2(1 + x^2)$. So

$$\begin{aligned} L &= \int_0^1 \sqrt{1 + 4x^2(1 + x^2)}\, dx = \int_0^1 \sqrt{4x^4 + 4x^2 + 1}\, dx = \int_0^1 \sqrt{(2x^2 + 1)^2}\, dx = \int_0^1 \left|2x^2 + 1\right| dx \\ &= \int_0^1 (2x^2 + 1)\, dx = \left[\tfrac{2}{3}x^3 + x\right]_0^1 = \left(\tfrac{2}{3} + 1\right) - 0 = \tfrac{5}{3} \end{aligned}$$

12. $36y^2 = (x^2-4)^3,\ y \geq 0 \quad \Rightarrow \quad y = \frac{1}{6}(x^2-4)^{3/2} \quad \Rightarrow \quad dy/dx = \frac{1}{6}\cdot\frac{3}{2}(x^2-4)^{1/2}(2x) = \frac{1}{2}x(x^2-4)^{1/2} \quad \Rightarrow$

$1+(dy/dx)^2 = 1+\frac{1}{4}x^2(x^2-4) = \frac{1}{4}x^4 - x^2 + 1 = \frac{1}{4}(x^4-4x^2+4) = \left[\frac{1}{2}(x^2-2)\right]^2$. So

$L = \int_2^3 \sqrt{\left[\frac{1}{2}(x^2-2)\right]^2}\,dx = \int_2^3 \frac{1}{2}(x^2-2)\,dx = \frac{1}{2}\left[\frac{1}{3}x^3-2x\right]_2^3 = \frac{1}{2}\left[(9-6)-\left(\frac{8}{3}-4\right)\right] = \frac{1}{2}\left(\frac{13}{3}\right) = \frac{13}{6}.$

13. $y = \dfrac{x^3}{3}+\dfrac{1}{4x} \quad \Rightarrow \quad \dfrac{dy}{dx} = x^2 - \dfrac{1}{4x^2} \quad \Rightarrow$

$1+\left(\dfrac{dy}{dx}\right)^2 = 1+\left(x^4-\dfrac{1}{2}+\dfrac{1}{16x^4}\right) = x^4+\dfrac{1}{2}+\dfrac{1}{16x^4} = \left(x^2+\dfrac{1}{4x^2}\right)^2$. So

$$L = \int_1^2 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_1^2 \left|x^2+\frac{1}{4x^2}\right|\,dx = \int_1^2 \left(x^2+\frac{1}{4x^2}\right)\,dx$$

$$= \left[\frac{1}{3}x^3 - \frac{1}{4x}\right]_1^2 = \left(\frac{8}{3}-\frac{1}{8}\right)-\left(\frac{1}{3}-\frac{1}{4}\right) = \frac{7}{3}+\frac{1}{8} = \frac{59}{24}$$

14. $x = \dfrac{y^4}{8}+\dfrac{1}{4y^2} \quad \Rightarrow \quad \dfrac{dx}{dy} = \dfrac{1}{2}y^3 - \dfrac{1}{2}y^{-3} \quad \Rightarrow$

$1+(dx/dy)^2 = 1+\frac{1}{4}y^6 - \frac{1}{2}+\frac{1}{4}y^{-6} = \frac{1}{4}y^6+\frac{1}{2}+\frac{1}{4}y^{-6} = \left(\frac{1}{2}y^3+\frac{1}{2}y^{-3}\right)^2$. So

$L = \int_1^2 \sqrt{\left(\frac{1}{2}y^3+\frac{1}{2}y^{-3}\right)^2}\,dy = \int_1^2 \left(\frac{1}{2}y^3+\frac{1}{2}y^{-3}\right)dy = \left[\frac{1}{8}y^4-\frac{1}{4}y^{-2}\right]_1^2 = \left(2-\frac{1}{16}\right)-\left(\frac{1}{8}-\frac{1}{4}\right)$

$= 2+\frac{1}{16} = \frac{33}{16}$

15. $y = \frac{1}{2}\ln(\sin 2x) \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{\cos 2x}{\sin 2x} = \cot 2x \quad \Rightarrow \quad 1+\left(\dfrac{dy}{dx}\right)^2 = 1+\cot^2 2x = \csc^2 2x$. So

$$L = \int_{\pi/8}^{\pi/6} \sqrt{\csc^2 2x}\,dx = \int_{\pi/8}^{\pi/6} |\csc 2x|\,dx = \int_{\pi/8}^{\pi/6} \csc 2x\,dx = \frac{1}{2}\int_{\pi/4}^{\pi/3} \csc u\,du \qquad \begin{bmatrix} u = 2x \\ du = 2\,dx \end{bmatrix}$$

$$= \frac{1}{2}\Big[\ln|\csc u - \cot u|\Big]_{\pi/4}^{\pi/3} = \frac{1}{2}\left[\ln\left(\frac{2}{\sqrt{3}}-\frac{1}{\sqrt{3}}\right)-\ln\left(\sqrt{2}-1\right)\right] = \frac{1}{2}\left[\ln\frac{1}{\sqrt{3}}-\ln\left(\sqrt{2}-1\right)\right]$$

16. $y = \ln(\cos x) \quad \Rightarrow \quad dy/dx = -\tan x \quad \Rightarrow \quad 1+(dy/dx)^2 = 1+\tan^2 x = \sec^2 x$. So

$L = \int_0^{\pi/3} \sqrt{\sec^2 x}\,dx = \int_0^{\pi/3} \sec x\,dx = \big[\ln|\sec x+\tan x|\big]_0^{\pi/3} = \ln\left(2+\sqrt{3}\right)-\ln(1+0) = \ln\left(2+\sqrt{3}\right).$

17. $y = \ln(\sec x) \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{\sec x\,\tan x}{\sec x} = \tan x \quad \Rightarrow \quad 1+\left(\dfrac{dy}{dx}\right)^2 = 1+\tan^2 x = \sec^2 x$. So

$L = \int_0^{\pi/4} \sqrt{\sec^2 x}\,dx = \int_0^{\pi/4} |\sec x|\,dx = \int_0^{\pi/4} \sec x\,dx = \Big[\ln|\sec x+\tan x|\Big]_0^{\pi/4}$

$= \ln\left(\sqrt{2}+1\right)-\ln(1+0) = \ln\left(\sqrt{2}+1\right)$

18. $x = e^y+\frac{1}{4}e^{-y} \quad \Rightarrow \quad dx/dy = e^y - \frac{1}{4}e^{-y} \quad \Rightarrow$

$1+(dx/dy)^2 = 1+\left(e^y-\frac{1}{4}e^{-y}\right)^2 = 1+\left(e^{2y}-\frac{1}{2}+\frac{1}{16}e^{-2y}\right) = e^{2y}+\frac{1}{2}+\frac{1}{16}e^{-2y} = \left(e^y+\frac{1}{4}e^{-y}\right)^2$. So

$$L = \int_0^1 \sqrt{\left(e^y+\tfrac{1}{4}e^{-y}\right)^2}\,dy = \int_0^1 \left|e^y+\tfrac{1}{4}e^{-y}\right|\,dy = \int_0^1 \left(e^y+\tfrac{1}{4}e^{-y}\right)\,dy = \left[e^y-\tfrac{1}{4}e^{-y}\right]_0^1$$

$$= e-\frac{1}{4}e^{-1}-\left(1-\frac{1}{4}\right) = e-\frac{1}{4e}-\frac{3}{4}$$

19. $x = \frac{1}{3}\sqrt{y}\,(y-3) = \frac{1}{3}y^{3/2} - y^{1/2} \quad\Rightarrow\quad dx/dy = \frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2} \quad\Rightarrow$

$1 + (dx/dy)^2 = 1 + \frac{1}{4}y - \frac{1}{2} + \frac{1}{4}y^{-1} = \frac{1}{4}y + \frac{1}{2} + \frac{1}{4}y^{-1} = \left(\frac{1}{2}y^{1/2} + \frac{1}{2}y^{-1/2}\right)^2$. So

$L = \int_1^9 \left(\frac{1}{2}y^{1/2} + \frac{1}{2}y^{-1/2}\right) dy = \frac{1}{2}\left[\frac{2}{3}y^{3/2} + 2y^{1/2}\right]_1^9 = \frac{1}{2}\left[\left(\frac{2}{3}\cdot 27 + 2\cdot 3\right) - \left(\frac{2}{3}\cdot 1 + 2\cdot 1\right)\right]$

$= \frac{1}{2}\left(24 - \frac{8}{3}\right) = \frac{1}{2}\left(\frac{64}{3}\right) = \frac{32}{3}$.

20. $y = 3 + \frac{1}{2}\cosh 2x \quad\Rightarrow\quad dy/dx = \sinh 2x \quad\Rightarrow\quad 1 + (dy/dx)^2 = 1 + \sinh^2(2x) = \cosh^2(2x)$. So

$L = \int_0^1 \sqrt{\cosh^2(2x)}\,dx = \int_0^1 \cosh 2x\,dx = \left[\frac{1}{2}\sinh 2x\right]_0^1 = \frac{1}{2}\sinh 2 - 0 = \frac{1}{2}\sinh 2.$

21. $y = \dfrac{1}{4}x^2 - \dfrac{1}{2}\ln x \quad\Rightarrow\quad \dfrac{dy}{dx} = \dfrac{1}{2}x - \dfrac{1}{2x} \quad\Rightarrow$

$1 + \left(\dfrac{dy}{dx}\right)^2 = 1 + \left(\dfrac{1}{4}x^2 - \dfrac{1}{2} + \dfrac{1}{4x^2}\right) = \dfrac{1}{4}x^2 + \dfrac{1}{2} + \dfrac{1}{4x^2} = \left(\dfrac{1}{2}x + \dfrac{1}{2x}\right)^2$. So

$$L = \int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_1^2 \left|\frac{1}{2}x + \frac{1}{2x}\right| dx = \int_1^2 \left(\frac{1}{2}x + \frac{1}{2x}\right) dx$$

$$= \left[\frac{1}{4}x^2 + \frac{1}{2}\ln|x|\right]_1^2 = \left(1 + \frac{1}{2}\ln 2\right) - \left(\frac{1}{4} + 0\right) = \frac{3}{4} + \frac{1}{2}\ln 2$$

22. $y = \sqrt{x - x^2} + \sin^{-1}\left(\sqrt{x}\right) \quad\Rightarrow\quad \dfrac{dy}{dx} = \dfrac{1-2x}{2\sqrt{x-x^2}} + \dfrac{1}{2\sqrt{x}\sqrt{1-x}} = \dfrac{2-2x}{2\sqrt{x}\sqrt{1-x}} = \sqrt{\dfrac{1-x}{x}} \quad\Rightarrow$

$1 + \left(\dfrac{dy}{dx}\right)^2 = 1 + \dfrac{1-x}{x} = \dfrac{1}{x}$. The curve has endpoints $(0,0)$ and $\left(1, \frac{\pi}{2}\right)$,

so $L = \int_0^1 \sqrt{1/x}\,dx = \lim\limits_{t\to 0^+} \int_t^1 \sqrt{1/x}\,dx = \lim\limits_{t\to 0^+} \left[2\sqrt{x}\right]_t^1 = \lim\limits_{t\to 0^+} \left[2\sqrt{1} - 2\sqrt{t}\right] = 2 - 0 = 2.$

23. $y = \ln(1 - x^2) \quad\Rightarrow\quad \dfrac{dy}{dx} = \dfrac{1}{1-x^2}\cdot(-2x) \quad\Rightarrow$

$1 + \left(\dfrac{dy}{dx}\right)^2 = 1 + \dfrac{4x^2}{(1-x^2)^2} = \dfrac{1 - 2x^2 + x^4 + 4x^2}{(1-x^2)^2} = \dfrac{1 + 2x^2 + x^4}{(1-x^2)^2} = \dfrac{(1+x^2)^2}{(1-x^2)^2} \quad\Rightarrow$

$\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2} = \sqrt{\left(\dfrac{1+x^2}{1-x^2}\right)^2} = \dfrac{1+x^2}{1-x^2} = -1 + \dfrac{2}{1-x^2}$ [by division] $= -1 + \dfrac{1}{1+x} + \dfrac{1}{1-x}$ [partial fractions].

So $L = \displaystyle\int_0^{1/2}\left(-1 + \frac{1}{1+x} + \frac{1}{1-x}\right) dx = \Big[-x + \ln|1+x| - \ln|1-x|\Big]_0^{1/2} = \left(-\tfrac{1}{2} + \ln\tfrac{3}{2} - \ln\tfrac{1}{2}\right) - 0 = \ln 3 - \tfrac{1}{2}$.

24. $y = 1 - e^{-x} \quad\Rightarrow\quad dy/dx = -(-e^{-x}) = e^{-x} \quad\Rightarrow\quad 1 + (dy/dx)^2 = 1 + e^{-2x}$. So

$$L = \int_0^2 \sqrt{1 + e^{-2x}}\,dx = \int_1^{e^{-2}} \sqrt{1 + u^2}\left(-\frac{1}{u}\,du\right) \qquad [u = e^{-x}]$$

$$\overset{23}{=} \left[\ln\left|\frac{1 + \sqrt{1+u^2}}{u}\right| - \sqrt{1+u^2}\right]_1^{e^{-2}} \qquad \text{[or substitute } u = \tan\theta\text{]}$$

$$= \ln\left|\frac{1 + \sqrt{1+e^{-4}}}{e^{-2}}\right| - \sqrt{1+e^{-4}} - \ln\left|\frac{1+\sqrt{2}}{1}\right| + \sqrt{2}$$

$$= \ln\left(1 + \sqrt{1+e^{-4}}\right) - \ln e^{-2} - \sqrt{1+e^{-4}} - \ln\left(1+\sqrt{2}\right) + \sqrt{2}$$

$$= \ln\left(1 + \sqrt{1+e^{-4}}\right) + 2 - \sqrt{1+e^{-4}} - \ln\left(1+\sqrt{2}\right) + \sqrt{2}$$

25. $y = \frac{1}{2}x^2 \quad \Rightarrow \quad dy/dx = x \quad \Rightarrow \quad 1 + (dy/dx)^2 = 1 + x^2$. So

$$L = \int_{-1}^{1} \sqrt{1+x^2}\,dx = 2\int_0^1 \sqrt{1+x^2}\,dx \quad [\text{by symmetry}] \quad \overset{21}{=} 2\left[\tfrac{x}{2}\sqrt{1+x^2} + \tfrac{1}{2}\ln\left(x + \sqrt{1+x^2}\,\right)\right]_0^1 \quad \begin{bmatrix}\text{or substitute}\\ x = \tan\theta\end{bmatrix}$$

$$= 2\left[\left(\tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\ln\left(1+\sqrt{2}\right)\right) - \left(0 + \tfrac{1}{2}\ln 1\right)\right] = \sqrt{2} + \ln\left(1+\sqrt{2}\,\right)$$

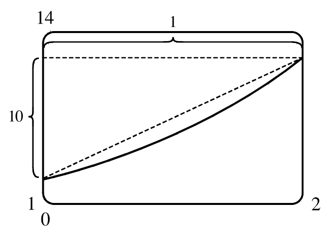
26. $x^2 = (y-4)^3 \quad \Rightarrow \quad x = (y-4)^{3/2} \quad [\text{for } x > 0] \quad \Rightarrow \quad dx/dy = \frac{3}{2}(y-4)^{1/2} \quad \Rightarrow$

$1 + (dx/dy)^2 = 1 + \frac{9}{4}(y-4) = \frac{9}{4}y - 8$. So

$$L = \int_5^8 \sqrt{\tfrac{9}{4}y - 8}\,dy = \int_{13/4}^{10} \sqrt{u}\,\left(\tfrac{4}{9}\,du\right) \quad \begin{bmatrix} u = \frac{9}{4}y - 8, \\ du = \frac{9}{4}\,dy \end{bmatrix} \quad = \tfrac{4}{9}\left[\tfrac{2}{3}u^{3/2}\right]_{13/4}^{10}$$

$$= \tfrac{8}{27}\left[10^{3/2} - \left(\tfrac{13}{4}\right)^{3/2}\right] \quad \left[\text{or } \tfrac{1}{27}\left(80\sqrt{10} - 13\sqrt{13}\,\right)\right]$$

27.

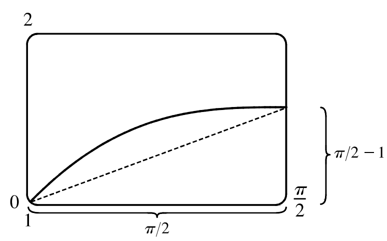


From the figure, the length of the curve is slightly larger than the hypotenuse of the triangle formed by the points $(1,2)$, $(1,12)$, and $(2,12)$. This length is about $\sqrt{10^2 + 1^2} \approx 10$, so we might estimate the length to be 10.

$y = x^2 + x^3 \quad \Rightarrow \quad y' = 2x + 3x^2 \quad \Rightarrow \quad 1 + (y')^2 = 1 + (2x + 3x^2)^2$.

So $L = \int_1^2 \sqrt{1 + (2x + 3x^2)^2}\,dx \approx 10.0556$.

28.

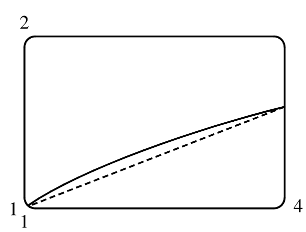


From the figure, the length of the curve is slightly larger than the hypotenuse of the triangle formed by the points $(1,1)$, $\left(\frac{\pi}{2}, 1\right)$, and $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$. This length is about $\sqrt{\left(\frac{\pi}{2}\right)^2 + \left(\frac{\pi}{2} - 1\right)^2} \approx 1.7$, so we might estimate the length to be 1.7. $\quad y = x + \cos x \quad \Rightarrow \quad y' = 1 - \sin x \quad \Rightarrow$

$1 + (y')^2 = 1 + (1 - \sin x)^2$. So

$L = \int_0^{\pi/2} \sqrt{1 + (1 - \sin x)^2}\,dx \approx 1.7294$.

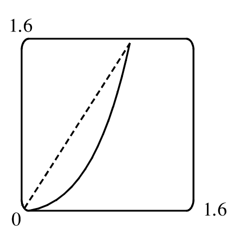
29.



From the figure, the length of the curve is slightly larger than the line segment joining the points $(1,1)$ and $\left(4, \sqrt[3]{4}\,\right)$. This length is $\sqrt{(4-1)^2 + \left(\sqrt[3]{4} - 1\right)^2} \approx 3.057$, so we might estimate the length of the curve to be 3.06. $\quad y = \sqrt[3]{x} = x^{1/3} \quad \Rightarrow \quad y' = \frac{1}{3}x^{-2/3} \quad \Rightarrow$

$1 + (y')^2 = 1 + \frac{1}{9}x^{-4/3}$. So $L = \int_1^4 \sqrt{1 + \frac{1}{9}x^{-4/3}}\,dx \approx 3.0609$.

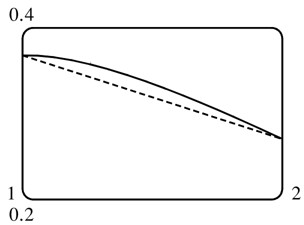
30.



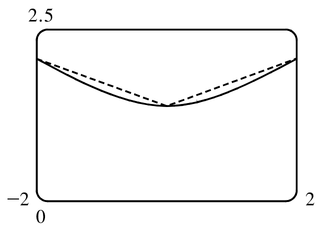
From the figure, the length of the curve is slightly larger than the line segment joining the points $(0,0)$ and $(1, \tan 1)$. This length is $\sqrt{(1-0)^2 + (\tan 1 - 0)^2} \approx 1.851$, so we might estimate the length of the curve to be 1.9. $\quad y = x\tan x \quad \Rightarrow$

$y' = x\sec^2 x + \tan x \quad \Rightarrow \quad 1 + (y')^2 = 1 + (x\sec^2 x + \tan x)^2$. So

$L = \int_0^1 \sqrt{1 + (x\sec^2 x + \tan x)^2}\,dx \approx 1.9799$.

31. 

From the figure, the length of the curve is slightly larger than the line segment joining the points $\left(1, e^{-1}\right)$ and $\left(2, 2e^{-2}\right)$. This length is $\sqrt{(2-1)^2 + (2e^{-2} - e^{-1})^2} \approx 1.0047$, so we might estimate the length of the curve to be 1.005. $y = xe^{-x} \;\Rightarrow\; y' = -xe^{-x} + e^{-x} \;\Rightarrow$ $1 + (y')^2 = 1 + (e^{-x} - xe^{-x})^2$. So $L = \int_1^2 \sqrt{1 + (e^{-x} - xe^{-x})^2}\, dx \approx 1.0054$.

32. 

From the figure, the curve is slightly larger than the sum of the lengths of the line segments joining the points $(-2, \ln 8)$ to $(0, \ln 4)$, and $(0, \ln 4)$ to $(2, \ln 8)$. These line segments each have length $\sqrt{(0+2)^2 + (\ln 4 - \ln 8)^2} \approx 2.1167$, so we might estimate the length of the curve to be $2(2.1167) = 4.2334$ or 4.25.

$$y = \ln(x^2 + 4) \;\Rightarrow\; y' = \frac{2x}{x^2 + 4} \;\Rightarrow\; 1 + (y')^2 = 1 + \frac{4x^2}{(x^2+4)^2}.$$

So $L = \displaystyle\int_{-2}^{2} \sqrt{1 + \frac{4x^2}{(x^2+4)^2}}\, dx \approx 4.2726$.

33. $y = x \sin x \;\Rightarrow\; dy/dx = x\cos x + (\sin x)(1) \;\Rightarrow\; 1 + (dy/dx)^2 = 1 + (x\cos x + \sin x)^2$. Let $f(x) = \sqrt{1 + (dy/dx)^2} = \sqrt{1 + (x\cos x + \sin x)^2}$. Then $L = \int_0^{2\pi} f(x)\, dx$. Since $n = 10$, $\Delta x = \frac{2\pi - 0}{10} = \frac{\pi}{5}$. Now

$$\begin{aligned} L \approx S_{10} &= \tfrac{\pi/5}{3}\left[f(0) + 4f\left(\tfrac{\pi}{5}\right) + 2f\left(\tfrac{2\pi}{5}\right) + 4f\left(\tfrac{3\pi}{5}\right) + 2f\left(\tfrac{4\pi}{5}\right) + 4f\left(\tfrac{5\pi}{5}\right) + 2f\left(\tfrac{6\pi}{5}\right)\right. \\ &\qquad \left. + 4f\left(\tfrac{7\pi}{5}\right) + 2f\left(\tfrac{8\pi}{5}\right) + 4f\left(\tfrac{9\pi}{5}\right) + f(2\pi)\right] \\ &\approx 15.498085 \end{aligned}$$

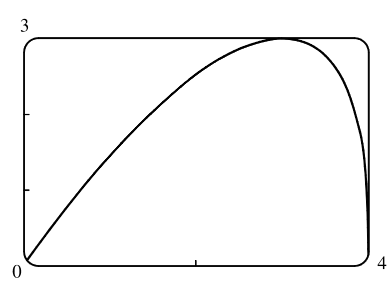
The value of the integral produced by a calculator is 15.374568 (to six decimal places).

34. $y = e^{-x^2} \;\Rightarrow\; dy/dx = e^{-x^2}(-2x) \;\Rightarrow\; L = \int_0^2 f(x)\, dx$, where $f(x) = \sqrt{1 + 4x^2 e^{-2x^2}}$. Since $n = 10$, $\Delta x = \frac{2-0}{10} = \frac{1}{5}$. Now

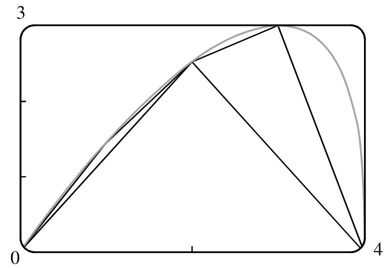
$$\begin{aligned} L \approx S_{10} &= \tfrac{1/5}{3}[f(0) + 4f(0.2) + 2f(0.4) + 4f(0.6) + 2f(0.8) + 4f(1) + 2f(1.2) \\ &\qquad + 4f(1.4) + 2f(1.6) + 4f(1.8) + f(2)] \\ &\approx 2.280559 \end{aligned}$$

The value of the integral produced by a calculator is 2.280526 (to six decimal places).

35. (a) Let $f(x) = y = x\sqrt[3]{4 - x}$ with $0 \le x \le 4$.



(b) The polygon with one side is just the line segment joining the points $(0, f(0)) = (0, 0)$ and $(4, f(4)) = (4, 0)$, and its length $L_1 = 4$.



The polygon with two sides joins the points $(0, 0)$, $(2, f(2)) = \left(2, 2\sqrt[3]{2}\right)$ and $(4, 0)$. Its length

$$L_2 = \sqrt{(2-0)^2 + \left(2\sqrt[3]{2} - 0\right)^2} + \sqrt{(4-2)^2 + \left(0 - 2\sqrt[3]{2}\right)^2} = 2\sqrt{4 + 2^{8/3}} \approx 6.43$$

Similarly, the inscribed polygon with four sides joins the points $(0, 0)$, $\left(1, \sqrt[3]{3}\right)$, $\left(2, 2\sqrt[3]{2}\right)$, $(3, 3)$, and $(4, 0)$, so its length

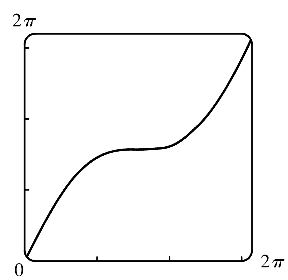
$$L_4 = \sqrt{1 + \left(\sqrt[3]{3}\right)^2} + \sqrt{1 + \left(2\sqrt[3]{2} - \sqrt[3]{3}\right)^2} + \sqrt{1 + \left(3 - 2\sqrt[3]{2}\right)^2} + \sqrt{1+9} \approx 7.50$$

(c) Using the arc length formula with $\dfrac{dy}{dx} = x\left[\frac{1}{3}(4-x)^{-2/3}(-1)\right] + \sqrt[3]{4-x} = \dfrac{12-4x}{3(4-x)^{2/3}}$, the length of the curve is

$$L = \int_0^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^4 \sqrt{1 + \left[\frac{12-4x}{3(4-x)^{2/3}}\right]^2}\, dx.$$

(d) According to a calculator, the length of the curve is $L \approx 7.7988$. The actual value is larger than any of the approximations in part (b). This is always true, since any approximating straight line between two points on the curve is shorter than the length of the curve between the two points.

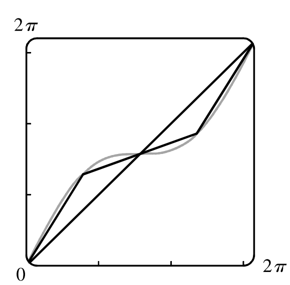
36. (a) Let $f(x) = y = x + \sin x$ with $0 \le x \le 2\pi$.



(b) The polygon with one side is just the line segment joining the points $(0, f(0)) = (0, 0)$ and $(2\pi, f(2\pi)) = (2\pi, 2\pi)$, and its length is $\sqrt{(2\pi - 0)^2 + (2\pi - 0)^2} = 2\sqrt{2}\,\pi \approx 8.9$.

The polygon with two sides joins the points $(0, 0)$, $(\pi, f(\pi)) = (\pi, \pi)$, and $(2\pi, 2\pi)$. Its length is

$$\begin{aligned}\sqrt{(\pi-0)^2 + (\pi-0)^2} + \sqrt{(2\pi-\pi)^2 + (2\pi-\pi)^2} &= \sqrt{2}\,\pi + \sqrt{2}\,\pi \\ &= 2\sqrt{2}\,\pi \approx 8.9\end{aligned}$$



Note from the diagram that the two approximations are the same because the sides of the two-sided polygon are in fact on the same line, since $f(\pi) = \pi = \frac{1}{2}f(2\pi)$.

[continued]

The four-sided polygon joins the points $(0,0)$, $\left(\frac{\pi}{2}, \frac{\pi}{2}+1\right)$, (π,π), $\left(\frac{3\pi}{2}, \frac{3\pi}{2}-1\right)$, and $(2\pi, 2\pi)$, so its length is

$$\sqrt{\left(\tfrac{\pi}{2}\right)^2+\left(\tfrac{\pi}{2}+1\right)^2}+\sqrt{\left(\tfrac{\pi}{2}\right)^2+\left(\tfrac{\pi}{2}-1\right)^2}+\sqrt{\left(\tfrac{\pi}{2}\right)^2+\left(\tfrac{\pi}{2}-1\right)^2}+\sqrt{\left(\tfrac{\pi}{2}\right)^2+\left(\tfrac{\pi}{2}+1\right)^2}\approx 9.4$$

(c) Using the arc length formula with $dy/dx = 1+\cos x$, the length of the curve is

$$L=\int_0^{2\pi}\sqrt{1+(1+\cos x)^2}\,dx=\int_0^{2\pi}\sqrt{2+2\cos x+\cos^2 x}\,dx$$

(d) The calculator approximates the integral as 9.5076. The actual length is larger than the approximations in part (b).

37. $y=e^x \quad\Rightarrow\quad dy/dx=e^x \quad\Rightarrow\quad 1+(dy/dx)^2 \quad\Rightarrow\quad 1+e^{2x} \quad\Rightarrow$

$$L=\int_0^2\sqrt{1+e^{2x}}\,dx=\int_1^{e^2}\sqrt{1+u^2}\left(\frac{1}{u}\,du\right)\qquad\begin{bmatrix}u=e^x,\\ du=e^x\,dx\end{bmatrix}$$

$$\overset{23}{=}\left[\sqrt{1+u^2}-\ln\left|\frac{1+\sqrt{1+u^2}}{u}\right|\right]_1^{e^2}=\left(\sqrt{1+e^4}-\ln\frac{1+\sqrt{1+e^4}}{e^2}\right)-\left(\sqrt{2}-\ln\frac{1+\sqrt{2}}{1}\right)$$

$$=\sqrt{1+e^4}-\ln(1+\sqrt{1+e^4}\,)+2-\sqrt{2}+\ln(1+\sqrt{2}\,)$$

An equivalent answer from a CAS is

$$-\sqrt{2}+\operatorname{arctanh}(\sqrt{2}/2)+\sqrt{e^4+1}-\operatorname{arctanh}(1/\sqrt{e^4+1}\,).$$

38. $y=x^{4/3} \quad\Rightarrow\quad dy/dx=\frac{4}{3}x^{1/3} \quad\Rightarrow\quad 1+(dy/dx)^2=1+\frac{16}{9}x^{2/3} \quad\Rightarrow$

$$L=\int_0^1\sqrt{1+\tfrac{16}{9}x^{2/3}}\,dx=\int_0^{4/3}\sqrt{1+u^2}\,\tfrac{81}{64}u^2\,du\qquad\begin{bmatrix}u=\frac{4}{3}x^{1/3},\ du=\frac{4}{9}x^{-2/3}\,dx,\\ dx=\frac{9}{4}x^{2/3}\,du=\frac{9}{4}\cdot\frac{9}{16}u^2\,du=\frac{81}{64}u^2\,du\end{bmatrix}$$

$$\overset{22}{=}\tfrac{81}{64}\left[\tfrac{1}{8}u(1+2u^2)\sqrt{1+u^2}-\tfrac{1}{8}\ln\left(u+\sqrt{1+u^2}\right)\right]_0^{4/3}=\tfrac{81}{64}\left[\tfrac{1}{6}\left(1+\tfrac{32}{9}\right)\sqrt{\tfrac{25}{9}}-\tfrac{1}{8}\ln\left(\tfrac{4}{3}+\sqrt{\tfrac{25}{9}}\right)\right]$$

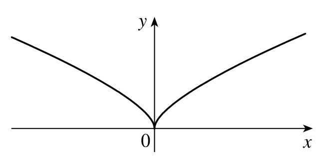
$$=\tfrac{81}{64}\left(\tfrac{1}{6}\cdot\tfrac{41}{9}\cdot\tfrac{5}{3}-\tfrac{1}{8}\ln 3\right)=\tfrac{205}{128}-\tfrac{81}{512}\ln 3\approx 1.4277586$$

39. The astroid $x^{2/3}+y^{2/3}=1$ has an equal length of arc in each quadrant. Thus, we can find the length of the curve in the first quadrant and then multiply by 4. The top half of the astroid has equation $y=(1-x^{2/3})^{3/2}$. Then

$dy/dx=-x^{-1/3}(1-x^{2/3})^{1/2} \quad\Rightarrow\quad 1+(dy/dx)^2=1+\left[-x^{-1/3}(1-x^{2/3})^{1/2}\right]^2=1+x^{-2/3}(1-x^{2/3})=x^{-2/3}$.

So the portion of the astroid in quadrant 1 has length $L=\int_0^1\sqrt{x^{-2/3}}\,dx=\int_0^1 x^{-1/3}\,dx=\left[\frac{3}{2}x^{2/3}\right]_0^1=\frac{3}{2}-0=\frac{3}{2}$. Thus, the astroid has length $4\left(\frac{3}{2}\right)=6$.

40. (a) Graph of $y^3=x^2$:



(b) $y=x^{2/3} \quad\Rightarrow\quad 1+\left(\dfrac{dy}{dx}\right)^2=1+\left(\frac{2}{3}x^{-1/3}\right)^2=1+\frac{4}{9}x^{-2/3}$. So $L=\int_0^1\sqrt{1+\frac{4}{9}x^{-2/3}}\,dx$ [an improper integral].

$x=y^{3/2} \quad\Rightarrow\quad 1+\left(\dfrac{dx}{dy}\right)^2=1+\left(\frac{3}{2}y^{1/2}\right)^2=1+\frac{9}{4}y$. So $L=\int_0^1\sqrt{1+\frac{9}{4}y}\,dy$.

The second integral equals $\frac{4}{9}\cdot\frac{2}{3}\left[\left(1+\frac{9}{4}y\right)^{3/2}\right]_0^1=\frac{8}{27}\left(\frac{13\sqrt{13}}{8}-1\right)=\frac{13\sqrt{13}-8}{27}$. [continued]

The first integral can be evaluated as follows:

$$\int_0^1 \sqrt{1+\tfrac{4}{9}x^{-2/3}}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{\sqrt{9x^{2/3}+4}}{3x^{1/3}}\,dx = \lim_{t\to 0^+}\int_{9t^{2/3}}^9 \frac{\sqrt{u+4}}{18}\,du \qquad \begin{bmatrix} u=9x^{2/3}, \\ du = 6x^{-1/3}\,dx \end{bmatrix}$$

$$= \int_0^9 \frac{\sqrt{u+4}}{18}\,du = \frac{1}{18}\cdot\left[\frac{2}{3}(u+4)^{3/2}\right]_0^9 = \frac{1}{27}(13^{3/2}-4^{3/2}) = \frac{13\sqrt{13}-8}{27}$$

(c) L = length of the arc of this curve from $(-1,1)$ to $(8,4)$

$$= \int_0^1 \sqrt{1+\frac{9}{4}y}\,dy + \int_0^4 \sqrt{1+\frac{9}{4}y}\,dy = \frac{13\sqrt{13}-8}{27} + \frac{8}{27}\left[\left(1+\frac{9}{4}y\right)^{3/2}\right]_0^4 \qquad \text{[from part (b)]}$$

$$= \frac{13\sqrt{13}-8}{27} + \frac{8}{27}\left(10\sqrt{10}-1\right) = \frac{13\sqrt{13}+80\sqrt{10}-16}{27}$$

41. $y = 2x^{3/2} \;\Rightarrow\; y' = 3x^{1/2} \;\Rightarrow\; 1+(y')^2 = 1+9x$. The arc length function with starting point $P_0(1,2)$ is

$s(x) = \int_1^x \sqrt{1+9t}\,dt = \left[\frac{2}{27}(1+9t)^{3/2}\right]_1^x = \frac{2}{27}\left[(1+9x)^{3/2} - 10\sqrt{10}\right]$.

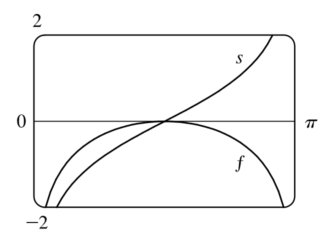
42. (a) $y = f(x) = \ln(\sin x) \;\Rightarrow\; y' = \dfrac{1}{\sin x}\cdot\cos x = \cot x \;\Rightarrow\; 1+(y')^2 = 1+\cot^2 x = \csc^2 x \;\Rightarrow$

$\sqrt{1+(y')^2} = \sqrt{\csc^2 x} = |\csc x|$. Therefore,

$$s(x) = \int_{\pi/2}^x \sqrt{1+[f'(t)]^2}\,dt = \int_{\pi/2}^x \csc t\,dt = \Big[\ln|\csc t - \cot t|\Big]_{\pi/2}^x$$

$$= \ln|\csc x - \cot x| - \ln|1-0| = \ln(\csc x - \cot x)$$

(b) Note that s is increasing on $(0,\pi)$ and that $x=0$ and $x=\pi$ are vertical asymptotes for both f and s.



The arc length function $s(x)$ uses $x=\pi/2$ as a starting point, so when $x<\pi/2$, it gives a length of arc moving in the negative x direction. Thus, $s(x)$ is negative when $x<\pi/2$.

43. $y = \sin^{-1}x + \sqrt{1-x^2} \;\Rightarrow\; y' = \dfrac{1}{\sqrt{1-x^2}} - \dfrac{x}{\sqrt{1-x^2}} = \dfrac{1-x}{\sqrt{1-x^2}} \;\Rightarrow$

$$1+(y')^2 = 1+\frac{(1-x)^2}{1-x^2} = \frac{1-x^2+1-2x+x^2}{1-x^2} = \frac{2-2x}{1-x^2} = \frac{2(1-x)}{(1+x)(1-x)} = \frac{2}{1+x} \;\Rightarrow$$

$\sqrt{1+(y')^2} = \sqrt{\dfrac{2}{1+x}}$. Thus, the arc length function with starting point $(0,1)$ is given by

$$s(x) = \int_0^x \sqrt{1+[f'(t)]^2}\,dt = \int_0^x \sqrt{\frac{2}{1+t}}\,dt = \sqrt{2}\left[2\sqrt{1+t}\right]_0^x = 2\sqrt{2}\left(\sqrt{1+x}-1\right).$$

44. (a) $s(x) = \int_a^x \sqrt{1+[f'(t)]^2}\,dt$ and $s(x) = \int_0^x \sqrt{3t+5}\,dt \;\Rightarrow\; 1+[f'(t)]^2 = 3t+5 \;\Rightarrow\; [f'(t)]^2 = 3t+4 \;\Rightarrow$

$f'(t) = \sqrt{3t+4}$ [since f is increasing]. So $f(t) = \int(3t+4)^{1/2}\,dt = \frac{2}{3}\cdot\frac{1}{3}(3t+4)^{3/2} + C$ and since f has y-intercept 2, $f(0) = \frac{2}{9}\cdot 8 + C$ and $f(0) = 2 \;\Rightarrow\; C = 2 - \frac{16}{9} = \frac{2}{9}$. Thus, $f(t) = \frac{2}{9}(3t+4)^{3/2} + \frac{2}{9}$.

