problem 6

Let $f(x)$ be defined by the sawtooth example

$f(x) = x$ for $-L < x < L$ with

$f(x+2L) = f(x)$ as examined in class and notes online. Compute the term-by-term derivative of your FS representation for $f(x)$ and compare it to the actual derivative

$f'(x) = 1$. Show all work and plot your results.

Solve

First derive the Fourier Series of the original function

Recall from problem 4

$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{dx}$, where $n = 0, 1, 2, 3, \ldots$ (26)

$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{dx}$, where $n = 1, 2, 3, \ldots$ (27)

Then plug $f(x) = x$ into function (26) and (27)

$$a_n = \frac{1}{L}\int_{-L}^{L} \mathrm{x}\cos\left(\frac{n\pi x}{L}\right)\mathrm{dx}$$

$$u = x \qquad v' = \cos\left(\frac{n\pi x}{L}\right)$$

$$u' = 1 \qquad v = \frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)$$

$$a_n = \frac{1}{L}\left(\left(\frac{L}{n\pi}\mathrm{x}\sin\left(\frac{n\pi x}{L}\right)\right)\Bigg|_{-L}^{L} - \int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\mathrm{dx}\right)$$
$$= \frac{1}{L}\left(\left(\frac{L}{n\pi}\mathrm{L}\sin\left(\frac{n\pi L}{L}\right)\right) - \left(\frac{L}{n\pi}(-\mathrm{L})\sin\left(\frac{n\pi(-L)}{L}\right)\right) - \left(-\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\right)\Bigg|_{-L}^{L}\right)$$
$$= \frac{1}{L}\left(\left(\frac{L^2}{n\pi}\sin(n\pi)\right) + \left(\frac{L^2}{n\pi}\sin(-n\pi)\right) - \left[\left(-\frac{L}{n\pi}\cos(n\pi)\right) - \left(-\frac{L}{n\pi}\cos(-n\pi)\right)\right]\right)$$

Since n=0,1,2,3,...

$$\sin(n\pi) = 0$$

So $a_n = \frac{1}{L}\left(\frac{L}{n\pi}\cos(n\pi) - \frac{L}{n\pi}\cos(-n\pi)\right) = 0$

$$b_n = \frac{1}{L}\int_{-L}^{L} \mathrm{x}\sin\left(\frac{n\pi x}{L}\right)\mathrm{dx}$$

$u = x \qquad v' = \sin\left(\frac{n\pi x}{L}\right)$

$u' = 1 \qquad v = -\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)$

$$b_n = \frac{1}{L}\left(\left(-\frac{L}{n\pi}\mathrm{x}\cos\left(\frac{n\pi x}{L}\right)\right)_{-L}^{L} - \int_{-L}^{L} -\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\mathrm{dx}\right)$$
$$= \frac{1}{L}\left(\left(-\frac{L}{n\pi}\mathrm{L}\cos\left(\frac{n\pi L}{L}\right)\right) - \left(-\frac{L}{n\pi}(-L)\cos\left(\frac{n\pi(-L)}{L}\right)\right) + \frac{L}{n\pi}\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\mathrm{dx}\right)$$
$$= \frac{1}{L}\left(\left(-\frac{L^2}{n\pi}\cos(n\pi)\right) - \left(\frac{L^2}{n\pi}\cos(-n\pi)\right) + \left[\left(\frac{L}{n\pi}\right)^2\sin\left(\frac{n\pi x}{L}\right)\right]_{-L}^{L}\right)$$
$$= \frac{1}{L}\left(-\frac{2L^2}{n\pi}\cos(n\pi) + \left(\frac{L}{n\pi}\right)^2[\sin(n\pi) - \sin(-n\pi)]\right)$$
$$= -\frac{1}{L}\frac{2L^2}{n\pi}\cos(n\pi)$$
$$= -\frac{2L}{n\pi}\cos(n\pi)$$

So the partial sum function $s_m(x) = \frac{a_0}{2} + \sum_{n=1}^{m}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right]$ becomes

$$s_m(x) = \sum_{n=1}^{m} -\frac{2L}{n\pi}\cos(n\pi)\sin\left(\frac{n\pi x}{L}\right) \qquad (28)$$

