6 ☐ INVERSE FUNCTIONS:

Exponential, Logarithmic, and Inverse Trigonometric Functions

6.1 Inverse Functions and Their Derivatives

1. (a) See Definition 1.

(b) It must pass the Horizontal Line Test.

2. (a) $f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y$ for any y in B. The domain of f^{-1} is B and the range of f^{-1} is A.

(b) See the steps in Box 5.

(c) Reflect the graph of f about the line $y = x$.

3. f is not one-to-one because $2 \neq 6$, but $f(2) = 2.0 = f(6)$.

4. f is one-to-one because it never takes on the same value twice.

5. We could draw a horizontal line that intersects the graph in more than one point. Thus, by the Horizontal Line Test, the function is not one-to-one.

6. No horizontal line intersects the graph more than once. Thus, by the Horizontal Line Test, the function is one-to-one.

7. No horizontal line intersects the graph more than once. Thus, by the Horizontal Line Test, the function is one-to-one.

8. We could draw a horizontal line that intersects the graph in more than one point. Thus, by the Horizontal Line Test, the function is not one-to-one.

9. The graph of $f(x) = 2x - 3$ is a line with slope 2. It passes the Horizontal Line Test, so f is one-to-one.

Algebraic solution: If $x_1 \neq x_2$, then $2x_1 \neq 2x_2 \quad \Rightarrow \quad 2x_1 - 3 \neq 2x_2 - 3 \quad \Rightarrow \quad f(x_1) \neq f(x_2)$, so f is one-to-one.

10. The graph of $f(x) = x^4 - 16$ is symmetric with respect to the y-axis. Pick any x-values equidistant from 0 to find two equal function values. For example, $f(-1) = -15$ and $f(1) = -15$, so f is not one-to-one.

11. No horizontal line intersects the graph of $r(t) = t^3 + 4$ more than once. Thus, by the Horizontal Line Test, the function is one-to-one.

Algebraic solution: If $t_1 \neq t_2$, then $t_1^3 \neq t_2^3 \quad \Rightarrow \quad t_1^3 + 4 \neq t_2^3 + 4 \quad \Rightarrow \quad r(t_1) \neq r(t_2)$, so r is one-to-one.

12. The graph of $g(x) = \sqrt[3]{x}$ passes the Horizontal Line Test, so g is one-to-one.

13. $g(x) = 1 - \sin x. \quad g(0) = 1$ and $g(\pi) = 1$, so g is not one-to-one.

14. The graph of $f(x) = x^4 - 1$ passes the Horizontal Line Test when x is restricted to the interval [0,10], so f is one-to-one.

15. A football will attain every height h up to its maximum height twice: once on the way up, and again on the way down. Thus, even if t_1 does not equal t_2, $f(t_1)$ may equal $f(t_2)$, so f is not 1-1.

16. f is not 1-1 because eventually we all stop growing and therefore, there are two times at which we have the same height.

17. (a) Since f is 1-1, $f(6) = 17 \quad \Leftrightarrow \quad f^{-1}(17) = 6$.

(b) Since f is 1-1, $f^{-1}(3) = 2 \quad \Leftrightarrow \quad f(2) = 3$.

18. First, we must determine x such that $f(x) = 3$. By inspection, we see that if $x = 1$, then $f(1) = 3$. Since f is 1-1 (f is an increasing function), it has an inverse, and $f^{-1}(3) = 1$. If f is a 1-1 function, then $f(f^{-1}(a)) = a$, so $f(f^{-1}(2)) = 2$.

19. $h(x) = x + \sqrt{x} \quad \Rightarrow \quad h'(x) = 1 + 1/(2\sqrt{x}\,) > 0$ on $(0, \infty)$. So h is increasing and hence, 1-1. By inspection, $h(4) = 4 + \sqrt{4} = 6$, so $h^{-1}(6) = 4$.

20. (a) f is 1-1 because it passes the Horizontal Line Test.

(b) Domain of $f = [-3, 3] =$ Range of f^{-1}. Range of $f = [-1, 3] =$ Domain of f^{-1}.

(c) Since $f(0) = 2$, $f^{-1}(2) = 0$.

(d) Since $f(-1.7) \approx 0$, $f^{-1}(0) \approx -1.7$.

21. We solve $C = \frac{5}{9}(F - 32)$ for F: $\frac{9}{5}C = F - 32 \quad \Rightarrow \quad F = \frac{9}{5}C + 32$. This gives us a formula for the inverse function, that is, the Fahrenheit temperature F as a function of the Celsius temperature C. $F \geq -459.67 \quad \Rightarrow \quad \frac{9}{5}C + 32 \geq -459.67 \quad \Rightarrow$ $\frac{9}{5}C \geq -491.67 \quad \Rightarrow \quad C \geq -273.15$, the domain of the inverse function.

22. $m = \dfrac{m_0}{\sqrt{1 - v^2/c^2}} \quad \Rightarrow \quad 1 - \dfrac{v^2}{c^2} = \dfrac{m_0^2}{m^2} \quad \Rightarrow \quad \dfrac{v^2}{c^2} = 1 - \dfrac{m_0^2}{m^2} \quad \Rightarrow \quad v^2 = c^2\left(1 - \dfrac{m_0^2}{m^2}\right) \quad \Rightarrow \quad v = c\sqrt{1 - \dfrac{m_0^2}{m^2}}$.

This formula gives us the speed v of the particle in terms of its mass m, that is, $v = f^{-1}(m)$.

23. $y = f(x) = 5 - 4x \quad \Rightarrow \quad 4x = 5 - y \quad \Rightarrow \quad x = \frac{1}{4}(5 - y)$. Interchange x and y: $y = \frac{1}{4}(5 - x)$. So $f^{-1}(x) = \frac{1}{4}(5 - x) = \frac{5}{4} - \frac{1}{4}x$.

24. We write $y = h(x) = \dfrac{6 - 3x}{5x + 7}$ and solve for x: $y(5x + 7) = 6 - 3x \quad \Rightarrow \quad 5xy + 7y = 6 - 3x \quad \Rightarrow$ $5xy + 3x = 6 - 7y \quad \Rightarrow \quad x(5y + 3) = 6 - 7y \quad \Rightarrow \quad x = \dfrac{6 - 7y}{5y + 3}$. Interchanging x and y gives $y = \dfrac{6 - 7x}{5x + 3}$, so $h^{-1}(x) = \dfrac{6 - 7x}{5x + 3}$.

25. First note that $f(x) = 1 - x^2$, $x \geq 0$, is one-to-one. We first write $y = 1 - x^2$, $x \geq 0$, and solve for x: $x^2 = 1 - y \quad \Rightarrow \quad x = \sqrt{1 - y}$ (since $x \geq 0$). Interchanging x and y gives $y = \sqrt{1 - x}$, so the inverse function is $f^{-1}(x) = \sqrt{1 - x}$.

26. Completing the square, we have $g(x) = x^2 - 2x = (x^2 - 2x + 1) - 1 = (x - 1)^2 - 1$ and, with the restriction $x \geq 1$, g is one-to-one. We write $y = (x - 1)^2 - 1$, $x \geq 1$, and solve for x: $x - 1 = \sqrt{y + 1}$ (since $x \geq 1 \quad \Leftrightarrow \quad x - 1 \geq 0$), so $x = 1 + \sqrt{y + 1}$. Interchanging x and y gives $y = 1 + \sqrt{x + 1}$, so $g^{-1}(x) = 1 + \sqrt{x + 1}$.

27. First write $y = g(x) = 2 + \sqrt{x+1}$ and note that $y \geq 2$. Solve for x: $y - 2 = \sqrt{x+1} \Rightarrow (y-2)^2 = x + 1 \Rightarrow$ $x = (y-2)^2 - 1$ $(y \geq 2)$. Interchanging x and y gives $y = (x-2)^2 - 1$, so $g^{-1}(x) = (x-2)^2 - 1$ with domain $x \geq 2$.

28. $y = f(x) = 1 + \sqrt{2+3x} \quad (y \geq 1) \Rightarrow y - 1 = \sqrt{2+3x} \Rightarrow (y-1)^2 = 2 + 3x \Rightarrow (y-1)^2 - 2 = 3x \Rightarrow$ $x = \frac{1}{3}(y-1)^2 - \frac{2}{3}$. Interchange x and y: $y = \frac{1}{3}(x-1)^2 - \frac{2}{3}$. So $f^{-1}(x) = \frac{1}{3}(x-1)^2 - \frac{2}{3}$. Note that the domain of f^{-1} is $x \geq 1$.

29. We solve $y = \left(2 + \sqrt[3]{x}\right)^5$ for x: $\sqrt[5]{y} = 2 + \sqrt[3]{x} \Rightarrow \sqrt[3]{x} = \sqrt[5]{y} - 2 \Rightarrow x = \left(\sqrt[5]{y} - 2\right)^3$. Interchanging x and y gives the inverse function $y = \left(\sqrt[5]{x} - 2\right)^3$.

30. For $f(x) = \dfrac{1 - \sqrt{x}}{1 + \sqrt{x}}$, the domain is $x \geq 0$. $f(0) = 1$ and as x increases, y decreases. As $x \to \infty$,

$\dfrac{1 - \sqrt{x}}{1 + \sqrt{x}} \cdot \dfrac{1/\sqrt{x}}{1/\sqrt{x}} = \dfrac{1/\sqrt{x} - 1}{1/\sqrt{x} + 1} \to \dfrac{-1}{1} = -1$, so the range of f is $-1 < y \leq 1$. Thus, the domain of f^{-1} is $-1 < x \leq 1$.

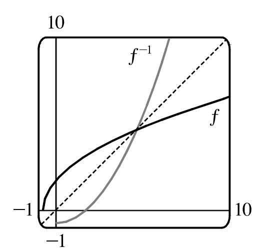
$$y = \frac{1 - \sqrt{x}}{1 + \sqrt{x}} \Rightarrow y(1 + \sqrt{x}) = 1 - \sqrt{x} \Rightarrow y + y\sqrt{x} = 1 - \sqrt{x} \Rightarrow \sqrt{x} + y\sqrt{x} = 1 - y \Rightarrow$$

$\sqrt{x}(1+y) = 1 - y \Rightarrow \sqrt{x} = \dfrac{1-y}{1+y} \Rightarrow x = \left(\dfrac{1-y}{1+y}\right)^2$. Interchange x and y: $y = \left(\dfrac{1-x}{1+x}\right)^2$. So

$f^{-1}(x) = \left(\dfrac{1-x}{1+x}\right)^2$ with $-1 < x \leq 1$.

31. $y = f(x) = \sqrt{4x+3}$ $(y \geq 0) \Rightarrow y^2 = 4x + 3 \Rightarrow x = \dfrac{y^2 - 3}{4}$.

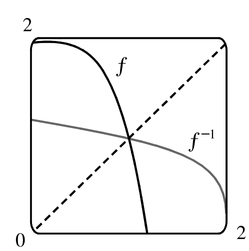
Interchange x and y: $y = \dfrac{x^2 - 3}{4}$. So $f^{-1}(x) = \dfrac{x^2 - 3}{4}$ $(x \geq 0)$. From the graph, we see that f and f^{-1} are reflections about the line $y = x$.



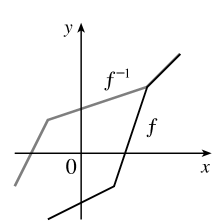
32. $y = f(x) = 2 - x^4$ $(x \geq 0) \Rightarrow x^4 = 2 - y \Rightarrow$

$x = \sqrt[4]{2-y}$ [since $x \geq 0$]. Interchange x and y: $y = \sqrt[4]{2-x}$.

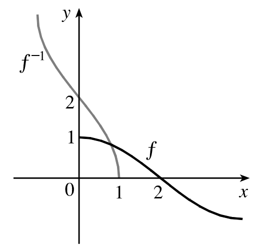
So $f^{-1}(x) = \sqrt[4]{2-x}$ $(x \leq 2)$. From the graph, we see that f and f^{-1} are reflections about the line $y = x$.



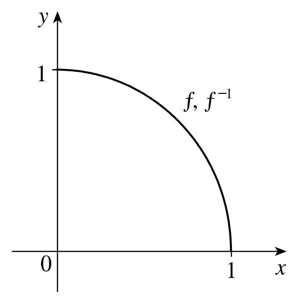
33. Reflect the graph of f about the line $y = x$. The points $(-1, -2)$, $(1, -1)$, $(2, 2)$, and $(3, 3)$ on f are reflected to $(-2, -1)$, $(-1, 1)$, $(2, 2)$, and $(3, 3)$ on f^{-1}.



34. Reflect the graph of f about the line $y = x$.

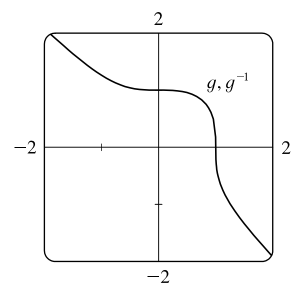


35. (a) $y = f(x) = \sqrt{1 - x^2}$ $\quad (0 \le x \le 1$ and note that $y \ge 0) \quad \Rightarrow$
$y^2 = 1 - x^2 \quad \Rightarrow \quad x^2 = 1 - y^2 \quad \Rightarrow \quad x = \sqrt{1 - y^2}$. So
$f^{-1}(x) = \sqrt{1 - x^2},\ 0 \le x \le 1.$ We see that $f^{-1}$and f are the same function.



(b) The graph of f is the portion of the circle $x^2 + y^2 = 1$ with $0 \le x \le 1$ and $0 \le y \le 1$ (quarter-circle in the first quadrant). The graph of f is symmetric with respect to the line $y = x$, so its reflection about $y = x$ is itself, that is, $f^{-1} = f$.

36. (a) $y = g(x) = \sqrt[3]{1 - x^3} \quad \Rightarrow \quad y^3 = 1 - x^3 \quad \Rightarrow \quad x^3 = 1 - y^3 \quad \Rightarrow$
$x = \sqrt[3]{1 - y^3}$. So $g^{-1}(x) = \sqrt[3]{1 - x^3}$. We see that g and g^{-1} are the same function.



(b) The graph of g is symmetric with respect to the line $y = x$, so its reflection about $y = x$ is itself, that is, $g^{-1} = g$.

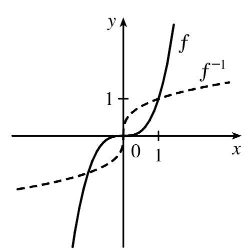
37. (a) $x_1 \ne x_2 \quad \Rightarrow \quad x_1^3 \ne x_2^3 \quad \Rightarrow \quad f(x_1) \ne f(x_2)$, so f is one-to-one.

(b) $f'(x) = 3x^2$ and $f(2) = 8 \quad \Rightarrow \quad f^{-1}(8) = 2$, so $(f^{-1})'(8) = 1/f'(f^{-1}(8)) = 1/f'(2) = \frac{1}{12}$.

(c) $y = x^3 \quad \Rightarrow \quad x = y^{1/3}$. Interchanging x and y gives $y = x^{1/3}$,
so $f^{-1}(x) = x^{1/3}$. Domain$\left(f^{-1}\right)$ = range$(f) = \mathbb{R}$.
Range(f^{-1}) = domain$(f) = \mathbb{R}$.

(d) $f^{-1}(x) = x^{1/3} \quad \Rightarrow \quad (f^{-1})'(x) = \frac{1}{3}x^{-2/3} \quad \Rightarrow$
$(f^{-1})'(8) = \frac{1}{3}\left(\frac{1}{4}\right) = \frac{1}{12}$ as in part (b).

(e)



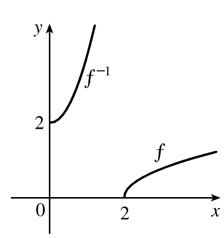
38. (a) $x_1 \ne x_2 \quad \Rightarrow \quad x_1 - 2 \ne x_2 - 2 \quad \Rightarrow \quad \sqrt{x_1 - 2} \ne \sqrt{x_2 - 2} \quad \Rightarrow \quad f(x_1) \ne f(x_2)$, so f is 1-1.

(b) $f(6) = 2$, so $f^{-1}(2) = 6$. Also $f'(x) = \dfrac{1}{2\sqrt{x-2}}$, so $(f^{-1})'(2) = \dfrac{1}{f'(f^{-1}(2))} = \dfrac{1}{f'(6)} = \dfrac{1}{1/4} = 4$.

(c) $y = \sqrt{x - 2} \quad \Rightarrow \quad y^2 = x - 2 \quad \Rightarrow \quad x = y^2 + 2$.
Interchange x and y: $y = x^2 + 2$. So $f^{-1}(x) = x^2 + 2$.
Domain $= [0, \infty)$, range $= [2, \infty)$.

(d) $f^{-1}(x) = x^2 + 2 \quad \Rightarrow \quad (f^{-1})'(x) = 2x \quad \Rightarrow \quad (f^{-1})'(2) = 4$.

(e)



39. (a) Since $x \geq 0$, $x_1 \neq x_2 \quad \Rightarrow \quad x_1^2 \neq x_2^2 \quad \Rightarrow \quad 9 - x_1^2 \neq 9 - x_2^2 \quad \Rightarrow \quad f(x_1) \neq f(x_2)$, so f is 1-1.

(b) $f'(x) = -2x$ and $f(1) = 8 \quad \Rightarrow \quad f^{-1}(8) = 1$, so $(f^{-1})'(8) = \dfrac{1}{f'(f^{-1}(8))} = \dfrac{1}{f'(1)} = \dfrac{1}{-2} = -\dfrac{1}{2}$.

(c) $y = 9 - x^2 \quad \Rightarrow \quad x^2 = 9 - y \quad \Rightarrow \quad x = \sqrt{9-y}$.

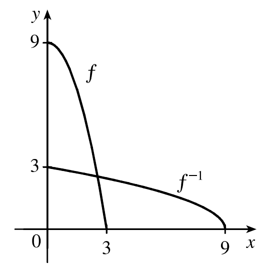
Interchange x and y: $y = \sqrt{9-x}$, so $f^{-1}(x) = \sqrt{9-x}$.

Domain$(f^{-1}) = $ range $(f) = [0, 9]$.

Range$(f^{-1}) = $ domain $(f) = [0, 3]$.

(d) $(f^{-1})'(x) = -1/\left(2\sqrt{9-x}\right) \quad \Rightarrow \quad (f^{-1})'(8) = -\frac{1}{2}$ as in part (b).

(e)



40. (a) $x_1 \neq x_2 \quad \Rightarrow \quad x_1 - 1 \neq x_2 - 1 \quad \Rightarrow \quad \dfrac{1}{x_1 - 1} \neq \dfrac{1}{x_2 - 1} \quad \Rightarrow \quad f(x_1) \neq f(x_2)$, so f is 1-1.

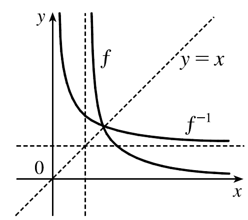
(b) $f^{-1}(2) = \frac{3}{2}$ since $f\left(\frac{3}{2}\right) = 2$. Also $f'(x) = -1/(x-1)^2$, so $(f^{-1})'(2) = 1/f'\left(\frac{3}{2}\right) = \frac{1}{-4} = -\frac{1}{4}$.

(c) $y = 1/(x-1) \quad \Rightarrow \quad x - 1 = 1/y \quad \Rightarrow \quad x = 1 + 1/y$. Interchange x and y: $y = 1 + 1/x$. So $f^{-1}(x) = 1 + 1/x$, $x > 0$ (since $y > 1$).

Domain $= (0, \infty)$, range $= (1, \infty)$.

(d) $(f^{-1})'(x) = -1/x^2$, so $(f^{-1})'(2) = -\frac{1}{4}$.

(e)



41. $f(x) = x^3 + 3\sin x + 2\cos x \quad \Rightarrow \quad f'(x) = 3x^2 + 3\cos x - 2\sin x$. Observe that $f(0) = 2$, so that $f^{-1}(2) = 0$.

By Exercise 83, we have $(f^{-1})'(2) = \dfrac{1}{f'(f^{-1}(2))} = \dfrac{1}{f'(0)} = \dfrac{1}{3(0)^2 + 3\cos 0 - 2\sin 0} = \dfrac{1}{3(1)} = \dfrac{1}{3}$.

42. $f(0) = 2 \quad \Rightarrow \quad f^{-1}(2) = 0$, and $f(x) = x^3 + 3\sin x + 2\cos x \quad \Rightarrow \quad f'(x) = 3x^2 + 3\cos x - 2\sin x$ and $f'(0) = 3$.

Thus, $(f^{-1})'(2) = \dfrac{1}{f'(f^{-1}(2))} = \dfrac{1}{f'(0)} = \dfrac{1}{3}$.

43. $f(0) = 3 \quad \Rightarrow \quad f^{-1}(3) = 0$, and $f(x) = 3 + x^2 + \tan(\pi x/2) \quad \Rightarrow \quad f'(x) = 2x + \frac{\pi}{2}\sec^2(\pi x/2)$ and

$f'(0) = \frac{\pi}{2} \cdot 1 = \frac{\pi}{2}$. Thus, $\left(f^{-1}\right)'(3) = 1/f'\left(f^{-1}(3)\right) = 1/f'(0) = 2/\pi$.

44. $f(1) = 3 \quad \Rightarrow \quad f^{-1}(3) = 1$, and $f(x) = \sqrt{x^3 + 4x + 4} \quad \Rightarrow \quad f'(x) = \dfrac{3x^2 + 4}{2\sqrt{x^3 + 4x + 4}}$ and $f'(1) = \dfrac{7}{6}$. Thus,

$(f^{-1})'(3) = \dfrac{1}{f'(f^{-1}(3))} = \dfrac{1}{f'(1)} = \dfrac{1}{7/6} = \dfrac{6}{7}$.

45. $f(4) = 5 \quad \Rightarrow \quad f^{-1}(5) = 4$. Thus, $(f^{-1})'(5) = \dfrac{1}{f'(f^{-1}(5))} = \dfrac{1}{f'(4)} = \dfrac{1}{2/3} = \dfrac{3}{2}$.

46. g is an increasing function, so it has an inverse. $g(2) = 8 \quad \Leftrightarrow \quad g^{-1}(8) = 2$. Thus,

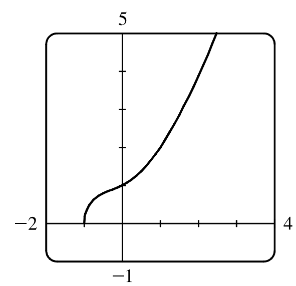
$(g^{-1})'(8) = \dfrac{1}{g'(g^{-1}(8))} = \dfrac{1}{g'(2)} = \dfrac{1}{5}$.

47. $f(x) = \int_3^x \sqrt{1+t^3}\,dt \quad\Rightarrow\quad f'(x) = \sqrt{1+x^3} > 0$, so f is an increasing function and it has an inverse. Since $f(3) = \int_3^3 \sqrt{1+t^3}\,dt = 0$, $f^{-1}(0) = 3$. Thus, $(f^{-1})'(0) = \dfrac{1}{f'(f^{-1}(0))} = \dfrac{1}{f'(3)} = \dfrac{1}{\sqrt{1+3^3}} = \dfrac{1}{\sqrt{28}}$.

48. $f(3) = 2 \quad\Rightarrow\quad f^{-1}(2) = 3$. Thus, $(f^{-1})'(2) = \dfrac{1}{f'(f^{-1}(2))} = \dfrac{1}{f'(3)} = 9$. Hence, $G(x) = \dfrac{1}{f^{-1}(x)} \quad\Rightarrow$

$$G'(x) = -\frac{(f^{-1})'(x)}{[f^{-1}(x)]^2} \quad\Rightarrow\quad G'(2) = -\frac{(f^{-1})'(2)}{[f^{-1}(2)]^2} = -\frac{9}{(3)^2} = -1.$$

49. We see that the graph of $y = f(x) = \sqrt{x^3+x^2+x+1}$ is increasing, so f is 1-1. Enter $x = \sqrt{y^3+y^2+y+1}$ and use your CAS to solve the equation for y. You will likely get two (irrelevant) solutions involving imaginary expressions, as well as one which can be simplified to



$$y = f^{-1}(x) = -\tfrac{\sqrt[3]{4}}{6}\left(\sqrt[3]{D - 27x^2 + 20} - \sqrt[3]{D + 27x^2 - 20} + \sqrt[3]{2}\right)$$

where $D = 3\sqrt{3}\sqrt{27x^4 - 40x^2 + 16}$ or, equivalently, $\dfrac{1}{6}\dfrac{M^{2/3} - 8 - 2M^{1/3}}{2M^{1/3}}$,

where $M = 108x^2 + 12\sqrt{48 - 120x^2 + 81x^4} - 80$.

50. Since $\sin(2n\pi) = 0$, $h(x) = \sin x$ is not one-to-one. $h'(x) = \cos x > 0$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, so h is increasing and hence 1-1 on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Let $y = f^{-1}(x) = \sin^{-1}x$ so that $\sin y = x$. Differentiating $\sin y = x$ implicitly with respect to x gives us $\cos y\,\dfrac{dy}{dx} = 1 \quad\Rightarrow\quad \dfrac{dy}{dx} = \dfrac{1}{\cos y}$. Now $\cos^2 y + \sin^2 y = 1 \quad\Rightarrow\quad \cos y = \pm\sqrt{1-\sin^2 y}$, but since $\cos y > 0$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we have $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-\sin^2 y}} = \dfrac{1}{\sqrt{1-x^2}}$.

51. (a) If the point (x, y) is on the graph of $y = f(x)$, then the point $(x - c, y)$ is that point shifted c units to the left. Since f is 1-1, the point (y, x) is on the graph of $y = f^{-1}(x)$ and the point corresponding to $(x - c, y)$ on the graph of f is $(y, x - c)$ on the graph of f^{-1}. Thus, the curve's reflection is shifted *down* the same number of units as the curve itself is shifted to the left. So an expression for the inverse function is $g^{-1}(x) = f^{-1}(x) - c$.

(b) If we compress (or stretch) a curve horizontally, the curve's reflection in the line $y = x$ is compressed (or stretched) *vertically* by the same factor. Using this geometric principle, we see that the inverse of $h(x) = f(cx)$ can be expressed as $h^{-1}(x) = (1/c)\,f^{-1}(x)$.

52. (a) We know that $g'(x) = \dfrac{1}{f'(g(x))}$. Thus,

$$g''(x) = -\frac{f''(g(x))\cdot g'(x)}{[f'(g(x))]^2} = -\frac{f''(g(x))\cdot[1/f'(g'(x))]}{[f'(g(x))]^2} = -\frac{f''(g(x))}{f'(g(x))[f'(g(x))]^2} = -\frac{f''(g(x))}{[f'(g(x))]^3}.$$

(b) f is increasing $\quad\Rightarrow\quad f'(g(x)) > 0 \quad\Rightarrow\quad [f'(g(x))]^3 > 0$. $\quad f$ is concave upward $\quad\Rightarrow\quad f''(g(x)) > 0$.

So $g''(x) = -\dfrac{f''(g(x))}{[f'(g(x))]^3} < 0$, which implies that g [f's inverse] is concave downward.

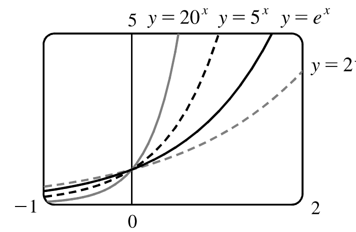
6.2 Exponential Functions and Their Derivatives

1. (a) $f(x) = b^x,\ \ b > 0$ (b) $\mathbb{R}$ (c) $(0, \infty)$ (d) See Figures 4(c), 4(b), and 4(a), respectively.

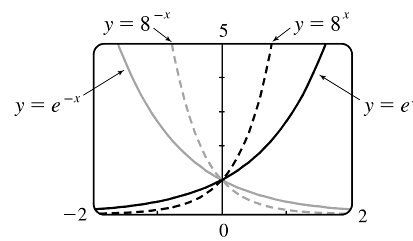
2. (a) The number e is the value of a such that the slope of the tangent line at $x = 0$ on the graph of $y = a^x$ is exactly 1.

(b) $e \approx 2.71828$ (c) $f(x) = e^x$

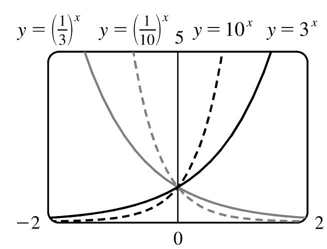
3. All of these graphs approach 0 as $x \to -\infty$, all of them pass through the point $(0, 1)$, and all of them are increasing and approach ∞ as $x \to \infty$. The larger the base, the faster the function increases for $x > 0$, and the faster it approaches 0 as $x \to -\infty$.



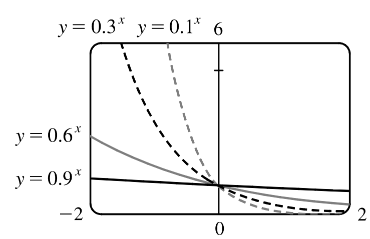
4. The graph of e^{-x} is the reflection of the graph of e^x about the y-axis, and the graph of 8^{-x} is the reflection of that of 8^x about the y-axis. The graph of 8^x increases more quickly than that of e^x for $x > 0$, and approaches 0 faster as $x \to -\infty$.



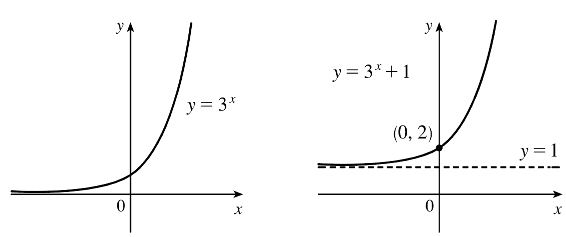
5. The functions with base greater than 1 (3^x and 10^x) are increasing, while those with base less than 1 $\left[\left(\frac{1}{3}\right)^x \text{ and } \left(\frac{1}{10}\right)^x\right]$ are decreasing. The graph of $\left(\frac{1}{3}\right)^x$ is the reflection of that of 3^x about the y-axis, and the graph of $\left(\frac{1}{10}\right)^x$ is the reflection of that of 10^x about the y-axis. The graph of 10^x increases more quickly than that of 3^x for $x > 0$, and approaches 0 faster as $x \to -\infty$.



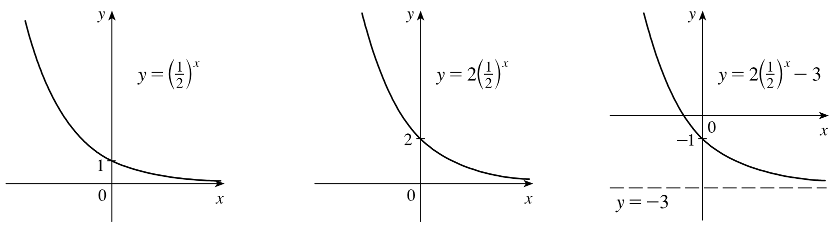
6. Each of the graphs approaches ∞ as $x \to -\infty$, and each approaches 0 as $x \to \infty$. The smaller the base, the faster the function grows as $x \to -\infty$, and the faster it approaches 0 as $x \to \infty$.



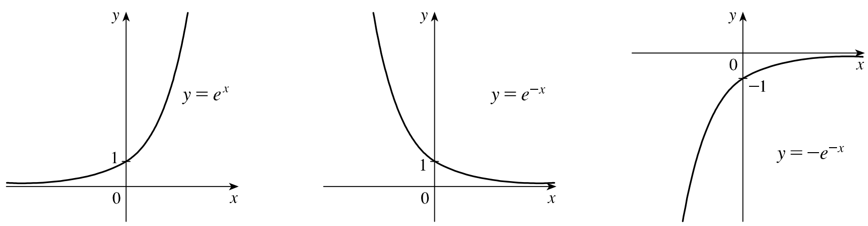
7. We start with the graph of $y = 3^x$ (Figure 12) and shift 1 unit upward to get the graph of $g(x) = 3^x + 1$.



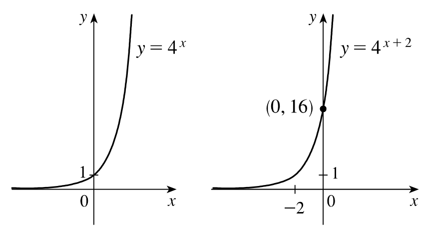
8. We start with the graph of $y = \left(\frac{1}{2}\right)^x$ (Figure 3) and stretch vertically by a factor of 2 to obtain the graph of $y = 2\left(\frac{1}{2}\right)^x$. Then we shift the graph 3 units downward to get the graph of $h(x) = 2\left(\frac{1}{2}\right)^x - 3$.



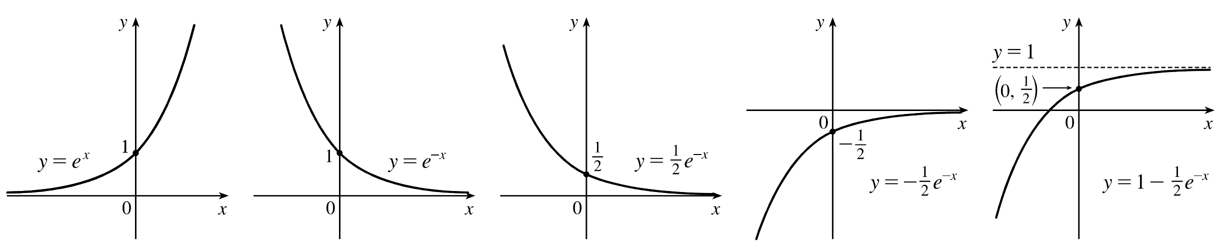
9. We start with the graph of $y = e^x$ (Figure 14) and reflect about the y-axis to get the graph of $y = e^{-x}$. Then we reflect the graph about the x-axis to get the graph of $y = -e^{-x}$.



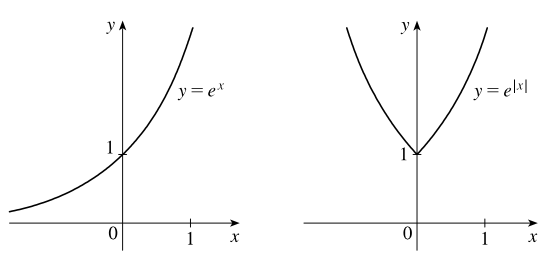
10. We start with the graph of $y = 4^x$ (Figure 3) and shift 2 units to the left to get the graph of $y = 4^{x+2}$.



11. We start with the graph of $y = e^x$ (Figure 14) and reflect about the y-axis to get the graph of $y = e^{-x}$. Then we compress the graph vertically by a factor of 2 to obtain the graph of $y = \frac{1}{2}e^{-x}$ and then reflect about the x-axis to get the graph of $y = -\frac{1}{2}e^{-x}$. Finally, we shift the graph one unit upward to get the graph of $y = 1 - \frac{1}{2}e^{-x}$.



12. We start with the graph of $y = e^x$ (Figure 14) and reflect the portion of the graph in the first quadrant about the y-axis to obtain the graph of $y = e^{|x|}$.



13. (a) To find the equation of the graph that results from shifting the graph of $y = e^x$ two units downward, we subtract 2 from the original function to get $y = e^x - 2$.

(b) To find the equation of the graph that results from shifting the graph of $y = e^x$ two units to the right, we replace x with $x - 2$ in the original function to get $y = e^{x-2}$.

(c) To find the equation of the graph that results from reflecting the graph of $y = e^x$ about the x-axis, we multiply the original function by -1 to get $y = -e^x$.

(d) To find the equation of the graph that results from reflecting the graph of $y = e^x$ about the y-axis, we replace x with $-x$ in the original function to get $y = e^{-x}$.

(e) To find the equation of the graph that results from reflecting the graph of $y = e^x$ about the x-axis and then about the y-axis, we first multiply the original function by -1 (to get $y = -e^x$) and then replace x with $-x$ in this equation to get $y = -e^{-x}$.

14. (a) This reflection consists of first reflecting the graph about the x-axis (giving the graph with equation $y = -e^x$) and then shifting this graph $2 \cdot 4 = 8$ units upward. So the equation is $y = -e^x + 8$.

(b) This reflection consists of first reflecting the graph about the y-axis (giving the graph with equation $y = e^{-x}$) and then shifting this graph $2 \cdot 2 = 4$ units to the right. So the equation is $y = e^{-(x-4)}$.

15. (a) The denominator is zero when $1 - e^{1-x^2} = 0 \quad\Leftrightarrow\quad e^{1-x^2} = 1 \quad\Leftrightarrow\quad 1 - x^2 = 0 \quad\Leftrightarrow\quad x = \pm 1$. Thus, the function $f(x) = \dfrac{1 - e^{x^2}}{1 - e^{1-x^2}}$ has domain $\{x \mid x \neq \pm 1\} = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

(b) The denominator is never equal to zero, so the function $f(x) = \dfrac{1 + x}{e^{\cos x}}$ has domain $\mathbb{R}$, or $(-\infty, \infty)$.

16. (a) The function $g(t) = \sqrt{10^t - 100}$ has domain $\{t \mid 10^t - 100 \geq 0\} = \{t \mid 10^t \geq 10^2\} = \{t \mid t \geq 2\} = [2, \infty)$.

(b) The sine and exponential functions have domain $\mathbb{R}$, so $g(t) = \sin(e^t - 1)$ also has domain $\mathbb{R}$.

17. Use $y = Cb^x$ with the points $(1, 6)$ and $(3, 24)$. $\quad 6 = Cb^1 \quad \left[C = \frac{6}{b}\right]$ and $24 = Cb^3 \quad\Rightarrow\quad 24 = \left(\dfrac{6}{b}\right) b^3 \quad\Rightarrow$ $4 = b^2 \quad\Rightarrow\quad b = 2$ [since $b > 0$] and $C = \frac{6}{2} = 3$. The function is $f(x) = 3 \cdot 2^x$.

18. Given the y-intercept $(0, 2)$, we have $y = Ca^x = 2a^x$. Using the point $\left(2, \frac{2}{9}\right)$ gives us $\frac{2}{9} = 2a^2 \quad\Rightarrow\quad \frac{1}{9} = a^2 \quad\Rightarrow$ $a = \frac{1}{3}$ [since $a > 0$]. The function is $f(x) = 2\left(\frac{1}{3}\right)^x$ or $f(x) = 2(3)^{-x}$.

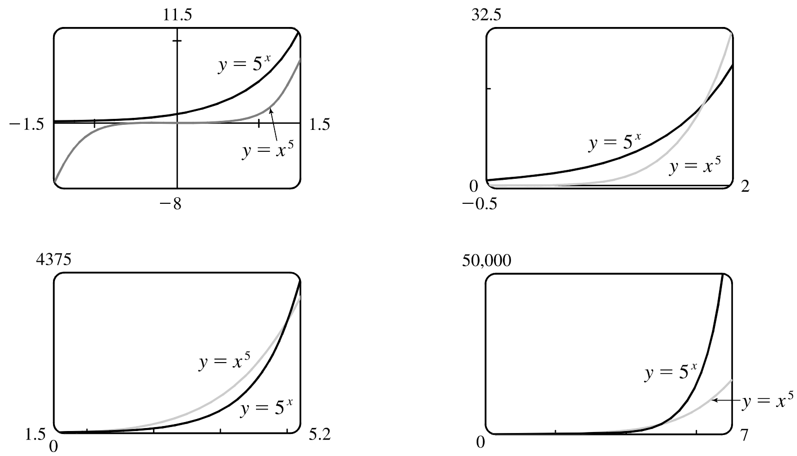
19. In this question, we know that $x = 1$, So for the function f:

$$x = 1 \;\Rightarrow\; f(1) = 1^2 = 1 \text{ m} \tag{1}$$

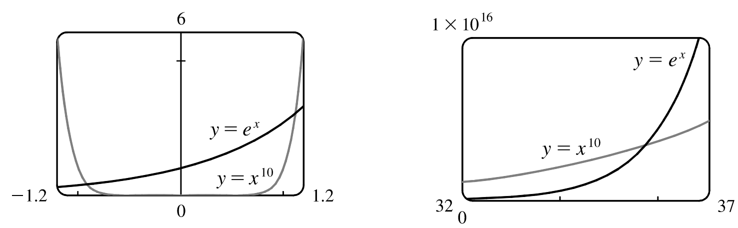
and for the function g:

$$x = 1 \;\Rightarrow\; g(1) = 2^1 = 2 \text{ m} \tag{2}$$

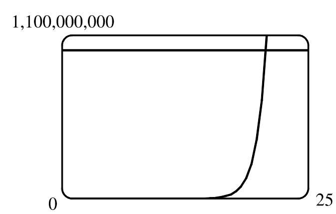
20. We see from the graphs that for x less than about 1.8, $g(x) = 5^x > f(x) = x^5$, and then near the point $(1.8, 17.1)$ the curves intersect. Then $f(x) > g(x)$ from $x \approx 1.8$ until $x = 5$. At $(5, 3125)$ there is another point of intersection, and for $x > 5$ we see that $g(x) > f(x)$. In fact, g increases much more rapidly than f beyond that point.



21. The graph of g finally surpasses that of f at $x \approx 35.8$.



22. We graph $y = e^x$ and $y = 1{,}000{,}000{,}000$ and determine where $e^x = 1 \times 10^9$. This seems to be true at $x \approx 20.723$, so $e^x > 1 \times 10^9$ for $x > 20.723$.



23. $\lim\limits_{x \to \infty} (1.001)^x = \infty$ by (3), since $1.001 > 1$.

24. By (3), if $b > 1$, $\lim\limits_{x \to -\infty} b^x = 0$, so $\lim\limits_{x \to -\infty} (1.001)^x = 0$.

25. Divide numerator and denominator by e^{3x}: $\lim\limits_{x \to \infty} \dfrac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}} = \lim\limits_{x \to \infty} \dfrac{1 - e^{-6x}}{1 + e^{-6x}} = \dfrac{1 - 0}{1 + 0} = 1$

26. Let $t = -x^2$. As $x \to \infty$, $t \to -\infty$. So $\lim\limits_{x \to \infty} e^{-x^2} = \lim\limits_{t \to -\infty} e^t = 0$ by (11).

27. Let $t = 3/(2 - x)$. As $x \to 2^+$, $t \to -\infty$. So $\lim\limits_{x \to 2^+} e^{3/(2-x)} = \lim\limits_{t \to -\infty} e^t = 0$ by (11).

28. Let $t = 3/(2 - x)$. As $x \to 2^-$, $t \to \infty$. So $\lim\limits_{x \to 2^-} e^{3/(2-x)} = \lim\limits_{t \to \infty} e^t = \infty$ by (11).

29. Since $-1 \leq \cos x \leq 1$ and $e^{-2x} > 0$, we have $-e^{-2x} \leq e^{-2x}\cos x \leq e^{-2x}$. We know that $\lim\limits_{x\to\infty}(-e^{-2x}) = 0$ and $\lim\limits_{x\to\infty}\left(e^{-2x}\right) = 0$, so by the Squeeze Theorem, $\lim\limits_{x\to\infty}(e^{-2x}\cos x) = 0$.

30. $\lim\limits_{x\to(\pi/2)^+} e^{\sec x} = 0$ since $\sec x \to -\infty$ as $x \to (\pi/2)^+$.

31. $f(t) = -2e^t \quad\Rightarrow\quad f'(t) = -2(e^t) = -2e^t$

32. $k(r) = e^r + r^e \quad\Rightarrow\quad k'(r) = e^r + er^{e-1}$

33. $f(x) = (3x^2 - 5x)e^x \quad\overset{\text{PR}}{\Rightarrow}$

$$\begin{aligned} f'(x) &= (3x^2 - 5x)(e^x)' + e^x(3x^2 - 5x)' = (3x^2 - 5x)e^x + e^x(6x - 5) \\ &= e^x[(3x^2 - 5x) + (6x - 5)] = e^x(3x^2 + x - 5) \end{aligned}$$

34. By the Quotient Rule, $y = \dfrac{e^x}{1 - e^x} \quad\Rightarrow\quad y' = \dfrac{(1 - e^x)e^x - e^x(-e^x)}{(1 - e^x)^2} = \dfrac{e^x - e^{2x} + e^{2x}}{(1 - e^x)^2} = \dfrac{e^x}{(1 - e^x)^2}$.

35. By (9), $y = e^{ax^3} \quad\Rightarrow\quad y' = e^{ax^3}\dfrac{d}{dx}(ax^3) = 3ax^2e^{ax^3}$.

36. $g(x) = e^{x^2 - x} \quad\overset{\text{CR}}{\Rightarrow}\quad g'(x) = e^{x^2 - x} \cdot \dfrac{d}{dx}(x^2 - x) = e^{x^2 - x}(2x - 1)$

37. $y = e^{\tan\theta} \quad\Rightarrow\quad y' = e^{\tan\theta}\dfrac{d}{d\theta}(\tan\theta) = (\sec^2\theta)e^{\tan\theta}$

38. Let $u = g(x) = e^x + 1$ and $y = f(u) = \sqrt[3]{u} = u^{1/3}$. Then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left(\frac{1}{3}u^{-2/3}\right)(e^x) = \left(\frac{1}{3\sqrt[3]{(e^x + 1)^2}}\right)(e^x) = \frac{e^x}{3\sqrt[3]{(e^x + 1)^2}}.$$

39. $f(x) = \dfrac{x^2e^x}{x^2 + e^x} \quad\overset{\text{QR}}{\Rightarrow}$

$$\begin{aligned} f'(x) &= \frac{(x^2 + e^x)\left[x^2e^x + e^x(2x)\right] - x^2e^x(2x + e^x)}{(x^2 + e^x)^2} = \frac{x^4e^x + 2x^3e^x + x^2e^{2x} + 2xe^{2x} - 2x^3e^x - x^2e^{2x}}{(x^2 + e^x)^2} \\ &= \frac{x^4e^x + 2xe^{2x}}{(x^2 + e^x)^2} = \frac{xe^x(x^3 + 2e^x)}{(x^2 + e^x)^2} \end{aligned}$$

40. $A(r) = \sqrt{r} \cdot e^{r^2 + 1} \Rightarrow$

$$\begin{aligned} A'(r) &= \sqrt{r} \cdot e^{r^2+1} \cdot \frac{d}{dr}\left(r^2 + 1\right) + e^{r^2+1} \cdot \frac{d}{dr}\left(\sqrt{r}\right) = \sqrt{r} \cdot e^{r^2+1} \cdot 2r + e^{r^2+1} \cdot \frac{1}{2\sqrt{r}} \\ &= e^{r^2+1}\left(2r\sqrt{r} + \frac{1}{2\sqrt{r}}\right) \quad\text{or}\quad e^{r^2+1}\left(\frac{4r^2 + 1}{2\sqrt{r}}\right) \end{aligned}$$

41. Using the Product Rule and the Chain Rule, $y = x^2e^{-3x} \quad\Rightarrow$

$y' = x^2e^{-3x}(-3) + e^{-3x}(2x) = e^{-3x}(-3x^2 + 2x) = xe^{-3x}(2 - 3x)$.

42. $f(t) = \tan(1 + e^{2t}) \quad\Rightarrow\quad f'(t) = \sec^2(1 + e^{2t}) \cdot (1 + e^{2t})' = 2e^{2t}\sec^2(1 + e^{2t})$

43. $f(t) = e^{at}\sin bt \quad\Rightarrow\quad f'(t) = e^{at}(\cos bt)\cdot b + (\sin bt)e^{at}\cdot a = e^{at}(b\cos bt + a\sin bt)$

44. $f(z) = e^{z/(z-1)} \quad\Rightarrow\quad f'(z) = e^{z/(z-1)}\dfrac{d}{dz}\dfrac{z}{z-1} = e^{z/(z-1)}\dfrac{(z-1)(1)-z(1)}{(z-1)^2} = -\dfrac{e^{z/(z-1)}}{(z-1)^2}$

45. By (9), $F(t) = e^{t\sin 2t} \quad\Rightarrow$

$F'(t) = e^{t\sin 2t}(t\sin 2t)' = e^{t\sin 2t}(t\cdot 2\cos 2t + \sin 2t\cdot 1) = e^{t\sin 2t}(2t\cos 2t + \sin 2t)$

46. $y = e^{\sin 2x} + \sin(e^{2x}) \quad\Rightarrow$

$y' = e^{\sin 2x}\dfrac{d}{dx}\sin 2x + \cos(e^{2x})\dfrac{d}{dx}e^{2x} = e^{\sin 2x}(\cos 2x)\cdot 2 + \cos(e^{2x})\,e^{2x}\cdot 2 = 2\cos 2x\,e^{\sin 2x} + 2e^{2x}\cos(e^{2x})$

47. $g(u) = e^{\sqrt{\sec u^2}} \quad\Rightarrow$

$$g'(u) = e^{\sqrt{\sec u^2}}\,\frac{d}{du}\sqrt{\sec u^2} = e^{\sqrt{\sec u^2}}\,\frac{1}{2}(\sec u^2)^{-1/2}\frac{d}{du}\sec u^2$$

$$= e^{\sqrt{\sec u^2}}\,\frac{1}{2\sqrt{\sec u^2}}\cdot\sec u^2\,\tan u^2\cdot 2u = u\sqrt{\sec u^2}\,\tan u^2\,e^{\sqrt{\sec u^2}}$$

48. $f(t) = e^{1/t}\sqrt{t^2-1} \quad\Rightarrow$

$$f'(t) = e^{1/t}\cdot\frac{1}{2\sqrt{t^2-1}}\cdot 2t + \sqrt{t^2-1}\cdot e^{1/t}\cdot\left(-\frac{1}{t^2}\right) \qquad \left[\frac{1}{t} = t^{-1};\ \frac{d}{dt}\left(t^{-1}\right) = -t^{-2} = -\frac{1}{t^2}\right]$$

$$= e^{1/t}\left(\frac{t}{\sqrt{t^2-1}} - \frac{\sqrt{t^2-1}}{t^2}\right) \quad\text{or}\quad e^{1/t}\left(\frac{t^3-t^2+1}{t^2\sqrt{t^2-1}}\right)$$

49. $g(x) = \sin\left(\dfrac{e^x}{1+e^x}\right) \quad\Rightarrow$

$$g'(x) = \cos\left(\frac{e^x}{1+e^x}\right)\cdot\frac{(1+e^x)e^x - e^x(e^x)}{(1+e^x)^2} = \cos\left(\frac{e^x}{1+e^x}\right)\cdot\frac{e^x(1+e^x-e^x)}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2}\cos\left(\frac{e^x}{1+e^x}\right)$$

50. $f(x) = e^{\sin^2(x^2)} \quad\Rightarrow\quad f'(x) = e^{\sin^2(x^2)}\cdot 2\sin(x^2)\cdot\cos(x^2)\cdot 2x = 4x\sin(x^2)\,\cos(x^2)e^{\sin^2(x^2)}$

51. $y = e^x\cos x + \sin x \quad\Rightarrow\quad y' = e^x(-\sin x) + (\cos x)(e^x) + \cos x = e^x(\cos x - \sin x) + \cos x$, so $y'(0) = e^0(\cos 0 - \sin 0) + \cos 0 = 1(1-0) + 1 = 2$. An equation of the tangent line to the curve $y = e^x\cos x + \sin x$ at the point $(0,1)$ is $y - 1 = 2(x-0)$ or $y = 2x+1$.

52. $y = \dfrac{1+x}{1+e^x} \quad\Rightarrow\quad y' = \dfrac{(1+e^x)(1)-(1+x)e^x}{(1+e^x)^2} = \dfrac{1+e^x-e^x-xe^x}{(1+e^x)^2} = \dfrac{1-xe^x}{(1+e^x)^2}$

At $\left(0,\frac{1}{2}\right)$, $y' = \dfrac{1}{(1+1)^2} = \dfrac{1}{4}$, and an equation of the tangent line is $y - \frac{1}{2} = \frac{1}{4}(x-0)$ or $y = \frac{1}{4}x + \frac{1}{2}$.

53. $\dfrac{d}{dx}(e^{x/y}) = \dfrac{d}{dx}(x-y) \quad\Rightarrow\quad e^{x/y}\cdot\dfrac{d}{dx}\left(\dfrac{x}{y}\right) = 1 - y' \quad\Rightarrow\quad e^{x/y}\cdot\dfrac{y\cdot 1 - x\cdot y'}{y^2} = 1 - y' \quad\Rightarrow$

$$e^{x/y}\cdot\frac{1}{y} - \frac{xe^{x/y}}{y^2}\cdot y' = 1 - y' \quad\Rightarrow\quad y' - \frac{xe^{x/y}}{y^2}\cdot y' = 1 - \frac{e^{x/y}}{y} \quad\Rightarrow\quad y'\left(1 - \frac{xe^{x/y}}{y^2}\right) = \frac{y - e^{x/y}}{y} \quad\Rightarrow$$

$$y' = \frac{\dfrac{y-e^{x/y}}{y}}{\dfrac{y^2 - xe^{x/y}}{y^2}} = \frac{y(y-e^{x/y})}{y^2 - xe^{x/y}}$$

54. $xe^y + ye^x = 1 \quad \Rightarrow \quad xe^y y' + e^y \cdot 1 + ye^x + e^x y' = 0 \quad \Rightarrow \quad y'(xe^y + e^x) = -e^y - ye^x \quad \Rightarrow \quad y' = -\dfrac{e^y + ye^x}{xe^y + e^x}$. At $(0, 1)$, $y' = -\dfrac{e + 1 \cdot 1}{0 + 1} = -(e + 1)$, so an equation for the tangent line is $y - 1 = -(e + 1)(x - 0)$, or $y = -(e + 1)x + 1$.

55. $y = e^x + e^{-x/2} \quad \Rightarrow \quad y' = e^x - \frac{1}{2}e^{-x/2} \quad \Rightarrow \quad y'' = e^x + \frac{1}{4}e^{-x/2}$, so

$2y'' - y' - y = 2\left(e^x + \frac{1}{4}e^{-x/2}\right) - \left(e^x - \frac{1}{2}e^{-x/2}\right) - \left(e^x + e^{-x/2}\right) = 0.$

56. $y = Ae^{-x} + Bxe^{-x} \quad \Rightarrow \quad y' = -Ae^{-x} + Be^{-x} - Bxe^{-x} = (B - A)e^{-x} - Bxe^{-x} \quad \Rightarrow$

$y'' = (A - B)e^{-x} - Be^{-x} + Bxe^{-x} = (A - 2B)e^{-x} + Bxe^{-x}$,

so $y'' + 2y' + y = (A - 2B)e^{-x} + Bxe^{-x} + 2\left[(B - A)e^{-x} - Bxe^{-x}\right] + Ae^{-x} + Bxe^{-x} = 0.$

57. $y = e^{rx} \quad \Rightarrow \quad y' = re^{rx} \quad \Rightarrow \quad y'' = r^2e^{rx}$, so if $y = e^{rx}$ satisfies the differential equation $y'' + 6y' + 8y = 0$, then $r^2e^{rx} + 6re^{rx} + 8e^{rx} = 0$; that is, $e^{rx}(r^2 + 6r + 8) = 0$. Since $e^{rx} > 0$ for all x, we must have $r^2 + 6r + 8 = 0$, or $(r + 2)(r + 4) = 0$, so $r = -2$ or -4.

58. $y = e^{\lambda x} \quad \Rightarrow \quad y' = \lambda e^{\lambda x} \quad \Rightarrow \quad y'' = \lambda^2 e^{\lambda x}$. Thus, $y + y' = y'' \quad \Leftrightarrow \quad e^{\lambda x} + \lambda e^{\lambda x} = \lambda^2 e^{\lambda x} \quad \Leftrightarrow$

$e^{\lambda x}(\lambda^2 - \lambda - 1) = 0 \quad \Leftrightarrow \quad \lambda = \frac{1 \pm \sqrt{5}}{2}$, since $e^{\lambda x} \neq 0$.

59. $f(x) = e^{2x} \quad \Rightarrow \quad f'(x) = 2e^{2x} \quad \Rightarrow \quad f''(x) = 2 \cdot 2e^{2x} = 2^2e^{2x} \quad \Rightarrow$

$f'''(x) = 2^2 \cdot 2e^{2x} = 2^3e^{2x} \quad \Rightarrow \quad \cdots \quad \Rightarrow \quad f^{(n)}(x) = 2^n e^{2x}$

60. $f(x) = xe^{-x} \quad \Rightarrow \quad f'(x) = x(-e^{-x}) + e^{-x} = (1 - x)e^{-x} \quad \Rightarrow$

$f''(x) = (1 - x)(-e^{-x}) + e^{-x}(-1) = (x - 2)e^{-x} \quad \Rightarrow \quad f'''(x) = (x - 2)(-e^{-x}) + e^{-x} = (3 - x)e^{-x} \quad \Rightarrow$

$f^{(4)}(x) = (3 - x)(-e^{-x}) + e^{-x}(-1) = (x - 4)e^{-x} \quad \Rightarrow \quad \cdots \quad \Rightarrow \quad f^{(n)}(x) = (-1)^n(x - n)e^{-x}.$

So $D^{1000}xe^{-x} = (x - 1000)e^{-x}$.

61. (a) $f(x) = e^x + x$ is continuous on $\mathbb{R}$ and $f(-1) = e^{-1} - 1 < 0 < 1 = f(0)$, so by the Intermediate Value Theorem, $e^x + x = 0$ has a solution in $(-1, 0)$.

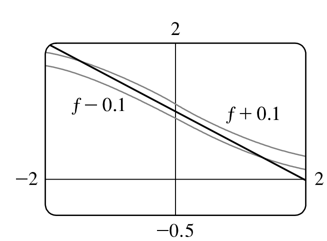
(b) $f(x) = e^x + x \quad \Rightarrow \quad f'(x) = e^x + 1$, so $x_{n+1} = x_n - \dfrac{e^{x_n} + x_n}{e^{x_n} + 1}$. Using $x_1 = -0.5$, we get $x_2 \approx -0.566311$, $x_3 \approx -0.567143 \approx x_4$, so the solution is -0.567143 to six decimal places.

62. $f(x) = \dfrac{2}{1 + e^x} \quad \Rightarrow \quad f'(x) = -\dfrac{2e^x}{(1 + e^x)^2}$, so $f(0) = 1$ and $f'(0) = -\dfrac{1}{2}$.

Thus, $f(x) \approx f(0) + f'(0)(x - 0) = 1 - \frac{1}{2}x$. We need

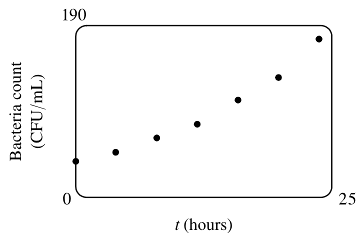
$\dfrac{2}{1 + e^x} - 0.1 < 1 - \dfrac{1}{2}x < \dfrac{2}{1 + e^x} + 0.1$, which is true when

$-1.423 < x < 1.423$. Note that to ensure the accuracy, we have rounded the smaller value up and the larger value down.



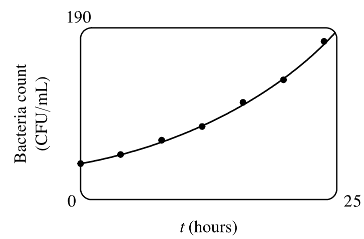
63. Half of 76.0 RNA copies per mL, corresponding to $t = 1$, is 38.0 RNA copies per mL. Using the graph of V in Figure 11, we estimate that it takes about 3.5 additional days for the patient's viral load to decrease to 38 RNA copies per mL.

64. (a)



(b) Using a graphing calculator, we obtain the exponential curve $f(t) = 36.89301(1.06614)^t$.

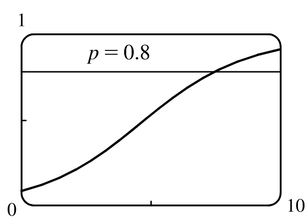
(c) Using the TRACE and zooming in, we find that the bacteria count doubles from 37 to 74 in about 10.87 hours.



65. (a) $\lim\limits_{t\to\infty} p(t) = \lim\limits_{t\to\infty} \dfrac{1}{1 + ae^{-kt}} = \dfrac{1}{1 + a \cdot 0} = 1$, since $k > 0 \quad\Rightarrow\quad -kt \to -\infty \quad\Rightarrow\quad e^{-kt} \to 0$. As time increases, the proportion of the population that has heard the rumor approaches 1; that is, everyone in the population has heard the rumor.

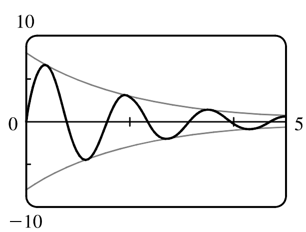
(b) $p(t) = (1 + ae^{-kt})^{-1} \quad\Rightarrow\quad \dfrac{dp}{dt} = -(1 + ae^{-kt})^{-2}(-kae^{-kt}) = \dfrac{kae^{-kt}}{(1 + ae^{-kt})^2}$

(c)



From the graph of $p(t) = (1 + 10e^{-0.5t})^{-1}$, it seems that $p(t) = 0.8$ (indicating that 80% of the population has heard the rumor) when $t \approx 7.4$ hours.

66. (a)



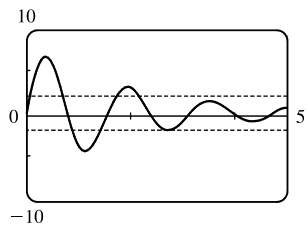
The displacement function is squeezed between the other two functions. This is because $-1 \le \sin 4t \le 1 \quad\Rightarrow\quad -8e^{-t/2} \le 8e^{-t/2} \sin 4t \le 8e^{-t/2}$.

(b) The maximum value of the displacement is about 6.6 cm, occurring at $t \approx 0.36$ s. It occurs just before the graph of the displacement function touches the graph of $8e^{-t/2}$ (when $t = \frac{\pi}{8} \approx 0.39$).

(c) The velocity of the object is the derivative of its displacement function, that is,

$$\frac{d}{dt}\left(8e^{-t/2}\sin 4t\right) = 8\left[e^{-t/2}\cos 4t(4) + \sin 4t\left(-\tfrac{1}{2}\right)e^{-t/2}\right]$$

If the displacement is zero, then we must have $\sin 4t = 0$ (since the exponential term in the displacement function is always positive). The first time that $\sin 4t = 0$ after $t = 0$ occurs at $t = \frac{\pi}{4}$. Substituting this into our expression for the velocity, and noting that the second term vanishes, we get $v\left(\frac{\pi}{4}\right) = 8e^{-\pi/8}\cos\left(4 \cdot \frac{\pi}{4}\right) \cdot 4 = -32e^{-\pi/8} \approx -21.6$ cm/s.

(d) 

The graph indicates that the displacement is less than 2 cm from equilibrium whenever t is larger than about 2.8.

67. $f(x) = \dfrac{e^x}{1+x^2}$, $[0, 3]$. $f'(x) = \dfrac{(1+x^2)e^x - e^x(2x)}{(1+x^2)^2} = \dfrac{e^x(x^2-2x+1)}{(1+x^2)^2} = \dfrac{e^x(x-1)^2}{(1+x^2)^2}$. $f'(x) = 0 \;\Rightarrow\; (x-1)^2 = 0 \;\Leftrightarrow\; x = 1$. $f'(x)$ exists for all real numbers since $1 + x^2$ is never equal to 0. $f(0) = 1$, $f(1) = e/2 \approx 1.359$, and $f(3) = e^3/10 \approx 2.009$. So $f(3) = e^3/10$ is the absolute maximum value and $f(0) = 1$ is the absolute minimum value.

68. $f(x) = xe^{x/2}$, $[-3, 1]$. $f'(x) = xe^{x/2}\left(\frac{1}{2}\right) + e^{x/2}(1) = e^{x/2}\left(\frac{1}{2}x + 1\right)$. $f'(x) = 0 \;\Leftrightarrow\; \frac{1}{2}x + 1 = 0 \;\Leftrightarrow\; x = -2$. $f(-3) = -3e^{-3/2} \approx -0.669$, $f(-2) = -2e^{-1} \approx -0.736$, and $f(1) = e^{1/2} \approx 1.649$. So $f(1) = e^{1/2}$ is the absolute maximum value and $f(-2) = -2/e$ is the absolute minimum value.

69. $f(x) = x - e^x \;\Rightarrow\; f'(x) = 1 - e^x = 0 \;\Leftrightarrow\; e^x = 1 \;\Leftrightarrow\; x = 0$. Now $f'(x) > 0$ for all $x < 0$ and $f'(x) < 0$ for all $x > 0$, so the absolute maximum value is $f(0) = 0 - 1 = -1$.

70. $g(x) = \dfrac{e^x}{x} \;\Rightarrow\; g'(x) = \dfrac{xe^x - e^x}{x^2} = 0 \;\Leftrightarrow\; e^x(x-1) = 0 \;\Rightarrow\; x = 1$. Now $g'(x) > 0 \;\Leftrightarrow\; \dfrac{xe^x - e^x}{x^2} > 0 \;\Leftrightarrow\; x - 1 > 0 \;\Leftrightarrow\; x > 1$ and $g'(x) < 0 \;\Leftrightarrow\; \dfrac{xe^x - e^x}{x^2} < 0 \;\Leftrightarrow\; x - 1 < 0 \;\Leftrightarrow\; x < 1$. Thus there is an absolute minimum value of $g(1) = e$ at $x = 1$.

71. (a) $f(x) = xe^{2x} \;\Rightarrow\; f'(x) = x(2e^{2x}) + e^{2x}(1) = e^{2x}(2x+1)$. Thus, $f'(x) > 0$ if $x > -\frac{1}{2}$ and $f'(x) < 0$ if $x < -\frac{1}{2}$. So f is increasing on $\left(-\frac{1}{2}, \infty\right)$ and f is decreasing on $\left(-\infty, -\frac{1}{2}\right)$.

(b) $f''(x) = e^{2x}(2) + (2x+1) \cdot 2e^{2x} = 2e^{2x}[1 + (2x+1)] = 2e^{2x}(2x+2) = 4e^{2x}(x+1)$. $f''(x) > 0 \;\Leftrightarrow\; x > -1$ and $f''(x) < 0 \;\Leftrightarrow\; x < -1$. Thus, f is concave upward on $(-1, \infty)$ and f is concave downward on $(-\infty, -1)$.

(c) There is an inflection point at $\left(-1, -e^{-2}\right)$, or $\left(-1, -1/e^2\right)$.

72. (a) $f(x) = \dfrac{e^x}{x^2} \;\Rightarrow\; f'(x) = \dfrac{x^2e^x - e^x(2x)}{(x^2)^2} = \dfrac{xe^x(x-2)}{x^4} = \dfrac{e^x(x-2)}{x^3}$. $f'(x) > 0 \;\Leftrightarrow\; x < 0$ or $x > 2$, so f is increasing on $(-\infty, 0)$ and $(2, \infty)$. $f'(x) < 0 \;\Leftrightarrow\; 0 < x < 2$, so f is decreasing on $(0, 2)$.

(b) $f''(x) = \dfrac{x^3[e^x \cdot 1 + (x-2)e^x] - e^x(x-2) \cdot 3x^2}{(x^3)^2} = \dfrac{x^2e^x[x(x-1) - 3(x-2)]}{x^6} = \dfrac{e^x(x^2-4x+6)}{x^4}$.

$x^2 - 4x + 6 = (x^2 - 4x + 4) + 2 = (x-2)^2 + 2 > 0$, so $f''(x) > 0$ and f is CU on $(-\infty, 0)$ and $(0, \infty)$.

(c) There are no changes in concavity and, hence, there are no points of inflection.

73. $y = f(x) = e^{-1/(x+1)}$ **A.** $D = \{x \mid x \neq -1\} = (-\infty, -1) \cup (-1, \infty)$ **B.** No x-intercept; y-intercept $= f(0) = e^{-1}$

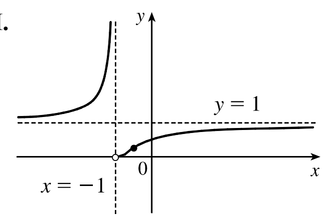
C. No symmetry **D.** $\lim\limits_{x \to \pm\infty} e^{-1/(x+1)} = 1$ since $-1/(x+1) \to 0$, so $y = 1$ is a HA. $\lim\limits_{x \to -1^+} e^{-1/(x+1)} = 0$ since $-1/(x+1) \to -\infty$, $\lim\limits_{x \to -1^-} e^{-1/(x+1)} = \infty$ since $-1/(x+1) \to \infty$, so $x = -1$ is a VA.

E. $f'(x) = e^{-1/(x+1)}/(x+1)^2 \quad \Rightarrow \quad f'(x) > 0$ for all x except 1, so f is increasing on $(-\infty, -1)$ and $(-1, \infty)$. **F.** No extreme values

G. $f''(x) = \dfrac{e^{-1/(x+1)}}{(x+1)^4} + \dfrac{e^{-1/(x+1)}(-2)}{(x+1)^3} = -\dfrac{e^{-1/(x+1)}(2x+1)}{(x+1)^4} \quad \Rightarrow$

$f''(x) > 0 \quad \Leftrightarrow \quad 2x + 1 < 0 \quad \Leftrightarrow \quad x < -\frac{1}{2}$, so f is CU on $(-\infty, -1)$ and $\left(-1, -\frac{1}{2}\right)$, and CD on $\left(-\frac{1}{2}, \infty\right)$. f has an IP at $\left(-\frac{1}{2}, e^{-2}\right)$.

H.



74. $y = f(x) = e^{-x} \sin x,\ 0 \leq x \leq 2\pi$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \quad \Leftrightarrow \quad \sin x = 0 \quad \Leftrightarrow \quad x = 0, \pi$, and 2π. **C.** No symmetry **D.** No asymptote **E.** $f'(x) = e^{-x} \cos x + \sin x \,(-e^{-x}) = e^{-x}(\cos x - \sin x)$.

$f'(x) = 0 \quad \Leftrightarrow \quad \cos x = \sin x \quad \Leftrightarrow \quad x = \frac{\pi}{4}, \frac{5\pi}{4}$. $f'(x) > 0$ if x is in $\left(0, \frac{\pi}{4}\right)$ or $\left(\frac{5\pi}{4}, 2\pi\right)$ [f is increasing] and $f'(x) < 0$ if x is in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ [f is decreasing]. **F.** Local maximum value $f\left(\frac{\pi}{4}\right)$ and local minimum value $f\left(\frac{5\pi}{4}\right)$

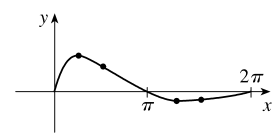
G. $f''(x) = e^{-x}(-\sin x - \cos x) + (\cos x - \sin x)(-e^{-x}) = e^{-x}(-2\cos x)$. $f''(x) > 0 \quad \Leftrightarrow \quad -2\cos x > 0 \quad \Leftrightarrow$

$\cos x < 0 \quad \Rightarrow \quad x$ is in $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ [f is CU] and $f''(x) < 0 \quad \Leftrightarrow$

$\cos x > 0 \quad \Rightarrow \quad x$ is in $\left(0, \frac{\pi}{2}\right)$ or $\left(\frac{3\pi}{2}, 2\pi\right)$ [f is CD].

IP at $\left(\frac{\pi}{2} + n\pi, f\left(\frac{\pi}{2} + n\pi\right)\right)$

H.



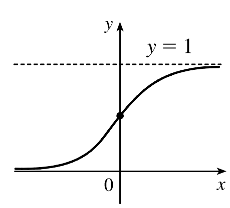
75. $y = 1/(1 + e^{-x})$ **A.** $D = \mathbb{R}$ **B.** No x-intercept; y-intercept $= f(0) = \frac{1}{2}$. **C.** No symmetry

D. $\lim\limits_{x \to \infty} 1/(1 + e^{-x}) = \frac{1}{1 + 0} = 1$ and $\lim\limits_{x \to -\infty} 1/(1 + e^{-x}) = 0$ since $\lim\limits_{x \to -\infty} e^{-x} = \infty$, so f has horizontal asymptotes $y = 0$ and $y = 1$. **E.** $f'(x) = -(1 + e^{-x})^{-2}(-e^{-x}) = e^{-x}/(1 + e^{-x})^2$. This is positive for all x, so f is increasing on $\mathbb{R}$.

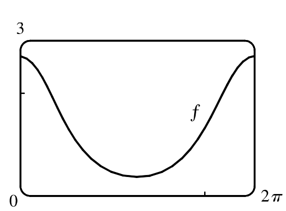
F. No extreme values **G.** $f''(x) = \dfrac{(1 + e^{-x})^2(-e^{-x}) - e^{-x}(2)(1 + e^{-x})(-e^{-x})}{(1 + e^{-x})^4} = \dfrac{e^{-x}(e^{-x} - 1)}{(1 + e^{-x})^3}$

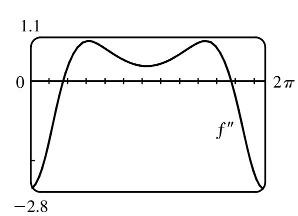
The second factor in the numerator is negative for $x > 0$ and positive for $x < 0$, and the other factors are always positive, so f is CU on $(-\infty, 0)$ and CD on $(0, \infty)$. IP at $\left(0, \frac{1}{2}\right)$

H.



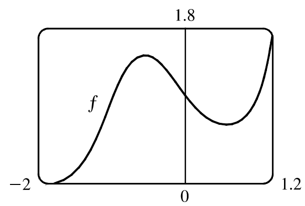
76. The function $f(x) = e^{\cos x}$ is periodic with period 2π, so we consider it only on the interval $[0, 2\pi]$. We see that it has local maxima of about $f(0) \approx 2.72$ and $f(2\pi) \approx 2.72$, and a local minimum of about $f(3.14) \approx 0.37$. To find the

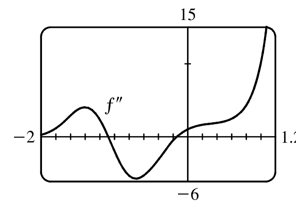




exact values, we calculate $f'(x) = -\sin x e^{\cos x}$. This is 0 when $-\sin x = 0 \iff x = 0, \pi$ or 2π (since we are only considering $x \in [0, 2\pi]$). Also $f'(x) > 0 \iff \sin x < 0 \iff 0 < x < \pi$. So $f(0) = f(2\pi) = e$ (both maxima) and $f(\pi) = e^{\cos \pi} = 1/e$ (minimum). To find the inflection points, we calculate and graph $f''(x) = \dfrac{d}{dx}\left(-\sin x\, e^{\cos x}\right) = -\cos x\, e^{\cos x} - \sin x(e^{\cos x})(-\sin x) = e^{\cos x}\left(\sin^2 x - \cos x\right)$. From the graph of $f''(x)$, we see that f has inflection points at $x \approx 0.90$ and at $x \approx 5.38$. These x-coordinates correspond to inflection points $(0.90, 1.86)$ and $(5.38, 1.86)$.

77. $f(x) = e^{x^3 - x} \to 0$ as $x \to -\infty$, and $f(x) \to \infty$ as $x \to \infty$. From the graph, it appears that f has a local minimum of about $f(0.58) = 0.68$, and a local maximum of about $f(-0.58) = 1.47$.



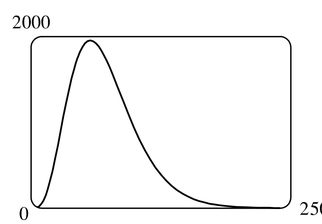


To find the exact values, we calculate $f'(x) = \left(3x^2 - 1\right)e^{x^3 - x}$, which is 0 when $3x^2 - 1 = 0 \iff x = \pm\frac{1}{\sqrt{3}}$. The negative solution corresponds to the local maximum $f\left(-\frac{1}{\sqrt{3}}\right) = e^{(-1/\sqrt{3})^3 - (-1/\sqrt{3})} = e^{2\sqrt{3}/9}$, and the positive solution corresponds to the local minimum $f\left(\frac{1}{\sqrt{3}}\right) = e^{(1/\sqrt{3})^3 - (1/\sqrt{3})} = e^{-2\sqrt{3}/9}$. To estimate the inflection points, we calculate and graph

$$f''(x) = \frac{d}{dx}\left[\left(3x^2 - 1\right)e^{x^3 - x}\right] = \left(3x^2 - 1\right)e^{x^3 - x}\left(3x^2 - 1\right) + e^{x^3 - x}(6x) = e^{x^3 - x}\left(9x^4 - 6x^2 + 6x + 1\right).$$

From the graph, it appears that $f''(x)$ changes sign (and thus f has inflection points) at $x \approx -0.15$ and $x \approx -1.09$. From the graph of f, we see that these x-values correspond to inflection points at about $(-0.15, 1.15)$ and $(-1.09, 0.82)$.

78. $S(t) = At^p e^{-kt}$ with $A = 0.01$, $p = 4$, and $k = 0.07$. We will find the zeros of f'' for $f(t) = t^p e^{-kt}$.



$$f'(t) = t^p(-ke^{-kt}) + e^{-kt}(pt^{p-1}) = e^{-kt}(-kt^p + pt^{p-1})$$
$$\begin{aligned} f''(t) &= e^{-kt}(-kpt^{p-1} + p(p-1)t^{p-2}) + (-kt^p + pt^{p-1})(-ke^{-kt}) \\ &= t^{p-2}e^{-kt}[-kpt + p(p-1) + k^2t^2 - kpt] \\ &= t^{p-2}e^{-kt}(k^2t^2 - 2kpt + p^2 - p) \end{aligned}$$

Using the given values of p and k gives us $f''(t) = t^2e^{-0.07t}(0.0049t^2 - 0.56t + 12)$. So $S''(t) = 0.01f''(t)$ and its zeros are $t = 0$ and the solutions of $0.0049t^2 - 0.56t + 12 = 0$, which are $t_1 = \frac{200}{7} \approx 28.57$ and $t_2 = \frac{600}{7} \approx 85.71$.
At t_1 minutes, the rate of increase of the level of medication in the bloodstream is at its greatest and at t_2 minutes, the rate of decrease is the greatest.

79. Let $a = 0.135$ and $b = -2.802$. Then $C(t) = ate^{bt} \Rightarrow C'(t) = a(t \cdot e^{bt} \cdot b + e^{bt} \cdot 1) = ae^{bt}(bt + 1)$. $C'(t) = 0 \iff bt + 1 = 0 \iff t = -\dfrac{1}{b} \approx 0.36$ h. $C(0) = 0$, $C(-1/b) = -\dfrac{a}{b}e^{-1} = -\dfrac{a}{be} \approx 0.0177$, and $C(3) = 3ae^{3b} \approx 0.00009$.

[continued]

The maximum average BAC during the first three hours is about 0.0177 g/dL and it occurs at approximately 0.36 h (21.4 min).

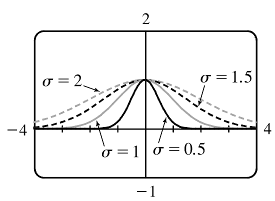
80. (a) As $|x| \to \infty$, $t = -x^2/(2\sigma^2) \to -\infty$, and $e^t \to 0$. The HA is $y = 0$. Since t takes on its maximum value at $x = 0$, so does e^t. Showing this result using derivatives, we have $f(x) = e^{-x^2/(2\sigma^2)} \Rightarrow f'(x) = e^{-x^2/(2\sigma^2)}(-x/\sigma^2)$. $f'(x) = 0 \Leftrightarrow x = 0$. Because f' changes from positive to negative at $x = 0$, $f(0) = 1$ is a local maximum. For inflection points, we find $f''(x) = -\dfrac{1}{\sigma^2}\left[e^{-x^2/(2\sigma^2)} \cdot 1 + xe^{-x^2/(2\sigma^2)}(-x/\sigma^2)\right] = \dfrac{-1}{\sigma^2}e^{-x^2/(2\sigma^2)}(1 - x^2/\sigma^2)$.

$f''(x) = 0 \Leftrightarrow x^2 = \sigma^2 \Leftrightarrow x = \pm\sigma$. $f''(x) < 0 \Leftrightarrow x^2 < \sigma^2 \Leftrightarrow -\sigma < x < \sigma$.

So f is CD on $(-\sigma, \sigma)$ and CU on $(-\infty, -\sigma)$ and (σ, ∞). IP at $(\pm\sigma, e^{-1/2})$.

(b) Since we have IP at $x = \pm\sigma$, the inflection points move away from the y-axis as σ increases.

(c)



From the graph, we see that as σ increases, the graph tends to spread out and there is more area between the curve and the x-axis.

81. $\displaystyle\int_0^1 (x^e + e^x)\,dx = \left[\frac{x^{e+1}}{e+1} + e^x\right]_0^1 = \left(\frac{1}{e+1} + e\right) - (0 + 1) = \frac{1}{e+1} + e - 1$

82. $\int_{-5}^{5} e\,dx = \Big[ex\Big]_{-5}^{5} = 5e - (-5e) = 10e$

83. $\displaystyle\int_0^2 \frac{dx}{e^{\pi x}} = \int_0^2 e^{-\pi x}\,dx = \left[-\frac{1}{\pi}e^{-\pi x}\right]_0^2 = -\frac{1}{\pi}e^{-2\pi} + \frac{1}{\pi}e^0 = \frac{1}{\pi}(1 - e^{-2\pi})$

84. Let $u = -t^4$. Then $du = -4t^3\,dt$ and $t^3\,dt = -\frac14\,du$, so $\int t^3 e^{-t^4}\,dt = \int e^u\left(-\frac14\,du\right) = -\frac14 e^u + C = -\frac14 e^{-t^4} + C$.

85. Let $u = 1 + e^x$. Then $du = e^x\,dx$, so $\int e^x\sqrt{1 + e^x}\,dx = \int \sqrt{u}\,du = \frac23 u^{3/2} + C = \frac23(1 + e^x)^{3/2} + C$.

86. $\displaystyle\int \frac{(1 + e^x)^2}{e^x}\,dx = \int \frac{1 + 2e^x + e^{2x}}{e^x}\,dx = \int (e^{-x} + 2 + e^x)\,dx = -e^{-x} + 2x + e^x + C$

87. $\int (e^x + e^{-x})^2\,dx = \int (e^{2x} + 2 + e^{-2x})\,dx = \frac12 e^{2x} + 2x - \frac12 e^{-2x} + C$

88. $\int e^x(4 + e^x)^5\,dx \begin{bmatrix} u = 4 + e^x, \\ du = e^x\,dx \end{bmatrix} = \int u^5\,du = \frac16 u^6 + C = \frac16(4 + e^x)^6 + C$

89. Let $x = 1 - e^u$. Then $dx = -e^u\,du$ and $e^u\,du = -dx$, so

$$\int \frac{e^u}{(1 - e^u)^2}\,du = \int \frac{1}{x^2}(-dx) = -\int x^{-2}\,dx = -(-x^{-1}) + C = \frac{1}{x} + C = \frac{1}{1 - e^u} + C.$$

90. Let $u = \sin\theta$. Then $du = \cos\theta\,d\theta$, so $\int e^{\sin\theta}\cos\theta\,d\theta = \int e^u\,du = e^u + C = e^{\sin\theta} + C$.

91. Let $u = 1/x$, so $du = -1/x^2\,dx$. When $x = 1$, $u = 1$; when $x = 2$, $u = \frac{1}{2}$. Thus,

$$\int_1^2 \frac{e^{1/x}}{x^2}\,dx = \int_1^{1/2} e^u\,(-du) = -\left[e^u\right]_1^{1/2} = -(e^{1/2} - e) = e - \sqrt{e}.$$

92. $$\int_0^1 \frac{\sqrt{1+e^{-x}}}{e^x}\,dx = \int_0^1 \sqrt{1+e^{-x}}\,e^{-x}\,dx = \int_2^{1+1/e} u^{1/2}\,(-du) \qquad \begin{bmatrix} u = 1+e^{-x}, \\ du = -e^{-x}\,dx \end{bmatrix}$$

$$= \left[-\tfrac{2}{3}u^{3/2}\right]_2^{1+1/e} = -\tfrac{2}{3}[(1+1/e)^{3/2} - 2^{3/2}] = \tfrac{4}{3}\sqrt{2} - \tfrac{2}{3}(1+1/e)^{3/2}$$

93. $f_{\text{avg}} = \frac{1}{2-0}\int_0^2 2xe^{-x^2}\,dx$

$= \frac{1}{2}\left[-e^{-x^2}\right]_0^2 = \frac{1}{2}(-e^{-4}+1)$

94. Area $= \int_{-1}^{1}[e^y - (y^2-2)]\,dy = \int_{-1}^{1}(e^y - y^2 + 2)\,dy = \left[e^y - \frac{1}{3}y^3 + 2y\right]_{-1}^{1}$

$= \left(e - \frac{1}{3} + 2\right) - \left(e^{-1} + \frac{1}{3} - 2\right) = e - e^{-1} - \frac{2}{3} + 4 = e - e^{-1} + \frac{10}{3}$

95. Area $= \int_0^1 \left(e^{3x} - e^x\right)dx = \left[\frac{1}{3}e^{3x} - e^x\right]_0^1 = \left(\frac{1}{3}e^3 - e\right) - \left(\frac{1}{3} - 1\right) = \frac{1}{3}e^3 - e + \frac{2}{3} \approx 4.644$

96. $f''(x) = 3e^x + 5\sin x \;\Rightarrow\; f'(x) = 3e^x - 5\cos x + C \;\Rightarrow\; 2 = f'(0) = 3 - 5 + C \;\Rightarrow\; C = 4$, so

$f'(x) = 3e^x - 5\cos x + 4 \;\Rightarrow\; f(x) = 3e^x - 5\sin x + 4x + D \;\Rightarrow\; 1 = f(0) = 3 + D \;\Rightarrow\; D = -2$,

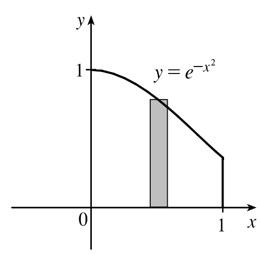
so $f(x) = 3e^x - 5\sin x + 4x - 2$.

