problem 4

Let $f(x)$ be defined by:

$$f = \begin{cases} 0 & -3 \le x \le -1 \\ 1 & -1 \le x \le 1 \\ 0 & 1 \le x \le 3 \end{cases} \qquad (16)$$

and suppose that $f(x+6) = f(x), i.e. f$ has period 6. Find Fourier Series representation for $f(x)$.

Plot the origional function $f(x)$ as well as its 5, 10 and 20th partial sums

$$s_m(x) = \frac{a_0}{2} + \sum_{n=1}^{m} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right] \qquad (17)$$

of the Fourier Series for $f(x)$. Use different colors and markers as well as a legend in your plot. Additionally, plot the frequency spectrum of $f(x)$ for $n = 1, \ldots, 20$. Finally, plot the absolute value of the \`error' $e_m(x) = f(x) - s_m(x)$ (versus *x*) for the same values of m=5, 10 and 20. Where does the largest error in the approximation come, i.e. for which values of *x* is the approximation poorest?

Solve:

From the problem we could found that period is 6, therefore L=3.

we could compute A_n and B_n by using the following equation:

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \mathrm{dx}, \text{ where } n = 0, 1, 2, 3, \ldots \qquad (18)$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{dx}, \text{ where } n = 1, 2, 3, \ldots \qquad (19)$$

Then we can plug equation (16) into (18) and (19)

$$\begin{aligned} a_n &= \frac{1}{3} \int_{-3}^{3} f(x) \cos\left(\frac{n\pi x}{3}\right) \mathrm{dx} \\ &= \frac{1}{3} \left(\int_{-3}^{-1} 0 * \cos\left(\frac{n\pi x}{3}\right) \mathrm{dx} + \int_{-1}^{1} 1 * \cos\left(\frac{n\pi x}{3}\right) \mathrm{dx} + \int_{-3}^{3} 0 * \cos\left(\frac{n\pi x}{3}\right) \mathrm{dx} \right) \\ &= \frac{1}{3} \int_{-1}^{1} \cos\left(\frac{n\pi x}{3}\right) \mathrm{dx} \\ &= \frac{1}{3} \frac{3}{n\pi} \sin\left(\frac{n\pi x}{3}\right)\Bigg|_{-1}^{1} \\ &= \frac{2}{n\pi} \sin\left(\frac{n\pi}{3}\right) \end{aligned} \qquad (20)$$

Where $n = 1, 2, 3, \ldots$

for a_0

$$a_0 = \frac{1}{3}\int_{-3}^{3} f(x)\cos\left(\frac{0*\pi x}{3}\right)\mathrm{dx}$$
$$= \frac{1}{3}\int_{-1}^{1} 1\,\mathrm{dx}$$
$$= \frac{2}{3}$$

$$b_n = \frac{1}{3}\int_{-3}^{3} f(x)\sin\left(\frac{n\pi x}{3}\right)\mathrm{dx} = \frac{1}{3}\int_{-1}^{1} \sin\left(\frac{n\pi x}{3}\right)\mathrm{dx}$$

Since sin function is odd function, the integral is been taken over a symmetric interval from -1 to 1. So the result of integration is 0.

So $b_n = 0$.

So we can fill equation (17) by using the coefficien a_0, a_n and b_n

$$s_m(x) = \frac{1}{3} + \sum_{n=1}^{m}\left[\frac{2}{n\pi}\sin\left(\frac{n\pi}{3}\right)\cos\left(\frac{n\pi x}{3}\right)\right] \qquad (21)$$

Then we could compute the sum for each n=5,10 and 20.