If we write it term by term

$$s_m(x) = -\frac{2L}{\pi}\cos(\pi)\sin\left(\frac{\pi x}{L}\right) - \frac{2L}{2\pi}\cos(2\pi)\sin\left(\frac{2\pi x}{L}\right) - \frac{2L}{3\pi}\cos(3\pi)\sin\left(\frac{3\pi x}{L}\right) + \ldots$$
$$\mathrm{fs}(x) = \frac{2L}{\pi}\left[-\cos(\pi)\sin\left(\frac{\pi x}{L}\right) - \frac{1}{2}\cos(2\pi)\sin\left(\frac{2\pi x}{L}\right) - \frac{1}{3}\cos(3\pi)\sin\left(\frac{3\pi x}{L}\right) + \ldots\right]$$
$$= \frac{2L}{\pi}\left[\sin\left(\frac{\pi x}{L}\right) - \frac{1}{2}\sin\left(\frac{2\pi x}{L}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{L}\right) - \ldots\right]$$

So if we take derivative of the [F.S.]

$$\mathrm{fs}'(x) = \frac{2L}{\pi}\left[\frac{\pi}{L}\cos\left(\frac{\pi x}{L}\right) - \frac{1}{2}\frac{2\pi}{L}\cos\left(\frac{2\pi x}{L}\right) + \frac{1}{3}\frac{3\pi}{L}\cos\left(\frac{3\pi x}{L}\right) - \ldots\right]$$
$$= \frac{2L}{\pi}\left[\frac{\pi}{L}\cos\left(\frac{\pi x}{L}\right) - \frac{\pi}{L}\cos\left(\frac{2\pi x}{L}\right) + \frac{\pi}{L}\cos\left(\frac{3\pi x}{L}\right) - \ldots\right]$$
$$= 2\left[\cos\left(\frac{\pi x}{L}\right) - \cos\left(\frac{2\pi x}{L}\right) + \cos\left(\frac{3\pi x}{L}\right) - \ldots\right]$$

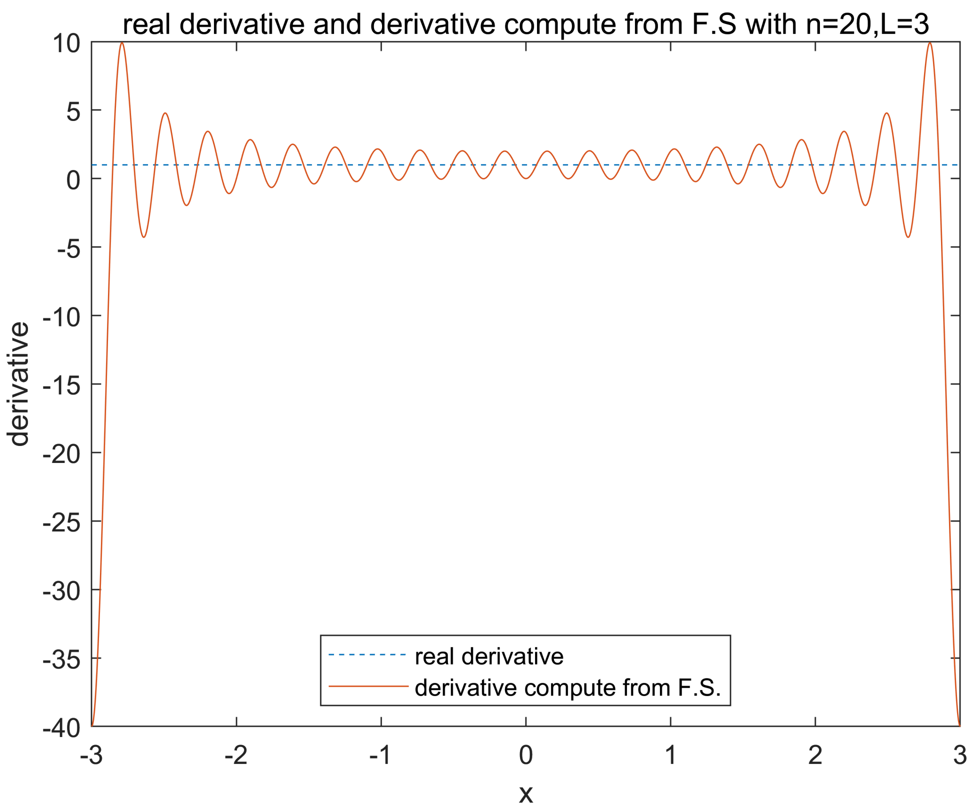
$$= 2\sum_{n=1}^{\infty}(-1)^{(n+1)}\cos\left(\frac{n\pi x}{L}\right) \qquad (29)$$

By observing equation (29) it seems that hardly convert to 1 which is the derivative of the original function f(x)=x.

