7. Solve the following problem analytically (separation of variables) and evaluate the analytical solution

at the grid points: x=0, 0.1, 0.2, ... , 0.9, 1 for t =0.015. Compare these results to your numerical solution in #6 above.

$u_t = u_{xx} \qquad 0 < x < 1 \;\; t > 0$

$u(0,t) = 0 \qquad t > 0$

$u(1,t) = 0 \qquad t > 0$

$u(x,0) = \sin(\pi x) \qquad 0 \le x \le 1$

$$U = X(x)T(t)$$
$$X(x)T'(t) = X''(x)T(t)$$
$$\frac{X(x)T'(t)}{X(x)T(t)} = \frac{X''(x)T(t)}{X(x)T(t)}$$
$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)}$$
$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = k$$

Then

$$\begin{cases} \dfrac{X''(x)}{X(x)} = k & (12) \\ \dfrac{T'(t)}{T(t)} = k & (13) \end{cases}$$

The solution of (13) is $T(t) = T(0)e^{\mathrm{kt}}$ (14)

With equation (12)

$X''(x) - \mathrm{kX}(x) = 0$ (15)

Guess with the solution $X(x) = e^{\mathrm{rx}}$

Plug the solution into (15):

$$r^2e^{\mathrm{rx}} - \mathrm{ke}^{\mathrm{rx}} = 0$$
$$e^{\mathrm{rx}}(r^2 - k) = 0$$

so $r^2 - k = 0$

$r^2 = k$

$r = \pm\sqrt{k}$

Then with sign of k, there are three case.

Case 1 $K > 0$

let $K = \lambda^2$

$r = \pm\lambda$

So the solution is:

$X(x) = \mathrm{Ae}^{\lambda x} + \mathrm{Be}^{-\lambda x}$

Case 2 $K < 0$

let $K = -\lambda^2$

$\frac{X(x)}{X''(x)} = -\lambda^2$

$X(x) = -\lambda^2 X''(x)$

$X(x) = \mathrm{Asin}(\lambda x) + \mathrm{Bcos}(\lambda x)$

Case 3 $K = 0$

$r = 0$

$X = 1$

By plug BCs in:

BCs $\begin{cases} u(0,t) = 0 \\ u(1,t) = 0 \quad 0 < t < \infty \end{cases}$

With Case 1 $K > 0, K = \lambda^2$

$u(x,t) = X(x)T(t) = (\mathrm{Ae}^{\lambda x} + \mathrm{Be}^{-\lambda x})T(0)e^{\lambda t}$

By plug $u(0,t) = 0$ and $u(1,t) = 0$ in

$u(0,t) = (\mathrm{Ae}^{\lambda 0} + \mathrm{Be}^{-\lambda 0})T(0)e^{\lambda^2 t} = (A + B)T(0)e^{\lambda^2 t} = 0$

$u(1,t) = (\mathrm{Ae}^{\lambda} + \mathrm{Be}^{-\lambda})T(0)e^{\lambda^2 t} = 0$

So $T(0) = 0$

Therefore $(\mathrm{A}e^{\lambda x} + \mathrm{B}e^{-\lambda x})T(0)e^{\lambda^2 t}$=0 in all cituation.

So Case 1 $K > 0$ is not true

Case 2 $K < 0, K = -\lambda^2$

$u(x,t) = X(x)T(t) = (\mathrm{A}\sin(\lambda x) + \mathrm{B}\cos(\lambda x))T(0)e^{-\lambda^2 t}$

By plug $u(0,t) = 0$ and $u(1,t) = 0$ in

$u(0,t) = (\mathrm{A}\sin(\lambda 0) + \mathrm{B}\cos(\lambda 0))T(0)e^{-\lambda^2 t} = \mathrm{B}\mathrm{T}(0)e^{-\lambda^2 t} = 0$

$u(1,t) = (\mathrm{A}\sin(\lambda) + \mathrm{B}\cos(\lambda))T(0)e^{-\lambda^2 t} = 0$

Since $T(0)$ cannot be 0 from Case 1

$\mathrm{B} = 0$.

$u(x,t) = X(x)T(t) = \mathrm{A}\sin(\lambda x)T(0)e^{-\lambda^2 t}$

$u(1,t) = \mathrm{A}\sin(\lambda)T(0)e^{-\lambda^2 t} = 0$

$\mathrm{A}\sin(\lambda) = 0$

$\lambda = n\pi$

Case 3 $K = 0$

$u(x,t) = X(x)T(t) = T(0)e^{\mathrm{kt}}$

$T(0) = 0$ if we plug $u(0,t) = 0$ in.

Therefore Case 2 $K < 0$ is true

And $u(x,t) = X(x)T(t) = \sum_{n=1}^{\infty} (A_n \sin(n\pi x))C_n e^{-(n\pi)^2 t}$

