Applications

Example 7 Total Sales

A small business sells \$20,000 worth of sports memorabilia during its first year. The owner of the business has set a goal of increasing annual sales by \$15,000 each year for 19 years. Assuming that this goal is met, find the total sales during the first 20 years this business is in operation.

Algebraic Solution

The annual sales form an arithmetic sequence in which $a_1 = 20{,}000$ and $d = 15{,}000$. So,

$$a_n = 20{,}000 + 15{,}000(n - 1)$$

and the nth term of the sequence is

$$a_n = 15{,}000n + 5000.$$

This implies that the 20th term of the sequence is

$$\begin{aligned} a_{20} &= 15{,}000(20) + 5000 \\ &= 300{,}000 + 5000 \\ &= 305{,}000. \end{aligned}$$

The sum of the first 20 terms of the sequence is

$$S_n = \frac{n}{2}(a_1 + a_n) \qquad \textit{n}\text{th partial sum formula}$$

$$= \frac{20}{2}(20{,}000 + 305{,}000) \qquad \text{Substitute 20 for } n\text{, 20,000 for } a_1\text{, and 305,000 for } a_n.$$

$$= 10(325{,}000) \qquad \text{Simplify.}$$

$$= 3{,}250{,}000. \qquad \text{Simplify.}$$

So, the total sales for the first 20 years are \$3,250,000.

CHECKPOINT Now try Exercise 85.

Numerical Solution

The annual sales form an arithmetic sequence in which $a_1 = 20{,}000$ and $d = 15{,}000$. So,

$$a_n = 20{,}000 + 15{,}000(n - 1)$$

and the nth term of the sequence is

$$a_n = 15{,}000n + 5000.$$

Use the *list editor* of a graphing utility to create a table that shows the sales for each of the first 20 years and the total sales for the first 20 years, as shown in Figure 8.7.

Figure 8.7

So, the total sales for the first 20 years are \$3,250,000.

Figure 8.8 shows the annual sales for the business in Example 7. Notice that the annual sales for the business follow a *linear growth* pattern. In other words, saying that a quantity increases arithmetically is the same as saying that it increases linearly.

Figure 8.8

If you go on to take a course in calculus, you will study sequences and series in detail. You will learn that sequences and series play a major role in the study of calculus.

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8.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1 and 2, fill in the blank.

1. The nth term of an arithmetic sequence has the form _____ .
2. The formula $S_n = \frac{n}{2}(a_1 + a_n)$ can be used to find the sum of the first n terms of an arithmetic sequence, called the _____ .
3. How do you know when a sequence is arithmetic?
4. Is 4 or 1 the common difference of the arithmetic sequence $a_n = 4n + 1$?

Procedures and Problem Solving

Determining Whether a Sequence Is Arithmetic **In Exercises 5–10, determine whether or not the sequence is arithmetic. If it is, find the common difference.**

5. $10, 8, 6, 4, 2, \ldots$
6. $4, 9, 14, 19, 24, \ldots$
7. $3, \frac{5}{2}, 2, \frac{3}{2}, 1, \ldots$
8. $\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{8}{3}, \frac{16}{3}, \ldots$
9. $3.7, 4.3, 4.9, 5.5, 6.1, \ldots$
10. $1^2, 2^2, 3^2, 4^2, 5^2, \ldots$

Writing the Terms of a Sequence **In Exercises 11–20, write the first five terms of the sequence. Determine whether or not the sequence is arithmetic. If it is, find the common difference. (Assume n begins with 1.)**

11. $a_n = 8 + 13n$
12. $a_n = 2^n + n$
13. $a_n = \dfrac{1}{n+1}$
14. $a_n = 1 + (n - 1)4$
15. $a_n = 150 - 7n$
16. $a_n = 2^{n-1}$
17. $a_n = 3 + 2(-1)^n$
18. $a_n = 3 - 4(n + 6)$
19. $a_n = (-1)^n$
20. $a_n = (-1)^{2n+1}$

Finding the nth Term of an Arithmetic Sequence **In Exercises 21–30, find a formula for a_n for the arithmetic sequence.**

21. $a_1 = 1, d = 3$
22. $a_1 = 15, d = 4$
23. $a_1 = 100, d = -8$
24. $a_1 = 0, d = -\frac{2}{3}$
25. $4, \frac{3}{2}, -1, -\frac{7}{2}, \ldots$
26. $10, 5, 0, -5, -10, \ldots$
27. $a_1 = 5, a_4 = 15$
28. $a_1 = -4, a_5 = 16$
29. $a_3 = 94, a_6 = 85$
30. $a_5 = 190, a_{10} = 115$

Writing the Terms of an Arithmetic Sequence **In Exercises 31–38, write the first five terms of the arithmetic sequence. Use the *table* feature of a graphing utility to verify your results.**

31. $a_1 = 5, d = 6$
32. $a_1 = 5, d = -\frac{3}{4}$
33. $a_1 = -10, d = -12$
34. $a_4 = 16, a_{10} = 46$
35. $a_8 = 26, a_{12} = 42$
36. $a_6 = -38, a_{11} = -73$
37. $a_3 = 19, a_{15} = -1.7$
38. $a_5 = 16, a_{14} = 38.5$

Writing Terms of an Arithmetic Sequence **In Exercises 39–42, write the first five terms of the arithmetic sequence. Find the common difference and write the nth term of the sequence as a function of n.**

39. $a_1 = 15, \quad a_{k+1} = a_k + 4$
40. $a_1 = 6, \quad a_{k+1} = a_k + 5$
41. $a_1 = \frac{3}{5}, \quad a_{k+1} = -\frac{1}{10} + a_k$
42. $a_1 = 1.5, \quad a_{k+1} = a_k - 2.5$

Using a Recursion Formula **In Exercises 43–46, the first two terms of the arithmetic sequence are given. Find the missing term. Use the *table* feature of a graphing utility to verify your results.**

43. $a_1 = 5, \quad a_2 = 11, \quad a_{10} =$
44. $a_1 = 3, \quad a_2 = 13, \quad a_9 =$
45. $a_1 = 4.2, \quad a_2 = 6.6, \quad a_7 =$
46. $a_1 = -0.7, \quad a_2 = -13.8, \quad a_8 =$

Graphing Terms of a Sequence **In Exercises 47–50, use a graphing utility to graph the first 10 terms of the sequence. (Assume n begins with 1.)**

47. $a_n = 15 - \frac{3}{2}n$
48. $a_n = -5 + 2n$
49. $a_n = 0.4n - 2$
50. $a_n = -1.3n + 7$

Finding Terms of a Sequence **In Exercises 51–56, use the *table* feature of a graphing utility to find the first 10 terms of the sequence. (Assume n begins with 1.)**

51. $a_n = 4n - 5$
52. $a_n = 17 + 3n$
53. $a_n = 20 - \frac{3}{4}n$
54. $a_n = \frac{4}{5}n + 12$
55. $a_n = 1.5 + 0.05n$
56. $a_n = 8 - 12.5n$

Finding the Sum of a Finite Arithmetic Sequence **In Exercises 57–64, find the sum of the finite arithmetic sequence.**

57. $2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 + 18 + 20$
58. $1 + 4 + 7 + 10 + 13 + 16 + 19$

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59. $-1 + (-3) + (-5) + (-7) + (-9)$

60. $-5 + (-3) + (-1) + 1 + 3 + 5$

61. Sum of the first 100 positive integers

62. Sum of the first 50 negative integers

63. Sum of the integers from -100 to 30

64. Sum of the integers from -10 to 50

Finding a Partial Sum of an Arithmetic Sequence **In Exercises 65–70, find the indicated *n*th partial sum of the arithmetic sequence.**

65. 8, 20, 32, 44, . . . , $n = 10$

66. 2, 8, 14, 20, . . . , $n = 25$

67. 0.5, 1.3, 2.1, 2.9, . . . , $n = 10$

68. 4.2, 3.7, 3.2, 2.7, . . . , $n = 12$

69. $a_1 = 100,\ a_{25} = 220,\quad n = 25$

70. $a_1 = 15,\ a_{100} = 307,\quad n = 100$

Finding a Partial Sum of an Arithmetic Sequence **In Exercises 71–76, find the partial sum without using a graphing utility.**

71. $\displaystyle\sum_{n=1}^{50} n$

72. $\displaystyle\sum_{n=1}^{100} 2n$

73. $\displaystyle\sum_{n=11}^{30} n - \sum_{n=1}^{10} n$

74. $\displaystyle\sum_{n=51}^{100} n - \sum_{n=1}^{50} n$

75. $\displaystyle\sum_{n=1}^{500} (n + 8)$

76. $\displaystyle\sum_{n=1}^{100} \frac{8 - 3n}{16}$

Finding a Partial Sum Using a Graphing Utility **In Exercises 77–82, use a graphing utility to find the partial sum.**

77. $\displaystyle\sum_{n=1}^{20} (2n + 1)$

78. $\displaystyle\sum_{n=0}^{50} (50 - 2n)$

79. $\displaystyle\sum_{n=1}^{100} \frac{n + 1}{2}$

80. $\displaystyle\sum_{n=0}^{100} \frac{4 - n}{4}$

81. $\displaystyle\sum_{i=1}^{60} \left(250 - \tfrac{2}{5}i\right)$

82. $\displaystyle\sum_{j=1}^{200} (10.5 + 0.025j)$

83. ***Why you should learn it*** *(p. 581)* A brick patio has the approximate shape of a trapezoid, as shown in the figure. The patio has 18 rows of bricks. The first row has 14 bricks and the 18th row has 31 bricks. How many bricks are in the patio?

84. **Performing Arts** An auditorium has 20 rows of seats. There are 20 seats in the first row, 21 seats in the second row, 22 seats in the third row, and so on. How many seats are there in all 20 rows?

85. **Sales** A hardware store makes a profit of \$30,000 during its first year. The store owner sets a goal of increasing profits by \$5000 each year for 4 years. Assuming that this goal is met, find the total profit during the first 5 years of business.

86. **Physics** An object with negligible air resistance is dropped from a plane. During the first second of fall, the object falls 16 feet; during the second second, it falls 48 feet; during the third second, it falls 80 feet; and during the fourth second, it falls 112 feet. Assume this pattern continues. How many feet will the object fall in 8 seconds?

87. MODELING DATA

The table shows the sales a_n (in billions of dollars) for Coca-Cola Enterprises from 2001 through 2008. (Source: Coca-Cola Enterprises, Inc.)

Year	Sales, a_n (in billions of dollars)
2001	15.7
2002	16.9
2003	17.3
2004	18.2
2005	18.7
2006	19.8
2007	20.9
2008	21.8

(a) Use the *regression* feature of a graphing utility to find an arithmetic sequence for the data. Let n represent the year, with $n = 1$ corresponding to 2001.

(b) Use the sequence from part (a) to approximate the annual sales for Coca-Cola Enterprises for the years 2001 through 2008. How well does the model fit the data?

(c) Use the sequence to find the total sales for Coca-Cola Enterprises over the period from 2001 through 2008.

(d) Use the sequence to predict the total sales over the period from 2009 through 2016. Is your total reasonable? Explain.

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88. MODELING DATA

The table shows the numbers a_n (in thousands) of master's degrees conferred in the United States from 2000 through 2007. (Source: U.S. National Center for Education Statistics)

Year	Master's degrees conferred, a_n (in thousands)
2000	457
2001	468
2002	482
2003	513
2004	559
2005	575
2006	594
2007	605

(a) Use the *regression* feature of a graphing utility to find an arithmetic sequence for the data. Let n represent the year, with $n = 0$ corresponding to 2000.

(b) Use the sequence from part (a) to approximate the numbers of master's degrees conferred for the years 2000 through 2007. How well does the model fit the data?

(c) Use the sequence to find the total number of master's degrees conferred over the period from 2000 through 2007.

(d) Use the sequence to predict the total number of master's degrees conferred over the period from 2008 through 2018. Is your total reasonable? Explain.

Conclusions

True or False? **In Exercises 89 and 90, determine whether the statement is true or false. Justify your answer.**

89. Given the nth term and the common difference of an arithmetic sequence, it is possible to find the $(n + 1)$th term.

90. If the only known information about a finite arithmetic sequence is its first term and its last term, then it is possible to find the sum of the sequence.

Finding Terms of a Sequence **In Exercises 91 and 92, find the first 10 terms of the sequence.**

91. $a_1 = x, d = 2x$

92. $a_1 = -y, d = 5y$

93. Think About It The sum of the first 20 terms of an arithmetic sequence with a common difference of 3 is 650. Find the first term.

94. Writing Explain how to use the first two terms of an arithmetic sequence to find the nth term.

95. Think About It The sum of the first n terms of an arithmetic sequence with first term a_1 and common difference d is S_n. Determine the sum when each term is increased by 5. Explain.

96. Think About It In each sequence, decide whether it is possible to fill in the blanks to form an arithmetic sequence. If so, find a recursion formula for the sequence. Explain how you found your answers.

(a) $-7, __, __, __, __, __, 11$

(b) $17, __, __, __, __, __, __, 59$

(c) $2, 6, __, __, 162$

(d) $4, 7.5, __, __, __, __, __, 28.5$

(e) $8, 12, __, __, __, 60.75$

97. Writing Carl Friedrich Gauss, a famous nineteenth century mathematician, was a child prodigy. It was said that when Gauss was 10 he was asked by his teacher to add the numbers from 1 to 100. Almost immediately, Gauss found the answer by mentally finding the summation. Write an explanation of how he arrived at his conclusion, and then find the formula for the sum of the first n natural numbers.

98. CAPSTONE Describe the characteristics of an arithmetic sequence. Give an example of a sequence that is arithmetic and one that is not arithmetic.

Finding the Sum of a Finite Arithmetic Sequence **In Exercises 99–102, find the sum using the method from Exercise 97.**

99. The first 200 natural numbers

100. The first 100 even natural numbers from 2 to 200, inclusive

101. The first 51 odd natural numbers from 1 to 101, inclusive

102. The first 100 multiples of 4 from 4 to 400, inclusive

Cumulative Mixed Review

Gauss-Jordan Elimination **In Exercises 103 and 104, use Gauss-Jordan elimination to solve the system of equations.**

103. $\begin{cases} 2x - y + 7z = -10 \\ 3x + 2y - 4z = 17 \\ 6x - 5y + z = -20 \end{cases}$

104. $\begin{cases} -x + 4y + 10z = 4 \\ 5x - 3y + z = 31 \\ 8x + 2y - 3z = -5 \end{cases}$

105. *Make a Decision* To work an extended application analyzing the amount of municipal waste recovered in the United States from 1983 through 2008, visit this textbook's *Companion Website*. (Data Source: U.S. Environmental Protection Agency)

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8.3 Geometric Sequences and Series

Geometric Sequences

In Section 8.2, you learned that a sequence whose consecutive terms have a common *difference* is an arithmetic sequence. In this section, you will study another important type of sequence called a **geometric sequence.** Consecutive terms of a geometric sequence have a common *ratio*.

Definition of Geometric Sequence

A sequence is **geometric** when the ratios of consecutive terms are the same. So, the sequence $a_1, a_2, a_3, a_4, \ldots, a_n, \ldots$ is geometric when there is a number r such that

$$\frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} = \cdots = r, \qquad r \neq 0.$$

The number r is the **common ratio** of the sequence.

Example 1 Examples of Geometric Sequences

a. The sequence whose nth term is 2^n is geometric. For this sequence, the common ratio between consecutive terms is 2.

$$\underbrace{2, 4, 8}_{\frac{4}{2} = 2}, 16, \ldots, 2^n, \ldots \qquad \text{Begin with } n = 1.$$

b. The sequence whose nth term is $4(3^n)$ is geometric. For this sequence, the common ratio between consecutive terms is 3.

$$\underbrace{12, 36}_{\frac{36}{12} = 3}, 108, 324, \ldots, 4(3^n), \ldots \qquad \text{Begin with } n = 1.$$

c. The sequence whose nth term is $\left(-\frac{1}{3}\right)^n$ is geometric. For this sequence, the common ratio between consecutive terms is $-\frac{1}{3}$.

$$\underbrace{-\frac{1}{3}, \frac{1}{9}}_{\frac{1/9}{-1/3} = -\frac{1}{3}}, -\frac{1}{27}, \frac{1}{81}, \ldots, \left(-\frac{1}{3}\right)^n, \ldots \qquad \text{Begin with } n = 1.$$

CHECKPOINT Now try Exercise 7.

The sequence

$$1, 4, 9, 16, \ldots$$

whose nth term is n^2, is *not* geometric. The ratio of the second term to the first term is

$$\frac{a_2}{a_1} = \frac{4}{1} = 4$$

but the ratio of the third term to the second term is

$$\frac{a_3}{a_2} = \frac{9}{4}.$$

What you should learn

- Recognize, write, and find the nth terms of geometric sequences.
- Find nth partial sums of geometric sequences.
- Find sums of infinite geometric series.
- Use geometric sequences to model and solve real-life problems.

Why you should learn it

Geometric sequences can reduce the amount of time it takes to find the sum of a sequence of numbers with a common ratio. For instance, Exercise 109 on page 598 shows how to use a geometric sequence to find the total vertical distance traveled by a bouncing ball.

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Study Tip

In Example 1, notice that each of the geometric sequences has an nth term of the form ar^n, where r is the common ratio of the sequence.

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The *n*th Term of a Geometric Sequence

The *n*th term of a geometric sequence has the form

$$a_n = a_1 r^{n-1}$$

where r is the common ratio of consecutive terms of the sequence. So, every geometric sequence can be written in the following form.

$$\begin{array}{ccccccccc} a_1, & a_2, & a_3, & a_4, & a_5, & \ldots, & a_n, & \ldots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \\ a_1, & a_1 r, & a_1 r^2, & a_1 r^3, & a_1 r^4, & \ldots, & a_1 r^{n-1}, & \ldots \end{array}$$

When you know the *n*th term of a geometric sequence, you can find the $(n + 1)$th term by multiplying by r. That is,

$$a_{n+1} = a_n r.$$

Example 2 Finding the Terms of a Geometric Sequence

Write the first five terms of the geometric sequence whose first term is

$$a_1 = 3$$

and whose common ratio is

$$r = 2.$$

Solution

Starting with 3, repeatedly multiply by 2 to obtain the following.

$a_1 = 3$	1st term
$a_2 = 3(2^1) = 6$	2nd term
$a_3 = 3(2^2) = 12$	3rd term
$a_4 = 3(2^3) = 24$	4th term
$a_5 = 3(2^4) = 48$	5th term

CHECKPOINT Now try Exercise 17.

Technology Tip

You can use a graphing utility to generate the geometric sequence in Example 2 by using the following steps.

3 [ENTER]

2 [×] [ANS]

Now press the *enter* key repeatedly to generate the terms of the sequence.

Example 3 Finding a Term of a Geometric Sequence

Find the 15th term of the geometric sequence whose first term is 20 and whose common ratio is 1.05.

Algebraic Solution

$a_n = a_1 r^{n-1}$	Formula for a geometric sequence
$a_{15} = 20(1.05)^{15-1}$	Substitute 20 for a_1, 1.05 for r, and 15 for n.
≈ 39.60	Use a calculator.

CHECKPOINT Now try Exercise 31.

Numerical Solution

For this sequence, $r = 1.05$ and $a_1 = 20$. So, $a_n = 20(1.05)^{n-1}$. Use the *table* feature of a graphing utility to create a table that shows the terms of the sequence, as shown in Figure 8.9.

The number in the 15th row is the 15th term of the sequence.

So, $a_{15} \approx 39.60$.

Figure 8.9

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Example 4 Finding a Term of a Geometric Sequence

Find a formula for the nth term of the following geometric sequence. What is the ninth term of the sequence?

$$5, 15, 45, \ldots$$

Solution

The common ratio of this sequence is

$$r = \frac{15}{5} = 3.$$

Because the first term is $a_1 = 5$, the formula must have the form

$$a_n = a_1 r^{n-1} = 5(3)^{n-1}.$$

You can determine the ninth term ($n = 9$) to be

$$\begin{aligned} a_9 &= 5(3)^{9-1} && \text{Substitute 9 for } n. \\ &= 5(6561) && \text{Use a calculator.} \\ &= 32{,}805. && \text{Simplify.} \end{aligned}$$

A graph of the first nine terms of the sequence is shown in Figure 8.10. Notice that the points lie on an exponential curve. This makes sense because a_n is an exponential function of n.

Figure 8.10

CHECKPOINT Now try Exercise 39.

When you know *any* two terms of a geometric sequence, you can use that information to find a formula for the nth term of the sequence.

Example 5 Finding a Term of a Geometric Sequence

The fourth term of a geometric sequence is 125, and the 10th term is

$$\frac{125}{64}.$$

Find the 14th term. (Assume that the terms of the sequence are positive.)

Solution

The 10th term is related to the fourth term by the equation

$$a_{10} = a_4 r^6. \qquad \text{Multiply 4th term by } r^{10-4}.$$

Because $a_{10} = 125/64$ and $a_4 = 125$, you can solve for r as follows.

$$\frac{125}{64} = 125r^6$$

$$\frac{1}{64} = r^6 \quad \Rightarrow \quad \frac{1}{2} = r$$

You can obtain the 14th term by multiplying the 10th term by r^4.

$$a_{14} = a_{10}r^4 = \frac{125}{64}\left(\frac{1}{2}\right)^4 = \frac{125}{1024}$$

CHECKPOINT Now try Exercise 45.

Technology Tip

Most graphing utilities have a built-in function that will display the terms of a geometric sequence. For instructions on how to use the *sequence* feature, see Appendix A; for specific keystrokes, go to this textbook's *Companion Website*.

Study Tip

Remember that r is the common ratio of consecutive terms of a geometric sequence. So, in Example 5,

$$\begin{aligned} a_{10} &= a_1 r^9 \\ &= a_1 \cdot r \cdot r \cdot r \cdot r^6 \\ &= a_1 \cdot \frac{a_2}{a_1} \cdot \frac{a_3}{a_2} \cdot \frac{a_4}{a_3} \cdot r^6 \end{aligned}$$

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The Sum of a Finite Geometric Sequence

The formula for the sum of a *finite* geometric sequence is as follows.

The Sum of a Finite Geometric Sequence (See the proof on page 633.)

The sum of the finite geometric sequence

$$a_1,\ a_1r,\ a_1r^2,\ a_1r^3,\ a_1r^4,\ .\ .\ .\ ,\ a_1r^{n-1}$$

with common ratio $r \neq 1$ is given by

$$S_n = \sum_{i=1}^{n} a_1 r^{i-1} = a_1\left(\frac{1-r^n}{1-r}\right).$$

Example 6 Finding the Sum of a Finite Geometric Sequence

Find the following sum.

$$\sum_{n=1}^{12} 4(0.3)^n$$

Solution

By writing out a few terms, you have

$$\sum_{n=1}^{12} 4(0.3)^n = 4(0.3)^1 + 4(0.3)^2 + 4(0.3)^3 + \cdot\ \cdot\ \cdot + 4(0.3)^{12}.$$

Now, because

$$a_1 = 4(0.3), \qquad r = 0.3, \qquad \text{and} \qquad n = 12$$

you can apply the formula for the sum of a finite geometric sequence to obtain

$$\sum_{n=1}^{12} 4(0.3)^n = a_1\left(\frac{1-r^n}{1-r}\right)$$ Formula for sum of a finite geometric sequence

$$= 4(0.3)\left[\frac{1-(0.3)^{12}}{1-0.3}\right]$$ Substitute 4(0.3) for a_1, 0.3 for r, and 12 for n.

$$\approx 1.71.$$ Use a calculator.

CHECKPOINT Now try Exercise 55.

When using the formula for the sum of a geometric sequence, be careful to check that the index begins at $i = 1$. For an index that begins at $i = 0$, you must adjust the formula for the nth partial sum. For instance, if the index in Example 6 had begun with $n = 0$, then the sum would have been

$$\sum_{n=0}^{12} 4(0.3)^n = 4(0.3)^0 + \sum_{n=1}^{12} 4(0.3)^n$$

$$= 4 + \sum_{n=1}^{12} 4(0.3)^n$$

$$\approx 4 + 1.71$$

$$= 5.71.$$

Technology Tip

Using the *sum sequence* feature of a graphing utility, you can calculate the sum of the sequence in Example 6 to be about 1.7142848, as shown below.

Calculate the sum beginning at $n = 0$. You should obtain a sum of about 5.7142848.

What's Wrong?

You use a graphing utility to find the sum

$$\sum_{n=1}^{6} 7(0.5)^n$$

as shown in the figure. What's wrong?

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Geometric Series

The sum of the terms of an infinite geometric sequence is called an **infinite geometric series** or simply a **geometric series.**

The formula for the sum of a *finite geometric sequence* can, depending on the value of r, be extended to produce a formula for the sum of an *infinite geometric series.* Specifically, if the common ratio r has the property that

$$|r| < 1$$

then it can be shown that r^n becomes arbitrarily close to zero as n increases without bound. Consequently,

$$a_1\left(\frac{1 - r^n}{1 - r}\right) \longrightarrow a_1\left(\frac{1 - 0}{1 - r}\right) \qquad \text{as} \qquad n \longrightarrow \infty.$$

This result is summarized as follows.

The Sum of an Infinite Geometric Series

If $|r| < 1$, then the infinite geometric series

$$a_1 + a_1r + a_1r^2 + a_1r^3 + \cdots + a_1r^{n-1} + \cdots$$

has the sum

$$S = \sum_{i=0}^{\infty} a_1r^i = \frac{a_1}{1 - r}.$$

Note that when

$$|r| \geq 1$$

the series does not have a sum.

Explore the Concept

Notice that the formula for the sum of an infinite geometric series requires that $|r| < 1$. What happens when $r = 1$ or $r = -1$? Give examples of infinite geometric series for which $|r| > 1$ and convince yourself that they do not have finite sums.

Explore the Concept

On most graphing utilities, it is not possible to find the sum of the series in Example 7(a) because you cannot enter ∞ as the upper limit of summation. Can you still find the sum using a graphing utility without entering ∞? If so, which partial sum will result in 10, the exact sum of the series?

Example 7 Finding the Sum of an Infinite Geometric Series

Find each sum.

a. $\displaystyle\sum_{n=0}^{\infty} 4(0.6)^n$

b. $3 + 0.3 + 0.03 + 0.003 + \cdots$

Solution

a. $$\sum_{n=0}^{\infty} 4(0.6)^n = 4 + 4(0.6) + 4(0.6)^2 + 4(0.6)^3 + \cdots + 4(0.6)^n + \cdots$$

$$= \frac{4}{1 - 0.6} \qquad \frac{a_1}{1 - r}$$

$$= 10$$

b. $$3 + 0.3 + 0.03 + 0.003 + \cdots = 3 + 3(0.1) + 3(0.1)^2 + 3(0.2)^3 + \cdots$$

$$= \frac{3}{1 - 0.1} \qquad \frac{a_1}{1 - r}$$

$$= \frac{10}{3}$$

$$\approx 3.33$$

CHECKPOINT Now try Exercise 75.

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Application

Example 8 Increasing Annuity

A deposit of \$50 is made on the first day of each month in an account that pays 6% compounded monthly. What is the balance at the end of 2 years? (This type of savings plan is called an **increasing annuity.**)

Personal Financial Advisor

Solution

Recall from Section 3.1 that the compound interest formula is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}. \qquad \text{Formula for compound interest}$$

To find the balance in the account after 24 months, consider each of the 24 deposits separately. The first deposit will gain interest for 24 months, and its balance will be

$$A_{24} = 50\left(1 + \frac{0.06}{12}\right)^{24}$$

$$= 50(1.005)^{24}.$$

The second deposit will gain interest for 23 months, and its balance will be

$$A_{23} = 50\left(1 + \frac{0.06}{12}\right)^{23}$$

$$= 50(1.005)^{23}.$$

The last deposit will gain interest for only 1 month, and its balance will be

$$A_1 = 50\left(1 + \frac{0.06}{12}\right)^{1}$$

$$= 50(1.005).$$

The total balance in the annuity will be the sum of the balances of the 24 deposits. Using the formula for the sum of a finite geometric sequence, with $a_1 = 50(1.005)$, $r = 1.005$, and $n = 24$, you have

$$S_n = a_1\left(\frac{1 - r^n}{1 - r}\right) \qquad \text{Formula for sum of a finite geometric sequence}$$

$$S_{24} = 50(1.005)\left[\frac{1 - (1.005)^{24}}{1 - 1.005}\right] \qquad \text{Substitute } 50(1.005) \text{ for } a_1, 1.005 \text{ for } r, \text{ and } 24 \text{ for } n.$$

$$\approx \$1277.96. \qquad \text{Simplify.}$$

You can use the *sum sequence* feature of a graphing utility to verify this result, as shown in Figure 8.11.

Figure 8.11

CHECKPOINT Now try Exercise 101.

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8.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1–3, fill in the blank(s).

1. A sequence is called a _____ sequence if the ratios of consecutive terms are the same. This ratio is called the _____ ratio.
2. The nth term of a geometric sequence has the form $a_n =$ _____ .
3. The sum of the terms of an infinite geometric sequence is called a _____ .

4. Can a geometric sequence have a common ratio of 0?
5. For what values of the common ratio r is it possible to find the sum of an infinite geometric series?
6. Which formula represents the sum of a *finite geometric sequence*? an *infinite geometric series*?

(a) $S = \dfrac{a_1}{1 - r}, \ |r| < 1$ (b) $S_n = a_1\left(\dfrac{1 - r^n}{1 - r}\right)$

Procedures and Problem Solving

Identifying a Geometric Sequence **In Exercises 7–16, determine whether or not the sequence is geometric. If it is, find the common ratio.**

7. 5, 15, 45, 135, . . .
8. 3, 12, 48, 192, . . .
9. 6, 18, 30, 42, . . .
10. 1, -2, 4, -8, . . .
11. 1, $-\frac{1}{2}$, $\frac{1}{4}$, $-\frac{1}{8}$, . . .
12. 5, 1, 0.2, 0.04, . . .
13. $\frac{1}{8}$, $\frac{1}{4}$, $\frac{1}{2}$, 1, . . .
14. 9, -6, 4, $-\frac{8}{3}$, . . .
15. 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, . . .
16. $\frac{1}{5}$, $\frac{2}{7}$, $\frac{3}{9}$, $\frac{4}{11}$, . . .

Finding the Terms of a Geometric Sequence **In Exercises 17–24, write the first five terms of the geometric sequence.**

17. $a_1 = 6, \quad r = 3$
18. $a_1 = 4, \quad r = 2$
19. $a_1 = 1, \quad r = \frac{1}{2}$
20. $a_1 = 2, \quad r = \frac{1}{3}$
21. $a_1 = 5, \quad r = -\frac{1}{10}$
22. $a_1 = 6, \quad r = -\frac{1}{4}$
23. $a_i = 1, \quad r = e$
24. $a_1 = 4, \quad r = \sqrt{3}$

Finding the nth Term of a Geometric Sequence **In Exercises 25–30, write the first five terms of the geometric sequence. Find the common ratio and write the nth term of the sequence as a function of n.**

25. $a_1 = 64, \ a_{k+1} = \frac{1}{2}a_k$
26. $a_1 = 81, \ a_{k+1} = \frac{1}{3}a_k$
27. $a_1 = 9, \ a_{k+1} = 2a_k$
28. $a_1 = 5, \ a_{k+1} = -3a_k$
29. $a_1 = 6, \ a_{k+1} = -\frac{3}{2}a_k$
30. $a_1 = 30, \ a_{k+1} = -\frac{2}{3}a_k$

Finding a Term of a Geometric Sequence **In Exercises 31–38, find the indicated term of the geometric sequence (a) using the *table* feature of a graphing utility and (b) algebraically.**

31. $a_1 = 6, r = \frac{1}{3}$, 12th term
32. $a_1 = 24, r = \frac{3}{2}$, 8th term
33. $a_1 = 8, r = -\frac{4}{3}$, 7th term
34. $a_1 = 8, r = -\frac{3}{4}$, 9th term
35. $a_1 = -1, r = 3$, 6th term
36. $a_1 = -\frac{1}{128}, r = 2$, 12th term
37. $a_1 = 500, r = -1.02$, 14th term
38. $a_1 = 1000, r = 1.005$, 11th term

Finding a Term of a Geometric Sequence **In Exercises 39–42, find a formula for the nth term of the geometric sequence. Then find the indicated term of the geometric sequence.**

39. 9th term: 7, 21, 63, . . .
40. 7th term: 3, 36, 432, . . .
41. 10th term: 5, 30, 180, . . .
42. 22nd term: 4, 8, 16, . . .

Finding a Term of a Geometric Sequence **In Exercises 43–46, find the indicated term of the geometric sequence.**

43. $a_1 = 4, a_4 = \frac{1}{2}$, 10th term
44. $a_1 = 5, a_3 = \frac{45}{4}$, 8th term
45. $a_2 = -18, a_5 = \frac{2}{3}$, 6th term
46. $a_3 = \frac{16}{3}, a_5 = \frac{64}{27}$, 7th term

Using a Graphing Utility **In Exercises 47–50, use a graphing utility to graph the first 10 terms of the sequence.**

47. $a_n = 12(-0.75)^{n-1}$
48. $a_n = 20(1.25)^{n-1}$
49. $a_n = 2(1.3)^{n-1}$
50. $a_n = 10(-1.2)^{n-1}$

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Finding a Sequence of Partial Sums **In Exercises 51 and 52, find the sequence of the first five partial sums S_1, S_2, S_3, S_4, and S_5 of the geometric sequence by adding terms.**

51. $8, -4, 2, -1, \frac{1}{2}, \ldots$ **52.** $8, 12, 18, 27, \frac{81}{2}, \ldots$

Finding a Sequence of Partial Sums **In Exercises 53 and 54, use a graphing utility to create a table showing the sequence of the first 10 partial sums $S_1, S_2, S_3, \ldots,$ and S_{10} for the series.**

53. $\sum_{n=1}^{\infty} 16\left(\frac{1}{2}\right)^{n-1}$ **54.** $\sum_{n=1}^{\infty} 4(0.2)^{n-1}$

Finding the Sum of a Finite Geometric Sequence **In Exercises 55–64, find the sum. Use a graphing utility to verify your result.**

55. $\sum_{n=1}^{9} 2^{n-1}$ **56.** $\sum_{n=1}^{9} (-2)^{n-1}$

57. $\sum_{i=1}^{7} 64\left(-\frac{1}{2}\right)^{i-1}$ **58.** $\sum_{i=1}^{6} 32\left(\frac{1}{4}\right)^{i-1}$

59. $\sum_{n=0}^{20} 3\left(\frac{3}{2}\right)^{n}$ **60.** $\sum_{n=0}^{15} 2\left(\frac{4}{3}\right)^{n}$

61. $\sum_{i=1}^{10} 8\left(-\frac{1}{4}\right)^{i-1}$ **62.** $\sum_{i=1}^{10} 5\left(-\frac{1}{3}\right)^{i-1}$

63. $\sum_{n=0}^{5} 300(1.06)^{n}$ **64.** $\sum_{n=0}^{6} 500(1.04)^{n}$

Using Summation Notation **In Exercises 65–68, use summation notation to write the sum.**

65. $5 + 15 + 45 + \cdots + 3645$

66. $7 + 14 + 28 + \cdots + 896$

67. $2 - \frac{1}{2} + \frac{1}{8} - \cdots + \frac{1}{2048}$

68. $15 - 3 + \frac{3}{5} - \cdots - \frac{3}{625}$

Finding the Sum of an Infinite Geometric Series **In Exercises 69–82, find the sum of the infinite geometric series, if possible. If not possible, explain why.**

69. $\sum_{n=0}^{\infty} 10\left(\frac{4}{5}\right)^{n}$ **70.** $\sum_{n=0}^{\infty} 6\left(\frac{2}{3}\right)^{n}$

71. $\sum_{n=0}^{\infty} 5\left(-\frac{1}{2}\right)^{n}$ **72.** $\sum_{n=0}^{\infty} 9\left(-\frac{2}{3}\right)^{n}$

73. $\sum_{n=1}^{\infty} 2\left(\frac{7}{3}\right)^{n-1}$ **74.** $\sum_{n=1}^{\infty} 8\left(\frac{5}{3}\right)^{n-1}$

75. $\sum_{n=0}^{\infty} 10(0.11)^{n}$ **76.** $\sum_{n=0}^{\infty} 5(0.45)^{n}$

77. $\sum_{n=0}^{\infty} -3(-0.9)^{n}$ **78.** $\sum_{n=0}^{\infty} -10(-0.2)^{n}$

79. $9 + 6 + 4 + \frac{8}{3} + \cdots$ **80.** $8 + 6 + \frac{9}{2} + \frac{27}{8} + \cdots$

81. $3 - 1 + \frac{1}{3} - \frac{1}{9} + \cdots$

82. $-6 + 5 - \frac{25}{6} + \frac{125}{36} - \cdots$

Writing a Repeating Decimal as a Rational Number **In Exercises 83–86, find the rational number representation of the repeating decimal.**

83. $0.\overline{36}$ **84.** $0.\overline{297}$

85. $1.2\overline{5}$ **86.** $1.3\overline{8}$

Identifying a Sequence **In Exercises 87–94, determine whether the sequence associated with the series is arithmetic or geometric. Find the common difference or ratio and find the sum of the first 15 terms.**

87. $8 + 16 + 32 + 64 + \cdots$

88. $17 + 14 + 11 + 8 + \cdots$

89. $90 + 30 + 10 + \frac{10}{3} + \cdots$

90. $\frac{5}{4} + \frac{7}{4} + \frac{9}{4} + \frac{11}{4} + \cdots$

91. $\sum_{n=1}^{\infty} 6n$ **92.** $\sum_{n=1}^{\infty} 2^{n-1}$

93. $\sum_{n=0}^{\infty} 6(0.8)^{n}$ **94.** $\sum_{n=0}^{\infty} \frac{n+4}{4}$

95. **Compound Interest** A principal of \$1000 is invested at 3% interest. Find the amount after 10 years if the interest is compounded (a) annually, (b) semiannually, (c) quarterly, (d) monthly, and (e) daily.

96. **Annuity** A deposit of \$100 is made at the beginning of each month in an account that pays 3% interest, compounded monthly. The balance A in the account at the end of 5 years is given by

$$A = 100\left(1 + \frac{0.03}{12}\right)^{1} + \cdots + 100\left(1 + \frac{0.03}{12}\right)^{60}.$$

Find A.

97. **Annuity** A deposit of P dollars is made at the beginning of each month in an account earning an annual interest rate r, compounded monthly. The balance A after t years is given by

$$A = P\left(1 + \frac{r}{12}\right) + P\left(1 + \frac{r}{12}\right)^{2} + \cdots + P\left(1 + \frac{r}{12}\right)^{12t}.$$

Show that the balance is given by

$$A = P\left[\left(1 + \frac{r}{12}\right)^{12t} - 1\right]\left(1 + \frac{12}{r}\right).$$

98. **Annuity** A deposit of P dollars is made at the beginning of each month in an account earning an annual interest rate r, compounded continuously. The balance A after t years is given by $A = Pe^{r/12} + Pe^{2r/12} + \cdots + Pe^{12tr/12}$. Show that the balance is given by

$$A = \frac{Pe^{r/12}(e^{rt} - 1)}{e^{r/12} - 1}.$$

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Annuities **In Exercises 99–102, consider making monthly deposits of P dollars in a savings account earning an annual interest rate r. Use the results of Exercises 97 and 98 to find the balances A after t years if the interest is compounded (a) monthly and (b) continuously.**

99. $P = \$50,\ r = 7\%,\ t = 20$ years

100. $P = \$75,\ r = 4\%,\ t = 25$ years

101. $P = \$100,\ r = 5\%,\ t = 40$ years

102. $P = \$20,\ r = 6\%,\ t = 50$ years

103. **Geometry** The sides of a square are 16 inches in length. A new square is formed by connecting the midpoints of the sides of the original square, and two of the resulting triangles are shaded. After this process is repeated five more times, determine the total area of the shaded region.

104. **Geometry** The sides of a square are 27 inches in length. New squares are formed by dividing the original square into nine squares. The center square is then shaded (see figure). After this process is repeated three more times, determine the total area of the shaded region.

105. **Physics** The temperature of water in an ice cube tray is 70°F when it is placed in a freezer. Its temperature n hours after being placed in the freezer is 20% less than 1 hour earlier.

(a) Find a formula for the nth term of the geometric sequence that gives the temperature of the water n hours after it is placed in the freezer.

(b) Find the temperatures of the water 6 hours and 12 hours after it is placed in the freezer.

(c) Use a graphing utility to graph the sequence to approximate the time required for the water to freeze.

106. **Fractals** In a *fractal*, a geometric figure is repeated at smaller and smaller scales. The sphereflake shown is a computer-generated fractal that was created by Eric Haines. The radius of the large sphere is 1. Attached to the large sphere are nine spheres of radius $\frac{1}{3}$. Attached to each of the smaller spheres are nine spheres of radius $\frac{1}{9}$. This process is continued infinitely.

Eric Haines

(a) Write a formula in series notation that gives the surface area of the sphereflake.

(b) Write a formula in series notation that gives the volume of the sphereflake.

(c) Is the surface area of the sphereflake finite or infinite? Is the volume finite or infinite? If either is finite, find the value.

107. **MODELING DATA**

The table shows the mid-year populations a_n of China (in millions) from 2002 through 2008. (Source: U.S. Census Bureau)

Year	Population, a_n
2002	1284.3
2003	1291.5
2004	1298.8
2005	1306.3
2006	1314.0
2007	1321.9
2008	1330.0

(a) Use the *exponential regression* feature of a graphing utility to find a geometric sequence that models the data. Let n represent the year, with $n = 2$ corresponding to 2002.

(b) Use the sequence from part (a) to describe the rate at which the population of China is growing.

(c) Use the sequence from part (a) to predict the population of China in 2015. The U.S. Census Bureau predicts the population of China will be 1393.4 million in 2015. How does this value compare with your prediction?

(d) Use the sequence from part (a) to determine when the population of China will reach 1.35 billion.

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108. **Manufacturing** The manufacturer of a new food processor plans to produce and sell 8000 units per year. Each year, 10% of all units sold become inoperative. So, 8000 units will be in use after 1 year, $[8000 + 0.9(8000)]$ units will be in use after 2 years, and so on.

(a) Write a formula in series notation for the number of units that will be operative after n years.

(b) Find the numbers of units that will be operative after 10 years, 15 years, and 20 years.

(c) If this trend continues indefinitely, will the number of units that will be operative be finite? If so, how many? If not, explain your reasoning.

109. ***Why you should learn it*** *(p. 589)* A ball is dropped from a height of 6 feet and begins bouncing, as shown in the figure. The height of each bounce is three-fourths the height of the previous bounce. Find the total vertical distance the ball travels before coming to rest.

Conclusions

True or False? **In Exercises 110–112, determine whether the statement is true or false. Justify your answer.**

110. For a geometric sequence in which the quotient of term a_7 and term a_5 is 16, the common ratio is 8.

111. The first n terms of a geometric sequence with a common ratio of 1 are the same as the first n terms of an arithmetic sequence with a common difference of 0 if both sequences have the same first term.

112. You can find the nth term of a geometric sequence by multiplying its common ratio by the first term of the sequence raised to the $(n - 1)$th power.

Finding the Terms of a Geometric Sequence **In Exercises 113 and 114, write the first five terms of the geometric sequence.**

113. $a_1 = 3, r = \dfrac{x}{2}$

114. $a_1 = \dfrac{1}{2}, r = 7x$

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Finding a Term of a Geometric Sequence **In Exercises 115 and 116, find the indicated term of the geometric sequence.**

115. $a_1 = 100, r = e^x$, 9th term

116. $a_1 = 4, r = \dfrac{4x}{3}$, 6th term

117. **Exploration** Use a graphing utility to graph each function. Identify the horizontal asymptote of the graph and determine its relationship to the sum.

(a) $f(x) = 6\left[\dfrac{1 - (0.5)^x}{1 - (0.5)}\right], \quad \displaystyle\sum_{n=0}^{\infty} 6\left(\frac{1}{2}\right)^n$

(b) $f(x) = 2\left[\dfrac{1 - (0.8)^x}{1 - (0.8)}\right], \quad \displaystyle\sum_{n=0}^{\infty} 2\left(\frac{4}{5}\right)^n$

118. **Writing** Write a brief paragraph explaining why the terms of a geometric sequence decrease in magnitude when $-1 < r < 1$.

119. **Writing** Write a brief paragraph explaining how to use the first two terms of a geometric sequence to find the nth term.

120. **CAPSTONE** The terms of a geometric sequence can be written as

$$a_1,\ a_2 = a_1r,\ a_3 = a_2r,\ a_4 = a_3r, \ldots.$$

Write each term of the sequence in terms of a_1 and r. Then, based on the pattern, write the nth term of the geometric sequence. What do you notice?

Cumulative Mixed Review

121. **Average Speed** A truck traveled at an average speed of 50 miles per hour on a 200-mile trip. On the return trip, the average speed was 42 miles per hour. Find the average speed for the round trip.

122. **Work Rate** Your friend can mow a lawn in 4 hours and you can mow it in 6 hours. How long will it take both of you to mow the lawn working together?

Finding a Determinant **In Exercises 123 and 124, find the determinant of the matrix.**

123. $\begin{bmatrix} -1 & 3 & 4 \\ -2 & 8 & 0 \\ 2 & 5 & -1 \end{bmatrix}$ **124.** $\begin{bmatrix} -1 & 0 & 4 \\ -4 & 3 & 5 \\ 0 & 2 & -3 \end{bmatrix}$

125. ***Make a Decision*** To work an extended application analyzing the monthly profits of a clothing manufacturer over a period of 36 months, visit this textbook's *Companion Website.*

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8.4 The Binomial Theorem

Binomial Coefficients

Recall that a *binomial* is a polynomial that has two terms. In this section, you will study a formula that provides a quick method of raising a binomial to a power. To begin, look at the expansion of

$$(x + y)^n$$

for several values of n.

$$(x + y)^0 = 1$$

$$(x + y)^1 = x + y$$

$$(x + y)^2 = x^2 + 2xy + y^2$$

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

$$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

$$(x + y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$$

There are several observations you can make about these expansions.

1. In each expansion, there are $n + 1$ terms.
2. In each expansion, x and y have symmetric roles. The powers of x decrease by 1 in successive terms, whereas the powers of y increase by 1.
3. The sum of the powers of each term is n. For instance, in the expansion of

   $$(x + y)^5$$

   the sum of the powers of each term is 5.

   $$(x + y)^5 = x^5 + \overbrace{5x^4y^1}^{4+1=5} + \overbrace{10x^3y^2}^{3+2=5} + 10x^2y^3 + 5x^1y^4 + y^5$$

4. The coefficients increase and then decrease in a symmetric pattern.

The coefficients of a binomial expansion are called **binomial coefficients.** To find them, you can use the **Binomial Theorem.**

What you should learn

- Use the Binomial Theorem to calculate binomial coefficients.
- Use binomial coefficients to write binomial expansions.
- Use Pascal's Triangle to calculate binomial coefficients.

Why you should learn it

You can use binomial coefficients to model and solve real-life problems. For instance, in Exercise 118 on page 606, you will use binomial coefficients to write the expansion of a model that represents the amount of child support collected in the United States.

The Binomial Theorem (See the proof on page 634.)

In the expansion of $(x + y)^n$

$$(x + y)^n = x^n + nx^{n-1}y + \cdots + {}_nC_r x^{n-r}y^r + \cdots + nxy^{n-1} + y^n$$

the coefficient of $x^{n-r}y^r$ is

$${}_nC_r = \frac{n!}{(n-r)!r!}.$$

The symbol

$$\binom{n}{r}$$

is often used in place of ${}_nC_r$ to denote binomial coefficients.

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Example 1 Finding Binomial Coefficients

Find each binomial coefficient.

a. ${}_8C_2$ **b.** $\binom{10}{3}$

c. ${}_7C_0$ **d.** $\binom{8}{8}$

Solution

a. ${}_8C_2 = \dfrac{8!}{6! \cdot 2!} = \dfrac{(8 \cdot 7) \cdot \cancel{6!}}{\cancel{6!} \cdot 2!} = \dfrac{8 \cdot 7}{2 \cdot 1} = 28$

b. $\binom{10}{3} = \dfrac{10!}{7! \cdot 3!} = \dfrac{(10 \cdot 9 \cdot 8) \cdot \cancel{7!}}{\cancel{7!} \cdot 3!} = \dfrac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120$

c. ${}_7C_0 = \dfrac{\cancel{7!}}{\cancel{7!} \cdot 0!} = 1$

d. $\binom{8}{8} = \dfrac{\cancel{8!}}{0! \cdot \cancel{8!}} = 1$

CHECKPOINT Now try Exercise 5.