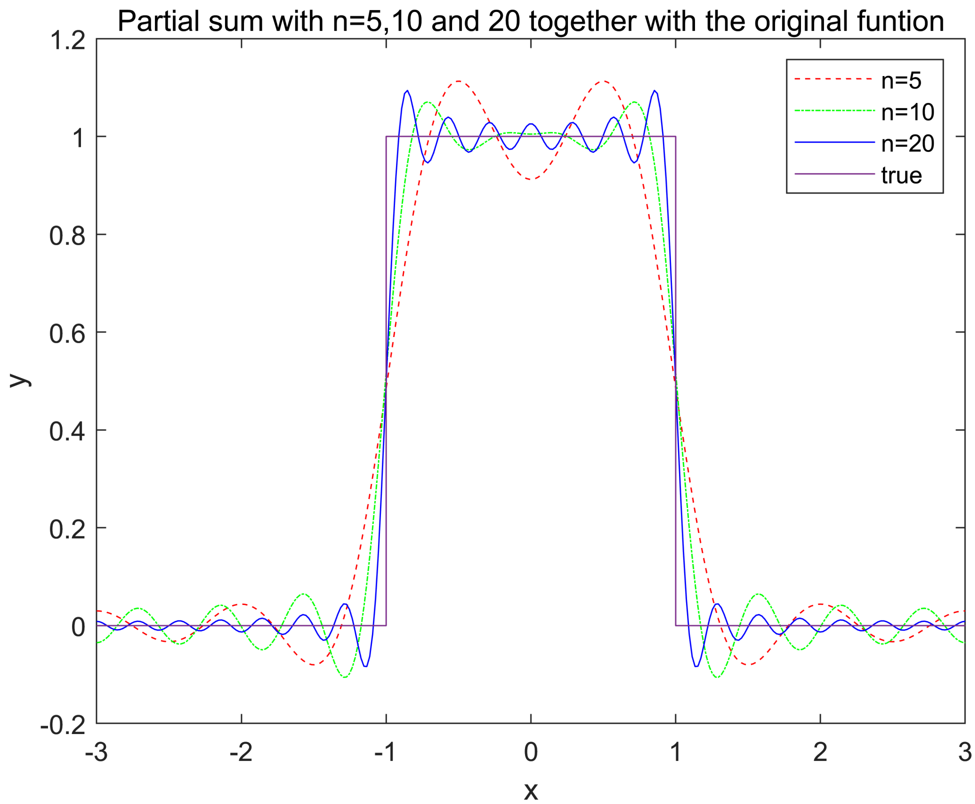
```
clear
format long
k = 300;%number of x
x = linspace(-3, 3, 300);%define x
n5 = sm(5, x);%sum of equation (21) with n=5
n10 = sm(10, x);%sum of equation (21) with n=10
n20 = sm(20, x);%sum of equation (21) with n=20

%original function
x1 = [-3, -1];%x value from -3 to -1
y1 = [0, 0];%y value from -3 to -1
x2 = x1 + 2;%x value from -1 to 1
y2 = y1 + 1;%y value from -1 to 1
x3 = x2 + 2;%x value from 1 to 3
y3 = y1;%y value from 1 to 3
xt=[x1 x2 x3];%get x1 x2 x3 together
yt=[y1 y2 y3];%get y1 y2 y3 together
%finishing produce the original function
```

Then plot n=5,10,20 with original function together.

```
%plot
plot(x, n5, '--', 'Color', 'r')
hold on
plot(x, n10, '-.', 'Color', 'g')
plot(x, n20, '-', 'Color', 'b')
plot(xt, yt)
legend('n=5', 'n=10', 'n=20', 'true')
xlabel('x')
```

```
ylabel('y')
title('Partial sum with n=5,10 and 20 together with the original funtion')
hold off
```




```
%finish plot
```

Fig 4.1 In the plot the stright solid line is the true function f(x), n=5 has the line '--', n=10 has line '-.-' also the curved solid line represents the n=20.

Recall the equation from problem 3

The spectrum can be calculate by

$$C_n = \sqrt{A_n^2 + B_n^2}$$

since $B_n = 0$, $C_n = \sqrt{A_n^2}$ (22)

```
n = 20;%define max n
spy = zeros(n, 1);
for j = 1:n
    spy(j) = sqrt(an(j)*an(j));%equation (22) and restoring each Cn
end
```

And plot

```
%plot Cn
```

```
plot(spy, '*')
title('Spectrum of S_m until n=20')
xlabel('n')
ylabel('C_n')
```