let $\widetilde{A}_n = A_n C_n$

$u(x,t) = \sum_{n=1}^{\infty} \widetilde{A}_n \sin(n\pi x) e^{-(n\pi)^2 t}$

Then plug in IC

IC: $u(x,0) = \sin(\pi x)$

$$u(x,0) = \sum_{n=1}^{\infty} \tilde{A}_n \sin(n\pi x) = \sin(\pi x)$$

So n=1 and $\tilde{A}_n = 1$

$$u(x,t) = \sin(\pi x)e^{-\pi^2 t} \qquad (16)$$

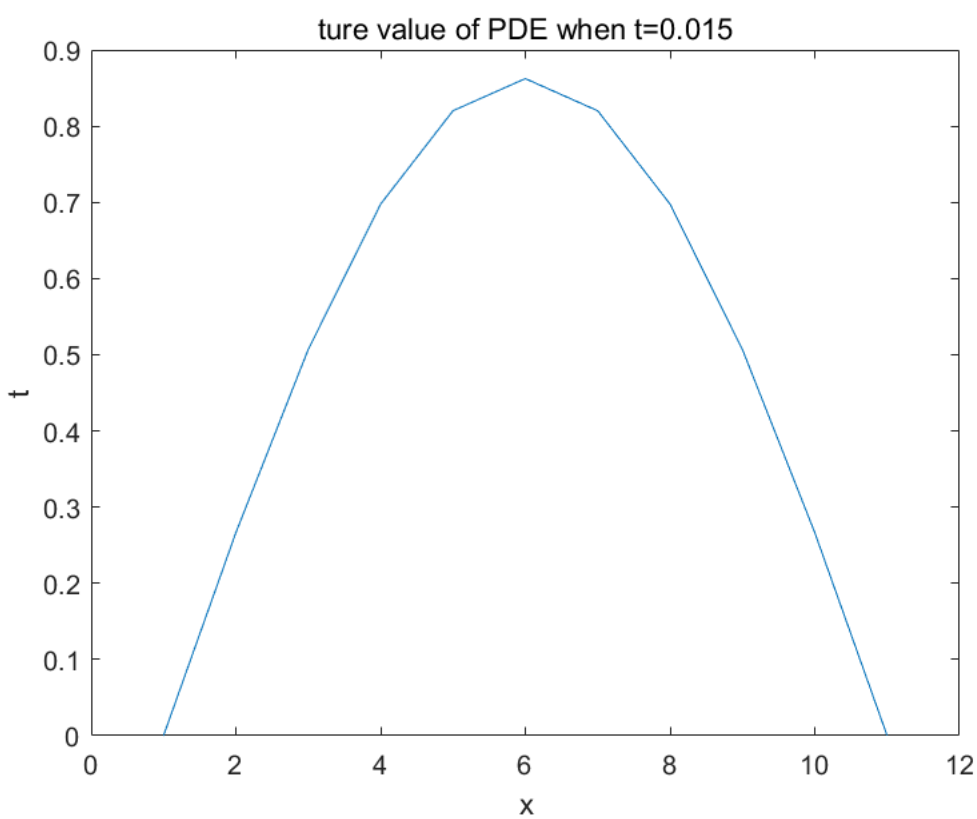
so for t=0.015

$$u(x,t) = \sin(\pi x)e^{-0.015\pi^2}$$

Then plot it.

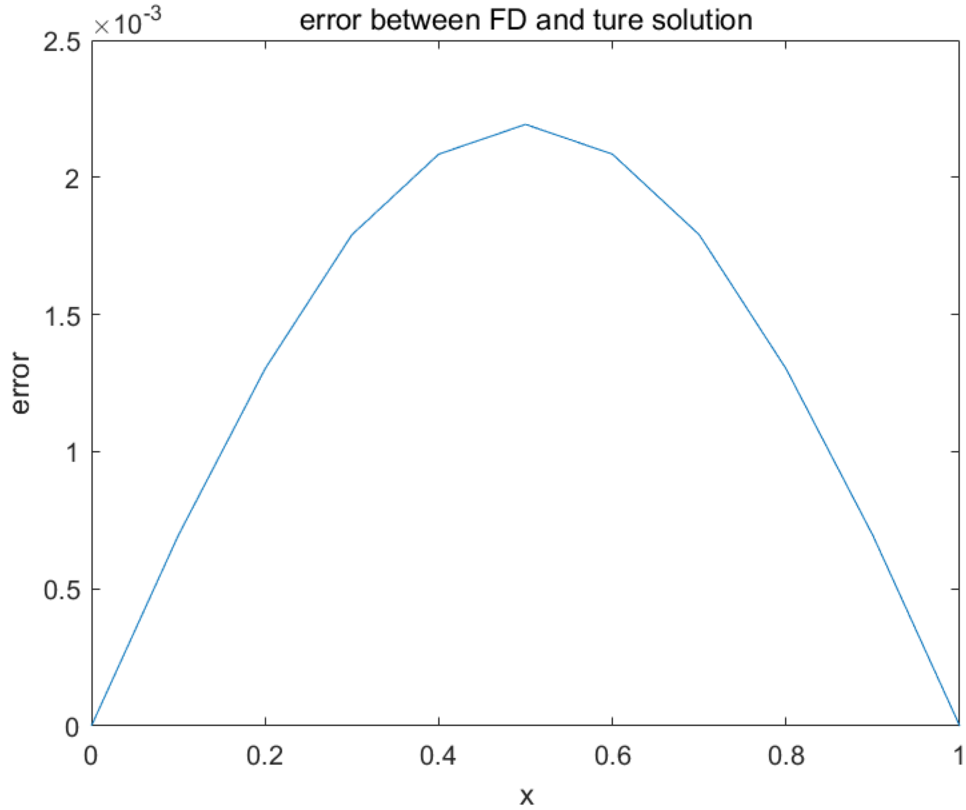
```
clear
H=0.1;%set H
N=1/H+1;%compute N
x=linspace(0,1,N);%generate same numbers of x with last question

ut=u(x,0.015);%compute the true value
plot(ut)%plot
title('true value of PDE when t=0.015')
xlabel('x')
ylabel('t')
```



From the graph we could see the true value are lower than the approximate value.

```
%get approximate solution from last problem
up = [0 0.2658 0.5056 0.6959 0.8181 0.8602 0.8181 0.6959 0.5056 0.2658 0];
%compute the difference
err = abs(ut - up);
plot(x, err)
title('error between FD and true solution')
xlabel('x')
ylabel('error')
```



From the plot we could clearly saw that the error with explicit method is small.

```
function [ut]=u(x,t)
%funtion of equation (16)
   ut=sin(pi*x)*exp(-pi^2*t);
end
```