(b) $s(x) = \int_0^x \sqrt{3t+5}\,dt = \left[\frac{2}{9}(3t+5)^{3/2}\right]_0^x = \frac{2}{9}(3x+5)^{3/2} - \frac{2}{9}(5)^{3/2}$.

$s(x) = 3 \iff \frac{2}{9}(3x+5)^{3/2} = 3 + \frac{2}{9}(5\sqrt{5}) \iff (3x+5)^{3/2} = \frac{27}{2} + 5\sqrt{5} \iff 3x+5 = \left(\frac{27}{2}+5\sqrt{5}\right)^{2/3} \Rightarrow$

$x_1 = \frac{1}{3}\left[\left(\frac{27}{2}+5\sqrt{5}\right)^{2/3} - 5\right]$. Thus, the point on the graph of f that is 3 units along the curve from the y-intercept is $(x_1, f(x_1)) \approx (1.159, 4.765)$.

45. The prey hits the ground when $y = 0 \iff 180 - \frac{1}{45}x^2 = 0 \iff x^2 = 45 \cdot 180 \Rightarrow x = \sqrt{8100} = 90$, since x must be positive. $y' = -\frac{2}{45}x \Rightarrow 1 + (y')^2 = 1 + \frac{4}{45^2}x^2$, so the distance traveled by the prey is

$$L = \int_0^{90} \sqrt{1 + \frac{4}{45^2}x^2}\,dx = \int_0^4 \sqrt{1+u^2}\left(\tfrac{45}{2}\,du\right) \qquad \begin{bmatrix} u = \frac{2}{45}x, \\ du = \frac{2}{45}\,dx \end{bmatrix}$$

$$\overset{21}{=} \tfrac{45}{2}\left[\tfrac{1}{2}u\sqrt{1+u^2} + \tfrac{1}{2}\ln\left(u+\sqrt{1+u^2}\right)\right]_0^4 = \tfrac{45}{2}\left[2\sqrt{17} + \tfrac{1}{2}\ln\left(4+\sqrt{17}\right)\right] = 45\sqrt{17} + \tfrac{45}{4}\ln\left(4+\sqrt{17}\right) \approx 209.1 \text{ m}$$

46. $y = 50 - \frac{1}{10}(x-15)^2 \Rightarrow y' = -\frac{1}{5}(x-15) \Rightarrow 1 + (y')^2 = 1 + \frac{1}{5^2}(x-15)^2$, so the distance traveled by the kite is

$$L = \int_0^{25} \sqrt{1 + \frac{1}{5^2}(x-15)^2}\,dx = \int_{-3}^{2} \sqrt{1+u^2}\,(5\,du) \qquad \begin{bmatrix} u = \frac{1}{5}(x-15), \\ du = \frac{1}{5}dx \end{bmatrix}$$

$$\overset{21}{=} 5\left[\frac{1}{2}u\sqrt{1+u^2} + \frac{1}{2}\ln\left(u+\sqrt{1+u^2}\right)\right]_{-3}^{2} = \frac{5}{2}\left[2\sqrt{5} + \ln\left(2+\sqrt{5}\right) + 3\sqrt{10} - \ln\left(-3+\sqrt{10}\right)\right]$$

≈ 43.05 m

47. The sine wave has amplitude 2 and period 30 since it goes through two periods in a distance of 60 cm, so its equation is $y = 2\sin\left(\frac{2\pi}{14}x\right) = 2\sin\left(\frac{\pi}{15}x\right)$. The width w of the flat metal sheet needed to make the panel is the arc length of the sine curve from $x = 0$ to $x = 60$ cm. We set up the integral to evaluate w using the arc length formula with $\frac{dy}{dx} = \frac{2\pi}{15}\cos\left(\frac{\pi}{15}x\right)$: $L = \int_0^{60}\sqrt{1 + \left[\frac{2\pi}{15}\cos\left(\frac{\pi}{15}x\right)\right]^2}\,dx = 2\int_0^{30}\sqrt{1 + \left[\frac{2\pi}{15}\cos\left(\frac{\pi}{15}x\right)\right]^2}\,dx$. This integral would be very difficult to evaluate exactly, so we use a CAS, and find that $L \approx 62.55$ cm.

48. (a) $y = a\cosh\left(\frac{x}{a}\right) \Rightarrow \frac{dy}{dx} = \sinh\left(\frac{x}{a}\right) \Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \sinh^2\left(\frac{x}{a}\right) = \cosh^2\left(\frac{x}{a}\right)$. So

$$L = \int_c^d \sqrt{\cosh^2\left(\frac{x}{a}\right)}\,dx = \int_c^d \cosh\left(\frac{x}{a}\right)dx = \left[a\sinh\left(\frac{x}{a}\right)\right]_c^d = a\left[\sinh\left(\frac{d}{a}\right) - \sinh\left(\frac{c}{a}\right)\right].$$

(b) $A = \int_c^d a\cosh\left(\frac{x}{a}\right)dx = \left[a^2\sinh\left(\frac{x}{a}\right)\right]_c^d = a^2\left[\sinh\left(\frac{d}{a}\right) - \sinh\left(\frac{c}{a}\right)\right]$. The ratio of the area under the catenary to its arc length is $\dfrac{A}{L} = \dfrac{a^2\left[\sinh\left(\frac{d}{a}\right) - \sinh\left(\frac{c}{a}\right)\right]}{a\left[\sinh\left(\frac{d}{a}\right) - \sinh\left(\frac{c}{a}\right)\right]} = a$. Thus, the ratio of the area under the catenary to its arc length over any interval is a, a constant, that does not depend on the interval.

49. $y = c + a\cosh\left(\frac{x}{a}\right) \Rightarrow y' = \sinh\left(\frac{x}{a}\right) \Rightarrow 1 + (y')^2 = 1 + \sinh^2\left(\frac{x}{a}\right) = \cosh^2\left(\frac{x}{a}\right)$. So

$$L = \int_{-b}^{b} \sqrt{\cosh^2\left(\frac{x}{a}\right)}\,dx = 2\int_0^b \cosh\left(\frac{x}{a}\right)dx = 2\left[a\sinh\left(\frac{x}{a}\right)\right]_0^b = 2a\sinh\left(\frac{b}{a}\right).$$

At $x = 0$, $y = c + a$, so $c + a = 9$. The poles are 20 m apart, so $b = 10$, and $L = 20.4 \implies 20.4 = 2asinh(b/a)$. By computer's calculation, we could see that $y = 20.4$ intersects $y = 2xsinh(10/x)$ at $x \approx 28.95$ for $x > 0$. So $a \approx 28.95$ and wire should be attached at a distance of $y = c + acosh(10/a) = 9 - a + acosh(10/a) \approx 10.74$ m above the ground.

50. Let $y = a - b\cosh cx$, where $a = 211.49$, $b = 20.96$, and $c = 0.03291765$. Then $y' = -bc\sinh cx \Rightarrow$ $1 + (y')^2 = 1 + b^2c^2\sinh^2(cx)$. So $L = \int_{-91.2}^{91.2} \sqrt{1 + b^2c^2\sinh^2(cx)}\,dx \approx 451.137 \approx 451$, to the nearest meter.

51. $f(x) = \frac{1}{4}e^x + e^{-x} \Rightarrow f'(x) = \frac{1}{4}e^x - e^{-x} \Rightarrow$

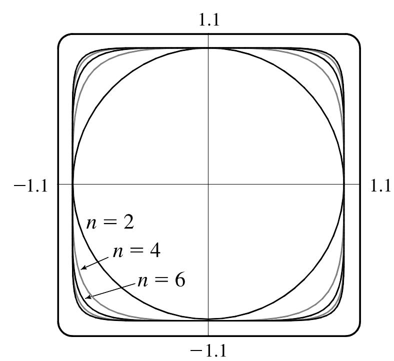
$1 + [f'(x)]^2 = 1 + \left(\frac{1}{4}e^x - e^{-x}\right)^2 = 1 + \frac{1}{16}e^{2x} - \frac{1}{2} + e^{-2x} = \frac{1}{16}e^{2x} + \frac{1}{2} + e^{-2x} = \left(\frac{1}{4}e^x + e^{-x}\right)^2 = [f(x)]^2$. The arc length of the curve $y = f(x)$ on the interval $[a, b]$ is $L = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx = \int_a^b \sqrt{[f(x)]^2}\,dx = \int_a^b f(x)\,dx$, which is the area under the curve $y = f(x)$ on the interval $[a, b]$.

52. By symmetry, the length of the curve in each quadrant is the same, so we'll find the length in the first quadrant and multiply by 4.

$x^{2k} + y^{2k} = 1 \Rightarrow y^{2k} = 1 - x^{2k} \Rightarrow y = (1 - x^{2k})^{1/(2k)}$

(in the first quadrant), so we use the arc length formula with

$$\frac{dy}{dx} = \frac{1}{2k}(1 - x^{2k})^{1/(2k)-1}(-2kx^{2k-1}) = -x^{2k-1}(1 - x^{2k})^{1/(2k)-1}$$



The total length is therefore

$$L_{2k} = 4\int_0^1 \sqrt{1 + [-x^{2k-1}(1 - x^{2k})^{1/(2k)-1}]^2}\,dx = 4\int_0^1 \sqrt{1 + x^{2(2k-1)}(1 - x^{2k})^{1/k-2}}\,dx$$

Now from the graph, we see that as k increases, the "corners" of these fat circles get closer to the points $(\pm1, \pm1)$ and $(\pm1, \mp1)$, and the "edges" of the fat circles approach the lines joining these four points. It seems plausible that as $k \to \infty$, the total length of the fat circle with $n = 2k$ will approach the length of the perimeter of the square with sides of length 2. This is supported by taking the limit as $k \to \infty$ of the equation of the fat circle in the first quadrant: $\lim\limits_{k\to\infty}(1 - x^{2k})^{1/(2k)} = 1$ for $0 \le x < 1$. So we guess that $\lim\limits_{k\to\infty} L_{2k} = 4 \cdot 2 = 8$.

53. $y = \int_1^x \sqrt{t^3 - 1}\,dt \Rightarrow dy/dx = \sqrt{x^3 - 1}$ [by FTC1] $\Rightarrow 1 + (dy/dx)^2 = 1 + \left(\sqrt{x^3 - 1}\right)^2 = x^3 \Rightarrow$

$L = \int_1^4 \sqrt{x^3}\,dx = \int_1^4 x^{3/2}\,dx = \frac{2}{5}\left[x^{5/2}\right]_1^4 = \frac{2}{5}(32 - 1) = \frac{62}{5} = 12.4$

DISCOVERY PROJECT Arc Length Contest

For advice on how to run the contest and a list of student entries, see the article "Arc Length Contest" by Larry Riddle in *The College Mathematics Journal*, Volume 29, No. 4, September 1998, pages 314–320.

8.2 Area of a Surface of Revolution

1. (a) $y = \sqrt[3]{x} = x^{1/3} \quad\Rightarrow\quad dy/dx = \frac{1}{3}x^{-2/3} \quad\Rightarrow\quad ds = \sqrt{1 + (dy/dx)^2}\,dx = \sqrt{1 + \frac{1}{9}x^{-4/3}}\,dx.$

By (7), $S = \int 2\pi y\,ds = \int_1^8 2\pi\sqrt[3]{x}\,\sqrt{1 + \frac{1}{9}x^{-4/3}}\,dx.$

(b) $y = \sqrt[3]{x} \quad\Rightarrow\quad x = y^3 \quad\Rightarrow\quad dx/dy = 3y^2 \quad\Rightarrow\quad ds = \sqrt{1 + (dx/dy)^2}\,dy = \sqrt{1 + 9y^4}\,dy.$

When $x = 1$, $y = 1$ and when $x = 8$, $y = 2$. Thus, $S = \int 2\pi y\,ds = \int_1^2 2\pi y\sqrt{1 + 9y^4}\,dy.$

2. (a) $x^2 = e^y \quad\Rightarrow\quad y = \ln x^2 = 2\ln x$ [for $x > 0$] $\quad\Rightarrow\quad \dfrac{dy}{dx} = \dfrac{2}{x} \quad\Rightarrow\quad ds = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}\,dx = \sqrt{1 + \dfrac{4}{x^2}}\,dx.$

By (7), $S = \displaystyle\int 2\pi y\,ds = \int_1^e 4\pi\ln x\sqrt{1 + \frac{4}{x^2}}\,dx.$

(b) $x^2 = e^y \quad\Rightarrow\quad x = \sqrt{e^y} \quad\Rightarrow\quad e^{y/2}$ [positive since $x > 0$] $\quad\Rightarrow\quad dx/dy = \frac{1}{2}e^{y/2} \quad\Rightarrow$

$ds = \sqrt{1 + (dx/dy)^2}\,dy = \sqrt{1 + \frac{1}{4}e^y}\,dy$. When $x = 1$, $y = 0$ and when $x = e$, $y = 2$. Thus,

$S = \int 2\pi y\,ds = \int_0^2 2\pi y\sqrt{1 + \frac{1}{4}e^y}\,dy.$

3. (a) $x = \ln(2y + 1) \quad\Rightarrow\quad e^x = 2y + 1 \quad\Rightarrow\quad y = \frac{1}{2}e^x - \frac{1}{2} \quad\Rightarrow\quad dy/dx = \frac{1}{2}e^x \quad\Rightarrow$

$ds = \sqrt{1 + (dy/dx)^2}\,dx = \sqrt{1 + \frac{1}{4}e^{2x}}\,dx$. When $y = 0$, $x = 0$ and when $y = 1$, $x = \ln 3$. Thus, by (7),

$S = \int 2\pi y\,ds = \int_0^{\ln 3} 2\pi\left(\frac{1}{2}e^x - \frac{1}{2}\right)\sqrt{1 + \frac{1}{4}e^{2x}}\,dx = \int_0^{\ln 3} \pi(e^x - 1)\sqrt{1 + \frac{1}{4}e^{2x}}\,dx.$

(b) $x = \ln(2y + 1) \quad\Rightarrow\quad dx/dy = \dfrac{2}{2y + 1} \quad\Rightarrow\quad ds = \sqrt{1 + (dx/dy)^2}\,dy = \sqrt{1 + 4/(2y + 1)^2}\,dy$. By (7),

$S = \int 2\pi y\,ds = \int_0^1 2\pi y\,\sqrt{1 + 4/(2y + 1)^2}\,dy.$

4. (a) $y = \tan^{-1}x \quad\Rightarrow\quad dy/dx = 1/(1 + x^2) \quad\Rightarrow\quad ds = \sqrt{1 + (dy/dx)^2}\,dx = \sqrt{1 + 1/(1 + x^2)^2}\,dx$. By (7),

$S = \int 2\pi y\,ds = \int_0^1 2\pi\tan^{-1}x\,\sqrt{1 + 1/(1 + x^2)^2}\,dx.$

(b) $y = \tan^{-1}x \quad\Rightarrow\quad x = \tan y \quad\Rightarrow\quad dx/dy = \sec^2 y \quad\Rightarrow\quad ds = \sqrt{1 + (dx/dy)^2}\,dy = \sqrt{1 + \sec^4 y}\,dy$. When $x = 0$,

$y = 0$ and when $x = 1$, $y = \frac{\pi}{4}$. Thus $S = \int 2\pi y\,ds = \int_0^{\pi/4} 2\pi y\,\sqrt{1 + \sec^4 y}\,dy.$

5. (a) $xy = 4 \;\Rightarrow\; y = \dfrac{4}{x} = 4x^{-1} \;\Rightarrow\; dy/dx = -4x^{-2} \;\Rightarrow\; ds = \sqrt{1 + (dy/dx)^2}\,dx = \sqrt{1 + 16/x^4}\,dx.$

By (8), $S = \int 2\pi x\,ds = \int_1^8 2\pi x\sqrt{1 + 16/x^4}\,dx.$

(b) $xy = 4 \;\Rightarrow\; x = \dfrac{4}{y} = 4y^{-1} \;\Rightarrow\; dx/dy = -4y^{-2} \;\Rightarrow\; ds = \sqrt{1 + (dx/dy)^2}\,dy = \sqrt{1 + 16/y^4}\,dy.$

When $x = 1$, $y = 4$ and when $x = 8$, $y = \frac{1}{2}$. By (8), $S = \displaystyle\int 2\pi x\,ds = \int_{1/2}^{4} \frac{8\pi}{y}\sqrt{1 + 16/y^4}\,dy.$

6. (a) $y = (x+1)^4 \;\Rightarrow\; dy/dx = 4(x+1)^3 \;\Rightarrow\; ds = \sqrt{1 + (dy/dx)^2}\,dx = \sqrt{1 + 16(x+1)^6}\,dx$. By (8),

$S = \int 2\pi x\,ds = \int_0^2 2\pi x\sqrt{1 + 16(x+1)^6}\,dx.$

(b) $y = (x+1)^4 \;\Rightarrow\; x = y^{1/4} - 1 \;\Rightarrow\; \dfrac{dx}{dy} = \dfrac{1}{4}y^{-3/4} \;\Rightarrow\; ds = \sqrt{1 + (dx/dy)^2}\,dy = \sqrt{1 + \frac{1}{16}y^{-3/2}}\,dy.$

When $x = 0$, $y = 1$ and when $x = 2$, $y = 81$. By (8), $S = \int 2\pi x\,ds = \int_1^{81} 2\pi(y^{1/4} - 1)\sqrt{1 + \frac{1}{16}y^{-3/2}}\,dy.$

7. (a) $y = 1 + \sin x \;\Rightarrow\; dy/dx = \cos x \;\Rightarrow\; ds = \sqrt{1 + (dy/dx)^2}\,dx = \sqrt{1 + \cos^2 x}\,dx.$

By (8), $S = \int 2\pi x\,ds = \int_0^{\pi/2} 2\pi x\sqrt{1 + \cos^2 x}\,dx.$

(b) $y = 1 + \sin x \;\Rightarrow\; x = \sin^{-1}(y-1) \;\Rightarrow\; \dfrac{dx}{dy} = \dfrac{1}{\sqrt{1 - (y-1)^2}} \;\Rightarrow$

$ds = \sqrt{1 + \left(\dfrac{dx}{dy}\right)^2}\,dy = \sqrt{1 + \dfrac{1}{1 - (y-1)^2}}\,dy$. When $x = 0$, $y = 1$ and when $x = \pi/2$, $y = 2$.

By (8), $S = \displaystyle\int 2\pi x\,ds = \int_1^2 2\pi \sin^{-1}(y-1)\sqrt{1 + \frac{1}{1 - (y-1)^2}}\,dy$ or $\displaystyle\int_1^2 2\pi \sin^{-1}(y-1)\sqrt{1 + \frac{1}{2y - y^2}}\,dy.$

8. (a) $x = e^{2y} \;\Rightarrow\; y = \dfrac{1}{2}\ln x \;\Rightarrow\; \dfrac{dy}{dx} = \dfrac{1}{2x} \;\Rightarrow\; ds = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}\,dx = \sqrt{1 + \dfrac{1}{4x^2}}\,dx.$

When $y = 0$, $x = 1$ and when $y = 2$, $x = e^4$. By (8), $S = \displaystyle\int 2\pi x\,ds = \int_1^{e^4} 2\pi x\sqrt{1 + \frac{1}{4x^2}}\,dx.$

(b) $x = e^{2y} \;\Rightarrow\; dx/dy = 2e^{2y} \;\Rightarrow\; ds = \sqrt{1 + (dx/dy)^2}\,dy = \sqrt{1 + 4e^{4y}}\,dy.$

By (8), $S = \int 2\pi x\,ds = \int_0^2 2\pi e^{2y}\sqrt{1 + 4e^{4y}}\,dy.$

9. $y = x^3 \;\Rightarrow\; y' = 3x^2$. So

$$\begin{aligned} S &= \textstyle\int_0^2 2\pi y\,\sqrt{1 + (y')^2}\,dx = 2\pi \int_0^2 x^3\,\sqrt{1 + 9x^4}\,dx \\ &= \tfrac{2\pi}{36}\textstyle\int_1^{145} \sqrt{u}\,du \qquad [u = 1 + 9x^4,\ du = 36x^3\,dx] \\ &= \tfrac{\pi}{18}\left[\tfrac{2}{3}u^{3/2}\right]_1^{145} = \tfrac{\pi}{27}\left(145\sqrt{145} - 1\right) \end{aligned}$$

10. $y=\sqrt{5-x} \quad\Rightarrow\quad y'=\frac{1}{2}(5-x)^{-1/2}(-1)=-1/(2\sqrt{5-x}\,)$. So

$$\begin{aligned} S &= \int_3^5 2\pi y\sqrt{1+(y')^2}\,dx = \int_3^5 2\pi\sqrt{5-x}\sqrt{1+\frac{1}{4(5-x)}}\,dx = 2\pi\int_3^5\sqrt{5-x+\tfrac{1}{4}}\,dx \\ &= 2\pi\int_3^5\sqrt{\tfrac{21}{4}-x}\,dx = 2\pi\int_{9/4}^{1/4}\sqrt{u}\,(-du) \qquad \begin{bmatrix} u=\frac{21}{4}-x, \\ du=-dx \end{bmatrix} \\ &= 2\pi\int_{1/4}^{9/4} u^{1/2}\,du = 2\pi\left[\tfrac{2}{3}u^{3/2}\right]_{1/4}^{9/4} = \tfrac{4\pi}{3}\left(\tfrac{27}{8}-\tfrac{1}{8}\right) = \tfrac{13\pi}{3} \end{aligned}$$

11. $y^2=x+1 \quad\Rightarrow\quad y=\sqrt{x+1}$ (for $0\le x\le 3$ and $1\le y\le 2$) $\quad\Rightarrow\quad y'=1/(2\sqrt{x+1}\,)$. So

$$\begin{aligned} S &= \int_0^3 2\pi y\sqrt{1+(y')^2}\,dx = 2\pi\int_0^3\sqrt{x+1}\sqrt{1+\frac{1}{4(x+1)}}\,dx = 2\pi\int_0^3\sqrt{x+1+\tfrac{1}{4}}\,dx \\ &= 2\pi\int_0^3\sqrt{x+\tfrac{5}{4}}\,dx = 2\pi\int_{5/4}^{17/4}\sqrt{u}\,du \qquad \begin{bmatrix} u=x+\frac{5}{4}, \\ du=dx \end{bmatrix} \\ &= 2\pi\left[\tfrac{2}{3}u^{3/2}\right]_{5/4}^{17/4} = 2\pi\cdot\frac{2}{3}\left(\frac{17^{3/2}}{8}-\frac{5^{3/2}}{8}\right) = \frac{\pi}{6}(17\sqrt{17}-5\sqrt{5}). \end{aligned}$$

12. $y=\sqrt{1+e^x} \quad\Rightarrow\quad y'=\frac{1}{2}(1+e^x)^{-1/2}(e^x)=\dfrac{e^x}{2\sqrt{1+e^x}} \quad\Rightarrow$

$$\sqrt{1+(y')^2}=\sqrt{1+\frac{e^{2x}}{4(1+e^x)}}=\sqrt{\frac{4+4e^x+e^{2x}}{4(1+e^x)}}=\sqrt{\frac{(e^x+2)^2}{4(1+e^x)}}=\frac{e^x+2}{2\sqrt{1+e^x}}.\text{ So}$$

$$\begin{aligned} S &= \int_0^1 2\pi y\sqrt{1+(y')^2}\,dx = 2\pi\int_0^1\sqrt{1+e^x}\,\frac{e^x+2}{2\sqrt{1+e^x}}\,dx = \pi\int_0^1(e^x+2)\,dx \\ &= \pi\left[e^x+2x\right]_0^1 = \pi[(e+2)-(1+0)] = \pi(e+1) \end{aligned}$$

13. $y=\cos\left(\frac{1}{2}x\right) \quad\Rightarrow\quad y'=-\frac{1}{2}\sin\left(\frac{1}{2}x\right)$. So

$$\begin{aligned} S &= \int_0^\pi 2\pi y\sqrt{1+(y')^2}\,dx = 2\pi\int_0^\pi\cos\left(\tfrac{1}{2}x\right)\sqrt{1+\tfrac{1}{4}\sin^2\left(\tfrac{1}{2}x\right)}\,dx \\ &= 2\pi\int_0^1\sqrt{1+\tfrac{1}{4}u^2}\,(2\,du) \qquad \begin{bmatrix} u=\sin\left(\frac{1}{2}x\right), \\ du=\frac{1}{2}\cos\left(\frac{1}{2}x\right)dx \end{bmatrix} \\ &= 2\pi\int_0^1\sqrt{4+u^2}\,du \overset{21}{=} 2\pi\left[\frac{u}{2}\sqrt{4+u^2}+2\ln\left(u+\sqrt{4+u^2}\right)\right]_0^1 \\ &= 2\pi\left[\left(\tfrac{1}{2}\sqrt{5}+2\ln(1+\sqrt{5}\,)\right)-(0+2\ln 2)\right] = \pi\sqrt{5}+4\pi\ln\left(\frac{1+\sqrt{5}}{2}\right) \end{aligned}$$

14. $y=\dfrac{x^3}{6}+\dfrac{1}{2x} \quad\Rightarrow\quad \dfrac{dy}{dx}=\dfrac{x^2}{2}-\dfrac{1}{2x^2} \quad\Rightarrow\quad \sqrt{1+\left(\dfrac{dy}{dx}\right)^2}=\sqrt{\dfrac{x^4}{4}+\dfrac{1}{2}+\dfrac{1}{4x^4}}=\sqrt{\left(\dfrac{x^2}{2}+\dfrac{1}{2x^2}\right)^2}=\dfrac{x^2}{2}+\dfrac{1}{2x^2}$. So

$$\begin{aligned} S &= \int_{1/2}^1 2\pi\left(\frac{x^3}{6}+\frac{1}{2x}\right)\left(\frac{x^2}{2}+\frac{1}{2x^2}\right)dx = 2\pi\int_{1/2}^1\left(\frac{x^5}{12}+\frac{x}{12}+\frac{x}{4}+\frac{1}{4x^3}\right)dx \\ &= 2\pi\int_{1/2}^1\left(\frac{x^5}{12}+\frac{x}{3}+\frac{x^{-3}}{4}\right)dx = 2\pi\left[\frac{x^6}{72}+\frac{x^2}{6}-\frac{x^{-2}}{8}\right]_{1/2}^1 \\ &= 2\pi\left[\left(\tfrac{1}{72}+\tfrac{1}{6}-\tfrac{1}{8}\right)-\left(\tfrac{1}{64\cdot 72}+\tfrac{1}{24}-\tfrac{1}{2}\right)\right] = 2\pi\left(\tfrac{263}{512}\right) = \tfrac{263}{256}\pi \end{aligned}$$

15. $x = \frac{1}{3}(y^2+2)^{3/2} \quad \Rightarrow \quad dx/dy = \frac{1}{2}(y^2+2)^{1/2}(2y) = y\sqrt{y^2+2} \quad \Rightarrow \quad 1 + (dx/dy)^2 = 1 + y^2(y^2+2) = (y^2+1)^2$.

So $S = 2\pi \int_1^2 y(y^2+1)\,dy = 2\pi\left[\frac{1}{4}y^4 + \frac{1}{2}y^2\right]_1^2 = 2\pi\left(4+2-\frac{1}{4}-\frac{1}{2}\right) = \frac{21\pi}{2}$.

16. $x = 1 + 2y^2 \quad \Rightarrow \quad 1 + (dx/dy)^2 = 1 + (4y)^2 = 1 + 16y^2$.

So $S = 2\pi \int_1^2 y\sqrt{1+16y^2}\,dy = \frac{\pi}{16}\int_1^2 (16y^2+1)^{1/2} 32y\,dy = \frac{\pi}{16}\left[\frac{2}{3}(16y^2+1)^{3/2}\right]_1^2 = \frac{\pi}{24}\left(65\sqrt{65} - 17\sqrt{17}\right)$.

17. $y = \frac{1}{3}x^{3/2} \quad \Rightarrow \quad y' = \frac{1}{2}x^{1/2} \quad \Rightarrow \quad 1 + (y')^2 = 1 + \frac{1}{4}x$. So

$$\begin{aligned} S &= \int_0^{12} 2\pi x\sqrt{1+(y')^2}\,dx = 2\pi\int_0^{12} x\sqrt{1+\tfrac{1}{4}x}\,dx = 2\pi\int_0^{12} x\,\tfrac{1}{2}\sqrt{4+x}\,dx \\ &= \pi\int_4^{16}(u-4)\sqrt{u}\,du \qquad \begin{bmatrix} u = x+4, \\ du = dx \end{bmatrix} \\ &= \pi\int_4^{16}\left(u^{3/2} - 4u^{1/2}\right)du = \pi\left[\tfrac{2}{5}u^{5/2} - \tfrac{8}{3}u^{3/2}\right]_4^{16} = \pi\left[\left(\tfrac{2}{5}\cdot 1024 - \tfrac{8}{3}\cdot 64\right) - \left(\tfrac{2}{5}\cdot 32 - \tfrac{8}{3}\cdot 8\right)\right] \\ &= \pi\left(\tfrac{2}{5}\cdot 992 - \tfrac{8}{3}\cdot 56\right) = \pi\left(\tfrac{5952-2240}{15}\right) = \tfrac{3712\pi}{15} \end{aligned}$$

18. $x^{2/3} + y^{2/3} = 1$, $0 \le y \le 1$. The curve is symmetric about the y-axis from $x = -1$ to $x = 1$, so we'll use the portion of the curve from $x = 0$ to $x = 1$. $\quad y^{2/3} = 1 - x^{2/3} \quad \Rightarrow \quad y = (1-x^{2/3})^{3/2} \quad \Rightarrow$

$y' = \frac{3}{2}(1-x^{2/3})^{1/2}\left(-\frac{2}{3}x^{-1/3}\right) = -\dfrac{\sqrt{1-x^{2/3}}}{x^{1/3}} \quad \Rightarrow \quad 1 + (y')^2 = 1 + \dfrac{1-x^{2/3}}{x^{2/3}} = \dfrac{x^{2/3}+1-x^{2/3}}{x^{2/3}} = x^{-2/3}$. So

$S = \int_0^1 2\pi x\sqrt{1+(y')^2}\,dx = 2\pi\int_0^1 x(x^{-1/3})\,dx = 2\pi\int_0^1 x^{2/3}\,dx = 2\pi\left[\frac{3}{5}x^{5/3}\right]_0^1 = 2\pi\left(\frac{3}{5}\right) = \frac{6\pi}{5}$.

19. $x = \sqrt{a^2-y^2} \quad \Rightarrow \quad dx/dy = \frac{1}{2}(a^2-y^2)^{-1/2}(-2y) = -y/\sqrt{a^2-y^2} \quad \Rightarrow$

$1 + \left(\dfrac{dx}{dy}\right)^2 = 1 + \dfrac{y^2}{a^2-y^2} = \dfrac{a^2-y^2}{a^2-y^2} + \dfrac{y^2}{a^2-y^2} = \dfrac{a^2}{a^2-y^2}$. So

$$S = \int_0^{a/2} 2\pi\sqrt{a^2-y^2}\,\frac{a}{\sqrt{a^2-y^2}}\,dy = 2\pi\int_0^{a/2} a\,dy = 2\pi a\big[\,y\,\big]_0^{a/2} = 2\pi a\left(\frac{a}{2} - 0\right) = \pi a^2.$$

Note that this is $\frac{1}{4}$ the surface area of a sphere of radius a, and the length of the interval $y = 0$ to $y = a/2$ is $\frac{1}{4}$ the length of the interval $y = -a$ to $y = a$.

20. $y = \frac{1}{4}x^2 - \frac{1}{2}\ln x \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{x}{2} - \dfrac{1}{2x} \quad \Rightarrow \quad 1 + \left(\dfrac{dy}{dx}\right)^2 = 1 + \dfrac{x^2}{4} - \dfrac{1}{2} + \dfrac{1}{4x^2} = \dfrac{x^2}{4} + \dfrac{1}{2} + \dfrac{1}{4x^2} = \left(\dfrac{x}{2} + \dfrac{1}{2x}\right)^2$. So

$$\begin{aligned} S &= \int_1^2 2\pi x\sqrt{\left(\frac{x}{2}+\frac{1}{2x}\right)^2}\,dx = 2\pi\int_1^2 x\left(\frac{x}{2}+\frac{1}{2x}\right)dx = \pi\int_1^2 (x^2+1)\,dx = \pi\left[\tfrac{1}{3}x^3 + x\right]_1^2 \\ &= \pi\left[\left(\tfrac{8}{3}+2\right) - \left(\tfrac{1}{3}+1\right)\right] = \tfrac{10}{3}\pi \end{aligned}$$

21. $y = e^{-x^2} \quad \Rightarrow \quad dy/dx = -2xe^{-x^2} \quad \Rightarrow \quad ds = \sqrt{1+(dy/dx)^2}\,dx = \sqrt{1+4x^2e^{-2x^2}}\,dx$. By (7),

$S = \int 2\pi y\,ds = \int_{-1}^1 2\pi e^{-x^2}\sqrt{1+4x^2e^{-2x^2}}\,dx \approx 11.0753$.

22. $xy = y^2 - 1 \;\Rightarrow\; x = y - y^{-1} \;\Rightarrow\; dx/dy = 1 + y^{-2} \;\Rightarrow\; ds = \sqrt{1 + (dx/dy)^2}\,dy = \sqrt{1 + (1 + y^{-2})^2}\,dy.$

By (7), $S = \int 2\pi y\,ds = \int_1^3 2\pi y\sqrt{1 + (1 + y^{-2})^2}\,dy \approx 40.8099.$

23. $x = y + y^3 \;\Rightarrow\; dx/dy = 1 + 3y^2 \;\Rightarrow\; ds = \sqrt{1 + (dx/dy)^2}\,dy = \sqrt{1 + (1 + 3y^2)^2}\,dy.$

By (8), $S = \int 2\pi x\,ds = \int_0^1 2\pi(y + y^3)\sqrt{1 + (1 + 3y^2)^2}\,dy \approx 13.5134.$

24. $y = x + \sin x \;\Rightarrow\; dy/dx = 1 + \cos x \;\Rightarrow\; ds = \sqrt{1 + (dy/dx)^2}\,dx = \sqrt{1 + (1 + \cos x)^2}\,dx.$

By (8), $S = \int 2\pi x\,ds = \int_0^{2\pi/3} 2\pi x\sqrt{1 + (1 + \cos x)^2}\,dx \approx 21.2980.$

25. $\ln y = x - y^2 \;\Rightarrow\; x = \ln y + y^2 \;\Rightarrow\; \dfrac{dx}{dy} = \dfrac{1}{y} + 2y \;\Rightarrow\; ds = \sqrt{1 + (dx/dy)^2}\,dy = \sqrt{1 + (y^{-1} + 2y)^2}\,dy.$

By (7), $S = \int 2\pi y\,ds = \int_1^4 2\pi y\sqrt{1 + (y^{-1} + 2y)^2}\,dy \approx 286.9239.$

26. $x = \cos^2 y \;\Rightarrow\; dx/dy = 2\cos y\,(-\sin y) \;\Rightarrow\; ds = \sqrt{1 + (dx/dy)^2}\,dy = \sqrt{1 + 4\sin^2 y\cos^2 y}\,dy.$

By (8), $S = \int 2\pi x\,ds = \int_0^{\pi/2} 2\pi\cos^2 y\sqrt{1 + 4\sin^2 y\cos^2 y}\,dy \approx 6.0008.$

27. $y = 1/x \;\Rightarrow\; ds = \sqrt{1 + (dy/dx)^2}\,dx = \sqrt{1 + (-1/x^2)^2}\,dx = \sqrt{1 + 1/x^4}\,dx \;\Rightarrow$

$$S = \int_1^2 2\pi\cdot\frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\,dx = 2\pi\int_1^2 \frac{\sqrt{x^4 + 1}}{x^3}\,dx = 2\pi\int_1^4 \frac{\sqrt{u^2 + 1}}{u^2}\left(\tfrac{1}{2}\,du\right) \qquad [u = x^2,\ du = 2x\,dx]$$

$$= \pi\int_1^4 \frac{\sqrt{1 + u^2}}{u^2}\,du \overset{24}{=} \pi\left[-\frac{\sqrt{1 + u^2}}{u} + \ln\left(u + \sqrt{1 + u^2}\right)\right]_1^4$$

$$= \pi\left[-\tfrac{\sqrt{17}}{4} + \ln\left(4 + \sqrt{17}\right) + \tfrac{\sqrt{2}}{1} - \ln\left(1 + \sqrt{2}\right)\right] = \frac{\pi}{4}\left[4\ln\left(\sqrt{17} + 4\right) - 4\ln\left(\sqrt{2} + 1\right) - \sqrt{17} + 4\sqrt{2}\right]$$

28. $y = \sqrt{x^2 + 1} \;\Rightarrow\; \dfrac{dy}{dx} = \dfrac{x}{\sqrt{x^2 + 1}} \;\Rightarrow\; ds = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}\,dx = \sqrt{1 + \dfrac{x^2}{x^2 + 1}}\,dx \;\Rightarrow$

$$S = \int_0^3 2\pi\sqrt{x^2 + 1}\sqrt{1 + \frac{x^2}{x^2 + 1}}\,dx = 2\pi\int_0^3 \sqrt{2x^2 + 1}\,dx = 2\sqrt{2}\,\pi\int_0^3 \sqrt{x^2 + \left(\tfrac{1}{\sqrt{2}}\right)^2}\,dx$$

$$\overset{21}{=} 2\sqrt{2}\pi\left[\tfrac{1}{2}x\sqrt{x^2 + \tfrac{1}{2}} + \tfrac{1}{4}\ln\left(x + \sqrt{x^2 + \tfrac{1}{2}}\right)\right]_0^3 = 2\sqrt{2}\pi\left[\tfrac{3}{2}\sqrt{9 + \tfrac{1}{2}} + \tfrac{1}{4}\ln\left(3 + \sqrt{9 + \tfrac{1}{2}}\right) - \tfrac{1}{4}\ln\tfrac{1}{\sqrt{2}}\right]$$

$$= 2\sqrt{2}\pi\left[\tfrac{3}{2}\sqrt{\tfrac{19}{2}} + \tfrac{1}{4}\ln\left(3 + \sqrt{\tfrac{19}{2}}\right) + \tfrac{1}{4}\ln\sqrt{2}\right] = 2\sqrt{2}\pi\left[\tfrac{3}{2}\tfrac{\sqrt{19}}{\sqrt{2}} + \tfrac{1}{4}\ln\left(3\sqrt{2} + \sqrt{19}\right)\right]$$

$$= 3\sqrt{19}\pi + \tfrac{\pi}{\sqrt{2}}\ln\left(3\sqrt{2} + \sqrt{19}\right)$$

29. $y = x^3$ and $0 \le y \le 1 \;\Rightarrow\; y' = 3x^2$ and $0 \le x \le 1$.

$$S = \int_0^1 2\pi x\sqrt{1 + (3x^2)^2}\,dx = 2\pi\int_0^3 \sqrt{1 + u^2}\,\tfrac{1}{6}\,du \qquad \left[\begin{array}{l} u = 3x^2, \\ du = 6x\,dx \end{array}\right]$$

$$= \tfrac{\pi}{3}\int_0^3 \sqrt{1 + u^2}\,du \overset{21}{=} \text{[or use CAS]}\ \tfrac{\pi}{3}\left[\tfrac{1}{2}u\sqrt{1 + u^2} + \tfrac{1}{2}\ln\left(u + \sqrt{1 + u^2}\right)\right]_0^3$$

$$= \tfrac{\pi}{3}\left[\tfrac{3}{2}\sqrt{10} + \tfrac{1}{2}\ln\left(3 + \sqrt{10}\right)\right] = \tfrac{\pi}{6}\left[3\sqrt{10} + \ln\left(3 + \sqrt{10}\right)\right]$$

30. $y = \ln(x+1)$, $0 \le x \le 1$. $ds = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}\,dx = \sqrt{1 + \left(\dfrac{1}{x+1}\right)^2}\,dx$, so

$$S = \int_0^1 2\pi x \sqrt{1 + \frac{1}{(x+1)^2}}\,dx = \int_1^2 2\pi(u-1)\sqrt{1 + \frac{1}{u^2}}\,du \qquad [u = x+1,\ du = dx]$$

$$= 2\pi \int_1^2 u \frac{\sqrt{1+u^2}}{u}\,du - 2\pi \int_1^2 \frac{\sqrt{1+u^2}}{u}\,du = 2\pi \int_1^2 \sqrt{1+u^2}\,du - 2\pi \int_1^2 \frac{\sqrt{1+u^2}}{u}\,du$$

$$\overset{21,23}{=} \text{[or use CAS]} \quad 2\pi\left[\tfrac{1}{2}u\sqrt{1+u^2} + \tfrac{1}{2}\ln\left(u + \sqrt{1+u^2}\right)\right]_1^2 - 2\pi\left[\sqrt{1+u^2} - \ln\left(\frac{1+\sqrt{1+u^2}}{u}\right)\right]_1^2$$

$$= 2\pi\left[\sqrt{5} + \tfrac{1}{2}\ln\left(2+\sqrt{5}\right) - \tfrac{1}{2}\sqrt{2} - \tfrac{1}{2}\ln\left(1+\sqrt{2}\right)\right] - 2\pi\left[\sqrt{5} - \ln\left(\tfrac{1+\sqrt{5}}{2}\right) - \sqrt{2} + \ln\left(1+\sqrt{2}\right)\right]$$

$$= 2\pi\left[\tfrac{1}{2}\ln\left(2+\sqrt{5}\right) + \ln\left(\tfrac{1+\sqrt{5}}{2}\right) + \tfrac{\sqrt{2}}{2} - \tfrac{3}{2}\ln\left(1+\sqrt{2}\right)\right]$$

31. $y = \frac{1}{5}x^5 \quad \Rightarrow \quad dy/dx = x^4 \quad \Rightarrow \quad 1 + (dy/dx)^2 = 1 + x^8 \quad \Rightarrow \quad S = \int_0^5 2\pi\left(\frac{1}{5}x^5\right)\sqrt{1+x^8}\,dx.$

Let $f(x) = \frac{2}{5}\pi x^5\sqrt{1+x^8}$. Since $n = 10$, $\Delta x = \frac{5-0}{10} = \frac{1}{2}$. Then

$$\begin{aligned} S \approx S_{10} &= \tfrac{1/2}{3}[f(0) + 4f(0.5) + 2f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + 2f(3) \\ &\qquad + 4f(3.5) + 2f(4) + 4f(4.5) + f(5)] \\ &\approx 1{,}230{,}507 \end{aligned}$$

The value of the integral produced by a calculator is approximately 1,227,192.

