problem 1

Strang 3.1

10. Use three hat function, with $h = \frac{1}{4}$, to solve $-u'' = 2$ with $u(0) = u(1) = 0$.Verify that the approximation *U* matches $u = x - x^2$ at the nodes.

Solve:

From the problem we know that $f(x) = 2$ (1)

Since $h = \frac{1}{4}$, $n = \frac{1}{h} - 1 = 3$

So the grid point on x are $x_0 = 0, x_1 = \frac{1}{4}, x_2 = \frac{1}{2}, x_3 = \frac{3}{4}, x_4 = 1$.

The hat function will be:

$$\phi_1(x) = \begin{cases} 4x & 0 \le x \le \frac{1}{4} \\ -4x + 2 & \frac{1}{4} \le x \le \frac{1}{2} \\ 0 & \frac{1}{2} \le x \le 1 \end{cases} \phi_2(x) = \begin{cases} 0 & 0 \le x \le \frac{1}{4} \\ 4x - 1 & \frac{1}{4} \le x \le \frac{1}{2} \\ -4x + 3 & \frac{1}{2} \le x \le \frac{3}{4} \\ 0 & \frac{3}{4} \le x \le 1 \end{cases} \phi_3(x) = \begin{cases} 0 & 0 \le x \le \frac{1}{2} \\ 4x - 2 & \frac{1}{2} \le x \le \frac{3}{4} \\ -4x + 4 & \frac{3}{4} \le x \le 1 \end{cases}$$

The graph of these three hat function are:

```
clear
Num_x_hat = 200;
hat_x = linspace(0,1,Num_x_hat);
OneFour = Num_x_hat/4;
TwoFour = Num_x_hat/2;
ThreeFour = 3*Num_x_hat/4;
%define the each stop point.
phi1 = [4*hat_x(1:OneFour) -4*hat_x(OneFour + 1:TwoFour) + 2 ...
    0*hat_x(TwoFour + 1:Num_x_hat)];
%phi1
phi2 = [0*hat_x(1:OneFour) 4*hat_x(OneFour + 1:TwoFour) - 1 ...
    -4*hat_x(TwoFour + 1:ThreeFour) + 3 0*hat_x(ThreeFour + 1:Num_x_hat)];
%phi2
phi3 = [0*hat_x(1:TwoFour) 4*hat_x(TwoFour + 1:ThreeFour) - 2 ...
    -4*hat_x(ThreeFour + 1:Num_x_hat) + 4];
%phi3
plot(hat_x, phi1)
hold on
plot(hat_x, phi2,'-.')
plot(hat_x, phi3,'--')
legend('\phi_1','\phi_2','\phi_3')
hold off
```