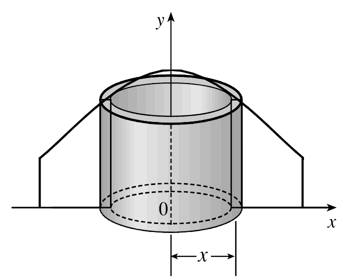
97. $V = \int_0^1 \pi(e^x)^2\,dx = \pi\int_0^1 e^{2x}\,dx = \frac{1}{2}\pi\left[e^{2x}\right]_0^1 = \frac{\pi}{2}\left(e^2 - 1\right)$

98. The shell has radius x, circumference $2\pi x$, and height e^{-x^2}, so $V = \displaystyle\int_0^1 2\pi x e^{-x^2}\,dx$.

Let $u = x^2$. Thus, $du = 2x\,dx$, so

$$V = \pi\int_0^1 e^{-u}\,du = \pi\left[-e^{-u}\right]_0^1 = \pi(1 - 1/e).$$





99. First Figure Let $u = \sqrt{x}$, so $x = u^2$ and $dx = 2u\,du$. When $x = 0$, $u = 0$; when $x = 1$, $u = 1$. Thus,

$A_1 = \int_0^1 e^{\sqrt{x}}\,dx = \int_0^1 e^u(2u\,du) = 2\int_0^1 ue^u\,du$.

Second Figure $A_2 = \int_0^1 2xe^x\,dx = 2\int_0^1 ue^u\,du$.

Third Figure Let $u = \sin x$, so $du = \cos x\,dx$. When $x = 0$, $u = 0$; when $x = \frac{\pi}{2}$, $u = 1$. Thus,

$A_3 = \int_0^{\pi/2} e^{\sin x}\sin 2x\,dx = \int_0^{\pi/2} e^{\sin x}(2\sin x\,\cos x)\,dx = \int_0^1 e^u(2u\,du) = 2\int_0^1 ue^u\,du$.

Since $A_1 = A_2 = A_3$, all three areas are equal.

100. Let $r(t) = ae^{bt}$ with $a = 450.268$ and $b = 1.12567$, and $n(t) =$ population after t hours. Since $r(t) = n'(t)$, $\int_0^3 r(t)\,dt = n(3) - n(0)$ is the total change in the population after three hours. Since we start with 400 bacteria, the population will be

$$n(3) = 400 + \textstyle\int_0^3 r(t)\,dt = 400 + \int_0^3 ae^{bt}\,dt = 400 + \dfrac{a}{b}\left[e^{bt}\right]_0^3 = 400 + \dfrac{a}{b}\left(e^{3b} - 1\right)$$
$$\approx 400 + 11{,}313 = 11{,}713 \text{ bacteria}$$

101. The rate is measured in liters per minute. Integrating from $t = 0$ minutes to $t = 60$ minutes will give us the total amount of oil that leaks out (in liters) during the first hour.

$$\int_0^{60} r(t)\,dt = \int_0^{60} 100e^{-0.01t}\,dt \qquad [u = -0.01t, du = -0.01dt]$$
$$= 100\int_0^{-0.6} e^u(-100\,du) = -10{,}000\left[e^u\right]_0^{-0.6} = -10{,}000(e^{-0.6} - 1) \approx 4511.9 \approx 4512 \text{ L}$$

102. The rate G is measured in kilograms per year. Integrating from $t = 0$ years (2000) to $t = 20$ years (2020) will give us the net change in biomass from 2000 to 2020.

$$\int_0^{20} \frac{60{,}000e^{-0.6t}}{(1 + 5e^{-0.6t})^2}\,dt = \int_6^{1+5e^{-12}} \frac{60{,}000}{u^2}\left(-\tfrac{1}{3}\,du\right) \qquad \begin{bmatrix} u = 1 + 5e^{-0.6t}, \\ du = -3e^{-0.6t}dt \end{bmatrix}$$
$$= \left[\frac{20{,}000}{u}\right]_6^{1+5e^{-12}} = \frac{20{,}000}{1 + 5e^{-12}} - \frac{20{,}000}{6} \approx 16{,}666$$

Thus, the predicted biomass for the year 2020 is approximately $25{,}000 + 16{,}666 = 41{,}666$ kg.

103. $$\int_0^{30} u(t)\,dt = \int_0^{30} \frac{r}{V}\,C_0e^{-rt/V}\,dt = C_0\int_1^{e^{-30r/V}} (-dx) \qquad \begin{bmatrix} x = e^{-rt/V}, \\ dx = -\frac{r}{V}e^{-rt/V}\,dt \end{bmatrix}$$
$$= C_0\Big[-x\Big]_1^{e^{-30r/V}} = C_0(-e^{-30r/V} + 1)$$

The integral $\int_0^{30} u(t)\,dt$ represents the total amount of urea removed from the blood in the first 30 minutes of dialysis.

104. (a) $\operatorname{erf}(x) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^x e^{-t^2}\,dt \quad\Rightarrow\quad \int_0^x e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}\operatorname{erf}(x)$. By Property 5 of definite integrals in Section 4.2,

$\int_0^b e^{-t^2}\,dt = \int_0^a e^{-t^2}\,dt + \int_a^b e^{-t^2}\,dt$, so

$$\int_a^b e^{-t^2}\,dt = \int_0^b e^{-t^2}\,dt - \int_0^a e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}\operatorname{erf}(b) - \frac{\sqrt{\pi}}{2}\operatorname{erf}(a) = \tfrac{1}{2}\sqrt{\pi}\left[\operatorname{erf}(b) - \operatorname{erf}(a)\right].$$

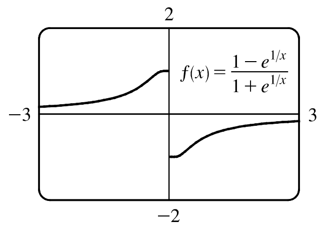
(b) $y = e^{x^2}\operatorname{erf}(x) \quad\Rightarrow\quad y' = 2xe^{x^2}\operatorname{erf}(x) + e^{x^2}\operatorname{erf}'(x) = 2xy + e^{x^2}\cdot\dfrac{2}{\sqrt{\pi}}e^{-x^2}$ [by FTC1] $= 2xy + \dfrac{2}{\sqrt{\pi}}$.

105. We use Theorem 6.1.7. Note that $f(0) = 3 + 0 + e^0 = 4$, so $f^{-1}(4) = 0$. Also $f'(x) = 1 + e^x$. Therefore,

$$\left(f^{-1}\right)'(4) = \frac{1}{f'(f^{-1}(4))} = \frac{1}{f'(0)} = \frac{1}{1 + e^0} = \frac{1}{2}.$$

106. We recognize this limit as the definition of the derivative of the function $f(x) = e^{\sin x}$ at $x = \pi$, since it is of the form $\displaystyle\lim_{x\to\pi}\frac{f(x) - f(\pi)}{x - \pi}$. Therefore, the limit is equal to $f'(\pi) = (\cos\pi)e^{\sin\pi} = -1\cdot e^0 = -1$.

107.



From the graph, it appears that f is an odd function (f is undefined for $x = 0$).

To prove this, we must show that $f(-x) = -f(x)$.

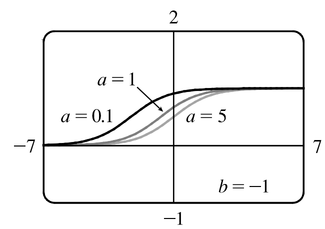
$$f(-x) = \frac{1 - e^{1/(-x)}}{1 + e^{1/(-x)}} = \frac{1 - e^{(-1/x)}}{1 + e^{(-1/x)}} = \frac{1 - \dfrac{1}{e^{1/x}}}{1 + \dfrac{1}{e^{1/x}}} \cdot \frac{e^{1/x}}{e^{1/x}} = \frac{e^{1/x} - 1}{e^{1/x} + 1}$$

$$= -\frac{1 - e^{1/x}}{1 + e^{1/x}} = -f(x)$$

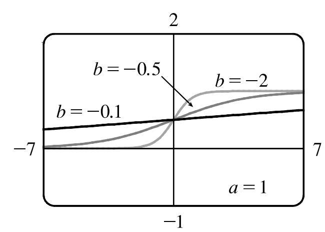
so f is an odd function.

108. We'll start with $b = -1$ and graph $f(x) = \dfrac{1}{1 + ae^{bx}}$ for $a = 0.1$, 1, and 5.

From the graph, we see that there is a horizontal asymptote $y = 0$ as $x \to -\infty$ and a horizontal asymptote $y = 1$ as $x \to \infty$. If $a = 1$, the y-intercept is $\left(0, \frac{1}{2}\right)$. As a gets smaller (close to 0), the graph of f moves left. As a gets larger, the graph of f moves right.

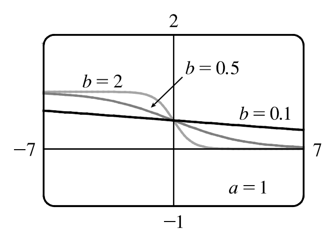


As b changes from -1 to 0, the graph of f is stretched horizontally. As b changes through large negative values, the graph of f is compressed horizontally. (This takes care of negatives values of b.)



If b is positive, the graph of f is reflected through the y-axis.

Last, if $b = 0$, the graph of f is the horizontal line $y = 1/(1 + a)$.



6.3 Logarithmic Functions

1. (a) It is defined as the inverse of the exponential function with base b, that is, $\log_b x = y \iff b^y = x$.

(b) $(0, \infty)$ (c) $\mathbb{R}$ (d) See Figure 1.

2. (a) The natural logarithm is the logarithm with base e, denoted $\ln x$.

(b) The common logarithm is the logarithm with base 10, denoted $\log x$.

(c) See Figure 3.

3. (a) $\log_3 81 = \log_3 3^4 = 4$ (b) $\log_3\left(\frac{1}{81}\right) = \log_3 3^{-4} = -4$ (c) $\log_9 3 = \log_9 9^{1/2} = \frac{1}{2}$

4. (a) $\ln \dfrac{1}{e^2} = \ln e^{-2} = -2$ (b) $\ln \sqrt{e} = \ln e^{1/2} = \frac{1}{2}$ (c) $\ln\left(\ln e^{e^{50}}\right) = \ln(e^{50}) = 50$

5. (a) $\log_2 30 - \log_2 15 = \log_2\left(\dfrac{30}{15}\right) = \log_2 2 = 1$

(b) $\log_3 10 - \log_3 5 - \log_3 18 = \log_3\left(\dfrac{10}{5}\right) - \log_3 18 = \log_3 2 - \log_3 18 = \log_3\left(\dfrac{2}{18}\right) = \log_3\left(\dfrac{1}{9}\right)$

$= \log_3 3^{-2} = -2$

(c) $2\log_5 100 - 4\log_5 50 = \log_5 100^2 - \log_5 50^4 = \log_5\left(\dfrac{100^2}{50^4}\right) = \log_5\left(\dfrac{10^4}{5^4 \cdot 10^4}\right) = \log_5 5^{-4} = -4$

6. (a) $e^{3\ln 2} = e^{\ln 2^3} = 2^3 = 8$ (b) $e^{-2\ln 5} = e^{\ln 5^{-2}} = 5^{-2} = \frac{1}{25}$ (c) $e^{\ln(\ln e^3)} = e^{\ln(3)} = 3$

7. (a) $\log_{10}\left(x^2y^3z\right) = \log_{10} x^2 + \log_{10} y^3 + \log_{10} z$ [Law 1]

$= 2\log_{10} x + 3\log_{10} y + \log_{10} z$ [Law 3]

(b) $\ln\left(\dfrac{x^4}{\sqrt{x^2-4}}\right) = \ln x^4 - \ln(x^2-4)^{1/2}$ [Law 2]

$= 4\ln x - \frac{1}{2}\ln[(x+2)(x-2)]$ [Law 3]

$= 4\ln x - \frac{1}{2}[\ln(x+2) + \ln(x-2)]$ [Law 1]

$= 4\ln x - \frac{1}{2}\ln(x+2) - \frac{1}{2}\ln(x-2)$

8. (a) $\ln\sqrt{\dfrac{3x}{x-3}} = \ln\left(\dfrac{3x}{x-3}\right)^{1/2} = \dfrac{1}{2}\ln\left(\dfrac{3x}{x-3}\right)$ [Law 3]

$= \frac{1}{2}[\ln 3 + \ln x - \ln(x-3)]$ [Laws 1 and 2]

$= \frac{1}{2}\ln 3 + \frac{1}{2}\ln x - \frac{1}{2}\ln(x-3)$

(b) $\log_2\left[(x^3+1)\sqrt[3]{(x-3)^2}\right] = \log_2(x^3+1) + \log_2\sqrt[3]{(x-3)^2}$ [Law 1]

$= \log_2(x^3+1) + \log_2(x-3)^{2/3}$

$= \log_2(x^3+1) + \frac{2}{3}\log_2(x-3)$ [Law 3]

9. (a) $\log_{10} 20 - \frac{1}{3}\log_{10} 1000 = \log_{10} 20 - \log_{10} 1000^{1/3} = \log_{10} 20 - \log_{10}\sqrt[3]{1000}$

$= \log_{10} 20 - \log_{10} 10 = \log_{10}\left(\frac{20}{10}\right) = \log_{10} 2$

(b) $\ln a - 2\ln b + 3\ln c = \ln a - \ln b^2 + \ln c^3 = \ln\dfrac{a}{b^2} + \ln c^3 = \ln\dfrac{ac^3}{b^2}$

10. (a) $\ln 10 + 2\ln 5 = \ln 10 + \ln 5^2$ [by Law 3]

$= \ln[(10)(25)]$ [by Law 1]

$= \ln 250$

(b) $\log_{10} 4 + \log_{10} a - \frac{1}{3}\log_{10}(a+1) = \log_{10}(4a) - \log_{10}(a+1)^{1/3} = \log_{10}\dfrac{4a}{\sqrt[3]{a+1}}$

11. (a) $3\ln(x-2) - \ln(x^2-5x+6) + 2\ln(x-3) = \ln(x-2)^3 - \ln[(x-2)(x-3)] + \ln(x-3)^2$

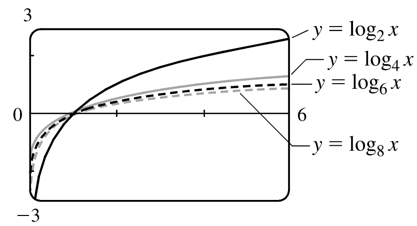
$= \ln\left[\dfrac{(x-2)^3(x-3)^2}{(x-2)(x-3)}\right] = \ln[(x-2)^2(x-3)]$

(b) $c\log_a x - d\log_a y + \log_a z = \log_a x^c - \log_a y^d + \log_a z = \log_a\left(\dfrac{x^c z}{y^d}\right)$

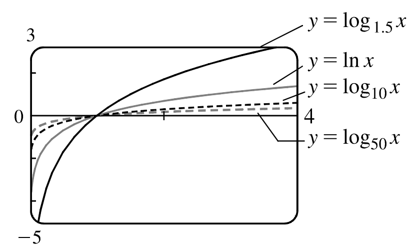
12. (a) $\log_5 10 = \dfrac{\ln 10}{\ln 5} \approx 1.430677$ (b) $\log_{15} 12 = \dfrac{\ln 12}{\ln 15} \approx 0.917600$

13. (a) $\log_3 12 = \dfrac{\ln 12}{\ln 3} \approx 2.261860$ (b) $\log_{12} 6 = \dfrac{\ln 6}{\ln 12} \approx 0.721057$

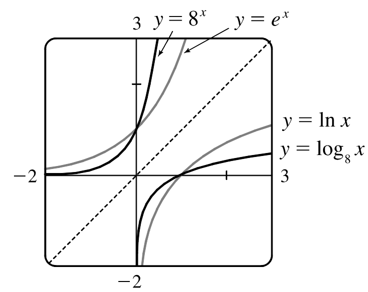
14. To graph the functions, we use $\log_2 x = \dfrac{\ln x}{\ln 2}$, $\log_4 x = \dfrac{\ln x}{\ln 4}$, etc. These graphs all approach $-\infty$ as $x \to 0^+$, and they all pass through the point $(1, 0)$. Also, they are all increasing, and all approach ∞ as $x \to \infty$. The smaller the base, the larger the rate of increase of the function (for $x > 1$) and the closer the approach to the y-axis (as $x \to 0^+$).



15. To graph these functions, we use $\log_{1.5} x = \dfrac{\ln x}{\ln 1.5}$ and $\log_{50} x = \dfrac{\ln x}{\ln 50}$. These graphs all approach $-\infty$ as $x \to 0^+$, and they all pass through the point $(1, 0)$. Also, they are all increasing, and all approach ∞ as $x \to \infty$. The functions with larger bases increase extremely slowly, and the ones with smaller bases do so somewhat more quickly. The functions with large bases approach the y-axis more closely as $x \to 0^+$.

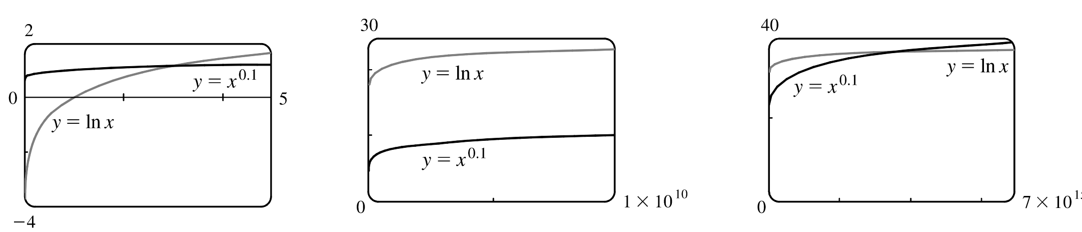


16. We see that the graph of $\ln x$ is the reflection of the graph of e^x about the line $y = x$, and that the graph of $\log_8 x$ is the reflection of the graph of 8^x about the same line. The graph of 8^x increases more quickly than that of e^x. Also note that $\log_8 x \to \infty$ as $x \to \infty$ more slowly than $\ln x$.



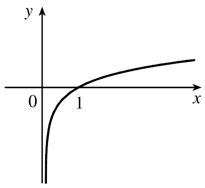
17. We need x such that $\log_2 x = 25$ cm $\iff$ $x = 2^{25} = 33\,554\,432$ cm $= 335.5443$ km

18.

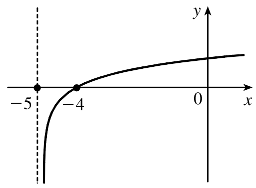


From the graphs, we see that $f(x) = x^{0.1} > g(x) = \ln x$ for approximately $0 < x < 3.06$, and then $g(x) > f(x)$ for $3.06 < x < 3.43 \times 10^{15}$ (approximately). At that point, the graph of f finally surpasses the graph of g for good.

19. (a) Shift the graph of $y = \log_{10} x$ five units to the left to obtain the graph of $y = \log_{10}(x+5)$. Note the vertical asymptote of $x = -5$.

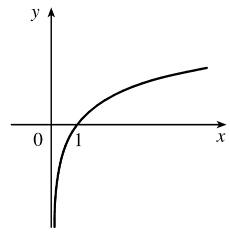


$y = \log_{10} x$

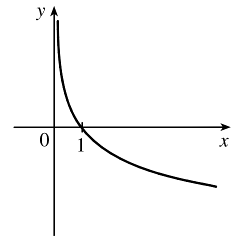


$y = \log_{10}(x+5)$

(b) Reflect the graph of $y = \ln x$ about the x-axis to obtain the graph of $y = -\ln x$.

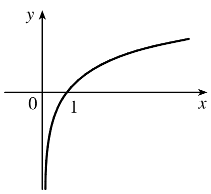


$y = \ln x$

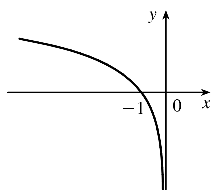


$y = -\ln x$

20. (a) Reflect the graph of $y = \ln x$ about the y-axis to obtain the graph of $y = \ln(-x)$.

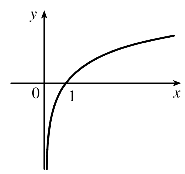


$y = \ln x$

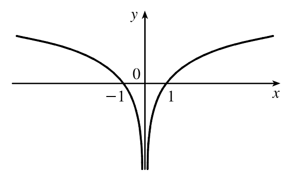


$y = \ln(-x)$

(b) Reflect the portion of the graph of $y = \ln x$ to the right of the y-axis about the y-axis. The graph of $y = \ln|x|$ is that reflection in addition to the original portion.



$y = \ln x$

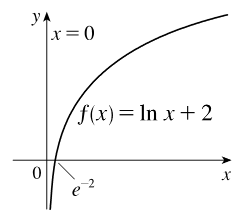


$y = \ln|x|$

21. (a) The domain of $f(x) = \ln x + 2$ is $x > 0$ and the range is $\mathbb{R}$.

(b) $y = 0 \;\Rightarrow\; \ln x + 2 = 0 \;\Rightarrow\; \ln x = -2 \;\Rightarrow\; x = e^{-2}$

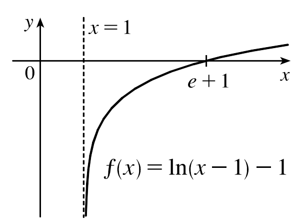
(c) We shift the graph of $y = \ln x$ two units upward.



22. (a) The domain of $f(x) = \ln(x-1) - 1$ is $x > 1$ and the range is $\mathbb{R}$.

(b) $y = 0 \;\Rightarrow\; \ln(x-1) - 1 = 0 \;\Rightarrow\; \ln(x-1) = 1 \;\Rightarrow$
$x - 1 = e^1 \;\Rightarrow\; x = e + 1$

(c) We shift the graph of $y = \ln x$ one unit to the right and one unit downward.



23. (a) $\ln(4x+2) = 3 \;\Rightarrow\; e^{\ln(4x+2)} = e^3 \;\Rightarrow\; 4x + 2 = e^3 \;\Rightarrow\; 4x = e^3 - 2 \;\Rightarrow\; x = \frac{1}{4}(e^3 - 2) \approx 4.521$

(b) $e^{2x-3} = 12 \;\Rightarrow\; \ln e^{2x-3} = \ln 12 \;\Rightarrow\; 2x - 3 = \ln 12 \;\Rightarrow\; 2x = 3 + \ln 12 \;\Rightarrow\; x = \frac{1}{2}(3 + \ln 12) \approx 2.742$

24. (a) $\log_2(x^2 - x - 1) = 2 \;\Rightarrow\; x^2 - x - 1 = 2^2 = 4 \;\Rightarrow\; x^2 - x - 5 = 0 \;\Rightarrow$

$$x = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-5)}}{2(1)} = \frac{1 \pm \sqrt{21}}{2}.$$

Solutions are $x_1 = \dfrac{1 - \sqrt{21}}{2} \approx -1.791$ and $x_2 = \dfrac{1 + \sqrt{21}}{2} \approx 2.791$.

(b) $1 + e^{4x+1} = 20 \quad \Rightarrow \quad e^{4x+1} = 19 \quad \Rightarrow \quad \ln e^{4x+1} = \ln 19 \quad \Rightarrow \quad 4x + 1 = \ln 19 \quad \Rightarrow \quad 4x = -1 + \ln 19 \quad \Rightarrow$
$x = \frac{1}{4}(-1 + \ln 19) \approx 0.486$

25. (a) $\ln x + \ln(x-1) = 0 \quad \Rightarrow \quad \ln[x(x-1)] = 0 \quad \Rightarrow \quad e^{\ln[x^2 - x]} = e^0 \quad \Rightarrow \quad x^2 - x = 1 \quad \Rightarrow \quad x^2 - x - 1 = 0$. The quadratic formula gives $x = \dfrac{1 \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \dfrac{1 \pm \sqrt{5}}{2}$, but we note that $\ln \dfrac{1-\sqrt{5}}{2}$ is undefined because $\dfrac{1-\sqrt{5}}{2} < 0$. Thus, $x = \dfrac{1+\sqrt{5}}{2} \approx 1.618$.

(b) $5^{1-2x} = 9 \quad \Rightarrow \quad \ln 5^{1-2x} = \ln 9 \quad \Rightarrow \quad (1-2x)\ln 5 = \ln 9 \quad \Rightarrow \quad 1 - 2x = \dfrac{\ln 9}{\ln 5} \quad \Rightarrow \quad x = \dfrac{1}{2} - \dfrac{\ln 9}{2\ln 5} \approx -0.183$

26. (a) $\ln(\ln x) = 0 \quad \Rightarrow \quad e^{\ln(\ln x)} = e^0 \quad \Rightarrow \quad \ln x = 1 \quad \Rightarrow \quad x = e \approx 2.718$

(b) $\dfrac{60}{1 + e^{-x}} = 4 \quad \Rightarrow \quad 60 = 4(1 + e^{-x}) \quad \Rightarrow \quad 15 = 1 + e^{-x} \quad \Rightarrow \quad 14 = e^{-x} \quad \Rightarrow \quad \ln 14 = \ln e^{-x} \quad \Rightarrow$
$\ln 14 = -x \quad \Rightarrow \quad x = -\ln 14 \approx -2.639$

27. (a) $e^{2x} - 3e^x + 2 = 0 \quad \Leftrightarrow \quad (e^x - 1)(e^x - 2) = 0 \quad \Leftrightarrow \quad e^x = 1$ or $e^x = 2 \quad \Leftrightarrow \quad x = \ln 1$ or $x = \ln 2$, so $x = 0$ or $\ln 2$.

(b) $e^{e^x} = 10 \quad \Leftrightarrow \quad \ln\left(e^{e^x}\right) = \ln 10 \quad \Leftrightarrow \quad e^x \ln e = e^x = \ln 10 \quad \Leftrightarrow \quad \ln e^x = \ln(\ln 10) \quad \Leftrightarrow \quad x = \ln\ln 10$

28. (a) $e^{3x+1} = k \quad \Leftrightarrow \quad 3x + 1 = \ln k \quad \Leftrightarrow \quad x = \frac{1}{3}(\ln k - 1)$

(b) $\log_2(mx) = c \quad \Leftrightarrow \quad mx = 2^c \quad \Leftrightarrow \quad x = 2^c/m$

29. (a) $\ln(1 + x^3) - 4 = 0 \quad \Leftrightarrow \quad \ln(1 + x^3) = 4 \quad \Leftrightarrow \quad 1 + x^3 = e^4 \quad \Leftrightarrow \quad x^3 = e^4 - 1 \quad \Leftrightarrow \quad x = \sqrt[3]{e^4 - 1} \approx 3.7704$.

(b) $2e^{1/x} = 42 \quad \Leftrightarrow \quad e^{1/x} = 21 \quad \Leftrightarrow \quad \dfrac{1}{x} = \ln 21 \quad \Leftrightarrow \quad x = \dfrac{1}{\ln 21} \approx 0.3285$.

30. (a) $2^{1-3x} = 99 \quad \Leftrightarrow \quad (1-3x)\ln 2 = \ln 99 \quad \Leftrightarrow \quad 1 - 3x = \dfrac{\ln 99}{\ln 2} \quad \Leftrightarrow \quad 3x = 1 - \dfrac{\ln 99}{\ln 2} \quad \Leftrightarrow$
$x = \dfrac{1}{3}\left(1 - \dfrac{\ln 99}{\ln 2}\right) \approx -1.8765$

(b) $\ln\left(\dfrac{x+1}{x}\right) = 2 \quad \Leftrightarrow \quad \dfrac{x+1}{x} = e^2 \quad \Leftrightarrow \quad x + 1 = e^2 x \quad \Leftrightarrow \quad (e^2 - 1)x = 1 \quad \Leftrightarrow \quad x = \dfrac{1}{e^2 - 1} \approx 0.1565$

31. (a) $\ln x < 0 \quad \Rightarrow \quad x < e^0 \quad \Rightarrow \quad x < 1$. Since the domain of $f(x) = \ln x$ is $x > 0$, the solution of the original inequality is $0 < x < 1$.

(b) $e^x > 5 \quad \Rightarrow \quad \ln e^x > \ln 5 \quad \Rightarrow \quad x > \ln 5$

32. (a) $1 < e^{3x-1} < 2 \quad \Rightarrow \quad \ln 1 < 3x - 1 < \ln 2 \quad \Rightarrow \quad 0 < 3x - 1 < \ln 2 \quad \Rightarrow \quad 1 < 3x < 1 + \ln 2 \quad \Rightarrow$
$\frac{1}{3} < x < \frac{1}{3}(1 + \ln 2)$

(b) $1 - 2\ln x < 3 \quad \Rightarrow \quad -2\ln x < 2 \quad \Rightarrow \quad \ln x > -1 \quad \Rightarrow \quad x > e^{-1}$

33. If I is the intensity of the 1989 San Francisco earthquake, then $\log_{10}(I/S) = 7.1 \quad \Rightarrow$
$\log_{10}(16I/S) = \log_{10} 16 + \log_{10}(I/S) = \log_{10} 16 + 7.1 \approx 8.3$.

34. Let I_1 and I_2 be the intensities of the music and the mower. Then $10\log_{10}\left(\dfrac{I_1}{I_0}\right) = 120$ and $10\log_{10}\left(\dfrac{I_2}{I_0}\right) = 106$, so

$$\log_{10}\left(\frac{I_1}{I_2}\right) = \log_{10}\left(\frac{I_1/I_0}{I_2/I_0}\right) = \log_{10}\left(\frac{I_1}{I_0}\right) - \log_{10}\left(\frac{I_2}{I_0}\right) = 12 - 10.6 = 1.4 \quad\Rightarrow\quad \frac{I_1}{I_2} = 10^{1.4} \approx 25.$$

35. (a) $n = f(t) = 100 \cdot 2^{t/3} \quad\Rightarrow\quad \dfrac{n}{100} = 2^{t/3} \quad\Rightarrow\quad \log_2\left(\dfrac{n}{100}\right) = \dfrac{t}{3} \quad\Rightarrow\quad t = 3\log_2\left(\dfrac{n}{100}\right)$. Using the Change of Base Formula, we can write this as $t = f^{-1}(n) = 3 \cdot \dfrac{\ln(n/100)}{\ln 2}$. This function tells us how long it will take to obtain n bacteria (given the number n).

(b) $n = 50{,}000 \quad\Rightarrow\quad t = f^{-1}(50{,}000) = 3 \cdot \dfrac{\ln\left(\frac{50{,}000}{100}\right)}{\ln 2} = 3\left(\dfrac{\ln 500}{\ln 2}\right) \approx 26.9$ hours

36. (a) We write $Q = Q_0(1 - e^{-t/a})$ and solve for t: $\dfrac{Q}{Q_0} = 1 - e^{-t/a} \quad\Rightarrow\quad e^{-t/a} = 1 - \dfrac{Q}{Q_0} \quad\Rightarrow$

$-\dfrac{t}{a} = \ln\left(1 - \dfrac{Q}{Q_0}\right) \quad\Rightarrow\quad t = -a\ln\left(1 - \dfrac{Q}{Q_0}\right)$. This formula gives the time (in seconds) needed after a discharge to obtain a given charge Q.

(b) We set $Q = 0.9Q_0$ and $a = 50$ to get $t = -50\ln\left(1 - \dfrac{0.9Q_0}{Q_0}\right) = -50\ln(0.1) \approx 115.1$ seconds. It will take approximately 115 seconds—just shy of two minutes—to recharge the capacitors to 90% of capacity.

37. $\lim\limits_{x\to 1^+} \ln(\sqrt{x} - 1) = -\infty$ since $\sqrt{x} - 1 \to 0^+$ as $x \to 1^+$.

38. As $x \to 2^-$, $8x - x^4 = x(8 - x^3) \to 0^+$ since x is positive and $8 - x^3 \to 0^+$. Thus, $\lim\limits_{x\to 2^-} \log_5(8x - x^4) = -\infty$.

39. $\lim\limits_{x\to 0} \ln(\cos x) = \ln 1 = 0$. [$\ln(\cos x)$ is continuous at $x = 0$ since it is the composite of two continuous functions.]

40. $\lim\limits_{x\to 0^+} \ln(\sin x) = -\infty$ since $\sin x \to 0^+$ as $x \to 0^+$.

41. $\lim\limits_{x\to\infty} [\ln(1 + x^2) - \ln(1 + x)] = \lim\limits_{x\to\infty} \ln\dfrac{1 + x^2}{1 + x} = \ln\left(\lim\limits_{x\to\infty} \dfrac{1 + x^2}{1 + x}\right) = \ln\left(\lim\limits_{x\to\infty} \dfrac{\frac{1}{x} + x}{\frac{1}{x} + 1}\right) = \infty$, since the limit in parentheses is ∞.

42. $\lim\limits_{x\to\infty} [\ln(2 + x) - \ln(1 + x)] = \lim\limits_{x\to\infty} \ln\left(\dfrac{2 + x}{1 + x}\right) = \lim\limits_{x\to\infty} \ln\left(\dfrac{2/x + 1}{1/x + 1}\right) = \ln\dfrac{1}{1} = \ln 1 = 0$

43. $f(x) = \ln(4 - x^2)$.

$D_f = \{x \mid 4 - x^2 > 0\} = \{x \mid x^2 < 4\} = \{x \mid |x| < 2\} = (-2, 2)$

44. $g(x) = \log_2(x^2 + 3x)$.

$$\begin{aligned} D_g &= \{x \mid x^2 + 3x > 0\} = \{x \mid x(x + 3) > 0\} = \{x \mid x < 0 \text{ and } x + 3 < 0\} \cup \{x \mid x > 0 \text{ and } x + 3 > 0\} \\ &= \{x \mid x < 0 \text{ and } x < -3\} \cup \{x \mid x > 0 \text{ and } x > -3\} = (-\infty, -3) \cup (0, \infty) \end{aligned}$$

45. (a) For $f(x) = \sqrt{3 - e^{2x}}$, we must have $3 - e^{2x} \geq 0 \;\Rightarrow\; e^{2x} \leq 3 \;\Rightarrow\; 2x \leq \ln 3 \;\Rightarrow\; x \leq \frac{1}{2}\ln 3$.

Thus, the domain of f is $(-\infty, \frac{1}{2}\ln 3]$.

(b) $y = f(x) = \sqrt{3 - e^{2x}}$ [note that $y \geq 0$] $\Rightarrow\; y^2 = 3 - e^{2x} \;\Rightarrow\; e^{2x} = 3 - y^2 \;\Rightarrow\; 2x = \ln(3 - y^2) \;\Rightarrow$ $x = \frac{1}{2}\ln(3 - y^2)$. Interchange x and y: $y = \frac{1}{2}\ln(3 - x^2)$. So $f^{-1}(x) = \frac{1}{2}\ln(3 - x^2)$. For the domain of f^{-1}, we must have $3 - x^2 > 0 \;\Rightarrow\; x^2 < 3 \;\Rightarrow\; |x| < \sqrt{3} \;\Rightarrow\; -\sqrt{3} < x < \sqrt{3} \;\Rightarrow\; 0 \leq x < \sqrt{3}$ since $x \geq 0$. Note that the domain of f^{-1}, $[0, \sqrt{3})$, equals the range of f.

46. (a) For $f(x) = \ln(2 + \ln x)$, we must have $2 + \ln x > 0 \;\Rightarrow\; \ln x > -2 \;\Rightarrow\; x > e^{-2}$. Thus, the domain of f is (e^{-2}, ∞).

(b) $y = f(x) = \ln(2 + \ln x) \;\Rightarrow\; e^y = 2 + \ln x \;\Rightarrow\; \ln x = e^y - 2 \;\Rightarrow\; x = e^{e^y - 2}$. Interchange x and y: $y = e^{e^x - 2}$. So $f^{-1}(x) = e^{e^x - 2}$. The domain of f^{-1}, as well as the range of f, is $\mathbb{R}$.

47. (a) We must have $e^x - 3 > 0 \;\Leftrightarrow\; e^x > 3 \;\Leftrightarrow\; x > \ln 3$. Thus, the domain of $f(x) = \ln(e^x - 3)$ is $(\ln 3, \infty)$.

(b) $y = \ln(e^x - 3) \;\Rightarrow\; e^y = e^x - 3 \;\Rightarrow\; e^x = e^y + 3 \;\Rightarrow\; x = \ln(e^y + 3)$, so $f^{-1}(x) = \ln(e^x + 3)$.

Now $e^x + 3 > 0 \;\Rightarrow\; e^x > -3$, which is true for any real x, so the domain of f^{-1} is $\mathbb{R}$.

48. (a) By (6), $e^{\ln 300} = 300$ and $\ln(e^{300}) = 300$.

(b) A calculator gives $e^{\ln 300} = 300$ and an error message for $\ln(e^{300})$ because e^{300} is larger than most calculators can evaluate.

49. We solve $y = 3\ln(x - 2)$ for x: $y/3 = \ln(x - 2) \;\Rightarrow\; e^{y/3} = x - 2 \;\Rightarrow\; x = 2 + e^{y/3}$. Interchanging x and y gives the inverse function $y = 2 + e^{x/3}$.

50. $y = g(x) = \log_4(x^3 + 2) \;\Rightarrow\; 4^y = x^3 + 2 \;\Rightarrow\; x^3 = 4^y - 2 \;\Rightarrow\; x = \sqrt[3]{4^y - 2}$. Interchange x and y: $y = \sqrt[3]{4^x - 2}$. So $g^{-1}(x) = \sqrt[3]{4^x - 2}$.

51. We solve $y = e^{1-x}$ for x: $\ln y = \ln e^{1-x} \;\Rightarrow\; \ln y = 1 - x \;\Rightarrow\; x = 1 - \ln y$. Interchanging x and y gives the inverse function $y = 1 - \ln x$.

52. $y = (\ln x)^2$, $x \geq 1$, $\ln x = \sqrt{y} \;\Rightarrow\; x = e^{\sqrt{y}}$. Interchange x and y: $y = e^{\sqrt{x}}$ is the inverse function.

53. $y = f(x) = 3^{2x-4} \;\Rightarrow\; \log_3 y = 2x - 4 \;\Rightarrow\; 2x = \log_3 y + 4 \;\Rightarrow\; x = \frac{1}{2}\log_3 y + 2$. Interchange x and y: $y = \frac{1}{2}\log_3 x + 2$. So $f^{-1}(x) = \frac{1}{2}\log_3 x + 2$.

54. We solve $y = \dfrac{1 - e^{-x}}{1 + e^{-x}}$ for x: $y(1 + e^{-x}) = 1 - e^{-x} \;\Rightarrow\; y + ye^{-x} = 1 - e^{-x} \;\Rightarrow\; e^{-x} + ye^{-x} = 1 - y \;\Rightarrow$ $e^{-x}(1 + y) = 1 - y \;\Rightarrow\; e^{-x} = \dfrac{1 - y}{1 + y} \;\Rightarrow\; -x = \ln\dfrac{1 - y}{1 + y} \;\Rightarrow\; x = -\ln\dfrac{1 - y}{1 + y}$ or, equivalently, $x = \ln\left(\dfrac{1 - y}{1 + y}\right)^{-1} = \ln\dfrac{1 + y}{1 - y}$. Interchanging x and y gives the inverse function $y = \ln\dfrac{1 + x}{1 - x}$.

55. $f(x) = e^{3x} - e^x \quad \Rightarrow \quad f'(x) = 3e^{3x} - e^x$. Thus, $f'(x) > 0 \quad \Leftrightarrow \quad 3e^{3x} > e^x \quad \Leftrightarrow \quad \dfrac{3e^{3x}}{e^x} > \dfrac{e^x}{e^x} \quad \Leftrightarrow \quad 3e^{2x} > 1 \quad \Leftrightarrow$

$e^{2x} > \frac{1}{3} \quad \Leftrightarrow \quad 2x > \ln\left(\frac{1}{3}\right) = -\ln 3 \quad \Leftrightarrow \quad x > -\frac{1}{2}\ln 3$, so f is increasing on $\left(-\frac{1}{2}\ln 3, \infty\right)$.

56. $y = 2e^x - e^{-3x} \quad \Rightarrow \quad y' = 2e^x + 3e^{-3x} \quad \Rightarrow \quad y'' = 2e^x - 9e^{-3x}$. Thus, $y'' < 0 \quad \Leftrightarrow \quad 2e^x < 9e^{-3x} \quad \Leftrightarrow$

$e^{4x} < \frac{9}{2} \quad \Leftrightarrow \quad 4x < \ln\frac{9}{2} \quad \Leftrightarrow \quad x < \frac{1}{4}\ln\frac{9}{2}$, so f is concave downward on $\left(-\infty, \frac{1}{4}\ln\frac{9}{2}\right)$.

57. (a) We have to show that $-f(x) = f(-x)$.

$$-f(x) = -\ln\left(x + \sqrt{x^2+1}\right) = \ln\left(\left(x + \sqrt{x^2+1}\right)^{-1}\right) = \ln\frac{1}{x + \sqrt{x^2+1}}$$

$$= \ln\left(\frac{1}{x + \sqrt{x^2+1}} \cdot \frac{x - \sqrt{x^2+1}}{x - \sqrt{x^2+1}}\right) = \ln\frac{x - \sqrt{x^2+1}}{x^2 - x^2 - 1} = \ln\left(\sqrt{x^2+1} - x\right) = f(-x)$$

Thus, f is an odd function.

(b) Let $y = \ln\left(x + \sqrt{x^2+1}\right)$. Then $e^y = x + \sqrt{x^2+1} \quad \Leftrightarrow \quad (e^y - x)^2 = x^2 + 1 \quad \Leftrightarrow \quad e^{2y} - 2xe^y + x^2 = x^2 + 1 \quad \Leftrightarrow$

$2xe^y = e^{2y} - 1 \quad \Leftrightarrow \quad x = \dfrac{e^{2y} - 1}{2e^y} = \frac{1}{2}(e^y - e^{-y})$. Thus, the inverse function is $f^{-1}(x) = \frac{1}{2}(e^x - e^{-x})$.

58. Let $\left(a, e^{-a}\right)$ be the point where the tangent meets the curve. The tangent has slope $-e^{-a}$ and is perpendicular to the line $2x - y = 8$, which has slope 2. So $-e^{-a} = -\frac{1}{2} \quad \Rightarrow \quad e^{-a} = \frac{1}{2} \quad \Rightarrow \quad e^a = 2 \quad \Rightarrow \quad a = \ln(e^a) = \ln 2$. Thus, the point on the curve is $\left(\ln 2, \frac{1}{2}\right)$ and the equation of the tangent is $y - \frac{1}{2} = -\frac{1}{2}(x - \ln 2)$ or $x + 2y = 1 + \ln 2$.

59. $x^{1/\ln x} = 2 \quad \Rightarrow \quad \ln(x^{1/\ln x}) = \ln(2) \quad \Rightarrow \quad \dfrac{1}{\ln x} \cdot \ln x = \ln 2 \quad \Rightarrow \quad 1 = \ln 2$, a contradiction, so the given equation has no solution. The function $f(x) = x^{1/\ln x} = (e^{\ln x})^{1/\ln x} = e^1 = e$ for all $x > 0$, so the function $f(x) = x^{1/\ln x}$ is the constant function $f(x) = e$.

60. (a) $\displaystyle\lim_{x\to\infty} x^{\ln x} = \lim_{x\to\infty} \left(e^{\ln x}\right)^{\ln x} = \lim_{x\to\infty} e^{(\ln x)^2} = \infty$ since $(\ln x)^2 \to \infty$ as $x \to \infty$.

(b) $\displaystyle\lim_{x\to 0^+} x^{-\ln x} = \lim_{x\to 0^+} \left(e^{\ln x}\right)^{-\ln x} = \lim_{x\to 0^+} e^{-(\ln x)^2} = 0$ since $-(\ln x)^2 \to -\infty$ as $x \to 0^+$.

(c) $\displaystyle\lim_{x\to 0^+} x^{1/x} = \lim_{x\to 0^+} \left(e^{\ln x}\right)^{1/x} = \lim_{x\to 0^+} e^{(\ln x/x)} = 0$ since $\dfrac{\ln x}{x} \to -\infty$ as $x \to 0^+$. Note that as $x \to 0^+$, $\ln x$ is a large negative number and x is a small positive number, so $(\ln x)/x \to -\infty$.

(d) $\displaystyle\lim_{x\to\infty} (\ln 2x)^{-\ln x} = \lim_{x\to\infty} \left[e^{\ln(\ln 2x)}\right]^{-\ln x} = \lim_{x\to\infty} e^{-\ln x \ln(\ln 2x)} = 0$ since $-\ln x \ln(\ln 2x) \to -\infty$ as $x \to \infty$.

61. (a) Let $\varepsilon > 0$ be given. We need N such that $|b^x - 0| < \varepsilon$ when $x < N$. But $b^x < \varepsilon \quad \Leftrightarrow \quad x < \log_b \varepsilon$. Let $N = \log_b \varepsilon$.

Then $x < N \quad \Rightarrow \quad x < \log_b \varepsilon \quad \Rightarrow \quad |b^x - 0| = b^x < \varepsilon$, so $\displaystyle\lim_{x\to-\infty} b^x = 0$.

(b) Let $M > 0$ be given. We need N such that $b^x > M$ when $x > N$. But $b^x > M \quad \Leftrightarrow \quad x > \log_b M$. Let $N = \log_b M$.

Then $x > N \quad \Rightarrow \quad x > \log_b M \quad \Rightarrow \quad b^x > M$, so $\displaystyle\lim_{x\to\infty} b^x = \infty$.

62. (a) $v(t) = ce^{-kt} \quad \Rightarrow \quad a(t) = v'(t) = -kce^{-kt} = -kv(t)$

(b) $v(0) = ce^0 = c$, so c is the initial velocity.

(c) $v(t) = ce^{-kt} = c/2 \quad \Rightarrow \quad e^{-kt} = \frac{1}{2} \quad \Rightarrow \quad -kt = \ln\frac{1}{2} = -\ln 2 \quad \Rightarrow \quad t = (\ln 2)/k$

63. $\ln(x^2 - 2x - 2) \le 0 \quad \Rightarrow \quad 0 < x^2 - 2x - 2 \le 1$. Now $x^2 - 2x - 2 \le 1$ gives $x^2 - 2x - 3 \le 0$ and hence $(x-3)(x+1) \le 0$. So $-1 \le x \le 3$. Now $0 < x^2 - 2x - 2 \quad \Rightarrow \quad x < 1 - \sqrt{3}$ or $x > 1 + \sqrt{3}$. Therefore, $\ln(x^2 - 2x - 2) \le 0 \quad \Leftrightarrow \quad -1 \le x < 1 - \sqrt{3} \;$ or $\; 1 + \sqrt{3} < x \le 3$.

64. (a) The primes less than 25 are 2, 3, 5, 7, 11, 13, 17, 19, and 23. There are 9 of them, so $\pi(25) = 9$. We use the sieve of Eratosthenes, and arrive at the figure at right. There are 25 numbers left over, so $\pi(100) = 25$.

	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~
~~51~~	~~52~~	53	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~	59	~~60~~
61	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	67	~~68~~	~~69~~	~~70~~
71	~~72~~	73	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	79	~~80~~
~~81~~	~~82~~	83	~~84~~	~~35~~	~~86~~	~~87~~	~~88~~	89	~~90~~
~~91~~	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	97	~~98~~	~~99~~	~~100~~

(b) Let $f(n) = \dfrac{\pi(n)}{n/\ln n}$. We compute $f(100) = \dfrac{25}{100/\ln 100} \approx 1.15$, $f(1000) \approx 1.16$, $f\left(10^4\right) \approx 1.13$, $f\left(10^5\right) \approx 1.10$, $f\left(10^6\right) \approx 1.08$, and $f\left(10^7\right) \approx 1.07$.

(c) By the Prime Number Theorem, the number of primes less than a billion, that is, $\pi(10^9)$, should be close to $10^9/\ln 10^9 \approx 48{,}254{,}942$. In fact, $\pi(10^9) = 50{,}847{,}543$, so our estimate is off by about 5.1%. Do not attempt this calculation at home.

6.4 Derivatives of Logarithmic Functions

1. The differentiation formula for logarithmic functions, $\dfrac{d}{dx}(\log_b x) = \dfrac{1}{x \ln b}$, is simplest when $b = e$ because $\ln e = 1$.

2. $g(t) = \ln(3 + t^2) \quad \Rightarrow \quad g'(t) = \dfrac{1}{3+t^2} \cdot \dfrac{d}{dt}(3 + t^2) = \dfrac{1}{3+t^2} \cdot 2t = \dfrac{2t}{3+t^2}$

3. $f(x) = \ln(x^2 + 3x + 5) \quad \Rightarrow \quad f'(x) = \dfrac{1}{x^2+3x+5} \cdot \dfrac{d}{dx}(x^2 + 3x + 5) = \dfrac{1}{x^2+3x+5} \cdot (2x+3) = \dfrac{2x+3}{x^2+3x+5}$

4. $f(x) = x \ln x - x \quad \Rightarrow \quad f'(x) = x \cdot \dfrac{1}{x} + (\ln x) \cdot 1 - 1 = 1 + \ln x - 1 = \ln x$

5. $f(x) = \sin(\ln x) \quad \Rightarrow \quad f'(x) = \cos(\ln x) \cdot \dfrac{d}{dx} \ln x = \cos(\ln x) \cdot \dfrac{1}{x} = \dfrac{\cos(\ln x)}{x}$

6. $f(x) = \ln(\sin^2 x) = \ln(\sin x)^2 = 2 \ln|\sin x| \quad \Rightarrow \quad f'(x) = 2 \cdot \dfrac{1}{\sin x} \cdot \cos x = 2 \cot x$

7. $f(x) = \ln\dfrac{1}{x} \quad \Rightarrow \quad f'(x) = \dfrac{1}{1/x} \dfrac{d}{dx}\left(\dfrac{1}{x}\right) = x\left(-\dfrac{1}{x^2}\right) = -\dfrac{1}{x}$.

Another solution: $f(x) = \ln\dfrac{1}{x} = \ln 1 - \ln x = -\ln x \quad \Rightarrow \quad f'(x) = -\dfrac{1}{x}$.

8. $y = \dfrac{1}{\ln x} = (\ln x)^{-1} \quad \Rightarrow \quad y' = -1(\ln x)^{-2} \cdot \dfrac{1}{x} = \dfrac{-1}{x(\ln x)^2}$

9. $g(x) = \ln(xe^{-2x}) = \ln x + \ln e^{-2x} = \ln x - 2x \quad \Rightarrow \quad g'(x) = \dfrac{1}{x} - 2$

10. $g(t) = \sqrt{1 + \ln t} \quad \Rightarrow \quad g'(t) = \frac{1}{2}(1 + \ln t)^{-1/2} \dfrac{d}{dt}(1 + \ln t) = \dfrac{1}{2\sqrt{1+\ln t}} \cdot \dfrac{1}{t} = \dfrac{1}{2t\sqrt{1+\ln t}}$

11. $F(t) = (\ln t)^2 \sin t \quad \Rightarrow \quad F'(t) = (\ln t)^2 \cos t + \sin t \cdot 2 \ln t \cdot \dfrac{1}{t} = \ln t \left(\ln t \, \cos t + \dfrac{2 \sin t}{t} \right)$

12. $p(t) = \ln \sqrt{t^2+1} \quad \Rightarrow \quad p'(t) = \dfrac{1}{\sqrt{t^2+1}} \cdot \dfrac{d}{dt}\left(\sqrt{t^2+1}\right) = \dfrac{1}{\sqrt{t^2+1}} \cdot \dfrac{2t}{2\sqrt{t^2+1}} = \dfrac{t}{t^2+1}$

Or: $p(t) = \ln \sqrt{t^2+1} = \ln(t^2+1)^{1/2} = \frac{1}{2}\ln(t^2+1) \quad \Rightarrow \quad p'(t) = \dfrac{1}{2} \cdot \dfrac{1}{t^2+1} \cdot 2t = \dfrac{t}{t^2+1}$

13. $y = \log_8(x^2+3x) \quad \Rightarrow \quad y' = \dfrac{1}{(x^2+3x)\ln 8} \cdot \dfrac{d}{dx}(x^2+3x) = \dfrac{1}{(x^2+3x)\ln 8} \cdot (2x+3) = \dfrac{2x+3}{(x^2+3x)\ln 8}$

14. $y = \log_{10} \sec x \quad \Rightarrow \quad y' = \dfrac{1}{\sec x\,(\ln 10)} \cdot \dfrac{d}{dx}(\sec x) = \dfrac{1}{\sec x\,(\ln 10)} \cdot \sec x \, \tan x = \dfrac{\tan x}{\ln 10}$

15. $f(u) = \dfrac{\ln u}{1 + \ln(2u)} \quad \Rightarrow$

$$f'(u) = \frac{[1+\ln(2u)] \cdot \frac{1}{u} - \ln u \cdot \frac{1}{2u} \cdot 2}{[1+\ln(2u)]^2} = \frac{\frac{1}{u}[1+\ln(2u) - \ln u]}{[1+\ln(2u)]^2} = \frac{1 + (\ln 2 + \ln u) - \ln u}{u[1+\ln(2u)]^2} = \frac{1+\ln 2}{u[1+\ln(2u)]^2}$$

16. $y = \ln(\csc x - \cot x) \quad \Rightarrow$

$$y' = \frac{1}{\csc x - \cot x} \frac{d}{dx}(\csc x - \cot x) = \frac{1}{\csc x - \cot x}(-\csc x \, \cot x + \csc^2 x) = \frac{\csc x(\csc x - \cot x)}{\csc x - \cot x} = \csc x$$

17. $f(x) = x^5 + 5^x \quad \Rightarrow \quad f'(x) = 5x^4 + 5^x \ln 5$

18. $g(x) = x \sin(2^x) \quad \Rightarrow \quad g'(x) = x \cos(2^x) \cdot 2^x \ln 2 + \sin(2^x) \cdot 1 = x 2^x \ln 2 \cos(2^x) + \sin(2^x)$

19. $T(z) = 2^z \log_2 z \quad \Rightarrow \quad T'(z) = 2^z \dfrac{1}{z \ln 2} + \log_2 z \cdot 2^z \ln 2 = 2^z \left(\dfrac{1}{z \ln 2} + \log_2 z \, (\ln 2) \right)$.

Note that $\log_2 z \, (\ln 2) = \dfrac{\ln z}{\ln 2} (\ln 2) = \ln z$ by the change of base formula. Thus, $T'(z) = 2^z \left(\dfrac{1}{z \ln 2} + \ln z \right)$.

20. $g(x) = e^{x^2 \ln x} \quad \Rightarrow$

$$g'(x) = e^{x^2 \ln x} \cdot \frac{d}{dx}(x^2 \ln x) = e^{x^2 \ln x} \left[x^2 \cdot \frac{1}{x} + (\ln x) \cdot 2x \right] = e^{x^2 \ln x}(x + 2x \ln x) = x\, e^{x^2 \ln x}(1 + 2 \ln x)$$

21. $g(t) = \ln \dfrac{t(t^2+1)^4}{\sqrt[3]{2t-1}} = \ln t + \ln(t^2+1)^4 - \ln \sqrt[3]{2t-1} = \ln t + 4\ln(t^2+1) - \frac{1}{3}\ln(2t-1) \quad \Rightarrow$

$$g'(t) = \frac{1}{t} + 4 \cdot \frac{1}{t^2+1} \cdot 2t - \frac{1}{3} \cdot \frac{1}{2t-1} \cdot 2 = \frac{1}{t} + \frac{8t}{t^2+1} - \frac{2}{3(2t-1)}$$

22. $y = \ln\sqrt{\dfrac{1+2x}{1-2x}} = \ln\sqrt{1+2x} - \ln\sqrt{1-2x} = \frac{1}{2}\ln(1+2x) - \frac{1}{2}\ln(1-2x) \quad\Rightarrow$

$$y' = \frac{1}{2}\cdot\frac{1}{1+2x}\cdot 2 - \frac{1}{2}\cdot\frac{1}{1-2x}\cdot(-2) = \frac{1}{1+2x} + \frac{1}{1-2x}$$

23. $y = \ln\left|3-2x^5\right| \quad\Rightarrow\quad y' = \dfrac{1}{3-2x^5}\cdot(-10x^4) = \dfrac{-10x^4}{3-2x^5}$

24. $y = \ln(e^{-x} + xe^{-x}) = \ln(e^{-x}(1+x)) = \ln(e^{-x}) + \ln(1+x) = -x + \ln(1+x) \quad\Rightarrow$

$$y' = -1 + \frac{1}{1+x} = \frac{-1-x+1}{1+x} = -\frac{x}{1+x}$$

25. $h(x) = e^{x^2+\ln x} = e^{x^2}\cdot e^{\ln x} = e^{x^2}\cdot x = xe^{x^2} \Rightarrow$

$$h'(x) = x\cdot\frac{d}{dx}\left(e^{x^2}\right) + e^{x^2}\cdot\frac{d}{dx}(x) = x\cdot e^{x^2}\cdot\frac{d}{dx}(x^2) + e^{x^2}\cdot 1 = x\cdot e^{x^2}\cdot 2x + e^{x^2}$$

$$= 2x^2e^{x^2} + e^{x^2} = e^{x^2}(2x^2+1)$$

26. $y = \log_2(x\log_5 x) \quad\Rightarrow$

$$y' = \frac{1}{(x\log_5 x)(\ln 2)}\,\frac{d}{dx}(x\log_5 x) = \frac{1}{(x\log_5 x)(\ln 2)}\left(x\cdot\frac{1}{x\ln 5} + \log_5 x\right) = \frac{1}{(x\log_5 x)(\ln 5)(\ln 2)} + \frac{1}{x(\ln 2)}.$$

Note that $\log_5 x(\ln 5) = \dfrac{\ln x}{\ln 5}(\ln 5) = \ln x$ by the change of base formula. Thus, $y' = \dfrac{1}{x\ln x\ln 2} + \dfrac{1}{x\ln 2} = \dfrac{1+\ln x}{x\ln x\ln 2}$.

27. Using Formula 7 and the Chain Rule, $G(x) = 4^{C/x} \quad\Rightarrow$

$$G'(x) = 4^{C/x}(\ln 4)\,\frac{d}{dx}\frac{C}{x} \quad \left[\frac{C}{x} = Cx^{-1}\right] \quad = 4^{C/x}(\ln 4)\left(-Cx^{-2}\right) = -C(\ln 4)\,\frac{4^{C/x}}{x^2}.$$

28. $F(t) = 3^{\cos 2t} \quad\Rightarrow\quad F'(t) = 3^{\cos 2t}\,\ln 3\,\dfrac{d}{dt}(\cos 2t) = -2(\sin 2t)\,3^{\cos 2t}\,\ln 3$

29. $\dfrac{d}{dx}\ln\!\left(x+\sqrt{x^2+1}\right) = \dfrac{1}{x+\sqrt{x^2+1}}\cdot\dfrac{d}{dx}\left(x+\sqrt{x^2+1}\right) = \dfrac{1}{x+\sqrt{x^2+1}}\cdot\left(1+\dfrac{2x}{2\sqrt{x^2+1}}\right)$

$$= \frac{1}{x+\sqrt{x^2+1}}\cdot\left(\frac{\sqrt{x^2+1}}{\sqrt{x^2+1}} + \frac{x}{\sqrt{x^2+1}}\right) = \frac{1}{x+\sqrt{x^2+1}}\cdot\left(\frac{x+\sqrt{x^2+1}}{\sqrt{x^2+1}}\right) = \frac{1}{\sqrt{x^2+1}}$$

30. $\dfrac{d}{dx}\ln\sqrt{\dfrac{1-\cos x}{1+\cos x}} = \dfrac{d}{dx}\left(\ln\sqrt{1-\cos x} - \ln\sqrt{1+\cos x}\right) = \dfrac{d}{dx}\left[\dfrac{1}{2}\ln(1-\cos x) - \dfrac{1}{2}\ln(1+\cos x)\right]$

$$= \frac{1}{2}\cdot\frac{1}{1-\cos x}\cdot\sin x - \frac{1}{2}\cdot\frac{1}{1+\cos x}\cdot(-\sin x)$$

$$= \frac{1}{2}\left(\frac{\sin x}{1-\cos x} + \frac{\sin x}{1+\cos x}\right) = \frac{1}{2}\left[\frac{\sin x\,(1+\cos x) + \sin x\,(1-\cos x)}{(1-\cos x)(1+\cos x)}\right]$$

$$= \frac{1}{2}\left(\frac{\sin x + \sin x\,\cos x + \sin x - \sin x\,\cos x}{1-\cos^2 x}\right) = \frac{1}{2}\left(\frac{2\sin x}{\sin^2 x}\right) = \frac{1}{\sin x} = \csc x$$

31. $y = \sqrt{x}\,\ln x \quad\Rightarrow\quad y' = \sqrt{x}\cdot\dfrac{1}{x} + (\ln x)\,\dfrac{1}{2\sqrt{x}} = \dfrac{2+\ln x}{2\sqrt{x}} \quad\Rightarrow$

$$y'' = \frac{2\sqrt{x}\,(1/x) - (2+\ln x)(1/\sqrt{x})}{(2\sqrt{x})^2} = \frac{2/\sqrt{x} - (2+\ln x)(1/\sqrt{x})}{4x} = \frac{2-(2+\ln x)}{\sqrt{x}(4x)} = -\frac{\ln x}{4x\sqrt{x}}$$

32. $y = \dfrac{\ln x}{1+\ln x} \quad\Rightarrow\quad y' = \dfrac{(1+\ln x)(1/x) - (\ln x)(1/x)}{(1+\ln x)^2} = \dfrac{1}{x(1+\ln x)^2} \quad\Rightarrow$

$$y'' = -\frac{\frac{d}{dx}[x(1+\ln x)^2]}{[x(1+\ln x)^2]^2} \quad \text{[Reciprocal Rule]} \quad = -\frac{x\cdot 2(1+\ln x)\cdot(1/x) + (1+\ln x)^2}{x^2(1+\ln x)^4}$$

$$= -\frac{(1+\ln x)[2+(1+\ln x)]}{x^2(1+\ln x)^4} = -\frac{3+\ln x}{x^2(1+\ln x)^3}$$

33. $y = \ln|\sec x| \quad\Rightarrow\quad y' = \dfrac{1}{\sec x}\dfrac{d}{dx}\sec x = \dfrac{1}{\sec x}\sec x\,\tan x = \tan x \quad\Rightarrow\quad y'' = \sec^2 x$

34. $y = \ln(1+\ln x) \quad\Rightarrow\quad y' = \dfrac{1}{1+\ln x}\cdot\dfrac{1}{x} = \dfrac{1}{x(1+\ln x)} \quad\Rightarrow$

$$y'' = -\frac{\frac{d}{dx}[x(1+\ln x)]}{[x(1+\ln x)]^2} \quad \text{[Reciprocal Rule]} \quad = -\frac{x(1/x)+(1+\ln x)(1)}{x^2(1+\ln x)^2} = -\frac{1+1+\ln x}{x^2(1+\ln x)^2} = -\frac{2+\ln x}{x^2(1+\ln x)^2}$$

35. $f(x) = \dfrac{x}{1-\ln(x-1)} \quad\Rightarrow$

$$f'(x) = \frac{[1-\ln(x-1)]\cdot 1 - x\cdot\frac{-1}{x-1}}{[1-\ln(x-1)]^2} = \frac{\frac{(x-1)[1-\ln(x-1)]+x}{x-1}}{[1-\ln(x-1)]^2} = \frac{x-1-(x-1)\ln(x-1)+x}{(x-1)[1-\ln(x-1)]^2}$$

$$= \frac{2x-1-(x-1)\ln(x-1)}{(x-1)[1-\ln(x-1)]^2}$$

$$\text{Dom}(f) = \{x \mid x-1>0 \quad\text{and}\quad 1-\ln(x-1)\neq 0\} = \{x \mid x>1 \quad\text{and}\quad \ln(x-1)\neq 1\}$$

$$= \left\{x \mid x>1 \quad\text{and}\quad x-1\neq e^1\right\} = \{x \mid x>1 \quad\text{and}\quad x\neq 1+e\} = (1,1+e)\cup(1+e,\infty)$$

36. $f(x) = \sqrt{2+\ln x} = (2+\ln x)^{1/2} \quad\Rightarrow\quad f'(x) = \dfrac{1}{2}(2+\ln x)^{-1/2}\cdot\dfrac{1}{x} = \dfrac{1}{2x\sqrt{2+\ln x}}$

$\text{Dom}(f) = \{x \mid 2+\ln x\geq 0\} = \{x \mid \ln x\geq -2\} = \{x \mid x\geq e^{-2}\} = [e^{-2},\infty)$.

37. $f(x) = \ln(x^2-2x) \quad\Rightarrow\quad f'(x) = \dfrac{1}{x^2-2x}(2x-2) = \dfrac{2(x-1)}{x(x-2)}$.

$\text{Dom}(f) = \{x \mid x(x-2)>0\} = (-\infty,0)\cup(2,\infty)$.

38. $f(x) = \ln\ln\ln x \quad\Rightarrow\quad f'(x) = \dfrac{1}{\ln\ln x}\cdot\dfrac{1}{\ln x}\cdot\dfrac{1}{x}$.

$\text{Dom}(f) = \{x \mid \ln\ln x>0\} = \{x \mid \ln x>1\} = \{x \mid x>e\} = (e,\infty)$.

39. $f(x) = \ln(x+\ln x) \quad\Rightarrow\quad f'(x) = \dfrac{1}{x+\ln x}\dfrac{d}{dx}(x+\ln x) = \dfrac{1}{x+\ln x}\left(1+\dfrac{1}{x}\right)$.

Substitute 1 for x to get $f'(1) = \dfrac{1}{1+\ln 1}\left(1+\dfrac{1}{1}\right) = \dfrac{1}{1+0}(1+1) = 1\cdot 2 = 2$.

40. $f(x) = \cos(\ln x^2) \quad\Rightarrow\quad f'(x) = -\sin(\ln x^2)\,\dfrac{d}{dx}\ln x^2 = -\sin(\ln x^2)\dfrac{1}{x^2}(2x) = -\dfrac{2\sin(\ln x^2)}{x}$.

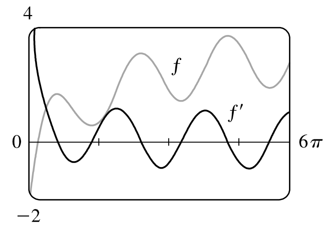
Substitute 1 for x to get $f'(1) = -\dfrac{2\sin(\ln 1^2)}{1} = -2\sin 0 = 0$.

41. $y = \ln(x^2 - 3x + 1) \quad \Rightarrow \quad y' = \dfrac{1}{x^2 - 3x + 1} \cdot (2x - 3) \quad \Rightarrow \quad y'(3) = \frac{1}{1} \cdot 3 = 3$, so an equation of a tangent line at $(3, 0)$ is $y - 0 = 3(x - 3)$, or $y = 3x - 9$.

42. $y = x^2 \ln x \quad \Rightarrow \quad y' = x^2 \cdot \dfrac{1}{x} + (\ln x)(2x) \quad \Rightarrow \quad y'(1) = 1 + 0 = 1$, so an equation of a tangent line at $(1, 0)$ is $y - 0 = 1(x - 1)$, or $y = x - 1$.

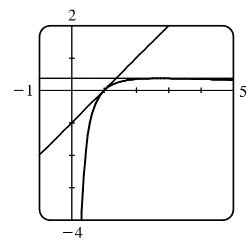
43. $f(x) = \sin x + \ln x \quad \Rightarrow \quad f'(x) = \cos x + 1/x$.

This is reasonable, because the graph shows that f increases when f' is positive, and $f'(x) = 0$ when f has a horizontal tangent.



44. $y = \dfrac{\ln x}{x} \quad \Rightarrow \quad y' = \dfrac{x(1/x) - \ln x}{x^2} = \dfrac{1 - \ln x}{x^2}$.

$y'(1) = \dfrac{1 - 0}{1^2} = 1$ and $y'(e) = \dfrac{1 - 1}{e^2} = 0 \quad \Rightarrow \quad$ equations of tangent lines are $y - 0 = 1(x - 1)$ or $y = x - 1$ and $y - 1/e = 0(x - e)$ or $y = 1/e$.



45. $f(x) = cx + \ln(\cos x) \quad \Rightarrow \quad f'(x) = c + \dfrac{1}{\cos x} \cdot (-\sin x) = c - \tan x$.

$f'(\frac{\pi}{4}) = 6 \quad \Rightarrow \quad c - \tan \frac{\pi}{4} = 6 \quad \Rightarrow \quad c - 1 = 6 \quad \Rightarrow \quad c = 7$.

46. $f(x) = \log_b(3x^2 - 2) \quad \Rightarrow \quad f'(x) = \dfrac{1}{(3x^2 - 2) \ln b} \cdot 6x$. $f'(1) = 3 \quad \Rightarrow \quad \dfrac{1}{\ln b} \cdot 6 = 3 \quad \Rightarrow \quad 2 = \ln b \quad \Rightarrow \quad b = e^2$.

47. $y = (x^2 + 2)^2 (x^4 + 4)^4 \quad \Rightarrow \quad \ln y = \ln[(x^2 + 2)^2 (x^4 + 4)^4] \quad \Rightarrow \quad \ln y = 2\ln(x^2 + 2) + 4\ln(x^4 + 4) \quad \Rightarrow$

$\dfrac{1}{y}\, y' = 2 \cdot \dfrac{1}{x^2 + 2} \cdot 2x + 4 \cdot \dfrac{1}{x^4 + 4} \cdot 4x^3 \quad \Rightarrow \quad y' = y\left(\dfrac{4x}{x^2 + 2} + \dfrac{16x^3}{x^4 + 4}\right) \quad \Rightarrow$

$y' = (x^2 + 2)^2 (x^4 + 4)^4 \left(\dfrac{4x}{x^2 + 2} + \dfrac{16x^3}{x^4 + 4}\right)$

48. $y = \dfrac{e^{-x} \cos^2 x}{x^2 + x + 1} \quad \Rightarrow \quad \ln y = \ln \dfrac{e^{-x} \cos^2 x}{x^2 + x + 1} \quad \Rightarrow$

$\ln y = \ln e^{-x} + \ln |\cos x|^2 - \ln(x^2 + x + 1) = -x + 2\ln|\cos x| - \ln(x^2 + x + 1) \quad \Rightarrow$

$\dfrac{1}{y}\, y' = -1 + 2 \cdot \dfrac{1}{\cos x}(-\sin x) - \dfrac{1}{x^2 + x + 1}(2x + 1) \quad \Rightarrow \quad y' = y\left(-1 - 2\tan x - \dfrac{2x + 1}{x^2 + x + 1}\right) \quad \Rightarrow$

$y' = -\dfrac{e^{-x} \cos^2 x}{x^2 + x + 1}\left(1 + 2\tan x + \dfrac{2x + 1}{x^2 + x + 1}\right)$

49. $y = \sqrt{\dfrac{x - 1}{x^4 + 1}} \quad \Rightarrow \quad \ln y = \ln\left(\dfrac{x - 1}{x^4 + 1}\right)^{1/2} \quad \Rightarrow \quad \ln y = \dfrac{1}{2}\ln(x - 1) - \dfrac{1}{2}\ln(x^4 + 1) \quad \Rightarrow$

$\dfrac{1}{y}\, y' = \dfrac{1}{2}\dfrac{1}{x - 1} - \dfrac{1}{2}\dfrac{1}{x^4 + 1} \cdot 4x^3 \quad \Rightarrow \quad y' = y\left(\dfrac{1}{2(x - 1)} - \dfrac{2x^3}{x^4 + 1}\right) \quad \Rightarrow \quad y' = \sqrt{\dfrac{x - 1}{x^4 + 1}}\left(\dfrac{1}{2x - 2} - \dfrac{2x^3}{x^4 + 1}\right)$

50. $y = \sqrt{x}\, e^{x^2-x}(x+1)^{2/3} \quad \Rightarrow \quad \ln y = \ln\left[x^{1/2}e^{x^2-x}(x+1)^{2/3}\right] \quad \Rightarrow$

$\ln y = \frac{1}{2}\ln x + (x^2-x) + \frac{2}{3}\ln(x+1) \quad \Rightarrow \quad \dfrac{1}{y}\,y' = \dfrac{1}{2}\cdot\dfrac{1}{x} + 2x - 1 + \dfrac{2}{3}\cdot\dfrac{1}{x+1} \quad \Rightarrow$

$y' = y\left(\dfrac{1}{2x} + 2x - 1 + \dfrac{2}{3x+3}\right) \quad \Rightarrow \quad y' = \sqrt{x}\, e^{x^2-x}(x+1)^{2/3}\left(\dfrac{1}{2x} + 2x - 1 + \dfrac{2}{3x+3}\right)$

51. $y = x^x \quad \Rightarrow \quad \ln y = \ln x^x \quad \Rightarrow \quad \ln y = x\ln x \quad \Rightarrow \quad y'/y = x(1/x) + (\ln x)\cdot 1 \quad \Rightarrow \quad y' = y(1+\ln x) \quad \Rightarrow$

$y' = x^x(1+\ln x)$

52. $y = x^{1/x} \quad \Rightarrow \quad \ln y = \dfrac{1}{x}\ln\, x \quad \Rightarrow \quad \dfrac{y'}{y} = -\dfrac{1}{x^2}\ln\, x + \dfrac{1}{x}\left(\dfrac{1}{x}\right) \quad \Rightarrow \quad y' = x^{1/x}\,\dfrac{1-\ln x}{x^2}$

53. $y = x^{\sin x} \quad \Rightarrow \quad \ln y = \ln x^{\sin x} \quad \Rightarrow \quad \ln y = \sin x \ln x \quad \Rightarrow \quad \dfrac{y'}{y} = (\sin x)\cdot\dfrac{1}{x} + (\ln x)(\cos x) \quad \Rightarrow$

$y' = y\left(\dfrac{\sin x}{x} + \ln x \cos x\right) \quad \Rightarrow \quad y' = x^{\sin x}\left(\dfrac{\sin x}{x} + \ln x\, \cos x\right)$

54. $y = \left(\sqrt{x}\right)^x \quad \Rightarrow \quad \ln y = \ln\left(\sqrt{x}\right)^x \quad \Rightarrow \quad \ln y = x\ln x^{1/2} \quad \Rightarrow \quad \ln y = \frac{1}{2}x\ln x \quad \Rightarrow \quad \dfrac{1}{y}\,y' = \dfrac{1}{2}x\cdot\dfrac{1}{x} + \ln x\cdot\dfrac{1}{2} \quad \Rightarrow$

$y' = y\left(\frac{1}{2} + \frac{1}{2}\ln x\right) \quad \Rightarrow \quad y' = \frac{1}{2}\left(\sqrt{x}\right)^x(1+\ln x)$

55. $y = (\cos x)^x \quad \Rightarrow \quad \ln y = \ln(\cos x)^x \quad \Rightarrow \quad \ln y = x\ln\cos x \quad \Rightarrow \quad \dfrac{1}{y}\,y' = x\cdot\dfrac{1}{\cos x}\cdot(-\sin x) + \ln\cos x\cdot 1 \quad \Rightarrow$

$y' = y\left(\ln\cos x - \dfrac{x\sin x}{\cos x}\right) \quad \Rightarrow \quad y' = (\cos x)^x(\ln\cos x - x\tan x)$

56. $y = (\sin x)^{\ln x} \quad \Rightarrow \quad \ln y = \ln(\sin x)^{\ln x} \quad \Rightarrow \quad \ln y = \ln x\cdot\ln\sin x \quad \Rightarrow \quad \dfrac{1}{y}\,y' = \ln x\cdot\dfrac{1}{\sin x}\cdot\cos x + \ln\sin x\cdot\dfrac{1}{x} \quad \Rightarrow$

$y' = y\left(\ln x\cdot\dfrac{\cos x}{\sin x} + \dfrac{\ln\sin x}{x}\right) \quad \Rightarrow \quad y' = (\sin x)^{\ln x}\left(\ln x\cot x + \dfrac{\ln\sin x}{x}\right)$

57. $y = x^{\ln x} \quad \Rightarrow \quad \ln y = \ln x\ln x = (\ln x)^2 \quad \Rightarrow \quad \dfrac{y'}{y} = 2\ln x\left(\dfrac{1}{x}\right) \quad \Rightarrow \quad y' = x^{\ln x}\left(\dfrac{2\ln x}{x}\right)$

58. $y = (\ln x)^{\cos x} \quad \Rightarrow \quad \ln y = \cos x\ln(\ln x) \quad \Rightarrow \quad \dfrac{y'}{y} = \cos x\cdot\dfrac{1}{\ln x}\cdot\dfrac{1}{x} + (\ln\ln x)(-\sin x) \quad \Rightarrow$

$y' = (\ln x)^{\cos x}\left(\dfrac{\cos x}{x\ln x} - \sin x\ln\ln x\right)$

59. $y = \ln(x^2+y^2) \quad \Rightarrow \quad y' = \dfrac{1}{x^2+y^2}\,\dfrac{d}{dx}\,(x^2+y^2) \quad \Rightarrow \quad y' = \dfrac{2x+2yy'}{x^2+y^2} \quad \Rightarrow \quad x^2y' + y^2y' = 2x + 2yy' \quad \Rightarrow$

$x^2y' + y^2y' - 2yy' = 2x \quad \Rightarrow \quad (x^2+y^2-2y)y' = 2x \quad \Rightarrow \quad y' = \dfrac{2x}{x^2+y^2-2y}$

60. $x^y = y^x \quad \Rightarrow \quad y\ln x = x\ln y \quad \Rightarrow \quad y\cdot\dfrac{1}{x} + (\ln x)\cdot y' = x\cdot\dfrac{1}{y}\cdot y' + \ln y \quad \Rightarrow \quad y'\ln x - \dfrac{x}{y}\,y' = \ln y - \dfrac{y}{x} \quad \Rightarrow$

$y' = \dfrac{\ln y - y/x}{\ln x - x/y}$

61. $f(x) = \ln(x-1) \;\Rightarrow\; f'(x) = \dfrac{1}{(x-1)} = (x-1)^{-1} \;\Rightarrow\; f''(x) = -(x-1)^{-2} \;\Rightarrow\; f'''(x) = 2(x-1)^{-3} \;\Rightarrow$

$f^{(4)}(x) = -2\cdot 3(x-1)^{-4} \;\Rightarrow\; \cdots \;\Rightarrow\; f^{(n)}(x) = (-1)^{n-1}\cdot 2\cdot 3\cdot 4\cdot \cdots \cdot (n-1)(x-1)^{-n} = (-1)^{n-1}\dfrac{(n-1)!}{(x-1)^n}$

62. $y = x^8 \ln x$, so $D^9 y = D^8 y' = D^8(8x^7 \ln x + x^7)$. But the eighth derivative of x^7 is 0, so we now have

$D^8(8x^7 \ln x) = D^7(8\cdot 7x^6 \ln x + 8x^6) = D^7(8\cdot 7x^6 \ln x) = D^6(8\cdot 7\cdot 6x^5 \ln x) = \cdots = D(8!\,x^0 \ln x) = 8!/x.$

63. $f(x) = \dfrac{\ln x}{\sqrt{x}} \;\Rightarrow\; f'(x) = \dfrac{\sqrt{x}\,(1/x) - (\ln x)[1/(2\sqrt{x}\,)]}{x} = \dfrac{2-\ln x}{2x^{3/2}} \;\Rightarrow$

$f''(x) = \dfrac{2x^{3/2}(-1/x) - (2-\ln x)(3x^{1/2})}{4x^3} = \dfrac{3\ln x - 8}{4x^{5/2}} > 0 \;\Leftrightarrow\; \ln x > \frac{8}{3} \;\Leftrightarrow\; x > e^{8/3}$, so f is CU on $(e^{8/3}, \infty)$

and CD on $(0, e^{8/3})$. The inflection point is $\left(e^{8/3}, \frac{8}{3}e^{-4/3}\right)$.

64. $f(x) = x\ln x$, $f'(x) = \ln x + 1 = 0$ when $\ln x = -1 \;\Leftrightarrow\; x = e^{-1}$. $f'(x) > 0 \;\Leftrightarrow\; \ln x + 1 > 0 \;\Leftrightarrow$

$\ln x > -1 \;\Leftrightarrow\; x > 1/e$. $f'(x) < 0 \;\Leftrightarrow\; \ln x + 1 < 0 \;\Leftrightarrow\; x < 1/e$. Therefore, there is an absolute minimum value of $f(1/e) = (1/e)\,\ln(1/e) = -1/e$.

65. $y = f(x) = \ln(\sin x)$

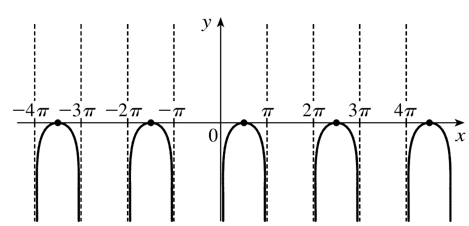
A. $D = \{x \text{ in } \mathbb{R} \mid \sin x > 0\} = \displaystyle\bigcup_{n=-\infty}^{\infty} (2n\pi, (2n+1)\,\pi) = \cdots \cup (-4\pi, -3\pi) \cup (-2\pi, -\pi) \cup (0, \pi) \cup (2\pi, 3\pi) \cup \cdots$

B. No y-intercept; x-intercepts: $f(x) = 0 \;\Leftrightarrow\; \ln(\sin x) = 0 \;\Leftrightarrow\; \sin x = e^0 = 1 \;\Leftrightarrow\; x = 2n\pi + \frac{\pi}{2}$ for each integer n. **C.** f is periodic with period 2π. **D.** $\lim\limits_{x\to(2n\pi)^+} f(x) = -\infty$ and $\lim\limits_{x\to[(2n+1)\pi]^-} f(x) = -\infty$, so the lines $x = n\pi$ are VAs for all integers n. **E.** $f'(x) = \dfrac{\cos x}{\sin x} = \cot x$, so $f'(x) > 0$ when $2n\pi < x < 2n\pi + \frac{\pi}{2}$ for each integer n, and $f'(x) < 0$ when $2n\pi + \frac{\pi}{2} < x < (2n+1)\pi$. Thus, f is increasing on $\left(2n\pi, 2n\pi + \frac{\pi}{2}\right)$ and decreasing on $\left(2n\pi + \frac{\pi}{2}, (2n+1)\pi\right)$ for each integer n.

F. Local maximum values $f\left(2n\pi + \frac{\pi}{2}\right) = 0$, no local minimum.

G. $f''(x) = -\csc^2 x < 0$, so f is CD on $(2n\pi, (2n+1)\pi)$ for each integer n. No IP

H.



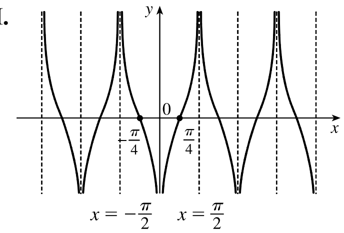
66. $y = \ln(\tan^2 x)$ **A.** $D = \{x \mid x \neq n\pi/2\}$ **B.** x-intercepts $n\pi + \frac{\pi}{4}$, no y-intercept. **C.** $f(-x) = f(x)$, so the curve is symmetric about the y-axis. Also $f(x+\pi) = f(x)$, so f is periodic with period π, and we consider parts D–G only for $-\frac{\pi}{2} < x < \frac{\pi}{2}$. **D.** $\lim\limits_{x\to 0} \ln(\tan^2 x) = -\infty$ and $\lim\limits_{x\to(\pi/2)^-} \ln(\tan^2 x) = \infty$, $\lim\limits_{x\to-(-\pi/2)^+} \ln(\tan^2 x) = \infty$, so $x = 0$, $x = \pm\frac{\pi}{2}$ are VA. **E.** $f'(x) = \dfrac{2\tan x \sec^2 x}{\tan^2 x} = 2\,\dfrac{\sec^2 x}{\tan x} > 0 \;\Leftrightarrow$

$\tan x > 0 \;\Leftrightarrow\; 0 < x < \frac{\pi}{2}$, so f is increasing on $\left(0, \frac{\pi}{2}\right)$ and decreasing on $\left(-\frac{\pi}{2}, 0\right)$. **F.** No maximum or minimum

G. $f'(x) = \dfrac{2}{\sin x \cos x} = \dfrac{4}{\sin 2x} \;\Rightarrow\; f''(x) = \dfrac{-8\cos 2x}{\sin^2 2x} < 0$

$\Leftrightarrow\; \cos 2x > 0 \Leftrightarrow\; -\frac{\pi}{4} < x < \frac{\pi}{4}$, so f is CD on $\left(-\frac{\pi}{4}, 0\right)$ and $\left(0, \frac{\pi}{4}\right)$ and CU on $\left(-\frac{\pi}{2}, -\frac{\pi}{4}\right)$ and $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. IP are $\left(\pm\frac{\pi}{4}, 0\right)$.

