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What is the solution to the vibrating string problem below

PDE $u_{tt} = -\alpha^2 u_{xx}$ $\quad 0 < x < L, \quad 0 < t < \infty$

BCs $\begin{cases} u(0,t) = 0 \\ u(L,t) = 0 \quad 0 < t < \infty \end{cases}$

IC $\begin{cases} u(x,0) = 0 \\ u_t(x,0) = \sin\left(\frac{3\pi x}{L}\right) \quad 0 \le x \le 1 \end{cases}$

Letting $\alpha = 1$ and $L = 1$ what does the graph of the solution look like for various values of time? Plot it in Matlab

Let $u(x,t) = X(x)T(t)$, then the PDE become:

$$\frac{\partial^2}{\partial x^2}[X(x)T(t)] = -\alpha^2 \frac{\partial^2}{\partial x^2}[X(x)T(t)]$$

$$X(x)T''(t) = -\alpha^2 X''(x)T(t)$$

Then divide both side $X(x)T(t)$

$$\frac{X(x)T''(t)}{X(x)T(t)} = -\alpha^2 \frac{X''(x)T(t)}{X(x)T(t)}$$

$$\frac{T(t)}{T'(t)} = -\alpha^2 \frac{X''(x)}{X(x)}$$

$$-\frac{1}{\alpha^2}\frac{T(t)}{T''(t)} = \frac{X''(x)}{X(x)}$$

$$-\frac{1}{\alpha^2}\frac{T(t)}{T'(t)} = \frac{X''(x)}{X(x)} = k$$

Then

$$\begin{cases} \dfrac{X''(x)}{X(x)} = k & (7) \\ -\dfrac{1}{\alpha^2}\dfrac{T''(t)}{T(t)} = k & (8) \end{cases}$$

Guess the solution $y(t) = e^{rt}$

$y'' + \mathrm{py}' + \mathrm{qy} = 0$

$r^2 e^{\mathrm{rt}} + \mathrm{pr}e^{\mathrm{rt}} + qe^{\mathrm{rt}} = 0$, which $p = 0, q = -k$

$e^{\mathrm{rt}}(r^2 + \mathrm{pr} + q) = 0$

since e^{rt}can not be 0

$(r^2 + \mathrm{pr} + q) = 0$

$r = -\frac{p \pm \sqrt{p^2 - 4q}}{2} = \pm \frac{\sqrt{-4q}}{2}$

For the situation that one root:

$r = -\frac{p}{2}$

$y = C_1 e^{\mathrm{rt}} + \mathrm{tC}_2 e^{\mathrm{rt}}$

For the situation that two roots:

$r_{1,2} = -\frac{p \pm \sqrt{p^2 - 4q}}{2}$

$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

For the situation that complex roots:

$r_{1,2} = a \pm \mathrm{bi} = \pm \frac{\sqrt{-4q}}{2} = \pm \frac{2i\sqrt{q}}{2} = \pm \sqrt{q}\, i$

$y = C_1 e^{\mathrm{at}} \cos(\mathrm{bt}) + C_2 e^{\mathrm{at}} \sin(\mathrm{bt})$

Case 1 $k > 0$, $k = \lambda^2$, which is the situation that r has two root.

$X''(x) = \lambda^2 X(x)$

$X(x) = C_1 e^{\lambda x} + C_2 e^{-\lambda x}$

Plug BCs in:

BCs $\begin{cases} u(0,t) = 0 \\ u(L,t) = 0 \quad 0 < t < \infty \end{cases}$

$X(0) = C_1 + C_2 = 0$

$C_1 = -C_2$

$X(L) = C_1 e^{\lambda L} + C_2 e^{-\lambda L} = -C_2 e^{\lambda L} + C_2 e^{-\lambda L} = C_2(e^{-\lambda L} - e^{\lambda L}) = 0$

So either $C_2 = 0$ or $(e^{-\lambda L} - e^{\lambda L}) = 0$

If $C_2 = 0, C_1 = 0$

Which $X(x) = 0$ and then $u(x,t) = 0$

So $C_2 = 0$ is not true, $(e^{-\lambda L} - e^{\lambda L}) = 0$

$e^{-\lambda L} - e^{\lambda L} = 0$,

$e^{-\lambda L} = e^{\lambda L}$

Which also means $X(x) = 0$ and then $u(x,t) = 0$

So $k > 0$ is not true.