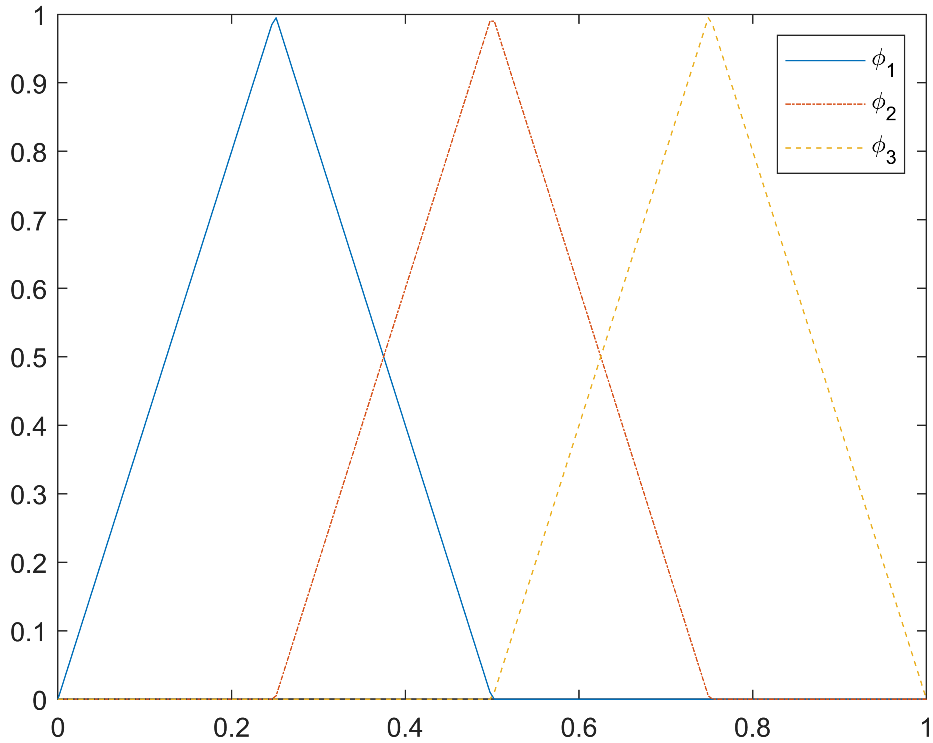

Fig 1.1

The plot shows three hat function by phi1 phi2 and phi3 in the different line style. Which phi1 is in solid line '—', phi2 is in dot line '._.' and phi3 is in '--'.

The Stiffness Martix has the formular:

$\mathrm{Ku} = b$ (2)

which $K = \begin{bmatrix} \langle \phi_1', \phi_1' \rangle & \langle \phi_2', \phi_1' \rangle & \langle \phi_3', \phi_1' \rangle \\ \langle \phi_1', \phi_2' \rangle & \langle \phi_2', \phi_2' \rangle & \langle \phi_3', \phi_2' \rangle \\ \langle \phi_1', \phi_3' \rangle & \langle \phi_2', \phi_3' \rangle & \langle \phi_3', \phi_3' \rangle \end{bmatrix}, u = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}, b = \begin{bmatrix} \langle f(x), \phi_1 \rangle \\ \langle f(x), \phi_2 \rangle \\ \langle f(x), \phi_3 \rangle \end{bmatrix}$

Find the derivative of $\phi_1 \phi_2 \phi_3$:

$$\phi_1'(x) = \begin{cases} 4 & 0 \le x \le \frac{1}{4} \\ -4 & \frac{1}{4} \le x \le \frac{1}{2} \\ 0 & \frac{1}{2} \le x \le 1 \end{cases} \phi_2'(x) = \begin{cases} 0 & 0 \le x \le \frac{1}{4} \\ 4 & \frac{1}{4} \le x \le \frac{1}{2} \\ -4 & \frac{1}{2} \le x \le \frac{3}{4} \\ 0 & \frac{3}{4} \le x \le 1 \end{cases} \phi_3'(x) = \begin{cases} 0 & 0 \le x \le \frac{1}{2} \\ 4 & \frac{1}{2} \le x \le \frac{3}{4} \\ -4 & \frac{3}{4} \le x \le 1 \end{cases} \quad (3)$$

Then compute (2) with (3) :

$$\langle \phi_1{}', \phi_1{}' \rangle = \int_0^1 (\phi_1{}'(x))^2 \mathrm{dx} = \int_0^{\frac{1}{4}} 4^2 \mathrm{dx} + \int_{\frac{1}{4}}^{\frac{1}{2}} (-4)^2 \mathrm{dx} + \int_{\frac{1}{2}}^{1} 0 \mathrm{dx} = 8$$

$$\langle \phi_2{}', \phi_2{}' \rangle = \int_0^1 (\phi_2{}'(x))^2 \mathrm{dx} = \int_0^{\frac{1}{4}} 0 \mathrm{dx} + \int_{\frac{1}{4}}^{\frac{1}{2}} 4^2 \mathrm{dx} + \int_{\frac{1}{2}}^{\frac{3}{4}} (-4)^2 \mathrm{dx} + \int_{\frac{3}{4}}^{1} 0 \mathrm{dx} = 8$$

