Math 6020
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Midterm Exam
Dr. Rowe

(a) Using pencil and paper, show that f(x) integrates to 1. Show your work.

$$\int_0^\infty f(x)dx = \int_0^\infty \frac{x}{\sigma^2} e^{\frac{-x^2}{2\sigma^2}} dx$$

$$u = \frac{-x^2}{2\sigma^2} \quad du = \frac{-x}{\sigma^2} dx$$

The integral will be

$$-\int e^u du$$

$$= -e^u$$

$$= \left[-e^{-\frac{x^2}{2\sigma^2}} \right]_0^\infty$$

$$= 1$$

b) Using pencil and paper, derive the expectation E(x) of this distribution. Show your work

$$E[x] = \int_0^\infty x f(x)dx = \int_0^\infty \frac{x^2}{\sigma^2} e^{\frac{-x^2}{2\sigma^2}} dx$$

$$u = x, v' = \frac{x}{\sigma^2} e^{\frac{-x^2}{2\sigma^2}}$$

$$u' = 1, v = -e^{\frac{-x^2}{2\sigma^2}}$$

$$I = \left(xe^{\frac{-x^2}{2\sigma^2}} \right)_0^\infty - \int_0^\infty - e^{\frac{-x^2}{2\sigma^2}} dx$$

$$let, u = \frac{x}{\sqrt{2}\sigma}, du = \frac{dx}{\sqrt{2}\sigma}, dx = du\sqrt{2}\sigma$$

$$I = \sqrt{2}\sigma \int_0^\infty e^{-u^2} du$$

$$let, u = \sqrt{t}, du = \frac{1}{2} t^{-\frac{1}{2}} dt$$

$$I = \sqrt{2}\sigma \frac{1}{2} \int_0^\infty e^{-t} t^{-\frac{1}{2}} dt$$

$$I = \sqrt{2}\sigma \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \sqrt{2\pi}\sigma \frac{1}{2} = \sigma\sqrt{\frac{\pi}{2}}$$

(c) Using pencil and paper, derive the expectation var(x) of this distribution. Show your work

$$\mathrm{var}(x) = E[x^2] - (E[x])^2$$

$$E[x^2] = \int_0^\infty x^2 f(x)dx = \int_0^\infty \frac{x^3}{\sigma^2} e^{\frac{-x^2}{2\sigma^2}} dx$$

$$u = x^2, v' = \frac{x}{\sigma^2} e^{\frac{-x^2}{2\sigma^2}}$$

$$u' = 2x, v = e^{\frac{-x^2}{2\sigma^2}}$$

$$I = \left(x^2 e^{\frac{-x^2}{2\sigma^2}} \right)_0^\infty - 2\int_0^\infty x e^{\frac{-x^2}{2\sigma^2}} dx$$

$$I = \left(x^2 e^{\frac{-x^2}{2\sigma^2}} \right)_0^\infty - 2\left(e^{\frac{-x^2}{2\sigma^2}} \right)_0^\infty$$

$$I = 2\sigma^2$$

$$\mathrm{var}(x) = E[x^2] - (E[x])^2$$

$$= 2\sigma^2 - \frac{\sigma^2 \pi}{2} = \left(2 - \frac{\pi}{2} \right)\sigma^2$$

(d) Using pencil and paper, convert the integral

$\int_0^\infty \frac{x}{\sigma^2} e^{-x^2/(2\sigma^2)} dx$ to an integral of the form $\int_0^1 h(u)du$

$$\int_0^\infty \frac{x}{\sigma^2} e^{-x^2/(2\sigma^2)} dx$$

Let $y = \frac{1}{x+1}$ so as $x \to 0, y \to 1$ and $x \to \infty, y \to 0$

$$x = \frac{1}{y} - 1$$

And $|J| = \frac{dx(y)}{dy} = \left| -y^{-2} \right| = y^{-2}$

$$h(y) = \frac{\frac{1}{y} - 1}{\sigma^2} e^{\frac{-(\frac{1}{y}-1)^2}{2\sigma^2}} y^{-2}$$

$$\int_0^\infty f(x)dx = \int_0^1 \frac{\frac{1}{y} - 1}{\sigma^2} e^{\frac{-(\frac{1}{y}-1)^2}{2\sigma^2}} y^{-2} dy$$

