problem 2

Strang 3.1

11. solve $-u'' = x$ with $u(0) = u(1) = 0$. Here $u(x)$ is cubic. Then solve approximately with two hat function and $h = \frac{1}{3}$. Where is the largest error?

Solve:

First solving $-u'' = x$ by using anti-derivatives:

$$u'' = -x$$

$$u' = -\int \mathrm{xdx} = -\frac{1}{2}x^2 + C_1$$

$$u = \int \left(-\frac{1}{2}x^2 + C_1\right)\mathrm{dx} = -\frac{1}{6}x^3 + C_1 x + C_2$$

by using the boundary condition $u(0) = u(1) = 0$

$$u(0) = C_2 = 0$$

$$u(1) = -\frac{1}{6} + C_1 = 0$$

So $C_1 = \frac{1}{6}$

So $u(x) = -\frac{1}{6}x^3 + \frac{1}{6}x$ (6)

Then with FEM

$$h = \frac{1}{3}$$

$$n = \frac{1}{h} - 1 = 2$$

So the grid point on x are $x_0 = 0, x_1 = \frac{1}{3}, x_2 = \frac{2}{3}, x_3 = 1$.

The hat function will be:

$$\phi_1(x) = \begin{cases} 3x & 0 \le x \le \frac{1}{3} \\ -3x + 2 & \frac{1}{3} \le x \le \frac{2}{3} \\ 0 & \frac{2}{3} \le x \le 1 \end{cases}, \phi_2(x) = \begin{cases} 0 & 0 \le x \le \frac{1}{3} \\ 3x - 1 & \frac{1}{3} \le x \le \frac{2}{3} \\ -3x + 3 & \frac{2}{3} \le x \le 1 \end{cases}$$

The graph of these three hat function are:

```
clear
Num_x_hat = 201;
hat_x = linspace(0, 1, Num_x_hat);
OneThird = Num_x_hat/3;
TwoThird = 2*Num_x_hat/3;
%define the each stop point.
phi1 = [3*hat_x(1:OneThird) -3*hat_x(OneThird + 1:TwoThird) + 2 ...
    0*hat_x(TwoThird + 1:Num_x_hat)];
%phi1
phi2 = [0*hat_x(1:OneThird) 3*hat_x(OneThird+1:TwoThird)-1 ...
    -3*hat_x(TwoThird + 1:Num_x_hat) + 3];
%phi2
%plot
plot(hat_x, phi1)
hold on
plot(hat_x, phi2,'--')
legend('\phi_1', '\phi_2')
hold off
```

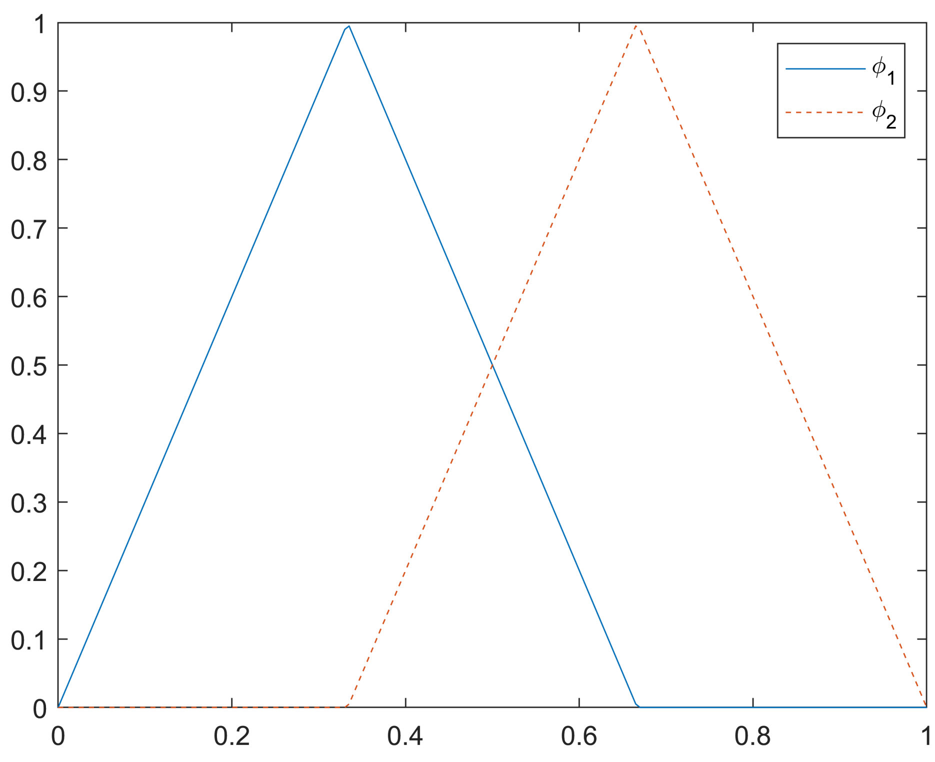

Fig 2.1

The plot shows three hat function by phi1 and phi2 in the different line style. Which phi1 is in solid line '—' and phi2 is in '--'.