When $r \neq 0$ and $r \neq n$, as in parts (a) and (b) of Example 1, there is a simple pattern for evaluating binomial coefficients that works because there will always be factorial terms that divide out from the expression.

$${}_8C_2 = \frac{\overbrace{8 \cdot 7}^{\text{2 factors}}}{\underbrace{2 \cdot 1}_{\text{2 factorial}}} \quad \text{and} \quad \binom{10}{3} = \frac{\overbrace{10 \cdot 9 \cdot 8}^{\text{3 factors}}}{\underbrace{3 \cdot 2 \cdot 1}_{\text{3 factorial}}}$$

Technology Tip

Most graphing utilities are programmed to evaluate ${}_nC_r$. For instructions on how to use the ${}_nC_r$ feature, see Appendix A; for specific keystrokes, go to this textbook's *Companion Website.*

Example 2 Finding Binomial Coefficients

Find each binomial coefficient using the pattern shown above.

a. ${}_7C_3$ **b.** ${}_7C_4$ **c.** ${}_{12}C_1$ **d.** ${}_{12}C_{11}$

Solution

a. ${}_7C_3 = \dfrac{7 \cdot \cancel{6} \cdot 5}{\cancel{3} \cdot \cancel{2} \cdot 1} = 35$

b. ${}_7C_4 = \dfrac{7 \cdot \cancel{6} \cdot 5 \cdot \cancel{4}}{\cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1} = 35$

c. ${}_{12}C_1 = \dfrac{12}{1} = 12$

d. ${}_{12}C_{11} = \dfrac{12!}{1! \cdot 11!} = \dfrac{(12) \cdot \cancel{11!}}{1! \cdot \cancel{11!}} = \dfrac{12}{1} = 12$

CHECKPOINT Now try Exercise 7.

Explore the Concept

Find each pair of binomial coefficients.

a. ${}_7C_0$, ${}_7C_7$ **d.** ${}_7C_1$, ${}_7C_6$

b. ${}_8C_0$, ${}_8C_8$ **e.** ${}_8C_1$, ${}_8C_7$

c. ${}_{10}C_0$, ${}_{10}C_{10}$ **f.** ${}_{10}C_1$, ${}_{10}C_9$

What do you observe about the pairs in (a), (b), and (c)? What do you observe about the pairs in (d), (e), and (f)? Write two conjectures from your observations. Develop a convincing argument for your two conjectures.

It is not a coincidence that the results in parts (a) and (b) of Example 2 are the same and that the results in parts (c) and (d) are the same. In general, it is true that

$${}_nC_r = {}_nC_{n-r}.$$

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Binomial Expansions

As mentioned at the beginning of this section, when you write out the coefficients for a binomial that is raised to a power, you are **expanding a binomial.** The formulas for binomial coefficients give you an easy way to expand binomials, as demonstrated in the next four examples.

Example 3 Expanding a Binomial

Write the expansion of the expression $(x + 1)^3$.

Solution

The binomial coefficients are

$${}_3C_0 = 1,\ {}_3C_1 = 3,\ {}_3C_2 = 3,\ \text{and}\ {}_3C_3 = 1.$$

Therefore, the expansion is as follows.

$$\begin{aligned}(x + 1)^3 &= (1)x^3 + (3)x^2(1) + (3)x(1^2) + (1)(1^3)\\ &= x^3 + 3x^2 + 3x + 1\end{aligned}$$

CHECKPOINT Now try Exercise 23.

To expand binomials representing *differences*, rather than sums, you alternate signs, as shown in the next example.

Example 4 Expanding a Binomial

Write the expansion of the expression $(x - 1)^3$.

Solution

Expand using the binomial coefficient from Example 3.

$$\begin{aligned}(x - 1)^3 &= [x + (-1)]^3\\ &= (1)x^3 + (3)x^2(-1) + (3)x(-1)^2 + (1)(-1)^3\\ &= x^3 - 3x^2 + 3x - 1\end{aligned}$$

CHECKPOINT Now try Exercise 27.

Example 5 Expanding Binomial Expressions

Write the expansion of each expression.

a. $(2x - 3)^4$ **b.** $(x - 2y)^4$

Solution

The binomial coefficients are

$${}_4C_0 = 1,\ {}_4C_1 = 4,\ {}_4C_2 = 6,\ {}_4C_3 = 4,\ \text{and}\ {}_4C_4 = 1.$$

Therefore, the expansions are as follows.

a. $$\begin{aligned}(2x - 3)^4 &= (1)(2x)^4 - (4)(2x)^3(3) + (6)(2x)^2(3^2) - (4)(2x)(3^3) + (1)(3^4)\\ &= 16x^4 - 96x^3 + 216x^2 - 216x + 81\end{aligned}$$

b. $$\begin{aligned}(x - 2y)^4 &= (1)x^4 - (4)x^3(2y) + (6)x^2(2y)^2 - (4)x(2y)^3 + (1)(2y)^4\\ &= x^4 - 8x^3y + 24x^2y^2 - 32xy^3 + 16y^4\end{aligned}$$

CHECKPOINT Now try Exercise 35.

Technology Tip

You can use a graphing utility to check the expansion in Example 5(a) by graphing the original binomial expression and the expansion in the same viewing window. The graphs should coincide, as shown below.

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Example 6 Expanding a Binomial

Write the expansion of the expression

$$(x^2 + 4)^3.$$

Solution

Expand using the binomial coefficients from Example 3.

$$\begin{aligned}(x^2 + 4)^3 &= (1)(x^2)^3 + (3)(x^2)^2(4) + (3)x^2(4^2) + (1)(4^3)\\ &= x^6 + 12x^4 + 48x^2 + 64\end{aligned}$$

CHECKPOINT Now try Exercise 37.

Sometimes you will need to find a specific term in a binomial expansion. Instead of writing out the entire expansion, you can use the fact that, from the Binomial Theorem, the $(r + 1)$th term is

$${}_nC_r\, x^{n-r}y^r.$$

For instance, to find the third term of the expression in Example 6, you could use the formula above with $n = 3$ and $r = 2$ to obtain

$$\begin{aligned}{}_3C_2(x^2)^{3-2} \cdot 4^2 &= 3(x^2) \cdot 16\\ &= 48x^2.\end{aligned}$$

Example 7 Finding a Term or Coefficient in a Binomial Expansion

a. Find the sixth term of $(a + 2b)^8$.

b. Find the coefficient of the term a^6b^5 in the expansion of $(2a - 5b)^{11}$.

Solution

a. Because the formula is for the $(r + 1)$th term, r is one less than the number of the term you need. So, to find the sixth term in this binomial expansion, use $r = 5$, $n = 8$, $x = a$ and $y = 2b$.

$$\begin{aligned}{}_nC_r x^{n-r}y^r &= {}_8C_5a^{8-5}(2b)^5\\ &= 56 \cdot a^3 \cdot (2b)^5\\ &= 56(2^5)a^3b^5\\ &= 1792a^3b^5\end{aligned}$$

b. Note that

$$(2a - 5b)^{11} = [2a + (-5b)]^{11}.$$

So,

$$n = 11, \quad r = 5, \quad x = 2a, \quad \text{and} \quad y = -5b.$$

Substitute these values to obtain

$$\begin{aligned}{}_nC_r\, x^{n-r}\, y^r &= {}_{11}C_5(2a)^6(-5b)^5\\ &= (462)(64a^6)(-3125b^5)\\ &= -92{,}400{,}000a^6b^5.\end{aligned}$$

So, the coefficient is $-92{,}400{,}000$.

CHECKPOINT Now try Exercises 55 and 67.

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Pascal's Triangle

There is a convenient way to remember the pattern for binomial coefficients. By arranging the coefficients in a triangular pattern, you obtain the following array, which is called **Pascal's Triangle.** This triangle is named after the famous French mathematician Blaise Pascal (1623–1662).

$$
\begin{array}{ccccccccccccccc}
 & & & & & & 1 & & & & & & & & \\
 & & & & & 1 & & 1 & & & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
$$

$4 + 6 = 10$

$15 + 6 = 21$

The first and last number in each row of Pascal's Triangle is 1. Every other number in each row is formed by adding the two numbers immediately above the number. Pascal noticed that the numbers in this triangle are precisely the same numbers as the coefficients of binomial expansions, as follows.

$$(x + y)^0 = 1 \qquad \text{0th row}$$

$$(x + y)^1 = 1x + 1y \qquad \text{1st row}$$

$$(x + y)^2 = 1x^2 + 2xy + 1y^2 \qquad \text{2nd row}$$

$$(x + y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3 \qquad \text{3rd row}$$

$$(x + y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4 \qquad \vdots$$

$$(x + y)^5 = 1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5$$

$$(x + y)^6 = 1x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + 1y^6$$

$$(x + y)^7 = 1x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + 1y^7$$

The top row of Pascal's Triangle is called the *zeroth row* because it corresponds to the binomial expansion

$$(x + y)^0 = 1.$$

Similarly, the next row is called the *first row* because it corresponds to the binomial expansion

$$(x + y)^1 = 1(x) + 1(y).$$

In general, the *nth row* of Pascal's Triangle gives the coefficients of $(x + y)^n$.

Explore the Concept

Complete the table and describe the result.

n	r	${}_nC_r$	${}_nC_{n-r}$
9	5		
7	1		
12	4		
6	0		
10	7		

What characteristics of Pascal's Triangle are illustrated by this table?

Example 8 Using Pascal's Triangle

Use the seventh row of Pascal's Triangle to find the binomial coefficients.

$${}_8C_0 \quad {}_8C_1 \quad {}_8C_2 \quad {}_8C_3 \quad {}_8C_4 \quad {}_8C_5 \quad {}_8C_6 \quad {}_8C_7 \quad {}_8C_8$$

Solution

$$
\begin{array}{ccccccccccccccccc}
 & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 & \longleftarrow \text{Seventh row} \\
1 & & 8 & & 28 & & 56 & & 70 & & 56 & & 28 & & 8 & & 1 \\
{}_8C_0 & & {}_8C_1 & & {}_8C_2 & & {}_8C_3 & & {}_8C_4 & & {}_8C_5 & & {}_8C_6 & & {}_8C_7 & & {}_8C_8
\end{array}
$$

CHECKPOINT Now try Exercise 71.

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8.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1 and 2, fill in the blanks.

1. The notation used to denote a binomial coefficient is _____ or _____ .
2. When you write out the coefficients for a binomial that is raised to a power, you are _____ a _____ .
3. List two ways to find binomial coefficients.
4. In the expression of $(x + y)^3$, what is the sum of the powers of the third term?

Procedures and Problem Solving

Finding Binomial Coefficients **In Exercises 5–16, find the binomial coefficient.**

5. ${}_7C_5$
6. ${}_8C_6$
7. ${}_{20}C_{15}$
8. ${}_{12}C_3$
9. ${}_{14}C_1$
10. ${}_{18}C_{17}$
11. $\binom{12}{0}$
12. $\binom{20}{20}$
13. $\binom{10}{4}$
14. $\binom{10}{6}$
15. $\binom{100}{98}$
16. $\binom{100}{2}$

Finding Binomial Coefficients **In Exercises 17–22, use a graphing utility to find ${}_nC_r$.**

17. ${}_{41}C_{36}$
18. ${}_{34}C_4$
19. ${}_{50}C_{48}$
20. ${}_{500}C_{498}$
21. ${}_{250}C_2$
22. ${}_{1000}C_2$

Expanding a Binomial **In Exercises 23–54, use the Binomial Theorem to expand and simplify the expression.**

23. $(x + 2)^4$
24. $(x + 1)^6$
25. $(a + 3)^3$
26. $(a + 2)^4$
27. $(y - 4)^3$
28. $(y - 2)^5$
29. $(x + y)^5$
30. $(x + y)^6$
31. $(r + 3s)^6$
32. $(4x + 3y)^4$
33. $(x - y)^5$
34. $(2x - y)^5$
35. $(1 - 4x)^3$
36. $(5 - 2y)^3$
37. $(x^2 + 2)^4$
38. $(3 - y^2)^3$
39. $(x^2 - 5)^5$
40. $(y^2 + 1)^6$
41. $(x^2 + y^2)^4$
42. $(x^2 + y^2)^6$
43. $(x^3 - y)^6$
44. $(2x^3 - y)^5$
45. $\left(\frac{1}{x} + y\right)^5$
46. $\left(\frac{1}{x} + 2y\right)^6$
47. $\left(\frac{2}{x} - y\right)^4$
48. $\left(\frac{2}{x} - 3y\right)^5$
49. $(4x - 1)^3 - 2(4x - 1)^4$
50. $(x + 3)^5 - 4(x + 3)^4$
51. $2(x - 3)^4 + 5(x - 3)^2$
52. $3(x + 1)^5 + 4(x + 1)^3$
53. $-3(x - 2)^3 - 4(x + 1)^6$
54. $5(x + 2)^5 - 2(x - 1)^2$

Finding a Term in a Binomial Expansion **In Exercises 55–62, find the specified *n*th term in the expansion of the binomial.**

55. $(x + 8)^{10}$, $n = 4$
56. $(x - 5)^6$, $n = 7$
57. $(x - 6y)^5$, $n = 3$
58. $(x - 10z)^7$, $n = 4$
59. $(4x + 3y)^9$, $n = 8$
60. $(5a + 6b)^5$, $n = 5$
61. $(10x - 3y)^{12}$, $n = 10$
62. $(7x + 2y)^{15}$, $n = 7$

Finding a Coefficient in a Binomial Expansion **In Exercises 63–70, find the coefficient *a* of the given term in the expansion of the binomial.**

	Binomial	*Term*
63.	$(x + 3)^{12}$	ax^5
64.	$(x + 4)^{12}$	ax^4
65.	$(x - 2y)^{10}$	ax^8y^2
66.	$(4x - y)^{10}$	ax^2y^8
67.	$(3x - 2y)^9$	ax^6y^3
68.	$(2x - 3y)^8$	ax^4y^4
69.	$(x^2 + y)^{10}$	ax^8y^6
70.	$(z^2 - 1)^{12}$	az^6

Using Pascal's Triangle **In Exercises 71–74, use Pascal's Triangle to find the binomial coefficient.**

71. ${}_7C_4$
72. ${}_6C_3$
73. ${}_6C_5$
74. ${}_5C_2$

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Using Pascal's Triangle **In Exercises 75–80, expand the binomial by using Pascal's Triangle to determine the coefficients.**

75. $(3t - 2v)^4$ **76.** $(5v - 2z)^4$

77. $(2x - 3y)^5$ **78.** $(5y + 2)^5$

79. $(x + 2y)^5$ **80.** $(2v + 3)^6$

Using the Binomial Theorem **In Exercises 81–84, use the Binomial Theorem to expand and simplify the expression.**

81. $\left(\sqrt{x} + 5\right)^3$ **82.** $\left(2\sqrt{t} - 1\right)^3$

83. $(x^{2/3} - y^{1/3})^3$ **84.** $(u^{3/5} + v^{1/5})^5$

Expanding an Expression **In Exercises 85–90, expand the expression in the difference quotient and simplify.**

$$\frac{f(x+h) - f(x)}{h}, \quad h \neq 0$$

85. $f(x) = x^3$ **86.** $f(x) = x^4$

87. $f(x) = x^6$ **88.** $f(x) = x^8$

89. $f(x) = \sqrt{x}$ **90.** $f(x) = \dfrac{1}{x}$

Expanding a Complex Number **In Exercises 91–104, use the Binomial Theorem to expand the complex number. Simplify your result. (Remember that $i = \sqrt{-1}$.)**

91. $(1 + i)^4$ **92.** $(4 - i)^5$

93. $(4 + i)^4$ **94.** $(2 - i)^5$

95. $(2 - 3i)^6$ **96.** $(3 - 2i)^6$

97. $\left(5 + \sqrt{-16}\right)^3$ **98.** $\left(5 + \sqrt{-9}\right)^3$

99. $\left(4 + \sqrt{3}i\right)^4$ **100.** $\left(5 - \sqrt{3}i\right)^4$

101. $\left(-\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i\right)^3$ **102.** $\left(\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i\right)^3$

103. $\left(\dfrac{1}{4} - \dfrac{\sqrt{3}}{4}i\right)^3$ **104.** $\left(\dfrac{1}{3} - \dfrac{\sqrt{3}}{3}i\right)^3$

Using the Binomial Theorem to Approximate **In Exercises 105–108, use the Binomial Theorem to approximate the quantity accurate to three decimal places. For example, in Exercise 105, use the expansion**

$$(1.02)^8 = (1 + 0.02)^8$$
$$= 1 + 8(0.02) + 28(0.02)^2 + \cdots.$$

105. $(1.02)^8$ **106.** $(2.005)^{10}$

107. $(2.99)^{12}$ **108.** $(1.98)^9$

Using the Binomial Theorem **In Exercises 109 and 110, use a graphing utility to graph f and g in the same viewing window. What is the relationship between the two graphs? Use the Binomial Theorem to write the polynomial function g in standard form.**

109. $f(x) = x^3 - 4x, \quad g(x) = f(x + 4)$

110. $f(x) = -x^4 + 4x^2 - 1, \quad g(x) = f(x - 3)$

Using a Graphing Utility **In Exercises 111 and 112, use a graphing utility to graph the functions in the given order and in the same viewing window. Compare the graphs. Which two functions have identical graphs, and why?**

111. (a) $f(x) = (1 - x)^3$

(b) $g(x) = 1 - 3x$

(c) $h(x) = 1 - 3x + 3x^2$

(d) $p(x) = 1 - 3x + 3x^2 - x^3$

112. (a) $f(x) = \left(1 - \frac{1}{2}x\right)^4$

(b) $g(x) = 1 - 2x + \frac{3}{2}x^2$

(c) $h(x) = 1 - 2x + \frac{3}{2}x^2 - \frac{1}{2}x^3$

(d) $p(x) = 1 - 2x + \frac{3}{2}x^2 - \frac{1}{2}x^3 + \frac{1}{16}x^4$

Finding a Probability **In Exercises 113–116, consider n independent trials of an experiment in which each trial has two possible outcomes, success or failure. The probability of a success on each trial is p and the probability of a failure is $q = 1 - p$. In this context, the term ${}_nC_k\, p^k q^{n-k}$ in the expansion of $(p + q)^n$ gives the probability of k successes in the n trials of the experiment.**

113. A fair coin is tossed seven times. To find the probability of obtaining four heads, evaluate the term

$${}_7C_4\left(\tfrac{1}{2}\right)^4\left(\tfrac{1}{2}\right)^3$$

in the expansion of $\left(\frac{1}{2} + \frac{1}{2}\right)^7$.

114. The probability of a baseball player getting a hit during any given time at bat is $\frac{1}{4}$. To find the probability that the player gets three hits during the next 10 times at bat, evaluate the term

$${}_{10}C_3\left(\tfrac{1}{4}\right)^3\left(\tfrac{3}{4}\right)^7$$

in the expansion of $\left(\frac{1}{4} + \frac{3}{4}\right)^{10}$.

115. The probability of a sales representative making a sale to any one customer is $\frac{1}{3}$. The sales representative makes eight contacts a day. To find the probability of making four sales, evaluate the term

$${}_8C_4\left(\tfrac{1}{3}\right)^4\left(\tfrac{2}{3}\right)^4$$

in the expansion of $\left(\frac{1}{3} + \frac{2}{3}\right)^8$.

116. To find the probability that the sales representative in Exercise 115 makes four sales when the probability of a sale to any one customer is $\frac{1}{2}$, evaluate the term

$${}_8C_4\left(\tfrac{1}{2}\right)^4\left(\tfrac{1}{2}\right)^4$$

in the expansion of $\left(\frac{1}{2} + \frac{1}{2}\right)^8$.

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117. MODELING DATA

The per capita consumption of bottled water f (in gallons) in the United States from 1990 through 2008 can be approximated by the model

$$f(t) = 0.044t^2 + 0.44t + 8.3, \quad 0 \le t \le 18$$

where t represents the year, with $t = 0$ corresponding to 1990 (see figure). (Source: Economic Research Service, U.S. Department of Agriculture)

(a) Adjust the model so that $t = 0$ corresponds to 2000 rather than 1990. To do this, shift the graph of f 10 units to the left and obtain $g(t) = f(t + 10)$. Write $g(t)$ in standard form.

(b) Use a graphing utility to graph f and g in the same viewing window.

118. ***Why you should learn it*** *(p. 599)* The amounts f (in millions of dollars) of child support collected in the United States from 1990 through 2007 can be approximated by the model

$$f(t) = 0.26t^2 + 1162.5t + 5531,$$
$$0 \le t \le 17$$

where t represents the year, with $t = 0$ corresponding to 1990 (see figure). (Source: U.S. Department of Health and Human Services)

(a) Adjust the model so that $t = 0$ corresponds to 2000 rather than 1990. To do this, shift the graph of f 10 units to the left and obtain $g(t) = f(t + 10)$. Write $g(t)$ in standard form.

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(b) Use a graphing utility to graph f and g in the same viewing window.

Conclusions

True or False? In Exercises 119–122, determine whether the statement is true or false. Justify your answer.

119. The Binomial Theorem can be used to produce each row of Pascal's Triangle.

120. A binomial that represents a difference cannot always be accurately expanded using the Binomial Theorem.

121. One of the terms in the expansion of $(x - 2y)^{12}$ is $7920x^4y^8$.

122. The x^{10}-term and the x^{14}-term in the expansion of $(x^2 + 3)^{12}$ have identical coefficients.

123. Think About It How many terms are in the expansion of $(x + y)^n$?

124. Writing In your own words, explain how to form the rows of Pascal's Triangle. Then form rows 8–10 of Pascal's Triangle.

125. Error Analysis You are a math instructor and receive the following solutions from one of your students on a quiz. Find the error(s) in each solution and write a short paragraph discussing ways that your student could avoid the error(s) in the future.

(a) Find the second term in the expansion of $(2x - 3y)^5$.

$\cancel{5(2x)^4(3y)^2 = 720x^4y^2}$

(b) Find the fourth term in the expansion of $\left(\frac{1}{2}x + 7y\right)^6$.

$\cancel{{}_6C_4\left(\frac{1}{2}x\right)^2(7y)^4 = 9003.75x^2y^4}$

126. CAPSTONE How do the expansions of $(x + y)^n$ and $(x - y)^n$ differ? Support your explanation with an example.

Proof In Exercises 127–130, prove the property for all integers r and n, where $0 \le r \le n$.

127. ${}_nC_r = {}_nC_{n-r}$

128. ${}_nC_0 - {}_nC_1 + {}_nC_2 - \cdots \pm {}_nC_n = 0$

129. ${}_{n+1}C_r = {}_nC_r + {}_nC_{r-1}$

130. The sum of the numbers in the nth row of Pascal's Triangle is 2^n.

Cumulative Mixed Review

Finding the Inverse of a Matrix In Exercises 131 and 132, find the inverse of the matrix.

131. $\begin{bmatrix} -1 & -4 \\ 1 & 2 \end{bmatrix}$

132. $\begin{bmatrix} 11 & -12 \\ 2 & -2 \end{bmatrix}$

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8.5 Counting Principles

Simple Counting Problems

The last two sections of this chapter present a brief introduction to some of the basic counting principles and their application to probability. In the next section, you will see that much of probability has to do with counting the number of ways an event can occur. The following two examples describe simple counting problems.

What you should learn

- Solve simple counting problems.
- Use the Fundamental Counting Principle to solve more complicated counting problems.
- Use permutations to solve counting problems.
- Use combinations to solve counting problems.

Why you should learn it

You can use counting principles to solve counting problems that occur in real life. For instance, in Exercise 65 on page 615, you are asked to use counting principles to determine in how many ways a player can select six numbers in a Powerball lottery.

Example 1 Selecting Pairs of Numbers at Random

Eight pieces of paper are numbered from 1 to 8 and placed in a box. One piece of paper is drawn from the box, its number is written down, and the piece of paper is *returned to the box*. Then, a second piece of paper is drawn from the box, and its number is written down. Finally, the two numbers are added together. In how many different ways can a sum of 12 be obtained?

Solution

To solve this problem, count the number of different ways that a sum of 12 can be obtained using two numbers from 1 to 8.

First number	4	5	6	7	8
Second number	8	7	6	5	4

From this list, you can see that a sum of 12 can occur in five different ways.

CHECKPOINT Now try Exercise 9.

Example 2 Selecting Pairs of Numbers at Random

Eight pieces of paper are numbered from 1 to 8 and placed in a box. Two pieces of paper are drawn from the box *at the same time*, and the numbers on the pieces of paper are written down and totaled. In how many different ways can a sum of 12 be obtained?

Solution

To solve this problem, count the number of different ways that a sum of 12 can be obtained using two *different* numbers from 1 to 8.

First number	4	5	7	8
Second number	8	7	5	4

So, a sum of 12 can be obtained in four different ways.

CHECKPOINT Now try Exercise 11.

The difference between the counting problems in Examples 1 and 2 can be described by saying that the random selection in Example 1 occurs **with replacement,** whereas the random selection in Example 2 occurs **without replacement,** which eliminates the possibility of choosing two 6's.

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The Fundamental Counting Principle

Examples 1 and 2 describe simple counting problems in which you can *list* each possible way that an event can occur. When it is possible, this is always the best way to solve a counting problem. However, some events can occur in so many different ways that it is not feasible to write out the entire list. In such cases, you must rely on formulas and counting principles. The most important of these is the **Fundamental Counting Principle.**

Fundamental Counting Principle

Let E_1 and E_2 be two events. The first event E_1 can occur in m_1 different ways. After E_1 has occurred, E_2 can occur in m_2 different ways. The number of ways that the two events can occur is $m_1 \cdot m_2$.

The Fundamental Counting Principle can be extended to three or more events. For instance, the number of ways that three events E_1, E_2, and E_3 can occur is

$$m_1 \cdot m_2 \cdot m_3.$$

Example 3 Using the Fundamental Counting Principle

How many different pairs of letters from the English alphabet are possible?

Solution

There are two events in this situation. The first event is the choice of the first letter, and the second event is the choice of the second letter. Because the English alphabet contains 26 letters, it follows that the number of two-letter pairs is

$$26 \cdot 26 = 676.$$

CHECKPOINT Now try Exercise 13.

Example 4 Using the Fundamental Counting Principle

Telephone numbers in the United States currently have 10 digits. The first three are the *area code* and the next seven are the *local telephone number*. How many different telephone numbers are possible within each area code? (Note that at this time, a local telephone number cannot begin with 0 or 1.)

Solution

Because the first digit cannot be 0 or 1, there are only eight choices for the first digit. For each of the other six digits, there are 10 choices.

So, the number of local telephone numbers that are possible within each area code is

$$8 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 8{,}000{,}000.$$

CHECKPOINT Now try Exercise 19.

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Permutations

One important application of the Fundamental Counting Principle is in determining the number of ways that n elements can be arranged (in order). An ordering of n elements is called a **permutation** of the elements.

Definition of Permutation

A **permutation** of n different elements is an ordering of the elements such that one element is first, one is second, one is third, and so on.

Example 5 Finding the Number of Permutations of n Elements

How many permutations of the following letters are possible?

A B C D E F

Solution

Consider the following reasoning.

First position:	Any of the *six* letters
Second position:	Any of the remaining *five* letters
Third position:	Any of the remaining *four* letters
Fourth position:	Any of the remaining *three* letters
Fifth position:	Either of the remaining *two* letters
Sixth position:	The *one* remaining letter

So, the numbers of choices for the six positions are as follows.

The total number of permutations of the six letters is

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720.$$

CHECKPOINT Now try Exercise 37.

Number of Permutations of n Elements

The number of permutations of n elements is given by

$$n \cdot (n - 1) \cdot \cdot \cdot 4 \cdot 3 \cdot 2 \cdot 1 = n!.$$

In other words, there are $n!$ different ways that n elements can be ordered.

It is useful, on occasion, to order a *subset* of a collection of elements rather than the entire collection. For example, you might want to choose and order r elements out of a collection of n elements. Such an ordering is called a **permutation of n elements taken r at a time.**

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Example 6 Counting Horse Race Finishes

Eight horses are running in a race. In how many different ways can these horses come in first, second, and third? (Assume that there are no ties.)

Jockey

Solution

Here are the different possibilities.

Win (first position):	*Eight* choices
Place (second position):	*Seven* choices
Show (third position):	*Six* choices

The numbers of choices for the three positions are as follows.

$$\overbrace{\underbrace{\square}_{8}\ \underbrace{\square}_{7}\ \underbrace{\square}_{6}}^{\text{Different orders of horses}}$$

So, using the Fundamental Counting Principle, you can determine that there are

$$8 \cdot 7 \cdot 6 = 336$$

different ways in which the eight horses can come in first, second, and third.

CHECKPOINT Now try Exercise 41.

Permutations of n Elements Taken r at a Time

The number of **permutations of n elements taken r at a time** is given by

$$\begin{aligned} {}_nP_r &= \frac{n!}{(n-r)!} \\ &= n(n-1)(n-2)\cdot\cdot\cdot(n-r+1). \end{aligned}$$

Using this formula, you can rework Example 6 to find that the number of permutations of eight horses taken three at a time is

$$\begin{aligned} {}_8P_3 &= \frac{8!}{(8-3)!} \\ &= \frac{8!}{5!} \\ &= \frac{8 \cdot 7 \cdot 6 \cdot \cancel{5!}}{\cancel{5!}} \\ &= 336 \end{aligned}$$

which is the same answer obtained in the example.

Technology Tip

Most graphing utilities are programmed to evaluate ${}_nP_r$. Figure 8.12 shows how one graphing utility evaluates the permutation ${}_8P_3$. For instructions on how to use the ${}_nP_r$ feature, see Appendix A; for specific keystrokes, go to this textbook's *Companion Website*.

Figure 8.12

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Remember that for permutations, order is important. So, to find the possible permutations of the letters

A, B, C, and D

taken three at a time, the permutations (A, B, D) and (B, A, D) would be counted as different because the *order* of the elements is different.

Consider, however, the possible permutations of the letters A, A, B, and C. The total number of permutations of the four letters would be

$${}_4P_4 = 4!.$$

However, not all of these arrangements would be *distinguishable* because there are two A's in the list. To find the number of distinguishable permutations, you can use the following formula.

Distinguishable Permutations

Suppose a set of n objects has n_1 of one kind of object, n_2 of a second kind, n_3 of a third kind, and so on, with

$$n = n_1 + n_2 + n_3 + \cdot \cdot \cdot + n_k.$$

The number of **distinguishable permutations** of the n objects is given by

$$\frac{n!}{n_1! \cdot n_2! \cdot n_3! \cdot \cdot \cdot \cdot \cdot n_k!}.$$

Example 7 Distinguishable Permutations

In how many distinguishable ways can the letters in BANANA be written?

Solution

This word has six letters, of which three are A's, two are N's, and one is a B. So, the number of distinguishable ways in which the letters can be written is

$$\frac{6!}{3! \cdot 2! \cdot 1!} = \frac{6 \cdot 5 \cdot 4 \cdot \cancel{3!}}{\cancel{3!} \cdot 2!} = 60.$$

The 60 different distinguishable permutations are as follows.

AAABNN	AAANBN	AAANNB	AABANN
AABNAN	AABNNA	AANABN	AANANB
AANBAN	AANBNA	AANNAB	AANNBA
ABAANN	ABANAN	ABANNA	ABNAAN
ABNANA	ABNNAA	ANAABN	ANAANB
ANABAN	ANABNA	ANANAB	ANANBA
ANBAAN	ANBANA	ANBNAA	ANNAAB
ANNABA	ANNBAA	BAAANN	BAANAN
BAANNA	BANAAN	BANANA	BANNAA
BNAAAN	BNAANA	BNANAA	BNNAAA
NAAABN	NAAANB	NAABAN	NAABNA
NAANAB	NAANBA	NABAAN	NABANA
NABNAA	NANAAB	NANABA	NANBAA
NBAAAN	NBAANA	NBANAA	NBNAAA
NNAAAB	NNAABA	NNABAA	NNBAAA

CHECKPOINT Now try Exercise 49.

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Combinations

When you count the number of possible permutations of a set of elements, order is important. As a final topic in this section, you will look at a method for selecting subsets of a larger set in which order *is not* important. Such subsets are called **combinations of *n* elements taken *r* at a time.** For instance, the combinations

$$\{A, B, C\} \quad \text{and} \quad \{B, A, C\}$$

are equivalent because both sets contain the same three elements, and the order in which the elements are listed is not important. So, you would count only one of the two sets. A common example of a combination is a card game in which the player is free to reorder the cards after they have been dealt.

Combinations of *n* Elements Taken *r* at a Time

The number of **combinations of *n* elements taken *r* at a time** is given by

$${}_nC_r = \frac{n!}{(n-r)!r!}.$$

Note that the formula for ${}_nC_r$ is the same one given for binomial coefficients.

Example 8 Combinations of *n* Elements Taken *r* at a Time

a. In how many different ways can three letters be chosen from the letters A, B, C, D, and E? (The order of the three letters is not important.)

b. A standard poker hand consists of five cards dealt from a deck of 52. How many different poker hands are possible? (After the cards are dealt, the player may reorder them, so order is not important.)

Solution

a. You can find the number of different ways in which the letters can be chosen by using the formula for the number of combinations of five elements taken three at a time, as follows.

$${}_5C_3 = \frac{5!}{2!3!} = \frac{5 \cdot \overset{2}{\cancel{4}} \cdot \cancel{3!}}{\cancel{2} \cdot 1 \cdot \cancel{3!}} = 10$$

b. You can find the number of different poker hands by using the formula for the number of combinations of 52 elements taken five at a time, as follows.

$${}_{52}C_5 = \frac{52!}{47!5!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot \cancel{47!}}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot \cancel{47!}} = 2{,}598{,}960$$

CHECKPOINT Now try Exercise 61.

When solving problems involving counting principles, you need to be able to distinguish among the various counting principles in order to determine which is necessary to solve the problem correctly. To do this, ask yourself the following questions.

1. Is the order of the elements important? *Permutation*

2. Are the chosen elements a subset of a larger set in which order is not important? *Combination*

3. Are there two or more separate events? *Fundamental Counting Principle*

Study Tip

In Example 8(a), you could also make a list of the different combinations of three letters chosen from five letters.

{A, B, C}	{A, B, D}
{A, B, E}	{A, C, D}
{A, C, E}	{A, D, E}
{B, C, D}	{B, C, E}
{B, D, E}	{C, D, E}

From this list, you can conclude that there are 10 different ways in which three letters can be chosen from five letters, which is the same answer obtained in Example 8(a).

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8.5 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1 and 2, fill in the blank.

1. The _____ states that if there are m_1 ways for one event to occur and m_2 ways for a second event to occur, then there are $m_1 \cdot m_2$ ways for both events to occur.
2. The number of _____ of n objects is given by $\dfrac{n!}{n_1! \cdot n_2! \cdot n_3! \cdot \cdot \cdot \cdot \cdot n_k!}$.
3. Is the ordering of n elements called a *permutation* or a *combination* of the elements?
4. What do n and r represent in the formula ${}_nP_r = \dfrac{n!}{(n-r)!}$?

Procedures and Problem Solving

Selecting Numbers at Random **In Exercises 5–12, determine the number of ways in which a computer can randomly generate one or more such integers, or pairs of integers, from 1 through 15.**

5. An odd integer
6. An even integer
7. A prime integer
8. An integer that is greater than 6
9. A pair of integers whose sum is 20
10. An integer that is divisible by 4
11. A pair of distinct integers whose sum is 20
12. An integer that is divisible by 3
13. **Audio Engineering** A customer can choose one of four amplifiers, one of six compact disc players, and one of five speaker models for an entertainment system. Determine the number of possible system configurations.
14. **Education** A college student is preparing a course schedule for the next semester. The student must select one of two mathematics courses, one of three science courses, and one of five courses from the social sciences and humanities. How many schedules are possible?
15. **Education** In how many ways can a 10-question true-false exam be answered? (Assume that no questions are omitted.)
16. **Physiology** In a physiology class, a student must dissect three different specimens. The student can select one of nine earthworms, one of four frogs, and one of seven fetal pigs. In how many ways can the student select the specimens?
17. **Fundamental Counting Principle** How many three-digit numbers can be formed under each condition?
    (a) The leading digit cannot be 0.
    (b) The leading digit cannot be 0 and no repetition of digits is allowed.
    (c) The leading digit cannot be 0 and the number must be a multple of 5.
18. **Fundamental Counting Principle** How many four-digit numbers can be formed under each condition?
    (a) The leading digit cannot be 0.
    (b) The leading digit cannot be 0 and the number must be less than 5000.
    (c) The leading digit cannot be 0 and the number must be even.
19. **Telecommunication** In 2010, the state of Nevada had two area codes. Using the information about telephone numbers given in Example 4, how many telephone numbers could Nevada's phone system have accommodated?
20. **Fundamental Counting Principle** In Pennsylvania, each standard automobile license plate number consists of three letters followed by a four-digit number. How many distinct Pennsylvania license plate numbers can be formed?
21. **Radio Broadcasting** Typically radio stations are identified by four "call letters." Radio stations east of the Mississippi River have call letters that start with the letter W and radio stations west of the Mississippi River have call letters that start with the letter K.
    (a) Find the number of different sets of radio station call letters that are possible in the United States.
    (b) Find the number of different sets of radio station call letters that are possible when the call letters must include a Q.
22. **Banking** ATM personal identification number (PIN) codes typically consist of four-digit sequences of numbers.
    (a) Find the total number of ATM codes possible.
    (b) Find the total number of ATM codes possible when the first digit is not 0.
23. **Operations Research** In 1963, the United States Postal Service launched the Zoning Improvement Plan (ZIP) Code to streamline the mail-delivery system. A ZIP code consists of a five-digit sequence of numbers.

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(a) Find the number of ZIP codes consisting of five digits.

(b) Find the number of ZIP codes consisting of five digits when the first digit is 1 or 2.

24. **Operations Research** In 1983, in order to identify small geographic segments, such as city blocks or a single building, within a delivery code, the post office began to use an expanded ZIP code called ZIP+4, which is composed of the original five-digit code plus a four-digit add-on code.

(a) Find the number of ZIP codes consisting of five digits followed by the four additional digits.

(b) Find the number of ZIP codes consisting of five digits followed by the four additional digits when the first number of the five-digit code is 1 or 2.

25. **Finding Permutations** Three couples have reserved seats in a row for a concert. In how many different ways can they be seated when (a) there are no seating restrictions? (b) each couple sits together?

26. **Finding Permutations** In how many orders can five girls and three boys walk through a doorway single file when (a) there are no restrictions? (b) the girls walk through before the boys?

Finding a Permutation **In Exercises 27–32, evaluate ${}_nP_r$ using the formula from this section.**

27. ${}_4P_4$ **28.** ${}_5P_5$

29. ${}_8P_3$ **30.** ${}_{20}P_2$

31. ${}_5P_4$ **32.** ${}_7P_4$

Finding a Permutation Using a Graphing Utility **In Exercises 33–36, evaluate ${}_nP_r$ using a graphing utility.**

33. ${}_{20}P_6$ **34.** ${}_{10}P_8$

35. ${}_{120}P_4$ **36.** ${}_{100}P_3$

37. **Photography** In how many ways can five children posing for a photograph line up in a row?

38. **Transportation Design** In how many ways can six people sit in a six-passenger minivan?

39. **Politics** The nine justices of the U.S. Supreme Court pose for a photograph while standing in a straight line, as opposed to the typical pose of two rows. How many different orders of the justices are possible for this photograph?

40. **Manufacturing** Four processes are involved in assembling a product, and they can be performed in any order. The management wants to test each order to determine which is the least time-consuming. How many different orders will have to be tested?

41. **Politics** From a pool of 12 candidates, the offices of president, vice-president, secretary, and treasurer will be filled. In how many ways can the offices be filled?

42. **Physical Education** How many different batting orders can a baseball coach create from a team of 15 players when there are nine positions to fill?

43. **Combinatorics** A school locker has a dial lock on which there are 37 numbers from 0 to 36. Find the total number of possible lock combinations when the lock requires a three-digit sequence of left-right-left and the numbers can be repeated.

44. **Athletics** Eight sprinters have qualified for the finals in the 100-meter dash at the NCAA national track meet. How many different orders of the top three finishes are possible? (Assume there are no ties.)

Finding Permutations **In Exercises 45 and 46, use the letters A, B, C, and D.**

45. Write all permutations of the letters.

46. Write all permutations of the letters when the letters B and C must remain between the letters A and D.

Finding Distinguishable Permutations **In Exercises 47–50, find the number of distinguishable permutations of the group of letters.**

47. A, A, G, E, E, E, M **48.** B, B, B, T, T, T, T, T

49. A, L, G, E, B, R, A **50.** M, I, S, S, I, S, S, I, P, P, I

Finding a Combination **In Exercises 51–56, evaluate ${}_nC_r$ using the formula from this section.**

51. ${}_5C_2$ **52.** ${}_6C_3$

53. ${}_4C_1$ **54.** ${}_5C_1$

55. ${}_{25}C_0$ **56.** ${}_{20}C_0$

Finding a Combination Using a Graphing Utility **In Exercises 57–60, evaluate ${}_nC_r$ using a graphing utility.**

57. ${}_{20}C_5$ **58.** ${}_{10}C_7$

59. ${}_{42}C_5$ **60.** ${}_{50}C_6$

Finding a Combination **In Exercises 61 and 62, use the letters A, B, C, D, E, and F.**

61. Write all possible selections of two letters that can be formed from the letters. (The order of the two letters is not important.)

62. Write all possible selections of three letters that can be formed from the letters. (The order of the three letters is not important.)

63. **Politics** As of March 2010, the U.S. Senate Committee on Indian Affairs had 15 members. Party affiliation is not a factor in selection. How many different committees are possible from the 100 U.S. senators?

64. **Education** You can answer any 12 questions from a total of 14 questions on an exam. In how many different ways can you select the questions?

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65. *Why you should learn it* *(p. 607)* Powerball is played with 59 white balls, numbered 1 through 59, and 39 red balls, numbered 1 through 39. Five white balls and one red ball, the Powerball, are drawn. In how many ways can a player select the six numbers?

66. **Geometry** Three points that are not collinear determine three lines. How many lines are determined by nine points, no three of which are collinear?

67. **Quality Engineering** A shipment of 30 flat screen televisions contains three defective units. In how many ways can a vending company purchase four of these units and receive (a) all good units, (b) two good units, and (c) at least two good units?

68. **Game Theory** You are dealt five cards from an ordinary deck of 52 playing cards. In how many ways can you get a full house? (A full house consists of three of one kind and two of another. For example, 8-8-8-5-5 and K-K-K-10-10 are full houses.)

69. **Game Theory** Five cards are chosen from a standard deck of 52 cards. How many five-card combinations contain two jacks and three aces?

70. **Law** A law office interviews paralegals for 10 openings. There are 13 paralegals with two years of experience and 20 paralegals with one year of experience. How many combinations of seven paralegals with two years of experience and three paralegals with one year of experience are possible?

71. **Administration** A six-member research committee is to be formed having one administrator, three faculty members, and two students. There are seven administrators, 12 faculty members, and 25 students in contention for the committee. How many six-member committees are possible?

72. **Psychology** The number of possible interpersonal relationships increases dramatically as the size of a group increases. Determine the numbers of different two-person relationships that are possible in groups of people of sizes (a) 3, (b) 8, (c) 12, and (d) 20.

Geometry **In Exercises 73–76, find the number of diagonals of the polygon. (A line segment connecting any two nonadjacent vertices is called a *diagonal* of a polygon.)**

73. Pentagon **74.** Hexagon

75. Octagon **76.** Nonagon

Solving an Equation **In Exercises 77–84, solve for *n*.**

77. $14 \cdot {}_nP_3 = {}_{n+2}P_4$ **78.** ${}_nP_5 = 18 \cdot {}_{n-2}P_4$

79. ${}_nP_4 = 10 \cdot {}_{n-1}P_3$ **80.** ${}_nP_6 = 12 \cdot {}_{n-1}P_5$

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81. ${}_{n+1}P_3 = 4 \cdot {}_nP_2$ **82.** ${}_{n+2}P_3 = 6 \cdot {}_{n+2}P_1$

83. $4 \cdot {}_{n+1}P_2 = {}_{n+2}P_3$ **84.** $5 \cdot {}_{n-1}P_1 = {}_nP_2$

Conclusions

True or False? **In Exercises 85 and 86, determine whether the statement is true or false. Justify your answer.**

85. The number of pairs of letters that can be formed from any of the first 13 letters in the alphabet (A–M), where repetitions are allowed, is an example of a permutation.

86. The number of permutations of n elements can be derived by using the Fundamental Counting Principle.

87. **Think About It** Can your calculator evaluate ${}_{100}P_{80}$? If not, explain why.

88. **CAPSTONE** Decide whether each scenario should be counted using permutations or combinations. Explain your reasoning. (Do not calculate.)

(a) Number of ways 10 people can line up in a row for concert tickets

(b) Number of different arrangements of three types of flowers from an array of 20 types

(c) Number of four-digit pin numbers for a debit card

(d) Number of two-scoop ice cream sundaes created from 31 different flavors

89. **Writing** Explain in your own words the meaning of ${}_nP_r$.

90. Without calculating the numbers, determine which of the following is greater. Explain.

(a) The number of combinations of 10 elements taken six at a time

(b) The number of permutations of 10 elements taken six at a time

Proof **In Exercises 91–94, prove the identity.**

91. ${}_nP_{n-1} = {}_nP_n$ **92.** ${}_nC_n = {}_nC_0$

93. ${}_nC_{n-1} = {}_nC_1$ **94.** ${}_nC_r = \dfrac{{}_nP_r}{r!}$

Cumulative Mixed Review

Solving an Equation **In Exercises 95 and 96, solve the equation. Round your answer to three decimal places, if necessary.**

95. $\log_2(x - 3) = 5$ **96.** $e^{x/3} = 16$

Using Cramer's Rule **In Exercises 97 and 98, use Cramer's Rule to solve the system of equations.**

97. $\begin{cases} -5x + 3y = -14 \\ 7x - 2y = 2 \end{cases}$ **98.** $\begin{cases} 8x + y = 35 \\ 6x + 2y = 10 \end{cases}$

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8.6 Probability

The Probability of an Event

Any happening whose result is uncertain is called an **experiment.** The possible results of the experiment are **outcomes,** the set of all possible outcomes of the experiment is the **sample space** of the experiment, and any subcollection of a sample space is an **event.**

For instance, when a six-sided die is tossed, the sample space can be represented by the numbers 1 through 6. For the experiment to be fair, each of the outcomes is *equally likely*.

To describe a sample space in such a way that each outcome is equally likely, you must sometimes distinguish between or among various outcomes in ways that appear artificial. Example 1 illustrates such a situation.

What you should learn

- Find probabilities of events.
- Find probabilities of mutually exclusive events.
- Find probabilities of independent events.

Why you should learn it

You can use probability to solve a variety of problems that occur in real life. For instance, in Exercise 54 on page 623, you are asked to use probability to help analyze the age distribution of unemployed workers.

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Example 1 Finding the Sample Space

Find the sample space for each of the following.

a. One coin is tossed.

b. Two coins are tossed.

c. Three coins are tossed.

Solution

a. Because the coin will land either heads up (denoted by H) or tails up (denoted by T), the sample space S is

$$S = \{H, T\}.$$

b. Because either coin can land heads up or tails up, the possible outcomes are as follows.

HH = heads up on both coins
HT = heads up on first coin and tails up on second coin
TH = tails up on first coin and heads up on second coin
TT = tails up on both coins

So, the sample space is

$$S = \{HH, HT, TH, TT\}.$$

Note that this list distinguishes between the two cases

HT and TH

even though these two outcomes appear to be similar.

c. Following the notation in part (b), the sample space is

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

Note that this list distinguishes among the cases

HHT, HTH, and THH

and among the cases

HTT, THT, and TTH.

CHECKPOINT Now try Exercise 9.

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To calculate the probability of an event, count the number of outcomes in the event and in the sample space. The *number of equally likely outcomes* in event E is denoted by $n(E)$, and the number of equally likely outcomes in the sample space S is denoted by $n(S)$. The probability that event E will occur is given by $n(E)/n(S)$.

