Γ. $X \sim \Gamma(n, \beta)$ $\beta > 0$ $Y = \frac{1}{X}$

$X = \frac{1}{Y}$ $\frac{\partial X}{\partial Y} = -(Y)^{-2}$

$$f_X(x) = \frac{\beta^n}{\Gamma(n)} x^{n-1} e^{-\beta x}$$

$$f_Y(y) = \frac{\beta^n}{\Gamma(n)} \left(\frac{1}{y}\right)^{n-1} e^{-\beta \frac{1}{y}} \cdot y^{-2}$$

$$= \frac{\beta^n}{\Gamma(n)} y^{-n-2} e^{-\beta y^{-1}}$$

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MSSC 6030 Final Exam Shuaizhou Wang

1. a) $u''(x) = x-2 \qquad u(0) = 0 \qquad u'(1) = 4$

$$\int u''(x)dx = \int (x-2)\,dx$$

$$u'(x) = \frac{1}{2}x^2 - 2x + C_1$$

$$\int u'(x)dx = \int \frac{1}{2}x^2 - 2x + C_1 \, dx$$

$$u(x) = \frac{1}{6}x^3 - x^2 + C_1 x + C_2$$

$u(0) = C_2 = 0 \qquad u'(1) = \frac{1}{2} - 2 + C_1 = 4$

$$C_1 = \frac{11}{2}$$

So $u(x) = \frac{1}{6}x^3 - x^2 + \frac{11}{2}x$

b.)i) $h = 0.25$

$x_0 = 0 \qquad f(x) = x-2$
$x_1 = 0.25$
$x_2 = 0.5$
$x_3 = 0.75$
$x_4 = 1$

$$u''(x) \approx \frac{1}{h}\left[\frac{u_{i+1}-u_i}{h} - \frac{u_i - u_{i-1}}{h}\right] = f(x)$$

$$= \frac{1}{h^2}\left[u_{i+1} - 2u_i + u_{i-1}\right] = x_i - 2$$

$$\left[u_{i+1} - 2u_i + u_{i-1}\right] = h^2(x_i - 2) \qquad (1)$$

$i=1 \qquad u_2 - 2u_1 + \overset{=0}{u_0} = h^2(x_1 - 2) \qquad -u_3 + u_2 = h^2(x_3+1) - 4h$

$i=2 \qquad u_3 - 2u_2 + u_1 = h^2(x_2 - 2) \qquad \Uparrow$

$i=3 \qquad u_4 - 2u_3 + u_2 = h^2(x_3 - 2) \qquad 4h + u_3 - 2u_3 + u_2 = h^2(x_3+1)$

$\Uparrow$

$u_0 = u(0) = 0 \qquad u'(1) = 4 \Rightarrow \frac{u_4 - u_3}{h} = 4 \Rightarrow u_4 = 4h + u_3$

So the matrix will be

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} h^2(x_1-2) \\ h^2(x_2-2) \\ h^2(x_3-2)-4h \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -7/64 \\ -3/32 \\ -69/64 \end{bmatrix}$$

Solve the equation

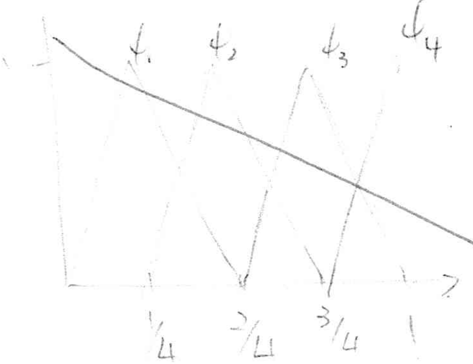
$$\Rightarrow \begin{bmatrix} -2 & 1 & 0 \\ 0 & -3 & 2 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -7/64 \\ -14/64 \\ -69/64 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} -2 & 1 & 0 \\ 0 & -3 & 2 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -7/64 \\ -14/64 \\ -113/64 \end{bmatrix} \Rightarrow \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 41/32 \\ 157/64 \\ 113/32 \end{bmatrix}$$

So $u(x_1) = \frac{41}{32}$ $u(x_2) = \frac{157}{64}$ $u(x_3) = \frac{113}{32}$

c) $h = 0.25$ So $n=3$. the hat function which need to be use is



$$\phi_1 = \begin{cases} 4x & 0 \le x \le \frac{1}{4} \\ -4x+2 & \frac{1}{4} < x \le \frac{1}{2} \\ 0 & \frac{1}{2} \le x \le 1 \end{cases} \qquad \phi_4 = \begin{cases} 0 & 0 \le x \le \frac{3}{4} \\ 4x-3 & \frac{3}{4} \le x \le 1 \end{cases}$$

$$\phi_2 = \begin{cases} 0 & 0 \le x \le \frac{1}{4} \\ 4x-1 & \frac{1}{4} \le x \le \frac{1}{2} \\ -4x+3 & \frac{1}{2} \le x \le \frac{3}{4} \\ 0 & \frac{3}{4} \le x \le 1 \end{cases}$$