$$\langle \phi_3{}', \phi_3{}' \rangle = \int_0^1 (\phi_3{}'(x))^2 \mathrm{dx} = \int_0^{\frac{1}{2}} 0 \mathrm{dx} + \int_{\frac{1}{2}}^{\frac{3}{4}} (4)^2 \mathrm{dx} + \int_{\frac{3}{4}}^{1} (-4)^2 \mathrm{dx} = 8$$

$$\langle \phi_1{}', \phi_2{}' \rangle = \langle \phi_2{}', \phi_1{}' \rangle = \int_0^1 \phi_1{}'(x)\phi_2{}'(x) \mathrm{dx} = \int_0^{\frac{1}{4}} 0 \mathrm{dx} + \int_{\frac{1}{4}}^{\frac{1}{2}} -16 \mathrm{dx} + \int_{\frac{1}{2}}^{\frac{3}{4}} 0 \mathrm{dx} + \int_{\frac{3}{4}}^{1} 0 \mathrm{dx} = -4$$

$$\langle \phi_1{}', \phi_3{}' \rangle = \langle \phi_3{}', \phi_1{}' \rangle = \int_0^1 \phi_1{}'(x)\phi_3{}'(x) \mathrm{dx} = \int_0^{\frac{1}{4}} 0 \mathrm{dx} + \int_{\frac{1}{4}}^{\frac{1}{2}} 0 \mathrm{dx} + \int_{\frac{1}{2}}^{\frac{3}{4}} 0 \mathrm{dx} + \int_{\frac{3}{4}}^{1} 0 \mathrm{dx} = 0$$

$$\langle \phi_2{}', \phi_3{}' \rangle = \langle \phi_3{}', \phi_2{}' \rangle = \int_0^1 \phi_2{}'(x)\phi_3{}'(x) \mathrm{dx} = \int_0^{\frac{1}{4}} 0 \mathrm{dx} + \int_{\frac{1}{4}}^{\frac{1}{2}} 0 \mathrm{dx} + \int_{\frac{1}{2}}^{\frac{3}{4}} -16 \mathrm{dx} + \int_{\frac{3}{4}}^{1} 0 \mathrm{dx} = -4$$

So $K = \begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix}$ (4)

Then compute b with (1):

$$\langle f(x), \phi_1 \rangle = \int_0^1 f(x)\phi_1(x) = \int_0^{\frac{1}{4}} 8x \mathrm{dx} + \int_{\frac{1}{4}}^{\frac{2}{4}} (-8x+4) \mathrm{dx} + \int_{\frac{1}{2}}^{1} 0 \mathrm{dx} = \frac{1}{2}$$

$$\langle f(x), \phi_2 \rangle = \int_0^1 f(x)\phi_2(x) = \int_0^{\frac{1}{4}} 0 \mathrm{dx} + \int_{\frac{1}{4}}^{\frac{2}{4}} (8x-2) \mathrm{dx} + \int_{\frac{1}{2}}^{\frac{3}{4}} (-8x+6) \mathrm{dx} + \int_{\frac{3}{4}}^{1} 0 \mathrm{dx} = \frac{1}{2}$$

$$\langle f(x), \phi_3 \rangle = \int_0^1 f(x)\phi_3(x) = \int_0^{\frac{1}{2}} 0 \mathrm{dx} + \int_{\frac{1}{2}}^{\frac{3}{4}} (8x-4) \mathrm{dx} + \int_{\frac{3}{4}}^{1} (-8x+8) \mathrm{dx} = \frac{1}{2}$$

so $b = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ (5)

