For the ODE like $\frac{dy}{dx} = f(x, y)$ and the point $y(x_0) = y_0$

The following method might be work, but some of them do not have very high accuracy.

Froward Euler method $y_{n+1} = (1+2h)y_n$ $\quad O(h)$

Backward Euler method $y_{n+1} = (1-2h)^{-1} y_n$ $\quad O(h)$

Trapezoid $y_{n+1} = \frac{(1+h)}{(1-h)} y_n$ $\quad O(h^3)$

By roughly analyse the method before we could use the following equation to predict the solution numerical.

$$y_{n+1} = y_n + h\phi(x_n, y_n, h) \tag{1}$$

And for each of three method above we will have:

$\phi_{FE} = 2y_n$, $\phi_{BE} = \frac{2hy_n}{1-2h}$ and $\phi_{TR} = \frac{2hy_n}{1-h}$

So the Runge Kutta method could be used in derive a higher order of prediction.

We first define an equation about the accuracy of equation 1

Def: If an ODE has a true solution $y(x)$, $\exists P, P \in Z^+$ that

$$y(x+h) - y(x) - h\phi = O(h^{p+1}) \tag{2}$$

Which mean for particular $h\phi(x_n, y_n, h)$, the accuracy of prediction is $O(h^{p+1})$.

Then take integral of ODE

$$\int_{x_n}^{x_{n+1}} dy = \int_{x_n}^{x_{n+1}} f(x, y)dx$$

$$y(x_{n+1}) - y(x_n) = \int_{x_n}^{x_{n+1}} f(x, y)dx$$

To predict The integral is main goal of solving the ODE.

And if we combine it with equation (2)

$$h\phi + O(h^{p+1}) = \int_{x_n}^{x_{n+1}} f(x, y)dx$$

So the closer we predict of the integral the more accuracy we will get. Also notice that: h is the width from x_n to x_{n+1}

we need use Newton-Cotes formula:

$$\int_a^b f(x)dx \approx \sum_{i=1}^{r} w_i f(x_i) \tag{3}$$

Which mean the sum of the product of width and value of the function.

So after we plug integral into equation (3):

$$\int_{x_n}^{x_{n+1}} f(x, y(x))dx = h\sum_{i=1}^{r} c_i f(x_n + \lambda_i h, y_n(x_n + \lambda_i h)) \qquad (4)$$

λ_i Is a small number, r is the number of point inside the interval h is the total width.

So apparently the more r we take the more accuracy we will get.
Also for each y point yn need to be predict from all of point above.
If we take r=2, the next y value need to be predict from the previous point, Where

$$y_{n+1} = y_n + h\phi(x_n, y_n, h)$$

So the equation for r=2 is:

$$y_{n+1} = y_n + h\phi(x_n, y_n, h)$$

$$h\phi(x_n, y_n, h) = \sum_{i=1}^{2} c_i K_i$$
$$K_1 = f(x_n, y_n)$$
$$K_2 = f(x_n + \lambda_2 h, y_n + h\mu_{21} K_1)$$

Also the next y value need to predict from k1 and k2, And so on
We finally take (4) into (2), we will get the Ruuge Kutta equation like:

$$y_{n+1} = y_n + h\phi(x_n, y_n, h)$$

where

$$\phi(x_n, y_n, h) = \sum_{i=1}^{r} c_i K_i$$
$$K_1 = f(x_n, y_n) \qquad (5)$$
$$K_i = f(x_n + \lambda_i h, y_n + h\sum_{j=1}^{i-1} \mu_{ij} K_j)$$
$$i = 2,3,\ldots,r$$

Now I will use r=2 for example

So (5) becomes:

$$y_{n+1} = y_n + h\phi(x_n, y_n, h)$$

where

$$h\phi(x_n, y_n, h) = \sum_{i=1}^{2} c_i K_i \qquad (6)$$
$$K_1 = f(x_n, y_n)$$
$$K_2 = f(x_n + \lambda_2 h, y_n + h\mu_{21} K_1)$$

So in these equations c_1, c_2, λ_2 and μ_{21} are the unknowns.

To build equations that solve these unknowns we will take Taylor expansion of $y(x_{n+1})$

$$y(x_{n+1}) = y_n + hy'_n + \frac{h^2}{2} y''_n + \frac{h^3}{3!} y'''_n + \frac{h^4}{4!} y^{(4)}{}_n + O(h^5) \tag{7}$$

And we could change $y'_n, y''_n, y'''_n, y^{(4)}{}_n$ to:

$$y'_n = f(x_n, y_n) = f_n \tag{8}$$

$$y''_n = \frac{d}{dx} f(x_n, y(x_n)) = f'_x(x_n, y_n) + f(x_n, y_n) f'_y(x_n, y_n) \tag{9}$$

$$y'''_n = f''_{xx} + 2f(x_n, y_n) f''_{xy} + f^2(x_n, y_n) f''_{yy} + f'_y[f'_x + f(x_n, y_n) f'_y] \tag{10}$$

Then we can plug (8),(9) and (10) into (7)

$$y(x_{n+1}) = y_n + hf_n + \frac{h^2}{2}\left[f'_x + f'_y f_n\right] + O(h^3) \tag{11}$$