$$\phi_3 = \begin{cases} 0 & 0 \le x \le \frac{1}{2} \\ 4x-2 & \frac{1}{2} \le x \le \frac{3}{4} \\ -4x+4 & \frac{3}{4} \le x \le 1 \end{cases}$$

c). $u''(x) = x - 2$ $u(0) = 0$ $u'(1) = 4$

$1/4$ $2/4$ $3/4$ $Lu = u''$ $f(x) = x - 2$

$$\langle LU_2, \varphi_j \rangle = \langle f, \varphi_j \rangle$$

$$\int_0^1 u''v \, dx = \int_0^1 f v \, dx$$

$$\int_0^1 \frac{d}{dx}(u')v \, dx = \int_0^1 f v \, dx$$

$$\left[u'v \Big|_{x=0}^{x=1} - \int_0^1 u'v' \, dx \right] = \int_0^1 f v \, dx$$

$$u'(1)v(1) - u'(0)v(0) - \int_0^1 u'v' \, dx = \int_0^1 f v \, dx$$

$$4v(1) - \int_0^1 u'v' \, dx = \int_0^1 f v \, dx$$

$$\int_0^1 u'v' \, dx = 4v(1) - \int_0^1 f v \, dx$$

2. a) i) $f(x) = x^2$ $[0,1]$ $L=1$

Fourier. $B_n = 2\int_0^1 x^2 \sin\left(\frac{n\pi x}{1}\right) dx$

$u = x^2$ $v' = \sin(n\pi x)$

$u' = 2x$ $v = -\frac{\cos(n\pi x)}{n\pi}$

$$I = -\frac{x^2\cos(n\pi x)}{n\pi} - \int -2x\frac{\cos(n\pi x)}{n\pi}\,dx$$

$$= -\frac{x^2\cos(n\pi x)}{n\pi} + \frac{2}{n\pi}\int x\cos(n\pi x).\,dx$$

$u = x$ $v' = \cos(n\pi x)$

$u' = 1$ $v = \frac{\sin(n\pi x)}{n\pi}$

$$= \frac{x\sin(n\pi x)}{n\pi} - \int \frac{\sin(n\pi x)}{n\pi}\,dx$$

$\Downarrow$

$-\frac{\cos(n\pi x)}{n^2\pi^2}$

$$I = 2\left.\frac{2\pi n\, x\sin(n\pi x) + 2 - n^2\pi^2x^2)\cos(n\pi x)}{n^3\pi^3}\right|_0^1$$

$$= 2\frac{2\pi n\sin(n\pi) + (2-\pi^2n^2)\cos(n\pi)-2}{n^3\pi^3}$$

So. Fourier Sine Series is

$$f(x) = \sum_{n=1}^{\infty} 2\frac{2\pi n\sin(n\pi)+(2-n^2\pi^2)\cos(n\pi)-2}{n^3\pi^3}\sin(n\pi x)$$

haar $h_0(t) = \begin{cases}1 & 0<t<1\\ 0\end{cases}$ $h_1(t) = \begin{cases}1 & 0<t<\frac{1}{2}\\ -1 & \frac{1}{2}<t<1\\ 0 & \text{else}\end{cases}$

$h_2(t) = \sqrt{2}\, h_1(2t)$ $h_3(t) = \sqrt{2}\, h_1(2t-1)$

$h_4(t) = 2h_1(t)$ $h_5(t) = 2h_1(4t-1)$

$h_6(t) = 2h_1(4t-2)$ $h_7(t) = 2h_1(4t-3)$

$$f(t) = \sum_{n=0}^{7} c_n h_n(t)$$

$$c_n = \langle h_n, f(t) \rangle$$

$$c_0 = \int_0^1 x^2\,dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}$$

$$c_1 = \int_0^{1/2} x^2\,dx + \int_{1/2}^{1} -x^2\,dx = \frac{1}{3}x^3\Big|_0^{1/2} + -\frac{1}{3}x^3\Big|_{1/2}^{1} = \frac{1}{24}\left(-\frac{1}{3} + \frac{1}{3}\frac{1}{8}\right)$$

$$= \frac{1}{12} - \frac{1}{3} = -\frac{3}{12}$$

$$= -\frac{1}{4}$$

$$c_2 = \int_0^{1/4} \sqrt{2}\,x^2\,dx + \int_{1/4}^{1/2} -\sqrt{2}\,x^2\,dx = \frac{1}{3}\sqrt{2}\,x^3\Big|_0^{1/4} - \frac{\sqrt{2}}{3}x^3\Big|_{1/4}^{1/2} = \frac{1}{3}\sqrt{2}\left(\frac{1}{4}\right)^3 + \left(-\frac{\sqrt{2}}{3}\frac{1}{8} + \frac{\sqrt{2}}{3}\frac{1}{64}\right)$$

$$= \frac{\sqrt{2}}{32}$$

$$c_3 = \int_{1/2}^{3/4} \sqrt{2}\,x^2\,dx + \int_{3/4}^{1} -\sqrt{2}\,x^2\,dx = \frac{19}{3 \cdot 2^{11/2}} + \left(-\frac{37}{3 \cdot 2^{11/2}}\right)$$