H.

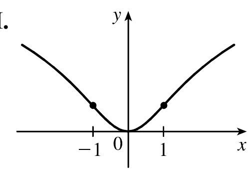


67. $y = f(x) = \ln(1 + x^2)$ **A.** $D = \mathbb{R}$ **B.** Both intercepts are 0. **C.** $f(-x) = f(x)$, so the curve is symmetric about the y-axis. **D.** $\lim\limits_{x\to\pm\infty} \ln(1 + x^2) = \infty$, no asymptotes. **E.** $f'(x) = \dfrac{2x}{1 + x^2} > 0 \;\Leftrightarrow$ $x > 0$, so f is increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$.

F. $f(0) = 0$ is a local and absolute minimum.

G. $f''(x) = \dfrac{2(1 + x^2) - 2x(2x)}{(1 + x^2)^2} = \dfrac{2(1 - x^2)}{(1 + x^2)^2} > 0 \;\Leftrightarrow$ $|x| < 1$, so f is CU on $(-1, 1)$, CD on $(-\infty, -1)$ and $(1, \infty)$. IP $(1, \ln 2)$ and $(-1, \ln 2)$.

H.



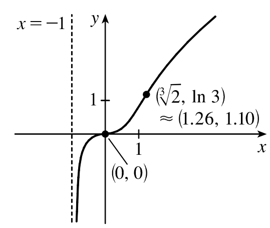
68. $y = f(x) = \ln(1 + x^3)$ **A.** $1 + x^3 > 0 \;\Leftrightarrow\; x^3 > -1 \;\Leftrightarrow\; x > -1$, so $D = (-1, \infty)$. **B.** y-intercept: $f(0) = \ln 1 = 0$; x-intercept: $f(x) = 0 \;\Leftrightarrow\; \ln(1 + x^3) = 0 \;\Leftrightarrow\; 1 + x^3 = e^0 \;\Leftrightarrow\; x^3 = 0 \;\Leftrightarrow\; x = 0$ **C.** No symmetry **D.** $\lim\limits_{x\to-1^+} f(x) = -\infty$, so $x = -1$ is a VA **E.** $f'(x) = \dfrac{3x^2}{1 + x^3}$. $f'(x) > 0$ on $(-1, 0)$ and $(0, \infty)$ $[f'(x) = 0$ at $x = 0]$, so by Exercise 3.3.79, f is increasing on $(-1, \infty)$. **F.** No extreme values

G. $f''(x) = \dfrac{(1 + x^3)(6x) - 3x^2(3x^2)}{(1 + x^3)^2}$

$$= \frac{3x[2(1 + x^3) - 3x^3]}{(1 + x^3)^2} = \frac{3x(2 - x^3)}{(1 + x^3)^2}$$

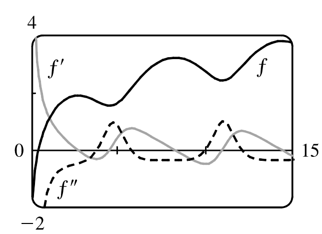
$f''(x) > 0 \;\Leftrightarrow\; 0 < x < \sqrt[3]{2}$, so f is CU on $\left(0, \sqrt[3]{2}\right)$ and f is CD on $(-1, 0)$ and $\left(\sqrt[3]{2}, \infty\right)$. IP at $(0, 0)$ and $\left(\sqrt[3]{2}, \ln 3\right)$

H.



69. We use the CAS to calculate $f'(x) = \dfrac{2 + \sin x + x\cos x}{2x + x\sin x}$ and

$f''(x) = \dfrac{2x^2 \sin x + 4\sin x - \cos^2 x + x^2 + 5}{x^2(\cos^2 x - 4\sin x - 5)}$. From the graphs, it seems that $f' > 0$ (and so f is increasing) on approximately the intervals $(0, 2.7)$, $(4.5, 8.2)$ and $(10.9, 14.3)$. It seems that f'' changes sign (indicating inflection points) at $x \approx 3.8, 5.7, 10.0$ and 12.0.

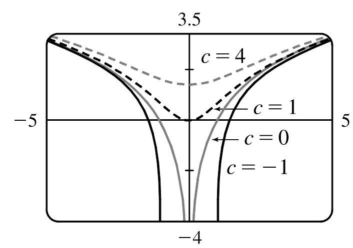


Looking back at the graph of $f(x) = \ln(2x + x\sin x)$, this implies that the inflection points have approximate coordinates $(3.8, 1.7)$, $(5.7, 2.1)$, $(10.0, 2.7)$, and $(12.0, 2.9)$.

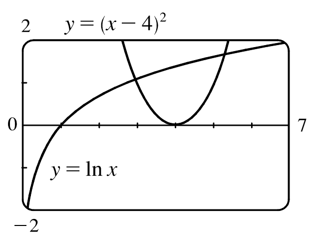
70. We see that if $c \le 0$, $f(x) = \ln(x^2 + c)$ is only defined for $x^2 > -c \;\Rightarrow\; |x| > \sqrt{-c}$, and

$\lim\limits_{x\to\sqrt{-c}^+} f(x) = \lim\limits_{x\to-\sqrt{-c}^-} f(x) = -\infty$, since $\ln y \to -\infty$ as $y \to 0$. Thus, for $c < 0$, there are vertical asymptotes at $x = \pm\sqrt{c}$, and as c decreases (that is, $|c|$ increases), the asymptotes get further apart. For $c = 0$, $\lim\limits_{x\to 0} f(x) = -\infty$, so there is a vertical asymptote at $x = 0$. If $c > 0$, there is no asymptote. To find the maxima, minima, and inflection points, we differentiate: $f(x) = \ln(x^2 + c) \;\Rightarrow\; f'(x) = \dfrac{1}{x^2 + c}(2x)$, so by the First Derivative Test there is a local and absolute minimum at $x = 0$. Differentiating again, we get

$f''(x) = \dfrac{1}{x^2+c}(2) + 2x\left[-\left(x^2+c\right)^{-2}(2x)\right] = \dfrac{2(c - x^2)}{(x^2+c)^2}$. Now if $c \le 0$, this is always negative, so f is concave down on both of the intervals on which it is defined. If $c > 0$, then f'' changes sign when $c = x^2 \;\Leftrightarrow\; x = \pm\sqrt{c}$. So for $c > 0$ there are inflection points at $\pm\sqrt{c}$, and as c increases, the inflection points get further apart.

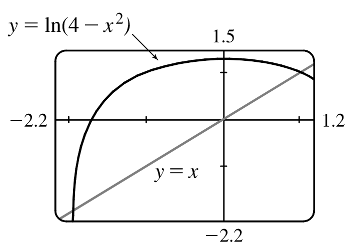


71.



From the graph, it appears that the curves $y = (x-4)^2$ and $y = \ln x$ intersect just to the left of $x = 3$ and to the right of $x = 5$, at about $x = 5.3$. Let $f(x) = \ln x - (x-4)^2$. Then $f'(x) = 1/x - 2(x-4)$, so Newton's method says that $x_{n+1} = x_n - f(x_n)/f'(x_n) = x_n - \dfrac{\ln x_n - (x_n - 4)^2}{1/x_n - 2(x_n - 4)}$. Taking $x_0 = 3$, we get $x_1 \approx 2.957738$, $x_2 \approx 2.958516 \approx x_3$, so the first solution is 2.958516, to six decimal places. Taking $x_0 = 5$, we get $x_1 \approx 5.290755$, $x_2 \approx 5.290718 \approx x_3$, so the second (and final) solution is 5.290718, to six decimal places.

72.

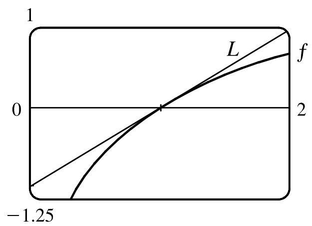


We use Newton's method with $f(x) = \ln(4 - x^2) - x$ and $f'(x) = \dfrac{1}{4 - x^2}(-2x) - 1 = -1 - \dfrac{2x}{4 - x^2}$. The formula is $x_{n+1} = x_n - f(x_n)/f'(x_n)$. From the graphs it seems that the solutions occur at approximately $x = -1.9$ and $x = 1.1$. However, if we use $x_1 = -1.9$ as an initial approximation to the first solution, we get $x_2 \approx -2.009611$, and $f(x) = \ln(x-2)^2 - x$ is undefined at this point, making it impossible to calculate x_3. We must use a more accurate first estimate, such as $x_1 = -1.95$. With this approximation, we get $x_1 = -1.95$, $x_2 \approx -1.1967495$, $x_3 \approx -1.964760$, $x_4 \approx x_5 \approx -1.964636$. Calculating the second solution gives $x_1 = 1.1$, $x_2 \approx 1.058649$, $x_3 \approx 1.058007$, $x_4 \approx x_5 \approx 1.058006$. So, correct to six decimal places, the two solutions of the equation $\ln(4 - x^2) = x$ are $x = -1.964636$ and $x = 1.058006$.

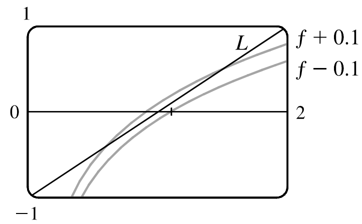
73. (a) Let $f(x) = \ln x \quad \Rightarrow \quad f'(x) = 1/x \quad \Rightarrow \quad f''(x) = -1/x^2$. The linear approximation to $\ln x$ near 1 is

$\ln x \approx f(1) + f'(1)(x-1) = \ln 1 + \frac{1}{1}(x-1) = x - 1$.

(b)

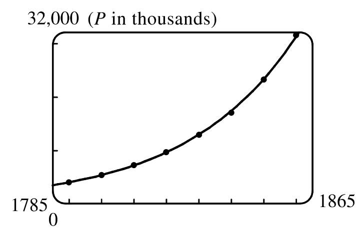


(c)



From the graph, it appears that the linear approximation is accurate to within 0.1 for x between about 0.62 and 1.51.

74. (a) $P = ab^t$ with $a = 4.502714 \times 10^{-20}$ and $b = 1.029953851$, where P is measured in thousands of people. The fit appears to be very good.



(b) **For 1800:** $m_1 = \dfrac{5308 - 3929}{1800 - 1790} = 137.9$, $m_2 = \dfrac{7240 - 5308}{1810 - 1800} = 193.2$.

So $P'(1800) \approx (m_1 + m_2)/2 = 165.55$ thousand people/year.

For 1850: $m_1 = \dfrac{23{,}192 - 17{,}063}{1850 - 1840} = 612.9$, $m_2 = \dfrac{31{,}443 - 23{,}192}{1860 - 1850} = 825.1$.

So $P'(1850) \approx (m_1 + m_2)/2 = 719$ thousand people/year.

(c) Using $P'(t) = ab^t \ln b$ (from Formula 7) with the values of a and b from part (a), we get $P'(1800) \approx 156.85$ thousand people/year and $P'(1850) \approx 686.07$. These estimates are somewhat less than the ones in part (b).

(d) $P(1870) \approx 41{,}946.56$. The difference of 3.4 million people is most likely due to the Civil War (1861–1865).

75. $\displaystyle\int_2^4 \frac{3}{x}\,dx = 3\int_2^4 \frac{1}{x}\,dx = 3\Big[\ln|x|\Big]_2^4 = 3(\ln 4 - \ln 2) = 3\ln\frac{4}{2} = 3\ln 2$

76. Let $u = 5x + 1$, so $du = 5\,dx$. When $x = 0$, $u = 1$; when $x = 3$, $u = 16$. Thus,

$$\int_0^3 \frac{dx}{5x+1} = \int_1^{16} \frac{1}{u}\left(\frac{1}{5}\,du\right) = \frac{1}{5}\Big[\ln|u|\Big]_1^{16} = \frac{1}{5}(\ln 16 - \ln 1) = \frac{1}{5}\ln 16.$$

77. $\displaystyle\int_1^2 \frac{dt}{8-3t} = \left[-\frac{1}{3}\ln|8-3t|\right]_1^2 = -\frac{1}{3}\ln 2 - \left(-\frac{1}{3}\ln 5\right) = \frac{1}{3}(\ln 5 - \ln 2) = \frac{1}{3}\ln\frac{5}{2}$

Or: Let $u = 8 - 3t$. Then $du = -3\,dt$, so

$$\int_1^2 \frac{dt}{8-3t} = \int_5^2 \frac{-\frac{1}{3}\,du}{u} = \left[-\frac{1}{3}\ln|u|\right]_5^2 = -\frac{1}{3}\ln 2 - \left(-\frac{1}{3}\ln 5\right) = \frac{1}{3}(\ln 5 - \ln 2) = \frac{1}{3}\ln\frac{5}{2}.$$

78. $\displaystyle\int_4^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx = \int_4^9 \left(x + 2 + \frac{1}{x}\right) dx = \left[\tfrac{1}{2}x^2 + 2x + \ln x\right]_4^9 = \frac{81}{2} + 18 + \ln 9 - (8 + 8 + \ln 4)$

$\displaystyle\qquad = \tfrac{85}{2} + \ln\tfrac{9}{4}$

79. $\displaystyle\int_1^3 \left(\frac{3x^2+4x+1}{x}\right) dx = \int_1^3 \left(3x+4+\frac{1}{x}\right) dx = \left[\frac{3}{2}x^2+4x+\ln|x|\right]_1^3 = \left(\frac{27}{2}+12+\ln 3\right) - \left(\frac{3}{2}+4+\ln 1\right)$

$= 20 + \ln 3$

80. Let $u = \ln t$. Then $du = \dfrac{1}{t}\,dt$, so $\displaystyle\int \frac{\cos(\ln t)}{t}\,dt = \int \cos u\,du = \sin u + C = \sin(\ln t) + C$.

81. Let $u = \ln x$. Then $du = \dfrac{dx}{x} \quad\Rightarrow\quad \displaystyle\int \frac{(\ln x)^2}{x}\,dx = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}(\ln x)^3 + C$.

82. Let $u = z^3 + 1$. Then $du = 3z^2\,dz$ and $\frac{1}{3}\,du = z^2\,dz$, so

$$\int \frac{z^2}{z^3+1}\,dz = \int \frac{1}{u}\left(\frac{1}{3}\,du\right) = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|z^3+1| + C.$$

83. $\displaystyle\int \frac{\sin 2x}{1+\cos^2 x}\,dx = 2\int \frac{\sin x\cos x}{1+\cos^2 x}\,dx = 2I$. Let $u = \cos x$. Then $du = -\sin x\,dx$, so

$$2I = -2\int \frac{u\,du}{1+u^2} = -2\cdot\tfrac{1}{2}\ln(1+u^2) + C = -\ln(1+u^2) + C = -\ln(1+\cos^2 x) + C.$$

Or: Let $u = 1 + \cos^2 x$.

84. Let $u = e^x + 1$. Then $du = e^x\,dx$, so $\displaystyle\int \frac{e^x}{e^x+1}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln(e^x+1) + C$.

85. $\displaystyle\int_0^4 2^s\,ds = \left[\frac{1}{\ln 2}2^s\right]_0^4 = \frac{16}{\ln 2} - \frac{1}{\ln 2} = \frac{15}{\ln 2}$

86. Let $u = x^2$. Then $du = 2x\,dx$, so $\displaystyle\int x2^{x^2}\,dx = \tfrac{1}{2}\int 2^u\,du = \frac{1}{2}\frac{2^u}{\ln 2} + C = \frac{1}{2\ln 2}2^{x^2} + C$.

87. (a) $\dfrac{d}{dx}(\ln|\sin x| + C) = \dfrac{1}{\sin x}\cos x = \cot x$

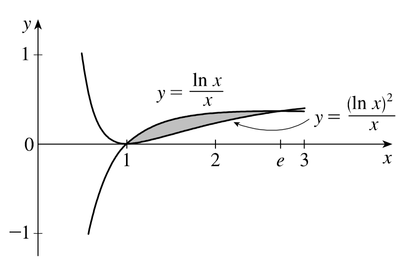
(b) Let $u = \sin x$. Then $du = \cos x\,dx$, so $\displaystyle\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln|\sin x| + C$.

88. $\dfrac{\ln x}{x} = \dfrac{(\ln x)^2}{x} \quad\Leftrightarrow\quad \ln x = (\ln x)^2 \quad\Leftrightarrow\quad 0 = (\ln x)^2 - \ln x \quad\Leftrightarrow$

$0 = \ln x(\ln x - 1) \quad\Leftrightarrow\quad \ln x = 0 \text{ or } 1 \quad\Leftrightarrow\quad x = e^0 \text{ or } e^1$ [1 or e]

$\displaystyle A = \int_1^e \left[\frac{\ln x}{x} - \frac{(\ln x)^2}{x}\right] dx = \left[\tfrac{1}{2}(\ln x)^2 - \tfrac{1}{3}(\ln x)^3\right]_1^e$

$= \left(\tfrac{1}{2} - \tfrac{1}{3}\right) - (0 - 0) = \tfrac{1}{6}$



89. The cross-sectional area is $\pi\left(1/\sqrt{x+1}\right)^2 = \pi/(x+1)$. Therefore, the volume is

$$\int_0^1 \frac{\pi}{x+1}\,dx = \pi[\ln(x+1)]_0^1 = \pi(\ln 2 - \ln 1) = \pi\ln 2.$$

90. Using cylindrical shells, we get $V = \int_0^3 \frac{2\pi x}{x^2+1}\,dx = \pi\big[\ln(1+x^2)\big]_0^3 = \pi \ln 10$.

91. $W = \int_{V_1}^{V_2} P\,dV = \int_{600}^{1000} \frac{C}{V}\,dV = C\int_{600}^{1000} \frac{1}{V}\,dV = C\Big[\ln|V|\Big]_{600}^{1000} = C(\ln 1000 - \ln 600) = C\ln\frac{1000}{600} = C\ln\frac{5}{3}$.

Initially, $PV = C$, where $P = 150$ kPa and $V = 600\text{ cm}^3$, so $C = (150)(600) = 90{,}000\text{ kPa}\cdot\text{cm}^3$. Thus,

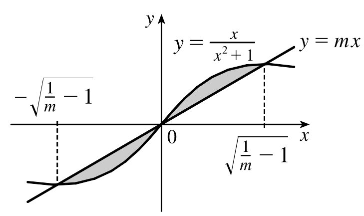
$$\begin{aligned} W &= 90{,}000\ln\tfrac{5}{3}\text{ kPa}\cdot\text{cm}^3 = 90{,}000\ln\tfrac{5}{3}\left(1000\,\frac{\text{N}}{\text{m}^2}\right)\left(\frac{1}{100}\text{ m}\right)^3 \\ &= 90\ln\tfrac{5}{3}\text{ N}\cdot\text{m} \approx 45.974\text{ J} \end{aligned}$$

92. $f''(x) = x^{-2},\ x > 0 \ \Rightarrow\ f'(x) = -1/x + C \ \Rightarrow\ f(x) = -\ln x + Cx + D.\ 0 = f(1) = C + D$ and $0 = f(2) = -\ln 2 + 2C + D = -\ln 2 + 2C - C = -\ln 2 + C \ \Rightarrow\ C = \ln 2$ and $D = -\ln 2$. So $f(x) = -\ln x + (\ln 2)x - \ln 2$.

93. $f(x) = 2x + \ln x \ \Rightarrow\ f'(x) = 2 + 1/x$. If $g = f^{-1}$, then $f(1) = 2 \ \Rightarrow\ g(2) = 1$, so $g'(2) = 1/f'(g(2)) = 1/f'(1) = \frac{1}{3}$.

94. $f(x) = e^x + \ln x \ \Rightarrow\ f'(x) = e^x + 1/x.\ h = f^{-1}$ and $f(1) = e \ \Rightarrow\ h(e) = 1$, so $h'(e) = 1/f'(1) = 1/(e+1)$.

95. The curve and the line will determine a region when they intersect at two or more points. So we solve the equation $x/(x^2+1) = mx \ \Rightarrow\ x = 0$ or $mx^2 + m - 1 = 0 \ \Rightarrow\ x = 0$ or $x = \dfrac{\pm\sqrt{-4(m)(m-1)}}{2m} = \pm\sqrt{\dfrac{1}{m} - 1}$.



Note that if $m = 1$, this has only the solution $x = 0$, and no region is determined. But if $1/m - 1 > 0 \ \Leftrightarrow\ 1/m > 1 \ \Leftrightarrow\ 0 < m < 1$, then there are two solutions. [Another way of seeing this is to observe that the slope of the tangent to $y = x/(x^2+1)$ at the origin is $y' = 1$ and therefore we must have $0 < m < 1$.] Note that we cannot just integrate between the positive and negative roots, since the curve and the line cross at the origin. Since mx and $x/(x^2+1)$ are both odd functions, the total area is twice the area between the curves on the interval $\left[0, \sqrt{1/m - 1}\right]$. So the total area enclosed is

$$\begin{aligned} 2\int_0^{\sqrt{1/m-1}} \left[\frac{x}{x^2+1} - mx\right] dx &= 2\left[\tfrac{1}{2}\ln(x^2+1) - \tfrac{1}{2}mx^2\right]_0^{\sqrt{1/m-1}} \\ &= \left[\ln\left(\frac{1}{m} - 1 + 1\right) - m\left(\frac{1}{m} - 1\right)\right] - (\ln 1 - 0) \\ &= \ln\left(\frac{1}{m}\right) + m - 1 = m - \ln m - 1 \end{aligned}$$

96. Let $m = n/x$. Then $n = xm$, and as $n \to \infty$, $m \to \infty$.

Therefore, $\lim\limits_{n\to\infty}\left(1+\dfrac{x}{n}\right)^n = \lim\limits_{m\to\infty}\left(1+\dfrac{1}{m}\right)^{mx} = \left[\lim\limits_{m\to\infty}\left(1+\dfrac{1}{m}\right)^m\right]^x = e^x$ by Equation 9.

97. If $f(x) = \ln(1+x)$, then $f'(x) = \dfrac{1}{1+x}$, so $f'(0) = 1$.

Thus, $\lim\limits_{x\to 0}\dfrac{\ln(1+x)}{x} = \lim\limits_{x\to 0}\dfrac{f(x)}{x} = \lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x-0} = f'(0) = 1$.

98. (a) $I = \log_2\left(\dfrac{2D}{W}\right) \quad\Rightarrow\quad \dfrac{dI}{dD}\quad [W \text{ constant}] \quad = \dfrac{1}{\left(\dfrac{2D}{W}\right)\ln 2}\cdot\dfrac{2}{W} = \dfrac{1}{D\ln 2}$

As D increases, the rate of change of difficulty decreases.

(b) $I = \log_2\left(\dfrac{2D}{W}\right) \Rightarrow \dfrac{dI}{dW}\quad [D \text{ constant}] \quad = \dfrac{1}{\left(\dfrac{2D}{W}\right)\ln 2}\cdot\left(-2DW^{-2}\right) = \dfrac{W}{2D\ln 2}\cdot\dfrac{-2D}{W^2} = -\dfrac{1}{W\ln 2}$

The negative sign indicates that difficulty decreases with increasing width. While the magnitude of the rate of change decreases with increasing width $\left(\text{that is, } \left|-\dfrac{1}{W\ln 2}\right| = \dfrac{1}{W\ln 2} \text{ decreases as } W \text{ increases}\right)$, the rate of change itself increases (gets closer to zero from the negative side) with increasing values of W.

(c) The answers to (a) and (b) agree with intuition. For fixed width, the difficulty of acquiring a target increases, but less and less so, as the distance to the target increases. Similarly, for a fixed distance to a target, the difficulty of acquiring the target decreases, but less and less so, as the width of the target increases.

99. $y = b^x \quad\Rightarrow\quad y' = b^x \ln b$, so the slope of the tangent line to the curve $y = b^x$ at the point (a, b^a) is $b^a \ln b$. An equation of this tangent line is then $y - b^a = b^a \ln b\,(x-a)$. If c is the x-intercept of this tangent line, then $0 - b^a = b^a \ln b\,(c-a) \quad\Rightarrow$ $-1 = \ln b\,(c-a) \quad\Rightarrow\quad \dfrac{-1}{\ln b} = c - a \quad\Rightarrow\quad |c-a| = \left|\dfrac{-1}{\ln b}\right| = \dfrac{1}{|\ln b|}$. The distance between $(a,0)$ and $(c,0)$ is $|c-a|$, and this distance is the constant $\dfrac{1}{|\ln b|}$ for any a. [*Note*: The absolute value is needed for the case $0 < b < 1$ because $\ln b$ is negative there. If $b > 1$, we can write $a - c = 1/(\ln b)$ as the constant distance between $(a,0)$ and $(c,0)$.]

100. $y = b^x \quad\Rightarrow\quad y' = b^x \ln b$, so the slope of the tangent line to the curve $y = b^x$ at the point (x_0, y_0) is $b^{x_0}\ln b$. An equation of this tangent line is then $y - y_0 = b^{x_0}\ln b\,(x - x_0)$. Since this tangent line must pass through $(0,0)$, we have $0 - y_0 = b^{x_0}\ln b\,(0 - x_0)$, or $y_0 = b^{x_0}(\ln b)\,x_0$. Since (x_0, y_0) is a point on the exponential curve $y = b^x$, we also have $y_0 = b^{x_0}$. Equating the expressions for y_0 gives $b^{x_0} = b^{x_0}(\ln b)\,x_0 \quad\Rightarrow\quad 1 = (\ln b)\,x_0 \quad\Rightarrow\quad x_0 = 1/(\ln b)$.

So $y_0 = b^{x_0} = e^{x_0 \ln b}$ [by Formula 6.3.7] $= e^{(1/(\ln b))\ln b} = e^1 = e$.

6.2* The Natural Logarithmic Function

1. (a) $\ln\sqrt{ab} = \ln(ab)^{1/2} = \frac{1}{2}\ln(ab) = \frac{1}{2}(\ln a + \ln b) = \frac{1}{2}\ln a + \frac{1}{2}\ln b$ [assuming that the variables are positive]

(b)
$$\begin{aligned}\ln\left(\frac{x^4}{\sqrt{x^2-4}}\right) &= \ln x^4 - \ln(x^2-4)^{1/2} && \text{[Law 2]}\\ &= 4\ln x - \tfrac{1}{2}\ln[(x+2)(x-2)] && \text{[Law 3]}\\ &= 4\ln x - \tfrac{1}{2}[\ln(x+2)+\ln(x-2)] && \text{[Law 1]}\\ &= 4\ln x - \tfrac{1}{2}\ln(x+2) - \tfrac{1}{2}\ln(x-2)\end{aligned}$$

2. (a) $\ln\dfrac{x^2}{y^3z^4} = \ln x^2 - \ln(y^3z^4) = 2\ln x - (\ln y^3 + \ln z^4) = 2\ln x - 3\ln y - 4\ln z$

(b)
$$\begin{aligned}\ln\sqrt{\frac{3x}{x-3}} = \ln\left(\frac{3x}{x-3}\right)^{1/2} &= \frac{1}{2}\ln\left(\frac{3x}{x-3}\right) && \text{[Law 3]}\\ &= \tfrac{1}{2}[\ln 3 + \ln x - \ln(x-3)] && \text{[Laws 1 and 2]}\\ &= \tfrac{1}{2}\ln 3 + \tfrac{1}{2}\ln x - \tfrac{1}{2}\ln(x-3)\end{aligned}$$

3. (a) $\ln a - 2\ln b + 3\ln c = \ln a - \ln b^2 + \ln c^3 = \ln\dfrac{a}{b^2} + \ln c^3 = \ln\dfrac{ac^3}{b^2}$

(b) $\ln 4 + \ln a - \frac{1}{3}\ln(a+1) = \ln(4\cdot a) - \ln(a+1)^{1/3} = \ln(4a) - \ln\sqrt[3]{a+1} = \ln\dfrac{4a}{\sqrt[3]{a+1}}$

4. (a)
$$\begin{aligned}\ln 10 + 2\ln 5 &= \ln 10 + \ln 5^2 && \text{[by Law 3]}\\ &= \ln[(10)(25)] && \text{[by Law 1]}\\ &= \ln 250\end{aligned}$$

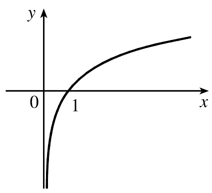
(b)
$$\begin{aligned}3\ln(x-2) - \ln(x^2-5x+6) + 2\ln(x-3) &= \ln(x-2)^3 - \ln[(x-2)(x-3)] + \ln(x-3)^2\\ &= \ln\left[\frac{(x-2)^3(x-3)^2}{(x-2)(x-3)}\right] = \ln[(x-2)^2(x-3)]\end{aligned}$$

5. (a) $\ln 3 + \frac{1}{3}\ln 8 = \ln 3 + \ln 8^{1/3} = \ln 3 + \ln 2 = \ln(3\cdot 2) = \ln 6$

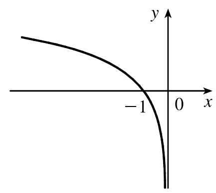
(b)
$$\begin{aligned}\tfrac{1}{3}\ln(x+2)^3 + \tfrac{1}{2}\left[\ln x - \ln(x^2+3x+2)^2\right] &= \ln[(x+2)^3]^{1/3} + \tfrac{1}{2}\ln\frac{x}{(x^2+3x+2)^2} && \text{[by Laws 3, 2]}\\ &= \ln(x+2) + \ln\frac{\sqrt{x}}{x^2+3x+2} && \text{[by Law 3]}\\ &= \ln\frac{(x+2)\sqrt{x}}{(x+1)(x+2)} && \text{[by Law 1]}\\ &= \ln\frac{\sqrt{x}}{x+1}\end{aligned}$$

Note that since $\ln x$ is defined for $x > 0$, we have $x+1$, $x+2$, and x^2+3x+2 all positive, and hence their logarithms are defined.

6. Reflect the graph of $y = \ln x$ about the y-axis to obtain the graph of $y = \ln(-x)$.

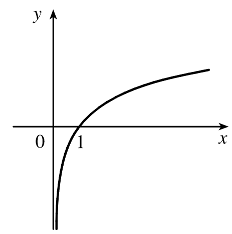


$y = \ln x$

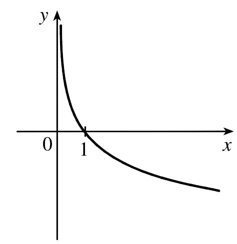


$y = \ln(-x)$

7. Reflect the graph of $y = \ln x$ about the x-axis to obtain the graph of $y = -\ln x$.

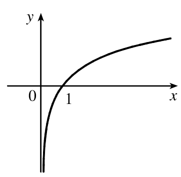


$y = \ln x$

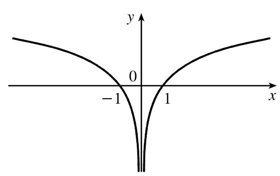


$y = -\ln x$

8. Reflect the portion of the graph of $y = \ln x$ to the right of the y-axis about the y-axis. The graph of $y = \ln|x|$ is that reflection in addition to the original portion.

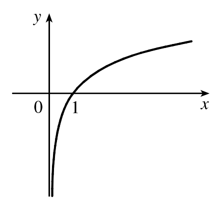


$y = \ln x$

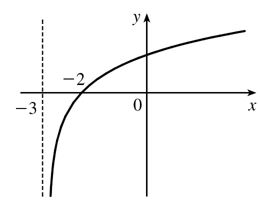


$y = \ln|x|$

9.



$y = \ln x$



$y = \ln(x + 3)$

10. $\lim\limits_{x \to 1^+} \ln(\sqrt{x} - 1) = -\infty$ since $\sqrt{x} - 1 \to 0^+$ as $x \to 1^+$.

11. $\lim\limits_{x \to 0} \ln(\cos x) = \ln 1 = 0$. [$\ln(\cos x)$ is continuous at $x = 0$ since it is the composite of two continuous functions.]

12. $\lim\limits_{x \to 0^+} \ln(\sin x) = -\infty$ since $\sin x \to 0^+$ as $x \to 0^+$.

13. $\lim\limits_{x \to \infty} [\ln(1 + x^2) - \ln(1 + x)] = \lim\limits_{x \to \infty} \ln \dfrac{1 + x^2}{1 + x} = \ln\left(\lim\limits_{x \to \infty} \dfrac{1 + x^2}{1 + x}\right) = \ln\left(\lim\limits_{x \to \infty} \dfrac{\frac{1}{x} + x}{\frac{1}{x} + 1}\right) = \infty$, since the limit in parentheses is ∞.

14. $\lim\limits_{x \to \infty} [\ln(2 + x) - \ln(1 + x)] = \lim\limits_{x \to \infty} \ln\left(\dfrac{2 + x}{1 + x}\right) = \lim\limits_{x \to \infty} \ln\left(\dfrac{2/x + 1}{1/x + 1}\right) = \ln \dfrac{1}{1} = \ln 1 = 0$

15. $f(x) = x^3 \ln x \quad \Rightarrow \quad f'(x) = x^3 \cdot \dfrac{1}{x} + (\ln x) \cdot 3x^2 = x^2 + 3x^2 \ln x = x^2(1 + 3\ln x)$

16. $f(x) = x \ln x - x \quad \Rightarrow \quad f'(x) = x \cdot \dfrac{1}{x} + (\ln x) \cdot 1 - 1 = 1 + \ln x - 1 = \ln x$

17. $f(x) = \ln(x^2 + 3x + 5) \quad \Rightarrow \quad f'(x) = \dfrac{1}{x^2 + 3x + 5} \cdot \dfrac{d}{dx}(x^2 + 3x + 5) = \dfrac{1}{x^2 + 3x + 5} \cdot (2x + 3) = \dfrac{2x + 3}{x^2 + 3x + 5}$

18. $g(t) = \ln(3 + t^2) \quad \Rightarrow \quad g'(t) = \dfrac{1}{3 + t^2} \cdot \dfrac{d}{dt}(3 + t^2) = \dfrac{1}{3 + t^2} \cdot 2t = \dfrac{2t}{3 + t^2}$

19. $f(x) = \sin(\ln x) \quad \Rightarrow \quad f'(x) = \cos(\ln x) \cdot \dfrac{d}{dx} \ln x = \cos(\ln x) \cdot \dfrac{1}{x} = \dfrac{\cos(\ln x)}{x}$

20. $Q(r) = \ln(\cos r) \quad \Rightarrow \quad Q'(r) = \dfrac{1}{\cos r} \cdot \dfrac{d}{dr} \cos r = \dfrac{1}{\cos r}(-\sin r) = -\tan r$

21. $f(x) = \ln \dfrac{1}{x} \quad \Rightarrow \quad f'(x) = \dfrac{1}{1/x} \dfrac{d}{dx}\left(\dfrac{1}{x}\right) = x\left(-\dfrac{1}{x^2}\right) = -\dfrac{1}{x}.$

Another solution: $f(x) = \ln \dfrac{1}{x} = \ln 1 - \ln x = -\ln x \quad \Rightarrow \quad f'(x) = -\dfrac{1}{x}.$

22. $y = \dfrac{1}{\ln x} = (\ln x)^{-1} \quad \Rightarrow \quad y' = -1(\ln x)^{-2} \cdot \dfrac{1}{x} = \dfrac{-1}{x(\ln x)^2}$

23. $f(x) = \sin x \, \ln(5x) \quad \Rightarrow \quad f'(x) = \sin x \cdot \dfrac{1}{5x} \cdot \dfrac{d}{dx}(5x) + \ln(5x) \cdot \cos x = \dfrac{\sin x \cdot 5}{5x} + \cos x \ln(5x) = \dfrac{\sin x}{x} + \cos x \, \ln(5x)$

24. $g(t) = \sqrt{1 + \ln t} \quad \Rightarrow \quad g'(t) = \frac{1}{2}(1 + \ln t)^{-1/2} \dfrac{d}{dt}(1 + \ln t) = \dfrac{1}{2\sqrt{1 + \ln t}} \cdot \dfrac{1}{t} = \dfrac{1}{2t\sqrt{1 + \ln t}}$

25. $F(t) = (\ln t)^2 \sin t \quad \Rightarrow \quad F'(t) = (\ln t)^2 \cos t + \sin t \cdot 2 \ln t \cdot \dfrac{1}{t} = \ln t \left(\ln t \, \cos t + \dfrac{2 \sin t}{t}\right)$

26. $p(t) = \ln \sqrt{t^2 + 1} \quad \Rightarrow \quad p'(t) = \dfrac{1}{\sqrt{t^2 + 1}} \cdot \dfrac{d}{dt}\left(\sqrt{t^2 + 1}\right) = \dfrac{1}{\sqrt{t^2 + 1}} \cdot \dfrac{2t}{2\sqrt{t^2 + 1}} = \dfrac{t}{t^2 + 1}$

Or: $p(t) = \ln \sqrt{t^2 + 1} = \ln(t^2 + 1)^{1/2} = \frac{1}{2} \ln(t^2 + 1) \quad \Rightarrow \quad p'(t) = \dfrac{1}{2} \cdot \dfrac{1}{t^2 + 1} \cdot 2t = \dfrac{t}{t^2 + 1}$

27. $y = (\ln \tan x)^2 \quad \Rightarrow \quad y' = 2(\ln \tan x) \cdot \dfrac{1}{\tan x} \cdot \sec^2 x = \dfrac{2(\ln \tan x) \, \sec^2 x}{\tan x}$ or $y' = \dfrac{2 \ln \tan x}{\sin x \cos x}$

28. $y = \ln(\tan^2 x) = \ln(\tan x)^2 = 2 \ln \tan x \quad \Rightarrow \quad y' = 2 \dfrac{1}{\tan x} \sec^2 x = 2 \dfrac{\cos x}{\sin x} \dfrac{1}{\cos^2 x} = \dfrac{2}{\sin x \, \cos x}$ [or $2 \csc x \, \sec x$]

29. $f(u) = \dfrac{\ln u}{1 + \ln(2u)} \quad \Rightarrow$

$$f'(u) = \frac{[1 + \ln(2u)] \cdot \frac{1}{u} - \ln u \cdot \frac{1}{2u} \cdot 2}{[1 + \ln(2u)]^2} = \frac{\frac{1}{u}[1 + \ln(2u) - \ln u]}{[1 + \ln(2u)]^2} = \frac{1 + (\ln 2 + \ln u) - \ln u}{u[1 + \ln(2u)]^2} = \frac{1 + \ln 2}{u[1 + \ln(2u)]^2}$$

30. $g(x) = \ln \dfrac{a - x}{a + x} = \ln(a - x) - \ln(a + x) \quad \Rightarrow$

$$g'(x) = \frac{1}{a - x}(-1) - \frac{1}{a + x} = \frac{-(a + x) - (a - x)}{(a - x)(a + x)} = \frac{-2a}{a^2 - x^2}$$

31. $g(t) = \ln \dfrac{t(t^2 + 1)^4}{\sqrt[3]{2t - 1}} = \ln t + \ln(t^2 + 1)^4 - \ln \sqrt[3]{2t - 1} = \ln t + 4 \ln(t^2 + 1) - \frac{1}{3} \ln(2t - 1) \quad \Rightarrow$

$$g'(t) = \frac{1}{t} + 4 \cdot \frac{1}{t^2 + 1} \cdot 2t - \frac{1}{3} \cdot \frac{1}{2t - 1} \cdot 2 = \frac{1}{t} + \frac{8t}{t^2 + 1} - \frac{2}{3(2t - 1)}$$

32. $y = \ln \sqrt{\dfrac{1 + 2x}{1 - 2x}} = \ln \sqrt{1 + 2x} - \ln \sqrt{1 - 2x} = \frac{1}{2} \ln(1 + 2x) - \frac{1}{2} \ln(1 - 2x) \quad \Rightarrow$

$$y' = \frac{1}{2} \cdot \frac{1}{1 + 2x} \cdot 2 - \frac{1}{2} \cdot \frac{1}{1 - 2x} \cdot (-2) = \frac{1}{1 + 2x} + \frac{1}{1 - 2x}$$

33. $y = \ln\left|3 - 2x^5\right| \quad \Rightarrow \quad y' = \dfrac{1}{3 - 2x^5} \cdot (-10x^4) = \dfrac{-10x^4}{3 - 2x^5}$

34. $y = \ln(\csc x - \cot x) \quad \Rightarrow$

$$y' = \frac{1}{\csc x - \cot x}\frac{d}{dx}(\csc x - \cot x) = \frac{1}{\csc x - \cot x}(-\csc x\,\cot x + \csc^2 x) = \frac{\csc x(\csc x - \cot x)}{\csc x - \cot x} = \csc x$$

35. $$\frac{d}{dx}\ln\left(x + \sqrt{x^2+1}\right) = \frac{1}{x + \sqrt{x^2+1}} \cdot \frac{d}{dx}\left(x + \sqrt{x^2+1}\right) = \frac{1}{x + \sqrt{x^2+1}} \cdot \left(1 + \frac{2x}{2\sqrt{x^2+1}}\right)$$

$$= \frac{1}{x + \sqrt{x^2+1}} \cdot \left(\frac{\sqrt{x^2+1}}{\sqrt{x^2+1}} + \frac{x}{\sqrt{x^2+1}}\right) = \frac{1}{x + \sqrt{x^2+1}} \cdot \left(\frac{x + \sqrt{x^2+1}}{\sqrt{x^2+1}}\right) = \frac{1}{\sqrt{x^2+1}}$$

36. $$\frac{d}{dx}\ln\sqrt{\frac{1-\cos x}{1+\cos x}} = \frac{d}{dx}\left(\ln\sqrt{1-\cos x} - \ln\sqrt{1+\cos x}\right) = \frac{d}{dx}\left[\frac{1}{2}\ln(1-\cos x) - \frac{1}{2}\ln(1+\cos x)\right]$$

$$= \frac{1}{2} \cdot \frac{1}{1-\cos x} \cdot \sin x - \frac{1}{2} \cdot \frac{1}{1+\cos x} \cdot (-\sin x)$$

$$= \frac{1}{2}\left(\frac{\sin x}{1-\cos x} + \frac{\sin x}{1+\cos x}\right) = \frac{1}{2}\left[\frac{\sin x\,(1+\cos x) + \sin x\,(1-\cos x)}{(1-\cos x)(1+\cos x)}\right]$$

$$= \frac{1}{2}\left(\frac{\sin x + \sin x\,\cos x + \sin x - \sin x\,\cos x}{1-\cos^2 x}\right) = \frac{1}{2}\left(\frac{2\sin x}{\sin^2 x}\right) = \frac{1}{\sin x} = \csc x$$

37. $y = \sqrt{x}\,\ln x \quad \Rightarrow \quad y' = \sqrt{x} \cdot \dfrac{1}{x} + (\ln x)\dfrac{1}{2\sqrt{x}} = \dfrac{2 + \ln x}{2\sqrt{x}} \quad \Rightarrow$

$$y'' = \frac{2\sqrt{x}\,(1/x) - (2+\ln x)(1/\sqrt{x}\,)}{(2\sqrt{x}\,)^2} = \frac{2/\sqrt{x} - (2+\ln x)(1/\sqrt{x}\,)}{4x} = \frac{2 - (2+\ln x)}{\sqrt{x}(4x)} = -\frac{\ln x}{4x\sqrt{x}}$$

38. $y = \dfrac{\ln x}{1 + \ln x} \quad \Rightarrow \quad y' = \dfrac{(1+\ln x)(1/x) - (\ln x)(1/x)}{(1+\ln x)^2} = \dfrac{1}{x(1+\ln x)^2} \quad \Rightarrow$

$$y'' = -\frac{\dfrac{d}{dx}[x(1+\ln x)^2]}{[x(1+\ln x)^2]^2} \quad \text{[Reciprocal Rule]} \quad = -\frac{x \cdot 2(1+\ln x) \cdot (1/x) + (1+\ln x)^2}{x^2(1+\ln x)^4}$$

$$= -\frac{(1+\ln x)[2 + (1+\ln x)]}{x^2(1+\ln x)^4} = -\frac{3+\ln x}{x^2(1+\ln x)^3}$$

39. $y = \ln|\sec x| \quad \Rightarrow \quad y' = \dfrac{1}{\sec x}\dfrac{d}{dx}\sec x = \dfrac{1}{\sec x}\sec x\,\tan x = \tan x \quad \Rightarrow \quad y'' = \sec^2 x$

40. $y = \ln(1 + \ln x) \quad \Rightarrow \quad y' = \dfrac{1}{1+\ln x} \cdot \dfrac{1}{x} = \dfrac{1}{x(1+\ln x)} \quad \Rightarrow$

$$y'' = -\frac{\dfrac{d}{dx}[x(1+\ln x)]}{[x(1+\ln x)]^2} \quad \text{[Reciprocal Rule]} \quad = -\frac{x(1/x) + (1+\ln x)(1)}{x^2(1+\ln x)^2} = -\frac{1+1+\ln x}{x^2(1+\ln x)^2} = -\frac{2+\ln x}{x^2(1+\ln x)^2}$$

41. $f(x) = \dfrac{x}{1 - \ln(x-1)} \quad \Rightarrow$

$$f'(x) = \frac{[1-\ln(x-1)] \cdot 1 - x \cdot \dfrac{-1}{x-1}}{[1-\ln(x-1)]^2} = \frac{\dfrac{(x-1)[1-\ln(x-1)] + x}{x-1}}{[1-\ln(x-1)]^2} = \frac{x - 1 - (x-1)\ln(x-1) + x}{(x-1)[1-\ln(x-1)]^2}$$

$$= \frac{2x - 1 - (x-1)\ln(x-1)}{(x-1)[1-\ln(x-1)]^2}$$

[continued]

$$\mathrm{Dom}(f) = \{x \mid x - 1 > 0 \quad \text{and} \quad 1 - \ln(x-1) \neq 0\} = \{x \mid x > 1 \quad \text{and} \quad \ln(x-1) \neq 1\}$$
$$= \left\{x \mid x > 1 \quad \text{and} \quad x - 1 \neq e^1\right\} = \{x \mid x > 1 \quad \text{and} \quad x \neq 1 + e\} = (1, 1+e) \cup (1+e, \infty)$$

42. $f(x) = \sqrt{2 + \ln x} = (2 + \ln x)^{1/2} \quad \Rightarrow \quad f'(x) = \frac{1}{2}(2 + \ln x)^{-1/2} \cdot \frac{1}{x} = \frac{1}{2x\sqrt{2 + \ln x}}$

$\mathrm{Dom}(f) = \{x \mid 2 + \ln x \geq 0\} = \{x \mid \ln x \geq -2\} = \{x \mid x \geq e^{-2}\} = [e^{-2}, \infty)$.

43. $f(x) = \ln(x^2 - 2x) \quad \Rightarrow \quad f'(x) = \frac{1}{x^2 - 2x}(2x - 2) = \frac{2(x-1)}{x(x-2)}$.

$\mathrm{Dom}(f) = \{x \mid x(x-2) > 0\} = (-\infty, 0) \cup (2, \infty)$.

44. $f(x) = \ln \ln \ln x \quad \Rightarrow \quad f'(x) = \frac{1}{\ln \ln x} \cdot \frac{1}{\ln x} \cdot \frac{1}{x}$.

$\mathrm{Dom}(f) = \{x \mid \ln \ln x > 0\} = \{x \mid \ln x > 1\} = \{x \mid x > e\} = (e, \infty)$.

45. $f(x) = \ln(x + \ln x) \quad \Rightarrow \quad f'(x) = \frac{1}{x + \ln x} \frac{d}{dx}(x + \ln x) = \frac{1}{x + \ln x}\left(1 + \frac{1}{x}\right)$.

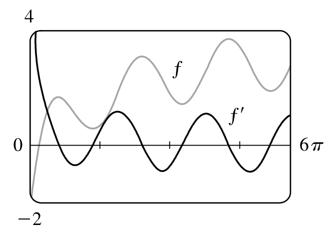
Substitute 1 for x to get $f'(1) = \frac{1}{1 + \ln 1}\left(1 + \frac{1}{1}\right) = \frac{1}{1+0}(1+1) = 1 \cdot 2 = 2$.

46. $f(x) = \frac{\ln x}{x} \quad \Rightarrow \quad f'(x) = \frac{x\left(\frac{1}{x}\right) - (\ln x)(1)}{x^2} = \frac{1 - \ln x}{x^2} \quad \Rightarrow \quad f''(x) = \frac{x^2\left(-\frac{1}{x}\right) - (1 - \ln x)(2x)}{(x^2)^2}$, so

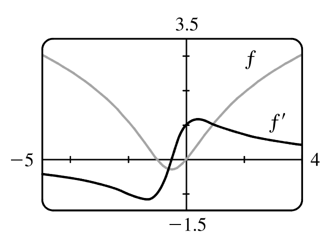
$f''(e) = \frac{-e - 0}{e^4} = -\frac{1}{e^3}$.

47. $f(x) = \sin x + \ln x \quad \Rightarrow \quad f'(x) = \cos x + 1/x$.

This is reasonable, because the graph shows that f increases when f' is positive, and $f'(x) = 0$ when f has a horizontal tangent.



48. $f(x) = \ln(x^2 + x + 1) \quad \Rightarrow \quad f'(x) = \frac{1}{x^2 + x + 1}(2x + 1)$. Notice from the graph that f is increasing when $f'(x)$ is positive.



49. $y = \sin(2 \ln x) \quad \Rightarrow \quad y' = \cos(2 \ln x) \cdot \frac{2}{x}$. At $(1, 0)$, $y' = \cos 0 \cdot \frac{2}{1} = 2$, so an equation of the tangent line is $y - 0 = 2 \cdot (x - 1)$, or $y = 2x - 2$.

50. $y = \ln(x^3 - 7) \quad \Rightarrow \quad y' = \frac{1}{x^3 - 7} \cdot 3x^2 \quad \Rightarrow \quad y'(2) = \frac{12}{8 - 7} = 12$, so an equation of a tangent line at $(2, 0)$ is $y - 0 = 12(x - 2)$ or $y = 12x - 24$.

51. $y = \ln(x^2 + y^2) \quad \Rightarrow \quad y' = \dfrac{1}{x^2 + y^2}\dfrac{d}{dx}(x^2 + y^2) \quad \Rightarrow \quad y' = \dfrac{2x + 2yy'}{x^2 + y^2} \quad \Rightarrow \quad x^2y' + y^2y' = 2x + 2yy' \quad \Rightarrow$

$x^2y' + y^2y' - 2yy' = 2x \quad \Rightarrow \quad (x^2 + y^2 - 2y)y' = 2x \quad \Rightarrow \quad y' = \dfrac{2x}{x^2 + y^2 - 2y}$

52. $\ln xy = \ln x + \ln y = y \sin x \quad \Rightarrow \quad 1/x + y'/y = y\cos x + y' \sin x \quad \Rightarrow \quad y'(1/y - \sin x) = y \cos x - 1/x \quad \Rightarrow$

$y' = \dfrac{y\cos x - 1/x}{1/y - \sin x} = \left(\dfrac{y}{x}\right)\dfrac{xy\cos x - 1}{1 - y\sin x}$

53. $f(x) = \ln(x-1) \quad \Rightarrow \quad f'(x) = \dfrac{1}{(x-1)} = (x-1)^{-1} \quad \Rightarrow \quad f''(x) = -(x-1)^{-2} \quad \Rightarrow \quad f'''(x) = 2(x-1)^{-3} \quad \Rightarrow$

$f^{(4)}(x) = -2 \cdot 3(x-1)^{-4} \quad \Rightarrow \quad \cdots \quad \Rightarrow \quad f^{(n)}(x) = (-1)^{n-1} \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot (n-1)(x-1)^{-n} = (-1)^{n-1}\dfrac{(n-1)!}{(x-1)^n}$

54. $y = x^8 \ln x$, so $D^9 y = D^8 y' = D^8(8x^7 \ln x + x^7)$. But the eighth derivative of x^7 is 0, so we now have

$D^8(8x^7 \ln x) = D^7(8 \cdot 7x^6 \ln x + 8x^6) = D^7(8 \cdot 7x^6 \ln x) = D^6(8 \cdot 7 \cdot 6x^5 \ln x) = \cdots = D(8!\, x^0 \ln x) = 8!/x$.

55. $y = f(x) = \ln(\sin x)$

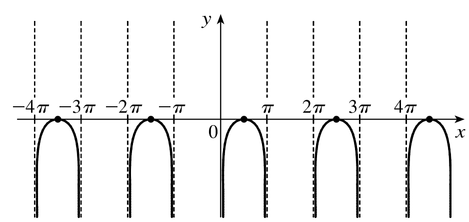
A. $D = \{x \text{ in } \mathbb{R} \mid \sin x > 0\} = \displaystyle\bigcup_{n=-\infty}^{\infty} (2n\pi, (2n+1)\pi) = \cdots \cup (-4\pi, -3\pi) \cup (-2\pi, -\pi) \cup (0, \pi) \cup (2\pi, 3\pi) \cup \cdots$

B. No y-intercept; x-intercepts: $f(x) = 0 \quad \Leftrightarrow \quad \ln(\sin x) = 0 \quad \Leftrightarrow \quad \sin x = e^0 = 1 \quad \Leftrightarrow \quad x = 2n\pi + \frac{\pi}{2}$ for each integer n. **C.** f is periodic with period 2π. **D.** $\displaystyle\lim_{x \to (2n\pi)^+} f(x) = -\infty$ and $\displaystyle\lim_{x \to [(2n+1)\pi]^-} f(x) = -\infty$, so the lines $x = n\pi$ are VAs for all integers n. **E.** $f'(x) = \dfrac{\cos x}{\sin x} = \cot x$, so $f'(x) > 0$ when $2n\pi < x < 2n\pi + \frac{\pi}{2}$ for each integer n, and $f'(x) < 0$ when $2n\pi + \frac{\pi}{2} < x < (2n+1)\pi$. Thus, f is increasing on $\left(2n\pi, 2n\pi + \frac{\pi}{2}\right)$ and decreasing on $\left(2n\pi + \frac{\pi}{2}, (2n+1)\pi\right)$ for each integer n.

F. Local maximum values $f\left(2n\pi + \frac{\pi}{2}\right) = 0$, no local minimum.

G. $f''(x) = -\csc^2 x < 0$, so f is CD on $(2n\pi, (2n+1)\pi)$ for each integer n. No IP

H.

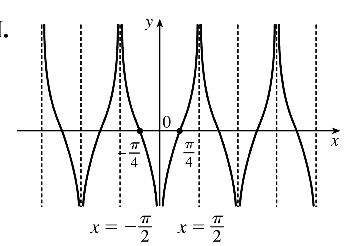


56. $y = \ln(\tan^2 x)$ **A.** $D = \{x \mid x \neq n\pi/2\}$ **B.** x-intercepts $n\pi + \frac{\pi}{4}$, no y-intercept. **C.** $f(-x) = f(x)$, so the curve is symmetric about the y-axis. Also $f(x + \pi) = f(x)$, so f is periodic with period π, and we consider parts D–G only for $-\frac{\pi}{2} < x < \frac{\pi}{2}$. **D.** $\displaystyle\lim_{x \to 0} \ln(\tan^2 x) = -\infty$ and $\displaystyle\lim_{x \to (\pi/2)^-} \ln(\tan^2 x) = \infty$, $\displaystyle\lim_{x \to -(-\pi/2)^+} \ln(\tan^2 x) = \infty$, so $x = 0$, $x = \pm\frac{\pi}{2}$ are VA. **E.** $f'(x) = \dfrac{2\tan x \sec^2 x}{\tan^2 x} = 2\,\dfrac{\sec^2 x}{\tan x} > 0 \quad \Leftrightarrow$ $\tan x > 0 \quad \Leftrightarrow \quad 0 < x < \frac{\pi}{2}$, so f is increasing on $\left(0, \frac{\pi}{2}\right)$ and decreasing on $\left(-\frac{\pi}{2}, 0\right)$. **F.** No maximum or minimum

G. $f'(x) = \dfrac{2}{\sin x \cos x} = \dfrac{4}{\sin 2x} \quad \Rightarrow \quad f''(x) = \dfrac{-8\cos 2x}{\sin^2 2x} < 0$

$\Leftrightarrow \quad \cos 2x > 0 \Leftrightarrow \quad -\frac{\pi}{4} < x < \frac{\pi}{4}$, so f is CD on $\left(-\frac{\pi}{4}, 0\right)$ and $\left(0, \frac{\pi}{4}\right)$ and CU on $\left(-\frac{\pi}{2}, -\frac{\pi}{4}\right)$ and $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. IP are $\left(\pm\frac{\pi}{4}, 0\right)$.

H.



57. $y = f(x) = \ln(1 + x^2)$ **A.** $D = \mathbb{R}$ **B.** Both intercepts are 0. **C.** $f(-x) = f(x)$, so the curve is symmetric about the y-axis. **D.** $\lim\limits_{x \to \pm\infty} \ln(1 + x^2) = \infty$, no asymptotes. **E.** $f'(x) = \dfrac{2x}{1 + x^2} > 0 \;\Leftrightarrow$ $x > 0$, so f is increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$.

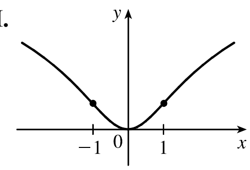
F. $f(0) = 0$ is a local and absolute minimum.

G. $f''(x) = \dfrac{2(1 + x^2) - 2x(2x)}{(1 + x^2)^2} = \dfrac{2(1 - x^2)}{(1 + x^2)^2} > 0 \;\Leftrightarrow$

$|x| < 1$, so f is CU on $(-1, 1)$, CD on $(-\infty, -1)$ and $(1, \infty)$.

IP $(1, \ln 2)$ and $(-1, \ln 2)$.

H.



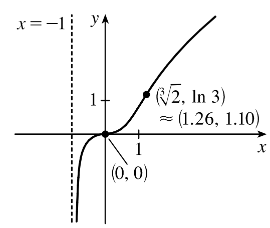
58. $y = f(x) = \ln(1 + x^3)$ **A.** $1 + x^3 > 0 \;\Leftrightarrow\; x^3 > -1 \;\Leftrightarrow\; x > -1$, so $D = (-1, \infty)$. **B.** y-intercept: $f(0) = \ln 1 = 0$; x-intercept: $f(x) = 0 \;\Leftrightarrow\; \ln(1 + x^3) = 0 \;\Leftrightarrow\; 1 + x^3 = e^0 \;\Leftrightarrow\; x^3 = 0 \;\Leftrightarrow\; x = 0$ **C.** No symmetry **D.** $\lim\limits_{x \to -1^+} f(x) = -\infty$, so $x = -1$ is a VA **E.** $f'(x) = \dfrac{3x^2}{1 + x^3}$. $f'(x) > 0$ on $(-1, 0)$ and $(0, \infty)$ $[f'(x) = 0$ at $x = 0]$, so by Exercise 3.3.79, f is increasing on $(-1, \infty)$. **F.** No extreme values

G. $f''(x) = \dfrac{(1 + x^3)(6x) - 3x^2(3x^2)}{(1 + x^3)^2}$

$$= \frac{3x[2(1 + x^3) - 3x^3]}{(1 + x^3)^2} = \frac{3x(2 - x^3)}{(1 + x^3)^2}$$

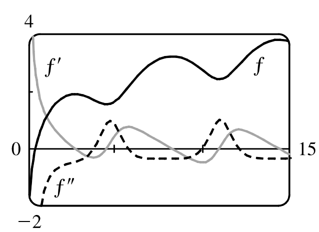
$f''(x) > 0 \;\Leftrightarrow\; 0 < x < \sqrt[3]{2}$, so f is CU on $\left(0, \sqrt[3]{2}\right)$ and f is CD on $(-1, 0)$ and $\left(\sqrt[3]{2}, \infty\right)$. IP at $(0, 0)$ and $\left(\sqrt[3]{2}, \ln 3\right)$

H.



59. We use the CAS to calculate $f'(x) = \dfrac{2 + \sin x + x\cos x}{2x + x\sin x}$ and

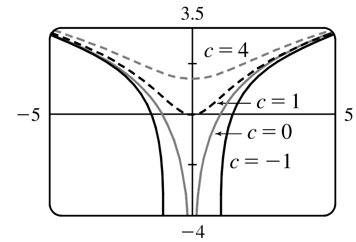
$f''(x) = \dfrac{2x^2 \sin x + 4\sin x - \cos^2 x + x^2 + 5}{x^2(\cos^2 x - 4\sin x - 5)}$. From the graphs, it seems that $f' > 0$ (and so f is increasing) on approximately the intervals $(0, 2.7)$, $(4.5, 8.2)$ and $(10.9, 14.3)$. It seems that f'' changes sign (indicating inflection points) at $x \approx 3.8, 5.7, 10.0$ and 12.0.



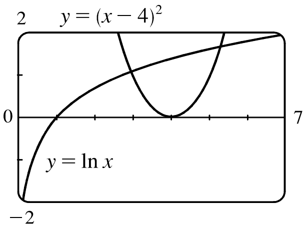
Looking back at the graph of $f(x) = \ln(2x + x\sin x)$, this implies that the inflection points have approximate coordinates $(3.8, 1.7)$, $(5.7, 2.1)$, $(10.0, 2.7)$, and $(12.0, 2.9)$.

60. We see that if $c \le 0$, $f(x) = \ln(x^2 + c)$ is only defined for $x^2 > -c \;\Rightarrow\; |x| > \sqrt{-c}$, and $\lim\limits_{x \to \sqrt{-c}^+} f(x) = \lim\limits_{x \to -\sqrt{-c}^-} f(x) = -\infty$, since $\ln y \to -\infty$ as $y \to 0$. Thus, for $c < 0$, there are vertical asymptotes at $x = \pm\sqrt{c}$, and as c decreases (that is, $|c|$ increases), the asymptotes get further apart. For $c = 0$, $\lim\limits_{x \to 0} f(x) = -\infty$, so there is a vertical asymptote at $x = 0$. If $c > 0$, there is no asymptote. To find the maxima, minima, and inflection points, we differentiate: $f(x) = \ln(x^2 + c) \;\Rightarrow\; f'(x) = \dfrac{1}{x^2 + c}\,(2x)$, so by the First Derivative Test there is a local and absolute minimum at $x = 0$. Differentiating again, we get

$f''(x) = \dfrac{1}{x^2+c}\,(2) + 2x\left[-\left(x^2+c\right)^{-2}(2x)\right] = \dfrac{2(c-x^2)}{(x^2+c)^2}$. Now if $c \le 0$, this is always negative, so f is concave down on both of the intervals on which it is defined. If $c > 0$, then f'' changes sign when $c = x^2$ $\Leftrightarrow$ $x = \pm\sqrt{c}$. So for $c > 0$ there are inflection points at $\pm\sqrt{c}$, and as c increases, the inflection points get further apart.

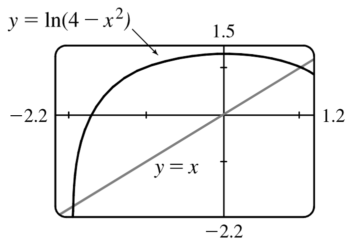


61.



From the graph, it appears that the curves $y = (x-4)^2$ and $y = \ln x$ intersect just to the left of $x = 3$ and to the right of $x = 5$, at about $x = 5.3$. Let $f(x) = \ln x - (x-4)^2$. Then $f'(x) = 1/x - 2(x-4)$, so Newton's method says that $x_{n+1} = x_n - f(x_n)/f'(x_n) = x_n - \dfrac{\ln x_n - (x_n-4)^2}{1/x_n - 2(x_n-4)}$. Taking $x_0 = 3$, we get $x_1 \approx 2.957738$, $x_2 \approx 2.958516 \approx x_3$, so the first solution is 2.958516, to six decimal places. Taking $x_0 = 5$, we get $x_1 \approx 5.290755$, $x_2 \approx 5.290718 \approx x_3$, so the second (and final) solution is 5.290718, to six decimal places.

62.



We use Newton's method with $f(x) = \ln(4-x^2) - x$ and $f'(x) = \dfrac{1}{4-x^2}\,(-2x) - 1 = -1 - \dfrac{2x}{4-x^2}$. The formula is $x_{n+1} = x_n - f(x_n)/f'(x_n)$. From the graphs it seems that the solutions occur at approximately $x = -1.9$ and $x = 1.1$. However, if we use $x_1 = -1.9$ as an initial approximation to the first solution, we get $x_2 \approx -2.009611$, and $f(x) = \ln(x-2)^2 - x$ is undefined at this point, making it impossible to calculate x_3. We must use a more accurate first estimate, such as $x_1 = -1.95$. With this approximation, we get $x_1 = -1.95$, $x_2 \approx -1.1967495$, $x_3 \approx -1.964760$, $x_4 \approx x_5 \approx -1.964636$. Calculating the second solution gives $x_1 = 1.1$, $x_2 \approx 1.058649$, $x_3 \approx 1.058007$, $x_4 \approx x_5 \approx 1.058006$. So, correct to six decimal places, the two solutions of the equation $\ln(4-x^2) = x$ are $x = -1.964636$ and $x = 1.058006$.

63. $y = (x^2+2)^2(x^4+4)^4 \;\Rightarrow\; \ln y = \ln[(x^2+2)^2(x^4+4)^4] \;\Rightarrow\; \ln y = 2\ln(x^2+2) + 4\ln(x^4+4) \;\Rightarrow$

$\dfrac{1}{y}\,y' = 2\cdot\dfrac{1}{x^2+2}\cdot 2x + 4\cdot\dfrac{1}{x^4+4}\cdot 4x^3 \;\Rightarrow\; y' = y\left(\dfrac{4x}{x^2+2} + \dfrac{16x^3}{x^4+4}\right) \;\Rightarrow$

$y' = (x^2+2)^2(x^4+4)^4\left(\dfrac{4x}{x^2+2} + \dfrac{16x^3}{x^4+4}\right)$

64. $y = \dfrac{(x+1)^4(x-5)^3}{(x-3)^8} \;\Rightarrow\; \ln|y| = 4\ln|x+1| + 3\ln|x-5| - 8\ln|x-3| \;\Rightarrow$

$\dfrac{y'}{y} = \dfrac{4}{x+1} + \dfrac{3}{x-5} - \dfrac{8}{x-3} \;\Rightarrow\; y' = \dfrac{(x+1)^4(x-5)^3}{(x-3)^8}\left(\dfrac{4}{x+1} + \dfrac{3}{x-5} - \dfrac{8}{x-3}\right)$

65. $y = \sqrt{\dfrac{x-1}{x^4+1}} \;\Rightarrow\; \ln y = \ln\left(\dfrac{x-1}{x^4+1}\right)^{1/2} \;\Rightarrow\; \ln y = \dfrac{1}{2}\ln(x-1) - \dfrac{1}{2}\ln(x^4+1) \;\Rightarrow$

$\dfrac{1}{y}\,y' = \dfrac{1}{2}\dfrac{1}{x-1} - \dfrac{1}{2}\dfrac{1}{x^4+1}\cdot 4x^3 \;\Rightarrow\; y' = y\left(\dfrac{1}{2(x-1)} - \dfrac{2x^3}{x^4+1}\right) \;\Rightarrow\; y' = \sqrt{\dfrac{x-1}{x^4+1}}\left(\dfrac{1}{2x-2} - \dfrac{2x^3}{x^4+1}\right)$

66. $y = \dfrac{(x^3+1)^4 \sin^2 x}{x^{1/3}} \quad \Rightarrow \quad \ln|y| = 4\ln\left|x^3+1\right| + 2\ln|\sin x| - \frac{1}{3}\ln|x|$. So $\dfrac{y'}{y} = 4\dfrac{3x^2}{x^3+1} + 2\dfrac{\cos x}{\sin x} - \dfrac{1}{3x} \quad \Rightarrow$

$$y' = \frac{(x^3+1)^4 \sin^2 x}{x^{1/3}}\left(\frac{12x^2}{x^3+1} + 2\cot x - \frac{1}{3x}\right).$$

67. $\displaystyle\int_2^4 \frac{3}{x}\,dx = 3\int_2^4 \frac{1}{x}\,dx = 3\Big[\ln|x|\Big]_2^4 = 3(\ln 4 - \ln 2) = 3\ln\frac{4}{2} = 3\ln 2$

68. Let $u = 5x+1$, so $du = 5\,dx$. When $x=0$, $u=1$; when $x=3$, $u=16$. Thus,

$$\int_0^3 \frac{dx}{5x+1} = \int_1^{16} \frac{1}{u}\left(\frac{1}{5}\,du\right) = \frac{1}{5}\Big[\ln|u|\Big]_1^{16} = \frac{1}{5}(\ln 16 - \ln 1) = \frac{1}{5}\ln 16.$$

69. $\displaystyle\int_1^2 \frac{dt}{8-3t} = \left[-\frac{1}{3}\ln|8-3t|\right]_1^2 = -\frac{1}{3}\ln 2 - \left(-\frac{1}{3}\ln 5\right) = \frac{1}{3}(\ln 5 - \ln 2) = \frac{1}{3}\ln\frac{5}{2}$

Or: Let $u = 8-3t$. Then $du = -3\,dt$, so

$$\int_1^2 \frac{dt}{8-3t} = \int_5^2 \frac{-\frac{1}{3}\,du}{u} = \left[-\frac{1}{3}\ln|u|\right]_5^2 = -\frac{1}{3}\ln 2 - \left(-\frac{1}{3}\ln 5\right) = \frac{1}{3}(\ln 5 - \ln 2) = \frac{1}{3}\ln\frac{5}{2}.$$

70. $\displaystyle\int_4^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx = \int_4^9 \left(x + 2 + \frac{1}{x}\right) dx = \left[\tfrac{1}{2}x^2 + 2x + \ln x\right]_4^9 = \frac{81}{2} + 18 + \ln 9 - (8+8+\ln 4)$

$\displaystyle\qquad = \tfrac{85}{2} + \ln\tfrac{9}{4}$

71. $\displaystyle\int_1^3 \left(\frac{3x^2+4x+1}{x}\right) dx = \int_1^3 \left(3x + 4 + \frac{1}{x}\right) dx = \left[\frac{3}{2}x^2 + 4x + \ln|x|\right]_1^3 = \left(\frac{27}{2} + 12 + \ln 3\right) - \left(\frac{3}{2} + 4 + \ln 1\right)$

$\qquad = 20 + \ln 3$

72. Let $u = \ln x$. Then $du = \dfrac{1}{x}\,dx$, so $\displaystyle\int_e^6 \frac{dx}{x\ln x} = \int_1^{\ln 6} \frac{1}{u}\,du = \Big[\ln|u|\Big]_1^{\ln 6} = \ln\ln 6 - \ln 1 = \ln\ln 6$

73. Let $u = \ln x$. Then $du = \dfrac{dx}{x} \quad \Rightarrow \quad \displaystyle\int \frac{(\ln x)^2}{x}\,dx = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}(\ln x)^3 + C.$

74. Let $u = z^3+1$. Then $du = 3z^2\,dz$ and $\frac{1}{3}\,du = z^2\,dz$, so

$$\int \frac{z^2}{z^3+1}\,dz = \int \frac{1}{u}\left(\frac{1}{3}\,du\right) = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln\left|z^3+1\right| + C.$$

75. $\displaystyle\int \frac{\sin 2x}{1+\cos^2 x}\,dx = 2\int \frac{\sin x\cos x}{1+\cos^2 x}\,dx = 2I$. Let $u = \cos x$. Then $du = -\sin x\,dx$, so

$$2I = -2\int \frac{u\,du}{1+u^2} = -2\cdot\tfrac{1}{2}\ln(1+u^2) + C = -\ln(1+u^2) + C = -\ln(1+\cos^2 x) + C.$$

Or: Let $u = 1 + \cos^2 x$.

76. Let $u = \ln t$. Then $du = \dfrac{1}{t}\,dt$, so $\displaystyle\int \frac{\cos(\ln t)}{t}\,dt = \int \cos u\,du = \sin u + C = \sin(\ln t) + C.$

77. (a) $\dfrac{d}{dx}\left(\ln|\sin x| + C\right) = \dfrac{1}{\sin x}\cos x = \cot x$

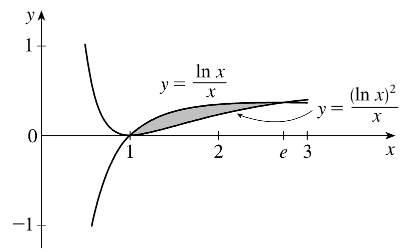
(b) Let $u = \sin x$. Then $du = \cos x\,dx$, so $\displaystyle\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln|\sin x| + C.$

78. $\dfrac{\ln x}{x} = \dfrac{(\ln x)^2}{x} \quad\Leftrightarrow\quad \ln x = (\ln x)^2 \quad\Leftrightarrow\quad 0 = (\ln x)^2 - \ln x \quad\Leftrightarrow$

$0 = \ln x(\ln x - 1) \quad\Leftrightarrow\quad \ln x = 0 \text{ or } 1 \quad\Leftrightarrow\quad x = e^0 \text{ or } e^1 \;\; [1 \text{ or } e]$

$$A = \int_1^e \left[\frac{\ln x}{x} - \frac{(\ln x)^2}{x}\right] dx = \left[\tfrac{1}{2}(\ln x)^2 - \tfrac{1}{3}(\ln x)^3\right]_1^e$$

$$= \left(\tfrac{1}{2} - \tfrac{1}{3}\right) - (0 - 0) = \tfrac{1}{6}$$



79. The cross-sectional area is $\pi\left(1/\sqrt{x+1}\right)^2 = \pi/(x+1)$. Therefore, the volume is

$$\int_0^1 \frac{\pi}{x+1}\,dx = \pi[\ln(x+1)]_0^1 = \pi(\ln 2 - \ln 1) = \pi \ln 2.$$

80. Using cylindrical shells, we get $V = \displaystyle\int_0^3 \frac{2\pi x}{x^2+1}\,dx = \pi\left[\ln(1+x^2)\right]_0^3 = \pi \ln 10.$

81. $W = \int_{V_1}^{V_2} P\,dV = \displaystyle\int_{600}^{1000} \frac{C}{V}\,dV = C\int_{600}^{1000} \frac{1}{V}\,dV = C\left[\ln|V|\right]_{600}^{1000} = C(\ln 1000 - \ln 600) = C\ln\tfrac{1000}{600} = C\ln\tfrac{5}{3}.$

Initially, $PV = C$, where $P = 150$ kPa and $V = 600\text{ cm}^3$, so $C = (150)(600) = 90{,}000\text{ kPa}\cdot\text{cm}^3$. Thus,

$$W = 90{,}000\ln\tfrac{5}{3}\text{ kPa}\cdot\text{cm}^3 = 90{,}000\ln\tfrac{5}{3}\left(1000\,\frac{\text{N}}{\text{m}^2}\right)\left(\frac{1}{100}\text{ m}\right)^3$$

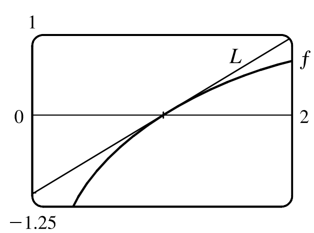
$$= 90\ln\tfrac{5}{3}\text{ N}\cdot\text{m} \approx 45.974\text{ J}$$

82. $f''(x) = x^{-2},\ x > 0 \quad\Rightarrow\quad f'(x) = -1/x + C \quad\Rightarrow\quad f(x) = -\ln x + Cx + D.\ \ 0 = f(1) = C + D$ and $0 = f(2) = -\ln 2 + 2C + D = -\ln 2 + 2C - C = -\ln 2 + C \quad\Rightarrow\quad C = \ln 2$ and $D = -\ln 2$. So $f(x) = -\ln x + (\ln 2)x - \ln 2$.

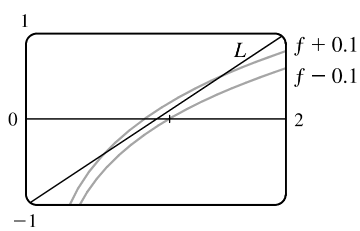
83. $f(x) = 2x + \ln x \quad\Rightarrow\quad f'(x) = 2 + 1/x$. If $g = f^{-1}$, then $f(1) = 2 \quad\Rightarrow\quad g(2) = 1$, so $g'(2) = 1/f'(g(2)) = 1/f'(1) = \frac{1}{3}$.

84. (a) Let $f(x) = \ln x \quad\Rightarrow\quad f'(x) = 1/x \quad\Rightarrow\quad f''(x) = -1/x^2$. The linear approximation to $\ln x$ near 1 is $\ln x \approx f(1) + f'(1)(x-1) = \ln 1 + \frac{1}{1}(x-1) = x - 1$.

(b)

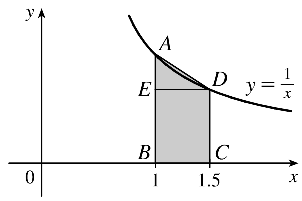


(c)



From the graph, it appears that the linear approximation is accurate to within 0.1 for x between about 0.62 and 1.51.

85. (a)



We interpret $\ln 1.5$ as the area under the curve $y = 1/x$ from $x = 1$ to $x = 1.5$. The area of the rectangle $BCDE$ is $\frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$. The area of the trapezoid $ABCD$ is $\frac{1}{2} \cdot \frac{1}{2}\left(1 + \frac{2}{3}\right) = \frac{5}{12}$. Thus, by comparing areas, we observe that $\frac{1}{3} < \ln 1.5 < \frac{5}{12}$.

(b) With $f(t) = 1/t$, $n = 10$, and $\Delta t = 0.05$, we have

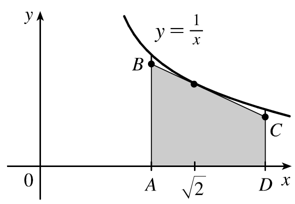
$$\begin{aligned} \ln 1.5 &= \textstyle\int_1^{1.5} (1/t)\,dt \approx (0.05)[f(1.025) + f(1.075) + \cdots + f(1.475)] \\ &= (0.05)\left[\tfrac{1}{1.025} + \tfrac{1}{1.075} + \cdots + \tfrac{1}{1.475}\right] \approx 0.4054 \end{aligned}$$

86. (a) $y = \dfrac{1}{t}$, $y' = -\dfrac{1}{t^2}$. The slope of AD is $\dfrac{1/2 - 1}{2 - 1} = -\dfrac{1}{2}$. Let c be the t-coordinate of the point on $y = \dfrac{1}{t}$ with slope $-\dfrac{1}{2}$.

Then $-\dfrac{1}{c^2} = -\dfrac{1}{2} \;\Rightarrow\; c^2 = 2 \;\Rightarrow\; c = \sqrt{2}$ since $c > 0$. Therefore the tangent line is given by

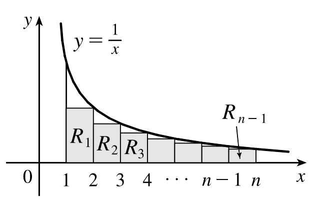
$y - \frac{1}{\sqrt{2}} = -\frac{1}{2}\left(t - \sqrt{2}\right) \;\Rightarrow\; y = -\frac{1}{2}t + \sqrt{2}$.

(b)

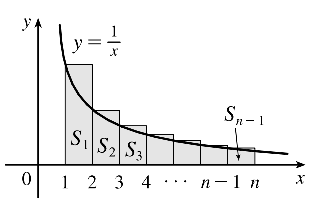


Since the graph of $y = 1/t$ is concave upward, the graph lies above the tangent line, that is, above the line segment BC. Now $|AB| = -\frac{1}{2} + \sqrt{2}$ and $|CD| = -1 + \sqrt{2}$. So the area of the trapezoid $ABCD$ is $\frac{1}{2}\left[\left(-\frac{1}{2} + \sqrt{2}\right) + \left(-1 + \sqrt{2}\right)1\right] = -\frac{3}{4} + \sqrt{2} \approx 0.6642$. So $\ln 2 >$ area of trapezoid $ABCD > 0.66$.

87.



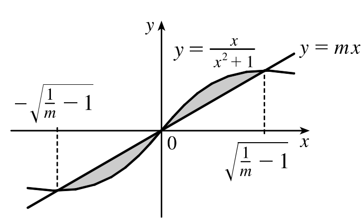
The area of R_i is $\dfrac{1}{i+1}$ and so $\dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n} < \displaystyle\int_1^n \frac{1}{t}\,dt = \ln n$.



The area of S_i is $\dfrac{1}{i}$ and so $1 + \dfrac{1}{2} + \cdots + \dfrac{1}{n-1} > \displaystyle\int_1^n \frac{1}{t}\,dt = \ln n$.

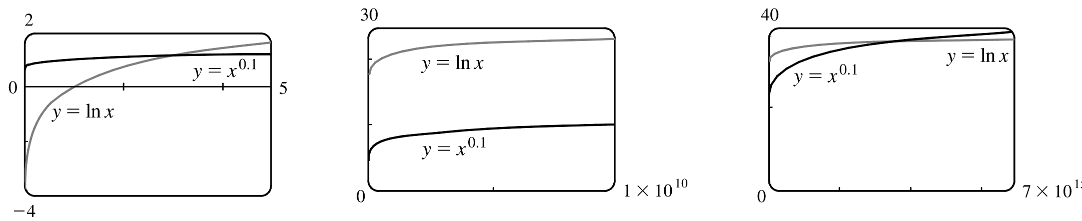
88. If $f(x) = \ln(x^r)$, then $f'(x) = (1/x^r)(rx^{r-1}) = r/x$. But if $g(x) = r\ln x$, then $g'(x) = r/x$. So f and g must differ by a constant: $\ln(x^r) = r\ln x + C$. Put $x = 1$: $\ln(1^r) = r\ln 1 + C \;\Rightarrow\; C = 0$, so $\ln(x^r) = r\ln x$.

89. The curve and the line will determine a region when they intersect at two or more points. So we solve the equation $x/(x^2 + 1) = mx \;\Rightarrow\; x = 0$ or $mx^2 + m - 1 = 0 \;\Rightarrow\; x = 0$ or $x = \dfrac{\pm\sqrt{-4(m)(m-1)}}{2m} = \pm\sqrt{\dfrac{1}{m} - 1}$.



Note that if $m = 1$, this has only the solution $x = 0$, and no region is determined. But if $1/m - 1 > 0 \quad \Leftrightarrow \quad 1/m > 1 \quad \Leftrightarrow \quad 0 < m < 1$, then there are two solutions. [Another way of seeing this is to observe that the slope of the tangent to $y = x/(x^2+1)$ at the origin is $y' = 1$ and therefore we must have $0 < m < 1$.] Note that we cannot just integrate between the positive and negative roots, since the curve and the line cross at the origin. Since mx and $x/(x^2+1)$ are both odd functions, the total area is twice the area between the curves on the interval $\left[0, \sqrt{1/m - 1}\right]$. So the total area enclosed is

$$\begin{aligned} 2\int_0^{\sqrt{1/m-1}} \left[\frac{x}{x^2+1} - mx\right] dx &= 2\left[\tfrac{1}{2}\ln(x^2+1) - \tfrac{1}{2}mx^2\right]_0^{\sqrt{1/m-1}} \\ &= \left[\ln\left(\frac{1}{m} - 1 + 1\right) - m\left(\frac{1}{m} - 1\right)\right] - (\ln 1 - 0) \\ &= \ln\left(\frac{1}{m}\right) + m - 1 = m - \ln m - 1 \end{aligned}$$

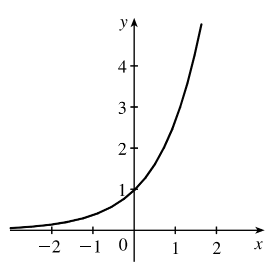
90. 

From the graphs, we see that $f(x) = x^{0.1} > g(x) = \ln x$ for approximately $0 < x < 3.06$, and then $g(x) > f(x)$ for $3.06 < x < 3.43 \times 10^{15}$ (approximately). At that point, the graph of f finally surpasses the graph of g for good.

91. If $f(x) = \ln(1+x)$, then $f'(x) = \dfrac{1}{1+x}$, so $f'(0) = 1$.

Thus, $\displaystyle\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = f'(0) = 1$.

6.3* The Natural Exponential Function

1. 

The function value at $x = 0$ is 1 and the slope at $x = 0$ is 1; that is, if $f(x) = e^x$, then $f'(0) = 1$.