Choosing some value of L and n, then test it numerically, then compare the result with $f'(x) = 1$.

Take L=3 and n=20 and testing as following:

```
clear
n = 20;L = 3;k = 1000;
%take roughly guess L=2 and n=20
%Then take k=1000 for number of interval point
x = linspace(-L, L, k);%define x from -L to L which is in one period
ts = ones(1, k);%derivative of the original function, f'(x)=1
o = each_term(n, x, L);%calculate the equation (29) by given n,l,and x
%plot
plot(x, ts, '--')
hold on
plot(x, o)
legend('real derivative', 'derivative compute from F.S.', 'location', 'south')
xlabel('x')
ylabel('derivative')
title('real derivative and derivative compute from F.S with n=20,L=3')
hold off
```




```
%finish plot
```

Fig 6.1 The plot shows the numerical test of derivative compute from F.S. n=20,L=3 are using in the computation. the solid cured line represents the Fourier Series derivative(equation 29), and the stright line means the real derivative of the original function.

From the plot we could see the derivative compute from F.S oscillate around 1. Also the amplitude of the oscillation is getting larger when the x=L and smaller in the middle. So we could conclude that in this case(n=20, L=3), derivative compute from F.S. are still far from the rea derivative.

At the case above it is only works for one case. To reduce the limitation, we may need further test.

Since we don't know the n and L, by further testing, we could fix one varible and then test the other.

Case 1 fix n and change L, compute equation (29) for each L

By examine the variance and mean of the each result we get from fixed n and L.

If the mean=1 and variance tend to 0, we may conclude the variance of L is usefull to help derivative get to real derivative.

```
numL = 500;%number of L will be test
jL = zeros(1, numL);%initial L
Lvar = zeros(1, numL);%initial variance of result from each L
Lmean = zeros(1, numL);%initial mean of result from each L
n = 50;%fix n=50
for j = 1:numL
    xl = linspace(-j, j, k);%define x
    oL = each_term(n, xl, j);%compute equation (29)
    jL(j) = j;%store l
    Lvar(j) = var(oL);%store variance
    Lmean(j) = mean(oL);%store mean
end
%plot
plot(jL, Lvar, '-')
hold on
plot(jL, Lmean, '--')
xlabel('L')
ylabel('var and mean')
title('variance and mean for different L')
legend('variance', 'mean')
hold off
```

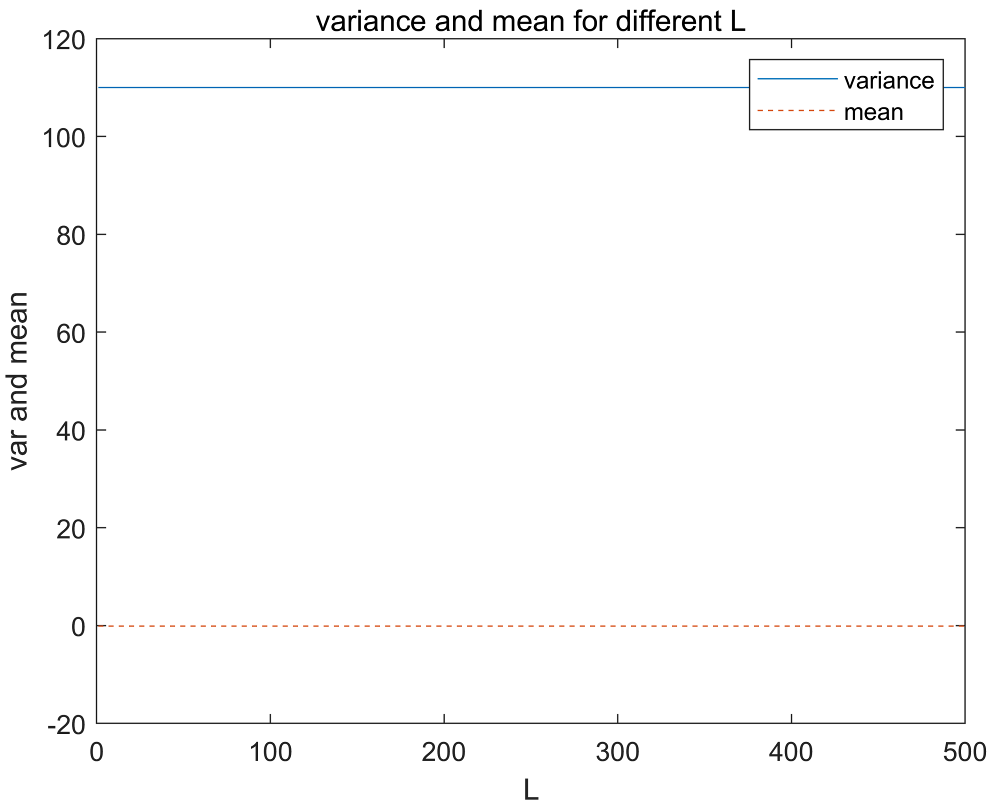


Fig 6.2 The plot show the mean(in '--' line style) and variance(in '-' line style) verses with the variable L.