32. $y = x\ln x \quad \Rightarrow \quad dy/dx = x \cdot \dfrac{1}{x} + \ln x = 1 + \ln x \quad \Rightarrow \quad 1 + (dy/dx)^2 = 1 + (1+\ln x)^2 \quad \Rightarrow$

$S = \int_1^2 2\pi x \ln x\,\sqrt{1 + (1+\ln x)^2}\,dx$. Let $f(x) = 2\pi x\ln x\,\sqrt{1+(1+\ln x)^2}$. Since $n = 10$, $\Delta x = \frac{2-1}{10} = \frac{1}{10}$. Then

$$\begin{aligned} S \approx S_{10} &= \tfrac{1/10}{3}[f(1) + 4f(1.1) + 2f(1.2) + 4f(1.3) + 2f(1.4) + 4f(1.5) + 2f(1.6) \\ &\qquad + 4f(1.7) + 2f(1.8) + 4f(1.9) + f(2)] \\ &\approx 7.248933 \end{aligned}$$

The value of the integral produced by a calculator is 7.248934 (to six decimal places).

33. $S = 2\pi \displaystyle\int_1^\infty y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\,dx = 2\pi \int_1^\infty \frac{\sqrt{x^4+1}}{x^3}\,dx$. Rather than trying to evaluate this integral, note that $\sqrt{x^4+1} > \sqrt{x^4} = x^2$ for $x > 0$. Thus, if the area is finite,

$S = 2\pi \displaystyle\int_1^\infty \frac{\sqrt{x^4+1}}{x^3}\,dx > 2\pi \int_1^\infty \frac{x^2}{x^3}\,dx = 2\pi \int_1^\infty \frac{1}{x}\,dx$. But we know that this integral diverges, so the area S is infinite.

34. $S = \int_0^\infty 2\pi y\,\sqrt{1 + (dy/dx)^2}\,dx = 2\pi \int_0^\infty e^{-x}\sqrt{1 + (-e^{-x})^2}\,dx \qquad [y = e^{-x},\ y' = -e^{-x}].$

Evaluate $I = \int e^{-x}\sqrt{1+(-e^{-x})^2}\,dx$ by using the substitution $u = -e^{-x}$, $du = e^{-x}\,dx$:

$I = \int \sqrt{1+u^2}\,du \overset{21}{=} \frac{1}{2}u\sqrt{1+u^2} + \frac{1}{2}\ln\left(u + \sqrt{1+u^2}\right) + C = \frac{1}{2}(-e^{-x})\sqrt{1+e^{-2x}} + \frac{1}{2}\ln\left(-e^{-x} + \sqrt{1+e^{-2x}}\right) + C.$

[continued]

Returning to the surface area integral, we have

$$\begin{aligned}
S &= 2\pi \lim_{t\to\infty} \int_0^t e^{-x}\sqrt{1+(-e^{-x})^2}\,dx = 2\pi \lim_{t\to\infty} \left[\tfrac{1}{2}(-e^{-x})\sqrt{1+e^{-2x}} + \tfrac{1}{2}\ln\!\left(-e^{-x}+\sqrt{1+e^{-2x}}\right)\right]_0^t \\
&= 2\pi \lim_{t\to\infty} \left\{\left[\tfrac{1}{2}(-e^{-t})\sqrt{1+e^{-2t}} + \tfrac{1}{2}\ln\!\left(-e^{-t}+\sqrt{1+e^{-2t}}\right)\right] - \left[\tfrac{1}{2}(-1)\sqrt{1+1} + \tfrac{1}{2}\ln\!\left(-1+\sqrt{1+1}\right)\right]\right\} \\
&= 2\pi \left\{\left[\tfrac{1}{2}(0)\sqrt{1} + \tfrac{1}{2}\ln\!\left(0+\sqrt{1}\right)\right] - \left[-\tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\ln\!\left(-1+\sqrt{2}\right)\right]\right\} \\
&= 2\pi\left\{[0] + \tfrac{1}{2}\left[\sqrt{2} - \ln\!\left(\sqrt{2}-1\right)\right]\right\} = \pi\left[\sqrt{2} - \ln\!\left(\sqrt{2}-1\right)\right]
\end{aligned}$$

35. As seen in the exercise figure, the loop of the curve $3ay^2 = x(a-x)^2$ extends from $x=0$ to $x=a$. The top half of the loop is given by $y = \sqrt{\dfrac{1}{3a}x(a-x)^2} = \dfrac{1}{\sqrt{3a}}\sqrt{x}\,|a-x| = \dfrac{1}{\sqrt{3a}}\sqrt{x}(a-x)$, since $x \le a$. Now,

$$\frac{dy}{dx} = \frac{1}{\sqrt{3a}}\left[\sqrt{x}\,(-1) + \frac{1}{2\sqrt{x}}(a-x)\right] = \frac{1}{\sqrt{3a}}\left[-\frac{2x}{2\sqrt{x}} + \frac{a-x}{2\sqrt{x}}\right] = \frac{1}{\sqrt{3a}}\left[\frac{a-3x}{2\sqrt{x}}\right] \quad\Rightarrow$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{(a-3x)^2}{12ax} = \frac{12ax}{12ax} + \frac{a^2-6ax+9x^2}{12ax} = \frac{a^2+6ax+9x^2}{12ax} = \frac{(a+3x)^2}{12ax}.$$

(a) $$\begin{aligned}
S &= \int 2\pi y\,ds = 2\pi\int_0^a \frac{\sqrt{x}(a-x)}{\sqrt{3a}}\sqrt{\frac{(a+3x)^2}{12ax}}\,dx = 2\pi\int_0^a \frac{(a-x)(a+3x)}{6a}\,dx \\
&= \frac{\pi}{3a}\int_0^a (a^2+2ax-3x^2)\,dx = \frac{\pi}{3a}\left[a^2x + ax^2 - x^3\right]_0^a = \frac{\pi}{3a}(a^3+a^3-a^3) \\
&= \frac{\pi}{3a}\cdot a^3 = \frac{\pi a^2}{3}
\end{aligned}$$

Note that the top half of the loop has been rotated about the x-axis, producing the full surface.

(b) We must rotate the full loop about the y-axis, so we get double the area obtained by rotating the top half of the loop:

$$\begin{aligned}
S &= 2\int_{x=0}^a 2\pi x\,ds = 4\pi\int_0^a x\,\frac{a+3x}{\sqrt{12ax}}\,dx = \frac{4\pi}{2\sqrt{3a}}\int_0^a x^{1/2}(a+3x)\,dx = \frac{2\pi}{\sqrt{3a}}\int_0^a (ax^{1/2}+3x^{3/2})\,dx \\
&= \frac{2\pi}{\sqrt{3a}}\left[\frac{2}{3}ax^{3/2} + \frac{6}{5}x^{5/2}\right]_0^a = \frac{2\pi\sqrt{3}}{3\sqrt{a}}\left(\frac{2}{3}a^{5/2} + \frac{6}{5}a^{5/2}\right) = \frac{2\pi\sqrt{3}}{3}\left(\frac{2}{3}+\frac{6}{5}\right)a^2 = \frac{2\pi\sqrt{3}}{3}\left(\frac{28}{15}\right)a^2 \\
&= \frac{56\pi\sqrt{3}a^2}{45}
\end{aligned}$$

36. In general, if the parabola $y = ax^2$, $-c \le x \le c$, is rotated about the y-axis, the surface area it generates is

$$\begin{aligned}
2\pi\int_0^c x\sqrt{1+(2ax)^2}\,dx &= 2\pi\int_0^{2ac} \frac{u}{2a}\sqrt{1+u^2}\,\frac{1}{2a}\,du \quad \begin{bmatrix} u = 2ax, \\ du = 2a\,dx \end{bmatrix} \\
&= \frac{\pi}{4a^2}\int_0^{2ac} \left(1+u^2\right)^{1/2} 2u\,du = \frac{\pi}{4a^2}\left[\tfrac{2}{3}\left(1+u^2\right)^{3/2}\right]_0^{2ac} = \frac{\pi}{6a^2}\left[\left(1+4a^2c^2\right)^{3/2} - 1\right]
\end{aligned}$$

Here $2c = 3$ m and $ac^2 = 1$ m, so $c = \frac{3}{2}$ and $a = \frac{4}{9}$. Thus, the surface area is

$$\begin{aligned}
S &= \frac{\pi}{6}\frac{81}{16}\left[\left(1 + 4\cdot\frac{16}{81}\cdot\frac{9}{4}\right)^{3/2} - 1\right] = \frac{81\pi}{96}\left[\left(1+\frac{16}{9}\right)^{3/2} - 1\right] \\
&\approx 9.62\ \mathrm{m}^2
\end{aligned}$$

37. (a) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1 \quad\Rightarrow\quad \dfrac{y\,(dy/dx)}{b^2}=-\dfrac{x}{a^2} \quad\Rightarrow\quad \dfrac{dy}{dx}=-\dfrac{b^2x}{a^2y} \quad\Rightarrow$

$$1+\left(\frac{dy}{dx}\right)^2=1+\frac{b^4x^2}{a^4y^2}=\frac{b^4x^2+a^4y^2}{a^4y^2}=\frac{b^4x^2+a^4b^2\left(1-x^2/a^2\right)}{a^4b^2\left(1-x^2/a^2\right)}=\frac{a^4b^2+b^4x^2-a^2b^2x^2}{a^4b^2-a^2b^2x^2}$$

$$=\frac{a^4+b^2x^2-a^2x^2}{a^4-a^2x^2}=\frac{a^4-\left(a^2-b^2\right)x^2}{a^2(a^2-x^2)}$$

The ellipsoid's surface area is twice the area generated by rotating the first-quadrant portion of the ellipse about the x-axis. Thus,

$$S=2\int_0^a 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=4\pi\int_0^a\frac{b}{a}\sqrt{a^2-x^2}\,\frac{\sqrt{a^4-(a^2-b^2)x^2}}{a\sqrt{a^2-x^2}}\,dx=\frac{4\pi b}{a^2}\int_0^a\sqrt{a^4-(a^2-b^2)x^2}\,dx$$

$$=\frac{4\pi b}{a^2}\int_0^{a\sqrt{a^2-b^2}}\sqrt{a^4-u^2}\,\frac{du}{\sqrt{a^2-b^2}}\quad\left[u=\sqrt{a^2-b^2}\,x\right]\quad\overset{30}{=}\frac{4\pi b}{a^2\sqrt{a^2-b^2}}\left[\frac{u}{2}\sqrt{a^4-u^2}+\frac{a^4}{2}\sin^{-1}\!\left(\frac{u}{a^2}\right)\right]_0^{a\sqrt{a^2-b^2}}$$

$$=\frac{4\pi b}{a^2\sqrt{a^2-b^2}}\left[\frac{a\sqrt{a^2-b^2}}{2}\sqrt{a^4-a^2(a^2-b^2)}+\frac{a^4}{2}\sin^{-1}\frac{\sqrt{a^2-b^2}}{a}\right]=2\pi\left[b^2+\frac{a^2b\sin^{-1}\dfrac{\sqrt{a^2-b^2}}{a}}{\sqrt{a^2-b^2}}\right]$$

(b) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1 \quad\Rightarrow\quad \dfrac{x\,(dx/dy)}{a^2}=-\dfrac{y}{b^2} \quad\Rightarrow\quad \dfrac{dx}{dy}=-\dfrac{a^2y}{b^2x} \quad\Rightarrow$

$$1+\left(\frac{dx}{dy}\right)^2=1+\frac{a^4y^2}{b^4x^2}=\frac{b^4x^2+a^4y^2}{b^4x^2}=\frac{b^4a^2(1-y^2/b^2)+a^4y^2}{b^4a^2(1-y^2/b^2)}=\frac{a^2b^4-a^2b^2y^2+a^4y^2}{a^2b^4-a^2b^2y^2}$$

$$=\frac{b^4-b^2y^2+a^2y^2}{b^4-b^2y^2}=\frac{b^4-(b^2-a^2)y^2}{b^2(b^2-y^2)}$$

The oblate spheroid's surface area is twice the area generated by rotating the first-quadrant portion of the ellipse about the y-axis. Thus,

$$S=2\int_0^b 2\pi\,x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy=4\pi\int_0^b\frac{a}{b}\sqrt{b^2-y^2}\,\frac{\sqrt{b^4-(b^2-a^2)y^2}}{b\sqrt{b^2-y^2}}\,dy$$

$$=\frac{4\pi a}{b^2}\int_0^b\sqrt{b^4-\left(b^2-a^2\right)y^2}\,dy=\frac{4\pi a}{b^2}\int_0^b\sqrt{b^4+\left(a^2-b^2\right)\,y^2}\,dy\qquad[\text{since } a>b]$$

$$=\frac{4\pi a}{b^2}\int_0^{b\sqrt{a^2-b^2}}\sqrt{b^4+u^2}\,\frac{du}{\sqrt{a^2-b^2}}\qquad\left[u=\sqrt{a^2-b^2}\,y\right]$$

$$\overset{21}{=}\frac{4\pi a}{b^2\sqrt{a^2-b^2}}\left[\frac{u}{2}\sqrt{b^4+u^2}+\frac{b^4}{2}\ln\!\left(u+\sqrt{b^4+u^2}\right)\right]_0^{b\sqrt{a^2-b^2}}$$

$$=\frac{4\pi a}{b^2\sqrt{a^2-b^2}}\left\{\left[\frac{b\sqrt{a^2-b^2}}{2}\,(ab)+\frac{b^4}{2}\ln\!\left(b\sqrt{a^2-b^2}+ab\right)\right]-\left[0+\frac{b^4}{2}\ln(b^2)\right]\right\}$$

$$=\frac{4\pi a}{b^2\sqrt{a^2-b^2}}\left[\frac{ab^2\sqrt{a^2-b^2}}{2}+\frac{b^4}{2}\ln\frac{b\sqrt{a^2-b^2}+ab}{b^2}\right]=2\pi a^2+\frac{2\pi ab^2}{\sqrt{a^2-b^2}}\ln\frac{\sqrt{a^2-b^2}+a}{b}$$

38. The upper half of the torus is generated by rotating the curve $(x-R)^2+y^2=r^2$, $y>0$, about the y-axis.

$$y\,\frac{dy}{dx}=-(x-R) \quad\Rightarrow\quad 1+\left(\frac{dy}{dx}\right)^2=1+\frac{(x-R)^2}{y^2}=\frac{y^2+(x-R)^2}{y^2}=\frac{r^2}{r^2-(x-R)^2}. \text{ Thus,}$$

$$S=2\int_{R-r}^{R+r}2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=4\pi\int_{R-r}^{R+r}\frac{rx}{\sqrt{r^2-(x-R)^2}}\,dx=4\pi r\int_{-r}^{r}\frac{u+R}{\sqrt{r^2-u^2}}\,du \qquad [u=x-R]$$

$$=4\pi r\int_{-r}^{r}\frac{u\,du}{\sqrt{r^2-u^2}}+4\pi Rr\int_{-r}^{r}\frac{du}{\sqrt{r^2-u^2}}=4\pi r\cdot 0+8\pi Rr\int_{0}^{r}\frac{du}{\sqrt{r^2-u^2}} \qquad \left[\begin{array}{c}\text{since the first integrand is odd}\\ \text{and the second is even}\end{array}\right]$$

$$=8\pi Rr\left[\sin^{-1}(u/r)\right]_0^r=8\pi Rr\left(\tfrac{\pi}{2}\right)=4\pi^2Rr$$

39. (a) The analogue of $f(x_i^*)$ in the derivation of (4) is now $c-f(x_i^*)$, so

$$S=\lim_{n\to\infty}\sum_{i=1}^{n}2\pi[c-f(x_i^*)]\sqrt{1+[f'(x_i^*)]^2}\,\Delta x=\int_a^b 2\pi[c-f(x)]\sqrt{1+[f'(x)]^2}\,dx.$$

(b) $y=x^{1/2} \quad\Rightarrow\quad y'=\frac{1}{2}x^{-1/2} \quad\Rightarrow\quad 1+(y')^2=1+1/4x$, so by part (a), $S=\int_0^4 2\pi\left(4-\sqrt{x}\right)\sqrt{1+1/(4x)}\,dx$.

Using a CAS, we get $S=2\pi\ln\left(\sqrt{17}+4\right)+\frac{\pi}{6}\left(31\sqrt{17}+1\right)\approx 80.6095$.

40. $y=x^3 \quad\Rightarrow\quad dy/dx=3x^2 \quad\Rightarrow\quad ds=\sqrt{1+(dy/dx)^2}\,dx=\sqrt{1+9x^4}\,dx$. Also, $x=y^{1/3} \quad\Rightarrow$ $dx/dy=\frac{1}{3}y^{-2/3} \quad\Rightarrow\quad ds=\sqrt{1+(dx/dy)^2}\,dy=\sqrt{1+\frac{1}{9}y^{-4/3}}\,dy$. When $x=1$, $y=1$ and when $x=2$, $y=8$.

(a) Since $x>-1$ over the x-interval $[1,2]$, the area of the surface obtained by rotating the curve about $x=-1$ is given by

$S=\int 2\pi(x+1)\,ds=\int_1^2 2\pi(x+1)\sqrt{1+9x^4}\,dx\approx 115.91$.

Alternative method: $S=\int 2\pi(x+1)\,ds=\int_1^8 2\pi(y^{1/3}+1)\sqrt{1+\frac{1}{9}y^{-4/3}}\,dy\approx 115.91$.

(b) Since $x<4$ over the x-interval $[1,2]$, the area of the surface obtained by rotating the curve about $x=4$ is given by

$S=\int 2\pi(4-x)\,ds=\int_1^2 2\pi(4-x)\sqrt{1+9x^4}\,dx\approx 106.60$.

Alternative method: $S=\int 2\pi(4-x)\,ds=\int_1^8 2\pi(4-y^{1/3})\sqrt{1+\frac{1}{9}y^{-4/3}}\,dy\approx 106.60$.

(c) Since $y>\frac{1}{2}$ over the y-interval $[1,8]$, the area of the surface obtained by rotating the curve about $y=\frac{1}{2}$ is given by

$S=\int 2\pi\left(y-\frac{1}{2}\right)ds=\int_1^2 2\pi\left(x^3-\frac{1}{2}\right)\sqrt{1+9x^4}\,dx\approx 177.23$.

Alternative method: $S=\int 2\pi\left(y-\frac{1}{2}\right)ds=\int_1^8 2\pi\left(y-\frac{1}{2}\right)\sqrt{1+\frac{1}{9}y^{-4/3}}\,dy\approx 177.23$.

(d) Since $y<10$ over the y-interval $[1,8]$, the area of the surface obtained by rotating the curve about $y=10$ is given by

$S=\int 2\pi(10-y)\,ds=\int_1^2 2\pi(10-x^3)\sqrt{1+9x^4}\,dx\approx 245.52$.

Alternative method: $S=\int 2\pi(10-y)\,ds=\int_1^8 2\pi(10-y)\sqrt{1+\frac{1}{9}y^{-4/3}}\,dy\approx 245.52$.

41. For the upper semicircle, $y = \sqrt{r^2 - x^2}$, $dy/dx = -x/\sqrt{r^2 - x^2}$. The surface area generated is

$$S_1 = \int_{-r}^{r} 2\pi(r - y)\, ds = \int_{-r}^{r} 2\pi\left(r - \sqrt{r^2 - x^2}\right)\sqrt{1 + \frac{x^2}{r^2 - x^2}}\, dx$$

$$= 4\pi \int_0^r \left(r - \sqrt{r^2 - x^2}\right)\frac{r}{\sqrt{r^2 - x^2}}\, dx = 4\pi \int_0^r \left(\frac{r^2}{\sqrt{r^2 - x^2}} - r\right) dx$$

For the lower semicircle, $y = -\sqrt{r^2 - x^2}$ and $\dfrac{dy}{dx} = \dfrac{x}{\sqrt{r^2 - x^2}}$, so $S_2 = 4\pi \displaystyle\int_0^r \left(\frac{r^2}{\sqrt{r^2 - x^2}} + r\right) dx$.

Thus, the total area is $S = S_1 + S_2 = 8\pi \displaystyle\int_0^r \left(\frac{r^2}{\sqrt{r^2 - x^2}}\right) dx = 8\pi\left[r^2 \sin^{-1}\left(\frac{x}{r}\right)\right]_0^r = 8\pi r^2\left(\frac{\pi}{2}\right) = 4\pi^2 r^2$.

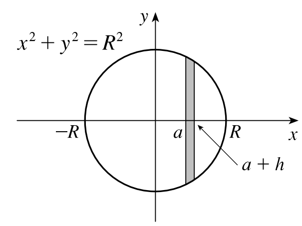
42. Rotate $y = \sqrt{R^2 - x^2}$ with $a \le x \le a + h$ about the x-axis to generate a zone of a sphere. $y = \sqrt{R^2 - x^2} \quad \Rightarrow$

$y' = \frac{1}{2}(R^2 - x^2)^{-1/2}(-2x) \quad \Rightarrow \quad ds = \sqrt{1 + \left(\dfrac{-x}{\sqrt{R^2 - x^2}}\right)^2}\, dx$. The surface area is

$$S = \int_a^{a+h} 2\pi y\, ds = 2\pi \int_a^{a+h} \sqrt{R^2 - x^2}\sqrt{1 + \frac{x^2}{R^2 - x^2}}\, dx$$

$$= 2\pi \int_a^{a+h} \sqrt{R^2 - x^2 + x^2}\, dx = 2\pi R\left[\, x\,\right]_a^{a+h}$$

$$= 2\pi R(a + h - a) = 2\pi Rh$$



43. Rotate $y = R$ with $0 \le x \le h$ about the x-axis to generate a zone of a cylinder. $y = R \quad \Rightarrow \quad y' = 0 \quad \Rightarrow$

$ds = \sqrt{1 + 0^2}\, dx = dx$. The surface area is $S = \int_0^h 2\pi y\, ds = 2\pi \int_0^h R\, dx = 2\pi R\left[\, x\,\right]_0^h = 2\pi Rh$.

44. Since $g(x) = f(x) + c$, we have $g'(x) = f'(x)$. Thus, the surface area generated by rotating the curve $g(x) = f(x) + c$ about the x-axis

$$S_g = \int_a^b 2\pi g(x)\sqrt{1 + [g'(x)]^2}\, dx = \int_a^b 2\pi[f(x) + c]\sqrt{1 + [f'(x)]^2}\, dx$$

$$= \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx + 2\pi c \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = S_f + 2\pi cL$$

45. $y = e^{x/2} + e^{-x/2} \quad \Rightarrow \quad y' = \frac{1}{2}e^{x/2} - \frac{1}{2}e^{-x/2} \quad \Rightarrow$

$1 + (y')^2 = 1 + \left(\frac{1}{2}e^{x/2} - \frac{1}{2}e^{-x/2}\right)^2 = 1 + \frac{1}{4}e^x - \frac{1}{2} + \frac{1}{4}e^{-x} = \frac{1}{4}e^x + \frac{1}{2} + \frac{1}{4}e^{-x} = \left(\frac{1}{2}e^{x/2} + \frac{1}{2}e^{-x/2}\right)^2$.

If we rotate the curve about the x-axis on the interval $a \le x \le b$, the resulting surface area is

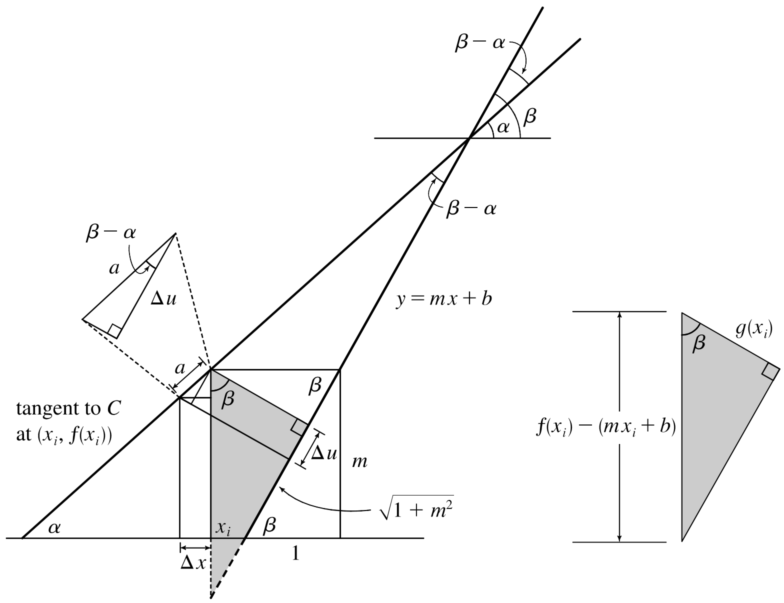
$S = \int_a^b 2\pi y\sqrt{1 + (y')^2}\, dx = 2\pi \int_a^b (e^{x/2} + e^{-x/2})\left(\frac{1}{2}e^{x/2} + \frac{1}{2}e^{-x/2}\right) dx = \pi \int_a^b (e^{x/2} + e^{-x/2})^2\, dx$, which is the same as the volume obtained by rotating the curve y about the x-axis on the interval $a \le x \le b$, namely, $V = \pi \int_a^b y^2\, dx$.

46. In the derivation of (4), we computed a typical contribution to the surface area to be $2\pi\, \dfrac{y_{i-1} + y_i}{2}\, |P_{i-1}P_i|$, the area of a frustum of a cone. When $f(x)$ is not necessarily positive, the approximations $y_i = f(x_i) \approx f(x_i^*)$ and $y_{i-1} = f(x_{i-1}) \approx f(x_i^*)$ must be replaced by $y_i = |f(x_i)| \approx |f(x_i^*)|$ and $y_{i-1} = |f(x_{i-1})| \approx |f(x_i^*)|$. Thus,

$2\pi\, \dfrac{y_{i-1} + y_i}{2}\, |P_{i-1}P_i| \approx 2\pi\, |f(x_i^*)|\, \sqrt{1 + [f'(x_i^*)]^2}\, \Delta x$. Continuing with the rest of the derivation as before, we obtain $S = \int_a^b 2\pi\, |f(x)|\, \sqrt{1 + [f'(x)]^2}\, dx$.

DISCOVERY PROJECT Rotating on a Slant

1.



In the figure, the segment a lying above the interval $[x_i - \Delta x, x_i]$ along the tangent to C has length $\Delta x \sec\alpha = \Delta x\sqrt{1+\tan^2\alpha} = \sqrt{1+[f'(x_i)]^2}\,\Delta x$. The segment from $(x_i, f(x_i))$ drawn perpendicular to the line $y = mx + b$ has length

$$g(x_i) = [f(x_i) - mx_i - b]\cos\beta = \frac{f(x_i) - mx_i - b}{\sec\beta} = \frac{f(x_i) - mx_i - b}{\sqrt{1+\tan^2\beta}} = \frac{f(x_i) - mx_i - b}{\sqrt{1+m^2}}$$

Also, $\cos(\beta - \alpha) = \dfrac{\Delta u}{\Delta x\ \sec\alpha} \quad \Rightarrow$

$$\Delta u = \Delta x\ \sec\alpha\ \cos(\beta-\alpha) = \Delta x\,\frac{\cos\beta\ \cos\alpha + \sin\beta\ \sin\alpha}{\cos\alpha} = \Delta x(\cos\beta + \sin\beta\ \tan\alpha)$$

$$= \Delta x\left[\frac{1}{\sqrt{1+m^2}} + \frac{m}{\sqrt{1+m^2}}f'(x_i)\right] = \frac{1+mf'(x_i)}{\sqrt{1+m^2}}\,\Delta x$$

Thus,
$$\text{Area}(\mathcal{R}) = \lim_{n\to\infty}\sum_{i=1}^{n} g(x_i)\,\Delta u = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{f(x_i) - mx_i - b}{\sqrt{1+m^2}}\cdot\frac{1+mf'(x_i)}{\sqrt{1+m^2}}\,\Delta x$$

$$= \frac{1}{1+m^2}\int_p^q [f(x) - mx - b][1 + mf'(x)]\,dx$$

2. From Problem 1 with $m = 1$, $f(x) = x + \sin x$, $mx + b = x - 2$, $p = 0$, and $q = 2\pi$,

$$\text{Area} = \frac{1}{1\,+\,1^2}\int_0^{2\pi}[x + \sin x - (x-2)]\,[1 + 1(1+\cos x)]\,dx = \tfrac{1}{2}\int_0^{2\pi}(\sin x + 2)(2 + \cos x)\,dx$$

$$= \tfrac{1}{2}\int_0^{2\pi}(2\sin x + \sin x\ \cos x + 4 + 2\cos x)\,dx = \tfrac{1}{2}\left[-2\cos x + \tfrac{1}{2}\sin^2 x + 4x + 2\sin x\right]_0^{2\pi}$$

$$= \tfrac{1}{2}[(-2 + 0 + 8\pi + 0) - (-2 + 0 + 0 + 0)] = \tfrac{1}{2}(8\pi) = 4\pi$$

3. $V = \displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\pi[g(x_i)]^2\,\Delta u = \lim_{n\to\infty}\sum_{i=1}^{n}\pi\left[\frac{f(x_i) - mx_i - b}{\sqrt{1+m^2}}\right]^2\frac{1+mf'(x_i)}{\sqrt{1+m^2}}\,\Delta x$

$$= \frac{\pi}{(1+m^2)^{3/2}}\int_p^q [f(x) - mx - b]^2[1 + mf'(x)]\,dx$$

4. $V = \dfrac{\pi}{(1+1^2)^{3/2}} \displaystyle\int_0^{2\pi} (x + \sin x - x + 2)^2 (1 + 1 + \cos x)\, dx$

$$= \frac{\pi}{2\sqrt{2}} \int_0^{2\pi} (\sin x + 2)^2 (\cos x + 2)\, dx = \frac{\pi}{2\sqrt{2}} \int_0^{2\pi} \left(\sin^2 x + 4\sin x + 4\right)(\cos x + 2)\ dx$$

$$= \frac{\pi}{2\sqrt{2}} \int_0^{2\pi} \left(\sin^2 x\, \cos x + 4\sin x\, \cos x + 4\cos x + 2\sin^2 x + 8\sin x + 8\right) dx$$

$$= \frac{\pi}{2\sqrt{2}} \left[\tfrac{1}{3}\sin^3 x + 2\sin^2 x + 4\sin x + x - \tfrac{1}{2}\sin 2x - 8\cos x + 8x\right]_0^{2\pi} \qquad [\text{since } 2\sin^2 x = 1 - \cos 2x]$$

$$= \frac{\pi}{2\sqrt{2}} [(2\pi - 8 + 16\pi) - (-8)] = \frac{9\sqrt{2}}{2}\pi^2$$

5. $S = \displaystyle\int_p^q 2\pi g(x)\sqrt{1 + [f'(x)]^2}\, dx = \frac{2\pi}{\sqrt{1+m^2}} \int_p^q [f(x) - mx - b]\sqrt{1 + [f'(x)]^2}\, dx$

6. From Problem 5 with $f(x) = \sqrt{x}$, $p = 0$, $q = 4$, $m = \frac{1}{2}$, and $b = 0$,

$$S = \frac{2\pi}{\sqrt{1 + \left(\frac{1}{2}\right)^2}} \int_0^4 \left(\sqrt{x} - \tfrac{1}{2}x\right)\sqrt{1 + \left(\frac{1}{2\sqrt{x}}\right)^2}\, dx \overset{\text{CAS}}{=} \frac{\pi}{\sqrt{5}}\left[\frac{\ln\left(\sqrt{17}+4\right)}{32} + \frac{37\sqrt{17}}{24} - \frac{1}{3}\right] \approx 8.554$$

8.3 Applications to Physics and Engineering

1. The weight density of water is $\rho = 1000 \text{ kg/m}^3$

(a) $P = \rho g d = (1000 \text{ kg/m}^3)(9.8 \text{ m/s}^2)(1 \text{ m}) = 9800 \text{ Pa}$

(b) $F = PA = (98{,}000 \text{ Pa})(1.5 \text{ m})(0.5 \text{ m}) = 7350 \text{ N}$

(c) The area of the ith strip is $0.5(\Delta x)$ and the pressure is $\rho g d = \rho g x_i$. Thus,

$$F = \int_0^1 \rho g x \cdot 0.5 dx = (9800)(0.5)\int_0^1 x dx = 4900[\frac{1}{2}x^2]|_0^1 = 4900(\frac{1}{2}) = 2450 \text{ N}$$

2. (a) $P = \rho g d = (820 \text{ kg/m}^3)(9.8 \text{ m/s}^2)(1.5 \text{ m}) = 12{,}054 \text{ Pa} \approx 12 \text{ kPa}$

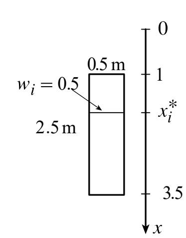
(b) $F = PA = (12{,}054 \text{ Pa})(8 \text{ m})(4 \text{ m}) \approx 3.86 \times 10^5 \text{ N}$ (A is the area at the bottom of the tank.)

(c) The area of the ith strip is $4(\Delta x)$ and the pressure is $\rho g d = \rho g\, x_i$. Thus,

$$F = \textstyle\int_0^{1.5} \rho g x \cdot 4\, dx = (820)(9.8) \cdot 4 \int_0^{1.5} x\, dx = 32{,}144\left[\frac{1}{2}x^2\right]_0^{3/2} = 16{,}072\left(\frac{9}{4}\right) \approx 3.62 \times 10^4 \text{ N}.$$

In Exercises 3–9, n is the number of subintervals of length Δx and x_i^* is a sample point in the ith subinterval $[x_{i-1}, x_i]$.

