Homework 1 - MSSC 6030: Spring 2020

Directions. All work is to be done in *complete sentences*. Assignments must be stapled with a printout of the assignment serving as the first page. Your name is to be written on the *back* of the final page of the assignment. Each problem must be on a *separate* sheet of paper. You are welcome to recycle paper, where one side is crossed out to avoid wasting paper, but your work MUST have **no more than one problem per page**. Each problem write-up must begin with the **full statement of the problem**. While you are encouraged to work through confusion with your classmates, your work must be written in your own words. The assignment is due in class on **Wednesday, February 26, 2020**.

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1. §3.1 - Problem 10 of Strang:

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2. §3.1 - Problem 11 of Strang:

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3. Graph the frequency spectrum of the following periodic functions:

   (a) $f(x) = \sin(x)$

   (b) $f(x) = \sin(x) + \cos(2x)$

   (c) $f(x) = \sin(x) + \cos(x) + 0.5\sin(3x)$

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4. Let $f(x)$ be defined by

$$f(x) = \begin{cases} 0, & -3 < x \leq -1 \\ 1, & -1 < x \leq 1 \\ 0 & 1 < x \leq 3 \end{cases}$$

   and suppose that $f(x+6) = f(x)$, i.e. f has period 6. Find Fourier Series representation for $f(x)$. Plot the original function $f(x)$ as well as its 5, 10, and 20th partial sums

$$s_m(x) = \frac{a_0}{2} + \sum_{n=1}^{m} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

   of the Fourier Series for $f(x)$. Use different colors and markers as well as a legend in your plot. Additionally, plot the frequency spectrum of $f(x)$ for $n = 1, \ldots, 20$. Finally, plot the **absolute value** of the 'error' $e_m(x) = f(x) - s_m(x)$ (versus x) for the same values of $m = 5, 10$, and 20. Where does the largest error in the approximation come, i.e. for which values of x is the approximation poorest?

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5. What is Gibb's Phenomenon? Research this and come up with an example where you can both demonstrate Gibb's phenomenon as well as compute the error predicted by Gibb's phenomenon precisely for your example. (Show that your FS representation can never improve at points of discontinuity to be better than BLANK.)

6. Let $f(x)$ be defined by the sawtooth example $f(x) = x$ for $-L < x < L$ with $f(x + 2L) = f(x)$ as examined in class and notes online. Compute the term-by-term derivative of your FS representation for $f(x)$ and compare it to the actual derivative $f'(x) = 1$. Show all work and plot your results.

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7. Compute the Fourier transform of $f(x)$ (provided that $a > 0$) $f(x) = \begin{cases} e^{-ax} & x \geq 0 \\ 0 & x < 0 \end{cases}$

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8. Solve the problem

$$\begin{array}{rll} \text{PDE} & u_t = \alpha^2 u_{xx}, & -\infty < x < \infty \\ \text{IC} & u(x,0) = e^{-x^2} & -\infty < x < \infty \end{array}$$

by using the Fourier transform.

Math 6030

Homework 2

problem 1

Strang 3.1

10. Use three hat function, with $h = \frac{1}{4}$, to solve $-u'' = 2$ with $u(0) = u(1) = 0$.Verify that the approximation *U* matches $u = x - x^2$ at the nodes.

Solve:

From the problem we know that $f(x) = 2$ (1)

Since $h = \frac{1}{4}$, $n = \frac{1}{h} - 1 = 3$

So the grid point on x are $x_0 = 0, x_1 = \frac{1}{4}, x_2 = \frac{1}{2}, x_3 = \frac{3}{4}, x_4 = 1$.

The hat function will be:

$$\phi_1(x) = \begin{cases} 4x & 0 \le x \le \frac{1}{4} \\ -4x + 2 & \frac{1}{4} \le x \le \frac{1}{2} \\ 0 & \frac{1}{2} \le x \le 1 \end{cases} \phi_2(x) = \begin{cases} 0 & 0 \le x \le \frac{1}{4} \\ 4x - 1 & \frac{1}{4} \le x \le \frac{1}{2} \\ -4x + 3 & \frac{1}{2} \le x \le \frac{3}{4} \\ 0 & \frac{3}{4} \le x \le 1 \end{cases} \phi_3(x) = \begin{cases} 0 & 0 \le x \le \frac{1}{2} \\ 4x - 2 & \frac{1}{2} \le x \le \frac{3}{4} \\ -4x + 4 & \frac{3}{4} \le x \le 1 \end{cases}$$

The graph of these three hat function are:

```
clear
Num_x_hat = 200;
hat_x = linspace(0,1,Num_x_hat);
OneFour = Num_x_hat/4;
TwoFour = Num_x_hat/2;
ThreeFour = 3*Num_x_hat/4;
%define the each stop point.
phi1 = [4*hat_x(1:OneFour) -4*hat_x(OneFour + 1:TwoFour) + 2 ...
    0*hat_x(TwoFour + 1:Num_x_hat)];
%phi1
phi2 = [0*hat_x(1:OneFour) 4*hat_x(OneFour + 1:TwoFour) - 1 ...
    -4*hat_x(TwoFour + 1:ThreeFour) + 3 0*hat_x(ThreeFour + 1:Num_x_hat)];
%phi2
phi3 = [0*hat_x(1:TwoFour) 4*hat_x(TwoFour + 1:ThreeFour) - 2 ...
    -4*hat_x(ThreeFour + 1:Num_x_hat) + 4];
%phi3
plot(hat_x, phi1)
hold on
plot(hat_x, phi2,'-.')
plot(hat_x, phi3,'--')
legend('\phi_1','\phi_2','\phi_3')
hold off
```

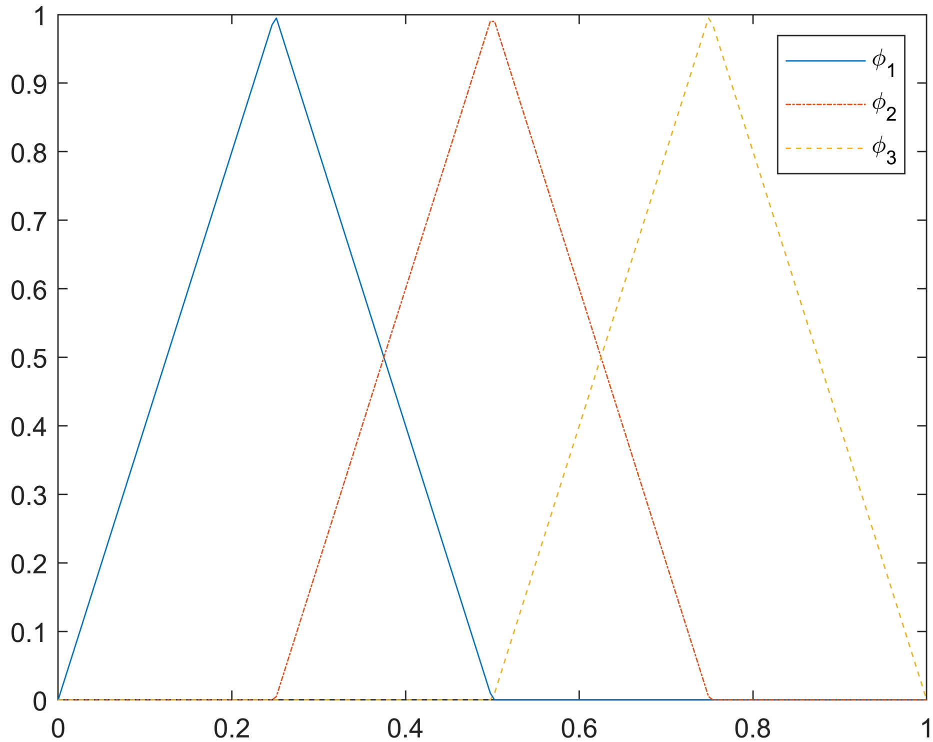


Fig 1.1

The plot shows three hat function by phi1 phi2 and phi3 in the different line style. Which phi1 is in solid line '—', phi2 is in dot line '._.' and phi3 is in '--'.

The Stiffness Martix has the formular:

$\mathrm{Ku} = b$ (2)

which $K = \begin{bmatrix} \langle \phi_1', \phi_1' \rangle & \langle \phi_2', \phi_1' \rangle & \langle \phi_3', \phi_1' \rangle \\ \langle \phi_1', \phi_2' \rangle & \langle \phi_2', \phi_2' \rangle & \langle \phi_3', \phi_2' \rangle \\ \langle \phi_1', \phi_3' \rangle & \langle \phi_2', \phi_3' \rangle & \langle \phi_3', \phi_3' \rangle \end{bmatrix}, u = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}, b = \begin{bmatrix} \langle f(x), \phi_1 \rangle \\ \langle f(x), \phi_2 \rangle \\ \langle f(x), \phi_3 \rangle \end{bmatrix}$

Find the derivative of $\phi_1 \phi_2 \phi_3$:

$$\phi_1'(x) = \begin{cases} 4 & 0 \le x \le \frac{1}{4} \\ -4 & \frac{1}{4} \le x \le \frac{1}{2} \\ 0 & \frac{1}{2} \le x \le 1 \end{cases} \phi_2'(x) = \begin{cases} 0 & 0 \le x \le \frac{1}{4} \\ 4 & \frac{1}{4} \le x \le \frac{1}{2} \\ -4 & \frac{1}{2} \le x \le \frac{3}{4} \\ 0 & \frac{3}{4} \le x \le 1 \end{cases} \phi_3'(x) = \begin{cases} 0 & 0 \le x \le \frac{1}{2} \\ 4 & \frac{1}{2} \le x \le \frac{3}{4} \\ -4 & \frac{3}{4} \le x \le 1 \end{cases} \qquad (3)$$

Then compute (2) with (3) :

$$\langle \phi_1', \phi_1' \rangle = \int_0^1 (\phi_1'(x))^2 \mathrm{dx} = \int_0^{\frac{1}{4}} 4^2 \mathrm{dx} + \int_{\frac{1}{4}}^{\frac{1}{2}} (-4)^2 \mathrm{dx} + \int_{\frac{1}{2}}^{1} 0 \mathrm{dx} = 8$$

$$\langle \phi_2', \phi_2' \rangle = \int_0^1 (\phi_2'(x))^2 \mathrm{dx} = \int_0^{\frac{1}{4}} 0 \mathrm{dx} + \int_{\frac{1}{4}}^{\frac{1}{2}} 4^2 \mathrm{dx} + \int_{\frac{1}{2}}^{\frac{3}{4}} (-4)^2 \mathrm{dx} + \int_{\frac{3}{4}}^{1} 0 \mathrm{dx} = 8$$

$$\langle \phi_3', \phi_3' \rangle = \int_0^1 (\phi_3'(x))^2 \mathrm{dx} = \int_0^{\frac{1}{2}} 0 \mathrm{dx} + \int_{\frac{1}{2}}^{\frac{3}{4}} (4)^2 \mathrm{dx} + \int_{\frac{3}{4}}^{1} (-4)^2 \mathrm{dx} = 8$$

$$\langle \phi_1', \phi_2' \rangle = \langle \phi_2', \phi_1' \rangle = \int_0^1 \phi_1'(x)\phi_2'(x) \mathrm{dx} = \int_0^{\frac{1}{4}} 0 \mathrm{dx} + \int_{\frac{1}{4}}^{\frac{1}{2}} -16 \mathrm{dx} + \int_{\frac{1}{2}}^{\frac{3}{4}} 0 \mathrm{dx} + \int_{\frac{3}{4}}^{1} 0 \mathrm{dx} = -4$$

$$\langle \phi_1', \phi_3' \rangle = \langle \phi_3', \phi_1' \rangle = \int_0^1 \phi_1'(x)\phi_3'(x) \mathrm{dx} = \int_0^{\frac{1}{4}} 0 \mathrm{dx} + \int_{\frac{1}{4}}^{\frac{1}{2}} 0 \mathrm{dx} + \int_{\frac{1}{2}}^{\frac{3}{4}} 0 \mathrm{dx} + \int_{\frac{3}{4}}^{1} 0 \mathrm{dx} = 0$$

$$\langle \phi_2', \phi_3' \rangle = \langle \phi_3', \phi_2' \rangle = \int_0^1 \phi_2'(x)\phi_3'(x) \mathrm{dx} = \int_0^{\frac{1}{4}} 0 \mathrm{dx} + \int_{\frac{1}{4}}^{\frac{1}{2}} 0 \mathrm{dx} + \int_{\frac{1}{2}}^{\frac{3}{4}} -16 \mathrm{dx} + \int_{\frac{3}{4}}^{1} 0 \mathrm{dx} = -4$$

