Numerical Methods For Tracing Null Geodesics in Kerr Spacetime

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Introduction

The first real image of a black hole was recently published, and while an amazing accomplishment, the quality still leaves much to the imagination. Here we discuss a method of producing realistic images using the geometry predicted by Einstein. The method outlined here was developed by Double Negative in collaboration with Kip Thorne for use in the movie: Interstellar. The task is simplified by only considering null geodesics, that is, the path that light will take in the presence of a massive body.

Theory

Increasingly general descriptions of the curvature of spacetime around black holes have been developed and the metrics that achieve this are simply summarized by the table below:

	Non-Rotating	Rotating
Uncharged	Schwarzschild	Kerr
Charged	Reissner-Nordstrom	Kerr-Newman

Here we explore the case of a rotating, uncharged black hole using the Kerr Metric. The inclusion of angular momentum effects such as frame dragging make this metric much more realistic than the more familiar Schwarzschild description. This metric is an exact solution to Einstein's Field equations and, in Boyer-Lindquist coordinates takes the form:

$$ds^2 = -\alpha^2 dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \varpi^2 (d\phi - \omega dt)^2$$

Where:

$$\rho = \sqrt{r^2 + a^2 \cos^2\theta}$$

$$\Delta = r^2 - 2r + a^2$$

$$\alpha = \frac{\rho\sqrt{\Delta}}{\Sigma}$$

$$\omega = \frac{2ar}{\Sigma^2}$$

$$\varpi = \frac{\Sigma \sin\theta}{\rho}$$

$$\Sigma = \sqrt{(r^2 + a^2)^2 - a^2 \Delta \sin^2\theta}$$

In order to trace the null geodesic we also need:

$$R = (r^2 + a^2 - ab)^2 - \Delta[(b-a)^2 + q]$$

$$\Theta = q - \cos^2\theta \left(\frac{b^2}{\sin^2\theta} - a^2\right)$$

With these equations we are prepared to begin creating the ray-tracing algorithm. We will take the mass of the black hole and speed of light to be unity, and the spin parameter to be *a.*

Ray-Tracing Formula

1. First specify the position of the camera: (r_c, θ_c, ϕ_c)
2. Specify the camera's speed (β) and direction of motion (*B*). For simplicity an equatorial, geodesic, orbit can be specified as:
$$B_r = B_\theta = 0, \quad B_\phi = 1, \ \beta = \frac{\varpi}{\alpha}((a + r_c^{\frac{3}{2}})^{-1} - \omega)$$
3. For each ray, specify the angles at which it intersects the camera: (θ_{cs}, ϕ_{cs})
4. Compute the Cartesian components of this incoming ray, in the camera's reference frame:
$$N_x = \sin\theta_{cs}\cos\theta_{cs}, \ N_y = \sin\theta_{cs}\sin\phi_{cs}, \ N_z = \cos\theta_{cs}$$
5. Now use the equations of relativistic aberration to compute the ray's direction of motion as measured by a fiducial observer (FIDO):
$$n_{Fx} = \frac{-\sqrt{1-\beta^2}N_x}{1-\beta N_y}, \ n_{Fy} = \frac{-N_y + \beta}{1-\beta N_y}, \ n_{Fz} = \frac{-\sqrt{1-\beta^2}N_z}{1-\beta N_y}$$
6. Convert these directions into the spherical, orthonormal basis of the FIDO:
$$n_{Fr} = \frac{B_\phi}{\kappa} n_{Fx} + B_r n_{Fy} + \frac{B_r B_\theta}{\kappa} n_{Fz}$$
$$n_{F\theta} = B_\theta n_{Fy} - \kappa n_{Fz}$$
$$n_{F\phi} = \frac{-B_r}{\kappa} n_{Fx} + B_\phi n_{Fy} + \frac{B_\theta B_\phi}{\kappa} n_{Fz}$$
$$\kappa = \sqrt{1 - B_\theta^2}$$
7. Compute the ray's canonical momenta:
$$p_t = -1, \quad p_r = E_F \frac{\rho}{\sqrt{\Delta}} n_{Fr}$$
$$p_\theta = E_F \rho n_{F\theta}, \quad p_\phi = E_F \varpi n_{F\phi}$$
Where the energy measured by the FIDO is: $E_F = (\alpha + \omega\varpi n_{F\phi})^{-1}$
8. Along with this energy, the axial angular momentum (*b*) and Carter constant (*q*) are conserved:
$$b = p_\phi, \quad q = p_\theta^2 + \cos^2\theta \left(\frac{b^2}{\sin^2\theta} - a^2\right)$$
9. Use the above values as initial conditions to solve the super-Hamiltonian, ray equations:
$$\frac{dr}{d\zeta} = \frac{\Delta p_r}{\rho^2}$$
$$\frac{d\theta}{d\zeta} = \frac{p_\theta}{\rho^2}$$
$$\frac{d\phi}{d\zeta} = -\frac{\partial}{\partial b}\left(\frac{R + \Delta\Theta}{2\Delta\rho^2}\right)$$
$$\frac{dp_r}{d\zeta} = -\frac{\partial}{\partial r}\left(-\frac{\Delta p_r^2}{2\rho^2} - \frac{p_\theta^2}{2\rho^2} + \frac{R + \Delta\Theta}{2\Delta\rho^2}\right)$$
$$\frac{dp_\theta}{d\zeta} = -\frac{\partial}{\partial \theta}\left(-\frac{\Delta p_r^2}{2\rho^2} - \frac{p_\theta^2}{2\rho^2} + \frac{R + \Delta\Theta}{2\Delta\rho^2}\right)$$

Due to the time irreversibility of the angular momentum of the black hole we numerically integrate the ODEs backwards from $\zeta = 0$ to $\zeta = \zeta_f$ where we let $\zeta_f \to -\infty$, or a large negative value.

