Problem 8

Solve the problem

PDE	$u_t = \alpha^2 u_{xx}$	$-\infty < x < \infty$
IC	$u(x,0) = e^{-x^2}$	$-\infty < x < \infty$

By using the Fourier transform

Solve:

To solve the PDE by using the Fourier transform, we need first take Fourier transform of the PDE equation and IC equation. Then we can solve the problem that after Fourier transform. At the last we need to take inverse Fourier transform to get real solution.

Step 1 take Fourier transform of PDE and IC.

$$F[u_t] = F[\alpha^2 u_{xx}]$$

$$\frac{\partial}{\partial t} F[u] = \alpha^2 \frac{\partial^2}{\partial x^2} F[u]$$

And then by properties of FT

$$\frac{\partial}{\partial t} F[u] = \alpha^2 (-\xi^2) F[u] \qquad (32)$$

Let $F[u] = U(\xi, t)$,equation (32) becomes

$$U_t = -\alpha^2 \xi^2 U \qquad (33)$$

Then take Fourier transform of IC:

$$F[u(x,0)] = F[e^{-x^2}]$$

$$U(\xi,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2} e^{-i\xi x} dx = \Phi(\xi)$$

Step 2 solve the problem under Fourier transform

If we take ξ to be constant. We could see the problem becomes an ODE problem

$$U_t = -\alpha^2 \xi^2 U$$

$$U(\xi, t = 0) = \Phi(\xi)$$

The solution of $U_t = -\alpha^2 \xi^2 U$ will be

$$U(t) = U(0) e^{-\alpha^2 \xi^2 t}$$

$$U(0) = \Phi(\xi)$$

Therefore the solution of ODE is

$$U(t) = \Phi(\xi) e^{-\alpha^2 \xi^2 t}$$

Step 3 take inverse Fourier Series of the solution.

$$F^{-1}[U(t,\xi)] = F^{-1}[\Phi(\xi)e^{-\alpha^2\xi^2 t}]$$

$$F^{-1}[U(t,\xi)] = F^{-1}[F[\phi(x)]F[g(t,x)]] \qquad (34)$$

Which $g(x,t)$ satisfies $F[g(t,x)] = e^{-\alpha^2\xi^2 t}$

By searching in the table

$$F\left[\frac{1}{\alpha\sqrt{2t}}e^{-\frac{x^2}{4\alpha^2 t}}\right] = e^{-\alpha^2\xi^2 t}$$

So $g(x,t) = \dfrac{1}{\alpha\sqrt{2t}}e^{-\frac{x^2}{4\alpha^2 t}}$

Since $F[f*g] = F[f]F[g]$ equation (34) will become

$$u(x,t) = F^{-1}[F[\phi(x)*g(t,x)]]$$
$$u(x,t) = \phi(x)*g(t,x)$$

$$u(x,t) = e^{-x^2} * \frac{1}{\alpha\sqrt{2t}}e^{-\frac{x^2}{4\alpha^2 t}}$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-y^2}\frac{1}{\alpha\sqrt{2t}}e^{-\frac{(x-y)^2}{4\alpha^2 t}}\,dy$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\frac{1}{\alpha\sqrt{2t}}\int_{-\infty}^{\infty} e^{-y^2}e^{-\frac{(x-y)^2}{4\alpha^2 t}}\,dy$$

$$u(x,t) = \frac{1}{2\alpha\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-y^2-\frac{(x-y)^2}{4\alpha^2 t}}\,dy$$

$$u(x,t) = \frac{1}{2\alpha\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-\frac{4\alpha^2 ty^2+(x-y)^2}{4\alpha^2 t}}\,dy$$

$$u(x,t) = \frac{1}{2\alpha\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-\frac{4\alpha^2 ty^2+x^2-2xy+y^2}{4\alpha^2 t}}\,dy$$

$$u(x,t) = \frac{1}{2\alpha\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(4\alpha^2 t+1)y^2+x^2-2xy}{4\alpha^2 t}}\,dy$$

$$u(x,t) = \frac{1}{2\alpha\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(4\alpha^2 t+1)y^2+x^2-2xy}{4\alpha^2 t}}\,dy$$

$$u(x,t) = \frac{1}{2\alpha\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(4\alpha^2 t+1)y^2+x^2-2xy}{4\alpha^2 t}}\,dy$$

$$u(x,t) = \frac{1}{2\alpha\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(4\alpha^2 t+1)y^2}{4\alpha^2 t}-\frac{x^2}{4\alpha^2 t}+\frac{2xy}{4\alpha^2 t}}\,dy$$

$$u(x,t)=\frac{1}{2\alpha\sqrt{\pi t}}\int_{-\infty}^{\infty}e^{-\frac{(4\alpha^2t+1)y^2}{4\alpha^2t}}e^{-\frac{x^2}{4\alpha^2t}}e^{\frac{2xy}{4\alpha^2t}}dy$$

$$u(x,t)=\frac{1}{2\alpha\sqrt{\pi t}}e^{-\frac{x^2}{4\alpha^2t}}\int_{-\infty}^{\infty}e^{-\frac{(4\alpha^2t+1)y^2}{4\alpha^2t}}e^{\frac{2xy}{4\alpha^2t}}dy$$