So $K = \begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix}$ (4)

Then compute b with (1):

$$\langle f(x), \phi_1 \rangle = \int_0^1 f(x)\phi_1(x) = \int_0^{\frac{1}{4}} 8x \mathrm{dx} + \int_{\frac{1}{4}}^{\frac{2}{4}} (-8x+4) \mathrm{dx} + \int_{\frac{1}{2}}^{1} 0 \mathrm{dx} = \frac{1}{2}$$

$$\langle f(x), \phi_2 \rangle = \int_0^1 f(x)\phi_2(x) = \int_0^{\frac{1}{4}} 0 \mathrm{dx} + \int_{\frac{1}{4}}^{\frac{2}{4}} (8x-2) \mathrm{dx} + \int_{\frac{1}{2}}^{\frac{3}{4}} (-8x+6) \mathrm{dx} + \int_{\frac{3}{4}}^{1} 0 \mathrm{dx} = \frac{1}{2}$$

$$\langle f(x), \phi_3 \rangle = \int_0^1 f(x)\phi_3(x) = \int_0^{\frac{1}{2}} 0 \mathrm{dx} + \int_{\frac{1}{2}}^{\frac{3}{4}} (8x-4) \mathrm{dx} + \int_{\frac{3}{4}}^{1} (-8x+8) \mathrm{dx} = \frac{1}{2}$$

so $b = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ (5)

So we could compute u from $\mathrm{Ku} = b$

```
h = 1/4;%define h
n = 1/h - 1;%compute n
x = linspace(0, 1, 100);%define x
xg = linspace(0,1,n + 2);%true x with 5 points
up = xg - xg.^2;%true y with 5 points
u = x - x.^2;%true solution
b = [1/2; 1/2; 1/2];%equation (5)
k = [8 -4 0;
    -4 8 -4;
     0 -4 8;];%equaiton (4)
coe = k\b;%solve equation (2)
coe = [0;coe;0]
```

```
coe = 5×1
         0
    0.1875
    0.2500
    0.1875
         0
```

```
error=abs(coe(:) - up(:))%compute the error
```

```
error = 5×1
10^-15 ×
         0
         0
    0.1110
    0.0278
         0
```

The solving the Ku=b are coe=[0 0.1875 0.2500 0.1875 0]

Also we could see the error of the result we compute are small enough.

So if we graph the solution and true value on the same page.

```
%plot
plot(x, u, '.')
hold on
plot(xg, up, '*')
plot(xg, coe, 'o', 'Markersize', 10, 'Color','r')
legend('true solution', 'true solution with grid points', 'estimate grid points')
xlabel('x')
ylabel('y')
title('plot of estimat grid point and true solution')
hold off
```

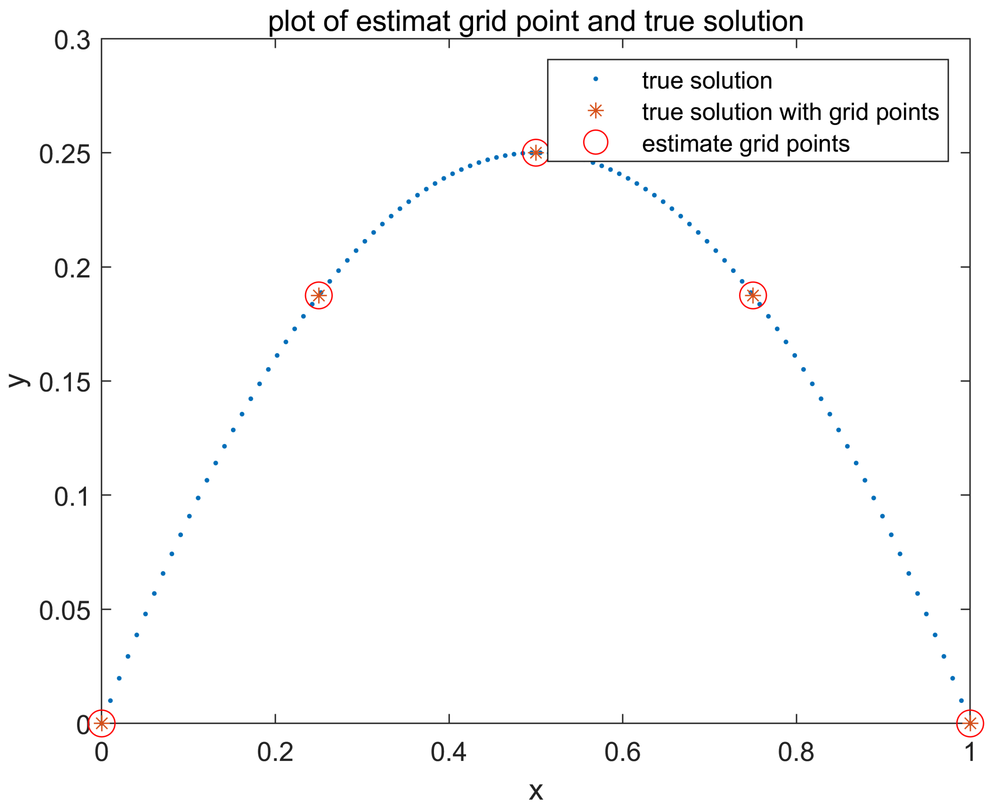


Fig 1.2 This plot shows the result we compute by FEM compare with true solution. On the graph they on the same spot.

problem 2

Strang 3.1

11. solve $-u'' = x$ with $u(0) = u(1) = 0$. Here $u(x)$ is cubic. Then solve approximately with two hat function and $h = \frac{1}{3}$. Where is the largest error?

Solve:

First solving $-u'' = x$ by using anti-derivatives:

$$u'' = -x$$

$$u' = -\int \mathrm{xdx} = -\frac{1}{2}x^2 + C_1$$

$$u = \int \left(-\frac{1}{2}x^2 + C_1\right)\mathrm{dx} = -\frac{1}{6}x^3 + C_1 x + C_2$$

by using the boundary condition $u(0) = u(1) = 0$

$$u(0) = C_2 = 0$$

$$u(1) = -\frac{1}{6} + C_1 = 0$$

So $C_1 = \frac{1}{6}$

So $u(x) = -\frac{1}{6}x^3 + \frac{1}{6}x$ (6)

Then with FEM

$$h = \frac{1}{3}$$

$$n = \frac{1}{h} - 1 = 2$$

So the grid point on x are $x_0 = 0, x_1 = \frac{1}{3}, x_2 = \frac{2}{3}, x_3 = 1$.

The hat function will be:

$$\phi_1(x) = \begin{cases} 3x & 0 \le x \le \frac{1}{3} \\ -3x + 2 & \frac{1}{3} \le x \le \frac{2}{3} \\ 0 & \frac{2}{3} \le x \le 1 \end{cases}, \phi_2(x) = \begin{cases} 0 & 0 \le x \le \frac{1}{3} \\ 3x - 1 & \frac{1}{3} \le x \le \frac{2}{3} \\ -3x + 3 & \frac{2}{3} \le x \le 1 \end{cases}$$

The graph of these three hat function are:

```
clear
Num_x_hat = 201;
hat_x = linspace(0, 1, Num_x_hat);
OneThird = Num_x_hat/3;
TwoThird = 2*Num_x_hat/3;
%define the each stop point.
phi1 = [3*hat_x(1:OneThird) -3*hat_x(OneThird + 1:TwoThird) + 2 ...
    0*hat_x(TwoThird + 1:Num_x_hat)];
%phi1
phi2 = [0*hat_x(1:OneThird) 3*hat_x(OneThird+1:TwoThird)-1 ...
    -3*hat_x(TwoThird + 1:Num_x_hat) + 3];
%phi2
%plot
plot(hat_x, phi1)
hold on
plot(hat_x, phi2,'--')
legend('\phi_1', '\phi_2')
hold off
```

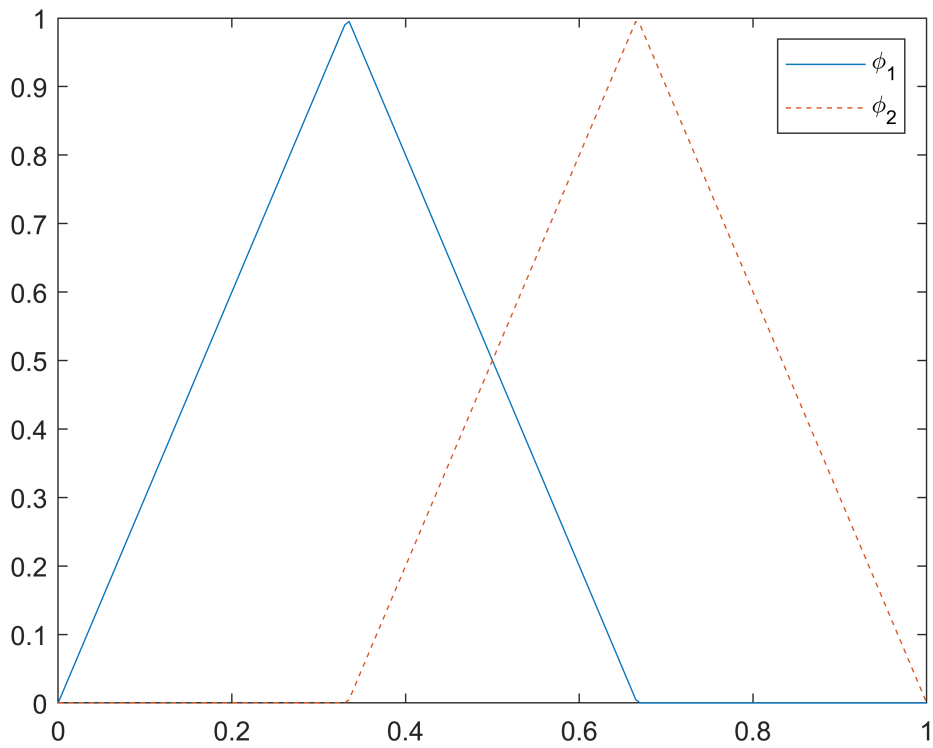


Fig 2.1

The plot shows three hat function by phi1 and phi2 in the different line style. Which phi1 is in solid line '—' and phi2 is in '--'.