Case 2 $k = 0$, $k = \lambda^2 = 0$,so $\lambda = 0$

$X(x) = C_1 e^{\lambda x} + \mathrm{x} C_2 e^{\lambda x} = C_1 + C_2 x$

Plug in BCs

BCs $\begin{cases} u(0,t) = 0 \\ u(L,t) = 0 \quad 0 < t < \infty \end{cases}$

$X(0) = C_1 + C_2 0 = C_1 = 0$

$X(L) = C_1 + C_2 L = C_2 L = 0$

So $C_2 = 0$

Therefor both $C_1$and C_2 equal to 0

$X(x) = 0$for any situation and $u(x,t) = 0$

So $k = 0$ is not true.

Case 3 $k < 0$, $k = -\lambda^2, r_{1,2} = \pm\sqrt{-k}\, i = \pm\lambda i$

$a = 0, b = \lambda$

$X(x) = C_1 \cos(\lambda x) + C_2 \sin(\lambda x)$

Plug in BCs

BCs $\begin{cases} u(0,t) = 0 \\ u(L,t) = 0 \quad 0 < t < \infty \end{cases}$

$X(0) = C_1\cos(0) + C_2\sin(0) = C_1 = 0$

$X(L) = C_2\sin(\lambda L) = 0$

$C_2 \neq 0$ since if $C_2 = C_1 = 0$, $X(x) = 0$ and $u(x,t) = 0$

So $\sin(\lambda L) = 0, \lambda L = n\pi, \lambda = \frac{n\pi}{L}$

$X(x) = C_n \sin\left(\frac{n\pi x}{L}\right)$

Then for $T(t), -\frac{1}{\alpha^2}\frac{T''(t)}{T(t)} = k$

$T''(t) = -\alpha^2 \mathrm{kT}(t)$

By using $k < 0$, $k = -\lambda^2$

$r_{1,2} = \pm\sqrt{-k\alpha^2}\, i = \pm \lambda\alpha i$

$a = 0, b = \lambda\alpha = \frac{n\pi\alpha}{L}$

$T(t) = a_n\cos\left(\frac{n\pi\alpha t}{L}\right) + b_n\sin\left(\frac{n\pi\alpha t}{L}\right)$

$u(x,t) = X(x)T(t) = X(x) = C_n\sin\left(\frac{n\pi x}{L}\right)\left[a_n\cos\left(\frac{n\pi\alpha t}{L}\right) + b_n\sin\left(\frac{n\pi\alpha t}{L}\right)\right]$

$= C_n\sin\left(\frac{n\pi x}{L}\right)a_n\cos\left(\frac{n\pi\alpha t}{L}\right) + C_n\sin\left(\frac{n\pi x}{L}\right)b_n\sin\left(\frac{n\pi\alpha t}{L}\right)$

Let $C_n a_n = A_n$ and $C_n b_n = B_n$

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[A_n\cos\left(\frac{n\pi\alpha t}{L}\right) + B_n\sin\left(\frac{n\pi\alpha t}{L}\right)\right]$$

Then plug IC in:

IC $\begin{cases} u(x,0) = 0 \\ u_t(x,0) = \sin\left(\frac{3\pi x}{L}\right) \quad 0 \le x \le 1 \end{cases}$

$$u(x,0) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)[A_n] = 0$$

Since $\sin\left(\frac{n\pi x}{L}\right) \neq 0, A_n = 0,$

$$u_t(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[\frac{n\pi\alpha}{L} B_n \cos\left(\frac{n\pi\alpha t}{L}\right)\right]$$

$$= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\frac{n\pi\alpha}{L}\left[B_n \cos\left(\frac{n\pi\alpha t}{L}\right)\right]$$

$$u_t(x,0) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\frac{n\pi\alpha}{L}\left[B_n \cos(0)\right] = \sin\left(\frac{3\pi x}{L}\right)$$

$$\sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\frac{n\pi\alpha}{L}\left[B_n\right] = \sin\left(\frac{3\pi x}{L}\right)$$