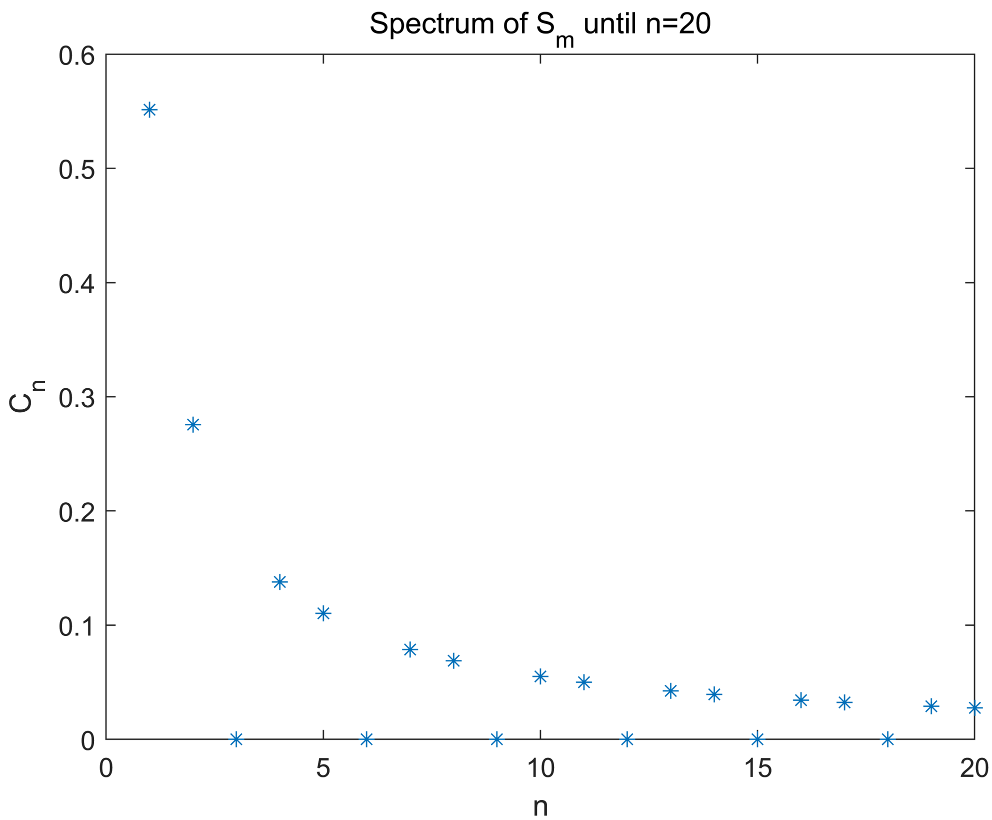


Fig 4.2 Spectrum of S_m until n=20

The error could be calculate by $e_m(x) = f(x) - s_m(x)$ (23)

```
%original function
A = zeros(1, k/3);
B = ones(1, k/3);
g = [A B A];
%finishing produce the original function
e5 = abs(g - n5);%equation (23)
e10 = abs(g - n10);
e20 = abs(g - n20);
```

Then plot them

```
plot(x, e5, 'o')
xlabel('x')
ylabel('absolute error')
title('absolute error vs x when n=5')
```