The Stiffness Martix has the formular:

$$\mathrm{Ku} = b \qquad (7)$$

which $K = \begin{bmatrix} \langle \phi_1', \phi_1' \rangle & \langle \phi_2', \phi_1' \rangle \\ \langle \phi_1', \phi_2' \rangle & \langle \phi_2', \phi_2' \rangle \end{bmatrix}$, $u = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$, $b = \begin{bmatrix} \langle f(x), \phi_1 \rangle \\ \langle f(x), \phi_2 \rangle \end{bmatrix}$

Find the derivative of ϕ_1 and ϕ_2:

$$\phi_1(x) = \begin{cases} 3 & 0 \le x \le \frac{1}{3} \\ -3 & \frac{1}{3} \le x \le \frac{2}{3} \\ 0 & \frac{2}{3} \le x \le 1 \end{cases}, \ \phi_2(x) = \begin{cases} 0 & 0 \le x \le \frac{1}{3} \\ 3 & \frac{1}{3} \le x \le \frac{2}{3} \\ -3 & \frac{2}{3} \le x \le 1 \end{cases} \qquad (8)$$

Then compute (7) with (8) :

$$\langle \phi_1', \phi_1' \rangle = \int_0^1 (\phi_1'(x))^2 \mathrm{dx} = \int_0^{\frac{1}{3}} 3^2 \mathrm{dx} + \int_{\frac{1}{3}}^{\frac{2}{3}} (-3)^2 \mathrm{dx} + \int_{\frac{2}{3}}^{1} 0 \mathrm{dx} = 6$$

$$\langle \phi_2', \phi_2' \rangle = \int_0^1 (\phi_2'(x))^2 \mathrm{dx} = \int_0^{\frac{1}{3}} 0 \mathrm{dx} + \int_{\frac{1}{3}}^{\frac{2}{3}} 3^2 \mathrm{dx} + \int_{\frac{2}{3}}^{1} (-3)^2 \mathrm{dx} = 6$$

$$\langle \phi_1', \phi_2' \rangle = \langle \phi_2', \phi_1' \rangle = \int_0^1 \phi_1'(x)\phi_2'(x) \mathrm{dx} = \int_0^{\frac{1}{3}} 0 \mathrm{dx} + \int_{\frac{1}{3}}^{\frac{2}{3}} -9 \mathrm{dx} + \int_{\frac{2}{3}}^{1} 0 \mathrm{dx} = -3$$

So $K = \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}$ (9)

Then compute b with $f(x) = x$:

$$\phi_1(x) = \begin{cases} 3x & 0 \le x \le \frac{1}{3} \\ -3x + 2 & \frac{1}{3} \le x \le \frac{2}{3} \\ 0 & \frac{2}{3} \le x \le 1 \end{cases}, \phi_2(x) = \begin{cases} 0 & 0 \le x \le \frac{1}{3} \\ 3x - 1 & \frac{1}{3} \le x \le \frac{2}{3} \\ -3x + 3 & \frac{2}{3} \le x \le 1 \end{cases}$$

$$\langle f(x), \phi_1 \rangle = \int_0^1 f(x)\phi_1(x) = \int_0^{\frac{1}{3}} 3x^2 \mathrm{dx} + \int_{\frac{1}{3}}^{\frac{2}{3}} (-3x^2 + 2x)\mathrm{dx} + \int_{\frac{2}{3}}^{1} 0\mathrm{dx}$$

$$= x^3|_0^{\frac{1}{3}} + (-x^3 + x^2)_{\frac{1}{3}}^{\frac{2}{3}} = \frac{1}{27} + \left(\left(-\frac{8}{27} + \frac{4}{9}\right) - \left(-\frac{1}{27} + \frac{1}{9}\right)\right)$$

$$= \frac{1}{27} + \left(\frac{4}{27} - \frac{2}{27}\right) = \frac{1}{9}$$

$$\langle f(x), \phi_2 \rangle = \int_0^1 f(x)\phi_2(x) = \int_0^{\frac{1}{3}} 0\mathrm{dx} + \int_{\frac{1}{3}}^{\frac{2}{3}} (3x^2 - \mathrm{xdx}) + \int_{\frac{2}{3}}^{1} (-3x^2 + 3x)\mathrm{dx}$$

$$= \left(x^3 - \frac{1}{2}x^2\right)_{\frac{1}{3}}^{\frac{2}{3}} + \left(-x^3 + \frac{3}{2}x^2\right)_{\frac{2}{3}}^{1} = \left(\left(\frac{8}{27} - \frac{1}{2} * \frac{4}{9}\right) - \left(\frac{1}{27} - \frac{1}{2} * \frac{1}{9}\right)\right) + \left(\left(-1 + \frac{3}{2}\right) - \left(-\frac{8}{27} + \frac{3}{2} * \frac{4}{9}\right)\right)$$

$$= \left(\frac{2}{27} + \frac{1}{54}\right) + \left(\frac{1}{2} - \frac{10}{27}\right) = \frac{5}{54} + \frac{7}{54} = \frac{12}{54} = \frac{2}{9}$$

so $b = \begin{bmatrix} \frac{1}{9} \\ \frac{2}{9} \end{bmatrix}$ (10)

So we could compute u from $\mathrm{Ku} = b$

```
h = 1/3;%define h
n = 1/h - 1;%compute n
Num_x = 99;%number of interval point
x = linspace(0, 1, Num_x);%define x
xg = linspace(0, 1, n + 2);%true x with 4 points
up = -1/6*xg.^3 + 1/6*xg;%true y with 4 points
u = -1/6*x.^3 + 1/6*x;%true solution
b = [1/9;2/9];%b from equation 10
k = [6 -3;
    -3 6];%K from equation 9
coe = k\b;%compute u from Ku=b
coe_final = [0;coe;0]
```

```
coe_final = 4×1
         0
    0.0494
    0.0617
         0
```

```
error = abs(coe_final(:) - up(:))%compute the error of four grid point
```

```
error = 4×1
10^-16 ×
         0
    0.1388
         0
         0
```

Therefore the solution is [0 0.0494 0.0617 0]

Also the error on the grid points are very small.

To compute the largest error we could plug the result into the equation:

$$f(x) = \alpha_1 \phi_1(x) + \alpha_2 \phi_2(x) \qquad (11)$$

```
x_1 = Num_x/3;%define the grid 0 to 1/3
x_2 = 2*Num_x/3;%define the grid 1/3 to 2/3
U_app = [3*x(1:x_1)*coe(1) ...
    (-3*x(x_1 + 1:x_2) + 2)*coe(1) + (3*x(x_1 + 1:x_2) - 1)*coe(2) ...
    (-3*x(x_2 + 1:Num_x) + 3)*coe(2)];%equation (11)
largest_error = max(abs(U_app - u))%compute the largest error
```

```
largest_error = 0.0116
```

Therefore we could see the largest error is 0.0116

```
%plot
plot(x, u, '.')
hold on
plot(xg, up, '*')
plot(xg, coe_final, 'o', 'Markersize', 10, "Color", 'r')
plot(x, U_app)
legend('true solution', 'true solution of the grid points', ...
    'estimate grid points', 'estimate solution', 'location', 'south')
xlabel('x')
ylabel('y')
title('plot of estimat grid point and true solution')
hold off
```

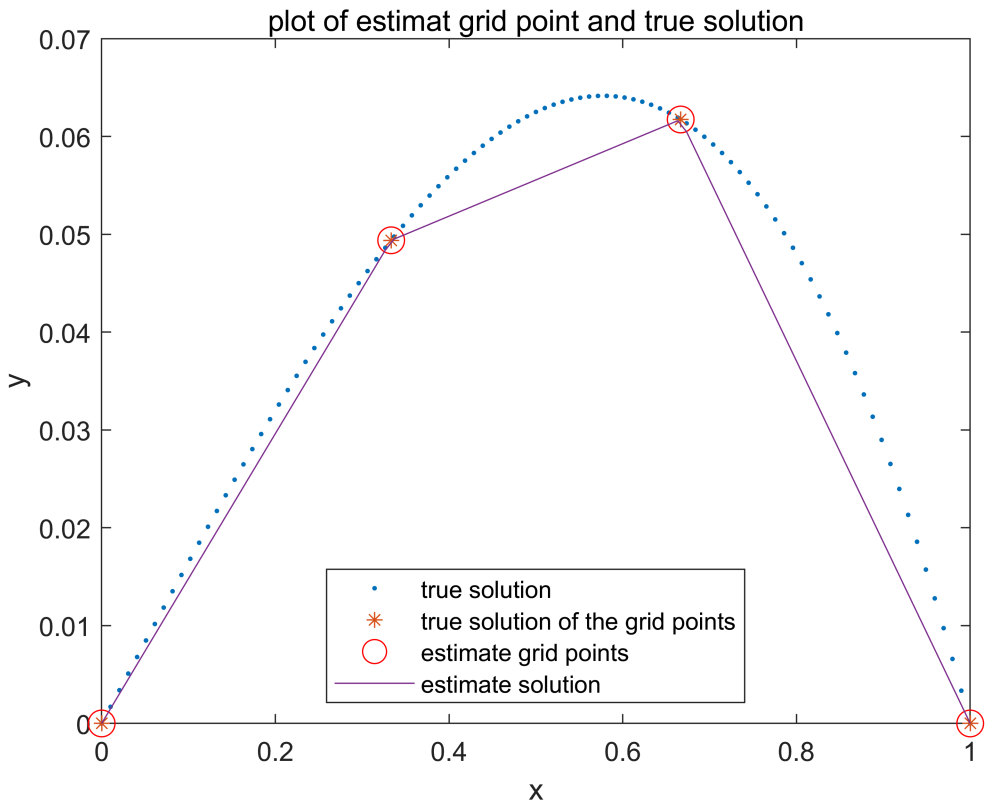


Fig 2.2 This plot shows the result we compute by FEM compare with true solution. The grid points, on the graph they on the same spot. The solid line represent function from FEM we coud see there still are difference from the true solution.