The difference of the approximate value and true value is:

$$R = y(x_{n+1}) - y_n - h[c_1 f(x_n, y_n) + c_2 f(x_n + \lambda_2 h, y_n + \mu_{21} h f_n)] \tag{12}$$

Also

$$f(x_n + \lambda_2 h, y_n + \mu_{21} hf_n) = f_n + f'_x \lambda_2 h + f'_y \mu_{21} \lambda_2 h \tag{13}$$

Then combine (12) (13) and (11):

$$R = hf_n + \frac{h^2}{2}\left[f'_x + f'_y f_n\right] + O(h^3) - h[c_1 f(x_n, y_n) + c_2 f_n + f'_x \lambda_2 h + f'_y \mu_{21} \lambda_2 h] \tag{14}$$

After deal with

$$R = (1 - c_1 - c_2) f_n h + (\frac{1}{2} - c_2 \lambda_2) f'_x h^2 + (\frac{1}{2} - c_2 \mu_{21}) f'_y f_n h^2 + O(h^3) \tag{15}$$

We want R to be smallest so we need the coefficients of f and h to be 0:
So we will get

$$1 - c_1 - c_2 = 0$$

$$\frac{1}{2} - c_2 \lambda_2 = 0$$

$$\frac{1}{2} - c_2 \mu_{21} = 0$$

Since we have 3 equaitons and 4 unknowns we will get infinite number of solution. So in order to make the compute simple, I choose:

$$c_2 = \frac{1}{2}$$

Therefore

$c_1 = \frac{1}{2}$ and $\begin{matrix} \lambda_2 = 1 \\ \mu_{21} = 1 \end{matrix}$

Then second order R-K method from equation(6) becomes

$$y_{n+1} = y_n + \frac{h}{2}(K_1 + K_2)$$

where

$$\begin{aligned} K_1 &= f(x_n, y_n) \\ K_2 &= f(x_n + h, y_n + hK_1) \end{aligned} \tag{16}$$

Which is also called modified Euler method.

With the same progress if we take r=3 or r=4, we will get third order and fourth order RK method:RK3

$$\begin{cases} y_{n+1} = y_n + \frac{h}{6}(K_1 + 4K_2 + K_3) \\ K_1 = f(x_n, y_n) \\ K_2 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}K_1) \\ K_3 = f(x_n + h, y_n - hK_1 + 2hK_2) \end{cases}$$

And RK4

$$\begin{cases} y_{n+1} = y_n + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4) \\ K_1 = f(x_n, y_n) \\ K_2 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}K_1) \\ K_3 = f(x_n + \frac{h}{2}, y_n + \frac{h}{2}K_2) \\ K_4 = f(x_n + h, y_n + hK_3) \end{cases}$$

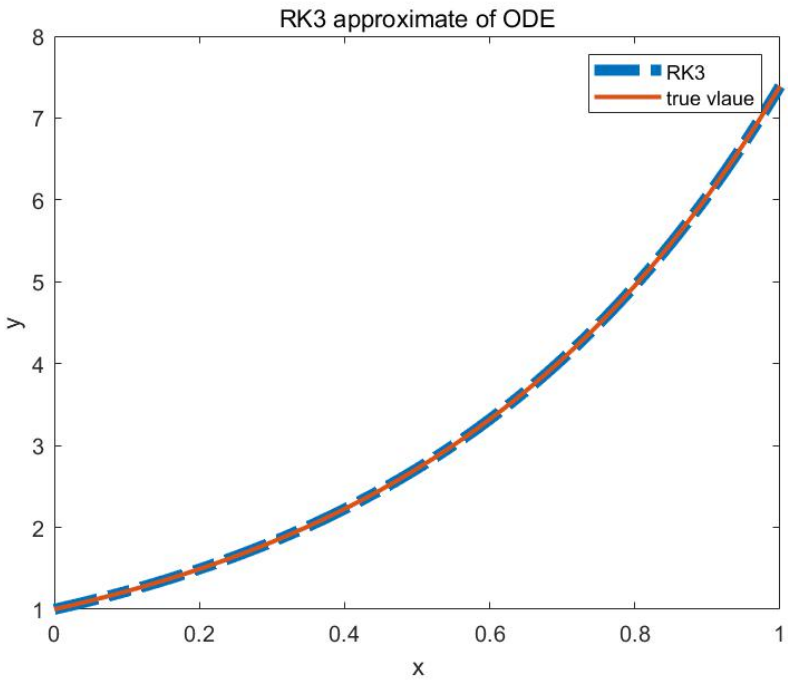
Since RK2 has accurate of O(h3), RK3 will be O(h4) and RK4 will be O(h5) of accuracy.

Example:

For the ODE: $y' = 2y$ and point $y(0) = 1$

The true solution is $y = e^{2x}$

And if we use the RK3 method:

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So from the plot the approximate from RK3 has coincide with the true value.
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sum(abs(yt(:)-y(:)))%total absolute error.
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ans = 1.4003e-04
Also the difference are very small.
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Activity:

For the same ODE:

$$y' = 2y \qquad y(0) = 1$$

Making a approximate Using RK4 and see if the RK4 has less error than RK3.