The Probability of an Event

If an event E has $n(E)$ equally likely outcomes and its sample space S has $n(S)$ equally likely outcomes, then the **probability** of event E is given by

$$P(E) = \frac{n(E)}{n(S)}.$$

Explore the Concept

Toss two coins 40 times and write down the number of heads that occur on each toss (0, 1, or 2). How many times did two heads occur? Without performing the experiment, how many times would you expect two heads to occur when two coins are tossed 1000 times?

Because the number of outcomes in an event must be less than or equal to the number of outcomes in the sample space, the probability of an event must be a number from 0 to 1, inclusive. That is,

$$0 \le P(E) \le 1$$

as indicated in Figure 8.13. If $P(E) = 0$, then event E *cannot occur,* and E is called an **impossible event.** If $P(E) = 1$, then event E *must occur,* and E is called a **certain event.**

Figure 8.13

Example 2 Finding the Probability of an Event

a. Two coins are tossed. What is the probability that both land heads up?

b. A card is drawn at random from a standard deck of playing cards. What is the probability that it is an ace?

Solution

a. Following the procedure in Example 1(b), let

$$E = \{HH\}$$

and

$$S = \{HH, HT, TH, TT\}.$$

The probability of getting two heads is

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{4}.$$

b. Because there are 52 cards in a standard deck of playing cards and there are four aces (one of each suit), the probability of drawing an ace is

$$P(E) = \frac{n(E)}{n(S)} = \frac{4}{52} = \frac{1}{13}.$$

CHECKPOINT Now try Exercise 13.

Note that a probability can be written as a fraction, a decimal, or a percent. For instance, in Example 2(a), the probability of getting two heads can be written as

$\frac{1}{4}$, 0.25, or 25%.

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You could have written out the sample space in Example 2(b) and simply counted the outcomes in the desired event. For larger sample spaces, however, using the counting principles discussed in Section 8.5 should save you time.

Example 3 Finding the Probability of an Event

Twelve-sided dice, as shown in Figure 8.14, can be constructed (in the shape of regular dodecahedrons) such that each of the numbers from 1 to 6 appears twice on each die. Show that these dice can be used in any game requiring ordinary six-sided dice without changing the probabilities of the various outcomes.

Figure 8.14

Solution

For an ordinary six-sided die, each of the numbers 1, 2, 3, 4, 5, and 6 occurs only once, so the probability of any particular number coming up is

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{6}.$$

For a 12-sided die, each number occurs twice, so the probability of any particular number coming up is

$$P(E) = \frac{n(E)}{n(S)} = \frac{2}{12} = \frac{1}{6}.$$

CHECKPOINT Now try Exercise 23.

Example 4 Random Selection

The numbers of colleges and universities in various regions of the United States in 2008 are shown in Figure 8.15. One institution is selected at random. What is the probability that the institution is in one of the three southern regions? (Source: U.S. National Center for Education Statistics)

Solution

From the figure, the total number of colleges and universities is 4347. Because there are

$$406 + 286 + 754 = 1446$$

colleges and universities in the three southern regions, the probability that the institution is in one of these regions is

$$P(E) = \frac{n(E)}{n(S)} = \frac{1446}{4347} \approx 0.333.$$

Figure 8.15

CHECKPOINT Now try Exercise 27.

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Mutually Exclusive Events

Two events A and B (from the same sample space) are **mutually exclusive** when A and B have no outcomes in common. In the terminology of sets, the intersection of A and B is the empty set, which implies that

$$P(A \cap B) = 0.$$

For instance, when two dice are tossed, the event A of rolling a total of 6 and the event B of rolling a total of 9 are mutually exclusive. To find the probability that one or the other of two mutually exclusive events will occur, you can *add* their individual probabilities.

Probability of the Union of Two Events

If A and B are events in the same sample space, then the probability of A *or* B occurring is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

If A and B are mutually exclusive, then

$$P(A \cup B) = P(A) + P(B).$$

Example 5 The Probability of a Union

One card is selected at random from a standard deck of 52 playing cards. What is the probability that the card is either a heart or a face card?

Solution

Because the deck has 13 hearts, the probability of selecting a heart (event A) is

$$P(A) = \frac{13}{52}.$$

Similarly, because the deck has 12 face cards, the probability of selecting a face card (event B) is

$$P(B) = \frac{12}{52}.$$

Because three of the cards are hearts and face cards (see Figure 8.16), it follows that

$$P(A \cap B) = \frac{3}{52}.$$

Figure 8.16

Finally, applying the formula for the probability of the union of two events, you can conclude that the probability of selecting a heart or a face card is

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= \frac{13}{52} + \frac{12}{52} - \frac{3}{52} \\ &= \frac{22}{52} \\ &\approx 0.42. \end{aligned}$$

CHECKPOINT Now try Exercise 41.

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Example 6 Probability of Mutually Exclusive Events

The human resources department of a company has compiled data on the numbers of employees who have been with the company for various periods of time. The results are shown in the table.

Years of service	Number of employees
0–4	157
5–9	89
10–14	74
15–19	63
20–24	42
25–29	38
30–34	37
35–39	21
40–44	8

Human Resources Manager

An employee is chosen at random. What is the probability that the employee has

a. 4 or fewer years of service?

b. 9 or fewer years of service?

Solution

a. To begin, add the numbers of employees.

$$157 + 89 + 74 + 63 + 42 + 38 + 37 + 21 + 8 = 529$$

So, the total number of employees is 529. Next, let event A represent choosing an employee with 0 to 4 years of service. From the table, you know there are 157 employees with 0 to 4 years of service. The probability of choosing an employee who has 4 or fewer years of service is

$$P(A) = \frac{157}{529} \approx 0.30.$$

b. Let event B represent choosing an employee with 5 to 9 years of service. From the table, you know that there are 89 employees with 5 to 9 years of service. Then

$$P(B) = \frac{89}{529}.$$

Because event A from part (a) and event B have no outcomes in common, you can conclude that these two events are mutually exclusive and that

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) \\ &= \frac{157}{529} + \frac{89}{529} \\ &= \frac{246}{529} \\ &\approx 0.47. \end{aligned}$$

So, the probability of choosing an employee who has 9 or fewer years of service is about 0.47.

CHECKPOINT Now try Exercise 47.

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Independent Events

Two events are **independent** when the occurrence of one has no effect on the occurrence of the other. For instance, rolling a total of 12 with two six-sided dice has no effect on the outcome of future rolls of the dice. (See Figure 8.17.) To find the probability that two independent events will occur, *multiply* the probabilities of each.

Figure 8.17

Probability of Independent Events

If A and B are **independent events,** then the probability that both A and B will occur is given by

$$P(A \text{ and } B) = P(A) \cdot P(B).$$

Example 7 Probability of Independent Events

A random number generator on a computer selects three integers from 1 to 20. What is the probability that all three numbers are less than or equal to 5?

Solution

The probability of selecting a number from 1 to 5 is

$$P(A) = \frac{5}{20} = \frac{1}{4}.$$

So, the probability that all three numbers are less than or equal to 5 is

$$P(A) \cdot P(A) \cdot P(A) = \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = \frac{1}{64}.$$

CHECKPOINT Now try Exercise 49.

Example 8 Probability of Independent Events

In 2009, approximately 13% of the adult population of the United States got most of their news from the Internet. In a survey, 10 people were chosen at random from the adult population. What is the probability that all 10 got most of their news from the Internet? (Source: CBS News/New York Times Poll)

Solution

Let A represent choosing an adult who gets most of his or her news from the Internet. The probability of choosing an adult who got most of his or her news from the Internet is 0.13, the probability of choosing a second adult who got most of his or her news from the Internet is 0.13, and so on. Because these events are independent, you can conclude that the probability that all 10 people got most of their news from the Internet is

$$[P(A)]^{10} = (0.13)^{10} \approx 0.000000001.$$

CHECKPOINT Now try Exercise 61.

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8.6 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1–4, fill in the blank(s).

1. The set of all possible outcomes of an experiment is called the _____ .
2. To determine the probability of an event, use the formula $P(E) = \dfrac{n(E)}{n(S)}$, where $n(E)$ is _____ and $n(S)$ is _____ .
3. If two events from the same sample space have no outcomes in common, then the two events are _____ .
4. If the occurrence of one event has no effect on the occurrence of a second event, then the events are _____ .
5. Write an inequality that represents the possible values of the probability $P(E)$ of an event.
6. What is the probability of an impossible event?
7. What is the probability of a certain event?
8. Match the probability formula with the correct probability name.

(a) Probability of the union of two events (i) $P(A \cup B) = P(A) + P(B)$

(b) Probability of mutually exclusive events (ii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

(c) Probability of independent events (iii) $P(A \text{ and } B) = P(A) \cdot P(B)$

Procedures and Problem Solving

Finding the Sample Space **In Exercises 9–12, determine the sample space for the experiment.**

9. A coin and a six-sided die are tossed.
10. A six-sided die is tossed twice and the results of roll 1 and roll 2 are recorded.
11. A taste tester has to rank three varieties of orange juice, A, B, and C, according to preference.
12. Two marbles are selected at random (without replacement) from a sack containing two red marbles, two blue marbles, and one yellow marble. The color of each marble is recorded.

Finding the Probability of an Event **In Exercises 13–16, find the probability for the experiment of tossing a coin three times. Use the sample space**

$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$.

13. The probability of getting exactly two tails
14. The probability of getting a head on the first toss
15. The probability of getting at least one head
16. The probability of getting at least two heads

Finding the Probability of an Event **In Exercises 17–20, find the probability for the experiment of selecting one card at random from a standard deck of 52 playing cards.**

17. The card is a face card.
18. The card is a black card.
19. The card is a face card in the suit of spades.
20. The card is a numbered card (2–10).

Finding the Probability of an Event **In Exercises 21–24, use the sample space from Exercise 10 to find the probability for the experiment of tossing a six-sided die twice.**

21. The sum is 6.
22. The sum is at least 8.
23. The sum is less than 11.
24. The sum is odd or prime.

Random Selection **In Exercises 25–28, one of a team's 2200 season ticket holders is selected at random to win a prize. The circle graph shows the ages of the season ticket holders. Find the probability of the event.**

Ages of Season Ticket Holders

25. The winner is younger than 19 years old.
26. The winner is older than 39 years old.

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27. The winner is 19 to 39 years old.

28. The winner is younger than 19 years old or older than 59 years old.

Using Combinations **In Exercises 29–32, find the probability for the experiment of drawing two marbles at random (without replacement) from a bag containing one green, two yellow, and three red marbles. (*Hint:* Use combinations to find the numbers of outcomes for the given event and sample space.)**

29. Both marbles are red.

30. Both marbles are yellow.

31. Neither marble is yellow.

32. The marbles are of different colors.

Finding the Probability of a Complement **The *complement of an event A* is the collection of all outcomes in the sample space that are *not* in *A*. If the probability of *A* is $P(A)$, then the probability of the complement A' is given by $P(A') = 1 - P(A)$. In Exercises 33–36, you are given the probability that an event *will* happen. Find the probability that the event *will not* happen.**

33. $P(E) = 0.75$

34. $P(E) = 0.\overline{2}$

35. $P(E) = \frac{2}{3}$

36. $P(E) = \frac{7}{8}$

Using the Probability of a Complement **In Exercises 37–40, you are given the probability that an event *will not* happen. Find the probability that the event *will* happen.**

37. $P(E') = 0.12$

38. $P(E') = 0.84$

39. $P(E') = \frac{13}{20}$

40. $P(E') = \frac{61}{100}$

The Probability of a Union **In Exercises 41–44, one card is selected at random from a standard deck of 52 playing cards. Use a formula to find the probability of the union of the two events.**

41. The card is a club or a king.

42. The card is a face card or a black card.

43. The card is a face card or a 2.

44. The card is a heart or a spade.

Probability of Mutually Exclusive Events **In Exercises 45–48, use the table, which shows the age groups of students in a college sociology class.**

Age	Number of students
18–19	11
20–21	18
22–30	2
31–40	1

A student from the class is randomly chosen for a project. Find the probability that the student is the given age.

45. 20 or 21 years old

46. 18 to 21 years old

47. Older than 21 years old

48. Younger than 31 years old

Probability of Independent Events **In Exercises 49–52, a random number generator selects three numbers from 1 through 10. Find the probability of the event.**

49. All three numbers are even.

50. All three numbers are less than or equal to 4.

51. Two numbers are less than 5 and the other number is 10.

52. One number is 2, 4, or 6, and the other two numbers are odd.

53. **Political Science** Taylor, Moore, and Perez are candidates for public office. It is estimated that Moore and Perez have about the same probability of winning, and Taylor is believed to be twice as likely to win as either of the others. Find the probability of each candidate's winning the election.

54. ***Why you should learn it*** *(p. 616)* In 2008, there were approximately 8.92 million unemployed workers in the United States. The circle graph shows the age profile of these unemployed workers. (Source: U.S. Bureau of Labor Statistics)

Ages of Unemployed Workers

(a) Estimate the number of unemployed workers in the 16–19 age group.

(b) What is the probability that a person selected at random from the population of unemployed workers is in the 25–44 age group?

(c) What is the probability that a person selected at random from the population of unemployed workers is 25 to 64 years old?

(d) What is the probability that a person selected at random from the population of unemployed workers is 45 or older?

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55. Education The levels of educational attainment of the United States population age 25 years or older in 2008 are shown in the circle graph. Use the fact that the population of people 25 years or older was 196.31 million in 2008. (Source: U.S. Census Bureau)

(a) Estimate the number of people 25 years old or older who have advanced degrees.

(b) Find the probability that a person 25 years old or older selected at random has earned a Bachelor's degree or higher.

(c) Find the probability that a person 25 years old or older selected at random has earned a high school diploma or gone on to post-secondary education.

56. Political Science One hundred college students were interviewed to determine their political party affiliations and whether they favored a balanced budget amendment to the Constitution. The results of the study are listed in the table, where *D* represents Democrat and *R* represents Republican.

	D	*R*	Total
Favor	23	32	55
Oppose	25	9	34
Unsure	7	4	11
Total	55	45	100

A person is selected at random from the sample. Find the probability that the person selected is (a) a person who doesn't favor the amendment, (b) a Republican, and (c) a Democrat who favors the amendment.

In Exercises 57–60, the sample spaces are large and you should use the counting principles discussed in Section 8.5.

57. Using Counting Principles On a game show you are given five digits to arrange in the proper order to form the price of a car. If you are correct, you win the car. What is the probability of winning if you (a) guess the position of each digit and (b) know the first digit and guess the others?

58. Using Counting Principles The deck for a card game is made up of 108 cards. Twenty-five each are red, yellow, blue, and green, and eight are wild cards. Each player is randomly dealt a seven-card hand. What is the probability that a hand will contain (a) exactly two wild cards, and (b) two wild cards, two red cards, and three blue cards?

59. Using Counting Principles A shipment of 12 microwave ovens contains three defective units. A vending company has ordered four of these units, and because all are packaged identically, the selection will be random. What is the probability that (a) all four units are good, (b) exactly two units are good, and (c) at least two units are good?

60. Using Counting Principles Two integers from 1 through 40 are chosen by a random number generator. What is the probability that (a) the numbers are both even, (b) one number is even and one is odd, (c) both numbers are less than 30, and (d) the same number is chosen twice?

✓ **61. Marketing** Suppose that the methods used by shoppers to pay for merchandise are as shown in the circle graph. Two shoppers are chosen at random. What is the probability that both shoppers paid for their purchases only in cash?

62. Aerospace Engineering A space vehicle has an independent backup system for one of its communication networks. The probability that either system will function satisfactorily for the duration of a flight is 0.985. What is the probability that during a given flight (a) both systems function satisfactorily, (b) at least one system functions satisfactorily, and (c) both systems fail?

63. Marketing A sales representative makes sales on approximately one-fifth of all calls. On a given day, the representative contacts six potential clients. What is the probability that a sale will be made with (a) all six contacts, (b) none of the contacts, and (c) at least one contact?

64. Genetics Assume that the probability of the birth of a child of a particular sex is 50%. In a family with four children, what is the probability that (a) all the children are boys, (b) all the children are the same sex, and (c) there is at least one boy?

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65. Estimating π A coin of diameter d is dropped onto a paper that contains a grid of squares d units on a side. (See figure.)

(a) Find the probability that the coin covers a vertex of one of the squares on the grid.

(b) Perform the experiment 100 times and use the results to approximate π.

66. Geometry You and a friend agree to meet at your favorite fast food restaurant between 5:00 and 6:00 P.M. The one who arrives first will wait 15 minutes for the other, after which the first person will leave (see figure). What is the probability that the two of you will actually meet, assuming that your arrival times are random within the hour?

Conclusions

True or False? In Exercises 67 and 68, determine whether the statement is true or false. Justify your answer.

67. If the probability of an outcome in a sample space is 1, then the probability of the other outcomes in the sample space is 0.

68. If A and B are independent events with nonzero probabilities, then A can occur when B occurs.

69. Exploration Consider a group of n people.

(a) Explain why the following pattern gives the probability that the n people have distinct birthdays.

$$n = 2: \quad \frac{365}{365} \cdot \frac{364}{365} = \frac{365 \cdot 364}{365^2}$$

$$n = 3: \quad \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} = \frac{365 \cdot 364 \cdot 363}{365^3}$$

(b) Use the pattern in part (a) to write an expression for the probability that four people ($n = 4$) have distinct birthdays.

(c) Let P_n be the probability that the n people have distinct birthdays. Verify that this probability can be obtained recursively by

$$P_1 = 1 \quad \text{and} \quad P_n = \frac{365 - (n - 1)}{365} P_{n-1}.$$

(d) Explain why $Q_n = 1 - P_n$ gives the probability that at least two people in a group of n people have the same birthday.

(e) Use the results of parts (c) and (d) to complete the table.

n	10	15	20	23	30	40	50
P_n							
Q_n							

(f) How many people must be in a group so that the probability of at least two of them having the same birthday is greater than $\frac{1}{2}$? Explain.

70. Writing Write a paragraph describing in your own words the difference between mutually exclusive events and independent events.

71. Think About It Let A and B be two events from the same sample space such that $P(A) = 0.76$ and $P(B) = 0.58$.

(a) Is it possible that A and B are mutually exclusive? Explain. Draw a diagram to support your answer.

(b) Is it possible that A' and B' are mutually exclusive? Explain. Draw a diagram to support your answer.

(c) Determine the possible range of $P(A \cup B)$.

72. CAPSTONE Give an example of each of the following events for the experiment of selecting one or more cards at random from a standard deck of 52 playing cards.

(a) A union of two mutually exclusive events

(b) A union of two events that are not mutually exclusive

(c) Independent events

(d) The complement of an event

Cumulative Mixed Review

Finding a Combination In Exercises 73–76, evaluate ${}_nC_r$. Verify your result using a graphing utility.

73. ${}_6C_2$

74. ${}_9C_5$

75. ${}_{11}C_8$

76. ${}_{16}C_{13}$

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8 Chapter Summary

	What did you learn?	Explanation and Examples	Review Exercises
8.1	**Use sequence notation to write the terms of sequences *(p. 570)*.**	$a_n = 7n - 4$; $a_1 = 7(1) - 4 = 3$, $a_2 = 7(2) - 4 = 10$, $a_3 = 7(3) - 4 = 17$, $a_4 = 7(4) - 4 = 24$	1–8
	Use factorial notation *(p. 573)*.	If n is a positive integer, then $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdot \cdot (n-1) \cdot n$.	9–12
	Use summation notation to write sums *(p. 574)*.	The sum of the first n terms of a sequence is represented by $\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + a_4 + \cdot \cdot \cdot + a_n$.	13–20
	Find sums of infinite series *(p. 575)*.	$\sum_{i=1}^{\infty} \frac{5}{10^i} = \frac{5}{10^1} + \frac{5}{10^2} + \frac{5}{10^3} + \frac{5}{10^4} + \frac{5}{10^5} + \cdot \cdot \cdot$ $= 0.5 + 0.05 + 0.005 + 0.0005 + 0.00005 + \cdot \cdot \cdot$ $= 0.55555 \ldots$ $= \frac{5}{9}$	21–24
	Use sequences and series to model and solve real-life problems *(p. 576)*.	A sequence can be used to model the resident population of the United States from 1980 through 2008. (See Example 10.)	25, 26
8.2	**Recognize, write, and find the *n*th terms of arithmetic sequences *(p. 581)*.**	$a_n = 9n + 5$; $a_1 = 9(1) + 5 = 14$, $a_2 = 9(2) + 5 = 23$, $a_3 = 9(3) + 5 = 32$, $a_4 = 9(4) + 5 = 41$	27–38
	Find *n*th partial sums of arithmetic sequences *(p. 584)*.	The sum of a finite arithmetic sequence with n terms is $S_n = \frac{n}{2}(a_1 + a_n)$.	39–42
	Use arithmetic sequences to model and solve real-life problems *(p. 585)*.	An arithmetic sequence can be used to find the total sales of a small business. (See Example 7.)	43, 44
8.3	**Recognize, write, and find the *n*th terms of geometric sequences *(p. 589)*.**	$a_n = 3(4^n)$; $a_1 = 3(4^1) = 12$, $a_2 = 3(4^2) = 48$, $a_3 = 3(4^3) = 192$, $a_4 = 3(4^4) = 768$	45–56
	Find *n*th partial sums of geometric sequences *(p. 592)*.	The sum of the finite geometric sequence $a_1, a_1r, a_1r^2, \ldots, a_1r^{n-1}$ with common ratio $r \neq 1$ is given by $S_n = \sum_{i=1}^{n} a_1r^{i-1} = a_1\left(\frac{1 - r^n}{1 - r}\right)$.	57–60
	Find sums of infinite geometric series *(p. 593)*.	If $\lvert r \rvert < 1$, then the infinite geometric series $a_1 + a_1r + a_1r^2 + \cdot \cdot \cdot + a_1r^{n-1} + \cdot \cdot \cdot$ has the sum $S = \sum_{i=0}^{\infty} a_1r^i = \frac{a_1}{1 - r}$.	61–64
	Use geometric sequences to model and solve real-life problems *(p. 594)*.	A finite geometric sequence can be used to find the balance in an annuity at the end of 2 years. (See Example 8.)	65, 66

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	What did you learn?	Explanation and Examples	Review Exercises
8.4	**Use the Binomial Theorem to calculate binomial coefficients *(p. 599).***	**The Binomial Theorem:** In the expansion of $(x + y)^n = x^n + nx^{n-1}y + \cdots + {}_nC_r x^{n-r}y^r + \cdots + nxy^{n-1} + y^n$, the coefficient of $x^{n-r}y^r$ is $${}_nC_r = \frac{n!}{(n-r)!r!}.$$	67–70
	Use binomial coefficients to write binomial expansions *(p. 601).*	$(x + 1)^3 = x^3 + 3x^2 + 3x + 1$ $(x - 1)^4 = x^4 - 4x^3 + 6x^2 - 4x + 1$	71–74
	Use Pascal's Triangle to calculate binomial coefficients *(p. 603).*	First several rows of Pascal's triangle: $$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$	75–78
8.5	**Solve simple counting problems *(p. 607).***	A computer randomly generates an integer from 1 through 15. The computer can generate an integer that is divisible by 3 in 5 ways (3, 6, 9, 12, and 15).	79, 80
	Use the Fundamental Counting Principle to solve more complicated counting problems *(p. 608).*	**Fundamental Counting Principle:** Let E_1 and E_2 be two events. The first event E_1 can occur in m_1 different ways. After E_1 has occurred, E_2 can occur in m_2 different ways. The number of ways that the two events can occur is $m_1 \cdot m_2$.	81, 82
	Use permutations to solve counting problems *(p. 609).*	A permutation of n different elements is an ordering of the elements such that one element is first, one is second, one is third, and so on. The number of permutations of n elements is given by $n \cdot (n-1) \cdot \cdots \cdot 4 \cdot 3 \cdot 2 \cdot 1 = n!.$ In other words, there are $n!$ different ways that n elements can be ordered. The number of permutations of n elements taken r at a time is $${}_nP_r = \frac{n!}{(n-r)!}.$$	83–88, 91
	Use combinations to solve counting problems *(p. 612).*	The number of combinations of n elements taken r at a time is $${}_nC_r = \frac{n!}{(n-r)!r!}.$$	89, 90, 92
8.6	**Find probabilities of events *(p. 616).***	If an event E has $n(E)$ equally likely outcomes and its sample space S has $n(S)$ equally likely outcomes, then the probability of event E is $$P(E) = \frac{n(E)}{n(S)}.$$	93, 94
	Find probabilities of mutually exclusive events *(p. 619).*	If A and B are events in the same sample space, then the probability of A or B occurring is $P(A \cup B) = P(A) + (B) - P(A \cap B).$ If A and B are mutually exclusive, then $P(A \cup B) = P(A) + P(B).$	95
	Find probabilities of independent events *(p. 621).*	If A and B are independent events, then the probability that both A and B will occur is $P(A \text{ and } B) = P(A) \cdot P(B)$.	96

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8 Review Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

8.1

Writing the Terms of a Sequence **In Exercises 1 and 2, write the first five terms of the sequence. (Assume *n* begins with 1.)**

1. $a_n = \dfrac{2^n}{2^n + 1}$ **2.** $a_n = \dfrac{(-1)^n}{n!}$

Finding the *n*th Term of a Sequence **In Exercises 3–6, write an expression for the *apparent* *n*th term of the sequence. (Assume *n* begins with 1.)**

3. $5, 10, 15, 20, 25, \ldots$ **4.** $50, 48, 46, 44, 42, \ldots$

5. $2, \frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \frac{2}{9}, \ldots$ **6.** $\frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \frac{7}{6}, \ldots$

A Recursive Sequence **In Exercises 7 and 8, write the first five terms of the sequence defined recursively.**

7. $a_1 = 9, a_{k+1} = a_k - 4$ **8.** $a_1 = 49, a_{k+1} = a_k + 6$

Simplifying Factorial Expressions **In Exercises 9–12, simplify the factorial expression.**

9. $\dfrac{18!}{20!}$ **10.** $\dfrac{10!}{8!}$

11. $\dfrac{(n+1)!}{(n-1)!}$ **12.** $\dfrac{2n!}{(n+1)!}$

Sigma Notation for Sums **In Exercises 13–16, find the sum.**

13. $\displaystyle\sum_{i=1}^{6} 5$ **14.** $\displaystyle\sum_{k=2}^{5} 4k$

15. $\displaystyle\sum_{j=1}^{4} \frac{6}{j^2}$ **16.** $\displaystyle\sum_{i=1}^{8} \frac{i}{i+1}$

Writing a Sum Using Sigma Notation **In Exercises 17–20, use sigma notation to write the sum. Then use a graphing utility to find the sum.**

17. $\dfrac{1}{2(1)} + \dfrac{1}{2(2)} + \dfrac{1}{2(3)} + \cdots + \dfrac{1}{2(20)}$

18. $2(1^2) + 2(2^2) + 2(3^2) + \cdots + 2(9^2)$

19. $\dfrac{1}{2} + \dfrac{2}{3} + \dfrac{3}{4} + \cdots + \dfrac{9}{10}$

20. $1 - \dfrac{1}{3} + \dfrac{1}{9} - \dfrac{1}{27} + \cdots$

Finding the Sum of a Series **In Exercises 21–24, find (a) the fourth partial sum and (b) the sum of the infinite series.**

21. $\displaystyle\sum_{k=1}^{\infty} \frac{5}{10^k}$ **22.** $\displaystyle\sum_{k=1}^{\infty} \frac{3}{2^k}$

23. $\displaystyle\sum_{k=1}^{\infty} 2(0.5)^k$ **24.** $\displaystyle\sum_{k=1}^{\infty} 4(0.25)^k$

25. **Compound Interest** A deposit of \$3000 is made in an account that earns 2% interest compounded quarterly. The balance in the account after *n* quarters is given by

$$a_n = 3000\left(1 + \frac{0.02}{4}\right)^n, \qquad n = 1, 2, 3, \ldots.$$

(a) Compute the first eight terms of this sequence.

(b) Find the balance in this account after 10 years by computing the 40th term of the sequence.

26. **Education** The average salaries a_n (in thousands of dollars) of college faculty members in the United States from 2001 through 2009 can be approximated by the model

$$a_n = 0.065n^2 + 1.43n + 58.2, \quad n = 1, 2, 3, \ldots, 9$$

where *n* is the year, with $n = 1$ corresponding to 2001. (Source: American Association of University Professors)

(a) Find the terms of this finite sequence for the given values of *n*.

(b) Use a graphing utility to graph the sequence for the given values of *n*.

(c) Use the graphing utility to construct a bar graph of the sequence for the given values of *n*.

(d) Use the sequence to predict the average salaries of college faculty members for the years 2010 through 2016. Do your results seem reasonable? Explain.

8.2

Determining Whether a Sequence Is Arithmetic **In Exercises 27–30, determine whether or not the sequence is arithmetic. If it is, find the common difference.**

27. $5, 3, 1, -1, -3, \ldots$ **28.** $0, 1, 3, 6, 10, \ldots$

29. $\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots$ **30.** $\frac{9}{9}, \frac{8}{9}, \frac{7}{9}, \frac{6}{9}, \frac{5}{9}, \ldots$

Writing the Terms of an Arithmetic Sequence **In Exercises 31–34, write the first five terms of the arithmetic sequence. Use the *table* feature of a graphing utility to verify your results.**

31. $a_1 = 3,\ d = 4$ **32.** $a_1 = 8,\ d = -2$

33. $a_4 = 10,\ a_{10} = 28$ **34.** $a_2 = 14,\ a_6 = 22$

Writing the Terms of an Arithmetic Sequence **In Exercises 35 and 36, write the first five terms of the arithmetic sequence. Find the common difference and write the *n*th term of the sequence as a function of *n*.**

35. $a_1 = 35,\ a_{k+1} = a_k - 3$ **36.** $a_1 = 15,\ a_{k+1} = a_k + \frac{5}{2}$

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Finding the *n*th Term of an Arithmetic Sequence **In Exercises 37 and 38, find a formula for a_n for the arithmetic sequence and find the sum of the first 25 terms of the sequence.**

37. $a_1 = 100, \quad d = -3$ **38.** $a_1 = 10, \quad a_3 = 28$

Finding a Partial Sum of an Arithmetic Sequence **In Exercises 39 and 40, find the partial sum. Use a graphing utility to verify your result.**

39. $\sum_{j=1}^{10} (2j - 3)$ **40.** $\sum_{k=1}^{11} \left(\frac{2}{3}k + 4\right)$

41. Find the sum of the first 50 positive multiples of 5.

42. Find the sum of the integers from 10 to 70.

43. **Accounting** The starting salary for an accountant is \$36,000 with a guaranteed salary increase of \$2250 per year. Determine (a) the salary during the fifth year and (b) the total compensation through 5 full years of employment.

44. **Agriculture** In his first trip around a field, a farmer makes 123 bales. In his second trip, he makes 11 fewer bales. Because each trip is shorter than the preceding trip, the farmer estimates that the same pattern will continue. Estimate the total number of bales made if there are another five trips around the field.

8.3

Identifying a Geometric Sequence **In Exercises 45–48, determine whether or not the sequence is geometric. If it is, find the common ratio.**

45. 5, 10, 20, 40, . . . **46.** $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$

47. 54, −18, 6, −2, . . . **48.** $\frac{1}{3}, -\frac{2}{3}, \frac{4}{3}, -\frac{8}{3}, \ldots$

Finding the Terms of a Geometric Sequence **In Exercises 49–52, write the first five terms of the geometric sequence.**

49. $a_1 = 4, \ r = -\frac{1}{4}$ **50.** $a_1 = 2, \ r = \frac{3}{2}$

51. $a_1 = 9, \ a_3 = 4$ **52.** $a_1 = 2, \ a_3 = 12$

Finding the *n*th Term of a Geometric Sequence **In Exercises 53 and 54, write the first five terms of the geometric sequence. Find the common ratio and write the *n*th term of the sequence as a function of *n*.**

53. $a_1 = 120, \ a_{k+1} = \frac{1}{3}a_k$ **54.** $a_1 = 200, \ a_{k+1} = 0.1a_k$

Finding a Term of a Geometric Sequence **In Exercises 55 and 56, find the indicated term of the geometric sequence (a) algebraically and (b) using the *table* feature of a graphing utility.**

55. $a_1 = 16, \quad a_2 = -8,$ 6th term

56. $a_3 = 6, \quad a_4 = 1,$ 9th term

Finding the Sum of a Finite Geometric Sequence **In Exercises 57–60, find the sum. Use a graphing utility to verify your result.**

57. $\sum_{i=1}^{7} 2^{i-1}$ **58.** $\sum_{i=1}^{5} 3^{i-1}$

59. $\sum_{n=1}^{7} (-4)^{n-1}$ **60.** $\sum_{n=1}^{4} 12\left(-\frac{1}{2}\right)^{n-1}$

Finding the Sum of an Infinite Geometric Series **In Exercises 61–64, find the sum of the infinite geometric series.**

61. $\sum_{i=1}^{\infty} 4\left(\frac{7}{8}\right)^{i-1}$ **62.** $\sum_{i=1}^{\infty} 6\left(\frac{1}{3}\right)^{i-1}$

63. $\sum_{k=1}^{\infty} 4\left(\frac{2}{3}\right)^{k-1}$ **64.** $\sum_{k=1}^{\infty} 1.3\left(\frac{1}{10}\right)^{k-1}$

65. **Finance** A company buys a fleet of six vans for \$130,000. During the next 5 years, the fleet will depreciate at a rate of 30% per year. (That is, at the end of each year, the depreciated value will be 70% of the value at the beginning of the year.)

(a) Find the formula for the *n*th term of a geometric sequence that gives the value of the fleet *t* full years after it was purchased.

(b) Find the depreciated value of the fleet at the end of 5 full years.

66. **Finance** A deposit of \$80 is made at the beginning of each month in an account that pays 4% interest, compounded monthly. The balance *A* in the account at the end of 4 years is given by

$$A = 80\left(1 + \frac{0.04}{12}\right)^1 + \cdots + 80\left(1 + \frac{0.04}{12}\right)^{48}.$$

Find *A*.

8.4

Finding Binomial Coefficients **In Exercises 67–70, find the binomial coefficient. Use a graphing utility to verify your result.**

67. ${}_{10}C_8$ **68.** ${}_{12}C_5$

69. ${}_{9}C_4$ **70.** ${}_{14}C_{12}$

Expanding a Binomial **In Exercises 71–74, use the Binomial Theorem to expand and simplify the expression.**

71. $(x + 5)^4$ **72.** $(y - 3)^3$

73. $(a - 4b)^5$ **74.** $(3x + y)^7$

Using Pascal's Triangle **In Exercises 75–78, use Pascal's Triangle to find the binomial coefficient.**

75. ${}_{6}C_3$ **76.** ${}_{9}C_7$

77. ${}_{8}C_4$ **78.** ${}_{10}C_5$

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8.5

79. Probability Slips of paper numbered 1 through 15 are placed in a hat. In how many ways can two numbers be drawn so that the sum of the numbers is 12? Assume the random selection is without replacement.

80. Aviation Ten people are boarding an aircraft. Two have tickets for first class and board before those in economy class. In how many ways can the ten people board the aircraft?

81. Education A college student is preparing a course schedule of four classes for the next semester. The student can choose from the open sections shown in the table.

Course	Sections
Math 100	001–004
Economics 110	001–003
English 105	001–006
Humanities 101	001–003

(a) Find the number of possible schedules that the student can create from the offerings.

(b) Find the number of possible schedules that the student can create from the offerings when two of the Math 100 sections are closed.

(c) Find the number of possible schedules that the student can create from the offerings when two of the Math 100 sections and four of the English 105 sections are closed.

82. Marketing A telemarketing firm is making calls to prospective customers by randomly dialing seven-digit phone numbers within an area code.

(a) Find the number of possible calls that the telemarketer can make.

(b) The telemarketing firm is calling only within exchanges that begin with a "7" or a "6." How many different calls are possible?

(c) The telemarketing firm is calling only within exchanges that do not begin with a "0" or a "1." How many different calls are possible?

Finding a Permutation **In Exercises 83 and 84, evaluate ${}_nP_r$ using the formula from Section 8.5. Use a graphing utility to verify your result.**

83. ${}_{12}P_{10}$ **84.** ${}_6P_4$

Finding Distinguishable Permutations **In Exercises 85 and 86, find the number of distinguishable permutations of the group of letters.**

85. C, A, L, C, U, L, U, S **86.** I, N, T, E, G, R, A, T, E

Solving an Equation **In Exercises 87 and 88, solve for n.**

87. ${}_{n+1}P_2 = 4 \cdot {}_nP_1$ **88.** $8 \cdot {}_nP_2 = {}_{n+1}P_3$

Finding a Combination **In Exercises 89 and 90, evaluate ${}_nC_r$ using the formula from Section 8.5. Use a graphing utility to verify your result.**

89. ${}_8C_6$ **90.** ${}_{50}C_{48}$

91. Athletics There are 12 bicyclists entered in a race. In how many different orders could the 12 bicyclists finish? (Assume there are no ties.)

92. Athletics From a pool of seven juniors and eleven seniors, four co-captains will be chosen for the football team. How many different combinations are possible if two juniors and two seniors are to be chosen?

8.6

93. Fashion A man has five pairs of socks (no two pairs are the same color). He randomly selects two socks from a drawer. What is the probability that he gets a matched pair?

94. Library Science A child returns a five-volume set of books to a bookshelf. The child is not able to read, and so cannot distinguish one volume from another. What is the probability that the books are shelved in the correct order?

95. Education A sample of college students, faculty members, and administrators were asked whether they favored a proposed increase in the annual activity fee to enhance student life on campus. The results of the study are shown in the table.

	Favor	Oppose	Total
Students	237	163	400
Faculty	37	38	75
Admin.	18	7	25
Total	292	208	500

A person is selected at random from the sample. Find each probability.

(a) The person is not in favor of the proposal.

(b) The person is a student.

(c) The person is a faculty member and is in favor of the proposal.

96. Probability A six-sided die is rolled six times. What is the probability that each side appears exactly once?

Conclusions

True or False? **In Exercises 97 and 98, determine whether the statement is true or false. Justify your answer.**

97. $\dfrac{(n+2)!}{n!} = (n+2)(n+1)$ **98.** $\displaystyle\sum_{k=1}^{8} 3k = 3\sum_{k=1}^{8} k$

99. Writing In your own words, explain what makes a sequence (a) arithmetic and (b) geometric.

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8 Chapter Test

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Take this test as you would take a test in class. After you are finished, check your work against the answers in the back of the book.

In Exercises 1–4, write the first five terms of the sequence.

1. $a_n = \left(-\frac{2}{3}\right)^{n-1}$ (Begin with $n = 1$.)

2. $a_1 = 12$ and $a_{k+1} = a_k + 4$

3. $b_n = \dfrac{(-1)^n x^n}{n}$ (Begin with $n = 1$.)

4. $b_n = \dfrac{(-1)^{2n+1} x^{2n+1}}{(2n+1)!}$ (Begin with $n = 1$.)

5. Simplify $\dfrac{11! \cdot 4!}{4! \cdot 7!}$.

6. Simplify $\dfrac{n!}{(n+1)!}$.

7. Simplify $\dfrac{2n!}{(n-1)!}$.

8. Write an expression for the *apparent* nth term of the sequence

2, 5, 10, 17, 26, ... (Assume n begins with 1.)

In Exercises 9 and 10, find a formula for the nth term of the sequence.

9. Arithmetic: $a_1 = 5000,\ d = -100$

10. Geometric: $a_1 = 4,\ a_{k+1} = \frac{1}{2}a_k$

11. Use sigma notation to write $\dfrac{2}{3(1)+1} + \dfrac{2}{3(2)+1} + \cdots + \dfrac{2}{3(12)+1}$.

12. Use sigma notation to write $2 + \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \frac{1}{128} + \cdots$.

In Exercises 13–15, find the sum.

13. $\displaystyle\sum_{n=1}^{7} (8n - 5)$

14. $\displaystyle\sum_{n=1}^{8} 24\left(\frac{1}{6}\right)^{n-1}$

15. $5 - 2 + \frac{4}{5} - \frac{8}{25} + \frac{16}{125} - \cdots$

16. Use the Binomial Theorem to expand and simplify $(2a - 5b)^4$.

In Exercises 17–20, evaluate the expression.

17. ${}_9C_3$

18. ${}_{20}C_3$

19. ${}_9P_2$

20. ${}_{70}P_3$

21. Solve for n in $4 \cdot {}_nP_3 = {}_{n+1}P_4$.

22. How many distinct license plates can be issued consisting of one letter followed by a three-digit number?

23. Four students are randomly selected from a class of 25 to answer questions from a reading assignment. In how many ways can the four be selected?

24. A card is drawn from a standard deck of 52 playing cards. Find the probability that it is a red face card.

25. In 2010, five of the eleven men's basketball teams in the Big Ten Conference participated in the NCAA Men's Basketball Championship Tournament. What is the probability that a random selection of five teams from the Big Ten Conference will be the same as the five teams that participated in the tournament?

26. Two integers from 1 to 60 are chosen by a random number generator. What is the probability that (a) both numbers are odd, (b) both numbers are less than 12, and (c) the same number is chosen twice?

27. A weather forecast indicates that the probability of snow is 75%. What is the probability that it will not snow?

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Proofs in Mathematics

Properties of Sums (p. 574)

1. $\sum_{i=1}^{n} c = cn, \qquad c$ is a constant.
2. $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i, \qquad c$ is a constant.
3. $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$
4. $\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$

Proof

Each of these properties follows directly from the properties of real numbers.

1. $\sum_{i=1}^{n} c = c + c + c + \cdots + c = cn \qquad n$ terms

The Distributive Property is used in the proof of Property 2.

2. $$\begin{aligned}\sum_{i=1}^{n} ca_i &= ca_1 + ca_2 + ca_3 + \cdots + ca_n \\ &= c(a_1 + a_2 + a_3 + \cdots + a_n) \\ &= c\sum_{i=1}^{n} a_i\end{aligned}$$

The proof of Property 3 uses the Commutative and Associative Properties of Addition.

3. $$\begin{aligned}\sum_{i=1}^{n} (a_i + b_i) &= (a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + \cdots + (a_n + b_n) \\ &= (a_1 + a_2 + a_3 + \cdots + a_n) + (b_1 + b_2 + b_3 + \cdots + b_n) \\ &= \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i\end{aligned}$$

The proof of Property 4 uses the Commutative and Associative Properties of Addition and the Distributive Property.

4. $$\begin{aligned}\sum_{i=1}^{n} (a_i - b_i) &= (a_1 - b_1) + (a_2 - b_2) + (a_3 - b_3) + \cdots + (a_n - b_n) \\ &= (a_1 + a_2 + a_3 + \cdots + a_n) + (-b_1 - b_2 - b_3 - \cdots - b_n) \\ &= (a_1 + a_2 + a_3 + \cdots + a_n) - (b_1 + b_2 + b_3 + \cdots + b_n) \\ &= \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i\end{aligned}$$

Infinite Series

The study of infinite series was considered a novelty in the fourteenth century. Logician Richard Suiseth, whose nickname was Calculator, solved this problem.

If throughout the first half of a given time interval a variation continues at a certain intensity; throughout the next quarter of the interval at double the intensity; throughout the following eighth at triple the intensity and so ad infinitum; The average intensity for the whole interval will be the intensity of the variation during the second subinterval (or double the intensity).

This is the same as saying that the sum of the infinite series

$$\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdots + \frac{n}{2^n} + \cdots$$

is 2.

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The Sum of a Finite Arithmetic Sequence (p. 584)

The sum of a finite arithmetic sequence with n terms is given by

$$S_n = \frac{n}{2}(a_1 + a_n).$$

Proof

Begin by generating the terms of the arithmetic sequence in two ways. In the first way, repeatedly add d to the first term to obtain

$$\begin{aligned} S_n &= a_1 + a_2 + a_3 + \cdots + a_{n-2} + a_{n-1} + a_n \\ &= a_1 + [a_1 + d] + [a_1 + 2d] + \cdots + [a_1 + (n-1)d]. \end{aligned}$$

In the second way, repeatedly subtract d from the nth term to obtain

$$\begin{aligned} S_n &= a_n + a_{n-1} + a_{n-2} + \cdots + a_3 + a_2 + a_1 \\ &= a_n + [a_n - d] + [a_n - 2d] + \cdots + [a_n - (n-1)d]. \end{aligned}$$

When you add these two versions of S_n, the multiples of d subtract out and you obtain

$$2S_n = (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \cdots + (a_1 + a_n) \qquad n \text{ terms}$$

$$2S_n = n(a_1 + a_n)$$

$$S_n = \frac{n}{2}(a_1 + a_n).$$

The Sum of a Finite Geometric Sequence (p. 592)

The sum of the finite geometric sequence

$$a_1,\ a_1r,\ a_1r^2,\ a_1r^3,\ a_1r^4,\ \ldots,\ a_1r^{n-1}$$

with common ratio $r \neq 1$ is given by

$$S_n = \sum_{i=1}^{n} a_1 r^{i-1} = a_1\left(\frac{1 - r^n}{1 - r}\right).$$

Proof

$$S_n = a_1 + a_1r + a_1r^2 + \cdots + a_1r^{n-2} + a_1r^{n-1}$$

$$rS_n = a_1r + a_1r^2 + a_1r^3 + \cdots + a_1r^{n-1} + a_1r^n \qquad \text{Multiply by } r.$$

Subtracting the second equation from the first yields

$$S_n - rS_n = a_1 - a_1r^n.$$

So, $S_n(1 - r) = a_1(1 - r^n)$, and, because $r \neq 1$, you have

$$S_n = a_1\left(\frac{1 - r^n}{1 - r}\right).$$

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The Binomial Theorem (p. 599)

In the expansion of $(x + y)^n$

$$(x + y)^n = x^n + nx^{n-1}y + \cdots + {}_nC_r x^{n-r}y^r + \cdots + nxy^{n-1} + y^n$$

the coefficient of $x^{n-r}y^r$ is

$${}_nC_r = \frac{n!}{(n-r)!r!}.$$

Proof

The Binomial Theorem can be proved quite nicely using **mathematical induction.** (For information about mathematical induction, see Appendix G on this textbook's *Companion Website*.) The steps are straightforward but look a little messy, so only an outline of the proof is presented.

1. For $n = 1$, you have

$$(x + y)^1 = x^1 + y^1 = {}_1C_0 x + {}_1C_1 y$$

and the formula is valid.

2. Assuming that the formula is true for $n = k$, the coefficient of $x^{k-r}y^r$ is

$${}_kC_r = \frac{k!}{(k-r)!r!} = \frac{k(k-1)(k-2)\cdots(k-r+1)}{r!}.$$

To show that the formula is true for $n = k + 1$, look at the coefficient of $x^{k+1-r}y^r$ in the expansion of

$$(x + y)^{k+1} = (x + y)^k(x + y).$$

From the right-hand side, you can determine that the term involving $x^{k+1-r}y^r$ is the sum of two products.

$$\begin{aligned}
&({}_kC_r x^{k-r}y^r)(x) + ({}_kC_{r-1}x^{k+1-r}y^{r-1})(y)\\
&= \left[\frac{k!}{(k-r)!r!} + \frac{k!}{(k+1-r)!(r-1)!}\right]x^{k+1-r}y^r\\
&= \left[\frac{(k+1-r)k!}{(k+1-r)!r!} + \frac{k!r}{(k+1-r)!r!}\right]x^{k+1-r}y^r\\
&= \left[\frac{k!(k+1-r+r)}{(k+1-r)!r!}\right]x^{k+1-r}y^r\\
&= \left[\frac{(k+1)!}{(k+1-r)!r!}\right]x^{k+1-r}y^r\\
&= {}_{k+1}C_r x^{k+1-r}y^r
\end{aligned}$$

So, by mathematical induction, the Binomial Theorem is valid for all positive integers n.

The Principle of Mathematical Induction

Let P_n be a statement involving the positive integer n. If

1. P_1 is true, and

2. the truth of P_k implies the truth of P_{k+1} for every positive integer k,

then P_n must be true for all positive integers n.