So we could compute u from $\mathrm{K}u = b$

```
h = 1/4;%define h
n = 1/h - 1;%compute n
x = linspace(0, 1, 100);%define x
xg = linspace(0,1,n + 2);%true x with 5 points
up = xg - xg.^2;%true y with 5 points
u = x - x.^2;%true solution
b = [1/2; 1/2; 1/2];%equation (5)
k = [8 -4 0;
    -4 8 -4;
     0 -4 8;];%equaiton (4)
coe = k\b;%solve equation (2)
coe = [0;coe;0]
```

```
coe = 5×1
         0
    0.1875
    0.2500
    0.1875
         0
```

```
error=abs(coe(:) - up(:))%compute the error
```

```
error = 5×1
```
10^{-15} ×
```
         0
         0
    0.1110
    0.0278
         0
```

The solving the Ku=b are coe=[0 0.1875 0.2500 0.1875 0]

Also we could see the error of the result we compute are small enough.

So if we graph the solution and true value on the same page.

```
%plot
plot(x, u, '.')
hold on
plot(xg, up, '*')
plot(xg, coe, 'o', 'Markersize', 10, 'Color','r')
legend('true solution', 'true solution with grid points', 'estimate grid points')
xlabel('x')
ylabel('y')
title('plot of estimat grid point and true solution')
hold off
```

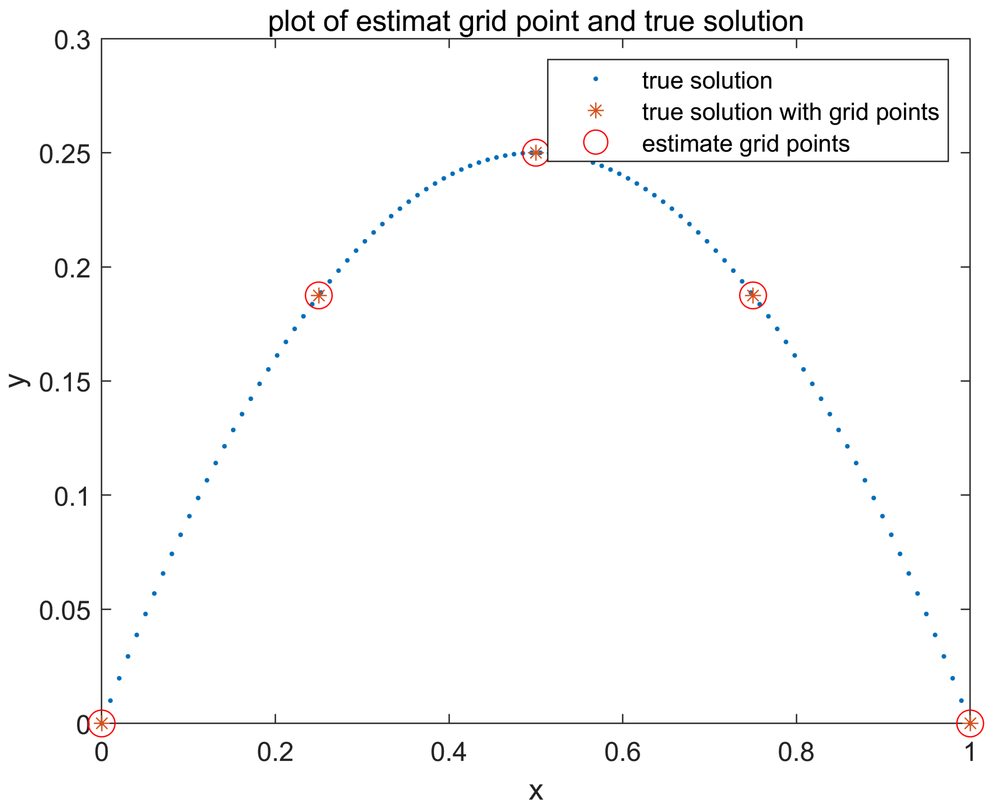

Fig 1.2 This plot shows the result we compute by FEM compare with true solution. On the graph they on the same spot.