2. (a) $e^{\ln 15} = 15$ by (4). (b) $e^{3\ln 2} = e^{\ln 2^3} = 2^3 = 8$ (c) $e^{-2\ln 5} = e^{\ln 5^{-2}} = 5^{-2} = \frac{1}{25}$

3. (a) $\ln \dfrac{1}{e^2} = \ln e^{-2} = -2$ (b) $\ln\sqrt{e} = \ln e^{1/2} = \frac{1}{2}$ (c) $\ln e^{\sin x} = \sin x$

4. (a) $\ln\left(\ln e^{e^{50}}\right) = \ln(e^{50}) = 50$ (b) $e^{\ln(\ln e^3)} = e^{\ln(3)} = 3$ (c) $e^{x + \ln x} = e^x e^{\ln x} = xe^x$

5. (a) $\ln(4x+2) = 3 \quad\Rightarrow\quad e^{\ln(4x+2)} = e^3 \quad\Rightarrow\quad 4x+2 = e^3 \quad\Rightarrow\quad 4x = e^3 - 2 \quad\Rightarrow\quad x = \frac{1}{4}(e^3-2) \approx 4.521$

(b) $e^{2x-3} = 12 \quad\Rightarrow\quad \ln e^{2x-3} = \ln 12 \quad\Rightarrow\quad 2x-3 = \ln 12 \quad\Rightarrow\quad 2x = 3 + \ln 12 \quad\Rightarrow\quad x = \frac{1}{2}(3+\ln 12) \approx 2.742$

6. (a) $\ln(x^2-1) = 3 \quad\Rightarrow\quad e^{\ln(x^2-1)} = e^3 \quad\Rightarrow\quad x^2 - 1 = e^3 \quad\Rightarrow\quad x^2 = e^3 + 1 \quad\Rightarrow\quad x = \pm\sqrt{e^3+1} \approx \pm 4.592$

(b) $1 + e^{4x+1} = 20 \quad\Rightarrow\quad e^{4x+1} = 19 \quad\Rightarrow\quad \ln e^{4x+1} = \ln 19 \quad\Rightarrow\quad 4x+1 = \ln 19 \quad\Rightarrow\quad 4x = -1 + \ln 19 \quad\Rightarrow$
$x = \frac{1}{4}(-1+\ln 19) \approx 0.486$

7. (a) $\ln x + \ln(x-1) = 0 \quad\Rightarrow\quad \ln[x(x-1)] = 0 \quad\Rightarrow\quad e^{\ln[x^2-x]} = e^0 \quad\Rightarrow\quad x^2 - x = 1 \quad\Rightarrow\quad x^2 - x - 1 = 0$. The quadratic formula gives $x = \dfrac{1 \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \dfrac{1 \pm \sqrt{5}}{2}$, but we note that $\ln \dfrac{1-\sqrt{5}}{2}$ is undefined because $\dfrac{1-\sqrt{5}}{2} < 0$. Thus, $x = \dfrac{1+\sqrt{5}}{2} \approx 1.618$.

(b) $e - e^{-2x} = 1 \quad\Rightarrow\quad e - 1 = e^{-2x} \quad\Rightarrow\quad \ln(e-1) = \ln e^{-2x} \quad\Rightarrow\quad \ln(e-1) = -2x \quad\Rightarrow$
$x = -\frac{1}{2}\ln(e-1) \approx -0.271$

8. (a) $\ln(\ln x) = 0 \quad\Rightarrow\quad e^{\ln(\ln x)} = e^0 \quad\Rightarrow\quad \ln x = 1 \quad\Rightarrow\quad x = e \approx 2.718$

(b) $\dfrac{60}{1+e^{-x}} = 4 \quad\Rightarrow\quad 60 = 4(1+e^{-x}) \quad\Rightarrow\quad 15 = 1 + e^{-x} \quad\Rightarrow\quad 14 = e^{-x} \quad\Rightarrow\quad \ln 14 = \ln e^{-x} \quad\Rightarrow$
$\ln 14 = -x \quad\Rightarrow\quad x = -\ln 14 \approx -2.639$

9. (a) $e^{2x} - 3e^x + 2 = 0 \quad\Leftrightarrow\quad (e^x-1)(e^x-2) = 0 \quad\Leftrightarrow\quad e^x = 1$ or $e^x = 2 \quad\Leftrightarrow\quad x = \ln 1$ or $x = \ln 2$, so $x = 0$ or $\ln 2$.

(b) $e^{e^x} = 10 \quad\Leftrightarrow\quad \ln\left(e^{e^x}\right) = \ln 10 \quad\Leftrightarrow\quad e^x \ln e = e^x = \ln 10 \quad\Leftrightarrow\quad \ln e^x = \ln(\ln 10) \quad\Leftrightarrow\quad x = \ln\ln 10$

10. (a) $e^{3x+1} = k \quad\Leftrightarrow\quad 3x+1 = \ln k \quad\Leftrightarrow\quad x = \frac{1}{3}(\ln k - 1)$

(b) $\ln(2x+1) = 2 - \ln x \quad\Rightarrow\quad \ln x + \ln(2x+1) = \ln e^2 \quad\Rightarrow\quad \ln[x(2x+1)] = \ln e^2 \quad\Rightarrow\quad 2x^2 + x = e^2 \quad\Rightarrow$
$2x^2 + x - e^2 = 0 \quad\Rightarrow\quad x = \dfrac{-1+\sqrt{1+8e^2}}{4}$ [since $x > 0$].

11. (a) $\ln(1+x^3) - 4 = 0 \quad\Leftrightarrow\quad \ln(1+x^3) = 4 \quad\Leftrightarrow\quad 1 + x^3 = e^4 \quad\Leftrightarrow\quad x^3 = e^4 - 1 \quad\Leftrightarrow\quad x = \sqrt[3]{e^4-1} \approx 3.7704$.

(b) $2e^{1/x} = 42 \quad\Leftrightarrow\quad e^{1/x} = 21 \quad\Leftrightarrow\quad \dfrac{1}{x} = \ln 21 \quad\Leftrightarrow\quad x = \dfrac{1}{\ln 21} \approx 0.3285$.

12. (a) $3^{1/(x-4)} = 7 \quad\Rightarrow\quad \ln 3^{1/(x-4)} = \ln 7 \quad\Rightarrow\quad \dfrac{1}{x-4}\ln 3 = \ln 7 \quad\Rightarrow\quad x - 4 = \dfrac{\ln 3}{\ln 7} \quad\Rightarrow\quad x = 4 + \dfrac{\ln 3}{\ln 7} \approx 4.5646$

(b) $\ln\left(\dfrac{x+1}{x}\right) = 2 \quad\Leftrightarrow\quad \dfrac{x+1}{x} = e^2 \quad\Leftrightarrow\quad x + 1 = e^2 x \quad\Leftrightarrow\quad (e^2-1)x = 1 \quad\Leftrightarrow\quad x = \dfrac{1}{e^2-1} \approx 0.1565$

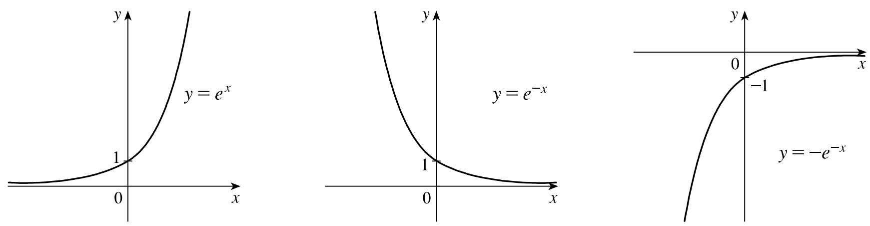
13. (a) $\ln x < 0 \quad\Rightarrow\quad x < e^0 \quad\Rightarrow\quad x < 1$. Since the domain of $f(x) = \ln x$ is $x > 0$, the solution of the original inequality is $0 < x < 1$.

(b) $e^x > 5 \quad\Rightarrow\quad \ln e^x > \ln 5 \quad\Rightarrow\quad x > \ln 5$

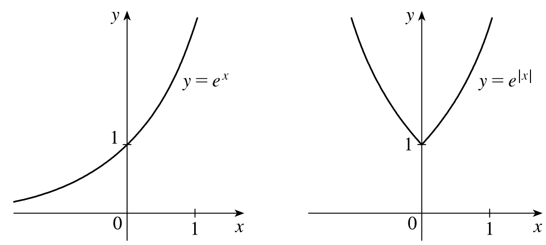
14. (a) $1 < e^{3x-1} < 2 \quad\Rightarrow\quad \ln 1 < 3x - 1 < \ln 2 \quad\Rightarrow\quad 0 < 3x - 1 < \ln 2 \quad\Rightarrow\quad 1 < 3x < 1 + \ln 2 \quad\Rightarrow$
$\frac{1}{3} < x < \frac{1}{3}(1 + \ln 2)$

(b) $1 - 2\ln x < 3 \quad\Rightarrow\quad -2\ln x < 2 \quad\Rightarrow\quad \ln x > -1 \quad\Rightarrow\quad x > e^{-1}$

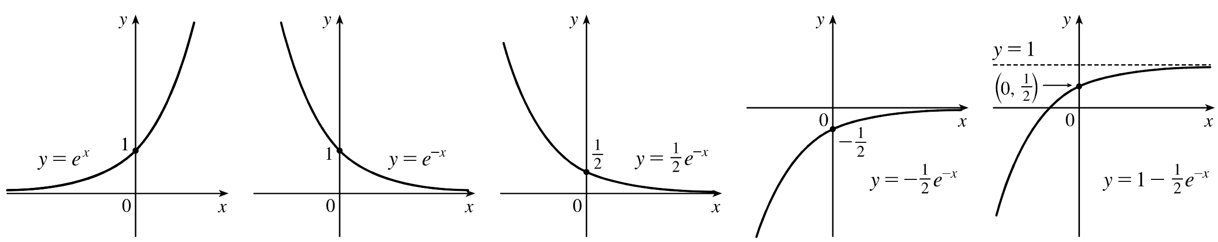
15. We start with the graph of $y = e^x$ (Figure 2) and reflect about the y-axis to get the graph of $y = e^{-x}$. Then we reflect the graph about the x-axis to get the graph of $y = -e^{-x}$.



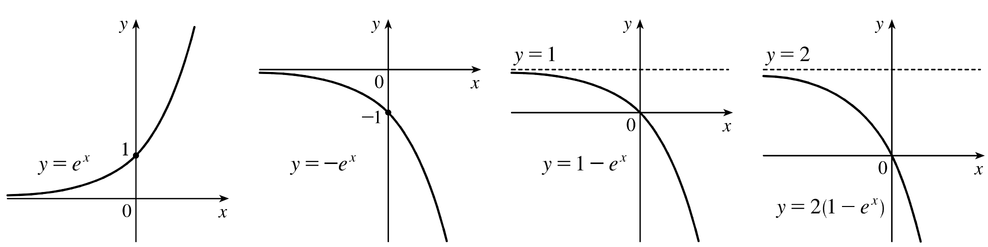
16. We start with the graph of $y = e^x$ (Figure 2) and reflect the portion of the graph in the first quadrant about the y-axis to obtain the graph of $y = e^{|x|}$.



17. We start with the graph of $y = e^x$ (Figure 2) and reflect about the y-axis to get the graph of $y = e^{-x}$. Then we compress the graph vertically by a factor of 2 to obtain the graph of $y = \frac{1}{2}e^{-x}$ and then reflect about the x-axis to get the graph of $y = -\frac{1}{2}e^{-x}$. Finally, we shift the graph one unit upward to get the graph of $y = 1 - \frac{1}{2}e^{-x}$.



18. We start with the graph of $y = e^x$ (Figure 2) and reflect about the x-axis to get the graph of $y = -e^x$. Then shift the graph upward one unit to get the graph of $y = 1 - e^x$. Finally, we stretch the graph vertically by a factor of 2 to obtain the graph of $y = 2(1 - e^x)$.



19. (a) For $f(x) = \sqrt{3 - e^{2x}}$, we must have $3 - e^{2x} \geq 0 \quad \Rightarrow \quad e^{2x} \leq 3 \quad \Rightarrow \quad 2x \leq \ln 3 \quad \Rightarrow \quad x \leq \frac{1}{2} \ln 3$.

Thus, the domain of f is $(-\infty, \frac{1}{2} \ln 3]$.

(b) $y = f(x) = \sqrt{3 - e^{2x}}$ [note that $y \geq 0$] $\Rightarrow \quad y^2 = 3 - e^{2x} \quad \Rightarrow \quad e^{2x} = 3 - y^2 \quad \Rightarrow \quad 2x = \ln(3 - y^2) \quad \Rightarrow$ $x = \frac{1}{2} \ln(3 - y^2)$. Interchange x and y: $y = \frac{1}{2} \ln(3 - x^2)$. So $f^{-1}(x) = \frac{1}{2} \ln(3 - x^2)$. For the domain of f^{-1}, we must have $3 - x^2 > 0 \quad \Rightarrow \quad x^2 < 3 \quad \Rightarrow \quad |x| < \sqrt{3} \quad \Rightarrow \quad -\sqrt{3} < x < \sqrt{3} \quad \Rightarrow \quad 0 \leq x < \sqrt{3}$ since $x \geq 0$. Note that the domain of f^{-1}, $[0, \sqrt{3}\,)$, equals the range of f.

20. (a) For $f(x) = \ln(2 + \ln x)$, we must have $2 + \ln x > 0 \quad \Rightarrow \quad \ln x > -2 \quad \Rightarrow \quad x > e^{-2}$. Thus, the domain of f is (e^{-2}, ∞).

(b) $y = f(x) = \ln(2 + \ln x) \quad \Rightarrow \quad e^y = 2 + \ln x \quad \Rightarrow \quad \ln x = e^y - 2 \quad \Rightarrow \quad x = e^{e^y - 2}$. Interchange x and y: $y = e^{e^x - 2}$. So $f^{-1}(x) = e^{e^x - 2}$. The domain of f^{-1}, as well as the range of f, is $\mathbb{R}$.

21. We solve $y = 3\ln(x - 2)$ for x: $y/3 = \ln(x - 2) \quad \Rightarrow \quad e^{y/3} = x - 2 \quad \Rightarrow \quad x = 2 + e^{y/3}$. Interchanging x and y gives the inverse function $y = 2 + e^{x/3}$.

22. $y = (\ln x)^2$, $x \geq 1$, $\ln x = \sqrt{y} \quad \Rightarrow \quad x = e^{\sqrt{y}}$. Interchange x and y: $y = e^{\sqrt{x}}$ is the inverse function.

23. We solve $y = e^{1-x}$ for x: $\ln y = \ln e^{1-x} \quad \Rightarrow \quad \ln y = 1 - x \quad \Rightarrow \quad x = 1 - \ln y$. Interchanging x and y gives the inverse function $y = 1 - \ln x$.

24. We solve $y = \dfrac{1 - e^{-x}}{1 + e^{-x}}$ for x: $y(1 + e^{-x}) = 1 - e^{-x} \quad \Rightarrow \quad y + ye^{-x} = 1 - e^{-x} \quad \Rightarrow \quad e^{-x} + ye^{-x} = 1 - y \quad \Rightarrow$ $e^{-x}(1 + y) = 1 - y \quad \Rightarrow \quad e^{-x} = \dfrac{1 - y}{1 + y} \quad \Rightarrow \quad -x = \ln\dfrac{1 - y}{1 + y} \quad \Rightarrow \quad x = -\ln\dfrac{1 - y}{1 + y}$ or, equivalently, $x = \ln\left(\dfrac{1 - y}{1 + y}\right)^{-1} = \ln\dfrac{1 + y}{1 - y}$. Interchanging x and y gives the inverse function $y = \ln\dfrac{1 + x}{1 - x}$.

25. Divide numerator and denominator by e^{3x}: $\displaystyle\lim_{x \to \infty} \frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}} = \lim_{x \to \infty} \frac{1 - e^{-6x}}{1 + e^{-6x}} = \frac{1 - 0}{1 + 0} = 1$

26. Let $t = -x^2$. As $x \to \infty$, $t \to -\infty$. So $\displaystyle\lim_{x \to \infty} e^{-x^2} = \lim_{t \to -\infty} e^t = 0$ by (6).

27. Let $t = 3/(2 - x)$. As $x \to 2^+$, $t \to -\infty$. So $\displaystyle\lim_{x \to 2^+} e^{3/(2-x)} = \lim_{t \to -\infty} e^t = 0$ by (6).

28. Let $t = 3/(2 - x)$. As $x \to 2^-$, $t \to \infty$. So $\displaystyle\lim_{x \to 2^-} e^{3/(2-x)} = \lim_{t \to \infty} e^t = \infty$ by (6).

29. Since $-1 \leq \cos x \leq 1$ and $e^{-2x} > 0$, we have $-e^{-2x} \leq e^{-2x} \cos x \leq e^{-2x}$. We know that $\displaystyle\lim_{x \to \infty} (-e^{-2x}) = 0$ and $\displaystyle\lim_{x \to \infty} (e^{-2x}) = 0$, so by the Squeeze Theorem, $\displaystyle\lim_{x \to \infty} (e^{-2x} \cos x) = 0$.

30. $\displaystyle\lim_{x\to(\pi/2)^+} e^{\sec x} = 0$ since $\sec x \to -\infty$ as $x \to (\pi/2)^+$.

31. $f(t) = -2e^t \quad\Rightarrow\quad f'(t) = -2(e^t) = -2e^t$

32. $k(r) = e^r + r^e \quad\Rightarrow\quad k'(r) = e^r + er^{e-1}$

33. $f(x) = (3x^2 - 5x)e^x \quad \overset{\text{PR}}{\Rightarrow}$

$$\begin{aligned} f'(x) &= (3x^2-5x)(e^x)' + e^x(3x^2-5x)' = (3x^2-5x)e^x + e^x(6x-5) \\ &= e^x[(3x^2-5x)+(6x-5)] = e^x(3x^2+x-5) \end{aligned}$$

34. By the Quotient Rule, $y = \dfrac{e^x}{1-e^x} \quad\Rightarrow\quad y' = \dfrac{(1-e^x)e^x - e^x(-e^x)}{(1-e^x)^2} = \dfrac{e^x - e^{2x} + e^{2x}}{(1-e^x)^2} = \dfrac{e^x}{(1-e^x)^2}$.

35. By (9), $y = e^{ax^3} \quad\Rightarrow\quad y' = e^{ax^3}\dfrac{d}{dx}(ax^3) = 3ax^2e^{ax^3}$.

36. $g(x) = e^{x^2-x} \quad \overset{\text{CR}}{\Rightarrow} \quad g'(x) = e^{x^2-x}\cdot\dfrac{d}{dx}(x^2-x) = e^{x^2-x}(2x-1)$

37. $y = e^{\tan\theta} \quad\Rightarrow\quad y' = e^{\tan\theta}\dfrac{d}{d\theta}(\tan\theta) = (\sec^2\theta)e^{\tan\theta}$

38. Let $u = g(x) = e^x + 1$ and $y = f(u) = \sqrt[3]{u} = u^{1/3}$. Then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \left(\frac{1}{3}u^{-2/3}\right)(e^x) = \left(\frac{1}{3\sqrt[3]{(e^x+1)^2}}\right)(e^x) = \frac{e^x}{3\sqrt[3]{(e^x+1)^2}}.$$

39. $f(x) = \dfrac{x^2e^x}{x^2+e^x} \quad \overset{\text{QR}}{\Rightarrow}$

$$\begin{aligned} f'(x) &= \frac{(x^2+e^x)\left[x^2e^x + e^x(2x)\right] - x^2e^x(2x+e^x)}{(x^2+e^x)^2} = \frac{x^4e^x + 2x^3e^x + x^2e^{2x} + 2xe^{2x} - 2x^3e^x - x^2e^{2x}}{(x^2+e^x)^2} \\ &= \frac{x^4e^x + 2xe^{2x}}{(x^2+e^x)^2} = \frac{xe^x(x^3+2e^x)}{(x^2+e^x)^2} \end{aligned}$$

40. $A(r) = \sqrt{r}\cdot e^{r^2+1} \Rightarrow$

$$\begin{aligned} A'(r) &= \sqrt{r}\cdot e^{r^2+1}\cdot\frac{d}{dr}\left(r^2+1\right) + e^{r^2+1}\cdot\frac{d}{dr}\left(\sqrt{r}\right) = \sqrt{r}\cdot e^{r^2+1}\cdot 2r + e^{r^2+1}\cdot\frac{1}{2\sqrt{r}} \\ &= e^{r^2+1}\left(2r\sqrt{r} + \frac{1}{2\sqrt{r}}\right) \text{ or } e^{r^2+1}\left(\frac{4r^2+1}{2\sqrt{r}}\right) \end{aligned}$$

41. Using the Product Rule and the Chain Rule, $y = x^2e^{-3x} \quad\Rightarrow$

$y' = x^2e^{-3x}(-3) + e^{-3x}(2x) = e^{-3x}(-3x^2+2x) = xe^{-3x}(2-3x)$.

42. $f(t) = \tan(1+e^{2t}) \quad\Rightarrow\quad f'(t) = \sec^2(1+e^{2t})\cdot(1+e^{2t})' = 2e^{2t}\sec^2(1+e^{2t})$

43. $f(t) = e^{at}\sin bt \quad\Rightarrow\quad f'(t) = e^{at}(\cos bt)\cdot b + (\sin bt)e^{at}\cdot a = e^{at}(b\cos bt + a\sin bt)$

44. $f(z) = e^{z/(z-1)} \quad \Rightarrow \quad f'(z) = e^{z/(z-1)} \dfrac{d}{dz} \dfrac{z}{z-1} = e^{z/(z-1)} \dfrac{(z-1)(1) - z(1)}{(z-1)^2} = -\dfrac{e^{z/(z-1)}}{(z-1)^2}$

45. By (9), $F(t) = e^{t \sin 2t} \quad \Rightarrow$

$F'(t) = e^{t \sin 2t}(t \sin 2t)' = e^{t \sin 2t}(t \cdot 2 \cos 2t + \sin 2t \cdot 1) = e^{t \sin 2t}(2t \cos 2t + \sin 2t)$

46. $y = e^{\sin 2x} + \sin(e^{2x}) \quad \Rightarrow$

$y' = e^{\sin 2x} \dfrac{d}{dx} \sin 2x + \cos(e^{2x}) \dfrac{d}{dx} e^{2x} = e^{\sin 2x}(\cos 2x) \cdot 2 + \cos(e^{2x})\, e^{2x} \cdot 2 = 2 \cos 2x\, e^{\sin 2x} + 2e^{2x} \cos(e^{2x})$

47. $g(u) = e^{\sqrt{\sec u^2}} \quad \Rightarrow$

$$g'(u) = e^{\sqrt{\sec u^2}} \frac{d}{du} \sqrt{\sec u^2} = e^{\sqrt{\sec u^2}} \frac{1}{2} (\sec u^2)^{-1/2} \frac{d}{du} \sec u^2$$

$$= e^{\sqrt{\sec u^2}} \frac{1}{2\sqrt{\sec u^2}} \cdot \sec u^2 \tan u^2 \cdot 2u = u\sqrt{\sec u^2} \tan u^2\, e^{\sqrt{\sec u^2}}$$

48. $f(t) = e^{1/t} \sqrt{t^2 - 1} \quad \Rightarrow$

$$f'(t) = e^{1/t} \cdot \frac{1}{2\sqrt{t^2-1}} \cdot 2t + \sqrt{t^2-1} \cdot e^{1/t} \cdot \left(-\frac{1}{t^2}\right) \qquad \left[\frac{1}{t} = t^{-1};\ \frac{d}{dt}\left(t^{-1}\right) = -t^{-2} = -\frac{1}{t^2}\right]$$

$$= e^{1/t}\left(\frac{t}{\sqrt{t^2-1}} - \frac{\sqrt{t^2-1}}{t^2}\right) \text{ or } e^{1/t}\left(\frac{t^3 - t^2 + 1}{t^2\sqrt{t^2-1}}\right)$$

49. $g(x) = \sin\left(\dfrac{e^x}{1 + e^x}\right) \quad \Rightarrow$

$$g'(x) = \cos\left(\frac{e^x}{1+e^x}\right) \cdot \frac{(1+e^x)e^x - e^x(e^x)}{(1+e^x)^2} = \cos\left(\frac{e^x}{1+e^x}\right) \cdot \frac{e^x(1+e^x-e^x)}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2} \cos\left(\frac{e^x}{1+e^x}\right)$$

50. $f(x) = e^{\sin^2(x^2)} \quad \Rightarrow \quad f'(x) = e^{\sin^2(x^2)} \cdot 2\sin(x^2) \cdot \cos(x^2) \cdot 2x = 4x \sin(x^2) \cos(x^2) e^{\sin^2(x^2)}$

51. $y = e^x \cos x + \sin x \quad \Rightarrow \quad y' = e^x(-\sin x) + (\cos x)(e^x) + \cos x = e^x(\cos x - \sin x) + \cos x$, so

$y'(0) = e^0(\cos 0 - \sin 0) + \cos 0 = 1(1-0) + 1 = 2$. An equation of the tangent line to the curve $y = e^x \cos x + \sin x$ at the point $(0, 1)$ is $y - 1 = 2(x - 0)$ or $y = 2x + 1$.

52. $y = \dfrac{1+x}{1+e^x} \quad \Rightarrow \quad y' = \dfrac{(1+e^x)(1) - (1+x)e^x}{(1+e^x)^2} = \dfrac{1 + e^x - e^x - xe^x}{(1+e^x)^2} = \dfrac{1 - xe^x}{(1+e^x)^2}$

At $\left(0, \frac{1}{2}\right)$, $y' = \dfrac{1}{(1+1)^2} = \dfrac{1}{4}$, and an equation of the tangent line is $y - \frac{1}{2} = \frac{1}{4}(x - 0)$ or $y = \frac{1}{4}x + \frac{1}{2}$.

53. $\dfrac{d}{dx}(e^{x/y}) = \dfrac{d}{dx}(x - y) \quad \Rightarrow \quad e^{x/y} \cdot \dfrac{d}{dx}\left(\dfrac{x}{y}\right) = 1 - y' \quad \Rightarrow \quad e^{x/y} \cdot \dfrac{y \cdot 1 - x \cdot y'}{y^2} = 1 - y' \quad \Rightarrow$

$$e^{x/y} \cdot \frac{1}{y} - \frac{xe^{x/y}}{y^2} \cdot y' = 1 - y' \quad \Rightarrow \quad y' - \frac{xe^{x/y}}{y^2} \cdot y' = 1 - \frac{e^{x/y}}{y} \quad \Rightarrow \quad y'\left(1 - \frac{xe^{x/y}}{y^2}\right) = \frac{y - e^{x/y}}{y} \quad \Rightarrow$$

$$y' = \frac{\dfrac{y - e^{x/y}}{y}}{\dfrac{y^2 - xe^{x/y}}{y^2}} = \frac{y(y - e^{x/y})}{y^2 - xe^{x/y}}$$

54. $xe^y + ye^x = 1 \quad\Rightarrow\quad xe^y y' + e^y \cdot 1 + ye^x + e^x y' = 0 \quad\Rightarrow\quad y'(xe^y + e^x) = -e^y - ye^x \quad\Rightarrow\quad y' = -\dfrac{e^y + ye^x}{xe^y + e^x}$. At $(0,1)$, $y' = -\dfrac{e + 1\cdot 1}{0 + 1} = -(e+1)$, so an equation for the tangent line is $y - 1 = -(e+1)(x-0)$, or $y = -(e+1)x + 1$.

55. $y = e^x + e^{-x/2} \quad\Rightarrow\quad y' = e^x - \frac{1}{2}e^{-x/2} \quad\Rightarrow\quad y'' = e^x + \frac{1}{4}e^{-x/2}$, so

$2y'' - y' - y = 2\left(e^x + \frac{1}{4}e^{-x/2}\right) - \left(e^x - \frac{1}{2}e^{-x/2}\right) - \left(e^x + e^{-x/2}\right) = 0.$

56. $y = Ae^{-x} + Bxe^{-x} \quad\Rightarrow\quad y' = -Ae^{-x} + Be^{-x} - Bxe^{-x} = (B-A)e^{-x} - Bxe^{-x} \quad\Rightarrow$

$y'' = (A-B)e^{-x} - Be^{-x} + Bxe^{-x} = (A-2B)e^{-x} + Bxe^{-x}$,

so $y'' + 2y' + y = (A-2B)e^{-x} + Bxe^{-x} + 2\left[(B-A)e^{-x} - Bxe^{-x}\right] + Ae^{-x} + Bxe^{-x} = 0$.

57. $y = e^{rx} \quad\Rightarrow\quad y' = re^{rx} \quad\Rightarrow\quad y'' = r^2 e^{rx}$, so if $y = e^{rx}$ satisfies the differential equation $y'' + 6y' + 8y = 0$, then $r^2e^{rx} + 6re^{rx} + 8e^{rx} = 0$; that is, $e^{rx}(r^2 + 6r + 8) = 0$. Since $e^{rx} > 0$ for all x, we must have $r^2 + 6r + 8 = 0$, or $(r+2)(r+4) = 0$, so $r = -2$ or -4.

58. $y = e^{\lambda x} \quad\Rightarrow\quad y' = \lambda e^{\lambda x} \quad\Rightarrow\quad y'' = \lambda^2 e^{\lambda x}$. Thus, $y + y' = y'' \quad\Leftrightarrow\quad e^{\lambda x} + \lambda e^{\lambda x} = \lambda^2 e^{\lambda x} \quad\Leftrightarrow$

$e^{\lambda x}(\lambda^2 - \lambda - 1) = 0 \quad\Leftrightarrow\quad \lambda = \frac{1 \pm \sqrt{5}}{2}$, since $e^{\lambda x} \neq 0$.

59. $f(x) = e^{2x} \quad\Rightarrow\quad f'(x) = 2e^{2x} \quad\Rightarrow\quad f''(x) = 2\cdot 2e^{2x} = 2^2 e^{2x} \quad\Rightarrow$

$f'''(x) = 2^2 \cdot 2e^{2x} = 2^3 e^{2x} \quad\Rightarrow\quad \cdots \quad\Rightarrow\quad f^{(n)}(x) = 2^n e^{2x}$

60. $f(x) = xe^{-x} \quad\Rightarrow\quad f'(x) = x(-e^{-x}) + e^{-x} = (1-x)e^{-x} \quad\Rightarrow$

$f''(x) = (1-x)(-e^{-x}) + e^{-x}(-1) = (x-2)e^{-x} \quad\Rightarrow\quad f'''(x) = (x-2)(-e^{-x}) + e^{-x} = (3-x)e^{-x} \quad\Rightarrow$

$f^{(4)}(x) = (3-x)(-e^{-x}) + e^{-x}(-1) = (x-4)e^{-x} \quad\Rightarrow\quad \cdots \quad\Rightarrow\quad f^{(n)}(x) = (-1)^n(x-n)e^{-x}$.

So $D^{1000}xe^{-x} = (x - 1000)e^{-x}$.

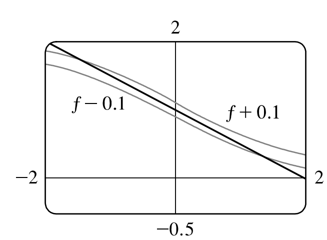
61. (a) $f(x) = e^x + x$ is continuous on $\mathbb{R}$ and $f(-1) = e^{-1} - 1 < 0 < 1 = f(0)$, so by the Intermediate Value Theorem, $e^x + x = 0$ has a solution in $(-1, 0)$.

(b) $f(x) = e^x + x \quad\Rightarrow\quad f'(x) = e^x + 1$, so $x_{n+1} = x_n - \dfrac{e^{x_n} + x_n}{e^{x_n} + 1}$. Using $x_1 = -0.5$, we get $x_2 \approx -0.566311$, $x_3 \approx -0.567143 \approx x_4$, so the solution is -0.567143 to six decimal places.

62. $f(x) = \dfrac{2}{1 + e^x} \quad\Rightarrow\quad f'(x) = -\dfrac{2e^x}{(1+e^x)^2}$, so $f(0) = 1$ and $f'(0) = -\dfrac{1}{2}$.

Thus, $f(x) \approx f(0) + f'(0)(x - 0) = 1 - \frac{1}{2}x$. We need

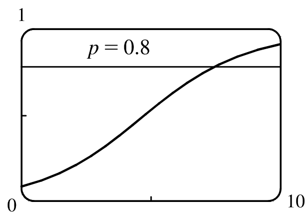
$\dfrac{2}{1+e^x} - 0.1 < 1 - \dfrac{1}{2}x < \dfrac{2}{1+e^x} + 0.1$, which is true when $-1.423 < x < 1.423$. Note that to ensure the accuracy, we have rounded the smaller value up and the larger value down.



63. (a) $\lim\limits_{t\to\infty} p(t) = \lim\limits_{t\to\infty} \dfrac{1}{1+ae^{-kt}} = \dfrac{1}{1+a\cdot 0} = 1$, since $k>0 \;\Rightarrow\; -kt \to -\infty \;\Rightarrow\; e^{-kt} \to 0$. As time increases, the proportion of the population that has heard the rumor approaches 1; that is, everyone in the population has heard the rumor.

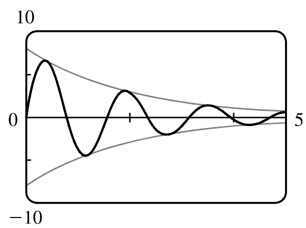
(b) $p(t) = (1+ae^{-kt})^{-1} \;\Rightarrow\; \dfrac{dp}{dt} = -(1+ae^{-kt})^{-2}(-kae^{-kt}) = \dfrac{kae^{-kt}}{(1+ae^{-kt})^2}$

(c)



From the graph of $p(t) = (1+10e^{-0.5t})^{-1}$, it seems that $p(t) = 0.8$ (indicating that 80% of the population has heard the rumor) when $t \approx 7.4$ hours.

64. (a)



The displacement function is squeezed between the other two functions. This is because $-1 \le \sin 4t \le 1 \;\Rightarrow\; -8e^{-t/2} \le 8e^{-t/2}\sin 4t \le 8e^{-t/2}$.

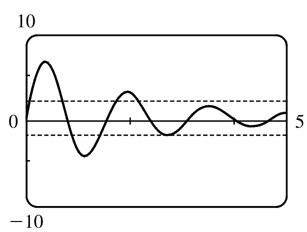
(b) The maximum value of the displacement is about 6.6 cm, occurring at $t \approx 0.36$ s. It occurs just before the graph of the displacement function touches the graph of $8e^{-t/2}$ (when $t = \frac{\pi}{8} \approx 0.39$).

(c) The velocity of the object is the derivative of its displacement function, that is,

$$\frac{d}{dt}\left(8e^{-t/2}\sin 4t\right) = 8\left[e^{-t/2}\cos 4t(4) + \sin 4t\left(-\tfrac{1}{2}\right)e^{-t/2}\right]$$

If the displacement is zero, then we must have $\sin 4t = 0$ (since the exponential term in the displacement function is always positive). The first time that $\sin 4t = 0$ after $t = 0$ occurs at $t = \frac{\pi}{4}$. Substituting this into our expression for the velocity, and noting that the second term vanishes, we get $v\left(\frac{\pi}{4}\right) = 8e^{-\pi/8}\cos\left(4\cdot\frac{\pi}{4}\right)\cdot 4 = -32e^{-\pi/8} \approx -21.6 \text{ cm/s}$.

(d)



The graph indicates that the displacement is less than 2 cm from equilibrium whenever t is larger than about 2.8.

65. $f(x) = \dfrac{e^x}{1+x^2}$, $[0,3]$. $f'(x) = \dfrac{(1+x^2)e^x - e^x(2x)}{(1+x^2)^2} = \dfrac{e^x(x^2-2x+1)}{(1+x^2)^2} = \dfrac{e^x(x-1)^2}{(1+x^2)^2}$. $f'(x) = 0 \;\Rightarrow\; (x-1)^2 = 0 \;\Leftrightarrow\; x = 1$. $f'(x)$ exists for all real numbers since $1+x^2$ is never equal to 0. $f(0) = 1$, $f(1) = e/2 \approx 1.359$, and $f(3) = e^3/10 \approx 2.009$. So $f(3) = e^3/10$ is the absolute maximum value and $f(0) = 1$ is the absolute minimum value.

66. $f(x) = xe^{x/2}$, $[-3, 1]$. $f'(x) = xe^{x/2}\left(\frac{1}{2}\right) + e^{x/2}(1) = e^{x/2}\left(\frac{1}{2}x + 1\right)$. $f'(x) = 0 \iff \frac{1}{2}x + 1 = 0 \iff x = -2$. $f(-3) = -3e^{-3/2} \approx -0.669$, $f(-2) = -2e^{-1} \approx -0.736$, and $f(1) = e^{1/2} \approx 1.649$. So $f(1) = e^{1/2}$ is the absolute maximum value and $f(-2) = -2/e$ is the absolute minimum value.

67. $f(x) = x - e^x \Rightarrow f'(x) = 1 - e^x = 0 \iff e^x = 1 \iff x = 0$. Now $f'(x) > 0$ for all $x < 0$ and $f'(x) < 0$ for all $x > 0$, so the absolute maximum value is $f(0) = 0 - 1 = -1$.

68. $g(x) = \dfrac{e^x}{x} \Rightarrow g'(x) = \dfrac{xe^x - e^x}{x^2} = 0 \iff e^x(x-1) = 0 \Rightarrow x = 1$. Now $g'(x) > 0 \iff \dfrac{xe^x - e^x}{x^2} > 0 \iff x - 1 > 0 \iff x > 1$ and $g'(x) < 0 \iff \dfrac{xe^x - e^x}{x^2} < 0 \iff x - 1 < 0 \iff x < 1$. Thus there is an absolute minimum value of $g(1) = e$ at $x = 1$.

69. (a) $f(x) = xe^{2x} \Rightarrow f'(x) = x(2e^{2x}) + e^{2x}(1) = e^{2x}(2x+1)$. Thus, $f'(x) > 0$ if $x > -\frac{1}{2}$ and $f'(x) < 0$ if $x < -\frac{1}{2}$. So f is increasing on $\left(-\frac{1}{2}, \infty\right)$ and f is decreasing on $\left(-\infty, -\frac{1}{2}\right)$.

(b) $f''(x) = e^{2x}(2) + (2x+1)\cdot 2e^{2x} = 2e^{2x}[1 + (2x+1)] = 2e^{2x}(2x+2) = 4e^{2x}(x+1)$. $f''(x) > 0 \iff x > -1$ and $f''(x) < 0 \iff x < -1$. Thus, f is concave upward on $(-1, \infty)$ and f is concave downward on $(-\infty, -1)$.

(c) There is an inflection point at $\left(-1, -e^{-2}\right)$, or $\left(-1, -1/e^2\right)$.

70. (a) $f(x) = \dfrac{e^x}{x^2} \Rightarrow f'(x) = \dfrac{x^2 e^x - e^x(2x)}{(x^2)^2} = \dfrac{xe^x(x-2)}{x^4} = \dfrac{e^x(x-2)}{x^3}$. $f'(x) > 0 \iff x < 0$ or $x > 2$, so f is increasing on $(-\infty, 0)$ and $(2, \infty)$. $f'(x) < 0 \iff 0 < x < 2$, so f is decreasing on $(0, 2)$.

(b) $f''(x) = \dfrac{x^3[e^x \cdot 1 + (x-2)e^x] - e^x(x-2)\cdot 3x^2}{(x^3)^2} = \dfrac{x^2 e^x[x(x-1) - 3(x-2)]}{x^6} = \dfrac{e^x(x^2 - 4x + 6)}{x^4}$.

$x^2 - 4x + 6 = (x^2 - 4x + 4) + 2 = (x-2)^2 + 2 > 0$, so $f''(x) > 0$ and f is CU on $(-\infty, 0)$ and $(0, \infty)$.

(c) There are no changes in concavity and, hence, there are no points of inflection.

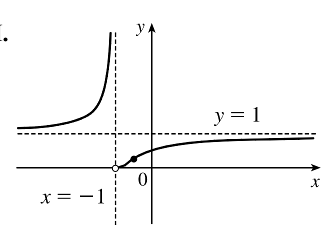
71. $y = f(x) = e^{-1/(x+1)}$ **A.** $D = \{x \mid x \neq -1\} = (-\infty, -1) \cup (-1, \infty)$ **B.** No x-intercept; y-intercept $= f(0) = e^{-1}$ **C.** No symmetry **D.** $\lim\limits_{x \to \pm\infty} e^{-1/(x+1)} = 1$ since $-1/(x+1) \to 0$, so $y = 1$ is a HA. $\lim\limits_{x \to -1^+} e^{-1/(x+1)} = 0$ since $-1/(x+1) \to -\infty$, $\lim\limits_{x \to -1^-} e^{-1/(x+1)} = \infty$ since $-1/(x+1) \to \infty$, so $x = -1$ is a VA.

E. $f'(x) = e^{-1/(x+1)}/(x+1)^2 \Rightarrow f'(x) > 0$ for all x except 1, so f is increasing on $(-\infty, -1)$ and $(-1, \infty)$. **F.** No extreme values

G. $f''(x) = \dfrac{e^{-1/(x+1)}}{(x+1)^4} + \dfrac{e^{-1/(x+1)}(-2)}{(x+1)^3} = -\dfrac{e^{-1/(x+1)}(2x+1)}{(x+1)^4} \Rightarrow$

$f''(x) > 0 \iff 2x + 1 < 0 \iff x < -\frac{1}{2}$, so f is CU on $(-\infty, -1)$ and $\left(-1, -\frac{1}{2}\right)$, and CD on $\left(-\frac{1}{2}, \infty\right)$. f has an IP at $\left(-\frac{1}{2}, e^{-2}\right)$.

H.

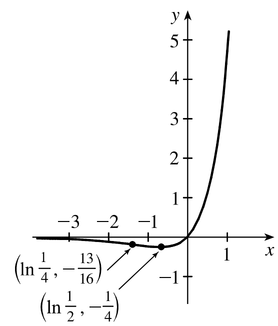


72. $y = f(x) = e^{2x} - e^x$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \Rightarrow e^{2x} = e^x \Rightarrow e^x = 1 \Rightarrow x = 0$. **C.** No symmetry **D.** $\lim\limits_{x\to-\infty} e^{2x} - e^x = 0$, so $y = 0$ is a HA. No VA. **E.** $f'(x) = 2e^{2x} - e^x = e^x(2e^x - 1)$, so $f'(x) > 0 \Leftrightarrow e^x > \frac{1}{2} \Leftrightarrow x > \ln\frac{1}{2} = -\ln 2$ and $f'(x) < 0 \Leftrightarrow e^x < \frac{1}{2} \Leftrightarrow x < \ln\frac{1}{2}$, so f is decreasing on $\left(-\infty, \ln\frac{1}{2}\right)$ and increasing on $\left(\ln\frac{1}{2}, \infty\right)$.

F. Local minimum value $f\left(\ln\frac{1}{2}\right) = e^{2\ln(1/2)} - e^{\ln(1/2)} = \left(\frac{1}{2}\right)^2 - \frac{1}{2} = -\frac{1}{4}$

G. $f''(x) = 4e^{2x} - e^x = e^x(4e^x - 1)$, so $f''(x) > 0 \Leftrightarrow e^x > \frac{1}{4} \Leftrightarrow x > \ln\frac{1}{4}$ and $f''(x) < 0 \Leftrightarrow x < \ln\frac{1}{4}$. Thus, f is CD on $\left(-\infty, \ln\frac{1}{4}\right)$ and CU on $\left(\ln\frac{1}{4}, \infty\right)$. IP at $\left(\ln\frac{1}{4}, \left(\frac{1}{4}\right)^2 - \frac{1}{4}\right) = \left(\ln\frac{1}{4}, -\frac{3}{16}\right)$

H.



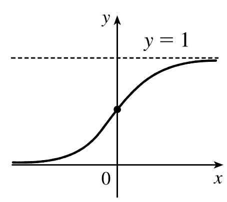
73. $y = 1/(1 + e^{-x})$ **A.** $D = \mathbb{R}$ **B.** No x-intercept; y-intercept $= f(0) = \frac{1}{2}$. **C.** No symmetry

D. $\lim\limits_{x\to\infty} 1/(1 + e^{-x}) = \frac{1}{1+0} = 1$ and $\lim\limits_{x\to-\infty} 1/(1 + e^{-x}) = 0$ since $\lim\limits_{x\to-\infty} e^{-x} = \infty$, so f has horizontal asymptotes $y = 0$ and $y = 1$. **E.** $f'(x) = -(1 + e^{-x})^{-2}(-e^{-x}) = e^{-x}/(1 + e^{-x})^2$. This is positive for all x, so f is increasing on $\mathbb{R}$.

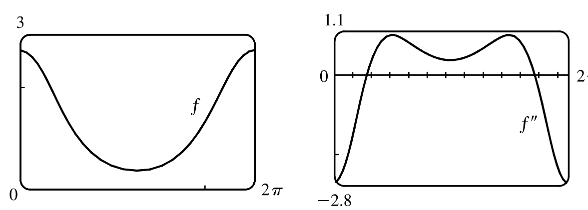
F. No extreme values **G.** $f''(x) = \dfrac{(1 + e^{-x})^2(-e^{-x}) - e^{-x}(2)(1 + e^{-x})(-e^{-x})}{(1 + e^{-x})^4} = \dfrac{e^{-x}(e^{-x} - 1)}{(1 + e^{-x})^3}$

The second factor in the numerator is negative for $x > 0$ and positive for $x < 0$, and the other factors are always positive, so f is CU on $(-\infty, 0)$ and CD on $(0, \infty)$. IP at $\left(0, \frac{1}{2}\right)$

H.

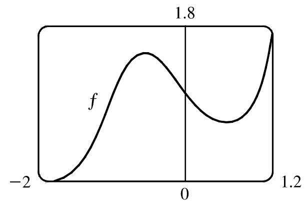


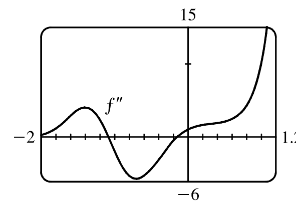
74. The function $f(x) = e^{\cos x}$ is periodic with period 2π, so we consider it only on the interval $[0, 2\pi]$. We see that it has local maxima of about $f(0) \approx 2.72$ and $f(2\pi) \approx 2.72$, and a local minimum of about $f(3.14) \approx 0.37$. To find the exact values, we calculate $f'(x) = -\sin x e^{\cos x}$. This is 0 when $-\sin x = 0 \Leftrightarrow x = 0, \pi$ or 2π (since we are only considering $x \in [0, 2\pi]$). Also $f'(x) > 0 \Leftrightarrow \sin x < 0 \Leftrightarrow 0 < x < \pi$. So $f(0) = f(2\pi) = e$ (both maxima) and $f(\pi) = e^{\cos\,\pi} = 1/e$ (minimum). To find the inflection points, we calculate and graph



$f''(x) = \dfrac{d}{dx}\left(-\sin x\, e^{\cos\, x}\right) = -\cos x\, e^{\cos x} - \sin x(e^{\cos x})(-\sin x) = e^{\cos x}\left(\sin^2 x - \cos x\right)$. From the graph of $f''(x)$, we see that f has inflection points at $x \approx 0.90$ and at $x \approx 5.38$. These x-coordinates correspond to inflection points $(0.90, 1.86)$ and $(5.38, 1.86)$.

75. $f(x) = e^{x^3 - x} \to 0$ as $x \to -\infty$, and $f(x) \to \infty$ as $x \to \infty$. From the graph, it appears that f has a local minimum of about $f(0.58) = 0.68$, and a local maximum of about $f(-0.58) = 1.47$.



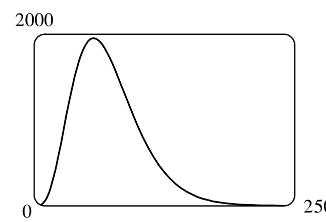


To find the exact values, we calculate $f'(x) = \left(3x^2 - 1\right)e^{x^3 - x}$, which is 0 when $3x^2 - 1 = 0 \quad \Leftrightarrow \quad x = \pm\frac{1}{\sqrt{3}}$. The negative solution corresponds to the local maximum $f\left(-\frac{1}{\sqrt{3}}\right) = e^{(-1/\sqrt{3})^3 \,-\, (-1/\sqrt{3})} = e^{2\sqrt{3}/9}$, and the positive solution corresponds to the local minimum $f\left(\frac{1}{\sqrt{3}}\right) = e^{(1/\sqrt{3})^3 \,-\, (1/\sqrt{3})} = e^{-2\sqrt{3}/9}$. To estimate the inflection points, we calculate and graph

$$f''(x) = \frac{d}{dx}\left[\left(3x^2 - 1\right)e^{x^3 - x}\right] = \left(3x^2 - 1\right)e^{x^3 - x}\left(3x^2 - 1\right) + e^{x^3 - x}(6x) = e^{x^3 - x}\left(9x^4 - 6x^2 + 6x + 1\right).$$

From the graph, it appears that $f''(x)$ changes sign (and thus f has inflection points) at $x \approx -0.15$ and $x \approx -1.09$. From the graph of f, we see that these x-values correspond to inflection points at about $(-0.15, 1.15)$ and $(-1.09, 0.82)$.

76. $S(t) = At^p e^{-kt}$ with $A = 0.01$, $p = 4$, and $k = 0.07$. We will find the zeros of f'' for $f(t) = t^p e^{-kt}$.



$$f'(t) = t^p(-ke^{-kt}) + e^{-kt}(pt^{p-1}) = e^{-kt}(-kt^p + pt^{p-1})$$

$$\begin{aligned} f''(t) &= e^{-kt}(-kpt^{p-1} + p(p-1)t^{p-2}) + (-kt^p + pt^{p-1})(-ke^{-kt}) \\ &= t^{p-2}e^{-kt}[-kpt + p(p-1) + k^2t^2 - kpt] \\ &= t^{p-2}e^{-kt}(k^2t^2 - 2kpt + p^2 - p) \end{aligned}$$

Using the given values of p and k gives us $f''(t) = t^2e^{-0.07t}(0.0049t^2 - 0.56t + 12)$. So $S''(t) = 0.01f''(t)$ and its zeros are $t = 0$ and the solutions of $0.0049t^2 - 0.56t + 12 = 0$, which are $t_1 = \frac{200}{7} \approx 28.57$ and $t_2 = \frac{600}{7} \approx 85.71$.

At t_1 minutes, the rate of increase of the level of medication in the bloodstream is at its greatest and at t_2 minutes, the rate of decrease is the greatest.

77. Let $a = 0.135$ and $b = -2.802$. Then $C(t) = ate^{bt} \quad \Rightarrow \quad C'(t) = a(t \cdot e^{bt} \cdot b + e^{bt} \cdot 1) = ae^{bt}(bt + 1)$. $\quad C'(t) = 0 \quad \Leftrightarrow$ $bt + 1 = 0 \quad \Leftrightarrow \quad t = -\dfrac{1}{b} \approx 0.36$ h. $\quad C(0) = 0$, $C(-1/b) = -\dfrac{a}{b}e^{-1} = -\dfrac{a}{be} \approx 0.0177$, and $C(3) = 3ae^{3b} \approx 0.00009$.

The maximum average BAC during the first three hours is about 0.0177 g/dL and it occurs at approximately 0.36 h (21.4 min).

78. (a) As $|x| \to \infty$, $t = -x^2/(2\sigma^2) \to -\infty$, and $e^t \to 0$. The HA is $y = 0$. Since t takes on its maximum value at $x = 0$, so does e^t. Showing this result using derivatives, we have $f(x) = e^{-x^2/(2\sigma^2)} \quad \Rightarrow \quad f'(x) = e^{-x^2/(2\sigma^2)}(-x/\sigma^2)$.

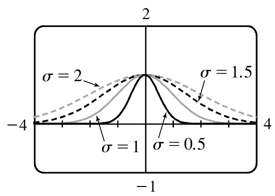
$f'(x) = 0 \quad \Leftrightarrow \quad x = 0$. Because f' changes from positive to negative at $x = 0$, $f(0) = 1$ is a local maximum. For inflection points, we find $f''(x) = -\dfrac{1}{\sigma^2}\left[e^{-x^2/(2\sigma^2)} \cdot 1 + xe^{-x^2/(2\sigma^2)}(-x/\sigma^2)\right] = \dfrac{-1}{\sigma^2}e^{-x^2/(2\sigma^2)}(1 - x^2/\sigma^2)$.

$f''(x) = 0 \quad \Leftrightarrow \quad x^2 = \sigma^2 \quad \Leftrightarrow \quad x = \pm\sigma$. $\quad f''(x) < 0 \quad \Leftrightarrow \quad x^2 < \sigma^2 \quad \Leftrightarrow \quad -\sigma < x < \sigma$.

So f is CD on $(-\sigma, \sigma)$ and CU on $(-\infty, -\sigma)$ and (σ, ∞). IP at $(\pm\sigma, e^{-1/2})$.

(b) Since we have IP at $x = \pm\sigma$, the inflection points move away from the y-axis as σ increases.

(c)



From the graph, we see that as σ increases, the graph tends to spread out and there is more area between the curve and the x-axis.

79. $\displaystyle\int_0^1 (x^e + e^x)\,dx = \left[\frac{x^{e+1}}{e+1} + e^x\right]_0^1 = \left(\frac{1}{e+1} + e\right) - (0+1) = \frac{1}{e+1} + e - 1$

80. $\int_{-5}^5 e\,dx = \Big[ex\Big]_{-5}^5 = 5e - (-5e) = 10e$

81. $\displaystyle\int_0^2 \frac{dx}{e^{\pi x}} = \int_0^2 e^{-\pi x}\,dx = \left[-\frac{1}{\pi}e^{-\pi x}\right]_0^2 = -\frac{1}{\pi}e^{-2\pi} + \frac{1}{\pi}e^0 = \frac{1}{\pi}(1 - e^{-2\pi})$

82. Let $u = -t^4$. Then $du = -4t^3\,dt$ and $t^3\,dt = -\frac{1}{4}\,du$, so $\int t^3 e^{-t^4}\,dt = \int e^u\left(-\frac{1}{4}\,du\right) = -\frac{1}{4}e^u + C = -\frac{1}{4}e^{-t^4} + C$.

83. Let $u = 1 + e^x$. Then $du = e^x\,dx$, so $\int e^x\sqrt{1+e^x}\,dx = \int \sqrt{u}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(1+e^x)^{3/2} + C$.

84. $\displaystyle\int \frac{(1+e^x)^2}{e^x}\,dx = \int \frac{1 + 2e^x + e^{2x}}{e^x}\,dx = \int (e^{-x} + 2 + e^x)\,dx = -e^{-x} + 2x + e^x + C$

85. $\int (e^x + e^{-x})^2\,dx = \int (e^{2x} + 2 + e^{-2x})\,dx = \frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x} + C$

86. $\int e^x(4+e^x)^5\,dx \quad \begin{bmatrix} u = 4 + e^x, \\ du = e^x\,dx \end{bmatrix} \quad = \int u^5\,du = \frac{1}{6}u^6 + C = \frac{1}{6}(4+e^x)^6 + C$

87. Let $x = 1 - e^u$. Then $dx = -e^u\,du$ and $e^u\,du = -dx$, so

$$\int \frac{e^u}{(1-e^u)^2}\,du = \int \frac{1}{x^2}(-dx) = -\int x^{-2}\,dx = -(-x^{-1}) + C = \frac{1}{x} + C = \frac{1}{1-e^u} + C.$$

88. Let $u = \sin\theta$. Then $du = \cos\theta\,d\theta$, so $\int e^{\sin\theta}\cos\theta\,d\theta = \int e^u\,du = e^u + C = e^{\sin\theta} + C$.

89. Let $u = 1/x$, so $du = -1/x^2\,dx$. When $x = 1$, $u = 1$; when $x = 2$, $u = \frac{1}{2}$. Thus,

$$\int_1^2 \frac{e^{1/x}}{x^2}\,dx = \int_1^{1/2} e^u\,(-du) = -\left[e^u\right]_1^{1/2} = -(e^{1/2} - e) = e - \sqrt{e}.$$

90. $\displaystyle\int_0^1 \frac{\sqrt{1+e^{-x}}}{e^x}\,dx = \int_0^1 \sqrt{1+e^{-x}}\,e^{-x}\,dx = \int_2^{1+1/e} u^{1/2}\,(-du) \qquad \begin{bmatrix} u = 1 + e^{-x}, \\ du = -e^{-x}\,dx \end{bmatrix}$

$$= \left[-\tfrac{2}{3}u^{3/2}\right]_2^{1+1/e} = -\tfrac{2}{3}[(1+1/e)^{3/2} - 2^{3/2}] = \tfrac{4}{3}\sqrt{2} - \tfrac{2}{3}(1+1/e)^{3/2}$$

91. $f_{\text{avg}} = \frac{1}{2-0}\int_0^2 2xe^{-x^2}\,dx$

$$= \tfrac{1}{2}\left[-e^{-x^2}\right]_0^2 = \tfrac{1}{2}(-e^{-4} + 1)$$

92. $\text{Area} = \int_{-1}^{1} [e^y - (y^2 - 2)]\, dy = \int_{-1}^{1} (e^y - y^2 + 2)\, dy = \left[e^y - \frac{1}{3}y^3 + 2y\right]_{-1}^{1}$

$= \left(e - \frac{1}{3} + 2\right) - \left(e^{-1} + \frac{1}{3} - 2\right) = e - e^{-1} - \frac{2}{3} + 4 = e - e^{-1} + \frac{10}{3}$

93. $\text{Area} = \int_0^1 \left(e^{3x} - e^x\right) dx = \left[\frac{1}{3}e^{3x} - e^x\right]_0^1 = \left(\frac{1}{3}e^3 - e\right) - \left(\frac{1}{3} - 1\right) = \frac{1}{3}e^3 - e + \frac{2}{3} \approx 4.644$

94. $f''(x) = 3e^x + 5\sin x \quad\Rightarrow\quad f'(x) = 3e^x - 5\cos x + C \quad\Rightarrow\quad 2 = f'(0) = 3 - 5 + C \quad\Rightarrow\quad C = 4$, so

$f'(x) = 3e^x - 5\cos x + 4 \quad\Rightarrow\quad f(x) = 3e^x - 5\sin x + 4x + D \quad\Rightarrow\quad 1 = f(0) = 3 + D \quad\Rightarrow\quad D = -2$,

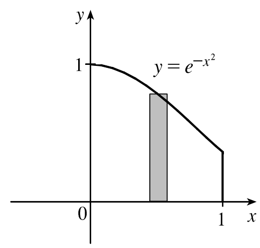
so $f(x) = 3e^x - 5\sin x + 4x - 2$.

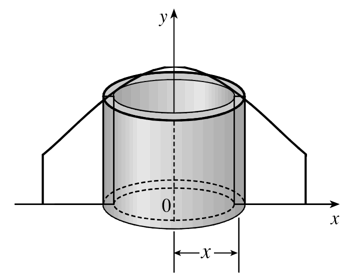
95. $V = \int_0^1 \pi(e^x)^2\, dx = \pi \int_0^1 e^{2x}\, dx = \frac{1}{2}\pi\left[e^{2x}\right]_0^1 = \frac{\pi}{2}\left(e^2 - 1\right)$

96. The shell has radius x, circumference $2\pi x$, and height e^{-x^2}, so $V = \displaystyle\int_0^1 2\pi x e^{-x^2}\, dx$.

Let $u = x^2$. Thus, $du = 2x\, dx$, so

$V = \pi \displaystyle\int_0^1 e^{-u}\, du = \pi\left[-e^{-u}\right]_0^1 = \pi(1 - 1/e)$.





97. **First Figure** Let $u = \sqrt{x}$, so $x = u^2$ and $dx = 2u\, du$. When $x = 0$, $u = 0$; when $x = 1$, $u = 1$. Thus,

$A_1 = \int_0^1 e^{\sqrt{x}}\, dx = \int_0^1 e^u (2u\, du) = 2\int_0^1 ue^u\, du$.

Second Figure $A_2 = \int_0^1 2xe^x\, dx = 2\int_0^1 ue^u\, du$.

Third Figure Let $u = \sin x$, so $du = \cos x\, dx$. When $x = 0$, $u = 0$; when $x = \frac{\pi}{2}$, $u = 1$. Thus,

$A_3 = \int_0^{\pi/2} e^{\sin x} \sin 2x\, dx = \int_0^{\pi/2} e^{\sin x}(2\sin x\, \cos x)\, dx = \int_0^1 e^u (2u\, du) = 2\int_0^1 ue^u\, du$.

Since $A_1 = A_2 = A_3$, all three areas are equal.

98. Let $r(t) = ae^{bt}$ with $a = 450.268$ and $b = 1.12567$, and $n(t) =$ population after t hours. Since $r(t) = n'(t)$, $\int_0^3 r(t)\, dt = n(3) - n(0)$ is the total change in the population after three hours. Since we start with 400 bacteria, the population will be

$$n(3) = 400 + \int_0^3 r(t)\, dt = 400 + \int_0^3 ae^{bt}\, dt = 400 + \frac{a}{b}\left[e^{bt}\right]_0^3 = 400 + \frac{a}{b}\left(e^{3b} - 1\right)$$

$$\approx 400 + 11{,}313 = 11{,}713 \text{ bacteria}$$

99. The rate is measured in liters per minute. Integrating from $t = 0$ minutes to $t = 60$ minutes will give us the total amount of oil that leaks out (in liters) during the first hour.

$\int_0^{60} r(t)\, dt = \int_0^{60} 100e^{-0.01t}\, dt \qquad [u = -0.01t, du = -0.01dt]$

$= 100\int_0^{-0.6} e^u(-100\, du) = -10{,}000\left[e^u\right]_0^{-0.6} = -10{,}000(e^{-0.6} - 1) \approx 4511.9 \approx 4512 \text{ L}$

100. The rate G is measured in kilograms per year. Integrating from $t = 0$ years (2000) to $t = 20$ years (2020) will give us the net change in biomass from 2000 to 2020.

$$\int_0^{20} \frac{60{,}000e^{-0.6t}}{(1+5e^{-0.6t})^2}\,dt = \int_6^{1+5e^{-12}} \frac{60{,}000}{u^2}\left(-\tfrac{1}{3}\,du\right) \qquad \begin{bmatrix} u = 1+5e^{-0.6t}, \\ du = -3e^{-0.6t}\,dt \end{bmatrix}$$

$$= \left[\frac{20{,}000}{u}\right]_6^{1+5e^{-12}} = \frac{20{,}000}{1+5e^{-12}} - \frac{20{,}000}{6} \approx 16{,}666$$

Thus, the predicted biomass for the year 2020 is approximately $25{,}000 + 16{,}666 = 41{,}666$ kg.

101. $$\int_0^{30} u(t)\,dt = \int_0^{30} \frac{r}{V}\,C_0e^{-rt/V}\,dt = C_0\int_1^{e^{-30r/V}}(-dx) \qquad \begin{bmatrix} x = e^{-rt/V}, \\ dx = -\frac{r}{V}e^{-rt/V}\,dt \end{bmatrix}$$

$$= C_0\Big[-x\Big]_1^{e^{-30r/V}} = C_0(-e^{-30r/V}+1)$$

The integral $\int_0^{30} u(t)\,dt$ represents the total amount of urea removed from the blood in the first 30 minutes of dialysis.

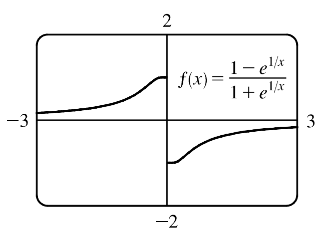
102. (a) $\operatorname{erf}(x) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^x e^{-t^2}\,dt \;\Rightarrow\; \int_0^x e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}\operatorname{erf}(x)$. By Property 5 of definite integrals in Section 4.2,

$\int_0^b e^{-t^2}\,dt = \int_0^a e^{-t^2}\,dt + \int_a^b e^{-t^2}\,dt$, so

$$\int_a^b e^{-t^2}\,dt = \int_0^b e^{-t^2}\,dt - \int_0^a e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}\operatorname{erf}(b) - \frac{\sqrt{\pi}}{2}\operatorname{erf}(a) = \tfrac{1}{2}\sqrt{\pi}\,[\operatorname{erf}(b) - \operatorname{erf}(a)].$$

(b) $y = e^{x^2}\operatorname{erf}(x) \;\Rightarrow\; y' = 2xe^{x^2}\operatorname{erf}(x) + e^{x^2}\operatorname{erf}'(x) = 2xy + e^{x^2}\cdot\dfrac{2}{\sqrt{\pi}}e^{-x^2}$ [by FTC1] $= 2xy + \dfrac{2}{\sqrt{\pi}}$.

103.



From the graph, it appears that f is an odd function (f is undefined for $x = 0$). To prove this, we must show that $f(-x) = -f(x)$.

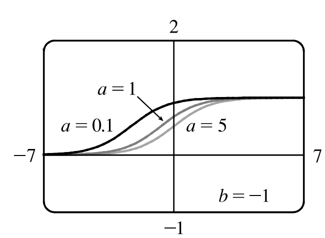
$$f(-x) = \frac{1-e^{1/(-x)}}{1+e^{1/(-x)}} = \frac{1-e^{(-1/x)}}{1+e^{(-1/x)}} = \frac{1-\dfrac{1}{e^{1/x}}}{1+\dfrac{1}{e^{1/x}}}\cdot\frac{e^{1/x}}{e^{1/x}} = \frac{e^{1/x}-1}{e^{1/x}+1}$$

$$= -\frac{1-e^{1/x}}{1+e^{1/x}} = -f(x)$$

so f is an odd function.

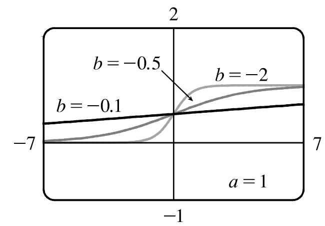
104. We'll start with $b = -1$ and graph $f(x) = \dfrac{1}{1+ae^{bx}}$ for $a = 0.1$, 1, and 5.

From the graph, we see that there is a horizontal asymptote $y = 0$ as $x \to -\infty$ and a horizontal asymptote $y = 1$ as $x \to \infty$. If $a = 1$, the y-intercept is $\left(0, \frac{1}{2}\right)$. As a gets smaller (close to 0), the graph of f moves left. As a gets larger, the graph of f moves right.

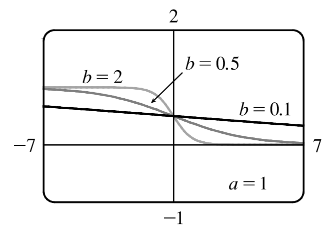


[continued]

As b changes from -1 to 0, the graph of f is stretched horizontally. As b changes through large negative values, the graph of f is compressed horizontally. (This takes care of negatives values of b.)



If b is positive, the graph of f is reflected through the y-axis.



Last, if $b = 0$, the graph of f is the horizontal line $y = 1/(1 + a)$.

105. Using the second law of logarithms and Equation 5, we have $\ln(e^x/e^y) = \ln e^x - \ln e^y = x - y = \ln(e^{x-y})$. Since ln is a one-to-one function, it follows that $e^x/e^y = e^{x-y}$.

106. Using the third law of logarithms and Equation 5, we have $\ln e^{rx} = rx = r \ln e^x = \ln (e^x)^r$. Since ln is a one-to-one function, it follows that $e^{rx} = (e^x)^r$.

6.4* General Logarithmic and Exponential Functions

1. (a) $b^x = e^{x \ln b}$

(b) The domain of $f(x) = b^x$ is $\mathbb{R}$.

(c) The range of $f(x) = b^x$ $[b \neq 1]$ is $(0, \infty)$.

(d) (i) See Figure 1. (ii) See Figure 3. (iii) See Figure 2.

2. (a) $\log_b x$ is the number y such that $b^y = x$.

(b) The domain of $f(x) = \log_b x$ is $(0, \infty)$.

(c) The range of $f(x) = \log_b x$ is $\mathbb{R}$.

(d) See Figure 11.

3. Since $b^x = e^{x \ln b}$, $4^{-\pi} = e^{-\pi \ln 4}$.

4. Since $b^x = e^{x \ln b}$, $x^{\sqrt{5}} = e^{\sqrt{5} \ln x}$.

5. Since $b^x = e^{x \ln b}$, $10^{x^2} = e^{x^2 \ln 10}$.

6. Since $b^x = e^{x \ln b}$, $(\tan x)^{\sec x} = e^{\sec x \ln \tan x}$.

7. (a) $\log_3 81 = \log_3 3^4 = 4$ (b) $\log_3\left(\frac{1}{81}\right) = \log_3 3^{-4} = -4$ (c) $\log_9 3 = \log_9 9^{1/2} = \frac{1}{2}$

8. *The Laws of Logarithms, used in several solutions for this section, are listed on Reference Page 4.*

(a) $\log_{10} \sqrt{10} = \log_{10} 10^{1/2} = \frac{1}{2}$ by the cancellation equation after (5).

(b) $$\begin{aligned} \log_{10} 40 + \log_{10} 2.5 &= \log_{10} [(40)(2.5)] && \text{[by Law 1]} \\ &= \log_{10} 100 \\ &= \log_{10} 10^2 = 2 && \text{[by the cancellation equation after (5)]} \end{aligned}$$

(c) $\log_2 30 - \log_2 15 = \log_2\left(\dfrac{30}{15}\right) = \log_2 2 = 1$

9. (a) $\log_3 10 - \log_3 5 - \log_3 18 = \log_3\left(\dfrac{10}{5}\right) - \log_3 18 = \log_3 2 - \log_3 18 = \log_3\left(\dfrac{2}{18}\right) = \log_3\left(\dfrac{1}{9}\right)$

$$= \log_3 3^{-2} = -2$$

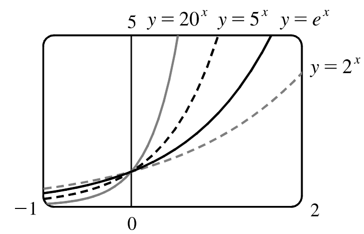
(b) $2\log_5 100 - 4\log_5 50 = \log_5 100^2 - \log_5 50^4 = \log_5\left(\dfrac{100^2}{50^4}\right) = \log_5\left(\dfrac{10^4}{5^4\cdot 10^4}\right) = \log_5 5^{-4} = -4$

10. (a) $\log_a \dfrac{1}{a} = -1$ since $a^{-1} = \dfrac{1}{a}$. $\quad$ [*Or:* $\log_a \dfrac{1}{a} = \log_a a^{-1} = -1$]

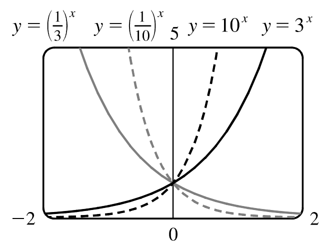
(b) $10^{(\log_{10} 4 + \log_{10} 7)} = 10^{\log_{10} 4}\cdot 10^{\log_{10} 7} = 4\cdot 7 = 28$

[*Or:* $10^{(\log_{10} 4 + \log_{10} 7)} = 10^{\log_{10}(4\cdot 7)} = 10^{\log_{10} 28} = 28$]

11. All of these graphs approach 0 as $x \to -\infty$, all of them pass through the point $(0, 1)$, and all of them are increasing and approach ∞ as $x \to \infty$. The larger the base, the faster the function increases for $x > 0$, and the faster it approaches 0 as $x \to -\infty$.

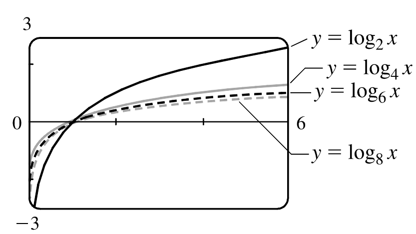


12. The functions with base greater than 1 (3^x and 10^x) are increasing, while those with base less than 1 $\left[\left(\frac{1}{3}\right)^x \text{ and } \left(\frac{1}{10}\right)^x\right]$ are decreasing. The graph of $\left(\frac{1}{3}\right)^x$ is the reflection of that of 3^x about the y-axis, and the graph of $\left(\frac{1}{10}\right)^x$ is the reflection of that of 10^x about the y-axis. The graph of 10^x increases more quickly than that of 3^x for $x > 0$, and approaches 0 faster as $x \to -\infty$.

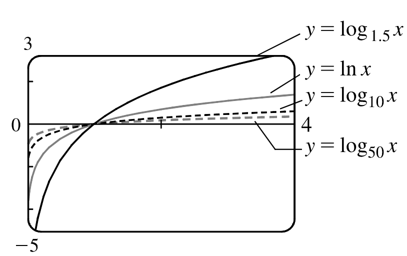


13. (a) $\log_5 10 = \dfrac{\ln 10}{\ln 5} \approx 1.430677$ $\quad$ (b) $\log_3 12 = \dfrac{\ln 12}{\ln 3} \approx 2.261860$ $\quad$ (c) $\log_{12} 6 = \dfrac{\ln 6}{\ln 12} \approx 0.721057$

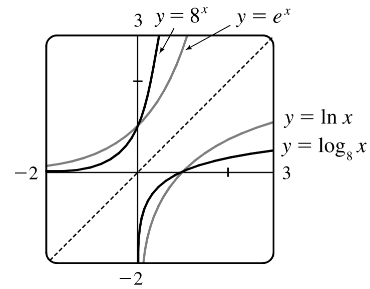
14. To graph the functions, we use $\log_2 x = \dfrac{\ln x}{\ln 2}$, $\log_4 x = \dfrac{\ln x}{\ln 4}$, etc. These graphs all approach $-\infty$ as $x \to 0^+$, and they all pass through the point $(1, 0)$. Also, they are all increasing, and all approach ∞ as $x \to \infty$. The smaller the base, the larger the rate of increase of the function (for $x > 1$) and the closer the approach to the y-axis (as $x \to 0^+$).



15. To graph these functions, we use $\log_{1.5} x = \dfrac{\ln x}{\ln 1.5}$ and $\log_{50} x = \dfrac{\ln x}{\ln 50}$. These graphs all approach $-\infty$ as $x \to 0^+$, and they all pass through the point $(1, 0)$. Also, they are all increasing, and all approach ∞ as $x \to \infty$. The functions with larger bases increase extremely slowly, and the ones with smaller bases do so somewhat more quickly. The functions with large bases approach the y-axis more closely as $x \to 0^+$.



16. We see that the graph of $\ln x$ is the reflection of the graph of e^x about the line $y = x$, and that the graph of $\log_8 x$ is the reflection of the graph of 8^x about the same line. The graph of 8^x increases more quickly than that of e^x. Also note that $\log_8 x \to \infty$ as $x \to \infty$ more slowly than $\ln x$.



17. Use $y = Cb^x$ with the points $(1, 6)$ and $(3, 24)$. $6 = Cb^1$ $\left[C = \frac{6}{b}\right]$ and $24 = Cb^3 \Rightarrow 24 = \left(\frac{6}{b}\right)b^3 \Rightarrow$ $4 = b^2 \Rightarrow b = 2$ [since $b > 0$] and $C = \frac{6}{2} = 3$. The function is $f(x) = 3 \cdot 2^x$.

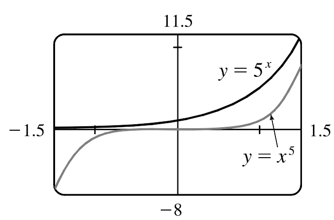
18. Given the y-intercept $(0, 2)$, we have $y = Ca^x = 2a^x$. Using the point $\left(2, \frac{2}{9}\right)$ gives us $\frac{2}{9} = 2a^2 \Rightarrow \frac{1}{9} = a^2 \Rightarrow$ $a = \frac{1}{3}$ [since $a > 0$]. The function is $f(x) = 2\left(\frac{1}{3}\right)^x$ or $f(x) = 2(3)^{-x}$.

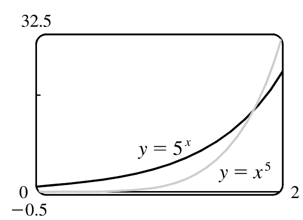
19. (a) 2 ft = 24 in, $f(24) = 24^2$ in = 576 in = 48 ft. $g(24) = 2^{24}$ in = $2^{24}/(12 \cdot 5280)$ mi ≈ 265 mi

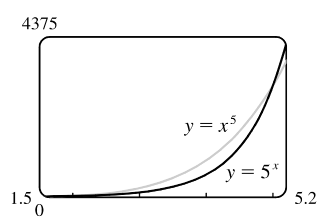
(b) 3 ft = 36 in, so we need x such that $\log_2 x = 36 \Leftrightarrow x = 2^{36} = 68{,}719{,}476{,}736$. In miles, this is

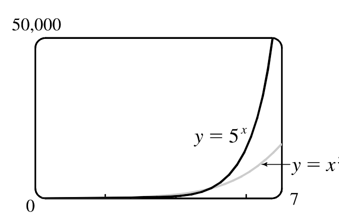
$$68{,}719{,}476{,}736 \text{ in} \cdot \frac{1 \text{ ft}}{12 \text{ in}} \cdot \frac{1 \text{ mi}}{5280 \text{ ft}} \approx 1{,}084{,}587.7 \text{ mi}.$$

20. We see from the graphs that for x less than about 1.8, $g(x) = 5^x > f(x) = x^5$, and then near the point $(1.8, 17.1)$ the curves intersect. Then $f(x) > g(x)$ from $x \approx 1.8$ until $x = 5$. At $(5, 3125)$ there is another point of intersection, and for $x > 5$ we see that $g(x) > f(x)$. In fact, g increases much more rapidly than f beyond that point.









21. $\lim\limits_{x\to\infty} (1.001)^x = \infty$ by Figure 1, since $1.001 > 1$.

22. By Figure 1, if $b > 1$, $\lim\limits_{x\to-\infty} b^x = 0$, so $\lim\limits_{x\to-\infty} (1.001)^x = 0$.

23. $\lim\limits_{t\to\infty} 2^{-t^2} = \lim\limits_{u\to-\infty} 2^u$ [where $u = -t^2$] $= 0$

24. Let $t = x^2 - 5x + 6$. As $x \to 3^+$, $t = (x-2)(x-3) \to 0^+$. $\lim\limits_{x \to 3^+} \log_{10}\left(x^2 - 5x + 6\right) = \lim\limits_{t \to 0^+} \log_{10} t = -\infty$

[analogous to (4) in Section 6.2*].

25. $f(x) = x^5 + 5^x \quad \Rightarrow \quad f'(x) = 5x^4 + 5^x \ln 5$

26. $g(x) = x\sin(2^x) \quad \Rightarrow \quad g'(x) = x\cos(2^x) \cdot 2^x \ln 2 + \sin(2^x) \cdot 1 = x2^x \ln 2 \cos(2^x) + \sin(2^x)$

27. Using Formula 4 and the Chain Rule, $G(x) = 4^{C/x} \quad \Rightarrow$

$$G'(x) = 4^{C/x}(\ln 4)\frac{d}{dx}\frac{C}{x} \quad \left[\frac{C}{x} = Cx^{-1}\right] \quad = 4^{C/x}(\ln 4)\left(-Cx^{-2}\right) = -C(\ln 4)\frac{4^{C/x}}{x^2}.$$

28. $F(t) = 3^{\cos 2t} \quad \Rightarrow \quad F'(t) = 3^{\cos 2t} \ln 3 \dfrac{d}{dt}(\cos 2t) = -2(\sin 2t)\, 3^{\cos 2t} \ln 3$

29. $L(v) = \tan(4^{v^2}) \quad \Rightarrow \quad L'(v) = \sec^2\left(4^{v^2}\right)\dfrac{d}{dv}\left(4^{v^2}\right) = \sec^2\left(4^{v^2}\right) \cdot 4^{v^2} \ln 4 \dfrac{d}{dv}(v^2) = 2v \ln 4 \sec^2\left(4^{v^2}\right) \cdot 4^{v^2}$

30. $G(u) = (1 + 10^{\ln u})^6 \quad \Rightarrow$

$$G'(u) = 6(1 + 10^{\ln u})^5 \frac{d}{du}(1 + 10^{\ln u}) = 6(1 + 10^{\ln u})^5\, 10^{\ln u} \ln 10 \frac{d}{du}(\ln u) = 6\ln 10(1 + 10^{\ln u})^5 \cdot 10^{\ln u}/u$$

31. $y = \log_8(x^2 + 3x) \quad \Rightarrow \quad y' = \dfrac{1}{(x^2 + 3x)\ln 8} \cdot \dfrac{d}{dx}(x^2 + 3x) = \dfrac{1}{(x^2 + 3x)\ln 8} \cdot (2x + 3) = \dfrac{2x + 3}{(x^2 + 3x)\ln 8}$

32. $f(x) = \log_{10}\sqrt{x} \quad \Rightarrow \quad f'(x) = \dfrac{1}{\sqrt{x}\,\ln 10}\dfrac{d}{dx}\sqrt{x} = \dfrac{1}{\sqrt{x}\,\ln 10}\dfrac{1}{2\sqrt{x}} = \dfrac{1}{2(\ln 10)x}$

Or: $f(x) = \log_{10}\sqrt{x} = \log_{10} x^{1/2} = \frac{1}{2}\log_{10} x \quad \Rightarrow \quad f'(x) = \dfrac{1}{2}\dfrac{1}{x\ln 10} = \dfrac{1}{2\,(\ln 10)\,x}$

33. $y = x\log_4 \sin x \quad \Rightarrow \quad y' = x \cdot \dfrac{1}{\sin x \ln 4} \cdot \cos x + \log_4 \sin x \cdot 1 = \dfrac{x\cot x}{\ln 4} + \log_4 \sin x$

34. $y = \log_2(x\log_5 x) \quad \Rightarrow$

$$y' = \frac{1}{(x\log_5 x)(\ln 2)}\frac{d}{dx}(x\log_5 x) = \frac{1}{(x\log_5 x)(\ln 2)}\left(x \cdot \frac{1}{x\ln 5} + \log_5 x\right) = \frac{1}{(x\log_5 x)(\ln 5)(\ln 2)} + \frac{1}{x(\ln 2)}.$$

Note that $\log_5 x(\ln 5) = \dfrac{\ln x}{\ln 5}(\ln 5) = \ln x$ by the change of base formula. Thus, $y' = \dfrac{1}{x\ln x\ln 2} + \dfrac{1}{x\ln 2} = \dfrac{1 + \ln x}{x\ln x\ln 2}$.

35. $y = x^x \quad \Rightarrow \quad \ln y = \ln x^x \quad \Rightarrow \quad \ln y = x\ln x \quad \Rightarrow \quad y'/y = x(1/x) + (\ln x) \cdot 1 \quad \Rightarrow \quad y' = y(1 + \ln x) \quad \Rightarrow$

$y' = x^x(1 + \ln x)$

36. $y = x^{1/x} \quad \Rightarrow \quad \ln y = \dfrac{1}{x}\ln x \quad \Rightarrow \quad \dfrac{y'}{y} = -\dfrac{1}{x^2}\ln x + \dfrac{1}{x}\left(\dfrac{1}{x}\right) \quad \Rightarrow \quad y' = x^{1/x}\dfrac{1 - \ln x}{x^2}$

37. $y = x^{\sin x} \quad \Rightarrow \quad \ln y = \ln x^{\sin x} \quad \Rightarrow \quad \ln y = \sin x \ln x \quad \Rightarrow \quad \dfrac{y'}{y} = (\sin x) \cdot \dfrac{1}{x} + (\ln x)(\cos x) \quad \Rightarrow$

$$y' = y\left(\frac{\sin x}{x} + \ln x\cos x\right) \quad \Rightarrow \quad y' = x^{\sin x}\left(\frac{\sin x}{x} + \ln x\,\cos x\right)$$

38. $y = \left(\sqrt{x}\right)^x \;\Rightarrow\; \ln y = \ln\left(\sqrt{x}\right)^x \;\Rightarrow\; \ln y = x\ln x^{1/2} \;\Rightarrow\; \ln y = \frac{1}{2}x\ln x \;\Rightarrow\; \dfrac{1}{y}y' = \dfrac{1}{2}x\cdot\dfrac{1}{x} + \ln x\cdot\dfrac{1}{2} \;\Rightarrow$

$y' = y\left(\frac{1}{2} + \frac{1}{2}\ln x\right) \;\Rightarrow\; y' = \frac{1}{2}\left(\sqrt{x}\right)^x(1+\ln x)$

39. $y = (\cos x)^x \;\Rightarrow\; \ln y = \ln(\cos x)^x \;\Rightarrow\; \ln y = x\ln\cos x \;\Rightarrow\; \dfrac{1}{y}y' = x\cdot\dfrac{1}{\cos x}\cdot(-\sin x) + \ln\cos x\cdot 1 \;\Rightarrow$

$y' = y\left(\ln\cos x - \dfrac{x\sin x}{\cos x}\right) \;\Rightarrow\; y' = (\cos x)^x(\ln\cos x - x\tan x)$

40. $y = (\sin x)^{\ln x} \;\Rightarrow\; \ln y = \ln(\sin x)^{\ln x} \;\Rightarrow\; \ln y = \ln x\cdot\ln\sin x \;\Rightarrow\; \dfrac{1}{y}y' = \ln x\cdot\dfrac{1}{\sin x}\cdot\cos x + \ln\sin x\cdot\dfrac{1}{x} \;\Rightarrow$

$y' = y\left(\ln x\cdot\dfrac{\cos x}{\sin x} + \dfrac{\ln\sin x}{x}\right) \;\Rightarrow\; y' = (\sin x)^{\ln x}\left(\ln x\cot x + \dfrac{\ln\sin x}{x}\right)$

41. $y = x^{\ln x} \;\Rightarrow\; \ln y = \ln x\ln x = (\ln x)^2 \;\Rightarrow\; \dfrac{y'}{y} = 2\ln x\left(\dfrac{1}{x}\right) \;\Rightarrow\; y' = x^{\ln x}\left(\dfrac{2\ln x}{x}\right)$

42. $y = (\ln x)^{\cos x} \;\Rightarrow\; \ln y = \cos x\ln(\ln x) \;\Rightarrow\; \dfrac{y'}{y} = \cos x\cdot\dfrac{1}{\ln x}\cdot\dfrac{1}{x} + (\ln\ln x)(-\sin x) \;\Rightarrow$

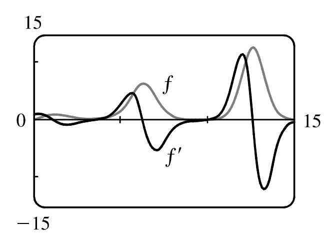
$y' = (\ln x)^{\cos x}\left(\dfrac{\cos x}{x\ln x} - \sin x\ln\ln x\right)$

43. $y = 10^x \;\Rightarrow\; y' = 10^x\ln 10$, so at $(1, 10)$, the slope of the tangent line is $10^1\ln 10 = 10\ln 10$, and its equation is $y - 10 = 10\ln 10(x-1)$, or $y = (10\ln 10)x + 10(1 - \ln 10)$.