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9 Topics in Analytic Geometry

Section 9.2, Example 5
Orbit of the Moon

9.1 Circles and Parabolas
9.2 Ellipses
9.3 Hyperbolas and Rotation of Conics
9.4 Parametric Equations
9.5 Polar Coordinates
9.6 Graphs of Polar Equations
9.7 Polar Equations of Conics

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9.1 Circles and Parabolas

Conics

Conic sections were discovered during the classical Greek period, 600 to 300 B.C. The early Greek studies were largely concerned with the geometric properties of conics. It was not until the early 17th century that the broad applicability of conics became apparent and played a prominent role in the early development of calculus.

A **conic section** (or simply **conic**) is the intersection of a plane and a double-napped cone. Notice in Figure 9.1 that in the formation of the four basic conics, the intersecting plane does not pass through the vertex of the cone.

Figure 9.1 *Basic Conics*

When the plane does pass through the vertex, the resulting figure is a **degenerate conic,** as shown in Figure 9.2.

Figure 9.2 *Degenerate Conics*

There are several ways to approach the study of conics. You could begin by defining conics in terms of the intersections of planes and cones, as the Greeks did, or you could define them algebraically, in terms of the general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

However, you will study a third approach, in which each of the conics is defined as a **locus** (collection) of points satisfying a certain geometric property. For example, the definition of a circle as *the collection of all points* (x, y) *that are equidistant from a fixed point* (h, k) leads to the standard equation of a circle

$$(x - h)^2 + (y - k)^2 = r^2. \qquad \text{Equation of circle}$$

What you should learn

- Recognize a conic as the intersection of a plane and a double-napped cone.
- Write equations of circles in standard form.
- Write equations of parabolas in standard form.
- Use the reflective property of parabolas to solve real-life problems.

Why you should learn it

Parabolas can be used to model and solve many types of real-life problems. For instance, in Exercise 103 on page 645, a parabola is used to design an entrance ramp for a highway.

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Circles

The definition of a circle as a locus of points is a more general definition of a circle as it applies to conics.

Definition of a Circle

A **circle** is the set of all points (x, y) in a plane that are equidistant from a fixed point (h, k), called the **center** of the circle. (See Figure 9.3.) The distance r between the center and any point (x, y) on the circle is the **radius.**

Figure 9.3

The Distance Formula can be used to obtain an equation of a circle whose center is (h, k) and whose radius is r.

$$\sqrt{(x-h)^2+(y-k)^2} = r \qquad \text{Distance Formula}$$

$$(x-h)^2+(y-k)^2 = r^2 \qquad \text{Square each side.}$$

Standard Form of the Equation of a Circle

The **standard form of the equation of a circle** is

$$(x-h)^2+(y-k)^2 = r^2.$$

The point (h, k) is the center of the circle, and the positive number r is the radius of the circle. The standard form of the equation of a circle whose center is the origin, $(h, k) = (0, 0)$, is

$$x^2+y^2 = r^2.$$

Example 1 Finding the Standard Equation of a Circle

The point $(1, 4)$ is on a circle whose center is at $(-2, -3)$, as shown in Figure 9.4. Write the standard form of the equation of the circle.

Figure 9.4

Solution

The radius of the circle is the distance between $(-2, -3)$ and $(1, 4)$.

$$r = \sqrt{[1-(-2)]^2+[4-(-3)]^2} \qquad \text{Use Distance Formula.}$$

$$= \sqrt{3^2+7^2} \qquad \text{Simplify.}$$

$$= \sqrt{58} \qquad \text{Radius}$$

The equation of the circle with center $(h, k) = (-2, -3)$ and radius $r = \sqrt{58}$ is

$$(x-h)^2+(y-k)^2 = r^2 \qquad \text{Standard form}$$

$$[x-(-2)]^2+[y-(-3)]^2 = \left(\sqrt{58}\right)^2 \qquad \text{Substitute for } h, k, \text{ and } r.$$

$$(x+2)^2+(y+3)^2 = 58. \qquad \text{Simplify.}$$

CHECKPOINT Now try Exercise 9.

Be careful when you are finding h and k from the standard equation of a circle. For instance, to find the correct h and k from the equation of the circle in Example 1, rewrite the quantities $(x+2)^2$ and $(y+3)^2$ using subtraction.

$$(x+2)^2 = [x-(-2)]^2 \quad \Rightarrow \quad h = -2$$

$$(y+3)^2 = [y-(-3)]^2 \quad \Rightarrow \quad k = -3$$

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Example 2 Sketching a Circle

Sketch the circle given by the equation

$$x^2 - 6x + y^2 - 2y + 6 = 0$$

and identify its center and radius.

Solution

Begin by writing the equation in standard form.

$x^2 - 6x + y^2 - 2y + 6 = 0$	Write original equation.
$(x^2 - 6x + 9) + (y^2 - 2y + 1) = -6 + 9 + 1$	Complete the squares.
$(x - 3)^2 + (y - 1)^2 = 4$	Write in standard form.

In this form, you can see that the graph is a circle whose center is the point (3, 1) and whose radius is $r = \sqrt{4} = 2$. Plot several points that are two units from the center. The points (5, 1), (3, 3), (1, 1), and (3, −1) are convenient. Draw a circle that passes through the four points, as shown in Figure 9.5.

Figure 9.5

CHECKPOINT Now try Exercise 29.

Technology Tip

You can use a graphing utility to confirm the result in Example 2 by graphing the upper and lower portions in the same viewing window. First, solve for y to obtain

$$y_1 = 1 + \sqrt{4 - (x - 3)^2}$$

and

$$y_2 = 1 - \sqrt{4 - (x - 3)^2}.$$

Then use a square setting, such as $-1 \le x \le 8$ and $-2 \le y \le 4$, to graph both equations.

Example 3 Finding the Intercepts of a Circle

Find the x- and y-intercepts of the graph of the circle given by the equation

$$(x - 4)^2 + (y - 2)^2 = 16.$$

Solution

To find any x-intercepts, let $y = 0$. To find any y-intercepts, let $x = 0$.

x-intercepts:

$(x - 4)^2 + (0 - 2)^2 = 16$	Substitute 0 for y.
$(x - 4)^2 = 12$	Simplify.
$x - 4 = \pm\sqrt{12}$	Take square root of each side.
$x = 4 \pm 2\sqrt{3}$	Add 4 to each side.

y-intercepts:

$(0 - 4)^2 + (y - 2)^2 = 16$	Substitute 0 for x.
$(y - 2)^2 = 0$	Simplify.
$y - 2 = 0$	Take square root of each side.
$y = 2$	Add 2 to each side.

So the x-intercepts are $(4 + 2\sqrt{3}, 0)$ and $(4 - 2\sqrt{3}, 0)$, and the y-intercept is (0, 2), as shown in Figure 9.6.

Figure 9.6

CHECKPOINT Now try Exercise 35.

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Parabolas

In Section 2.1, you learned that the graph of the quadratic function $f(x) = ax^2 + bx + c$ is a parabola that opens upward or downward. The following definition of a parabola is more general in the sense that it is independent of the orientation of the parabola.

Definition of a Parabola

A **parabola** is the set of all points (x, y) in a plane that are equidistant from a fixed line, the **directrix,** and a fixed point, the **focus,** not on the line. (See Figure 9.7.) The midpoint between the focus and the directrix is the **vertex,** and the line passing through the focus and the vertex is the **axis** of the parabola.

Figure 9.7

Note in Figure 9.7 that a parabola is symmetric with respect to its axis. Using the definition of a parabola, you can derive the following **standard form of the equation of a parabola** whose directrix is parallel to the x-axis or to the y-axis.

Standard Equation of a Parabola (See the proof on page 707.)

The **standard form of the equation of a parabola** with vertex at (h, k) is as follows.

$(x - h)^2 = 4p(y - k),\ p \neq 0$ — Vertical axis; directrix: $y = k - p$

$(y - k)^2 = 4p(x - h),\ p \neq 0$ — Horizontal axis; directrix: $x = h - p$

The focus lies on the axis p units (*directed distance*) from the vertex. If the vertex is at the origin $(0, 0)$, then the equation takes one of the following forms.

$x^2 = 4py$ — Vertical axis

$y^2 = 4px$ — Horizontal axis

See Figure 9.8.

$(x - h)^2 = 4p(y - k)$ — **(a)** ***Vertical axis:*** $p > 0$

$(x - h)^2 = 4p(y - k)$ — **(b)** ***Vertical axis:*** $p < 0$

$(y - k)^2 = 4p(x - h)$ — **(c)** ***Horizontal axis:*** $p > 0$

$(y - k)^2 = 4p(x - h)$ — **(d)** ***Horizontal axis:*** $p < 0$

Figure 9.8

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Example 4 Finding the Standard Equation of a Parabola

Find the standard form of the equation of the parabola with vertex at the origin and focus (0, 4).

Solution

The axis of the parabola is vertical, passing through (0, 0) and (0, 4), as shown in Figure 9.9. The standard form is $x^2 = 4py$, where $p = 4$. So, the equation is $x^2 = 16y$, or $y = \frac{1}{16}x^2$.

Figure 9.9

CHECKPOINT Now try Exercise 51.

Example 5 Finding the Focus of a Parabola

Find the focus of the parabola given by

$$y = -\tfrac{1}{2}x^2 - x + \tfrac{1}{2}.$$

Solution

To find the focus, convert to standard form by completing the square.

$y = -\frac{1}{2}x^2 - x + \frac{1}{2}$	Write original equation.
$-2y = x^2 + 2x - 1$	Multiply each side by -2.
$1 - 2y = x^2 + 2x$	Add 1 to each side.
$1 + 1 - 2y = x^2 + 2x + 1$	Complete the square.
$2 - 2y = x^2 + 2x + 1$	Combine like terms.
$-2(y - 1) = (x + 1)^2$	Write in standard form.

Comparing this equation with

$$(x - h)^2 = 4p(y - k)$$

you can conclude that $h = -1$, $k = 1$, and $p = -\frac{1}{2}$. Because p is negative, the parabola opens downward, as shown in Figure 9.10. Therefore, the focus of the parabola is

$$(h, k + p) = \left(-1, \tfrac{1}{2}\right). \qquad \text{Focus}$$

Figure 9.10

CHECKPOINT Now try Exercise 69.

Example 6 Finding the Standard Equation of a Parabola

Find the standard form of the equation of the parabola with vertex (1, 0) and focus (2, 0).

Solution

Because the axis of the parabola is horizontal, passing through (1, 0) and (2, 0), consider the equation

$$(y - k)^2 = 4p(x - h)$$

where $h = 1$, $k = 0$, and $p = 2 - 1 = 1$. So, the standard form is

$$(y - 0)^2 = 4(1)(x - 1) \quad \Rightarrow \quad y^2 = 4(x - 1).$$

You can use a graphing utility to confirm this equation. To do this, let

$$y_1 = \sqrt{4(x - 1)} \quad \text{and} \quad y_2 = -\sqrt{4(x - 1)}$$

as shown in Figure 9.11.

Figure 9.11

CHECKPOINT Now try Exercise 85.

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Reflective Property of Parabolas

A line segment that passes through the focus of a parabola and has endpoints on the parabola is called a **focal chord.** The specific focal chord perpendicular to the axis of the parabola is called the **latus rectum.**

Parabolas occur in a wide variety of applications. For instance, a parabolic reflector can be formed by revolving a parabola about its axis. The resulting surface has the property that all incoming rays parallel to the axis are reflected through the focus of the parabola. This is the principle behind the construction of the parabolic mirrors used in reflecting telescopes. Conversely, the light rays emanating from the focus of a parabolic reflector used in a flashlight are all parallel to one another, as shown in Figure 9.12.

Figure 9.12

A line is **tangent** to a parabola at a point on the parabola when the line intersects, but does not cross, the parabola at the point. Tangent lines to parabolas have special properties related to the use of parabolas in constructing reflective surfaces.

Reflective Property of a Parabola

The tangent line to a parabola at a point P makes equal angles with the following two lines (see Figure 9.13).

1. The line passing through P and the focus
2. The axis of the parabola

Figure 9.13

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Example 7 Finding the Tangent Line at a Point on a Parabola

Find the equation of the tangent line to the parabola given by $y = x^2$ at the point $(1, 1)$.

Solution

For this parabola, $p = \frac{1}{4}$ and the focus is $\left(0, \frac{1}{4}\right)$, as shown in Figure 9.14. You can find the y-intercept $(0, b)$ of the tangent line by equating the lengths of the two sides of the isosceles triangle shown in Figure 9.14:

$$d_1 = \frac{1}{4} - b$$

and

$$d_2 = \sqrt{(1-0)^2 + \left(1 - \frac{1}{4}\right)^2} = \frac{5}{4}.$$

Figure 9.14

Note that $d_1 = \frac{1}{4} - b$ rather than $b - \frac{1}{4}$. The order of subtraction for the distance is important because the distance must be positive. Setting $d_1 = d_2$ produces

$$\frac{1}{4} - b = \frac{5}{4}$$

$$b = -1.$$

So, the slope of the tangent line is

$$m = \frac{1 - (-1)}{1 - 0} = 2$$

and the equation of the tangent line in slope-intercept form is

$$y = 2x - 1.$$

CHECKPOINT Now try Exercise 93.

Technology Tip

Try using a graphing utility to confirm the result of Example 7. By graphing

$$y_1 = x^2 \quad \text{and} \quad y_2 = 2x - 1$$

in the same viewing window, you should be able to see that the line touches the parabola at the point $(1, 1)$.

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9.1 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1–4, fill in the blank(s).

1. A _____ is the intersection of a plane and a double-napped cone.
2. A collection of points satisfying a geometric property can also be referred to as a _____ of points.
3. A _____ is the set of all points (x, y) in a plane that are equidistant from a fixed point, called the _____ .
4. A _____ is the set of all points (x, y) in a plane that are equidistant from a fixed line, called the _____ , and a fixed point, called the _____ , not on the line.
5. What does the equation $(x - h)^2 + (y - k)^2 = r^2$ represent? What do h, k, and r represent?
6. The tangent line to a parabola at a point P makes equal angles with what two lines?

Procedures and Problem Solving

Finding the Standard Equation of a Circle **In Exercises 7–12, find the standard form of the equation of the circle with the given characteristics.**

7. Center at origin; radius: 4
8. Center at origin; radius: $4\sqrt{2}$
9. Center: $(3, 7)$; point on circle: $(1, 0)$
10. Center: $(6, -3)$; point on circle: $(-2, 4)$
11. Center: $(-3, -1)$; diameter: $2\sqrt{7}$
12. Center: $(5, -6)$; diameter: $4\sqrt{3}$

Identifying the Center and Radius of a Circle **In Exercises 13–18, identify the center and radius of the circle.**

13. $x^2 + y^2 = 49$
14. $x^2 + y^2 = 64$
15. $(x + 2)^2 + (y - 7)^2 = 16$
16. $(x + 9)^2 + (y + 1)^2 = 36$
17. $(x - 1)^2 + y^2 = 15$
18. $x^2 + (y + 12)^2 = 40$

Writing the Equation of a Circle in Standard Form **In Exercises 19–26, write the equation of the circle in standard form. Then identify its center and radius.**

19. $\frac{1}{4}x^2 + \frac{1}{4}y^2 = 1$
20. $\frac{1}{9}x^2 + \frac{1}{9}y^2 = 1$
21. $\frac{4}{3}x^2 + \frac{4}{3}y^2 = 1$
22. $\frac{9}{2}x^2 + \frac{9}{2}y^2 = 1$
23. $x^2 + y^2 - 2x + 6y + 9 = 0$
24. $x^2 + y^2 - 10x - 6y + 25 = 0$
25. $4x^2 + 4y^2 + 12x - 24y + 41 = 0$
26. $9x^2 + 9y^2 + 54x - 36y + 17 = 0$

Sketching a Circle **In Exercises 27–34, sketch the circle. Identify its center and radius.**

27. $x^2 = 16 - y^2$
28. $y^2 = 81 - x^2$
29. $x^2 + 4x + y^2 + 4y - 1 = 0$
30. $x^2 - 6x + y^2 + 6y + 14 = 0$
31. $x^2 - 14x + y^2 + 8y + 40 = 0$
32. $x^2 + 6x + y^2 - 12y + 41 = 0$
33. $x^2 + 2x + y^2 - 35 = 0$
34. $x^2 + y^2 + 10y + 9 = 0$

Finding the Intercepts of a Circle **In Exercises 35–40, find the x- and y-intercepts of the graph of the circle.**

35. $(x - 2)^2 + (y + 3)^2 = 9$
36. $(x + 5)^2 + (y - 4)^2 = 25$
37. $x^2 - 2x + y^2 - 6y - 27 = 0$
38. $x^2 + 8x + y^2 + 2y + 9 = 0$
39. $(x - 6)^2 + (y + 3)^2 = 16$
40. $(x + 7)^2 + (y - 8)^2 = 4$
41. **Seismology** An earthquake was felt up to 52 miles from its epicenter. You were located 40 miles west and 30 miles south of the epicenter.
    (a) Let the epicenter be at the point $(0, 0)$. Find the standard equation that describes the outer boundary of the earthquake.
    (b) Would you have felt the earthquake?
    (c) Verify your answer to part (b) by graphing the equation of the outer boundary of the earthquake and plotting your location. How far were you from the outer boundary of the earthquake?
42. **Landscape Design** A landscaper has installed a circular sprinkler that covers an area of 2000 square feet.
    (a) Find the radius of the region covered by the sprinkler. Round your answer to three decimal places.
    (b) The landscaper increases the area covered to 2500 square feet by increasing the water pressure. How much longer is the radius?

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Matching an Equation with a Graph **In Exercises 43–48, match the equation with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]**

(a)

(b)

(c)

(d)

(e)

(f)

43. $y^2 = -4x$ **44.** $x^2 = 2y$

45. $x^2 = -8y$ **46.** $y^2 = -12x$

47. $(y - 1)^2 = 4(x - 3)$ **48.** $(x + 3)^2 = -2(y - 1)$

Finding the Standard Equation of a Parabola **In Exercises 49–60, find the standard form of the equation of the parabola with the given characteristic(s) and vertex at the origin.**

49.

50.

51. Focus: $\left(0, -\frac{3}{2}\right)$ **52.** Focus: $\left(\frac{5}{2}, 0\right)$

53. Focus: $(-2, 0)$ **54.** Focus: $(0, -2)$

55. Directrix: $y = 1$ **56.** Directrix: $y = -3$

57. Directrix: $x = 2$ **58.** Directrix: $x = -3$

59. Horizontal axis and passes through the point $(4, 6)$

60. Vertical axis and passes through the point $(-3, -3)$

Finding the Vertex, Focus, and Directrix of a Parabola **In Exercises 61–78, find the vertex, focus, and directrix of the parabola and sketch its graph. Use a graphing utility to verify your graph.**

61. $y = \frac{1}{2}x^2$ **62.** $y = -2x^2$

63. $y^2 = -6x$ **64.** $y^2 = 3x$

65. $x^2 + 6y = 0$ **66.** $x + y^2 = 0$

67. $(x + 1)^2 + 8(y + 2) = 0$

68. $(x - 5) + (y + 4)^2 = 0$

69. $y^2 + 6y + 8x + 25 = 0$

70. $y^2 - 4y - 4x = 0$

71. $\left(x + \frac{3}{2}\right)^2 = 4(y - 2)$ **72.** $\left(x + \frac{1}{2}\right)^2 = 4(y - 1)$

73. $y = \frac{1}{4}(x^2 - 2x + 5)$ **74.** $x = \frac{1}{4}(y^2 + 2y + 33)$

75. $x^2 + 4x + 6y - 2 = 0$ **76.** $x^2 - 2x + 8y + 9 = 0$

77. $y^2 + x + y = 0$ **78.** $y^2 - 4x - 4 = 0$

Finding the Standard Equation of a Parabola **In Exercises 79–90, find the standard form of the equation of the parabola with the given characteristics.**

79.

80.

81.

82.

83. Vertex: $(-2, 0)$; focus: $\left(-\frac{3}{2}, 0\right)$

84. Vertex: $(3, -3)$; focus: $\left(3, -\frac{9}{4}\right)$

85. Vertex: $(5, 2)$; focus: $(3, 2)$

86. Vertex: $(-1, 2)$; focus: $(-1, 0)$

87. Vertex: $(0, 4)$; directrix: $y = 2$

88. Vertex: $(-2, 1)$; directrix: $x = 1$

89. Focus: $(2, 2)$; directrix: $x = -2$

90. Focus: $(0, 0)$; directrix: $y = 8$

Determining the Point of Tangency **In Exercises 91 and 92, the equations of a parabola and a tangent line to the parabola are given. Use a graphing utility to graph both in the same viewing window. Determine the coordinates of the point of tangency.**

	Parabola	*Tangent Line*
91.	$y^2 - 8x = 0$	$x - y + 2 = 0$
92.	$x^2 + 12y = 0$	$x + y - 3 = 0$

Finding the Tangent Line at a Point on a Parabola **In Exercises 93–96, find an equation of the tangent line to the parabola at the given point and find the x-intercept of the line.**

93. $x^2 = 2y$, $(4, 8)$

94. $x^2 = 2y$, $\left(-3, \frac{9}{2}\right)$

95. $y = -2x^2$, $(-1, -2)$

96. $y = -2x^2$, $(2, -8)$

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97. Architectural Design A simply supported beam is 64 feet long and has a load at the center (see figure). The deflection (bending) of the beam at its center is 1 inch. The shape of the deflected beam is parabolic.

(a) Find an equation of the parabola. (Assume that the origin is at the center of the beam.)

(b) How far from the center of the beam is the deflection equal to $\frac{1}{2}$ inch?

98. Architectural Design Repeat Exercise 97 when the length of the beam is 36 feet and the deflection of the beam at its center is 2 inches.

99. Mechanical Engineering The filament of an automobile headlight is at the focus of a parabolic reflector, which sends light out in a straight beam (see figure).

(a) The filament of the headlight is 1.5 inches from the vertex. Find an equation for the cross section of the reflector.

(b) The reflector is 8 inches wide. Find the depth of the reflector.

100. Environmental Science Water is flowing from a horizontal pipe 48 feet above the ground. The falling stream of water has the shape of a parabola whose vertex $(0, 48)$ is at the end of the pipe (see figure). The stream of water strikes the ocean at the point $(10\sqrt{3}, 0)$. Find the equation of the path taken by the water.

101. MODELING DATA

A cable of the Golden Gate Bridge is suspended (in the shape of a parabola) between two towers that are 1280 meters apart. The top of each tower is 152 meters above the roadway. The cable touches the roadway midway between the towers.

(a) Draw a sketch of the cable. Locate the origin of a rectangular coordinate system at the center of the roadway. Label the coordinates of the known points.

(b) Write an equation that models the cable.

(c) Complete the table by finding the height y of the suspension cable over the roadway at a distance of x meters from the center of the bridge.

x	0	200	400	500	600
y					

102. Transportation Design Roads are often designed with parabolic surfaces to allow rain to drain off. A particular road that is 32 feet wide is 0.4 foot higher in the center than it is on the sides (see figure).

(a) Find an equation of the parabola with its vertex at the origin that models the road surface.

(b) How far from the center of the road is the road surface 0.1 foot lower than in the middle?

103. ***Why you should learn it*** *(p. 636)* Road engineers design a parabolic entrance ramp from a straight street to an interstate highway (see figure). Find an equation of the parabola.

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104. **Astronomy** A satellite in a 100-mile-high circular orbit around Earth has a velocity of approximately 17,500 miles per hour. When this velocity is multiplied by $\sqrt{2}$, the satellite has the minimum velocity necessary to escape Earth's gravity, and follows a parabolic path with the center of Earth as the focus (see figure).

(a) Find the escape velocity of the satellite.

(b) Find an equation of its path (assume the radius of Earth is 4000 miles).

Projectile Motion **In Exercises 105 and 106, consider the path of a projectile projected horizontally with a velocity of v feet per second at a height of s feet, where the model for the path is**

$$x^2 = -\frac{v^2}{16}(y - s).$$

In this model (in which air resistance is disregarded), y is the height (in feet) of the projectile and x is the horizontal distance (in feet) the projectile travels.

105. A ball is thrown from the top of a 100-foot tower with a velocity of 28 feet per second.

(a) Find the equation of the parabolic path.

(b) How far does the ball travel horizontally before striking the ground?

106. A cargo plane is flying at an altitude of 30,000 feet and a speed of 540 miles per hour. A supply crate is dropped from the plane. How many *feet* will the crate travel horizontally before it hits the ground?

Finding the Tangent Line at a Point on a Circle **In Exercises 107–110, find an equation of the tangent line to the circle at the indicated point. Recall from geometry that the tangent line to a circle is perpendicular to the radius of the circle at the point of tangency.**

	Circle	*Point*
107.	$x^2 + y^2 = 25$	$(3, -4)$
108.	$x^2 + y^2 = 169$	$(-5, 12)$
109.	$x^2 + y^2 = 12$	$(2, -2\sqrt{2})$
110.	$x^2 + y^2 = 24$	$(-2\sqrt{5}, 2)$

Conclusions

True or False? **In Exercises 111–117, determine whether the statement is true or false. Justify your answer.**

111. The equation $x^2 + (y + 5)^2 = 25$ represents a circle with its center at the origin and a radius of 5.

112. The graph of the equation $x^2 + y^2 = r^2$ will have x-intercepts $(\pm r, 0)$ and y-intercepts $(0, \pm r)$.

113. A circle is a degenerate conic.

114. It is possible for a parabola to intersect its directrix.

115. The point which lies on the graph of a parabola closest to its focus is the vertex of the parabola.

116. The directrix of the parabola $x^2 = y$ intersects, or is tangent to, the graph of the parabola at its vertex, $(0, 0)$.

117. If the vertex and focus of a parabola are on a horizontal line, then the directrix of the parabola is a vertical line.

118. **CAPSTONE** In parts (a)–(d), describe in words how a plane could intersect with the double-napped cone to form the conic section (see figure).

(a) Circle (b) Ellipse

(c) Parabola (d) Hyperbola

119. **Think About It** The equation $x^2 + y^2 = 0$ is a degenerate conic. Sketch the graph of this equation and identify the degenerate conic. Describe the intersection of the plane with the double-napped cone for this particular conic.

Think About It **In Exercises 120 and 121, change the equation so that its graph matches the description.**

120. $(y - 3)^2 = 6(x + 1)$; upper half of parabola

121. $(y + 1)^2 = 2(x - 2)$; lower half of parabola

Cumulative Mixed Review

Approximating Relative Minimum and Maximum Values **In Exercises 122–125, use a graphing utility to approximate any relative minimum or maximum values of the function.**

122. $f(x) = 3x^3 - 4x + 2$

123. $f(x) = 2x^2 + 3x$

124. $f(x) = x^4 + 2x + 2$

125. $f(x) = x^5 - 3x - 1$

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9.2 Ellipses

Introduction

The third type of conic is called an **ellipse.** It is defined as follows.

Definition of an Ellipse

An **ellipse** is the set of all points (x, y) in a plane, the sum of whose distances from two distinct fixed points (**foci**) is constant. [See Figure 9.15(a).]

(a) **(b)**

Figure 9.15

The line through the foci intersects the ellipse at two points called **vertices.** The chord joining the vertices is the **major axis,** and its midpoint is the **center** of the ellipse. The chord perpendicular to the major axis at the center is the **minor axis.** [See Figure 9.15(b).]

You can visualize the definition of an ellipse by imagining two thumbtacks placed at the foci, as shown in Figure 9.16. If the ends of a fixed length of string are fastened to the thumbtacks and the string is drawn taut with a pencil, then the path traced by the pencil will be an ellipse.

Figure 9.16

$$2\sqrt{b^2 + c^2} = 2a$$
$$b^2 + c^2 = a^2$$

Figure 9.17

To derive the standard form of the equation of an ellipse, consider the ellipse in Figure 9.17 with the following points.

Center: (h, k) Vertices: $(h \pm a, k)$ Foci: $(h \pm c, k)$

Note that the center is the midpoint of the segment joining the foci.

The sum of the distances from any point on the ellipse to the two foci is constant. Using a vertex point, this constant sum is

$$(a + c) + (a - c) = 2a \qquad \text{Length of major axis}$$

or simply the length of the major axis.

What you should learn

- Write equations of ellipses in standard form.
- Use properties of ellipses to model and solve real-life problems.
- Find eccentricities of ellipses.

Why you should learn it

Ellipses can be used to model and solve many types of real-life problems. For instance, Exercise 58 on page 654 shows how the focal properties of an ellipse are used by a lithotripter machine to break up kidney stones.

Urologist

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Now, if you let (x, y) be *any* point on the ellipse, then the sum of the distances between (x, y) and the two foci must also be $2a$. That is,

$$\sqrt{[x-(h-c)]^2+(y-k)^2}+\sqrt{[x-(h+c)]^2+(y-k)^2}=2a$$

which, after expanding and regrouping, reduces to

$$(a^2-c^2)(x-h)^2+a^2(y-k)^2=a^2(a^2-c^2).$$

Finally, in Figure 9.17, you can see that

$$b^2=a^2-c^2$$

which implies that the equation of the ellipse is

$$b^2(x-h)^2+a^2(y-k)^2=a^2b^2$$

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1.$$

You would obtain a similar equation in the derivation by starting with a vertical major axis. Both results are summarized as follows.

Standard Equation of an Ellipse

The **standard form of the equation of an ellipse** with center (h, k) and major and minor axes of lengths $2a$ and $2b$, respectively, where $0 < b < a$, is

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1 \qquad \text{Major axis is horizontal.}$$

$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1. \qquad \text{Major axis is vertical.}$$

The foci lie on the major axis, c units from the center, with

$$c^2=a^2-b^2.$$

If the center is at the origin $(0, 0)$, then the equation takes one of the following forms.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \text{Major axis is horizontal.}$$

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1 \qquad \text{Major axis is vertical.}$$

Explore the Concept

On page 647, it was noted that an ellipse can be drawn using two thumbtacks, a string of fixed length (greater than the distance between the two tacks), and a pencil. Try doing this. Vary the length of the string and the distance between the thumbtacks. Explain how to obtain ellipses that are almost circular. Explain how to obtain ellipses that are long and narrow.

Figure 9.18 shows both the vertical and horizontal orientations for an ellipse.

Major axis is horizontal. *Major axis is vertical.*

Figure 9.18

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Example 1 Finding the Standard Equation of an Ellipse

Find the standard form of the equation of the ellipse having foci at

$$(0, 1) \quad \text{and} \quad (4, 1)$$

and a major axis of length 6, as shown in Figure 9.19.

Figure 9.19

Solution

By the Midpoint Formula, the center of the ellipse is (2, 1) and the distance from the center to one of the foci is $c = 2$. Because $2a = 6$, you know that $a = 3$. Now, from $c^2 = a^2 - b^2$, you have

$$b = \sqrt{a^2 - c^2} = \sqrt{9 - 4} = \sqrt{5}.$$

Because the major axis is horizontal, the standard equation is

$$\frac{(x-2)^2}{3^2} + \frac{(y-1)^2}{(\sqrt{5})^2} = 1.$$

CHECKPOINT Now try Exercise 31.

Example 2 Sketching an Ellipse

Sketch the ellipse given by

$$4x^2 + y^2 = 36$$

and identify the center and vertices.

Algebraic Solution

$$4x^2 + y^2 = 36 \qquad \text{Write original equation.}$$

$$\frac{4x^2}{36} + \frac{y^2}{36} = \frac{36}{36} \qquad \text{Divide each side by 36.}$$

$$\frac{x^2}{3^2} + \frac{y^2}{6^2} = 1 \qquad \text{Write in standard form.}$$

The center of the ellipse is (0, 0). Because the denominator of the y^2-term is larger than the denominator of the x^2-term, you can conclude that the major axis is vertical. Moreover, because $a = 6$, the vertices are $(0, -6)$ and $(0, 6)$. Finally, because $b = 3$, the endpoints of the minor axis are $(-3, 0)$ and $(3, 0)$, as shown in Figure 9.20.

Figure 9.20

CHECKPOINT Now try Exercise 37.

Graphical Solution

Solve the equation of the ellipse for y as follows.

$$4x^2 + y^2 = 36$$

$$y^2 = 36 - 4x^2$$

$$y = \pm\sqrt{36 - 4x^2}$$

Then use a graphing utility to graph

$$y_1 = \sqrt{36 - 4x^2}$$

and

$$y_2 = -\sqrt{36 - 4x^2}$$

in the same viewing window, as shown in Figure 9.21. Be sure to use a square setting.

Figure 9.21

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Example 3 Graphing an Ellipse

Graph the ellipse given by $x^2 + 4y^2 + 6x - 8y + 9 = 0$.

Solution

Begin by writing the original equation in standard form. In the third step, note that 9 and 4 are added to *both* sides of the equation when completing the squares.

$$x^2 + 4y^2 + 6x - 8y + 9 = 0 \qquad \text{Write original equation.}$$

$$(x^2 + 6x + \square) + 4(y^2 - 2y + \square) = -9 \qquad \text{Group terms and factor 4 out of } y\text{-terms.}$$

$$(x^2 + 6x + 9) + 4(y^2 - 2y + 1) = -9 + 9 + 4(1) \qquad \text{Complete the square.}$$

$$(x + 3)^2 + 4(y - 1)^2 = 4 \qquad \text{Write in completed square form.}$$

$$\frac{(x + 3)^2}{2^2} + \frac{(y - 1)^2}{1^2} = 1 \qquad \text{Write in standard form.}$$

Now you see that the center is $(h, k) = (-3, 1)$. Because the denominator of the x-term is $a^2 = 2^2$, the endpoints of the major axis lie two units to the right and left of the center. Similarly, because the denominator of the y-term is $b^2 = 1^2$, the endpoints of the minor axis lie one unit up and down from the center. The graph of this ellipse is shown in Figure 9.22.

Figure 9.22

CHECKPOINT Now try Exercise 41.

Example 4 Analyzing an Ellipse

Find the center, vertices, and foci of the ellipse $4x^2 + y^2 - 8x + 4y - 8 = 0$.

Solution

By completing the square, you can write the original equation in standard form.

$$4x^2 + y^2 - 8x + 4y - 8 = 0 \qquad \text{Write original equation.}$$

$$4(x^2 - 2x + \square) + (y^2 + 4y + \square) = 8 \qquad \text{Group terms and factor 4 out of } x\text{-terms.}$$

$$4(x^2 - 2x + 1) + (y^2 + 4y + 4) = 8 + 4(1) + 4 \qquad \text{Complete the square.}$$

$$4(x - 1)^2 + (y + 2)^2 = 16 \qquad \text{Write in completed square form.}$$

$$\frac{(x - 1)^2}{2^2} + \frac{(y + 2)^2}{4^2} = 1 \qquad \text{Write in standard form.}$$

So, the major axis is vertical, where $h = 1$, $k = -2$, $a = 4$, $b = 2$, and

$$c = \sqrt{a^2 - b^2} = \sqrt{16 - 4} = \sqrt{12} = 2\sqrt{3}.$$

Therefore, you have the following.

Center: $(1, -2)$

Vertices: $(1, -6)$

$(1, 2)$

Foci: $(1, -2 - 2\sqrt{3})$

$(1, -2 + 2\sqrt{3})$

The graph of the ellipse is shown in Figure 9.23.

Figure 9.23

CHECKPOINT Now try Exercise 43.

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Application

Ellipses have many practical and aesthetic uses. For instance, machine gears, supporting arches, and acoustic designs often involve elliptical shapes. The orbits of satellites and planets are also ellipses. Example 5 investigates the elliptical orbit of the moon about Earth.

Example 5 An Application Involving an Elliptical Orbit

The moon travels about Earth in an elliptical orbit with Earth at one focus, as shown in Figure 9.24. The major and minor axes of the orbit have lengths of 768,800 kilometers and 767,640 kilometers, respectively. Find the greatest and least distances (the *apogee* and *perigee*) from Earth's center to the moon's center. Then graph the orbit of the moon on a graphing utility.

Figure 9.24

Study Tip

Note in Example 5 and Figure 9.24 that Earth *is not* the center of the moon's orbit.

Astronaut

Solution

Because $2a = 768{,}800$ and $2b = 767{,}640$, you have

$$a = 384{,}400 \qquad \text{and} \qquad b = 383{,}820$$

which implies that

$$\begin{aligned} c &= \sqrt{a^2 - b^2} \\ &= \sqrt{384{,}400^2 - 383{,}820^2} \\ &\approx 21{,}108. \end{aligned}$$

So, the greatest distance between the center of Earth and the center of the moon is

$$\begin{aligned} a + c &\approx 384{,}400 + 21{,}108 \\ &= 405{,}508 \text{ kilometers} \end{aligned}$$

and the least distance is

$$\begin{aligned} a - c &\approx 384{,}400 - 21{,}108 \\ &= 363{,}292 \text{ kilometers.} \end{aligned}$$

To graph the orbit of the moon on a graphing utility, first let $a = 384{,}400$ and $b = 383{,}820$ in the standard form of an equation of an ellipse centered at the origin, and then solve for y.

$$\frac{x^2}{384{,}400^2} + \frac{y^2}{383{,}820^2} = 1 \quad \Rightarrow \quad y = \pm 383{,}820\sqrt{1 - \frac{x^2}{384{,}400^2}}$$

Graph the upper and lower portions in the same viewing window, as shown in Figure 9.25.

Figure 9.25

CHECKPOINT Now try Exercise 59.

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Eccentricity

One of the reasons it was difficult for early astronomers to detect that the orbits of the planets are ellipses is that the foci of the planetary orbits are relatively close to their centers, and so the orbits are nearly circular. To measure the ovalness of an ellipse, you can use the concept of **eccentricity.**

Definition of Eccentricity

The **eccentricity** e of an ellipse is given by the ratio $e = \dfrac{c}{a}$.

Note that $0 < e < 1$ for *every* ellipse.

To see how this ratio is used to describe the shape of an ellipse, note that because the foci of an ellipse are located along the major axis between the vertices and the center, it follows that

$$0 < c < a.$$

For an ellipse that is nearly circular, the foci are close to the center and the ratio c/a is close to 0 [see Figure 9.26(a)]. On the other hand, for an elongated ellipse, the foci are close to the vertices and the ratio c/a is close to 1 [see Figure 9.26(b)].

(a) **(b)**

Figure 9.26

The orbit of the moon has an eccentricity of

$$e \approx 0.0549 \qquad \text{Eccentricity of the moon}$$

and the eccentricities of the eight planetary orbits are as follows.

Planet	Eccentricity, e
Mercury	0.2056
Venus	0.0068
Earth	0.0167
Mars	0.0934
Jupiter	0.0484
Saturn	0.0542
Uranus	0.0472
Neptune	0.0086

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9.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1–4, fill in the blank(s).

1. An _____ is the set of all points (x, y) in a plane, the sum of whose distances from two distinct fixed points called _____ is constant.
2. The chord joining the vertices of an ellipse is called the _____ , and its midpoint is the _____ of the ellipse.
3. The chord perpendicular to the major axis at the center of an ellipse is called the _____ of the ellipse.
4. The eccentricity e of an ellipse is given by the ratio $e =$ _____.

In Exercises 5–8, consider the ellipse given by $\frac{x^2}{2^2} + \frac{y^2}{8^2} = 1$.

5. Is the major axis horizontal or vertical?
6. What is the length of the major axis?
7. What is the length of the minor axis?
8. Is the ellipse elongated or nearly circular?

Procedures and Problem Solving

Identifying the Equation of an Ellipse **In Exercises 9–12, match the equation with its graph. [The graphs are labeled (a), (b), (c), and (d).]**

(a)

(b)

(c)

(d)

9. $\frac{x^2}{4} + \frac{y^2}{9} = 1$
10. $\frac{x^2}{9} + \frac{y^2}{4} = 1$
11. $\frac{(x-2)^2}{16} + (y+1)^2 = 1$
12. $\frac{(x+2)^2}{9} + \frac{(y+2)^2}{4} = 1$

Using the Standard Equation of an Ellipse **In Exercises 13–18, find the center, vertices, foci, and eccentricity of the ellipse, and sketch its graph. Use a graphing utility to verify your graph.**

13. $\frac{x^2}{64} + \frac{y^2}{9} = 1$
14. $\frac{x^2}{16} + \frac{y^2}{81} = 1$
15. $\frac{(x-4)^2}{16} + \frac{(y+1)^2}{25} = 1$
16. $\frac{(x+3)^2}{12} + \frac{(y-2)^2}{16} = 1$
17. $\frac{(x+5)^2}{\frac{9}{4}} + (y-1)^2 = 1$
18. $(x+2)^2 + \frac{(y+4)^2}{\frac{1}{4}} = 1$

An Ellipse Centered at the Origin **In Exercises 19–26, find the standard form of the equation of the ellipse with the given characteristics and center at the origin.**

19.

20.

21. Vertices: $(\pm 3, 0)$; foci: $(\pm 2, 0)$
22. Vertices: $(0, \pm 8)$; foci: $(0, \pm 4)$
23. Foci: $(\pm 5, 0)$; major axis of length 14
24. Foci: $(\pm 2, 0)$; major axis of length 10
25. Vertices: $(0, \pm 5)$; passes through the point $(4, 2)$
26. Vertical major axis; passes through points $(0, 6)$ and $(3, 0)$

Finding the Standard Equation of an Ellipse **In Exercises 27–36, find the standard form of the equation of the ellipse with the given characteristics.**

27.

28.

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29. Vertices: $(0, 2), (8, 2)$; minor axis of length 2

30. Foci: $(0, 0), (4, 0)$; major axis of length 6

31. Foci: $(0, 0), (0, 8)$; major axis of length 36

32. Center: $(2, -1)$; vertex: $\left(2, \frac{1}{2}\right)$; minor axis of length 2

33. Vertices: $(3, 1), (3, 9)$; minor axis of length 6

34. Center: $(3, 2)$; $a = 3c$; foci: $(1, 2), (5, 2)$

35. Center: $(0, 4)$; $a = 2c$; vertices: $(-4, 4), (4, 4)$

36. Vertices: $(5, 0), (5, 12)$; endpoints of the minor axis: $(0, 6), (10, 6)$

Using the Standard Equation of an Ellipse **In Exercises 37–48, (a) find the standard form of the equation of the ellipse, (b) find the center, vertices, foci, and eccentricity of the ellipse, and (c) sketch the ellipse. Use a graphing utility to verify your graph.**

37. $x^2 + 9y^2 = 36$

38. $16x^2 + y^2 = 16$

39. $49x^2 + 4y^2 - 196 = 0$

40. $4x^2 + 49y^2 - 196 = 0$

41. $9x^2 + 4y^2 + 36x - 24y + 36 = 0$

42. $9x^2 + 4y^2 - 54x + 40y + 37 = 0$

43. $6x^2 + 2y^2 + 18x - 10y + 2 = 0$

44. $x^2 + 4y^2 - 6x + 20y - 2 = 0$

45. $16x^2 + 25y^2 - 32x + 50y + 16 = 0$

46. $9x^2 + 25y^2 - 36x - 50y + 61 = 0$

47. $12x^2 + 20y^2 - 12x + 40y - 37 = 0$

48. $36x^2 + 9y^2 + 48x - 36y + 43 = 0$

Finding Eccentricity **In Exercises 49–52, find the eccentricity of the ellipse.**

49. $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

50. $\dfrac{x^2}{25} + \dfrac{y^2}{49} = 1$

51. $x^2 + 9y^2 - 10x + 36y + 52 = 0$

52. $4x^2 + 3y^2 - 8x + 18y + 19 = 0$

53. **Using Eccentricity** Find an equation of the ellipse with vertices $(\pm 5, 0)$ and eccentricity $e = \frac{3}{5}$.

54. **Using Eccentricity** Find an equation of the ellipse with vertices $(0, \pm 8)$ and eccentricity $e = \frac{1}{2}$.

55. **Using Eccentricity** Find an equation of the ellipse with foci $(\pm 3, 0)$ and eccentricity $e = \frac{4}{5}$.

56. **Architecture** Statuary Hall is an elliptical room in the United States Capitol Building in Washington, D.C. The room is also referred to as the Whispering Gallery because a person standing at one focus of the room can hear even a whisper spoken by a person standing at the other focus. Given that the dimensions of Statuary Hall are 46 feet wide by 97 feet long, find an equation for the shape of the floor surface of the hall. Determine the distance between the foci.

57. **Architecture** A fireplace arch is to be constructed in the shape of a semiellipse. The opening is to have a height of 2 feet at the center and a width of 6 feet along the base (see figure). The contractor draws the outline of the ellipse on the wall by the method discussed on page 647. Give the required positions of the tacks and the length of the string.

58. ***Why you should learn it*** *(p. 647)* A lithotripter machine uses an elliptical reflector to break up kidney stones nonsurgically. A spark plug in the reflector generates energy waves at one focus of an ellipse. The reflector directs these waves toward the kidney stone positioned at the other focus of the ellipse with enough energy to break up the stone,as shown in the figure. The lengths of the major and minor axes of the ellipse are 280 millimeters and 160 millimeters, respectively. How far is the spark from the kidney stone?

59. **Astronomy** Halley's comet has an elliptical orbit with the sun at one focus. The eccentricity of the orbit is approximately 0.97. The length of the major axis of the orbit is about 35.88 astronomical units. (An astronomical unit is about 93 million miles.) Find the standard form of the equation of the orbit. Place the center of the orbit at the origin and place the major axis on the x-axis.

60. **Geometry** The area of the ellipse in the figure is twice the area of the circle. How long is the major axis? (*Hint:* The area of an ellipse is given by $A = \pi ab$.)

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61. **Aeronautics** The first artificial satellite to orbit Earth was Sputnik I (launched by the former Soviet Union in 1957). Its highest point above Earth's surface was 947 kilometers, and its lowest point was 228 kilometers. The center of Earth was a focus of the elliptical orbit, and the radius of Earth is 6378 kilometers (see figure). Find the eccentricity of the orbit.

62. **Geometry** A line segment through a focus with endpoints on an ellipse, perpendicular to the major axis, is called a **latus rectum** of the ellipse. Therefore, an ellipse has two latera recta. Knowing the length of the latera recta is helpful in sketching an ellipse because this information yields other points on the curve (see figure). Show that the length of each latus rectum is $2b^2/a$.

Using Latera Recta **In Exercises 63–66, sketch the ellipse using the latera recta (see Exercise 62).**

63. $\dfrac{x^2}{4} + \dfrac{y^2}{1} = 1$

64. $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$

65. $9x^2 + 4y^2 = 36$

66. $5x^2 + 3y^2 = 15$

Conclusions

True or False? **In Exercises 67 and 68, determine whether the statement is true or false. Justify your answer.**

67. It is easier to distinguish the graph of an ellipse from the graph of a circle when the eccentricity of the ellipse is large (close to 1).

68. The area of a circle with diameter $d = 2r = 8$ is greater than the area of an ellipse with major axis $2a = 8$.

69. **Think About It** Consider the ellipse

$$\frac{x^2}{328} + \frac{y^2}{327} = 1.$$

Is this ellipse better described as *elongated* or *nearly circular*? Explain your reasoning.

70. **CAPSTONE** Consider the ellipse shown.

(a) Identify the center, vertices, and foci of the ellipse.

(b) Write the standard form of the equation of the ellipse.

(c) Find the eccentricity e of the ellipse.

71. **Think About It** At the beginning of this section, it was noted that an ellipse can be drawn using two thumbtacks, a string of fixed length (greater than the distance between the two tacks), and a pencil (see Figure 9.16). When the ends of the string are fastened at the tacks and the string is drawn taut with a pencil, the path traced by the pencil is an ellipse.

(a) What is the length of the string in terms of a?

(b) Explain why the path is an ellipse.

72. **Error Analysis** Describe the error in finding the distance between the foci.

$$c^2 = a^2 + b^2$$
$$= 64 + 36$$
$$= 100$$

So, $c = 10$ and the distance between the foci is 20 inches.

73. **Think About It** Find the equation of an ellipse such that for any point on the ellipse, the sum of the distances from the points (2, 2) and (10, 2) is 36.

74. **Proof** Show that $a^2 = b^2 + c^2$ for the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $a > 0$, $b > 0$, and the distance from the center of the ellipse (0, 0) to a focus is c.

Cumulative Mixed Review

Identifying a Sequence **In Exercises 75–78, determine whether the sequence is arithmetic, geometric, or neither.**

75. 66, 55, 44, 33, 22, . . .

76. 80, 40, 20, 10, 5, . . .