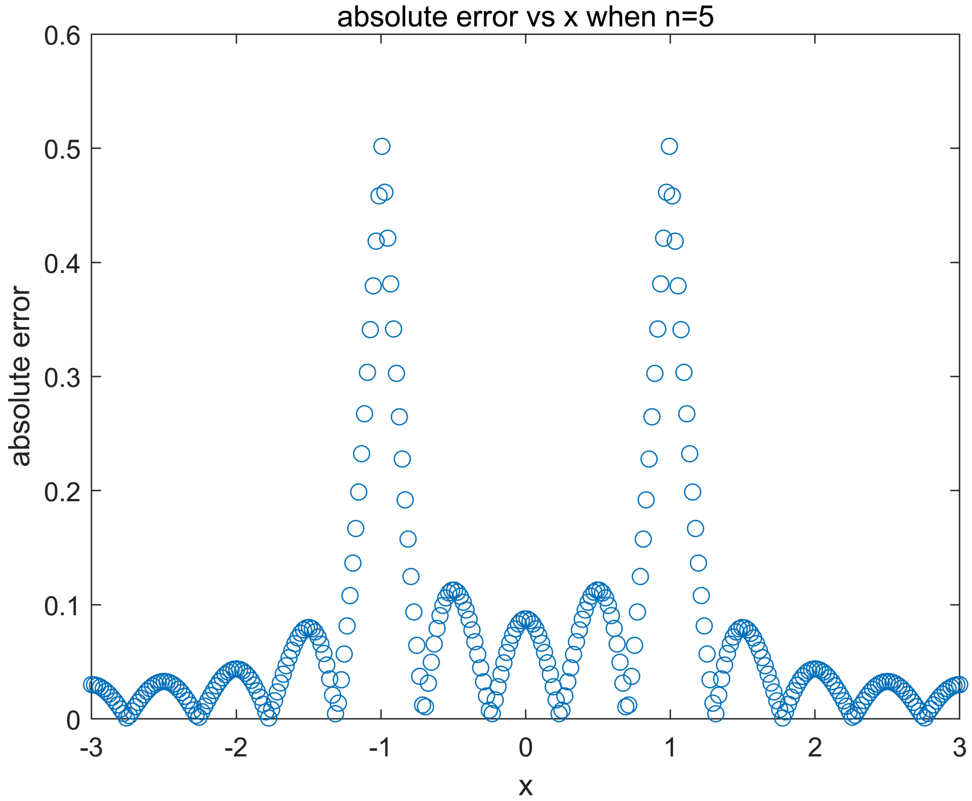


Fig 4.3a The plot shows absolute error when n=5, we could see a big error around -1 and 1

```
plot(x, e10, 'o')
xlabel('x')
ylabel('absolute error')
title('absolute error vs x when n=10')
```

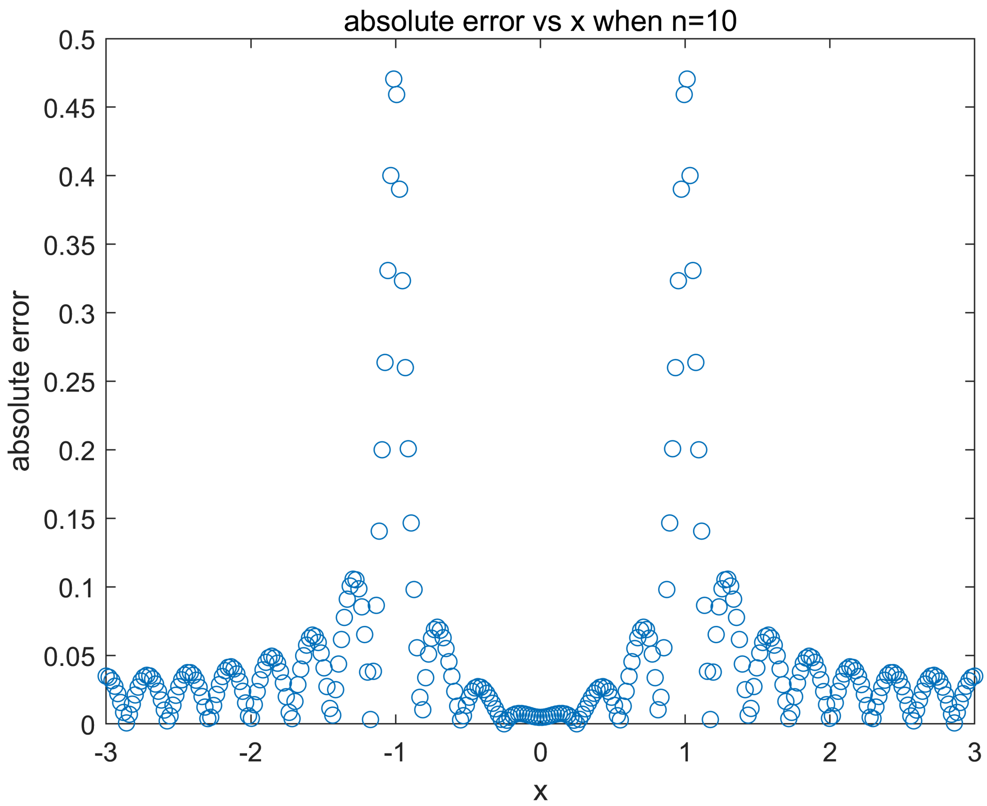


Fig 4.3b

```
plot(x, e20, 'o')
xlabel('x')
ylabel('absolute error')
title('absolute error vs x when n=20')
```