The Stiffness Martix has the formular:

$$\mathrm{Ku} = b \qquad (7)$$

which $K = \begin{bmatrix} \langle \phi_1', \phi_1' \rangle & \langle \phi_2', \phi_1' \rangle \\ \langle \phi_1', \phi_2' \rangle & \langle \phi_2', \phi_2' \rangle \end{bmatrix}$, $u = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$, $b = \begin{bmatrix} \langle f(x), \phi_1 \rangle \\ \langle f(x), \phi_2 \rangle \end{bmatrix}$

Find the derivative of ϕ_1 and ϕ_2:

$$\phi_1(x) = \begin{cases} 3 & 0 \le x \le \frac{1}{3} \\ -3 & \frac{1}{3} \le x \le \frac{2}{3} \\ 0 & \frac{2}{3} \le x \le 1 \end{cases}, \ \phi_2(x) = \begin{cases} 0 & 0 \le x \le \frac{1}{3} \\ 3 & \frac{1}{3} \le x \le \frac{2}{3} \\ -3 & \frac{2}{3} \le x \le 1 \end{cases} \qquad (8)$$

Then compute (7) with (8) :

$$\langle \phi_1', \phi_1' \rangle = \int_0^1 (\phi_1'(x))^2 \mathrm{dx} = \int_0^{\frac{1}{3}} 3^2 \mathrm{dx} + \int_{\frac{1}{3}}^{\frac{2}{3}} (-3)^2 \mathrm{dx} + \int_{\frac{2}{3}}^{1} 0 \mathrm{dx} = 6$$

$$\langle \phi_2', \phi_2' \rangle = \int_0^1 (\phi_2'(x))^2 \mathrm{dx} = \int_0^{\frac{1}{3}} 0 \mathrm{dx} + \int_{\frac{1}{3}}^{\frac{2}{3}} 3^2 \mathrm{dx} + \int_{\frac{2}{3}}^{1} (-3)^2 \mathrm{dx} = 6$$

$$\langle \phi_1', \phi_2' \rangle = \langle \phi_2', \phi_1' \rangle = \int_0^1 \phi_1'(x)\phi_2'(x) \mathrm{dx} = \int_0^{\frac{1}{3}} 0 \mathrm{dx} + \int_{\frac{1}{3}}^{\frac{2}{3}} -9 \mathrm{dx} + \int_{\frac{2}{3}}^{1} 0 \mathrm{dx} = -3$$

So $K = \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix}$ (9)

Then compute b with $f(x) = x$:

$$\phi_1(x) = \begin{cases} 3x & 0 \le x \le \frac{1}{3} \\ -3x + 2 & \frac{1}{3} \le x \le \frac{2}{3} \\ 0 & \frac{2}{3} \le x \le 1 \end{cases}, \phi_2(x) = \begin{cases} 0 & 0 \le x \le \frac{1}{3} \\ 3x - 1 & \frac{1}{3} \le x \le \frac{2}{3} \\ -3x + 3 & \frac{2}{3} \le x \le 1 \end{cases}$$

$$\langle f(x), \phi_1 \rangle = \int_0^1 f(x)\phi_1(x) = \int_0^{\frac{1}{3}} 3x^2 \mathrm{dx} + \int_{\frac{1}{3}}^{\frac{2}{3}} (-3x^2 + 2x)\mathrm{dx} + \int_{\frac{2}{3}}^{1} 0\mathrm{dx}$$

$$= x^3|_0^{\frac{1}{3}} + (-x^3 + x^2)_{\frac{1}{3}}^{\frac{2}{3}} = \frac{1}{27} + \left(\left(-\frac{8}{27} + \frac{4}{9}\right) - \left(-\frac{1}{27} + \frac{1}{9}\right)\right)$$

$$= \frac{1}{27} + \left(\frac{4}{27} - \frac{2}{27}\right) = \frac{1}{9}$$

$$\langle f(x), \phi_2 \rangle = \int_0^1 f(x)\phi_2(x) = \int_0^{\frac{1}{3}} 0\mathrm{dx} + \int_{\frac{1}{3}}^{\frac{2}{3}} (3x^2 - \mathrm{xdx}) + \int_{\frac{2}{3}}^{1} (-3x^2 + 3x)\mathrm{dx}$$

$$= \left(x^3 - \frac{1}{2}x^2\right)_{\frac{1}{3}}^{\frac{2}{3}} + \left(-x^3 + \frac{3}{2}x^2\right)_{\frac{2}{3}}^{1} = \left(\left(\frac{8}{27} - \frac{1}{2} * \frac{4}{9}\right) - \left(\frac{1}{27} - \frac{1}{2} * \frac{1}{9}\right)\right) + \left(\left(-1 + \frac{3}{2}\right) - \left(-\frac{8}{27} + \frac{3}{2} * \frac{4}{9}\right)\right)$$

$$= \left(\frac{2}{27} + \frac{1}{54}\right) + \left(\frac{1}{2} - \frac{10}{27}\right) = \frac{5}{54} + \frac{7}{54} = \frac{12}{54} = \frac{2}{9}$$

so $b = \begin{bmatrix} \frac{1}{9} \\ \frac{2}{9} \end{bmatrix}$ (10)