$$= -\frac{18}{3 \cdot 2^{11/2}} = -\frac{3\sqrt{2}}{32}$$

$$c_4 = \int_0^{1/8} 2x^2 + \int_{1/8}^{1/4} -2x^2 = \frac{1}{768} + \left(-\frac{7}{768}\right) = -\frac{6}{768} = -\frac{1}{128}$$

$$c_5 = \int_{1/4}^{3/8} 2x^2 + \int_{3/8}^{1/2} -2x^2 = \frac{19}{768} - \frac{37}{768} = -\frac{18}{768} = -\frac{3}{128}$$

$$c_6 = \int_{1/2}^{5/8} 2x^2 + \int_{5/8}^{3/4} -2x^2 = \frac{61}{768} - \frac{91}{768} = -\frac{30}{768} = -\frac{5}{128}$$

$$c_7 = \int_{3/4}^{7/8} 2x^2 + \int_{7/8}^{1} -2x^2 = \frac{127}{768} - \frac{169}{768} = -\frac{42}{768} = -\frac{7}{128}$$

$$f(t) = \frac{1}{3} + \sum_{n=0}^{7} c_n h_n(t)$$

iii) $g(x) = \begin{cases} 0 & 0 \le x \le 1/2 \\ 1 & 1/2 \le x \le 1 \end{cases}$

Fourier. $B_n = 2\int_0^{1/2} 0 \, dx + 2\int_{1/2}^{1} \sin(n\pi x)\,dx$

$$= 2\left(\frac{-\cos(n\pi x)}{n\pi}\right)\Bigg|_{1/2}^{1}$$

$$= \frac{2\left(\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi)\right)}{n\pi}$$

$$g(x) = \sum_{n=1}^{\infty} \frac{2\left(\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi)\right)}{n\pi} \sin(n\pi x)$$

Haar. $C_n = \langle h_n, g(t) \rangle$

$C_0 = \int_{1/2}^{1} 1 \, dt = t\Big|_{1/2}^{1} = \frac{1}{2}$

$C_1 = \int_{1/2}^{1} -1 \, dt = -t\Big|_{1/2}^{1} = -1 - \left(-\frac{1}{2}\right) = -\frac{1}{2}$

$C_2 = 0 \qquad C_3 = \int_{1/2}^{3/4} \sqrt{2}\, dt + \int_{3/4}^{1} -\sqrt{2}\, dt = \sqrt{2}\left(\frac{1}{4} - \frac{1}{4}\right) = 0$

$C_4 = 0 \qquad C_5 = 0$

$C_6 = \int_{1/2}^{5/8} 2 + \int_{5/8}^{3/4} -2 = 2\left(\frac{1}{8} - \frac{1}{8}\right) = 0$

$C_7 = \int_{3/4}^{7/8} 2 + \int_{7/8}^{1} -2 = 2\left(\frac{1}{8} - \frac{1}{8}\right) = 0$

$f(t) = \sum_{n=0}^{7} C_n h_n(t) = \frac{1}{2} + \left(-\frac{1}{2}\right) h_1(t)$

4. with step h.

the taylor expansion of $y(x+h)$ is

$$y(x) + hy'(x) + \frac{h^2}{2!} y''(x) + \frac{h^3}{3!} y'''(x) + \cdots$$

For any numerical method.

the prediction of point x_{n+1} is y_{n+1}

the accuracy is $y(x+h) - y_{n+1}$

Since the prediction of y_{n+1} will need $y(x)$ (which $y(x) = y_0$ is a provided po

$$y(x+h) - y_{n+1} = hy'(x) + \frac{h^2}{2!} y''(x) + \frac{h^3}{3!} y'''(x) + \cdots$$

$$- h\phi(x, y, h)$$

which $h\phi$ is function that predict the $[y_{n+1} - y(x)]$

~~So~~ the how the order of (h^n) left ~~has been~~ means the accurate of the method

Since y', y'' .... can be determine from $f(x)$

So. the term left from the substraction.

has order $(h)^{n+1}$ means the method has nth order accurate.

6. F D.

① has larger error, $o(h)$

② Can be solve 1^{st} and 2^{nd} order of ODE with a point given, for ~~two~~ 2^{nd} order ODE need two point or one point and one free end

Numerical method.

FEM.

① will be used on Solving 2^{nd} order of ODE and PDE
② Has larger computation ③ numerical method.

Separation of Variable.

① will be used on Solving PDE with IC ~~and~~ or BC.
② True solution will be fund, ③ with some kind of IC, BC will be hard to find the solution.

Monte Carlo:

① Integration: Definite integral

② PDE with IC or BC
③ numerical method ④ take a lot computation by computer.

Fourier Transform.

① Analytical solution will fund
② PDE with 1^{st} or 2^{nd} order.
③ difficult to use if the step of Inverse of Fourier Transform cannot be fund

④ Take lots of word to do the convolution.

7. The accuracy of numerical method can be determined by subtract the approximate value with the taylor expansion of the true value.

And for solution, compute the total absolute error ~~betee~~ between the solution and discretized true solution.