(e) Generate 10^6 uniform (0,1) random numbers and evaluate the integral in part (d)

(Matlab code midterme.mlx)

```
clear
n = 10e6; %number of random number
u = rand(n, 1); %n numbers of uniform
sig = 10; %sigma=10
theta = sum(g(u,sig))/n %calculate integral
```

```
theta = 1.0005
```

```
function [hy] = g(x,sig)
%h(u)
    hy = (1./x-1)./sig^2.*exp(-(1./x-1).^2/(2*sig^2)).*x.^(-2);
end
```

(f) Using pencil and paper, convert the integral

$\int_0^\infty \frac{x^2}{\sigma^2} e^{-x^2/(2\sigma^2)} dx$ to an integral of the form $\int_0^1 h(u)du$

Let $y = \frac{1}{x+1}$ so as $x \to 0, y \to 1$ and $x \to \infty, y \to 0$

$$x = \frac{1}{y} - 1$$

And $|J| = \frac{dx(y)}{dy} = \left| -y^{-2} \right| = y^{-2}$

$$h(y) = \frac{\frac{1}{y} - 1}{\sigma^2} e^{\frac{-\left(\frac{1}{y}-1\right)^2}{2\sigma^2}} y^{-2}$$

$$\int_0^\infty \frac{x^2}{\sigma^2} e^{-x^2/(2\sigma^2)} dx = \int_0^1 \frac{\left(\frac{1}{y} - 1\right)^2}{\sigma^2} e^{\frac{-\left(\frac{1}{y}-1\right)^2}{2\sigma^2}} y^{-2} dy$$

(g) Generate 10^6 uniform (0,1) random numbers and evaluate the integral in part (f)

(Matlab code midtermg.mlx)

```
clear
n = 10e6;
u = rand(n, 1);
sig=10;
theta = sum(g(u,sig))/n
```

```
theta = 12.5333
```

```
true_value = sqrt(pi/2)*sig
```

```
true_value = 12.5331
```

```
function [hy] = g(x,sig)
    hy = (1./x-1).^2./sig^2.*exp(-(1./x-1).^2/(2*sig^2)).*x.^(-2);
end
```

(h)

Let $y = \frac{1}{x+1}$ so as $x \to 0, y \to 1$ and $x \to \infty, y \to 0$

$$x = \frac{1}{y} - 1$$

And $|J| = \frac{dx(y)}{dy} = \left|-y^{-2}\right| = y^{-2}$

$$h(y) = \frac{\frac{1}{y} - 1}{\sigma^2} e^{\frac{-(\frac{1}{y}-1)^2}{2\sigma^2}} y^{-2}$$

$$\int_0^\infty \frac{x(x-\mu)^2}{\sigma^2} e^{\frac{-x^2}{2\sigma^2}} dx = \int_0^1 \frac{\left(\frac{1}{y} - 1 - \mu\right)^2 \left(\frac{1}{y} - 1\right)}{\sigma^2} e^{\frac{-\left(\frac{1}{y}-1\right)^2}{2\sigma^2}} y^{-2} dy$$

(i) Generate 10^6 uniform (0,1) random numbers and evaluate the integral in part (f)

(Matlab code midtermi.mlx)
From the (f) and (g) true value of mean of f(x) which is E(x) equals to 12.5331

```
clear
rng('default')
n = 10e6;
u = rand(n, 1);
sig = 10;
mu = 12.5331;
theta = sum(g(u,sig,mu))/n
```

```
theta = 42.9242
```

```
true_value = (2-pi/2)*sig^2
```

```
true_value = 42.9204
```

```
function [hy] = g(x,sig,mu)
    hy =
(1./x-1).*(1./x-1-mu).^2./sig^2.*exp(-(1./x-1).^2/(2*sig^2)).*x.^(-2);
end
```

(j) Compare theoretical value in (a) to (d), (b) to (f), and (c) to (h)

	theoretical value(symbol)	theoretical value(number)	Numerical value
Integral of f(x) (a) To (d)	1	1	1.0005
E(x) (b) To (f)	$\sigma\sqrt{\frac{\pi}{2}}$	12.5331	12.5333
Var(x) (c) To (h)	$\left(2-\frac{\pi}{2}\right)\sigma^2$	42.9204	42.9242

Which we can see the value are close enough.

