PROBLEM 2: This problem will use Fourier Sine series (using terms of the form $sin(n\pi x)$) and Haar wavelets (at most 8 terms).

(a) (20 pts) Approximate the following functions on $[0, 1]$. You may use the computer to verify your computations but you must also show the work by hand.

(i) $f(x) = x^2$

(ii) $g(x) = \begin{cases} 0 & 0 \leq x \leq \frac{1}{2} \\ 1 & \frac{1}{2} \leq x \leq 1 \end{cases}$

2. a) i) $f(x) = x^2$ $[0,1]$ $L = 1$

Fourier. $B_n = 2\int_0^1 x^2 \sin\left(\frac{n\pi x}{1}\right) dx$

$u = x^2$ $v' = \sin(n\pi x)$

$u' = 2x$ $v = -\frac{\cos(n\pi x)}{n\pi}$

$$I = -\frac{x^2\cos(n\pi x)}{n\pi} - \int -2x\frac{\cos(n\pi x)}{n\pi}\, dx$$

$$= -\frac{x^2\cos(n\pi x)}{n\pi} + \frac{2}{n\pi}\int x\cos(n\pi x)\, dx$$

$u = x$ $v' = \cos(n\pi x)$

$u' = 1$ $v = \frac{\sin(n\pi x)}{n\pi}$

$$= \frac{x\sin(n\pi x)}{n\pi} - \int \frac{\sin(n\pi x)}{n\pi}\, dx$$

$$-\frac{\cos(n\pi x)}{n^2\pi^2}$$

$$I = 2\frac{2\pi n\, x\sin(n\pi x) + (2 - n^2\pi^2x^2)\cos(n\pi x)}{n^3\pi^3}\Bigg|_0^1$$

$$= 2\frac{2\pi n\sin(n\pi) + (2-\pi^2n^2)\cos(n\pi) - 2}{n^3\pi^3}$$

So. Fourier Sine Series is

$$f(x) = \sum_{n=1}^{\infty} 2\frac{2\pi n\sin(n\pi) + (2-n^2\pi^2)\cos(n\pi) - 2}{n^3\pi^3}\sin(n\pi x)$$

haar $h_0(t) = \begin{cases} 1 & 0<t<1 \\ 0 & \end{cases}$ $h_1(t) = \begin{cases} 1 & 0<t<\frac{1}{2} \\ -1 & \frac{1}{2}<t<1 \\ 0 & \text{else} \end{cases}$

$h_2(t) = \sqrt{2}\, h_1(2t)$ $h_3(t) = \sqrt{2}\, h_1(2t-1)$

$h_4(t) = 2h_1(t)$ $h_5(t) = 2h_1(4t-1)$

$h_6(t) = 2h_1(4t-2)$ $h_7(t) = 2h_1(4t-3)$

$$f(t) = \sum_{n=0}^{7} c_n h_n(t)$$

$$c_n = \langle h_n, f(t) \rangle$$

$$c_0 = \int_0^1 x^2\,dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}$$

$$c_1 = \int_0^{1/2} x^2\,dx + \int_{1/2}^1 -x^2\,dx = \frac{1}{3}x^3\Big|_0^{1/2} + -\frac{1}{3}x^3\Big|_{1/2}^1 = \frac{1}{24}\left(-\frac{1}{3}+\frac{1}{3}\frac{1}{8}\right)$$

$$= \frac{1}{12} - \frac{1}{3} = -\frac{3}{12}$$

$$= -\frac{1}{4}$$

$$c_2 = \int_0^{1/4} \sqrt{2}\,x^2\,dx + \int_{1/4}^{1/2} -\sqrt{2}\,x^2\,dx = \frac{1}{3}\sqrt{2}\,x^3\Big|_0^{1/4}$$

$$-\frac{\sqrt{2}}{3}x^3\Big|_{1/4}^{1/2} = \frac{1}{3}\sqrt{2}\left(\frac{1}{4}\right)^3 + \left(-\frac{\sqrt{2}}{3}\frac{1}{8} + \frac{\sqrt{2}}{3}\frac{1}{64}\right)$$

$$= \frac{-\sqrt{2}}{32}$$

$$c_3 = \int_{1/2}^{3/4} \sqrt{2}\,x^2\,dx + \int_{3/4}^{1} -\sqrt{2}\,x^2\,dx = \frac{19}{3.2^{11/2}} + \left(-\frac{37}{3.2^{11/2}}\right)$$

$$= -\frac{18}{3.2^{11/2}} = -\frac{3\sqrt{2}}{32}$$

$$c_4 = \int_0^{1/8} 2x^2 + \int_{1/8}^{1/4} -2x^2 = \frac{1}{768} + \left(-\frac{7}{768}\right) = -\frac{6}{768} = -\frac{1}{128}$$

$$c_5 = \int_{1/4}^{3/8} 2x^2 + \int_{3/8}^{1/2} -2x^2 = \frac{19}{768} - \frac{37}{768} = -\frac{18}{768} = -\frac{3}{128}$$

$$c_6 = \int_{1/2}^{5/8} 2x^2 + \int_{5/8}^{3/4} -2x^2 = \frac{61}{768} - \frac{91}{768} = -\frac{30}{768} = -\frac{5}{128}$$

$$c_7 = \int_{3/4}^{7/8} 2x^2 + \int_{7/8}^{1} -2x^2 = \frac{127}{768} - \frac{169}{768} = -\frac{42}{768} = -\frac{7}{128}$$

$$f(t) = \frac{1}{3} + \sum_{n=0}^{7} c_n h_n(t)$$

ii) $g(x) = \begin{cases} 0 & 0 \le x \le 1/2 \\ 1 & 1/2 \le x \le 1 \end{cases}$

Fourier. $B_n = 2\int_0^{1/2} 0\,dx + 2\int_{1/2}^{1} \sin(n\pi x)\,dx$

$$= 2\left(\frac{-\cos(n\pi x)}{n\pi}\right)\Bigg|_{1/2}^{1}$$

$$= \frac{2\left(\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi)\right)}{n\pi}$$

$$g(x) = \sum_{n=1}^{\infty} \frac{2\left(\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi)\right)}{n\pi} \sin(n\pi x)$$

Haar. $C_n = \langle h_n, g(t) \rangle$

$C_0 = \int_{1/2}^{1} 1\,dt = t\Big|_{1/2}^{1} = \frac{1}{2}$

$C_1 = \int_{1/2}^{1} -1\,dt = -t\Big|_{1/2}^{1} = -1 - \left(-\frac{1}{2}\right) = -\frac{1}{2}$

$C_2 = 0 \qquad C_3 = \int_{1/2}^{3/4} \sqrt{2}\,dt + \int_{3/4}^{1} -\sqrt{2}\,dt = \sqrt{2}\left(\frac{1}{4} - \frac{1}{4}\right) = 0$

$C_4 = 0 \qquad C_5 = 0$

$C_6 = \int_{1/2}^{5/8} 2 + \int_{5/8}^{3/4} -2 = 2\left(\frac{1}{8} - \frac{1}{8}\right) = 0$

$C_7 = \int_{3/4}^{7/8} 2 + \int_{7/8}^{1} -2 = 2\left(\frac{1}{8} - \frac{1}{8}\right) = 0$

$$g(t) = \sum_{n=0}^{7} C_n h_n(t) = \frac{1}{2} + \left(-\frac{1}{2}\right) h_1(t)$$