Problem 3

Graph the frequency spectrum of the following periodic functions:

(a) $f(x) = \sin(x)$

(b) $f(x) = \sin(x) + \cos(2x)$

(c) $f(x) = \sin(x) + \cos(2x) + 0.5\sin(3x)$

Solve:

(a) $f(x) = \sin(x)$ (12)

Periodic function could be written as Fourier Series equation:

$$f(x) = \sum_{n=0}^{\infty} \left[A_n \cos(nx) + B_n \sin(nx) \right] \tag{13}$$

By insert equation (12) to (13). We could find:

$A_1 = 0$, $B_1 = 1$,

Since the equation of spectrum C_n is:

$$C_n = \sqrt{A_n^{\ 2} + B_n^{\ 2}} \tag{14}$$

So if take result of A and B to equation (14)

$C_1 = \sqrt{B_1^{\ 2}} = \sqrt{1^2} = 1$

The plot is:

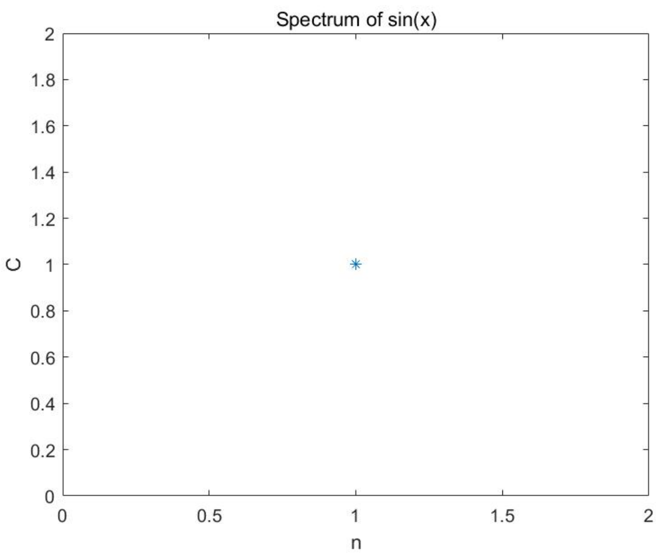


Fig 3.1 The plot of spectrum of sin(x) on the point n=1 is 1

(b) $f(x) = \sin(x) + \cos(2x)$ (15)

From the equation (13) we could get

$A_1 = 0$, $B_1 = 1$,

$A_2 = 1$, $B_1 = 0$,

So we could plug this two conditions to (14):

$$C_1 = \sqrt{{A_1}^2 + {B_1}^2} = \sqrt{1^2} = 1$$

$$C_2 = \sqrt{{A_2}^2 + {B_2}^2} = \sqrt{1^2} = 1$$

So the graph of the spectrum is:

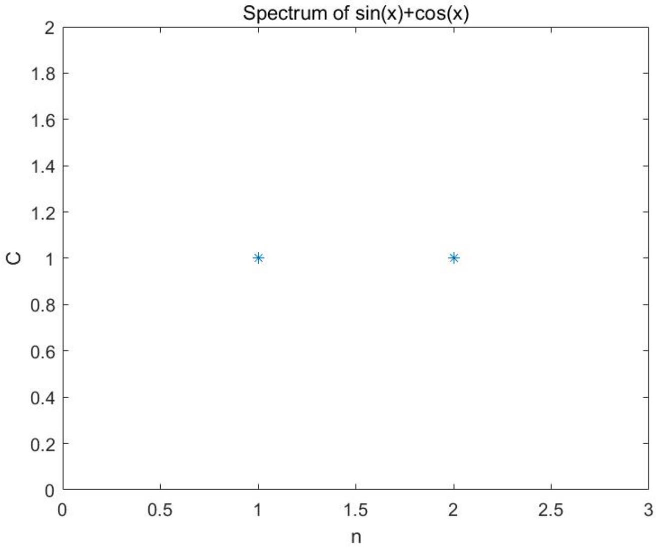


Fig3.2 The plot of spectrum of sin(x)+cos(x) on the point n=1 is 1, on the point n=2 is 1

(c) $f(x) = \sin(x) + \cos(2x) + 0.5\sin(3x)$ (15)

From the equation (13) we could get

$A_1 = 1$, $B_1 = 1$,

$A_2 = 0$, $B_1 = 0$,

$A_3 = 0$, $B_3 = 0.5$,

So we could plug this two conditions to (14):

$$C_1 = \sqrt{{A_1}^2 + {B_1}^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$C_3 = \sqrt{{A_3}^2 + {B_3}^2} = \sqrt{(0.5)^2} = \frac{1}{2}$$

So the graph of the spectrum is:

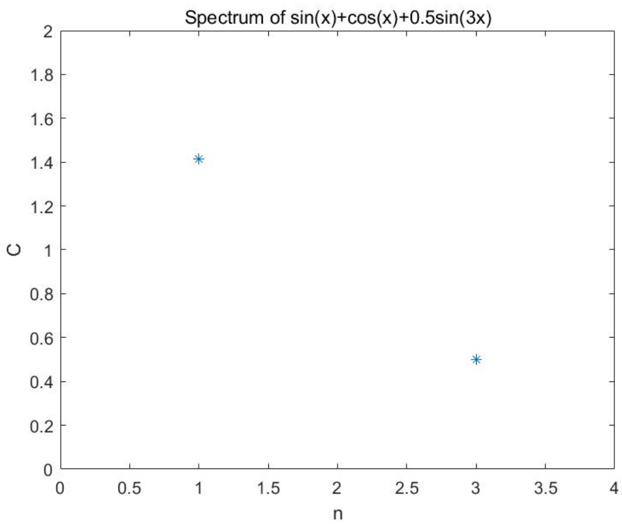


Fig 3.2 The plot of spectrum of sin(x)+cos(x) on the point n=1 is sqrt(2), on the point n=3 is 0.5

problem 4

Let $f(x)$ be defined by:

$$f=\begin{cases}0 & -3\le x\le -1\\ 1 & -1\le x\le 1\\ 0 & 1\le x\le 3\end{cases} \qquad (16)$$

and suppose that $f(x+6)=f(x), i.e. f$ has period 6. Find Fourier Series representation for $f(x)$.

Plot the origional function $f(x)$ as well as its 5, 10 and 20th partial sums

$$s_m(x)=\frac{a_0}{2}+\sum_{n=1}^{m}\left[a_n\cos\left(\frac{n\pi x}{L}\right)+b_n\sin\left(\frac{n\pi x}{L}\right)\right] \qquad (17)$$

of the Fourier Series for $f(x)$. Use different colors and markers as well as a legend in your plot. Additionally, plot the frequency spectrum of $f(x)$ for $n=1,\ldots,20$. Finally, plot the absolute value of the \`error' $e_m(x)=f(x)-s_m(x)$ (versus *x*) for the same values of m=5, 10 and 20. Where does the largest error in the approximation come, i.e. for which values of *x* is the approximation poorest?

Solve:

From the problem we could found that period is 6, therefore L=3.

we could compute A_n and B_n by using the following equation:

$$a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{dx}, \text{ where } n=0,1,2,3,\ldots \qquad (18)$$

$$b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{dx}, \text{ where } n=1,2,3,\ldots \qquad (19)$$

Then we can plug equation (16) into (18) and (19)

$$\begin{aligned}
a_n&=\frac{1}{3}\int_{-3}^{3}f(x)\cos\left(\frac{n\pi x}{3}\right)\mathrm{dx}\\
&=\frac{1}{3}\left(\int_{-3}^{-1}0*\cos\left(\frac{n\pi x}{3}\right)\mathrm{dx}+\int_{-1}^{1}1*\cos\left(\frac{n\pi x}{3}\right)\mathrm{dx}+\int_{-3}^{3}0*\cos\left(\frac{n\pi x}{3}\right)\mathrm{dx}\right)\\
&=\frac{1}{3}\int_{-1}^{1}\cos\left(\frac{n\pi x}{3}\right)\mathrm{dx}\\
&=\frac{1}{3}\frac{3}{n\pi}\sin\left(\frac{n\pi x}{3}\right)\Bigg|_{-1}^{1}\\
&=\frac{2}{n\pi}\sin\left(\frac{n\pi}{3}\right)
\end{aligned} \qquad (20)$$

Where $n=1,2,3,\ldots$

for a_0

$$a_0 = \frac{1}{3}\int_{-3}^{3} f(x)\cos\left(\frac{0 * \pi x}{3}\right)\mathrm{dx}$$
$$= \frac{1}{3}\int_{-1}^{1} 1\,\mathrm{dx}$$
$$= \frac{2}{3}$$

$$b_n = \frac{1}{3}\int_{-3}^{3} f(x)\sin\left(\frac{n\pi x}{3}\right)\mathrm{dx} = \frac{1}{3}\int_{-1}^{1} \sin\left(\frac{n\pi x}{3}\right)\mathrm{dx}$$

Since sin function is odd function, the integral is been taken over a symmetric interval from -1 to 1. So the result of integration is 0.

So $b_n = 0$.

So we can fill equation (17) by using the coefficien a_0, a_n and b_n

$$s_m(x) = \frac{1}{3} + \sum_{n=1}^{m}\left[\frac{2}{n\pi}\sin\left(\frac{n\pi}{3}\right)\cos\left(\frac{n\pi x}{3}\right)\right] \qquad (21)$$

Then we could compute the sum for each n=5,10 and 20.

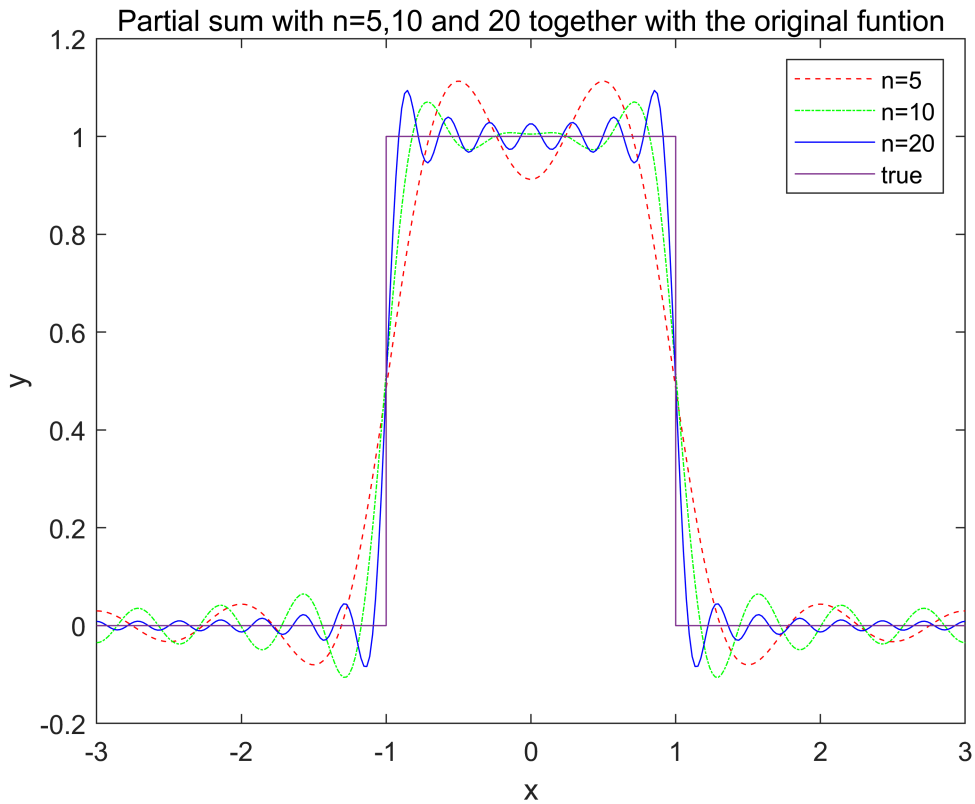
```
clear
format long
k = 300;%number of x
x = linspace(-3, 3, 300);%define x
n5 = sm(5, x);%sum of equation (21) with n=5
n10 = sm(10, x);%sum of equation (21) with n=10
n20 = sm(20, x);%sum of equation (21) with n=20

%original function
x1 = [-3, -1];%x value from -3 to -1
y1 = [0, 0];%y value from -3 to -1
x2 = x1 + 2;%x value from -1 to 1
y2 = y1 + 1;%y value from -1 to 1
x3 = x2 + 2;%x value from 1 to 3
y3 = y1;%y value from 1 to 3
xt=[x1 x2 x3];%get x1 x2 x3 together
yt=[y1 y2 y3];%get y1 y2 y3 together
%finishing produce the original function
```

Then plot n=5,10,20 with original function together.

```
%plot
plot(x, n5, '--', 'Color', 'r')
hold on
plot(x, n10, '-.', 'Color', 'g')
plot(x, n20, '-', 'Color', 'b')
plot(xt, yt)
legend('n=5', 'n=10', 'n=20', 'true')
xlabel('x')
```

```
ylabel('y')
title('Partial sum with n=5,10 and 20 together with the original funtion')
hold off
```




```
%finish plot
```

Fig 4.1 In the plot the stright solid line is the true function f(x), n=5 has the line '--', n=10 has line '-.-' also the curved solid line represents the n=20.

