Homework 3 - MSSC 6030: Spring 2020

Directions. All work is to be done in *complete sentences*. Assignments must be stapled with a printout of the assignment serving as the first page. Your name is to be written on the *back* of the final page of the assignment. Each problem must be on a *separate* sheet of paper. You are welcome to recycle paper, where one side is crossed out to avoid wasting paper, but your work MUST have **no more than one problem per page**. Each problem write-up must begin with the **full statement of the problem**. While you are encouraged to work through confusion with your classmates, your work must be written in your own words. **The assignment is due in dropbox on Wednesday, April 1, 2020 by 3:15pm.**

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1. Use the method of Separation of Variables to find the solution to the IBVP below. Graph the solution look like for various values of time using MATLAB.

$$\begin{array}{ll} \text{PDE} & u_t = u_{xx}, \qquad 0 < x < 1, \quad 0 < t < \infty \\ \text{BCs} & \begin{cases} u(0,t) = 0 & \\ u(1,t) = 0 & 0 < t < \infty \end{cases} \\ \text{IC} & u(x,0) = \sin(2\pi x) + \frac{1}{3}\sin(4\pi x) + \frac{1}{5}\sin(6\pi x), \qquad 0 \le x \le 1 \end{array}$$

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2. Repeat problem #1 above now with the IC

$$u(x,0) = \phi(x) = 1 \qquad 0 \le x \le 1$$

Plot the solution for various times t with increasing numbers of terms in your sum as well.

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3. What is the solution to the vibrating string problem below

$$\begin{array}{ll} \text{PDE} & u_{tt} = \alpha^2 u_{xx}, \qquad 0 < x < L, \quad 0 < t < \infty \\ \text{BCs} & \begin{cases} u(0,t) = 0 & \\ u(L,t) = 0 & 0 < t < \infty \end{cases} \\ \text{IC} & \begin{cases} u(x,0) = 0 & \\ u_t(x,0) = \sin\left(\frac{3\pi x}{L}\right), & 0 \le x \le 1 \end{cases} \end{array}$$

Letting $\alpha = 1$ and $L = 1$, what does the graph of the solution look like for various values of time? Plot it in MATLAB.

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4. Use 2D Finite Differences to solve the following BVP

$$u_{xx} + u_{yy} = 0, \qquad 0 < x < 1, \;\; 0 < y < 1$$

subject to $u(x,y) = 0$ on the top, left, and ride sides of the square domain with $u(x,y) = sin(\pi x)$ for $y = 0$ (i.e. the bottom of the square). Use 5 grid points (3 interior points) in each of the x and y directions. Code up your FD method into Matlab and plot the solution. Does your FD solution improve with a finer grid? Is it possible to use too many points? Discuss. Think about how you could compute the 'true' analytic solution.

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5. Use 2D Finite Differences to solve the following BVP

$$u_{xx} + u_{yy} + 2u = 0, \qquad 0 < x < 1, \ 0 < y < 1$$

subject to the boundary condition $u(x, y) = \sin((x + y)\pi)$ on the boundary. Use 6 grid points (4 interior points) in each of the x and y directions. Code up your FD method into Matlab and plot the solution. Test how your solution changes with the grid size.

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6. Find the finite-difference solution of the heat-conduction problem

$$\begin{array}{llcll} \text{PDE:} & u_t & = & u_{xx}, & 0 < x < 1, \quad t > 0 \\ \\ \text{BCs:} & u(0,t) & = & 0 & t > 0 \\ & u(1,t) & = & 0 & t > 0 \\ \\ \text{IC:} & u(x,0) & = & \sin(\pi x), & 0 \le x \le 1 \end{array}$$

for $t = 0.005,\ 0.010,\ 0.015$ by the *explicit method*. Let $h = \delta x = 0.1$. Plot the solution at $x = 0, 0.1, 0.2, 0.3, \ldots, 0.9, 1$ for $t = 0.015$.

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7. Solve the following problem analytically (separation of variables) and evaluate the analytical solution at the grid points: $x = 0, 0.1, 0.2, \ldots, 0.9, 1$ for $t = 0.015$. Compare these results to your numerical solution in #6 above.

$$\begin{array}{llcll} \text{PDE:} & u_t & = & u_{xx}, & 0 < x < 1, \quad t > 0 \\ \\ \text{BCs:} & u(0,t) & = & 0 & t > 0 \\ & u(1,t) & = & 0 & t > 0 \\ \\ \text{IC:} & u(x,0) & = & \sin(\pi x), & 0 \le x \le 1. \end{array}$$

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8. Consider the problem:

$$\begin{array}{llcll} \text{PDE:} & u_t & = & u_{xx}, & 0 < x < 1, \quad t > 0 \\ \\ \text{BCs:} & u(0,t) & = & 0 & t > 0 \\ & u(1,t) & = & 0 & t > 0 \\ \\ \text{IC:} & u(x,0) & = & \sin(\pi x), & 0 \le x \le 1 \end{array}$$

Solve this problem using the method described in class *(Implicit FD method)* using various values of lambda including $\lambda = 0, 1/4, 1/2, 3/4, 1$ and experiment with step sizes in x and t to check accuracy. Remember that if you use $\lambda = 0$ there are specific guidelines about how small the ratio of $\frac{k}{h^2}$ must be. Compare your results to the previous two problems (#6 and #7).

Plot your solutions for various times (up to at least $t = 0.05$) and compare to the true solution. You may find it instructive to also plot the error between the true and approximate solutions.

Print out your coefficient matrix as well as the RHS vector for a small number of grid points to make sure it looks like you think it does.