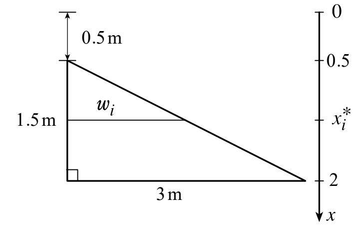
3. Set up a vertical x-axis as shown, with $x = 0$ at the water's surface and x increasing in the downward direction. Then the area of the ith rectangular strip is $\frac{1}{2}\Delta x$ and the pressure on the strip is $\rho g x_i^*$. Thus, the hydrostatic force on the strip is $\rho g x_i^* \cdot \frac{1}{2}\Delta x$ and the total hydrostatic force $\approx \sum_{i=1}^n \rho g x_i^* \cdot \frac{1}{2}\Delta x$. The total force



$$F = \lim_{n\to\infty}\sum_{i=1}^n \rho g x_i^* \cdot \frac{1}{2}\Delta x = \int_1^{3.5} \rho g x \cdot \frac{1}{2} dx = \frac{1}{2}\rho g[\frac{1}{2}x^2]|_1^{3.5}$$

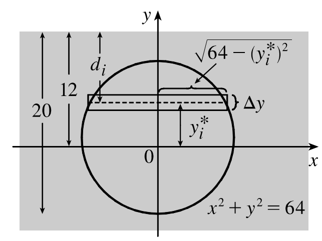
$$= \frac{1}{4}(9800)(11.25) = 27{,}562.5 \text{ N}$$

4. Set up a vertical axis as shown. Then the area of the ith rectangular strip is $2(x_i^* - 0.5)\,\Delta x$. $\left[\text{By similar triangles, } \dfrac{w_i}{x_i^* - 0.5} = \dfrac{3}{1.5}, \text{ so } w_i = 2(x_i^* - 0.5).\right]$ The pressure on the strip is $\rho g x_i^*$, so the hydrostatic force on the strip is $\rho g x_i^* \cdot 2(x_i^* - 0.5)\Delta x$ and the total hydrostatic force on the plate $\approx \sum\limits_{i=1}^{n} \rho g x_i^* \cdot 2(x_i^* - 0.5)\Delta x$. The total force is



$$\begin{aligned} F &= lim_{n\to\infty} \sum_{i=1}^{n} \rho g x_i^* \cdot 2(x_i^* - 0.5)\Delta x = \int_{0.5}^{2} \rho g x \cdot 2(x - 0.5)dx = 2\rho g \int_{0.5}^{2} (x^2 - \frac{1}{2}x)dx \\ &= 2\rho g[\frac{1}{3}x^3 - \frac{1}{4}x^2]|_{0.5}^{2} = 2(9800)(1.6875) = 33{,}075\text{ N} \end{aligned}$$

5. Set up a coordinate system as shown. Then the area of the ith rectangular strip is $2\sqrt{8^2 - (y_i^*)^2}\,\Delta y$. The pressure on the strip is $\delta d_i = \rho g(12 - y_i^*)$, so the hydrostatic force on the strip is $\rho g(12 - y_i^*)\,2\sqrt{64 - (y_i^*)^2}\,\Delta y$ and the total hydrostatic force on the plate $\approx \sum\limits_{i=1}^{n} \rho g(12 - y_i^*)\,2\sqrt{64 - (y_i^*)^2}\,\Delta y$.

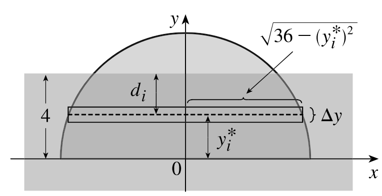


The total force $F = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \rho g(12 - y_i^*)\,2\sqrt{64 - (y_i^*)^2}\,\Delta y = \int_{-8}^{8} \rho g(12 - y)\,2\,\sqrt{64 - y^2}\,dy$

$= 2\rho g \cdot 12 \int_{-8}^{8} \sqrt{64 - y^2}\,dy - 2\rho g \int_{-8}^{8} y\sqrt{64 - y^2}\,dy.$

The second integral is 0 because the integrand is an odd function. The first integral is the area of a semicircular disk with radius 8. Thus, $F = 24\rho g\left(\frac{1}{2}\pi(8)^2\right) = 768\pi\rho g \approx 768\pi(1000)(9.8) \approx 2.36 \times 10^7$ N.

6. Set up a coordinate system as shown. Then the area of the ith rectangular strip is $2\,\sqrt{6^2 - (y_i^*)^2}\,\Delta y$. The pressure on the strip is $\delta d_i = \rho g(4 - y_i^*)$, so the hydrostatic force on the strip is $\rho g(4 - y_i^*)\,2\,\sqrt{36 - (y_i^*)^2}\,\Delta y$ and the hydrostatic force on the plate $\approx \sum\limits_{i=1}^{n} \rho g(4 - y_i^*)\,2\,\sqrt{36 - (y_i^*)^2}\,\Delta y$. The total force $F = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \rho g(4 - y_i^*)\,2\sqrt{36 - (y_i^*)^2}\,\Delta y = \int_0^4 \rho g(4 - y)\,2\sqrt{36 - y^2}\,dy = 8\rho g I_1 - 2\rho g I_2.$



$I_1 = \int_0^4 \sqrt{36 - y^2}\,dy = \int_0^{\alpha} \sqrt{36 - 36\sin^2\theta}\,(6\cos\theta\,d\theta)$ $\quad \begin{bmatrix} y = 6\sin\theta, \\ dy = 6\cos\theta\,d\theta \\ \alpha = \sin^{-1}(2/3) \end{bmatrix}$

$= \int_0^{\alpha} 36\cos^2\theta\,d\theta = \int_0^{\alpha} 36 \cdot \frac{1}{2}(1 + \cos 2\theta)\,d\theta = 18\big[\theta + \frac{1}{2}\sin 2\theta\big]_0^{\alpha}$

$= 18\big(\alpha + \frac{1}{2}\sin 2\alpha\big) = 18(\alpha + \sin\alpha\,\cos\alpha).$

$I_2 = \int_0^4 y\sqrt{36 - y^2}\,dy = \int_{36}^{20} \sqrt{u}\,(-\frac{1}{2}du)$ $\quad \begin{bmatrix} u = 36 - y^2, \\ du = -2y\,dy \end{bmatrix}$

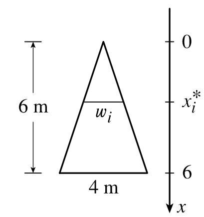
$= -\frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_{36}^{20} = -\frac{1}{3}(20^{3/2} - 216) = 72 - \frac{40}{3}\sqrt{5}.$

[continued]

Thus,

$$F = 8\rho g \cdot 18(\alpha + \sin\alpha\,\cos\alpha) - 2\rho g\left(72 - \tfrac{40}{3}\sqrt{5}\right) = 144\rho g\left(\sin^{-1}\tfrac{2}{3} + \tfrac{2}{3}\tfrac{\sqrt{5}}{3}\right) - 2\rho g\left(72 - \tfrac{40}{3}\sqrt{5}\right)$$
$$= \rho g\left(144\sin^{-1}\tfrac{2}{3} + \tfrac{176}{3}\sqrt{5} - 144\right) \approx 9.04 \times 10^5\text{ N}\quad [\rho = 1000,\ g \approx 9.8].$$

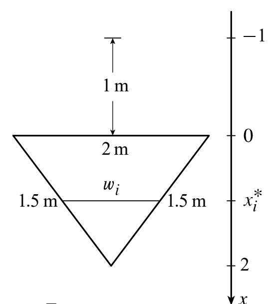
7. Set up a vertical x-axis as shown. By similar triangles, $w_i/4 = x_i^*/6$, so $w_i = \frac{2}{3}x_i^*$, and the area of the ith rectangular strip is $\frac{2}{3}x_i^*\,\Delta x$. The pressure on the ith strip is $\rho g x_i^*$, so the hydrostatic force on the strip is $\rho g x_i^* \cdot \frac{2}{3}x_i^*\,\Delta x$, and the hydrostatic force on the plate is $\approx \sum_{i=1}^{n} \rho g x_i^* \cdot \frac{2}{3}x_i^*\,\Delta x$. The total force is



$$F = \lim_{n\to\infty}\sum_{i=1}^{n} \rho g x_i^* \cdot \tfrac{2}{3}x_i^*\,\Delta x = \tfrac{2}{3}\rho g\int_0^6 x^2\,dx = \tfrac{2}{3}\rho g\left[\tfrac{1}{3}x^3\right]_0^6 = \tfrac{2}{9}\rho g(216 - 0)$$
$$= 48\rho g = 48(1000)(9.8) = 470{,}400\text{ N}$$

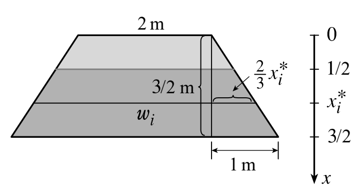
8. Draw a diagram with a vertical axis as shown. The height of the triangle is $\sqrt{(1.5)^2 - (1)^2} = \sqrt{5/4} = \sqrt{5}/2$.

By similar triangles, $\frac{w_i}{2} = \frac{x_i^*}{\sqrt{5}/2}$, so $w_i = \frac{4x_i^*}{\sqrt{5}}$, and the area of the *i* th rectangular strip is $(4/\sqrt{5}x_i^*)\Delta x$. The pressure on the *i* th strip is $\rho(1 + x_i^*)$, so the hydrostatic force on the strip is $4/\sqrt{5}\rho x_i^*(1 + x_i^*)\Delta x$, and the hydrostatic force on the plate is $\approx \sum_{i=1}^{n} 4/\sqrt{5}\rho x_i^*(1 + x_i^*)\Delta x$. The total force is



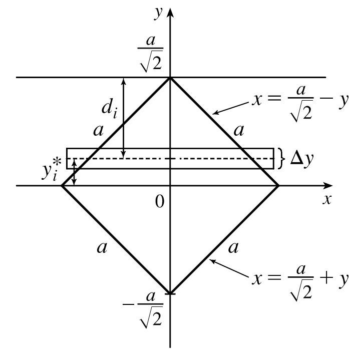
$$F = \lim_{n\to\infty}\sum_{i=1}^{n} 4/\sqrt{5}\rho x_i^*(1 + x_i^*)\Delta x = 4/\sqrt{5}\rho\int_0^{\sqrt{5}/2} x(1 + x)\,dx = 4/\sqrt{5}\rho\int_0^{\sqrt{5}/2}(x + x^2)\,dx$$
$$= \frac{4}{\sqrt{5}}\rho\left[\frac{1}{2}\left(\frac{\sqrt{5}}{2}\right)^2 + \frac{1}{3}\left(\frac{\sqrt{5}}{2}\right)^3\right] = \frac{4}{\sqrt{5}}(1000)(9.8)\left[\frac{5}{8} + \frac{5\sqrt{5}}{24}\right] \approx 1.91 \cdot 10^4\text{ N}$$

9. Set up a vertical x-axis as shown. Then the area of the ith rectangular strip is $w_i\,\Delta x = \left(2 + 1 \cdot \frac{2}{3}x_i^*\right)\Delta x$. The pressure on the strip is $\rho g(x_i^* - 0.5)$, so the hydrostatic force on the strip is $\rho g(x_i^* - 0.5)(2 + \frac{2}{3}x_i^*)\Delta x$ and the hydrostatic force on the plate $\approx \sum_{i=1}^{n} \rho g(x_i^* - 0.5)(2 + \frac{2}{3}x_i^*)\Delta x$. The total force is



$$F = lim_{n\to\infty}\sum_{i=1}^{n} \rho g(x_i^* - 0.5)(2 + \frac{2}{3}x_i^*)\Delta x = \int_{0.5}^{1.5} \rho g(x - 0.5)(2 + \frac{2}{3}x)dx$$
$$= \rho g\int_{0.5}^{1.5}(\frac{2}{3}x^2 + \frac{5}{3}x - 1)dx = \rho g[\frac{2}{9}x^3 + \frac{5}{6}x^2 - x]|_{0.5}^{1.5} = (9800)(\frac{25}{18}) \approx 13{,}611\text{ N}$$

10. Set up coordinate axes as shown in the figure. For the *top half,* the length of the ith strip is $2\left(a/\sqrt{2}-y_i^*\right)$ and its area is $2\left(a/\sqrt{2}-y_i^*\right)\Delta y$. The pressure on this strip is approximately $\delta d_i = \delta\left(a/\sqrt{2}-y_i^*\right)$ and so the force on the strip is approximately $2\delta\left(a/\sqrt{2}-y_i^*\right)^2\Delta y$. The total force is



$$F_1 = \lim_{n\to\infty}\sum_{i=1}^{n} 2\delta\left(\frac{a}{\sqrt{2}}-y_i^*\right)^2\Delta y = 2\delta\int_0^{a/\sqrt{2}}\left(\frac{a}{\sqrt{2}}-y\right)^2 dy$$

$$= 2\delta\left[-\frac{1}{3}\left(\frac{a}{\sqrt{2}}-y\right)^3\right]_0^{a/\sqrt{2}} = -\frac{2}{3}\delta\left[0-\left(\frac{a}{\sqrt{2}}\right)^3\right] = \frac{2\delta}{3}\frac{a^3}{2\sqrt{2}} = \frac{\sqrt{2}\,a^3\delta}{6}$$

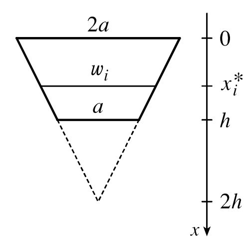
For the *bottom half,* the length is $2\left(a/\sqrt{2}+y_i^*\right)$ and the total force is

$$F_2 = \lim_{n\to\infty}\sum_{i=1}^{n} 2\delta\left(\frac{a}{\sqrt{2}}+y_i^*\right)\left(\frac{a}{\sqrt{2}}-y_i^*\right)\Delta y = 2\delta\int_{-a/\sqrt{2}}^{0}\left(\frac{a^2}{2}-y^2\right)dy = 2\delta\left[\tfrac{1}{2}a^2y-\tfrac{1}{3}y^3\right]_{-a/\sqrt{2}}^{0}$$

$$= 2\delta\left[0-\left(-\frac{\sqrt{2}\,a^3}{4}+\frac{\sqrt{2}\,a^3}{12}\right)\right] = 2\delta\left(\frac{\sqrt{2}\,a^3}{6}\right) = \frac{2\sqrt{2}\,a^3\delta}{6} \qquad [F_2 = 2F_1]$$

Thus, the total force $F = F_1 + F_2 = \dfrac{3\sqrt{2}\,a^3\delta}{6} = \dfrac{\sqrt{2}\,a^3\delta}{2}$.

11. Set up a vertical x-axis as shown. Then the area of the ith rectangular strip is $\frac{a}{h}(2h-x_i^*)\,\Delta x$. $\left[\text{By similar triangles, } \dfrac{w_i}{2h-x_i^*} = \dfrac{2a}{2h}, \text{ so } w_i = \dfrac{a}{h}(2h-x_i^*).\right]$ The pressure on the strip is δx_i^*, so the hydrostatic force on the plate $\approx \sum_{i=1}^{n}\delta x_i^*\,\frac{a}{h}(2h-x_i^*)\,\Delta x$. The total force is



$$F = \lim_{n\to\infty}\sum_{i=1}^{n}\delta x_i^*\,\frac{a}{h}(2h-x_i^*)\,\Delta x = \delta\,\frac{a}{h}\int_0^h x(2h-x)\,dx = \frac{a\delta}{h}\int_0^h\left(2hx-x^2\right)dx$$

$$= \frac{a\delta}{h}\left[hx^2-\tfrac{1}{3}x^3\right]_0^h = \frac{a\delta}{h}\left(h^3-\tfrac{1}{3}h^3\right) = \frac{a\delta}{h}\left(\frac{2h^3}{3}\right) = \tfrac{2}{3}\delta a h^2$$

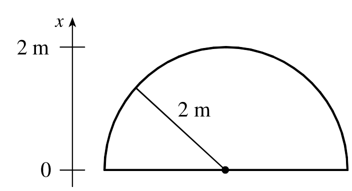
12. $F = \int_0^2 \rho g(10-x)2\sqrt{4-x^2}\,dx$

$= 20\rho g\int_0^2\sqrt{4-x^2}\,dx - \rho g\int_0^2\sqrt{4-x^2}\,2x\,dx$

$= 20\rho g\frac{1}{4}\pi(2^2) - \rho g\int_0^4 u^{1/2}\,du \qquad [u = 4-x^2,\ du = -2x\,dx]$

$= 20\pi\rho g - \frac{2}{3}\rho g\left[u^{3/2}\right]_0^4 = 20\pi\rho g - \frac{16}{3}\rho g = \rho g\left(20\pi-\frac{16}{3}\right)$

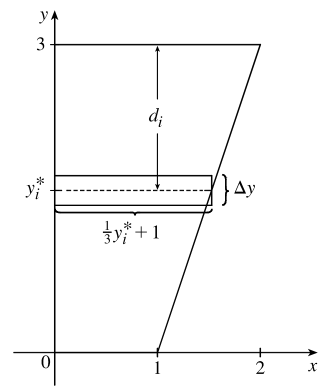
$= (1000)(9.8)\left(20\pi-\frac{16}{3}\right) \approx 5.63\times10^5$ N



13. The solution is similar to the solution for Example 2. The pressure on the strip is approximately $\rho g d_i = 753 \cdot 9.8 \cdot (\frac{5}{4} - y_i^*) = 7379.4(\frac{5}{4} - y_i^*)$ and the total force is

$$\begin{aligned} F &= lim_{n\to\infty} \sum_{i=1}^{n} 7379.4 \cdot (\frac{5}{4} - y_i^*) \cdot 2 \cdot \sqrt{\frac{25}{16} - (y_i^*)^2}\Delta y \\ &= 7379.4 \cdot 2 \int_{-\frac{5}{4}}^{\frac{5}{4}} (\frac{5}{4} - y)\sqrt{\frac{25}{16} - y^2 dy} \\ &= 7379.4 \cdot \frac{5}{4} \cdot 2 \int_{-\frac{5}{4}}^{\frac{5}{4}} \sqrt{\frac{25}{16} - y^2 dy} - 7379.4 \cdot 2 \int_{-\frac{5}{4}}^{\frac{5}{4}} y\sqrt{\frac{25}{16} - y^2 dy} \\ &\approx 45{,}279 \text{ N} \end{aligned}$$

14. (a) Set up a coordinate system as shown. The slanted side of the trough has equation $y = 3x - 3$ or $x = \frac{1}{3}y + 1$, so the area of the ith rectangular strip is $\left(\frac{1}{3}y_i^* + 1\right)\Delta y$. The pressure on the strip is $\rho g d_i = \rho g(3 - y_i^*)$, so the hydrostatic force on the strip is $\rho g(3 - y_i^*)\left(\frac{1}{3}y_i^* + 1\right)\Delta y$ and the total force on the end of the trough is



$$F = \lim_{n\to\infty} \sum_{i=1}^{n} \rho g(3 - y_i^*)\left(\frac{1}{3}y_i^* + 1\right)\Delta y = \rho g \int_0^3 (3 - y)\left(\frac{1}{3}y + 1\right) dy$$

$$= \rho g \int_0^3 \left(3 - \frac{1}{3}y^2\right) dy = \rho g\left[3y - \frac{1}{9}y^3\right]_0^3 = \rho g(9 - 3)$$

$$\approx (925)(9.8)(6) = 54{,}390 \text{ N}$$

(b) When filled to a depth of 1.2 m, the pressure on the ith rectangular strip is $\rho g(1.2 - y_i^*)$, so the hydrostatic force on the strip is $\rho g(1.2 - y_i^*)\left(\frac{1}{3}y_i^* + 1\right)\Delta y$ and the total force on the end of the trough is

$$F = \lim_{n\to\infty} \sum_{i=1}^{n} \rho g(1.2 - y_i^*)\left(\frac{1}{3}y_i^* + 1\right)\Delta y = \rho g \int_0^{1.2} (1.2 - y)\left(\frac{1}{3}y + 1\right) dy = \rho g \int_0^{1.2} \left(1.2 - 0.6y - \frac{1}{3}y^2\right) dy$$

$$= \rho g\left[1.2y - 0.3y^2 - \frac{1}{9}y^3\right]_0^{1.2} = (925)(9.8)(1.44 - 0.432 - 0.192) = 7397.04 \text{ N}$$

Note that this is about 14% of the force for the completely filled trough.

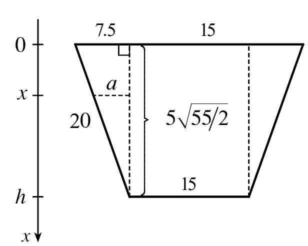
15. (a) The top of the cube has depth $d = 1 \text{ m} - 20 \text{ cm} = 80 \text{ cm} = 0.8 \text{ m}$.

$$F = \rho g d A \approx (1000)(9.8)(0.8)(0.2)^2 = 313.6 \approx 314 \text{ N}$$

(b) The area of a strip is $0.2\,\Delta x$ and the pressure on it is $\rho g x_i^*$.

$$F = \int_{0.8}^{1} \rho g x(0.2)\,dx = 0.2\rho g\left[\frac{1}{2}x^2\right]_{0.8}^{1} = (0.2\rho g)(0.18) = 0.036\rho g = 0.036(1000)(9.8) = 352.8 \approx 353 \text{ N}$$

16. The height of the dam is $h = \sqrt{20 - 7.5^2}\cos 30^\circ = 5\sqrt{\frac{165}{4}}$.



The width of the trapezoid is $w = 15 + 2a$.

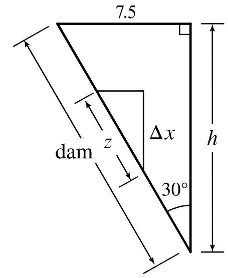
By similar triangles, $\frac{7.5}{h} = \frac{a}{h-x} \Rightarrow a = \frac{7.5}{h}(h - x)$. Thus,

$w = 15 + 2 \cdot \frac{7.5}{h}(h - x) = 15 + \frac{15}{h} \cdot h - \frac{15}{h} \cdot x = 15 + 15 - \frac{15x}{h} = 30 - \frac{15x}{h}$.

[continued]

From the small triangle in the second figure, $\cos 30^\circ = \dfrac{\Delta x}{z} \Rightarrow$

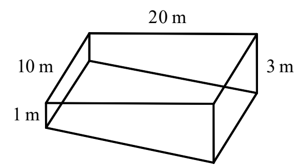
$z = \Delta x \sec 30^\circ = 2\,\Delta x/\sqrt{3}$.



$$F = \int_0^h \delta x \left(30 - \frac{15x}{h}\right) \frac{2}{\sqrt{3}} dx = \frac{60\delta}{\sqrt{3}} \int_0^h x\,dx - \frac{30\delta}{h\sqrt{3}} \int_0^h x^2\,dx$$

$$= \frac{60\delta}{\sqrt{3}} \frac{h^2}{2} - \frac{30\delta}{h\sqrt{3}} \frac{h^3}{3} = \frac{20\delta h^2}{\sqrt{3}} = \frac{20(1000)}{\sqrt{3}} \cdot \frac{4125}{16} \approx 2.98 \cdot 10^6 \text{ kg}$$

17. (a) *Shallow end:* The area of a strip is $10\,\Delta x$ and the pressure on it is $\rho g x_i$.



$$F = \int_0^1 \rho g x 10\,dx = 10\rho g\left[\tfrac{1}{2}x^2\right]_0^1 = 10\rho g \cdot \tfrac{1}{2} = 5\rho g$$

$$= 5(9800) = 49{,}000 \text{ N}$$

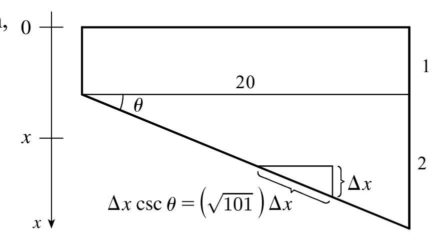
Deep end: $F = \int_0^3 \rho g x 10\,dx = 10\rho g[\dfrac{1}{2}]|_0^3 = 10\rho g \dfrac{9}{2} = 45\rho g = 45(9800) = 441{,}000$ N

Sides: For the first 1 m, the length of a side is constant at 20 m. For $1 < x \leq 3$, we can use similar triangles to find the length a: $\dfrac{a}{20} = \dfrac{3-x}{2} \quad \Rightarrow \quad a = 20 \cdot \dfrac{3-x}{2}$.

$$F = \int_0^1 \rho g x 20 dx + \int_1^3 \rho g x(20)\frac{3-x}{2} dx = 20\rho g[\frac{1}{2}x^2]|_0^1 + 10\rho g \int_1^3 (3x - x^2) dx = 10\rho g + 10\rho g[\frac{3}{2}x^2 - \frac{1}{3}x^3]|_1^3$$

$$= 10\rho g + 10\rho g[(\frac{27}{2} - 9) - (\frac{3}{2} - \frac{1}{3})] = 10\rho g + \frac{100}{3}\rho g = \frac{130}{3}\rho g = \frac{130}{3}(9800) \approx 424{,}667 \text{ N}$$

(b) Bottom of the pool for any right triangle with hypotenuse on the bottom,



$\sin\theta = \dfrac{\Delta x}{\text{hypotenuse}} \quad \Rightarrow$

hypotenuse $= \Delta x \csc\theta = \Delta x \dfrac{\sqrt{20^2 + 2^2}}{2} = \sqrt{101}\Delta x$.

$$F = \int_3^9 \rho g x 10\sqrt{101} dx = 10\sqrt{101}\rho g[\frac{1}{2}x^2]|_1^3$$

$$= 10\sqrt{101}\rho g(\frac{9}{2} - \frac{1}{2}) \approx \quad 3{,}939{,}551 \text{ N}$$

18. Partition the interval $[a, b]$ by points x_i as usual and choose $x_i^* \in [x_{i-1}, x_i]$ for each i. The ith horizontal strip of the immersed plate is approximated by a rectangle of height Δx_i and width $w(x_i^*)$, so its area is $A_i \approx w(x_i^*)\,\Delta x_i$. For small Δx_i, the pressure P_i on the ith strip is almost constant and $P_i \approx \rho g x_i^*$ by Equation 1. The hydrostatic force F_i acting on the ith strip is $F_i = P_i A_i \approx \rho g x_i^* w(x_i^*)\,\Delta x_i$. Adding these forces and taking the limit as $n \to \infty$, we obtain the hydrostatic force on the immersed plate:

$$F = \lim_{n\to\infty} \sum_{i=1}^n F_i = \lim_{n\to\infty} \sum_{i=1}^n \rho g x_i^* w(x_i^*)\,\Delta x_i = \int_a^b \rho g x w(x)\,dx$$

19. From Exercise 18, we have $F = \int_a^b \rho g x w(x) dx = \int_{2.2}^{2.8} (9.8)(1000) x w(x) dx$. From the table, we see that $\Delta x = 0.1$, so using Simpson's Rule to estimate F, we get

$$F \approx (9.8)(1000)\frac{0.1}{3}[2.2w(2.2) + 4(2.3)w(2.3) + 2(2.4)w(2.4) + 4(2.5)w(2.5)$$
$$+ 2(2.6)w(2.6) + 4(2.7)w(2.7) + 2.8w(2.8)]$$
$$= \frac{980}{3}[2.2(0.4) + 9.2(0.5) + 4.8(1.0) + 10(1.2) + 5.2(1.1) + 10.8(1.3) + 2.8(1.3)]$$
$$= \frac{980}{3}(45.68) \approx 14{,}922 \text{ N}$$

20. (a) From Equation 8, $\overline{x} = \frac{1}{A}\int_a^b xw(x)\,dx \quad\Rightarrow\quad A\overline{x} = \int_a^b xw(x)\,dx \quad\Rightarrow\quad \rho g A\overline{x} = \rho g\int_a^b xw(x)\,dx \quad\Rightarrow$

$(\rho g\overline{x})A = \int_a^b \rho g\, xw(x)\,dx = F$ by Exercise 18.

(b) For the figure in Exercise 10, let the coordinates of the centroid $(\overline{x},\overline{y}) = \left(a/\sqrt{2},0\right)$.

$$F = (\rho g\overline{x})A = \rho g\,\frac{a}{\sqrt{2}}\,a^2 = \delta\,\frac{\sqrt{2}\,a}{2}\,a^2 = \frac{\sqrt{2}\,a^3\delta}{2}.$$

21. The moment M of the system about the origin is $M = \sum\limits_{i=1}^{2} m_i x_i = m_1x_1 + m_2x_2 = 6\cdot 10 + 9\cdot 30 = 330.$

The mass m of the system is $m = \sum\limits_{i=1}^{2} m_i = m_1 + m_2 = 6 + 9 = 15.$

The center of mass of the system is $\overline{x} = M/m = \frac{330}{15} = 22.$

22. The moment M is $m_1x_1 + m_2x_2 + m_3x_3 = 12(-3) + 15(2) + 20(8) = 154.$ The mass m is $m_1 + m_2 + m_3 = 12 + 15 + 20 = 47.$ The center of mass is $\overline{x} = M/m = \frac{154}{47}.$

23. The mass is $m = \sum\limits_{i=1}^{3} m_i = 5 + 8 + 7 = 20.$ The moment about the x-axis is $M_x = \sum\limits_{i=1}^{3} m_i y_i = 5(1) + 8(4) + 7(-2) = 23.$

The moment about the y-axis is $M_y = \sum\limits_{i=1}^{3} m_i x_i = 5(3) + 8(0) + 7(-5) = -20.$ The center of mass is

$$(\overline{x},\overline{y}) = \left(\frac{M_y}{m},\frac{M_x}{m}\right) = \left(-\frac{20}{20},\frac{23}{20}\right) = (-1, 1.15).$$

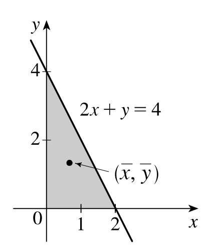
24. The mass is $m = \sum\limits_{i=1}^{4} m_i = 4 + 3 + 6 + 3 = 16.$

The moment about the x-axis is $M_x = \sum\limits_{i=1}^{4} m_i y_i = 4(1) + 3(-1) + 6(2) + 3(-5) = -2.$

The moment about the y-axis is $M_y = \sum\limits_{i=1}^{4} m_i x_i = 4(6) + 3(3) + 6(-2) + 3(-2) = 15.$

The center of mass is $(\overline{x},\overline{y}) = \left(\dfrac{M_y}{m},\dfrac{M_x}{m}\right) = \left(\dfrac{15}{16},-\dfrac{2}{16}\right) = \left(\dfrac{15}{16},-\dfrac{1}{8}\right).$

25. The region in the figure is "left-heavy" and "bottom-heavy," so we know that $\overline{x} < 1$ and $\overline{y} < 2$, and we might guess that $\overline{x} = 0.7$ and $\overline{y} = 1.3$. The line $2x + y = 4$ can be expressed as $y = 4 - 2x$, so $A = \int_0^2 (4 - 2x)\,dx = \left[4x - x^2\right]_0^2 = 8 - 4 = 4.$

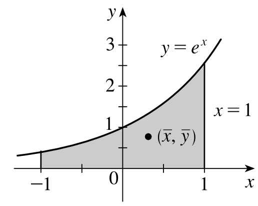


$$\overline{x} = \frac{1}{A}\int_0^2 x(4-2x)\,dx = \frac{1}{4}\int_0^2 (4x - 2x^2)\,dx = \frac{1}{4}\left[2x^2 - \tfrac{2}{3}x^3\right]_0^2 = \frac{1}{4}\left(8 - \tfrac{16}{3}\right) = \frac{2}{3}.$$

$$\overline{y} = \frac{1}{A}\int_0^2 \tfrac{1}{2}(4-2x)^2dx = \frac{1}{4}\int_0^2 \tfrac{1}{2}\cdot 4(2-x)^2dx = \frac{1}{2}\left[-\tfrac{1}{3}(2-x)^3\right]_0^2 = -\frac{1}{6}(0-8) = \frac{4}{3}.$$

Thus, the centroid is $(\overline{x},\overline{y}) = \left(\frac{2}{3},\frac{4}{3}\right).$

26. The region in the figure is "right-heavy" and "bottom-heavy," so we know that $\overline{x} > 0$ and $\overline{y} < 1$, and we might guess that $\overline{x} = 0.3$ and $\overline{y} = 0.8$.



$A = \int_{-1}^{1} e^x\,dx = \left[e^x\right]_{-1}^{1} = e - e^{-1}$.

$$\overline{x} = \frac{1}{A}\int_{-1}^{1} xe^x\,dx = \frac{1}{e - e^{-1}}\left[xe^x - e^x\right]_{-1}^{1} \quad \text{[by parts]}$$

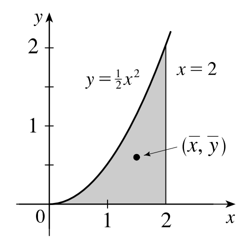
$$= \frac{1}{e - e^{-1}}\left[(e - e) - (-e^{-1} - e^{-1})\right] = \frac{2e^{-1}}{e - e^{-1}} \cdot \frac{e}{e} = \frac{2}{e^2 - 1}$$

$$\overline{y} = \frac{1}{A}\int_{-1}^{1} \frac{1}{2}(e^x)^2\,dx = \frac{1}{2(e - e^{-1})}\int_{-1}^{1} e^{2x}\,dx = \frac{1}{2(e - e^{-1})} \cdot \frac{1}{2}\left[e^{2x}\right]_{-1}^{1} = \frac{e^2 - e^{-2}}{4(e - e^{-1})} \cdot \frac{e^2}{e^2} = \frac{e^4 - 1}{4e(e^2 - 1)}$$

$$= \frac{(e^2 + 1)(e^2 - 1)}{4e(e^2 - 1)} = \frac{e^2 + 1}{4e}$$

Thus, the centroid is $(\overline{x}, \overline{y}) = \left(\dfrac{2}{e^2 - 1}, \dfrac{e^2 + 1}{4e}\right) \approx (0.31, 0.77)$.

27. The region in the figure is "right-heavy" and "bottom-heavy," so we know that $\overline{x} > 1$ and $\overline{y} < 1$, and we might guess that $\overline{x} = 1.5$ and $\overline{y} = 0.5$.



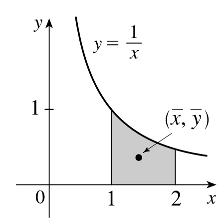
$A = \int_0^2 \frac{1}{2}x^2\,dx = \left[\frac{1}{6}x^3\right]_0^2 = \frac{8}{6} - 0 = \frac{4}{3}$.

$$\overline{x} = \frac{1}{A}\int_0^2 x \cdot \tfrac{1}{2}x^2 dx = \tfrac{3}{4}\int_0^2 \tfrac{1}{2}x^3 dx = \tfrac{3}{8}\int_0^2 x^3 dx = \tfrac{3}{8}\left[\tfrac{1}{4}x^4\right]_0^2 = \tfrac{3}{32}(16 - 0) = \tfrac{3}{2}.$$

$$\overline{y} = \frac{1}{A}\int_0^2 \tfrac{1}{2}\left(\tfrac{1}{2}x^2\right)^2 dx = \tfrac{3}{4}\int_0^2 \tfrac{1}{8}x^4\,dx = \tfrac{3}{32}\left[\tfrac{1}{5}x^5\right]_0^2 = \tfrac{3}{160}(32 - 0) = \tfrac{3}{5}.$$

Thus, the centroid is $(\overline{x}, \overline{y}) = \left(\frac{3}{2}, \frac{3}{5}\right)$.

28. The region in the figure is "left-heavy" and "bottom-heavy," so we know $\overline{x} < 1.5$ and $\overline{y} < 0.5$, and we might guess that $\overline{x} = 1.4$ and $\overline{y} = 0.4$.

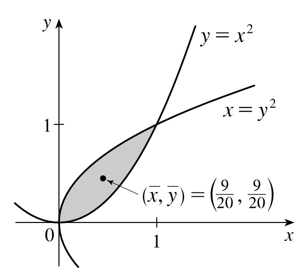


$A = \int_1^2 \frac{1}{x}\,dx = [\ln x]_1^2 = \ln 2$. $\overline{x} = \frac{1}{A}\int_1^2 x \cdot \frac{1}{x}\,dx = \frac{1}{A}[x]_1^2 = \frac{1}{A} = \frac{1}{\ln 2}$.

$\overline{y} = \frac{1}{A}\int_1^2 \frac{1}{2}\left(\frac{1}{x}\right)^2\,dx = \frac{1}{2A}\int_1^2 x^{-2}\,dx = \frac{1}{2A}\left[-\frac{1}{x}\right]_1^2 = \frac{1}{2\ln 2}\left(-\frac{1}{2} + 1\right) = \frac{1}{4\ln 2}$.

Thus, the centroid is $(\overline{x}, \overline{y}) = \left(\frac{1}{\ln 2}, \frac{1}{4\ln 2}\right) \approx (1.44, 0.36)$.

29. $A = \int_0^1 (x^{1/2} - x^2)\,dx = \left[\frac{2}{3}x^{3/2} - \frac{1}{3}x^3\right]_0^1 = \left(\frac{2}{3} - \frac{1}{3}\right) - 0 = \frac{1}{3}$.



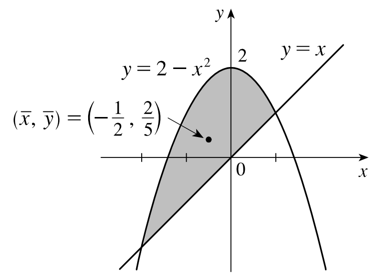
$\overline{x} = \frac{1}{A}\int_0^1 x(x^{1/2} - x^2)\,dx = 3\int_0^1 (x^{3/2} - x^3)\,dx = 3\left[\frac{2}{5}x^{5/2} - \frac{1}{4}x^4\right]_0^1$

$= 3\left(\frac{2}{5} - \frac{1}{4}\right) = 3\left(\frac{3}{20}\right) = \frac{9}{20}$.

$\overline{y} = \frac{1}{A}\int_0^1 \frac{1}{2}\left[(x^{1/2})^2 - (x^2)^2\right]dx = 3\left(\frac{1}{2}\right)\int_0^1 (x - x^4)\,dx$

$= \frac{3}{2}\left[\frac{1}{2}x^2 - \frac{1}{5}x^5\right]_0^1 = \frac{3}{2}\left(\frac{1}{2} - \frac{1}{5}\right) = \frac{3}{2}\left(\frac{3}{10}\right) = \frac{9}{20}$.

Thus, the centroid is $(\overline{x}, \overline{y}) = \left(\frac{9}{20}, \frac{9}{20}\right)$.