$$\frac{3\pi\alpha}{L}\left[B_3\right] = 1$$

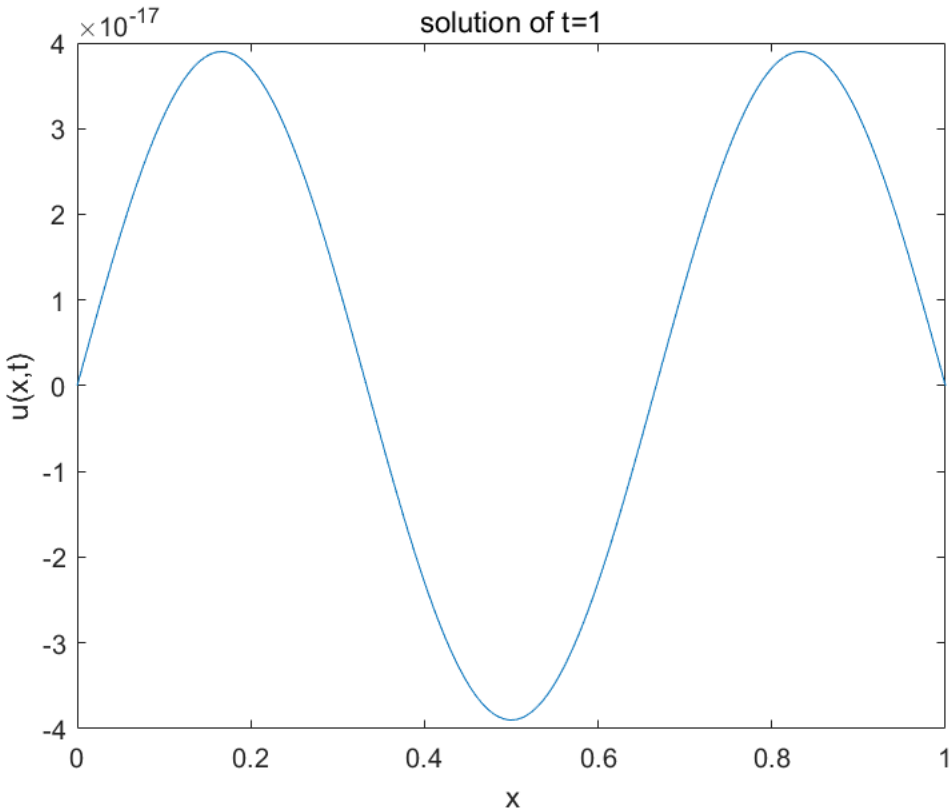
$$B_3 = \frac{L}{3\pi\alpha}$$

So $u(x,t) = \frac{L}{3\pi\alpha}\sin\left(\frac{3\pi x}{L}\right)\sin\left(\frac{3\pi\alpha t}{L}\right)$

