1

Use the method of Separation of Variables to find the solution to the IBVP below. Graph the solution look like for various values of time using Matlab.

PDE $\quad u_t = u_{xx} \qquad 0 < x < 1, \quad 0 < t < \infty$

BCs $\quad \begin{cases} u(0,1) = 0 \\ u(1,t) = 0 \quad 0 < t < \infty \end{cases}$

IC $\quad u(x,0) = \sin(2\pi x) + \frac{1}{3}\sin(4\pi x) + \frac{1}{5}\sin(6\pi x) \qquad 0 \le x \le 1$

Let $u(x,t) = X(x)T(t)$, then the PDE become:

$$\frac{\partial}{\partial t}[X(x)T(t)] = \frac{\partial^2}{\partial x^2}[X(x)T(t)]$$

$$X(x)T'(t) = X''(x)T(t)$$

Then divide both side $X(x)T(t)$

$$\frac{X(x)T'(t)}{X(x)T(t)} = \frac{X''(x)T(t)}{X(x)T(t)}$$

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)}$$

The only result the equation satified is both RHS and LHS equals to constant k.

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)} = k$$

Then

$$\begin{cases} \dfrac{X''(x)}{X(x)} = k \quad (1) \\ \dfrac{T'(t)}{T(t)} = k \quad (2) \end{cases}$$

The solution of (2) is $T(t) = T(0)e^{\mathrm{kt}}$ (3)

With equation (1)

$X''(x) - \mathrm{kX}(x) = 0$ (4)

Guess with the solution $X(x) = e^{\mathrm{rx}}$

Plug the solution into (4):

$r^2 e^{\mathrm{rx}} - \mathrm{ke}^{\mathrm{rx}} = 0$
$e^{\mathrm{rx}}(r^2 - k) = 0$

so $r^2 - k = 0$

$r^2 = k$

$r = \pm\sqrt{k}$

Then with sign of k, there are three case.

Case 1 $K > 0$

let $K = \lambda^2$

$r = \pm\lambda$

So the solution is:

$X(x) = \mathrm{Ae}^{\lambda x} + \mathrm{Be}^{-\lambda x}$

Case 2 $K < 0$

let $K = -\lambda^2$

$\frac{X(x)}{X''(x)} = -\lambda^2$

$X(x) = -\lambda^2 X''(x)$

$X(x) = \mathrm{Asin}(\lambda x) + \mathrm{Bcos}(\lambda x)$

Case 3 $K = 0$

$r = 0$

$X = 1$

By plug BCs in:

BCs $\begin{cases} u(0,1) = 0 \\ u(1,t) = 0 \quad 0 < t < \infty \end{cases}$

With Case 1 $K > 0, K = \lambda^2$

$u(x,t) = X(x)T(t) = (\mathrm{A}e^{\lambda x} + \mathrm{B}e^{-\lambda x})T(0)e^{\lambda t}$

By plug $u(0,1) = 0$ and $u(1,t) = 0$ in

$u(0,1) = (\mathrm{A}e^{\lambda 0} + \mathrm{B}e^{-\lambda 0})T(0)e^{\lambda} = (A+B)T(0)e^{\lambda} = 0$

$u(1,t) = (\mathrm{A}e^{\lambda} + \mathrm{B}e^{-\lambda})T(0)e^{\lambda t} = 0$

So $T(0) = 0$

Therefore $(\mathrm{A}e^{\lambda x} + \mathrm{B}e^{-\lambda x})T(0)e^{\lambda t}$=0 in all cituation.

So Case 1 $K > 0$ is not true

Case 2 $K < 0, K = -\lambda^2$

$u(x,t) = X(x)T(t) = (\mathrm{A}\sin(\lambda x) + \mathrm{B}\cos(\lambda x))T(0)e^{-\lambda^2 t}$

By plug $u(0,1) = 0$ and $u(1,t) = 0$ in

$u(0,1) = (\mathrm{A}\sin(\lambda 0) + \mathrm{B}\cos(\lambda 0))T(0)e^{-\lambda^2} = \mathrm{B}\mathrm{T}(0)e^{-\lambda^2} = 0$

$u(1,t) = (\mathrm{A}\sin(\lambda) + \mathrm{B}\cos(\lambda))T(0)e^{-\lambda^2 t} = 0$

Since $T(0)$ cannot be 0 from Case 1, $\mathrm{B} = 0$.