30. The curves intersect when $2-x^2=x \quad\Leftrightarrow\quad 0=x^2+x-2 \quad\Leftrightarrow$
$0=(x+2)(x-1) \quad\Leftrightarrow\quad x=-2 \text{ or } x=1$.

$A=\int_{-2}^{1}(2-x^2-x)\,dx=\left[2x-\frac{1}{3}x^3-\frac{1}{2}x^2\right]_{-2}^{1}=\frac{7}{6}-\left(-\frac{10}{3}\right)=\frac{9}{2}$.

$\overline{x}=\frac{1}{A}\int_{-2}^{1}x(2-x^2-x)\,dx=\frac{2}{9}\int_{-2}^{1}(2x-x^3-x^2)\,dx$

$=\frac{2}{9}\left[x^2-\frac{1}{4}x^4-\frac{1}{3}x^3\right]_{-2}^{1}=\frac{2}{9}\left(\frac{5}{12}-\frac{8}{3}\right)=-\frac{1}{2}$.

$\overline{y}=\frac{1}{A}\int_{-2}^{1}\frac{1}{2}\left[(2-x^2)^2-x^2\right]dx=\frac{2}{9}\cdot\frac{1}{2}\int_{-2}^{1}(4-5x^2+x^4)\,dx$

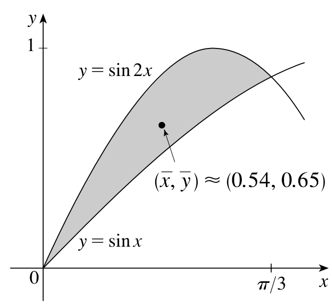
$=\frac{1}{9}\left[4x-\frac{5}{3}x^3+\frac{1}{5}x^5\right]_{-2}^{1}=\frac{1}{9}\left[\frac{38}{15}-\left(-\frac{16}{15}\right)\right]=\frac{2}{5}$.

Thus, the centroid is $(\overline{x},\overline{y})=\left(-\frac{1}{2},\frac{2}{5}\right)$.

31. $A=\int_0^{\pi/3}(\sin 2x-\sin x)\,dx=\left[-\frac{1}{2}\cos 2x+\cos x\right]_0^{\pi/3}=\left(\frac{1}{4}+\frac{1}{2}\right)-\left(-\frac{1}{2}+1\right)=\frac{1}{4}$.

$$\overline{x}=\frac{1}{A}\int_0^{\pi/3}x(\sin 2x-\sin x)\,dx=4\left[x\left(-\frac{1}{2}\cos 2x+\cos x\right)-\left(-\frac{1}{4}\sin 2x+\sin x\right)\right]_0^{\pi/3} \qquad \left[\begin{array}{c}\text{by parts with } u=x \text{ and}\\ dv=(\sin 2x-\sin x)\,dx\end{array}\right]$$

$$=4\left[\frac{\pi}{3}\left(\frac{1}{4}+\frac{1}{2}\right)-\left(-\frac{\sqrt{3}}{8}+\frac{\sqrt{3}}{2}\right)\right]=\pi-\frac{3\sqrt{3}}{2}.$$



$$\overline{y}=\frac{1}{A}\int_0^{\pi/3}\tfrac{1}{2}(\sin^2 2x-\sin^2 x)\,dx$$

$=4\int_0^{\pi/3}\frac{1}{2}(4\sin^2 x\cos^2 x-\sin^2 x)\,dx \qquad \left[\begin{array}{c}\text{double-angle}\\ \text{identity}\end{array}\right]$

$=2\int_0^{\pi/3}\left[4\sin^2 x\left(1-\sin^2 x\right)-\sin^2 x\right]dx$

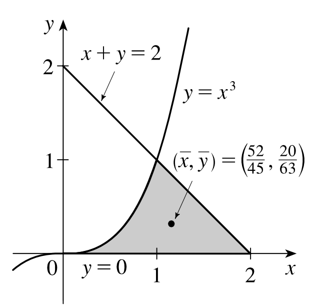
$=6\int_0^{\pi/3}\sin^2 x\,dx-8\int_0^{\pi/3}\sin^4 x\,dx$

Now, $\displaystyle\int_0^{\pi/3}\sin^2 x\,dx \overset{63}{=} \left[\frac{1}{2}x-\frac{1}{4}\sin 2x\right]_0^{\pi/3}=\frac{\pi}{6}-\frac{\sqrt{3}}{8}$ and

$$\int_0^{\pi/3}\sin^4 x\,dx=\frac{1}{4}\left[\frac{3}{2}x-\sin 2x+\frac{1}{8}\sin 4x\right]_0^{\pi/3} \qquad [\text{by Example 7.2.4}]$$

$$=\frac{1}{4}\left[\frac{\pi}{2}-\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{16}\right]=\frac{\pi}{8}-\frac{9\sqrt{3}}{64}$$

So $\overline{y}=6\left(\dfrac{\pi}{6}-\dfrac{\sqrt{3}}{8}\right)-8\left(\dfrac{\pi}{8}-\dfrac{9\sqrt{3}}{64}\right)=\dfrac{3\sqrt{3}}{8}$. Thus, the centroid is $(\overline{x},\overline{y})=\left(\pi-\dfrac{3\sqrt{3}}{2},\dfrac{3\sqrt{3}}{8}\right)\approx(0.54,0.65)$.



32. $A=\int_0^1 x^3\,dx+\int_1^2(2-x)\,dx=\left[\frac{1}{4}x^4\right]_0^1+\left[2x-\frac{1}{2}x^2\right]_1^2$

$=\frac{1}{4}+(4-2)-\left(2-\frac{1}{2}\right)=\frac{3}{4}$.

$\overline{x}=\frac{1}{A}\left[\int_0^1 x(x^3)\,dx+\int_1^2 x(2-x)\,dx\right]=\frac{4}{3}\left[\int_0^1 x^4\,dx+\int_1^2(2x-x^2)\,dx\right]$

$=\frac{4}{3}\left\{\left[\frac{1}{5}x^5\right]_0^1+\left[x^2-\frac{1}{3}x^3\right]_1^2\right\}=\frac{4}{3}\left[\frac{1}{5}+\left(4-\frac{8}{3}\right)-\left(1-\frac{1}{3}\right)\right]$

$=\frac{4}{3}\left(\frac{13}{15}\right)=\frac{52}{45}$.

[continued]

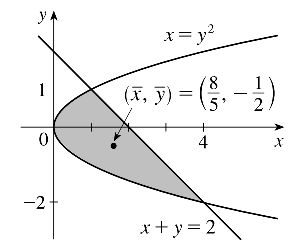
$\overline{y} = \frac{1}{A}\left[\int_0^1 \frac{1}{2}(x^3)^2\,dx + \int_1^2 \frac{1}{2}(2-x)^2\,dx\right] = \frac{2}{3}\left[\int_0^1 x^6 dx + \int_1^2 (x-2)^2\,dx\right] = \frac{2}{3}\left\{\left[\frac{1}{7}x^7\right]_0^1 + \left[\frac{1}{3}(x-2)^3\right]_1^2\right\}$

$= \frac{2}{3}\left(\frac{1}{7} - 0 + 0 + \frac{1}{3}\right) = \frac{2}{3}\left(\frac{10}{21}\right) = \frac{20}{63}.$

Thus, the centroid is $(\overline{x}, \overline{y}) = \left(\frac{52}{45}, \frac{20}{63}\right)$.

33. The curves intersect when $2 - y = y^2 \quad\Leftrightarrow\quad 0 = y^2 + y - 2 \quad\Leftrightarrow$

$0 = (y+2)(y-1) \quad\Leftrightarrow\quad y = -2$ or $y = 1$.



$A = \int_{-2}^{1}(2 - y - y^2)\,dy = \left[2y - \frac{1}{2}y^2 - \frac{1}{3}y^3\right]_{-2}^{1} = \frac{7}{6} - \left(-\frac{10}{3}\right) = \frac{9}{2}.$

$\overline{x} = \frac{1}{A}\int_{-2}^{1} \frac{1}{2}[(2-y)^2 - (y^2)^2]\,dy = \frac{2}{9}\cdot\frac{1}{2}\int_{-2}^{1}(4 - 4y + y^2 - y^4)\,dy$

$= \frac{1}{9}\left[4y - 2y^2 + \frac{1}{3}y^3 - \frac{1}{5}y^5\right]_{-2}^{1} = \frac{1}{9}\left[\frac{32}{15} - \left(-\frac{184}{15}\right)\right] = \frac{8}{5}.$

$\overline{y} = \frac{1}{A}\int_{-2}^{1} y(2 - y - y^2)\,dy = \frac{2}{9}\int_{-2}^{1}(2y - y^2 - y^3)\,dy$

$= \frac{2}{9}\left[y^2 - \frac{1}{3}y^3 - \frac{1}{4}y^4\right]_{-2}^{1} = \frac{2}{9}\left(\frac{5}{12} - \frac{8}{3}\right) = -\frac{1}{2}.$

Thus, the centroid is $(\overline{x}, \overline{y}) = \left(\frac{8}{5}, -\frac{1}{2}\right)$.

34. An equation of the line is $y = -\frac{3}{2}x + 3$. $A = \frac{1}{2}(2)(3) = 3$, so $m = \rho A = 4(3) = 12$.

$M_x = \rho\int_0^2 \frac{1}{2}\left(-\frac{3}{2}x + 3\right)^2\,dx = \frac{1}{2}\rho\int_0^2\left(\frac{9}{4}x^2 - 9x + 9\right)dx = \frac{1}{2}(4)\left[\frac{3}{4}x^3 - \frac{9}{2}x^2 + 9x\right]_0^2 = 2(6 - 18 + 18) = 12.$

$M_y = \rho\int_0^2 x\left(-\frac{3}{2}x + 3\right)dx = \rho\int_0^2\left(-\frac{3}{2}x^2 + 3x\right)dx = 4\left[-\frac{1}{2}x^3 + \frac{3}{2}x^2\right]_0^2 = 4(-4 + 6) = 8.$

$\overline{x} = \dfrac{M_y}{m} = \dfrac{8}{12} = \dfrac{2}{3}$ and $\overline{y} = \dfrac{M_x}{m} = \dfrac{12}{12} = 1$. Thus, the center of mass is $(\overline{x}, \overline{y}) = \left(\frac{2}{3}, 1\right)$. Since ρ is constant, the center of mass is also the centroid.

35. The quarter-circle has equation $y = \sqrt{4^2 - x^2}$ for $0 \le x \le 4$ and the line has equation $y = -2$.

$A = \frac{1}{4}\pi(4)^2 + 2(4) = 4\pi + 8 = 4(\pi + 2)$, so $m = \rho A = 6\cdot 4(\pi + 2) = 24(\pi + 2)$.

$M_x = \rho\int_0^4 \frac{1}{2}\left[\left(\sqrt{16 - x^2}\right)^2 - (-2)^2\right]dx = \frac{1}{2}\rho\int_0^4(16 - x^2 - 4)\,dx = \frac{1}{2}(6)\left[12x - \frac{1}{3}x^3\right]_0^4 = 3\left(48 - \frac{64}{3}\right) = 80.$

$M_y = \rho\int_0^4 x\left[\sqrt{16 - x^2} - (-2)\right]dx = \rho\int_0^4 x\sqrt{16 - x^2}\,dx + \rho\int_0^4 2x\,dx = 6\left[-\frac{1}{3}(16 - x^2)^{3/2}\right]_0^4 + 6\left[x^2\right]_0^4$

$= 6\left(0 + \frac{64}{3}\right) + 6(16) = 224.$

$\overline{x} = \dfrac{M_y}{m} = \dfrac{224}{24(\pi + 2)} = \dfrac{28}{3(\pi + 2)}$ and $\overline{y} = \dfrac{M_x}{m} = \dfrac{80}{24(\pi + 2)} = \dfrac{10}{3(\pi + 2)}$.

Thus, the center of mass is $\left(\dfrac{28}{3(\pi + 2)}, \dfrac{10}{3(\pi + 2)}\right) \approx (1.82, 0.65)$.

36. We'll use $n = 8$, so $\Delta x = \frac{b - a}{n} = \frac{8 - 0}{8} = 1$.

$$\begin{aligned} A = \int_0^8 f(x)\,dx \approx S_{10} &= \tfrac{1}{3}[f(0) + 4f(1) + 2f(2) + 4f(3) + 2f(4) + 4f(5) + 2f(6) + 4f(7) + f(8)] \\ &\approx \tfrac{1}{3}[0 + 4(2.0) + 2(2.6) + 4(2.3) + 2(2.2) + 4(3.3) + 2(4.0) + 4(3.2) + 0] \\ &= \tfrac{1}{3}(60.8) = 20.2\overline{6} \quad \left[\text{or } \tfrac{304}{15}\right] \end{aligned}$$

[continued]

Now
$$\int_0^8 x\,f(x)\,dx \approx \tfrac{1}{3}[0\cdot f(0) + 4\cdot 1\cdot f(1) + 2\cdot 2\cdot f(2) + 4\cdot 3\cdot f(3) + 2\cdot 4\cdot f(4) + 4\cdot 5\cdot f(5) + 2\cdot 6\cdot f(6) + 4\cdot 7\cdot f(7) + 8\cdot f(8)]$$
$$\approx \tfrac{1}{3}[0 + 8 + 10.4 + 27.6 + 17.6 + 66 + 48 + 89.6 + 0]$$
$$= \tfrac{1}{3}(267.2) = 89.0\overline{6}\ \left[\text{or } \tfrac{1336}{15}\right], \text{ so } \overline{x} = \tfrac{1}{A}\int_0^8 x f(x)\,dx \approx 4.39.$$

Also,
$$\int_0^8 [f(x)]^2\,dx \approx \tfrac{1}{3}[0^2 + 4(2.0)^2 + 2(2.6)^2 + 4(2.3)^2 + 2(2.2)^2 + 4(3.3)^2 + 2(4.0)^2 + 4(3.2)^2 + 0^2]$$
$$= \tfrac{1}{3}(176.88) = 58.96, \text{ so } \overline{y} = \tfrac{1}{A}\int_0^8 \tfrac{1}{2}[f(x)]^2\,dx \approx 1.45.$$

Thus, the centroid is $(\overline{x}, \overline{y}) \approx (4.4, 1.5)$.

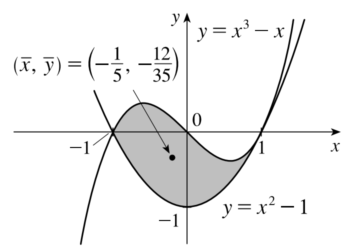
37. $A = \int_{-1}^{1}[(x^3 - x) - (x^2 - 1)]\,dx = \int_{-1}^{1}(1 - x^2)\,dx$ $\left[\begin{array}{c}\text{odd-degree terms}\\ \text{drop out}\end{array}\right]$

$= 2\int_0^1 (1 - x^2)\,dx = 2\left[x - \tfrac{1}{3}x^3\right]_0^1 = 2\left(\tfrac{2}{3}\right) = \tfrac{4}{3}.$

$\overline{x} = \tfrac{1}{A}\int_{-1}^{1} x(x^3 - x - x^2 + 1)\,dx = \tfrac{3}{4}\int_{-1}^{1}(x^4 - x^2 - x^3 + x)\,dx$

$= \tfrac{3}{4}\int_{-1}^{1}(x^4 - x^2)\,dx = \tfrac{3}{4}\cdot 2\int_0^1 (x^4 - x^2)\,dx$

$= \tfrac{3}{2}\left[\tfrac{1}{5}x^5 - \tfrac{1}{3}x^3\right]_0^1 = \tfrac{3}{2}\left(-\tfrac{2}{15}\right) = -\tfrac{1}{5}.$



$\overline{y} = \tfrac{1}{A}\int_{-1}^{1}\tfrac{1}{2}[(x^3 - x)^2 - (x^2 - 1)^2]\,dx = \tfrac{3}{4}\cdot\tfrac{1}{2}\int_{-1}^{1}(x^6 - 2x^4 + x^2 - x^4 + 2x^2 - 1)\,dx$

$= \tfrac{3}{8}\cdot 2\int_0^1 (x^6 - 3x^4 + 3x^2 - 1)\,dx = \tfrac{3}{4}\left[\tfrac{1}{7}x^7 - \tfrac{3}{5}x^5 + x^3 - x\right]_0^1 = \tfrac{3}{4}\left(-\tfrac{16}{35}\right) = -\tfrac{12}{35}.$

Thus, the centroid is $(\overline{x}, \overline{y}) = \left(-\tfrac{1}{5}, -\tfrac{12}{35}\right)$.

38. The curves intersect at $x = a \approx -1.315974$ and $x = b \approx 0.53727445$.

$A = \int_a^b [(2 - x^2) - e^x]\,dx = \left[2x - \tfrac{1}{3}x^3 - e^x\right]_a^b \approx 1.452014.$

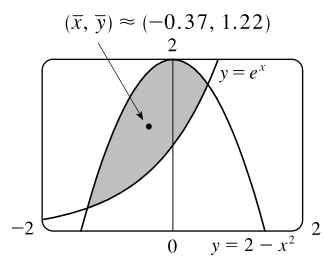
$\overline{x} = \tfrac{1}{A}\int_a^b x(2 - x^2 - e^x)\,dx = \tfrac{1}{A}\left[x^2 - \tfrac{1}{4}x^4 - xe^x + e^x\right]_a^b$

≈ -0.374293

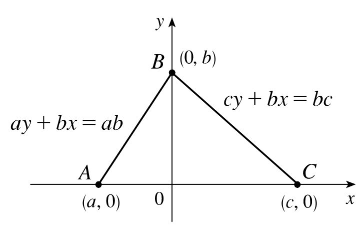
$\overline{y} = \tfrac{1}{A}\int_a^b \tfrac{1}{2}[(2 - x^2)^2 - (e^x)^2]\,dx = \tfrac{1}{2A}\int_a^b (4 - 4x^2 + x^4 - e^{2x})\,dx$

$= \tfrac{1}{2A}\left[4x - \tfrac{4}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{2}e^{2x}\right]_a^b \approx 1.218131$

Thus, the centroid is $(\overline{x}, \overline{y}) \approx (-0.37, 1.22)$.



39. Choose x- and y-axes so that the base (one side of the triangle) lies along the x-axis with the other vertex along the positive y-axis as shown. From geometry, we know the medians intersect at a point $\tfrac{2}{3}$ of the way from each vertex (along the median) to the opposite side. The median from B goes to the midpoint $\left(\tfrac{1}{2}(a + c), 0\right)$ of side AC, so the point of intersection of the medians is $\left(\tfrac{2}{3}\cdot\tfrac{1}{2}(a + c), \tfrac{1}{3}b\right) = \left(\tfrac{1}{3}(a + c), \tfrac{1}{3}b\right)$.



This can also be verified by finding the equations of two medians, and solving them simultaneously to find their point of intersection. Now let us compute the location of the centroid of the triangle. The area is $A = \tfrac{1}{2}(c - a)b$.

[continued]

$$\overline{x} = \frac{1}{A}\left[\int_a^0 x \cdot \frac{b}{a}(a-x)\,dx + \int_0^c x \cdot \frac{b}{c}(c-x)\,dx\right] = \frac{1}{A}\left[\frac{b}{a}\int_a^0 (ax - x^2)\,dx + \frac{b}{c}\int_0^c (cx - x^2)\,dx\right]$$

$$= \frac{b}{Aa}\left[\frac{1}{2}ax^2 - \frac{1}{3}x^3\right]_a^0 + \frac{b}{Ac}\left[\frac{1}{2}cx^2 - \frac{1}{3}x^3\right]_0^c = \frac{b}{Aa}\left[-\frac{1}{2}a^3 + \frac{1}{3}a^3\right] + \frac{b}{Ac}\left[\frac{1}{2}c^3 - \frac{1}{3}c^3\right]$$

$$= \frac{2}{a\,(c-a)} \cdot \frac{-a^3}{6} + \frac{2}{c\,(c-a)} \cdot \frac{c^3}{6} = \frac{1}{3\,(c-a)}(c^2 - a^2) = \frac{a+c}{3}$$

and
$$\overline{y} = \frac{1}{A}\left[\int_a^0 \frac{1}{2}\left(\frac{b}{a}(a-x)\right)^2 dx + \int_0^c \frac{1}{2}\left(\frac{b}{c}(c-x)\right)^2 dx\right]$$

$$= \frac{1}{A}\left[\frac{b^2}{2a^2}\int_a^0 (a^2 - 2ax + x^2)\,dx + \frac{b^2}{2c^2}\int_0^c (c^2 - 2cx + x^2)\,dx\right]$$

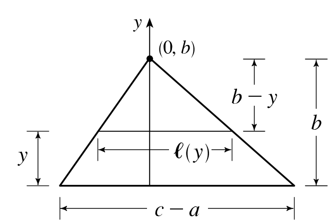
$$= \frac{1}{A}\left[\frac{b^2}{2a^2}\left[a^2x - ax^2 + \tfrac{1}{3}x^3\right]_a^0 + \frac{b^2}{2c^2}\left[c^2x - cx^2 + \tfrac{1}{3}x^3\right]_0^c\right]$$

$$= \frac{1}{A}\left[\frac{b^2}{2a^2}\left(-a^3 + a^3 - \tfrac{1}{3}a^3\right) + \frac{b^2}{2c^2}\left(c^3 - c^3 + \tfrac{1}{3}c^3\right)\right] = \frac{1}{A}\left[\frac{b^2}{6}(-a+c)\right] = \frac{2}{(c-a)\,b} \cdot \frac{(c-a)b^2}{6} = \frac{b}{3}$$

Thus, the centroid is $(\overline{x}, \overline{y}) = \left(\dfrac{a+c}{3}, \dfrac{b}{3}\right)$, as claimed.

Remarks: Actually the computation of $\overline{y}$ is all that is needed. By considering each side of the triangle in turn to be the base, we see that the centroid is $\frac{1}{3}$ of the way from each side to the opposite vertex and must therefore be the intersection of the medians.

The computation of $\overline{y}$ in this problem (and many others) can be simplified by using horizontal rather than vertical approximating rectangles. If the length of a thin rectangle at coordinate y is $\ell(y)$, then its area is $\ell(y)\,\Delta y$, its mass is $\rho\ell(y)\,\Delta y$, and its moment about the x-axis is $\Delta M_x = \rho y \ell(y)\,\Delta y$. Thus,



$$M_x = \int \rho y \ell(y)\,dy \qquad \text{and} \qquad \overline{y} = \frac{\int \rho y \ell(y)\,dy}{\rho A} = \frac{1}{A}\int y\ell(y)\,dy$$

In this problem, $\ell(y) = \dfrac{c-a}{b}(b-y)$ by similar triangles, so

$$\overline{y} = \frac{1}{A}\int_0^b \frac{c-a}{b}\,y(b-y)\,dy = \frac{2}{b^2}\int_0^b (by - y^2)\,dy = \frac{2}{b^2}\left[\tfrac{1}{2}by^2 - \tfrac{1}{3}y^3\right]_0^b = \frac{2}{b^2} \cdot \frac{b^3}{6} = \frac{b}{3}$$

Notice that only one integral is needed when this method is used.

40. The rectangle to the left of the y-axis has centroid $\left(-\frac{1}{2}, 1\right)$ and area 2. The triangle to the right of the y-axis has area 2 and centroid $\left(\frac{2}{3}, \frac{2}{3}\right)$ [by Exercise 39, the centroid is two-thirds of the way from the vertex $(0, 0)$ to the point $(1, 1)$].

$$\overline{x} = \frac{M_y}{m} = \frac{1}{m}\sum_{i=1}^{2} m_i x_i = \frac{1}{2+2}\left[2\left(-\tfrac{1}{2}\right) + 2\left(\tfrac{2}{3}\right)\right] = \tfrac{1}{4}\left(\tfrac{1}{3}\right) = \tfrac{1}{12}.$$

$$\overline{y} = \frac{M_x}{m} = \frac{1}{m}\sum_{i=1}^{2} m_i y_i = \frac{1}{2+2}\left[2(1) + 2\left(\tfrac{2}{3}\right)\right] = \tfrac{1}{4}\left(\tfrac{10}{3}\right) = \tfrac{5}{6}.$$ Thus, the centroid is $(\overline{x}, \overline{y}) = \left(\frac{1}{12}, \frac{5}{6}\right)$.

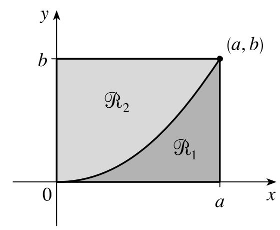
41. Divide the lamina into two triangles and one rectangle with respective masses of 2, 2 and 4, so that the total mass is 8. Using the result of Exercise 39, the triangles have centroids $\left(-1, \frac{2}{3}\right)$ and $\left(1, \frac{2}{3}\right)$. The centroid of the rectangle (its center) is $\left(0, -\frac{1}{2}\right)$.

So, using Formulas 6 and 7, we have $\overline{y} = \dfrac{M_x}{m} = \dfrac{1}{m}\sum\limits_{i=1}^{3} m_i\, y_i = \frac{1}{8}\left[2\left(\frac{2}{3}\right) + 2\left(\frac{2}{3}\right) + 4\left(-\frac{1}{2}\right)\right] = \frac{1}{8}\left(\frac{2}{3}\right) = \frac{1}{12}$, and $\overline{x} = 0$,

since the lamina is symmetric about the line $x = 0$. Thus, the centroid is $(\overline{x}, \overline{y}) = \left(0, \frac{1}{12}\right)$.

42. The parabola has equation $y = kx^2$ and passes through (a, b),

so $b = ka^2 \;\Rightarrow\; k = \dfrac{b}{a^2}$ and hence, $y = \dfrac{b}{a^2}x^2$.



$\mathcal{R}_1$ has area $A_1 = \displaystyle\int_0^a \frac{b}{a^2}x^2\,dx = \frac{b}{a^2}\left[\frac{1}{3}x^3\right]_0^a = \frac{b}{a^2}\left(\frac{a^3}{3}\right) = \frac{1}{3}ab.$

Since $\mathcal{R}$ has area ab, $\mathcal{R}_2$ has area $A_2 = ab - \frac{1}{3}ab = \frac{2}{3}ab$.

For $\mathcal{R}_1$:

$$\overline{x}_1 = \frac{1}{A_1}\int_0^a x\left(\frac{b}{a^2}\,x^2\right)dx = \frac{3}{ab}\,\frac{b}{a^2}\int_0^a x^3\,dx = \frac{3}{a^3}\left[\frac{1}{4}x^4\right]_0^a = \frac{3}{a^3}\left(\frac{1}{4}a^4\right) = \frac{3}{4}a$$

$$\overline{y}_1 = \frac{1}{A_1}\int_0^a \frac{1}{2}\left(\frac{b}{a^2}\,x^2\right)^2 dx = \frac{3}{ab}\,\frac{b^2}{2a^4}\int_0^a x^4\,dx = \frac{3b}{2a^5}\left[\frac{1}{5}x^5\right]_0^a = \frac{3b}{2a^5}\left(\frac{1}{5}a^5\right) = \frac{3}{10}b$$

Thus, the centroid for $\mathcal{R}_1$ is $(\overline{x}_1, \overline{y}_1) = \left(\frac{3}{4}a, \frac{3}{10}b\right)$.

For $\mathcal{R}_2$:

$$\overline{x}_2 = \frac{1}{A_2}\int_0^a x\left(b - \frac{b}{a^2}\,x^2\right)dx = \frac{3}{2ab}\int_0^a b\left(x - \frac{1}{a^2}\,x^3\right)dx = \frac{3}{2a}\left[\frac{1}{2}x^2 - \frac{1}{4a^2}x^4\right]_0^a$$

$$= \frac{3}{2a}\left(\frac{a^2}{2} - \frac{a^2}{4}\right) = \frac{3}{2a}\left(\frac{a^2}{4}\right) = \frac{3}{8}a$$

$$\overline{y}_2 = \frac{1}{A_2}\int_0^a \frac{1}{2}\left[(b)^2 - \left(\frac{b}{a^2}\,x^2\right)^2\right]dx = \frac{3}{2ab}\,\frac{1}{2}\int_0^a b^2\left(1 - \frac{1}{a^4}\,x^4\right)dx = \frac{3b}{4a}\left[x - \frac{1}{5a^4}\,x^5\right]_0^a$$

$$= \frac{3b}{4a}\left(a - \frac{1}{5}a\right) = \frac{3b}{4a}\left(\frac{4a}{5}\right) = \frac{3}{5}b$$

Thus, the centroid for $\mathcal{R}_2$ is $(\overline{x}_2, \overline{y}_2) = \left(\frac{3}{8}a, \frac{3}{5}b\right)$. Note the relationships: $A_2 = 2A_1$, $\overline{x}_1 = 2\,\overline{x}_2$, $\overline{y}_2 = 2\,\overline{y}_1$.

43. $\int_a^b (cx + d)\, f(x)\,dx = \int_a^b cx\, f(x)\,dx + \int_a^b d\,f(x)\,dx = c\int_a^b x\,f(x)\,dx + d\int_a^b f(x)\,dx = c\overline{x}A + d\int_a^b f(x)\,dx$ [by (8)]

$= c\overline{x}\int_a^b f(x)\,dx + d\int_a^b f(x)\,dx = (c\overline{x} + d)\int_a^b f(x)\,dx$

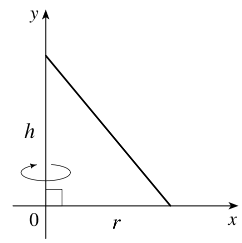
44. A sphere can be generated by rotating a semicircle about its diameter. The center of mass travels a distance

$2\pi\overline{y} = 2\pi\left(\dfrac{4r}{3\pi}\right)$ [from Example 4] $= \dfrac{8r}{3}$, so by the Theorem of Pappus, the volume of the sphere is

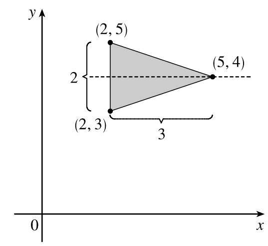
$V = Ad = \dfrac{\pi r^2}{2}\cdot\dfrac{8r}{3} = \dfrac{4}{3}\pi r^3.$

45. A cone of height h and radius r can be generated by rotating a right triangle about one of its legs as shown. By Exercise 39, $\overline{x} = \frac{1}{3}r$, so by the Theorem of Pappus, the volume of the cone is

$$V = Ad = \left(\tfrac{1}{2} \cdot \text{base} \cdot \text{height}\right) \cdot (2\pi\overline{x}) = \tfrac{1}{2}rh \cdot 2\pi\left(\tfrac{1}{3}r\right) = \tfrac{1}{3}\pi r^2 h.$$



46. From the symmetry in the figure, $\overline{y} = 4$. So the distance traveled by the centroid when rotating the triangle about the x-axis is $d = 2\pi \cdot 4 = 8\pi$. The area of the triangle is $A = \frac{1}{2}bh = \frac{1}{2}(2)(3) = 3$. By the Theorem of Pappus, the volume of the resulting solid is $Ad = 3(8\pi) = 24\pi$.



47. The curve C is the quarter-circle $y = \sqrt{16 - x^2}$, $0 \le x \le 4$. Its length L is $\frac{1}{4}(2\pi \cdot 4) = 2\pi$.

Now $y' = \frac{1}{2}(16 - x^2)^{-1/2}(-2x) = \dfrac{-x}{\sqrt{16 - x^2}} \quad\Rightarrow\quad 1 + (y')^2 = 1 + \dfrac{x^2}{16 - x^2} = \dfrac{16}{16 - x^2} \quad\Rightarrow$

$ds = \sqrt{1 + (y')^2}\,dx = \dfrac{4}{\sqrt{16 - x^2}}\,dx$, so

$\overline{x} = \dfrac{1}{L}\displaystyle\int x\,ds = \dfrac{1}{2\pi}\int_0^4 4x(16 - x^2)^{-1/2}\,dx = \dfrac{4}{2\pi}\Big[-(16 - x^2)^{1/2}\Big]_0^4 = \dfrac{2}{\pi}(0 + 4) = \dfrac{8}{\pi}$ and

$\overline{y} = \dfrac{1}{L}\displaystyle\int y\,ds = \dfrac{1}{2\pi}\int_0^4 \sqrt{16 - x^2} \cdot \dfrac{4}{\sqrt{16 - x^2}}\,dx = \dfrac{4}{2\pi}\int_0^4 dx = \dfrac{2}{\pi}\Big[x\Big]_0^4 = \dfrac{2}{\pi}(4 - 0) = \dfrac{8}{\pi}$. Thus, the centroid

is $\left(\dfrac{8}{\pi}, \dfrac{8}{\pi}\right)$. Note that the centroid does not lie on the curve, but does lie on the line $y = x$, as expected, due to the symmetry of the curve.

48. (a) From Exercise 47, we have $\overline{y} = (1/L)\int y\,ds \quad\Leftrightarrow\quad \overline{y}L = \int y\,ds$. The surface area is

$S = \int 2\pi y\,ds = 2\pi\int y\,ds = 2\pi(\overline{y}L) = L(2\pi\overline{y})$, which is the product of the arc length of C and the distance traveled by the centroid of C.

(b) From Exercise 47, $L = 2\pi$ and $\overline{y} = \frac{8}{\pi}$. By the Second Theorem of Pappus, the surface area is

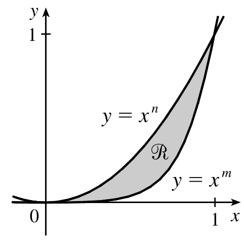
$S = L(2\pi\overline{y}) = 2\pi(2\pi \cdot \frac{8}{\pi}) = 32\pi$.

A geometric formula for the surface area of a half-sphere is $S = 2\pi r^2$. With $r = 4$, we get $S = 32\pi$, which agrees with our first answer.

49. The circle has arc length (circumference) $L = 2\pi r$. As in Example 7, the distance traveled by the centroid during a rotation is $d = 2\pi R$. Therefore, by the Second Theorem of Pappus, the surface area is

$$S = Ld = (2\pi r)(2\pi R) = 4\pi^2 rR$$

50. (a) Let $0 \le x \le 1$. If $n < m$, then $x^n > x^m$; that is, raising x to a larger power produces a smaller number.



(b) Using Formulas 9 and the fact that the area of $\mathcal{R}$ is

$A = \int_0^1 (x^n - x^m)\,dx = \dfrac{1}{n+1} - \dfrac{1}{m+1} = \dfrac{m-n}{(n+1)(m+1)}$, we get

$$\overline{x} = \frac{(n+1)(m+1)}{m-n} \int_0^1 x[x^n - x^m]\,dx = \frac{(n+1)(m+1)}{m-n} \int_0^1 \left(x^{n+1} - x^{m+1}\right) dx$$

$$= \frac{(n+1)(m+1)}{m-n}\left[\frac{1}{n+2} - \frac{1}{m+2}\right] = \frac{(n+1)(m+1)}{(n+2)(m+2)}$$

and

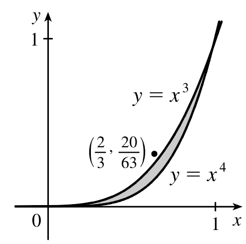
$$\overline{y} = \frac{(n+1)(m+1)}{m-n} \int_0^1 \tfrac{1}{2}\left[(x^n)^2 - (x^m)^2\right] dx = \frac{(n+1)(m+1)}{2(m-n)} \int_0^1 \left(x^{2n} - x^{2m}\right) dx$$

$$= \frac{(n+1)(m+1)}{2(m-n)}\left[\frac{1}{2n+1} - \frac{1}{2m+1}\right] = \frac{(n+1)(m+1)}{(2n+1)(2m+1)}$$

(c) If we take $n = 3$ and $m = 4$, then

$$(\overline{x}, \overline{y}) = \left(\frac{4\cdot 5}{5\cdot 6}, \frac{4\cdot 5}{7\cdot 9}\right) = \left(\frac{2}{3}, \frac{20}{63}\right)$$

which lies outside $\mathcal{R}$ since $\left(\frac{2}{3}\right)^3 = \frac{8}{27} < \frac{20}{63}$. This is the simplest of many possibilities.



51. Suppose the region lies between two curves $y = f(x)$ and $y = g(x)$ where $f(x) \ge g(x)$, as illustrated in Figure 13. Choose points x_i with $a = x_0 < x_1 < \cdots < x_n = b$ and choose x_i^* to be the midpoint of the ith subinterval; that is, $x_i^* = \overline{x}_i = \frac{1}{2}(x_{i-1} + x_i)$. Then the centroid of the ith approximating rectangle R_i is its center $C_i = \left(\overline{x}_i, \frac{1}{2}[f(\overline{x}_i) + g(\overline{x}_i)]\right)$. Its area is $[f(\overline{x}_i) - g(\overline{x}_i)]\,\Delta x$, so its mass is $\rho[f(\overline{x}_i) - g(\overline{x}_i)]\,\Delta x$. Thus, $M_y(R_i) = \rho[f(\overline{x}_i) - g(\overline{x}_i)]\,\Delta x \cdot \overline{x}_i = \rho\overline{x}_i\,[f(\overline{x}_i) - g(\overline{x}_i)]\,\Delta x$ and $M_x(R_i) = \rho[f(\overline{x}_i) - g(\overline{x}_i)]\,\Delta x \cdot \frac{1}{2}[f(\overline{x}_i) + g(\overline{x}_i)] = \rho \cdot \frac{1}{2}\left[f(\overline{x}_i)^2 - g(\overline{x}_i)^2\right]\Delta x$. Summing over i and taking the limit as $n \to \infty$, we get $M_y = \lim\limits_{n\to\infty} \sum_i \rho\overline{x}_i\,[f(\overline{x}_i) - g(\overline{x}_i)]\,\Delta x = \rho\int_a^b x[f(x) - g(x)]\,dx$ and

$M_x = \lim\limits_{n\to\infty} \sum_i \rho \cdot \frac{1}{2}\left[f(\overline{x}_i)^2 - g(\overline{x}_i)^2\right]\Delta x = \rho\int_a^b \frac{1}{2}\left[f(x)^2 - g(x)^2\right] dx.$

Thus, $\overline{x} = \dfrac{M_y}{m} = \dfrac{M_y}{\rho A} = \dfrac{1}{A}\displaystyle\int_a^b x[f(x) - g(x)]\,dx$ and $\overline{y} = \dfrac{M_x}{m} = \dfrac{M_x}{\rho A} = \dfrac{1}{A}\displaystyle\int_a^b \tfrac{1}{2}\left[f(x)^2 - g(x)^2\right] dx.$

DISCOVERY PROJECT Complementary Coffee Cups

1. Cup A has volume $V_A = \int_0^h \pi[f(y)]^2\,dy$ and cup B has volume

$$V_B = \int_0^h \pi[k - f(y)]^2\,dy = \int_0^h \pi\{k^2 - 2kf(y) + [f(y)]^2\}\,dy$$
$$= \left[\pi k^2 y\right]_0^h - 2\pi k \int_0^h f(y)\,dy + \int_0^h \pi[f(y)]^2\,dy = \pi k^2 h - 2\pi k A_1 + V_A$$

Thus, $V_A = V_B \iff \pi k(kh - 2A_1) = 0 \iff k = 2(A_1/h)$; that is, k is twice the average value of f on the interval $[0, h]$.