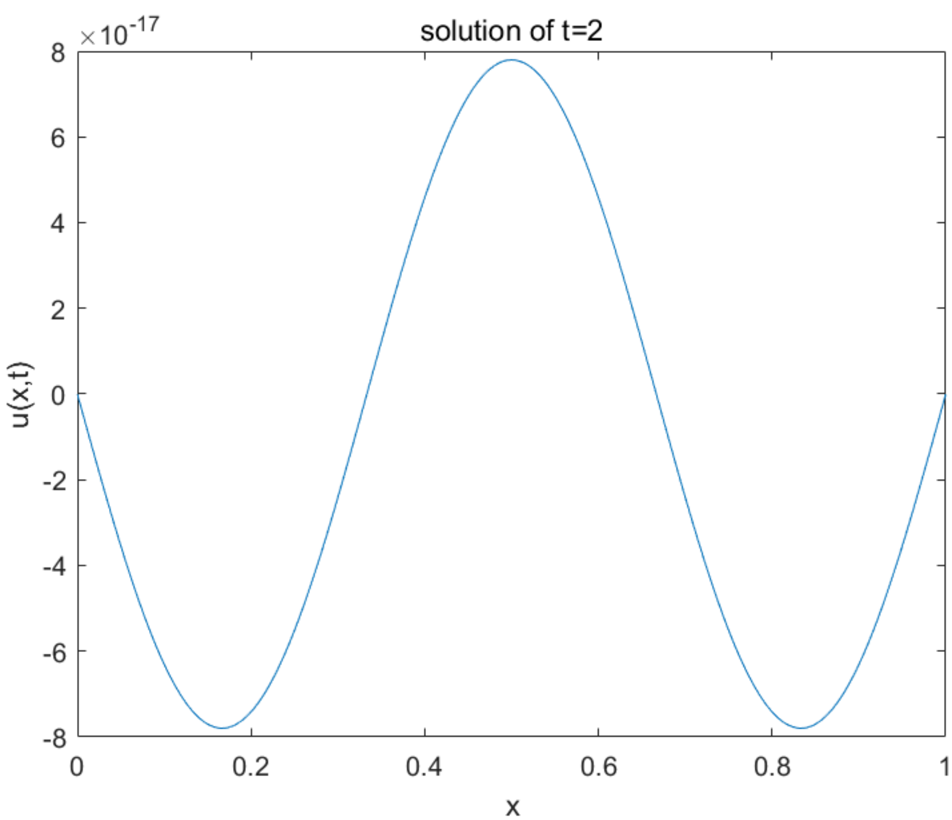
Letting $\alpha = 1$ and $L = 1$

$$u(x,t) = \frac{1}{3\pi}\sin(3\pi x)\sin(3\pi t) \qquad (9)$$

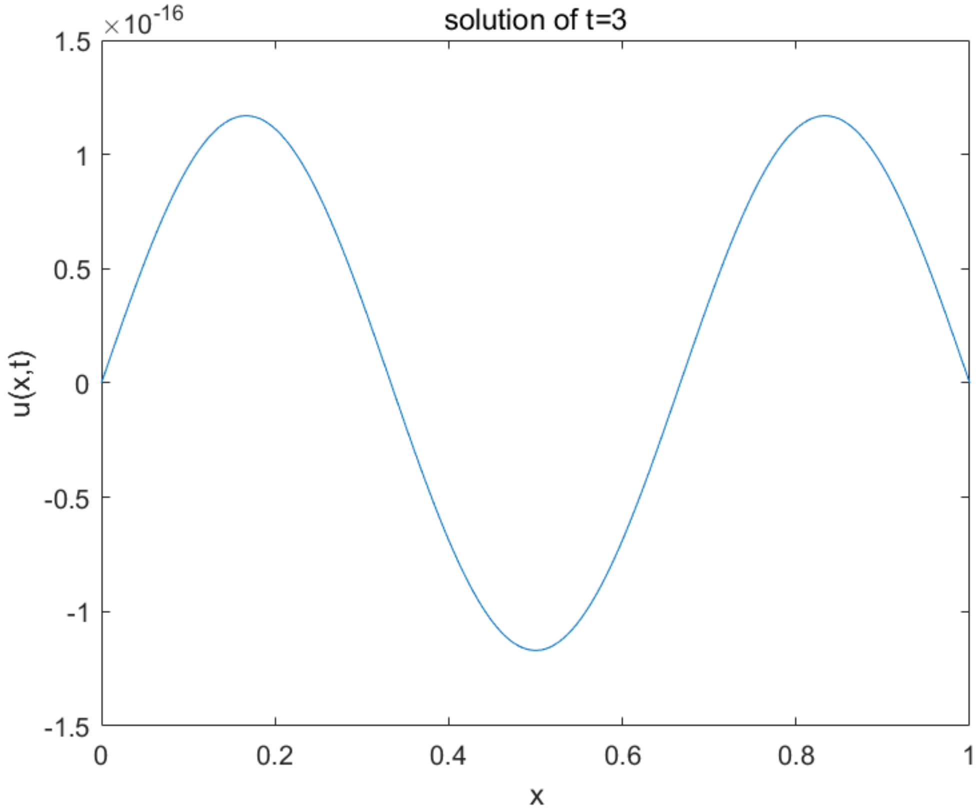
```
clear
x=linspace(0,1,500);%generate x

plot(x,u(x,1))
xlabel("x");ylabel("u(x,t)");title("solution of t=1");
```