$u(x,t) = X(x)T(t) = \mathrm{A}\sin(\lambda x)T(0)e^{-\lambda^2 t}$

$u(1,t) = \mathrm{A}\sin(\lambda)T(0)e^{-\lambda^2 t} = 0$

$\mathrm{A}\sin(\lambda) = 0$

$\lambda = n\pi$

Case 3 $K = 0$

$u(x,t) = X(x)T(t) = T(0)e^{\mathrm{kt}}$

$T(0) = 0$ if we plug $u(0,1) = 0$ in.

Therefore Case 2 $K < 0$ is true

And $u(x,t) = X(x)T(t) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) C_n e^{-(n\pi)^2 t}$

let $A_n = \tilde{A}_n C_n$

$$u(x,t)=\sum_{n=1}^{\infty}\tilde{A}_n\sin(n\pi x)e^{-(n\pi)^2t}$$

Then plug in IC

IC: $u(x,0)=\sin(2\pi x)+\frac{1}{3}\sin(4\pi x)+\frac{1}{5}\sin(6\pi x)$

$$u(x,0)=\sum_{n=1}^{\infty}\tilde{A}_n\sin(n\pi x)=\sin(2\pi x)+\frac{1}{3}\sin(4\pi x)+\frac{1}{5}\sin(6\pi x)$$

Multiply $\sin(m\pi x)$ on both side and then integral, m=1,2,3,...

$$\left\langle\sum_{n=1}^{\infty}\tilde{A}_n\sin(n\pi x),\sin(m\pi x)\right\rangle=\left\langle\sin(2\pi x)+\frac{1}{3}\sin(4\pi x)+\frac{1}{5}\sin(6\pi x),\sin(m\pi x)\right\rangle$$

$$\text{LHS}=\sum_{n=1}^{\infty}\tilde{A}_n\int_0^1\sin(n\pi x)\sin(m\pi x)\text{dx}=\begin{cases}0 & \text{if } m\neq n\\ \frac{1}{2} & \text{if } m=n\end{cases}$$

$$\text{RHS}=\int_0^1\left(\sin(2\pi x)+\frac{1}{3}\sin(4\pi x)+\frac{1}{5}\sin(6\pi x)\right)\sin(m\pi x)\text{dx}$$

$$\text{RHS}=\int_0^1\sin(2\pi x)\sin(m\pi x)\text{dx}+\int_0^1\frac{1}{3}\sin(4\pi x)\sin(m\pi x)\text{dx}+\int_0^1\frac{1}{5}\sin(6\pi x)\sin(m\pi x)\text{dx}$$

$$\int_0^1\sin(2\pi x)\sin(m\pi x)\text{dx}=\frac{\tilde{A}_m}{2}$$

$$\int_0^1\frac{1}{3}\sin(4\pi x)\sin(m\pi x)\text{dx}=\frac{\tilde{A}_m}{2}$$

$$\int_0^1\frac{1}{5}\sin(6\pi x)\sin(m\pi x)\text{dx}=\frac{\tilde{A}_m}{2}$$

Simplifiy

$$\int_0^1\sin(2\pi x)\sin(2\pi x)\text{dx}=\frac{\tilde{A}_2}{2}=\frac{1}{2}$$

$$\int_0^1\frac{1}{3}\sin(4\pi x)\sin(4\pi x)\text{dx}=\frac{\tilde{A}_4}{2}=\frac{1}{3}\frac{1}{2}=\frac{1}{6}$$

$$\int_0^1\frac{1}{5}\sin(6\pi x)\sin(6\pi x)\text{dx}=\frac{\tilde{A}_6}{2}=\frac{1}{5}\frac{1}{2}=\frac{1}{10}$$

So $\tilde{A}_2=1$, $\tilde{A}_4=\frac{1}{3}$, $\tilde{A}_6=\frac{1}{5}$, and $\tilde{A}_n=0$ with other n