44. $f(x) = x^{\cos x} = (e^{\ln x})^{\cos x} \;\Rightarrow$

$$f'(x) = e^{\ln x\cos x}\left[\ln x(-\sin x) + \cos x\left(\frac{1}{x}\right)\right] = x^{\cos x}\left[\frac{\cos x}{x} - \sin x\ln x\right]$$

This is reasonable, because the graph shows that f increases when $f'(x)$ is positive.



45. $\displaystyle\int_0^4 2^s\,ds = \left[\frac{1}{\ln 2}2^s\right]_0^4 = \frac{16}{\ln 2} - \frac{1}{\ln 2} = \frac{15}{\ln 2}$

46. $\int\left(x^5 + 5^x\right)dx = \frac{1}{6}x^6 + \frac{1}{\ln 5}5^x + C$

47. $\displaystyle\int\frac{\log_{10}x}{x}\,dx = \int\frac{(\ln x)/(\ln 10)}{x}\,dx = \frac{1}{\ln 10}\int\frac{\ln x}{x}\,dx$. Now put $u = \ln x$, so $du = \dfrac{1}{x}\,dx$, and the expression becomes $\displaystyle\frac{1}{\ln 10}\int u\,du = \frac{1}{\ln 10}\left(\tfrac{1}{2}u^2 + C_1\right) = \frac{1}{2\ln 10}(\ln x)^2 + C.$

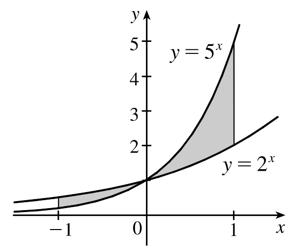
Or: The substitution $u = \log_{10}x$ gives $du = \dfrac{dx}{x\ln 10}$ and we get $\displaystyle\int\frac{\log_{10}x}{x}\,dx = \tfrac{1}{2}\ln 10(\log_{10}x)^2 + C.$

48. Let $u = x^2$. Then $du = 2x\,dx$, so $\displaystyle\int x2^{x^2}\,dx = \tfrac{1}{2}\int 2^u\,du = \frac{1}{2}\frac{2^u}{\ln 2} + C = \frac{1}{2\ln 2}2^{x^2} + C.$

49. Let $u = \sin\theta$. Then $du = \cos\theta\,d\theta$ and $\displaystyle\int 3^{\sin\theta}\cos\theta\,d\theta = \int 3^u\,du = \frac{3^u}{\ln 3} + C = \frac{1}{\ln 3}3^{\sin\theta} + C$.

50. Let $u = 2^x + 1$. Then $du = 2^x\ln 2\,dx$, so $\displaystyle\int \frac{2^x}{2^x+1}\,dx = \int \frac{1}{u}\frac{du}{\ln 2} = \frac{1}{\ln 2}\ln|u| + C = \frac{1}{\ln 2}\ln(2^x+1) + C$.

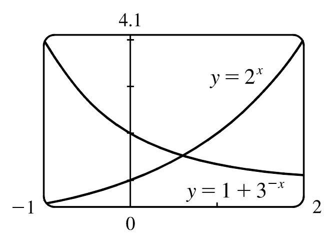
51. $$A = \int_{-1}^{0}(2^x - 5^x)\,dx + \int_0^1 (5^x - 2^x)\,dx = \left[\frac{2^x}{\ln 2} - \frac{5^x}{\ln 5}\right]_{-1}^{0} + \left[\frac{5^x}{\ln 5} - \frac{2^x}{\ln 2}\right]_0^1$$
$$= \left(\frac{1}{\ln 2} - \frac{1}{\ln 5}\right) - \left(\frac{1/2}{\ln 2} - \frac{1/5}{\ln 5}\right) + \left(\frac{5}{\ln 5} - \frac{2}{\ln 2}\right) - \left(\frac{1}{\ln 5} - \frac{1}{\ln 2}\right)$$
$$= \frac{16}{5\ln 5} - \frac{1}{2\ln 2}$$



52. Using disks, the volume is $V = \int_0^1 \pi[10^{-x}]^2\,dx = \pi\int_0^1 10^{-2x}\,dx$. To evaluate the integral, we let $u = -2x \Rightarrow du = -2\,dx$, $x = 0 \Rightarrow u = 0$, and $x = 1 \Rightarrow u = -2$, so we have

$$V = -\frac{\pi}{2}\int_0^{-2} 10^u\,du = -\frac{\pi}{2}\left[\frac{1}{\ln 10}10^u\right]_0^{-2} = -\frac{\pi}{2\ln 10}(10^{-2} - 1) = \frac{99\pi}{200\ln 10}$$

53. We see that the graphs of $y = 2^x$ and $y = 1 + 3^{-x}$ intersect at $x \approx 0.6$. We let $f(x) = 2^x - 1 - 3^{-x}$ and calculate $f'(x) = 2^x\ln 2 + 3^{-x}\ln 3$, and using the formula $x_{n+1} = x_n - f(x_n)/f'(x_n)$ (Newton's method), we get $x_1 = 0.6$, $x_2 \approx x_3 \approx 0.600967$. So, correct to six decimal places, the solution occurs at $x = 0.600967$.



54. $x^y = y^x \Rightarrow y\ln x = x\ln y \Rightarrow y\cdot\frac{1}{x} + (\ln x)\cdot y' = x\cdot\frac{1}{y}\cdot y' + \ln y \Rightarrow y'\ln x - \frac{x}{y}y' = \ln y - \frac{y}{x} \Rightarrow$
$y' = \dfrac{\ln y - y/x}{\ln x - x/y}$

55. $y = g(x) = \log_4(x^3+2) \Rightarrow 4^y = x^3 + 2 \Rightarrow x^3 = 4^y - 2 \Rightarrow x = \sqrt[3]{4^y - 2}$. Interchange x and y: $y = \sqrt[3]{4^x - 2}$. So $g^{-1}(x) = \sqrt[3]{4^x - 2}$.

56. $\displaystyle\lim_{x\to 0^+} x^{-\ln x} = \lim_{x\to 0^+}\left(e^{\ln x}\right)^{-\ln x} = \lim_{x\to 0^+} e^{-(\ln x)^2} = 0$ since $-(\ln x)^2 \to -\infty$ as $x \to 0^+$.

57. If I is the intensity of the 1989 San Francisco earthquake, then $\log_{10}(I/S) = 7.1 \Rightarrow$
$\log_{10}(16I/S) = \log_{10}16 + \log_{10}(I/S) = \log_{10}16 + 7.1 \approx 8.3$.

58. Let I_1 and I_2 be the intensities of the music and the mower. Then $10\log_{10}\left(\dfrac{I_1}{I_0}\right) = 120$ and $10\log_{10}\left(\dfrac{I_2}{I_0}\right) = 106$, so

$$\log_{10}\left(\frac{I_1}{I_2}\right) = \log_{10}\left(\frac{I_1/I_0}{I_2/I_0}\right) = \log_{10}\left(\frac{I_1}{I_0}\right) - \log_{10}\left(\frac{I_2}{I_0}\right) = 12 - 10.6 = 1.4 \Rightarrow \frac{I_1}{I_2} = 10^{1.4} \approx 25.$$

59. We find I with the loudness formula from Exercise 58, substituting $I_0 = 10^{-12}$ and $L = 50$:

$50 = 10\log_{10}\dfrac{I}{10^{-12}} \quad\Leftrightarrow\quad 5 = \log_{10}\dfrac{I}{10^{-12}} \quad\Leftrightarrow\quad 10^5 = \dfrac{I}{10^{-12}} \quad\Leftrightarrow\quad I = 10^{-7}\text{ watt/m}^2$. Now we differentiate L with respect to I: $L = 10\log_{10}\dfrac{I}{I_0} \quad\Rightarrow\quad \dfrac{dL}{dI} = 10\dfrac{1}{(I/I_0)\ln 10}\left(\dfrac{1}{I_0}\right) = \dfrac{10}{\ln 10}\left(\dfrac{1}{I}\right)$. Substituting $I = 10^{-7}$, we get

$L'(50) = \dfrac{10}{\ln 10}\left(\dfrac{1}{10^{-7}}\right) = \dfrac{10^8}{\ln 10} \approx 4.34 \times 10^7\ \dfrac{\text{dB}}{\text{watt/m}^2}$.

60. (a) $I(x) = I_0 a^x \quad\Rightarrow\quad I'(x) = I_0(\ln a)a^x = (I_0 a^x)\ \ln a = I(x)\ \ln a$

(b) We substitute $I_0 = 8$, $a = 0.38$ and $x = 20$ into the first expression for $I'(x)$ above:

$I'(20) = 8(\ln 0.38)(0.38)^{20} \approx -3.05 \times 10^{-8}$.

(c) The average value of the function $I(x)$ between $x = 0$ and $x = 20$ is

$$\frac{\int_0^{20} I(x)\,dx}{20-0} = \frac{1}{20}\int_0^{20} 8(0.38)^x\,dx = \frac{2}{5}\left[\frac{(0.38)^x}{\ln 0.38}\right]_0^{20} = \frac{2(0.38^{20}-1)}{5\ln 0.38} \approx 0.41.$$

61. (a) $I = \log_2\left(\dfrac{2D}{W}\right) \quad\Rightarrow\quad \dfrac{dI}{dD}\quad [W\text{ constant}] \quad = \dfrac{1}{\left(\dfrac{2D}{W}\right)\ln 2}\cdot\dfrac{2}{W} = \dfrac{1}{D\ln 2}$

As D increases, the rate of change of difficulty decreases.

(b) $I = \log_2\left(\dfrac{2D}{W}\right) \Rightarrow \dfrac{dI}{dW}\quad [D\text{ constant}] \quad = \dfrac{1}{\left(\dfrac{2D}{W}\right)\ln 2}\cdot\left(-2DW^{-2}\right) = \dfrac{W}{2D\ln 2}\cdot\dfrac{-2D}{W^2} = -\dfrac{1}{W\ln 2}$

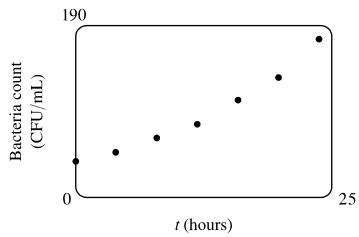
The negative sign indicates that difficulty decreases with increasing width. While the magnitude of the rate of change decreases with increasing width $\left(\text{that is, } \left|-\dfrac{1}{W\ln 2}\right| = \dfrac{1}{W\ln 2} \text{ decreases as } W \text{ increases}\right)$, the rate of change itself increases (gets closer to zero from the negative side) with increasing values of W.

(c) The answers to (a) and (b) agree with intuition. For fixed width, the difficulty of acquiring a target increases, but less and less so, as the distance to the target increases. Similarly, for a fixed distance to a target, the difficulty of acquiring the target decreases, but less and less so, as the width of the target increases.

62. $P(t) = (1.43653 \times 10^9)\cdot(1.01395)^t \quad\Rightarrow\quad P'(t) = (1.43653 \times 10^9)\cdot(1.01395)^t(\ln 1.01395)$. The units for $P'(t)$ are millions of people per year. The rates of increase for 1920, 1950, and 2000 are $P'(20) \approx 26.25$, $P'(50) \approx 39.78$, and $P'(100) \approx 79.53$, respectively.

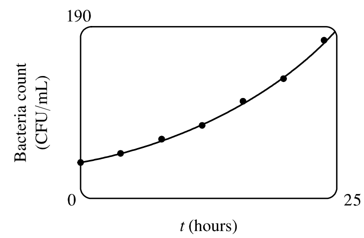
63. Half of 76.0 RNA copies per mL, corresponding to $t = 1$, is 38.0 RNA copies per mL. Using the graph of V in Figure 9, we estimate that it takes about 3.5 additional days for the patient's viral load to decrease to 38 RNA copies per mL.

64. (a)

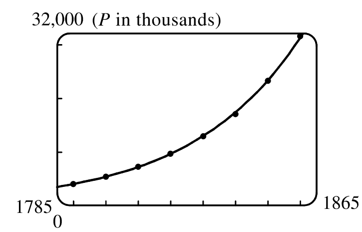


(b) Using a graphing calculator, we obtain the exponential curve $f(t) = 36.89301(1.06614)^t$.

(c) Using the TRACE and zooming in, we find that the bacteria count doubles from 37 to 74 in about 10.87 hours.



65. (a) $P = ab^t$ with $a = 4.502714 \times 10^{-20}$ and $b = 1.029953851$, where P is measured in thousands of people. The fit appears to be very good.



(b) **For 1800:** $m_1 = \dfrac{5308 - 3929}{1800 - 1790} = 137.9$, $m_2 = \dfrac{7240 - 5308}{1810 - 1800} = 193.2$.

So $P'(1800) \approx (m_1 + m_2)/2 = 165.55$ thousand people/year.

For 1850: $m_1 = \dfrac{23{,}192 - 17{,}063}{1850 - 1840} = 612.9$, $m_2 = \dfrac{31{,}443 - 23{,}192}{1860 - 1850} = 825.1$.

So $P'(1850) \approx (m_1 + m_2)/2 = 719$ thousand people/year.

(c) Using $P'(t) = ab^t \ln b$ (from Formula 4) with the values of a and b from part (a), we get $P'(1800) \approx 156.85$ thousand people/year and $P'(1850) \approx 686.07$. These estimates are somewhat less than the ones in part (b).

(d) $P(1870) \approx 41{,}946.56$. The difference of 3.4 million people is most likely due to the Civil War (1861–1865).

66. Let $m = n/x$. Then $n = xm$, and as $n \to \infty$, $m \to \infty$.

Therefore, $\displaystyle\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = \lim_{m\to\infty}\left(1 + \frac{1}{m}\right)^{mx} = \left[\lim_{m\to\infty}\left(1 + \frac{1}{m}\right)^m\right]^x = e^x$ by Equation 9.

67. $y = b^x \Rightarrow y' = b^x \ln b$, so the slope of the tangent line to the curve $y = b^x$ at the point (a, b^a) is $b^a \ln b$. An equation of this tangent line is then $y - b^a = b^a \ln b\,(x - a)$. If c is the x-intercept of this tangent line, then $0 - b^a = b^a \ln b\,(c - a) \Rightarrow$ $-1 = \ln b\,(c - a) \Rightarrow \dfrac{-1}{\ln b} = c - a \Rightarrow |c - a| = \left|\dfrac{-1}{\ln b}\right| = \dfrac{1}{|\ln b|}$. The distance between $(a, 0)$ and $(c, 0)$ is $|c - a|$, and this distance is the constant $\dfrac{1}{|\ln b|}$ for any a. [*Note*: The absolute value is needed for the case $0 < b < 1$ because $\ln b$ is negative there. If $b > 1$, we can write $a - c = 1/(\ln b)$ as the constant distance between $(a, 0)$ and $(c, 0)$.]

68. $y = b^x \Rightarrow y' = b^x \ln b$, so the slope of the tangent line to the curve $y = b^x$ at the point (x_0, y_0) is $b^{x_0} \ln b$. An equation of this tangent line is then $y - y_0 = b^{x_0} \ln b\,(x - x_0)$. Since this tangent line must pass through $(0, 0)$, we have

$0 - y_0 = b^{x_0} \ln b\,(0 - x_0)$, or $y_0 = b^{x_0}\,(\ln b)\,x_0$. Since (x_0, y_0) is a point on the exponential curve $y = b^x$, we also have $y_0 = b^{x_0}$. Equating the expressions for y_0 gives $b^{x_0} = b^{x_0}\,(\ln b)\,x_0 \;\Rightarrow\; 1 = (\ln b)\,x_0 \;\Rightarrow\; x_0 = 1/(\ln b)$.

So $y_0 = b^{x_0} = e^{x_0 \ln b}$ [by combining Property 6.3*.4 with Law 3 of Theorem 6.2*.3] $= e^{(1/(\ln b))\ln b} = e^1 = e$.

69. Using Definition 1 and the second law of exponents for e^x, we have $b^{x-y} = e^{(x-y)\ln b} = e^{x\ln b - y\ln b} = \dfrac{e^{x\ln b}}{e^{y\ln b}} = \dfrac{b^x}{b^y}$.

70. Using Definition 1, the first law of logarithms, and the first law of exponents for e^x, we have

$(ab)^x = e^{x\ln(ab)} = e^{x(\ln a + \ln b)} = e^{x\ln a + x\ln b} = e^{x\ln a}e^{x\ln b} = a^x b^x$.

71. Let $\log_b x = r$ and $\log_b y = s$. Then $b^r = x$ and $b^s = y$.

(a) $xy = b^r b^s = b^{r+s} \;\Rightarrow\; \log_b(xy) = r + s = \log_b x + \log_b y$

(b) $\dfrac{x}{y} = \dfrac{b^r}{b^s} = b^{r-s} \;\Rightarrow\; \log_b \dfrac{x}{y} = r - s = \log_b x - \log_b y$

(c) $x^y = (b^r)^y = b^{ry} \;\Rightarrow\; \log_b(x^y) = ry = y\log_b x$

6.5 Exponential Growth and Decay

1. The relative growth rate is $\dfrac{1}{P}\dfrac{dP}{dt} = 0.4159$, so $\dfrac{dP}{dt} = 0.4159P$ and by Theorem 2,

$P(t) = P(0)e^{0.4159t} = 3.8e^{0.4159t}$ million cells. Thus, $P(2) = 3.8e^{0.4159(2)} \approx 8.7$ million cells.

2. (a) By Theorem 2, $P(t) = P(0)e^{kt} = 50e^{kt}$. In 20 minutes $\left(\frac{1}{3}\text{ hour}\right)$, there are 100 cells, so $P\left(\frac{1}{3}\right) = 50e^{k/3} = 100 \;\Rightarrow$

$e^{k/3} = 2 \;\Rightarrow\; k/3 = \ln 2 \;\Rightarrow\; k = 3\ln 2 = \ln(2^3) = \ln 8$.

(b) $P(t) = 50e^{(\ln 8)t} = 50 \cdot 8^t$

(c) $P(6) = 50 \cdot 8^6 = 50 \cdot 2^{18} = 13{,}107{,}200$ cells

(d) $\dfrac{dP}{dt} = kP \;\Rightarrow\; P'(6) = kP(6) = (\ln 8)P(6) \approx 27{,}255{,}656$ cells/h

(e) $P(t) = 10^6 \;\Leftrightarrow\; 50 \cdot 8^t = 1{,}000{,}000 \;\Leftrightarrow\; 8^t = 20{,}000 \;\Leftrightarrow\; t\ln 8 = \ln 20{,}000 \;\Leftrightarrow\; t = \dfrac{\ln 20{,}000}{\ln 8} \approx 4.76$ h

3. (a) By Theorem 2, $P(t) = P(0)e^{kt} = 50e^{kt}$. Now $P(1.5) = 50e^{k(1.5)} = 975 \;\Rightarrow\; e^{1.5k} = \frac{975}{50} \;\Rightarrow$

$1.5k = \ln 19.5 \;\Rightarrow\; k = \frac{1}{1.5}\ln 19.5 \approx 1.9803$. So $P(t) \approx 50e^{1.9803t}$ cells.

(b) Using 1.9803 for k, we get $P(3) = 50e^{1.9803(3)} = 19{,}013.85 \approx 19{,}014$ cells.

(c) $\dfrac{dP}{dt} = kP \;\Rightarrow\; P'(3) = k \cdot P(3) = 1.9803 \cdot 19{,}014$ [from parts (a) and (b)] $= 37{,}653.4 \approx 37{,}653$ cells/h

(d) $P(t) = 50e^{1.9803t} = 250{,}000 \;\Rightarrow\; e^{1.9803t} = \dfrac{250{,}000}{50} \;\Rightarrow\; e^{1.9803t} = 5000 \;\Rightarrow\; 1.9803t = \ln 5000 \;\Rightarrow$

$t = \dfrac{\ln 5000}{1.9803} \approx 4.30$ h

4. (a) $y(t) = y(0)e^{kt} \Rightarrow y(2) = y(0)e^{2k} = 400$ and $y(6) = y(0)e^{6k} = 25{,}600$. Dividing these equations, we get

$e^{6k}/e^{2k} = 25{,}600/400 \Rightarrow e^{4k} = 64 \Rightarrow 4k = \ln 2^6 = 6\ln 2 \Rightarrow k = \frac{3}{2}\ln 2 \approx 1.0397$, about 104% per hour.

(b) $400 = y(0)e^{2k} \Rightarrow y(0) = 400/e^{2k} \Rightarrow y(0) = 400/e^{3\ln 2} = 400/\left(e^{\ln 2}\right)^3 = 400/2^3 = 50$.

(c) $y(t) = y(0)e^{kt} = 50e^{(3/2)(\ln 2)t} = 50(e^{\ln 2})^{(3/2)t} \Rightarrow y(t) = 50(2)^{1.5t}$

(d) $y(4.5) = 50(2)^{1.5(4.5)} = 50(2)^{6.75} \approx 5382$ bacteria

(e) $\dfrac{dy}{dt} = ky = \left(\dfrac{3}{2}\ln 2\right)(50(2)^{6.75})$ [from parts (a) and (b)] ≈ 5596 bacteria/h

(f) $y(t) = 50{,}000 \Rightarrow 50{,}000 = 50(2)^{1.5t} \Rightarrow 1000 = (2)^{1.5t} \Rightarrow \ln 1000 = 1.5t\ln 2 \Rightarrow$

$t = \dfrac{\ln 1000}{1.5\ln 2} \approx 6.64$ h

5. (a) Let the population (in millions) in the year t be $P(t)$. Since the initial time is the year 1750, we substitute $t - 1750$ for t in Theorem 2, so the exponential model gives $P(t) = P(1750)e^{k(t-1750)}$. Then $P(1800) = 980 = 790e^{k(1800-1750)} \Rightarrow$ $\frac{980}{790} = e^{k(50)} \Rightarrow \ln\frac{980}{790} = 50k \Rightarrow k = \frac{1}{50}\ln\frac{980}{790} \approx 0.0043104$. So with this model, we have $P(1900) = 790e^{k(1900-1750)} \approx 1508$ million, and $P(1950) = 790e^{k(1950-1750)} \approx 1871$ million. Both of these estimates are much too low.

(b) In this case, the exponential model gives $P(t) = P(1850)e^{k(t-1850)} \Rightarrow P(1900) = 1650 = 1260e^{k(1900-1850)} \Rightarrow$ $\ln\frac{1650}{1260} = k(50) \Rightarrow k = \frac{1}{50}\ln\frac{1650}{1260} \approx 0.005393$. So with this model, we estimate $P(1950) = 1260e^{k(1950-1850)} \approx 2161$ million. This is still too low, but closer than the estimate of $P(1950)$ in part (a).

(c) The exponential model gives $P(t) = P(1900)e^{k(t-1900)} \Rightarrow P(1950) = 2560 = 1650e^{k(1950-1900)} \Rightarrow$ $\ln\frac{2560}{1650} = k(50) \Rightarrow k = \frac{1}{50}\ln\frac{2560}{1650} \approx 0.008785$. With this model, we estimate $P(2000) = 1650e^{k(2000-1900)} \approx 3972$ million. This is much too low. The discrepancy is explained by the fact that the world birth rate (average yearly number of births per person) is about the same as always, whereas the mortality rate (especially the infant mortality rate) is much lower, owing mostly to advances in medical science and to the wars in the first part of the twentieth century. The exponential model assumes, among other things, that the birth and mortality rates will remain constant.

6. (a) Let $P(t)$ be the population (in millions) in the year t. Since the initial time is the year 1950, we substitute $t - 1950$ for t in Theorem 2, and find that the exponential model gives $P(t) = P(1950)e^{k(t-1950)} \Rightarrow$ $P(1960) = 100 = 83e^{k(1960-1950)} \Rightarrow \frac{100}{83} = e^{10k} \Rightarrow k = \frac{1}{10}\ln\frac{100}{83} \approx 0.0186$. With this model, we estimate $P(1980) = 83e^{k(1980-1950)} = 83e^{30k} \approx 145$ million, which is an underestimate of the actual population of 150 million.

(b) As in part (a), $P(t) = P(1960)e^{k(t-1960)} \;\Rightarrow\; P(1980) = 150 = 100e^{20k} \;\Rightarrow\; 20k = \ln\frac{150}{100} \;\Rightarrow$

$k = \frac{1}{20}\ln\frac{3}{2} \approx 0.0203$. Thus, $P(2000) = 100e^{40k} = 225$ million, which is an overestimate of the actual population of 214 million.

(c) As in part (a), $P(t) = P(1980)e^{k(t-1980)} \;\Rightarrow\; P(2000) = 214 = 150e^{20k} \;\Rightarrow\; 20k = \ln\frac{214}{150} \;\Rightarrow$

$k = \frac{1}{20}\ln\frac{214}{150} \approx 0.0178$. Thus, $P(2010) = 150e^{30k} \approx 256$, which is an overestimate of the actual population of 243 million.

(d) Using the model in part (c), $P(2025) = 150e^{(2025-1980)k} = 150e^{45k} \approx 334$ million. This prediction is likely too high. The model gave an overestimate for 2010, and the amount of overestimation is likely to compound as time increases.

7. (a) If $y = [N_2O_5]$ then by Theorem 2, $\dfrac{dy}{dt} = -0.0005y \;\Rightarrow\; y(t) = y(0)e^{-0.0005t} = Ce^{-0.0005t}$.

(b) $y(t) = Ce^{-0.0005t} = 0.9C \;\Rightarrow\; e^{-0.0005t} = 0.9 \;\Rightarrow\; -0.0005t = \ln 0.9 \;\Rightarrow\; t = -2000\ln 0.9 \approx 211$ s

8. (a) The mass remaining after t days is $y(t) = y(0)\,e^{kt} = 50e^{kt}$. Since the half-life is 28 days, $y(28) = 50e^{28k} = 25 \;\Rightarrow$

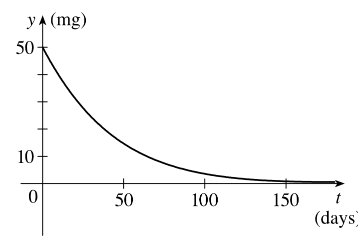
$e^{28k} = \frac{1}{2} \;\Rightarrow\; 28k = \ln\frac{1}{2} \;\Rightarrow\; k = -(\ln 2)/28$, so $y(t) = 50e^{-(\ln 2)t/28} = 50 \cdot 2^{-t/28}$.

(b) $y(40) = 50 \cdot 2^{-40/28} \approx 18.6$ mg

(c) $y(t) = 2 \;\Rightarrow\; 2 = 50 \cdot 2^{-t/28} \;\Rightarrow\; \frac{2}{50} = 2^{-t/28} \;\Rightarrow$

$(-t/28)\ln 2 = \ln\frac{1}{25} \;\Rightarrow\; t = \left(-28\ln\frac{1}{25}\right)/\ln 2 \approx 130$ days

(d)



9. (a) If $y(t)$ is the mass (in mg) remaining after t years, then $y(t) = y(0)e^{kt} = 100e^{kt}$.

$y(30) = 100e^{30k} = \frac{1}{2}(100) \;\Rightarrow\; e^{30k} = \frac{1}{2} \;\Rightarrow\; k = -(\ln 2)/30 \;\Rightarrow\; y(t) = 100e^{-(\ln 2)t/30} = 100 \cdot 2^{-t/30}$

(b) $y(100) = 100 \cdot 2^{-100/30} \approx 9.92$ mg

(c) $100e^{-(\ln 2)t/30} = 1 \;\Rightarrow\; -(\ln 2)t/30 = \ln\frac{1}{100} \;\Rightarrow\; t = -30\,\frac{\ln 0.01}{\ln 2} \approx 199.3$ years

10. (a) If $y(t)$ is the mass after t days and $y(0) = A$, then $y(t) = Ae^{kt}$.

$y(300) = Ae^{300k} = 0.643A \;\Rightarrow\; e^{300k} = 0.643 \;\Rightarrow\; k = \frac{1}{300}\ln 0.643$. To find the half-life, we set the mass after t days equal to one-half of the original mass. Hence, $Ae^{(1/300)(\ln 0.643)t} = \frac{1}{2}A \;\Leftrightarrow\; \frac{1}{300}(\ln 0.643)\,t = \ln\frac{1}{2} \;\Leftrightarrow$

$t = \dfrac{300\ln\frac{1}{2}}{\ln 0.643} \approx 471$ days.

(b) $Ae^{(1/300)(\ln 0.643)t} = \frac{1}{3}A \;\Leftrightarrow\; \frac{1}{300}(\ln 0.643)\,t = \ln\frac{1}{3} \;\Leftrightarrow\; t = \dfrac{300\ln\frac{1}{3}}{\ln 0.643} \approx 746$ days

11. Let $y(t)$ be the level of radioactivity. Thus, $y(t) = y(0)e^{-kt}$ and k is determined by using the half-life:

$$y(5730) = \tfrac{1}{2}y(0) \quad\Rightarrow\quad y(0)e^{-k(5730)} = \tfrac{1}{2}y(0) \quad\Rightarrow\quad e^{-5730k} = \tfrac{1}{2} \quad\Rightarrow\quad -5730k = \ln\tfrac{1}{2} \quad\Rightarrow\quad k = -\frac{\ln\frac{1}{2}}{5730} = \frac{\ln 2}{5730}.$$

If 74% of the ^{14}C remains, then we know that $y(t) = 0.74y(0) \quad\Rightarrow\quad 0.74 = e^{-t(\ln 2)/5730} \quad\Rightarrow\quad \ln 0.74 = -\dfrac{t\ln 2}{5730} \quad\Rightarrow$

$$t = -\frac{5730(\ln 0.74)}{\ln 2} \approx 2489 \approx 2500 \text{ years.}$$

12. From Exercise 11, we have the model $y(t) = y(0)e^{-kt}$ with $k = (\ln 2)/5730$. Thus,

$y(68{,}000{,}000) = y(0)e^{-68{,}000{,}000k} \approx y(0)\cdot 0 = 0$. There would be an undetectable amount of ^{14}C remaining for a 68-million-year-old dinosaur.

Now let $y(t) = 0.1\%\,y(0)$, so $0.001y(0) = y(0)e^{-kt} \quad\Rightarrow\quad 0.001 = e^{-kt} \quad\Rightarrow\quad \ln 0.001 = -kt \quad\Rightarrow$

$t = \dfrac{\ln 0.001}{-k} = \dfrac{\ln 0.001}{-(\ln 2)/5730} \approx 57{,}104$, which is the maximum age of a fossil that we could date using ^{14}C.

13. Let t measure time since a dinosaur died in millions of years, and let $y(t)$ be the amount of ^{40}K in the dinosaur's bones at time t. Then $y(t) = y(0)e^{-kt}$ and k is determined by the half-life: $y(1250) = \frac{1}{2}y(0) \quad\Rightarrow\quad y(0)e^{-k(1250)} = \frac{1}{2}y(0) \quad\Rightarrow$

$e^{-1250k} = \frac{1}{2} \quad\Rightarrow\quad -1250k = \ln\frac{1}{2} \quad\Rightarrow\quad k = -\dfrac{\ln\frac{1}{2}}{1250} = \dfrac{\ln 2}{1250}$. To determine if a dinosaur dating of 68 million years is possible, we find that $y(68) = y(0)e^{-k(68)} \approx 0.963y(0)$, indicating that about 96% of the ^{40}K is remaining, which is clearly detectable. To determine the maximum age of a fossil by using ^{40}K, we solve $y(t) = 0.1\%y(0)$ for t.

$y(0)e^{-kt} = 0.001y(0) \quad\Leftrightarrow\quad e^{-kt} = 0.001 \quad\Leftrightarrow\quad -kt = \ln 0.001 \quad\Leftrightarrow\quad t = \dfrac{\ln 0.001}{-(\ln 2)/1250} \approx 12{,}457$ million, or 12.457 billion years.

14. From the information given, we know that $\dfrac{dy}{dx} = 2y \quad\Rightarrow\quad y = Ce^{2x}$ by Theorem 2. To calculate C we use the point $(0,5)$:

$5 = Ce^{2(0)} \quad\Rightarrow\quad C = 5$. Thus, the equation of the curve is $y = 5e^{2x}$.

15. (a) Using Newton's Law of Cooling, $\dfrac{dT}{dt} = k(T - T_s)$, we have $\dfrac{dT}{dt} = k(T - 22)$. Now let $y = T - 22$, so

$y(0) = T(0) - 22 = 85 - 22 = 63$, so y is a solution of the initial-value problem $dy/dt = ky$ with $y(0) = 63$ and by Theorem 2 we have $y(t) = y(0)e^{kt} = 63\,e^{kt}$.

$y(30) = 63e^{30k} = 65 - 22 \implies e^{30k} = \frac{43}{63} \implies k = \frac{1}{30}\ln\frac{43}{63}$, so $y(t) = 63e^{\frac{1}{30}t\ln\frac{43}{63}}$ and $y(45) = 63e^{\frac{45}{30}\ln\frac{43}{63}} \approx$ 35.5°C. Thus, $T(45) \approx 35.5 + 22 = 57.5^\circ$C.

(b) $T(t) = 40 \implies y(t) = 18$. $y(t) = 63e^{\frac{1}{30}t\ln\frac{43}{63}} = 18 \implies e^{\frac{1}{30}t\ln\frac{43}{63}} = \frac{18}{63} \implies \frac{1}{30}t\ln\frac{43}{63} = \ln\frac{2}{7} \implies t = \frac{30\ln\frac{2}{7}}{\ln\frac{43}{63}} \approx 98$ min.

16. Let $T(t)$ be the temperature of the body t hours after 1:30 PM. Then $T(0) = 32.5$ and $T(1) = 30.3$. Using Newton's Law of Cooling, $\dfrac{dT}{dt} = k(T - T_s)$, we have $\dfrac{dT}{dt} = k(T - 20)$. Now let $y = T - 20$, so $y(0) = T(0) - 20 = 32.5 - 20 = 12.5$, so y is a solution to the initial value problem $dy/dt = ky$ with $y(0) = 12.5$ and by Theorem 2 we have $y(t) = y(0)e^{kt} = 12.5e^{kt}$.

$y(1) = 30.3 - 20 \Rightarrow 10.3 = 12.5e^{k(1)} \Rightarrow e^k = \frac{10.3}{12.5} \Rightarrow k = \ln\frac{10.3}{12.5}$. The murder occurred when $y(t) = 37 - 20 \Rightarrow 12.5e^{kt} = 17 \Rightarrow e^{kt} = \frac{17}{12.5} \Rightarrow kt = \ln\frac{17}{12.5} \Rightarrow t = \left(\ln\frac{17}{12.5}\right)/\ln\frac{10.3}{12.5} \approx -1.588$ h ≈ -95 minutes. Thus, the murder took place about 95 minutes before 1:30 PM, or 11:55 AM.

17. $\dfrac{dT}{dt} = k(T - 20)$. Letting $y = T - 20$, we get $\dfrac{dy}{dt} = ky$, so $y(t) = y(0)e^{kt}$. $y(0) = T(0) - 20 = 5 - 20 = -15$, so $y(25) = y(0)e^{25k} = -15e^{25k}$, and $y(25) = T(25) - 20 = 10 - 20 = -10$, so $-15e^{25k} = -10 \Rightarrow e^{25k} = \frac{2}{3}$. Thus, $25k = \ln\left(\frac{2}{3}\right)$ and $k = \frac{1}{25}\ln\left(\frac{2}{3}\right)$, so $y(t) = y(0)e^{kt} = -15e^{(1/25)\ln(2/3)t}$. More simply, $e^{25k} = \frac{2}{3} \Rightarrow e^k = \left(\frac{2}{3}\right)^{1/25} \Rightarrow e^{kt} = \left(\frac{2}{3}\right)^{t/25} \Rightarrow y(t) = -15 \cdot \left(\frac{2}{3}\right)^{t/25}$.

(a) $T(50) = 20 + y(50) = 20 - 15 \cdot \left(\frac{2}{3}\right)^{50/25} = 20 - 15 \cdot \left(\frac{2}{3}\right)^2 = 20 - \frac{20}{3} = 13.\bar{3}\,^\circ\mathrm{C}$

(b) $15 = T(t) = 20 + y(t) = 20 - 15 \cdot \left(\frac{2}{3}\right)^{t/25} \Rightarrow 15 \cdot \left(\frac{2}{3}\right)^{t/25} = 5 \Rightarrow \left(\frac{2}{3}\right)^{t/25} = \frac{1}{3} \Rightarrow$

$(t/25)\ln\left(\frac{2}{3}\right) = \ln\left(\frac{1}{3}\right) \Rightarrow t = 25\ln\left(\frac{1}{3}\right)/\ln\left(\frac{2}{3}\right) \approx 67.74$ min.

18. $\dfrac{dT}{dt} = k(T - 20)$. Let $y = T - 20$. Then $\dfrac{dy}{dt} = ky$, so $y(t) = y(0)e^{kt}$. $y(0) = T(0) - 20 = 95 - 20 = 75$, so $y(t) = 75e^{kt}$. When $T(t) = 70$, $\dfrac{dT}{dt} = -1^\circ\mathrm{C/min}$. Equivalently, $\dfrac{dy}{dt} = -1$ when $y(t) = 50$. Thus, $-1 = \dfrac{dy}{dt} = ky(t) = 50k$ and $50 = y(t) = 75e^{kt}$. The first relation implies $k = -1/50$, so the second relation says $50 = 75e^{-t/50}$. Thus, $e^{-t/50} = \frac{2}{3} \Rightarrow -t/50 = \ln\left(\frac{2}{3}\right) \Rightarrow t = -50\ln\left(\frac{2}{3}\right) \approx 20.27$ min.

19. (a) Let $P(h)$ be the pressure at altitude h. Then $dP/dh = kP \Rightarrow P(h) = P(0)e^{kh} = 101.3e^{kh}$.

$P(1000) = 101.3e^{1000k} = 87.14 \Rightarrow 1000k = \ln\left(\frac{87.14}{101.3}\right) \Rightarrow k = \frac{1}{1000}\ln\left(\frac{87.14}{101.3}\right) \Rightarrow$

$P(h) = 101.3\,e^{\frac{1}{1000}h\ln\left(\frac{87.14}{101.3}\right)}$, so $P(3000) = 101.3e^{3\ln\left(\frac{87.14}{101.3}\right)} \approx 64.5$ kPa.

(b) $P(6187) = 101.3\,e^{\frac{6187}{1000}\ln\left(\frac{87.14}{101.3}\right)} \approx 39.9$ kPa

20. (a) Using $A = A_0\left(1+\dfrac{r}{n}\right)^{nt}$ with $A_0 = 2500$, $r = 0.045$, and $t = 3$, we have:

(i) Annually: $n = 1$ $\quad A = 2500\left(1+\dfrac{0.045}{1}\right)^{1\cdot 3} = \$2852.92$

(ii) Quarterly: $n = 4$ $\quad A = 2500\left(1+\dfrac{0.045}{4}\right)^{4\cdot 3} = \$2859.19$

(iii) Monthly: $n = 12$ $\quad A = 2500\left(1+\dfrac{0.045}{12}\right)^{12\cdot 3} = \$2860.62$

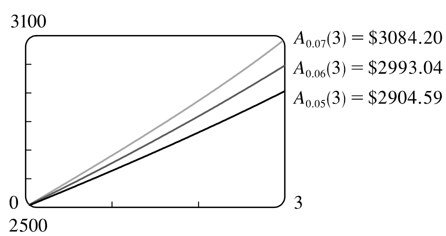
(iv) Weekly: $n = 52$ $\quad A = 2500\left(1+\dfrac{0.045}{52}\right)^{52\cdot 3} = \$2861.17$

(v) Daily: $n = 365$ $\quad A = 2500\left(1+\dfrac{0.045}{365}\right)^{365\cdot 3} = \$2861.32$

(vi) Hourly: $n = 365\cdot 24$ $\quad A = 2500\left(1+\dfrac{0.045}{365\cdot 24}\right)^{365\cdot 24\cdot 3} = \$2861.34$

(vii) Continuously: $\quad A = 2500e^{(0.045)3} = \$2861.34$

(b)



21. (a) Using $A = A_0\left(1+\dfrac{r}{n}\right)^{nt}$ with $A_0 = 4000$, $r = 0.0175$, and $t = 5$, we have:

(i) Annually: $n = 1$ $\quad A = 4000\left(1+\dfrac{0.0175}{1}\right)^{1\cdot 5} = \$4362.47$

(ii) Semiannually: $n = 2$ $\quad A = 4000\left(1+\dfrac{0.0175}{2}\right)^{2\cdot 5} = \$4364.11$

(iii) Monthly: $n = 12$ $\quad A = 4000\left(1+\dfrac{0.0175}{12}\right)^{12\cdot 5} = \$4365.49$

(iv) Weekly: $n = 52$ $\quad A = 4000\left(1+\dfrac{0.0175}{52}\right)^{52\cdot 5} = \$4365.70$

(v) Daily: $n = 365$ $\quad A = 4000\left(1+\dfrac{0.0175}{365}\right)^{365\cdot 5} = \$4365.76$

(vi) Continuously: $\quad A = 4000e^{(0.0175)5} = \$4365.77$

(b) $dA/dt = 0.0175A$ and $A(0) = 4000$.

22. (a) $A_0e^{0.03t} = 2A_0 \quad\Leftrightarrow\quad e^{0.03t} = 2 \quad\Leftrightarrow\quad 0.03t = \ln 2 \quad\Leftrightarrow\quad t = \frac{100}{3}\ln 2 \approx 23.10$, so the investment will double in about 23.10 years.

(b) The annual interest rate in $A = A_0(1+r)^t$ is r. From part (a), we have $A = A_0 e^{0.03t}$. These amounts must be equal, so $(1+r)^t = e^{0.03t} \quad \Rightarrow \quad 1 + r = e^{0.03} \quad \Rightarrow \quad r = e^{0.03} - 1 \approx 0.0305 = 3.05\%$, which is the equivalent annual interest rate.

APPLIED PROJECT Controlling Red Blood Cell Loss During Surgery

1. Let $R(t)$ be the volume of RBCs (in liters) at time t (in hours). Since the total volume of blood is 5 L, the concentration of RBCs is $R/5$. The patient bleeds 2 L of blood in 4 hours, so

$$\frac{dR}{dt} = -\frac{2L}{4h} \cdot \frac{R}{5} = -\frac{1}{10}R$$

From Section 6.5, we know that $dR/dt = kR$ has solution $R(t) = R(0)e^{kt}$. In this case, $R(0) = 45\%$ of $5 = \frac{9}{4}$ and $k = -\frac{1}{10}$, so $R(t) = \frac{9}{4}e^{-t/10}$. At the end of the operation, the volume of RBCs is $R(4) = \frac{9}{4}e^{-0.4} \approx 1.51$ L.

2. Let V be the volume of blood that is extracted and replaced with saline solution. Let $R_A(t)$ be the volume of RBCs with the ANH procedure. Then $R_A(0)$ is 45% of $(5 - V)$, or $\frac{9}{20}(5 - V)$, and hence $R_A(t) = \frac{9}{20}(5 - V)e^{-t/10}$. We want $R_A(4) \geq 25\%$ of $5 \quad \Leftrightarrow \quad \frac{9}{20}(5 - V)e^{-0.4} \geq \frac{5}{4} \quad \Leftrightarrow \quad 5 - V \geq \frac{25}{9}e^{0.4} \quad \Leftrightarrow \quad V \leq 5 - \frac{25}{9}e^{0.4} \approx 0.86$ L. To maximize the effect of the ANH procedure, the surgeon should remove 0.86 L of blood and replace it with saline solution.

3. The RBC loss *without* the ANH procedure is $R(0) - R(4) = \frac{9}{4} - \frac{9}{4}e^{-0.4} \approx 0.74$ L. The RBC loss *with* the ANH procedure is $R_A(0) - R_A(4) = \frac{9}{20}(5 - V) - \frac{9}{20}(5 - V)e^{-0.4} = \frac{9}{20}(5 - V)(1 - e^{-0.4})$. Now let $V = 5 - \frac{25}{9}e^{0.4}$ [from Problem 2] to get $R_A(0) - R_A(4) = \frac{9}{20}\left[5 - \left(5 - \frac{25}{9}e^{0.4}\right)\right](1 - e^{0.4}) = \frac{9}{20} \cdot \frac{25}{9}e^{0.4}(1 - e^{0.4}) = \frac{5}{4}(e^{0.4} - 1) \approx 0.61$ L. Thus, the ANH procedure reduces the RBC loss by about $0.74 - 0.61 = 0.13$ L (about 4.4 fluid ounces).

6.6 Inverse Trigonometric Functions

1. (a) $\sin^{-1}(0.5) = \frac{\pi}{6}$ because $\sin\frac{\pi}{6} = 0.5$ and $\frac{\pi}{6}$ is in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (the range of $\sin^{-1}$).

(b) $\cos^{-1}(-1) = \pi$ because $\cos\pi = -1$ and π is in the interval $[0, \pi]$ (the range of $\cos^{-1}$).

2. (a) $\tan^{-1}\sqrt{3} = \frac{\pi}{3}$ because $\tan\frac{\pi}{3} = \sqrt{3}$ and $\frac{\pi}{3}$ is in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (the range of $\tan^{-1}$).

(b) $\sec^{-1} 2 = \frac{\pi}{3}$ because $\sec\frac{\pi}{3} = 2$ and $\frac{\pi}{3}$ is in $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$ (the range of $\sec^{-1}$).

3. (a) $\csc^{-1}\sqrt{2} = \frac{\pi}{4}$ because $\csc\frac{\pi}{4} = \sqrt{2}$ and $\frac{\pi}{4}$ is in $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$ (the range of $\csc^{-1}$).

(b) $\cos^{-1}\left(\sqrt{3}/2\right) = \frac{\pi}{6}$ because $\cos\frac{\pi}{6} = \sqrt{3}/2$ and $\frac{\pi}{6}$ is in $[0, \pi]$.

4. (a) $\cot^{-1}\left(-\sqrt{3}\right) = \frac{5\pi}{6}$ because $\cot\frac{5\pi}{6} = -\sqrt{3}$ and $\frac{5\pi}{6}$ is in $(0, \pi)$ (the range of $\cot^{-1}$).

(b) $\arcsin 1 = \frac{\pi}{2}$ because $\sin\frac{\pi}{2} = 1$ and $\frac{\pi}{2}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (the range of arcsin).

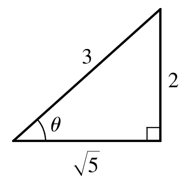
5. (a) In general, $\tan(\arctan x) = x$ for any real number x. Thus, $\tan(\arctan 10) = 10$.

(b) $\arcsin(\sin(5\pi/4)) = \arcsin\left(-1/\sqrt{2}\right) = -\frac{\pi}{4}$ because $\sin\left(-\frac{\pi}{4}\right) = -1/\sqrt{2}$ and $-\frac{\pi}{4}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

6. (a) $\tan^{-1}\left(\tan\dfrac{3\pi}{4}\right) = \tan^{-1}(-1) = -\dfrac{\pi}{4}$ (b) $\cos\left(\arcsin\dfrac{1}{2}\right) = \cos\left(\dfrac{\pi}{6}\right) = \dfrac{\sqrt{3}}{2}$

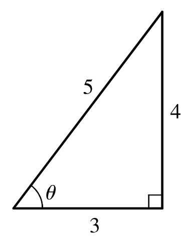
7. Let $\theta = \sin^{-1}\left(\frac{2}{3}\right)$ [see the figure].

Then $\tan\left(\sin^{-1}\left(\frac{2}{3}\right)\right) = \tan\theta = \dfrac{2}{\sqrt{5}}$.



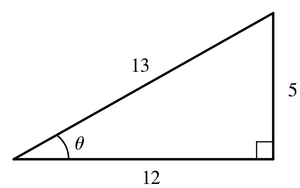
8. Let $\theta = \arccos\frac{3}{5}$ [see the figure].

Then $\csc\left(\arccos\frac{3}{5}\right) = \csc\theta = \frac{5}{4}$.



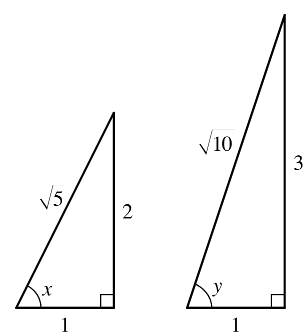
9. Let $\theta = \sin^{-1}\left(\frac{5}{13}\right)$ [see the figure].

$$\cos\left(2\sin^{-1}\left(\tfrac{5}{13}\right)\right) = \cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$= \left(\tfrac{12}{13}\right)^2 - \left(\tfrac{5}{13}\right)^2 = \tfrac{144}{169} - \tfrac{25}{169} = \tfrac{119}{169}$$



10. Let $x = \tan^{-1} 2$ and $y = \tan^{-1} 3$. Then

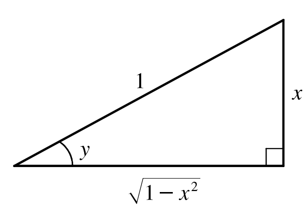
$$\cos\left(\tan^{-1} 2 + \tan^{-1} 3\right) = \cos(x + y) = \cos x\, \cos y - \sin x\, \sin y$$
$$= \frac{1}{\sqrt{5}}\frac{1}{\sqrt{10}} - \frac{2}{\sqrt{5}}\frac{3}{\sqrt{10}}$$
$$= \frac{-5}{\sqrt{50}} = \frac{-5}{5\sqrt{2}} = \frac{-1}{\sqrt{2}}$$



11. Let $y = \sin^{-1} x$. Then $-\frac{\pi}{2} \le y \le \frac{\pi}{2} \Rightarrow \cos y \ge 0$, so $\cos(\sin^{-1} x) = \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$.

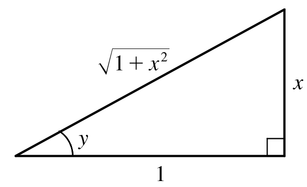
12. Let $y = \sin^{-1} x$. Then $\sin y = x$, so from the triangle (which illustrates the case $y > 0$), we see that

$\tan(\sin^{-1} x) = \tan y = \dfrac{x}{\sqrt{1 - x^2}}$.



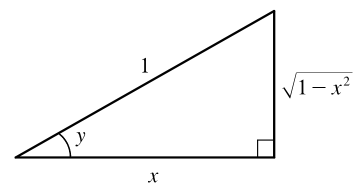
13. Let $y = \tan^{-1} x$. Then $\tan y = x$, so from the triangle (which illustrates the case $y > 0$), we see that

$$\sin(\tan^{-1} x) = \sin y = \frac{x}{\sqrt{1+x^2}}.$$

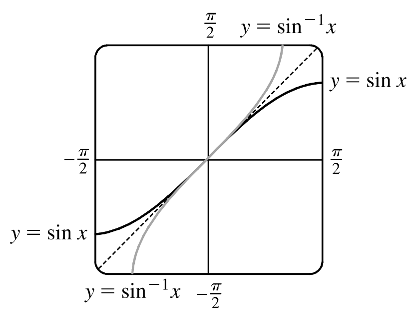


14. Let $y = \arccos x$. Then $\cos y = x$, so from the triangle (which illustrates the case $y > 0$), we see that

$$\begin{aligned}\sin(2\arccos x) &= \sin 2y = 2\sin y\, \cos y \\ &= 2(\sqrt{1-x^2}\,)(x) = 2x\,\sqrt{1-x^2}\end{aligned}$$

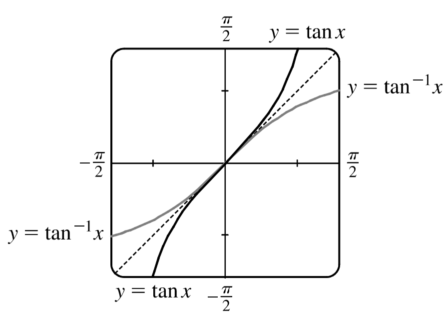


15.



The graph of $\sin^{-1} x$ is the reflection of the graph of $\sin x$ about the line $y = x$.

16.



The graph of $\tan^{-1} x$ is the reflection of the graph of $\tan x$ about the line $y = x$.

17. Let $y = \cos^{-1} x$. Then $\cos y = x$ and $0 \le y \le \pi \quad \Rightarrow \quad -\sin y\,\dfrac{dy}{dx} = 1 \quad \Rightarrow$

$$\frac{dy}{dx} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1-\cos^2 y}} = -\frac{1}{\sqrt{1-x^2}}.$$ [Note that $\sin y \ge 0$ for $0 \le y \le \pi$.]

18. (a) Let $a = \sin^{-1} x$ and $b = \cos^{-1} x$. Then $\cos a = \sqrt{1-\sin^2 a} = \sqrt{1-x^2}$ since $\cos a \ge 0$ for $-\frac{\pi}{2} \le a \le \frac{\pi}{2}$. Similarly, $\sin b = \sqrt{1-x^2}$. So

$$\begin{aligned}\sin(\sin^{-1} x + \cos^{-1} x) &= \sin(a+b) = \sin a\cos b + \cos a \sin b = x \cdot x + \sqrt{1-x^2}\,\sqrt{1-x^2} \\ &= x^2 + (1-x^2) = 1\end{aligned}$$

But $-\frac{\pi}{2} \le \sin^{-1} x + \cos^{-1} x \le \frac{3\pi}{2}$, and so $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$.

(b) We differentiate $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ with respect to x, and get

$$\frac{1}{\sqrt{1-x^2}} + \frac{d}{dx}\left(\cos^{-1}x\right) = 0 \quad\Rightarrow\quad \frac{d}{dx}\left(\cos^{-1}x\right) = -\frac{1}{\sqrt{1-x^2}}.$$

19. Let $y = \cot^{-1}x$. Then $\cot y = x \quad\Rightarrow\quad -\csc^2 y\,\dfrac{dy}{dx} = 1 \quad\Rightarrow\quad \dfrac{dy}{dx} = -\dfrac{1}{\csc^2 y} = -\dfrac{1}{1+\cot^2 y} = -\dfrac{1}{1+x^2}$.

20. Let $y = \sec^{-1}x$. Then $\sec y = x$ and $y \in \left(0, \frac{\pi}{2}\right] \cup \left[\pi, \frac{3\pi}{2}\right)$. Differentiate with respect to x:

$\sec y\,\tan y\left(\dfrac{dy}{dx}\right) = 1 \quad\Rightarrow\quad \dfrac{dy}{dx} = \dfrac{1}{\sec y\,\tan y} = \dfrac{1}{\sec y\,\sqrt{\sec^2 y - 1}} = \dfrac{1}{x\,\sqrt{x^2-1}}$. Note that $\tan^2 y = \sec^2 y - 1 \quad\Rightarrow$

$\tan y = \sqrt{\sec^2 y - 1}$ since $\tan y > 0$ when $0 < y < \frac{\pi}{2}$ or $\pi < y < \frac{3\pi}{2}$.

21. Let $y = \csc^{-1}x$. Then $\csc y = x \quad\Rightarrow\quad -\csc y\,\cot y\,\dfrac{dy}{dx} = 1 \quad\Rightarrow$

$\dfrac{dy}{dx} = -\dfrac{1}{\csc y\,\cot y} = -\dfrac{1}{\csc y\,\sqrt{\csc^2 y - 1}} = -\dfrac{1}{x\,\sqrt{x^2-1}}$. Note that $\cot y \geq 0$ on the domain of $\csc^{-1}x$.

22. $y = \tan^{-1}(x^2) \quad\Rightarrow\quad y' = \dfrac{1}{1+(x^2)^2}\cdot\dfrac{d}{dx}(x^2) = \dfrac{1}{1+x^4}\cdot 2x = \dfrac{2x}{1+x^4}$

23. $f(x) = \sin^{-1}(5x) \Rightarrow f'(x) = \dfrac{1}{\sqrt{1-(5x)^2}}\cdot\dfrac{d}{dx}(5x) = \dfrac{5}{\sqrt{1-25x^2}}$

24. $g(x) = \arccos\sqrt{x} \quad\Rightarrow\quad g'(x) = -\dfrac{1}{\sqrt{1-\left(\sqrt{x}\right)^2}}\,\dfrac{d}{dx}\sqrt{x} = -\dfrac{1}{\sqrt{1-x}}\left(\dfrac{1}{2}x^{-1/2}\right) = -\dfrac{1}{2\sqrt{x}\,\sqrt{1-x}}$

25. $y = (\tan^{-1}x)^2 \quad\Rightarrow\quad y' = 2(\tan^{-1}x)^1\cdot\dfrac{d}{dx}(\tan^{-1}x) = 2\tan^{-1}x\cdot\dfrac{1}{1+x^2} = \dfrac{2\tan^{-1}x}{1+x^2}$

26. $g(x) = \sec^{-1}(e^x) \Rightarrow g'(x) = \dfrac{1}{e^x\sqrt{(e^x)^2-1}}\cdot\dfrac{d}{dx}(e^x) = \dfrac{1}{e^x\sqrt{e^{2x}-1}}\cdot e^x = \dfrac{1}{\sqrt{e^{2x}-1}}$

27. $y = \tan^{-1}\sqrt{x-1} \quad\Rightarrow$

$$y' = \frac{1}{1+\left(\sqrt{x-1}\right)^2}\cdot\frac{d}{dt}\left(\sqrt{x-1}\right) = \frac{1}{1+(x-1)}\cdot\frac{1}{2\sqrt{x-1}} = \frac{1}{x}\cdot\frac{1}{2\sqrt{x-1}} = \frac{1}{2x\sqrt{x-1}}$$

28. $g(t) = \ln(\arctan(t^4)) \quad\Rightarrow$

$$\begin{aligned} g'(t) &= \frac{1}{\arctan(t^4)}\cdot\frac{d}{dt}(\arctan(t^4)) = \frac{1}{\arctan(t^4)}\cdot\frac{1}{1+(t^4)^2}\cdot\frac{d}{dt}\left(t^4\right) \\ &= \frac{1}{\arctan(t^4)}\cdot\frac{1}{1+t^8}\cdot 4t^3 = \frac{4t^3}{(1+t^8)\arctan(t^4)} \end{aligned}$$

29. $y = \arctan(\cos\theta) \quad\Rightarrow\quad y' = \dfrac{1}{1+(\cos\theta)^2}\,(-\sin\theta) = -\dfrac{\sin\theta}{1+\cos^2\theta}$

30. $y = \tan^{-1}\bigl(x - \sqrt{1+x^2}\,\bigr) \quad\Rightarrow$

$$\begin{aligned} y' &= \frac{1}{1+\bigl(x-\sqrt{x^2+1}\,\bigr)^2}\left(1-\frac{x}{\sqrt{x^2+1}}\right) = \frac{1}{1+x^2-2x\sqrt{x^2+1}+x^2+1}\left(\frac{\sqrt{x^2+1}-x}{\sqrt{x^2+1}}\right) \\ &= \frac{\sqrt{x^2+1}-x}{2\bigl(1+x^2-x\sqrt{x^2+1}\,\bigr)\sqrt{x^2+1}} = \frac{\sqrt{x^2+1}-x}{2\bigl[\sqrt{x^2+1}\,(1+x^2)-x(x^2+1)\bigr]} = \frac{\sqrt{x^2+1}-x}{2\bigl[(1+x^2)\bigl(\sqrt{x^2+1}-x\bigr)\bigr]} \\ &= \frac{1}{2(1+x^2)} \end{aligned}$$

31. $f(z) = e^{\arcsin(z^2)} \quad\Rightarrow\quad f'(z) = e^{\arcsin(z^2)}\cdot\dfrac{d}{dz}\,[\arcsin(z^2)] = e^{\arcsin(z^2)}\cdot\dfrac{1}{\sqrt{1-(z^2)^2}}\cdot 2z = \dfrac{2ze^{\arcsin(z^2)}}{\sqrt{1-z^4}}$

32. $R(t) = \arcsin(1/t) \quad\Rightarrow$

$$\begin{aligned} R'(t) &= \frac{1}{\sqrt{1-(1/t)^2}}\,\frac{d}{dt}\,\frac{1}{t} = \frac{1}{\sqrt{1-1/t^2}}\left(-\frac{1}{t^2}\right) = -\frac{1}{\sqrt{1-1/t^2}}\,\frac{1}{\sqrt{t^4}} = -\frac{1}{\sqrt{t^4-t^2}} \\ &= -\frac{1}{\sqrt{t^2(t^2-1)}} = -\frac{1}{|\,t\,|\sqrt{t^2-1}} \end{aligned}$$

33. $h(t) = \cot^{-1}(t) + \cot^{-1}(1/t) \quad\Rightarrow$

$$h'(t) = -\frac{1}{1+t^2} - \frac{1}{1+(1/t)^2}\cdot\frac{d}{dt}\,\frac{1}{t} = -\frac{1}{1+t^2} - \frac{t^2}{t^2+1}\cdot\left(-\frac{1}{t^2}\right) = -\frac{1}{1+t^2} + \frac{1}{t^2+1} = 0.$$

Note that this makes sense because $h(t) = \dfrac{\pi}{2}$ for $t > 0$ and $h(t) = \dfrac{3\pi}{2}$ for $t < 0$.

34. $y = \cos^{-1}(\sin^{-1}t) \quad\Rightarrow\quad y' = -\dfrac{1}{\sqrt{1-(\sin^{-1}t)^2}}\cdot\dfrac{d}{dt}\,\sin^{-1}t = -\dfrac{1}{\sqrt{1-(\sin^{-1}t)^2}}\cdot\dfrac{1}{\sqrt{1-t^2}}$

35. $y = x\sin^{-1}x + \sqrt{1-x^2} \quad\Rightarrow$

$$y' = x\cdot\frac{1}{\sqrt{1-x^2}} + (\sin^{-1}x)(1) + \frac{1}{2}(1-x^2)^{-1/2}(-2x) = \frac{x}{\sqrt{1-x^2}} + \sin^{-1}x - \frac{x}{\sqrt{1-x^2}} = \sin^{-1}x$$

36. $y = \arctan\sqrt{\dfrac{1-x}{1+x}} = \arctan\left(\dfrac{1-x}{1+x}\right)^{1/2} \quad\Rightarrow$

$$\begin{aligned} y' &= \frac{1}{1+\left(\sqrt{\dfrac{1-x}{1+x}}\right)^2}\cdot\frac{d}{dx}\left(\frac{1-x}{1+x}\right)^{1/2} = \frac{1}{1+\dfrac{1-x}{1+x}}\cdot\frac{1}{2}\left(\frac{1-x}{1+x}\right)^{-1/2}\cdot\frac{(1+x)(-1)-(1-x)(1)}{(1+x)^2} \\ &= \frac{1}{\dfrac{1+x}{1+x}+\dfrac{1-x}{1+x}}\cdot\frac{1}{2}\left(\frac{1+x}{1-x}\right)^{1/2}\cdot\frac{-2}{(1+x)^2} = \frac{1+x}{2}\cdot\frac{1}{2}\cdot\frac{(1+x)^{1/2}}{(1-x)^{1/2}}\cdot\frac{-2}{(1+x)^2} \\ &= \frac{-1}{2(1-x)^{1/2}(1+x)^{1/2}} = \frac{-1}{2\sqrt{1-x^2}} \end{aligned}$$

37. $y = \tan^{-1}\left(\dfrac{x}{a}\right) + \ln\sqrt{\dfrac{x-a}{x+a}} = \tan^{-1}\left(\dfrac{x}{a}\right) + \dfrac{1}{2}\ln\left(\dfrac{x-a}{x+a}\right) \quad\Rightarrow$

$$y' = \frac{1}{1+\left(\dfrac{x}{a}\right)^2}\cdot\frac{1}{a} + \frac{1}{2}\cdot\frac{1}{\dfrac{x-a}{x+a}}\cdot\frac{(x+a)\cdot 1-(x-a)\cdot 1}{(x+a)^2} = \frac{1}{a+\dfrac{x^2}{a}} + \frac{1}{2}\cdot\frac{x+a}{x-a}\cdot\frac{2a}{(x+a)^2}$$

$$= \frac{1}{a+\dfrac{x^2}{a}}\cdot\frac{a}{a} + \frac{a}{(x-a)(x+a)} = \frac{a}{x^2+a^2} + \frac{a}{x^2-a^2}$$

38. $f(x) = \arcsin(e^x) \quad\Rightarrow\quad f'(x) = \dfrac{1}{\sqrt{1-(e^x)^2}}\cdot e^x = \dfrac{e^x}{\sqrt{1-e^{2x}}}$.

$\text{Domain}(f) = \{x \mid -1 \le e^x \le 1\} = \{x \mid 0 < e^x \le 1\} = (-\infty, 0]$.

$\text{Domain}(f') = \left\{x \mid 1-e^{2x} > 0\right\} = \left\{x \mid e^{2x} < 1\right\} = \{x \mid 2x < 0\} = (-\infty, 0)$.

39. $g(x) = \cos^{-1}(3-2x) \quad\Rightarrow\quad g'(x) = -\dfrac{1}{\sqrt{1-(3-2x)^2}}(-2) = \dfrac{2}{\sqrt{1-(3-2x)^2}}$.

$\text{Domain}(g) = \{x \mid -1 \le 3-2x \le 1\} = \{x \mid -4 \le -2x \le -2\} = \{x \mid 2 \ge x \ge 1\} = [1, 2]$.

$$\begin{aligned}\text{Domain}(g') &= \left\{x \mid 1-(3-2x)^2 > 0\right\} = \left\{x \mid (3-2x)^2 < 1\right\} = \{x \mid |3-2x| < 1\} \\ &= \{x \mid -1 < 3-2x < 1\} = \{x \mid -4 < -2x < -2\} = \{x \mid 2 > x > 1\} = (1, 2)\end{aligned}$$

40. $\dfrac{d}{dx}\tan^{-1}(x^2y) = \dfrac{d}{dx}(x+xy^2) \quad\Rightarrow\quad \dfrac{1}{1+(x^2y)^2}(x^2y' + y\cdot 2x) = 1 + x\cdot 2y\,y' + y^2\cdot 1 \quad\Rightarrow$

$$\frac{x^2}{1+x^4y^2}\,y' - 2xy\,y' = 1+y^2-\frac{2xy}{1+x^4y^2} \quad\Rightarrow\quad y'\left(\frac{x^2}{1+x^4y^2}-2xy\right) = 1+y^2-\frac{2xy}{1+x^4y^2} \quad\Rightarrow$$

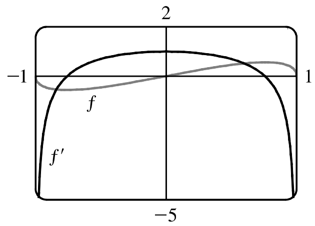
$$y' = \frac{1+y^2-\dfrac{2xy}{1+x^4y^2}}{\dfrac{x^2}{1+x^4y^2}-2xy} \quad\text{or}\quad y' = \frac{1+x^4y^2+y^2+x^4y^4-2xy}{x^2-2xy-2x^5y^3}$$

41. $g(x) = x\sin^{-1}\left(\dfrac{x}{4}\right) + \sqrt{16-x^2} \quad\Rightarrow\quad g'(x) = \sin^{-1}\left(\dfrac{x}{4}\right) + \dfrac{x}{4\sqrt{1-(x/4)^2}} - \dfrac{x}{\sqrt{16-x^2}} = \sin^{-1}\left(\dfrac{x}{4}\right) \quad\Rightarrow$

$g'(2) = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

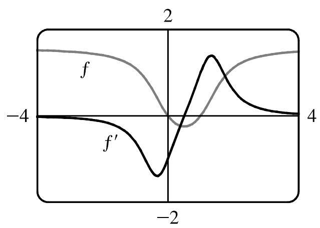
42. $y = 3\arccos\dfrac{x}{2} \quad\Rightarrow\quad y' = 3\left[-\dfrac{1}{\sqrt{1-(x/2)^2}}\right]\left(\dfrac{1}{2}\right)$, so at $(1, \pi)$, $y' = -\dfrac{3}{2\sqrt{1-\frac{1}{4}}} = -\sqrt{3}$. An equation of the tangent line is $y - \pi = -\sqrt{3}\,(x-1)$, or $y = -\sqrt{3}\,x + \pi + \sqrt{3}$.

43. $f(x) = \sqrt{1-x^2}\arcsin x \quad\Rightarrow\quad f'(x) = \sqrt{1-x^2}\cdot\dfrac{1}{\sqrt{1-x^2}} + \arcsin x\cdot\dfrac{1}{2}\left(1-x^2\right)^{-1/2}(-2x) = 1 - \dfrac{x\arcsin x}{\sqrt{1-x^2}}$



Note that $f' = 0$ where the graph of f has a horizontal tangent. Also note that f' is negative when f is decreasing and f' is positive when f is increasing.

44. $f(x) = \arctan(x^2 - x) \quad\Rightarrow\quad f'(x) = \dfrac{1}{1+(x^2-x)^2}\cdot\dfrac{d}{dx}(x^2-x) = \dfrac{2x-1}{1+(x^2-x)^2}$



Note that $f' = 0$ where the graph of f has a horizontal tangent. Also note that f' is negative when f is decreasing and f' is positive when f is increasing.

45. $\lim\limits_{x\to -1^+}\sin^{-1}x = \sin^{-1}(-1) = -\frac{\pi}{2}$

46. Let $t = \dfrac{1+x^2}{1+2x^2}$. As $x\to\infty$, $t = \dfrac{1+x^2}{1+2x^2} = \dfrac{1/x^2+1}{1/x^2+2} \to \dfrac{1}{2}$.

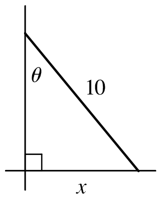
$\lim\limits_{x\to\infty}\arccos\left(\dfrac{1+x^2}{1+2x^2}\right) = \lim\limits_{t\to 1/2}\arccos t = \arccos\frac{1}{2} = \frac{\pi}{3}$.

47. Let $t = e^x$. As $x\to\infty$, $t\to\infty$. $\lim\limits_{x\to\infty}\arctan(e^x) = \lim\limits_{t\to\infty}\arctan t = \frac{\pi}{2}$ by (8).

48. Let $t = \ln x$. As $x\to 0^+$, $t\to -\infty$. $\lim\limits_{x\to 0^+}\tan^{-1}(\ln x) = \lim\limits_{t\to -\infty}\tan^{-1}t = -\frac{\pi}{2}$ by (8).

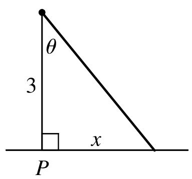
49. If $y = f^{-1}(x)$, then $f(y) = x$. Differentiating implicitly with respect to x and remembering that y is a function of x, we get $f'(y)\dfrac{dy}{dx} = 1$, so $\dfrac{dy}{dx} = \dfrac{1}{f'(y)} \quad\Rightarrow\quad \left(f^{-1}\right)'(x) = \dfrac{1}{f'(f^{-1}(x))}$.

50. $f(4) = 5 \quad\Rightarrow\quad f^{-1}(5) = 4$. By Exercise 83, $\left(f^{-1}\right)'(5) = \dfrac{1}{f'(f^{-1}(5))} = \dfrac{1}{f'(4)} = \dfrac{1}{2/3} = \dfrac{3}{2}$.

51. 

$\dfrac{dx}{dt} = 2$ ft/s, $\sin\theta = \dfrac{x}{10} \quad\Rightarrow\quad \theta = \sin^{-1}\!\left(\dfrac{x}{10}\right)$, $\dfrac{d\theta}{dx} = \dfrac{1/10}{\sqrt{1-(x/10)^2}}$,

$\dfrac{d\theta}{dt} = \dfrac{d\theta}{dx}\dfrac{dx}{dt} = \dfrac{1/10}{\sqrt{1-(x/10)^2}}\,(2)$ rad/s, $\left.\dfrac{d\theta}{dt}\right]_{x=6} = \dfrac{2/10}{\sqrt{1-(6/10)^2}}$ rad/s $= \frac{1}{4}$ rad/s

52. 

$\dfrac{d\theta}{dt} = 4$ rev/min $= 8\pi\cdot 60$ rad/h. From the diagram, we see that $\tan\theta = \dfrac{x}{3} \quad\Rightarrow\quad \theta = \tan^{-1}\!\left(\dfrac{x}{3}\right)$.

Thus, $8\pi\cdot 60 = \dfrac{d\theta}{dt} = \dfrac{d\theta}{dx}\dfrac{dx}{dt} = \dfrac{1/3}{1+(x/3)^2}\dfrac{dx}{dt}$. So $\dfrac{dx}{dt} = 8\pi\cdot 60\cdot 3\left[1+\left(\dfrac{x}{3}\right)^2\right]$ km/h, and

at $x=1$, $\dfrac{dx}{dt} = 8\pi\cdot 60\cdot 3\left[1+\frac{1}{9}\right]$ km/h $= 1600\pi$ km/h.

53. $y = f(x) = \sin^{-1}(x/(x+1))$ **A.** $D = \{x \mid -1 \le x/(x+1) \le 1\}$. For $x > -1$ we have $-x-1 \le x \le x+1 \quad\Leftrightarrow$ $2x \ge -1 \quad\Leftrightarrow\quad x \ge -\frac{1}{2}$, so $D = \left[-\frac{1}{2}, \infty\right)$. **B.** Intercepts are 0 **C.** No symmetry

D. $\displaystyle\lim_{x\to\infty} \sin^{-1}\!\left(\frac{x}{x+1}\right) = \lim_{x\to\infty} \sin^{-1}\!\left(\frac{1}{1+1/x}\right) = \sin^{-1} 1 = \tfrac{\pi}{2}$, so $y = \frac{\pi}{2}$ is a HA.

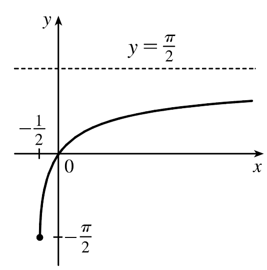
E. $f'(x) = \dfrac{1}{\sqrt{1-[x/(x+1)]^2}}\,\dfrac{(x+1)-x}{(x+1)^2} = \dfrac{1}{(x+1)\sqrt{2x+1}} > 0$,

so f is increasing on $\left(-\frac{1}{2}, \infty\right)$. **F.** No local maximum or minimum,

$f\left(-\frac{1}{2}\right) = \sin^{-1}(-1) = -\frac{\pi}{2}$ is an absolute minimum

G. $f''(x) = -\dfrac{\sqrt{2x+1} + (x+1)/\sqrt{2x+1}}{(x+1)^2(2x+1)}$

$= -\dfrac{3x+2}{(x+1)^2(2x+1)^{3/2}} < 0$ on D, so f is CD on $\left(-\frac{1}{2}, \infty\right)$.

H. 

54. $y = f(x) = \tan^{-1}\!\left(\dfrac{x-1}{x+1}\right)$ **A.** $D = \{x \mid x \ne -1\}$ **B.** x-intercept $= 1$, y-intercept $= f(0) = \tan^{-1}(-1) = -\frac{\pi}{4}$

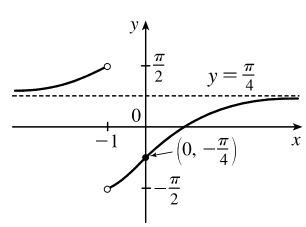
C. No symmetry **D.** $\displaystyle\lim_{x\to\pm\infty} \tan^{-1}\!\left(\frac{x-1}{x+1}\right) = \lim_{x\to\pm\infty} \tan^{-1}\!\left(\frac{1-1/x}{1+1/x}\right) = \tan^{-1} 1 = \tfrac{\pi}{4}$, so $y = \frac{\pi}{4}$ is a HA.

Also $\displaystyle\lim_{x\to -1^+} \tan^{-1}\!\left(\frac{x-1}{x+1}\right) = -\frac{\pi}{2}$ and $\displaystyle\lim_{x\to -1^-} \tan^{-1}\!\left(\frac{x-1}{x+1}\right) = \frac{\pi}{2}$.

E. $f'(x) = \dfrac{1}{1+[(x-1)/(x+1)]^2}\,\dfrac{(x+1)-(x-1)}{(x+1)^2} = \dfrac{2}{(x+1)^2+(x-1)^2} = \dfrac{1}{x^2+1} > 0$,

so f is increasing on $(-\infty, -1)$ and $(-1, \infty)$. **F.** No extreme values

G. $f''(x) = -2x/\left(x^2+1\right)^2 > 0 \quad\Leftrightarrow\quad x < 0$, so f is CU on $(-\infty, -1)$ and $(-1, 0)$, and CD on $(0, \infty)$. IP at $\left(0, -\frac{\pi}{4}\right)$

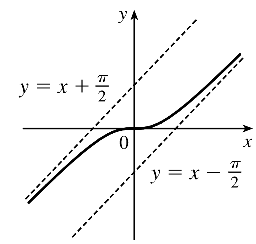
H. 

55. $y = f(x) = x - \tan^{-1}x$ **A.** $D = \mathbb{R}$ **B.** Intercepts are 0 **C.** $f(-x) = -f(x)$, so the curve is symmetric about the origin. **D.** $\lim_{x\to\infty}(x - \tan^{-1}x) = \infty$ and $\lim_{x\to-\infty}(x - \tan^{-1}x) = -\infty$, no HA.

But $f(x) - \left(x - \frac{\pi}{2}\right) = -\tan^{-1}x + \frac{\pi}{2} \to 0$ as $x \to \infty$, and $f(x) - \left(x + \frac{\pi}{2}\right) = -\tan^{-1}x - \frac{\pi}{2} \to 0$ as $x \to -\infty$, so $y = x \pm \frac{\pi}{2}$ are slant asymptotes. **E.** $f'(x) = 1 - \dfrac{1}{x^2+1} = \dfrac{x^2}{x^2+1} > 0$, so f is increasing on $\mathbb{R}$. **F.** No extrema

G. $f''(x) = \dfrac{(1+x^2)(2x) - x^2(2x)}{(1+x^2)^2} = \dfrac{2x}{(1+x^2)^2} > 0 \quad\Leftrightarrow\quad x > 0$, so f is CU on $(0,\infty)$, CD on $(-\infty, 0)$. IP at $(0,0)$.

H.



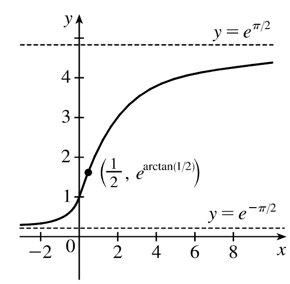
56. $y = f(x) = e^{\arctan x}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = e^0 = 1$; no x-intercept since $e^{\arctan x}$ is positive for all x. **C.** No symmetry **D.** $\lim_{x\to-\infty} f(x) = e^{-\pi/2}$ [≈ 0.21], so $y = e^{-\pi/2}$ is a HA. $\lim_{x\to\infty} f(x) = e^{\pi/2}$ [≈ 4.81], so $y = e^{\pi/2}$ is a HA. **E.** $f'(x) = e^{\arctan x}\left(\dfrac{1}{1+x^2}\right)$. $f'(x) > 0$ for all x, so f is increasing on $\mathbb{R}$. **F.** No extreme values

$$\textbf{G.}\ f''(x) = \frac{(1+x^2)e^{\arctan x}\left(\dfrac{1}{1+x^2}\right) - e^{\arctan x}(2x)}{(1+x^2)^2} = \frac{e^{\arctan x}(1-2x)}{(1+x^2)^2}$$

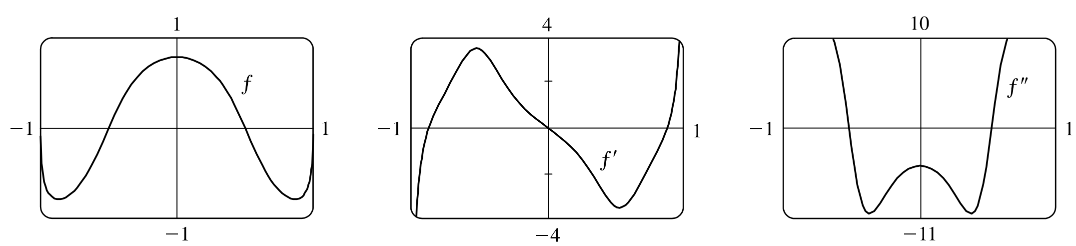
$f''(x) > 0$ for $x < \frac{1}{2}$, so f is CU on $\left(-\infty, \frac{1}{2}\right)$ and f is CD on $\left(\frac{1}{2}, \infty\right)$.

IP at $\left(\frac{1}{2}, e^{\arctan 1/2}\right) \approx (0.5, 1.59)$

H.

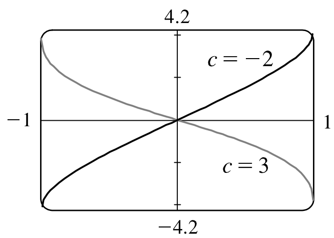


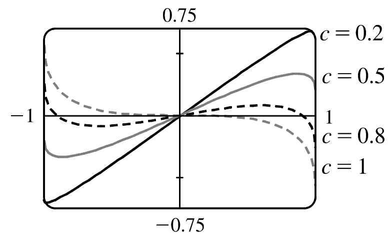
57. $f(x) = \arctan(\cos(3\arcsin x))$. We use a CAS to compute f' and f'', and to graph f, f', and f'':



From the graph of f', it appears that the only maximum occurs at $x = 0$ and there are minima at $x = \pm 0.87$. From the graph of f'', it appears that there are inflection points at $x = \pm 0.52$.

58. First note that the function $f(x) = x - c\sin^{-1}x$ is only defined on the interval $[-1, 1]$, since $\sin^{-1}$ is only defined on that interval. We differentiate to get $f'(x) = 1 - c/\sqrt{1-x^2}$. Now if $c \le 0$, then $f'(x) \ge 1$, so there is no extremum and f is increasing on its domain. If $c > 1$, then $f'(x) < 0$, so there is no local extremum and f is decreasing on its domain, and if $c = 1$, then there is still no extremum, since $f'(x)$ does not change sign at $x = 0$. So we can only have local extrema if $0 < c < 1$. In this case, f is increasing where $f'(x) > 0 \iff \sqrt{1-x^2} > c \iff |x| < \sqrt{1-c^2}$, and decreasing where $\sqrt{1-c^2} < |x| \le 1$. f has a maximum at $x = \sqrt{1-c^2}$ and a minimum at $x = -\sqrt{1-c^2}$.





59. $f(x) = \dfrac{2x^2+5}{x^2+1} = \dfrac{2(x^2+1)+3}{x^2+1} = 2 + \dfrac{3}{x^2+1} \quad \Rightarrow \quad F(x) = 2x + 3\tan^{-1}x + C$

60. $g'(t) = \dfrac{2}{\sqrt{1-t^2}}$ implies that $g'(t)$ is defined for t in $(-1, 1)$ and $g(t) = 2\sin^{-1}t + C$ for $-1 < t < 1$. By continuity, we can extend the domain of g to $[-1, 1]$. Now $g(1) = 5 \quad \Rightarrow \quad 2\sin^{-1}1 + C = 5 \quad \Rightarrow \quad C = 5 - 2\left(\frac{\pi}{2}\right) = 5 - \pi$, so $g(t) = 2\sin^{-1}t + 5 - \pi$.