Let

$$k=-\frac{(4\alpha^2t+1)}{4\alpha^2t}$$

$$l=\frac{2x}{4\alpha^2t}$$

The integral becomes

$$u(x,t)=\frac{1}{2\alpha\sqrt{\pi t}}e^{-\frac{x^2}{4\alpha^2t}}\int_{-\infty}^{\infty}e^{ky^2}e^{ly}dy$$

$$u(x,t)=\frac{1}{2\alpha\sqrt{\pi t}}e^{-\frac{x^2}{4\alpha^2t}}\int_{-\infty}^{\infty}e^{ky^2+ly}dy$$

$$u(x,t)=\frac{1}{2\alpha\sqrt{\pi t}}e^{-\frac{x^2}{4\alpha^2t}}\int_{-\infty}^{\infty}e^{\frac{4k^2y^2+4kly}{4k}}dy$$

$$u(x,t)=\frac{1}{2\alpha\sqrt{\pi t}}e^{-\frac{x^2}{4\alpha^2t}}\int_{-\infty}^{\infty}e^{\frac{4k^2y^2+l^2+4kly-l^2}{4k}}dy$$

$$u(x,t)=\frac{1}{2\alpha\sqrt{\pi t}}e^{-\frac{x^2}{4\alpha^2t}}\int_{-\infty}^{\infty}e^{\frac{(2ky+l)^2-l^2}{4k}}dy$$

$$u(x,t)=\frac{1}{2\alpha\sqrt{\pi t}}e^{-\frac{x^2}{4\alpha^2t}}\int_{-\infty}^{\infty}e^{\frac{(2ky+l)^2}{4k}-\frac{l^2}{4k}}dy$$

$$u(x,t)=\frac{1}{2\alpha\sqrt{\pi t}}e^{-\frac{x^2}{4\alpha^2t}}\int_{-\infty}^{\infty}e^{\frac{(2ky+l)^2}{4k}}e^{-\frac{l^2}{4k}}dy$$

$$u(x,t)=\frac{1}{2\alpha\sqrt{\pi t}}e^{-\frac{x^2}{4\alpha^2t}}e^{-\frac{l^2}{4k}}\int_{-\infty}^{\infty}e^{\frac{(2ky+l)^2}{4k}}dy$$

Take $\frac{1}{4k}=-a$ in order to integral.

Integral now becomes

$$u(x,t)=\frac{1}{2\alpha\sqrt{\pi t}}e^{-\frac{x^2}{4\alpha^2t}}e^{-\frac{l^2}{4k}}\int_{-\infty}^{\infty}e^{-a(2ky+l)^2}dy$$

$$u(x,t)=\frac{1}{2\alpha\sqrt{\pi t}}e^{-\frac{x^2}{4\alpha^2t}}e^{-\frac{l^2}{4k}}\frac{\sqrt{\pi}}{2\sqrt{a}k}$$

$$u(x,t)=\frac{1}{2\alpha\sqrt{\pi t}}e^{-\frac{x^2}{4\alpha^2t}}e^{-\frac{l^2}{4k}}\frac{\sqrt{\pi}}{2\sqrt{a}k}$$

$$u(x,t)=\frac{1}{2\alpha\sqrt{\pi t}}e^{-\frac{x^2}{4\alpha^2t}}e^{-\frac{l^2}{4k}}\frac{\sqrt{\pi}}{2\sqrt{-\frac{1}{4k}}k}$$

$$k = -\frac{(4\alpha^2 t + 1)}{4\alpha^2 t}$$

with

$$l = \frac{2x}{4\alpha^2 t}$$

After simplified.

The result is

$$u(x,t) = \frac{1}{2\alpha\sqrt{\pi t}} \frac{2\alpha\sqrt{\pi t}}{\sqrt{4\alpha^2 t + 1}} e^{4\alpha^2 t + 1}$$

$$u(x,t) = \frac{1}{\sqrt{4\alpha^2 t + 1}} e^{-\frac{x^2}{4\alpha^2 t + 1}}$$