Recall the equation from problem 3

The spectrum can be calculate by

$$C_n = \sqrt{A_n^2 + B_n^2}$$

since $B_n = 0$, $C_n = \sqrt{A_n^2}$ (22)

```
n = 20;%define max n
spy = zeros(n, 1);
for j = 1:n
    spy(j) = sqrt(an(j)*an(j));%equation (22) and restoring each Cn
end
```

And plot

```
%plot Cn
```

```
plot(spy, '*')
title('Spectrum of S_m until n=20')
xlabel('n')
ylabel('C_n')
```

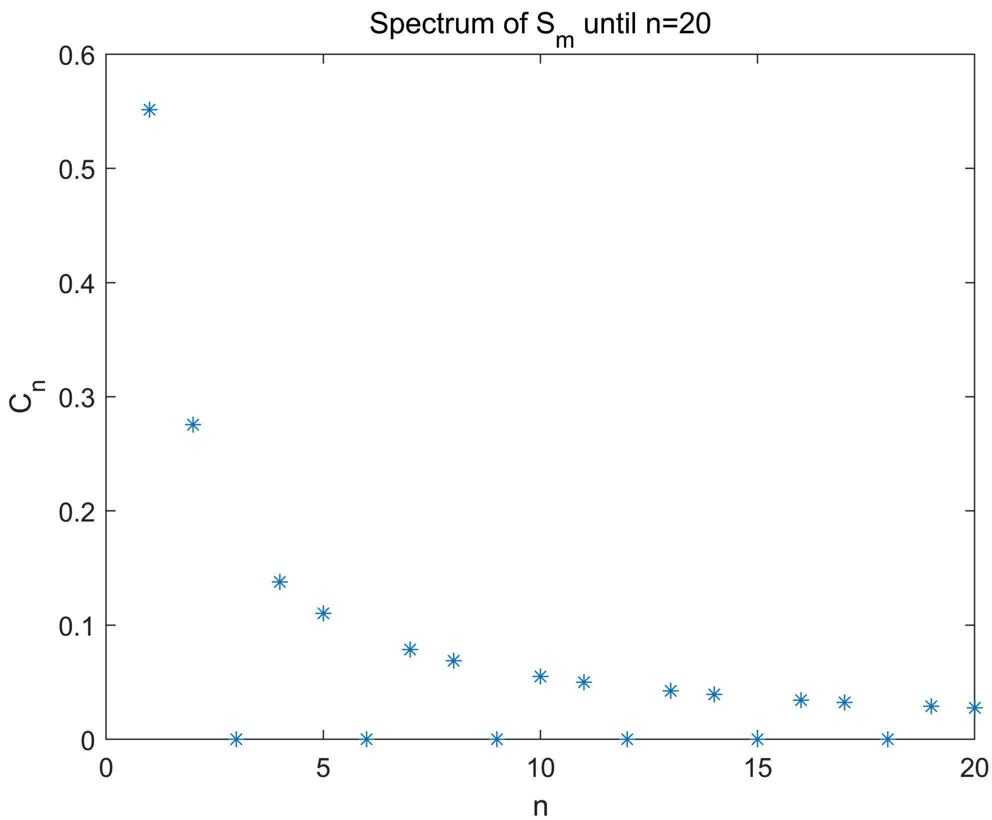


Fig 4.2 Spectrum of S_m until n=20

The error could be calculate by $e_m(x) = f(x) - s_m(x)$ (23)

```
%original function
A = zeros(1, k/3);
B = ones(1, k/3);
g = [A B A];
%finishing produce the original function
e5 = abs(g - n5);%equation (23)
e10 = abs(g - n10);
e20 = abs(g - n20);
```

Then plot them

```
plot(x, e5, 'o')
xlabel('x')
ylabel('absolute error')
title('absolute error vs x when n=5')
```

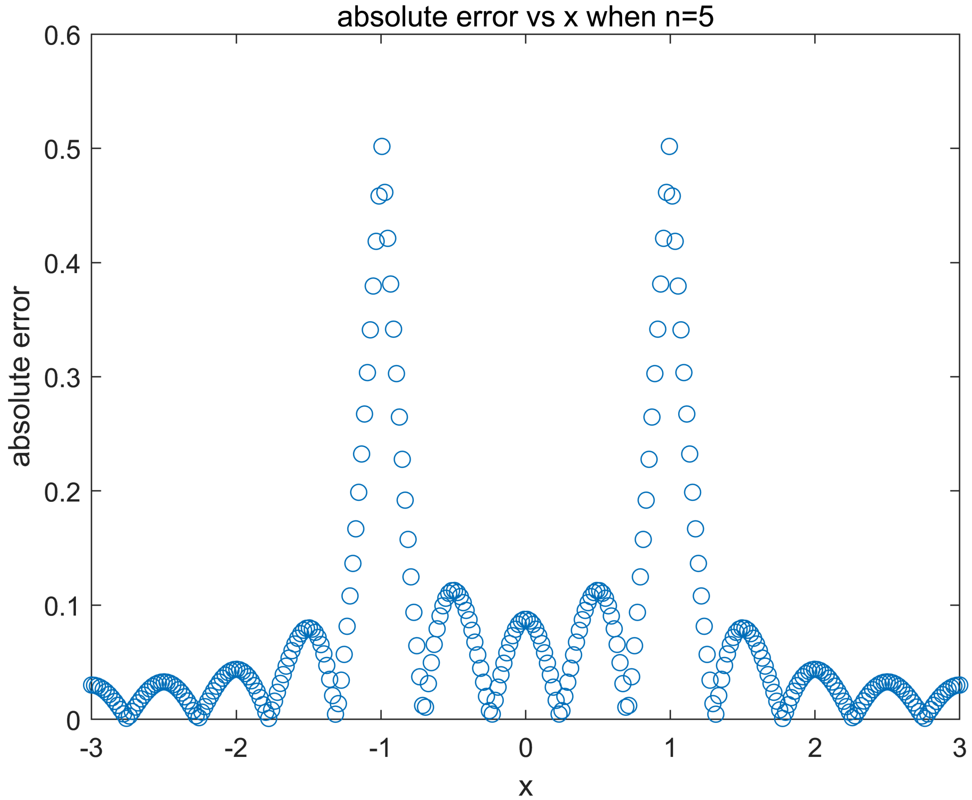


Fig 4.3a The plot shows absolute error when n=5, we could see a big error around -1 and 1

```
plot(x, e10, 'o')
xlabel('x')
ylabel('absolute error')
title('absolute error vs x when n=10')
```

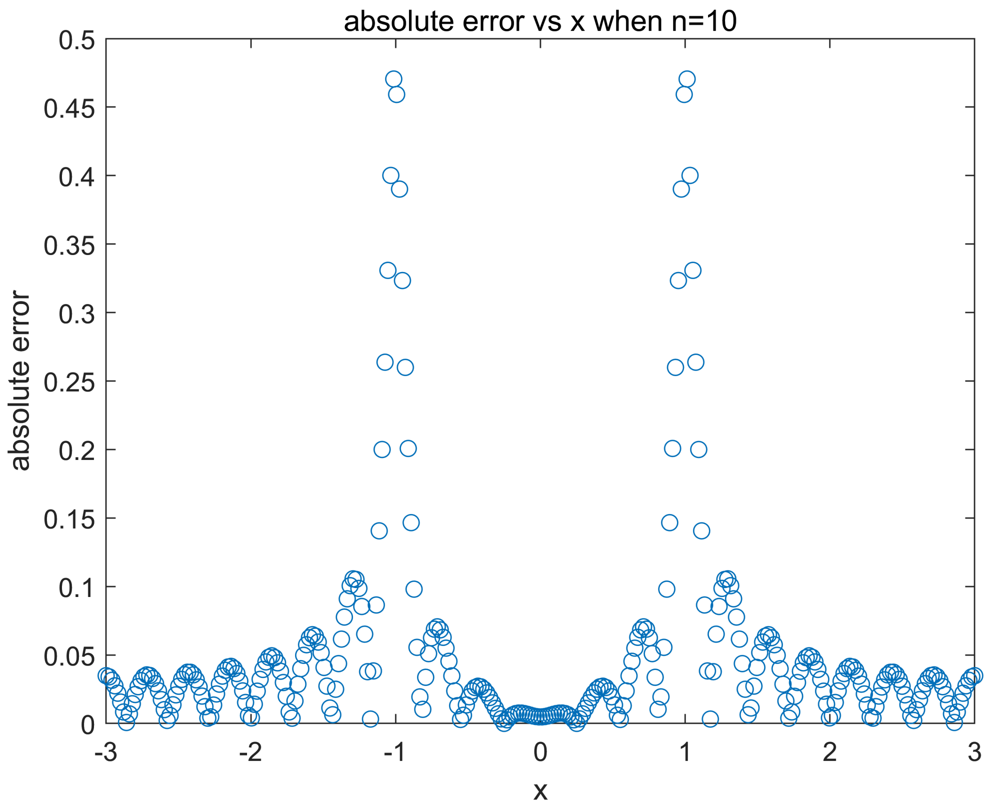


Fig 4.3b

```
plot(x, e20, 'o')
xlabel('x')
ylabel('absolute error')
title('absolute error vs x when n=20')
```

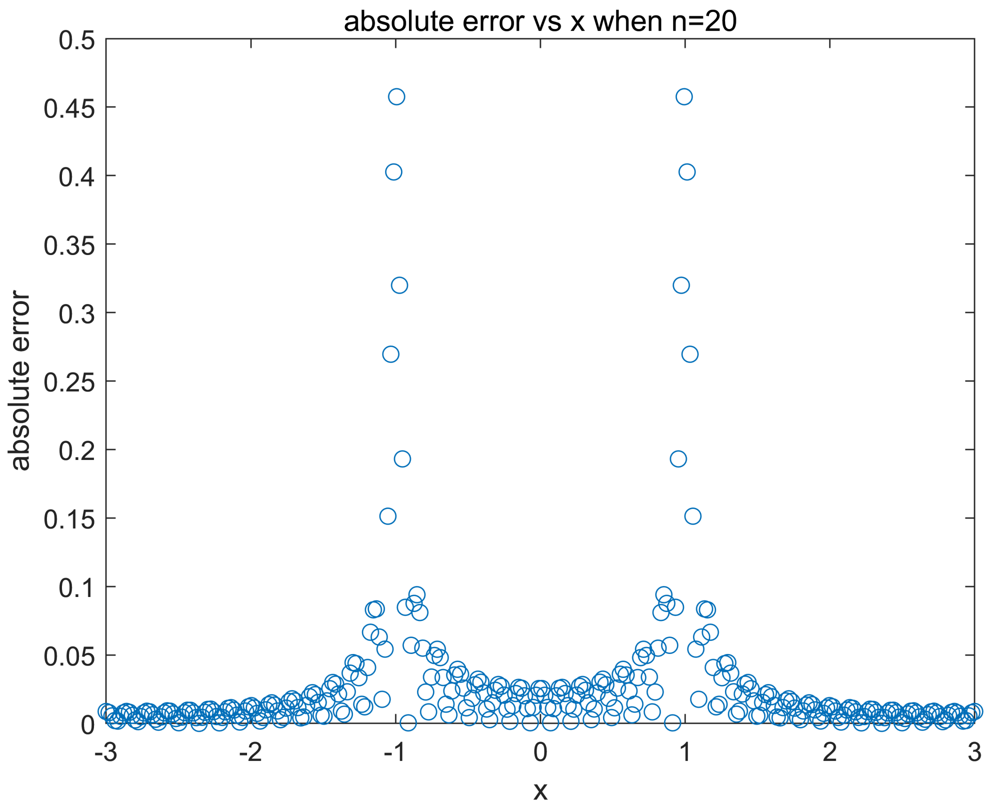


Fig 4.3c

Then the sum of each error are:

```
errn5 = sum(e5)
```

```
errn5 =
  23.526405630863092
```

```
errn10 = sum(e10)
```

```
errn10 =
  15.892651600874025
```

```
errn20 = sum(e20)
```

```
errn20 =
   9.026425762739727
```

The total error for n=5 is aroud 23.5, for n=10 is about 15.9, and for n=20 is about 9.0.