61. $\displaystyle\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{8}{1+x^2}\,dx = \Big[8\arctan x\Big]_{1/\sqrt{3}}^{\sqrt{3}} = 8\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = 8\left(\frac{\pi}{6}\right) = \frac{4\pi}{3}$

62. $\displaystyle\int_{-1/\sqrt{2}}^{1/\sqrt{2}} \frac{6}{\sqrt{1-p^2}}\,dp = 6\Big[\sin^{-1}p\Big]_{-1/\sqrt{2}}^{1/\sqrt{2}} = 6\left[\sin^{-1}\frac{1}{\sqrt{2}} - \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)\right] = 6\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = 6\left(\frac{\pi}{2}\right) = 3\pi$

63. Let $u = \sin^{-1}x$, so $du = \dfrac{dx}{\sqrt{1-x^2}}$. When $x = 0$, $u = 0$; when $x = \frac{1}{2}$, $u = \frac{\pi}{6}$. Thus,

$$\int_0^{1/2} \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/6} u\,du = \left[\frac{u^2}{2}\right]_0^{\pi/6} = \frac{\pi^2}{72}.$$

64. Let $u = 4x$. Then $du = 4\,dx$, so

$$\int_0^{\sqrt{3}/4} \frac{dx}{1+16x^2} = \frac{1}{4}\int_0^{\sqrt{3}} \frac{1}{1+u^2}\,du = \tfrac{1}{4}\left[\tan^{-1}u\right]_0^{\sqrt{3}} = \tfrac{1}{4}\left(\tan^{-1}\sqrt{3} - \tan^{-1}0\right) = \tfrac{1}{4}\left(\tfrac{\pi}{3} - 0\right) = \tfrac{\pi}{12}.$$

65. Let $u = 1 + x^2$. Then $du = 2x\,dx$, so

$$\begin{aligned}\int \frac{1+x}{1+x^2}\,dx &= \int \frac{1}{1+x^2}\,dx + \int \frac{x}{1+x^2}\,dx = \tan^{-1}x + \int \frac{\frac{1}{2}\,du}{u} = \tan^{-1}x + \tfrac{1}{2}\ln|u| + C \\ &= \tan^{-1}x + \tfrac{1}{2}\ln\left|1+x^2\right| + C = \tan^{-1}x + \tfrac{1}{2}\ln(1+x^2) + C \quad [\text{since } 1+x^2 > 0].\end{aligned}$$

66. Let $u = -\cos x$. Then $du = \sin x\,dx$, so

$$\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}\,dx = \int_{-1}^{0} \frac{1}{1+u^2}\,du = \left[\tan^{-1} u\right]_{-1}^{0} = \tan^{-1} 0 - \tan^{-1}(-1) = 0 - \left(-\tfrac{\pi}{4}\right) = \tfrac{\pi}{4}.$$

67. Let $u = \arctan x$. Then $du = \dfrac{1}{x^2+1}\,dx$, so $\displaystyle\int \frac{(\arctan x)^2}{x^2+1}\,dx = \int u^2\,du = \tfrac{1}{3}u^3 + C = \tfrac{1}{3}(\arctan x)^3 + C.$

68. Let $u = \arctan x$. Then $du = \dfrac{1}{1+x^2}\,dx = \dfrac{1}{x^2+1}\,dx$, so

$$\int \frac{1}{(x^2+1)\arctan x}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln|\arctan x| + C.$$

69. Let $u = \arcsin x$. Then $du = \dfrac{1}{\sqrt{1-x^2}}\,dx$, so $\displaystyle\int \frac{e^{\arcsin x}}{\sqrt{1-x^2}}\,dx = \int e^u\,du = e^u + C = e^{\arcsin x} + C.$

70. Let $u = \frac{1}{2}x$. Then $du = \frac{1}{2}\,dx \quad \Rightarrow$

$$\int \frac{1}{x\sqrt{x^2-4}}\,dx = \int \frac{1}{2x\sqrt{(x/2)^2-1}}\,dx = \int \frac{2\,du}{4u\sqrt{u^2-1}} = \frac{1}{2}\int \frac{du}{u\sqrt{u^2-1}} = \tfrac{1}{2}\sec^{-1} u + C = \tfrac{1}{2}\sec^{-1}\left(\tfrac{1}{2}x\right) + C.$$

71. Let $u = t^3$. Then $du = 3t^2\,dt$ and $\displaystyle\int \frac{t^2}{\sqrt{1-t^6}}\,dt = \int \frac{\frac{1}{3}\,du}{\sqrt{1-u^2}} = \tfrac{1}{3}\sin^{-1} u + C = \tfrac{1}{3}\sin^{-1}(t^3) + C.$

72. Let $u = e^x$, so $du = e^x\,dx$. When $x = 0$, $u = 1$; when $x = 1$, $u = e$. Thus,

$$\int_0^1 \frac{e^x}{1+e^{2x}}\,dx = \int_1^e \frac{1}{1+u^2}\,du = \left[\tan^{-1} u\right]_1^e = \tan^{-1} e - \tan^{-1} 1 = \tan^{-1} e - \tfrac{\pi}{4}.$$

73. Let $u = \sqrt{x}$. Then $du = \dfrac{dx}{2\sqrt{x}}$ and $\displaystyle\int \frac{dx}{\sqrt{x}\,(1+x)} = \int \frac{2\,du}{1+u^2} = 2\tan^{-1} u + C = 2\tan^{-1}\sqrt{x} + C.$

74. Let $u = x^2$. Then $du = 2x\,dx$, so $\displaystyle\int \frac{x}{1+x^4}\,dx = \int \frac{\frac{1}{2}\,du}{1+u^2} = \tfrac{1}{2}\tan^{-1} u + C = \tfrac{1}{2}\tan^{-1}(x^2) + C.$

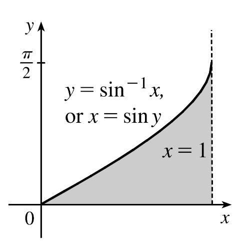
75. Let $u = x/a$. Then $du = dx/a$, so

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \int \frac{1}{a\sqrt{1-(x/a)^2}}\,dx = \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1}\left(\frac{x}{a}\right) + C.$$

76. We use the disk method: $\displaystyle A = \int_0^2 \pi\left[\frac{1}{\sqrt{x^2+4}}\right]^2 dx = \pi\int_0^2 \frac{1}{x^2+4}\,dx$. By Formula 14, this is equal to

$\pi\left[\frac{1}{2}\tan^{-1}(x/2)\right]_0^2 = \frac{\pi}{2}\left(\frac{\pi}{4} - 0\right) = \frac{\pi^2}{8}.$

77.



The integral represents the area below the curve $y = \sin^{-1} x$ on the interval $x \in [0,1]$. The bounding curves are $y = \sin^{-1} x \quad \Leftrightarrow \quad x = \sin y$, $y = 0$ and $x = 1$. We see that y ranges between $\sin^{-1} 0 = 0$ and $\sin^{-1} 1 = \frac{\pi}{2}$. So we have to integrate the function $x = 1 - \sin y$ between $y = 0$ and $y = \frac{\pi}{2}$:

$\int_0^1 \sin^{-1} x\,dx = \int_0^{\pi/2}(1 - \sin y)\,dy = \left(\frac{\pi}{2} + \cos\frac{\pi}{2}\right) - (0 + \cos 0) = \frac{\pi}{2} - 1.$

78. Let $a = \arctan x$ and $b = \arctan y$. Then by the addition formula for the tangent (see Reference Page 2 in the textbook),

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - (\tan a)(\tan b)} = \frac{\tan(\arctan x) + \tan(\arctan y)}{1 - \tan(\arctan x)\,\tan(\arctan y)} \quad\Rightarrow\quad \tan(a+b) = \frac{x+y}{1-xy} \quad\Rightarrow$$

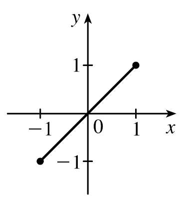
$\arctan x + \arctan y = a + b = \arctan\left(\dfrac{x+y}{1-xy}\right)$, since $-\frac{\pi}{2} < \arctan x + \arctan y < \frac{\pi}{2}$.

79. (a) $\arctan\frac{1}{2} + \arctan\frac{1}{3} = \arctan\left(\dfrac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}\right) = \arctan 1 = \dfrac{\pi}{4}$

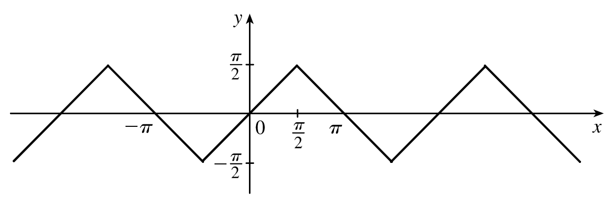
(b) $2\arctan\frac{1}{3} + \arctan\frac{1}{7} = \left(\arctan\frac{1}{3} + \arctan\frac{1}{3}\right) + \arctan\frac{1}{7} = \arctan\left(\dfrac{\frac{1}{3}+\frac{1}{3}}{1-\frac{1}{3}\cdot\frac{1}{3}}\right) + \arctan\frac{1}{7}$

$$= \arctan\tfrac{3}{4} + \arctan\tfrac{1}{7} = \arctan\left(\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\cdot\frac{1}{7}}\right) = \arctan 1 = \frac{\pi}{4}$$

80. (a) $f(x) = \sin(\sin^{-1}x)$



(b) $g(x) = \sin^{-1}(\sin x)$



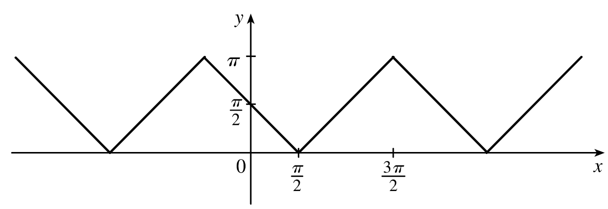
(c) $g'(x) = \dfrac{d}{dx}\sin^{-1}(\sin x) = \dfrac{1}{\sqrt{1-\sin^2 x}}\cos x = \dfrac{\cos x}{\sqrt{\cos^2 x}} = \dfrac{\cos x}{|\cos x|}$

(d) $h(x) = \cos^{-1}(\sin x)$, so

$h'(x) = -\dfrac{\cos x}{\sqrt{1-\sin^2 x}} = -\dfrac{\cos x}{|\cos x|}$.

Notice that $h(x) = \frac{\pi}{2} - g(x)$ because

$\sin^{-1}t + \cos^{-1}t = \frac{\pi}{2}$ for all t.



81. Let $f(x) = 2\sin^{-1}x - \cos^{-1}(1-2x^2)$. Then $f'(x) = \dfrac{2}{\sqrt{1-x^2}} - \dfrac{4x}{\sqrt{1-(1-2x^2)^2}} = \dfrac{2}{\sqrt{1-x^2}} - \dfrac{4x}{2x\sqrt{1-x^2}} = 0$

[since $x \geq 0$]. Thus $f'(x) = 0$ for all $x \in [0,1)$. Thus $f(x) = C$. To find C let $x = 0$. Thus

$2\sin^{-1}(0) - \cos^{-1}(1) = 0 = C$. Therefore we see that $f(x) = 2\sin^{-1}x - \cos^{-1}(1-2x^2) = 0 \quad\Rightarrow$

$2\sin^{-1}x = \cos^{-1}(1-2x^2)$.

82. Let $f(x) = \arcsin\left(\dfrac{x-1}{x+1}\right) - 2\arctan\sqrt{x} + \frac{\pi}{2}$. Note that the domain of f is $[0,\infty)$. Thus

$f'(x) = \dfrac{1}{\sqrt{1-\left(\dfrac{x-1}{x+1}\right)^2}}\,\dfrac{(x+1)-(x-1)}{(x+1)^2} - \dfrac{2}{1+x}\cdot\dfrac{1}{2\sqrt{x}} = \dfrac{1}{\sqrt{x}\,(x+1)} - \dfrac{1}{\sqrt{x}\,(x+1)} = 0$. Then $f(x) = C$.

To find C, we let $x = 0 \quad\Rightarrow\quad \arcsin(-1) - 2\arctan 0 + \frac{\pi}{2} = C \quad\Rightarrow\quad -\frac{\pi}{2} - 0 + \frac{\pi}{2} = 0 = C$. Thus, $f(x) = 0 \quad\Rightarrow$

$\arcsin\left(\dfrac{x-1}{x+1}\right) = 2\arctan\sqrt{x} - \frac{\pi}{2}$.

83. $y = \sec^{-1}x \quad\Rightarrow\quad \sec y = x \quad\Rightarrow\quad \sec y \tan y \, \dfrac{dy}{dx} = 1 \quad\Rightarrow\quad \dfrac{dy}{dx} = \dfrac{1}{\sec y \tan y}$. Now $\tan^2 y = \sec^2 y - 1 = x^2 - 1$, so

$\tan y = \pm\sqrt{x^2 - 1}$. For $y \in \left[0, \frac{\pi}{2}\right)$, $x \geq 1$, so $\sec y = x = |x|$ and $\tan y \geq 0 \quad\Rightarrow\quad \dfrac{dy}{dx} = \dfrac{1}{x\sqrt{x^2-1}} = \dfrac{1}{|x|\sqrt{x^2-1}}$.

For $y \in \left(\frac{\pi}{2}, \pi\right]$, $x \leq -1$, so $|x| = -x$ and $\tan y = -\sqrt{x^2 - 1} \quad\Rightarrow$

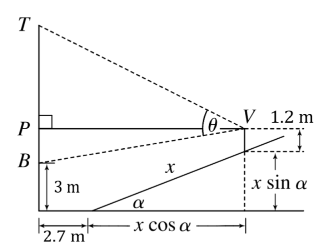
$$\frac{dy}{dx} = \frac{1}{\sec y\ \tan y} = \frac{1}{x\left(-\sqrt{x^2-1}\right)} = \frac{1}{(-x)\sqrt{x^2-1}} = \frac{1}{|x|\sqrt{x^2-1}}$$

84. (a) Since $|\arctan(1/x)| < \frac{\pi}{2}$, we have $0 \leq |x \arctan(1/x)| \leq \frac{\pi}{2}|x| \to 0$ as $x \to 0$. So, by the Squeeze Theorem, $\lim\limits_{x\to 0} f(x) = 0 = f(0)$, so f is continuous at 0.

(b) Here $\dfrac{f(x) - f(0)}{x - 0} = \dfrac{x\arctan(1/x) - 0}{x} = \arctan\left(\dfrac{1}{x}\right)$. So (see Exercise 62 in Section 2.2 for a discussion of left- and right-hand derivatives) $f'_-(0) = \lim\limits_{x\to 0^-} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x\to 0^-} \arctan\left(\dfrac{1}{x}\right) = \lim\limits_{y\to -\infty} \arctan y = -\dfrac{\pi}{2}$, while

$f'_+(0) = \lim\limits_{x\to 0^+} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x\to 0^+} \arctan\left(\dfrac{1}{x}\right) = \lim\limits_{y\to\infty} \arctan y = \dfrac{\pi}{2}$. So $f'(0)$ does not exist.

APPLIED PROJECT Where to Sit at the Movies

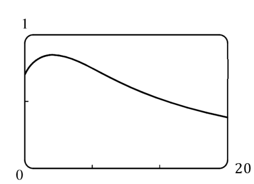


1. $|VP| = 2.7 + x\cos\alpha$, $|PT| = 10.5 - (1.2 + x\sin\alpha) = 9.3 - x\sin\alpha$, and $|PB| = (1.2 + x\sin\alpha) - 3 = x\sin\alpha - 1.8$. So, using the Pythagorean Theorem, we have

$|VT| = \sqrt{|VP|^2 + |PT|^2} = \sqrt{(2.7 + x\cos\alpha)^2 + (9.3 - x\sin\alpha)^2} = a$, and

$|VB| = \sqrt{|VP|^2 + |PB|^2} = \sqrt{(2.7 + x\cos\alpha)^2 + (x\sin\alpha - 1.8)^2} = b$. Using the Law of Cosines on $\triangle VBT$, we get

$7.5^2 = a^2 + b^2 - 2ab\cos\theta \Leftrightarrow \quad \theta = \arccos\left(\dfrac{a^2 + b^2 - 56.25}{2ab}\right)$, as required.



2. From the graph of θ, it appears that the value of x which maximizes θ is $x \approx 2.3$ m. Assuming that the first row is at $x = 0$, the row closest to this value of x is the fourth row, at $x = 2.7$ m, and from the graph, the viewing angle in this row seems to be about 0.84 radians, or about $48.2°$.

3. With a CAS,we type in the definition of θ, substitute in the proper values of a and b in terms of x and $\alpha = 20° = \frac{\pi}{9}$ radians, and then use the differentiation command to find the derivative. We use a numerical root finder and find that the root of the equation $d\theta/dx = 0$ is $x \approx 2.3$, as approximated in Problem 2.

4. From the graph in Problem 2, it seems that the average value of the function on the interval [0,18] is about 0.6. We can use a CAS to approximate $\frac{1}{18}\int_0^{18}\theta(x)\,dx \approx 0.606 \approx 34.7^\circ$. (The calculation is much faster if we reduce the number of digits of accuracy required.) The minimum value is $\theta(18) \approx 0.36$ and, from Problem 2, the maximum value is about 0.84.

6.7 Hyperbolic Functions

1. (a) $\sinh 0 = \frac{1}{2}(e^0 - e^{-0}) = 0$ (b) $\cosh 0 = \frac{1}{2}(e^0 + e^{-0}) = \frac{1}{2}(1+1) = 1$

2. (a) $\tanh 0 = \dfrac{(e^0 - e^{-0})/2}{(e^0 + e^{-0})/2} = 0$ (b) $\tanh 1 = \dfrac{e^1 - e^{-1}}{e^1 + e^{-1}} = \dfrac{e^2 - 1}{e^2 + 1} \approx 0.76159$

3. (a) $\cosh(\ln 5) = \frac{1}{2}(e^{\ln 5} + e^{-\ln 5}) = \frac{1}{2}\left(5 + (e^{\ln 5})^{-1}\right) = \frac{1}{2}(5 + 5^{-1}) = \frac{1}{2}\left(5 + \frac{1}{5}\right) = \frac{13}{5}$

(b) $\cosh 5 = \frac{1}{2}(e^5 + e^{-5}) \approx 74.20995$

4. (a) $\sinh 4 = \frac{1}{2}(e^4 - e^{-4}) \approx 27.28992$

(b) $\sinh(\ln 4) = \frac{1}{2}(e^{\ln 4} - e^{-\ln 4}) = \frac{1}{2}\left(4 - (e^{\ln 4})^{-1}\right) = \frac{1}{2}(4 - 4^{-1}) = \frac{1}{2}\left(4 - \frac{1}{4}\right) = \frac{15}{8}$

5. (a) $\operatorname{sech} 0 = \dfrac{1}{\cosh 0} = \dfrac{1}{1} = 1$ (b) $\cosh^{-1} 1 = 0$ because $\cosh 0 = 1$.

6. (a) $\sinh 1 = \frac{1}{2}(e^1 - e^{-1}) \approx 1.17520$

(b) Using Equation 3, we have $\sinh^{-1} 1 = \ln\left(1 + \sqrt{1^2 + 1}\right) = \ln\left(1 + \sqrt{2}\right) \approx 0.88137$.

7. $8\sinh x + 5\cosh x = 8\left(\dfrac{e^x - e^{-x}}{2}\right) + 5\left(\dfrac{e^x + e^{-x}}{2}\right) = \dfrac{8}{2}e^x - \dfrac{8}{2}e^{-x} + \dfrac{5}{2}e^x + \dfrac{5}{2}e^{-x} = \dfrac{13}{2}e^x - \dfrac{3}{2}e^{-x}$

8. $2e^{2x} + 3e^{-2x} = a\sinh 2x + b\cosh 2x \quad\Rightarrow\quad 2e^{2x} + 3e^{-2x} = a\left(\dfrac{e^{2x} - e^{-2x}}{2}\right) + b\left(\dfrac{e^{2x} + e^{-2x}}{2}\right) \quad\Rightarrow$

$2e^{2x} + 3e^{-2x} = \dfrac{a}{2}e^{2x} - \dfrac{a}{2}e^{-2x} + \dfrac{b}{2}e^{2x} + \dfrac{b}{2}e^{-2x} \quad\Rightarrow\quad 2e^{2x} + 3e^{-2x} = \dfrac{a+b}{2}e^{2x} + \dfrac{-a+b}{2}e^{-2x} \quad\Rightarrow$

$\dfrac{a+b}{2} = 2$ and $\dfrac{-a+b}{2} = 3 \quad\Rightarrow\quad a + b = 4$ and $-a + b = 6 \quad\Rightarrow\quad 2b = 10 \quad\Rightarrow\quad b = 5$ and $a = -1$.

Thus, $2e^{2x} + 3e^{-2x} = -\sinh 2x + 5\cosh 2x$.

9. $\sinh(\ln x) = \dfrac{1}{2}(e^{\ln x} - e^{-\ln x}) = \dfrac{1}{2}\left(x - e^{\ln x^{-1}}\right) = \dfrac{1}{2}(x - x^{-1}) = \dfrac{1}{2}\left(x - \dfrac{1}{x}\right) = \dfrac{1}{2}\left(\dfrac{x^2 - 1}{x}\right) = \dfrac{x^2 - 1}{2x}$

10. $$\begin{aligned}\cosh(4\ln x) = \cosh(\ln x^4) &= \frac{1}{2}\left(e^{\ln x^4} + e^{-\ln x^4}\right) = \frac{1}{2}\left(x^4 + e^{\ln x^{-4}}\right) = \frac{1}{2}(x^4 + x^{-4}) \\ &= \frac{1}{2}\left(x^4 + \frac{1}{x^4}\right) = \frac{1}{2}\left(\frac{x^8 + 1}{x^4}\right) = \frac{x^8 + 1}{2x^4}\end{aligned}$$

11. $\sinh(-x) = \frac{1}{2}[e^{-x} - e^{-(-x)}] = \frac{1}{2}(e^{-x} - e^{x}) = -\frac{1}{2}(e^{-x} - e^{x}) = -\sinh x$

12. $\cosh(-x) = \frac{1}{2}[e^{-x} + e^{-(-x)}] = \frac{1}{2}(e^{-x} + e^{x}) = \frac{1}{2}(e^{x} + e^{-x}) = \cosh x$

13. $\cosh x + \sinh x = \frac{1}{2}(e^{x} + e^{-x}) + \frac{1}{2}(e^{x} - e^{-x}) = \frac{1}{2}(2e^{x}) = e^{x}$

14. $\cosh x - \sinh x = \frac{1}{2}(e^{x} + e^{-x}) - \frac{1}{2}(e^{x} - e^{-x}) = \frac{1}{2}(2e^{-x}) = e^{-x}$

15.
$$\begin{aligned}\sinh x\,\cosh y + \cosh x\,\sinh y &= \left[\tfrac{1}{2}(e^{x} - e^{-x})\right]\left[\tfrac{1}{2}(e^{y} + e^{-y})\right] + \left[\tfrac{1}{2}(e^{x} + e^{-x})\right]\left[\tfrac{1}{2}(e^{y} - e^{-y})\right] \\ &= \tfrac{1}{4}[(e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y}) + (e^{x+y} - e^{x-y} + e^{-x+y} - e^{-x-y})] \\ &= \tfrac{1}{4}(2e^{x+y} - 2e^{-x-y}) = \tfrac{1}{2}[e^{x+y} - e^{-(x+y)}] = \sinh(x+y)\end{aligned}$$

16.
$$\begin{aligned}\cosh x\,\cosh y + \sinh x\,\sinh y &= \left[\tfrac{1}{2}(e^{x} + e^{-x})\right]\left[\tfrac{1}{2}(e^{y} + e^{-y})\right] + \left[\tfrac{1}{2}(e^{x} - e^{-x})\right]\left[\tfrac{1}{2}(e^{y} - e^{-y})\right] \\ &= \tfrac{1}{4}\left[(e^{x+y} + e^{x-y} + e^{-x+y} + e^{-x-y}) + (e^{x+y} - e^{x-y} - e^{-x+y} + e^{-x-y})\right] \\ &= \tfrac{1}{4}(2e^{x+y} + 2e^{-x-y}) = \tfrac{1}{2}\left[e^{x+y} + e^{-(x+y)}\right] = \cosh(x+y)\end{aligned}$$

17. Divide both sides of the identity $\cosh^2 x - \sinh^2 x = 1$ by $\sinh^2 x$:

$$\frac{\cosh^2 x}{\sinh^2 x} - \frac{\sinh^2 x}{\sinh^2 x} = \frac{1}{\sinh^2 x} \quad \Leftrightarrow \quad \coth^2 x - 1 = \operatorname{csch}^2 x.$$

18.
$$\begin{aligned}\tanh(x+y) &= \frac{\sinh(x+y)}{\cosh(x+y)} = \frac{\sinh x\,\cosh y + \cosh x\,\sinh y}{\cosh x\,\cosh y + \sinh x\,\sinh y} = \frac{\dfrac{\sinh x\,\cosh y}{\cosh x\,\cosh y} + \dfrac{\cosh x\,\sinh y}{\cosh x\,\cosh y}}{\dfrac{\cosh x\,\cosh y}{\cosh x\,\cosh y} + \dfrac{\sinh x\,\sinh y}{\cosh x\,\cosh y}} \\ &= \frac{\tanh x + \tanh y}{1 + \tanh x\,\tanh y}\end{aligned}$$

19. Putting $y = x$ in the result from Exercise 15, we have

$\sinh 2x = \sinh(x+x) = \sinh x\,\cosh x + \cosh x\,\sinh x = 2\sinh x\,\cosh x$.

20. Putting $y = x$ in the result from Exercise 16, we have

$\cosh 2x = \cosh(x+x) = \cosh x\,\cosh x + \sinh x\,\sinh x = \cosh^2 x + \sinh^2 x$.

21. $\tanh(\ln x) = \dfrac{\sinh(\ln x)}{\cosh(\ln x)} = \dfrac{(e^{\ln x} - e^{-\ln x})/2}{(e^{\ln x} + e^{-\ln x})/2} = \dfrac{x - (e^{\ln x})^{-1}}{x + (e^{\ln x})^{-1}} = \dfrac{x - x^{-1}}{x + x^{-1}} = \dfrac{x - 1/x}{x + 1/x} = \dfrac{(x^2 - 1)/x}{(x^2 + 1)/x} = \dfrac{x^2 - 1}{x^2 + 1}$

22. $\dfrac{1 + \tanh x}{1 - \tanh x} = \dfrac{1 + (\sinh x)/\cosh x}{1 - (\sinh x)/\cosh x} = \dfrac{\cosh x + \sinh x}{\cosh x - \sinh x} = \dfrac{\frac{1}{2}(e^{x} + e^{-x}) + \frac{1}{2}(e^{x} - e^{-x})}{\frac{1}{2}(e^{x} + e^{-x}) - \frac{1}{2}(e^{x} - e^{-x})} = \dfrac{e^{x}}{e^{-x}} = e^{2x}$

Or: Using the results of Exercises 13 and 14, $\dfrac{\cosh x + \sinh x}{\cosh x - \sinh x} = \dfrac{e^{x}}{e^{-x}} = e^{2x}$

23. By Exercise 13, $(\cosh x + \sinh x)^n = (e^{x})^n = e^{nx} = \cosh nx + \sinh nx$.

24. $\coth x = \dfrac{1}{\tanh x} \;\Rightarrow\; \coth x = \dfrac{1}{\tanh x} = \dfrac{1}{12/13} = \dfrac{13}{12}.$

$\operatorname{sech}^2 x = 1 - \tanh^2 x = 1 - \left(\frac{12}{13}\right)^2 = \frac{25}{169} \;\Rightarrow\; \operatorname{sech} x = \frac{5}{13}$ [sech, like cosh, is positive].

$\cosh x = \dfrac{1}{\operatorname{sech} x} \;\Rightarrow\; \cosh x = \dfrac{1}{5/13} = \dfrac{13}{5}.$

$\tanh x = \dfrac{\sinh x}{\cosh x} \;\Rightarrow\; \sinh x = \tanh x \cosh x \;\Rightarrow\; \sinh x = \dfrac{12}{13} \cdot \dfrac{13}{5} = \dfrac{12}{5}.$

$\operatorname{csch} x = \dfrac{1}{\sinh x} \;\Rightarrow\; \operatorname{csch} x = \dfrac{1}{12/5} = \dfrac{5}{12}.$

25. $\operatorname{sech} x = \dfrac{1}{\cosh x} \;\Rightarrow\; \operatorname{sech} x = \dfrac{1}{5/3} = \dfrac{3}{5}.$

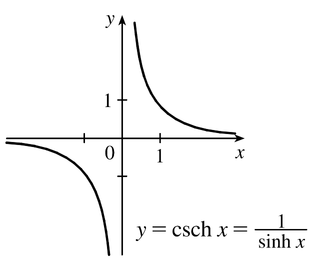
$\cosh^2 x - \sinh^2 x = 1 \;\Rightarrow\; \sinh^2 x = \cosh^2 x - 1 = \left(\frac{5}{3}\right)^2 - 1 = \frac{16}{9} \;\Rightarrow\; \sinh x = \frac{4}{3}$ [because $x > 0$].

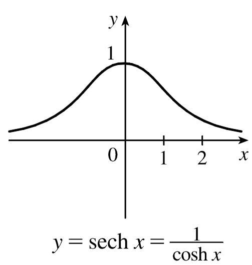
$\operatorname{csch} x = \dfrac{1}{\sinh x} \;\Rightarrow\; \operatorname{csch} x = \dfrac{1}{4/3} = \dfrac{3}{4}.$

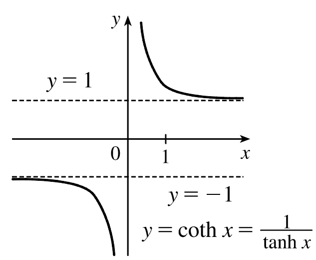
$\tanh x = \dfrac{\sinh x}{\cosh x} \;\Rightarrow\; \tanh x = \dfrac{4/3}{5/3} = \dfrac{4}{5}.$

$\coth x = \dfrac{1}{\tanh x} \;\Rightarrow\; \coth x = \dfrac{1}{4/5} = \dfrac{5}{4}.$

26. (a)







27. (a) $\displaystyle\lim_{x\to\infty} \tanh x = \lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} \cdot \frac{e^{-x}}{e^{-x}} = \lim_{x\to\infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} = \frac{1-0}{1+0} = 1$

(b) $\displaystyle\lim_{x\to-\infty} \tanh x = \lim_{x\to-\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} \cdot \frac{e^x}{e^x} = \lim_{x\to-\infty} \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{0-1}{0+1} = -1$

(c) $\displaystyle\lim_{x\to\infty} \sinh x = \lim_{x\to\infty} \frac{e^x - e^{-x}}{2} = \infty$

(d) $\displaystyle\lim_{x\to-\infty} \sinh x = \lim_{x\to-\infty} \frac{e^x - e^{-x}}{2} = -\infty$

(e) $\displaystyle\lim_{x\to\infty} \operatorname{sech} x = \lim_{x\to\infty} \frac{2}{e^x + e^{-x}} = 0$

(f) $\displaystyle\lim_{x\to\infty} \coth x = \lim_{x\to\infty} \frac{e^x + e^{-x}}{e^x - e^{-x}} \cdot \frac{e^{-x}}{e^{-x}} = \lim_{x\to\infty} \frac{1 + e^{-2x}}{1 - e^{-2x}} = \frac{1+0}{1-0} = 1$ [*Or:* Use part (a).]

(g) $\displaystyle\lim_{x\to 0^+} \coth x = \lim_{x\to 0^+} \frac{\cosh x}{\sinh x} = \infty$, since $\sinh x \to 0$ through positive values and $\cosh x \to 1$.

(h) $\displaystyle\lim_{x\to 0^-} \coth x = \lim_{x\to 0^-} \frac{\cosh x}{\sinh x} = -\infty$, since $\sinh x \to 0$ through negative values and $\cosh x \to 1$.

(i) $\displaystyle\lim_{x\to -\infty} \operatorname{csch} x = \lim_{x\to -\infty} \frac{2}{e^x - e^{-x}} = 0$

(j) $\displaystyle\lim_{x\to\infty} \frac{\sinh x}{e^x} = \lim_{x\to\infty} \frac{e^x - e^{-x}}{2e^x} = \lim_{x\to\infty} \frac{1 - e^{-2x}}{2} = \frac{1-0}{2} = \frac{1}{2}$

28. (a) $\displaystyle\frac{d}{dx}(\cosh x) = \frac{d}{dx}\left[\tfrac{1}{2}(e^x + e^{-x})\right] = \tfrac{1}{2}(e^x - e^{-x}) = \sinh x$

(b) $\displaystyle\frac{d}{dx}(\tanh x) = \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right) = \frac{\cosh x\,\cosh x - \sinh x\,\sinh x}{\cosh^2 x} = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$

(c) $\displaystyle\frac{d}{dx}(\operatorname{csch} x) = \frac{d}{dx}\left(\frac{1}{\sinh x}\right) = -\frac{\cosh x}{\sinh^2 x} = -\frac{1}{\sinh x}\cdot\frac{\cosh x}{\sinh x} = -\operatorname{csch} x\,\coth x$

(d) $\displaystyle\frac{d}{dx}(\operatorname{sech} x) = \frac{d}{dx}\left(\frac{1}{\cosh x}\right) = -\frac{\sinh x}{\cosh^2 x} = -\frac{1}{\cosh x}\cdot\frac{\sinh x}{\cosh x} = -\operatorname{sech} x\,\tanh x$

(e) $\displaystyle\frac{d}{dx}(\coth x) = \frac{d}{dx}\left(\frac{\cosh x}{\sinh x}\right) = \frac{\sinh x\,\sinh x - \cosh x\,\cosh x}{\sinh^2 x} = \frac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} = -\frac{1}{\sinh^2 x} = -\operatorname{csch}^2 x$

29. Let $y = \sinh^{-1} x$. Then $\sinh y = x$ and, by Example 1(a), $\cosh^2 y - \sinh^2 y = 1 \;\Rightarrow\;$ [with $\cosh y > 0$]
$\cosh y = \sqrt{1 + \sinh^2 y} = \sqrt{1 + x^2}$. So by Exercise 13, $e^y = \sinh y + \cosh y = x + \sqrt{1+x^2} \;\Rightarrow$
$y = \ln\!\left(x + \sqrt{1+x^2}\right)$.

30. Let $y = \cosh^{-1} x$. Then $\cosh y = x$ and $y \geq 0$, so $\sinh y = \sqrt{\cosh^2 y - 1} = \sqrt{x^2 - 1}$. So, by Exercise 13,
$e^y = \cosh y + \sinh y = x + \sqrt{x^2-1} \;\Rightarrow\; y = \ln\!\left(x + \sqrt{x^2-1}\right)$.
Another method: Write $x = \cosh y = \frac{1}{2}\left(e^y + e^{-y}\right)$ and solve a quadratic, as in Example 3.

31. (a) Let $y = \tanh^{-1} x$. Then $\displaystyle x = \tanh y = \frac{\sinh y}{\cosh y} = \frac{(e^y - e^{-y})/2}{(e^y + e^{-y})/2}\cdot\frac{e^y}{e^y} = \frac{e^{2y}-1}{e^{2y}+1} \;\Rightarrow\; xe^{2y} + x = e^{2y} - 1 \;\Rightarrow$

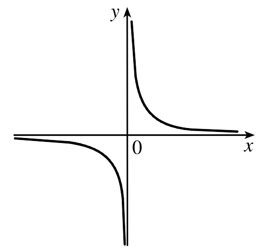
$$1 + x = e^{2y} - xe^{2y} \;\Rightarrow\; 1 + x = e^{2y}(1-x) \;\Rightarrow\; e^{2y} = \frac{1+x}{1-x} \;\Rightarrow\; 2y = \ln\!\left(\frac{1+x}{1-x}\right) \;\Rightarrow$$

$$y = \frac{1}{2}\ln\!\left(\frac{1+x}{1-x}\right).$$

(b) Let $y = \tanh^{-1} x$. Then $x = \tanh y$, so from Exercise 22 we have

$$e^{2y} = \frac{1 + \tanh y}{1 - \tanh y} = \frac{1+x}{1-x} \;\Rightarrow\; 2y = \ln\!\left(\frac{1+x}{1-x}\right) \;\Rightarrow\; y = \frac{1}{2}\ln\!\left(\frac{1+x}{1-x}\right).$$

32. (a) (i) $y = \operatorname{csch}^{-1} x \quad \Leftrightarrow \quad \operatorname{csch} y = x \quad (x \neq 0)$



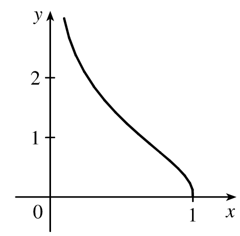
(ii) We sketch the graph of csch^{-1} by reflecting the graph of csch (see Exercise 26) about the line $y = x$.

(iii) Let $y = \operatorname{csch}^{-1} x$. Then $x = \operatorname{csch} y = \dfrac{2}{e^y - e^{-y}} \quad \Rightarrow \quad xe^y - xe^{-y} = 2 \quad \Rightarrow$

$x(e^y)^2 - 2e^y - x = 0 \quad \Rightarrow \quad e^y = \dfrac{1 \pm \sqrt{x^2+1}}{x}$. But $e^y > 0$, so for $x > 0$,

$e^y = \dfrac{1 + \sqrt{x^2+1}}{x}$ and for $x < 0$, $e^y = \dfrac{1 - \sqrt{x^2+1}}{x}$. Thus, $\operatorname{csch}^{-1} x = \ln\left(\dfrac{1}{x} + \dfrac{\sqrt{x^2+1}}{|x|}\right)$.

(b) (i) $y = \operatorname{sech}^{-1} x \quad \Leftrightarrow \quad \operatorname{sech} y = x$ and $y > 0$.



(ii) We sketch the graph of sech^{-1} by reflecting the graph of sech (see Exercise 26) about the line $y = x$.

(iii) Let $y = \operatorname{sech}^{-1} x$, so $x = \operatorname{sech} y = \dfrac{2}{e^y + e^{-y}} \quad \Rightarrow \quad xe^y + xe^{-y} = 2 \quad \Rightarrow$

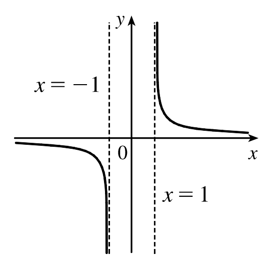
$x\left(e^y\right)^2 - 2e^y + x = 0 \quad \Leftrightarrow \quad e^y = \dfrac{1 \pm \sqrt{1-x^2}}{x}$. But $y > 0 \quad \Rightarrow \quad e^y > 1$.

This rules out the minus sign because $\dfrac{1 - \sqrt{1-x^2}}{x} > 1 \quad \Leftrightarrow \quad 1 - \sqrt{1-x^2} > x \quad \Leftrightarrow \quad 1 - x > \sqrt{1-x^2} \quad \Leftrightarrow$

$1 - 2x + x^2 > 1 - x^2 \quad \Leftrightarrow \quad x^2 > x \quad \Leftrightarrow \quad x > 1$, but $x = \operatorname{sech} y \leq 1$.

Thus, $e^y = \dfrac{1 + \sqrt{1-x^2}}{x} \quad \Rightarrow \quad \operatorname{sech}^{-1} x = \ln\left(\dfrac{1 + \sqrt{1-x^2}}{x}\right)$.

(c) (i) $y = \coth^{-1} x \quad \Leftrightarrow \quad \coth y = x$



(ii) We sketch the graph of $\coth^{-1}$ by reflecting the graph of coth (see Exercise 26) about the line $y = x$.

(iii) Let $y = \coth^{-1} x$. Then $x = \coth y = \dfrac{e^y + e^{-y}}{e^y - e^{-y}} \quad \Rightarrow$

$xe^y - xe^{-y} = e^y + e^{-y} \quad \Rightarrow \quad (x-1)e^y = (x+1)e^{-y} \quad \Rightarrow \quad e^{2y} = \dfrac{x+1}{x-1} \quad \Rightarrow$

$2y = \ln \dfrac{x+1}{x-1} \quad \Rightarrow \quad \coth^{-1} x = \dfrac{1}{2} \ln \dfrac{x+1}{x-1}$

33. (a) Let $y = \cosh^{-1} x$. Then $\cosh y = x$ and $y \geq 0 \quad \Rightarrow \quad \sinh y \dfrac{dy}{dx} = 1 \quad \Rightarrow$

$\dfrac{dy}{dx} = \dfrac{1}{\sinh y} = \dfrac{1}{\sqrt{\cosh^2 y - 1}} = \dfrac{1}{\sqrt{x^2-1}}$ [since $\sinh y \geq 0$ for $y \geq 0$]. *Or:* Use Formula 4.

(b) Let $y = \tanh^{-1} x$. Then $\tanh y = x \quad \Rightarrow \quad \operatorname{sech}^2 y \dfrac{dy}{dx} = 1 \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{1}{\operatorname{sech}^2 y} = \dfrac{1}{1 - \tanh^2 y} = \dfrac{1}{1-x^2}$.

Or: Use Formula 5.

(c) Let $y = \coth^{-1} x$. Then $\coth y = x \quad \Rightarrow \quad -\operatorname{csch}^2 y \dfrac{dy}{dx} = 1 \quad \Rightarrow \quad \dfrac{dy}{dx} = -\dfrac{1}{\operatorname{csch}^2 y} = \dfrac{1}{1 - \coth^2 y} = \dfrac{1}{1-x^2}$

by Exercise 17.

34. (a) Let $y = \operatorname{sech}^{-1} x$. Then $\operatorname{sech} y = x \;\Rightarrow\; -\operatorname{sech} y \,\tanh y \,\dfrac{dy}{dx} = 1 \;\Rightarrow$

$$\frac{dy}{dx} = -\frac{1}{\operatorname{sech} y \,\tanh y} = -\frac{1}{\operatorname{sech} y\,\sqrt{1-\operatorname{sech}^2 y}} = -\frac{1}{x\sqrt{1-x^2}}.$$ [Note that $y > 0$ and so $\tanh y > 0$.]

(b) Let $y = \operatorname{csch}^{-1} x$. Then $\operatorname{csch} y = x \;\Rightarrow\; -\operatorname{csch} y \,\coth y \,\dfrac{dy}{dx} = 1 \;\Rightarrow\; \dfrac{dy}{dx} = -\dfrac{1}{\operatorname{csch} y\,\coth y}$. By Exercise 17,

$\coth y = \pm\sqrt{\operatorname{csch}^2 y + 1} = \pm\sqrt{x^2+1}$. If $x > 0$, then $\coth y > 0$, so $\coth y = \sqrt{x^2+1}$. If $x < 0$, then $\coth y < 0$,

so $\coth y = -\sqrt{x^2+1}$. In either case we have $\dfrac{dy}{dx} = -\dfrac{1}{\operatorname{csch} y\,\coth y} = -\dfrac{1}{|x|\sqrt{x^2+1}}$.

35. $f(x) = \cosh 3x \;\Rightarrow\; f'(x) = \sinh(3x)\cdot\dfrac{d}{dx}(3x) = \sinh(3x)\cdot 3 = 3\sinh 3x$

36. $f(x) = e^x\cosh x \;\overset{\text{PR}}{\Rightarrow}\; f'(x) = e^x\sinh x + (\cosh x)e^x = e^x(\sinh x + \cosh x)$, or, using Exercise 13, $e^x(e^x) = e^{2x}$.

37. $h(x) = \sinh(x^2) \;\Rightarrow\; h'(x) = \cosh(x^2)\,\dfrac{d}{dx}(x^2) = 2x\cosh(x^2)$

38. $g(x) = \sinh^2 x = (\sinh x)^2 \;\Rightarrow\; g'(x) = 2(\sinh x)^1\dfrac{d}{dx}(\sinh x) = 2\sinh x\,\cosh x$, or, using Exercise 19, $\sinh 2x$.

39. $G(t) = \sinh(\ln t) \;\Rightarrow\; G'(t) = \cosh(\ln t)\,\dfrac{d}{dt}\ln t = \dfrac{1}{2}\left(e^{\ln t} + e^{-\ln t}\right)\left(\dfrac{1}{t}\right) = \dfrac{1}{2t}\left(t + \dfrac{1}{t}\right) = \dfrac{1}{2t}\left(\dfrac{t^2+1}{t}\right) = \dfrac{t^2+1}{2t^2}$

Or: $G(t) = \sinh(\ln t) = \dfrac{1}{2}(e^{\ln t} - e^{-\ln t}) = \dfrac{1}{2}\left(t - \dfrac{1}{t}\right) \;\Rightarrow\; G'(t) = \dfrac{1}{2}\left(1 + \dfrac{1}{t^2}\right) = \dfrac{t^2+1}{2t^2}$

40. $F(t) = \ln(\sinh t) \;\Rightarrow\; F'(t) = \dfrac{1}{\sinh t}\,\dfrac{d}{dt}\sinh t = \dfrac{1}{\sinh t}\cosh t = \coth t$

41. $f(x) = \tanh\sqrt{x} \;\Rightarrow\; f'(x) = \operatorname{sech}^2\sqrt{x}\,\dfrac{d}{dx}\sqrt{x} = \operatorname{sech}^2\sqrt{x}\left(\dfrac{1}{2\sqrt{x}}\right) = \dfrac{\operatorname{sech}^2\sqrt{x}}{2\sqrt{x}}$

42. $H(v) = e^{\tanh 2v} \;\Rightarrow\; H'(v) = e^{\tanh 2v}\cdot\dfrac{d}{dv}\tanh 2v = e^{\tanh 2v}\cdot\operatorname{sech}^2(2v)\cdot 2 = 2e^{\tanh 2v}\operatorname{sech}^2(2v)$

43. $y = \operatorname{sech} x\,\tanh x \;\overset{\text{PR}}{\Rightarrow}\; y' = \operatorname{sech} x\cdot\operatorname{sech}^2 x + \tanh x\cdot(-\operatorname{sech} x\,\tanh x) = \operatorname{sech}^3 x - \operatorname{sech} x\,\tanh^2 x$

44. $y = \operatorname{sech}(\tanh x) \;\Rightarrow\; y' = -\operatorname{sech}(\tanh x)\,\tanh(\tanh x)\cdot\dfrac{d}{dx}(\tanh x) = -\operatorname{sech}(\tanh x)\,\tanh(\tanh x)\cdot\operatorname{sech}^2 x$

45. $g(t) = t\coth\sqrt{t^2+1} \;\overset{\text{PR}}{\Rightarrow}$

$$g'(t) = t\left[-\operatorname{csch}^2\sqrt{t^2+1}\left(\tfrac{1}{2}(t^2+1)^{-1/2}\cdot 2t\right)\right] + \left(\coth\sqrt{t^2+1}\right)(1) = \coth\sqrt{t^2+1} - \frac{t^2}{\sqrt{t^2+1}}\operatorname{csch}^2\sqrt{t^2+1}$$

46. $f(t) = \dfrac{1+\sinh t}{1-\sinh t} \quad \overset{\text{QR}}{\Rightarrow}$

$$f'(t) = \frac{(1-\sinh t)\,\cosh t - (1+\sinh t)(-\cosh t)}{(1-\sinh t)^2} = \frac{\cosh t - \sinh t\,\cosh t + \cosh t + \sinh t\,\cosh t}{(1-\sinh t)^2}$$
$$= \frac{2\cosh t}{(1-\sinh t)^2}$$

47. $f(x) = \sinh^{-1}(-2x) \quad \Rightarrow \quad f'(x) = \dfrac{1}{\sqrt{1+(-2x)^2}} \cdot \dfrac{d}{dx}(-2x) = -\dfrac{2}{\sqrt{1+4x^2}}$

48. $g(x) = \tanh^{-1}(x^3) \quad \Rightarrow \quad g'(x) = \dfrac{1}{1-(x^3)^2} \cdot \dfrac{d}{dx}(x^3) = \dfrac{3x^2}{1-x^6}$

49. $y = \cosh^{-1}(\sec\theta) \quad \Rightarrow$

$$y' = \frac{1}{\sqrt{\sec^2\theta - 1}} \cdot \frac{d}{d\theta}(\sec\theta) = \frac{1}{\sqrt{\tan^2\theta}} \cdot \sec\theta\,\tan\theta = \frac{1}{\tan\theta} \cdot \sec\theta\,\tan\theta \quad [\text{since } 0 \le \theta < \pi/2] \quad = \sec\theta$$

50. $y = \operatorname{sech}^{-1}(\sin\theta) \quad \Rightarrow$

$$y' = -\frac{1}{\sin\theta\sqrt{1-\sin^2\theta}} \cdot \frac{d}{d\theta}(\sin\theta) = -\frac{1}{\sin\theta\sqrt{\cos^2\theta}} \cdot \cos\theta$$
$$= -\frac{1}{\sin\theta \cdot \cos\theta} \cdot \cos\theta \quad [\text{since } 0 < \theta < \pi/2] \quad = -\frac{1}{\sin\theta} = -\csc\theta$$

51. $G(u) = \cosh^{-1}\sqrt{1+u^2} \quad \Rightarrow$

$$G'(u) = \frac{1}{\sqrt{\left(\sqrt{1+u^2}\right)^2 - 1}} \cdot \frac{d}{du}\left(\sqrt{1+u^2}\right) = \frac{1}{\sqrt{(1+u^2)-1}} \cdot \frac{2u}{2\sqrt{1+u^2}} = \frac{u}{\sqrt{u^2}\cdot\sqrt{1+u^2}}$$
$$= \frac{u}{u\sqrt{1+u^2}} \quad [\text{since } u > 0] \quad = \frac{1}{\sqrt{1+u^2}}$$

52. $y = x\tanh^{-1}x + \ln\sqrt{1-x^2} = x\tanh^{-1}x + \frac{1}{2}\ln(1-x^2) \quad \Rightarrow$

$$y' = \tanh^{-1}x + \frac{x}{1-x^2} + \frac{1}{2}\left(\frac{1}{1-x^2}\right)(-2x) = \tanh^{-1}x$$

53. $y = x\sinh^{-1}(x/3) - \sqrt{9+x^2} \quad \Rightarrow$

$$y' = \sinh^{-1}\left(\frac{x}{3}\right) + x\frac{1/3}{\sqrt{1+(x/3)^2}} - \frac{2x}{2\sqrt{9+x^2}} = \sinh^{-1}\left(\frac{x}{3}\right) + \frac{x}{\sqrt{9+x^2}} - \frac{x}{\sqrt{9+x^2}} = \sinh^{-1}\left(\frac{x}{3}\right)$$

54. $\dfrac{1+\tanh x}{1-\tanh x} = \dfrac{1+(\sinh x)/\cosh x}{1-(\sinh x)/\cosh x} = \dfrac{\cosh x + \sinh x}{\cosh x - \sinh x} = \dfrac{e^x}{e^{-x}}$ [by Exercises 13 and 14] $= e^{2x}$, so

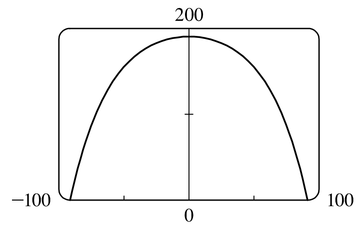
$\sqrt[4]{\dfrac{1+\tanh x}{1-\tanh x}} = \sqrt[4]{e^{2x}} = e^{x/2}$. Thus, $\dfrac{d}{dx}\sqrt[4]{\dfrac{1+\tanh x}{1-\tanh x}} = \dfrac{d}{dx}(e^{x/2}) = \dfrac{1}{2}e^{x/2}$.

55. $\dfrac{d}{dx}\arctan(\tanh x) = \dfrac{1}{1+(\tanh x)^2}\,\dfrac{d}{dx}(\tanh x) = \dfrac{\operatorname{sech}^2 x}{1+\tanh^2 x} = \dfrac{1/\cosh^2 x}{1+(\sinh^2 x)/\cosh^2 x}$

$$= \frac{1}{\cosh^2 x + \sinh^2 x} = \frac{1}{\cosh 2x} \quad [\text{by Exercise 20}] \quad = \operatorname{sech} 2x$$

56. (a) Let $a = 0.03291765$. A graph of the central curve,

$y = f(x) = 211.49 - 20.96 \cosh ax$, is shown.



(b) $f(0) = 211.49 - 20.96 \cosh 0 = 211.49 - 20.96(1) = 190.53$ m.

(c) $y = 100 \quad \Rightarrow \quad 100 = 211.49 - 20.96 \cosh ax \quad \Rightarrow$

$20.96 \cosh ax = 111.49 \quad \Rightarrow \quad \cosh ax = \dfrac{111.49}{20.96} \quad \Rightarrow$

$ax = \pm \cosh^{-1} \dfrac{111.49}{20.96} \quad \Rightarrow \quad x = \pm \dfrac{1}{a} \cosh^{-1} \dfrac{111.49}{20.96} \approx \pm 71.56$ m. The points are approximately $(\pm 71.56, 100)$.

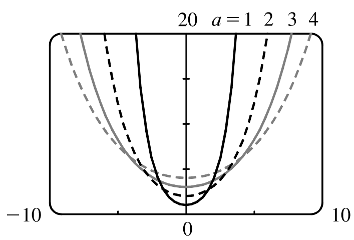
(d) $f(x) = 211.49 - 20.96 \cosh ax \quad \Rightarrow \quad f'(x) = -20.96 \sinh ax \cdot a$.

$$f'\left(\pm\frac{1}{a}\cosh^{-1}\frac{111.49}{20.96}\right) = -20.96a \sinh\left[a\left(\pm\frac{1}{a}\cosh^{-1}\frac{111.49}{20.96}\right)\right] = -20.96a \sinh\left(\pm\cosh^{-1}\frac{111.49}{20.96}\right) \approx \mp 3.6.$$

So the slope at $(71.56, 100)$ is about -3.6 and the slope at $(-71.56, 100)$ is about 3.6.

57. As the depth d of the water gets large, the fraction $\dfrac{2\pi d}{L}$ gets large, and from Figure 5 or Exercise 27(a), $\tanh\left(\dfrac{2\pi d}{L}\right)$ approaches 1. Thus, $v = \sqrt{\dfrac{gL}{2\pi} \tanh\left(\dfrac{2\pi d}{L}\right)} \approx \sqrt{\dfrac{gL}{2\pi}(1)} = \sqrt{\dfrac{gL}{2\pi}}$.

58.



For $y = a \cosh(x/a)$ with $a > 0$, we have the y-intercept equal to a. As a increases, the graph flattens.

59. (a) $y = 20 \cosh(x/20) - 15 \quad \Rightarrow \quad y' = 20 \sinh(x/20) \cdot \frac{1}{20} = \sinh(x/20)$. Since the right pole is positioned at $x = 7$, we have $y'(7) = \sinh \frac{7}{20} \approx 0.3572$.

(b) If α is the angle between the tangent line and the x-axis, then $\tan \alpha =$ slope of the line $= \sinh \frac{7}{20}$, so $\alpha = \tan^{-1}\left(\sinh \frac{7}{20}\right) \approx 0.343 \text{ rad} \approx 19.66°$. Thus, the angle between the line and the pole is $\theta = 90° - \alpha \approx 70.34°$.

60. We differentiate the function twice, then substitute into the differential equation: $y = \dfrac{T}{\rho g} \cosh \dfrac{\rho g x}{T} \quad \Rightarrow$

$\dfrac{dy}{dx} = \dfrac{T}{\rho g} \sinh\left(\dfrac{\rho g x}{T}\right) \dfrac{\rho g}{T} = \sinh \dfrac{\rho g x}{T} \quad \Rightarrow \quad \dfrac{d^2 y}{dx^2} = \cosh\left(\dfrac{\rho g x}{T}\right) \dfrac{\rho g}{T} = \dfrac{\rho g}{T} \cosh \dfrac{\rho g x}{T}$. We evaluate the two sides separately: LHS $= \dfrac{d^2 y}{dx^2} = \dfrac{\rho g}{T} \cosh \dfrac{\rho g x}{T}$ and RHS $= \dfrac{\rho g}{T} \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2} = \dfrac{\rho g}{T} \sqrt{1 + \sinh^2 \dfrac{\rho g x}{T}} = \dfrac{\rho g}{T} \cosh \dfrac{\rho g x}{T}$, by the identity proved in Example 1(a).

61. (a) From Exercise 60, the shape of the cable is given by $y = f(x) = \dfrac{T}{\rho g} \cosh\left(\dfrac{\rho g x}{T}\right)$. The shape is symmetric about the y-axis, so the lowest point is $(0, f(0)) = \left(0, \dfrac{T}{\rho g}\right)$ and the poles are at $x = \pm 100$. We want to find T when the lowest point is 60 m, so $\dfrac{T}{\rho g} = 60 \quad \Rightarrow \quad T = 60\rho g = (60\text{ m})(2\text{ kg/m})(9.8\text{ m/s}^2) = 1176\,\dfrac{\text{kg-m}}{\text{s}^2}$, or 1176 N (newtons).

The height of each pole is $f(100) = \dfrac{T}{\rho g}\cosh\left(\dfrac{\rho g \cdot 100}{T}\right) = 60\cosh\left(\dfrac{100}{60}\right) \approx 164.50\text{ m}.$

(b) If the tension is doubled from T to $2T$, then the low point is doubled since $\dfrac{T}{\rho g} = 60 \quad \Rightarrow \quad \dfrac{2T}{\rho g} = 120$. The height of the poles is now $f(100) = \dfrac{2T}{\rho g}\cosh\left(\dfrac{\rho g \cdot 100}{2T}\right) = 120\,\cosh\left(\dfrac{100}{120}\right) \approx 164.13$ m, just a slight decrease.

62. (a) $\displaystyle\lim_{t\to\infty} v(t) = \lim_{t\to\infty}\sqrt{\frac{mg}{k}}\tanh\left(t\sqrt{\frac{gk}{m}}\right) = \sqrt{\frac{mg}{k}}\lim_{t\to\infty}\tanh\left(t\sqrt{\frac{gk}{m}}\right) = \sqrt{\frac{mg}{k}}\cdot 1 \quad \left[\begin{matrix}\text{as } t\to\infty,\\ t\sqrt{gk/m}\to\infty\end{matrix}\right] \quad = \sqrt{\frac{mg}{k}}$

(b) Belly-to-earth: $g = 9.8$, $k = 0.515$, $m = 60$, so the terminal velocity is $\sqrt{\dfrac{60(9.8)}{0.515}} \approx 33.79$ m/s.

Feet-first: $g = 9.8$, $k = 0.067$, $m = 60$, so the terminal velocity is $\sqrt{\dfrac{60(9.8)}{0.067}} \approx 93.68$ m/s.

63. (a) $y = A\sinh mx + B\cosh mx \quad \Rightarrow \quad y' = mA\cosh mx + mB\sinh mx \quad \Rightarrow$

$y'' = m^2A\sinh mx + m^2B\cosh mx = m^2(A\sinh mx + B\cosh mx) = m^2y$

(b) From part (a), a solution of $y'' = 9y$ is $y(x) = A\sinh 3x + B\cosh 3x$. Now $-4 = y(0) = A\sinh 0 + B\cosh 0 = B$, so $B = -4$. Also, $y'(x) = 3A\cosh 3x - 12\sinh 3x$, so $6 = y'(0) = 3A \quad \Rightarrow \quad A = 2$. Thus, $y = 2\sinh 3x - 4\cosh 3x$.

64. $\cosh x = \cosh[\ln(\sec\theta + \tan\theta)] = \dfrac{1}{2}\left[e^{\ln(\sec\theta+\tan\theta)} + e^{-\ln(\sec\theta+\tan\theta)}\right] = \dfrac{1}{2}\left[\sec\theta + \tan\theta + \dfrac{1}{\sec\theta + \tan\theta}\right]$

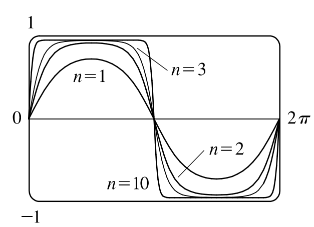
$= \dfrac{1}{2}\left[\sec\theta + \tan\theta + \dfrac{\sec\theta - \tan\theta}{(\sec\theta + \tan\theta)(\sec\theta - \tan\theta)}\right] = \dfrac{1}{2}\left[\sec\theta + \tan\theta + \dfrac{\sec\theta - \tan\theta}{\sec^2\theta - \tan^2\theta}\right]$

$= \frac{1}{2}(\sec\theta + \tan\theta + \sec\theta - \tan\theta) = \sec\theta$

65. The tangent to $y = \cosh x$ has slope 1 when $y' = \sinh x = 1 \quad \Rightarrow \quad x = \sinh^{-1} 1 = \ln\left(1 + \sqrt{2}\right)$, by Equation 3.

Since $\sinh x = 1$ and $y = \cosh x = \sqrt{1 + \sinh^2 x}$, we have $\cosh x = \sqrt{2}$. The point is $\left(\ln\left(1 + \sqrt{2}\right), \sqrt{2}\right)$.

66. $f_n(x) = \tanh(n\sin x)$, where n is a positive integer. Note that $f_n(x + 2\pi) = f_n(x)$; that is, f_n is periodic with period 2π. Also, from Figure 3, $-1 < \tanh x < 1$, so we can choose a viewing rectangle of $[0, 2\pi] \times [-1, 1]$. From the graph, we see that $f_n(x)$ becomes more rectangular looking as n increases. As n becomes large, the graph of f_n approaches the graph of $y = 1$ on the intervals $(2k\pi, (2k+1)\pi)$ and $y = -1$ on the intervals $((2k-1)\pi, 2k\pi)$.



67. Let $u = \cosh x$. Then $du = \sinh x\,dx$, so $\int \sinh x\,\cosh^2 x\,dx = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}\cosh^3 x + C$.

68. Let $u = 1 + 4x$. Then $du = 4\,dx$, so $\int \sinh(1 + 4x)\,dx = \frac{1}{4}\int \sinh u\,du = \frac{1}{4}\cosh u + C = \frac{1}{4}\cosh(1 + 4x) + C$.

69. Let $u = \sqrt{x}$. Then $du = \dfrac{dx}{2\sqrt{x}}$ and $\displaystyle\int \frac{\sinh\sqrt{x}}{\sqrt{x}}\,dx = \int \sinh u \cdot 2\,du = 2\cosh u + C = 2\cosh\sqrt{x} + C.$

70. Let $u = \cosh x$. Then $du = \sinh x\,dx$, and $\displaystyle\int \tanh x\,dx = \int \frac{\sinh x}{\cosh x}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln(\cosh x) + C.$

71. $\displaystyle\int \frac{\cosh x}{\cosh^2 x - 1}\,dx = \int \frac{\cosh x}{\sinh^2 x}\,dx = \int \frac{\cosh x}{\sinh x}\cdot\frac{1}{\sinh x}\,dx = \int \coth x\ \operatorname{csch} x\,dx = -\operatorname{csch} x + C$

72. Let $u = 2 + \tanh x$. Then $du = \operatorname{sech}^2 x\,dx$, so

$$\int \frac{\operatorname{sech}^2 x}{2+\tanh x}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln|2+\tanh x| + C = \ln(2+\tanh x) + C\ \ [\text{since } 2+\tanh x > 1].$$

73. Let $t = 3u$. Then $dt = 3\,du$ and

$$\int_4^6 \frac{1}{\sqrt{t^2-9}}\,dt = \int_{4/3}^2 \frac{1}{\sqrt{9u^2-9}}\,3\,du = \int_{4/3}^2 \frac{du}{\sqrt{u^2-1}} = \Big[\cosh^{-1}u\Big]_{4/3}^2 = \cosh^{-1}2 - \cosh^{-1}\left(\tfrac{4}{3}\right) \quad \text{or}$$

$$= \Big[\cosh^{-1}u\Big]_{4/3}^2 = \Big[\ln\big(u+\sqrt{u^2-1}\big)\Big]_{4/3}^2 = \ln\big(2+\sqrt{3}\big) - \ln\left(\frac{4+\sqrt{7}}{3}\right) = \ln\left(\frac{6+3\sqrt{3}}{4+\sqrt{7}}\right)$$

74. Let $u = 4t$. Then $du = 4\,dt$ and

$$\int_0^1 \frac{1}{\sqrt{16t^2+1}}\,dt = \int_0^4 \frac{\frac{1}{4}\,du}{\sqrt{u^2+1}} = \frac{1}{4}\Big[\sinh^{-1}u\Big]_0^4 = \frac{1}{4}\Big[\ln\Big(u+\sqrt{u^2+1}\Big)\Big]_0^4 = \tfrac{1}{4}\Big[\ln\Big(4+\sqrt{17}\Big) - \ln 1\Big]$$

$$= \tfrac{1}{4}\ln\big(4+\sqrt{17}\big)$$

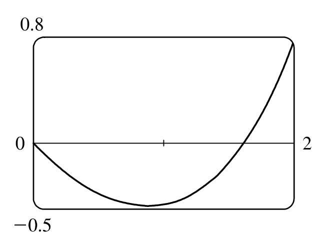
75. Let $u = e^x$. Then $du = e^x\,dx$ and $\displaystyle\int \frac{e^x}{1-e^{2x}}\,dx = \int \frac{du}{1-u^2} = \tanh^{-1}u + C = \tanh^{-1}e^x + C$

$\left[\text{or } \dfrac{1}{2}\ln\left(\dfrac{1+e^x}{1-e^x}\right) + C\right].$

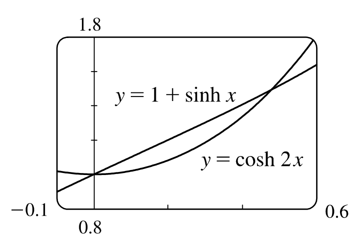
76. We want $\int_0^1 \sinh cx\,dx = 1$. To calculate the integral, we put $u = cx$, so $du = c\,dx$, the upper limit becomes c, and the equation becomes

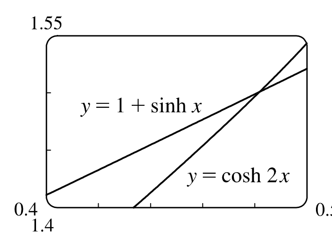
$\dfrac{1}{c}\displaystyle\int_0^c \sinh u\,du = 1 \quad\Leftrightarrow\quad \dfrac{1}{c}[\cosh c - 1] = 1 \quad\Leftrightarrow\quad \cosh c - 1 = c.$

We plot the function $f(c) = \cosh c - c - 1$, and see that its positive root lies at approximately $c = 1.62$. So the equation $\int_0^1 \sinh cx\,dx = 1$ holds for $c \approx 1.62$.



77. (a) From the graphs, we estimate that the two curves $y = \cosh 2x$ and $y = 1 + \sinh x$ intersect at $x = 0$ and at $x = a \approx 0.481$.





(b) We have found the two solutions of the equation $\cosh 2x = 1 + \sinh x$ to be $x = 0$ and $x = a \approx 0.481$. Note from the first graph that $1 + \sinh x > \cosh 2x$ on the interval $(0, a)$, so the area between the two curves is

$$\begin{aligned} A &= \int_0^a (1 + \sinh x - \cosh 2x)\,dx = \left[x + \cosh x - \tfrac{1}{2}\sinh 2x\right]_0^a \\ &= \left[a + \cosh a - \tfrac{1}{2}\sinh 2a\right] - \left[0 + \cosh 0 - \tfrac{1}{2}\sinh 0\right] \approx 0.0402 \end{aligned}$$

78. The area of the triangle with vertices O, P, and $(\cosh t, 0)$ is $\frac{1}{2}\sinh t \cosh t$, and the area under the curve $x^2 - y^2 = 1$, from $x = 1$ to $x = \cosh t$, is $\int_1^{\cosh t} \sqrt{x^2 - 1}\,dx$. Therefore, the area of the shaded region is $A(t) = \frac{1}{2}\sinh t \cosh t - \int_1^{\cosh t} \sqrt{x^2 - 1}\,dx$. So, by FTC1,

$$\begin{aligned} A'(t) &= \tfrac{1}{2}\left(\cosh^2 t + \sinh^2 t\right) - \sqrt{\cosh^2 t - 1}\,\sinh t = \tfrac{1}{2}\left(\cosh^2 t + \sinh^2 t\right) - \sqrt{\sinh^2 t}\,\sinh t \\ &= \tfrac{1}{2}\left(\cosh^2 t + \sinh^2 t\right) - \sinh^2 t = \tfrac{1}{2}\left(\cosh^2 t - \sinh^2 t\right) = \tfrac{1}{2}(1) = \tfrac{1}{2} \end{aligned}$$

Thus $A(t) = \frac{1}{2}t + C$, since $A'(t) = \frac{1}{2}$. To calculate C, we let $t = 0$. Thus, $A(0) = \frac{1}{2}\sinh 0 \cosh 0 - \int_1^{\cosh 0} \sqrt{x^2 - 1}\,dx = \frac{1}{2}(0) + C \quad \Rightarrow \quad C = 0$. Thus $A(t) = \frac{1}{2}t$.

79. If $ae^x + be^{-x} = \alpha\cosh(x + \beta)$ [or $\alpha\sinh(x + \beta)$], then $ae^x + be^{-x} = \frac{\alpha}{2}\left(e^{x+\beta} \pm e^{-x-\beta}\right) = \frac{\alpha}{2}\left(e^x e^\beta \pm e^{-x}e^{-\beta}\right) = \left(\frac{\alpha}{2}e^\beta\right)e^x \pm \left(\frac{\alpha}{2}e^{-\beta}\right)e^{-x}$. Comparing coefficients of e^x and e^{-x}, we have $a = \frac{\alpha}{2}e^\beta$ **(1)** and $b = \pm\frac{\alpha}{2}e^{-\beta}$ **(2)**. We need to find α and β. Dividing equation **(1)** by equation **(2)** gives us $\frac{a}{b} = \pm e^{2\beta} \quad \Rightarrow \quad (\star) \quad 2\beta = \ln\left(\pm\frac{a}{b}\right) \quad \Rightarrow \quad \beta = \frac{1}{2}\ln\left(\pm\frac{a}{b}\right)$. Solving equations **(1)** and **(2)** for e^β gives us $e^\beta = \dfrac{2a}{\alpha}$ and $e^\beta = \pm\dfrac{\alpha}{2b}$, so $\dfrac{2a}{\alpha} = \pm\dfrac{\alpha}{2b} \quad \Rightarrow \quad \alpha^2 = \pm 4ab \quad \Rightarrow \quad \alpha = 2\sqrt{\pm ab}$.

$(\star)$ If $\frac{a}{b} > 0$, we use the $+$ sign and obtain a cosh function, whereas if $\frac{a}{b} < 0$, we use the $-$ sign and obtain a sinh function.

In summary, if a and b have the same sign, we have $ae^x + be^{-x} = 2\sqrt{ab}\cosh\left(x + \frac{1}{2}\ln\frac{a}{b}\right)$, whereas, if a and b have the opposite sign, then $ae^x + be^{-x} = 2\sqrt{-ab}\,\sinh\left(x + \frac{1}{2}\ln\left(-\frac{a}{b}\right)\right)$.

6.8 Indeterminate Forms and l'Hospital's Rule

Note: The use of l'Hospital's Rule is indicated by an H above the equal sign: $\overset{\text{H}}{=}$

1. (a) $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)}$ is an indeterminate form of type $\dfrac{0}{0}$.

(b) $\displaystyle\lim_{x\to a} \frac{f(x)}{p(x)} = 0$ because the numerator approaches 0 while the denominator becomes large.

(c) $\displaystyle\lim_{x\to a} \frac{h(x)}{p(x)} = 0$ because the numerator approaches a finite number while the denominator becomes large.

(d) If $\lim_{x\to a} p(x) = \infty$ and $f(x) \to 0$ through positive values, then $\lim_{x\to a} \dfrac{p(x)}{f(x)} = \infty$. [For example, take $a = 0$, $p(x) = 1/x^2$, and $f(x) = x^2$.] If $f(x) \to 0$ through negative values, then $\lim_{x\to a} \dfrac{p(x)}{f(x)} = -\infty$. [For example, take $a = 0$, $p(x) = 1/x^2$, and $f(x) = -x^2$.] If $f(x) \to 0$ through both positive and negative values, then the limit might not exist. [For example, take $a = 0$, $p(x) = 1/x^2$, and $f(x) = x$.]

(e) $\lim_{x\to a} \dfrac{p(x)}{q(x)}$ is an indeterminate form of type $\dfrac{\infty}{\infty}$.

2. (a) $\lim_{x\to a} [f(x)p(x)]$ is an indeterminate form of type $0 \cdot \infty$.

(b) When x is near a, $p(x)$ is large and $h(x)$ is near 1, so $h(x)p(x)$ is large. Thus, $\lim_{x\to a} [h(x)p(x)] = \infty$.

(c) When x is near a, $p(x)$ and $q(x)$ are both large, so $p(x)q(x)$ is large. Thus, $\lim_{x\to a} [p(x)q(x)] = \infty$.

3. (a) When x is near a, $f(x)$ is near 0 and $p(x)$ is large, so $f(x) - p(x)$ is large negative. Thus, $\lim_{x\to a} [f(x) - p(x)] = -\infty$.

(b) $\lim_{x\to a} [p(x) - q(x)]$ is an indeterminate form of type $\infty - \infty$.

(c) When x is near a, $p(x)$ and $q(x)$ are both large, so $p(x) + q(x)$ is large. Thus, $\lim_{x\to a} [p(x) + q(x)] = \infty$.

4. (a) $\lim_{x\to a} [f(x)]^{g(x)}$ is an indeterminate form of type 0^0.

(b) If $y = [f(x)]^{p(x)}$, then $\ln y = p(x) \ln f(x)$. When x is near a, $p(x) \to \infty$ and $\ln f(x) \to -\infty$, so $\ln y \to -\infty$. Therefore, $\lim_{x\to a} [f(x)]^{p(x)} = \lim_{x\to a} y = \lim_{x\to a} e^{\ln y} = 0$, provided f^p is defined.

(c) $\lim_{x\to a} [h(x)]^{p(x)}$ is an indeterminate form of type 1^∞.

(d) $\lim_{x\to a} [p(x)]^{f(x)}$ is an indeterminate form of type ∞^0.

(e) If $y = [p(x)]^{q(x)}$, then $\ln y = q(x) \ln p(x)$. When x is near a, $q(x) \to \infty$ and $\ln p(x) \to \infty$, so $\ln y \to \infty$. Therefore, $\lim_{x\to a} [p(x)]^{q(x)} = \lim_{x\to a} y = \lim_{x\to a} e^{\ln y} = \infty$.

(f) $\lim_{x\to a} \sqrt[q(x)]{p(x)} = \lim_{x\to a} [p(x)]^{1/q(x)}$ is an indeterminate form of type ∞^0.

5. From the graphs of f and g, we see that $\lim_{x\to 2} f(x) = 0$ and $\lim_{x\to 2} g(x) = 0$, so l'Hospital's Rule applies.

$$\lim_{x\to 2} \frac{f(x)}{g(x)} \overset{\text{H}}{=} \lim_{x\to 2} \frac{f'(x)}{g'(x)} = \frac{\lim_{x\to 2} f'(x)}{\lim_{x\to 2} g'(x)} = \frac{f'(2)}{g'(2)} = \frac{1.8}{\frac{4}{5}} = \frac{9}{4}$$

6. From the graphs of f and g, we see that $\lim_{x\to 2} f(x) = 0$ and $\lim_{x\to 2} g(x) = 0$, so l'Hospital's Rule applies.

$$\lim_{x\to 2} \frac{f(x)}{g(x)} \overset{\text{H}}{=} \lim_{x\to 2} \frac{f'(x)}{g'(x)} = \frac{\lim_{x\to 2} f'(x)}{\lim_{x\to 2} g'(x)} = \frac{f'(2)}{g'(2)} = \frac{1.5}{-1} = -\frac{3}{2}$$

7. f and $g = e^x - 1$ are differentiable and $g' = e^x \neq 0$ on an open interval that contains 0. $\lim\limits_{x\to 0} f(x) = 0$ and $\lim\limits_{x\to 0} g(x) = 0$, so we have the indeterminate form $\frac{0}{0}$ and can apply l'Hospital's Rule.

$$\lim_{x\to 0} \frac{f(x)}{e^x - 1} \overset{\text{H}}{=} \lim_{x\to 0} \frac{f'(x)}{e^x} = \frac{1}{1} = 1$$

Note that $\lim\limits_{x\to 0} f'(x) = 1$ since the graph of f has the same slope as the line $y = x$ at $x = 0$.

8. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 3} \dfrac{x-3}{x^2-9} = \lim\limits_{x\to 3} \dfrac{x-3}{(x+3)(x-3)} = \lim\limits_{x\to 3} \dfrac{1}{x+3} = \dfrac{1}{3+3} = \dfrac{1}{6}$

Note: Alternatively, we could apply l'Hospital's Rule.

9. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 4} \dfrac{x^2-2x-8}{x-4} = \lim\limits_{x\to 4} \dfrac{(x-4)(x+2)}{x-4} = \lim\limits_{x\to 4}(x+2) = 4+2 = 6$

Note: Alternatively, we could apply l'Hospital's Rule.

10. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to -2} \dfrac{x^3+8}{x+2} \overset{\text{H}}{=} \lim\limits_{x\to -2} \dfrac{3x^2}{1} = 3(-2)^2 = 12$

Note: Alternatively, we could factor and simplify.

11. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 1} \dfrac{x^7-1}{x^3-1} \overset{\text{H}}{=} \lim\limits_{x\to 1} \dfrac{7x^6}{3x^2} = \dfrac{7}{3}$

Note: Alternatively, we could factor and simplify.

12. This limit has the form $\frac{0}{0}$.

$$\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} = \lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2} = \lim_{x\to 4} \frac{x-4}{(x-4)\left(\sqrt{x}+2\right)} = \lim_{x\to 4} \frac{1}{\sqrt{x}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{4}$$

Note: Alternatively, we could apply l'Hospital's Rule.

13. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to \pi/4} \dfrac{\sin x - \cos x}{\tan x - 1} \overset{\text{H}}{=} \lim\limits_{x\to \pi/4} \dfrac{\cos x + \sin x}{\sec^2 x} = \lim\limits_{x\to 4} \dfrac{\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}}{\left(\sqrt{2}\right)^2} = \dfrac{\sqrt{2}}{2}$

14. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 0} \dfrac{\tan 3x}{\sin 2x} \overset{\text{H}}{=} \lim\limits_{x\to 0} \dfrac{3\sec^2 3x}{2\cos 2x} = \dfrac{3(1)^2}{2(1)} = \dfrac{3}{2}$

15. This limit has the form $\frac{0}{0}$. $\lim\limits_{t\to 0} \dfrac{e^{2t}-1}{\sin t} \overset{\text{H}}{=} \lim\limits_{t\to 0} \dfrac{2e^{2t}}{\cos t} = \dfrac{2(1)}{1} = 2$

16. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 0} \dfrac{x^2}{1-\cos x} \overset{\text{H}}{=} \lim\limits_{x\to 0} \dfrac{2x}{\sin x} = \lim\limits_{x\to 0} \dfrac{2}{(\sin x)/x} = \dfrac{2}{1} = 2$

17. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 1} \dfrac{\sin(x-1)}{x^3+x-2} \overset{\text{H}}{=} \lim\limits_{x\to 1} \dfrac{\cos(x-1)}{3x^2+1} = \dfrac{\cos 0}{3(1)^2+1} = \dfrac{1}{4}$

18. The limit can be evaluated by substituting π for θ. $\lim\limits_{\theta\to \pi} \dfrac{1+\cos\theta}{1-\cos\theta} = \dfrac{1+(-1)}{1-(-1)} = \dfrac{0}{2} = 0$

19. This limit has the form $\frac{\infty}{\infty}$. $\displaystyle\lim_{x\to\infty}\frac{\sqrt{x}}{1+e^x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{\frac{1}{2\sqrt{x}}}{e^x} = \lim_{x\to\infty}\frac{1}{e^x\cdot 2\sqrt{x}} = 0$

20. This limit has the form $\frac{\infty}{\infty}$. $\displaystyle\lim_{x\to\infty}\frac{x+x^2}{1-2x^2} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{1+2x}{-4x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{2}{-4} = -\frac{1}{2}$.

A better method is to divide the numerator and the denominator by x^2: $\displaystyle\lim_{x\to\infty}\frac{x+x^2}{1-2x^2} = \lim_{x\to\infty}\frac{\frac{1}{x}+1}{\frac{1}{x^2}-2} = \frac{0+1}{0-2} = -\frac{1}{2}$.

21. $\displaystyle\lim_{x\to 0^+}[(\ln x)/x] = -\infty$ since $\ln x \to -\infty$ as $x \to 0^+$ and dividing by small values of x just increases the magnitude of the quotient $(\ln x)/x$. L'Hospital's Rule does not apply.

22. This limit has the form $\frac{\infty}{\infty}$. $\displaystyle\lim_{x\to\infty}\frac{\ln\sqrt{x}}{x^2} = \lim_{x\to\infty}\frac{\frac{1}{2}\ln x}{x^2} \overset{\text{H}}{=} \lim_{x\to\infty} = \frac{\frac{1}{2x}}{2x} = \lim_{x\to\infty}\frac{1}{4x^2} = 0$

23. This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{x\to 3}\frac{\ln(x/3)}{3-x} \overset{\text{H}}{=} \lim_{x\to 3}\frac{1/(x/3)\cdot(1/3)}{-1} = \lim_{x\to 3}\left(-\frac{1}{x}\right) = -\frac{1}{3}$

24. This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{t\to 0}\frac{8^t-5^t}{t} \overset{\text{H}}{=} \lim_{t\to 0}\frac{8^t\ln 8-5^t\ln 5}{1} = \ln 8-\ln 5 = \ln\frac{8}{5}$

25. This limit has the form $\frac{0}{0}$.

$$\begin{aligned}\lim_{x\to 0}\frac{\sqrt{1+2x}-\sqrt{1-4x}}{x} &\overset{\text{H}}{=} \lim_{x\to 0}\frac{\frac{1}{2}(1+2x)^{-1/2}\cdot 2-\frac{1}{2}(1-4x)^{-1/2}(-4)}{1}\\ &= \lim_{x\to 0}\left(\frac{1}{\sqrt{1+2x}}+\frac{2}{\sqrt{1-4x}}\right) = \frac{1}{\sqrt{1}}+\frac{2}{\sqrt{1}} = 3\end{aligned}$$

26. This limit has the form $\frac{\infty}{\infty}$.

$$\lim_{u\to\infty}\frac{e^{u/10}}{u^3} \overset{\text{H}}{=} \lim_{u\to\infty}\frac{e^{u/10}\cdot\frac{1}{10}}{3u^2} \overset{\text{H}}{=} \frac{1}{30}\lim_{u\to\infty}\frac{e^{u/10}\cdot\frac{1}{10}}{2u} \overset{\text{H}}{=} \frac{1}{600}\lim_{u\to\infty}\frac{e^{u/10}\cdot\frac{1}{10}}{1} = \frac{1}{6000}\lim_{u\to\infty}e^{u/10} = \infty$$

27. This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{x\to 0}\frac{e^x+e^{-x}-2}{e^x-x-1} \overset{\text{H}}{=} \lim_{x\to 0}\frac{e^x-e^{-x}}{e^x-1} \overset{\text{H}}{=} \lim_{x\to 0}\frac{e^x+e^{-x}}{e^x} = \frac{1+1}{1} = 2$

28. This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{x\to 0}\frac{\sinh x-x}{x^3} \overset{\text{H}}{=} \lim_{x\to 0}\frac{\cosh x-1}{3x^2} \overset{\text{H}}{=} \lim_{x\to 0}\frac{\sinh x}{6x} \overset{\text{H}}{=} \lim_{x\to 0}\frac{\cosh x}{6} = \frac{1}{6}$

29. This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{x\to 0}\frac{\tanh x}{\tan x} \overset{\text{H}}{=} \lim_{x\to 0}\frac{\operatorname{sech}^2 x}{\sec^2 x} = \frac{\operatorname{sech}^2 0}{\sec^2 0} = \frac{1}{1} = 1$

30. This limit has the form $\frac{0}{0}$.

$$\begin{aligned}\lim_{x\to 0}\frac{x-\sin x}{x-\tan x} &\overset{\text{H}}{=} \lim_{x\to 0}\frac{1-\cos x}{1-\sec^2 x} \overset{\text{H}}{=} \lim_{x\to 0}\frac{-(-\sin x)}{-2\sec x\,(\sec x\,\tan x)} = -\frac{1}{2}\lim_{x\to 0}\frac{\sin x\left(\dfrac{\cos x}{\sin x}\right)}{\sec^2 x}\\ &= -\tfrac{1}{2}\lim_{x\to 0}\cos^3 x = -\tfrac{1}{2}(1)^3 = -\tfrac{1}{2}\end{aligned}$$

Another method is to write the limit as $\displaystyle\lim_{x\to 0}\frac{1-\dfrac{\sin x}{x}}{1-\dfrac{\tan x}{x}}$.