So we could compute u from $\mathrm{Ku} = b$

```
h = 1/3;%define h
n = 1/h - 1;%compute n
Num_x = 99;%number of interval point
x = linspace(0, 1, Num_x);%define x
xg = linspace(0, 1, n + 2);%true x with 4 points
up = -1/6*xg.^3 + 1/6*xg;%true y with 4 points
u = -1/6*x.^3 + 1/6*x;%true solution
b = [1/9;2/9];%b from equation 10
k = [6 -3;
    -3 6];%K from equation 9
coe = k\b;%compute u from Ku=b
coe_final = [0;coe;0]
```

```
coe_final = 4×1
         0
    0.0494
    0.0617
         0
```

```
error = abs(coe_final(:) - up(:))%compute the error of four grid point
```

error = 4×1
10^{-16} ×

```
         0
    0.1388
         0
         0
```

Therefore the solution is [0 0.0494 0.0617 0]

Also the error on the grid points are very small.

To compute the largest error we could plug the result into the equation:

$$f(x) = \alpha_1\phi_1(x) + \alpha_2\phi_2(x) \qquad (11)$$

```
x_1 = Num_x/3;%define the grid 0 to 1/3
x_2 = 2*Num_x/3;%define the grid 1/3 to 2/3
U_app = [3*x(1:x_1)*coe(1) ...
    (-3*x(x_1 + 1:x_2) + 2)*coe(1) + (3*x(x_1 + 1:x_2) - 1)*coe(2) ...
    (-3*x(x_2 + 1:Num_x) + 3)*coe(2)];%equation (11)
largest_error = max(abs(U_app - u))%compute the largest error
```

```
largest_error = 0.0116
```

Therefore we could see the largest error is 0.0116

```
%plot
plot(x, u, '.')
hold on
plot(xg, up, '*')
plot(xg, coe_final, 'o', 'Markersize', 10, "Color", 'r')
plot(x, U_app)
legend('true solution', 'true solution of the grid points', ...
    'estimate grid points', 'estimate solution', 'location', 'south')
xlabel('x')
ylabel('y')
title('plot of estimat grid point and true solution')
hold off
```

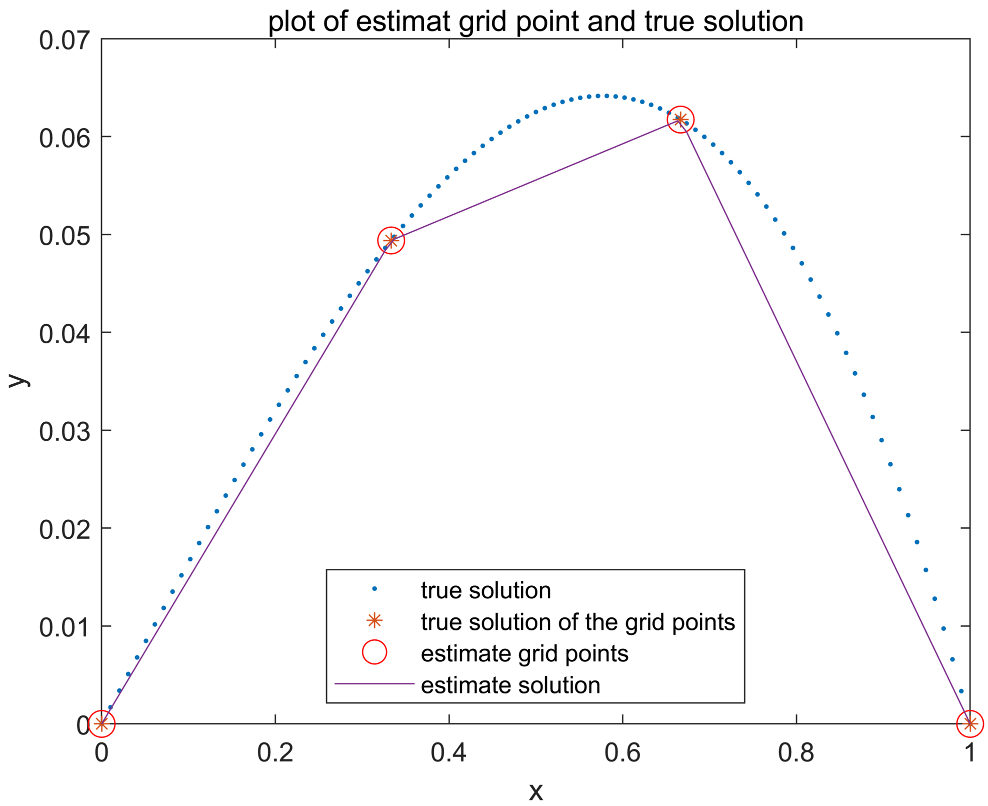

Fig 2.2 This plot shows the result we compute by FEM compare with true solution. The grid points, on the graph they on the same spot. The solid line represent function from FEM we coud see there still are difference from the true solution.