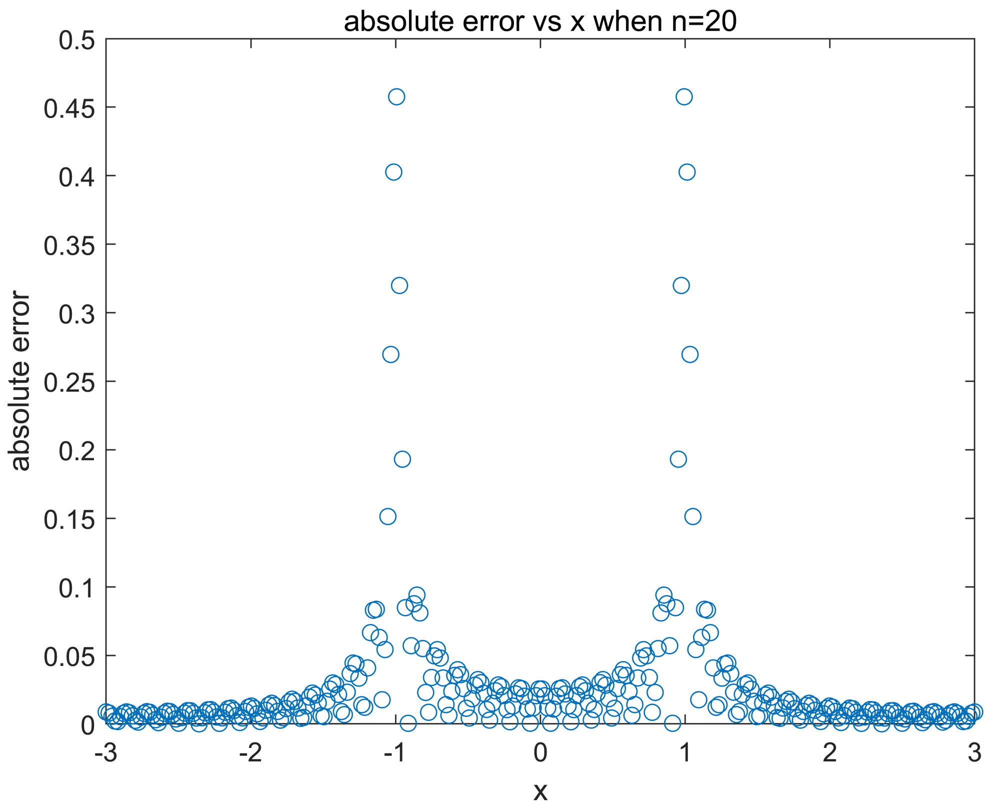


Fig 4.3c

Then the sum of each error are:

```
errn5 = sum(e5)
```

```
errn5 =
  23.526405630863092
```

```
errn10 = sum(e10)
```

```
errn10 =
  15.892651600874025
```

```
errn20 = sum(e20)
```

```
errn20 =
   9.026425762739727
```

The total error for n=5 is aroud 23.5, for n=10 is about 15.9, and for n=20 is about 9.0.

So as a conclusion the largest error between n=5,10 and 20 are at the point -1 and 1 where is the stopped point in the original function.

Also when the number of partial sum takes, the sum of error are also been reduced, but due to the Gibb's Phenomenon, the approching on the point -1 and 1 will not have a enough error no matter how large of n we use.

```
function [s] = sm(m, x)
%function of calculate partial sum from equation (21)
    s=0;%initial s
    for j =1:m
        s = s + an(j).*cos(j*pi.*x/3);%take the sum up to n
    end
    s = s + 1/3;%add a_0
end

function [a] = an(n)
%function of an from equation (20)
    a = 2/(n*pi)*sin(n*pi/3);
end
```