77. $\frac{1}{4}, \frac{1}{2}, 1, 2, 4, \ldots$

78. $-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \ldots$

Finding the Sum of a Finite Geometric Sequence **In Exercises 79 and 80, find the sum.**

79. $\displaystyle\sum_{n=0}^{6} 3^n$

80. $\displaystyle\sum_{n=1}^{10} 4\left(\frac{3}{4}\right)^{n-1}$

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9.3 Hyperbolas and Rotation of Conics

Introduction

The definition of a **hyperbola** is similar to that of an ellipse. The difference is that for an ellipse, the *sum* of the distances between the foci and a point on the ellipse is constant; whereas for a hyperbola, the *difference* of the distances between the foci and a point on the hyperbola is constant.

Definition of a Hyperbola

A **hyperbola** is the set of all points (x, y) in a plane, the difference of whose distances from two distinct fixed points, the **foci,** is a positive constant. [See Figure 9.27(a).]

(a) **(b)**

Figure 9.27

The graph of a hyperbola has two disconnected parts called the **branches.** The line through the two foci intersects the hyperbola at two points called the **vertices.** The line segment connecting the vertices is the **transverse axis,** and the midpoint of the transverse axis is the **center** of the hyperbola [see Figure 9.27(b)]. The development of the **standard form of the equation of a hyperbola** is similar to that of an ellipse. Note, however, that a, b, and c are related differently for hyperbolas than for ellipses. For a hyperbola, the distance between the foci and the center is greater than the distance between the vertices and the center.

Standard Equation of a Hyperbola

The **standard form of the equation of a hyperbola** with center at (h, k) is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \qquad \text{Transverse axis is horizontal.}$$

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1. \qquad \text{Transverse axis is vertical.}$$

The vertices are a units from the center, and the foci are c units from the center. Moreover, $c^2 = a^2 + b^2$. If the center of the hyperbola is at the origin $(0, 0)$, then the equation takes one of the following forms.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad \text{Transverse axis is horizontal.}$$

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \qquad \text{Transverse axis is vertical.}$$

What you should learn

- Write equations of hyperbolas in standard form.
- Find asymptotes of and graph hyperbolas.
- Use properties of hyperbolas to solve real-life problems.
- Classify conics from their general equations.
- Rotate the coordinate axes to eliminate the xy-term in equations of conics.

Why you should learn it

Hyperbolas can be used to model and solve many types of real-life problems. For instance, in Exercise 50 on page 666, hyperbolas are used to locate the position of an explosion that was recorded by three listening stations.

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Figure 9.28 shows both the horizontal and vertical orientations for a hyperbola.

Transverse axis is horizontal. ***Transverse axis is vertical.***

Figure 9.28

Example 1 Finding the Standard Equation of a Hyperbola

Find the standard form of the equation of the hyperbola with foci $(-1, 2)$ and $(5, 2)$ and vertices $(0, 2)$ and $(4, 2)$.

Solution

By the Midpoint Formula, the center of the hyperbola occurs at the point $(2, 2)$. Furthermore, $c = 3$ and $a = 2$, and it follows that

$$\begin{aligned} b &= \sqrt{c^2 - a^2} \\ &= \sqrt{3^2 - 2^2} \\ &= \sqrt{9 - 4} \\ &= \sqrt{5}. \end{aligned}$$

So, the hyperbola has a horizontal transverse axis, and the standard form of the equation of the hyperbola is

$$\frac{(x-2)^2}{2^2} - \frac{(y-2)^2}{(\sqrt{5})^2} = 1.$$

This equation simplifies to

$$\frac{(x-2)^2}{4} - \frac{(y-2)^2}{5} = 1.$$

Figure 9.29 shows the hyperbola.

Figure 9.29

CHECKPOINT Now try Exercise 39.

Technology Tip

You can use a graphing utility to graph a hyperbola by graphing the upper and lower portions in the same viewing window. To do this, you must solve the equation for y before entering it into the graphing utility. When graphing equations of conics, it can be difficult to solve for y, which is why it is very important to know the algebra used to solve equations for y.

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Asymptotes of a Hyperbola

Each hyperbola has two **asymptotes** that intersect at the center of the hyperbola. The asymptotes pass through the corners of a rectangle of dimensions $2a$ by $2b$, with its center at (h, k), as shown in Figure 9.30.

Figure 9.30

Asymptotes of a Hyperbola

$$y = k \pm \frac{b}{a}(x - h)$$ Asymptotes for horizontal transverse axis

$$y = k \pm \frac{a}{b}(x - h)$$ Asymptotes for vertical transverse axis

The **conjugate axis** of a hyperbola is the line segment of length $2b$ joining $(h, k + b)$ and $(h, k - b)$ when the transverse axis is horizontal, and the line segment of length $2b$ joining $(h + b, k)$ and $(h - b, k)$ when the transverse axis is vertical.

Example 2 Sketching a Hyperbola

Sketch the hyperbola whose equation is

$$4x^2 - y^2 = 16.$$

Algebraic Solution

$$4x^2 - y^2 = 16$$ Write original equation.

$$\frac{4x^2}{16} - \frac{y^2}{16} = \frac{16}{16}$$ Divide each side by 16.

$$\frac{x^2}{2^2} - \frac{y^2}{4^2} = 1$$ Write in standard form.

Because the x^2-term is positive, you can conclude that the transverse axis is horizontal. So, the vertices occur at $(-2, 0)$ and $(2, 0)$, the endpoints of the conjugate axis occur at $(0, -4)$ and $(0, 4)$, and you can sketch the rectangle shown in Figure 9.31. Finally, by drawing the asymptotes

$$y = 2x \quad \text{and} \quad y = -2x$$

through the corners of this rectangle, you can complete the sketch, as shown in Figure 9.32.

Figure 9.31

Figure 9.32

CHECKPOINT Now try Exercise 21.

Graphical Solution

Solve the equation of the hyperbola for y, as follows.

$$4x^2 - y^2 = 16$$
$$4x^2 - 16 = y^2$$
$$\pm\sqrt{4x^2 - 16} = y$$

Then use a graphing utility to graph

$$y_1 = \sqrt{4x^2 - 16} \quad \text{and} \quad y_2 = -\sqrt{4x^2 - 16}$$

in the same viewing window, as shown in Figure 9.33. Be sure to use a square setting.

Figure 9.33

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Example 3 Finding the Asymptotes of a Hyperbola

Sketch the hyperbola given by

$$4x^2 - 3y^2 + 8x + 16 = 0$$

and find the equations of its asymptotes.

Solution

$4x^2 - 3y^2 + 8x + 16 = 0$	Write original equation.
$4(x^2 + 2x) - 3y^2 = -16$	Subtract 16 from each side and factor.
$4(x^2 + 2x + 1) - 3y^2 = -16 + 4(1)$	Complete the square.
$4(x + 1)^2 - 3y^2 = -12$	Write in completed square form.
$\dfrac{y^2}{2^2} - \dfrac{(x + 1)^2}{(\sqrt{3})^2} = 1$	Write in standard form.

From this equation you can conclude that the hyperbola has a vertical transverse axis, is centered at $(-1, 0)$, has vertices $(-1, 2)$ and $(-1, -2)$, and has a conjugate axis with endpoints $(-1 - \sqrt{3}, 0)$ and $(-1 + \sqrt{3}, 0)$. To sketch the hyperbola, draw a rectangle through these four points. The asymptotes are the lines passing through the corners of the rectangle, as shown in Figure 9.34. Finally, using $a = 2$ and $b = \sqrt{3}$, you can conclude that the equations of the asymptotes are

$$y = \frac{2}{\sqrt{3}}(x + 1) \quad \text{and} \quad y = -\frac{2}{\sqrt{3}}(x + 1).$$

Figure 9.34

You can verify your sketch using a graphing utility, as shown in Figure 9.35. Notice that the graphing utility does not draw the asymptotes. When you trace along the branches, however, you will see that the values of the hyperbola approach the asymptotes.

Figure 9.35

CHECKPOINT Now try Exercise 25.

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Example 4 Using Asymptotes to Find the Standard Equation

Find the standard form of the equation of the hyperbola having vertices $(3, -5)$ and $(3, 1)$ and having asymptotes

$$y = 2x - 8 \quad \text{and} \quad y = -2x + 4$$

as shown in Figure 9.36.

Figure 9.36

Solution

By the Midpoint Formula, the center of the hyperbola is $(3, -2)$. Furthermore, the hyperbola has a vertical transverse axis with $a = 3$. From the original equations, you can determine the slopes of the asymptotes to be

$$m_1 = 2 = \frac{a}{b} \quad \text{and} \quad m_2 = -2 = -\frac{a}{b}$$

and because $a = 3$, you can conclude that $b = \frac{3}{2}$. So, the standard form of the equation of the hyperbola is

$$\frac{(y+2)^2}{3^2} - \frac{(x-3)^2}{\left(\frac{3}{2}\right)^2} = 1.$$

CHECKPOINT Now try Exercise 45.

As with ellipses, the *eccentricity* of a hyperbola is

$$e = \frac{c}{a} \qquad \text{Eccentricity}$$

and because $c > a$ it follows that $e > 1$. When the eccentricity is large, the branches of the hyperbola are nearly flat, as shown in Figure 9.37(a). When the eccentricity is close to 1, the branches of the hyperbola are more pointed, as shown in Figure 9.37(b).

(a) **(b)**

Figure 9.37

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Applications

The following application was developed during World War II. It shows how the properties of hyperbolas can be used in radar and other detection systems.

Example 5 An Application Involving Hyperbolas

Two microphones, 1 mile apart, record an explosion. Microphone A receives the sound 2 seconds before microphone B. Where did the explosion occur?

Solution

Assuming sound travels at 1100 feet per second, you know that the explosion took place 2200 feet farther from B than from A, as shown in Figure 9.38. The locus of all points that are 2200 feet closer to A than to B is one branch of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where

$$c = \frac{5280}{2} = 2640$$

and

$$a = \frac{2200}{2} = 1100.$$

Figure 9.38

So, $b^2 = c^2 - a^2 = 2640^2 - 1100^2 = 5{,}759{,}600$, and you can conclude that the explosion occurred somewhere on the right branch of the hyperbola

$$\frac{x^2}{1{,}210{,}000} - \frac{y^2}{5{,}759{,}600} = 1.$$

CHECKPOINT Now try Exercise 49.

Another interesting application of conic sections involves the orbits of comets in our solar system. Of the 610 comets identified prior to 1970, 245 have elliptical orbits, 295 have parabolic orbits, and 70 have hyperbolic orbits. The center of the sun is a focus of each of these orbits, and each orbit has a vertex at the point where the comet is closest to the sun, as shown in Figure 9.39. Undoubtedly, there are many comets with parabolic or hyperbolic orbits that have not been identified. You get to see such comets only *once*. Comets with elliptical orbits, such as Halley's comet, are the only ones that remain in our solar system.

Figure 9.39

If p is the distance between the vertex and the focus in meters, and v is the velocity of the comet at the vertex in meters per second, then the type of orbit is determined as follows.

1. Ellipse: $v < \sqrt{2GM/p}$
2. Parabola: $v = \sqrt{2GM/p}$
3. Hyperbola: $v > \sqrt{2GM/p}$

In each of the above, $M \approx 1.989 \times 10^{30}$ kilograms (the mass of the sun) and $G \approx 6.67 \times 10^{-11}$ cubic meter per kilogram-second squared (the universal gravitational constant).

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General Equations of Conics

Classifying a Conic from Its General Equation

The graph of $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is one of the following.

1. Circle:	$A = C$	$A \neq 0$
2. Parabola:	$AC = 0$	$A = 0$ or $C = 0$, but not both.
3. Ellipse:	$AC > 0$	A and C have like signs.
4. Hyperbola:	$AC < 0$	A and C have unlike signs.

The test above is valid when the graph is a *conic*. The test does not apply to equations such as

$$x^2 + y^2 = -1$$

whose graphs are not conics.

Example 6 Classifying Conics from General Equations

Classify the graph of each equation.

a. $4x^2 - 9x + y - 5 = 0$

b. $4x^2 - y^2 + 8x - 6y + 4 = 0$

c. $2x^2 + 4y^2 - 4x + 12y = 0$

d. $2x^2 + 2y^2 - 8x + 12y + 2 = 0$

Solution

a. For the equation $4x^2 - 9x + y - 5 = 0$, you have

$$AC = 4(0) = 0. \qquad \text{Parabola}$$

So, the graph is a parabola.

b. For the equation $4x^2 - y^2 + 8x - 6y + 4 = 0$, you have

$$AC = 4(-1) < 0. \qquad \text{Hyperbola}$$

So, the graph is a hyperbola.

c. For the equation $2x^2 + 4y^2 - 4x + 12y = 0$, you have

$$AC = 2(4) > 0. \qquad \text{Ellipse}$$

So, the graph is an ellipse.

d. For the equation $2x^2 + 2y^2 - 8x + 12y + 2 = 0$, you have

$$A = C = 2. \qquad \text{Circle}$$

So, the graph is a circle.

CHECKPOINT Now try Exercise 55.

Study Tip

Notice in Example 6(a) that there is no y^2-term in the equation. Therefore, $C = 0$.

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Rotation

You have learned that the equation of a conic with axes parallel to one of the coordinates axes has a standard form that can be written in the general form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0. \qquad \text{Horizontal or vertical axis}$$

You will now study the equations of conics whose axes are rotated so that they are not parallel to either the x-axis or the y-axis. The general equation for such conics contains an xy-term.

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \qquad \text{Equation in } xy\text{-plane}$$

To eliminate this xy-term, you can use a procedure called **rotation of axes.** The objective is to rotate the x- and y-axes until they are parallel to the axes of the conic. The rotated axes are denoted as the x'-axis and the y'-axis, as shown in Figure 9.40.

Figure 9.40

After the rotation, the equation of the conic in the new $x'y'$-plane will have the form

$$A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0. \qquad \text{Equation in } x'y'\text{-plane}$$

Because this equation has no xy-term, you can obtain a standard form by completing the square. The following theorem identifies how much to rotate the axes to eliminate the xy-term and also the equations for determining the new coefficients A', C', D', E', and F'.

Rotation of Axes to Eliminate an xy-Term (See the proof on page 708.)

The general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

can be rewritten as

$$A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0$$

by rotating the coordinate axes through an angle θ, where

$$\cot 2\theta = \frac{A - C}{B}.$$

The coefficients of the new equation are obtained by making the substitutions

$$x = x' \cos\theta - y' \sin\theta \qquad \text{and} \qquad y = x' \sin\theta + y' \cos\theta.$$

Note that the substitutions

$$x = x' \cos\theta - y' \sin\theta \quad \text{and} \quad y = x' \sin\theta + y' \cos\theta$$

were developed to eliminate the $x'y'$-term in the rotated system. You can use this to check your work. In other words, when your final equation contains an $x'y'$-term, you know that you have made a mistake.

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Example 7 Rotation of Axes for a Hyperbola

Rotate the axes to eliminate the xy-term in the equation

$$xy - 1 = 0.$$

Then write the equation in standard form and sketch its graph.

Solution

Because $A = 0$, $B = 1$, and $C = 0$, you have

$$\cot 2\theta = \frac{A - C}{B} = 0$$

which implies that $2\theta = \frac{\pi}{2}$, or $\theta = \frac{\pi}{4}$. The equation in the $x'y'$-system is obtained by making the substitutions

$$\begin{aligned} x &= x' \cos\frac{\pi}{4} - y' \sin\frac{\pi}{4} \\ &= x'\left(\frac{1}{\sqrt{2}}\right) - y'\left(\frac{1}{\sqrt{2}}\right) \\ &= \frac{x' - y'}{\sqrt{2}} \end{aligned}$$

and

$$\begin{aligned} y &= x' \sin\frac{\pi}{4} + y' \cos\frac{\pi}{4} \\ &= x'\left(\frac{1}{\sqrt{2}}\right) + y'\left(\frac{1}{\sqrt{2}}\right) \\ &= \frac{x' + y'}{\sqrt{2}}. \end{aligned}$$

The equation in the $x'y'$-system is obtained by substituting these expressions into the equation $xy - 1 = 0$.

$$\left(\frac{x' - y'}{\sqrt{2}}\right)\left(\frac{x' + y'}{\sqrt{2}}\right) - 1 = 0$$

$$\frac{(x')^2 - (y')^2}{2} - 1 = 0$$

$$\frac{(x')^2}{(\sqrt{2})^2} - \frac{(y')^2}{(\sqrt{2})^2} = 1 \qquad \text{Write in standard form.}$$

In the $x'y'$-system, this is a hyperbola centered at the origin with vertices at $(\pm\sqrt{2}, 0)$, as shown in Figure 9.41. To find the coordinates of the vertices in the xy-system, substitute the coordinates $(\pm\sqrt{2}, 0)$ into the equations

$$x = \frac{x' - y'}{\sqrt{2}} \qquad \text{and} \qquad y = \frac{x' + y'}{\sqrt{2}}.$$

This substitution yields the vertices $(1, 1)$ and $(-1, -1)$ in the xy-system. Note also that the asymptotes of the hyperbola have equations

$$y' = \pm x'$$

which correspond to the original x- and y-axes.

CHECKPOINT Now try Exercise 73.

Vertices:
In $x'y'$-system: $(\sqrt{2}, 0), (-\sqrt{2}, 0)$
In xy-system: $(1, 1), (-1, -1)$
Figure 9.41

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9.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1–4, fill in the blank(s).

1. A _____ is the set of all points (x, y) in a plane, the difference of whose distances from two distinct fixed points is a positive constant.
2. The line segment connecting the vertices of a hyperbola is called the _____ , and the midpoint of the line segment is the _____ of the hyperbola.
3. The general form of the equation of a conic is given by _____ .
4. The procedure used to eliminate the xy-term in a general second-degree equation is called _____ of _____.
5. Which of the following equations of a hyperbola have a horizontal transverse axis? a vertical transverse axis?

(a) $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$ (b) $\dfrac{(y-k)^2}{a^2} - \dfrac{(x-h)^2}{b^2} = 1$

(c) $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$ (d) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

6. How many asymptotes does a hyperbola have? Where do these asymptotes intersect?

Procedures and Problem Solving

Matching an Equation with a Graph **In Exercises 7–10, match the equation with its graph. [The graphs are labeled (a), (b), (c), and (d).]**

(a)

(b)

(c)

(d)

7. $\dfrac{y^2}{9} - \dfrac{x^2}{25} = 1$
8. $\dfrac{y^2}{25} - \dfrac{x^2}{9} = 1$
9. $\dfrac{(x-1)^2}{16} - \dfrac{y^2}{4} = 1$
10. $\dfrac{(x+1)^2}{16} - \dfrac{(y-2)^2}{9} = 1$

Finding the Center, Vertices, Foci, and Asymptotes of a Hyperbola **In Exercises 11–20, find the center, vertices, foci, and asymptotes of the hyperbola, and sketch its graph using the asymptotes as an aid. Use a graphing utility to verify your graph.**

11. $x^2 - y^2 = 1$
12. $\dfrac{x^2}{9} - \dfrac{y^2}{25} = 1$
13. $\dfrac{y^2}{1} - \dfrac{x^2}{4} = 1$
14. $\dfrac{y^2}{9} - \dfrac{x^2}{1} = 1$
15. $\dfrac{y^2}{25} - \dfrac{x^2}{81} = 1$
16. $\dfrac{x^2}{36} - \dfrac{y^2}{4} = 1$
17. $\dfrac{(x-1)^2}{4} - \dfrac{(y+2)^2}{1} = 1$
18. $\dfrac{(x+3)^2}{144} - \dfrac{(y-2)^2}{25} = 1$
19. $\dfrac{(y+5)^2}{\frac{1}{9}} - \dfrac{(x-1)^2}{\frac{1}{4}} = 1$
20. $\dfrac{(y-1)^2}{\frac{1}{4}} - \dfrac{(x+3)^2}{\frac{1}{16}} = 1$

Sketching a Hyperbola **In Exercises 21–30, (a) find the standard form of the equation of the hyperbola, (b) find the center, vertices, foci, and asymptotes of the hyperbola, and (c) sketch the hyperbola. Use a graphing utility to verify your graph.**

✓ 21. $4x^2 - 9y^2 = 36$
22. $25x^2 - 4y^2 = 100$
23. $2x^2 - 3y^2 = 6$
24. $6y^2 - 3x^2 = 18$
✓ 25. $9x^2 - y^2 - 36x - 6y + 18 = 0$
26. $x^2 - 9y^2 + 36y - 72 = 0$
27. $x^2 - 9y^2 + 2x - 54y - 80 = 0$
28. $16y^2 - x^2 + 2x + 64y + 63 = 0$
29. $9y^2 - x^2 + 2x + 54y + 62 = 0$
30. $9x^2 - y^2 + 54x + 10y + 55 = 0$

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Finding the Standard Equation of a Hyperbola **In Exercises 31–36, find the standard form of the equation of the hyperbola with the given characteristics and center at the origin.**

31. Vertices: $(0, \pm 2)$; foci: $(0, \pm 4)$

32. Vertices: $(\pm 3, 0)$; foci: $(\pm 6, 0)$

33. Vertices: $(\pm 1, 0)$; asymptotes: $y = \pm 5x$

34. Vertices: $(0, \pm 3)$; asymptotes: $y = \pm 3x$

35. Foci: $(0, \pm 8)$; asymptotes: $y = \pm 4x$

36. Foci: $(\pm 10, 0)$; asymptotes: $y = \pm \frac{3}{4}x$

Finding the Standard Equation of a Hyperbola **In Exercises 37–48, find the standard form of the equation of the hyperbola with the given characteristics.**

37. Vertices: $(2, 0), (6, 0)$; foci: $(0, 0), (8, 0)$

38. Vertices: $(2, 3), (2, -3)$; foci: $(2, 5), (2, -5)$

✓ **39.** Vertices: $(4, 1), (4, 9)$; foci: $(4, 0), (4, 10)$

40. Vertices: $(-2, 1), (2, 1)$; foci: $(-3, 1), (3, 1)$

41. Vertices: $(2, 3), (2, -3)$;
passes through the point $(0, 5)$

42. Vertices: $(-2, 1), (2, 1)$;
passes through the point $(5, 4)$

43. Vertices: $(0, 4), (0, 0)$;
passes through the point $\left(\sqrt{5}, -1\right)$

44. Vertices: $(1, 2), (1, -2)$;
passes through the point $\left(0, \sqrt{5}\right)$

✓ **45.** Vertices: $(1, 2), (3, 2)$;
asymptotes: $y = x,\ y = 4 - x$

46. Vertices: $(3, 0), (3, -6)$;
asymptotes: $y = x - 6,\ y = -x$

47. Vertices: $(0, 2), (6, 2)$; asymptotes: $y = \frac{2}{3}x,\ y = 4 - \frac{2}{3}x$

48. Vertices: $(3, 0), (3, 4)$; asymptotes: $y = \frac{2}{3}x,\ y = 4 - \frac{2}{3}x$

✓ **49.** **Meteorology** You and a friend live 4 miles apart (on the same "east-west" street) and are talking on the phone. You hear a clap of thunder from lightning in a storm, and 18 seconds later your friend hears the thunder. Find an equation that gives the possible places where the lightning could have occurred. (Assume that the coordinate system is measured in feet and that sound travels at 1100 feet per second.)

50. ***Why you should learn it*** *(p. 656)* Three listening stations located at $(3300, 0)$, $(3300, 1100)$, and $(-3300, 0)$ monitor an explosion. The last two stations detect the explosion 1 second and 4 seconds after the first, respectively. Determine the coordinates of the explosion. (Assume that the coordinate system is measured in feet and that sound travels at 1100 feet per second.)

51. **Art and Design** The base for the pendulum of a clock has the shape of a hyperbola (see figure).

(a) Write an equation of the cross section of the base.

(b) Each unit in the coordinate plane represents $\frac{1}{2}$ foot. Find the width of the base 4 inches from the bottom.

52. **MODELING DATA**

Long distance radio navigation for aircraft and ships uses synchronized pulses transmitted by widely separated transmitting stations. These pulses travel at the speed of light (186,000 miles per second). The difference in the times of arrival of these pulses at an aircraft or ship is constant on a hyperbola having the transmitting stations as foci. Assume that two stations, 300 miles apart, are positioned on a rectangular coordinate system at coordinates $(-150, 0)$ and $(150, 0)$, and that a ship is traveling on a hyperbolic path with coordinates $(x, 75)$ (see figure).

(a) Find the x-coordinate of the position of the ship when the time difference between the pulses from the transmitting stations is 1000 microseconds (0.001 second).

(b) Determine the distance between the ship and station 1 when the ship reaches the shore.

(c) The captain of the ship wants to enter a bay located between the two stations. The bay is 30 miles from station 1. What should be the time difference between the pulses?

(d) The ship is 60 miles offshore when the time difference in part (c) is obtained. What is the position of the ship?

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53. Hyperbolic Mirror A hyperbolic mirror (used in some telescopes) has the property that a light ray directed at a focus will be reflected to the other focus. The focus of a hyperbolic mirror (see figure) has coordinates (24, 0). Find the vertex of the mirror given that the mount at the top edge of the mirror has coordinates (24, 24).

54. Photography A panoramic photo can be taken using a hyperbolic mirror. The camera is pointed toward the vertex of the mirror and the camera's optical center is positioned at one focus of the mirror (see figure). An equation for the cross-section of the mirror is

$$\frac{x^2}{25} - \frac{y^2}{16} = 1.$$

Find the distance from the camera's optical center to the mirror.

Classifying a Conic from a General Equation **In Exercises 55–64, classify the graph of the equation as a circle, a parabola, an ellipse, or a hyperbola.**

55. $9x^2 + 4y^2 - 18x + 16y - 119 = 0$

56. $x^2 + y^2 - 4x - 6y - 23 = 0$

57. $16x^2 - 9y^2 + 32x + 54y - 209 = 0$

58. $x^2 + 4x - 8y + 20 = 0$

59. $y^2 + 12x + 4y + 28 = 0$

60. $4x^2 + 25y^2 + 16x + 250y + 541 = 0$

61. $x^2 + y^2 + 2x - 6y = 0$

62. $y^2 - x^2 + 2x - 6y - 8 = 0$

63. $x^2 - 6x - 2y + 7 = 0$

64. $9x^2 + 4y^2 - 90x + 8y + 228 = 0$

Matching an Equation with a Graph **In Exercises 65–70, match the graph with its equation. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]**

65. $xy + 4 = 0$

66. $x^2 + 2xy + y^2 = 0$

67. $-2x^2 + 3xy + 2y^2 + 3 = 0$

68. $x^2 - xy + 3y^2 - 5 = 0$

69. $3x^2 + 2xy + y^2 - 10 = 0$

70. $x^2 - 4xy + 4y^2 + 10x - 30 = 0$

Finding a Point in a Rotated Coordinate System **In Exercises 71 and 72, the $x'y'$-coordinate system has been rotated θ degrees from the xy-coordinate system. The coordinates of a point in the xy-coordinate system are given. Find the coordinates of the point in the rotated coordinate system.**

71. $\theta = 90°$, (0, 3)

72. $\theta = 45°$, (3, 3)

Rotation of Axes **In Exercises 73–80, rotate the axes to eliminate the xy-term in the equation. Then write the equation in standard form. Sketch the graph of the resulting equation, showing both sets of axes.**

73. $xy + 1 = 0$

74. $xy - 2 = 0$

75. $x^2 - 4xy + y^2 + 1 = 0$

76. $xy + x - 2y + 3 = 0$

77. $5x^2 - 6xy + 5y^2 - 12 = 0$

78. $13x^2 + 6\sqrt{3}xy + 7y^2 - 16 = 0$

79. $3x^2 - 2\sqrt{3}xy + y^2 + 2x + 2\sqrt{3}y = 0$

80. $16x^2 - 24xy + 9y^2 - 60x - 80y + 100 = 0$

Graphing a Conic **In Exercises 81–84, use a graphing utility to graph the conic. Determine the angle θ through which the axes are rotated. Explain how you used the graphing utility to obtain the graph.**

81. $x^2 + 3xy + y^2 = 20$

82. $x^2 - 4xy + 2y^2 = 8$

83. $17x^2 + 32xy - 7y^2 = 75$

84. $40x^2 + 36xy + 25y^2 = 52$

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Sketching the Graph of a Degenerate Conic **In Exercises 85–88, sketch (if possible) the graph of the degenerate conic.**

85. $y^2 - 16x^2 = 0$

86. $x^2 + y^2 - 2x + 6y + 10 = 0$

87. $x^2 + 2xy + y^2 - 1 = 0$

88. $x^2 - 10xy + y^2 = 0$

Conclusions

True or False? **In Exercises 89–93, determine whether the statement is true or false. Justify your answer.**

89. In the standard form of the equation of a hyperbola, the larger the ratio of b to a, the larger the eccentricity of the hyperbola.

90. In the standard form of the equation of a hyperbola, the trivial solution of two intersecting lines occurs when $b = 0$.

91. If $D \neq 0$ and $E \neq 0$, then the graph of $x^2 - y^2 + Dx + Ey = 0$ is a hyperbola.

92. If the asymptotes of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \text{ where } a, b > 0$$

intersect at right angles, then $a = b$.

93. After using a rotation of axes to eliminate the xy-term from an equation of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

the coefficients of the x^2- and y^2-terms remain A and C, respectively.

94. **Think About It** Consider a hyperbola centered at the origin with a horizontal transverse axis. Use the definition of a hyperbola to derive its standard form.

95. **Writing** Explain how the central rectangle of a hyperbola can be used to sketch its asymptotes.

96. **Exploration** Use the figure to show that

$$|d_2 - d_1| = 2a.$$

97. **Think About It** Find the equation of the hyperbola for any point on which the difference between its distances from the points $(2, 2)$ and $(10, 2)$ is 6.

98. **Proof** Show that $c^2 = a^2 + b^2$ for the equation of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where the distance from the center of the hyperbola $(0, 0)$ to a focus is c.

99. **Proof** Prove that the graph of the equation

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

is one of the following (except in degenerate cases).

Conic	*Condition*
(a) Circle	$A = C$
(b) Parabola	$A = 0$ or $C = 0$ (but not both)
(c) Ellipse	$AC > 0$
(d) Hyperbola	$AC < 0$

100. **CAPSTONE** Given the hyperbolas

$$\frac{x^2}{16} - \frac{y^2}{9} = 1 \quad \text{and} \quad \frac{y^2}{9} - \frac{x^2}{16} = 1$$

describe any common characteristics that the hyperbolas share, as well as any differences in the graphs of the hyperbolas. Verify your results by using a graphing utility to graph both hyperbolas in the same viewing window.

Cumulative Mixed Review

Operations with Polynomials **In Exercises 101–104, perform the indicated operation.**

101. Subtract: $(x^3 - 3x^2) - (6 - 2x - 4x^2)$

102. Multiply: $\left(3x - \frac{1}{2}\right)(x + 4)$

103. Divide: $\dfrac{x^3 - 3x + 4}{x + 2}$

104. Expand: $[(x + y) + 3]^2$

Factoring a Polynomial **In Exercises 105–110, factor the polynomial completely.**

105. $x^3 - 16x$

106. $x^2 + 14x + 49$

107. $2x^3 - 24x^2 + 72x$

108. $6x^3 - 11x^2 - 10x$

109. $16x^3 + 54$

110. $4 - x + 4x^2 - x^3$

Graphing a Function **In Exercises 111–118, graph the function.**

111. $f(x) = |x + 3|$

112. $f(x) = |x - 4| + 1$

113. $g(x) = \sqrt{4 - x^2}$

114. $g(x) = \sqrt{3x - 2}$

115. $h(t) = -(t - 2)^3 + 3$

116. $h(t) = \frac{1}{2}(t + 4)^3$

117. $f(t) = [\![t - 5]\!] + 1$

118. $f(t) = -2[\![t]\!] + 3$

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9.4 Parametric Equations

Plane Curves

Up to this point, you have been representing a graph by a single equation involving *two* variables such as x and y. In this section, you will study situations in which it is useful to introduce a *third* variable to represent a curve in the plane.

To see the usefulness of this procedure, consider the path of an object that is propelled into the air at an angle of 45°. When the initial velocity of the object is 48 feet per second, it can be shown that the object follows the parabolic path

$$y = -\frac{x^2}{72} + x \qquad \text{Rectangular equation}$$

as shown in Figure 9.42. However, this equation does not tell the whole story. Although it does tell you *where* the object has been, it does not tell you *when* the object was at a given point (x, y) on the path. To determine this time, you can introduce a third variable t, called a **parameter.** It is possible to write both x and y as functions of t to obtain the **parametric equations**

$$x = 24\sqrt{2}t \qquad \text{Parametric equation for } x$$

$$y = -16t^2 + 24\sqrt{2}t. \qquad \text{Parametric equation for } y$$

From this set of equations you can determine that at time $t = 0$, the object is at the point $(0, 0)$. Similarly, at time $t = 1$, the object is at the point

$$\left(24\sqrt{2}, 24\sqrt{2} - 16\right)$$

and so on.

Rectangular equation:
$y = -\frac{x^2}{72} + x$

Parametric equations:
$x = 24\sqrt{2}t$
$y = -16t^2 + 24\sqrt{2}t$

Curvilinear motion: two variables for position, one variable for time
Figure 9.42

For this particular motion problem, x and y are continuous functions of t, and the resulting path is a **plane curve.** (Recall that a *continuous function* is one whose graph can be traced without lifting the pencil from the paper.)

Definition of a Plane Curve

If f and g are continuous functions of t on an interval I, then the set of ordered pairs

$$(f(t), g(t))$$

is a **plane curve** C. The equations given by

$$x = f(t) \qquad \text{and} \qquad y = g(t)$$

are **parametric equations** for C, and t is the **parameter.**

What you should learn

- Evaluate sets of parametric equations for given values of the parameter.
- Graph curves that are represented by sets of parametric equations.
- Rewrite sets of parametric equations as single rectangular equations by eliminating the parameter.
- Find sets of parametric equations for graphs.

Why you should learn it

Parametric equations are useful for modeling the path of an object. For instance, in Exercise 62 on page 676, a set of parametric equations is used to model the path of a football.

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Graphs of Plane Curves

One way to sketch a curve represented by a pair of parametric equations is to plot points in the xy-plane. Each set of coordinates (x, y) is determined from a value chosen for the parameter t. By plotting the resulting points in the order of *increasing* values of t, you trace the curve in a specific direction. This is called the **orientation** of the curve.

Technology Tip

Most graphing utilities have a *parametric* mode. So, another way to graph a curve represented by a pair of parametric equations is to use a graphing utility, as shown in Example 2. For instructions on how to use the *parametric* mode, see Appendix A; for specific keystrokes, go to this textbook's *Companion Website*.

Example 1 Sketching a Plane Curve

Sketch the curve given by the parametric equations

$$x = t^2 - 4 \quad \text{and} \quad y = \frac{t}{2}, \qquad -2 \le t \le 3.$$

Describe the orientation of the curve.

Solution

Using values of t in the interval, the parametric equations yield the points (x, y) shown in the table.

t	-2	-1	0	1	2	3
x	0	-3	-4	-3	0	5
y	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$

Figure 9.43

By plotting these points in the order of increasing t, you obtain the curve shown in Figure 9.43. The arrows on the curve indicate its orientation as t increases from -2 to 3. So, when a particle moves on this curve, it would start at $(0, -1)$ and then move along the curve to the point $\left(5, \frac{3}{2}\right)$.

CHECKPOINT Now try Exercise 9(a) and (b).

Note that the graph shown in Figure 9.43 does not define y as a function of x. This points out one benefit of parametric equations—they can be used to represent graphs that are more general than graphs of functions.

Two different sets of parametric equations can have the same graph. For example, the set of parametric equations

$$x = 4t^2 - 4 \quad \text{and} \quad y = t, \quad -1 \le t \le \tfrac{3}{2}$$

has the same graph as the set given in Example 1. However, by comparing the values of t in Figures 9.43 and 9.44, you can see that this second graph is traced out more *rapidly* (considering t as time) than the first graph. So, in applications, different parametric representations can be used to represent various *speeds* at which objects travel along a given path.

Figure 9.44

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Example 2 Using a Graphing Utility in Parametric Mode

Use a graphing utility to graph the curves represented by the parametric equations. Using the graph and the Vertical Line Test, for which curve is y a function of x?

a. $x = t^2, y = t^3$

b. $x = t, y = t^3$

c. $x = t^2, y = t$

Solution

Begin by setting the graphing utility to *parametric* mode. When choosing a viewing window, you must set not only minimum and maximum values of x and y, but also minimum and maximum values of t.

a. Enter the parametric equations for x and y, as shown in Figure 9.45. Use the viewing window shown in Figure 9.46. The curve is shown in Figure 9.47. From the graph, you can see that y *is not* a function of x.

Figure 9.45

Figure 9.46

Figure 9.47

b. Enter the parametric equations for x and y, as shown in Figure 9.48. Use the viewing window shown in Figure 9.49. The curve is shown in Figure 9.50. From the graph, you can see that y *is* a function of x.

Figure 9.48

Figure 9.49

Figure 9.50

c. Enter the parametric equations for x and y, as shown in Figure 9.51. Use the viewing window shown in Figure 9.52. The curve is shown in Figure 9.53. From the graph, you can see that y *is not* a function of x.

Figure 9.51

Figure 9.52

Figure 9.53

CHECKPOINT Now try Exercise 9(c).

Explore the Concept

Use a graphing utility set in *parametric* mode to graph the curve

$$x = t \quad \text{and} \quad y = 1 - t^2.$$

Set the viewing window so that $-4 \le x \le 4$ and $-12 \le y \le 2$. Now, graph the curve with various settings for t. Use the following.

a. $0 \le t \le 3$

b. $-3 \le t \le 0$

c. $-3 \le t \le 3$

Compare the curves given by the different t settings. Repeat this experiment using $x = -t$. How does this change the results?

Technology Tip

Notice in Example 2 that in order to set the viewing windows of parametric graphs, you have to scroll down to enter the Ymax and Yscl values.

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Eliminating the Parameter

Many curves that are represented by sets of parametric equations have graphs that can also be represented by rectangular equations (in x and y). The process of finding the rectangular equation is called **eliminating the parameter.**

Parametric equations	→	Solve for t in one equation.	→	Substitute in second equation.	→	Rectangular equation
$x = t^2 - 4$ $y = \frac{1}{2}t$		$t = 2y$		$x = (2y)^2 - 4$		$x = 4y^2 - 4$

Now you can recognize that the equation $x = 4y^2 - 4$ represents a parabola with a horizontal axis and vertex at $(-4, 0)$.

When converting equations from parametric to rectangular form, you may need to alter the domain of the rectangular equation so that its graph matches the graph of the parametric equations. This situation is demonstrated in Example 3.

Study Tip

It is important to realize that eliminating the parameter is primarily an aid to curve sketching. When the parametric equations represent the path of a moving object, the graph alone is not sufficient to describe the object's motion. You still need the parametric equations to determine the *position*, *direction*, and *speed* at a given time.

Example 3 Eliminating the Parameter

Identify the curve represented by the equations

$$x = \frac{1}{\sqrt{t+1}} \quad \text{and} \quad y = \frac{t}{t+1}.$$

Solution

Solving for t in the equation for x produces

$$x^2 = \frac{1}{t+1} \quad \Rightarrow \quad \frac{1}{x^2} = t + 1 \quad \Rightarrow \quad \frac{1}{x^2} - 1 = t.$$

Substituting in the equation for y, you obtain the rectangular equation

$$y = \frac{t}{t+1} = \frac{\frac{1}{x^2} - 1}{\frac{1}{x^2} - 1 + 1} = \frac{\frac{1 - x^2}{x^2}}{\frac{1}{x^2}} \cdot \frac{x^2}{x^2} = 1 - x^2.$$

From the rectangular equation, you can recognize that the curve is a parabola that opens downward and has its vertex at $(0, 1)$, as shown in Figure 9.54. The rectangular equation is defined for all values of x. The parametric equation for x, however, is defined only when $t > -1$. From the graph of the parametric equations, you can see that x is always positive, as shown in Figure 9.55. So, you should restrict the domain of x to positive values, as shown in Figure 9.56.

Figure 9.54

Figure 9.55

Figure 9.56

CHECKPOINT Now try Exercise 9(d).

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Finding Parametric Equations for a Graph

You have been studying techniques for sketching the graph represented by a set of parametric equations. Now consider the *reverse* problem—that is, how can you find a set of parametric equations for a given graph or a given physical description? From the discussion following Example 1, you know that such a representation is not unique. That is, the equations

$$x = 4t^2 - 4 \quad \text{and} \quad y = t, \ -1 \le t \le \frac{3}{2}$$

produced the same graph as the equations

$$x = t^2 - 4 \quad \text{and} \quad y = \frac{t}{2}, \ -2 \le t \le 3.$$

This is further demonstrated in Example 4.

Example 4 Finding Parametric Equations for a Given Graph

Find a set of parametric equations to represent the graph of $y = 1 - x^2$ using the parameters (a) $t = x$ and (b) $t = 1 - x$.

Solution

a. Letting $t = x$, you obtain the following parametric equations.

$x = t$ Parametric equation for x

$y = 1 - t^2$ Parametric equation for y

The graph of these equations is shown in Figure 9.57.

Figure 9.57

b. Letting $t = 1 - x$, you obtain the following parametric equations.

$x = 1 - t$ Parametric equation for x

$y = 1 - (1 - t)^2 = 2t - t^2$ Parametric equation for y

The graph of these equations is shown in Figure 9.58. Note that the graphs in Figures 9.57 and 9.58 have opposite orientations.

Figure 9.58

CHECKPOINT Now try Exercise 47.

What's Wrong?

You use a graphing utility in *parametric* mode to graph the parametric equations in Example 4(a). You use a standard viewing window and expect to obtain a parabola similar to Figure 9.57. Your result is shown below. What's wrong?

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9.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1 and 2, fill in the blank(s).

1. If f and g are continuous functions of t on an interval I, then the set of ordered pairs $(f(t), g(t))$ is a _____ C. The equations given by $x = f(t)$ and $y = g(t)$ are _____ for C, and t is the _____ .

2. The _____ of a curve is the direction in which the curve is traced out for increasing values of the parameter.

3. Given a set of parametric equations, how do you find the corresponding rectangular equation?

4. What point on the plane curve represented by the parametric equations $x = t$ and $y = t$ corresponds to $t = 3$?

Procedures and Problem Solving

Identifying the Graph of Parametric Equations **In Exercises 5–8, match the set of parametric equations with its graph. [The graphs are labeled (a), (b), (c), and (d).]**

(a)

(b)

(c)

(d)

5. $x = t,\ y = t + 2$

6. $x = t^2, y = t - 2$

7. $x = \sqrt{t}, y = t$

8. $x = \dfrac{1}{t}, y = t + 25$

9. **Using Parametric Equations** Consider the parametric equations $x = \sqrt{t}$ and $y = 2 - t$.

(a) Create a table of x- and y-values using $t = 0, 1, 2, 3,$ and 4.

(b) Plot the points (x, y) generated in part (a) and sketch a graph of the parametric equations for $t \geq 0$. Describe the orientation of the curve.

(c) Use a graphing utility to graph the curve represented by the parametric equations.

(d) Find the rectangular equation by eliminating the parameter. Sketch its graph. How does the graph differ from those in parts (b) and (c)?

10. **Using Parametric Equations** Consider the parametric equations $x = 4\cos^2 t$ and $y = 4\sin t$.

(a) Create a table of x- and y-values using $t = -\pi/2$, $-\pi/4$, 0, $\pi/4$, and $\pi/2$.

(b) Plot the points (x, y) generated in part (a) and sketch a graph of the parametric equations for $-\dfrac{\pi}{2} \leq t \leq \dfrac{\pi}{2}$. Describe the orientation of the curve.

(c) Use a graphing utility to graph the curve represented by the parametric equations.

(d) Find the rectangular equation by eliminating the parameter. Sketch its graph. How does the graph differ from those in parts (b) and (c)?

Identifying Parametric Equations for a Plane Curve **In Exercises 11 and 12, determine which set of parametric equations represents the graph shown.**

11. (a) $x = t^2$
$y = 2t + 1$

(b) $x = 2t + 1$
$y = t^2$

(c) $x = 2t - 1$
$y = t^2$

(d) $x = -2t + 1$
$y = t^2$

12. (a) $x = 2 - \cos\theta$
$y = 3 - \sin\theta$

(b) $x = 3 - \cos\theta$
$y = 2 + \sin\theta$

(c) $x = 2 + \cos\theta$
$y = 3 - \sin\theta$

(d) $x = 3 + \cos\theta$
$y = 2 - \sin\theta$

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Sketching a Plane Curve and Eliminating the Parameter **In Exercises 13–28, sketch the curve represented by the parametric equations (indicate the orientation of the curve). Use a graphing utility to confirm your result. Then eliminate the parameter and write the corresponding rectangular equation whose graph represents the curve. Adjust the domain of the resulting rectangular equation, if necessary.**

13. $x = t, y = -4t$

14. $x = t, y = \frac{1}{2}t$

15. $x = 3t - 3, y = 2t + 1$

16. $x = 3 - 2t, y = 2 + 3t$

17. $x = \frac{1}{4}t, y = t^2$

18. $x = t, y = t^3$

19. $x = t + 2, y = t^2$

20. $x = \sqrt{t}, y = 1 - t$

21. $x = 2t, y = |t - 2|$

22. $x = |t - 1|, y = t + 2$

23. $x = 2\cos\theta, y = 3\sin\theta$

24. $x = \cos\theta, y = 4\sin\theta$

25. $x = e^{-t}, y = e^{3t}$

26. $x = e^{2t}, y = e^t$

27. $x = t^3, y = 3\ln t$

28. $x = \ln 2t, y = 2t^2$

Using a Graphing Utility in Parametric Mode **In Exercises 29–34, use a graphing utility to graph the curve represented by the parametric equations.**

29. $x = 4 + 3\cos\theta$
$y = -2 + \sin\theta$

30. $x = 4 + 3\cos\theta$
$y = -2 + 2\sin\theta$

31. $x = 4\sec\theta$
$y = 2\tan\theta$

32. $x = \sec\theta$
$y = \tan\theta$

33. $x = t/2$
$y = \ln(t^2 + 1)$

34. $x = 10 - 0.01e^t$
$y = 0.4t^2$

Comparing Plane Curves **In Exercises 35 and 36, determine how the plane curves differ from each other.**

35. (a) $x = t$
$y = 2t + 1$

(b) $x = \cos\theta$
$y = 2\cos\theta + 1$

(c) $x = e^{-t}$
$y = 2e^{-t} + 1$

(d) $x = e^t$
$y = 2e^t + 1$

36. (a) $x = 2\sqrt{t}$
$y = 4 - \sqrt{t}$

(b) $x = 2\sqrt[3]{t}$
$y = 4 - \sqrt[3]{t}$

(c) $x = 2(t + 1)$
$y = 3 - t$

(d) $x = -2t^2$
$y = 4 + t^2$

Eliminating the Parameter **In Exercises 37–40, eliminate the parameter and obtain the standard form of the rectangular equation.**

37. Line through (x_1, y_1) and (x_2, y_2):
$x = x_1 + t(x_2 - x_1)$
$y = y_1 + t(y_2 - y_1)$

38. Circle: $x = h + r\cos\theta, y = k + r\sin\theta$

39. Ellipse: $x = h + a\cos\theta, y = k + b\sin\theta$

40. Hyperbola: $x = h + a\sec\theta, y = k + b\tan\theta$

Finding Parametric Equations for a Given Graph **In Exercises 41–44, use the results of Exercises 37–40 to find a set of parametric equations for the line or conic.**

41. Line: passes through $(3, 1)$ and $(-2, 6)$

42. Circle: center: $(3, -2)$; radius: 4

43. Ellipse: vertices: $(\pm 5, 0)$; foci: $(\pm 3, 0)$

44. Hyperbola: vertices: $(\pm 2, 0)$; foci: $(\pm 4, 0)$

Finding Parametric Equations for a Given Graph **In Exercises 45–52, find a set of parametric equations to represent the graph of the given rectangular equation using the parameters (a) $t = x$ and (b) $t = 2 - x$.**

45. $y = 5x - 3$

46. $y = 4 - 7x$

47. $y = \dfrac{1}{x}$

48. $y = \dfrac{1}{2x}$

49. $y = 6x^2 - 5$

50. $y = x^3 + 2x$

51. $y = e^x$

52. $y = \ln(x + 4)$

Using a Graphing Utility **In Exercises 53–56, use a graphing utility to graph the curve represented by the parametric equations.**

53. Witch of Agnesi: $x = 2\cot\theta, y = 2\sin^2\theta$

54. Folium of Descartes: $x = \dfrac{3t}{1 + t^3}, y = \dfrac{3t^2}{1 + t^3}$

55. Cycloid: $x = \theta + \sin\theta, \quad y = 1 - \cos\theta$

56. Prolate cycloid: $x = 2\theta - 4\sin\theta, \quad y = 2 - 4\cos\theta$

Identifying a Graph **In Exercises 57–60, match the parametric equations with the correct graph. [The graphs are labeled (a), (b), (c), and (d).]**

(a)

(b)

(c)

(d)

57. Lissajous curve: $x = 2\cos\theta, y = \sin 2\theta$

58. Evolute of ellipse: $x = 2\cos^3\theta, y = 4\sin^3\theta$

59. Involute of circle: $x = \frac{1}{2}(\cos\theta + \theta\sin\theta)$
$y = \frac{1}{2}(\sin\theta - \theta\cos\theta)$

60. Serpentine curve: $x = \frac{1}{2}\cot\theta, y = 4\sin\theta\cos\theta$

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Projectile Motion **In Exercises 61 and 62, consider a projectile launched at a height of h feet above the ground at an angle of θ with the horizontal. The initial velocity is v_0 feet per second and the path of the projectile is modeled by the parametric equations**

$$x = (v_0 \cos \theta)t \text{ and } y = h + (v_0 \sin \theta)t - 16t^2.$$

61. MODELING DATA

The center field fence in Yankee Stadium is 7 feet high and 408 feet from home plate. A baseball is hit at a point 3 feet above the ground. It leaves the bat at an angle of θ degrees with the horizontal at a speed of 100 miles per hour (see figure).