So $u(x,t) = \sum_{n=1}^{\infty} \tilde{A}_n \sin(n\pi x) e^{-(n\pi)^2 t} = \sin(2\pi x) e^{-(2\pi)^2 t} + \frac{1}{3}\sin(4\pi x) e^{-(4\pi)^2 t} + \frac{1}{5}\sin(6\pi x) e^{-(6\pi)^2 t}$ (5)

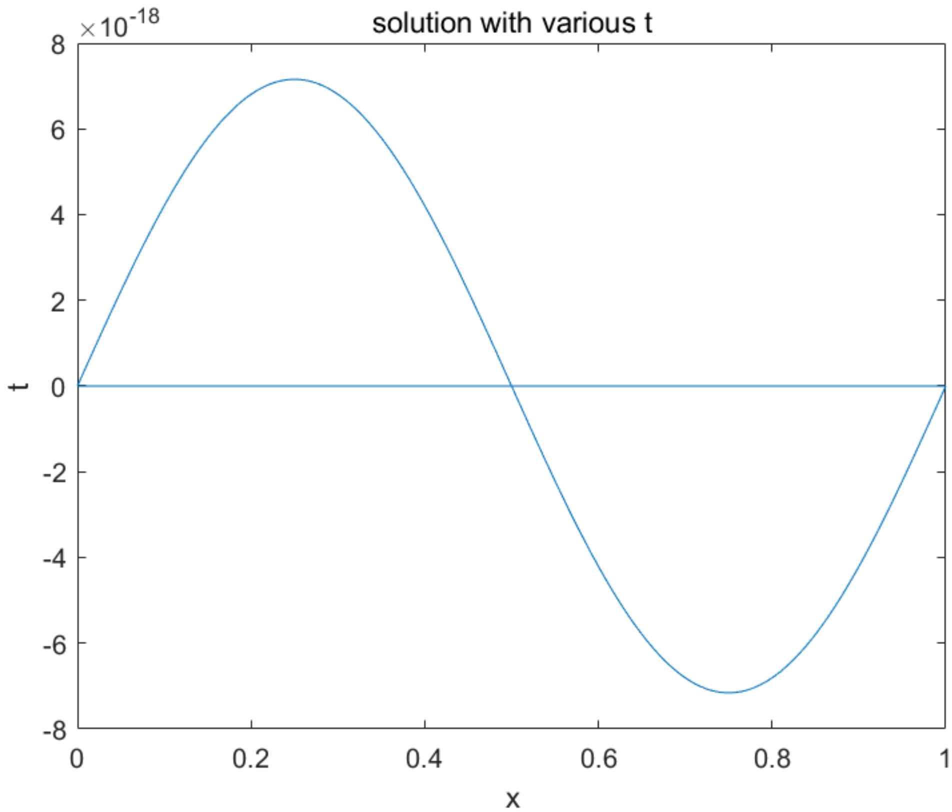
$u_t = -(2\pi)^2 \sin(2\pi x) e^{-(2\pi)^2 t} - (4\pi)^2 \frac{1}{3}\sin(4\pi x) e^{-(4\pi)^2 t} - (6\pi)^2 \frac{1}{5}\sin(6\pi x) e^{-(6\pi)^2 t}$

$= -4\pi^2 \sin(2\pi x) e^{-(2\pi)^2 t} - 16\pi^2 \frac{1}{3}\sin(4\pi x) e^{-(4\pi)^2 t} - 36\pi^2 \frac{1}{5}\sin(6\pi x) e^{-(6\pi)^2 t}$

```
clear
x=linspace(0,1,500);%generate x according to PDE interval (0,1)

for t=1:50
    %loop for t
    plot(x,u(x,t))%plot with each t
    hold on
end

title("solution with various t")
xlabel("x")
ylabel("u(x,t)")
```




```
function [uf]=u(x,t)
% equation(5)
    uf=sin(2*pi*x).*exp(-4*pi^2*t) ...
```

```
    +1/3*sin(4*pi*x).*exp(-16*pi^2*t) ...
    +1/5*sin(6*pi*x).*exp(-36*pi^2*t);
end
```