31. This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{x\to 0}\frac{\sin^{-1}x}{x}\overset{\text{H}}{=}\lim_{x\to 0}\frac{1/\sqrt{1-x^2}}{1}=\lim_{x\to 0}\frac{1}{\sqrt{1-x^2}}=\frac{1}{1}=1$

32. This limit has the form $\frac{\infty}{\infty}$. $\displaystyle\lim_{x\to\infty}\frac{(\ln x)^2}{x}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{2(\ln x)(1/x)}{1}=2\lim_{x\to\infty}\frac{\ln x}{x}\overset{\text{H}}{=}2\lim_{x\to\infty}\frac{1/x}{1}=2(0)=0$

33. This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{x\to 0}\frac{x3^x}{3^x-1}\overset{\text{H}}{=}\lim_{x\to 0}\frac{x3^x\ln 3+3^x}{3^x\ln 3}=\lim_{x\to 0}\frac{3^x(x\ln 3+1)}{3^x\ln 3}=\lim_{x\to 0}\frac{x\ln 3+1}{\ln 3}=\frac{1}{\ln 3}$

34. This limit has the form $\frac{0}{0}$.

$$\lim_{x\to 0}\frac{e^x+e^{-x}-2\cos x}{x\sin x}\overset{\text{H}}{=}\lim_{x\to 0}\frac{e^x-e^{-x}+2\sin x}{x\cos x+\sin x}\overset{\text{H}}{=}\lim_{x\to 0}\frac{e^x+e^{-x}+2\cos x}{-x\sin x+\cos x+\cos x}=\frac{1+1+2}{0+1+1}=\frac{4}{2}=2$$

35. This limit can be evaluated by substituting 0 for x. $\displaystyle\lim_{x\to 0}\frac{\ln(1+x)}{\cos x+e^x-1}=\frac{\ln 1}{1+1-1}=\frac{0}{1}=0$

36. This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{x\to 1}\frac{x\sin(x-1)}{2x^2-x-1}\overset{\text{H}}{=}\lim_{x\to 1}\frac{x\cos(x-1)+\sin(x-1)}{4x-1}=\frac{\cos 0}{4-1}=\frac{1}{3}$

37. This limit has the form $\frac{0}{\infty}$, so l'Hospital's Rule doesn't apply. As $x\to 0^+$, $\arctan 2x\to 0$ and $\ln x\to -\infty$, so $\displaystyle\lim_{x\to 0^+}\frac{\arctan 2x}{\ln x}=0$.

38. This limit has the form $\frac{0}{0}$.

$$\begin{aligned}\lim_{x\to 0}\frac{x^2\sin x}{\sin x-x}&\overset{\text{H}}{=}\lim_{x\to 0}\frac{x^2\cos x+2x\sin x}{\cos x-1}\overset{\text{H}}{=}\lim_{x\to 0}\frac{-x^2\sin x+2x\cos x+2x\cos x+2\sin x}{-\sin x}\\&=\lim_{x\to 0}\frac{(2-x^2)\sin x+4x\cos x}{-\sin x}\\&\overset{\text{H}}{=}\lim_{x\to 0}\frac{(2-x^2)\cos x-2x\sin x-4x\sin x+4\cos x}{-\cos x}=\frac{2(1)-0-0+4(1)}{-1}=-6\end{aligned}$$

39. This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{x\to 1}\frac{x^a-1}{x^b-1}$ [for $b\neq 0$] $\displaystyle\overset{\text{H}}{=}\lim_{x\to 1}\frac{ax^{a-1}}{bx^{b-1}}=\frac{a(1)}{b(1)}=\frac{a}{b}$

40. This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{x\to\infty}\frac{e^{-x}}{(\pi/2)-\tan^{-1}x}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{-e^{-x}}{-\dfrac{1}{1+x^2}}=\lim_{x\to\infty}\frac{1+x^2}{e^x}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{2x}{e^x}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{2}{e^x}=0$

41. This limit has the form $\frac{0}{0}$.

$$\lim_{x\to 0}\frac{\cos x-1+\frac{1}{2}x^2}{x^4}\overset{\text{H}}{=}\lim_{x\to 0}\frac{-\sin x+x}{4x^3}\overset{\text{H}}{=}\lim_{x\to 0}\frac{-\cos x+1}{12x^2}\overset{\text{H}}{=}\lim_{x\to 0}\frac{\sin x}{24x}\overset{\text{H}}{=}\lim_{x\to 0}\frac{\cos x}{24}=\frac{1}{24}$$

42. This limit has the form $\frac{0}{0}$.

$$\begin{aligned}\lim_{x\to 0}\frac{x-\sin x}{x\sin(x^2)}&\overset{\text{H}}{=}\lim_{x\to 0}\frac{1-\cos x}{x\cos(x^2)\cdot 2x+\sin(x^2)}=\lim_{x\to 0}\frac{1-\cos x}{2x^2\cos(x^2)+\sin(x^2)}\\&\overset{\text{H}}{=}\lim_{x\to 0}\frac{\sin x}{2x^2\left[-\sin(x^2)\cdot 2x\right]+\cos(x^2)\cdot 4x+\cos(x^2)\cdot 2x}=\lim_{x\to 0}\frac{\sin x}{6x\cos(x^2)-4x^3\sin(x^2)}\\&\overset{\text{H}}{=}\lim_{x\to 0}\frac{\cos x}{6x\left[-\sin(x^2)\cdot 2x\right]+\cos(x^2)\cdot 6-4x^3\cos(x^2)\cdot 2x-\sin(x^2)\cdot 12x^2}=\frac{1}{0+1\cdot 6-0-0}=\frac{1}{6}\end{aligned}$$

43. This limit has the form $\infty \cdot 0$. We'll change it to the form $\frac{0}{0}$.

$$\lim_{x\to\infty} x\sin(\pi/x) = \lim_{x\to\infty} \frac{\sin(\pi/x)}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{\cos(\pi/x)(-\pi/x^2)}{-1/x^2} = \pi \lim_{x\to\infty} \cos(\pi/x) = \pi(1) = \pi$$

44. This limit has the form $\infty \cdot 0$. We'll change it to the form $\frac{\infty}{\infty}$.

$$\lim_{x\to\infty} \sqrt{x}\, e^{-x/2} = \lim_{x\to\infty} \frac{\sqrt{x}}{e^{x/2}} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{\frac{1}{2}x^{-1/2}}{\frac{1}{2}e^{x/2}} = \lim_{x\to\infty} \frac{1}{\sqrt{x}\, e^{x/2}} = 0$$

45. This limit has the form $0 \cdot \infty$. We'll change it to the form $\frac{0}{0}$. $\lim_{x\to 0} \sin 5x \csc 3x = \lim_{x\to 0} \frac{\sin 5x}{\sin 3x} \overset{\text{H}}{=} \lim_{x\to 0} \frac{5\cos 5x}{3\cos 3x} = \frac{5\cdot 1}{3\cdot 1} = \frac{5}{3}$

46. This limit has the form $(-\infty)\cdot 0$.

$$\lim_{x\to -\infty} x\ln\left(1 - \frac{1}{x}\right) = \lim_{x\to -\infty} \frac{\ln\left(1 - \frac{1}{x}\right)}{\frac{1}{x}} \overset{\text{H}}{=} \lim_{x\to -\infty} \frac{\frac{1}{1 - 1/x}\cdot \frac{1}{x^2}}{-\frac{1}{x^2}} = \lim_{x\to -\infty} \frac{-1}{1 - \frac{1}{x}} = \frac{-1}{1} = -1$$

47. This limit has the form $\infty \cdot 0$. $\lim_{x\to\infty} x^3 e^{-x^2} = \lim_{x\to\infty} \frac{x^3}{e^{x^2}} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{3x^2}{2xe^{x^2}} = \lim_{x\to\infty} \frac{3x}{2e^{x^2}} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{3}{4xe^{x^2}} = 0$

48. This limit has the form $\infty \cdot 0$.

$\lim_{x\to\infty} x^{3/2}\sin(1/x) = \lim_{x\to\infty} x^{1/2} \cdot \frac{\sin(1/x)}{1/x} = \lim_{t\to 0^+} \frac{1}{\sqrt{t}} \frac{\sin t}{t}$ [where $t = 1/x$] $= \infty$ since as $t \to 0^+$, $\frac{1}{\sqrt{t}} \to \infty$

and $\frac{\sin t}{t} \to 1$.

49. This limit has the form $0 \cdot (-\infty)$.

$$\lim_{x\to 1^+} \ln x\, \tan(\pi x/2) = \lim_{x\to 1^+} \frac{\ln x}{\cot(\pi x/2)} \overset{\text{H}}{=} \lim_{x\to 1^+} \frac{1/x}{(-\pi/2)\csc^2(\pi x/2)} = \frac{1}{(-\pi/2)(1)^2} = -\frac{2}{\pi}$$

50. This limit has the form $0 \cdot \infty$. $\lim_{x\to(\pi/2)^-} \cos x\, \sec 5x = \lim_{x\to(\pi/2)^-} \frac{\cos x}{\cos 5x} \overset{\text{H}}{=} \lim_{x\to(\pi/2)^-} \frac{-\sin x}{-5\sin 5x} = \frac{-1}{-5} = \frac{1}{5}$

51. This limit has the form $\infty - \infty$.

$$\lim_{x\to 1}\left(\frac{x}{x-1} - \frac{1}{\ln x}\right) = \lim_{x\to 1} \frac{x\ln x - (x-1)}{(x-1)\ln x} \overset{\text{H}}{=} \lim_{x\to 1} \frac{x(1/x) + \ln x - 1}{(x-1)(1/x) + \ln x} = \lim_{x\to 1} \frac{\ln x}{1 - (1/x) + \ln x}$$

$$\overset{\text{H}}{=} \lim_{x\to 1} \frac{1/x}{1/x^2 + 1/x} \cdot \frac{x^2}{x^2} = \lim_{x\to 1} \frac{x}{1+x} = \frac{1}{1+1} = \frac{1}{2}$$

52. This limit has the form $\infty - \infty$. $\lim_{x\to 0}(\csc x - \cot x) = \lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right) = \lim_{x\to 0} \frac{1 - \cos x}{\sin x} \overset{\text{H}}{=} \lim_{x\to 0} \frac{\sin x}{\cos x} = 0$

53. This limit has the form $\infty - \infty$.

$$\lim_{x\to 0^+}\left(\frac{1}{x} - \frac{1}{e^x - 1}\right) = \lim_{x\to 0^+} \frac{e^x - 1 - x}{x(e^x - 1)} \overset{\text{H}}{=} \lim_{x\to 0^+} \frac{e^x - 1}{xe^x + e^x - 1} \overset{\text{H}}{=} \lim_{x\to 0^+} \frac{e^x}{xe^x + e^x + e^x} = \frac{1}{0+1+1} = \frac{1}{2}$$

54. This limit has the form $\infty - \infty$.

$$\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\tan^{-1}x}\right)=\lim_{x\to 0^+}\frac{\tan^{-1}x-x}{x\tan^{-1}x}\overset{\text{H}}{=}\lim_{x\to 0^+}\frac{1/(1+x^2)-1}{x/(1+x^2)+\tan^{-1}x}=\lim_{x\to 0^+}\frac{1-(1+x^2)}{x+(1+x^2)\tan^{-1}x}$$

$$=\lim_{x\to 0^+}\frac{-x^2}{x+(1+x^2)\tan^{-1}x}\overset{\text{H}}{=}\lim_{x\to 0^+}\frac{-2x}{1+(1+x^2)(1/(1+x^2))+(\tan^{-1}x)(2x)}$$

$$=\lim_{x\to 0^+}\frac{-2x}{2+2x\tan^{-1}x}=\frac{0}{2+0}=0$$

55. This limit has the form $\infty - \infty$.

$$\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\tan x}\right)=\lim_{x\to 0^+}\frac{\tan x-x}{x\tan x}\overset{\text{H}}{=}\lim_{x\to 0^+}\frac{\sec^2x-1}{x\sec^2x+\tan x}\overset{\text{H}}{=}\lim_{x\to 0^+}\frac{2\sec x\cdot\sec x\tan x}{x\cdot 2\sec x\cdot\sec x\tan x+\sec^2x+\sec^2x}$$

$$=\frac{0}{0+1+1}=0$$

56. The limit has the form $\infty - \infty$ and we will change the form to a product by factoring out x.

$$\lim_{x\to\infty}(x-\ln x)=\lim_{x\to\infty}x\left(1-\frac{\ln x}{x}\right)=\infty \text{ since } \lim_{x\to\infty}\frac{\ln x}{x}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{1/x}{1}=0.$$

57. $y=x^{\sqrt{x}} \quad\Rightarrow\quad \ln y=\sqrt{x}\,\ln x$, so

$$\lim_{x\to 0^+}\ln y=\lim_{x\to 0^+}\sqrt{x}\,\ln x=\lim_{x\to 0^+}\frac{\ln x}{x^{-1/2}}\overset{\text{H}}{=}\lim_{x\to 0^+}\frac{1/x}{-\frac{1}{2}x^{-3/2}}=-2\lim_{x\to 0^+}\sqrt{x}=0\quad\Rightarrow$$

$$\lim_{x\to 0^+}x^{\sqrt{x}}=\lim_{x\to 0^+}e^{\ln y}=e^0=1.$$

58. $y=(\tan 2x)^x \quad\Rightarrow\quad \ln y=x\cdot\ln\tan 2x$, so

$$\lim_{x\to 0^+}\ln y=\lim_{x\to 0^+}x\cdot\ln\tan 2x=\lim_{x\to 0^+}\frac{\ln\tan 2x}{1/x}\overset{\text{H}}{=}\lim_{x\to 0^+}\frac{(1/\tan 2x)(2\sec^2 2x)}{-1/x^2}=\lim_{x\to 0^+}\frac{-2x^2\cos 2x}{\sin 2x\,\cos^2 2x}$$

$$=\lim_{x\to 0^+}\frac{2x}{\sin 2x}\cdot\lim_{x\to 0^+}\frac{-x}{\cos 2x}=1\cdot 0=0\quad\Rightarrow$$

$$\lim_{x\to 0^+}(\tan 2x)^x=\lim_{x\to 0^+}e^{\ln y}=e^0=1.$$

59. $y=(1-2x)^{1/x} \quad\Rightarrow\quad \ln y=\dfrac{1}{x}\ln(1-2x)$, so $\displaystyle\lim_{x\to 0}\ln y=\lim_{x\to 0}\frac{\ln(1-2x)}{x}\overset{\text{H}}{=}\lim_{x\to 0}\frac{-2/(1-2x)}{1}=-2\quad\Rightarrow$

$$\lim_{x\to 0}(1-2x)^{1/x}=\lim_{x\to 0}e^{\ln y}=e^{-2}.$$

60. $y=\left(1+\dfrac{a}{x}\right)^{bx} \quad\Rightarrow\quad \ln y=bx\ln\left(1+\dfrac{a}{x}\right)$, so

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}\frac{b\ln(1+a/x)}{1/x}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{b\left(\dfrac{1}{1+a/x}\right)\left(-\dfrac{a}{x^2}\right)}{-1/x^2}=\lim_{x\to\infty}\frac{ab}{1+a/x}=ab\quad\Rightarrow$$

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx}=\lim_{x\to\infty}e^{\ln y}=e^{ab}.$$

61. $y = x^{1/(1-x)} \quad\Rightarrow\quad \ln y = \dfrac{1}{1-x}\ln x$, so $\lim\limits_{x\to1^+}\ln y = \lim\limits_{x\to1^+}\dfrac{1}{1-x}\ln x = \lim\limits_{x\to1^+}\dfrac{\ln x}{1-x} \overset{\text{H}}{=} \lim\limits_{x\to1^+}\dfrac{1/x}{-1} = -1 \quad\Rightarrow$

$\lim\limits_{x\to1^+} x^{1/(1-x)} = \lim\limits_{x\to1^+} e^{\ln y} = e^{-1} = \dfrac{1}{e}.$

62. $y = (e^x + 10x)^{1/x} \quad\Rightarrow\quad \ln y = \dfrac{1}{x}\ln(e^x + 10x)$, so

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty}\frac{1}{x}\ln(e^x+10x) = \lim_{x\to\infty}\frac{\ln(e^x+10x)}{x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{\dfrac{1}{e^x+10x}\cdot(e^x+10)}{1}$$

$$= \lim_{x\to\infty}\frac{e^x+10}{e^x+10x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{e^x}{e^x+10} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{e^x}{e^x} = \lim_{x\to\infty}(1) = 1 \quad\Rightarrow$$

$\lim\limits_{x\to\infty}(e^x+10x)^{1/x} = \lim\limits_{x\to\infty} e^{\ln y} = e^1 = e$

63. $y = x^{1/x} \quad\Rightarrow\quad \ln y = (1/x)\ln x \quad\Rightarrow\quad \lim\limits_{x\to\infty}\ln y = \lim\limits_{x\to\infty}\dfrac{\ln x}{x} \overset{\text{H}}{=} \lim\limits_{x\to\infty}\dfrac{1/x}{1} = 0 \quad\Rightarrow$

$\lim\limits_{x\to\infty} x^{1/x} = \lim\limits_{x\to\infty} e^{\ln y} = e^0 = 1$

64. $y = x^{e^{-x}} \quad\Rightarrow\quad \ln y = e^{-x}\ln x \quad\Rightarrow\quad \lim\limits_{x\to\infty}\ln y = \lim\limits_{x\to\infty}\dfrac{\ln x}{e^x} \overset{\text{H}}{=} \lim\limits_{x\to\infty}\dfrac{1/x}{e^x} = \lim\limits_{x\to\infty}\dfrac{1}{xe^x} = 0 \quad\Rightarrow$

$\lim\limits_{x\to\infty} x^{e^{-x}} = \lim\limits_{x\to\infty} e^{\ln y} = e^0 = 1$

65. $y = (4x+1)^{\cot x} \quad\Rightarrow\quad \ln y = \cot x\,\ln(4x+1)$, so $\lim\limits_{x\to0^+}\ln y = \lim\limits_{x\to0^+}\dfrac{\ln(4x+1)}{\tan x} \overset{\text{H}}{=} \lim\limits_{x\to0^+}\dfrac{\dfrac{4}{4x+1}}{\sec^2 x} = 4 \quad\Rightarrow$

$\lim\limits_{x\to0^+}(4x+1)^{\cot x} = \lim\limits_{x\to0^+} e^{\ln y} = e^4.$

66. $y = (1-\cos x)^{\sin x} \quad\Rightarrow\quad \ln y = \sin x\ln(1-\cos x)$, so

$$\lim_{x\to0^+}\ln y = \lim_{x\to0^+}\sin x\ln(1-\cos x) = \lim_{x\to0^+}\frac{\ln(1-\cos x)}{\csc x} \overset{\text{H}}{=} \lim_{x\to0^+}\frac{\dfrac{1}{1-\cos x}\cdot\sin x}{-\csc x\cot x}$$

$$= -\lim_{x\to0^+}\frac{\sin x}{(1-\cos x)\csc x\cot x} = -\lim_{x\to0^+}\frac{\sin x}{\csc x\cot x - \cot^2 x}\cdot\left(\frac{\sin^2 x}{\sin^2 x}\right)$$

$$= -\lim_{x\to0^+}\frac{\sin^3 x}{\cos x - \cos^2 x} = -\lim_{x\to0^+}\frac{\sin^3 x}{(1-\cos x)\cos x}$$

$$\overset{\text{H}}{=} -\lim_{x\to0^+}\frac{3\sin^2 x\cos x}{(1-\cos x)(-\sin x)+\cos x\,(\sin x)} = -\lim_{x\to0^+}\frac{3\sin x\cos x}{(\cos x - 1)+\cos x} = -\frac{0}{0+1} = 0 \quad\Rightarrow$$

$\lim\limits_{x\to0^+}(1-\cos x)^{\sin x} = \lim\limits_{x\to0^+} e^{\ln y} = e^0 = 1$

67. $y = (1+\sin 3x)^{1/x} \quad\Rightarrow\quad \ln y = \dfrac{1}{x}\ln(1+\sin 3x) \quad\Rightarrow$

$$\lim_{x\to0^+}\ln y = \lim_{x\to0^+}\frac{\ln(1+\sin 3x)}{x} \overset{\text{H}}{=} \lim_{x\to0^+}\frac{[1/(1+\sin 3x)]\cdot 3\cos 3x}{1} = \lim_{x\to0^+}\frac{3\cos 3x}{1+\sin 3x} = \frac{3\cdot1}{1+0} = 3 \quad\Rightarrow$$

$\lim\limits_{x\to0^+}(1+\sin 3x)^{1/x} = \lim\limits_{x\to0^+} e^{\ln y} = e^3$

68. $y = (\cos x)^{1/x^2} \quad \Rightarrow \quad \ln y = \dfrac{1}{x^2}\ln\cos x$, so

$$\lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{1}{x^2}\ln\cos x = \lim_{x\to 0}\frac{\ln\cos x}{x^2} \overset{\text{H}}{=} \lim_{x\to 0}\frac{\dfrac{1}{\cos x}\cdot(-\sin x)}{2x} = \lim_{x\to 0}\frac{-\tan x}{2x} \overset{\text{H}}{=} \lim_{x\to 0}\frac{-\sec^2 x}{2} = -\frac{1}{2} \quad \Rightarrow$$

$$\lim_{x\to 0}(\cos x)^{1/x^2} = \lim_{x\to 0} e^{\ln y} = e^{-1/2} = \frac{1}{\sqrt{e}}$$

69. The given limit is $\displaystyle\lim_{x\to 0^+}\frac{x^x - 1}{\ln x + x - 1}$. Note that $y = x^x \quad \Rightarrow \quad \ln y = x\ln x$, so

$$\lim_{x\to 0^+}\ln y = \lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\ln x}{\dfrac{1}{x}} \overset{\text{H}}{=} \lim_{x\to 0^+}\frac{\dfrac{1}{x}}{-\dfrac{1}{x^2}} = \lim_{x\to 0^+}(-x) = 0 \quad \Rightarrow \quad \lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{\ln y} = e^0 = 1.$$

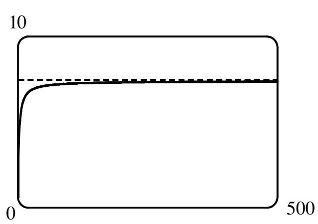
Therefore, the numerator of the given limit has limit $1 - 1 = 0$ as $x \to 0^+$. The denominator of the given limit $\to -\infty$ as $x \to 0^+$ since $\ln x \to -\infty$ as $x \to 0^+$. Thus, $\displaystyle\lim_{x\to 0^+}\frac{x^x - 1}{\ln x + x - 1} = 0$.

70. $y = \left(\dfrac{2x-3}{2x+5}\right)^{2x+1} \quad \Rightarrow \quad \ln y = (2x+1)\,\ln\left(\dfrac{2x-3}{2x+5}\right) \quad \Rightarrow$

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty}\frac{\ln(2x-3) - \ln(2x+5)}{1/(2x+1)} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{2/(2x-3) - 2/(2x+5)}{-2/(2x+1)^2} = \lim_{x\to\infty}\frac{-8(2x+1)^2}{(2x-3)(2x+5)}$$

$$= \lim_{x\to\infty}\frac{-8(2+1/x)^2}{(2-3/x)(2+5/x)} = -8 \quad \Rightarrow \quad \lim_{x\to\infty}\left(\frac{2x-3}{2x+5}\right)^{2x+1} = \lim_{x\to\infty} e^{\ln y} = e^{-8}$$

71.



From the graph, if $x = 500$, $y \approx 7.36$. The limit has the form 1^∞.

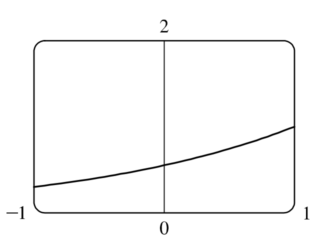
Now $y = \left(1 + \dfrac{2}{x}\right)^x \quad \Rightarrow \quad \ln y = x\ln\left(1 + \dfrac{2}{x}\right) \quad \Rightarrow$

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty}\frac{\ln(1+2/x)}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{\dfrac{1}{1+2/x}\left(-\dfrac{2}{x^2}\right)}{-1/x^2}$$

$$= 2\lim_{x\to\infty}\frac{1}{1+2/x} = 2(1) = 2 \quad \Rightarrow$$

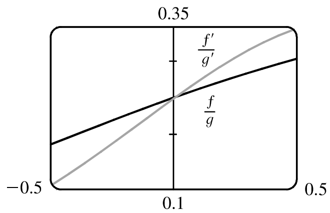
$$\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x = \lim_{x\to\infty} e^{\ln y} = e^2 \quad [\approx 7.39]$$

72.



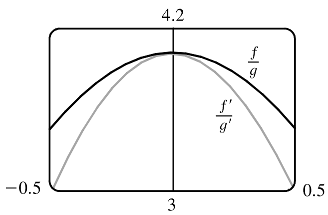
From the graph, as $x \to 0$, $y \approx 0.55$. The limit has the form $\frac{0}{0}$.

$$\lim_{x\to 0}\frac{5^x - 4^x}{3^x - 2^x} \overset{\text{H}}{=} \lim_{x\to 0}\frac{5^x\ln 5 - 4^x\ln 4}{3^x\ln 3 - 2^x\ln 2} = \frac{\ln 5 - \ln 4}{\ln 3 - \ln 2} = \frac{\ln\frac{5}{4}}{\ln\frac{3}{2}} \quad [\approx 0.55]$$

73. 

From the graph, it appears that $\lim\limits_{x\to 0}\dfrac{f(x)}{g(x)} = \lim\limits_{x\to 0}\dfrac{f'(x)}{g'(x)} = 0.25$.

We calculate $\lim\limits_{x\to 0}\dfrac{f(x)}{g(x)} = \lim\limits_{x\to 0}\dfrac{e^x - 1}{x^3 + 4x} \overset{\text{H}}{=} \lim\limits_{x\to 0}\dfrac{e^x}{3x^2 + 4} = \dfrac{1}{4}$.

74. 

From the graph, it appears that $\lim\limits_{x\to 0}\dfrac{f(x)}{g(x)} = \lim\limits_{x\to 0}\dfrac{f'(x)}{g'(x)} = 4$. We calculate

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{2x\sin x}{\sec x - 1} \overset{\text{H}}{=} \lim_{x\to 0}\frac{2(x\cos x + \sin x)}{\sec x\,\tan x}$$

$$\overset{\text{H}}{=} \lim_{x\to 0}\frac{2(-x\sin x + \cos x + \cos x)}{\sec x(\sec^2 x) + \tan x(\sec x\,\tan x)} = \frac{4}{1} = 4$$

75. $\lim\limits_{x\to\infty}\dfrac{e^x}{x^n} \overset{\text{H}}{=} \lim\limits_{x\to\infty}\dfrac{e^x}{nx^{n-1}} \overset{\text{H}}{=} \lim\limits_{x\to\infty}\dfrac{e^x}{n(n-1)x^{n-2}} \overset{\text{H}}{=} \cdots \overset{\text{H}}{=} \lim\limits_{x\to\infty}\dfrac{e^x}{n!} = \infty$

76. This limit has the form $\frac{\infty}{\infty}$. $\lim\limits_{x\to\infty}\dfrac{\ln x}{x^p} \overset{\text{H}}{=} \lim\limits_{x\to\infty}\dfrac{1/x}{px^{p-1}} = \lim\limits_{x\to\infty}\dfrac{1}{px^p} = 0$ since $p > 0$.

77. $\lim\limits_{x\to\infty}\dfrac{x}{\sqrt{x^2+1}} \overset{\text{H}}{=} \lim\limits_{x\to\infty}\dfrac{1}{\frac{1}{2}(x^2+1)^{-1/2}(2x)} = \lim\limits_{x\to\infty}\dfrac{\sqrt{x^2+1}}{x}$. Repeated applications of l'Hospital's Rule result in the original limit or the limit of the reciprocal of the function. Another method is to try dividing the numerator and denominator by x: $\lim\limits_{x\to\infty}\dfrac{x}{\sqrt{x^2+1}} = \lim\limits_{x\to\infty}\dfrac{x/x}{\sqrt{x^2/x^2 + 1/x^2}} = \lim\limits_{x\to\infty}\dfrac{1}{\sqrt{1 + 1/x^2}} = \dfrac{1}{1} = 1$

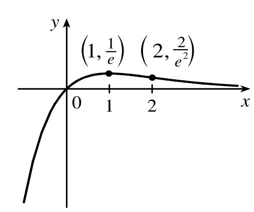
78. $\lim\limits_{x\to(\pi/2)^-}\dfrac{\sec x}{\tan x} \overset{\text{H}}{=} \lim\limits_{x\to(\pi/2)^-}\dfrac{\sec x\,\tan x}{\sec^2 x} = \lim\limits_{x\to(\pi/2)^-}\dfrac{\tan x}{\sec x}$. Repeated applications of l'Hospital's Rule result in the original limit or the limit of the reciprocal of the function. Another method is to simplify first:

$$\lim_{x\to(\pi/2)^-}\frac{\sec x}{\tan x} = \lim_{x\to(\pi/2)^-}\frac{1/\cos x}{\sin x/\cos x} = \lim_{x\to(\pi/2)^-}\frac{1}{\sin x} = \frac{1}{1} = 1$$

79. $y = f(x) = xe^{-x}$ **A.** $D = \mathbb{R}$ **B.** Intercepts are 0 **C.** No symmetry

D. $\lim\limits_{x\to\infty} xe^{-x} = \lim\limits_{x\to\infty}\dfrac{x}{e^x} \overset{\text{H}}{=} \lim\limits_{x\to\infty}\dfrac{1}{e^x} = 0$, so $y = 0$ is a HA. $\lim\limits_{x\to-\infty} xe^{-x} = -\infty$

E. $f'(x) = e^{-x} - xe^{-x} = e^{-x}(1-x) > 0 \Leftrightarrow x < 1$, so f is increasing on $(-\infty, 1)$ and decreasing on $(1, \infty)$. **F.** Absolute and local maximum value $f(1) = 1/e$. **G.** $f''(x) = e^{-x}(x-2) > 0 \Leftrightarrow x > 2$, so f is CU on $(2, \infty)$ and CD on $(-\infty, 2)$. IP at $(2, 2/e^2)$

H. 

80. $y = f(x) = \dfrac{\ln x}{x^2}$ **A.** $D = (0, \infty)$ **B.** y-intercept: none; x-intercept: $f(x) = 0 \Leftrightarrow \ln x = 0 \Leftrightarrow x = 1$

C. No symmetry **D.** $\lim\limits_{x\to 0^+} f(x) = -\infty$, so $x = 0$ is a VA; $\lim\limits_{x\to\infty}\dfrac{\ln x}{x^2} \overset{\text{H}}{=} \lim\limits_{x\to\infty}\dfrac{1/x}{2x} = 0$, so $y = 0$ is a HA.

[continued]

E. $f'(x) = \dfrac{x^2(1/x) - (\ln x)(2x)}{(x^2)^2} = \dfrac{x(1 - 2\ln x)}{x^4} = \dfrac{1 - 2\ln x}{x^3}$. $f'(x) > 0 \;\Leftrightarrow\; 1 - 2\ln x > 0 \;\Leftrightarrow\; \ln x < \frac{1}{2} \;\Rightarrow$

$0 < x < e^{1/2}$ and $f'(x) < 0 \;\Rightarrow\; x > e^{1/2}$, so f is increasing on $\left(0, \sqrt{e}\right)$ and decreasing on $\left(\sqrt{e}, \infty\right)$.

F. Local maximum value $f(e^{1/2}) = \dfrac{1/2}{e} = \dfrac{1}{2e}$

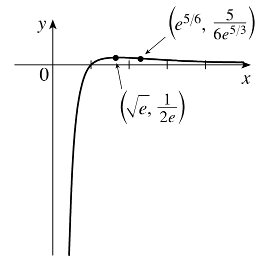
G. $f''(x) = \dfrac{x^3(-2/x) - (1 - 2\ln x)(3x^2)}{(x^3)^2}$

$= \dfrac{x^2[-2 - 3(1 - 2\ln x)]}{x^6} = \dfrac{-5 + 6\ln x}{x^4}$

$f''(x) > 0 \;\Leftrightarrow\; -5 + 6\ln x > 0 \;\Leftrightarrow\; \ln x > \frac{5}{6} \;\Rightarrow\; x > e^{5/6}$ [f is CU]

and $f''(x) < 0 \;\Leftrightarrow\; 0 < x < e^{5/6}$ [f is CD]. IP at $(e^{5/6}, 5/(6e^{5/3}))$

H.



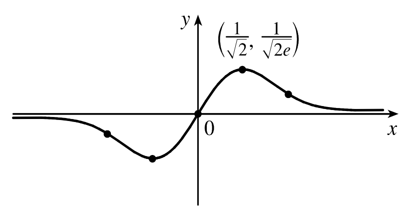
81. $y = f(x) = xe^{-x^2}$ **A.** $D = \mathbb{R}$ **B.** Intercepts are 0 **C.** $f(-x) = -f(x)$, so the curve is symmetric about the origin. **D.** $\displaystyle\lim_{x\to\pm\infty} xe^{-x^2} = \lim_{x\to\pm\infty} \frac{x}{e^{x^2}} \overset{\text{H}}{=} \lim_{x\to\pm\infty} \frac{1}{2xe^{x^2}} = 0$, so $y = 0$ is a HA.

E. $f'(x) = e^{-x^2} - 2x^2e^{-x^2} = e^{-x^2}(1 - 2x^2) > 0 \;\Leftrightarrow\; x^2 < \frac{1}{2} \;\Leftrightarrow\; |x| < \frac{1}{\sqrt{2}}$, so f is increasing on $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and decreasing on $\left(-\infty, -\frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}}, \infty\right)$. **F.** Local maximum value $f\left(\frac{1}{\sqrt{2}}\right) = 1/\sqrt{2e}$, local minimum value $f\left(-\frac{1}{\sqrt{2}}\right) = -1/\sqrt{2e}$ **G.** $f''(x) = -2xe^{-x^2}(1 - 2x^2) - 4xe^{-x^2} = 2xe^{-x^2}(2x^2 - 3) > 0 \;\Leftrightarrow$

$x > \sqrt{\frac{3}{2}}$ or $-\sqrt{\frac{3}{2}} < x < 0$, so f is CU on $\left(\sqrt{\frac{3}{2}}, \infty\right)$ and $\left(-\sqrt{\frac{3}{2}}, 0\right)$ and CD on $\left(-\infty, -\sqrt{\frac{3}{2}}\right)$ and $\left(0, \sqrt{\frac{3}{2}}\right)$.

IP are $(0, 0)$ and $\left(\pm\sqrt{\frac{3}{2}}, \pm\sqrt{\frac{3}{2}}\, e^{-3/2}\right)$.

H.



82. $y = f(x) = e^x/x$ **A.** $D = \{x \mid x \neq 0\}$ **B.** No intercept **C.** No symmetry

D. $\displaystyle\lim_{x\to\infty} \frac{e^x}{x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{e^x}{1} = \infty, = \lim_{x\to-\infty} \frac{e^x}{x} = 0$, so $y = 0$ is a HA. $\displaystyle\lim_{x\to 0^+} \frac{e^x}{x} = \infty$, $\displaystyle\lim_{x\to 0^-} \frac{e^x}{x} = -\infty$, so $x = 0$ is a VA.

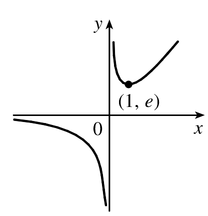
E. $f'(x) = \dfrac{xe^x - e^x}{x^2} > 0 \;\Leftrightarrow\; (x - 1)e^x > 0 \;\Leftrightarrow\; x > 1$, so f is increasing on $(1, \infty)$, and decreasing on $(-\infty, 0)$ and $(0, 1)$. **F.** $f(1) = e$ is a local minimum value.

G. $f''(x) = \dfrac{x^2(xe^x) - 2x(xe^x - e^x)}{x^4} = \dfrac{e^x(x^2 - 2x + 2)}{x^3} > 0 \;\Leftrightarrow\; x > 0$

since $x^2 - 2x + 2 > 0$ for all x. So f is CU on $(0, \infty)$ and CD on $(-\infty, 0)$.

No IP

H.



83. $y = f(x) = \dfrac{1}{x} + \ln x$ **A.** $D = (0, \infty)$ [same as $\ln x$] **B.** No y-intercept; no x-intercept [$1/x > |\ln x|$ on $(0, 1)$, and $1/x$ and $\ln x$ are both positive on $(1, \infty)$] **C.** No symmetry **D.** $\displaystyle\lim_{x\to 0^+} f(x) = \infty$, so $x = 0$ is a VA.

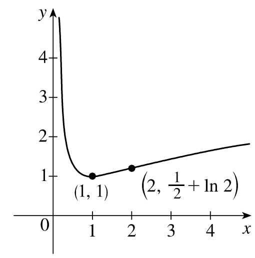
[continued]

E. $f'(x) = -\dfrac{1}{x^2} + \dfrac{1}{x} = \dfrac{x-1}{x^2}$. $f'(x) > 0$ for $x > 1$, so f is increasing on $(1, \infty)$ and f is decreasing on $(0, 1)$.

F. Local minimum value $f(1) = 1$ **G.** $f''(x) = \dfrac{2}{x^3} - \dfrac{1}{x^2} = \dfrac{2-x}{x^3}$.

$f''(x) > 0$ for $0 < x < 2$, so f is CU on $(0, 2)$, and f is CD on $(2, \infty)$. IP at $\left(2, \frac{1}{2} + \ln 2\right)$

H.



84. $y = f(x) = (x^2 - 3)e^{-x}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = -3$; x-intercepts: $f(x) = 0 \Leftrightarrow x^2 - 3 = 0 \Leftrightarrow x = \pm\sqrt{3}$ **C.** No symmetry **D.** $\lim\limits_{x\to\infty} (x^2 - 3)e^{-x} = \lim\limits_{x\to\infty} \dfrac{x^2 - 3}{e^x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{2x}{e^x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{2}{e^x} = 0$, so $y = 0$ is a HA.

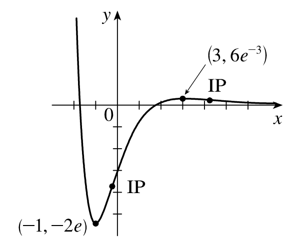
E. $f'(x) = (x^2 - 3)(-e^{-x}) + e^{-x}(2x) = -e^{-x}[(x^2 - 3) - 2x] = -e^{-x}(x - 3)(x + 1)$. $f'(x) > 0 \Leftrightarrow -1 < x < 3$ and $f'(x) < 0 \Leftrightarrow x < -1$ or $x > 3$, so f is increasing on $(-1, 3)$ and decreasing on $(-\infty, -1)$ and $(3, \infty)$.

F. Local maximum value $f(3) = 6e^{-3}$; local minimum value $f(-1) = -2e$

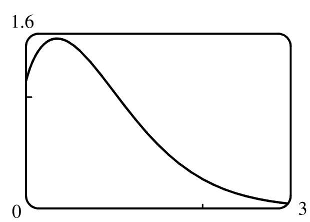
G. $f''(x) = (-e^{-x})(2x - 2) + (x^2 - 2x - 3)(e^{-x}) = e^{-x}[-(2x - 2) + (x^2 - 2x - 3)] = e^{-x}(x^2 - 4x - 1)$.

$f''(x) = 0 \Leftrightarrow x = \frac{4 \pm \sqrt{20}}{2} = 2 \pm \sqrt{5}$, so $f''(x) > 0 \Leftrightarrow x < 2 - \sqrt{5}$ or $x > 2 + \sqrt{5}$ and $f''(x) < 0 \Leftrightarrow 2 - \sqrt{5} < x < 2 + \sqrt{5}$, so f is CU on $\left(-\infty, 2 - \sqrt{5}\right)$ and $\left(2 + \sqrt{5}, \infty\right)$ and f is CD on $\left(2 - \sqrt{5}, 2 + \sqrt{5}\right)$. IP at $\left(2 - \sqrt{5}, f\left(2 - \sqrt{5}\right)\right) \approx (-0.24, -3.73)$ and $\left(2 + \sqrt{5}, f\left(2 + \sqrt{5}\right)\right) \approx (4.24, 0.22)$

H.



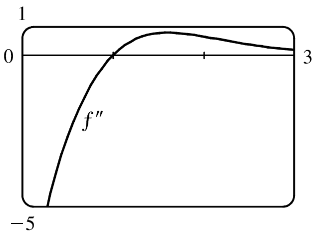
85. (a) $f(x) = x^{-x}$



(b) $y = f(x) = x^{-x}$. We note that $\ln f(x) = \ln x^{-x} = -x \ln x = -\dfrac{\ln x}{1/x}$, so

$$\lim_{x\to 0^+} \ln f(x) \overset{\text{H}}{=} \lim_{x\to 0^+} -\frac{1/x}{-x^{-2}} = \lim_{x\to 0^+} x = 0. \text{ Thus } \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{\ln f(x)} = e^0 = 1.$$

(c) From the graph, it appears that there is a local and absolute maximum of about $f(0.37) \approx 1.44$. To find the exact value, we differentiate: $f(x) = x^{-x} = e^{-x \ln x} \Rightarrow f'(x) = e^{-x \ln x}\left[-x\left(\dfrac{1}{x}\right) + \ln x(-1)\right] = -x^{-x}(1 + \ln x)$. This is 0 only when $1 + \ln x = 0 \Leftrightarrow x = e^{-1}$. Also $f'(x)$ changes from positive to negative at e^{-1}. So the maximum value is $f(1/e) = (1/e)^{-1/e} = e^{1/e}$.

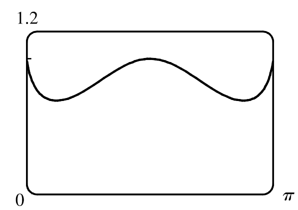
(d) 

We differentiate again to get

$$f''(x) = -x^{-x}(1/x) + (1+\ln x)^2(x^{-x}) = x^{-x}[(1+\ln x)^2 - 1/x]$$

From the graph of $f''(x)$, it seems that $f''(x)$ changes from negative to positive at $x = 1$, so we estimate that f has an IP at $x = 1$.

86. (a) $f(x) = (\sin x)^{\sin x}$ is continuous where $\sin x > 0$, that is, on intervals of the form $(2n\pi, (2n+1)\pi)$, so we have graphed f on $(0, \pi)$.



(b) $y = (\sin x)^{\sin x} \quad \Rightarrow \quad \ln y = \sin x \, \ln \sin x$, so

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \sin x \, \ln \sin x = \lim_{x\to 0^+} \frac{\ln \sin x}{\csc x} \overset{\text{H}}{=} \lim_{x\to 0^+} \frac{\cot x}{-\csc x \, \cot x}$$

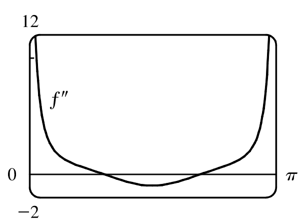
$$= \lim_{x\to 0^+} (-\sin x) = 0 \quad \Rightarrow \quad \lim_{x\to 0^+} y = e^0 = 1.$$

(c) It appears that we have a local maximum at $(1.57, 1)$ and local minima at $(0.38, 0.69)$ and $(2.76, 0.69)$.

$$y = (\sin x)^{\sin x} \quad \Rightarrow \quad \ln y = \sin x \, \ln \sin x \quad \Rightarrow \quad \frac{y'}{y} = (\sin x)\left(\frac{\cos x}{\sin x}\right) + (\ln \sin x)\, \cos x = \cos x\,(1 + \ln \sin x) \quad \Rightarrow$$

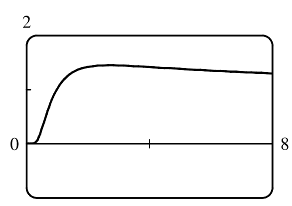
$y' = (\sin x)^{\sin x}(\cos x)(1 + \ln \sin x)$. $y' = 0 \quad \Rightarrow \quad \cos x = 0$ or $\ln \sin x = -1 \quad \Rightarrow \quad x_2 = \frac{\pi}{2}$ or $\sin x = e^{-1}$.

On $(0, \pi)$, $\sin x = e^{-1} \quad \Rightarrow \quad x_1 = \sin^{-1}(e^{-1})$ and $x_3 = \pi - \sin^{-1}(e^{-1})$. Approximating these points gives us $(x_1, f(x_1)) \approx (0.3767, 0.6922)$, $(x_2, f(x_2)) \approx (1.5708, 1)$, and $(x_3, f(x_3)) \approx (2.7649, 0.6922)$. The approximations confirm our estimates.

(d) 

From the graph, we see that $f''(x) = 0$ at $x \approx 0.94$ and $x \approx 2.20$. Since f'' changes sign at these values, they are x-coordinates of inflection points.

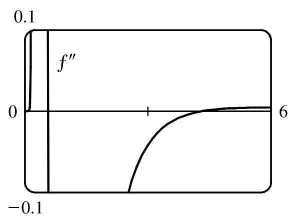
87. (a) $f(x) = x^{1/x}$



(b) Recall that $a^b = e^{b \ln a}$. $\lim_{x\to 0^+} x^{1/x} = \lim_{x\to 0^+} e^{(1/x)\ln x}$. As $x \to 0^+$, $\dfrac{\ln x}{x} \to -\infty$, so $x^{1/x} = e^{(1/x)\ln x} \to 0$. This indicates that there is a hole at $(0, 0)$. As $x \to \infty$, we have the indeterminate form ∞^0. $\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} e^{(1/x)\ln x}$, but $\lim_{x\to\infty} \dfrac{\ln x}{x} \overset{\text{H}}{=} \lim_{x\to\infty} \dfrac{1/x}{1} = 0$, so $\lim_{x\to\infty} x^{1/x} = e^0 = 1$. This indicates that $y = 1$ is a HA.

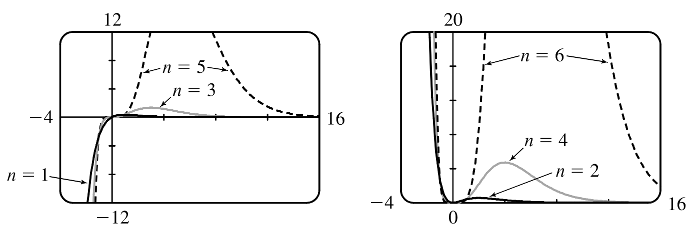
(c) Estimated maximum: $(2.72, 1.45)$. No estimated minimum. We use logarithmic differentiation to find any critical numbers. $y = x^{1/x} \;\Rightarrow\; \ln y = \dfrac{1}{x}\,\text{lnx} \;\Rightarrow\; \dfrac{y'}{y} = \dfrac{1}{x} \cdot \dfrac{1}{x} + (\ln x)\left(-\dfrac{1}{x^2}\right) \;\Rightarrow\; y' = x^{1/x}\left(\dfrac{1-\ln x}{x^2}\right) = 0 \;\Rightarrow$ $\ln x = 1 \;\Rightarrow\; x = e$. For $0 < x < e$, $y' > 0$ and for $x > e$, $y' < 0$, so $f(e) = e^{1/e}$ is a local maximum value. This point is approximately $(2.7183, 1.4447)$, which agrees with our estimate.

(d)



From the graph, we see that $f''(x) = 0$ at $x \approx 0.58$ and $x \approx 4.37$. Since f'' changes sign at these values, they are x-coordinates of inflection points.

88.

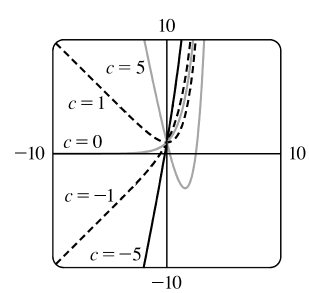


The first figure shows representative examples of $f(x) = x^n e^{-x}$ with n odd. n is even in the second figure. All curves pass through the origin and approach $y = 0$ as $x \to \infty$.

$f'(x) = \dfrac{x^n(n-x)}{xe^x} = 0 \;\Leftrightarrow\; x = n$ or $x = 0$ (the latter for $n > 1$). At $x = 0$, we have a local minimum for n even. At $x = n$, we have a local maximum for all n. As n increases, $(n, f(n))$ gets farther away from the origin.

$f''(x) = \dfrac{x^n\left(x^2 - 2nx + n^2 - n\right)}{x^2e^x} = 0 \;\Leftrightarrow\; x = n \pm \sqrt{n}$ or $x = 0$ (the latter for $n > 2$). As n increases, the IP move farther away from the origin—they are symmetric about the line $x = n$.

89. $f(x) = e^x - cx \;\Rightarrow\; f'(x) = e^x - c = 0 \;\Leftrightarrow\; e^x = c \;\Leftrightarrow\; x = \ln c$, $c > 0$. $f''(x) = e^x > 0$, so f is CU on $(-\infty, \infty)$. $\displaystyle\lim_{x\to\infty}(e^x - cx) = \lim_{x\to\infty}\left[x\left(\frac{e^x}{x} - c\right)\right] = L_1$. Now $\displaystyle\lim_{x\to\infty}\frac{e^x}{x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{e^x}{1} = \infty$, so $L_1 = \infty$, regardless of the value of c. For $L = \displaystyle\lim_{x\to-\infty}(e^x - cx)$, $e^x \to 0$, so L is determined by $-cx$. If $c > 0$, $-cx \to \infty$, and $L = \infty$. If $c < 0$, $-cx \to -\infty$, and $L = -\infty$. Thus, f has an absolute minimum for $c > 0$. As c increases, the minimum points $(\ln c, c - c\ln c)$ get farther away from the origin.



90. (a) $\displaystyle\lim_{t\to\infty} v = \lim_{t\to\infty}\frac{mg}{c}\left(1 - e^{-ct/m}\right) = \frac{mg}{c}\lim_{t\to\infty}\left(1 - e^{-ct/m}\right) = \frac{mg}{c}(1 - 0)$ [because $-ct/m \to -\infty$ as $t \to \infty$]

$= \dfrac{mg}{c}$, which is the speed the object approaches as time goes on, the so-called limiting velocity.

(b) $\lim\limits_{c\to 0^+} v = \lim\limits_{c\to 0^+} \dfrac{mg}{c}(1-e^{-ct/m}) = mg \lim\limits_{c\to 0^+} \dfrac{1-e^{-ct/m}}{c}$ [form is $\frac{0}{0}$]

$$\overset{\text{H}}{=} mg \lim_{c\to 0^+} \frac{(-e^{-ct/m})\cdot(-t/m)}{1} = \frac{mgt}{m} \lim_{c\to 0^+} e^{-ct/m} = gt(1) = gt$$

The velocity of a falling object in a vacuum is directly proportional to the amount of time it falls.

91. First we will find $\lim\limits_{n\to\infty}\left(1+\dfrac{r}{n}\right)^{nt}$, which is of the form 1^∞. $y = \left(1+\dfrac{r}{n}\right)^{nt} \Rightarrow \ln y = nt\ln\left(1+\dfrac{r}{n}\right)$, so

$\lim\limits_{n\to\infty} \ln y = \lim\limits_{n\to\infty} nt\ln\left(1+\dfrac{r}{n}\right) = t\lim\limits_{n\to\infty} \dfrac{\ln(1+r/n)}{1/n} \overset{\text{H}}{=} t\lim\limits_{n\to\infty} \dfrac{(-r/n^2)}{(1+r/n)(-1/n^2)} = t\lim\limits_{n\to\infty}\dfrac{r}{1+i/n} = tr \Rightarrow$

$\lim\limits_{n\to\infty} y = e^{rt}$. Thus, as $n\to\infty$, $A = A_0\left(1+\dfrac{r}{n}\right)^{nt} \to A_0e^{rt}$.

92. (a) $r=3$, $\rho=0.05 \Rightarrow P = \dfrac{1-10^{-\rho r^2}}{\rho r^2\ln 10} = \dfrac{1-10^{-0.45}}{0.45\ln 10} \approx 0.62$, or about 62%.

(b) $r=2$, $\rho=0.05 \Rightarrow P = \dfrac{1-10^{-0.2}}{0.2\ln 10} \approx 0.80$, or about 80%.

Yes, it makes sense. Since measured brightness decreases with light entering farther from the center of the pupil, a smaller pupil radius means that the average brightness measurements are higher than when including light entering at larger radii.

(c) $\lim\limits_{r\to 0^+} P = \lim\limits_{r\to 0^+} \dfrac{1-10^{-\rho r^2}}{\rho r^2\ln 10} \overset{\text{H}}{=} \lim\limits_{r\to 0^+} \dfrac{-10^{-\rho r^2}(\ln 10)(-2\rho r)}{2\rho r(\ln 10)} = \lim\limits_{r\to 0^+}\dfrac{1}{10^{\rho r^2}} = 1$, or 100%.

We might expect that 100% of the brightness is sensed at the very center of the pupil, so a limit of 1 would make sense in this context if the radius r could approach 0. This result isn't physically possible because there are limitations on how small the pupil can shrink.

93. (a) $\lim\limits_{t\to\infty} P(t) = \lim\limits_{t\to\infty} \dfrac{M}{1+Ae^{-kt}} = \dfrac{M}{1+A\cdot 0} = M$

It is to be expected that a population that is growing will eventually reach the maximum population size that can be supported.

(b) $\lim\limits_{M\to\infty} P(t) = \lim\limits_{M\to\infty} \dfrac{M}{1+\dfrac{M-P_0}{P_0}e^{-kt}} = \lim\limits_{M\to\infty} \dfrac{M}{1+\left(\dfrac{M}{P_0}-1\right)e^{-kt}} \overset{\text{H}}{=} \lim\limits_{M\to\infty} \dfrac{1}{\dfrac{1}{P_0}e^{-kt}} = P_0e^{kt}$

P_0e^{kt} is an exponential function.

94. (a) $\lim\limits_{R\to r^+} v = \lim\limits_{R\to r^+}\left[-c\left(\dfrac{r}{R}\right)^2\ln\left(\dfrac{r}{R}\right)\right] = -cr^2\lim\limits_{R\to r^+}\left[\left(\dfrac{1}{R}\right)^2\ln\left(\dfrac{r}{R}\right)\right] = -cr^2\cdot\dfrac{1}{r^2}\cdot\ln 1 = -c\cdot 0 = 0$

As the insulation of a metal cable becomes thinner, the velocity of an electrical impulse in the cable approaches zero.

(b) $\lim\limits_{r\to 0^+} v = \lim\limits_{r\to 0^+}\left[-c\left(\dfrac{r}{R}\right)^2\ln\left(\dfrac{r}{R}\right)\right] = -\dfrac{c}{R^2}\lim\limits_{r\to 0^+}\left[r^2\ln\left(\dfrac{r}{R}\right)\right]$ [form is $0\cdot\infty$]

$$= -\frac{c}{R^2}\lim_{r\to 0^+}\frac{\ln\left(\dfrac{r}{R}\right)}{\dfrac{1}{r^2}} \quad [\text{form is } \infty/\infty] \quad \overset{\text{H}}{=} -\frac{c}{R^2}\lim_{r\to 0^+}\frac{\dfrac{R}{r}\cdot\dfrac{1}{R}}{\dfrac{-2}{r^3}} = -\frac{c}{R^2}\lim_{r\to 0^+}\left(-\frac{r^2}{2}\right) = 0$$

As the radius of the metal cable approaches zero, the velocity of an electrical impulse in the cable approaches zero.

95. $\displaystyle \lim_{x\to 0}\frac{1}{x^2}\int_0^x \frac{2t}{\sqrt{t^3+1}}\,dt = \lim_{x\to 0}\frac{\int_0^x \frac{2t}{\sqrt{t^3+1}}\,dt}{x^2}\ \left[\text{form } \tfrac{0}{0}\right] \overset{\text{H}}{=} \lim_{x\to 0}\frac{\frac{2x}{\sqrt{x^3+1}}}{2x} = \lim_{x\to 0}\frac{1}{\sqrt{x^3+1}} = 1$

96. $\displaystyle \lim_{x\to\infty}\frac{1}{x^2}\int_0^x \ln(1+e^t)\,dt = \lim_{x\to\infty}\frac{\int_0^x \ln(1+e^t)\,dt}{x^2}\ \left[\text{form } \tfrac{\infty}{\infty}\right] \overset{\text{H}}{=} \lim_{x\to\infty}\frac{\ln(1+e^x)}{2x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{\frac{e^x}{1+e^x}}{2}$

$$= \lim_{x\to\infty}\frac{e^x}{2(1+e^x)} = \lim_{x\to\infty}\frac{e^x/e^x}{2(1+e^x)/e^x} = \lim_{x\to\infty}\frac{1}{2\left(\frac{1}{e^x}+1\right)} = \frac{1}{2(0+1)} = \frac{1}{2}$$

97. Both numerator and denominator approach 0 as $x \to 0$, so we use l'Hospital's Rule (and FTC1):

$$\lim_{x\to 0}\frac{S(x)}{x^3} = \lim_{x\to 0}\frac{\int_0^x \sin(\pi t^2/2)dt}{x^3} \overset{\text{H}}{=} \lim_{x\to 0}\frac{\sin(\pi x^2/2)}{3x^2} \overset{\text{H}}{=} \lim_{x\to 0}\frac{\pi x\cos(\pi x^2/2)}{6x} = \frac{\pi}{6}\cdot\cos 0 = \frac{\pi}{6}$$

98. Both numerator and denominator approach 0 as $a \to 0$, so we use l'Hospital's Rule. (Note that we are differentiating with respect to a, since that is the quantity which is changing.) We also use the Fundamental Theorem of Calculus, Part 1:

$$\lim_{a\to 0} T(x,t) = \lim_{a\to 0}\frac{C\int_0^a e^{-(x-u)^2/(4kt)}\,du}{a\sqrt{4\pi kt}} \overset{\text{H}}{=} \lim_{a\to 0}\frac{Ce^{-(x-a)^2/(4kt)}}{\sqrt{4\pi kt}} = \frac{Ce^{-x^2/(4kt)}}{\sqrt{4\pi kt}}$$

99. We see that both numerator and denominator approach 0, so we can use l'Hospital's Rule:

$$\begin{aligned}\lim_{x\to a}\frac{\sqrt{2a^3x-x^4}-a\sqrt[3]{aax}}{a-\sqrt[4]{ax^3}} &\overset{\text{H}}{=} \lim_{x\to a}\frac{\frac{1}{2}(2a^3x-x^4)^{-1/2}(2a^3-4x^3)-a\left(\frac{1}{3}\right)(aax)^{-2/3}a^2}{-\frac{1}{4}(ax^3)^{-3/4}(3ax^2)}\\ &= \frac{\frac{1}{2}(2a^3a-a^4)^{-1/2}(2a^3-4a^3)-\frac{1}{3}a^3(a^2a)^{-2/3}}{-\frac{1}{4}(aa^3)^{-3/4}(3aa^2)}\\ &= \frac{(a^4)^{-1/2}(-a^3)-\frac{1}{3}a^3(a^3)^{-2/3}}{-\frac{3}{4}a^3(a^4)^{-3/4}} = \frac{-a-\frac{1}{3}a}{-\frac{3}{4}} = \tfrac{4}{3}\left(\tfrac{4}{3}a\right) = \tfrac{16}{9}a\end{aligned}$$

100. Let the radius of the circle be r. We see that $A(\theta)$ is the area of the whole figure (a sector of the circle with radius 1), minus the area of $\triangle OPR$. But the area of the sector of the circle is $\frac{1}{2}r^2\theta$ (see Reference Page 1), and the area of the triangle is $\frac{1}{2}r\,|PQ| = \frac{1}{2}r(r\sin\theta) = \frac{1}{2}r^2\sin\theta$. So we have $A(\theta) = \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta = \frac{1}{2}r^2(\theta-\sin\theta)$. Now by elementary trigonometry, $B(\theta) = \frac{1}{2}\,|QR|\,|PQ| = \frac{1}{2}(r-|OQ|)\,|PQ| = \frac{1}{2}(r-r\cos\theta)(r\sin\theta) = \frac{1}{2}r^2(1-\cos\theta)\sin\theta$.

So the limit we want is

$$\begin{aligned}\lim_{\theta\to 0^+}\frac{A(\theta)}{B(\theta)} &= \lim_{\theta\to 0^+}\frac{\frac{1}{2}r^2(\theta-\sin\theta)}{\frac{1}{2}r^2(1-\cos\theta)\sin\theta} \overset{\text{H}}{=} \lim_{\theta\to 0^+}\frac{1-\cos\theta}{(1-\cos\theta)\cos\theta+\sin\theta\,(\sin\theta)}\\ &= \lim_{\theta\to 0^+}\frac{1-\cos\theta}{\cos\theta-\cos^2\theta+\sin^2\theta} \overset{\text{H}}{=} \lim_{\theta\to 0^+}\frac{\sin\theta}{-\sin\theta-2\cos\theta\,(-\sin\theta)+2\sin\theta\,(\cos\theta)}\\ &= \lim_{\theta\to 0^+}\frac{\sin\theta}{-\sin\theta+4\sin\theta\,\cos\theta} = \lim_{\theta\to 0^+}\frac{1}{-1+4\cos\theta} = \frac{1}{-1+4\cos 0} = \frac{1}{3}\end{aligned}$$

101. The limit, $L = \lim\limits_{x\to\infty}\left[x - x^2\ln\left(\dfrac{1+x}{x}\right)\right] = \lim\limits_{x\to\infty}\left[x - x^2\ln\left(\dfrac{1}{x}+1\right)\right]$. Let $t = 1/x$, so as $x\to\infty$, $t\to 0^+$.

$$L = \lim_{t\to 0^+}\left[\frac{1}{t} - \frac{1}{t^2}\ln(t+1)\right] = \lim_{t\to 0^+}\frac{t-\ln(t+1)}{t^2} \overset{\text{H}}{=} \lim_{t\to 0^+}\frac{1-\dfrac{1}{t+1}}{2t} = \lim_{t\to 0^+}\frac{t/(t+1)}{2t} = \lim_{t\to 0^+}\frac{1}{2\,(t+1)} = \frac{1}{2}$$

Note: Starting the solution by factoring out x or x^2 leads to a more complicated solution.

102. $y = [f(x)]^{g(x)} \Rightarrow \ln y = g(x)\ln f(x)$. Since f is a positive function, $\ln f(x)$ is defined. Now $\lim\limits_{x\to a}\ln y = \lim\limits_{x\to a} g(x)\ln f(x) = -\infty$ since $\lim\limits_{x\to a} g(x) = \infty$ and $\lim\limits_{x\to a} f(x) = 0 \Rightarrow \lim\limits_{x\to a}\ln f(x) = -\infty$. Thus, if $t = \ln y$, $\lim\limits_{x\to a} y = \lim\limits_{t\to -\infty} e^t = 0$. Note that the limit, $\lim\limits_{x\to a} g(x)\ln f(x)$, is *not* of the form $\infty\cdot 0$.

103. (a) We look for functions f and g whose individual limits are ∞ as $x\to 0$, but whose quotient has a limit of 7 as $x\to 0$.

One such pair of functions is $f(x) = \dfrac{7}{x^2}$ and $g(x) = \dfrac{1}{x^2}$. We have $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} g(x) = \infty$, and

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{7/x^2}{1/x^2} = \lim_{x\to 0} 7 = 7.$$

(b) We look for functions f and g whose individual limits are ∞ as $x\to 0$, but whose difference has a limit of 7 as $x\to 0$.

One such pair of functions is $f(x) = \dfrac{1}{x^2} + 7$ and $g(x) = \dfrac{1}{x^2}$. We have $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} g(x) = \infty$, and

$$\lim_{x\to 0}[f(x) - g(x)] = \lim_{x\to 0}\left[\left(\frac{1}{x^2}+7\right) - \frac{1}{x^2}\right] = \lim_{x\to 0} 7 = 7.$$

104. $L = \lim\limits_{x\to 0}\left(\dfrac{\sin 2x}{x^3} + a + \dfrac{b}{x^2}\right) = \lim\limits_{x\to 0}\dfrac{\sin 2x + ax^3 + bx}{x^3} \overset{\text{H}}{=} \lim\limits_{x\to 0}\dfrac{2\cos 2x + 3ax^2 + b}{3x^2}$. As $x\to 0$, $3x^2\to 0$, and $(2\cos 2x + 3ax^2 + b)\to b+2$, so the last limit exists only if $b+2=0$, that is, $b=-2$. Thus,

$$\lim_{x\to 0}\frac{2\cos 2x + 3ax^2 - 2}{3x^2} \overset{\text{H}}{=} \lim_{x\to 0}\frac{-4\sin 2x + 6ax}{6x} \overset{\text{H}}{=} \lim_{x\to 0}\frac{-8\cos 2x + 6a}{6} = \frac{6a-8}{6},$$ which is equal to 0 if and only if $a = \frac{4}{3}$. Hence, $L = 0$ if and only if $b = -2$ and $a = \frac{4}{3}$.

105. (a) We show that $\lim\limits_{x\to 0}\dfrac{f(x)}{x^n} = 0$ for every integer $n \geq 0$. Let $y = \dfrac{1}{x^2}$. Then

$$\lim_{x\to 0}\frac{f(x)}{x^{2n}} = \lim_{x\to 0}\frac{e^{-1/x^2}}{(x^2)^n} = \lim_{y\to\infty}\frac{y^n}{e^y} \overset{\text{H}}{=} \lim_{y\to\infty}\frac{ny^{n-1}}{e^y} \overset{\text{H}}{=} \cdots \overset{\text{H}}{=} \lim_{y\to\infty}\frac{n!}{e^y} = 0 \Rightarrow$$

$$\lim_{x\to 0}\frac{f(x)}{x^n} = \lim_{x\to 0} x^n\frac{f(x)}{x^{2n}} = \lim_{x\to 0} x^n \lim_{x\to 0}\frac{f(x)}{x^{2n}} = 0.$$ Thus, $f'(0) = \lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0}\dfrac{f(x)}{x} = 0$.

(b) Using the Chain Rule and the Quotient Rule we see that $f^{(n)}(x)$ exists for $x \neq 0$. In fact, we prove by induction that for each $n \geq 0$, there is a polynomial p_n and a non-negative integer k_n with $f^{(n)}(x) = p_n(x)f(x)/x^{k_n}$ for $x \neq 0$. This is true for $n = 0$; suppose it is true for the nth derivative. Then $f'(x) = f(x)(2/x^3)$, so

$$\begin{aligned} f^{(n+1)}(x) &= \left[x^{k_n}\left[p_n'(x)\,f(x) + p_n(x)\,f'(x)\right] - k_n x^{k_n-1} p_n(x)\,f(x)\right] x^{-2k_n} \\ &= \left[x^{k_n} p_n'(x) + p_n(x)(2/x^3) - k_n x^{k_n-1} p_n(x)\right] f(x) x^{-2k_n} \\ &= \left[x^{k_n+3} p_n'(x) + 2p_n(x) - k_n x^{k_n+2}\, p_n(x)\right] f(x) x^{-(2k_n+3)} \end{aligned}$$

which has the desired form.

Now we show by induction that $f^{(n)}(0) = 0$ for all n. By part (a), $f'(0) = 0$. Suppose that $f^{(n)}(0) = 0$. Then

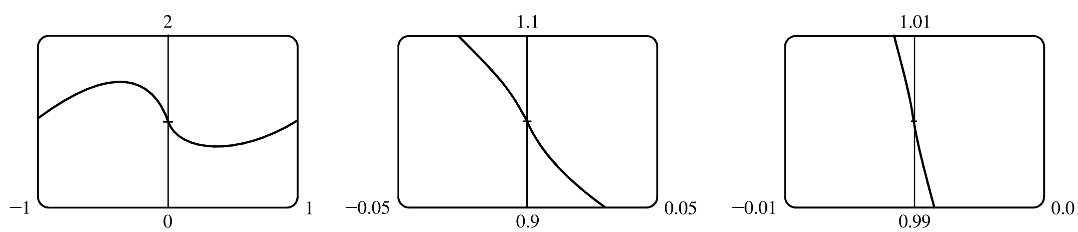
$$\begin{aligned} f^{(n+1)}(0) &= \lim_{x\to 0} \frac{f^{(n)}(x) - f^{(n)}(0)}{x - 0} = \lim_{x\to 0} \frac{f^{(n)}(x)}{x} = \lim_{x\to 0} \frac{p_n(x)\,f(x)/x^{k_n}}{x} = \lim_{x\to 0} \frac{p_n(x)\,f(x)}{x^{k_n+1}} \\ &= \lim_{x\to 0} p_n(x) \lim_{x\to 0} \frac{f(x)}{x^{k_n+1}} = p_n(0)\cdot 0 = 0 \qquad \text{[by part (a)]} \end{aligned}$$

106. (a) For f to be continuous, we need $\lim\limits_{x\to 0} f(x) = f(0) = 1$. We note that for $x \neq 0$, $\ln f(x) = \ln|x|^x = x\ln|x|$.

So $\lim\limits_{x\to 0} \ln f(x) = \lim\limits_{x\to 0} x\ln|x| = \lim\limits_{x\to 0} \dfrac{\ln|x|}{1/x} \overset{\text{H}}{=} \lim\limits_{x\to 0} \dfrac{1/x}{-1/x^2} = 0$. Therefore, $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} e^{\ln f(x)} = e^0 = 1$.

So f is continuous at 0.

(b) From the graphs, it appears that f is differentiable at 0.



(c) To find f', we use logarithmic differentiation: $\ln f(x) = x\ln|x| \;\Rightarrow\; \dfrac{f'(x)}{f(x)} = x\left(\dfrac{1}{x}\right) + \ln|x| \;\Rightarrow$

$f'(x) = f(x)(1 + \ln|x|) = |x|^x(1 + \ln|x|)$, $x \neq 0$. Now $f'(x) \to -\infty$ as $x \to 0$ [since $|x|^x \to 1$ and $(1 + \ln|x|) \to -\infty$], so the curve has a vertical tangent at $(0, 1)$ and is therefore not differentiable there. The fact cannot be seen in the graphs in part (b) because $\ln|x| \to -\infty$ very slowly as $x \to 0$.

6 Review

TRUE-FALSE QUIZ

1. True. If f is one-to-one, with domain $\mathbb{R}$, then $f^{-1}(f(6)) = 6$ by the first cancellation equation [see (6.1.4)].

2. False. By Theorem 6.1.7, $(f^{-1})'(6) = \dfrac{1}{f'\left(f^{-1}(6)\right)}$, *not* $\dfrac{1}{f'(6)}$ unless $f^{-1}(6) = 6$.

3. False. For example, $\cos\frac{\pi}{2} = \cos\left(-\frac{\pi}{2}\right)$, so $\cos x$ is not 1-1.

4. False. It is true that $\tan\frac{3\pi}{4} = -1$, but since the range of $\tan^{-1}$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we must have $\tan^{-1}(-1) = -\frac{\pi}{4}$.

5. True. The function $y = \ln x$ is increasing on $(0, \infty)$, so if $0 < a < b$, then $\ln a < \ln b$.

6. True. Since $a^x = e^{x \ln a}$, $\pi^{\sqrt{5}} = e^{\sqrt{5}\ln \pi}$.

7. True. We can divide by e^x since $e^x \neq 0$ for every x.

8. False. For example, $\ln(1+1) = \ln 2$, but $\ln 1 + \ln 1 = 0$. In fact $\ln a + \ln b = \ln(ab)$.

9. False. Let $x = e$. Then $(\ln x)^6 = (\ln e)^6 = 1^6 = 1$, but $6\ln x = 6\ln e = 6 \cdot 1 = 6 \neq 1 = (\ln x)^6$. What *is* true, however, is that $\ln(x^6) = 6\ln x$ for $x > 0$.

10. False. $\dfrac{d}{dx}(10^x) = 10^x \ln 10$, which is not equal to $x10^{x-1}$.

11. False. $\ln 10$ is a constant, so its derivative, $\dfrac{d}{dx}(\ln 10)$, is 0, not $\frac{1}{10}$.

12. True. $y = e^{3x} \quad\Rightarrow\quad \ln y = 3x \quad\Rightarrow\quad x = \frac{1}{3}\ln y \quad\Rightarrow\quad$ the inverse function is $y = \frac{1}{3}\ln x$.

13. False. The "-1" is not an exponent; it is an indication of an inverse function.

14. False. For example, $\tan^{-1} 20$ is defined; $\sin^{-1} 20$ and $\cos^{-1} 20$ are not.

15. True. See Figure 2 in Section 6.7.

16. True. $-\displaystyle\int_1^{10} \frac{dx}{x} = -\left[\ln|x|\right]_1^{10} = -\ln 10 + \ln 1 = \ln 10^{-1} + 0 = \ln\tfrac{1}{10}$

17. True. $\displaystyle\int_2^{16} \frac{dx}{x} = \left[\ln|x|\right]_2^{16} = \ln 16 - \ln 2 = \ln\frac{16}{2} = \ln 8 = \ln 2^3 = 3\ln 2$

18. False. L'Hospital's Rule does not apply since $\displaystyle\lim_{x\to\pi^-} \frac{\tan x}{1 - \cos x} = \frac{0}{2} = 0$.

19. False. Let $f(x) = 1 + \dfrac{1}{x}$ and $g(x) = x$. Then $\displaystyle\lim_{x\to\infty} f(x) = 1$ and $\displaystyle\lim_{x\to\infty} g(x) = \infty$, but

$\displaystyle\lim_{x\to\infty} [f(x)]^{g(x)} = \lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = e$, not 1.

EXERCISES

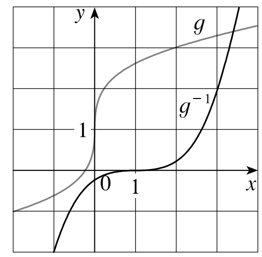
1. No. f is not 1-1 because the graph of f fails the Horizontal Line Test.

2. (a) g is one-to-one because it passes the Horizontal Line Test.

(b) When $y = 2$, $x \approx 0.2$. So $g^{-1}(2) \approx 0.2$.

(c) The range of g is $[-1, 3.5]$, which is the same as the domain of g^{-1}.

(d) We reflect the graph of g through the line $y = x$ to obtain the graph of g^{-1}.



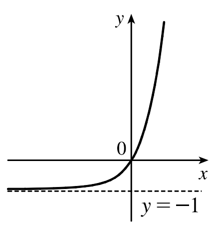
3. (a) $f^{-1}(3) = 7$ since $f(7) = 3$.

(b) $(f^{-1})'(3) = \dfrac{1}{f'(f^{-1}(3))} = \dfrac{1}{f'(7)} = \dfrac{1}{8}$

4. We write $y = \dfrac{2x+3}{1-5x}$ and solve for x: $y(1-5x) = 2x+3 \quad\Rightarrow\quad y - 5xy = 2x + 3 \quad\Rightarrow\quad y - 3 = 2x + 5xy \quad\Rightarrow$

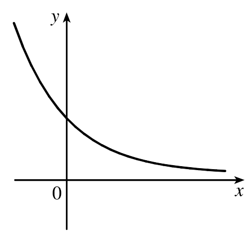
$y - 3 = x(2+5y) \quad\Rightarrow\quad x = \dfrac{y-3}{2+5y}$. Interchanging x and y gives $y = \dfrac{x-3}{2+5x}$, so $f^{-1}(x) = \dfrac{x-3}{2+5x}$.

5.

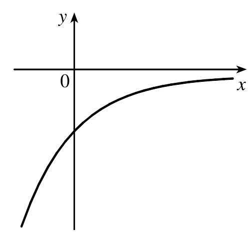


$y = 5^x - 1$

6.

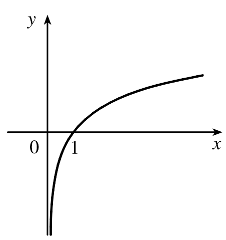


$y = e^{-x}$

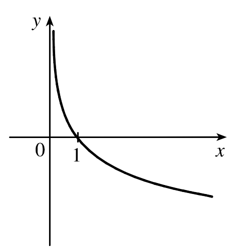


$y = -e^{-x}$

7. Reflect the graph of $y = \ln x$ about the x-axis to obtain the graph of $y = -\ln x$.

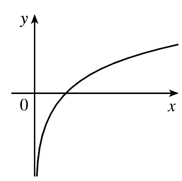


$y = \ln x$

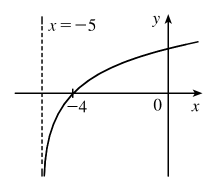


$y = -\ln x$

8. $y = \ln(x+5)$. Start with the graph of $y = \ln x$ and shift 5 units to the left.

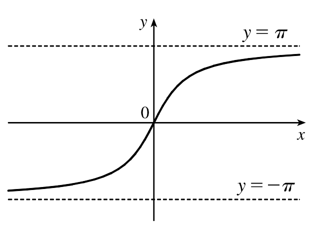


$y = \ln x$



$y = \ln(x+5)$

9.



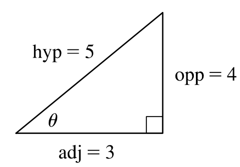
$y = 2\arctan x$

10. We have seen that if $b > 1$, then $b^x > x^b$ for sufficiently large x. (See Exercise 6.2.20.) In general, we could show that $\lim_{x\to\infty}\left(b^x/x^b\right) = \infty$ by using l'Hospital's Rule repeatedly. Also, $\log_b x$ increases much more slowly than either x^b or b^x. [Compare the graph of $\log_b x$ with those of x^b and b^x, or use l'Hospital's Rule to show that $\lim_{x\to\infty}\left[(\log_b x)/x^b\right] = 0$.] So for large x, $\log_b x < x^b < b^x$.

11. (a) $e^{2\ln 5} = e^{\ln 5^2} = 5^2 = 25$

(b) $\log_6 4 + \log_6 54 = \log_6(4 \cdot 54) = \log_6 216 = \log_6 6^3 = 3$

(c) Let $\theta = \arcsin\frac{4}{5}$, so $\sin\theta = \frac{4}{5}$. Draw a right triangle with angle θ as shown in the figure. By the Pythagorean Theorem, the adjacent side has length 3, and $\tan\left(\arcsin\dfrac{4}{5}\right) = \tan\theta = \dfrac{\text{opp}}{\text{adj}} = \dfrac{4}{3}$.



12. (a) $\ln\dfrac{1}{e^3} = \ln e^{-3} = -3$

(b) $\sin(\tan^{-1} 1) = \sin\dfrac{\pi}{4} = \dfrac{\sqrt{2}}{2}$

(c) $10^{-3\log 4} = 10^{\log 4^{-3}} = 4^{-3} = \dfrac{1}{4^3} = \dfrac{1}{64}$

13. $e^{2x} = 3 \;\Rightarrow\; \ln(e^{2x}) = \ln 3 \;\Rightarrow\; 2x = \ln 3 \;\Rightarrow\; x = \frac{1}{2}\ln 3 \approx 0.549$

14. $\ln x^2 = 5 \;\Rightarrow\; e^{\ln x^2} = e^5 \;\Rightarrow\; x^2 = e^5 \;\Rightarrow\; x = \pm\sqrt{e^5} \approx \pm 12.182$

15. $e^{e^x} = 10 \;\Rightarrow\; \ln\left(e^{e^x}\right) = \ln 10 \;\Rightarrow\; e^x = \ln 10 \;\Rightarrow\; \ln e^x = \ln(\ln 10) \;\Rightarrow\; x = \ln(\ln 10) \approx 0.834$

16. $\cos^{-1}x = 2 \;\Rightarrow\; \cos(\cos^{-1}x) = \cos 2 \;\Rightarrow\; x = \cos 2 \approx -0.416$

17. $\tan^{-1}(3x^2) = \dfrac{\pi}{4} \;\Rightarrow\; \tan(\tan^{-1}(3x^2)) = \tan\dfrac{\pi}{4} \;\Rightarrow\; 3x^2 = 1 \;\Rightarrow\; x^2 = \dfrac{1}{3} \;\Rightarrow\; x = \pm\dfrac{1}{\sqrt{3}} \approx \pm 0.577$

18. $\ln(1 + e^{-x}) = 3 \;\Rightarrow\; 1 + e^{-x} = e^3 \;\Rightarrow\; e^{-x} = e^3 - 1 \;\Rightarrow\; \ln e^{-x} = \ln(e^3 - 1) \;\Rightarrow\; -x = \ln(e^3 - 1) \;\Rightarrow\; x = -\ln(e^3 - 1)$

19. $\ln x - 1 = \ln(5 + x) - 4 \;\Rightarrow\; \ln x - \ln(5 + x) = -4 + 1 \;\Rightarrow\; \ln\dfrac{x}{5 + x} = -3 \;\Rightarrow\; e^{\ln(x/(5+x))} = e^{-3} \;\Rightarrow\; \dfrac{x}{5 + x} = e^{-3} \;\Rightarrow\; x = 5e^{-3} + xe^{-3} \;\Rightarrow\; x - xe^{-3} = 5e^{-3} \;\Rightarrow\; x(1 - e^{-3}) = 5e^{-3} \;\Rightarrow\; x = \dfrac{5e^{-3}}{1 - e^{-3}}$

or, multiplying by $\dfrac{e^3}{e^3}$, we have $x = \dfrac{5}{e^3 - 1} \approx 0.262$.