(a) Write a set of parametric equations that model the path of the baseball.

(b) Use a graphing utility to graph the path of the baseball when $\theta = 15°$. Is the hit a home run?

(c) Use the graphing utility to graph the path of the baseball when $\theta = 23°$. Is the hit a home run?

(d) Find the minimum angle required for the hit to be a home run.

62. *Why you should learn it* *(p. 669)* The quarterback of a football team releases a pass at a height of 7 feet above the playing field, and the football is caught by a receiver at a height of 4 feet, 30 yards directly downfield. The pass is released at an angle of 35° with the horizontal.

(a) Write a set of parametric equations for the path of the football.

(b) Find the speed of the football when it is released.

(c) Use a graphing utility to graph the path of the football and approximate its maximum height.

(d) Find the time the receiver has to position himself after the quarterback releases the football.

Conclusions

True or False? **In Exercises 63–66, determine whether the statement is true or false. Justify your answer.**

63. The two sets of parametric equations $x = t, y = t^2 + 1$ and $x = 3t,\ y = 9t^2 + 1$ correspond to the same rectangular equation.

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64. Because the graphs of the parametric equations $x = t^2$, $y = t^2$ and $x = t, y = t$ both represent the line $y = x$, they are the same plane curve.

65. If y is a function of t and x is a function of t, then y must be a function of x.

66. The parametric equations $x = at + h$ and $y = bt + k$, where $a \neq 0$ and $b \neq 0$, represent a circle centered at (h, k) when $a = b$.

67. **Think About It** The graph of the parametric equations $x = t^3$ and $y = t - 1$ is shown below. Would the graph change for the equations $x = (-t)^3$ and $y = -t - 1$? If so, how would it change?

68. CAPSTONE The curve shown is represented by the parametric equations

$$x = 6 \cos \theta \quad \text{and} \quad y = 6 \sin \theta, \quad 0 \leq \theta \leq 6.$$

(a) Describe the orientation of the curve.

(b) Determine a range of θ that gives the graph of a circle.

(c) Write a set of parametric equations representing the curve so that the curve traces from the same point as the original curve, but in the opposite direction.

(d) How does the original curve change when cosine and sine are interchanged?

Cumulative Mixed Review

Testing for Evenness and Oddness **In Exercises 69–72, check for symmetry with respect to both axes and to the origin. Then determine whether the function is even, odd, or neither.**

69. $f(x) = \dfrac{4x^2}{x^2 + 1}$

70. $f(x) = \sqrt{x}$

71. $y = e^x$

72. $(x - 2)^2 = y + 4$

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9.5 Polar Coordinates

Introduction

So far, you have been representing graphs of equations as collections of points (x, y) in the rectangular coordinate system, where x and y represent the directed distances from the coordinate axes to the point (x, y). In this section, you will study a second coordinate system called the **polar coordinate system.**

To form the polar coordinate system in the plane, fix a point O, called the **pole** (or **origin**), and construct from O an initial ray called the **polar axis,** as shown in Figure 9.59. Then each point P in the plane can be assigned **polar coordinates** (r, θ) as follows.

1. $r =$ *directed distance* from O to P

2. $\theta =$ *directed angle*, counterclockwise from the polar axis to segment $\overline{OP}$

Figure 9.59

What you should learn

- Plot points and find multiple representations of points in the polar coordinate system.
- Convert points from rectangular to polar form and vice versa.
- Convert equations from rectangular to polar form and vice versa.

Why you should learn it

Polar coordinates offer a different mathematical perspective on graphing. For instance, in Exercises 9–16 on page 681, you will see that a polar coordinate can be written in more than one way.

Example 1 Plotting Points in the Polar Coordinate System

a. The point $(r, \theta) = \left(2, \frac{\pi}{3}\right)$ lies two units from the pole on the terminal side of the angle $\theta = \frac{\pi}{3}$, as shown in Figure 9.60.

b. The point $(r, \theta) = \left(3, -\frac{\pi}{6}\right)$ lies three units from the pole on the terminal side of the angle $\theta = -\frac{\pi}{6}$, as shown in Figure 9.61.

c. The point $(r, \theta) = \left(3, \frac{11\pi}{6}\right)$ coincides with the point $\left(3, -\frac{\pi}{6}\right)$, as shown in Figure 9.62.

Figure 9.60

Figure 9.61

Figure 9.62

CHECKPOINT Now try Exercise 9.

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In rectangular coordinates, each point (x, y) has a unique representation. This is not true for polar coordinates. For instance, the coordinates

$$(r, \theta) \qquad \text{and} \qquad (r, \theta + 2\pi)$$

represent the same point, as illustrated in Example 1. Another way to obtain multiple representations of a point is to use negative values for r. Because r is a *directed distance*, the coordinates

$$(r, \theta) \qquad \text{and} \qquad (-r, \theta + \pi)$$

represent the same point. In general, the point (r, θ) can be represented as

$$(r, \theta) = (r, \theta \pm 2n\pi) \qquad \text{or} \qquad (r, \theta) = (-r, \theta \pm (2n + 1)\pi)$$

where n is any integer. Moreover, the pole is represented by

$$(0, \theta)$$

where θ is any angle.

Explore the Concept

Set your graphing utility to *polar* mode. Then graph the equation $r = 3$. (Use a viewing window in which $0 \le \theta \le 2\pi$, $-6 \le x \le 6$, and $-4 \le y \le 4$.) You should obtain a circle of radius 3.

a. Use the *trace* feature to cursor around the circle. Can you locate the point $(3, 5\pi/4)$?

b. Can you locate other representations of the point $(3, 5\pi/4)$? If so, explain how you did it.

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Example 2 Multiple Representations of Points

Plot the point

$$\left(3, -\frac{3\pi}{4}\right)$$

and find three additional polar representations of this point, using

$$-2\pi < \theta < 2\pi.$$

Solution

The point is shown in Figure 9.63. Three other representations are as follows.

$$\left(3, -\frac{3\pi}{4} + 2\pi\right) = \left(3, \frac{5\pi}{4}\right) \qquad \text{Add } 2\pi \text{ to } \theta.$$

$$\left(-3, -\frac{3\pi}{4} - \pi\right) = \left(-3, -\frac{7\pi}{4}\right) \qquad \text{Replace } r \text{ by } -r\text{; subtract } \pi \text{ from } \theta.$$

$$\left(-3, -\frac{3\pi}{4} + \pi\right) = \left(-3, \frac{\pi}{4}\right) \qquad \text{Replace } r \text{ by } -r\text{; add } \pi \text{ to } \theta.$$

Figure 9.63

CHECKPOINT Now try Exercise 11.

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Coordinate Conversion

To establish the relationship between polar and rectangular coordinates, let the polar axis coincide with the positive x-axis and the pole with the origin, as shown in Figure 9.64. Because (x, y) lies on a circle of radius r, it follows that $r^2 = x^2 + y^2$. Moreover, for $r > 0$, the definitions of the trigonometric functions imply that

$$\tan\theta = \frac{y}{x}, \qquad \cos\theta = \frac{x}{r}, \qquad \text{and} \qquad \sin\theta = \frac{y}{r}.$$

You can show that the same relationships hold for $r < 0$.

Figure 9.64

Coordinate Conversion

The polar coordinates (r, θ) are related to the rectangular coordinates (x, y) as follows.

Polar-to-Rectangular	*Rectangular-to-Polar*
$x = r\cos\theta$	$\tan\theta = \dfrac{y}{x}$
$y = r\sin\theta$	$r^2 = x^2 + y^2$

Example 3 Polar-to-Rectangular Conversion

Convert the point $(2, \pi)$ to rectangular coordinates.

Solution

For the point $(r, \theta) = (2, \pi)$, you have the following.

$$x = r\cos\theta = 2\cos\pi = -2$$

$$y = r\sin\theta = 2\sin\pi = 0$$

The rectangular coordinates are $(x, y) = (-2, 0)$. (See Figure 9.65.)

Figure 9.65

CHECKPOINT Now try Exercise 17.

Example 4 Rectangular-to-Polar Conversion

Convert the point $(-1, 1)$ to polar coordinates.

Solution

For the second-quadrant point $(x, y) = (-1, 1)$, you have

$$\tan\theta = \frac{y}{x} = \frac{1}{-1} = -1$$

$$\theta = \frac{3\pi}{4}.$$

Because θ lies in the same quadrant as (x, y), use positive r.

$$r = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$$

So, *one* set of polar coordinates is

$$(r, \theta) = \left(\sqrt{2}, \frac{3\pi}{4}\right)$$

as shown in Figure 9.66.

Figure 9.66

CHECKPOINT Now try Exercise 35.

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Equation Conversion

By comparing Examples 3 and 4, you can see that point conversion from the polar to the rectangular system is straightforward, whereas point conversion from the rectangular to the polar system is more involved. For equations, the opposite is true. To convert a rectangular equation to polar form, you simply replace x by $r \cos \theta$ and y by $r \sin \theta$. For instance, the rectangular equation $y = x^2$ can be written in polar form as follows.

$$y = x^2 \qquad \text{Rectangular equation}$$

$$r \sin \theta = (r \cos \theta)^2 \qquad \text{Polar equation}$$

$$r = \sec \theta \tan \theta \qquad \text{Simplest form}$$

On the other hand, converting a polar equation to rectangular form requires considerable ingenuity.

Example 5 demonstrates several polar-to-rectangular conversions that enable you to sketch the graphs of some polar equations.

Example 5 Converting Polar Equations to Rectangular Form

Describe the graph of each polar equation and find the corresponding rectangular equation.

a. $r = 2$ **b.** $\theta = \dfrac{\pi}{3}$ **c.** $r = \sec \theta$

Solution

a. The graph of the polar equation $r = 2$ consists of all points that are two units from the pole. In other words, this graph is a circle centered at the origin with a radius of 2, as shown in Figure 9.67. You can confirm this by converting to rectangular form, using the relationship $r^2 = x^2 + y^2$.

$$\underbrace{r = 2}_{\text{Polar equation}} \Rightarrow r^2 = 2^2 \Rightarrow \underbrace{x^2 + y^2 = 2^2}_{\text{Rectangular equation}}$$

Figure 9.67

b. The graph of the polar equation

$$\theta = \frac{\pi}{3}$$

consists of all points on the line that makes an angle of $\pi/3$ with the positive x-axis, as shown in Figure 9.68. To convert to rectangular form, you make use of the relationship $\tan \theta = y/x$.

$$\underbrace{\theta = \frac{\pi}{3}}_{\text{Polar equation}} \Rightarrow \tan \theta = \sqrt{3} \Rightarrow \underbrace{y = \sqrt{3}x}_{\text{Rectangular equation}}$$

Figure 9.68

c. The graph of the polar equation

$$r = \sec \theta$$

is not evident by simple inspection, so you convert to rectangular form by using the relationship $r \cos \theta = x$.

$$\underbrace{r = \sec \theta}_{\text{Polar equation}} \Rightarrow r \cos \theta = 1 \Rightarrow \underbrace{x = 1}_{\text{Rectangular equation}}$$

Now you can see that the graph is a vertical line, as shown in Figure 9.69.

Figure 9.69

CHECKPOINT Now try Exercise 91.

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9.5 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1 and 2, fill in the blank(s).

1. The origin of the polar coordinate system is called the _____ .
2. For the point (r, θ), r is the _____ from O to P and θ is the _____ counterclockwise from the polar axis to segment $\overline{OP}$.
3. How are the rectangular coordinates (x, y) related to the polar coordinates (r, θ)?
4. Do the polar coordinates $(1, \pi)$ and the rectangular coordinates $(-1, 0)$ represent the same point?

Procedures and Problem Solving

Finding Rectangular Coordinates **In Exercises 5–8, a point in polar coordinates is given. Find the corresponding rectangular coordinates for the point.**

5. $\left(4, \frac{\pi}{2}\right)$

6. $\left(3, \frac{3\pi}{2}\right)$

7. $\left(-1, \frac{5\pi}{4}\right)$

8. $(0, -\pi)$

Plotting Points in the Polar Coordinate System **In Exercises 9–16, plot the point given in polar coordinates and find three additional polar representations of the point, using $-2\pi < \theta < 2\pi$.**

9. $\left(3, \frac{5\pi}{6}\right)$
10. $\left(2, \frac{3\pi}{4}\right)$
11. $\left(-1, -\frac{\pi}{3}\right)$
12. $\left(-3, -\frac{7\pi}{6}\right)$
13. $\left(\sqrt{3}, \frac{5\pi}{6}\right)$
14. $\left(5\sqrt{2}, -\frac{11\pi}{6}\right)$
15. $\left(\frac{3}{2}, -\frac{3\pi}{2}\right)$
16. $\left(0, -\frac{\pi}{4}\right)$

Polar-to-Rectangular Conversion **In Exercises 17–26, plot the point given in polar coordinates and find the corresponding rectangular coordinates for the point.**

17. $\left(4, -\frac{\pi}{3}\right)$
18. $\left(2, \frac{7\pi}{6}\right)$
19. $\left(-1, -\frac{3\pi}{4}\right)$
20. $\left(16, \frac{5\pi}{2}\right)$
21. $\left(0, -\frac{7\pi}{6}\right)$
22. $\left(0, \frac{5\pi}{4}\right)$
23. $(\sqrt{2}, 2.36)$
24. $(2\sqrt{2}, 4.71)$
25. $(-5, -2.36)$
26. $(-3, -1.57)$

Using a Graphing Utility to Find Rectangular Coordinates **In Exercises 27–34, use a graphing utility to find the rectangular coordinates of the point given in polar coordinates. Round your results to two decimal places.**

27. $\left(2, \frac{2\pi}{9}\right)$
28. $\left(4, \frac{11\pi}{9}\right)$
29. $(-4.5, 1.3)$
30. $(8.25, 3.5)$
31. $(2.5, 1.58)$
32. $(5.4, 2.85)$
33. $(-4.1, -0.5)$
34. $(8.2, -3.2)$

Rectangular-to-Polar Conversion **In Exercises 35–44, plot the point given in rectangular coordinates and find two sets of polar coordinates for the point for $0 \le \theta < 2\pi$.**

35. $(-7, 0)$
36. $(0, -5)$
37. $(1, 1)$
38. $(-3, -3)$
39. $(-3, 4)$
40. $(3, -1)$
41. $(-\sqrt{3}, -\sqrt{3})$
42. $(\sqrt{3}, -1)$
43. $(6, 9)$
44. $(5, 12)$

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Using a Graphing Utility to Find Polar Coordinates **In Exercises 45–50, use a graphing utility to find one set of polar coordinates for the point given in rectangular coordinates. (There are many correct answers.)**

45. $(3, -2)$
46. $(-5, 2)$
47. $(\sqrt{3}, 2)$
48. $(3\sqrt{2}, 3\sqrt{2})$
49. $(\frac{5}{2}, \frac{4}{3})$
50. $(\frac{7}{4}, \frac{3}{2})$

Converting a Rectangular Equation to Polar Form **In Exercises 51–68, convert the rectangular equation to polar form. Assume $a > 0$.**

51. $x^2 + y^2 = 9$
52. $x^2 + y^2 = 16$
53. $y = 4$
54. $y = x$
55. $x = 8$
56. $x = a$
57. $3x - y + 2 = 0$
58. $3x + 5y - 2 = 0$
59. $xy = 4$
60. $2xy = 1$
61. $(x^2 + y^2)^2 = 9(x^2 - y^2)$
62. $y^2 - 8x - 16 = 0$
63. $x^2 + y^2 - 6x = 0$
64. $x^2 + y^2 - 8y = 0$
65. $x^2 + y^2 - 2ax = 0$
66. $x^2 + y^2 - 2ay = 0$
67. $y^2 = x^3$
68. $x^2 = y^3$

Converting a Polar Equation to Rectangular Form **In Exercises 69–88, convert the polar equation to rectangular form.**

69. $r = 4 \sin \theta$
70. $r = 2 \cos \theta$
71. $\theta = \dfrac{2\pi}{3}$
72. $\theta = \dfrac{5\pi}{3}$
73. $\theta = \dfrac{5\pi}{6}$
74. $\theta = \dfrac{11\pi}{6}$
75. $\theta = \dfrac{\pi}{2}$
76. $\theta = \pi$
77. $r = 4$
78. $r = 10$
79. $r = -3 \csc \theta$
80. $r = 2 \sec \theta$
81. $r^2 = \cos \theta$
82. $r^2 = \sin 2\theta$
83. $r = 2 \sin 3\theta$
84. $r = 3 \cos 2\theta$
85. $r = \dfrac{1}{1 - \cos \theta}$
86. $r = \dfrac{2}{1 + \sin \theta}$
87. $r = \dfrac{6}{2 - 3 \sin \theta}$
88. $r = \dfrac{6}{2 \cos \theta - 3 \sin \theta}$

Converting a Polar Equation to Rectangular Form **In Exercises 89–94, describe the graph of the polar equation and find the corresponding rectangular equation. Sketch its graph.**

89. $r = 6$
90. $r = 8$
91. $\theta = \dfrac{\pi}{4}$
92. $\theta = \dfrac{7\pi}{6}$
93. $r = 3 \sec \theta$
94. $r = 2 \csc \theta$

Conclusions

True or False? **In Exercises 95 and 96, determine whether the statement is true or false. Justify your answer.**

95. If (r_1, θ_1) and (r_2, θ_2) represent the same point in the polar coordinate system, then $|r_1| = |r_2|$.

96. If (r, θ_1) and (r, θ_2) represent the same point in the polar coordinate system, then $\theta_1 = \theta_2 + 2\pi n$ for some integer n.

97. **Think About It**

(a) Show that the distance between the points (r_1, θ_1) and (r_2, θ_2) is

$$\sqrt{r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_1 - \theta_2)}.$$

(b) Describe the positions of the points relative to each other for $\theta_1 = \theta_2$. Simplify the Distance Formula for this case. Is the simplification what you expected? Explain.

(c) Simplify the Distance Formula for $\theta_1 - \theta_2 = 90°$. Is the simplification what you expected? Explain.

(d) Choose two points in the polar coordinate system and find the distance between them. Then choose different polar representations of the same two points and apply the Distance Formula again. Discuss the result.

98. **CAPSTONE** In the rectangular coordinate system, each point (x, y) has a unique representation. Explain why this is not true for a point (r, θ) in the polar coordinate system.

99. **Think About It** Convert the polar equation $r = \cos \theta + 3 \sin \theta$ to rectangular form and identify the graph.

100. **Think About It** Convert the polar equation $r = 2(h \cos \theta + k \sin \theta)$ to rectangular form and verify that it is the equation of a circle. Find the radius of the circle and the rectangular coordinates of the center of the circle.

Cumulative Mixed Review

Solving a Triangle Using the Law of Sines or Cosines **In Exercises 101–104, use the Law of Sines or the Law of Cosines to solve the triangle.**

101. $a = 13, b = 19, c = 25$
102. $A = 24°, a = 10, b = 6$
103. $A = 56°, C = 38°, c = 12$
104. $B = 71°, a = 21, c = 29$

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9.6 Graphs of Polar Equations

Introduction

In previous chapters you sketched graphs in rectangular coordinate systems. You began with the basic point-plotting method. Then you used sketching aids such as a graphing utility, symmetry, intercepts, asymptotes, periods, and shifts to further investigate the natures of the graphs. This section approaches curve sketching in the polar coordinate system similarly.

What you should learn

- Graph polar equations by point plotting.
- Use symmetry and zeros as sketching aids.
- Recognize special polar graphs.

Why you should learn it

Several common figures, such as the circle in Exercise 10 on page 689, are easier to graph in the polar coordinate system than in the rectangular coordinate system.

Example 1 Graphing a Polar Equation by Point Plotting

Sketch the graph of the polar equation $r = 4 \sin \theta$ by hand.

Solution

The sine function is periodic, so you can get a full range of r-values by considering values of θ in the interval $0 \le \theta \le 2\pi$, as shown in the table.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{3\pi}{2}$	$\frac{11\pi}{6}$	2π
r	0	2	$2\sqrt{3}$	4	$2\sqrt{3}$	2	0	-2	-4	-2	0

By plotting these points, as shown in Figure 9.70, it appears that the graph is a circle of radius 2 whose center is the point $(x, y) = (0, 2)$.

Figure 9.70

CHECKPOINT Now try Exercise 27.

You can confirm the graph found in Example 1 in three ways.

1. *Convert to Rectangular Form* Multiply each side of the polar equation by r and convert the result to rectangular form.
2. *Use a Polar Coordinate Mode* Set your graphing utility to *polar* mode and graph the polar equation. (Use $0 \le \theta \le \pi$, $-6 \le x \le 6$, and $-4 \le y \le 4$.)
3. *Use a Parametric Mode* Set your graphing utility to *parametric* mode and graph $x = (4 \sin t) \cos t$ and $y = (4 \sin t) \sin t$.

Most graphing utilities have a *polar* graphing mode. If yours doesn't, you can rewrite the polar equation $r = f(\theta)$ in parametric form, using t as a parameter, as follows.

$$x = f(t) \cos t \qquad \text{and} \qquad y = f(t) \sin t$$

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Symmetry and Zeros

In Figure 9.70, note that as θ increases from 0 to 2π the graph is traced out twice. Moreover, note that the graph is *symmetric with respect to the line* $\theta = \pi/2$. Had you known about this symmetry and retracing ahead of time, you could have used fewer points. The three important types of symmetry to consider in polar curve sketching are shown in Figure 9.71.

Symmetry with Respect to the Line $\theta = \frac{\pi}{2}$ ***Symmetry with Respect to the Polar Axis*** ***Symmetry with Respect to the Pole***

Figure 9.71

Testing for Symmetry in Polar Coordinates

The graph of a polar equation is symmetric with respect to the following when the given substitution yields an equivalent equation.

1. The line $\theta = \frac{\pi}{2}$: Replace (r, θ) by $(r, \pi - \theta)$ or $(-r, -\theta)$.
2. The polar axis: Replace (r, θ) by $(r, -\theta)$ or $(-r, \pi - \theta)$.
3. The pole: Replace (r, θ) by $(r, \pi + \theta)$ or $(-r, \theta)$.

You can determine the symmetry of the graph of $r = 4 \sin \theta$ (see Example 1) as follows.

1. Replace (r, θ) by $(-r, -\theta)$:

$$-r = 4 \sin(-\theta) \quad \Rightarrow \quad r = -4 \sin(-\theta) = 4 \sin \theta$$

2. Replace (r, θ) by $(r, -\theta)$:

$$r = 4 \sin(-\theta) = -4 \sin \theta$$

3. Replace (r, θ) by $(-r, \theta)$:

$$-r = 4 \sin \theta \quad \Rightarrow \quad r = -4 \sin \theta$$

So, the graph of $r = 4 \sin \theta$ is symmetric with respect to the line $\theta = \pi/2$.

Study Tip

Recall from Section 4.2 that the sine function is odd. That is,

$$\sin(-\theta) = -\sin \theta.$$

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Example 2 Using Symmetry to Sketch a Polar Graph

Use symmetry to sketch the graph of

$$r = 3 + 2\cos\theta$$

by hand.

Solution

Replacing (r, θ) by $(r, -\theta)$ produces

$$r = 3 + 2\cos(-\theta) = 3 + 2\cos\theta. \qquad \cos(-u) = \cos u$$

So, by using the even trigonometric identity, you can conclude that the curve is symmetric with respect to the polar axis. Plotting the points in the table and using polar axis symmetry, you obtain the graph shown in Figure 9.72. This graph is called a **limaçon.**

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r	5	$3 + \sqrt{3}$	4	3	2	$3 - \sqrt{3}$	1

Figure 9.72

Use a graphing utility to confirm this graph.

CHECKPOINT Now try Exercise 31.

The three tests for symmetry in polar coordinates on page 684 are sufficient to guarantee symmetry, but they are not necessary. For instance, Figure 9.73 shows the graph of

$$r = \theta + 2\pi. \qquad \text{Spiral of Archimedes}$$

From the figure, you can see that the graph is symmetric with respect to the line $\theta = \pi/2$. Yet the tests on page 684 fail to indicate symmetry because neither of the following replacements yields an equivalent equation.

Figure 9.73

Original Equation	*Replacement*	*New Equation*
$r = \theta + 2\pi$	(r, θ) by $(-r, -\theta)$	$-r = -\theta + 2\pi$
$r = \theta + 2\pi$	(r, θ) by $(r, \pi - \theta)$	$r = -\theta + 3\pi$

The equations discussed in Examples 1 and 2 are of the form

$$r = f(\sin\theta) \qquad \text{Example 1}$$

and

$$r = g(\cos\theta). \qquad \text{Example 2}$$

The graph of the first equation is symmetric with respect to the line $\theta = \pi/2$, and the graph of the second equation is symmetric with respect to the polar axis. This observation can be generalized to yield the following *quick tests for symmetry.*

Quick Tests for Symmetry in Polar Coordinates

1. The graph of $r = f(\sin\theta)$ is symmetric with respect to the line $\theta = \frac{\pi}{2}$.
2. The graph of $r = g(\cos\theta)$ is symmetric with respect to the polar axis.

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An additional aid to sketching graphs of polar equations involves knowing the θ-values for which $r = 0$. In Example 1, $r = 0$ when $\theta = 0$. Some curves reach their zeros at more than one point, as shown in Example 3.

Example 3 Analyzing a Polar Graph

Analyze the graph of

$$r = 2\cos 3\theta.$$

Solution

Symmetry: With respect to the polar axis

Zeros of r: $r = 0$ when $3\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$

or $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
r	2	$\sqrt{2}$	0	$-\sqrt{2}$	-2	$-\sqrt{2}$	0

By plotting these points and using the specified symmetry and zeros, you can obtain the graph shown in Figure 9.74. This graph is called a **rose curve,** and each loop on the graph is called a *petal*. Note how the entire curve is generated as θ increases from 0 to π.

Figure 9.74

CHECKPOINT Now try Exercise 35.

Explore the Concept

Notice that the rose curve in Example 3 has three petals. How many petals do the rose curves $r = 2\cos 4\theta$ and $r = 2\sin 3\theta$ have? Determine the numbers of petals for the curves $r = 2\cos n\theta$ and $r = 2\sin n\theta$, where n is a positive integer.

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Special Polar Graphs

Several important types of graphs have equations that are simpler in polar form than in rectangular form. For example, the circle

$$r = 4 \sin \theta$$

in Example 1 has the more complicated rectangular equation

$$x^2 + (y - 2)^2 = 4.$$

Several other types of graphs that have simple polar equations are shown below.

Limaçons

$r = a \pm b \cos \theta$, $r = a \pm b \sin \theta \quad (a > 0, b > 0)$

$\frac{a}{b} < 1$	$\frac{a}{b} = 1$	$1 < \frac{a}{b} < 2$	$\frac{a}{b} \geq 2$
Limaçon with inner loop	Cardioid (heart-shaped)	Dimpled limaçon	Convex limaçon

Rose Curves

n petals when n is odd, $2n$ petals when n is even $(n \geq 2)$

$r = a \cos n\theta$	$r = a \cos n\theta$	$r = a \sin n\theta$	$r = a \sin n\theta$
Rose curve	Rose curve	Rose curve	Rose curve

Circles and Lemniscates

$r = a \cos \theta$	$r = a \sin \theta$	$r^2 = a^2 \sin 2\theta$	$r^2 = a^2 \cos 2\theta$
Circle	Circle	Lemniscate	Lemniscate

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The quick tests for symmetry presented on page 685 are especially useful when graphing rose curves. Because rose curves have the form $r = f(\sin\theta)$ or the form $r = g(\cos\theta)$, you know that a rose curve will be either symmetric with respect to the line $\theta = \pi/2$ or symmetric with respect to the polar axis.

Example 4 Analyzing a Rose Curve

Analyze the graph of $r = 3\cos 2\theta$.

Solution

Type of curve: Rose curve with $2n = 4$ petals

Symmetry: With respect to the polar axis, the line $\theta = \dfrac{\pi}{2}$, and the pole

Zeros of r: $r = 0$ when $\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}$

Using a graphing utility, enter the equation, as shown in Figure 9.75 (with $0 \le \theta \le 2\pi$). You should obtain the graph shown in Figure 9.76.

Figure 9.75

Figure 9.76

CHECKPOINT Now try Exercise 39.

Example 5 Analyzing a Lemniscate

Analyze the graph of $r^2 = 9\sin 2\theta$.

Solution

Type of curve: Lemniscate

Symmetry: With respect to the pole

Zeros of r: $r = 0$ when $\theta = 0, \dfrac{\pi}{2}$

Using a graphing utility, enter the equation, as shown in Figure 9.77 (with $0 \le \theta \le 2\pi$). You should obtain the graph shown in Figure 9.78.

Figure 9.77

Figure 9.78

CHECKPOINT Now try Exercise 45.

What's Wrong?

You use a graphing utility in *polar* mode to confirm the result in Example 5 and obtain the graph shown below (with $0 \le \theta \le 2\pi$). What's wrong?

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9.6 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1–4, fill in the blank.

1. The equation $r = 2 + \cos\theta$ represents a _____ .

2. The equation $r = 2\cos\theta$ represents a _____ .

3. The equation $r^2 = 4\sin 2\theta$ represents a _____ .

4. The equation $r = 1 + \sin\theta$ represents a _____ .

5. How can you test whether the graph of a polar equation is symmetric to the line $\theta = \dfrac{\pi}{2}$?

6. Is the graph of $r = 3 + 4\cos\theta$ symmetric with respect to the line $\theta = \dfrac{\pi}{2}$ or to the polar axis?

Procedures and Problem Solving

Identifying Types of Polar Graphs **In Exercises 7–12, identify the type of polar graph.**

7.

8.

9.

10.

11.

12.

Finding the Equation of a Polar Curve **In Exercises 13–16, determine the equation of the polar curve whose graph is shown.**

13.

14.

(a) $r = 1 - 2\sin\theta$
(b) $r = 1 + 2\sin\theta$
(c) $r = 1 + 2\cos\theta$
(d) $r = 1 - 2\cos\theta$

(a) $r = 2 - \cos\theta$
(b) $r = 2 - \sin\theta$
(c) $r = 2 + \cos\theta$
(d) $r = 2 + \sin\theta$

15.

16.

(a) $r = 2\cos 4\theta$
(b) $r = \cos 4\theta$
(c) $r = 2\cos 2\theta$
(d) $r = 2\cos\dfrac{\theta}{2}$

(a) $r = 2\sin 6\theta$
(b) $r = 2\cos\left(\dfrac{3\theta}{2}\right)$
(c) $r = 2\sin\left(\dfrac{3\theta}{2}\right)$
(d) $r = 2\sin 3\theta$

Testing for Symmetry **In Exercises 17–24, test for symmetry with respect to $\theta = \pi/2$, the polar axis, and the pole.**

17. $r = 5 + 4\cos\theta$

18. $r = 16\cos 3\theta$

19. $r = \dfrac{2}{1 + \sin\theta}$

20. $r = \dfrac{2}{1 - \cos\theta}$

21. $r = 6\sin\theta$

22. $r = 4\csc\theta\cos\theta$

23. $r^2 = 16\sin 2\theta$

24. $r^2 = 36\sin 2\theta$

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Sketching the Graph of a Polar Equation **In Exercises 25–34, sketch the graph of the polar equation. Use a graphing utility to verify your graph.**

25. $r = 5$

26. $\theta = -\dfrac{5\pi}{3}$

✓ **27.** $r = 3 \sin \theta$

28. $r = 2 \cos \theta$

29. $r = 3(1 - \cos \theta)$

30. $r = 4(1 + \sin \theta)$

✓ **31.** $r = 3 - 4 \cos \theta$

32. $r = 1 - 2 \sin \theta$

33. $r = 4 + 5 \sin \theta$

34. $r = 3 + 6 \cos \theta$

Analyzing a Polar Graph **In Exercises 35–38, identify and sketch the graph of the polar equation. Identify any symmetry and zeros of *r*. Use a graphing utility to verify your results.**

✓ **35.** $r = 5 \cos 3\theta$

36. $r = -\sin 5\theta$

37. $r = 7 \sin 2\theta$

38. $r = 3 \cos 5\theta$

Analyzing a Special Polar Graph **In Exercises 39–52, use a graphing utility to graph the polar equation. Describe your viewing window.**

✓ **39.** $r = 8 \cos 2\theta$

40. $r = \cos 2\theta$

41. $r = 2(5 - \sin \theta)$

42. $r = 6 - 4 \sin \theta$

43. $r = \dfrac{3}{\sin \theta - 2 \cos \theta}$

44. $r = \dfrac{6}{2 \sin \theta - 3 \cos \theta}$

✓ **45.** $r^2 = 4 \cos 2\theta$

46. $r^2 = 9 \sin \theta$

47. $r = 8 \sin \theta \cos^2 \theta$

48. $r = 2 \cos(3\theta - 2)$

49. $r = 2 \csc \theta + 6$

50. $r = 4 - \sec \theta$

51. $r = e^{2\theta}$

52. $r = e^{\theta/2}$

Using a Graphing Utility to Graph a Polar Equation **In Exercises 53–58, use a graphing utility to graph the polar equation. Find an interval for θ for which the graph is traced *only once*.**

53. $r = 3 - 4 \cos \theta$

54. $r = 2(1 - 2 \sin \theta)$

55. $r = 2 \cos \dfrac{3\theta}{2}$

56. $r = 3 \sin \dfrac{5\theta}{2}$

57. $r^2 = 16 \sin 2\theta$

58. $r^2 = \dfrac{1}{\theta}$

Using a Graphing Utility to Graph a Polar Equation **In Exercises 59–62, use a graphing utility to graph the polar equation and show that the indicated line is an asymptote of the graph.**

	Name of Graph	*Polar Equation*	*Asymptote*
59.	Conchoid	$r = 2 - \sec \theta$	$x = -1$
60.	Conchoid	$r = 2 + \csc \theta$	$y = 1$
61.	Hyperbolic spiral	$r = \dfrac{3}{\theta}$	$y = 3$
62.	Strophoid	$r = 2 \cos 2\theta \sec \theta$	$x = -2$

Conclusions

True or False? **In Exercises 63 and 64, determine whether the statement is true or false. Justify your answer.**

63. The graph of $r = 10 \sin 5\theta$ is a rose curve with five petals.

64. A rose curve will always have symmetry with respect to the line $\theta = \pi/2$.

65. **Exploration** The graph of $r = f(\theta)$ is rotated about the pole through an angle ϕ. Show that the equation of the rotated graph is $r = f(\theta - \phi)$.

66. **Exploration** Consider the graph of $r = f(\sin \theta)$.

(a) Show that when the graph is rotated counterclockwise $\pi/2$ radians about the pole, the equation of the rotated graph is $r = f(-\cos \theta)$.

(b) Show that when the graph is rotated counterclockwise π radians about the pole, the equation of the rotated graph is $r = f(-\sin \theta)$.

(c) Show that when the graph is rotated counterclockwise $3\pi/2$ radians about the pole, the equation of the rotated graph is $r = f(\cos \theta)$.

Writing an Equation for Special Polar Graphs **In Exercises 67 and 68, use the results of Exercises 65 and 66.**

67. Write an equation for the limaçon $r = 2 - \sin \theta$ after it has been rotated through each given angle.

(a) $\dfrac{\pi}{4}$ (b) $\dfrac{\pi}{2}$ (c) π (d) $\dfrac{3\pi}{2}$

68. Write an equation for the rose curve $r = 2 \sin 2\theta$ after it has been rotated through each given angle.

(a) $\dfrac{\pi}{6}$ (b) $\dfrac{\pi}{2}$ (c) $\dfrac{2\pi}{3}$ (d) π

69. **Exploration** Use a graphing utility to graph the polar equation $r = 2 + k \sin \theta$ for $k = 0$, $k = 1$, $k = 2$, and $k = 3$. Identify each graph.

70. **CAPSTONE** Explain why some polar curves have equations that are simpler in polar form than in rectangular form. Besides a circle, give an example of a curve that is simpler in polar form than in rectangular form. Give an example of a curve that is simpler in rectangular form than in polar form.

Cumulative Mixed Review

Finding the Zeros of a Rational Function **In Exercises 71–74, find the zeros (if any) of the rational function.**

71. $f(x) = \dfrac{x^2 - 9}{x + 1}$

72. $f(x) = 6 + \dfrac{4}{x^2 + 4}$

73. $f(x) = 5 - \dfrac{3}{x - 2}$

74. $f(x) = \dfrac{x^3 - 27}{x^2 + 4}$

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9.7 Polar Equations of Conics

Alternative Definition of Conics and Polar Equations

In Sections 9.2 and 9.3, you learned that the rectangular equations of ellipses and hyperbolas take simple forms when the origin lies at the *center*. As it happens, there are many important applications of conics in which it is more convenient to use one of the *foci* as the origin. In this section, you will learn that polar equations of conics take simple forms when one of the foci lies at the pole.

To begin, consider the following alternative definition of a conic that uses the concept of eccentricity (a measure of the flatness of the conic).

Alternative Definition of a Conic

The locus of a point in the plane that moves such that its distance from a fixed point (focus) is in a constant ratio to its distance from a fixed line (directrix) is a **conic.** The constant ratio is the **eccentricity** of the conic and is denoted by e. Moreover, the conic is an **ellipse** when $0 < e < 1$, a **parabola** when $e = 1$, and a **hyperbola** when $e > 1$. (See Figure 9.79.)

What you should learn

- Define conics in terms of eccentricities, and write and graph equations of conics in polar form.
- Use equations of conics in polar form to model real-life problems.

Why you should learn it

The orbits of planets and satellites can be modeled by polar equations. For instance, in Exercise 60 on page 697, you will use a polar equation to model the orbit of a satellite.

Jupiter

In Figure 9.79, note that for each type of conic, the focus is at the pole.

Ellipse: $0 < e < 1$

$$\frac{PF}{PQ} < 1$$

Parabola: $e = 1$

$$\frac{PF}{PQ} = 1$$

Hyperbola: $e > 1$

$$\frac{PF}{PQ} = \frac{P'F}{P'Q'} > 1$$

Figure 9.79

The benefit of locating a focus of a conic at the pole is that the equation of the conic becomes simpler.

Polar Equations of Conics (See the proof on page 709.)

The graph of a polar equation of the form

1. $r = \dfrac{ep}{1 \pm e\cos\theta}$

or

2. $r = \dfrac{ep}{1 \pm e\sin\theta}$

is a conic, where $e > 0$ is the eccentricity and $|p|$ is the distance between the focus (pole) and the directrix.

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An equation of the form

$$r = \frac{ep}{1 \pm e\cos\theta} \qquad \text{Vertical directrix}$$

corresponds to a conic with a vertical directrix and symmetry with respect to the polar axis. An equation of the form

$$r = \frac{ep}{1 \pm e\sin\theta} \qquad \text{Horizontal directrix}$$

corresponds to a conic with a horizontal directrix and symmetry with respect to the line $\theta = \pi/2$. Moreover, the converse is also true—that is, any conic with a focus at the pole and having a horizontal or vertical directrix can be represented by one of the given equations.

Example 1 Identifying a Conic from Its Equation

Identify the type of conic represented by the equation

$$r = \frac{15}{3 - 2\cos\theta}.$$

Algebraic Solution

To identify the type of conic, rewrite the equation in the form $r = ep/(1 \pm e\cos\theta)$.

$$r = \frac{15}{3 - 2\cos\theta}$$

$$= \frac{5}{1 - (2/3)\cos\theta} \qquad \text{Divide numerator and denominator by 3.}$$

Because $e = \frac{2}{3} < 1$, you can conclude that the graph is an ellipse.

CHECKPOINT Now try Exercise 15.

Graphical Solution

Use a graphing utility in *polar* mode and be sure to use a square setting, as shown in Figure 9.80.

Figure 9.80

For the ellipse in Figure 9.80, the major axis is horizontal and the vertices lie at $(r, \theta) = (15, 0)$ and $(r, \theta) = (3, \pi)$. So, the length of the *major* axis is $2a = 18$. To find the length of the *minor* axis, you can use the equations $e = c/a$ and $b^2 = a^2 - c^2$ to conclude that

$$b^2 = a^2 - c^2$$

$$= a^2 - (ea)^2$$

$$= a^2(1 - e^2). \qquad \text{Ellipse}$$

Because $e = \frac{2}{3}$, you have

$$b^2 = 9^2\left[1 - \left(\tfrac{2}{3}\right)^2\right] = 45,$$

which implies that

$$b = \sqrt{45} = 3\sqrt{5}.$$

So, the length of the minor axis is $2b = 6\sqrt{5}$. A similar analysis for hyperbolas yields

$$b^2 = c^2 - a^2$$

$$= (ea)^2 - a^2$$

$$= a^2(e^2 - 1). \qquad \text{Hyperbola}$$

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Example 2 Analyzing the Graph of a Polar Equation

Analyze the graph of the polar equation

$$r = \frac{32}{3 + 5\sin\theta}.$$

Solution

Dividing the numerator and denominator by 3 produces

$$r = \frac{32/3}{1 + (5/3)\sin\theta}.$$

Because $e = \frac{5}{3} > 1$, the graph is a hyperbola. The transverse axis of the hyperbola lies on the line $\theta = \pi/2$, and the vertices occur at $(r, \theta) = (4, \pi/2)$ and $(r, \theta) = (-16, 3\pi/2)$. Because the length of the transverse axis is 12, you can see that $a = 6$. To find b, write

$$b^2 = a^2(e^2 - 1) = 6^2\left[\left(\frac{5}{3}\right)^2 - 1\right] = 64.$$

So, $b = 8$. You can use a and b to determine that the asymptotes are $y = 10 \pm \frac{3}{4}x$, as shown in Figure 9.81.

Figure 9.81

CHECKPOINT Now try Exercise 27.

In the next example, you are asked to find a polar equation for a specified conic. To do this, let p be the distance between the pole and the directrix.

1. Horizontal directrix above the pole: $r = \dfrac{ep}{1 + e\sin\theta}$
2. Horizontal directrix below the pole: $r = \dfrac{ep}{1 - e\sin\theta}$
3. Vertical directrix to the right of the pole: $r = \dfrac{ep}{1 + e\cos\theta}$
4. Vertical directrix to the left of the pole: $r = \dfrac{ep}{1 - e\cos\theta}$

Explore the Concept

Try using a graphing utility in *polar* mode to verify the four orientations shown at the left.

Example 3 Finding the Polar Equation of a Conic

Find the polar equation of the parabola whose focus is the pole and whose directrix is the line $y = 3$.

Solution

From Figure 9.82, you can see that the directrix is horizontal and above the pole. Moreover, because the eccentricity of a parabola is $e = 1$ and the distance between the pole and the directrix is $p = 3$, you have the equation

$$r = \frac{ep}{1 + e\sin\theta} = \frac{3}{1 + \sin\theta}.$$

Figure 9.82

CHECKPOINT Now try Exercise 37.

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Application

Kepler's Laws (listed below), named after the German astronomer Johannes Kepler (1571–1630), can be used to describe the orbits of the planets about the sun.

1. Each planet moves in an elliptical orbit with the sun as a focus.
2. A ray from the sun to the planet sweeps out equal areas of the ellipse in equal times.
3. The square of the period (the time it takes for a planet to orbit the sun) is proportional to the cube of the mean distance between the planet and the sun.

Although Kepler simply stated these laws on the basis of observation, they were later validated by Isaac Newton (1642–1727). In fact, Newton was able to show that each law can be deduced from a set of universal laws of motion and gravitation that govern the movement of all heavenly bodies, including comets and satellites. This is illustrated in the next example, which involves the comet named after the English mathematician and physicist Edmund Halley (1656–1742).

If you use Earth as a reference with a period of 1 year and a distance of 1 astronomical unit (an *astronomical unit* is defined as the mean distance between Earth and the sun, or about 93 million miles), then the proportionality constant in Kepler's third law is 1. For example, because Mars has a mean distance to the sun of $d \approx 1.524$ astronomical units, its period P is given by

$$d^3 = P^2.$$

So, the period of Mars is $P \approx 1.88$ years.

Astronomer

Example 4 Halley's Comet

Halley's comet has an elliptical orbit with an eccentricity of $e \approx 0.967$. The length of the major axis of the orbit is approximately 35.88 astronomical units. Find a polar equation for the orbit. How close does Halley's comet come to the sun?

Solution

Using a vertical major axis, as shown in Figure 9.83, choose an equation of the form

$$r = \frac{ep}{1 + e \sin \theta}.$$

Because the vertices of the ellipse occur at $\theta = \pi/2$ and $\theta = 3\pi/2$, you can determine the length of the major axis to be the sum of the r-values of the vertices. That is,

$$2a = \frac{0.967p}{1 + 0.967} + \frac{0.967p}{1 - 0.967} \approx 29.79p \approx 35.88.$$

So, $p \approx 1.204$ and

$$ep \approx (0.967)(1.204) \approx 1.164.$$

Using this value of ep in the equation, you have

$$r = \frac{1.164}{1 + 0.967 \sin \theta}$$

where r is measured in astronomical units. To find the closest point to the sun (the focus), substitute $\theta = \pi/2$ into this equation to obtain

$$r = \frac{1.164}{1 + 0.967 \sin(\pi/2)} \approx 0.59 \text{ astronomical units} \approx 55{,}000{,}000 \text{ miles}.$$

CHECKPOINT Now try Exercise 55.

Figure 9.83

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9.7 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

1. Fill in the blank: The locus of a point in the plane that moves such that its distance from a fixed point (focus) is in a constant ratio to its distance from a fixed line (directrix) is a _____ .

2. Match the conic with its eccentricity.

(a) $0 < e < 1$ (i) ellipse

(b) $e = 1$ (ii) hyperbola

(c) $e > 1$ (iii) parabola

3. A conic has a polar equation of the form $r = \dfrac{ep}{1 + e\cos\theta}$. Is the directrix vertical or horizontal?

4. A conic with a horizontal directrix has a polar equation of the form $r = \dfrac{ep}{1 - e\sin\theta}$. Is the directrix above or below the pole?