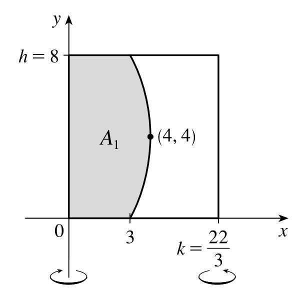
2. From Problem 1, $V_A = V_B \iff kh = 2A_1 \iff A_1 + A_2 = 2A_1 \iff A_2 = A_1$.

3. Let $\overline{x}_1$ and $\overline{x}_2$ denote the x-coordinates of the centroids of A_1 and A_2, respectively. By Pappus's Theorem,

$V_A = 2\pi\overline{x}_1 A_1$ and $V_B = 2\pi(k - \overline{x}_2)A_2$, so $V_A = V_B \iff \overline{x}_1 A_1 = kA_2 - \overline{x}_2 A_2 \iff kA_2 = \overline{x}_1 A_1 + \overline{x}_2 A_2 \overset{(\star)}{\iff}$ $kA_2 = \frac{1}{2}k\,(A_1 + A_2) \iff \frac{1}{2}kA_2 = \frac{1}{2}kA_1 \iff A_2 = A_1$, as shown in Problem 2. [$(\star)$ The sum of the moments of the regions of areas A_1 and A_2 about the y-axis equals the moment of the entire k-by-h rectangle about the y-axis.]

So, since $A_1 + A_2 = kh$, we have $V_A = V_B \iff A_1 = A_2 \iff A_1 = \frac{1}{2}(A_1 + A_2) \iff A_1 = \frac{1}{2}(kh) \iff$ $k = 2\,(A_1/h)$, as shown in Problem 1.

4. We'll use a cup that is $h = 8$ cm high with a diameter of 6 cm on the top and the bottom and symmetrically bulging to a diameter of 8 cm in the middle (all inside dimensions).



For an equation, we'll use a parabola with a vertex at $(4, 4)$; that is, $x = a(y-4)^2 + 4$. To find a, use the point $(3, 0)$: $3 = a(0-4)^2 + 4 \;\Rightarrow\; -1 = 16a \;\Rightarrow\; a = -\frac{1}{16}$. To find k, we'll use the relationship in Problem 1, so we need A_1.

$$A_1 = \int_0^8 \left[-\tfrac{1}{16}(y-4)^2 + 4\right] dy = \int_{-4}^4 \left(-\tfrac{1}{16}u^2 + 4\right) du \quad [u = y - 4]$$
$$= 2\int_0^4 \left(-\tfrac{1}{16}u^2 + 4\right) du = 2\left[-\tfrac{1}{48}u^3 + 4u\right]_0^4 = 2\left(-\tfrac{4}{3} + 16\right) = \tfrac{88}{3}.$$

Thus, $k = 2(A_1/h) = 2\left(\frac{88/3}{8}\right) = \frac{22}{3}$.

So with $h = 8$ and curve $x = -\frac{1}{16}(y-4)^2 + 4$, we have

$$V_A = \int_0^8 \pi\left[-\tfrac{1}{16}(y-4)^2 + 4\right]^2 dy = \pi\int_{-4}^4 \left(-\tfrac{1}{16}u^2 + 4\right)^2 du \quad [u = y - 4] \quad = 2\pi\int_0^4 \left(\tfrac{1}{256}u^4 - \tfrac{1}{2}u^2 + 16\right) du$$
$$= 2\pi\left[\tfrac{1}{1280}u^5 - \tfrac{1}{6}u^3 + 16u\right]_0^4 = 2\pi\left(\tfrac{4}{5} - \tfrac{32}{3} + 64\right) = 2\pi\left(\tfrac{812}{15}\right) = \tfrac{1624}{15}\pi$$

This is approximately 340 cm^3 or 11.5 fl. oz. And with $k = \frac{22}{3}$, we know from Problem 1 that cup B holds the same amount.

8.4 Applications to Economics and Biology

1. By the Net Change Theorem, $C(4000) - C(0) = \int_0^{4000} C'(x)\,dx \quad \Rightarrow$

$$C(4000) = 18{,}000 + \int_0^{4000} (0.82 - 0.000\,03x + 0.000\,000\,003x^2)\,dx$$
$$= 18{,}000 + \left[0.82x - 0.000\,015x^2 + 0.000\,000\,001x^3\right]_0^{4000} = 18{,}000 + 3104 = \$21{,}104$$

2. By the Net Change Theorem,

$$R(10{,}000) - R(5000) = \int_{5000}^{10{,}000} R'(x)\,dx = \int_{5000}^{10{,}000} (48 - 0.0012x)\,dx = \left[48x - 0.0006x^2\right]_{5000}^{10{,}000}$$
$$= 420{,}000 - 225{,}000 = \$195{,}000$$

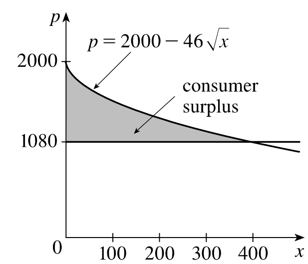
3. By the Net Change Theorem, $C(50) - C(0) = \int_0^{50} (0.6 + 0.008x)\,dx \quad \Rightarrow$

$C(50) = 100 + \left[0.6x + 0.004x^2\right]_0^{50} = 100 + (40 - 0) = 140$, or \$140,000.

Similarly, $C(100) - C(50) = \left[0.6x + 0.004x^2\right]_{50}^{100} = 100 - 40 = 60$, or \$60,000.

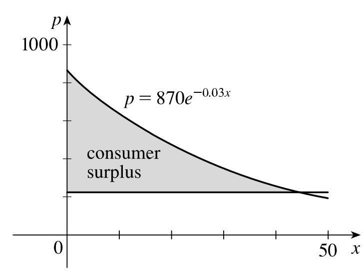
4. Consumer surplus $= \int_0^{400} [p(x) - p(400)]\,dx = \int_0^{400} [(2000 - 46\sqrt{x}\,) - 1080]\,dx$

$$= \int_0^{400} (920 - 46\sqrt{x}\,)\,dx = 46\int_0^{400} (20 - x^{1/2})\,dx$$
$$= 46\left[20x - \tfrac{2}{3}x^{3/2}\right]_0^{400} = 46\left(8000 - \tfrac{2}{3}\cdot 8000\right)$$
$$= 46\cdot\tfrac{1}{3}\cdot 8000 \approx \$122{,}666.67$$



5. Consumer surplus $= \int_0^{45} [p(x) - p(45)]\,dx$

$$= \int_0^{45} \left(870e^{-0.03x} - 870e^{-0.03(45)}\right)dx$$
$$= 870\left[-\frac{1}{0.03}e^{-0.03x} - e^{-1.35}x\right]_0^{45}$$
$$= 870\left(-\frac{1}{0.03}e^{-1.35} - 45e^{-1.35} + \frac{1}{0.03}\right) \approx \$11{,}332.78$$



6. $p = 2.80 \quad \Rightarrow \quad 6 - \dfrac{x}{3500} = 2.80 \quad \Rightarrow \quad \dfrac{x}{3500} = 3.2 \quad \Rightarrow \quad x = 11{,}200$

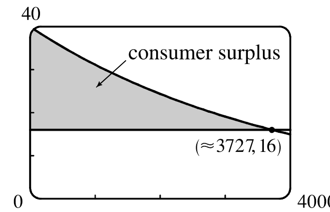
$$\text{Consumer surplus} = \int_0^{11{,}200} [p(x) - 2.80]\,dx = \int_0^{11{,}200} \left(6 - \frac{x}{3500} - 2.80\right)dx$$
$$= \int_0^{11{,}200} \left(3.2 - \frac{x}{3500}\right)dx = \left[3.2x - \frac{x^2}{7000}\right]_0^{11{,}200} = 35{,}840 - 17{,}920 = \$17{,}920$$

7. Since the demand increases by 30 for each dollar the price is lowered, the demand function, $p(x)$, is linear with slope $-\frac{1}{30}$. Also, $p(210) = 18$, so an equation for the demand is $p - 18 = -\frac{1}{30}(x - 210)$ or $p = -\frac{1}{30}x + 25$. A selling price of \$15 implies that $15 = -\frac{1}{30}x + 25 \quad \Rightarrow \quad \frac{1}{30}x = 10 \quad \Rightarrow \quad x = 300$.

Consumer surplus $= \int_0^{300} \left(-\frac{1}{30}x + 25 - 15\right)dx = \int_0^{300} \left(-\frac{1}{30}x + 10\right)dx = \left[-\frac{1}{60}x^2 + 10x\right]_0^{300} = \$1500$

8. $p(x) = \dfrac{800{,}000e^{-x/5000}}{x + 20{,}000} = 16 \quad\Rightarrow\quad x = x_1 \approx 3727.04.$

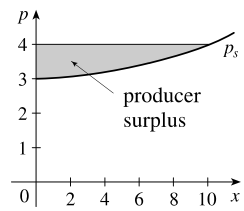
Consumer surplus $= \int_0^{x_1} [p(x) - 16]\, dx \approx \$37{,}753$



9. $p_S(x) = 3 + 0.01x^2. \quad P = p_S(10) = 3 + 1 = 4.$

$$\text{Producer surplus} = \int_0^{10} [P - p_S(x)]\, dx = \int_0^{10} \left[4 - 3 - 0.01x^2\right] dx$$

$$= \left[x - \tfrac{0.01}{3}x^3\right]_0^{10} \approx 10 - 3.33 = \$6.67$$



10. $P = p_S(x) \quad\Rightarrow\quad 625 = 125 + 0.002x^2 \quad\Rightarrow\quad 500 = \frac{1}{500}x^2 \quad\Rightarrow\quad x^2 = 500^2 \quad\Rightarrow\quad x = 500.$

$$\text{Producer surplus} = \int_0^{500} [P - p_S(x)]\, dx = \int_0^{500} [625 - (125 + 0.002x^2)]\, dx = \int_0^{500} \left(500 - \tfrac{1}{500}x^2\right) dx$$

$$= \left[500x - \tfrac{1}{1500}x^3\right]_0^{500} = 500^2 - \tfrac{1}{1500}(500^3) \approx \$166{,}666.67$$

11. $p = \sqrt{30 + 0.01xe^{0.001x}} = 30$ when $x \approx 3278.5$ (using a graphing calculator or other computing device). Then the producer surplus is approximately $\int_0^{3278.5} \left[30 - \sqrt{30 + 0.01xe^{0.001x}}\,\right] dx \approx \$55{,}735$.

12. (a) Demand curve $p_D(x) =$ supply curve $p_S(x) \quad\Leftrightarrow\quad 50 - \frac{1}{20}x = 20 + \frac{1}{10}x \quad\Leftrightarrow\quad 30 = \frac{3}{20}x \quad\Leftrightarrow\quad x = 200.$

$p_D(200) = 50 - \frac{1}{20}(200) = 40$, so the market for this good is in equilibrium when the quantity is 200 and the price is \$40.

(b) At equilibrium, the

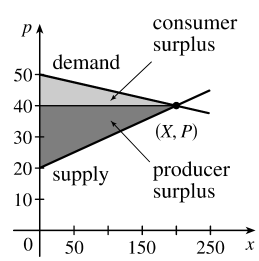
$$\text{Consumer surplus} = \int_0^{200} [p_D(x) - 40]\, dx = \int_0^{200} \left(50 - \tfrac{1}{20}x - 40\right) dx$$

$$= \left[10x - \tfrac{1}{40}x^2\right]_0^{200} = \$1000$$

and the

$$\text{Producer surplus} = \int_0^{200} [40 - p_S(x)]\, dx = \int_0^{200} \left(40 - 20 - \tfrac{1}{10}x\right) dx$$

$$= \left[20x - \tfrac{1}{20}x^2\right]_0^{200} = \$2000$$



13. (a) Demand function $p(x) =$ supply function $p_S(x) \quad\Leftrightarrow\quad 228.4 - 18x = 27x + 57.4 \quad\Leftrightarrow\quad 171 = 45x \quad\Leftrightarrow$

$x = \frac{19}{5}$ [3.8 thousand]. $\quad p(3.8) = 228.4 - 18(3.8) = 160$. The market for the stereos is in equilibrium when the quantity is 3800 and the price is \$160.

(b) $$\text{Consumer surplus} = \int_0^{3.8} [p(x) - 160]\, dx = \int_0^{3.8} (228.4 - 18x - 160)\, dx = \int_0^{3.8} (68.4 - 18x)\, dx$$

$$= \Big[68.4x - 9x^2\Big]_0^{3.8} = 68.4(3.8) - 9(3.8)^2 = 129.96$$

$$\text{Producer surplus} = \int_0^{3.8} [160 - p_S(x)]\, dx = \int_0^{3.8} [160 - (27x + 57.4)]\, dx = \int_0^{3.8} (102.6 - 27x)\, dx$$

$$= \Big[102.6x - 13.5x^2\Big]_0^{3.8} = 102.6(3.8) - 13.5(3.8)^2 = 194.94$$

Thus, the maximum total surplus for the stereos is $129.96 + 194.94 = 324.9$, or \$324,900.

14. $p(x) = p_S(x) \quad \Leftrightarrow \quad 312e^{-0.14x} = 26e^{0.2x} \quad \Leftrightarrow \quad \dfrac{312}{26} = \dfrac{e^{0.2x}}{e^{-0.14x}} \quad \Leftrightarrow \quad 12 = e^{0.34x} \quad \Leftrightarrow \quad \ln 12 = 0.34x \quad \Leftrightarrow$

$x = X = \dfrac{\ln 12}{0.34}$. $X \approx 7.3085$ (in thousands) and $p(X) \approx 112.1465$.

Consumer surplus $= \int_0^X [p(x) - p(X)]\,dx \approx \int_0^{7.3085} (312e^{-0.14x} - 112.1465)\,dx \approx 607.896$

Producer surplus $= \int_0^X [p_S(X) - p_S(x)]\,dx \approx \int_0^{7.3085} (112.1465 - 26e^{0.2x})\,dx \approx 388.896$

Maximum total surplus $\approx 607.896 + 388.896 = 996.792$, or \$996,792.

Note: Since $p(X) = p_S(X)$, the maximum total surplus could be found by calculating $\int_0^X [p(x) - p_S(x)]\,dx$.

15. $f(8) - f(4) = \int_4^8 f'(t)\,dt = \int_4^8 \sqrt{t}\,dt = \left[\frac{2}{3}t^{3/2}\right]_4^8 = \frac{2}{3}\left(16\sqrt{2} - 8\right) \approx \$9.75$ million

16. The total revenue R obtained in the first four years is

$$R = \int_0^4 f(t)\,dt = \int_0^4 9000\sqrt{1+2t}\,dt = \int_1^9 9000u^{1/2}\left(\tfrac{1}{2}\,du\right) \qquad [u = 1+2t,\ du = 2\,dt]$$
$$= 4500\left[\tfrac{2}{3}u^{3/2}\right]_1^9 = 3000(27-1) = \$78{,}000$$

17. Future value $= \int_0^T f(t)\,e^{r(T-t)}\,dt = \int_0^6 8000e^{0.04t}\,e^{0.062(6-t)}\,dt = 8000\int_0^6 e^{0.04t}\,e^{0.372-0.062t}\,dt$

$$= 8000\int_0^6 e^{0.372-0.022t}\,dt = 8000e^{0.372}\int_0^6 e^{-0.022t}\,dt = 8000e^{0.372}\left[\frac{e^{-0.022t}}{-0.022}\right]_0^6$$
$$= \frac{8000e^{0.372}}{-0.022}(e^{-0.132} - 1) \approx \$65{,}230.48$$

18. Present value $= \int_0^T f(t)\,e^{-rt}\,dt = \int_0^6 8000e^{0.04t}\,e^{-0.062t}\,dt = 8000\int_0^6 e^{-0.022t}\,dt = 8000\left[\dfrac{e^{-0.022t}}{-0.022}\right]_0^6$

$$= \frac{8000}{-0.022}\,(e^{-0.132} - 1) \approx \$44{,}966.91$$

19. $N = \displaystyle\int_a^b Ax^{-k}\,dx = A\left[\frac{x^{-k+1}}{-k+1}\right]_a^b = \frac{A}{1-k}\,(b^{1-k} - a^{1-k})$.

Similarly, $\displaystyle\int_a^b Ax^{1-k}\,dx = A\left[\frac{x^{2-k}}{2-k}\right]_a^b = \frac{A}{2-k}\,(b^{2-k} - a^{2-k})$.

Thus, $\overline{x} = \displaystyle\frac{1}{N}\int_a^b Ax^{1-k}\,dx = \frac{[A/(2-k)](b^{2-k} - a^{2-k})}{[A/(1-k)](b^{1-k} - a^{1-k})} = \frac{(1-k)(b^{2-k} - a^{2-k})}{(2-k)(b^{1-k} - a^{1-k})}$.

20. $n(9) - n(5) = \displaystyle\int_5^9 (2200 + 10e^{0.8t})\,dt = \left[2200t + \frac{10e^{0.8t}}{0.8}\right]_5^9 = \left[2200t\right]_5^9 + \tfrac{25}{2}\left[e^{0.8t}\right]_5^9$

$$= 2200(9-5) + 12.5(e^{7.2} - e^4) \approx 24{,}860$$

21. $F = \dfrac{\pi P R^4}{8\eta l} = \dfrac{\pi(4000)(0.008)^4}{8(0.027)(2)} \approx 1.19 \times 10^{-4}\ \text{cm}^3/\text{s}$

22. If the flux remains constant, then $\dfrac{\pi P_0 R_0^4}{8\eta l} = \dfrac{\pi P R^4}{8\eta l} \quad \Rightarrow \quad P_0 R_0^4 = P R^4 \quad \Rightarrow \quad \dfrac{P}{P_0} = \left(\dfrac{R_0}{R}\right)^4$.

$R = \frac{3}{4}R_0 \quad \Rightarrow \quad \dfrac{P}{P_0} = \left(\dfrac{R_0}{\frac{3}{4}R_0}\right)^4 \quad \Rightarrow \quad P = P_0\left(\frac{4}{3}\right)^4 \approx 3.1605P_0 > 3P_0$; that is, the blood pressure is more than tripled.

23. From (3), $F = \dfrac{A}{\int_0^T c(t)\,dt} = \dfrac{6}{20I}$, where

$$I = \int_0^{10} te^{-0.6t}\,dt = \left[\frac{1}{(-0.6)^2}\,(-0.6t-1)\,e^{-0.6t}\right]_0^{10} \quad \begin{bmatrix}\text{integrating}\\ \text{by parts}\end{bmatrix} = \tfrac{1}{0.36}(-7e^{-6}+1)$$

Thus, $F = \dfrac{6(0.36)}{20(1-7e^{-6})} = \dfrac{0.108}{1-7e^{-6}} \approx 0.1099$ L/s or 6.594 L/min.

24. As in Example 2, we will estimate the cardiac output using Simpson's Rule with $\Delta t = (16-0)/8 = 2$.

$$\begin{aligned}\textstyle\int_0^{16} c(t)\,dt &\approx \tfrac{2}{3}[c(0)+4c(2)+2c(4)+4c(6)+2c(8)+4c(10)+2c(12)+4c(14)+c(16)]\\ &= \tfrac{2}{3}[0+4(4.1)+2(8.9)+4(8.5)+2(6.7)+4(4.3)+2(2.5)+4(1.2)+0.2]\\ &= \tfrac{2}{3}(108.8) = 72.5\overline{3}\ \text{mg}\cdot\text{s/L}\end{aligned}$$

Therefore, $F \approx \dfrac{A}{72.5\overline{3}} = \dfrac{5.5}{72.5\overline{3}} \approx 0.0758$ L/s or 4.55 L/min.

25. As in Example 2, we will estimate the cardiac output using Simpson's Rule with $\Delta t = (16-0)/8 = 2$.

$$\begin{aligned}\textstyle\int_0^{16} c(t)\,dt &\approx \tfrac{2}{3}[c(0)+4c(2)+2c(4)+4c(6)+2c(8)+4c(10)+2c(12)+4c(14)+c(16)]\\ &\approx \tfrac{2}{3}[0+4(6.1)+2(7.4)+4(6.7)+2(5.4)+4(4.1)+2(3.0)+4(2.1)+1.5]\\ &= \tfrac{2}{3}(109.1) = 72.7\overline{3}\ \text{mg}\cdot\text{s/L}\end{aligned}$$

Therefore, $F \approx \dfrac{A}{72.7\overline{3}} = \dfrac{7}{72.7\overline{3}} \approx 0.0962$ L/s or 5.77 L/min.

8.5 Probability

1. (a) $\int_{50{,}000}^{65{,}000} f(x)\,dx$ is the probability that a randomly chosen tire will have a lifetime between $50{,}000$ and $65{,}000$ kilometers.

(b) $\int_{40{,}000}^{\infty} f(x)\,dx$ is the probability that a randomly chosen tire will have a lifetime of at least $40{,}000$ kilometers.

2. (a) The probability that you drive to school in less than 15 minutes is $\int_0^{15} f(t)\,dt$.

(b) The probability that it takes you more than half an hour to get to school is $\int_{30}^{\infty} f(t)\,dt$.

3. (a) In general, we must satisfy the two conditions that are mentioned before Example 1—namely, **(1)** $f(x) \geq 0$ for all x, and **(2)** $\int_{-\infty}^{\infty} f(x)\,dx = 1$. For $0 \leq x \leq 1$, $f(x) = 30x^2(1-x)^2 \geq 0$ and $f(x) = 0$ for all other values of x, so $f(x) \geq 0$ for all x. Also,

$$\begin{aligned}\textstyle\int_{-\infty}^{\infty} f(x)\,dx &= \textstyle\int_0^1 30x^2(1-x)^2\,dx = \int_0^1 30x^2(1-2x+x^2)\,dx = \int_0^1 (30x^2-60x^3+30x^4)\,dx\\ &= \left[10x^3-15x^4+6x^5\right]_0^1 = 10-15+6 = 1\end{aligned}$$

Therefore, f is a probability density function.

(b) $P\left(X \leq \frac{1}{3}\right) = \int_{-\infty}^{1/3} f(x)\,dx = \int_0^{1/3} 30x^2(1-x)^2\,dx = \left[10x^3-15x^4+6x^5\right]_0^{1/3} = \frac{10}{27} - \frac{15}{81} + \frac{6}{243} = \frac{17}{81}$

4. (a) In general, we must satisfy the two conditions that are mentioned before Example 1—namely, **(1)** $f(x) \geq 0$ for all x, and **(2)** $\int_{-\infty}^{\infty} f(x)\,dx = 1$. For $f(x) = \dfrac{e^{3-x}}{(1+e^{3-x})^2}$, the numerator and denominator are both positive, so $f(x) \geq 0$ for all x.

Also,

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx = \lim_{t\to-\infty}\int_t^0 \frac{e^{3-x}}{(1+e^{3-x})^2}\,dx + \lim_{s\to\infty}\int_0^s \frac{e^{3-x}}{(1+e^{3-x})^2}\,dx$$

$$= \lim_{t\to-\infty}\int_{x=t}^{0}\frac{-du}{u^2} + \lim_{s\to\infty}\int_{x=0}^{s}\frac{-du}{u^2} \qquad \begin{bmatrix} u = 1+e^{3-x}, \\ du = -e^{3-x}\,dx \end{bmatrix}$$

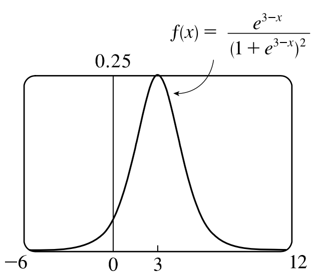
$$= \lim_{t\to-\infty}\left[\frac{1}{u}\right]_{x=t}^{0} + \lim_{s\to\infty}\left[\frac{1}{u}\right]_{x=0}^{s} = \lim_{t\to-\infty}\left[\frac{1}{1+e^{3-x}}\right]_t^0 + \lim_{s\to\infty}\left[\frac{1}{1+e^{3-x}}\right]_0^s$$

$$= \lim_{t\to-\infty}\left(\frac{1}{1+e^3} - \frac{1}{1+e^{3-t}}\right) + \lim_{s\to\infty}\left(\frac{1}{1+e^{3-s}} - \frac{1}{1+e^3}\right) = \frac{1}{1+e^3} - 0 + 1 - \frac{1}{1+e^3} = 1.$$

Therefore, f is a probability density function.

(b) $P(3 \leq X \leq 4) = \displaystyle\int_3^4 f(x)\,dx = \left[\frac{1}{1+e^{3-x}}\right]_3^4$ [from part (a)] $= \dfrac{1}{1+e^{-1}} - \dfrac{1}{1+1} \approx 0.231$

(c)



The graph of f appears to be symmetric about the line $x = 3$, so the mean appears to be 3. Similarly, half the area under the graph of f appears to lie to the right of $x = 3$, so the median also appears to be 3.

5. (a) In general, we must satisfy the two conditions that are mentioned before Example 1—namely, **(1)** $f(x) \geq 0$ for all x, and **(2)** $\int_{-\infty}^{\infty} f(x)\,dx = 1$. If $c \geq 0$, then $f(x) \geq 0$, so condition (1) is satisfied. For condition (2), we see that

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty}\frac{c}{1+x^2}\,dx \text{ and}$$

$$\int_0^{\infty}\frac{c}{1+x^2}\,dx = \lim_{t\to\infty}\int_0^t \frac{c}{1+x^2}\,dx = c\lim_{t\to\infty}\left[\tan^{-1}x\right]_0^t = c\lim_{t\to\infty}\tan^{-1}t = c\left(\frac{\pi}{2}\right)$$

Similarly, $\displaystyle\int_{-\infty}^{0}\frac{c}{1+x^2}\,dx = c\left(\frac{\pi}{2}\right)$, so $\displaystyle\int_{-\infty}^{\infty}\frac{c}{1+x^2}\,dx = 2c\left(\frac{\pi}{2}\right) = c\pi$.

Since $c\pi$ must equal 1, we must have $c = 1/\pi$ so that f is a probability density function.

(b) $P(-1 < X < 1) = \displaystyle\int_{-1}^{1}\frac{1/\pi}{1+x^2}\,dx = \frac{2}{\pi}\int_0^1\frac{1}{1+x^2}\,dx = \frac{2}{\pi}\left[\tan^{-1}x\right]_0^1 = \frac{2}{\pi}\left(\frac{\pi}{4} - 0\right) = \frac{1}{2}$

6. (a) For $0 \leq x \leq 3$, we have $f(x) = k(3x - x^2)$, which is nonnegative if and only if $k \geq 0$. Also, $\int_{-\infty}^{\infty} f(x)\,dx = \int_0^3 k(3x - x^2)\,dx = k\left[\frac{3}{2}x^2 - \frac{1}{3}x^3\right]_0^3 = k\left(\frac{27}{2} - 9\right) = \frac{9}{2}k$. Now $\frac{9}{2}k = 1 \;\Rightarrow\; k = \frac{2}{9}$. Therefore, f is a probability density function if and only if $k = \frac{2}{9}$.

(b) Let $k = \frac{2}{9}$.

$P(X > 1) = \int_1^\infty f(x)\,dx = \int_1^3 \frac{2}{9}(3x - x^2)\,dx = \frac{2}{9}\left[\frac{3}{2}x^2 - \frac{1}{3}x^3\right]_1^3 = \frac{2}{9}\left[\left(\frac{27}{2} - 9\right) - \left(\frac{3}{2} - \frac{1}{3}\right)\right] = \frac{2}{9}\left(\frac{10}{3}\right) = \frac{20}{27}$.

(c) The mean $\mu = \int_{-\infty}^\infty x f(x)\,dx = \int_0^3 x\left[\frac{2}{9}(3x - x^2)\right]dx = \frac{2}{9}\int_0^3 (3x^2 - x^3)\,dx$
$= \frac{2}{9}\left[x^3 - \frac{1}{4}x^4\right]_0^3 = \frac{2}{9}\left(27 - \frac{81}{4}\right) = \frac{2}{9}\left(\frac{27}{4}\right) = \frac{3}{2}$.

7. (a) In general, we must satisfy the two conditions that are mentioned before Example 1—namely, **(1)** $f(x) \geq 0$ for all x, and **(2)** $\int_{-\infty}^\infty f(x)\,dx = 1$. Since $f(x) = 0$ or $f(x) = 0.1$, condition (1) is satisfied. For condition (2), we see that $\int_{-\infty}^\infty f(x)\,dx = \int_0^{10} 0.1\,dx = \left[\frac{1}{10}x\right]_0^{10} = 1$. Thus, $f(x)$ is a probability density function for the spinner's values.

(b) Since all the numbers between 0 and 10 are equally likely to be selected, we expect the mean to be halfway between the endpoints of the interval; that is, $x = 5$.

$$\mu = \int_{-\infty}^\infty x f(x)\,dx = \int_0^{10} x(0.1)\,dx = \left[\tfrac{1}{20}x^2\right]_0^{10} = \tfrac{100}{20} = 5, \quad \text{as expected.}$$

8. (a) As in the preceding exercise, **(1)** $f(x) \geq 0$ and **(2)** $\int_{-\infty}^\infty f(x)\,dx = \int_0^{10} f(x)\,dx = \frac{1}{2}(10)(0.2)$ [area of a triangle] $= 1$. So $f(x)$ is a probability density function.

(b) (i) $P(X < 3) = \int_0^3 f(x)\,dx = \frac{1}{2}(3)(0.1) = \frac{3}{20} = 0.15$

(ii) We first compute $P(X > 8)$ and then subtract that value and our answer in (i) from 1 (the total probability).
$P(X > 8) = \int_8^{10} f(x)\,dx = \frac{1}{2}(2)(0.1) = \frac{2}{20} = 0.10$. So $P(3 \leq X \leq 8) = 1 - 0.15 - 0.10 = 0.75$.

(c) We find equations of the lines from $(0, 0)$ to $(6, 0.2)$ and from $(6, 0.2)$ to $(10, 0)$, and find that

$$f(x) = \begin{cases} \frac{1}{30}x & \text{if } 0 \leq x < 6 \\ -\frac{1}{20}x + \frac{1}{2} & \text{if } 6 \leq x < 10 \\ 0 & \text{otherwise} \end{cases}$$

$$\mu = \int_{-\infty}^\infty x f(x)\,dx = \int_0^6 x\left(\tfrac{1}{30}x\right)dx + \int_6^{10} x\left(-\tfrac{1}{20}x + \tfrac{1}{2}\right)dx = \left[\tfrac{1}{90}x^3\right]_0^6 + \left[-\tfrac{1}{60}x^3 + \tfrac{1}{4}x^2\right]_6^{10}$$
$$= \tfrac{216}{90} + \left(-\tfrac{1000}{60} + \tfrac{100}{4}\right) - \left(-\tfrac{216}{60} + \tfrac{36}{4}\right) = \tfrac{16}{3} = 5.\overline{3}$$

9. We need to find m so that $\int_m^\infty f(t)\,dt = \frac{1}{2} \;\Rightarrow\; \lim\limits_{x\to\infty} \int_m^x \frac{1}{5}e^{-t/5}\,dt = \frac{1}{2} \;\Rightarrow\; \lim\limits_{x\to\infty}\left[\frac{1}{5}(-5)e^{-t/5}\right]_m^x = \frac{1}{2} \;\Rightarrow$
$(-1)(0 - e^{-m/5}) = \frac{1}{2} \;\Rightarrow\; e^{-m/5} = \frac{1}{2} \;\Rightarrow\; -m/5 = \ln\frac{1}{2} \;\Rightarrow\; m = -5\ln\frac{1}{2} = 5\ln 2 \approx 3.47$ min.

10. (a) $\mu = 1000 \;\Rightarrow\; f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{1000}e^{-t/1000} & \text{if } t \geq 0 \end{cases}$

(i) $P(0 \leq X \leq 200) = \int_0^{200} \frac{1}{1000}e^{-t/1000}\,dt = \left[-e^{-t/1000}\right]_0^{200} = -e^{-1/5} + 1 \approx 0.181$

(ii) $P(X > 800) = \int_{800}^\infty \frac{1}{1000}e^{-t/1000}\,dt = \lim\limits_{x\to\infty}\left[-e^{-t/1000}\right]_{800}^x = 0 + e^{-4/5} \approx 0.449$

(b) We need to find m so that $\int_m^\infty f(t)\,dt = \frac{1}{2} \;\Rightarrow\; \lim\limits_{x\to\infty}\int_m^x \frac{1}{1000}e^{-t/1000}\,dt = \frac{1}{2} \;\Rightarrow\; \lim\limits_{x\to\infty}\left[-e^{-t/1000}\right]_m^x = \frac{1}{2} \;\Rightarrow$
$0 + e^{-m/1000} = \frac{1}{2} \;\Rightarrow\; -m/1000 = \ln\frac{1}{2} \;\Rightarrow\; m = -1000\ln\frac{1}{2} = 1000\ln 2 \approx 693.1$ h.

11. (a) An exponential density function with $\mu = 1.6$ is $f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{1.6}e^{-t/1.6} & \text{if } t \geq 0 \end{cases}$.

The probability that a customer waits less than a second is

$$P(X < 1) = \int_0^1 f(t)\,dt = \int_0^1 \tfrac{1}{1.6}e^{-t/1.6}\,dt = \left[-e^{-t/1.6}\right]_0^1 = -e^{-1/1.6} + 1 \approx 0.465.$$

(b) The probability that a customer waits more than 3 seconds is

$$P(X > 3) = \int_3^\infty f(t)\,dt = \lim_{s\to\infty}\int_3^s f(t)\,dt = \lim_{s\to\infty}\left[-e^{-t/1.6}\right]_3^s = \lim_{s\to\infty}\left(-e^{-s/1.6} + e^{-3/1.6}\right) = e^{-3/1.6} \approx 0.153.$$

Or: Calculate $1 - \int_0^3 f(t)\,dt$.

(c) We want to find b such that $P(X > b) = 0.05$. From part (b), $P(X > b) = e^{-b/1.6}$. Solving $e^{-b/1.6} = 0.05$ gives us

$-\frac{b}{1.6} = \ln 0.05 \quad\Rightarrow\quad b = -1.6\ln 0.05 \approx 4.79$ seconds.

Or: Solve $\int_0^b f(t)\,dt = 0.95$ for b.

12. (a) We first find an antiderivative of $g(t) = t^2\,e^{at}$.

$$\begin{aligned}\int t^2\,e^{at}\,dt &= \frac{1}{a}\,t^2\,e^{at} - \int \frac{2}{a}\,t\,e^{at}\,dt && \left[\begin{matrix} u = t^2, & dv = e^{at}\,dt \\ du = 2t\,dt, & v = \frac{1}{a}e^{at}\end{matrix}\right] \\ &= \frac{1}{a}\,t^2\,e^{at} - \frac{2}{a}\left[\frac{1}{a}\,t\,e^{at} - \int \frac{1}{a}\,e^{at}\,dt\right] && \left[\begin{matrix} u = t, & dv = e^{at}\,dt \\ du = dt, & v = \frac{1}{a}e^{at}\end{matrix}\right] \\ &= \frac{1}{a}\,t^2\,e^{at} - \frac{2}{a^2}\,t\,e^{at} + \frac{2}{a^3}\,e^{at} + C = \frac{1}{a}\,e^{at}\left(t^2 - \frac{2}{a}t + \frac{2}{a^2}\right) + C \\ &= -20e^{-0.05t}(t^2 + 40t + 800) + C && [\text{with } a = -0.05]\end{aligned}$$

$$\begin{aligned}P(0 \leq X \leq 48) &= \int_0^{48} f(t)\,dt = \frac{1}{15{,}676}\int_0^{48} g(t)\,dt = \frac{1}{15{,}676}\left[-20e^{-0.05t}(t^2 + 40t + 800)\right]_0^{48} \\ &= \frac{-20}{15{,}676}(5024e^{-2.4} - 800) \approx 0.439.\end{aligned}$$

(b) $$\begin{aligned}P(X > 36) = P(36 < X \leq 150) &= \frac{1}{15{,}676}\int_{36}^{150} g(t)\,dt = \frac{1}{15{,}676}\left[-20e^{-0.05t}(t^2 + 40t + 800)\right]_{36}^{150} \\ &= \frac{-20}{15{,}676}(29{,}300e^{-7.5} - 3536e^{-1.8}) \approx 0.725\end{aligned}$$

13. (a) $f(t) = \begin{cases} \frac{1}{1600}t & \text{if } 0 \leq t \leq 40 \\ \frac{1}{20} - \frac{1}{1600}t & \text{if } 40 < t \leq 80 \\ 0 & \text{otherwise}\end{cases}$

$$\begin{aligned}P(30 \leq T \leq 60) &= \int_{30}^{60} f(t)\,dt = \int_{30}^{40} \frac{t}{1600}\,dt + \int_{40}^{60}\left(\frac{1}{20} - \frac{t}{1600}\right)dt = \left[\frac{t^2}{3200}\right]_{30}^{40} + \left[\frac{t}{20} - \frac{t^2}{3200}\right]_{40}^{60} \\ &= \left(\frac{1600}{3200} - \frac{900}{3200}\right) + \left(\frac{60}{20} - \frac{3600}{3200}\right) - \left(\frac{40}{20} - \frac{1600}{3200}\right) = -\frac{1300}{3200} + 1 = \frac{19}{32}\end{aligned}$$

The probability that the amount of REM sleep is between 30 and 60 minutes is $\frac{19}{32} \approx 59.4\%$.

(b) $\mu = \int_{-\infty}^{\infty} t\,f(t)\,dt = \int_0^{40} t\left(\frac{t}{1600}\right)dt + \int_{40}^{80} t\left(\frac{1}{20} - \frac{t}{1600}\right)dt = \left[\frac{t^3}{4800}\right]_0^{40} + \left[\frac{t^2}{40} - \frac{t^3}{4800}\right]_{40}^{80}$

$$= \frac{64{,}000}{4800} + \left(\frac{6400}{40} - \frac{512{,}000}{4800}\right) - \left(\frac{1600}{40} - \frac{64{,}000}{4800}\right) = -\frac{384{,}000}{4800} + 120 = 40$$

The mean amount of REM sleep is 40 minutes.