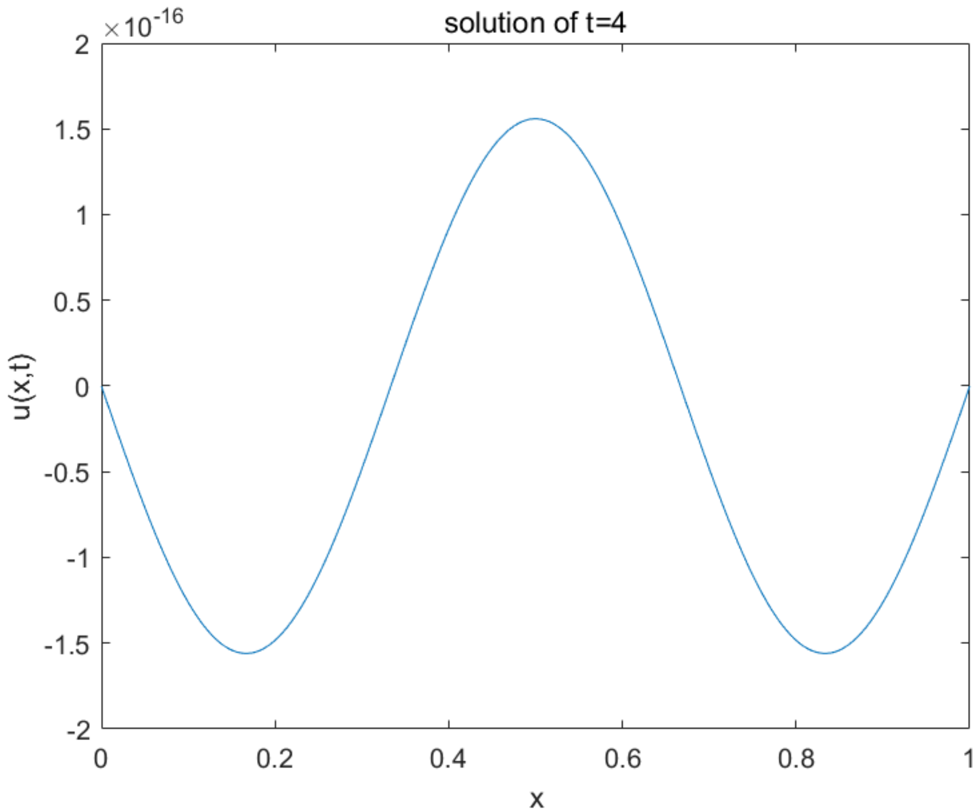

```
plot(x,u(x,2))
xlabel("x");ylabel("u(x,t)");title("solution of t=2");
```



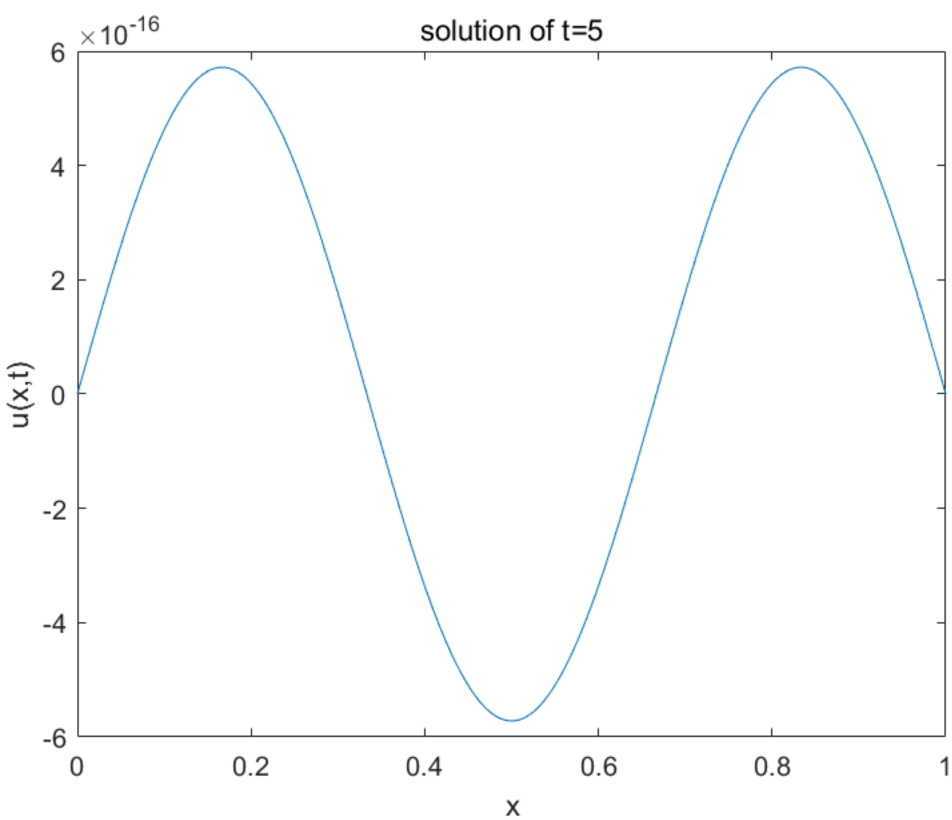
```
plot(x,u(x,3))
xlabel("x");ylabel("u(x,t)");title("solution of t=3");
```




```
plot(x,u(x,4))
xlabel("x");ylabel("u(x,t)");title("solution of t=4");
```




```
plot(x,u(x,5))
xlabel("x");ylabel("u(x,t)");title("solution of t=5");
```