Procedures and Problem Solving

Identifying a Conic **In Exercises 5–8, use a graphing utility to graph the polar equation for (a) $e = 1$, (b) $e = 0.5$, and (c) $e = 1.5$. Identify the conic for each equation.**

5. $r = \dfrac{2e}{1 + e\cos\theta}$

6. $r = \dfrac{2e}{1 - e\cos\theta}$

7. $r = \dfrac{2e}{1 - e\sin\theta}$

8. $r = \dfrac{2e}{1 + e\sin\theta}$

Identifying the Polar Equation of a Conic **In Exercises 9–14, match the polar equation with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]**

(a)

(b)

(c)

(d)

(e)

(f)

9. $r = \dfrac{4}{1 - \cos\theta}$

10. $r = \dfrac{3}{2 - \cos\theta}$

11. $r = \dfrac{3}{2 + \cos\theta}$

12. $r = \dfrac{4}{1 - 3\sin\theta}$

13. $r = \dfrac{3}{1 + 2\sin\theta}$

14. $r = \dfrac{4}{1 + \sin\theta}$

Identifying a Conic from Its Equation **In Exercises 15–24, identify the type of conic represented by the equation. Use a graphing utility to confirm your result.**

15. $r = \dfrac{3}{1 - \cos\theta}$

16. $r = \dfrac{7}{1 + \sin\theta}$

17. $r = \dfrac{4}{4 - \cos\theta}$

18. $r = \dfrac{7}{7 + \sin\theta}$

19. $r = \dfrac{8}{4 + 3\sin\theta}$

20. $r = \dfrac{9}{3 - 2\cos\theta}$

21. $r = \dfrac{6}{2 + \sin\theta}$

22. $r = \dfrac{5}{-1 + 2\cos\theta}$

23. $r = \dfrac{3}{4 - 8\cos\theta}$

24. $r = \dfrac{10}{3 + 9\sin\theta}$

Analyzing the Graph of a Polar Equation **In Exercises 25–30, identify the type of conic represented by the polar equation and analyze its graph. Then use a graphing utility to graph the polar equation.**

25. $r = \dfrac{-5}{1 - \sin\theta}$

26. $r = \dfrac{-1}{2 + 4\sin\theta}$

27. $r = \dfrac{14}{14 + 17\sin\theta}$

28. $r = \dfrac{12}{2 - \cos\theta}$

29. $r = \dfrac{3}{-4 + 2\cos\theta}$

30. $r = \dfrac{4}{1 - 2\cos\theta}$

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Graphing a Rotated Conic **In Exercises 31–36, use a graphing utility to graph the rotated conic.**

31. $r = \dfrac{3}{1 - \cos(\theta - \pi/4)}$ (See Exercise 15.)

32. $r = \dfrac{7}{7 + \sin(\theta - \pi/3)}$ (See Exercise 18.)

33. $r = \dfrac{4}{4 - \cos(\theta + 3\pi/4)}$ (See Exercise 17.)

34. $r = \dfrac{9}{3 - 2\cos(\theta + \pi/2)}$ (See Exercise 20.)

35. $r = \dfrac{8}{4 + 3\sin(\theta + \pi/6)}$ (See Exercise 19.)

36. $r = \dfrac{5}{-1 + 2\cos(\theta + 2\pi/3)}$ (See Exercise 22.)

Finding the Polar Equation of a Conic **In Exercises 37–52, find a polar equation of the conic with its focus at the pole.**

	Conic	*Eccentricity*	*Directrix*
37.	Parabola	$e = 1$	$x = -1$
38.	Parabola	$e = 1$	$y = -4$
39.	Ellipse	$e = \frac{1}{2}$	$y = 1$
40.	Ellipse	$e = \frac{3}{4}$	$y = -4$
41.	Hyperbola	$e = 2$	$x = 1$
42.	Hyperbola	$e = \frac{3}{2}$	$x = -1$

	Conic	*Vertex or Vertices*
43.	Parabola	$\left(1, -\dfrac{\pi}{2}\right)$
44.	Parabola	$(8, 0)$
45.	Parabola	$(5, \pi)$
46.	Parabola	$\left(10, \dfrac{\pi}{2}\right)$
47.	Ellipse	$(2, 0), (10, \pi)$
48.	Ellipse	$\left(2, \dfrac{\pi}{2}\right), \left(4, \dfrac{3\pi}{2}\right)$
49.	Ellipse	$(20, 0), (4, \pi)$
50.	Hyperbola	$\left(1, \dfrac{3\pi}{2}\right), \left(9, \dfrac{3\pi}{2}\right)$
51.	Hyperbola	$(2, 0), (8, 0)$
52.	Hyperbola	$\left(4, \dfrac{\pi}{2}\right), \left(1, \dfrac{\pi}{2}\right)$

53. **Astronomy** The planets travel in elliptical orbits with the sun at one focus. Assume that the focus is at the pole, the major axis lies on the polar axis, and the length of the major axis is $2a$ (see figure). Show that the polar equation of the orbit of a planet is

$$r = \frac{(1 - e^2)a}{1 - e\cos\theta}$$

where e is the eccentricity.

54. **Astronomy** Use the result of Exercise 53 to show that the minimum distance (*perihelion*) from the sun to a planet is $r = a(1 - e)$ and that the maximum distance (*aphelion*) is $r = a(1 + e)$.

Astronomy **In Exercises 55–58, use the results of Exercises 53 and 54 to find the polar equation of the orbit of the planet and the perihelion and aphelion distances.**

55. Earth $a = 9.2956 \times 10^7$ miles, $e = 0.0167$

56. Mercury $a = 3.5983 \times 10^7$ miles, $e = 0.2056$

57. Venus $a = 6.7283 \times 10^7$ miles, $e = 0.0068$

58. Jupiter $a = 7.7841 \times 10^8$ kilometers, $e = 0.0484$

59. **Astronomy** Use the results of Exercises 53 and 54, where for the planet Neptune, $a = 4.498 \times 10^9$ kilometers and $e = 0.0086$ and for the dwarf planet Pluto, $a = 5.906 \times 10^9$ kilometers and $e = 0.2488$.

(a) Find the polar equation of the orbit of each planet.

(b) Find the perihelion and aphelion distances for each planet.

(c) Use a graphing utility to graph both Neptune's and Pluto's equations of orbit in the same viewing window.

(d) Is Pluto ever closer to the sun than Neptune? Until recently, Pluto was considered the ninth planet. Why was Pluto called the ninth planet and Neptune the eighth planet?

(e) Do the orbits of Neptune and Pluto intersect? Will Neptune and Pluto ever collide? Why or why not?

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60. ***Why you should learn it*** *(p. 691)* On November 27, 1963, the United States launched a satellite named *Explorer 18*. Its low and high points above the surface of Earth were about 119 miles and 122,800 miles, respectively (see figure). The center of Earth is at one focus of the orbit.

(a) Find the polar equation of the orbit (assume the radius of Earth is 4000 miles).

(b) Find the distance between the surface of Earth and the satellite when $\theta = 60°$.

(c) Find the distance between the surface of Earth and the satellite when $\theta = 30°$.

Conclusions

True or False? **In Exercises 61–64, determine whether the statement is true or false. Justify your answer.**

61. The graph of $r = 4/(-3 - 3\sin\theta)$ has a horizontal directrix above the pole.

62. The conic represented by the following equation is an ellipse.

$$r^2 = \frac{16}{9 - 4\cos\left(\theta + \frac{\pi}{4}\right)}$$

63. For values of $e > 1$ and $0 \le \theta \le 2\pi$, the graphs of the following equations are the same.

$$r = \frac{ex}{1 - e\cos\theta} \quad \text{and} \quad r = \frac{e(-x)}{1 + e\cos\theta}$$

64. The graph of $r = \dfrac{5}{1 - \sin[\theta - (\pi/4)]}$ can be obtained by rotating the graph of $r = \dfrac{5}{1 + \sin\theta}$ about the pole.

65. **Verifying a Polar Equation** Show that the polar equation of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{is} \quad r^2 = \frac{b^2}{1 - e^2\cos^2\theta}.$$

66. **Verifying a Polar Equation** Show that the polar equation of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{is} \quad r^2 = \frac{-b^2}{1 - e^2\cos^2\theta}.$$

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Writing a Polar Equation **In Exercises 67–72, use the results of Exercises 65 and 66 to write the polar form of the equation of the conic.**

67. $\dfrac{x^2}{169} + \dfrac{y^2}{144} = 1$

68. $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$

69. $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$

70. $\dfrac{x^2}{36} - \dfrac{y^2}{4} = 1$

71. Hyperbola — One focus: $(5, 0)$; Vertices: $(4, 0), (4, \pi)$

72. Ellipse — One focus: $(4, 0)$; Vertices: $(5, 0), (5, \pi)$

73. **Exploration** Consider the polar equation

$$r = \frac{4}{1 - 0.4\cos\theta}.$$

(a) Identify the conic without graphing the equation.

(b) Without graphing the following polar equations, describe how each differs from the given polar equation. Use a graphing utility to verify your results.

$$r = \frac{4}{1 + 0.4\cos\theta}, \quad r = \frac{4}{1 - 0.4\sin\theta}$$

74. **Exploration** The equation

$$r = \frac{ep}{1 \pm e\sin\theta}$$

is the equation of an ellipse with $0 < e < 1$. What happens to the lengths of both the major axis and the minor axis when the value of e remains fixed and the value of p changes? Use an example to explain your reasoning.

75. **Think About It** What conic does the polar equation given by $r = a\sin\theta + b\cos\theta$ represent?

76. **CAPSTONE** In your own words, define the term *eccentricity* and explain how it can be used to classify conics. Then explain how you can use the values of b and c to determine whether a polar equation of the form

$$r = \frac{a}{b + c\sin\theta}$$

represents an ellipse, a parabola, or a hyperbola.

Cumulative Mixed Review

Evaluating a Trigonometric Expression **In Exercises 77–80, find the value of the trigonometric function given that u and v are in Quadrant IV and $\sin u = -\frac{3}{5}$ and $\cos v = 1/\sqrt{2}$.**

77. $\cos(u + v)$

78. $\sin(u + v)$

79. $\sin(u - v)$

80. $\cos(u - v)$

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9 Chapter Summary

	What did you learn?	Explanation and Examples	Review Exercises
9.1	**Recognize a conic as the intersection of a plane and a double-napped cone *(p. 636)*.**	In the formation of the four basic conics, the intersecting plane does not pass through the vertex of the cone. (See Figure 9.1.) When the plane does pass through the vertex, the resulting figure is a degenerate conic, such as a point or a line. (See Figure 9.2.)	1, 2
	Write equations of circles in standard form *(p. 637)*.	The standard form of the equation of a circle with center at (h, k) is $(x - h)^2 + (y - k)^2 = r^2$. The standard form of the equation of a circle whose center is the origin, $(h, k) = (0, 0)$, is $x^2 + y^2 = r^2$.	3–14
	Write equations of parabolas in standard form *(p. 639)*.	The standard form of the equation of a parabola with vertex at (h, k) is as follows. $(x - h)^2 = 4p(y - k),\ p \neq 0$ Vertical axis $(y - k)^2 = 4p(x - h),\ p \neq 0$ Horizontal axis	15–24
	Use the reflective property of parabolas to solve real-life problems *(p. 641)*.	The tangent line to a parabola at a point P makes equal angles with (1) the line passing through P and the focus, and (2) the axis of the parabola. (See Figure 9.13.)	25, 26
9.2	**Write equations of ellipses in standard form *(p. 647)*.**	$\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$ Horizontal major axis $\dfrac{(x-h)^2}{b^2} + \dfrac{(y-k)^2}{a^2} = 1$ Vertical major axis	27–38
	Use properties of ellipses to model and solve real-life problems *(p. 651)*.	The properties of ellipses can be used to find the greatest and smallest distances from Earth's center to the moon's center. (See Example 5.)	39, 40
	Find eccentricities of ellipses *(p. 652)*.	The eccentricity e of an ellipse is given by $e = \dfrac{c}{a}$.	41, 42
9.3	**Write equations of hyperbolas in standard form *(p. 656)*.**	$\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$ Horizontal transverse axis $\dfrac{(y-k)^2}{a^2} - \dfrac{(x-h)^2}{b^2} = 1$ Vertical transverse axis	43–46
	Find asymptotes of and graph hyperbolas *(p. 658)*.	$y = k \pm \dfrac{b}{a}(x - h)$ Horizontal transverse axis $y = k \pm \dfrac{a}{b}(x - h)$ Vertical transverse axis	47–52
	Use properties of hyperbolas to solve real-life problems *(p. 661)*.	The properties of hyperbolas can be used in radar and other detection systems. (See Example 5.)	53, 54
	Classify conics from their general equations *(p. 662)*.	The graph of $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is • a circle when $A = C$, with $A \neq 0$. • a parabola when $AC = 0$, with $A = 0$ or $C = 0$ (but not both). • an ellipse when $AC > 0$. • a hyperbola when $AC < 0$.	55–58
	Rotate the coordinate axes to eliminate the *xy*-term in equations of conics *(p. 663)*.	The equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ can be rewritten as $A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0$ by rotating the coordinate axes through an angle θ, where $\cot 2\theta = (A - C)/B$.	59–62

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	What did you learn?	Explanation and Examples	Review Exercises
9.4	**Evaluate sets of parametric equations for given values of the parameter *(p. 669)*.**	If f and g are continuous functions of t on an interval I, then the set of ordered pairs $(f(t), g(t))$ is a plane curve C. The equations $x = f(t)$ and $y = g(t)$ are parametric equations for C, and t is the parameter.	63, 64
	Graph curves that are represented by sets of parametric equations *(p. 670)*, and rewrite sets of parametric equations as single rectangular equations by eliminating the parameter *(p. 672)*.	One way to sketch a curve represented by parametric equations is to plot points in the xy-plane. Each set of coordinates (x, y) is determined from a value chosen for the parameter t. To eliminate the parameter in a pair of parametric equations, solve for t in one equation, and substitute that value of t into the other equation. The result is the corresponding rectangular equation.	65–82
	Find sets of parametric equations for graphs *(p. 673)*.	When finding a set of parametric equations for a given graph, remember that the parametric equations are not unique.	83–94
9.5	**Plot points and find multiple representations of points in the polar coordinate system *(p. 677)*.**		95–100
	Convert points from rectangular to polar form and vice versa *(p. 679)*.	The polar coordinates (r, θ) are related to the rectangular coordinates (x, y) as follows. Polar-to-Rectangular: $x = r\cos\theta,\ y = r\sin\theta$ Rectangular-to-Polar: $\tan\theta = \dfrac{y}{x},\ r^2 = x^2 + y^2$	101–110
	Convert equations from rectangular to polar form and vice versa *(p. 680)*.	To convert a rectangular equation to polar form, replace x by $r\cos\theta$ and y by $r\sin\theta$. Converting a polar equation to rectangular form is more complex.	111–126
9.6	**Graph polar equations by point plotting *(p. 683)*, and use symmetry and zeros as sketching aids *(p. 684)*.**	The graph of a polar equation is symmetric with respect to the following when the given substitution yields an equivalent equation. **1.** Line $\theta = \pi/2$: Replace (r, θ) by $(r, \pi - \theta)$ or $(-r, -\theta)$. **2.** Polar axis: Replace (r, θ) by $(r, -\theta)$ or $(-r, \pi - \theta)$. **3.** Pole: Replace (r, θ) by $(r, \pi + \theta)$ or $(-r, \theta)$.	127–140
	Recognize special polar graphs *(p. 687)*.	Several types of graphs, such as limaçons, rose curves, circles, and lemniscates, have equations that are simpler in polar form than in rectangular form. (See page 687.)	133–140
9.7	**Define conics in terms of eccentricities, and write and graph equations of conics in polar form *(p. 691)*.**	The eccentricity of a conic is denoted by e. The conic is an ellipse when $0 < e < 1$, a parabola when $e = 1$, and a hyperbola when $e > 1$. The graph of a polar equation of the form **1.** $r = (ep)/(1 \pm e\cos\theta)$ or **2.** $r = (ep)/(1 \pm e\sin\theta)$ is a conic, where $e > 0$ is the eccentricity and $\lvert p\rvert$ is the distance between the focus (pole) and the directrix.	141–150
	Use equations of conics in polar form to model real-life problems *(p. 694)*.	An equation of a conic in polar form can be used to model the orbit of Halley's comet. (See Example 4.)	151, 152

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9 Review Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

9.1

Forming a Conic Section **In Exercises 1 and 2, state the type of conic formed by the intersection of the plane and the double-napped cone.**

1.

2.

Finding the Standard Equation of a Circle **In Exercises 3–6, find the standard form of the equation of the circle with the given characteristics.**

3. Center at origin; point on the circle: $(-3, -4)$

4. Center at origin; point on the circle: $(8, -15)$

5. Endpoints of a diameter: $(-1, 2)$ and $(5, 6)$

6. Endpoints of a diameter: $(-2, 3)$ and $(6, -5)$

Writing the Equation of a Circle in Standard Form **In Exercises 7–10, write the equation of the circle in standard form. Then identify its center and radius.**

7. $\frac{1}{2}x^2 + \frac{1}{2}y^2 = 18$

8. $\frac{3}{4}x^2 + \frac{3}{4}y^2 = 1$

9. $16x^2 + 16y^2 - 16x + 24y - 3 = 0$

10. $4x^2 + 4y^2 + 32x - 24y + 51 = 0$

Sketching a Circle **In Exercises 11 and 12, sketch the circle. Identify its center and radius.**

11. $x^2 + y^2 + 4x + 6y - 3 = 0$

12. $x^2 + y^2 + 8x - 10y - 8 = 0$

Finding the Intercepts of a Circle **In Exercises 13 and 14, find the x- and y-intercepts of the graph of the circle.**

13. $(x - 3)^2 + (y + 1)^2 = 7$

14. $(x + 5)^2 + (y - 6)^2 = 27$

Finding the Vertex, Focus, and Directrix of a Parabola **In Exercises 15–18, find the vertex, focus, and directrix of the parabola, and sketch its graph.**

15. $4x - y^2 = 0$

16. $y = -\frac{1}{8}x^2$

17. $\frac{1}{2}y^2 + 18x = 0$

18. $\frac{1}{4}y - 8x^2 = 0$

Finding the Standard Equation of a Parabola **In Exercises 19–22, find the standard form of the equation of the parabola with the given characteristics.**

19. Vertex: $(0, 0)$
Focus: $(4, 0)$

20. Vertex: $(2, 0)$
Focus: $(0, 0)$

21.

22.

Finding the Tangent Line at a Point on a Parabola **In Exercises 23 and 24, find an equation of the tangent line to the parabola at the given point and find the x-intercept of the line.**

23. $x^2 = -2y$, $(2, -2)$

24. $y^2 = -2x$, $(-8, -4)$

25. **Architecture** A parabolic archway (see figure) is 12 meters high at the vertex. At a height of 10 meters, the width of the archway is 8 meters. How wide is the archway at ground level?

Figure for 25

Figure for 26

26. **Architecture** A church window (see figure) is bounded on top by a parabola and below by the arc of a circle.

(a) Find equations of the parabola and the circle.

(b) Use a graphing utility to create a table showing the vertical distances d between the circle and the parabola for various values of x.

x	0	1	2	3	4
d					

9.2

Using the Standard Equation of an Ellipse **In Exercises 27–30, find the center, vertices, foci, and eccentricity of the ellipse and sketch its graph. Use a graphing utility to verify your graph.**

27. $\dfrac{x^2}{4} + \dfrac{y^2}{16} = 1$

28. $\dfrac{x^2}{9} + \dfrac{y^2}{8} = 1$

29. $\dfrac{(x + 1)^2}{25} + \dfrac{(y - 2)^2}{49} = 1$

30. $\dfrac{(x - 5)^2}{1} + \dfrac{(y + 3)^2}{36} = 1$

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Using the Standard Equation of an Ellipse **In Exercises 31–34, (a) find the standard form of the equation of the ellipse, (b) find the center, vertices, foci, and eccentricity of the ellipse, and (c) sketch the ellipse. Use a graphing utility to verify your graph.**

31. $16x^2 + 9y^2 - 32x + 72y + 16 = 0$

32. $4x^2 + 25y^2 + 16x - 150y + 141 = 0$

33. $3x^2 + 8y^2 + 12x - 112y + 403 = 0$

34. $x^2 + 20y^2 - 5x + 120y + 185 = 0$

Finding the Standard Equation of an Ellipse **In Exercises 35–38, find the standard form of the equation of the ellipse with the given characteristics.**

35. Vertices: $(\pm 5, 0)$; foci: $(\pm 4, 0)$

36. Vertices: $(0, \pm 6)$; passes through the point $(2, 2)$

37. Vertices: $(-3, 0), (7, 0)$; foci: $(0, 0), (4, 0)$

38. Vertices: $(2, 0), (2, 4)$; foci: $(2, 1), (2, 3)$

39. **Architecture** A semielliptical archway is to be formed over the entrance to an estate. The arch is to be set on pillars that are 10 feet apart and is to have a height (atop the pillars) of 4 feet. Where should the foci be placed in order to sketch the arch?

40. **Architecture** You are building a wading pool that is in the shape of an ellipse. Your plans give an equation for the elliptical shape of the pool measured in feet as

$$\frac{x^2}{324} + \frac{y^2}{196} = 1.$$

Find the longest distance across the pool, the shortest distance, and the distance between the foci.

41. **Astronomy** Saturn moves in an elliptical orbit with the sun at one focus. The least distance and the greatest distance of the planet from the sun are 1.3495×10^9 and 1.5045×10^9 kilometers, respectively. Find the eccentricity of the orbit, defined by $e = c/a$.

42. **Astronomy** Mercury moves in an elliptical orbit with the sun at one focus. The eccentricity of Mercury's orbit is $e = 0.2056$. The length of the major axis is 72 million miles. Find the standard equation of Mercury's orbit. Place the center of the orbit at the origin and the major axis on the x-axis.

9.3

Finding the Standard Equation of a Hyperbola **In Exercises 43–46, find the standard form of the equation of the hyperbola with the given characteristics.**

43. Vertices: $(\pm 4, 0)$; foci: $(\pm 6, 0)$

44. Vertices: $(0, \pm 1)$; foci: $(0, \pm 2)$

45. Foci: $(0, 0), (8, 0)$; asymptotes: $y = \pm 2(x - 4)$

46. Foci: $(3, \pm 2)$; asymptotes: $y = \pm 2(x - 3)$

Sketching a Hyperbola **In Exercises 47–52, (a) find the standard form of the equation of the hyperbola, (b) find the center, vertices, foci, and eccentricity of the hyperbola, and (c) sketch the hyperbola.**

47. $5y^2 - 4x^2 = 20$

48. $x^2 - y^2 = \frac{9}{4}$

49. $9x^2 - 16y^2 - 18x - 32y - 151 = 0$

50. $-4x^2 + 25y^2 - 8x + 150y + 121 = 0$

51. $y^2 - 4x^2 - 2y - 48x + 59 = 0$

52. $9x^2 - y^2 - 72x + 8y + 119 = 0$

53. **Marine Navigation** Radio transmitting station A is located 200 miles east of transmitting station B. A ship is in an area to the north and 40 miles west of station A. Synchronized radio pulses transmitted to the ship at 186,000 miles per second by the two stations are received 0.0005 second sooner from station A than from station B. How far north is the ship?

54. **Physics** Two of your friends live 4 miles apart on the same "east-west" street, and you live halfway between them. You are having a three-way phone conversation when you hear an explosion. Six seconds later your friend to the east hears the explosion, and your friend to the west hears it 8 seconds after you do. Find equations of two hyperbolas that would locate the explosion. (Assume that the coordinate system is measured in feet and that sound travels at 1100 feet per second.)

Classifying a Conic from a General Equation **In Exercises 55–58, classify the graph of the equation as a circle, a parabola, an ellipse, or a hyperbola.**

55. $3x^2 + 2y^2 - 12x + 12y + 29 = 0$

56. $4x^2 + 4y^2 - 4x + 8y - 11 = 0$

57. $5x^2 - 2y^2 + 10x - 4y + 17 = 0$

58. $-4y^2 + 5x + 3y + 7 = 0$

Rotation of Axes **In Exercises 59–62, rotate the axes to eliminate the xy-term in the equation. Then write the equation in standard form. Sketch the graph of the resulting equation, showing both sets of axes.**

59. $xy - 3 = 0$

60. $x^2 - 4xy + y^2 + 9 = 0$

61. $5x^2 - 2xy + 5y^2 - 12 = 0$

62. $4x^2 + 8xy + 4y^2 + 7\sqrt{2}x + 9\sqrt{2}y = 0$

9.4

Sketching the Graph of Parametric Equations **In Exercises 63 and 64, complete the table for the set of parametric equations. Plot the points (x, y) and sketch a graph of the parametric equations.**

63. $x = 3t - 2$
$y = 7 - 4t$

t	-2	-1	0	1	2	3
x						
y						

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64. $x = \sqrt{t}$
$y = 8 - t$

t	0	1	2	3	4
x					
y					

Sketching a Plane Curve and Eliminating the Parameter **In Exercises 65–70, sketch the curve represented by the parametric equations (indicate the orientation of the curve). Then eliminate the parameter and write the corresponding rectangular equation whose graph represents the curve. Adjust the domain of the resulting rectangular equation, if necessary.**

65. $x = 2t$, $y = 4t$

66. $x = 4t + 1$, $y = 2 - 3t$

67. $x = t^2 + 2$, $y = 4t^2 - 3$

68. $x = \ln 4t$, $y = t^2$

69. $x = t^3$, $y = \frac{1}{2}t^2$

70. $x = \frac{4}{t}$, $y = t^2 - 1$

Using a Graphing Utility in Parametric Mode **In Exercises 71–82, use a graphing utility to graph the curve represented by the parametric equations.**

71. $x = \sqrt[3]{t}, y = t$

72. $x = t, y = \sqrt[3]{t}$

73. $x = \frac{1}{t}, y = t$

74. $x = t, y = \frac{1}{t}$

75. $x = 2t, y = 4t$

76. $x = t^2, y = \sqrt{t}$

77. $x = 1 + 4t, y = 2 - 3t$

78. $x = t + 4, y = t^2$

79. $x = 3, y = t$

80. $x = t, y = 2$

81. $x = 6 \cos \theta$, $y = 6 \sin \theta$

82. $x = 3 + 3 \cos \theta$, $y = 2 + 5 \sin \theta$

Finding Parametric Equations for a Given Graph **In Exercises 83–86, find a set of parametric equations to represent the graph of the given rectangular equation using the parameters (a) $t = x$ and (b) $t = 1 - x$.**

83. $y = 6x + 2$

84. $y = 10 - x$

85. $y = x^2 + 2$

86. $y = 2x^3 + 5x$

Finding Parametric Equations for a Line **In Exercises 87–90, find a set of parametric equations for the line that passes through the given points. (There are many correct answers.)**

87. $(3, 5), (8, 5)$

88. $(2, -1), (2, 4)$

89. $(-1, 6), (10, 0)$

90. $(0, 0), \left(\frac{5}{2}, 6\right)$

Athletics **In Exercises 91–94, the quarterback of a football team releases a pass at a height of 7 feet above the playing field, and the football is caught at a height of 4 feet, 30 yards directly downfield. The pass is released at an angle of 35° with the horizontal. The parametric equations for the path of the football are given by $x = 0.82v_0t$ and $y = 7 + 0.57v_0t - 16t^2$, where v_0 is the speed of the football (in feet per second) when it is released.**

91. Find the speed of the football when it is released.

92. Write a set of parametric equations for the path of the ball.

93. Use a graphing utility to graph the path of the ball and approximate its maximum height.

94. Find the time the receiver has to position himself after the quarterback releases the ball.

9.5

Plotting Points in the Polar Coordinate System **In Exercises 95–100, plot the point given in polar coordinates and find three additional polar representations of the point, using $-2\pi < \theta < 2\pi$.**

95. $\left(2, \frac{\pi}{4}\right)$

96. $\left(-5, -\frac{\pi}{3}\right)$

97. $\left(-2, -\frac{11\pi}{6}\right)$

98. $\left(1, \frac{5\pi}{6}\right)$

99. $(-7, 4.19)$

100. $\left(\sqrt{3}, 2.62\right)$

Polar-to-Rectangular Conversion **In Exercises 101–106, plot the point given in polar coordinates and find the corresponding rectangular coordinates for the point.**

101. $\left(5, -\frac{7\pi}{6}\right)$

102. $\left(-4, \frac{2\pi}{3}\right)$

103. $\left(2, -\frac{5\pi}{3}\right)$

104. $\left(-1, \frac{11\pi}{6}\right)$

105. $\left(3, \frac{3\pi}{4}\right)$

106. $\left(0, \frac{\pi}{2}\right)$

Rectangular-to-Polar Conversion **In Exercises 107–110, plot the point given in rectangular coordinates and find *two* sets of polar coordinates for the point for $0 \le \theta < 2\pi$.**

107. $(0, -9)$

108. $(-3, 4)$

109. $(5, -5)$

110. $\left(-3, -\sqrt{3}\right)$

Converting a Rectangular Equation to Polar Form **In Exercises 111–118, convert the rectangular equation to polar form.**

111. $x^2 + y^2 = 81$

112. $x^2 + y^2 = 48$

113. $x^2 + y^2 - 4x = 0$

114. $x^2 + y^2 - 6y = 0$

115. $xy = 5$

116. $xy = -2$

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117. $4x^2 + y^2 = 1$

118. $2x^2 + 3y^2 = 1$

Converting a Polar Equation to Rectangular Form **In Exercises 119–126, convert the polar equation to rectangular form.**

119. $r = 5$

120. $r = 12$

121. $r = 3\cos\theta$

122. $r = 8\sin\theta$

123. $r^2 = \cos 2\theta$

124. $r^2 = \sin\theta$

125. $\theta = \dfrac{5\pi}{6}$

126. $\theta = \dfrac{4\pi}{3}$

9.6

Sketching the Graph of a Polar Equation **In Exercises 127–132, sketch the graph of the polar equation by hand. Then use a graphing utility to verify your graph.**

127. $r = 5$

128. $r = 3$

129. $\theta = \dfrac{\pi}{2}$

130. $\theta = -\dfrac{5\pi}{6}$

131. $r = 5\cos\theta$

132. $r = 2\sin\theta$

Analyzing a Polar Graph **In Exercises 133–140, identify and then sketch the graph of the polar equation. Identify any symmetry and zeros of *r*. Use a graphing utility to verify your graph.**

133. $r = 5 + 4\cos\theta$

134. $r = 1 + 4\sin\theta$

135. $r = 3 - 5\sin\theta$

136. $r = 2 - 6\cos\theta$

137. $r = -3\cos 2\theta$

138. $r = \cos 5\theta$

139. $r^2 = 5\sin 2\theta$

140. $r^2 = \cos 2\theta$

9.7

Identifying a Conic from Its Equation **In Exercises 141–146, identify the type of conic represented by the equation. Then use a graphing utility to graph the polar equation.**

141. $r = \dfrac{1}{1 + 2\sin\theta}$

142. $r = \dfrac{6}{1 + \sin\theta}$

143. $r = \dfrac{4}{5 - 3\cos\theta}$

144. $r = \dfrac{6}{-1 + 4\cos\theta}$

145. $r = \dfrac{5}{6 + 2\sin\theta}$

146. $r = \dfrac{3}{4 - 4\cos\theta}$

Finding the Polar Equation of a Conic **In Exercises 147–150, find a polar equation of the conic with its focus at the pole.**

147. Parabola, vertex: $(2, \pi)$

148. Parabola, vertex: $(2, \pi/2)$

149. Ellipse, vertices: $(5, 0)$, $(1, \pi)$

150. Hyperbola, vertices: $(1, 0)$, $(7, 0)$

151. **Astronomy** The planet Mars has an elliptical orbit with an eccentricity of $e \approx 0.093$. The length of the major axis of the orbit is approximately 3.05 astronomical units. Find a polar equation for the orbit and its perihelion and aphelion distances.

152. **Astronomy** An asteroid takes a parabolic path with Earth as its focus. It is about 6,000,000 miles from Earth at its closest approach. Write the polar equation of the path of the asteroid with its vertex at $\theta = -\pi/2$. Find the distance between the asteroid and Earth when $\theta = -\pi/3$.

Conclusions

True or False? **In Exercises 153 and 154, determine whether the statement is true or false. Justify your answer.**

153. The graph of $\frac{1}{4}x^2 - y^4 = 1$ represents the equation of a hyperbola.

154. There is only one set of parametric equations that represents the line $y = 3 - 2x$.

Writing **In Exercises 155 and 156, an equation and four variations are given. In your own words, describe how the graph of each of the variations differs from the graph of the original equation.**

155. $y^2 = 8x$

(a) $(y - 2)^2 = 8x$ (b) $y^2 = 8(x + 1)$

(c) $y^2 = -8x$ (d) $y^2 = 4x$

156. $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

(a) $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ (b) $\dfrac{x^2}{4} + \dfrac{y^2}{4} = 1$

(c) $\dfrac{x^2}{4} + \dfrac{y^2}{25} = 1$ (d) $\dfrac{(x - 3)^2}{4} + \dfrac{y^2}{9} = 1$

157. The graph of the parametric equations $x = 2\sec t$ and $y = 3\tan t$ is shown in the figure. Would the graph change for the equations $x = 2\sec(-t)$ and $y = 3\tan(-t)$? If so, how would it change?

Figure for 157

Figure for 158

158. The path of a moving object is modeled by the parametric equations $x = 4\cos t$ and $y = 3\sin t$, where t is time (see figure). How would the path change for each of the following?

(a) $x = 4\cos 2t$, $y = 3\sin 2t$

(b) $x = 5\cos t$, $y = 3\sin t$

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9 Chapter Test

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Take this test as you would take a test in class. After you are finished, check your work against the answers in the back of the book.

In Exercises 1–3, graph the conic and identify any vertices and foci.

1. $y^2 - 8x = 0$ **2.** $y^2 - 4x + 4 = 0$ **3.** $x^2 - 4y^2 - 4x = 0$

4. Find the standard form of the equation of the parabola with focus $(8, -2)$ and directrix $x = 4$, and sketch the parabola.

5. Find the standard form of the equation of the ellipse shown at the right.

Figure for 5

6. Find the standard form of the equation of the hyperbola with vertices $(0, \pm 3)$ and asymptotes $y = \pm\frac{3}{2}x$.

7. Use a graphing utility to graph the conic $x^2 - \dfrac{y^2}{4} = 1$. Describe your viewing window.

8. (a) Determine the number of degrees the axis must be rotated to eliminate the xy-term of the conic $x^2 + 6xy + y^2 - 6 = 0$.

(b) Graph the conic in part (a) and use a graphing utility to confirm your result.

In Exercises 9–11, sketch the curve represented by the parametric equations. Then eliminate the parameter and write the corresponding rectangular equation whose graph represents the curve.

9. $x = t^2 - 6$, $y = \dfrac{1}{2}t - 1$

10. $x = \sqrt{t^2 + 2}$, $y = \dfrac{t}{4}$

11. $x = 2 + 3\cos\theta$, $y = 2\sin\theta$

In Exercises 12–14, find a set of parametric equations to represent the graph of the given rectangular equation using the parameters (a) $t = x$ and (b) $t = 2 - x$.

12. $4x + y = 7$ **13.** $y = \dfrac{3}{x}$ **14.** $y = x^2 + 10$

15. Convert the polar coordinates $\left(-2, \dfrac{5\pi}{6}\right)$ to rectangular form.

16. Convert the rectangular coordinates $(2, -2)$ to polar form and find two additional polar representations of this point. (There are many correct answers.)

17. Convert the rectangular equation $x^2 + y^2 - 3x = 0$ to polar form.

18. Convert the polar equation $r = 2\sin\theta$ to rectangular form.

In Exercises 19–21, identify the conic represented by the polar equation algebraically. Then use a graphing utility to graph the polar equation.

19. $r = 2 + 3\sin\theta$ **20.** $r = \dfrac{1}{1 - \cos\theta}$ **21.** $r = \dfrac{4}{2 + 3\sin\theta}$

22. Find a polar equation of an ellipse with its focus at the pole, an eccentricity of $e = \frac{1}{4}$, and directrix at $y = 4$.

23. Find a polar equation of a hyperbola with its focus at the pole, an eccentricity of $e = \frac{5}{4}$, and directrix at $y = 2$.

24. For the polar equation $r = 8\cos 3\theta$, find the maximum value of $|r|$ and any zeros of r. Verify your answers numerically.

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7–9 Cumulative Test

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Take this test to review the material in Chapters 7–9. After you are finished, check your work against the answers in the back of the book.

In Exercises 1–4, use any method to solve the system of equations.

1. $\begin{cases} -x - 3y = 5 \\ 4x + 2y = 10 \end{cases}$

2. $\begin{cases} 2x - y^2 = 0 \\ x - y = 4 \end{cases}$

3. $\begin{cases} 2x - 3y + z = 13 \\ -4x + y - 2z = -6 \\ x - 3y + 3z = 12 \end{cases}$

4. $\begin{cases} x - 4y + 3z = 5 \\ 5x + 2y - z = 1 \\ -2x - 8y = 30 \end{cases}$

In Exercises 5–8, perform the matrix operations given

$$A = \begin{bmatrix} -3 & 0 & -4 \\ 2 & 4 & 5 \\ -4 & 8 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 5 & 2 \\ 6 & -3 & 3 \\ 0 & 4 & -2 \end{bmatrix}.$$

5. $3A - 2B$ **6.** $5A + 3B$ **7.** AB **8.** BA

9. Find (a) the inverse of A (if it exists) and (b) the determinant of A.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 7 & -10 \\ -5 & -7 & -15 \end{bmatrix}$$

10. Use a determinant to find the area of the triangle with vertices $(0, 0)$, $(6, 2)$, and $(8, 10)$.

11. Write the first five terms of each sequence a_n. (Assume that n begins with 1.)

(a) $a_n = \dfrac{(-1)^{n+1}}{2n + 3}$ (b) $a_n = 3(2)^{n-1}$

In Exercises 12–15, find the sum. Use a graphing utility to verify your result.

12. $\displaystyle\sum_{k=1}^{6} (7k - 2)$ **13.** $\displaystyle\sum_{k=1}^{4} \frac{2}{k^2 + 4}$ **14.** $\displaystyle\sum_{n=0}^{10} 9\left(\frac{3}{4}\right)^n$ **15.** $\displaystyle\sum_{n=0}^{50} 100\left(-\frac{1}{2}\right)^n$

In Exercises 16–18, find the sum of the infinite geometric series.

16. $\displaystyle\sum_{n=0}^{\infty} 3\left(-\frac{3}{5}\right)^n$ **17.** $\displaystyle\sum_{n=1}^{\infty} 5(-0.02)^n$ **18.** $4 - 2 + 1 - \frac{1}{2} + \frac{1}{4} - \cdots$

19. Find each binomial coefficient.

(a) ${}_{20}C_{18}$ (b) $\dbinom{20}{2}$

In Exercises 20–23, use the Binomial Theorem to expand and simplify the expression.

20. $(x + 3)^4$ **21.** $(2x + y^2)^5$

22. $(x - 2y)^6$ **23.** $(3a - 4b)^8$

In Exercises 24–27, find the number of distinguishable permutations of the group of letters.

24. L, I, O, N, S **25.** S, E, A, B, E, E, S

26. B, O, B, B, L, E, H, E, A, D **27.** I, N, T, U, I, T, I, O, N

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In Exercises 28–31, identify the conic and sketch its graph.

28. $\dfrac{(y+3)^2}{36} - \dfrac{(x-5)^2}{121} = 1$

29. $\dfrac{(x-2)^2}{4} + \dfrac{(y+1)^2}{9} = 1$

30. $y^2 - x^2 = 16$

31. $x^2 + y^2 - 2x - 4y + 1 = 0$

In Exercises 32–34, find the standard form of the equation of the conic.

32.

33.

34.

35. Use a graphing utility to graph $x^2 - 4xy + 2y^2 = 6$. Determine the angle θ through which the axes are rotated.

In Exercises 36–38, (a) sketch the curve represented by the parametric equations, (b) use a graphing utility to verify your graph, and (c) eliminate the parameter and write the corresponding rectangular equation whose graph represents the curve. Adjust the domain of the resulting rectangular equation, if necessary.

36. $x = 2t + 1$
$y = t^2$

37. $x = \cos\theta$
$y = 2\sin^2\theta$

38. $x = 4\ln t$
$y = \frac{1}{2}t^2$

In Exercises 39–42, find a set of parametric equations to represent the graph of the given rectangular equation using the parameters (a) $t = x$ and (b) $t = \dfrac{x}{2}$.

39. $y = 3x - 2$

40. $x^2 - y = 16$

41. $y = \dfrac{2}{x}$

42. $y = \dfrac{e^{2x}}{e^{2x} + 1}$

In Exercises 43–46, plot the point given in polar coordinates and find three additional polar representations of the point, using $-2\pi < \theta < 2\pi$.

43. $\left(8, \dfrac{5\pi}{6}\right)$

44. $\left(5, -\dfrac{3\pi}{4}\right)$

45. $\left(-2, \dfrac{5\pi}{4}\right)$

46. $\left(-3, -\dfrac{11\pi}{6}\right)$

47. Convert the rectangular equation $4x + 4y + 1 = 0$ to polar form.

48. Convert the polar equation $r = 4\cos\theta$ to rectangular form.

49. Convert the polar equation $r = \dfrac{2}{4 - 5\cos\theta}$ to rectangular form.

In Exercises 50–52, identify the type of polar graph represented by the polar equation. Then use a graphing utility to graph the polar equation.

50. $r = -\dfrac{\pi}{6}$

51. $r = 3 - 2\sin\theta$

52. $r = 2 + 5\cos\theta$

53. The salary for the first year of a job is \$32,500. During the next 14 years, the salary increases by 5% each year. Determine the total compensation over the 15-year period.

54. On a game show, the digits 3, 4, and 5 must be arranged in the proper order to form the price of an appliance. If they are arranged correctly, then the contestant wins the appliance. What is the probability of winning if the contestant knows that the price is at least \$400?

55. A parabolic archway is 16 meters high at the vertex. At a height of 14 meters, the width of the archway is 12 meters, as shown in the figure at the right. How wide is the archway at ground level?

Figure for 55

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Proofs in Mathematics

Standard Equation of a Parabola (p. 639)

The standard form of the equation of a parabola with vertex at (h, k) is as follows.

$(x - h)^2 = 4p(y - k), \quad p \neq 0$ — Vertical axis, directrix: $y = k - p$

$(y - k)^2 = 4p(x - h), \quad p \neq 0$ — Horizontal axis, directrix: $x = h - p$

The focus lies on the axis p units (*directed distance*) from the vertex. If the vertex is at the origin $(0, 0)$, then the equation takes one of the following forms.

$x^2 = 4py$ — Vertical axis

$y^2 = 4px$ — Horizontal axis

Parabolic Paths

There are many natural occurrences of parabolas in real life. For instance, the famous astronomer Galileo discovered in the 17th century that an object that is projected upward and obliquely to the pull of gravity travels in a parabolic path. Examples of this are the center of gravity of a jumping dolphin and the path of water molecules in a drinking fountain.

Proof

For the case in which the directrix is parallel to the x-axis and the focus lies above the vertex, as shown in the top figure, if (x, y) is any point on the parabola, then, by definition, it is equidistant from the focus

$$(h, k + p)$$

and the directrix

$$y = k - p.$$

So, you have

$$\sqrt{(x - h)^2 + [y - (k + p)]^2} = y - (k - p)$$

$$(x - h)^2 + [y - (k + p)]^2 = [y - (k - p)]^2$$

$$(x - h)^2 + y^2 - 2y(k + p) + (k + p)^2 = y^2 - 2y(k - p) + (k - p)^2$$

$$(x - h)^2 + y^2 - 2ky - 2py + k^2 + 2pk + p^2 = y^2 - 2ky + 2py + k^2 - 2pk + p^2$$

$$(x - h)^2 - 2py + 2pk = 2py - 2pk$$

$$(x - h)^2 = 4p(y - k).$$

Parabola with vertical axis

For the case in which the directrix is parallel to the y-axis and the focus lies to the right of the vertex, as shown in the bottom figure, if (x, y) is any point on the parabola, then, by definition, it is equidistant from the focus

$$(h + p, k)$$

and the directrix

$$x = h - p.$$

So, you have

$$\sqrt{[x - (h + p)]^2 + (y - k)^2} = x - (h - p)$$

$$[x - (h + p)]^2 + (y - k)^2 = [x - (h - p)]^2$$

$$x^2 - 2x(h + p) + (h + p)^2 + (y - k)^2 = x^2 - 2x(h - p) + (h - p)^2$$

$$x^2 - 2hx - 2px + h^2 + 2ph + p^2 + (y - k)^2 = x^2 - 2hx + 2px + h^2 - 2ph + p^2$$

$$-2px + 2ph + (y - k)^2 = 2px - 2ph$$

$$(y - k)^2 = 4p(x - h).$$

Parabola with horizontal axis

Note that when the vertex of a parabola is at the origin, the two equations above simplify to $x^2 = 4py$ and $y^2 = 4px$, respectively.

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Rotation of Axes to Eliminate an *xy*-Term (p. 663)

The general second-degree equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ can be rewritten as

$$A'(x')^2 + C'(y')^2 + D'x' + E'y' + F' = 0$$

by rotating the coordinate axes through an angle θ, where

$$\cot 2\theta = \frac{A - C}{B}.$$

The coefficients of the new equation are obtained by making the substitutions $x = x' \cos\theta - y' \sin\theta$ and $y = x' \sin\theta + y' \cos\theta$.

Proof

You need to discover how the coordinates in the *xy*-system are related to the coordinates in the $x'y'$-system. To do this, choose a point $P(x, y)$ in the original system and attempt to find its coordinates (x', y') in the rotated system. In either system, the distance r between the point P and the origin is the same. So, the equations for x, y, x', and y' are those given in the figures. Using the formulas for the sine and cosine of the difference of two angles, you have the following.

Rotated: $x' = r\cos(\alpha - \theta)$
$y' = r\sin(\alpha - \theta)$

$$\begin{aligned} x' &= r\cos(\alpha - \theta) \\ &= r(\cos\alpha\cos\theta + \sin\alpha\sin\theta) \\ &= r\cos\alpha\cos\theta + r\sin\alpha\sin\theta \\ &= x\cos\theta + y\sin\theta \end{aligned}$$

$$\begin{aligned} y' &= r\sin(\alpha - \theta) \\ &= r(\sin\alpha\cos\theta - \cos\alpha\sin\theta) \\ &= r\sin\alpha\cos\theta - r\cos\alpha\sin\theta \\ &= y\cos\theta - x\sin\theta \end{aligned}$$

Solving this system for x and y yields

$$x = x'\cos\theta - y'\sin\theta \qquad \text{and} \qquad y = x'\sin\theta + y'\cos\theta.$$

Finally, by substituting these values for x and y into the original equation and collecting terms, you obtain

Original: $x = r\cos\alpha$
$y = r\sin\alpha$

$$\begin{aligned} A' &= A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta \\ C' &= A\sin^2\theta - B\cos\theta\sin\theta + C\cos^2\theta \\ D' &= D\cos\theta + E\sin\theta \\ E' &= -D\sin\theta + E\cos\theta \\ F' &= F. \end{aligned}$$

To eliminate the $x'y'$-term, you must select θ such that $B' = 0$.

$$\begin{aligned} B' &= 2(C - A)\sin\theta\cos\theta + B(\cos^2\theta - \sin^2\theta) \\ &= (C - A)\sin 2\theta + B\cos 2\theta \\ &= B(\sin 2\theta)\left(\frac{C - A}{B} + \cot 2\theta\right) = 0, \quad \sin 2\theta \neq 0 \end{aligned}$$

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When $B = 0$, no rotation is necessary because the xy-term is not present in the original equation. When $B \neq 0$, the only way to make $B' = 0$ is to let

$$\cot 2\theta = \frac{A - C}{B}, \quad B \neq 0.$$

So, you have established the desired results.

Polar Equations of Conics (p. 691)

The graph of a polar equation of the form

1. $r = \dfrac{ep}{1 \pm e\cos\theta}$ or **2.** $r = \dfrac{ep}{1 \pm e\sin\theta}$

is a conic, where $e > 0$ is the eccentricity and $|p|$ is the distance between the focus (pole) and the directrix.

Proof

A proof for

$$r = \frac{ep}{1 + e\cos\theta}$$

with $p > 0$ is shown here. The proofs of the other cases are similar. In the figure, consider a vertical directrix, p units to the right of the focus $F(0, 0)$. If $P(r, \theta)$ is a point on the graph of

$$r = \frac{ep}{1 + e\cos\theta}$$

then the distance between P and the directrix is

$$\begin{aligned} PQ &= |p - x| \\ &= |p - r\cos\theta| \\ &= \left|p - \left(\frac{ep}{1 + e\cos\theta}\right)\cos\theta\right| \\ &= \left|p\left(1 - \frac{e\cos\theta}{1 + e\cos\theta}\right)\right| \\ &= \left|\frac{p}{1 + e\cos\theta}\right| \\ &= \left|\frac{r}{e}\right|. \end{aligned}$$

Moreover, because the distance between P and the pole is simply $PF = |r|$, the ratio of PF to PQ is

$$\frac{PF}{PQ} = \frac{|r|}{\left|\frac{r}{e}\right|} = |e| = e$$

and, by definition, the graph of the equation must be a conic.