So as a conclusion the largest error between n=5,10 and 20 are at the point -1 and 1 where is the stopped point in the original function.

Also when the number of partial sum takes, the sum of error are also been reduced, but due to the Gibb's Phenomenon, the approching on the point -1 and 1 will not have a enough error no matter how large of n we use.

```
function [s] = sm(m, x)
%function of calculate partial sum from equation (21)
    s=0;%initial s
    for j =1:m
        s = s + an(j).*cos(j*pi.*x/3);%take the sum up to n
    end
    s = s + 1/3;%add a_0
end

function [a] = an(n)
%function of an from equation (20)
    a = 2/(n*pi)*sin(n*pi/3);
end
```

problem 5

What is Gibb's Phenomenon? Research this and come up with an example where you can both demonstrate Gibb's phenomenon as well as compute the error predicted by Gibb's phenomenon precisely for your example. (Show that your FS representation can never improve at points of discontinuity to be better than BLANK.)

Solve:

Gibb's Phenomenon is the phenomenon that the Fourier Series can not represent well at the jump point or corner of period function.

The example used here are the same as the last problem

$$f = \begin{cases} 0 & -3 \le x \le -1 \\ 1 & -1 \le x \le 1 \\ 0 & 1 \le x \le 3 \end{cases} \qquad (24)$$

The test are only focus on the interval [-1, 1].

From the last problem we know that, the error can be compute as:

$$e_m(x) = f(x) - s_m(x) \qquad (25)$$

Then we can test for different num of partial sum and compute the equation (25) and find the largest the result.

```
clear
num_n = 1000;%number of partial sum will be tested
```

```
num_n =
        1000
```

```
k = num_n;%number of interval
x = linspace(-1, 1, k);%define x

jx = zeros(1, num_n);
maxx = zeros(1, num_n);
%loop for each number of n then calculate the nth sum of Fourier Series
for j = 1:num_n
    jx(j) = j;
    maxx(j) = max(sm(j, x)) - 1;%compute the nth sum of Fourier Series.
    % Then compute the difference with 1
    %Then take the largest on
end
Mean_FS_error = mean(maxx)%roughly calculate the averge of all off the error
```

```
ans =
   0.083156516012034
```

```

```

Then if we plot them

```
plot(jx, maxx)
xlabel('n')
ylabel('error')
```

```
title('error from Fourier Series at the edge of the function vs the nth Fourier Series')
```

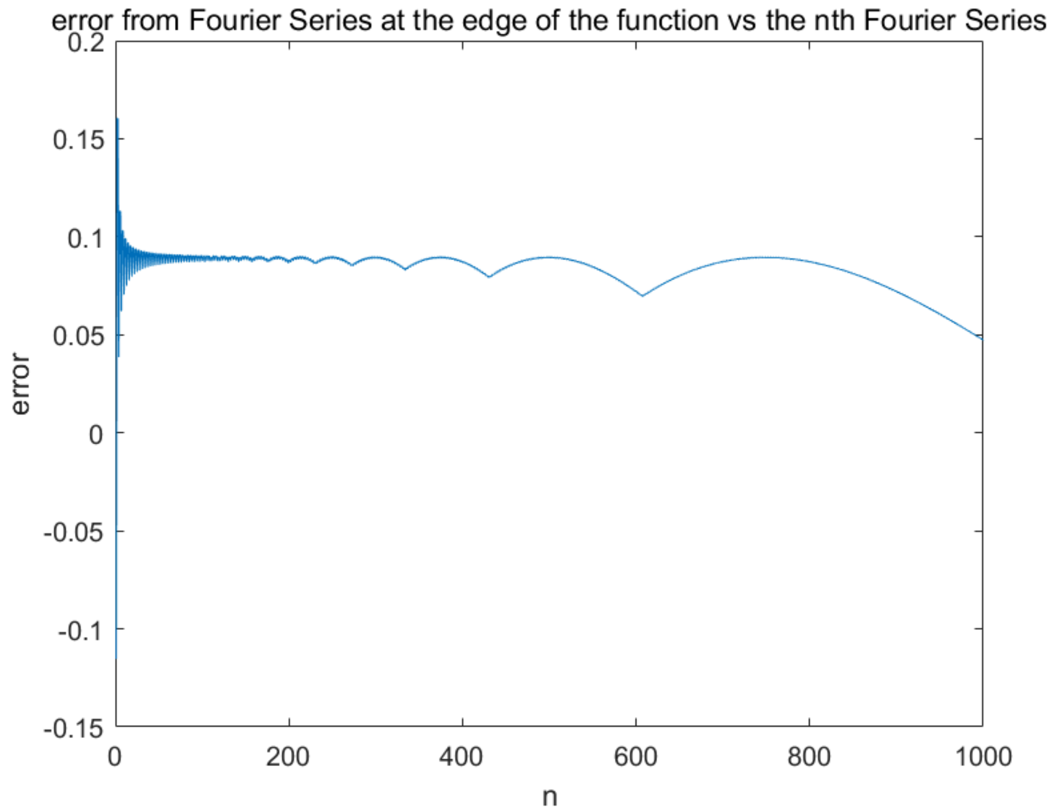


Fig 5.1 The plot of error with different n

From the graph we could see the difference between FS and true function are varbriating between 0.05 and 0.1. Also as much n we take the amplitude will increasing.

```
function [s] = sm(m, x)
%function of calculate partial sum from equation (21)
    s = 0;
    for j = 1:m
        s = s + an(j).*cos(j*pi.*x/3);
    end
    s = s + 1/3;
end

function [a] = an(n)
    a = 2/(n*pi)*sin(n*pi/3);
end
```

problem 6

Let $f(x)$ be defined by the sawtooth example

$f(x) = x$ for $-L < x < L$ with

$f(x+2L) = f(x)$ as examined in class and notes online. Compute the term-by-term derivative of your FS representation for $f(x)$ and compare it to the actual derivative

$f'(x) = 1$. Show all work and plot your results.

Solve

First derive the Fourier Series of the original function

Recall from problem 4

$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{dx}$, where $n = 0, 1, 2, 3, \ldots$ (26)

$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{dx}$, where $n = 1, 2, 3, \ldots$ (27)

Then plug $f(x) = x$ into function (26) and (27)

$$a_n = \frac{1}{L}\int_{-L}^{L} \mathrm{x}\cos\left(\frac{n\pi x}{L}\right)\mathrm{dx}$$

$$u = x \qquad v' = \cos\left(\frac{n\pi x}{L}\right)$$

$$u' = 1 \qquad v = \frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)$$

$$a_n = \frac{1}{L}\left(\left(\frac{L}{n\pi}\mathrm{x}\sin\left(\frac{n\pi x}{L}\right)\right)\Bigg|_{-L}^{L} - \int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\mathrm{dx}\right)$$
$$= \frac{1}{L}\left(\left(\frac{L}{n\pi}\mathrm{L}\sin\left(\frac{n\pi L}{L}\right)\right) - \left(\frac{L}{n\pi}(-\mathrm{L})\sin\left(\frac{n\pi(-L)}{L}\right)\right) - \left(-\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\right)\Bigg|_{-L}^{L}\right)$$
$$= \frac{1}{L}\left(\left(\frac{L^2}{n\pi}\sin(n\pi)\right) + \left(\frac{L^2}{n\pi}\sin(-n\pi)\right) - \left[\left(-\frac{L}{n\pi}\cos(n\pi)\right) - \left(-\frac{L}{n\pi}\cos(-n\pi)\right)\right]\right)$$

Since n=0,1,2,3,...

$$\sin(n\pi) = 0$$

So $a_n = \frac{1}{L}\left(\frac{L}{n\pi}\cos(n\pi) - \frac{L}{n\pi}\cos(-n\pi)\right) = 0$

$$b_n = \frac{1}{L}\int_{-L}^{L} \mathrm{x}\sin\left(\frac{n\pi x}{L}\right)\mathrm{dx}$$

$u = x \qquad v' = \sin\left(\frac{n\pi x}{L}\right)$

$u' = 1 \qquad v = -\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)$

$$b_n = \frac{1}{L}\left(\left(-\frac{L}{n\pi}\mathrm{x}\cos\left(\frac{n\pi x}{L}\right)\right)_{-L}^{L} - \int_{-L}^{L} -\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\mathrm{dx}\right)$$
$$= \frac{1}{L}\left(\left(-\frac{L}{n\pi}\mathrm{L}\cos\left(\frac{n\pi L}{L}\right)\right) - \left(-\frac{L}{n\pi}(-L)\cos\left(\frac{n\pi(-L)}{L}\right)\right) + \frac{L}{n\pi}\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\mathrm{dx}\right)$$
$$= \frac{1}{L}\left(\left(-\frac{L^2}{n\pi}\cos(n\pi)\right) - \left(\frac{L^2}{n\pi}\cos(-n\pi)\right) + \left[\left(\frac{L}{n\pi}\right)^2\sin\left(\frac{n\pi x}{L}\right)\right]_{-L}^{L}\right)$$
$$= \frac{1}{L}\left(-\frac{2L^2}{n\pi}\cos(n\pi) + \left(\frac{L}{n\pi}\right)^2[\sin(n\pi) - \sin(-n\pi)]\right)$$
$$= -\frac{1}{L}\frac{2L^2}{n\pi}\cos(n\pi)$$
$$= -\frac{2L}{n\pi}\cos(n\pi)$$

So the partial sum function $s_m(x) = \frac{a_0}{2} + \sum_{n=1}^{m}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right]$ becomes

$$s_m(x) = \sum_{n=1}^{m} -\frac{2L}{n\pi}\cos(n\pi)\sin\left(\frac{n\pi x}{L}\right) \qquad (28)$$