(k) Using uniform random numbers, compute the expectation of a function g(x) that Dr. Rowe can't get with pencil and paper.
(Matlab code midtermk.mlx)

$$E[x]=\int_0^\infty xf(x)dx$$

$$E\left[\frac{x}{\sigma^2}e^{-x^2/(2\sigma^2)}\right]=\int_0^\infty \frac{x^2}{\sigma^2}e^{-x^2/(2\sigma^2)}dx$$

$$E\left[\frac{x^2}{\sigma^2}e^{-x^2/(2\sigma^2)}\right]=\int_0^\infty \frac{x^3}{\sigma^2}e^{-x^2/(2\sigma^2)}dx$$

$$E\left[\frac{x(x-\mu)^2}{\sigma^2}e^{\frac{-x^2}{2\sigma^2}}\right]=\int_0^\infty \frac{x^2(x-\mu)^2}{\sigma^2}e^{\frac{-x^2}{2\sigma^2}}dx$$

```
clear
rng('default')
n = 10e6;
u = rand(n, 1);
sig=10;
mu=12.5331;
expectation_d = sum(g_Ed(u,sig))/n
```

```
expectation_d = 12.5226
```

```
expectation_f = sum(g_Ef(u,sig))/n
```

```
expectation_f = 199.8426
```

```
expectation_h = sum(g_Eh(u,sig,mu))/n
```

```
expectation_h = 715.8116
```

```
function [hy] = g_Ed(x,sig)
%expectation of (d)
    hy = (1./x-1).^2./sig^2.*exp(-(1./x-1).^2/(2*sig^2)).*x.^(-2);
end

function [hy] = g_Ef(x,sig)
%expectation of (f)
    hy = (1./x-1).^3./sig^2.*exp(-(1./x-1).^2/(2*sig^2)).*x.^(-2);
end

function [hy] = g_Eh(x,sig,mu)
%expectation of (h)
    hy =
(1./x-1).^2.*(1./x-1-mu).^2./sig^2.*exp(-(1./x-1).^2/(2*sig^2)).*x.^(-2)
;
end
```

(l) Verify your expectation of your function g(x) with numerical integration using rectangles (Matlab code midterml.mlx)

```
clear
num = 10e6;
k = 1;
x = linspace(0,1,num);
sig = 10;
mu = 12.5331;

integral_of_d = sum(g_Ed(x(2:num),sig)/num)
```

```
integral_of_d = 12.5331
```

```
integral_of_f = sum(g_Ef(x(2:num),sig)/num)
```

```
integral_of_f = 200.0000
```

```
integral_of_h = sum(g_Eh(x(2:num),sig,mu)/num)
```

```
integral_of_h = 715.3906
```

```
function [hy] = g_Ed(x,sig)
%expectation of (d)
    hy = (1./x-1).^2./sig^2.*exp(-(1./x-1).^2/(2*sig^2)).*x.^(-2);
end

function [hy] = g_Ef(x,sig)
%expectation of (f)
    hy = (1./x-1).^3./sig^2.*exp(-(1./x-1).^2/(2*sig^2)).*x.^(-2);
end

function [hy] = g_Eh(x,sig,mu)
%expectation of (h)
    hy =
(1./x-1).^2.*(1./x-1-mu).^2./sig^2.*exp(-(1./x-1).^2/(2*sig^2)).*x.^(-2)
;
end
```

	Monte Carlo Numerical integration	Numerical integration
(d)	12.5226	12.5331
(f)	199.8426	200.0000
(h)	715.8116	715.3906

We can see the value are close enough.