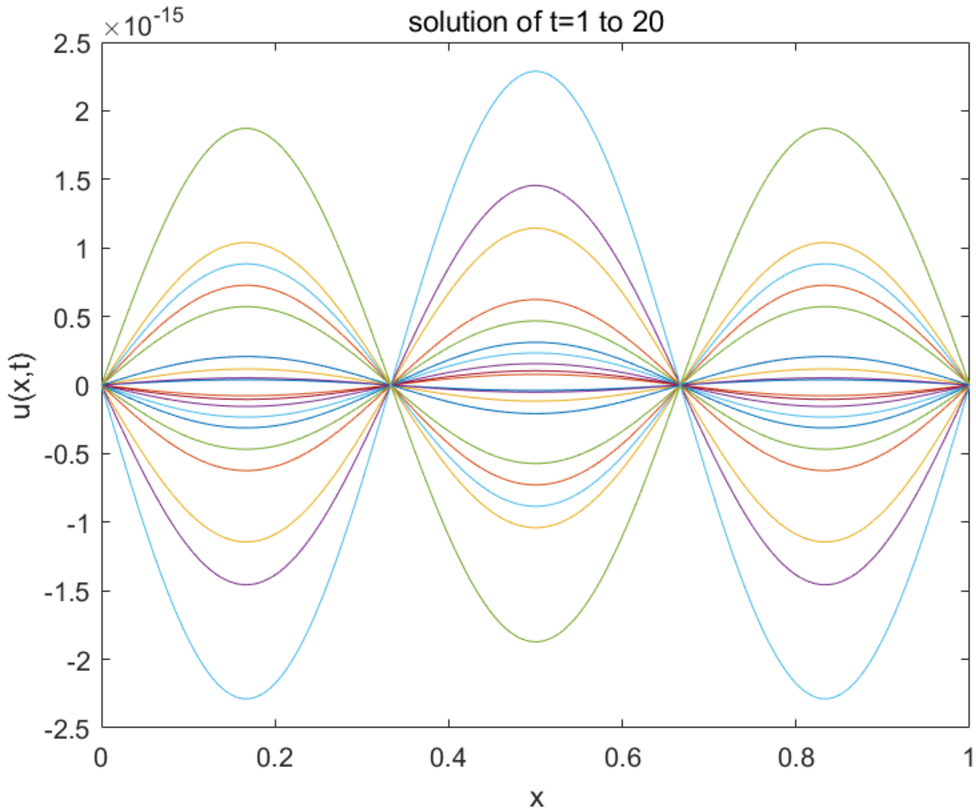


```
%plot for t=1 to 5
```

from the plot above we could see the wave vibrate around 0

So if we test for further t

```
for t=1:20
    %loop for t
    plot(x,u(x,t))%plot with each t
    hold on
end
xlabel("x");ylabel("u(x,t)");title("solution of t=1 to 20");
```




```
function [s]=u(x,t)
%equation (9)
    s=1/(3*pi)*sin(3*pi*x).*sin(3*pi*t);
end
```