Limitations

The implementation outlined computes the path of a single photon; however, it is possible to generalize to a circular or elliptical beam or ray bundle. The choice of Boyer-Lindquist coordinates prevents the camera from descending into the black hole. In order to do so one must switch to outgoing Kerr coordinates. It is left to the reader to implement the detection of a ray colliding with an object, ie. the event horizon, or accretion disk. It is worth noting that event horizon collisions can be detected from the constant properties of the ray and do not need to be actively checked. The colour of the ray can also be adjusted via the blueshift formula:

$$\frac{f_c}{f'} = \frac{\sqrt{1-\beta^2}}{1-\beta N_y}\frac{1-b\omega}{\alpha}$$

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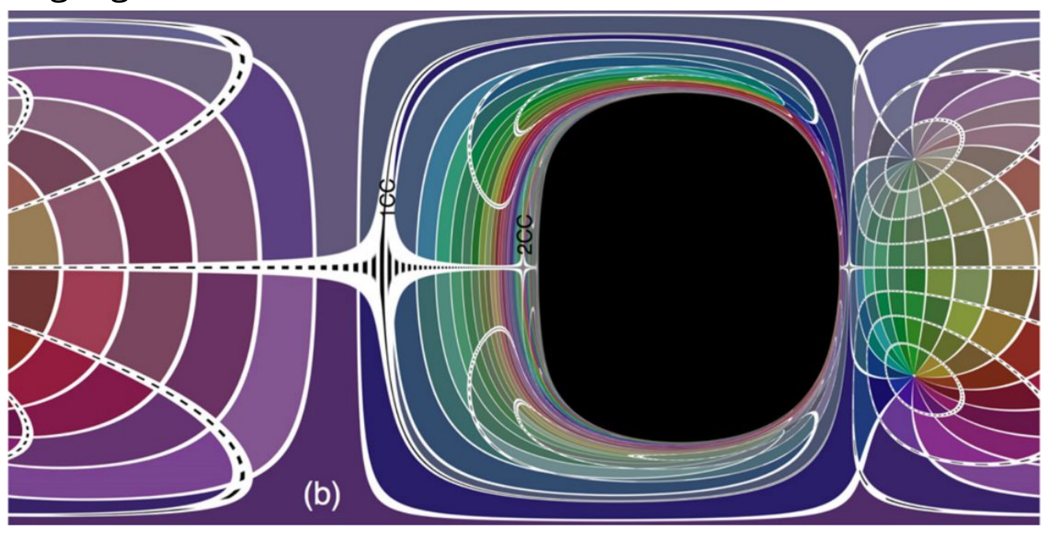
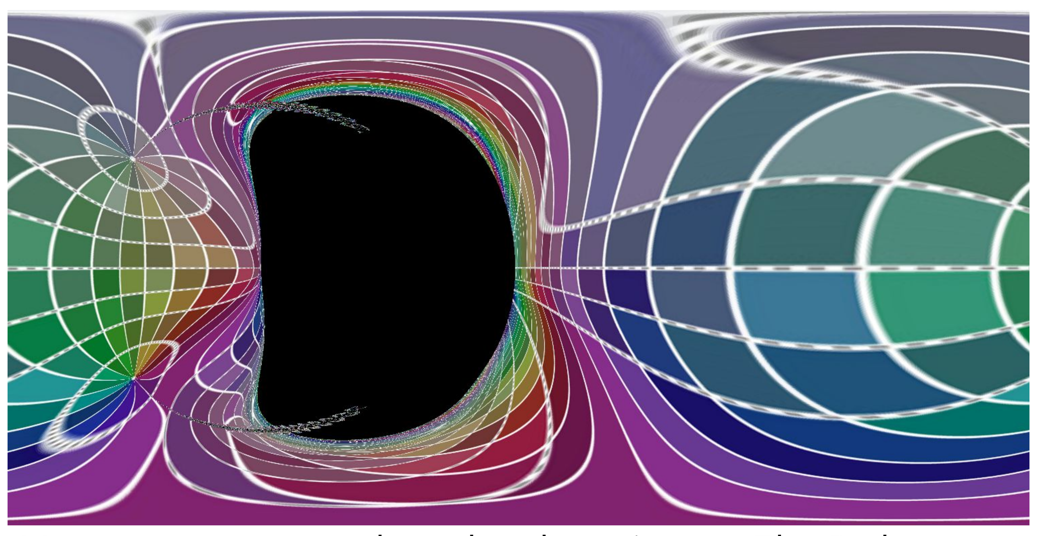
Results

Grid used as the background in the following simulations to highlight distortion



Simulation of spinning black hole with a/M = 0.999 at a distance of 2.60M, created during the production of interstellar.



My attempt to reproduce the above image. The Python code written to do so is at https://github.com/obulka/kerr_black_hole. You can see the repeated colours in the image, near the horizon, caused by lensing.

References

1. James O et al. 2015 Gravitational lensing by spinning black holes in astrophysics, and in the movie Interstellar IOP Publishing
2. Chen B et al. 2015 Algorithms and Programs for Strong Gravitational lensing in KAB The Astrophysical Journal Supplement Series
3. Schwarzschild reference: https://github.com/rantonels/starless