14. (a) With $\mu = 175$ and $\sigma = 7$, we have $P(165 \le x \le 185) = \int_{165}^{185} \frac{1}{7\sqrt{2\pi}} \exp\left(-\frac{(x-175)^2}{2\cdot 7^2}\right)dx \approx 0.847$ (using a calculator or computer to estimate the integral).

(b) $P(X > 180) = 1 - P(-\infty \le X \le 180) \approx 1 - 0.7627 = 0.2373$, so 23.73% of the adult male population is more than 180 cm tall.

15. $P(X \ge 5) = \int_5^{\infty} \frac{1}{1.9\sqrt{2\pi}} \exp\left(-\frac{(x-4.3)^2}{2\cdot 1.9^2}\right)dx$. To avoid the improper integral we approximate it by the integral from 5 to 50. Thus, $P(X \ge 5) \approx \int_5^{50} \frac{1}{1.9\sqrt{2\pi}} \exp\left(-\frac{(x-4.3)^2}{2\cdot 1.9^2}\right)dx \approx 0.356$, so there is about a 35.6 percent of the households that throw out at least 5 kg of paper a week.

16. (a) $P(0 \le X \le 480) = \displaystyle\int_0^{480} \frac{1}{12\sqrt{2\pi}} \exp\left(-\frac{(x-500)^2}{2\cdot 12^2}\right)dx \approx 0.0478$ (using a calculator or computer to estimate the integral), so there is about a 4.78% chance that a particular box contains less than 480 g of cereal.

(b) We need to find μ so that $P(0 \le X < 500) = 0.05$. Using our calculator or computer to find $P(0 \le X \le 500)$ for various values of μ, we find that if $\mu = 519.73$, $P = 0.05007$; and if $\mu = 519.74$, $P = 0.04998$. So a good target weight is at least 519.74 g.

17. (a) $P(0 \le X \le 100) = \displaystyle\int_0^{100} \frac{1}{8\sqrt{2\pi}} \exp\left(-\frac{(x-112)^2}{2\cdot 8^2}\right)dx \approx 0.0668$ (using a calculator or computer to estimate the integral), so there is about a 6.68% chance that a randomly chosen vehicle is traveling at a legal speed.

(b) $P(X \ge 125) = \displaystyle\int_{125}^{\infty} \frac{1}{8\sqrt{2\pi}} \exp\left(-\frac{(x-112)^2}{2\cdot 8^2}\right)dx = \int_{125}^{\infty} f(x)\,dx$. In this case, we could use a calculator or computer to estimate either $\int_{125}^{300} f(x)\,dx$ or $1 - \int_0^{125} f(x)\,dx$. Both are approximately 0.0521, so about 5.21% of the motorists are targeted.

18. $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)} \quad\Rightarrow\quad f'(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)} \dfrac{-2(x-\mu)}{2\sigma^2} = \dfrac{-1}{\sigma^3\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}(x-\mu) \quad\Rightarrow$

$$f''(x) = \frac{-1}{\sigma^3\sqrt{2\pi}}\left[e^{-(x-\mu)^2/(2\sigma^2)}\cdot 1 + (x-\mu)e^{-(x-\mu)^2/(2\sigma^2)}\frac{-2(x-\mu)}{2\sigma^2}\right]$$

$$= \frac{-1}{\sigma^3\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}\left[1 - \frac{(x-\mu)^2}{\sigma^2}\right] = \frac{1}{\sigma^5\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}\left[(x-\mu)^2 - \sigma^2\right]$$

$f''(x) < 0 \;\Rightarrow\; (x-\mu)^2 - \sigma^2 < 0 \;\Rightarrow\; |x-\mu| < \sigma \;\Rightarrow\; -\sigma < x-\mu < \sigma \;\Rightarrow\; \mu-\sigma < x < \mu+\sigma$ and similarly, $f''(x) > 0 \;\Rightarrow\; x < \mu-\sigma$ or $x > \mu+\sigma$. Thus, f changes concavity and has inflection points at $x = \mu \pm \sigma$.

19. $P(\mu - 2\sigma \leq X \leq \mu + 2\sigma) = \displaystyle\int_{\mu-2\sigma}^{\mu+2\sigma} \frac{1}{\sigma\sqrt{2\pi}} \exp\!\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx$. Substituting $t = \dfrac{x-\mu}{\sigma}$ and $dt = \dfrac{1}{\sigma}\,dx$ gives us

$\displaystyle\int_{-2}^{2} \frac{1}{\sigma\sqrt{2\pi}} e^{-t^2/2}(\sigma\,dt) = \frac{1}{\sqrt{2\pi}} \int_{-2}^{2} e^{-t^2/2}\,dt \approx 0.9545.$

20. Let $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ ce^{-cx} & \text{if } x \geq 0 \end{cases}$ where $c = 1/\mu$. By using parts, tables, or a CAS, we find that

(1): $\int xe^{bx}\,dx = (e^{bx}/b^2)(bx - 1)$

(2): $\int x^2e^{bx}\,dx = (e^{bx}/b^3)(b^2x^2 - 2bx + 2)$

Now

$$\begin{aligned} \sigma^2 &= \textstyle\int_{-\infty}^{\infty}(x-\mu)^2 f(x)\,dx = \int_{-\infty}^{0}(x-\mu)^2 f(x)\,dx + \int_{0}^{\infty}(x-\mu)^2 f(x)\,dx \\ &= 0 + \lim_{t\to\infty} c\textstyle\int_0^t (x-\mu)^2 e^{-cx}\,dx = c \cdot \lim_{t\to\infty} \int_0^t \left(x^2e^{-cx} - 2x\mu e^{-cx} + \mu^2 e^{-cx}\right) dx \end{aligned}$$

Next we use (2) and (1) with $b = -c$ to get

$$\sigma^2 = c \lim_{t\to\infty} \left[-\frac{e^{-cx}}{c^3}\left(c^2x^2 + 2cx + 2\right) - 2\mu\frac{e^{-cx}}{c^2}(-cx - 1) + \mu^2\frac{e^{-cx}}{-c}\right]_0^t$$

Using l'Hospital's Rule several times, along with the fact that $\mu = 1/c$, we get

$$\sigma^2 = c\left[0 - \left(-\frac{2}{c^3} + \frac{2}{c}\cdot\frac{1}{c^2} + \frac{1}{c^2}\cdot\frac{1}{-c}\right)\right] = c\left(\frac{1}{c^3}\right) = \frac{1}{c^2} \quad\Rightarrow\quad \sigma = \frac{1}{c} = \mu$$

21. (a) First $p(r) = \dfrac{4}{a_0^3} r^2 e^{-2r/a_0} \geq 0$ for $r \geq 0$. Next,

$$\int_{-\infty}^{\infty} p(r)\,dr = \int_0^{\infty} \frac{4}{a_0^3} r^2 e^{-2r/a_0}\,dr = \frac{4}{a_0^3} \lim_{t\to\infty} \int_0^t r^2 e^{-2r/a_0}\,dr$$

By using parts, tables, or a CAS [or as in Exercise 20] , we find that $\int x^2e^{bx}\,dx = (e^{bx}/b^3)(b^2x^2 - 2bx + 2)$. $(\star)$

Next, we use $(\star)$ (with $b = -2/a_0$) and l'Hospital's Rule to get $\dfrac{4}{a_0^3}\left[\dfrac{a_0^3}{-8}(-2)\right] = 1$. This satisfies the second condition for a function to be a probability density function.

(b) Using l'Hospital's Rule, $\dfrac{4}{a_0^3} \displaystyle\lim_{r\to\infty} \frac{r^2}{e^{2r/a_0}} = \frac{4}{a_0^3} \lim_{r\to\infty} \frac{2r}{(2/a_0)e^{2r/a_0}} = \frac{2}{a_0^2} \lim_{r\to\infty} \frac{2}{(2/a_0)e^{2r/a_0}} = 0.$

To find the maximum of p, we differentiate:

$$p'(r) = \frac{4}{a_0^3}\left[r^2 e^{-2r/a_0}\left(-\frac{2}{a_0}\right) + e^{-2r/a_0}(2r)\right] = \frac{4}{a_0^3} e^{-2r/a_0}(2r)\left(-\frac{r}{a_0} + 1\right)$$

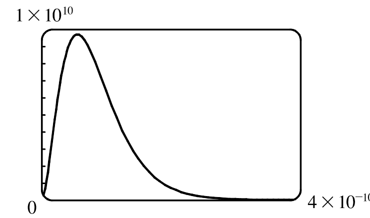
$p'(r) = 0 \quad\Leftrightarrow\quad r = 0$ or $1 = \dfrac{r}{a_0} \quad\Leftrightarrow\quad r = a_0 \quad [a_0 \approx 5.59 \times 10^{-11}$ m].

$p'(r)$ changes from positive to negative at $r = a_0$, so $p(r)$ has its maximum value at $r = a_0$.

(c) It is fairly difficult to find a viewing rectangle, but knowing the maximum value from part (b) helps.

$$p(a_0) = \frac{4}{a_0^3}\, a_0^2 e^{-2a_0/a_0} = \frac{4}{a_0}\, e^{-2} \approx 9{,}684{,}098{,}979$$

With a maximum of nearly 10 billion and a total area under the curve of 1, we know that the "hump" in the graph must be extremely narrow.



(d) $P(r) = \displaystyle\int_0^r \frac{4}{a_0^3} s^2 e^{-2s/a_0}\, ds \quad\Rightarrow\quad P(4a_0) = \int_0^{4a_0} \frac{4}{a_0^3} s^2 e^{-2s/a_0}\, ds$. Using ($\star$) from part (a) [with $b = -2/a_0$],

$$P(4a_0) = \frac{4}{a_0^3}\left[\frac{e^{-2s/a_0}}{-8/a_0^3}\left(\frac{4}{a_0^2}s^2 + \frac{4}{a_0}s + 2\right)\right]_0^{4a_0} = \frac{4}{a_0^3}\left(\frac{a_0^3}{-8}\right)[e^{-8}(64 + 16 + 2) - 1(2)] = -\tfrac{1}{2}(82e^{-8} - 2)$$

$$= 1 - 41e^{-8} \approx 0.986$$

(e) $\mu = \displaystyle\int_{-\infty}^{\infty} rp(r)\, dr = \frac{4}{a_0^3} \lim_{t\to\infty} \int_0^t r^3 e^{-2r/a_0}\, dr$. Integrating by parts three times or using a CAS, we find that

$\displaystyle\int x^3 e^{bx}\, dx = \frac{e^{bx}}{b^4}\left(b^3x^3 - 3b^2x^2 + 6bx - 6\right)$. So with $b = -\dfrac{2}{a_0}$, we use l'Hospital's Rule, and get

$$\mu = \frac{4}{a_0^3}\left[-\frac{a_0^4}{16}(-6)\right] = \tfrac{3}{2}a_0.$$

8 Review

TRUE-FALSE QUIZ

1. True. The graph of $y = f(x) + c$ is obtained by vertically translating $y = f(x)$ by c units. The arc length over the interval $a \le x \le b$ will be unchanged by this transformation.

2. False. Suppose $g(x) = 1$ and $h(x) = g(x) + 2 = 3$. Rotating the graph of $g(x)$ about the x-axis over the interval $[0, 2]$ will produce an open cylinder with radius 1 and height 2, so its surface area will be $2\pi(1)(2) = 4\pi$. Similarly, rotating $h(x)$ in the same way will generate an open cylinder with radius 3 and height 2, so its surface area will be $2\pi(3)(2) = 12\pi$.

3. False. Suppose $f(x) = x$ and $g(x) = 1$ so that $f(x) \le g(x)$ in the interval $[0, 1]$. $f(x)$ is a straight line so its arc length over $[0, 1]$ is the distance between its endpoints and is given by $\sqrt{(1-0)^2 + [f(1) - f(0)]^2} = \sqrt{2}$. Similarly, the arc length of $g(x)$ over $[0, 1]$ is 1, which is less than $\sqrt{2}$.

4. False. $y = x^3 \quad\Rightarrow\quad dy/dx = 3x^2 \quad\Rightarrow\quad L = \displaystyle\int_0^1 \sqrt{1 + (dy/dx)^2}\, dx = \int_0^1 \sqrt{1 + (3x^2)^2}\, dx = \int_0^1 \sqrt{1 + 9x^4}\, dx$

5. True. The smallest possible length of arc between the points $(0, 0)$ and $(3, 4)$ is the length of a straight line segment connecting the two points. This length is $\sqrt{(3-0)^2 + (4-0)^2} = \sqrt{25} = 5$.

6. True. By Equation 8.3.9, the centroid of a lamina depends only on its area and the curves $y = f(x)$ and $y = g(x)$, which define its shape.

7. True. The hydrostatic pressure depends only on the depth of the fluid, d, and the fluid's weight density, δ, as given by $P = \delta d$. See margin note next to Equation 8.3.1.

8. True. The total probability must be 1. See Equation 8.5.2 and the preceding discussion.

EXERCISES

1. $y = 4(x-1)^{3/2} \;\Rightarrow\; \dfrac{dy}{dx} = 6(x-1)^{1/2} \;\Rightarrow\; 1 + \left(\dfrac{dy}{dx}\right)^2 = 1 + 36(x-1) = 36x - 35$. Thus,

$$L = \int_1^4 \sqrt{36x-35}\,dx = \int_1^{109} \sqrt{u}\,\left(\frac{1}{36}\,du\right) \qquad \begin{bmatrix} u = 36x-35, \\ du = 36\,dx \end{bmatrix}$$
$$= \tfrac{1}{36}\left[\tfrac{2}{3}u^{3/2}\right]_1^{109} = \tfrac{1}{54}\left(109\sqrt{109} - 1\right)$$

2. $y = 2\ln\left(\sin\tfrac{1}{2}x\right) \;\Rightarrow\; \dfrac{dy}{dx} = 2 \cdot \dfrac{1}{\sin\left(\frac{1}{2}x\right)} \cdot \cos\left(\tfrac{1}{2}x\right) \cdot \tfrac{1}{2} = \cot\left(\tfrac{1}{2}x\right) \;\Rightarrow\; 1 + \left(\dfrac{dy}{dx}\right)^2 = 1 + \cot^2\left(\tfrac{1}{2}x\right) = \csc^2\left(\tfrac{1}{2}x\right)$.

Thus,

$$L = \int_{\pi/3}^{\pi} \sqrt{\csc^2\left(\tfrac{1}{2}x\right)}\,dx = \int_{\pi/3}^{\pi} \left|\csc\left(\tfrac{1}{2}x\right)\right| dx = \int_{\pi/3}^{\pi} \csc\left(\tfrac{1}{2}x\right) dx = \int_{\pi/6}^{\pi/2} \csc u\,(2\,du) \qquad \begin{bmatrix} u = \frac{1}{2}x, \\ du = \frac{1}{2}\,dx \end{bmatrix}$$
$$= 2\Big[\ln\left|\csc u - \cot u\right|\Big]_{\pi/6}^{\pi/2} = 2\left[\ln\left|\csc\tfrac{\pi}{2} - \cot\tfrac{\pi}{2}\right| - \ln\left|\csc\tfrac{\pi}{6} - \cot\tfrac{\pi}{6}\right|\right]$$
$$= 2\Big[\ln|1-0| - \ln\left|2-\sqrt{3}\right|\Big] = -2\ln\left(2-\sqrt{3}\right) \approx 2.63$$

3. $12x = 4y^3 + 3y^{-1} \;\Rightarrow\; x = \tfrac{1}{3}y^3 + \tfrac{1}{4}y^{-1} \;\Rightarrow\; \dfrac{dx}{dy} = y^2 - \tfrac{1}{4}y^{-2} \;\Rightarrow$

$1 + \left(\dfrac{dx}{dy}\right)^2 = 1 + y^4 - \tfrac{1}{2} + \tfrac{1}{16}y^{-4} = y^4 + \tfrac{1}{2} + \tfrac{1}{16}y^{-4} = (y^2 + \tfrac{1}{4}y^{-2})^2$. Thus,

$$L = \int_1^3 \sqrt{\left(y^2 + \tfrac{1}{4}y^{-2}\right)^2}\,dy = \int_1^3 \left|y^2 + \tfrac{1}{4}y^{-2}\right| dy = \int_1^3 \left(y^2 + \tfrac{1}{4}y^{-2}\right) dy = \left[\tfrac{1}{3}y^3 - \tfrac{1}{4}y^{-1}\right]_1^3$$
$$= \left(9 - \tfrac{1}{12}\right) - \left(\tfrac{1}{3} - \tfrac{1}{4}\right) = \tfrac{106}{12} = \tfrac{53}{6}$$

4. (a) $y = \dfrac{x^4}{16} + \dfrac{1}{2x^2} = \dfrac{1}{16}x^4 + \dfrac{1}{2}x^{-2} \;\Rightarrow\; \dfrac{dy}{dx} = \dfrac{1}{4}x^3 - x^{-3} \;\Rightarrow$

$1 + (dy/dx)^2 = 1 + \left(\tfrac{1}{4}x^3 - x^{-3}\right)^2 = 1 + \tfrac{1}{16}x^6 - \tfrac{1}{2} + x^{-6} = \tfrac{1}{16}x^6 + \tfrac{1}{2} + x^{-6} = \left(\tfrac{1}{4}x^3 + x^{-3}\right)^2$.

Thus, $L = \int_1^2 \left(\tfrac{1}{4}x^3 + x^{-3}\right) dx = \left[\tfrac{1}{16}x^4 - \tfrac{1}{2}x^{-2}\right]_1^2 = \left(1 - \tfrac{1}{8}\right) - \left(\tfrac{1}{16} - \tfrac{1}{2}\right) = \tfrac{21}{16}$.

(b) $S = \int_1^2 2\pi x\, ds = \int_1^2 2\pi x\sqrt{1 + (dy/dx)^2}\, dx = \int_1^2 2\pi x\left(\frac{1}{4}x^3 + x^{-3}\right) dx = 2\pi \int_1^2 \left(\frac{1}{4}x^4 + x^{-2}\right) dx$

$= 2\pi\left[\frac{1}{20}x^5 - \frac{1}{x}\right]_1^2 = 2\pi\left[\left(\frac{32}{20} - \frac{1}{2}\right) - \left(\frac{1}{20} - 1\right)\right] = 2\pi\left(\frac{8}{5} - \frac{1}{2} - \frac{1}{20} + 1\right) = 2\pi\left(\frac{41}{20}\right) = \frac{41}{10}\pi$

5. (a) $y = \dfrac{2}{x+1} \quad\Rightarrow\quad y' = \dfrac{-2}{(x+1)^2} \quad\Rightarrow\quad 1 + (y')^2 = 1 + \dfrac{4}{(x+1)^4}.$

For $0 \leq x \leq 3$, $L = \int_0^3 \sqrt{1 + (y')^2}\, dx = \int_0^3 \sqrt{1 + 4/(x+1)^4}\, dx \approx 3.5121$.

(b) The area of the surface obtained by rotating C about the x-axis is

$$S = \int_0^3 2\pi y\, ds = 2\pi \int_0^3 \frac{2}{x+1}\sqrt{1 + 4/(x+1)^4}\, dx \approx 22.1391.$$

(c) The area of the surface obtained by rotating C about the y-axis is

$S = \int_0^3 2\pi x\, ds = 2\pi \int_0^3 x\sqrt{1 + 4/(x+1)^4}\, dx \approx 29.8522.$

6. (a) $y = x^2 \quad\Rightarrow\quad 1 + (y')^2 = 1 + 4x^2$. Rotate about the y-axis for $0 \leq x \leq 1$:

$$S = \int_0^1 2\pi x\sqrt{1 + 4x^2}\, dx = \int_1^5 \frac{\pi}{4}\sqrt{u}\, du \quad [u = 1 + 4x^2] \;=\; \frac{\pi}{6}\left[u^{3/2}\right]_1^5 = \frac{\pi}{6}(5^{3/2} - 1)$$

(b) $y = x^2 \quad\Rightarrow\quad 1 + (y')^2 = 1 + 4x^2$. Rotate about the x-axis for $0 \leq x \leq 1$:

$$S = 2\pi \int_0^1 x^2\sqrt{1 + 4x^2}\, dx = 2\pi \int_0^2 \frac{1}{4}u^2\sqrt{1 + u^2}\,\frac{1}{2}\, du \quad [u = 2x] \quad = \frac{\pi}{4}\int_0^2 u^2\sqrt{1 + u^2}\, du$$

$$= \frac{\pi}{4}\left[\frac{1}{8}u(1 + 2u^2)\sqrt{1 + u^2} - \frac{1}{8}\ln\left|u + \sqrt{1 + u^2}\right|\right]_0^2 \qquad [u = \tan\theta \text{ or use Formula 22}]$$

$$= \frac{\pi}{4}\left[\frac{1}{4}(9)\sqrt{5} - \frac{1}{8}\ln\left(2 + \sqrt{5}\right) - 0\right] = \frac{\pi}{32}\left[18\sqrt{5} - \ln\left(2 + \sqrt{5}\right)\right]$$

7. $y = \sin x \quad\Rightarrow\quad y' = \cos x \quad\Rightarrow\quad 1 + (y')^2 = 1 + \cos^2 x$. Let $f(x) = \sqrt{1 + \cos^2 x}$. Then

$$\begin{aligned} L = \int_0^\pi f(x)\, dx &\approx S_{10} \\ &= \frac{(\pi - 0)/10}{3}\left[f(0) + 4f\left(\tfrac{\pi}{10}\right) + 2f\left(\tfrac{2\pi}{10}\right) + 4f\left(\tfrac{3\pi}{10}\right) + 2f\left(\tfrac{4\pi}{10}\right)\right. \\ &\qquad \left. + 4f\left(\tfrac{5\pi}{10}\right) + 2f\left(\tfrac{6\pi}{10}\right) + 4f\left(\tfrac{7\pi}{10}\right) + 2f\left(\tfrac{8\pi}{10}\right) + 4f\left(\tfrac{9\pi}{10}\right) + f(\pi)\right] \\ &\approx 3.8202 \end{aligned}$$

8. (a) $y = \sin x \quad\Rightarrow\quad y' = \cos x \quad\Rightarrow\quad 1 + (y')^2 = 1 + \cos^2 x.$

$S = \int_0^\pi 2\pi y\, ds = \int_0^\pi 2\pi \sin x\sqrt{1 + \cos^2 x}\, dx.$

(b) $S \approx 14.4260$

9. $y = \int_1^x \sqrt{\sqrt{t} - 1}\, dt \quad\Rightarrow\quad dy/dx = \sqrt{\sqrt{x} - 1} \quad\Rightarrow\quad 1 + (dy/dx)^2 = 1 + \left(\sqrt{x} - 1\right) = \sqrt{x}.$

Thus, $L = \int_1^{16} \sqrt{\sqrt{x}}\, dx = \int_1^{16} x^{1/4}\, dx = \frac{4}{5}\left[x^{5/4}\right]_1^{16} = \frac{4}{5}(32 - 1) = \frac{124}{5}.$

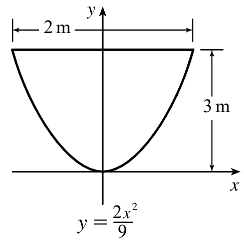
10. $S = \int_1^{16} 2\pi x\, ds = 2\pi \int_1^{16} x \cdot x^{1/4}\, dx = 2\pi \int_1^{16} x^{5/4}\, dx = 2\pi \cdot \frac{4}{9}\left[x^{9/4}\right]_1^{16} = \frac{8\pi}{9}(512 - 1) = \frac{4088}{9}\pi$

11. As in Example 8.3.1, $\frac{a}{1-x} = \frac{\frac{1}{2}}{1} \implies a = \frac{1}{2}(1-x)$ and $w = 1 + 2a = 2 - x$. Thus,

$$F = \int_0^1 \rho g x(2-x)dx = \rho g \int_0^1 x(2-x)dx = \rho g[x^2 - \frac{1}{3}x^3]|_0^1 = (9800)(\frac{2}{3}) \approx 6{,}533 \text{ N}$$

12. $y = \frac{8}{9}x^2 \implies x = \frac{3\sqrt{2y}}{4}$

$$\begin{aligned} F &= \int_0^2 \rho g(2-y)2\frac{3\sqrt{2y}}{4}\,dy \\ &= \rho g\frac{3}{2}\int_0^2 (2-y)\sqrt{2y}\,dy \\ &\approx 31{,}360 \text{ N} \end{aligned}$$

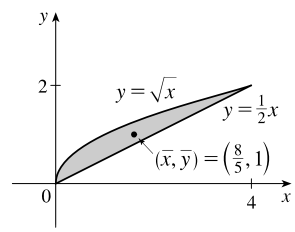


13. The area of the triangular region is $A = \frac{1}{2}(2)(4) = 4$. An equation of the line is $y = \frac{1}{2}x$ or $x = 2y$.

$$\overline{x} = \frac{1}{A}\int_0^2 \frac{1}{2}[f(y)]^2\,dy = \frac{1}{4}\int_0^2 \frac{1}{2}(2y)^2\,dy = \frac{1}{8}\int_0^2 4y^2\,dy = \frac{1}{8}\left[\frac{4}{3}y^3\right]_0^2 = \frac{1}{6}(8) = \frac{4}{3}$$

$$\overline{y} = \frac{1}{A}\int_0^2 y\,f(y)\,dy = \frac{1}{4}\int_0^2 y(2y)\,dy = \frac{1}{2}\int_0^2 y^2\,dy = \frac{1}{2}\left[\frac{1}{3}y^3\right]_0^2 = \frac{1}{6}(8) = \frac{4}{3}$$

The centroid of the region is $\left(\frac{4}{3}, \frac{4}{3}\right)$.



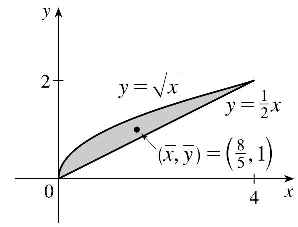
14. An equation of the line is $y = 8 - x$. An equation of the quarter-circle is $y = -\sqrt{8^2 - x^2}$ with $0 \le x \le 8$. The area of the region is $A = \frac{1}{2}(8)(8) + \frac{1}{4}\pi(8)^2 = 32 + 16\pi = 16(2+\pi)$.

$$\begin{aligned} \overline{x} &= \frac{1}{A}\int_0^8 x[f(x) - g(x)]\,dx = \frac{1}{A}\int_0^8 x\left[(8-x) + \sqrt{64-x^2}\right]dx \\ &= \frac{1}{A}\int_0^8 \left[8x - x^2 + x(64-x^2)^{1/2}\right]dx = \frac{1}{A}\left[4x^2 - \frac{1}{3}x^3 - \frac{1}{3}(64-x^2)^{3/2}\right]_0^8 \\ &= \frac{1}{A}\left[\left(256 - \frac{512}{3} - 0\right) - \left(0 - 0 - \frac{512}{3}\right)\right] = \frac{256}{16(2+\pi)} = \frac{16}{2+\pi} \end{aligned}$$

$$\begin{aligned} \overline{y} &= \frac{1}{A}\int_0^8 \frac{1}{2}\{[f(x)]^2 - [g(x)]^2\}\,dx = \frac{1}{2A}\int_0^8 \left[(8-x)^2 - (-\sqrt{64-x^2}\,)^2\right]dx \\ &= \frac{1}{2A}\int_0^8 \left[64 - 16x + x^2 - (64-x^2)\right]dx = \frac{1}{2A}\int_0^8 (2x^2 - 16x)\,dx \\ &= \frac{1}{A}\int_0^8 (x^2 - 8x)\,dx = \frac{1}{A}\left[\frac{1}{3}x^3 - 4x^2\right]_0^8 = \frac{1}{A}\left(\frac{512}{3} - 256\right) \\ &= \frac{1}{16(2+\pi)}\left(-\frac{256}{3}\right) = -\frac{16}{3(2+\pi)} \end{aligned}$$

The centroid of the region is $\left(\frac{16}{2+\pi}, -\frac{16}{3(2+\pi)}\right) \approx (3.11, -1.04)$.

15. $A = \int_0^4 \left(\sqrt{x} - \frac{1}{2}x\right) dx = \left[\frac{2}{3}x^{3/2} - \frac{1}{4}x^2\right]_0^4 = \frac{16}{3} - 4 = \frac{4}{3}$



$$\overline{x} = \frac{1}{A}\int_0^4 x\left(\sqrt{x} - \frac{1}{2}x\right) dx = \frac{3}{4}\int_0^4 \left(x^{3/2} - \frac{1}{2}x^2\right) dx$$

$$= \frac{3}{4}\left[\frac{2}{5}x^{5/2} - \frac{1}{6}x^3\right]_0^4 = \frac{3}{4}\left(\frac{64}{5} - \frac{64}{6}\right) = \frac{3}{4}\left(\frac{64}{30}\right) = \frac{8}{5}$$

$$\overline{y} = \frac{1}{A}\int_0^4 \frac{1}{2}\left[\left(\sqrt{x}\right)^2 - \left(\frac{1}{2}x\right)^2\right] dx = \frac{3}{4}\int_0^4 \frac{1}{2}\left(x - \frac{1}{4}x^2\right) dx = \frac{3}{8}\left[\frac{1}{2}x^2 - \frac{1}{12}x^3\right]_0^4 = \frac{3}{8}\left(8 - \frac{16}{3}\right) = \frac{3}{8}\left(\frac{8}{3}\right) = 1$$

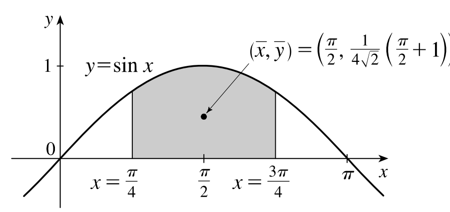
Thus, the centroid is $(\overline{x}, \overline{y}) = \left(\frac{8}{5}, 1\right)$.

16. From the symmetry of the region, $\overline{x} = \frac{\pi}{2}$. $A = \int_{\pi/4}^{3\pi/4} \sin x\, dx = \left[-\cos x\right]_{\pi/4}^{3\pi/4} = \frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right) = \sqrt{2}$

$$\overline{y} = \frac{1}{A}\int_{\pi/4}^{3\pi/4} \frac{1}{2}\sin^2 x\, dx = \frac{1}{A}\int_{\pi/4}^{3\pi/4} \frac{1}{4}\left(1 - \cos 2x\right) dx$$

$$= \frac{1}{4\sqrt{2}}\left[x - \frac{1}{2}\sin 2x\right]_{\pi/4}^{3\pi/4}$$

$$= \frac{1}{4\sqrt{2}}\left[\frac{3\pi}{4} - \frac{1}{2}(-1) - \frac{\pi}{4} + \frac{1}{2}\cdot 1\right] = \frac{1}{4\sqrt{2}}\left(\frac{\pi}{2} + 1\right)$$



Thus, the centroid is $(\overline{x}, \overline{y}) = \left(\frac{\pi}{2}, \frac{1}{4\sqrt{2}}\left(\frac{\pi}{2} + 1\right)\right) \approx (1.57, 0.45)$.

17. The centroid of this circle, $(1, 0)$, travels a distance $2\pi(1)$ when the lamina is rotated about the y-axis. The area of the circle is $\pi(1)^2$. So by the Theorem of Pappus, $V = A(2\pi\overline{x}) = \pi(1)^2 2\pi(1) = 2\pi^2$.

18. The semicircular region has an area of $\frac{1}{2}\pi r^2$, and sweeps out a sphere of radius r when rotated about the x-axis. $\overline{x} = 0$ because of symmetry about the line $x = 0$. And by the Theorem of Pappus, $V = A(2\pi\overline{y}) \Rightarrow$ $\frac{4}{3}\pi r^3 = \frac{1}{2}\pi r^2(2\pi\overline{y}) \Rightarrow \overline{y} = \frac{4}{3\pi}r$. Thus, the centroid is $(\overline{x}, \overline{y}) = \left(0, \frac{4}{3\pi}r\right)$.

19. $x = 100 \Rightarrow P = 2000 - 0.1(100) - 0.01(100)^2 = 1890$

$$\text{Consumer surplus} = \int_0^{100}\left[p(x) - P\right] dx = \int_0^{100}\left(2000 - 0.1x - 0.01x^2 - 1890\right) dx$$

$$= \left[110x - 0.05x^2 - \frac{0.01}{3}x^3\right]_0^{100} = 11{,}000 - 500 - \frac{10{,}000}{3} \approx \$7166.67$$

20. $\int_0^{24} c(t)\, dt \approx S_{12} = \frac{24 - 0}{12 \cdot 3}[1(0) + 4(1.9) + 2(3.3) + 4(5.1) + 2(7.6) + 4(7.1) + 2(5.8)$
$+ 4(4.7) + 2(3.3) + 4(2.1) + 2(1.1) + 4(0.5) + 1(0)]$
$= \frac{2}{3}(127.8) = 85.2\ \text{mg}\cdot\text{s/L}$

Therefore, $F \approx A/85.2 = 6/85.2 \approx 0.0704$ L/s or 4.225 L/min.

21. (a) At equilibrium, $p(x) = s(x) \Rightarrow 8.8 - 0.02(x) = 2.5 + 0.01x \Rightarrow x = 210$, and the corresponding price is $p(210) = 8.8 - 0.02(210) = \$4.60/\text{kg}$.

(b) The maximum total surplus occurs at market equilibrium, that is, when $x = 210$.

$$\begin{aligned}\text{Total surplus} &= \int_0^{210} [p(x) - s(x)]\,dx = \int_0^{210} [(8.8 - 0.02x) - (2.5 + 0.01x)]\,dx \\ &= \int_0^{210} [(6.3 - 0.03x)\,dx = [6.3x - 0.015x^2)]_0^{210} = 1323 - 661.5 = 661.5\end{aligned}$$

The maximum total surplus is \$661.500.

22. $P(250 \le X \le 280) = \displaystyle\int_{250}^{280} \frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/(2\sigma^2)}\, dx = \int_{250}^{280} \frac{1}{15\sqrt{2\pi}} \exp\!\left(\frac{-(x-268)^2}{2 \cdot 15^2}\right) dx \approx 0.673.$

Thus, the percentage of pregnancies that last between 250 and 280 days is about 67.3%.

23. (a) The probability density function is $f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{8}e^{-t/8} & \text{if } t \ge 0 \end{cases}$

$P(0 \le X \le 3) = \int_0^3 \frac{1}{8}e^{-t/8}\,dt = \left[-e^{-t/8}\right]_0^3 = -e^{-3/8} + 1 \approx 0.3127$

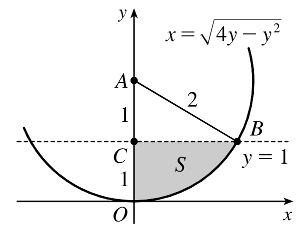
(b) $P(X > 10) = \int_{10}^{\infty} \frac{1}{8}e^{-t/8}\,dt = \lim\limits_{x\to\infty} \left[-e^{-t/8}\right]_{10}^{x} = \lim\limits_{x\to\infty} (-e^{-x/8} + e^{-10/8}) = 0 + e^{-5/4} \approx 0.2865$

(c) We need to find m such that $P(X \ge m) = \frac{1}{2} \Rightarrow \int_m^{\infty} \frac{1}{8}e^{-t/8}\,dt = \frac{1}{2} \Rightarrow \lim\limits_{x\to\infty} \left[-e^{-t/8}\right]_m^x = \frac{1}{2} \Rightarrow$

$\lim\limits_{x\to\infty} (-e^{-x/8} + e^{-m/8}) = \frac{1}{2} \Rightarrow e^{-m/8} = \frac{1}{2} \Rightarrow -m/8 = \ln\frac{1}{2} \Rightarrow m = -8\ln\frac{1}{2} = 8\ln 2 \approx 5.55$ minutes.