If we write it term by term

$$s_m(x) = -\frac{2L}{\pi}\cos(\pi)\sin\left(\frac{\pi x}{L}\right) - \frac{2L}{2\pi}\cos(2\pi)\sin\left(\frac{2\pi x}{L}\right) - \frac{2L}{3\pi}\cos(3\pi)\sin\left(\frac{3\pi x}{L}\right) + \ldots$$
$$\mathrm{fs}(x) = \frac{2L}{\pi}\left[-\cos(\pi)\sin\left(\frac{\pi x}{L}\right) - \frac{1}{2}\cos(2\pi)\sin\left(\frac{2\pi x}{L}\right) - \frac{1}{3}\cos(3\pi)\sin\left(\frac{3\pi x}{L}\right) + \ldots\right]$$
$$= \frac{2L}{\pi}\left[\sin\left(\frac{\pi x}{L}\right) - \frac{1}{2}\sin\left(\frac{2\pi x}{L}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{L}\right) - \ldots\right]$$

So if we take derivative of the [F.S.]

$$\mathrm{fs}'(x) = \frac{2L}{\pi}\left[\frac{\pi}{L}\cos\left(\frac{\pi x}{L}\right) - \frac{1}{2}\frac{2\pi}{L}\cos\left(\frac{2\pi x}{L}\right) + \frac{1}{3}\frac{3\pi}{L}\cos\left(\frac{3\pi x}{L}\right) - \ldots\right]$$
$$= \frac{2L}{\pi}\left[\frac{\pi}{L}\cos\left(\frac{\pi x}{L}\right) - \frac{\pi}{L}\cos\left(\frac{2\pi x}{L}\right) + \frac{\pi}{L}\cos\left(\frac{3\pi x}{L}\right) - \ldots\right]$$
$$= 2\left[\cos\left(\frac{\pi x}{L}\right) - \cos\left(\frac{2\pi x}{L}\right) + \cos\left(\frac{3\pi x}{L}\right) - \ldots\right]$$

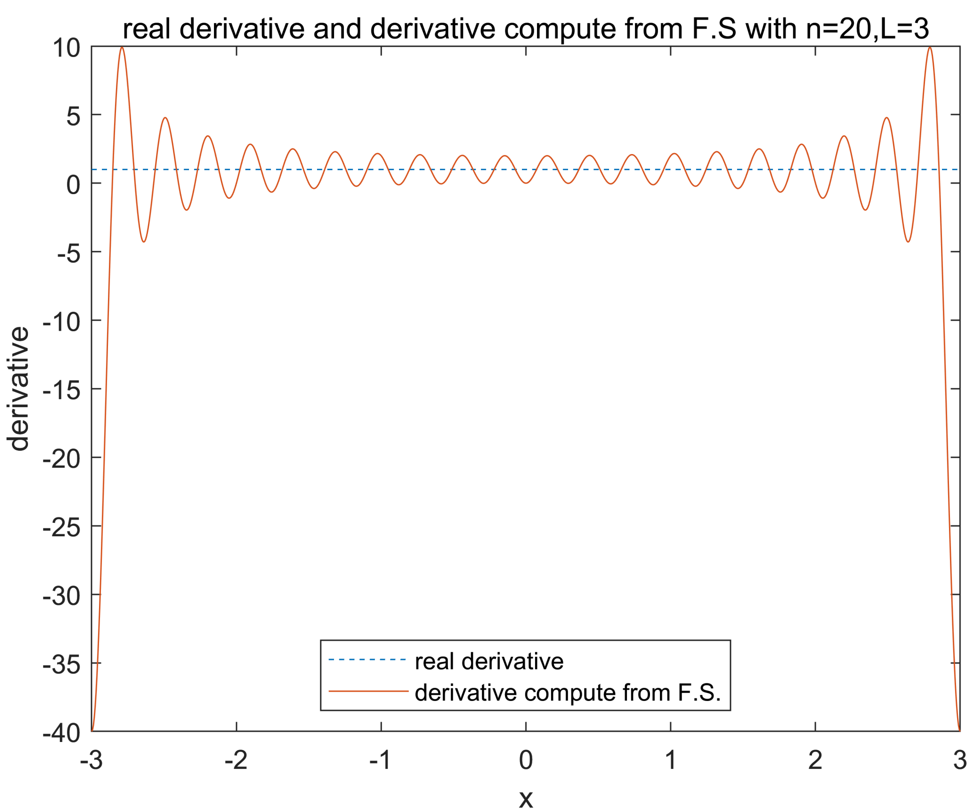
$$= 2\sum_{n=1}^{\infty}(-1)^{(n+1)}\cos\left(\frac{n\pi x}{L}\right) \qquad (29)$$

By observing equation (29) it seems that hardly convert to 1 which is the derivative of the original function f(x)=x.

Choosing some value of L and n, then test it numerically, then compare the result with $f'(x) = 1$.

Take L=3 and n=20 and testing as following:

```
clear
n = 20;L = 3;k = 1000;
%take roughly guess L=2 and n=20
%Then take k=1000 for number of interval point
x = linspace(-L, L, k);%define x from -L to L which is in one period
ts = ones(1, k);%derivative of the original function, f'(x)=1
o = each_term(n, x, L);%calculate the equation (29) by given n,l,and x
%plot
plot(x, ts, '--')
hold on
plot(x, o)
legend('real derivative', 'derivative compute from F.S.', 'location', 'south')
xlabel('x')
ylabel('derivative')
title('real derivative and derivative compute from F.S with n=20,L=3')
hold off
```




```
%finish plot
```

Fig 6.1 The plot shows the numerical test of derivative compute from F.S. n=20,L=3 are using in the computation. the solid cured line represents the Fourier Series derivative(equation 29), and the stright line means the real derivative of the original function.

From the plot we could see the derivative compute from F.S oscillate around 1. Also the amplitude of the oscillation is getting larger when the x=L and smaller in the middle. So we could conclude that in this case(n=20, L=3), derivative compute from F.S. are still far from the rea derivative.

At the case above it is only works for one case. To reduce the limitation, we may need further test.

Since we don't know the n and L, by further testing, we could fix one varible and then test the other.

Case 1 fix n and change L, compute equation (29) for each L

By examine the variance and mean of the each result we get from fixed n and L.

If the mean=1 and variance tend to 0, we may conclude the variance of L is usefull to help derivative get to real derivative.

```
numL = 500;%number of L will be test
jL = zeros(1, numL);%initial L
Lvar = zeros(1, numL);%initial variance of result from each L
Lmean = zeros(1, numL);%initial mean of result from each L
n = 50;%fix n=50
for j = 1:numL
    xl = linspace(-j, j, k);%define x
    oL = each_term(n, xl, j);%compute equation (29)
    jL(j) = j;%store l
    Lvar(j) = var(oL);%store variance
    Lmean(j) = mean(oL);%store mean
end
%plot
plot(jL, Lvar, '-')
hold on
plot(jL, Lmean, '--')
xlabel('L')
ylabel('var and mean')
title('variance and mean for different L')
legend('variance', 'mean')
hold off
```

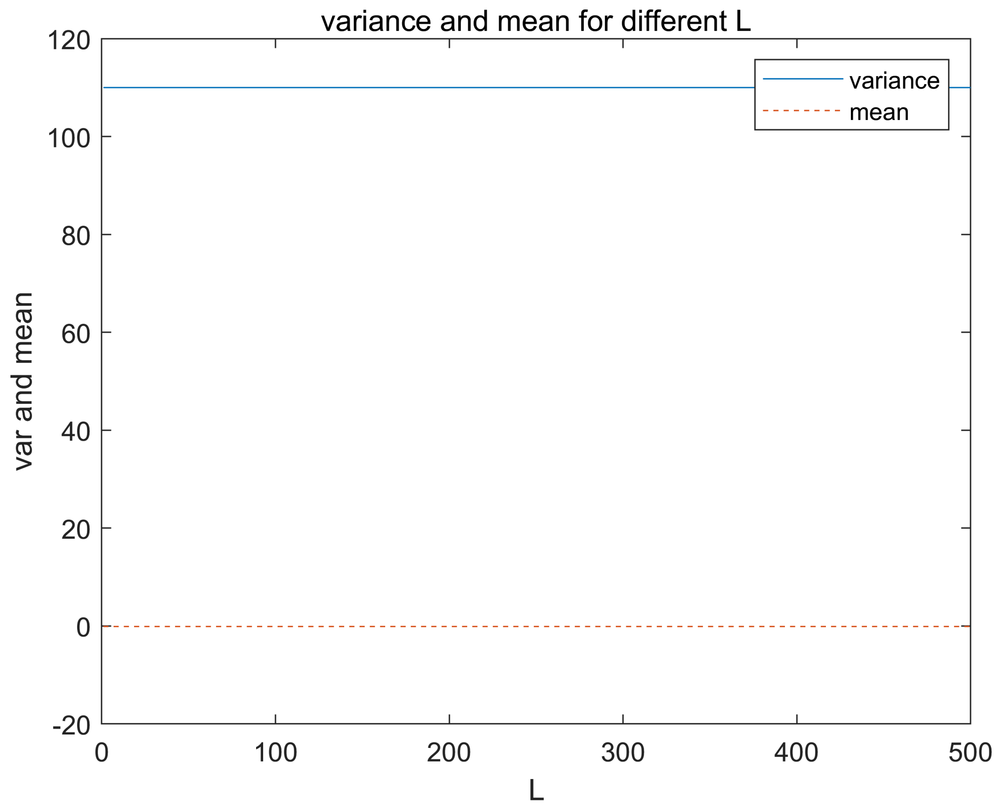


Fig 6.2 The plot show the mean(in '--' line style) and variance(in '-' line style) verses with the variable L.

We would see both of the mean and variance are straight line, also variance of each result is high.

Case 2 fix L and change n then do the same test with case 1

```
numn = 500;%number of n will be test
jn = zeros(1, numn);%initial n
nvar = zeros(1, numn);%initial the variance
nmean = zeros(1, numn);%initial the mean
L = 3;%fix L=3
xn = linspace(-L, L, k);%define x
for jj = 1:numn
    on = each_term(jj, xn, L);%compute equation(29)
    jn(jj) = jj;%store n
    nvar(jj) = var(on);%store variance
    nmean(jj) = mean(on);%store n
end
%plot
plot(jn, nvar, '-')
hold on
plot(jn, nmean, '--')
xlabel('n')
ylabel('var and mean')
title('variance and mean for different n')
legend('variance', 'mean')
```

```
hold off
```

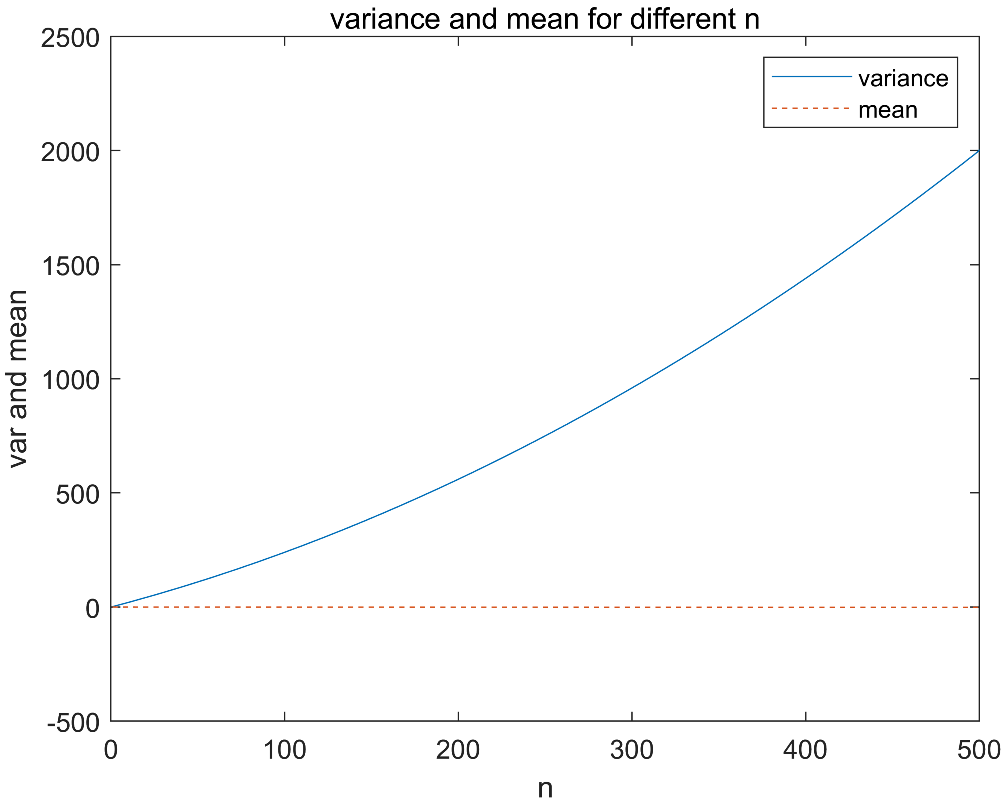


Fig 6.3 The plot show the mean(in '--' line style) and variance(in '-' line style) verses with the variable n.

