PROBLEM 1: Consider the boundary value problem $u''(x) = x - 2$, where $u(0) = 0$, and $u'(1) = 4$ for $0 \leq x \leq 1$.

(a) (5 pts) Solve the IVP by hand to find the true solution $u(x)$.

1. a) $u''(x) = x - 2 \qquad u(0) = 0 \qquad u'(1) = 4$

$$\int u''(x)\,dx = \int (x-2)\,dx$$

$$u'(x) = \frac{1}{2}x^2 - 2x + C_1$$

$$\int u'(x)\,dx = \int \frac{1}{2}x^2 - 2x + C_1 \,dx$$

$$u(x) = \frac{1}{6}x^3 - x^2 + C_1 x + C_2$$

$u(0) = C_2 = 0 \qquad u'(1) = \frac{1}{2} - 2 + C_1 = 4$

$$C_1 = \frac{11}{2}$$

So $u(x) = \frac{1}{6}x^3 - x^2 + \frac{11}{2}x$

(b) (25 pts) Solve the IVP using the method of *Finite Differences.*

(i) Using $h = 0.25$, write out all steps of the solution method by hand and justifying all entries in the resulting matrix equation.

(ii) Then, write code in *Matlab* to solve your problem from (i) to determine the solution $u(x)$. Compare your result to that from (a) at the grid nodes.

(iii) Generalize your code to solve the problem with $h = \frac{1}{20}$.

b.ii) $h = 0.25$

$$x_0 = 0 \qquad f(x) = x-2$$
$$x_1 = 0.25$$
$$x_2 = 0.5$$
$$x_3 = 0.75$$
$$x_4 = 1$$

$$u''(x) \approx \frac{1}{h}\left[\frac{u_{i+1}-u_i}{h} - \frac{u_i - u_{i-1}}{h}\right] = f(x)$$
$$= \frac{1}{h^2}\left[u_{i+1} - 2u_i + u_{i-1}\right] = x_i - 2$$
$$\left[u_{i+1} - 2u_i + u_{i-1}\right] = h^2(x_i - 2) \qquad (1)$$

$i=1$ $\quad u_2 - 2u_1 + \overset{=0}{u_0} = h^2(x_1 - 2)$ $\qquad -u_3 + u_2 = h^2(x_3+1) - 4h$

$i=2$ $\quad u_3 - 2u_2 + u_1 = h^2(x_2 - 2)$ $\qquad \Uparrow$

$i=3$ $\quad u_4 - 2u_3 + u_2 = h^2(x_3 - 2)$ $\qquad 4h + u_3 - 2u_3 + u_2 = h^2(x_3+1)$ $\quad \Uparrow$

$u_0 = u(0) = 0 \qquad u'(1) = 4 \Rightarrow \frac{u_4 - u_3}{h} = 4 \Rightarrow u_4 = 4h + u_3$

So the matrix will be

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} h^2(x_1 - 2) \\ h^2(x_2 - 2) \\ h^2(x_3 - 2) - 4h \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -7/64 \\ -3/32 \\ -69/64 \end{bmatrix}$$

Solve the equation

$$\Rightarrow \begin{bmatrix} -2 & 1 & 0 \\ 0 & -3 & 2 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -7/64 \\ -19/64 \\ -69/64 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} -2 & 1 & 0 \\ 0 & -3 & 2 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -7/64 \\ -19/64 \\ -113/64 \end{bmatrix} \Rightarrow \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 41/32 \\ 157/64 \\ 113/32 \end{bmatrix}$$

So $\quad u(x_1) = \frac{41}{32} \qquad u(x_2) = \frac{157}{64} \qquad u(x_3) = \frac{113}{32}$

c)