PROBLEMS PLUS

1. $x^2 + y^2 \le 4y \quad \Leftrightarrow \quad x^2 + (y-2)^2 \le 4$, so S is part of a circle, as shown in the diagram. The area of S is



$$\begin{aligned}\int_0^1 \sqrt{4y - y^2}\, dy &\overset{113}{=} \left[\tfrac{y-2}{2}\sqrt{4y-y^2} + 2\cos^{-1}\!\left(\tfrac{2-y}{2}\right)\right]_0^1 \quad [a = 2]\\ &= -\tfrac{1}{2}\sqrt{3} + 2\cos^{-1}\!\left(\tfrac{1}{2}\right) - 2\cos^{-1} 1\\ &= -\tfrac{\sqrt{3}}{2} + 2\left(\tfrac{\pi}{3}\right) - 2(0) = \tfrac{2\pi}{3} - \tfrac{\sqrt{3}}{2}\end{aligned}$$

Another method (without calculus): Note that $\theta = \angle CAB = \frac{\pi}{3}$, so the area is

$$(\text{area of sector } OAB) - (\text{area of } \triangle ABC) = \tfrac{1}{2}\left(2^2\right)\tfrac{\pi}{3} - \tfrac{1}{2}(1)\sqrt{3} = \tfrac{2\pi}{3} - \tfrac{\sqrt{3}}{2}$$

2. $y = \pm\sqrt{x^3 - x^4} \quad \Rightarrow$ The loop of the curve is symmetric about $y = 0$, and therefore $\overline{y} = 0$. At each point x where $0 \le x \le 1$, the lamina has a vertical length of $\sqrt{x^3 - x^4} - \left(-\sqrt{x^3 - x^4}\right) = 2\sqrt{x^3 - x^4}$. Therefore,

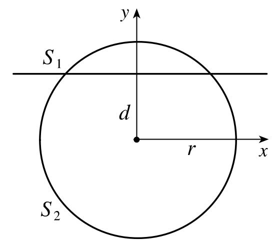
$$\overline{x} = \frac{\int_0^1 x \cdot 2\sqrt{x^3 - x^4}\, dx}{\int_0^1 2\sqrt{x^3 - x^4}\, dx} = \frac{\int_0^1 x\sqrt{x^3 - x^4}\, dx}{\int_0^1 \sqrt{x^3 - x^4}\, dx}.$$ We evaluate the integrals separately:

$$\begin{aligned}\int_0^1 x\sqrt{x^3 - x^4}\, dx &= \int_0^1 x^{5/2}\sqrt{1-x}\, dx\\ &= \int_0^{\pi/2} 2\sin^6\theta\cos\theta\sqrt{1-\sin^2\theta}\, d\theta \qquad \left[\begin{matrix}\sin\theta = \sqrt{x}, \cos\theta\, d\theta = dx/(2\sqrt{x}\,),\\ 2\sin\theta\,\cos\theta\, d\theta = dx\end{matrix}\right]\\ &= \int_0^{\pi/2} 2\sin^6\theta\cos^2\theta\, d\theta = \int_0^{\pi/2} 2\left[\tfrac{1}{2}\left(1-\cos 2\theta\right)\right]^3 \tfrac{1}{2}(1+\cos 2\theta) d\theta\\ &= \int_0^{\pi/2} \tfrac{1}{8}(1 - 2\cos 2\theta + 2\cos^3 2\theta - \cos^4 2\theta)\, d\theta\\ &= \int_0^{\pi/2} \tfrac{1}{8}\left[1 - 2\cos 2\theta + 2\cos 2\theta(1-\sin^2 2\theta) - \tfrac{1}{4}(1+\cos 4\theta)^2\right] d\theta\\ &= \tfrac{1}{8}\left[\theta - \tfrac{1}{3}\sin^3 2\theta\right]_0^{\pi/2} - \tfrac{1}{32}\int_0^{\pi/2}\left(1 + 2\cos 4\theta + \cos^2 4\theta\right) d\theta\\ &= \tfrac{\pi}{16} - \tfrac{1}{32}\left[\theta + \tfrac{1}{2}\sin 4\theta\right]_0^{\pi/2} - \tfrac{1}{64}\int_0^{\pi/2}(1+\cos 8\theta)\, d\theta\\ &= \tfrac{\pi}{16} - \tfrac{1}{32}\left[\theta + \tfrac{1}{2}\sin 4\theta\right]_0^{\pi/2} - \tfrac{1}{64}\int_0^{\pi/2}(1+\cos 8\theta)\, d\theta\\ &= \tfrac{3\pi}{64} - \tfrac{1}{64}\left[\theta + \tfrac{1}{8}\sin 8\theta\right]_0^{\pi/2} = \tfrac{5\pi}{128}\end{aligned}$$

$$\begin{aligned}\int_0^1 \sqrt{x^3 - x^4}\, dx &= \int_0^1 x^{3/2}\sqrt{1-x}\, dx = \int_0^{\pi/2} 2\sin^4\theta\cos\theta\sqrt{1-\sin^2\theta}\, d\theta \qquad [\sin\theta = \sqrt{x}\,]\\ &= \int_0^{\pi/2} 2\sin^4\theta\cos^2\theta\, d\theta = \int_0^{\pi/2} 2\cdot\tfrac{1}{4}(1-\cos 2\theta)^2\cdot\tfrac{1}{2}(1+\cos 2\theta)\, d\theta\\ &= \int_0^{\pi/2} \tfrac{1}{4}(1 - \cos 2\theta - \cos^2 2\theta + \cos^3 2\theta)\, d\theta\\ &= \int_0^{\pi/2} \tfrac{1}{4}\left[1 - \cos 2\theta - \tfrac{1}{2}(1+\cos 4\theta) + \cos 2\theta(1-\sin^2 2\theta)\right] d\theta\\ &= \tfrac{1}{4}\left[\tfrac{\theta}{2} - \tfrac{1}{8}\sin 4\theta - \tfrac{1}{6}\sin^3 2\theta\right]_0^{\pi/2} = \tfrac{\pi}{16}\end{aligned}$$

Therefore, $\overline{x} = \dfrac{5\pi/128}{\pi/16} = \dfrac{5}{8}$, and $(\overline{x}, \overline{y}) = \left(\frac{5}{8}, 0\right)$.

3. (a) The two spherical zones, whose surface areas we will call S_1 and S_2, are generated by rotation about the y-axis of circular arcs, as indicated in the figure. The arcs are the upper and lower portions of the circle $x^2 + y^2 = r^2$ that are obtained when the circle is cut with the line $y = d$. The portion of the upper arc in the first quadrant is sufficient to generate the upper spherical zone. That portion of the arc can be described by the relation $x = \sqrt{r^2 - y^2}$ for $d \le y \le r$. Thus, $dx/dy = -y/\sqrt{r^2 - y^2}$ and



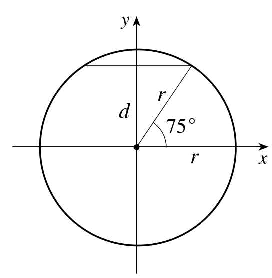
$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \sqrt{1 + \frac{y^2}{r^2 - y^2}}\, dy = \sqrt{\frac{r^2}{r^2 - y^2}}\, dy = \frac{r\, dy}{\sqrt{r^2 - y^2}}$$

From Formula 8.2.8 we have

$$S_1 = \int_d^r 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_d^r 2\pi \sqrt{r^2 - y^2}\, \frac{r\, dy}{\sqrt{r^2 - y^2}} = \int_d^r 2\pi r\, dy = 2\pi r(r - d)$$

Similarly, we can compute $S_2 = \int_{-r}^{d} 2\pi x \sqrt{1 + (dx/dy)^2}\, dy = \int_{-r}^{d} 2\pi r\, dy = 2\pi r(r + d)$. Note that $S_1 + S_2 = 4\pi r^2$, the surface area of the entire sphere.

(b) $r = 6370$ km and $d = r\,(\sin 75°) \approx 6153$ km, so the surface area of the Arctic Ocean is about $2\pi r(r - d) \approx 2\pi(6370)(217) \approx 8.69 \times 10^6\ \text{km}^2$.

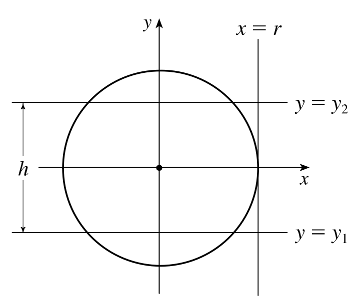


(c) The area on the sphere lies between planes $y = y_1$ and $y = y_2$, where $y_2 - y_1 = h$. Thus, we compute the surface area on the sphere to be $S = \int_{y_1}^{y_2} 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_{y_1}^{y_2} 2\pi r\, dy = 2\pi r(y_2 - y_1) = 2\pi r h$.

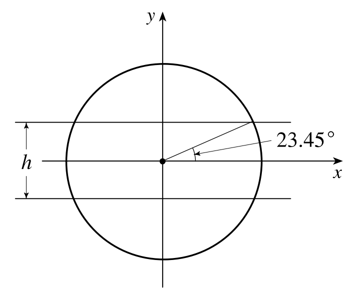
This equals the lateral area of a cylinder of radius r and height h, since such a cylinder is obtained by rotating the line $x = r$ about the y-axis, so the surface area of the cylinder between the planes $y = y_1$ and $y = y_2$ is

$$A = \int_{y_1}^{y_2} 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_{y_1}^{y_2} 2\pi r \sqrt{1 + 0^2}\, dy$$

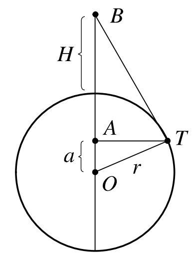
$$= 2\pi r y \Big|_{y = y_1}^{y_2} = 2\pi r(y_2 - y_1) = 2\pi r h$$



(d) $h = 2r\sin 23.45° \approx 3152$ mi, so the surface area of the Torrid Zone is $2\pi rh \approx 2\pi(3960)(3152) \approx 7.84 \times 10^7$ mi^2.



4. (a) Since the right triangles OAT and OTB are similar, we have $\dfrac{r+H}{r} = \dfrac{r}{a} \quad\Rightarrow$



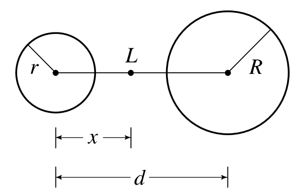
$a = \dfrac{r^2}{r+H}$. The surface area visible from B is $S = \int_a^r 2\pi x\sqrt{1+(dx/dy)^2}\,dy$.

From $x^2+y^2=r^2$, we get $\dfrac{d}{dy}(x^2+y^2) = \dfrac{d}{dy}(r^2) \quad\Rightarrow\quad 2x\,\dfrac{dx}{dy} + 2y = 0 \quad\Rightarrow$

$\dfrac{dx}{dy} = -\dfrac{y}{x}$ and $1 + \left(\dfrac{dx}{dy}\right)^2 = \dfrac{x^2+y^2}{x^2} = \dfrac{r^2}{x^2}$. Thus,

$$S = \int_a^r 2\pi x \cdot \frac{r}{x}\,dy = 2\pi r(r-a) = 2\pi r\left(r - \frac{r^2}{r+H}\right) = 2\pi r^2\left(1 - \frac{r}{r+H}\right) = 2\pi r^2 \cdot \frac{H}{r+H} = \frac{2\pi r^2 H}{r+H}.$$

(b) Assume $R \geq r$. If a light is placed at point L, at a distance x from the center of the sphere of radius r, then from part (a) we find that the total illuminated area A on the two spheres is [with $r + H = x$ and $r + H = d - x$].



$$A(x) = \frac{2\pi r^2(x-r)}{x} + \frac{2\pi R^2(d-x-R)}{d-x} \quad [r \leq x \leq d-R]. \qquad \frac{A(x)}{2\pi} = r^2\left(1-\frac{r}{x}\right) + R^2\left(1 - \frac{R}{d-x}\right),$$

so $A'(x) = 0 \quad\Leftrightarrow\quad 0 = r^2 \cdot \dfrac{r}{x^2} + R^2 \cdot \dfrac{-R}{(d-x)^2} \quad\Leftrightarrow\quad \dfrac{r^3}{x^2} = \dfrac{R^3}{(d-x)^2} \quad\Leftrightarrow\quad \dfrac{(d-x)^2}{x^2} = \dfrac{R^3}{r^3} \quad\Leftrightarrow$

$$\left(\frac{d}{x}-1\right)^2 = \left(\frac{R}{r}\right)^3 \quad\Rightarrow\quad \frac{d}{x} - 1 = \left(\frac{R}{r}\right)^{3/2} \quad\Leftrightarrow\quad \frac{d}{x} = 1 + \left(\frac{R}{r}\right)^{3/2} \quad\Leftrightarrow\quad x = x^* = \frac{d}{1+(R/r)^{3/2}}.$$

Now $A'(x) = 2\pi\left(\dfrac{r^3}{x^2} - \dfrac{R^3}{(d-x)^2}\right) \quad\Rightarrow\quad A''(x) = 2\pi\left(-\dfrac{2r^3}{x^3} - \dfrac{2R^3}{(d-x)^3}\right)$ and $A''(x^*) < 0$, so we have a local maximum at $x = x^*$.

However, x^* may not be an allowable value of x—we must show that x^* is between r and $d - R$.

(1) $x^* \geq r \quad\Leftrightarrow\quad \dfrac{d}{1+(R/r)^{3/2}} \geq r \quad\Leftrightarrow\quad d \geq r + R\sqrt{R/r}$

(2) $x^* \leq d - R \quad\Leftrightarrow\quad \dfrac{d}{1+(R/r)^{3/2}} \leq d - R \quad\Leftrightarrow\quad d \leq d - R + d\left(\dfrac{R}{r}\right)^{3/2} - R\left(\dfrac{R}{r}\right)^{3/2} \quad\Leftrightarrow$

$R + R\left(\dfrac{R}{r}\right)^{3/2} \leq d\left(\dfrac{R}{r}\right)^{3/2} \quad\Leftrightarrow\quad d \geq \dfrac{R}{(R/r)^{3/2}} + R = R + r\sqrt{r/R}$, but

$R + r\sqrt{r/R} \leq R + r$, and since $d > r + R$ [given], we conclude that $x^* \leq d - R$.

[continued]

Thus, from (1) and (2), x^* is not an allowable value of x if $d < r + R\sqrt{R/r}$.

So A may have a maximum at $x = r$, x^*, or $d - R$.

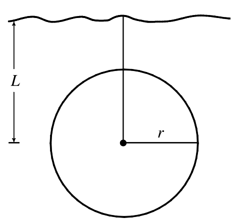
$$A(r) = \frac{2\pi R^2(d - r - R)}{d - r} \qquad \text{and} \qquad A(d - R) = \frac{2\pi r^2(d - r - R)}{d - R}$$

$A(r) > A(d - R) \;\Leftrightarrow\; \dfrac{R^2}{d - r} > \dfrac{r^2}{d - R} \;\Leftrightarrow\; R^2(d - R) > r^2(d - r) \;\Leftrightarrow\; R^2 d - R^3 > r^2 d - r^3 \;\Leftrightarrow$ $R^2 d - r^2 d > R^3 - r^3 \;\Leftrightarrow\; d(R - r)(R + r) > (R - r)(R^2 + Rr + r^2) \;\Leftrightarrow\; d > (R^2 + Rr + r^2)/(R + r) \;\Leftrightarrow$ $d > [(R + r)^2 - Rr]/(R + r) \;\Leftrightarrow\; d > R + r - Rr/(R + r)$. Now $R + r - Rr/(R + r) < R + r$, and we know that $d > R + r$, so we conclude that $A(r) > A(d - R)$.

In conclusion, A has an absolute maximum at $x = x^*$ provided $d \geq r + R\sqrt{R/r}$; otherwise, A has its maximum at $x = r$.

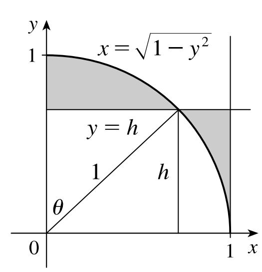
5. (a) Choose a vertical x-axis pointing downward with its origin at the surface. In order to calculate the pressure at depth z, consider n subintervals of the interval $[0, z]$ by points x_i and choose a point $x_i^* \in [x_{i-1}, x_i]$ for each i. The thin layer of water lying between depth x_{i-1} and depth x_i has a density of approximately $\rho(x_i^*)$, so the weight of a piece of that layer with unit cross-sectional area is $\rho(x_i^*)g\,\Delta x$. The total weight of a column of water extending from the surface to depth z (with unit cross-sectional area) would be approximately $\sum\limits_{i=1}^{n} \rho(x_i^*)g\,\Delta x$. The estimate becomes exact if we take the limit as $n \to \infty$; weight (or force) per unit area at depth z is $W = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \rho(x_i^*)g\,\Delta x$. In other words, $P(z) = \int_0^z \rho(x)g\,dx$. More generally, if we make no assumptions about the location of the origin, then $P(z) = P_0 + \int_0^z \rho(x)g\,dx$, where P_0 is the pressure at $x = 0$. Differentiating, we get $dP/dz = \rho(z)g$.

(b)



$$\begin{aligned}
F &= \int_{-r}^{r} P(L + x)\cdot 2\sqrt{r^2 - x^2}\,dx \\
&= \int_{-r}^{r} \left(P_0 + \int_0^{L+x} \rho_0 e^{z/H} g\,dz\right)\cdot 2\sqrt{r^2 - x^2}\,dx \\
&= P_0 \int_{-r}^{r} 2\sqrt{r^2 - x^2}\,dx + \rho_0 g H \int_{-r}^{r} \left(e^{(L+x)/H} - 1\right)\cdot 2\sqrt{r^2 - x^2}\,dx \\
&= (P_0 - \rho_0 g H)\int_{-r}^{r} 2\sqrt{r^2 - x^2}\,dx + \rho_0 g H \int_{-r}^{r} e^{(L+x)/H}\cdot 2\sqrt{r^2 - x^2}\,dx \\
&= (P_0 - \rho_0 g H)(\pi r^2) + \rho_0 g H e^{L/H} \int_{-r}^{r} e^{x/H}\cdot 2\sqrt{r^2 - x^2}\,dx
\end{aligned}$$

6. The problem can be reduced to finding the line which minimizes the shaded area in the diagram. An equation of the circle in the first quadrant is $x = \sqrt{1 - y^2}$. So the shaded area is



$$\begin{aligned}
A(h) &= \int_0^h \left(1 - \sqrt{1 - y^2}\right) dy + \int_h^1 \sqrt{1 - y^2}\,dy \\
&= \int_0^h \left(1 - \sqrt{1 - y^2}\right) dy - \int_1^h \sqrt{1 - y^2}\,dy
\end{aligned}$$

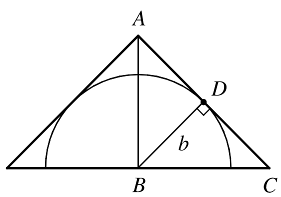
$A'(h) = 1 - \sqrt{1 - h^2} - \sqrt{1 - h^2}$ [by FTC] $= 1 - 2\sqrt{1 - h^2}$

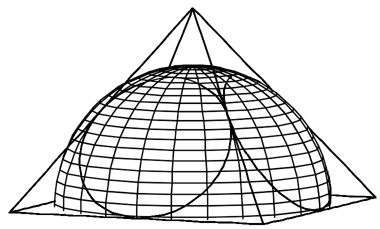
$A' = 0 \;\Leftrightarrow\; \sqrt{1 - h^2} = \frac{1}{2} \;\Rightarrow\; 1 - h^2 = \frac{1}{4} \;\Rightarrow\; h^2 = \frac{3}{4} \;\Rightarrow\; h = \frac{\sqrt{3}}{2}$.

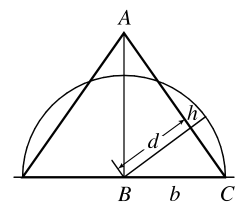
$A''(h) = -2 \cdot \frac{1}{2}(1-h^2)^{-1/2}(-2h) = \dfrac{2h}{\sqrt{1-h^2}} > 0$, so $h = \dfrac{\sqrt{3}}{2}$ gives a minimum value of A.

Note: Another strategy is to use the angle θ as the variable (see the diagram above) and show that $A = \theta + \cos\theta - \frac{\pi}{4} - \frac{1}{2}\sin 2\theta$, which is minimized when $\theta = \frac{\pi}{6}$.

7. To find the height of the pyramid, we use similar triangles. The first figure shows a cross-section of the pyramid passing through the top and through two opposite corners of the square base. Now $|BD| = b$, since it is a radius of the sphere, which has diameter $2b$ since it is tangent to the opposite sides of the square base. Also, $|AD| = b$ since $\triangle ADB$ is isosceles. So the height is $|AB| = \sqrt{b^2 + b^2} = \sqrt{2}\,b$.





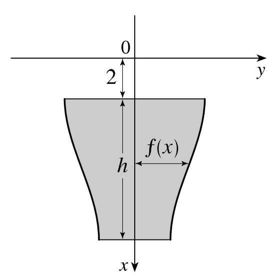


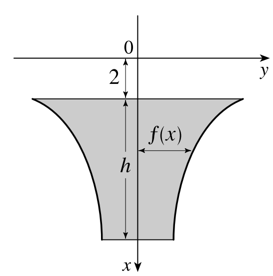
We first observe that the shared volume is equal to half the volume of the sphere, minus the sum of the four equal volumes (caps of the sphere) cut off by the triangular faces of the pyramid. See Exercise 5.2.61 for a derivation of the formula for the volume of a cap of a sphere. To use the formula, we need to find the perpendicular distance h of each triangular face from the surface of the sphere. We first find the distance d from the center of the sphere to one of the triangular faces. The third figure shows a cross-section of the pyramid through the top and through the midpoints of opposite sides of the square base. From similar triangles we find that

$$\frac{d}{b} = \frac{|AB|}{|AC|} = \frac{\sqrt{2}\,b}{\sqrt{b^2 + \left(\sqrt{2}\,b\right)^2}} \quad \Rightarrow \quad d = \frac{\sqrt{2}\,b^2}{\sqrt{3b^2}} = \frac{\sqrt{6}}{3}\,b$$

So $h = b - d = b - \frac{\sqrt{6}}{3}b = \frac{3-\sqrt{6}}{3}b$. So, using the formula $V = \pi h^2(r - h/3)$ from Exercise 5.2.61 with $r = b$, we find that the volume of each of the caps is $\pi\left(\frac{3-\sqrt{6}}{3}b\right)^2\left(b - \frac{3-\sqrt{6}}{3\cdot 3}b\right) = \frac{15-6\sqrt{6}}{9}\cdot\frac{6+\sqrt{6}}{9}\pi b^3 = \left(\frac{2}{3} - \frac{7}{27}\sqrt{6}\right)\pi b^3$. So, using our first observation, the shared volume is $V = \frac{1}{2}\left(\frac{4}{3}\pi b^3\right) - 4\left(\frac{2}{3} - \frac{7}{27}\sqrt{6}\right)\pi b^3 = \left(\frac{28}{27}\sqrt{6} - 2\right)\pi b^3$.

8. Orient the positive x-axis as in the figure. Suppose that the plate has height h and is symmetric about the x-axis. At depth x below the water ($2 \le x \le 2+h$), let the width of the plate be $2f(x)$. Now each of the n horizontal strips has height h/n and the ith strip ($1 \le i \le n$) goes from





$x = 2 + \left(\dfrac{i-1}{n}\right)h$ to $x = 2 + \left(\dfrac{i}{n}\right)h$. The hydrostatic force on the ith strip is $F(i) = \displaystyle\int_{2+[(i-1)/n]h}^{2+(i/n)h} 62.5x[2f(x)]\,dx$.

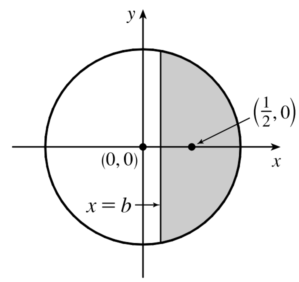
[continued]

If we now let $x[2f(x)] = k$ (a constant) so that $f(x) = k/(2x)$, then

$$F(i) = \int_{2+[(i-1)/n]h}^{2+(i/n)h} 62.5k\,dx = 62.5k\left[x\right]_{2+[(i-1)/n]h}^{2+(i/n)h} = 62.5k\left[\left(2+\frac{i}{n}h\right)-\left(2+\frac{i-1}{n}h\right)\right] = 62.5k\left(\frac{h}{n}\right)$$

So the hydrostatic force on the ith strip is independent of i, that is, the force on each strip is the same. So the plate can be shaped as shown in the figure. (In fact, the required condition is satisfied whenever the plate has width C/x at depth x, for some constant C. Many shapes are possible.)

9. We can assume that the cut is made along a vertical line $x = b > 0$, that the disk's boundary is the circle $x^2 + y^2 = 1$, and that the center of mass of the smaller piece (to the right of $x = b$) is $\left(\frac{1}{2}, 0\right)$. We wish to find b to two decimal places. We have $\dfrac{1}{2} = \overline{x} = \dfrac{\int_b^1 x \cdot 2\sqrt{1-x^2}\,dx}{\int_b^1 2\sqrt{1-x^2}\,dx}$. Evaluating the



numerator gives us $-\int_b^1 (1-x^2)^{1/2}(-2x)\,dx = -\frac{2}{3}\left[(1-x^2)^{3/2}\right]_b^1 = -\frac{2}{3}\left[0-(1-b^2)^{3/2}\right] = \frac{2}{3}(1-b^2)^{3/2}$.

Using Formula 30 in the table of integrals, we find that the denominator is

$\left[x\sqrt{1-x^2}+\sin^{-1}x\right]_b^1 = \left(0+\frac{\pi}{2}\right)-\left(b\sqrt{1-b^2}+\sin^{-1}b\right)$. Thus, we have $\dfrac{1}{2} = \overline{x} = \dfrac{\frac{2}{3}(1-b^2)^{3/2}}{\frac{\pi}{2}-b\sqrt{1-b^2}-\sin^{-1}b}$, or,

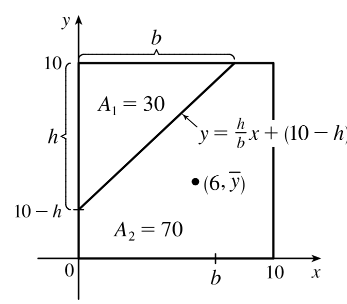
equivalently, $\frac{2}{3}(1-b^2)^{3/2} = \frac{\pi}{4} - \frac{1}{2}b\sqrt{1-b^2} - \frac{1}{2}\sin^{-1}b$. Solving this equation numerically with a calculator or CAS, we obtain $b \approx 0.138173$, or $b = 0.14$ m to two decimal places.

10. $A_1 = 30 \quad\Rightarrow\quad \frac{1}{2}bh = 30 \quad\Rightarrow\quad bh = 60$.

$\overline{x} = 6 \quad\Rightarrow\quad \dfrac{1}{A_2}\displaystyle\int_0^{10} xf(x)\,dx = 6 \quad\Rightarrow$

$\displaystyle\int_0^b x\left(\frac{h}{b}x+10-h\right)dx + \int_b^{10} x(10)\,dx = 6(70) \quad\Rightarrow$

$\displaystyle\int_0^b \left(\frac{h}{b}x^2+10x-hx\right)dx + 10\cdot\tfrac{1}{2}\left[x^2\right]_b^{10} = 420 \quad\Rightarrow$



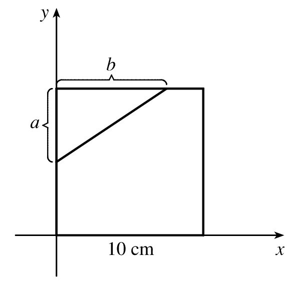
$\left[\dfrac{h}{3b}x^3+5x^2-\dfrac{h}{2}x^2\right]_0^b + 5(100-b^2) = 420 \quad\Rightarrow\quad \frac{1}{3}hb^2+5b^2-\frac{1}{2}hb^2+500-5b^2 = 420 \quad\Rightarrow\quad 80 = \frac{1}{6}hb^2 \quad\Rightarrow$

$480 = (hb)b \quad\Rightarrow\quad 480 = 60b \quad\Rightarrow\quad b = 8$. So $h = \frac{60}{8} = \frac{15}{2}$ and an equation of the line is

$y = \dfrac{15/2}{8}x + \left(10-\dfrac{15}{2}\right) = \dfrac{15}{16}x+\dfrac{5}{2}$. Now

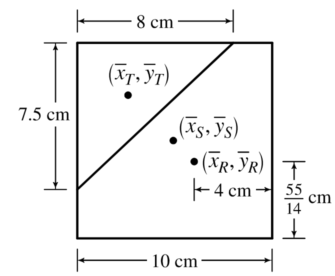
$$\begin{aligned}\overline{y} &= \frac{1}{A_2}\int_0^{10}\tfrac{1}{2}[f(x)]^2\,dx = \frac{1}{70\cdot 2}\left[\int_0^8\left(\frac{15}{16}x+\frac{5}{2}\right)^2dx + \int_8^{10}(10)^2\,dx\right]\\ &= \tfrac{1}{140}\left[\int_0^8\left(\tfrac{225}{256}x^2+\tfrac{75}{16}x+\tfrac{25}{4}\right)dx + 100(10-8)\right] = \tfrac{1}{140}\left(\left[\tfrac{225}{768}x^3+\tfrac{75}{32}x^2+\tfrac{25}{4}x\right]_0^8+200\right)\\ &= \tfrac{1}{140}(150+150+50+200) = \tfrac{550}{140} = \tfrac{55}{14}\end{aligned}$$

[continued]



Another solution: Assume that the right triangle cut from the square has legs a cm and b cm long as shown. The triangle has area 30 cm^2, so $\frac{1}{2}ab = 30$ and $ab = 60$. We place the square in the first quadrant of the xy-plane as shown, and we let T, R, and S denote the triangle, the remaining portion of the square, and the full square, respectively. By symmetry, the centroid of S is $(5, 5)$. By Exercise 8.3.39, the centroid of T is $\left(\dfrac{b}{3}, 10 - \dfrac{a}{3}\right)$.

We are given that the centroid of R is $(6, c)$, where c is to be determined. We take the density of the square to be 1, so that areas can be used as masses. Then T has mass $m_T = 30$, S has mass $m_S = 100$, and R has mass $m_R = m_S - m_T = 70$. By reasoning as in Exercises 40 and 41 of Section 8.3, we view S as consisting of a mass m_T at the centroid $(\overline{x}_T, \overline{y}_T)$ of T and a mass R at the centroid $(\overline{x}_R, \overline{y}_R)$ of R. Then $\overline{x}_S = \dfrac{m_T\overline{x}_T + m_R\overline{x}_R}{m_T + m_R}$ and $\overline{y}_S = \dfrac{m_T\,\overline{y}_T + m_R\overline{y}_R}{m_T + m_R}$; that is,

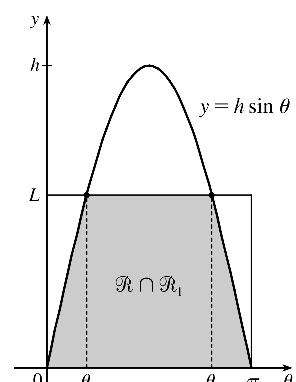


$5 = \dfrac{30(b/3) + 70(6)}{100}$ and $5 = \dfrac{30(10 - a/3) + 70c}{100}$.

Solving the first equation for b, we get $b = 8$ cm. Since $ab = 60$ cm^2, it follows that $a = \frac{60}{8} = 7.5$ cm. Now the second equation says that $70c = 200 + 10a$, so $7c = 20 + a = \frac{55}{2}$ and $c = \frac{55}{14} = 3.9\overline{285714}$ cm. The solution is depicted in the figure.

11. If $h = L$, then $P = \dfrac{\text{area under } y = L\sin\theta}{\text{area of rectangle}} = \dfrac{\int_0^\pi L\sin\theta\,d\theta}{\pi L} = \dfrac{[-\cos\theta]_0^\pi}{\pi} = \dfrac{-(-1)+1}{\pi} = \dfrac{2}{\pi}$.

If $h = \frac{1}{2}L$, we replace L with $\frac{1}{2}L$ in the above calculation to get $P = \frac{1}{2}\left(\frac{2}{\pi}\right) = \frac{1}{\pi}$.

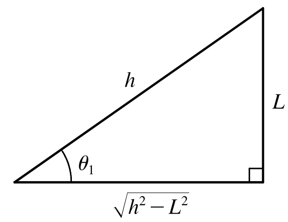


12. (a) The total set of possibilities can be identified with the rectangular region $\mathscr{R} = \{(\theta, y) \mid 0 \le y < L, 0 \le \theta < \pi\}$. Even when $h > L$, the needle intersects at least one line if and only if $y \le h\sin\theta$. Let $\mathscr{R}_1 = \{(\theta, y) \mid 0 \le y \le h\sin\theta, 0 \le \theta < \pi\}$. When $h \le L$, $\mathscr{R}_1$ is contained in $\mathscr{R}$, but that is no longer true when $h > L$. Thus, the probability that the needle intersects a line becomes

$$P = \frac{\text{area}(\mathscr{R} \cap \mathscr{R}_1)}{\text{area}(\mathscr{R})} = \frac{\text{area}(\mathscr{R} \cap \mathscr{R}_1)}{\pi L}$$

When $h > L$, the curve $y = h\sin\theta$ intersects the line $y = L$ twice — at $\left(\sin^{-1}(L/h), L\right)$ and at $\left(\pi - \sin^{-1}(L/h), L\right)$. Set $\theta_1 = \sin^{-1}(L/h)$ and $\theta_2 = \pi - \theta_1$. Then

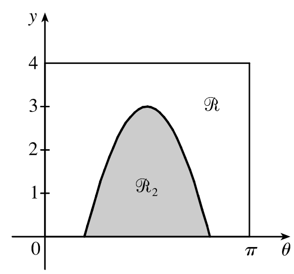
$$\begin{aligned}\text{area}(\mathcal{R}\cap\mathcal{R}_1) &= \int_0^{\theta_1} h\sin\theta\,d\theta + \int_{\theta_1}^{\theta_2} L\,d\theta + \int_{\theta_2}^{\pi} h\sin\theta\,d\theta \\ &= 2\int_0^{\theta_1} h\sin\theta\,d\theta + L(\theta_2-\theta_1) = 2h\left[-\cos\theta\right]_0^{\theta_1} + L(\pi-2\theta_1) \\ &= 2h(1-\cos\theta_1) + L(\pi-2\theta_1) \\ &= 2h\left(1-\frac{\sqrt{h^2-L^2}}{h}\right) + L\left[\pi - 2\sin^{-1}\left(\frac{L}{h}\right)\right] \\ &= 2h - 2\sqrt{h^2-L^2} + \pi L - 2L\sin^{-1}\left(\frac{L}{h}\right)\end{aligned}$$



We are told that $L = 4$ and $h = 7$, so $\text{area}(\mathcal{R}\cap\mathcal{R}_1) = 14 - 2\sqrt{33} + 4\pi - 8\sin^{-1}\left(\frac{4}{7}\right) \approx 10.21128$ and $P = \frac{1}{4\pi}\,\text{area}(\mathcal{R}\cap\mathcal{R}_1) \approx 0.812588$. (By comparison, $P = \frac{2}{\pi} \approx 0.636620$ when $h = L$, as shown in the solution to Problem 11.)

(b) The needle intersects at least two lines when $y + L \le h\sin\theta$; that is, when $y \le h\sin\theta - L$. Set $\mathcal{R}_2 = \{(\theta, y) \mid 0 \le y \le h\sin\theta - L, 0 \le \theta < \pi\}$. Then the probability that the needle intersects at least two lines is

$$P_2 = \frac{\text{area}(\mathcal{R}\cap\mathcal{R}_2)}{\text{area}(\mathcal{R})} = \frac{\text{area}(\mathcal{R}\cap\mathcal{R}_2)}{\pi L}$$



When $L = 4$ and $h = 7$, $\mathcal{R}_2$ is contained in $\mathcal{R}$ (see the figure). Thus,

$$\begin{aligned}P_2 &= \frac{1}{4\pi}\,\text{area}(\mathcal{R}_2) = \frac{1}{4\pi}\int_{\sin^{-1}(4/7)}^{\pi-\sin^{-1}(4/7)} (7\sin\theta - 4)\,d\theta = \tfrac{1}{4\pi}\cdot 2\int_{\sin^{-1}(4/7)}^{\pi/2} (7\sin\theta-4)\,d\theta \\ &= \frac{1}{2\pi}\left[-7\cos\theta - 4\theta\right]_{\sin^{-1}(4/7)}^{\pi/2} = \frac{1}{2\pi}\left[0 - 2\pi + 7\frac{\sqrt{33}}{7} + 4\sin^{-1}\left(\tfrac{4}{7}\right)\right] = \frac{\sqrt{33} + 4\sin^{-1}\left(\frac{4}{7}\right) - 2\pi}{2\pi} \\ &\approx 0.301497\end{aligned}$$

(c) The needle intersects at least three lines when $y + 2L \le h\sin\theta$: that is, when $y \le h\sin\theta - 2L$. Set $\mathcal{R}_3 = \{(\theta, y) \mid 0 \le y \le h\sin\theta - 2L, 0 \le \theta < \pi\}$. Then the probability that the needle intersects at least three lines is $P_3 = \dfrac{\text{area}(\mathcal{R}\cap\mathcal{R}_3)}{\text{area}(\mathcal{R})} = \dfrac{\text{area}(\mathcal{R}\cap\mathcal{R}_3)}{\pi L}$. (At this point, the generalization to P_n, n any positive integer, should be clear.)

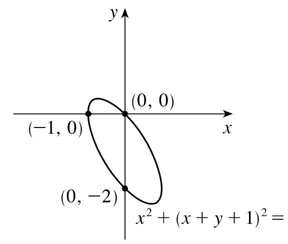
Under the given assumption,

$$\begin{aligned}P_3 &= \frac{1}{\pi L}\,\text{area}(\mathcal{R}_3) = \frac{1}{\pi L}\int_{\sin^{-1}(2L/h)}^{\pi-\sin^{-1}(2L/h)} (h\sin\theta - 2L)\,d\theta = \frac{2}{\pi L}\int_{\sin^{-1}(2L/h)}^{\pi/2} (h\sin\theta - 2L)\,d\theta \\ &= \frac{2}{\pi L}\left[-h\cos\theta - 2L\theta\right]_{\sin^{-1}(2L/h)}^{\pi/2} = \frac{2}{\pi L}\left[-\pi L + \sqrt{h^2-4L^2} + 2L\sin^{-1}(2L/h)\right]\end{aligned}$$

Note that the probability that a needle touches exactly one line is $P_1 - P_2$, the probability that it touches exactly two lines is $P_2 - P_3$, and so on.

13. Solve for y: $x^2 + (x+y+1)^2 = 1 \quad \Rightarrow \quad (x+y+1)^2 = 1 - x^2 \quad \Rightarrow \quad x + y + 1 = \pm\sqrt{1-x^2} \quad \Rightarrow$

$y = -x - 1 \pm \sqrt{1-x^2}$.



$$A = \int_{-1}^{1} \left[\left(-x - 1 + \sqrt{1-x^2}\right) - \left(-x - 1 - \sqrt{1-x^2}\right)\right] dx$$

$$= \int_{-1}^{1} 2\sqrt{1-x^2}\, dx = 2\left(\tfrac{\pi}{2}\right) \quad \left[\begin{array}{c}\text{area of}\\ \text{semicircle}\end{array}\right] \quad = \pi$$

$$\overline{x} = \frac{1}{A}\int_{-1}^{1} x \cdot 2\sqrt{1-x^2}\, dx = 0 \quad [\text{odd integrand}]$$

$$\overline{y} = \frac{1}{A}\int_{-1}^{1} \frac{1}{2}\left[\left(-x-1+\sqrt{1-x^2}\right)^2 - \left(-x-1-\sqrt{1-x^2}\right)^2\right] dx = \frac{1}{\pi}\int_{-1}^{1} \frac{1}{2}\left(-4x\sqrt{1-x^2} - 4\sqrt{1-x^2}\right) dx$$

$$= -\frac{2}{\pi}\int_{-1}^{1}\left(x\sqrt{1-x^2} + \sqrt{1-x^2}\right) dx = -\frac{2}{\pi}\int_{-1}^{1} x\sqrt{1-x^2}\, dx - \frac{2}{\pi}\int_{-1}^{1}\sqrt{1-x^2}\, dx$$

$$= -\frac{2}{\pi}(0) \quad [\text{odd integrand}] \quad -\frac{2}{\pi}\left(\frac{\pi}{2}\right) \quad \left[\begin{array}{c}\text{area of}\\ \text{semicircle}\end{array}\right] \quad = -1$$

Thus, as expected, the centroid is $(\overline{x}, \overline{y}) = (0, -1)$. We might expect this result since the centroid of an ellipse is located at its center.