20. $\log_5(c^x) = d \;\Rightarrow\; x\log_5 c = d \;\Rightarrow\; x = \dfrac{d}{\log_5 c}.$

Or: $\log_5(c^x) = d \;\Rightarrow\; 5^d = c^x \;\Rightarrow\; \ln 5^d = \ln c^x \;\Rightarrow\; d\ln 5 = x\ln c \;\Rightarrow\; x = \dfrac{d\ln 5}{\ln c}.$

21. $f(t) = t^2\ln t \;\Rightarrow\; f'(t) = t^2\cdot\dfrac{1}{t} + (\ln t)(2t) = t + 2t\ln t$ or $t(1 + 2\ln t)$

22. $g(t) = \dfrac{e^t}{1+e^t} \;\Rightarrow\; g'(t) = \dfrac{(1+e^t)e^t - e^t(e^t)}{(1+e^t)^2} = \dfrac{e^t}{(1+e^t)^2}$

23. $h(\theta) = e^{\tan 2\theta} \;\Rightarrow\; h'(\theta) = e^{\tan 2\theta}\cdot\sec^2 2\theta\cdot 2 = 2\sec^2(2\theta)\,e^{\tan 2\theta}$

24. $h(u) = 10^{\sqrt{u}} \;\Rightarrow\; h'(u) = 10^{\sqrt{u}}\cdot\ln 10\cdot\dfrac{1}{2\sqrt{u}} = \dfrac{(\ln 10)10^{\sqrt{u}}}{2\sqrt{u}}$

25. $y = \ln|\sec 5x + \tan 5x| \;\Rightarrow$

$$y' = \frac{1}{\sec 5x + \tan 5x}(\sec 5x\,\tan 5x\cdot 5 + \sec^2 5x\cdot 5) = \frac{5\sec 5x\,(\tan 5x + \sec 5x)}{\sec 5x + \tan 5x} = 5\sec 5x$$

26. $y = e^{mx}\cos nx \;\Rightarrow$

$$y' = e^{mx}(\cos nx)' + \cos nx\,(e^{mx})' = e^{mx}(-\sin nx\cdot n) + \cos nx\,(e^{mx}\cdot m) = e^{mx}(m\cos nx - n\sin nx)$$

27. $y = \ln(\sec^2 x) = 2\ln|\sec x| \;\Rightarrow\; y' = (2/\sec x)(\sec x\tan x) = 2\tan x$

28. $y = \sqrt{t\ln(t^4)} \;\Rightarrow$

$$y' = \frac{1}{2}[t\ln(t^4)]^{-1/2}\,\frac{d}{dt}\,[t\ln(t^4)] = \frac{1}{2\sqrt{t\ln(t^4)}}\cdot\left[1\cdot\ln(t^4) + t\cdot\frac{1}{t^4}\cdot 4t^3\right] = \frac{1}{2\sqrt{t\ln(t^4)}}\cdot[\ln(t^4) + 4] = \frac{\ln(t^4)+4}{2\sqrt{t\ln(t^4)}}$$

Or: Since y is only defined for $t > 0$, we can write $y = \sqrt{t\cdot 4\ln t} = 2\sqrt{t\ln t}$. Then

$y' = 2\cdot\dfrac{1}{2\sqrt{t\ln t}}\cdot\left(1\cdot\ln t + t\cdot\dfrac{1}{t}\right) = \dfrac{\ln t + 1}{\sqrt{t\ln t}}$. This agrees with our first answer since

$$\frac{\ln(t^4)+4}{2\sqrt{t\ln(t^4)}} = \frac{4\ln t + 4}{2\sqrt{t\cdot 4\ln t}} = \frac{4(\ln t + 1)}{2\cdot 2\sqrt{t\ln t}} = \frac{\ln t + 1}{\sqrt{t\ln t}}.$$

29. $y = \dfrac{e^{1/x}}{x^2} \;\Rightarrow\; y' = \dfrac{x^2(e^{1/x})' - e^{1/x}\,(x^2)'}{(x^2)^2} = \dfrac{x^2(e^{1/x})(-1/x^2) - e^{1/x}(2x)}{x^4} = \dfrac{-e^{1/x}(1+2x)}{x^4}$

30. $y = (\arcsin 2x)^2 \;\Rightarrow\; y' = 2(\arcsin 2x)\cdot(\arcsin 2x)' = 2\arcsin 2x\cdot\dfrac{1}{\sqrt{1-(2x)^2}}\cdot 2 = \dfrac{4\arcsin 2x}{\sqrt{1-4x^2}}$

31. $y = 5\arctan\dfrac{1}{x} \;\Rightarrow\; y' = 5\cdot\dfrac{1}{1+\left(\dfrac{1}{x}\right)^2}\cdot\dfrac{d}{dx}\left(\dfrac{1}{x}\right) = \dfrac{5}{1+\dfrac{1}{x^2}}\left(-\dfrac{1}{x^2}\right) = -\dfrac{5}{x^2+1}$

32. $y = x\sec^{-1}x \;\Rightarrow\; y' = x\cdot\dfrac{1}{x\sqrt{x^2-1}} + \left(\sec^{-1}x\right)\cdot 1 = \dfrac{1}{\sqrt{x^2-1}} + \sec^{-1}x$

33. $y = 3^{x \ln x} \quad \Rightarrow \quad y' = 3^{x \ln x}(\ln 3)\,\dfrac{d}{dx}(x \ln x) = 3^{x \ln x}(\ln 3)\left(x \cdot \dfrac{1}{x} + \ln x \cdot 1\right) = 3^{x \ln x}(\ln 3)(1 + \ln x)$

34. $y = e^{\cos x} + \cos(e^x) \quad \Rightarrow \quad y' = -\sin x\, e^{\cos x} - e^x \sin(e^x)$

35. $y = x \tan^{-1} x - \frac{1}{2} \ln(1 + x^2) \quad \Rightarrow$

$$y' = x \cdot \frac{1}{1 + x^2} + \left(\tan^{-1} x\right) \cdot 1 - \frac{1}{2}\left(\frac{2x}{1 + x^2}\right) = \frac{x}{1 + x^2} + \tan^{-1} x - \frac{x}{1 + x^2} = \tan^{-1} x$$

36. $F(z) = \log_{10}(1 + z^2) \quad \Rightarrow \quad F'(z) = \dfrac{1}{(\ln 10)(1 + z^2)} \cdot 2z = \dfrac{2z}{(\ln 10)(1 + z^2)}$

37. $y = x \sinh(x^2) \quad \Rightarrow \quad y' = x \cosh(x^2) \cdot 2x + \sinh(x^2) \cdot 1 = 2x^2 \cosh(x^2) + \sinh(x^2)$

38. $y = (\cos x)^x \quad \Rightarrow \quad \ln y = \ln(\cos x)^x = x \ln \cos x \quad \Rightarrow \quad \dfrac{y'}{y} = x \cdot \dfrac{1}{\cos x} \cdot (-\sin x) + \ln \cos x \cdot 1 \quad \Rightarrow$

$y' = (\cos x)^x (\ln \cos x - x \tan x)$

39. $y = \ln(\arcsin x^2) \quad \Rightarrow \quad y' = \dfrac{1}{\arcsin x^2} \cdot \dfrac{d}{dx}\,(\arcsin x^2) = \dfrac{1}{\arcsin x^2} \cdot \dfrac{1}{\sqrt{1 - (x^2)^2}} \cdot 2x = \dfrac{2x}{(\arcsin x^2)\sqrt{1 - x^4}}$

40. $y = \arctan\left(\arcsin \sqrt{x}\right) \quad \Rightarrow \quad y' = \dfrac{1}{1 + \left(\arcsin \sqrt{x}\right)^2} \cdot \dfrac{1}{\sqrt{1 - x}} \cdot \dfrac{1}{2\sqrt{x}}$

41. $y = \ln\left(\dfrac{1}{x}\right) + \dfrac{1}{\ln x} = \ln x^{-1} + (\ln x)^{-1} = -\ln x + (\ln x)^{-1} \quad \Rightarrow \quad y' = -1 \cdot \dfrac{1}{x} + (-1)(\ln x)^{-2} \cdot \dfrac{1}{x} = -\dfrac{1}{x} - \dfrac{1}{x\,(\ln x)^2}$

42. $y = \ln\left|\dfrac{x^2 - 4}{2x + 5}\right| = \ln\left|x^2 - 4\right| - \ln|2x + 5| \quad \Rightarrow \quad y' = \dfrac{2x}{x^2 - 4} - \dfrac{2}{2x + 5}$ or $\dfrac{2(x + 1)(x + 4)}{(x + 2)(x - 2)(2x + 5)}$

43. $y = \ln(\cosh 3x) \quad \Rightarrow \quad y' = (1/\cosh 3x)(\sinh 3x)(3) = 3 \tanh 3x$

44. $y = \dfrac{(x^2 + 1)^4}{(2x + 1)^3 (3x - 1)^5} \quad \Rightarrow$

$$\ln y = \ln \frac{(x^2 + 1)^4}{(2x + 1)^3 (3x - 1)^5} = \ln(x^2 + 1)^4 - \ln[(2x + 1)^3 (3x - 1)^5] = 4 \ln(x^2 + 1) - [\ln(2x + 1)^3 + \ln(3x - 1)^5]$$

$$= 4 \ln(x^2 + 1) - 3 \ln(2x + 1) - 5 \ln(3x - 1) \quad \Rightarrow$$

$$\frac{y'}{y} = 4 \cdot \frac{1}{x^2 + 1} \cdot 2x - 3 \cdot \frac{1}{2x + 1} \cdot 2 - 5 \cdot \frac{1}{3x - 1} \cdot 3 \quad \Rightarrow \quad y' = \frac{(x^2 + 1)^4}{(2x + 1)^3 (3x - 1)^5}\left(\frac{8x}{x^2 + 1} - \frac{6}{2x + 1} - \frac{15}{3x - 1}\right).$$

[The answer could be simplified to $y' = -\dfrac{(x^2 + 56x + 9)(x^2 + 1)^3}{(2x + 1)^4 (3x - 1)^6}$, but this is unnecessary.]

45. $y = \cosh^{-1}(\sinh x) \quad \Rightarrow \quad y' = (\cosh x)/\sqrt{\sinh^2 x - 1}$

46. $y = x\tanh^{-1}\sqrt{x} \quad\Rightarrow\quad y' = \tanh^{-1}\sqrt{x} + x\dfrac{1}{1-(\sqrt{x})^2}\dfrac{1}{2\sqrt{x}} = \tanh^{-1}\sqrt{x} + \dfrac{\sqrt{x}}{2(1-x)}$

47. $y = \cos\!\left(e^{\sqrt{\tan 3x}}\right) \quad\Rightarrow$

$$y' = -\sin\!\left(e^{\sqrt{\tan 3x}}\right)\cdot\left(e^{\sqrt{\tan 3x}}\right)' = -\sin\!\left(e^{\sqrt{\tan 3x}}\right)\,e^{\sqrt{\tan 3x}}\cdot\tfrac{1}{2}(\tan 3x)^{-1/2}\cdot\sec^2(3x)\cdot 3$$

$$= \frac{-3\sin\!\left(e^{\sqrt{\tan 3x}}\right)e^{\sqrt{\tan 3x}}\,\sec^2(3x)}{2\sqrt{\tan 3x}}$$

48. $\dfrac{d}{dx}\left(\dfrac{1}{2}\tan^{-1}x + \dfrac{1}{4}\ln\dfrac{(x+1)^2}{x^2+1}\right) = \dfrac{d}{dx}\left(\dfrac{1}{2}\tan^{-1}x + \dfrac{1}{2}\ln|x+1| - \dfrac{1}{4}\ln(x^2+1)\right)$

$$= \frac{1}{2}\frac{1}{x^2+1} + \frac{1}{2}\frac{1}{x+1} - \frac{1}{4}\frac{2x}{x^2+1} = \frac{1}{2}\left(\frac{1}{x^2+1} - \frac{x}{x^2+1} + \frac{1}{x+1}\right)$$

$$= \frac{1}{2}\left(\frac{1-x}{x^2+1} + \frac{1}{x+1}\right) = \frac{1}{2}\left(\frac{1-x^2}{(x^2+1)(1+x)} + \frac{x^2+1}{(x^2+1)(1+x)}\right)$$

$$= \frac{1}{2}\frac{2}{(x^2+1)(1+x)} = \frac{1}{(1+x)(x^2+1)}$$

49. $f(x) = e^{g(x)} \quad\Rightarrow\quad f'(x) = e^{g(x)}g'(x)$

50. $f(x) = g(e^x) \quad\Rightarrow\quad f'(x) = g'(e^x)\,e^x$

51. $f(x) = \ln|g(x)| \quad\Rightarrow\quad f'(x) = \dfrac{1}{g(x)}g'(x) = \dfrac{g'(x)}{g(x)}$

52. $f(x) = g(\ln x) \quad\Rightarrow\quad f'(x) = g'(\ln x)\cdot\dfrac{1}{x} = \dfrac{g'(\ln x)}{x}$

53. $f(x) = 2^x \quad\Rightarrow\quad f'(x) = 2^x\ln 2 \quad\Rightarrow\quad f''(x) = 2^x(\ln 2)^2 \quad\Rightarrow\quad\cdots\quad\Rightarrow\quad f^{(n)}(x) = 2^x(\ln 2)^n$

54. $f(x) = \ln(2x) = \ln 2 + \ln x \quad\Rightarrow\quad f'(x) = x^{-1}$, $f''(x) = -x^{-2}$, $f'''(x) = 2x^{-3}$, $f^{(4)}(x) = -2\cdot 3x^{-4}, \ldots,$

$f^{(n)}(x) = (-1)^{n-1}(n-1)!\,x^{-n}$

55. We first show it is true for $n = 1$: $f'(x) = e^x + xe^x = (x+1)e^x$. We now assume it is true for $n = k$:

$f^{(k)}(x) = (x+k)e^x$. With this assumption, we must show it is true for $n = k+1$:

$f^{(k+1)}(x) = \dfrac{d}{dx}\left[f^{(k)}(x)\right] = \dfrac{d}{dx}\left[(x+k)e^x\right] = e^x + (x+k)e^x = \left[x+(k+1)\right]e^x.$

Therefore, $f^{(n)}(x) = (x+n)e^x$ by mathematical induction.

56. Using implicit differentiation, $y = x + \arctan y \quad\Rightarrow\quad y' = 1 + \dfrac{1}{1+y^2}y' \quad\Rightarrow\quad y'\left(1 - \dfrac{1}{1+y^2}\right) = 1 \quad\Rightarrow$

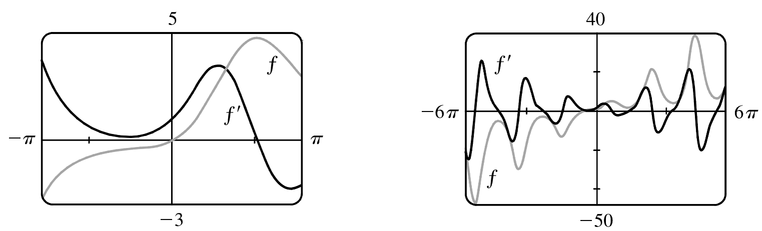
$y'\left(\dfrac{y^2}{1+y^2}\right) = 1 \quad\Rightarrow\quad y' = \dfrac{1+y^2}{y^2} = \dfrac{1}{y^2} + 1.$

57. $y = (2+x)e^{-x} \;\Rightarrow\; y' = (2+x)(-e^{-x}) + e^{-x} \cdot 1 = e^{-x}[-(2+x)+1] = e^{-x}(-x-1)$. At $(0,2)$, $y' = 1(-1) = -1$, so an equation of the tangent line is $y - 2 = -1(x-0)$, or $y = -x+2$.

58. $y = f(x) = x\ln x \;\Rightarrow\; f'(x) = \ln x + 1$, so the slope of the tangent at (e,e) is $f'(e) = 2$ and an equation is $y - e = 2(x-e)$ or $y = 2x - e$.

59. $y = [\ln(x+4)]^2 \;\Rightarrow\; y' = 2[\ln(x+4)]^1 \cdot \dfrac{1}{x+4} \cdot 1 = 2\,\dfrac{\ln(x+4)}{x+4}$ and $y' = 0 \;\Leftrightarrow\; \ln(x+4) = 0 \;\Leftrightarrow$ $x + 4 = e^0 \;\Rightarrow\; x+4 = 1 \;\Leftrightarrow\; x = -3$, so the tangent is horizontal at the point $(-3, 0)$.

60. $f(x) = xe^{\sin x} \;\Rightarrow\; f'(x) = x[e^{\sin x}(\cos x)] + e^{\sin x}(1) = e^{\sin x}(x\cos x + 1)$. As a check on our work, we notice from the graphs that $f'(x) > 0$ when f is increasing. Also, we see in the larger viewing rectangle a certain similarity in the graphs of f and f': the sizes of the oscillations of f and f' are linked.



61. (a) The line $x - 4y = 1$ has slope $\frac{1}{4}$. A tangent to $y = e^x$ has slope $\frac{1}{4}$ when $y' = e^x = \frac{1}{4} \;\Rightarrow\; x = \ln\frac{1}{4} = -\ln 4$. Since $y = e^x$, the y-coordinate is $\frac{1}{4}$ and the point of tangency is $\left(-\ln 4, \frac{1}{4}\right)$. Thus, an equation of the tangent line is $y - \frac{1}{4} = \frac{1}{4}(x + \ln 4)$ or $y = \frac{1}{4}x + \frac{1}{4}(\ln 4 + 1)$.

(b) The slope of the tangent at the point (a, e^a) is $\left.\dfrac{d}{dx}e^x\right|_{x=a} = e^a$. Thus, an equation of the tangent line is $y - e^a = e^a(x-a)$. We substitute $x = 0$, $y = 0$ into this equation, since we want the line to pass through the origin: $0 - e^a = e^a(0-a) \;\Leftrightarrow\; -e^a = e^a(-a) \;\Leftrightarrow\; a = 1$. So an equation of the tangent line at the point $(a, e^a) = (1, e)$ is $y - e = e(x-1)$ or $y = ex$.

62. (a) $\displaystyle\lim_{t\to\infty} C(t) = \lim_{t\to\infty}[K(e^{-at} - e^{-bt})] = K\lim_{t\to\infty}(e^{-at} - e^{-bt}) = K(0-0) = 0$ because $-at \to -\infty$ and $-bt \to -\infty$ as $t \to \infty$.

(b) $C(t) = K(e^{-at} - e^{-bt}) \;\Rightarrow\; C'(t) = K(e^{-at}(-a) - e^{-bt}(-b)) = K(-ae^{-at} + be^{-bt})$

(c) $C'(t) = 0 \;\Leftrightarrow\; be^{-bt} = ae^{-at} \;\Leftrightarrow\; \dfrac{b}{a} = e^{(-a+b)t} \;\Leftrightarrow\; \ln\dfrac{b}{a} = (b-a)t \;\Leftrightarrow\; t = \dfrac{\ln(b/a)}{b-a}$

63. $\displaystyle\lim_{x\to\infty} e^{-3x} = 0$ since $-3x \to -\infty$ as $x \to \infty$ and $\displaystyle\lim_{t\to-\infty} e^t = 0$.

64. $\displaystyle\lim_{x\to 10^-} \ln(100 - x^2) = -\infty$ since as $x \to 10^-$, $(100 - x^2) \to 0^+$.

65. Let $t = 2/(x-3)$. As $x \to 3^-$, $t \to -\infty$. $\lim\limits_{x\to 3^-} e^{2/(x-3)} = \lim\limits_{t\to -\infty} e^t = 0$

66. If $y = x^3 - x = x(x^2-1)$, then as $x \to \infty$, $y \to \infty$. $\lim\limits_{x\to\infty} \arctan(x^3 - x) = \lim\limits_{y\to\infty} \arctan y = \frac{\pi}{2}$.

67. As $x \to 0$, $\cosh x \to 1$, so $\lim\limits_{x\to 0} \ln(\cosh x) = 0$.

68. $-1 \le \sin x \le 1 \quad \Rightarrow \quad -e^{-x} \le e^{-x}\sin x \le e^{-x}$. Now $\lim\limits_{x\to\infty} \left(\pm e^{-x}\right) = 0$, so by the Squeeze Theorem,

$\lim\limits_{x\to\infty} e^{-x}\sin x = 0$.

69. $\displaystyle \lim_{x\to\infty} \frac{1+2^x}{1-2^x} = \lim_{x\to\infty} \frac{(1+2^x)/2^x}{(1-2^x)/2^x} = \lim_{x\to\infty} \frac{1/2^x + 1}{1/2^x - 1} = \frac{0+1}{0-1} = -1$

70. Let $t = x/4$, so $x = 4t$. As $x \to \infty$, $t \to \infty$. $\displaystyle \lim_{x\to\infty} \left(1 + \frac{4}{x}\right)^x = \lim_{t\to\infty} \left(1 + \frac{1}{t}\right)^{4t} = \left[\lim_{t\to\infty} \left(1 + \frac{1}{t}\right)^t\right]^4 = e^4$

71. This limit has the form $\frac{0}{0}$. $\displaystyle \lim_{x\to 0} \frac{e^x - 1}{\tan x} \overset{\text{H}}{=} \lim_{x\to 0} \frac{e^x}{\sec^2 x} = \frac{1}{1} = 1$

72. This limit has the form $\frac{0}{0}$. $\displaystyle \lim_{x\to 0} \frac{1-\cos x}{x^2 + x} \overset{\text{H}}{=} \lim_{x\to 0} \frac{\sin x}{2x+1} = \frac{0}{1} = 0$

73. This limit has the form $\frac{0}{0}$. $\displaystyle \lim_{x\to 0} \frac{e^{2x} - e^{-2x}}{\ln(x+1)} \overset{\text{H}}{=} \lim_{x\to 0} \frac{2e^{2x} + 2e^{-2x}}{1/(x+1)} = \frac{2+2}{1} = 4$

74. This limit has the form $\frac{\infty}{\infty}$. $\displaystyle \lim_{x\to\infty} \frac{e^{2x} - e^{-2x}}{\ln(x+1)} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{2e^{2x} + 2e^{-2x}}{1/(x+1)} = \lim_{x\to\infty} 2(x+1)(e^{2x} + e^{-2x}) = \infty$

since $2(x+1) \to \infty$ and $(e^{2x} + e^{-2x}) \to \infty$ as $x \to \infty$.

75. This limit has the form $\infty \cdot 0$.

$$\begin{aligned}
\lim_{x\to -\infty} (x^2 - x^3)e^{2x} &= \lim_{x\to -\infty} \frac{x^2 - x^3}{e^{-2x}} \quad \left[\tfrac{\infty}{\infty} \text{ form}\right] \quad \overset{\text{H}}{=} \lim_{x\to -\infty} \frac{2x - 3x^2}{-2e^{-2x}} \quad \left[\tfrac{\infty}{\infty} \text{ form}\right] \\
&\overset{\text{H}}{=} \lim_{x\to -\infty} \frac{2-6x}{4e^{-2x}} \quad \left[\tfrac{\infty}{\infty} \text{ form}\right] \quad \overset{\text{H}}{=} \lim_{x\to -\infty} \frac{-6}{-8e^{-2x}} = 0
\end{aligned}$$

76. This limit has the form $0 \cdot (-\infty)$. $\displaystyle \lim_{x\to 0^+} x^2 \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x^2} \overset{\text{H}}{=} \lim_{x\to 0^+} \frac{1/x}{-2/x^3} = \lim_{x\to 0^+} \left(-\tfrac{1}{2}x^2\right) = 0$

77. This limit has the form $\infty - \infty$.

$$\begin{aligned}
\lim_{x\to 1^+} \left(\frac{x}{x-1} - \frac{1}{\ln x}\right) &= \lim_{x\to 1^+} \left(\frac{x\ln x - x + 1}{(x-1)\ln x}\right) \overset{\text{H}}{=} \lim_{x\to 1^+} \frac{x\cdot(1/x) + \ln x - 1}{(x-1)\cdot(1/x) + \ln x} = \lim_{x\to 1^+} \frac{\ln x}{1 - 1/x + \ln x} \\
&\overset{\text{H}}{=} \lim_{x\to 1^+} \frac{1/x}{1/x^2 + 1/x} = \frac{1}{1+1} = \frac{1}{2}
\end{aligned}$$

78. $y = (\tan x)^{\cos x} \Rightarrow \ln y = \cos x \ln \tan x$, so

$$\lim_{x\to(\pi/2)^-} \ln y = \lim_{x\to(\pi/2)^-} \frac{\ln \tan x}{\sec x} \overset{\text{H}}{=} \lim_{x\to(\pi/2)^-} \frac{(1/\tan x)\sec^2 x}{\sec x\, \tan x} = \lim_{x\to(\pi/2)^-} \frac{\sec x}{\tan^2 x} = \lim_{x\to(\pi/2)^-} \frac{\cos x}{\sin^2 x} = \frac{0}{1^2} = 0,$$

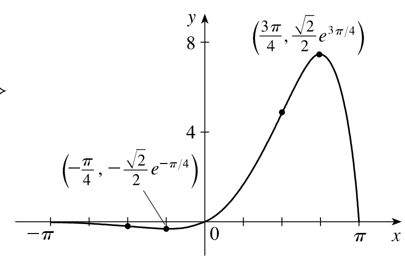
so $\lim\limits_{x\to(\pi/2)^-} (\tan x)^{\cos x} = \lim\limits_{x\to(\pi/2)^-} e^{\ln y} = e^0 = 1$.

79. $y = f(x) = e^x \sin x$, $-\pi \le x \le \pi$ **A.** $D = [-\pi, \pi]$ **B.** y-intercept: $f(0) = 0$; $f(x) = 0 \Leftrightarrow \sin x = 0 \Rightarrow x = -\pi, 0, \pi$. **C.** No symmetry **D.** No asymptote **E.** $f'(x) = e^x \cos x + \sin x \cdot e^x = e^x(\cos x + \sin x)$. $f'(x) = 0 \Leftrightarrow -\cos x = \sin x \Leftrightarrow -1 = \tan x \Rightarrow x = -\frac{\pi}{4}, \frac{3\pi}{4}$. $f'(x) > 0$ for $-\frac{\pi}{4} < x < \frac{3\pi}{4}$ and $f'(x) < 0$ for $-\pi < x < -\frac{\pi}{4}$ and $\frac{3\pi}{4} < x < \pi$, so f is increasing on $\left(-\frac{\pi}{4}, \frac{3\pi}{4}\right)$ and f is decreasing on $\left(-\pi, -\frac{\pi}{4}\right)$ and $\left(\frac{3\pi}{4}, \pi\right)$.

F. Local minimum value $f\left(-\frac{\pi}{4}\right) = (-\sqrt{2}/2)e^{-\pi/4} \approx -0.32$ and local maximum value $f\left(\frac{3\pi}{4}\right) = \left(\sqrt{2}/2\right) e^{3\pi/4} \approx 7.46$

G. $f''(x) = e^x(-\sin x + \cos x) + (\cos x + \sin x)e^x = e^x(2\cos x) > 0 \Rightarrow -\frac{\pi}{2} < x < \frac{\pi}{2}$ and $f''(x) < 0 \Rightarrow -\pi < x < -\frac{\pi}{2}$ and $\frac{\pi}{2} < x < \pi$, so f is CU on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and f is CD on $\left(-\pi, -\frac{\pi}{2}\right)$ and $\left(\frac{\pi}{2}, \pi\right)$. There are inflection points at $\left(-\frac{\pi}{2}, -e^{-\pi/2}\right)$ and $\left(\frac{\pi}{2}, e^{\pi/2}\right)$.

H.



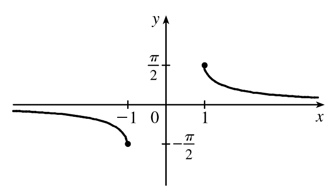
80. $y = f(x) = \sin^{-1}(1/x)$ **A.** $D = \{x \mid -1 \le 1/x \le 1\} = (-\infty, -1] \cup [1, \infty)$. **B.** No intercept **C.** $f(-x) = -f(x)$, symmetric about the origin **D.** $\lim\limits_{x\to\pm\infty} \sin^{-1}(1/x) = \sin^{-1}(0) = 0$, so $y = 0$ is a HA.

E. $f'(x) = \dfrac{1}{\sqrt{1 - (1/x)^2}}\left(-\dfrac{1}{x^2}\right) = \dfrac{-1}{\sqrt{x^4 - x^2}} < 0$, so f is decreasing on $(-\infty, -1)$ and $(1, \infty)$.

F. No local extreme value, but $f(1) = \frac{\pi}{2}$ is the absolute maximum value and $f(-1) = -\frac{\pi}{2}$ is the absolute minimum value.

G. $f''(x) = \dfrac{4x^3 - 2x}{2(x^4 - x^2)^{3/2}} = \dfrac{x(2x^2 - 1)}{(x^4 - x^2)^{3/2}} > 0$ for $x > 1$ and $f''(x) < 0$ for $x < -1$, so f is CU on $(1, \infty)$ and CD on $(-\infty, -1)$. No IP

H.

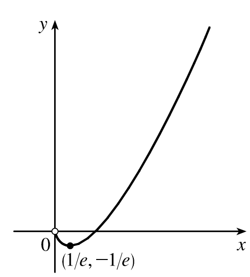


81. $y = f(x) = x \ln x$ **A.** $D = (0, \infty)$ **B.** No y-intercept; x-intercept 1. **C.** No symmetry **D.** No asymptote [Note that the graph approaches the point $(0, 0)$ as $x \to 0^+$.]

E. $f'(x) = x(1/x) + (\ln x)(1) = 1 + \ln x$, so $f'(x) \to -\infty$ as $x \to 0^+$ and $f'(x) \to \infty$ as $x \to \infty$. $f'(x) = 0 \Leftrightarrow \ln x = -1 \Leftrightarrow x = e^{-1} = 1/e$. $f'(x) > 0$ for $x > 1/e$, so f is decreasing on $(0, 1/e)$ and increasing on $(1/e, \infty)$. **F.** Local minimum: $f(1/e) = -1/e$. No local maximum.

G. $f''(x) = 1/x$, so $f''(x) > 0$ for $x > 0$. The graph is CU on $(0, \infty)$ and there is no IP.

H.

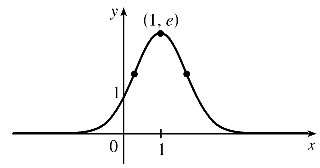


82. $y = f(x) = e^{2x-x^2}$ **A.** $D = \mathbb{R}$ **B.** y-intercept 1; no x-intercept **C.** No symmetry **D.** $\lim\limits_{x\to\pm\infty} e^{2x-x^2} = 0$, so $y = 0$ is a HA. **E.** $y = f(x) = e^{2x-x^2} \Rightarrow f'(x) = 2(1-x)e^{2x-x^2} > 0 \Leftrightarrow x < 1$, so f is increasing on $(-\infty, 1)$ and decreasing on $(1, \infty)$. **F.** $f(1) = e$ is a local and absolute maximum value.

G. $f''(x) = 2\left(2x^2 - 4x + 1\right)e^{2x-x^2} = 0 \Leftrightarrow x = 1 \pm \frac{\sqrt{2}}{2}$.

$f''(x) > 0 \Leftrightarrow x < 1 - \frac{\sqrt{2}}{2}$ or $x > 1 + \frac{\sqrt{2}}{2}$, so f is CU on $\left(-\infty, 1 - \frac{\sqrt{2}}{2}\right)$ and $\left(1 + \frac{\sqrt{2}}{2}, \infty\right)$, and CD on $\left(1 - \frac{\sqrt{2}}{2}, 1 + \frac{\sqrt{2}}{2}\right)$. IP at $\left(1 \pm \frac{\sqrt{2}}{2}, \sqrt{e}\right)$

H.



83. $y = f(x) = (x-2)e^{-x}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = -2$; x-intercept: $f(x) = 0 \Leftrightarrow x = 2$

C. No symmetry **D.** $\lim\limits_{x\to\infty} \dfrac{x-2}{e^x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{1}{e^x} = 0$, so $y = 0$ is a HA. No VA

E. $f'(x) = (x-2)(-e^{-x}) + e^{-x}(1) = e^{-x}[-(x-2)+1] = (3-x)e^{-x}$.

$f'(x) > 0$ for $x < 3$, so f is increasing on $(-\infty, 3)$ and decreasing on $(3, \infty)$.

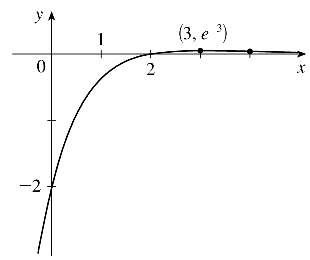
F. Local maximum value $f(3) = e^{-3}$, no local minimum value

G. $f''(x) = (3-x)(-e^{-x}) + e^{-x}(-1) = e^{-x}[-(3-x) + (-1)]$

$= (x-4)e^{-x} > 0$

for $x > 4$, so f is CU on $(4, \infty)$ and CD on $(-\infty, 4)$. IP at $(4, 2e^{-4})$

H.



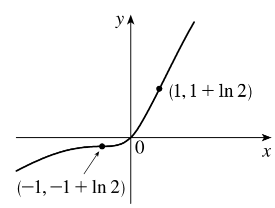
84. $y = f(x) = x + \ln(x^2+1)$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0 + \ln 1 = 0$; x-intercept: $f(x) = 0 \Leftrightarrow \ln(x^2+1) = -x \Leftrightarrow x^2 + 1 = e^{-x} \Rightarrow x = 0$ since the graphs of $y = x^2+1$ and $y = e^{-x}$ intersect only at $x = 0$.

C. No symmetry **D.** No asymptote **E.** $f'(x) = 1 + \dfrac{2x}{x^2+1} = \dfrac{x^2+2x+1}{x^2+1} = \dfrac{(x+1)^2}{x^2+1}$. $f'(x) > 0$ if $x \neq -1$ and f is increasing on $\mathbb{R}$. **F.** No local extreme values

G. $f''(x) = \dfrac{(x^2+1)2 - 2x(2x)}{(x^2+1)^2} = \dfrac{2[(x^2+1) - 2x^2]}{(x^2+1)^2} = \dfrac{2(1-x^2)}{(x^2+1)^2}$.

$f''(x) > 0 \Leftrightarrow -1 < x < 1$ and $f''(x) < 0 \Leftrightarrow x < -1$ or $x > 1$, so f is CU on $(-1, 1)$ and f is CD on $(-\infty, -1)$ and $(1, \infty)$. IP at $(-1, -1 + \ln 2)$ and $(1, 1 + \ln 2)$

H.

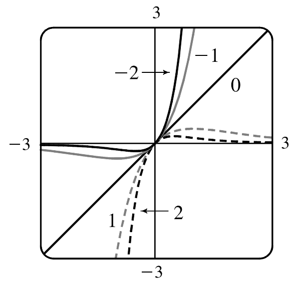


85. If $c < 0$, then $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} xe^{-cx} = \lim\limits_{x\to-\infty} \dfrac{x}{e^{cx}} \overset{\text{H}}{=} \lim\limits_{x\to-\infty} \dfrac{1}{ce^{cx}} = 0$, and $\lim\limits_{x\to\infty} f(x) = \infty$.

If $c > 0$, then $\lim\limits_{x\to-\infty} f(x) = -\infty$, and $\lim\limits_{x\to\infty} f(x) \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{1}{ce^{cx}} = 0$.

If $c = 0$, then $f(x) = x$, so $\lim\limits_{x\to\pm\infty} f(x) = \pm\infty$, respectively.

So we see that $c = 0$ is a transitional value. We now exclude the case $c = 0$, since we know how the function behaves in that case. To find the maxima and minima of f, we differentiate: $f(x) = xe^{-cx} \Rightarrow$ $f'(x) = x(-ce^{-cx}) + e^{-cx} = (1 - cx)e^{-cx}$. This is 0 when $1 - cx = 0 \Leftrightarrow x = 1/c$. If $c < 0$ then this represents a minimum value of $f(1/c) = 1/(ce)$, since $f'(x)$ changes from negative to positive at $x = 1/c$; and if $c > 0$, it represents a maximum value. As $|c|$ increases, the maximum or minimum point gets closer to the origin. To find the inflection points, we differentiate again: $f'(x) = e^{-cx}(1 - cx) \quad \Rightarrow$ $f''(x) = e^{-cx}(-c) + (1 - cx)(-ce^{-cx}) = (cx - 2)ce^{-cx}$. This changes sign when $cx - 2 = 0 \quad \Leftrightarrow \quad x = 2/c$. So as $|c|$ increases, the points of inflection get closer to the origin.

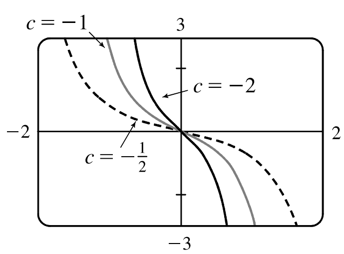


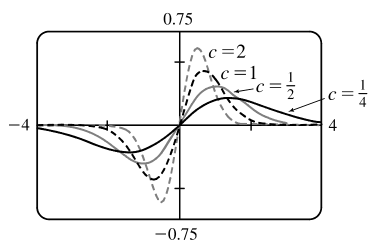
86. We exclude the case $c = 0$, since in that case $f(x) = 0$ for all x. To find the maxima and minima, we differentiate:

$$f(x) = cxe^{-cx^2} \quad \Rightarrow \quad f'(x) = c\left[xe^{-cx^2}(-2cx) + e^{-cx^2}(1)\right] = ce^{-cx^2}(-2cx^2 + 1)$$

This is 0 where $-2cx^2 + 1 = 0 \quad \Leftrightarrow \quad x = \pm 1/\sqrt{2c}$. So if $c > 0$, there are two maxima or minima, whose x-coordinates approach 0 as c increases. The negative solution gives a minimum and the positive solution gives a maximum, by the First Derivative Test. By substituting back into the equation, we see that $f\left(\pm 1/\sqrt{2c}\right) = c\left(\pm 1/\sqrt{2c}\right) e^{-c\left(\pm 1/\sqrt{2c}\right)^2} = \pm\sqrt{c/2e}$. So as c increases, the extreme points become more pronounced. Note that if $c > 0$, then $\lim\limits_{x \to \pm\infty} f(x) = 0$. If $c < 0$, then there are no extreme values, and $\lim\limits_{x \to \pm\infty} f(x) = \mp\infty$.

To find the points of inflection, we differentiate again: $f'(x) = ce^{-cx^2}\left(-2cx^2 + 1\right) \quad \Rightarrow$ $f''(x) = c\left[e^{-cx^2}(-4cx) + (-2cx^2 + 1)(-2cxe^{-cx^2})\right] = -2c^2xe^{-cx^2}(3 - 2cx^2)$. This is 0 at $x = 0$ and where $3 - 2cx^2 = 0 \quad \Leftrightarrow \quad x = \pm\sqrt{3/(2c)} \quad \Rightarrow \quad$ IP at $\left(\pm\sqrt{3/(2c)}, \pm\sqrt{3c/2}\, e^{-3/2}\right)$. If $c > 0$ there are three inflection points, and as c increases, the x-coordinates of the nonzero inflection points approach 0. If $c < 0$, there is only one inflection point, the origin.





87. $s(t) = Ae^{-ct}\cos(\omega t + \delta) \quad \Rightarrow$

$$v(t) = s'(t) = A\{e^{-ct}\left[-\omega\sin(\omega t + \delta)\right] + \cos(\omega t + \delta)(-ce^{-ct})\} = -Ae^{-ct}\left[\omega\sin(\omega t + \delta) + c\cos(\omega t + \delta)\right] \quad \Rightarrow$$

$$\begin{aligned} a(t) = v'(t) &= -A\{e^{-ct}[\omega^2\cos(\omega t + \delta) - c\omega\sin(\omega t + \delta)] + [\omega\sin(\omega t + \delta) + c\cos(\omega t + \delta)](-ce^{-ct})\} \\ &= -Ae^{-ct}[\omega^2\cos(\omega t + \delta) - c\omega\sin(\omega t + \delta) - c\omega\sin(\omega t + \delta) - c^2\cos(\omega t + \delta)] \\ &= -Ae^{-ct}[(\omega^2 - c^2)\cos(\omega t + \delta) - 2c\omega\sin(\omega t + \delta)] = Ae^{-ct}[(c^2 - \omega^2)\cos(\omega t + \delta) + 2c\omega\sin(\omega t + \delta)] \end{aligned}$$

88. (a) Let $f(x) = \ln x + x - 3$. Then $f'(x) = 1/x + 1 >$ [for $x > 0$)] and $f(2) \approx -0.307$ and $f(e) \approx 0.718$.

f is differentiable on $(2, e)$, continuous on $[2, e]$ and $f(2) < 0$, $f(e) > 0$. Therefore, by the Intermediate Value Theorem there exists a number c in $(2, e)$ such that $f(c) = 0$. Thus, there is one solution. But $f'(x) > 0$ for $x \in (2, e)$, so f is increasing on $(2, e)$, which means that there is exactly one solution.

(b) We use Newton's method with $f(x) = \ln x + x - 3$, $f'(x) = 1/x + 1$, and $x_1 = 2$.

$x_2 = x_1 - \dfrac{\ln x_1 + x_1 - 3}{1/x_1 + 1} = 2 - \dfrac{\ln 2 + 2 - 3}{1/2 + 1} \approx 2.20457$. Similarly, $x_3 \approx 2.20794$, $x_4 = 2.20794$. Thus, the solution of the equation, correct to four decimal places, is 2.2079.

89. (a) $y(t) = y(0)e^{kt} = 200e^{kt} \quad \Rightarrow \quad y(0.5) = 200e^{0.5k} = 360 \quad \Rightarrow \quad e^{0.5k} = 1.8 \quad \Rightarrow \quad 0.5k = \ln 1.8 \quad \Rightarrow$

$k = 2 \ln 1.8 = \ln(1.8)^2 = \ln 3.24 \quad \Rightarrow \quad y(t) = 200e^{(\ln 3.24)t} = 200(3.24)^t$

(b) $y(4) = 200(3.24)^4 \approx 22{,}040$ cells

(c) $y'(t) = 200(3.24)^t \cdot \ln 3.24$, so $y'(4) = 200(3.24)^4 \cdot \ln 3.24 \approx 25{,}910$ cells per hour

(d) $200(3.24)^t = 10{,}000 \quad \Rightarrow \quad (3.24)^t = 50 \quad \Rightarrow \quad t \ln 3.24 = \ln 50 \quad \Rightarrow \quad t = (\ln 50)/(\ln 3.24) \approx 3.33$ hours

90. (a) If $y(t)$ is the mass remaining after t years, then $y(t) = y(0)e^{kt} = 100e^{kt}$. $y(5.24) = 100e^{5.24k} = \frac{1}{2} \cdot 100 \quad \Rightarrow$

$e^{5.24k} = \frac{1}{2} \quad \Rightarrow \quad 5.24k = -\ln 2 \quad \Rightarrow \quad k = -\frac{1}{5.24} \ln 2 \quad \Rightarrow \quad y(t) = 100e^{-(\ln 2)t/5.24} = 100 \cdot 2^{-t/5.24}$. Thus,

$y(20) = 100 \cdot 2^{-20/5.24} \approx 7.1$ mg.

(b) $100 \cdot 2^{-t/5.24} = 1 \quad \Rightarrow \quad 2^{-t/5.24} = \dfrac{1}{100} \quad \Rightarrow \quad -\dfrac{t}{5.24} \ln 2 = \ln \dfrac{1}{100} \quad \Rightarrow \quad t = 5.24 \dfrac{\ln 100}{\ln 2} \approx 34.8$ years

91. Let $P(t) = \dfrac{64}{1 + 31e^{-0.7944t}} = \dfrac{A}{1 + Be^{ct}} = A(1 + Be^{ct})^{-1}$, where $A = 64$, $B = 31$, and $c = -0.7944$.

$$P'(t) = -A(1 + Be^{ct})^{-2}(Bce^{ct}) = -ABce^{ct}(1 + Be^{ct})^{-2}$$

$$\begin{aligned} P''(t) &= -ABce^{ct}[-2(1 + Be^{ct})^{-3}(Bce^{ct})] + (1 + Be^{ct})^{-2}(-ABc^2e^{ct}) \\ &= -ABc^2e^{ct}(1 + Be^{ct})^{-3}[-2Be^{ct} + (1 + Be^{ct})] = -\frac{ABc^2e^{ct}(1 - Be^{ct})}{(1 + Be^{ct})^3} \end{aligned}$$

The population is increasing most rapidly when its graph changes from CU to CD; that is, when $P''(t) = 0$ in this case.

$P''(t) = 0 \quad \Rightarrow \quad Be^{ct} = 1 \quad \Rightarrow \quad e^{ct} = \dfrac{1}{B} \quad \Rightarrow \quad ct = \ln \dfrac{1}{B} \quad \Rightarrow \quad t = \dfrac{\ln(1/B)}{c} = \dfrac{\ln(1/31)}{-0.7944} \approx 4.32$ days. Note that

$P\left(\dfrac{1}{c} \ln \dfrac{1}{B}\right) = \dfrac{A}{1 + Be^{c(1/c)\ln(1/B)}} = \dfrac{A}{1 + Be^{\ln(1/B)}} = \dfrac{A}{1 + B(1/B)} = \dfrac{A}{1 + 1} = \dfrac{A}{2}$, one-half the limit of P as $t \to \infty$.

92. Let $t = 4u$. Then $dt = 4\,du$ and

$$\int_0^4 \frac{1}{16 + t^2}\,dt = \int_0^1 \frac{1}{16 + 16u^2} \cdot 4\,du = \frac{1}{4}\int_0^1 \frac{du}{1 + u^2} = \frac{1}{4}\Big[\tan^{-1} u\Big]_0^1 = \tfrac{1}{4}(\tan^{-1} 1 - \tan^{-1} 0) = \tfrac{1}{4}\left(\tfrac{\pi}{4} - 0\right) = \tfrac{\pi}{16}.$$

93. Let $u = -2y^2$. Then $du = -4y\,dy$ and $\int_0^1 ye^{-2y^2}\,dy = \int_0^{-2} e^u\left(-\frac{1}{4}\,du\right) = -\frac{1}{4}\left[e^u\right]_0^{-2} = -\frac{1}{4}(e^{-2} - 1) = \frac{1}{4}(1 - e^{-2})$.

94. $\displaystyle\int_2^5 \frac{dr}{1+2r} = \frac{1}{2}\left[\ln|1+2r|\right]_2^5 = \frac{1}{2}(\ln 11 - \ln 5) = \frac{1}{2}\ln\frac{11}{5}$

95. Let $u = e^x$, so $du = e^x\,dx$. When $x = 0, u = 1$; when $x = 1, u = e$. Thus,

$$\int_0^1 \frac{e^x}{1+e^{2x}}\,dx = \int_1^e \frac{1}{1+u^2}\,du = \left[\arctan u\right]_1^e = \arctan e - \arctan 1 = \arctan e - \frac{\pi}{4}.$$

96. Let $u = \sin x$. Then $du = \cos x\,dx$, so

$$\int_0^{\pi/2} \frac{\cos x}{1+\sin^2 x}\,dx = \int_0^1 \frac{1}{1+u^2}\,du = \left[\tan^{-1} u\right]_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}.$$

97. Let $u = \sqrt{x}$. Then $du = \dfrac{dx}{2\sqrt{x}} \quad\Rightarrow\quad \displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int e^u\,du = 2e^u + C = 2e^{\sqrt{x}} + C.$

98. Let $u = \ln x$. Then $du = \dfrac{1}{x}\,dx$, so $\displaystyle\int \frac{\sin(\ln x)}{x}\,dx = \int \sin u\,du = -\cos u + C = -\cos(\ln x) + C.$

99. Let $u = x^2 + 2x$. Then $du = (2x+2)\,dx = 2(x+1)\,dx$ and

$$\int \frac{x+1}{x^2+2x}\,dx = \int \frac{\frac{1}{2}\,du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln\left|x^2+2x\right| + C.$$

100. Let $u = 1 + \cot x$. Then $du = -\csc^2 x\,dx$, so $\displaystyle\int \frac{\csc^2 x}{1+\cot x}\,dx = \int \frac{1}{u}\,(-du) = -\ln|u| + C = -\ln|1+\cot x| + C.$

101. Let $u = \ln(\cos x)$. Then $du = \dfrac{-\sin x}{\cos x}\,dx = -\tan x\,dx \quad\Rightarrow$

$\int \tan x \ln(\cos x)\,dx = -\int u\,du = -\frac{1}{2}u^2 + C = -\frac{1}{2}[\ln(\cos x)]^2 + C.$

102. Let $u = x^2$. Then $du = 2x\,dx$, so $\displaystyle\int \frac{x}{\sqrt{1-x^4}}\,dx = \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\sin^{-1} u + C = \frac{1}{2}\sin^{-1}\left(x^2\right) + C.$

103. Let $u = \tan\theta$. Then $du = \sec^2\theta\,d\theta$ and $\displaystyle\int 2^{\tan\theta}\sec^2\theta\,d\theta = \int 2^u\,du = \frac{2^u}{\ln 2} + C = \frac{2^{\tan\theta}}{\ln 2} + C.$

104. $\displaystyle\int \sinh au\,du = \frac{1}{a}\cosh au + C$

105. $\displaystyle\int_{-2}^{-1} \frac{z^2+1}{z}\,dz = \int_{-2}^{-1}\left(z + \frac{1}{z}\right)dz = \left[\frac{1}{2}z^2 + \ln|z|\right]_{-2}^{-1} = \left(\frac{1}{2} + \ln|-1|\right) - \left(2 + \ln|-2|\right) = -\frac{3}{2} - \ln 2$

106. $1 + e^{2x} > e^{2x} \quad\Rightarrow\quad \sqrt{1+e^{2x}} > \sqrt{e^{2x}} = e^x \quad\Rightarrow\quad \int_0^1 \sqrt{1+e^{2x}}\,dx \geq \int_0^1 e^x\,dx = \left.e^x\right]_0^1 = e - 1$

107. $\cos x \leq 1 \quad\Rightarrow\quad e^x\cos x \leq e^x \quad\Rightarrow\quad \int_0^1 e^x\cos x\,dx \leq \int_0^1 e^x\,dx = \left.e^x\right]_0^1 = e - 1$

108. For $0 \le x \le 1$, $0 \le \sin^{-1}x \le \frac{\pi}{2}$, so $\int_0^1 x\sin^{-1}x\,dx \le \int_0^1 x\left(\frac{\pi}{2}\right)dx = \left.\frac{\pi}{4}x^2\right]_0^1 = \frac{\pi}{4}$.

109. $f(x) = \displaystyle\int_1^{\sqrt{x}} \frac{e^s}{s}\,ds \quad\Rightarrow\quad f'(x) = \frac{d}{dx}\int_1^{\sqrt{x}} \frac{e^s}{s}\,ds = \frac{e^{\sqrt{x}}}{\sqrt{x}}\frac{d}{dx}\sqrt{x} = \frac{e^{\sqrt{x}}}{\sqrt{x}}\frac{1}{2\sqrt{x}} = \frac{e^{\sqrt{x}}}{2x}$

110. $f(x) = \displaystyle\int_{\ln x}^{2x} e^{-t^2}\,dt \quad\Rightarrow$

$$f'(x) = \frac{d}{dx}\int_{\ln x}^{2x} e^{-t^2}\,dt = -\frac{d}{dx}\int_0^{\ln x} e^{-t^2}\,dt + \frac{d}{dx}\int_0^{2x} e^{-t^2}\,dt = -e^{-(\ln x)^2}\left(\frac{1}{x}\right) + e^{-(2x)^2}(2) = -\frac{e^{-(\ln x)^2}}{x} + 2e^{-4x^2}$$

111. (a) $f_{\text{avg}} = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx = \frac{1}{5-1}\int_1^5 \frac{\ln x}{x}\,dx = \frac{1}{4}\int_0^{\ln 5} u\,du \qquad \begin{bmatrix} u = \ln x, \\ du = (1/x)\,dx \end{bmatrix}$

$= \frac{1}{4}\left[\frac{1}{2}u^2\right]_0^{\ln 5} = \frac{1}{8}(\ln 5)^2$

(b) $f(x) = \dfrac{\ln x}{x}$, $[1, 5]$. $f'(x) = \dfrac{x(1/x) - (\ln x)\cdot 1}{x^2} = \dfrac{1 - \ln x}{x^2} = 0 \quad\Leftrightarrow\quad 1 - \ln x = 0 \quad\Leftrightarrow\quad \ln x = 1 \quad\Leftrightarrow\quad x = e.$

$f(1) = 0$, $f(e) = 1/e \approx 0.37$, and $f(5) = \frac{1}{5}\ln 5 \approx 0.32$. So $f(e) = 1/e$ is the absolute maximum value and $f(1) = 0$ is the absolute minimum value.

112. $A = \int_{-2}^0 (e^{-x} - e^x)\,dx + \int_0^1 \left(e^x - e^{-x}\right)dx = \left[-e^{-x} - e^x\right]_{-2}^0 + \left[e^x + e^{-x}\right]_0^1$

$= [(-1-1) - (-e^2 - e^{-2})] + [(e + e^{-1}) - (1+1)] = e^2 + e + e^{-1} + e^{-2} - 4$

113. $V = \displaystyle\int_0^1 \frac{2\pi x}{1+x^4}\,dx$ by cylindrical shells. Let $u = x^2 \quad\Rightarrow\quad du = 2x\,dx$. Then

$V = \displaystyle\int_0^1 \frac{\pi}{1+u^2}\,du = \pi\left[\tan^{-1}u\right]_0^1 = \pi\left(\tan^{-1}1 - \tan^{-1}0\right) = \pi\left(\frac{\pi}{4}\right) = \frac{\pi^2}{4}.$

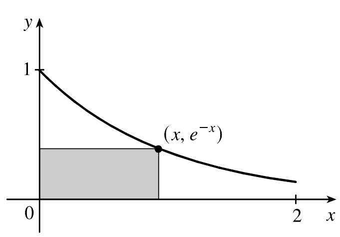
114. $f(x) = x + x^2 + e^x \quad\Rightarrow\quad f'(x) = 1 + 2x + e^x$ and $f(0) = 1 \quad\Rightarrow\quad g(1) = 0$ [where $g = f^{-1}$],

so $g'(1) = \dfrac{1}{f'(g(1))} = \dfrac{1}{f'(0)} = \dfrac{1}{2}$.

115. $f(x) = \ln x + \tan^{-1}x \quad\Rightarrow\quad f(1) = \ln 1 + \tan^{-1}1 = \frac{\pi}{4} \quad\Rightarrow\quad g\left(\frac{\pi}{4}\right) = 1$ [where $g = f^{-1}$].

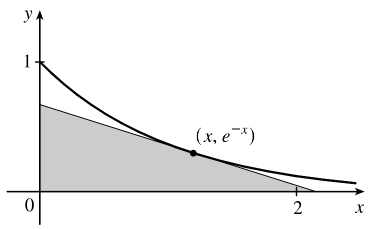
$f'(x) = \dfrac{1}{x} + \dfrac{1}{1+x^2}$, so $g'\left(\dfrac{\pi}{4}\right) = \dfrac{1}{f'(1)} = \dfrac{1}{3/2} = \dfrac{2}{3}$.

116.



The area of such a rectangle is just the product of its sides, that is, $A(x) = x \cdot e^{-x}$. We want to find the maximum of this function, so we differentiate:

$A'(x) = x(-e^{-x}) + e^{-x}(1) = e^{-x}(1-x)$. This is 0 only at $x = 1$, and changes from positive to negative there, so by the First Derivative Test this gives a local maximum. So the largest area is $A(1) = 1/e$.

117. 

We find the equation of a tangent to the curve $y = e^{-x}$, so that we can find the x- and y-intercepts of this tangent, and then we can find the area of the triangle. The slope of the tangent at the point $\left(a, e^{-a}\right)$ is given by $\left.\dfrac{d}{dx}e^{-x}\right]_{x=a} = -e^{-a}$, and so the equation of the tangent is $y - e^{-a} = -e^{-a}(x-a) \quad\Leftrightarrow$ $y = e^{-a}(a - x + 1)$.

The y-intercept of this line is $y = e^{-a}(a - 0 + 1) = e^{-a}(a+1)$. To find the x-intercept we set $y = 0 \quad\Rightarrow$ $e^{-a}(a - x + 1) = 0 \quad\Rightarrow\quad x = a + 1$. So the area of the triangle is $A(a) = \frac{1}{2}\left[e^{-a}(a+1)\right](a+1) = \frac{1}{2}e^{-a}(a+1)^2$. We differentiate this with respect to a: $A'(a) = \frac{1}{2}\left[e^{-a}(2)(a+1) + (a+1)^2 e^{-a}(-1)\right] = \frac{1}{2}e^{-a}\left(1 - a^2\right)$. This is 0 at $a = \pm 1$, and the root $a = 1$ gives a maximum, by the First Derivative Test. So the maximum area of the triangle is $A(1) = \frac{1}{2}e^{-1}(1+1)^2 = 2e^{-1} = 2/e$.

118. Using Theorem 4.2.4 with $a = 0$, $b = 1$, $\Delta x = \dfrac{1}{n}$, and $x_i = 0 + i(1/n) = i/n$, we have $\int_0^1 e^x\,dx = \lim\limits_{n\to\infty} \dfrac{1}{n}\sum\limits_{i=1}^{n} e^{i/n}$.

This series is a geometric series with $a = r = e^{1/n}$, so $\sum\limits_{i=1}^{n} e^{i/n} = e^{1/n}\dfrac{\left(e^{1/n}\right)^n - 1}{e^{1/n} - 1} = e^{1/n}\dfrac{e - 1}{e^{1/n} - 1} \quad\Rightarrow$

$\displaystyle\int_0^1 e^x\,dx = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} e^{i/n} = \lim_{n\to\infty}(e-1)e^{1/n}\frac{1/n}{e^{1/n} - 1}$. As $n \to \infty$, $1/n \to 0^+$, so $e^{1/n} \to e^0 = 1$.

Let $t = 1/n$. Then $e^{1/n} - 1 = e^t - 1 \to 0^+$ as $n \to \infty$, so l'Hospital's Rule gives $\lim\limits_{t\to 0^+}\dfrac{t}{e^t - 1} = \lim\limits_{t\to 0^+}\dfrac{1}{e^t} = 1$ and we have

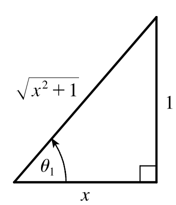
$$\int_0^1 e^x\,dx = \left[\lim_{t\to 0^+}(e-1)e^t\right]\left[\lim_{t\to 0^+}\frac{t}{e^t - 1}\right] = e - 1.$$

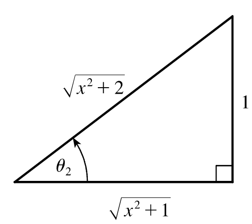
119. $\lim\limits_{x\to -1} F(x) = \lim\limits_{x\to -1}\dfrac{b^{x+1} - a^{x+1}}{x+1} \overset{\text{H}}{=} \lim\limits_{x\to -1}\dfrac{b^{x+1}\ln b - a^{x+1}\ln a}{1} = \ln b - \ln a = F(-1)$, so F is continuous at -1.

120. Let $\theta_1 = \operatorname{arccot} x$, so $\cot\theta_1 = x = x/1$.

So $\sin(\operatorname{arccot} x) = \sin\theta_1 = \dfrac{1}{\sqrt{x^2+1}}$.

Let $\theta_2 = \arctan\left[\dfrac{1}{\sqrt{x^2+1}}\right]$, so $\tan\theta_2 = \dfrac{1}{\sqrt{x^2+1}}$.





Hence, $\cos\{\arctan[\sin(\operatorname{arccot} x)]\} = \cos\theta_2 = \dfrac{\sqrt{x^2+1}}{\sqrt{x^2+2}} = \sqrt{\dfrac{x^2+1}{x^2+2}}$.

121. Using FTC1, we differentiate both sides of the given equation, $\int_1^x f(t)\,dt = (x-1)e^{2x} + \int_1^x e^{-t}f(t)\,dt$, and get

$f(x) = (x-1)\cdot e^{2x}\cdot 2 + e^{2x} + e^{-x}f(x) \quad\Rightarrow\quad f(x)\left(1 - e^{-x}\right) = 2(x-1)e^{2x} + e^{2x} \quad\Rightarrow$

$f(x) = \dfrac{e^{2x}[1 + 2(x-1)]}{1 - e^{-x}} = \dfrac{e^{2x}(2x-1)}{1 - e^{-x}}$.

122. The area $A(t) = \int_0^t \sin(x^2)\,dx$, and the area $B(t) = \frac{1}{2}t\sin(t^2)$. Since $\lim\limits_{t\to 0^+} A(t) = 0 = \lim\limits_{t\to 0^+} B(t)$, we can use l'Hospital's Rule:

$$\lim_{t\to 0^+} \frac{A(t)}{B(t)} \overset{\text{H}}{=} \lim_{t\to 0^+} \frac{\sin(t^2)}{\frac{1}{2}\sin(t^2) + \frac{1}{2}t[2t\cos(t^2)]} \qquad \text{[by FTC1 and the Product Rule]}$$

$$\overset{\text{H}}{=} \lim_{t\to 0^+} \frac{2t\cos(t^2)}{t\cos(t^2) - 2t^3\sin(t^2) + 2t\cos(t^2)} = \lim_{t\to 0^+} \frac{2\cos(t^2)}{3\cos(t^2) - 2t^2\sin(t^2)} = \frac{2}{3-0} = \frac{2}{3}$$

☐ PROBLEMS PLUS

1. Let $y = f(x) = e^{-x^2}$. The area of the rectangle under the curve from $-x$ to x is $A(x) = 2xe^{-x^2}$ where $x \geq 0$. We maximize $A(x)$: $A'(x) = 2e^{-x^2} - 4x^2e^{-x^2} = 2e^{-x^2}\left(1 - 2x^2\right) = 0 \quad \Rightarrow \quad x = \frac{1}{\sqrt{2}}$. This gives a maximum since $A'(x) > 0$ for $0 \leq x < \frac{1}{\sqrt{2}}$ and $A'(x) < 0$ for $x > \frac{1}{\sqrt{2}}$. We next determine the points of inflection of $f(x)$. Notice that $f'(x) = -2xe^{-x^2} = -A(x)$. So $f''(x) = -A'(x)$ and hence, $f''(x) < 0$ for $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$ and $f''(x) > 0$ for $x < -\frac{1}{\sqrt{2}}$ and $x > \frac{1}{\sqrt{2}}$. So $f(x)$ changes concavity at $x = \pm\frac{1}{\sqrt{2}}$, and the two vertices of the rectangle of largest area are at the inflection points.

2. We use proof by contradiction. Suppose that $\log_2 5$ is a rational number. Then $\log_2 5 = m/n$ where m and n are positive integers $\Rightarrow \quad 2^{m/n} = 5 \quad \Rightarrow \quad 2^m = 5^n$. But this is impossible since 2^m is even and 5^n is odd. So $\log_2 5$ is irrational.

3. $\ln\left(x^2 - 2x - 2\right) \leq 0 \quad \Rightarrow \quad x^2 - 2x - 2 \leq e^0 = 1 \quad \Rightarrow \quad x^2 - 2x - 3 \leq 0 \quad \Rightarrow \quad (x-3)(x+1) \leq 0 \quad \Rightarrow \quad x \in [-1, 3]$.
Since the argument must be positive, $x^2 - 2x - 2 > 0 \quad \Rightarrow \quad \left[x - \left(1 - \sqrt{3}\right)\right]\left[x - \left(1 + \sqrt{3}\right)\right] > 0 \quad \Rightarrow$
$x \in \left(-\infty, 1 - \sqrt{3}\right) \cup \left(1 + \sqrt{3}, \infty\right)$. The intersection of these intervals is $\left[-1, 1 - \sqrt{3}\right) \cup \left(1 + \sqrt{3}, 3\right]$.

4.
$$\begin{aligned}
\frac{1}{\log_2 x} + \frac{1}{\log_3 x} + \frac{1}{\log_5 x} &= \frac{1}{\frac{\log x}{\log 2}} + \frac{1}{\frac{\log x}{\log 3}} + \frac{1}{\frac{\log x}{\log 5}} && \text{[Change of Base formula]} \\
&= \frac{\log 2}{\log x} + \frac{\log 3}{\log x} + \frac{\log 5}{\log x} \\
&= \frac{\log 2 + \log 3 + \log 5}{\log x} = \frac{\log(2 \cdot 3 \cdot 5)}{\log x} && \text{[Law 1 of Logarithms]} \\
&= \frac{\log 30}{\log x} = \frac{1}{\frac{\log x}{\log 30}} = \frac{1}{\log_{30} x} && \text{[Change of Base formula]}
\end{aligned}$$

5. $f(x)$ has the form $e^{g(x)}$, so it will have an absolute maximum (minimum) where g has an absolute maximum (minimum).

$$g(x) = 10|x-2| - x^2 = \begin{cases} 10(x-2) - x^2 & \text{if } x - 2 > 0 \\ 10[-(x-2)] - x^2 & \text{if } x - 2 < 0 \end{cases} = \begin{cases} -x^2 + 10x - 20 & \text{if } x > 2 \\ -x^2 - 10x + 20 & \text{if } x < 2 \end{cases} \quad \Rightarrow$$

$$g'(x) = \begin{cases} -2x + 10 & \text{if } x > 2 \\ -2x - 10 & \text{if } x < 2 \end{cases}$$

$g'(x) = 0$ if $x = -5$ or $x = 5$, and $g'(2)$ does not exist, so the critical numbers of g are -5, 2, and 5. Since $g''(x) = -2$ for all $x \neq 2$, g is concave downward on $(-\infty, 2)$ and $(2, \infty)$, and g will attain its absolute maximum at one of the critical numbers. Since $g(-5) = 45$, $g(2) = -4$, and $g(5) = 5$, we see that $f(-5) = e^{45}$ is the absolute maximum value of f. Also, $\lim\limits_{x \to \infty} g(x) = -\infty$, so $\lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to \infty} e^{g(x)} = 0$. But $f(x) > 0$ for all x, so there is no absolute minimum value of f.

6. For $I = \int_0^4 xe^{(x-2)^4}\,dx$, let $u = x - 2$ so that $x = u + 2$ and $dx = du$. Then

$I = \int_{-2}^{2}(u+2)e^{u^4}\,du = \int_{-2}^{2} ue^{u^4}\,du + \int_{-2}^{2} 2e^{u^4}\,du = 0$ [by 4.5.6] $+\, 2\int_0^4 e^{(x-2)^4}\,dx = 2k$.

7. Consider the statement that $\dfrac{d^n}{dx^n}(e^{ax}\sin bx) = r^n e^{ax}\sin(bx + n\theta)$. For $n = 1$,

$\dfrac{d}{dx}(e^{ax}\sin bx) = ae^{ax}\sin bx + be^{ax}\cos bx$, and

$$re^{ax}\sin(bx+\theta) = re^{ax}[\sin bx\cos\theta + \cos bx\sin\theta] = re^{ax}\left(\frac{a}{r}\sin bx + \frac{b}{r}\cos bx\right) = ae^{ax}\sin bx + be^{ax}\cos bx$$

since $\tan\theta = \dfrac{b}{a} \;\Rightarrow\; \sin\theta = \dfrac{b}{r}$ and $\cos\theta = \dfrac{a}{r}$. So the statement is true for $n = 1$.

Assume it is true for $n = k$. Then

$$\begin{aligned}\frac{d^{k+1}}{dx^{k+1}}(e^{ax}\sin bx) &= \frac{d}{dx}\left[r^k e^{ax}\sin(bx+k\theta)\right] = r^k ae^{ax}\sin(bx+k\theta) + r^k e^{ax} b\cos(bx+k\theta)\\ &= r^k e^{ax}[a\sin(bx+k\theta) + b\cos(bx+k\theta)]\end{aligned}$$

But

$\sin[bx + (k+1)\theta] = \sin[(bx+k\theta)+\theta] = \sin(bx+k\theta)\cos\theta + \sin\theta\cos(bx+k\theta) = \frac{a}{r}\sin(bx+k\theta) + \frac{b}{r}\cos(bx+k\theta)$.

Hence, $a\sin(bx+k\theta) + b\cos(bx+k\theta) = r\sin[bx+(k+1)\theta]$. So

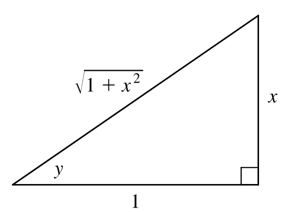
$$\frac{d^{k+1}}{dx^{k+1}}(e^{ax}\sin bx) = r^k e^{ax}[a\sin(bx+k\theta) + b\cos(bx+k\theta)] = r^k e^{ax}[r\sin(bx+(k+1)\theta)] = r^{k+1}e^{ax}[\sin(bx+(k+1)\theta)].$$

Therefore, the statement is true for all n by mathematical induction.

8. Let $y = \tan^{-1}x$. Then $\tan y = x$, so from the triangle we see that

$\sin(\tan^{-1}x) = \sin y = \dfrac{x}{\sqrt{1+x^2}}$. Using this fact we have that

$\sin(\tan^{-1}(\sinh x)) = \dfrac{\sinh x}{\sqrt{1+\sinh^2 x}} = \dfrac{\sinh x}{\cosh x} = \tanh x$.



Hence, $\sin^{-1}(\tanh x) = \sin^{-1}(\sin(\tan^{-1}(\sinh x))) = \tan^{-1}(\sinh x)$.

9. We first show that $\dfrac{x}{1+x^2} < \tan^{-1}x$ for $x > 0$. Let $f(x) = \tan^{-1}x - \dfrac{x}{1+x^2}$. Then

$f'(x) = \dfrac{1}{1+x^2} - \dfrac{1(1+x^2) - x(2x)}{(1+x^2)^2} = \dfrac{(1+x^2)-(1-x^2)}{(1+x^2)^2} = \dfrac{2x^2}{(1+x^2)^2} > 0$ for $x > 0$. So $f(x)$ is increasing

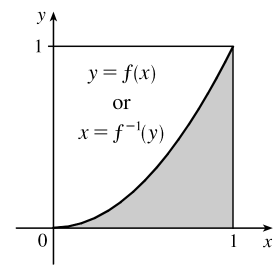
on $(0,\infty)$. Hence, $0 < x \;\Rightarrow\; 0 = f(0) < f(x) = \tan^{-1}x - \dfrac{x}{1+x^2}$. So $\dfrac{x}{1+x^2} < \tan^{-1}x$ for $0 < x$. We next show that

$\tan^{-1}x < x$ for $x > 0$. Let $h(x) = x - \tan^{-1}x$. Then $h'(x) = 1 - \dfrac{1}{1+x^2} = \dfrac{x^2}{1+x^2} > 0$. Hence, $h(x)$ is increasing

on $(0,\infty)$. So for $0 < x$, $0 = h(0) < h(x) = x - \tan^{-1}x$. Hence, $\tan^{-1}x < x$ for $x > 0$, and we conclude that

$\dfrac{x}{1+x^2} < \tan^{-1}x < x$ for $x > 0$.

10. The shaded region has area $\int_0^1 f(x)\,dx = \frac{1}{3}$. The integral $\int_0^1 f^{-1}(y)\,dy$ gives the area of the unshaded region, which we know to be $1 - \frac{1}{3} = \frac{2}{3}$. So $\int_0^1 f^{-1}(y)\,dy = \frac{2}{3}$.



11. By the Fundamental Theorem of Calculus, $f(x) = \int_1^x \sqrt{1+t^3}\,dt \quad \Rightarrow \quad f'(x) = \sqrt{1+x^3} > 0$ for $x > -1$. So f is increasing on $(-1, \infty)$ and hence is one-to-one. Note that $f(1) = 0$, so $f^{-1}(1) = 0 \quad \Rightarrow$ $(f^{-1})'(0) = 1/f'(1) = \frac{1}{\sqrt{2}}$.

12. $y = \dfrac{x}{\sqrt{a^2-1}} - \dfrac{2}{\sqrt{a^2-1}} \arctan \dfrac{\sin x}{a + \sqrt{a^2-1} + \cos x}$. Let $k = a + \sqrt{a^2-1}$. Then

$$\begin{aligned}
y' &= \frac{1}{\sqrt{a^2-1}} - \frac{2}{\sqrt{a^2-1}} \cdot \frac{1}{1 + \sin^2 x/(k+\cos x)^2} \cdot \frac{\cos x(k+\cos x) + \sin^2 x}{(k+\cos x)^2} \\
&= \frac{1}{\sqrt{a^2-1}} - \frac{2}{\sqrt{a^2-1}} \cdot \frac{k\cos x + \cos^2 x + \sin^2 x}{(k+\cos x)^2 + \sin^2 x} = \frac{1}{\sqrt{a^2-1}} - \frac{2}{\sqrt{a^2-1}} \cdot \frac{k\cos x + 1}{k^2 + 2k\cos x + 1} \\
&= \frac{k^2 + 2k\cos x + 1 - 2k\cos x - 2}{\sqrt{a^2-1}\,(k^2 + 2k\cos x + 1)} = \frac{k^2 - 1}{\sqrt{a^2-1}\,(k^2 + 2k\cos x + 1)}
\end{aligned}$$

But $k^2 = 2a^2 + 2a\sqrt{a^2-1} - 1 = 2a\left(a + \sqrt{a^2-1}\right) - 1 = 2ak - 1$, so $k^2 + 1 = 2ak$, and $k^2 - 1 = 2(ak - 1)$.

So $y' = \dfrac{2(ak-1)}{\sqrt{a^2-1}\,(2ak + 2k\cos x)} = \dfrac{ak-1}{\sqrt{a^2-1}k\,(a+\cos x)}$. But $ak - 1 = a^2 + a\sqrt{a^2-1} - 1 = k\sqrt{a^2-1}$,

so $y' = 1/(a + \cos x)$.

13. If $L = \lim\limits_{x\to\infty} \left(\dfrac{x+a}{x-a}\right)^x$, then L has the indeterminate form 1^∞, so

$$\begin{aligned}
\ln L &= \lim_{x\to\infty} \ln\left(\frac{x+a}{x-a}\right)^x = \lim_{x\to\infty} x\ln\left(\frac{x+a}{x-a}\right) = \lim_{x\to\infty} \frac{\ln(x+a) - \ln(x-a)}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{\dfrac{1}{x+a} - \dfrac{1}{x-a}}{-1/x^2} \\
&= \lim_{x\to\infty} \left[\frac{(x-a)-(x+a)}{(x+a)(x-a)} \cdot \frac{-x^2}{1}\right] = \lim_{x\to\infty} \frac{2ax^2}{x^2 - a^2} = \lim_{x\to\infty} \frac{2a}{1 - a^2/x^2} = 2a
\end{aligned}$$

Hence, $\ln L = 2a$, so $L = e^{2a}$. From the original equation, we want $L = e^1 \quad \Rightarrow \quad 2a = 1 \quad \Rightarrow \quad a = \frac{1}{2}$.

14. We first present some preliminary results that we will invoke when calculating the limit.

(1) If $y = (1+ax)^x$, then $\ln y = x\ln(1+ax)$, and $\lim\limits_{x\to 0^+} \ln y = \lim\limits_{x\to 0^+} x\ln(1+ax) = 0$. Thus, $\lim\limits_{x\to 0^+} (1+ax)^x = e^0 = 1$.

(2) If $y = (1+ax)^x$, then $\ln y = x\ln(1+ax)$, and implicitly differentiating gives us $\dfrac{y'}{y} = x \cdot \dfrac{a}{1+ax} + \ln(1+ax) \quad \Rightarrow$

[continued]

$y' = y\left[\dfrac{ax}{1+ax} + \ln(1+ax)\right]$. Thus, $y = (1+ax)^x \quad\Rightarrow\quad y' = (1+ax)^x\left[\dfrac{ax}{1+ax} + \ln(1+ax)\right]$.

(3) If $y = \dfrac{ax}{1+ax}$, then $y' = \dfrac{(1+ax)a - ax(a)}{(1+ax)^2} = \dfrac{a + a^2x - a^2x}{(1+ax)^2} = \dfrac{a}{(1+ax)^2}$.

$$\begin{aligned}
\lim_{x\to\infty}\frac{(x+2)^{1/x} - x^{1/x}}{(x+3)^{1/x} - x^{1/x}} &= \lim_{x\to\infty}\frac{x^{1/x}[(1+2/x)^{1/x} - 1]}{x^{1/x}[(1+3/x)^{1/x} - 1]} && \text{[factor out } x^{1/x}\text{]}\\
&= \lim_{x\to\infty}\frac{(1+2/x)^{1/x} - 1}{(1+3/x)^{1/x} - 1}\\
&= \lim_{t\to 0^+}\frac{(1+2t)^t - 1}{(1+3t)^t - 1} && \text{[let } t = 1/x\text{, form } 0/0 \text{ by } \mathbf{(1)}\text{]}\\
&\overset{\text{H}}{=} \lim_{t\to 0^+}\frac{(1+2t)^t\left[\dfrac{2t}{1+2t} + \ln(1+2t)\right]}{(1+3t)^t\left[\dfrac{3t}{1+3t} + \ln(1+3t)\right]} && \text{[by } \mathbf{(2)}\text{]}\\
&= \lim_{t\to 0^+}\frac{(1+2t)^t}{(1+3t)^t}\cdot\lim_{t\to 0^+}\frac{\dfrac{2t}{1+2t} + \ln(1+2t)}{\dfrac{3t}{1+3t} + \ln(1+3t)}\\
&= \frac{1}{1}\cdot\lim_{t\to 0^+}\frac{\dfrac{2t}{1+2t} + \ln(1+2t)}{\dfrac{3t}{1+3t} + \ln(1+3t)} && \text{[by } \mathbf{(1)}\text{, now form } 0/0\text{]}\\
&\overset{\text{H}}{=} \lim_{t\to 0^+}\frac{\dfrac{2}{(1+2t)^2} + \dfrac{2}{1+2t}}{\dfrac{3}{(1+3t)^2} + \dfrac{3}{1+3t}} && \text{[by } \mathbf{(3)}\text{]}\\
&= \frac{2+2}{3+3} = \frac{4}{6} = \frac{2}{3}
\end{aligned}$$

15. As in Exercise 4.3.70, assume that the integrand is defined at $t = 0$ so that it is continuous there. By l'Hospital's Rule and the Fundamental Theorem, using the notation $\exp(y) = e^y$,

$$\begin{aligned}
\lim_{x\to 0}\frac{\int_0^x (1-\tan 2t)^{1/t}\,dt}{x} &\overset{\text{H}}{=} \lim_{x\to 0}\frac{(1-\tan 2x)^{1/x}}{1} = \exp\left[\ln\left(\lim_{x\to 0}(1-\tan 2x)^{1/x}\right)\right] = \exp\left(\lim_{x\to 0}\frac{\ln(1-\tan 2x)}{x}\right)\\
&\overset{\text{H}}{=} \exp\left(\lim_{x\to 0}\frac{-2\sec^2 2x}{1-\tan 2x}\right) = \exp\left(\frac{-2\cdot 1^2}{1-0}\right) = e^{-2}
\end{aligned}$$

16. *Case (i) (first graph):* For $x + y \geq 0$, that is, $y \geq -x$, $|x+y| = x + y \leq e^x \quad\Rightarrow\quad y \leq e^x - x$.

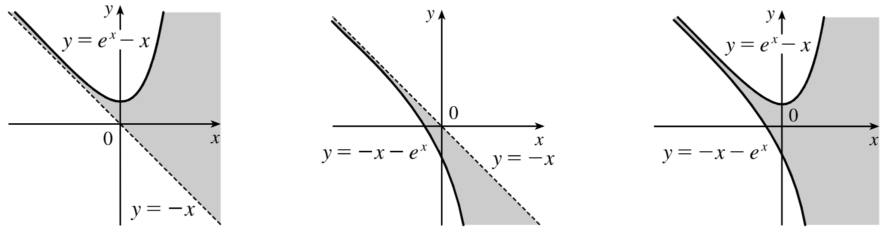
Note that $y = e^x - x$ is always above the line $y = -x$ and that $y = -x$ is a slant asymptote.

Case (ii) (second graph): For $x + y < 0$, that is, $y < -x$, $|x+y| = -x - y \leq e^x \quad\Rightarrow\quad y \geq -x - e^x$.

Note that $-x - e^x$ is always below the line $y = -x$ and $y = -x$ is a slant asymptote.

Putting the two pieces together gives the third graph.

[continued]



17. Both sides of the inequality are positive, so $\cosh(\sinh x) < \sinh(\cosh x)$

$\Leftrightarrow\quad \cosh^2(\sinh x) < \sinh^2(\cosh x) \quad\Leftrightarrow\quad \sinh^2(\sinh x) + 1 < \sinh^2(\cosh x)$

$\Leftrightarrow\quad 1 < [\sinh(\cosh x) - \sinh(\sinh x)][\sinh(\cosh x) + \sinh(\sinh x)]$

$$\Leftrightarrow\quad 1 < \left[\sinh\!\left(\frac{e^x + e^{-x}}{2}\right) - \sinh\!\left(\frac{e^x - e^{-x}}{2}\right)\right]\left[\sinh\!\left(\frac{e^x + e^{-x}}{2}\right) + \sinh\!\left(\frac{e^x - e^{-x}}{2}\right)\right]$$

$\Leftrightarrow\quad 1 < [2\cosh(e^x/2)\sinh(e^x/2)][2\sinh(e^x/2)\cosh(e^x/2)]$ [use the addition formulas and cancel]

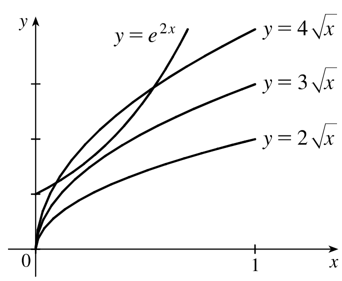
$\Leftrightarrow\quad 1 < [2\sinh(e^x/2)\cosh(e^x/2)][2\sinh(e^x/2)\cosh(e^x/2)] \quad\Leftrightarrow\quad 1 < \sinh\, e^x \sinh e^{-x}$,

by the half-angle formula. Now both e^x and e^{-x} are positive, and $\sinh y > y$ for $y > 0$, since $\sinh 0 = 0$ and $(\sinh y - y)' = \cosh y - 1 > 0$ for $x > 0$, so $1 = e^x e^{-x} < \sinh e^x \sinh e^{-x}$. So, following this chain of reasoning backward, we arrive at the desired result.

18. First, we recognize some symmetry in the inequality: $\dfrac{e^{x+y}}{xy} \geq e^2 \quad\Leftrightarrow\quad \dfrac{e^x}{x}\cdot\dfrac{e^y}{y} \geq e \cdot e$. This suggests that we need to show that $\dfrac{e^x}{x} \geq e$ for $x > 0$. If we can do this, then the inequality $\dfrac{e^y}{y} \geq e$ is true, and the given inequality follows. $f(x) = \dfrac{e^x}{x} \quad\Rightarrow$

$f'(x) = \dfrac{xe^x - e^x}{x^2} = \dfrac{e^x(x-1)}{x^2} = 0 \quad\Rightarrow\quad x = 1$. By the First Derivative Test, we have a minimum of $f(1) = e$, so $e^x/x \geq e$ for all x.

19.



Let $f(x) = e^{2x}$ and $g(x) = k\sqrt{x}$ $[k > 0]$. From the graphs of f and g, we see that f will intersect g exactly once when f and g share a tangent line. Thus, we must have $f = g$ and $f' = g'$ at $x = a$.

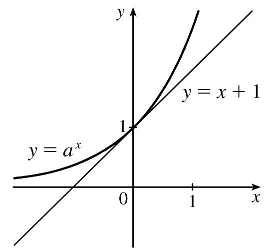
$$f(a) = g(a) \quad\Rightarrow\quad e^{2a} = k\sqrt{a} \quad (\star)$$

and $\quad f'(a) = g'(a) \quad\Rightarrow\quad 2e^{2a} = \dfrac{k}{2\sqrt{a}} \quad\Rightarrow\quad e^{2a} = \dfrac{k}{4\sqrt{a}}$.

So we must have $k\sqrt{a} = \dfrac{k}{4\sqrt{a}} \quad\Rightarrow\quad \left(\sqrt{a}\right)^2 = \dfrac{k}{4k} \quad\Rightarrow\quad a = \frac{1}{4}$. From $(\star)$, $e^{2(1/4)} = k\sqrt{1/4} \quad\Rightarrow$

$k = 2e^{1/2} = 2\sqrt{e} \approx 3.297$.

20. We see that at $x = 0$, $f(x) = a^x = 1 + x = 1$, so if $y = a^x$ is to lie above $y = 1 + x$, the two curves must just touch at $(0, 1)$, that is, we must have $f'(0) = 1$. [To see this analytically, note that $a^x \geq 1 + x \;\Rightarrow\; a^x - 1 \geq x \;\Rightarrow\; \dfrac{a^x - 1}{x} \geq 1$ for $x > 0$, so $f'(0) = \lim\limits_{x \to 0^+} \dfrac{a^x - 1}{x} \geq 1$. Similarly, for $x < 0$, $a^x - 1 \geq x \;\Rightarrow\; \dfrac{a^x - 1}{x} \leq 1$, so $f'(0) = \lim\limits_{x \to 0^-} \dfrac{a^x - 1}{x} \leq 1$.

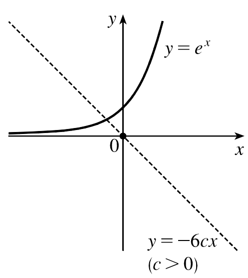


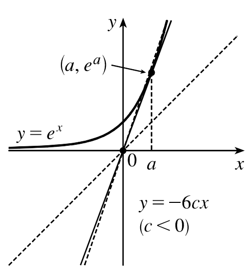
Since $1 \leq f'(0) \leq 1$, we must have $f'(0) = 1$.] But $f'(x) = a^x \ln a \;\Rightarrow\; f'(0) = \ln a$, so we have $\ln a = 1 \;\Leftrightarrow\; a = e$.

Another method: The inequality certainly holds for $x \leq -1$, so consider $x > -1$, $x \neq 0$. Then $a^x \geq 1 + x \;\Rightarrow\; a \geq (1 + x)^{1/x}$ for $x > 0 \;\Rightarrow\; a \geq \lim\limits_{x \to 0^+} (1 + x)^{1/x} = e$, by Equation 6.4.8. Also, $a^x \geq 1 + x \;\Rightarrow\; a \leq (1 + x)^{1/x}$ for $x < 0 \;\Rightarrow\; a \leq \lim\limits_{x \to 0^-} (1 + x)^{1/x} = e$. So since $e \leq a \leq e$, we must have $a = e$.

21. Suppose that the curve $y = a^x$ intersects the line $y = x$. Then $a^{x_0} = x_0$ for some $x_0 > 0$, and hence $a = x_0^{1/x_0}$. We find the maximum value of $g(x) = x^{1/x}$, $x > 0$, because if a is larger than the maximum value of this function, then the curve $y = a^x$ does not intersect the line $y = x$. $g'(x) = e^{(1/x)\ln x}\left(-\dfrac{1}{x^2} \ln x + \dfrac{1}{x} \cdot \dfrac{1}{x}\right) = x^{1/x}\left(\dfrac{1}{x^2}\right)(1 - \ln x)$. This is 0 only where $x = e$, and for $0 < x < e$, $f'(x) > 0$, while for $x > e$, $f'(x) < 0$, so g has an absolute maximum of $g(e) = e^{1/e}$. So if $y = a^x$ intersects $y = x$, we must have $0 < a \leq e^{1/e}$. Conversely, suppose that $0 < a \leq e^{1/e}$. Then $a^e \leq e$, so the graph of $y = a^x$ lies below or touches the graph of $y = x$ at $x = e$. Also $a^0 = 1 > 0$, so the graph of $y = a^x$ lies above that of $y = x$ at $x = 0$. Therefore, by the Intermediate Value Theorem, the graphs of $y = a^x$ and $y = x$ must intersect somewhere between $x = 0$ and $x = e$.

22.





$y = cx^3 + e^x \;\Rightarrow\; y' = 3cx^2 + e^x \;\Rightarrow\; y'' = 6cx + e^x$. The curve will have inflection points when y'' changes sign. $y'' = 0 \;\Rightarrow\; -6cx = e^x$, so y'' will change sign when the line $y = -6cx$ intersects the curve $y = e^x$ (but is not tangent to it). Note that if $c = 0$, the curve is just $y = e^x$, which has no inflection point.

The first figure shows that for $c > 0$, $y = -6cx$ will intersect $y = e^x$ once, so $y = cx^3 + e^x$ will have one inflection point.

The second figure shows that for $c < 0$, the line $y = -6cx$ can intersect the curve $y = e^x$ in two points (two inflection points), be tangent to it (no inflection point), or not intersect it (no inflection point). The tangent line at (a, e^a) has slope e^a, but from the diagram we see that the slope is $\dfrac{e^a}{a}$. So $\dfrac{e^a}{a} = e^a \;\Rightarrow\; a = 1$. Thus, the slope is e. The line $y = -6cx$ must have slope greater than e, so $-6c > e \;\Rightarrow\; c < -e/6$.

Therefore, the curve $y = cx^3 + e^x$ will have one inflection point if $c > 0$ and two inflection points if $c < -e/6$.