We would see both of the mean and variance are straight line, also variance of each result is high.

Case 2 fix L and change n then do the same test with case 1

```
numn = 500;%number of n will be test
jn = zeros(1, numn);%initial n
nvar = zeros(1, numn);%initial the variance
nmean = zeros(1, numn);%initial the mean
L = 3;%fix L=3
xn = linspace(-L, L, k);%define x
for jj = 1:numn
    on = each_term(jj, xn, L);%compute equation(29)
    jn(jj) = jj;%store n
    nvar(jj) = var(on);%store variance
    nmean(jj) = mean(on);%store n
end
%plot
plot(jn, nvar, '-')
hold on
plot(jn, nmean, '--')
xlabel('n')
ylabel('var and mean')
title('variance and mean for different n')
legend('variance', 'mean')
```

```
hold off
```

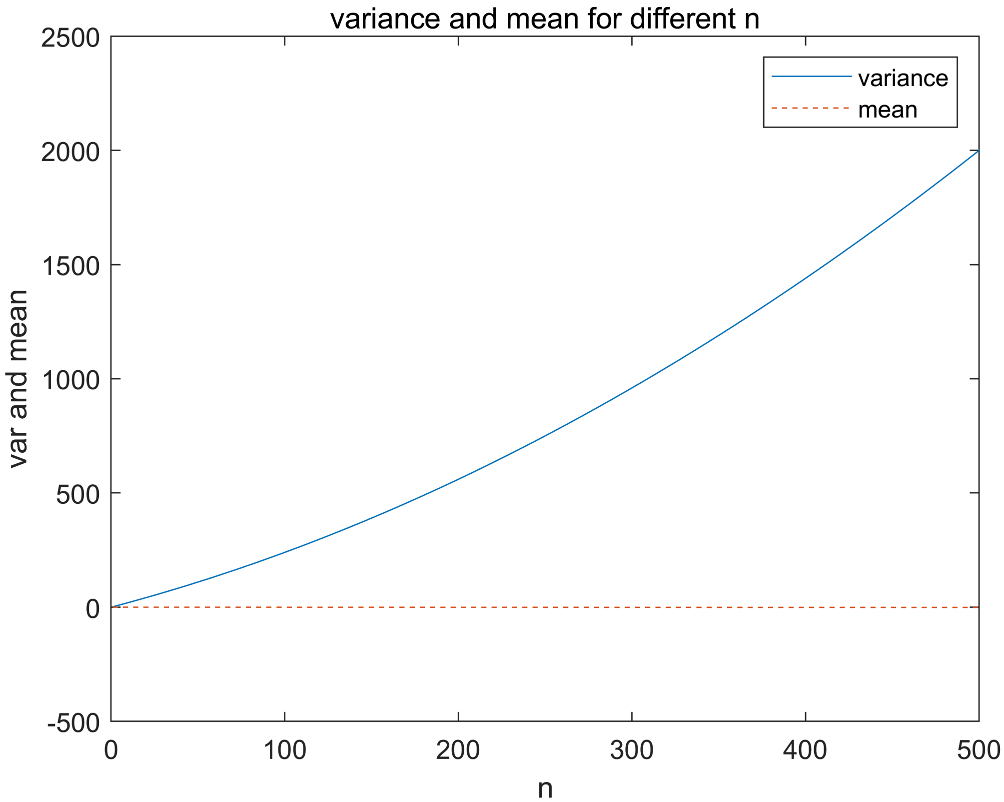


Fig 6.3 The plot show the mean(in '--' line style) and variance(in '-' line style) verses with the variable n.

The mean does not change and variance increases more and more fast.

```
function [o] = each_term(n, x, L)
%function of compute the derivative of
%Fouries Series according to %equation (29)
    o = 0;
    for j = 1:n
        o = o + (-1)^(j + 1).*cos(j*pi.*x/L);
        %the summation part of equation (29)
    end
    o = o*2;
end
```