The mean does not change and variance increases more and more fast.

```
function [o] = each_term(n, x, L)
%function of compute the derivative of
%Fouries Series according to %equation (29)
    o = 0;
    for j = 1:n
        o = o + (-1)^(j + 1).*cos(j*pi.*x/L);
        %the summation part of equation (29)
    end
    o = o*2;
end
```

Problem 7

Compute the Fourier transform of $f(x)$ provide that $a > 0$)

$$f(x) = \begin{cases} e^{-ax} & x \ge 0 \\ 0 & x < 0 \end{cases} \tag{30}$$

In order to take Fourier transform we need the formula:

$$F[f] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx \tag{31}$$

Then we can plug function (30) in to (31),becomes

$$F[f] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx = \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{0} 0 * e^{-i\xi x} dx + \int_{0}^{\infty} e^{-ax} e^{-i\xi x} dx \right)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-(a+i\xi)x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{a+i\xi} e^{-(a+i\xi)x} \right]_0^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{-1}{a+i\xi} \cdot 0 + \frac{1}{a+i\xi} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{a+i\xi}$$

Problem 8

Solve the problem

PDE	$u_t = \alpha^2 u_{xx}$	$-\infty < x < \infty$
IC	$u(x,0) = e^{-x^2}$	$-\infty < x < \infty$

By using the Fourier transform

Solve:

To solve the PDE by using the Fourier transform, we need first take Fourier transform of the PDE equation and IC equation. Then we can solve the problem that after Fourier transform. At the last we need to take inverse Fourier transform to get real solution.

Step 1 take Fourier transform of PDE and IC.

$$F[u_t] = F[\alpha^2 u_{xx}]$$

$$\frac{\partial}{\partial t} F[u] = \alpha^2 \frac{\partial^2}{\partial x^2} F[u]$$

And then by properties of FT

$$\frac{\partial}{\partial t} F[u] = \alpha^2 (-\xi^2) F[u] \qquad (32)$$

Let $F[u] = U(\xi, t)$,equation (32) becomes

$$U_t = -\alpha^2 \xi^2 U \qquad (33)$$

Then take Fourier transform of IC:

$$F[u(x,0)] = F[e^{-x^2}]$$

$$U(\xi,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2} e^{-i\xi x} dx = \Phi(\xi)$$

Step 2 solve the problem under Fourier transform

If we take ξ to be constant. We could see the problem becomes an ODE problem

$$U_t = -\alpha^2 \xi^2 U$$

$$U(\xi, t=0) = \Phi(\xi)$$

The solution of $U_t = -\alpha^2 \xi^2 U$ will be

$$U(t) = U(0) e^{-\alpha^2 \xi^2 t}$$

$$U(0) = \Phi(\xi)$$

Therefore the solution of ODE is

$$U(t) = \Phi(\xi) e^{-\alpha^2 \xi^2 t}$$

Step 3 take inverse Fourier Series of the solution.

$$F^{-1}[U(t,\xi)] = F^{-1}[\Phi(\xi)e^{-\alpha^2\xi^2 t}]$$

$$F^{-1}[U(t,\xi)] = F^{-1}[F[\phi(x)]F[g(t,x)]] \qquad (34)$$

Which $g(x,t)$ satisfies $F[g(t,x)] = e^{-\alpha^2\xi^2 t}$

By searching in the table

$$F\left[\frac{1}{\alpha\sqrt{2t}}e^{-\frac{x^2}{4\alpha^2 t}}\right] = e^{-\alpha^2\xi^2 t}$$

So $g(x,t) = \dfrac{1}{\alpha\sqrt{2t}}e^{-\frac{x^2}{4\alpha^2 t}}$

Since $F[f*g] = F[f]F[g]$ equation (34) will become

$$u(x,t) = F^{-1}[F[\phi(x)*g(t,x)]]$$
$$u(x,t) = \phi(x)*g(t,x)$$

$$u(x,t) = e^{-x^2} * \frac{1}{\alpha\sqrt{2t}}e^{-\frac{x^2}{4\alpha^2 t}}$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-y^2}\frac{1}{\alpha\sqrt{2t}}e^{-\frac{(x-y)^2}{4\alpha^2 t}}dy$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\frac{1}{\alpha\sqrt{2t}}\int_{-\infty}^{\infty} e^{-y^2}e^{-\frac{(x-y)^2}{4\alpha^2 t}}dy$$

$$u(x,t) = \frac{1}{2\alpha\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-y^2-\frac{(x-y)^2}{4\alpha^2 t}}dy$$

$$u(x,t) = \frac{1}{2\alpha\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-\frac{4\alpha^2 t y^2+(x-y)^2}{4\alpha^2 t}}dy$$

$$u(x,t) = \frac{1}{2\alpha\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-\frac{4\alpha^2 t y^2+x^2-2xy+y^2}{4\alpha^2 t}}dy$$

$$u(x,t) = \frac{1}{2\alpha\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(4\alpha^2 t+1)y^2+x^2-2xy}{4\alpha^2 t}}dy$$

$$u(x,t) = \frac{1}{2\alpha\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(4\alpha^2 t+1)y^2+x^2-2xy}{4\alpha^2 t}}dy$$

$$u(x,t) = \frac{1}{2\alpha\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(4\alpha^2 t+1)y^2+x^2-2xy}{4\alpha^2 t}}dy$$

$$u(x,t) = \frac{1}{2\alpha\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(4\alpha^2 t+1)y^2}{4\alpha^2 t}-\frac{x^2}{4\alpha^2 t}+\frac{2xy}{4\alpha^2 t}}dy$$

$$u(x,t)=\frac{1}{2\alpha\sqrt{\pi t}}\int_{-\infty}^{\infty}e^{-\frac{(4\alpha^2t+1)y^2}{4\alpha^2t}}e^{-\frac{x^2}{4\alpha^2t}}e^{\frac{2xy}{4\alpha^2t}}dy$$

$$u(x,t)=\frac{1}{2\alpha\sqrt{\pi t}}e^{-\frac{x^2}{4\alpha^2t}}\int_{-\infty}^{\infty}e^{-\frac{(4\alpha^2t+1)y^2}{4\alpha^2t}}e^{\frac{2xy}{4\alpha^2t}}dy$$

Let

$$k=-\frac{(4\alpha^2t+1)}{4\alpha^2t}$$

$$l=\frac{2x}{4\alpha^2t}$$

The integral becomes

$$u(x,t)=\frac{1}{2\alpha\sqrt{\pi t}}e^{-\frac{x^2}{4\alpha^2t}}\int_{-\infty}^{\infty}e^{ky^2}e^{ly}dy$$

$$u(x,t)=\frac{1}{2\alpha\sqrt{\pi t}}e^{-\frac{x^2}{4\alpha^2t}}\int_{-\infty}^{\infty}e^{ky^2+ly}dy$$

$$u(x,t)=\frac{1}{2\alpha\sqrt{\pi t}}e^{-\frac{x^2}{4\alpha^2t}}\int_{-\infty}^{\infty}e^{\frac{4k^2y^2+4kly}{4k}}dy$$

$$u(x,t)=\frac{1}{2\alpha\sqrt{\pi t}}e^{-\frac{x^2}{4\alpha^2t}}\int_{-\infty}^{\infty}e^{\frac{4k^2y^2+l^2+4kly-l^2}{4k}}dy$$

$$u(x,t)=\frac{1}{2\alpha\sqrt{\pi t}}e^{-\frac{x^2}{4\alpha^2t}}\int_{-\infty}^{\infty}e^{\frac{(2ky+l)^2-l^2}{4k}}dy$$

$$u(x,t)=\frac{1}{2\alpha\sqrt{\pi t}}e^{-\frac{x^2}{4\alpha^2t}}\int_{-\infty}^{\infty}e^{\frac{(2ky+l)^2}{4k}-\frac{l^2}{4k}}dy$$

$$u(x,t)=\frac{1}{2\alpha\sqrt{\pi t}}e^{-\frac{x^2}{4\alpha^2t}}\int_{-\infty}^{\infty}e^{\frac{(2ky+l)^2}{4k}}e^{-\frac{l^2}{4k}}dy$$

$$u(x,t)=\frac{1}{2\alpha\sqrt{\pi t}}e^{-\frac{x^2}{4\alpha^2t}}e^{-\frac{l^2}{4k}}\int_{-\infty}^{\infty}e^{\frac{(2ky+l)^2}{4k}}dy$$

Take $\frac{1}{4k}=-a$ in order to integral.

Integral now becomes

$$u(x,t)=\frac{1}{2\alpha\sqrt{\pi t}}e^{-\frac{x^2}{4\alpha^2t}}e^{-\frac{l^2}{4k}}\int_{-\infty}^{\infty}e^{-a(2ky+l)^2}dy$$

$$u(x,t)=\frac{1}{2\alpha\sqrt{\pi t}}e^{-\frac{x^2}{4\alpha^2t}}e^{-\frac{l^2}{4k}}\frac{\sqrt{\pi}}{2\sqrt{ak}}$$

$$u(x,t)=\frac{1}{2\alpha\sqrt{\pi t}}e^{-\frac{x^2}{4\alpha^2t}}e^{-\frac{l^2}{4k}}\frac{\sqrt{\pi}}{2\sqrt{ak}}$$

$$u(x,t)=\frac{1}{2\alpha\sqrt{\pi t}}e^{-\frac{x^2}{4\alpha^2t}}e^{-\frac{l^2}{4k}}\frac{\sqrt{\pi}}{2\sqrt{-\frac{1}{4k}k}}$$

$$k = -\frac{(4\alpha^2 t + 1)}{4\alpha^2 t}$$

with

$$l = \frac{2x}{4\alpha^2 t}$$

After simplified.

The result is

$$u(x,t) = \frac{1}{2\alpha\sqrt{\pi t}} \frac{2\alpha\sqrt{\pi t}}{\sqrt{4\alpha^2 t + 1}} e^{4\alpha^2 t + 1}$$

$$u(x,t) = \frac{1}{\sqrt{4\alpha^2 t + 1}} e^{-\frac{x^2}{4\alpha^2 t + 1}}$$