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Progressive Summary (Chapters 3–9)

This chart outlines the topics that have been covered so far in this text. Progressive Summary charts appear after Chapters 2, 3, 6, and 9. In each Progressive Summary, new topics encountered for the first time appear in red.

TRANSCENDENTAL FUNCTIONS

Exponential, Logarithmic, Trigonometric, Inverse Trigonometric

Rewriting

Exponential form ↔ Logarithmic form
Condense/expand logarithmic expressions
Simplify trigonometric expressions
Prove trigonometric identities
Use conversion formulas
Operations with vectors
Powers and roots of complex numbers

Solving

Equation	*Strategy*
Exponential	Take logarithm of each side
Logarithmic	Exponentiate each side
Trigonometric	Isolate function Factor, use inverse function
Multiple angle or high powers	Use trigonometric identities

Analyzing

Graphically	*Algebraically*
Intercepts	Domain, Range
Asymptotes	Transformations
Minimum values	Composition
Maximum values	Inverse Properties
	Amplitude, period
	Reference angles

Numerically

Table of values

SYSTEMS AND SERIES

Systems, Sequences, Series

Rewriting

Row operations for systems of equations
Partial fraction decomposition
Operations with matrices
Matrix form of a system of equations
*n*th term of a sequence
Summation form of a series

Solving

Equation	*Strategy*
System of linear equations	Substitution Elimination Gaussian Gauss-Jordan Inverse matrices Cramer's Rule

Analyzing

Systems:
Intersecting, parallel, and coincident lines, determinants
Sequences:
Graphing utility in *dot* mode, *n*th term, partial sums, summation formulas

OTHER TOPICS

Conics, Parametric and Polar Equations

Rewriting

Standard forms of conics
Eliminate parameters
Rectangular form ↔ Parametric form
Rectangular form ↔ Polar form

Solving

Equation	*Strategy*
Conics	Convert to standard form Convert to polar form

Analyzing

Conics:
Table of values, vertices, foci, axes, symmetry, asymptotes, translations, eccentricity
Parametric forms:
Point plotting, eliminate parameters
Polar forms:
Point plotting, special equations, symmetry, zeros, eccentricity, maximum *r*-values, directrix

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10 Analytic Geometry in Three Dimensions

Section 10.3, Example 4
Volume of a Parallelepiped

10.1 The Three-Dimensional Coordinate System
10.2 Vectors in Space
10.3 The Cross Product of Two Vectors
10.4 Lines and Planes in Space

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10.1 The Three-Dimensional Coordinate System

The Three-Dimensional Coordinate System

Recall that the Cartesian plane is determined by two perpendicular number lines called the x-axis and the y-axis. These axes, together with their point of intersection (the origin), allow you to develop a two-dimensional coordinate system for identifying points in a plane. To identify a point in space, you must introduce a third dimension to the model. The geometry of this three-dimensional model is called **solid analytic geometry.**

You can construct a **three-dimensional coordinate system** by passing a z-axis perpendicular to both the x- and y-axes at the origin. Figure 10.1 shows the positive portion of each coordinate axis. Taken as pairs, the axes determine three **coordinate planes:** the ***xy*-plane,** the ***xz*-plane,** and the ***yz*-plane.** These three coordinate planes separate the three-dimensional coordinate system into eight **octants.** The first octant is the one in which all three coordinates are positive. In this three-dimensional system, a point P in space is determined by an ordered triple (x, y, z), where x, y, and z are as follows.

x = directed distance from yz-plane to P

y = directed distance from xz-plane to P

z = directed distance from xy-plane to P

What you should learn

- Plot points in the three-dimensional coordinate system.
- Find distances between points in space and find midpoints of line segments joining points in space.
- Write equations of spheres in standard form and find traces of surfaces in space.

Why you should learn it

The three-dimensional coordinate system can be used to graph equations that model surfaces in space, such as the spherical shape of Earth, as shown in Exercise 88 on page 718.

Figure 10.1

Figure 10.2

A three-dimensional coordinate system can have either a **left-handed** or a **right-handed** orientation. In this text, you will work exclusively with right-handed systems, as illustrated in Figure 10.2. In a right-handed system, Octants II, III, and IV are found by rotating counterclockwise around the positive z-axis. Octant V is below Octant I. Octants VI, VII, and VIII are then found by rotating counterclockwise around the negative z-axis.

Example 1 Plotting Points in Space

To plot the point $(2, -3, 3)$, notice that

$$x = 2, \quad y = -3, \quad \text{and} \quad z = 3.$$

To help visualize the point, locate the point $(2, -3)$ in the xy-plane (denoted by a cross in Figure 10.3). The point $(2, -3, 3)$ lies three units above the cross. You can plot other points, such as

$$(-2, 6, 2), \quad (1, 4, 0), \quad \text{and} \quad (2, 2, -3)$$

in a similar manner, as shown in Figure 10.3.

CHECKPOINT Now try Exercise 15.

Figure 10.3

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The Distance and Midpoint Formulas

Many of the formulas established for the two-dimensional coordinate system can be extended to three dimensions. For example, to find the distance between two points in space, you can use the Pythagorean Theorem twice, as shown in Figure 10.4.

Figure 10.4

Distance Formula in Space

The distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) given by the **Distance Formula in Space** is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Example 2 Finding the Distance Between Two Points in Space

Find the distance between

$$(0, 1, 3) \quad \text{and} \quad (1, 4, -2).$$

Solution

$$\begin{aligned} d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} && \text{Distance Formula in Space} \\ &= \sqrt{(1 - 0)^2 + (4 - 1)^2 + (-2 - 3)^2} && \text{Substitute.} \\ &= \sqrt{1 + 9 + 25} && \text{Simplify.} \\ &= \sqrt{35} && \text{Simplify.} \end{aligned}$$

CHECKPOINT Now try Exercise 29.

Notice the similarity between the Distance Formulas in the plane and in space. The Midpoint Formulas in the plane and in space are also similar.

Midpoint Formula in Space

The midpoint of the line segment joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) given by the **Midpoint Formula in Space** is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

Example 3 Using the Midpoint Formula in Space

Find the midpoint of the line segment joining

$$(5, -2, 3) \quad \text{and} \quad (0, 4, 4).$$

Solution

Using the Midpoint Formula in Space, the midpoint is

$$\left(\frac{5 + 0}{2}, \frac{-2 + 4}{2}, \frac{3 + 4}{2}\right) = \left(\frac{5}{2}, 1, \frac{7}{2}\right)$$

as shown in Figure 10.5.

Figure 10.5

CHECKPOINT Now try Exercise 45.

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The Equation of a Sphere

A **sphere** with center (h, k, j) and radius r is defined as the set of all points (x, y, z) such that the distance between (x, y, z) and (h, k, j) is r, as shown in Figure 10.6. Using the Distance Formula, this condition can be written as

$$\sqrt{(x-h)^2 + (y-k)^2 + (z-j)^2} = r.$$

By squaring each side of this equation, you obtain the standard equation of a sphere.

Figure 10.6

Standard Equation of a Sphere

The **standard equation of a sphere** with center (h, k, j) and radius r is given by

$$(x-h)^2 + (y-k)^2 + (z-j)^2 = r^2.$$

Notice the similarity of this formula to the equation of a circle in the plane.

$$(x-h)^2 + (y-k)^2 + (z-j)^2 = r^2 \qquad \text{Equation of sphere in space}$$

$$(x-h)^2 + (y-k)^2 = r^2 \qquad \text{Equation of circle in the plane}$$

As is true with the equation of a circle, the equation of a sphere is simplified when the center lies at the origin. In this case, the equation is

$$x^2 + y^2 + z^2 = r^2. \qquad \text{Sphere with center at origin}$$

Explore the Concept

Find the equation of the sphere that has the points $(3, -2, 6)$ and $(-1, 4, 2)$ as endpoints of a diameter. Explain how this problem gives you a chance to use all three formulas discussed so far in this section: the Distance Formula in Space, the Midpoint Formula in Space, and the standard equation of a sphere.

Example 4 Finding the Equation of a Sphere

Find the standard equation of the sphere with center $(2, 4, 3)$ and radius 3. Does this sphere intersect the xy-plane?

Solution

The equation of the sphere with center $(h, k, j) = (2, 4, 3)$ and radius $r = 3$ is

$$(x-h)^2 + (y-k)^2 + (z-j)^2 = r^2 \qquad \text{Standard equation}$$

$$(x-2)^2 + (y-4)^2 + (z-3)^2 = 3^2. \qquad \text{Substitute.}$$

From the graph shown in Figure 10.7, you can see that the center of the sphere lies three units above the xy-plane. Because the sphere has a radius of 3, you can conclude that it does intersect the xy-plane—at the point $(2, 4, 0)$.

Figure 10.7

CHECKPOINT Now try Exercise 55.

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Example 5 Finding the Center and Radius of a Sphere

Find the center and radius of the sphere given by

$$x^2 + y^2 + z^2 - 2x + 4y - 6z + 8 = 0.$$

Solution

To obtain the standard equation of this sphere, complete the square as follows.

$$x^2 + y^2 + z^2 - 2x + 4y - 6z + 8 = 0$$

$$\left(x^2 - 2x + \square\right) + \left(y^2 + 4y + \square\right) + \left(z^2 - 6z + \square\right) = -8$$

$$(x^2 - 2x + 1) + (y^2 + 4y + 4) + (z^2 - 6z + 9) = -8 + 1 + 4 + 9$$

$$(x - 1)^2 + (y + 2)^2 + (z - 3)^2 = \left(\sqrt{6}\right)^2$$

So, the center of the sphere is $(1, -2, 3)$, and its radius is $\sqrt{6}$. See Figure 10.8.

CHECKPOINT Now try Exercise 65.

Figure 10.8

Note in Example 5 that the points satisfying the equation of the sphere are "surface points," not "interior points." In general, the collection of points satisfying an equation involving x, y, and z is called a **surface in space.**

Finding the intersection of a surface with one of the three coordinate planes (or with a plane parallel to one of the three coordinate planes) helps one visualize the surface. Such an intersection is called a **trace** of the surface. For example, the xy-trace of a surface consists of all points that are common to both the surface *and* the xy-plane.

Example 6 Finding a Trace of a Surface

Sketch the xy-trace of the sphere given by $(x - 3)^2 + (y - 2)^2 + (z + 4)^2 = 5^2$.

Solution

To find the xy-trace of this surface, use the fact that every point in the xy-plane has a z-coordinate of zero. By substituting $z = 0$ into the original equation, the resulting equation will represent the intersection of the surface with the xy-plane.

$(x - 3)^2 + (y - 2)^2 + (z + 4)^2 = 5^2$	Write original equation.
$(x - 3)^2 + (y - 2)^2 + (0 + 4)^2 = 5^2$	Substitute 0 for z.
$(x - 3)^2 + (y - 2)^2 + 16 = 25$	Simplify.
$(x - 3)^2 + (y - 2)^2 = 9$	Subtract 16 from each side.
$(x - 3)^2 + (y - 2)^2 = 3^2$	Equation of circle

From this form, you can see that the xy-trace is a circle of radius 3, as shown in Figure 10.9.

Figure 10.9

CHECKPOINT Now try Exercise 75.

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10.1 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1–6, fill in the blank(s).

1. A _____ coordinate system can be formed by passing a z-axis perpendicular to both the x-axis and the y-axis at the origin.
2. The coordinate planes of a three-dimensional coordinate system separate the coordinate system into eight _____ .
3. The distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) can be found using the _____ in Space.
4. A _____ is the set of all points (x, y, z) such that the distance between (x, y, z) and a fixed point (h, k, j) is r.
5. A _____ in _____ is the collection of points satisfying an equation involving x, y, and z.
6. The intersection of a surface with one of the three coordinate planes is called a _____ of the surface.
7. What does the equation $(x - h)^2 + (y - k)^2 + (z - j)^2 = r^2$ represent? What do h, k, j, and r represent?
8. How do you find the yz-trace of the sphere given by $(x - 2)^2 + (y + 1)^2 + (z - 2)^2 = 7^2$?

Procedures and Problem Solving

Approximating the Coordinates of a Point **In Exercises 9–12, approximate the coordinates of the points.**

9.

10.

11.

12.

Plotting Points in Space **In Exercises 13–18, plot both points in the same three-dimensional coordinate system.**

13. (a) $(-2, 1, 3)$
    (b) $(-1, 2, 1)$
14. (a) $(3, 0, 0)$
    (b) $(-3, -2, -1)$
15. (a) $(3, -1, 0)$
    (b) $(-4, 2, 2)$
16. (a) $(0, 4, -3)$
    (b) $(4, 0, 4)$
17. (a) $(3, -2, 5)$
    (b) $\left(\frac{3}{2}, 4, -2\right)$
18. (a) $(5, -2, 2)$
    (b) $(5, 2, -2)$

Finding the Coordinates of a Point **In Exercises 19–22, find the coordinates of the point.**

19. The point is located three units behind the yz-plane, three units to the right of the xz-plane, and five units above the xy-plane.
20. The point is located six units in front of the yz-plane, two units to the left of the xz-plane, and one unit below the xy-plane.
21. The point is located on the x-axis and 11 units in front of the yz-plane.
22. The point is located in the yz-plane, two units to the right of the xz-plane, and seven units above the xy-plane.

Determining the Octant **In Exercises 23–28, determine the octant(s) in which (x, y, z) is located so that the condition(s) is (are) satisfied.**

23. $x > 0, y > 0, z < 0$
24. $x < 0, y > 0, z < 0$
25. $z > 0$
26. $y < 0$
27. $xy < 0$
28. $yz > 0$

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Finding the Distance Between Two Points in Space **In Exercises 29–36, find the distance between the points.**

29.

30.

31. (0, 0, 0), (5, 2, 6)

32. (1, 0, 0), (7, 0, 4)

33. (−1, 4, −2), (6, 0, −9)

34. (1, 1, −7), (−2, −3, −7)

35. (0, −2, 0), (1, 0, −10)

36. (2, −4, 0), (0, 6, −3)

Using the Pythagorean Theorem **In Exercises 37–40, find the lengths of the sides of the right triangle. Show that these lengths satisfy the Pythagorean Theorem.**

37.

38.

39. (0, 0, 0), (2, 2, 1), (2, −4, 4)

40. (1, 0, 1), (1, 3, 1), (1, 0, 3)

Finding the Side Lengths of a Triangle **In Exercises 41–44, find the lengths of the sides of the triangle with the indicated vertices, and determine whether the triangle is a right triangle, an isosceles triangle, or neither.**

41. (1, −3, −2), (5, −1, 2), (−1, 1, 2)

42. (5, 3, 4), (7, 1, 3), (3, 5, 3)

43. (4, −1, −2), (8, 1, 2), (2, 3, 2)

44. (1, −2, −1), (3, 0, 0), (3, −6, 3)

Using the Midpoint Formula in Space **In Exercises 45–50, find the midpoint of the line segment joining the points.**

45. (3, −6, 10), (−3, 4, 4)

46. (1, 5, −1), (2, 2, 2)

47. (6, −2, 5), (−4, 2, 7)

48. (−3, 5, 5), (−6, 4, 8)

49. (−2, 8, 10), (7, −4, 2)

50. (9, −5, 1), (9, −2, −4)

Finding the Equation of a Sphere **In Exercises 51–60, find the standard form of the equation of the sphere with the given characteristics.**

51.

52.

53.

54.

55. Center: (0, 4, 3); radius: 4

56. Center: (2, −1, 8); radius: 7

57. Center: (−3, 7, 5); diameter: 10

58. Center: (0, 5, −9); diameter: 8

59. Endpoints of a diameter: (3, 0, 0), (0, 0, 6)

60. Endpoints of a diameter: (2, −2, 2), (−1, 4, 6)

Finding the Center and Radius of a Sphere **In Exercises 61–74, find the center and radius of the sphere.**

61. $x^2 + y^2 + z^2 - 6x = 0$

62. $x^2 + y^2 + z^2 - 9y = 0$

63. $x^2 + y^2 + z^2 - 4x + 2y = 0$

64. $x^2 + y^2 + z^2 - x - y - z = 0$

65. $x^2 + y^2 + z^2 - 4x + 2y - 6z + 10 = 0$

66. $x^2 + y^2 + z^2 - 6x + 4y + 9 = 0$

67. $x^2 + y^2 + z^2 + 4x - 8z + 19 = 0$

68. $x^2 + y^2 + z^2 - 8y - 6z + 13 = 0$

69. $9x^2 + 9y^2 + 9z^2 - 18x - 6y - 72z + 73 = 0$

70. $2x^2 + 2y^2 + 2z^2 - 2x - 6y - 4z + 5 = 0$

71. $4x^2 + 4y^2 + 4z^2 - 8x + 16y - 1 = 0$

72. $9x^2 + 9y^2 + 9z^2 - 18x + 36y + 54z - 126 = 0$

73. $9x^2 + 9y^2 + 9z^2 - 6x + 18y + 1 = 0$

74. $4x^2 + 4y^2 + 4z^2 - 4x - 32y + 8z + 33 = 0$

Finding a Trace of a Surface **In Exercises 75–80, sketch the graph of the equation and sketch the specified trace.**

75. $(x - 1)^2 + y^2 + z^2 = 36$; xz-trace

76. $x^2 + (y + 3)^2 + z^2 = 25$; yz-trace

77. $(x + 2)^2 + (y - 3)^2 + z^2 = 9$; yz-trace

78. $x^2 + (y - 1)^2 + (z + 1)^2 = 4$; xy-trace

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79. $x^2 + y^2 + z^2 - 2x - 4z + 1 = 0$; yz-trace

80. $x^2 + y^2 + z^2 - 4y - 6z - 12 = 0$; xz-trace

Graphing a Sphere **In Exercises 81–84, use a three-dimensional graphing utility to graph the sphere.**

81. $x^2 + y^2 + z^2 - 6x - 8y - 10z + 46 = 0$

82. $x^2 + y^2 + z^2 + 6y - 8z + 21 = 0$

83. $4x^2 + 4y^2 + 4z^2 - 8x - 16y + 8z - 25 = 0$

84. $9x^2 + 9y^2 + 9z^2 + 18x - 18y + 36z + 35 = 0$

85. **Crystallography** Crystals are classified according to their symmetry. Crystals shaped like cubes are classified as isometric. The vertices of an isometric crystal mapped onto a three-dimensional coordinate system are shown in the figure. Determine (x, y, z).

Figure for 85

Figure for 86

86. **Crystallography** Crystals shaped like rectangular prisms are classified as tetragonal. The vertices of a tetragonal crystal mapped onto a three-dimensional coordinate system are shown in the figure. Determine (x, y, z).

87. **Architecture** A spherical building has a diameter of 205 feet. The center of the building is placed at the origin of a three-dimensional coordinate system. What is the equation of the sphere?

88. ***Why you should learn it*** *(p. 712)* Assume that Earth is a sphere with a radius of 3963 miles. The center of Earth is placed at the origin of a three-dimensional coordinate system (see figure).

(a) What is the equation of the sphere?

(b) Lines of longitude that run north-south could be represented by what trace(s)? What shape would each of these traces form?

(c) Lines of latitude that run east-west could be represented by what trace(s)? What shape would each of these traces form?

(d) The prime meridian would represent what trace of the equation of the sphere from part (a)?

(e) The equator would represent what trace of the equation of the sphere from part (a)?

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Conclusions

True or False? **In Exercises 89 and 90, determine whether the statement is true or false. Justify your answer.**

89. In the ordered triple (x, y, z) that represents point P in space, x is the directed distance from the xy-plane to P.

90. The surface consisting of all points (x, y, z) in space that are the same distance r from the point (h, k, j) has a circle as its xy-trace.

91. **Think About It** What is the z-coordinate of any point in the xy-plane? What is the y-coordinate of any point in the xz-plane? What is the x-coordinate of any point in the yz-plane?

92. **CAPSTONE** In two-dimensional coordinate geometry, the graph of the equation $ax + by + c = 0$ is a line.

(a) In three-dimensional coordinate geometry, what is the graph of the equation $ax + by + cz = 0$?

(b) A sphere intersects the yz-plane. Describe the trace.

(c) A plane intersects the xy-plane. Describe the trace.

(d) How are the midpoint formulas in the plane and in space different?

93. A line segment has (x_1, y_1, z_1) as one endpoint and (x_m, y_m, z_m) as its midpoint. Find the other endpoint (x_2, y_2, z_2) of the line segment in terms of x_1, y_1, z_1, x_m, y_m, and z_m.

94. Use the result of Exercise 93 to find the coordinates of one endpoint of a line segment when the coordinates of the other endpoint and the midpoint are (3, 0, 2) and (5, 8, 7), respectively.

Cumulative Mixed Review

Completing the Square **In Exercises 95–100, solve the quadratic equation by completing the square.**

95. $v^2 + 3v - 2 = 0$

96. $z^2 - 7z - 19 = 0$

97. $x^2 - 5x + 5 = 0$

98. $x^2 + 3x - 1 = 0$

99. $4y^2 + 4y - 9 = 0$

100. $2x^2 + 5x - 8 = 0$

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10.2 Vectors in Space

Vectors in Space

Physical forces and velocities are not confined to the plane, so it is natural to extend the concept of vectors from two-dimensional space to three-dimensional space. In space, vectors are denoted by ordered triples

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle. \qquad \text{Component form}$$

The **zero vector** is denoted by

$$\mathbf{0} = \langle 0, 0, 0 \rangle. \qquad \text{Zero vector}$$

Using the unit vectors $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$ in the direction of the positive z-axis, the **standard unit vector notation** for $\mathbf{v}$ is

$$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k} \qquad \text{Unit vector form}$$

as shown in Figure 10.10. If $\mathbf{v}$ is represented by the directed line segment from $P(p_1, p_2, p_3)$ to $Q(q_1, q_2, q_3)$, as shown in Figure 10.11, then the **component form** of $\mathbf{v}$ is produced by subtracting the coordinates of the initial point from the coordinates of the terminal point

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle = \langle q_1 - p_1, q_2 - p_2, q_3 - p_3 \rangle.$$

Figure 10.10

Figure 10.11

What you should learn

- Find the component forms of, the unit vectors in the same direction of, the magnitudes of, the dot products of, and the angles between vectors in space.
- Determine whether vectors in space are parallel or orthogonal.
- Use vectors in space to solve real-life problems.

Why you should learn it

Vectors in space can be used to represent many physical forces, such as tension in the chains used to support a bird bath, as shown in Exercise 71 on page 725.

Vectors in Space

1. Two vectors are **equal** if and only if their corresponding components are equal.
2. The **magnitude** (or **length**) of $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ is
$$\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2}.$$
3. A **unit vector u** in the direction of **v** is
$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}, \quad \mathbf{v} \neq \mathbf{0}.$$
4. The **sum** of $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is
$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle. \qquad \text{Vector addition}$$
5. The **scalar multiple** of the real number c and $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ is
$$c\mathbf{u} = \langle cu_1, cu_2, cu_3 \rangle. \qquad \text{Scalar multiplication}$$
6. The **dot product** of $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is
$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3. \qquad \text{Dot product}$$

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Example 1 Finding the Component Form of a Vector

Find the component form and magnitude of the vector **v** having initial point (3, 4, 2) and terminal point (3, 6, 4). Then find a unit vector in the direction of **v**.

Solution

The component form of **v** is

$$\mathbf{v} = \langle 3 - 3, 6 - 4, 4 - 2 \rangle = \langle 0, 2, 2 \rangle$$

which implies that its magnitude is

$$\|\mathbf{v}\| = \sqrt{0^2 + 2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}.$$

The unit vector in the direction of **v** is

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{2\sqrt{2}}\langle 0, 2, 2 \rangle = \left\langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \left\langle 0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle.$$

CHECKPOINT Now try Exercise 11.

Example 2 Finding the Dot Product of Two Vectors

The dot product of $\langle 4, 0, 1 \rangle$ and $\langle -1, 3, 2 \rangle$ is

$$\langle 4, 0, 1 \rangle \cdot \langle -1, 3, 2 \rangle = 4(-1) + 0(3) + 1(2) = -4 + 0 + 2 = -2.$$

Note that the dot product of two vectors is a real number, not a vector.

CHECKPOINT Now try Exercise 41.

As was discussed in Section 6.4, the **angle between two nonzero vectors** is the angle θ, $0 \le \theta \le \pi$, between their respective standard position vectors, as shown in Figure 10.12. This angle can be found using the dot product. (Note that the angle between the zero vector and another vector is not defined.)

Figure 10.12

Angle Between Two Vectors

If θ is the angle between two nonzero vectors **u** and **v**, then $\cos\theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}$.

If the dot product of two nonzero vectors is zero, then the angle between the vectors is 90°. Such vectors are called **orthogonal.** For instance, the standard unit vectors **i**, **j**, and **k** are orthogonal to each other.

Example 3 Finding the Angle Between Two Vectors

Find the angle between $\mathbf{u} = \langle 1, 0, 2 \rangle$ and $\mathbf{v} = \langle 3, 1, 0 \rangle$.

Solution

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{\langle 1, 0, 2 \rangle \cdot \langle 3, 1, 0 \rangle}{\|\langle 1, 0, 2 \rangle\|\,\|\langle 3, 1, 0 \rangle\|} = \frac{3}{\sqrt{50}}$$

This implies that the angle between the two vectors is

$$\theta = \arccos \frac{3}{\sqrt{50}} \approx 64.9°.$$

See Figure 10.13.

Figure 10.13

CHECKPOINT Now try Exercise 45.

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Parallel Vectors

Recall from the definition of scalar multiplication that positive scalar multiples of a nonzero vector **v** have the same direction as **v**, whereas negative multiples have the direction opposite that of **v**. In general, two nonzero vectors **u** and **v** are **parallel** when there is some scalar c such that $\mathbf{u} = c\mathbf{v}$. For instance, in Figure 10.14, the vectors **u**, **v**, and **w** are parallel because $\mathbf{u} = 2\mathbf{v}$ and $\mathbf{w} = -\mathbf{v}$.

Figure 10.14

Technology Tip

Some graphing utilities have the capability of performing vector operations, such as the dot product. Consult the user's guide for your graphing utility for specific instructions.

Example 4 Parallel Vectors

Vector **w** has initial point $(1, -2, 0)$ and terminal point $(3, 2, 1)$. Which of the following vectors is parallel to **w**?

a. $\mathbf{u} = \langle 4, 8, 2\rangle$ **b.** $\mathbf{v} = \left\langle -1, -2, -\frac{1}{2}\right\rangle$ **c.** $\mathbf{z} = \langle 4, 8, 4\rangle$

Solution

Begin by writing **w** in component form.

$$\mathbf{w} = \langle 3 - 1, 2 - (-2), 1 - 0\rangle = \langle 2, 4, 1\rangle$$

a. Because

$$\begin{aligned}\mathbf{u} &= \langle 4, 8, 2\rangle \\ &= 2\langle 2, 4, 1\rangle \\ &= 2\mathbf{w}\end{aligned}$$

you can conclude that **u** *is* parallel to **w**.

b. Because

$$\begin{aligned}\mathbf{v} &= \left\langle -1, -2, -\frac{1}{2}\right\rangle \\ &= -\frac{1}{2}\langle 2, 4, 1\rangle \\ &= -\frac{1}{2}\mathbf{w}\end{aligned}$$

you can conclude that **v** *is* parallel to **w**.

c. In this case, you need to find a scalar c such that $\langle 4, 8, 4\rangle = c\langle 2, 4, 1\rangle$. However, equating corresponding components produces $c = 2$ for the first two components and $c = 4$ for the third. So, the equation has no solution, and the vectors **z** and **w** are *not* parallel.

CHECKPOINT Now try Exercise 49.

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You can use vectors to determine whether three points are collinear (lie on the same line). The points P, Q, and R are **collinear** if and only if the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are parallel.

Example 5 Using Vectors to Determine Collinear Points

Determine whether the points $P(2, -1, 4)$, $Q(5, 4, 6)$, and $R(-4, -11, 0)$ are collinear.

Solution

The component forms of $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are

$$\overrightarrow{PQ} = \langle 5 - 2, 4 - (-1), 6 - 4\rangle = \langle 3, 5, 2\rangle$$

and

$$\overrightarrow{PR} = \langle -4 - 2, -11 - (-1), 0 - 4\rangle = \langle -6, -10, -4\rangle.$$

Because $\overrightarrow{PR} = -2\overrightarrow{PQ}$, you can conclude that they are parallel. Therefore, the points P, Q, and R lie on the same line, as shown in Figure 10.15.

Figure 10.15

CHECKPOINT Now try Exercise 57.

Example 6 Finding the Terminal Point of a Vector

The initial point of the vector $\mathbf{v} = \langle 4, 2, -1\rangle$ is $P(3, -1, 6)$. What is the terminal point of this vector?

Solution

Using the component form of the vector whose initial point is $P(3, -1, 6)$ and whose terminal point is $Q(q_1, q_2, q_3)$, you can write

$$\begin{aligned}\overrightarrow{PQ} &= \langle q_1 - p_1, q_2 - p_2, q_3 - p_3\rangle \\ &= \langle q_1 - 3, q_2 + 1, q_3 - 6\rangle \\ &= \langle 4, 2, -1\rangle.\end{aligned}$$

This implies that

$$q_1 - 3 = 4, \quad q_2 + 1 = 2, \quad \text{and} \quad q_3 - 6 = -1.$$

The solutions of these three equations are

$$q_1 = 7, \quad q_2 = 1, \quad \text{and} \quad q_3 = 5.$$

So, the terminal point is $Q(7, 1, 5)$.

CHECKPOINT Now try Exercise 63.

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Application

In Section 6.3, you saw how to use vectors to solve an equilibrium problem in a plane. The next example shows how to use vectors to solve an equilibrium problem in space.

Example 7 Solving an Equilibrium Problem

A weight of 480 pounds is supported by three ropes. As shown in Figure 10.16, the weight is located at

$$S(0, 2, -1).$$

The ropes are tied to the points

$$P(2, 0, 0), \quad Q(0, 4, 0), \quad \text{and} \quad R(-2, 0, 0).$$

Find the force (or tension) on each rope.

Figure 10.16

Solution

The (downward) force of the weight is represented by the vector

$$\mathbf{w} = \langle 0, 0, -480 \rangle.$$

The force vectors corresponding to the ropes are as follows.

$$\mathbf{u} = \|\mathbf{u}\| \frac{\overrightarrow{SP}}{\|\overrightarrow{SP}\|} = \|\mathbf{u}\| \frac{\langle 2-0, 0-2, 0-(-1) \rangle}{3} = \|\mathbf{u}\| \left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right\rangle$$

$$\mathbf{v} = \|\mathbf{v}\| \frac{\overrightarrow{SQ}}{\|\overrightarrow{SQ}\|} = \|\mathbf{v}\| \frac{\langle 0-0, 4-2, 0-(-1) \rangle}{\sqrt{5}} = \|\mathbf{v}\| \left\langle 0, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$$

$$\mathbf{z} = \|\mathbf{z}\| \frac{\overrightarrow{SR}}{\|\overrightarrow{SR}\|} = \|\mathbf{z}\| \frac{\langle -2-0, 0-2, 0-(-1) \rangle}{3} = \|\mathbf{z}\| \left\langle -\frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right\rangle$$

For the system to be in equilibrium, it must be true that

$$\mathbf{u} + \mathbf{v} + \mathbf{z} + \mathbf{w} = \mathbf{0}$$

or

$$\mathbf{u} + \mathbf{v} + \mathbf{z} = -\mathbf{w}.$$

This yields the following system of linear equations.

$$\begin{cases} \frac{2}{3}\|\mathbf{u}\| \qquad\qquad - \frac{2}{3}\|\mathbf{z}\| = 0 \\ -\frac{2}{3}\|\mathbf{u}\| + \frac{2}{\sqrt{5}}\|\mathbf{v}\| - \frac{2}{3}\|\mathbf{z}\| = 0 \\ \frac{1}{3}\|\mathbf{u}\| + \frac{1}{\sqrt{5}}\|\mathbf{v}\| + \frac{1}{3}\|\mathbf{z}\| = 480 \end{cases}$$

Using the techniques demonstrated in Chapter 7, you can find the solution of the system to be

$$\|\mathbf{u}\| = 360.0$$

$$\|\mathbf{v}\| \approx 536.7$$

$$\|\mathbf{z}\| = 360.0.$$

So, the rope attached at point P has 360 pounds of tension, the rope attached at point Q has about 536.7 pounds of tension, and the rope attached at point R has 360 pounds of tension.

CHECKPOINT Now try Exercise 71.

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10.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1–3, fill in the blank.

1. The _____ vector is denoted by $\mathbf{0} = \langle 0, 0, 0 \rangle$.
2. The standard unit vector notation for a vector $\mathbf{v}$ in space is given by _____ .
3. Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are _____ when there is some scalar c such that $\mathbf{u} = c\mathbf{v}$.
4. A vector $\mathbf{v}$ is represented by the directed line segment from initial point $P(p_1, p_2, p_3)$ to terminal point $Q(q_1, q_2, q_3)$. What does $\langle q_1 - p_1, q_2 - p_2, q_3 - p_3 \rangle$ represent?
5. Write an expression for the magnitude of vector $\mathbf{v}$, $\|\mathbf{v}\|$, where $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$.
6. What is the dot product of two vectors that are orthogonal?

Procedures and Problem Solving

Finding the Component Form of a Vector **In Exercises 7–10, (a) find the component form of the vector v and (b) sketch the vector with its initial point at the origin.**

7.

8.

9.

10.

Finding the Component Form of a Vector **In Exercises 11–14, (a) write the component form of the vector v, (b) find the magnitude of v, and (c) find a unit vector in the direction of v.**

	Initial point	*Terminal point*
11.	$(-6, 4, -2)$	$(1, -1, 3)$
12.	$(-7, 3, 5)$	$(0, 0, 2)$
13.	$(-1, 2, -4)$	$(1, 4, -4)$
14.	$(0, -1, 0)$	$(0, 2, 1)$

Sketching Vectors **In Exercises 15–18, sketch each scalar multiple of v.**

15. $\mathbf{v} = \langle 1, 1, 3 \rangle$
    (a) $2\mathbf{v}$ (b) $-\mathbf{v}$ (c) $\frac{3}{2}\mathbf{v}$ (d) $0\mathbf{v}$
16. $\mathbf{v} = \langle -1, 2, 2 \rangle$
    (a) $-\mathbf{v}$ (b) $2\mathbf{v}$ (c) $\frac{1}{2}\mathbf{v}$ (d) $\frac{5}{2}\mathbf{v}$
17. $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$
    (a) $2\mathbf{v}$ (b) $-\mathbf{v}$ (c) $\frac{5}{2}\mathbf{v}$ (d) $0\mathbf{v}$
18. $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$
    (a) $4\mathbf{v}$ (b) $-2\mathbf{v}$ (c) $\frac{1}{2}\mathbf{v}$ (d) $0\mathbf{v}$

Using Vector Operations **In Exercises 19–24, find the vector z, given $\mathbf{u} = \langle -1, 3, 2 \rangle$, $\mathbf{v} = \langle 1, -2, -2 \rangle$, and $\mathbf{w} = \langle 5, 0, -5 \rangle$. Use a graphing utility to verify your answer.**

19. $\mathbf{z} = \mathbf{u} - 2\mathbf{v}$
20. $\mathbf{z} = 7\mathbf{u} + \mathbf{v} - \frac{1}{5}\mathbf{w}$
21. $2\mathbf{z} - 4\mathbf{u} = \mathbf{w}$
22. $\mathbf{u} + \mathbf{v} + \mathbf{z} = \mathbf{0}$
23. $\mathbf{z} = 2\mathbf{u} - 3\mathbf{v} + \frac{1}{2}\mathbf{w}$
24. $\mathbf{z} = 3\mathbf{w} - 2\mathbf{v} + \mathbf{u}$

Finding the Magnitude of a Vector **In Exercises 25–32, find the magnitude of v.**

25. $\mathbf{v} = \langle 7, 8, 7 \rangle$
26. $\mathbf{v} = \langle -2, 0, -5 \rangle$
27. $\mathbf{v} = \langle 1, -2, 4 \rangle$
28. $\mathbf{v} = \langle -1, 0, 3 \rangle$
29. $\mathbf{v} = 2\mathbf{i} - 4\mathbf{j} + \mathbf{k}$
30. $\mathbf{v} = \mathbf{i} + 3\mathbf{j} - \mathbf{k}$
31. Initial point: $(1, -3, 4)$; terminal point: $(1, 0, -1)$
32. Initial point: $(0, -1, 0)$; terminal point: $(1, 2, -2)$

Finding a Unit Vector **In Exercises 33–36, find a unit vector (a) in the direction of u and (b) in the direction opposite of u.**

33. $\mathbf{u} = 5\mathbf{i} - 12\mathbf{k}$
34. $\mathbf{u} = 3\mathbf{i} - 4\mathbf{k}$
35. $\mathbf{u} = 8\mathbf{i} + 3\mathbf{j} - \mathbf{k}$
36. $\mathbf{u} = -3\mathbf{i} + 5\mathbf{j} + 10\mathbf{k}$

Using a Graphing Utility **In Exercises 37–40, use a graphing utility to determine the specified quantity where $\mathbf{u} = \langle -1, 3, 4 \rangle$ and $\mathbf{v} = \langle 5, 4.5, -6 \rangle$.**

37. $6\mathbf{u} - 4\mathbf{v}$
38. $2\mathbf{u} + \frac{5}{2}\mathbf{v}$
39. $\|\mathbf{u} + \mathbf{v}\|$
40. $\dfrac{\mathbf{v}}{\|\mathbf{v}\|}$

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Finding the Dot Product of Two Vectors **In Exercises 41–44, find the dot product of u and v.**

41. $\mathbf{u} = \langle 4, 4, -1 \rangle$
$\mathbf{v} = \langle 2, -5, -8 \rangle$

42. $\mathbf{u} = \langle 3, -1, 6 \rangle$
$\mathbf{v} = \langle 4, -10, 1 \rangle$

43. $\mathbf{u} = 2\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$
$\mathbf{v} = 9\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

44. $\mathbf{u} = 3\mathbf{j} - 6\mathbf{k}$
$\mathbf{v} = 6\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}$

Finding the Angle Between Two Vectors **In Exercises 45–48, find the angle θ between the vectors.**

45. $\mathbf{u} = \langle 0, 2, 2 \rangle$
$\mathbf{v} = \langle 3, 0, -4 \rangle$

46. $\mathbf{u} = \langle -1, 3, 0 \rangle$
$\mathbf{v} = \langle 1, 2, -1 \rangle$

47. $\mathbf{u} = 10\mathbf{i} + 40\mathbf{j}$
$\mathbf{v} = -3\mathbf{j} + 8\mathbf{k}$

48. $\mathbf{u} = 8\mathbf{j} - 20\mathbf{k}$
$\mathbf{v} = 10\mathbf{i} - 5\mathbf{k}$

The Relationship of Two Vectors **In Exercises 49–54, determine whether u and v are orthogonal, parallel, or neither.**

49. $\mathbf{u} = \langle -12, 6, 15 \rangle$
$\mathbf{v} = \langle 8, -4, -10 \rangle$

50. $\mathbf{u} = \langle -1, 3, -1 \rangle$
$\mathbf{v} = \langle 2, -1, 5 \rangle$

51. $\mathbf{u} = \langle 0, 1, 6 \rangle$
$\mathbf{v} = \langle 1, -1, -2 \rangle$

52. $\mathbf{u} = \langle 0, 4, -1 \rangle$
$\mathbf{v} = \langle 1, 0, 0 \rangle$

53. $\mathbf{u} = -2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$
$\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$

54. $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$
$\mathbf{v} = -\mathbf{i} - \mathbf{j} - \mathbf{k}$

Using Vectors to Determine Collinear Points **In Exercises 55–58, use vectors to determine whether the points are collinear.**

55. $(5, 4, 1), (7, 3, -1), (4, 5, 3)$

56. $(-2, 7, 4), (-4, 8, 1), (0, 6, 7)$

57. $(1, 3, 2), (-1, 2, 5), (3, 4, -1)$

58. $(0, 4, 4), (-1, 5, 6), (-2, 6, 7)$

Using Vectors to Classify a Triangle **In Exercises 59–62, the vertices of a triangle are given. Determine whether the triangle is an acute triangle, an obtuse triangle, or a right triangle. Explain your reasoning.**

59. $(1, 2, 0), (0, 0, 0), (-2, 1, 0)$

60. $(-3, 0, 0), (0, 0, 0), (1, 2, 3)$

61. $(2, -3, 4), (0, 1, 2), (-1, 2, 0)$

62. $(2, -7, 3), (-1, 5, 8), (4, 6, -1)$

Finding the Terminal Point of a Vector **In Exercises 63–66, the vector v and its initial point are given. Find the terminal point.**

63. $\mathbf{v} = \langle 2, -4, 7 \rangle$
Initial point: $(1, 5, 0)$

64. $\mathbf{v} = \langle 4, -1, -1 \rangle$
Initial point: $(6, -4, 3)$

65. $\mathbf{v} = \left\langle 4, \frac{3}{2}, -\frac{1}{4} \right\rangle$
Initial point: $\left(2, 1, -\frac{3}{2}\right)$

66. $\mathbf{v} = \left\langle \frac{5}{2}, -\frac{1}{2}, 4 \right\rangle$
Initial point: $\left(3, 2, -\frac{1}{2}\right)$

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Finding a Scalar **In Exercises 67 and 68, determine the values of c that satisfy the given conditions.**

67. $\|c\mathbf{u}\| = 3, \mathbf{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

68. $\|c\mathbf{u}\| = 12, \mathbf{u} = -2\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$

Finding a Vector **In Exercises 69 and 70, write the component form of v.**

69. Vector **v** lies in the yz-plane, has magnitude 4, and makes an angle of 45° with the positive y-axis.

70. Vector **v** lies in the xz-plane, has magnitude 10, and makes an angle of 60° with the positive z-axis.

71. ***Why you should learn it*** *(p. 719)* A round bird bath weighs 10 pounds and has a radius of 12 inches. It is supported by three equally spaced 30-inch chains attached to a tree branch (see figure). Find the tension in each chain.

Figure for 71

Figure for 72

72. **Forces in Equilibrium** The weight of a crate is 500 newtons. Find the tension in each of the supporting cables shown in the figure.

Conclusions

True or False? **In Exercises 73 and 74, determine whether the statement is true or false. Justify your answer.**

73. If the dot product of two nonzero vectors is zero, then the angle between the vectors is a right angle.

74. If $\overrightarrow{AB}$ and $\overrightarrow{AC}$ represent parallel vectors, then points A, B, and C are collinear.

75. **Think About It** The initial and terminal points of **v** are (x_1, y_1, z_1) and (x, y, z), respectively. Describe the set of all points (x, y, z) such that $\|\mathbf{v}\| = 4$.

76. **CAPSTONE** Let **u** and **v** be nonzero vectors and let s and t be nonzero real numbers. The vectors $t\mathbf{v}$, $\mathbf{u} + t\mathbf{v}$, and $s\mathbf{u} + t\mathbf{v}$ are represented by three directed line segments from the same initial point. Describe the geometric figure generated by the terminal points of these directed line segments. Explain your reasoning.

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10.3 The Cross Product of Two Vectors

The Cross Product

Many applications in physics, engineering, and geometry involve finding a vector in space that is orthogonal to two given vectors. In this section, you will study a product that will yield such a vector. It is called the **cross product,** and it is most conveniently defined and calculated using the standard unit vector form.

Definition of Cross Product of Two Vectors in Space

Let

$$\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} \qquad \text{and} \qquad \mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

be vectors in space. The **cross product** of $\mathbf{u}$ and $\mathbf{v}$ is the vector

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.$$

What you should learn

- Find cross products of vectors in space.
- Use geometric properties of cross products of vectors in space.
- Use triple scalar products to find volumes of parallelepipeds.

Why you should learn it

The cross product of two vectors in space has many applications in physics and engineering. For instance, in Exercise 59 on page 732, the cross product is used to find the torque on the crank of a bicycle's brake.

It is important to note that this definition applies only to three-dimensional vectors. The cross product is not defined for two-dimensional vectors.

A convenient way to calculate $\mathbf{u} \times \mathbf{v}$ is to use the following *determinant form* with cofactor expansion. (This 3×3 determinant form is used simply to help remember the formula for the cross product—it is technically not a determinant because the entries of the corresponding matrix are not all real numbers.)

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \quad \begin{array}{l} \text{Put } \mathbf{u} \text{ in Row 2.} \\ \text{Put } \mathbf{v} \text{ in Row 3.} \end{array}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}\mathbf{k}$$

$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k}$$

$$= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

Note the minus sign in front of the $\mathbf{j}$-component. Recall from Section 7.7 that each of the three 2×2 determinants can be evaluated by using the following pattern.

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

Here is an example.

$$\begin{vmatrix} 2 & 4 \\ 3 & -1 \end{vmatrix} = 2(-1) - 4(3) = -2 - 12 = -14$$

Explore the Concept

Find each cross product and sketch the result. What can you conclude?

a. $\mathbf{i} \times \mathbf{j}$ **b.** $\mathbf{i} \times \mathbf{k}$ **c.** $\mathbf{j} \times \mathbf{k}$

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Example 1 Finding Cross Products

Given $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, find each cross product.

a. $\mathbf{u} \times \mathbf{v}$

b. $\mathbf{v} \times \mathbf{u}$

c. $\mathbf{v} \times \mathbf{v}$

Solution

a.
$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix} \\ &= \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}\mathbf{k} \\ &= (4 - 1)\mathbf{i} - (2 - 3)\mathbf{j} + (1 - 6)\mathbf{k} \\ &= 3\mathbf{i} + \mathbf{j} - 5\mathbf{k}\end{aligned}$$

b.
$$\begin{aligned}\mathbf{v} \times \mathbf{u} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix} \\ &= \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix}\mathbf{k} \\ &= (1 - 4)\mathbf{i} - (3 - 2)\mathbf{j} + (6 - 1)\mathbf{k} \\ &= -3\mathbf{i} - \mathbf{j} + 5\mathbf{k}\end{aligned}$$

Note that this result is the negative of that in part (a).

c.
$$\mathbf{v} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 2 \\ 3 & 1 & 2 \end{vmatrix} = \mathbf{0}$$

CHECKPOINT Now try Exercise 13.

The results obtained in Example 1 suggest some interesting algebraic properties of the cross product. For instance,

$$\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u}) \quad \text{and} \quad \mathbf{v} \times \mathbf{v} = \mathbf{0}.$$

These properties, and several others, are summarized in the following list.

Algebraic Properties of the Cross Product (See the proof on page 747.)

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in space and let c be a scalar.

1. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$
3. $c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$
4. $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$
5. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$
6. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$

Technology Tip

Some graphing utilities have the capability of performing vector operations, such as the cross product. Consult the user's guide for your graphing utility for specific instructions.

Explore the Concept

Calculate $\mathbf{u} \times \mathbf{v}$ and $-(\mathbf{v} \times \mathbf{u})$ for several values of $\mathbf{u}$ and $\mathbf{v}$. What do your results imply? Interpret your results geometrically.

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Geometric Properties of the Cross Product

The first property listed on the preceding page indicates that the cross product is *not commutative*. In particular, this property indicates that the vectors $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ have equal lengths but opposite directions. The following list gives some other *geometric* properties of the cross product of two vectors.

Geometric Properties of the Cross Product (See the proof on page 748.)

Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in space, and let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$.

1. $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.
2. $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\| \sin \theta$
3. $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are scalar multiples of each other.
4. $\|\mathbf{u} \times \mathbf{v}\|$ = area of parallelogram having $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides.

Both $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ are perpendicular to the plane determined by $\mathbf{u}$ and $\mathbf{v}$. One way to remember the orientations of the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$ is to compare them with the unit vectors

$$\mathbf{i}, \quad \mathbf{j}, \quad \text{and} \quad \mathbf{k} = \mathbf{i} \times \mathbf{j}$$

as shown in Figure 10.17. The three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$ form a *right-handed system*.

Figure 10.17

Example 2 Using the Cross Product

Find a unit vector that is orthogonal to both

$$\mathbf{u} = 3\mathbf{i} - 4\mathbf{j} + \mathbf{k} \quad \text{and} \quad \mathbf{v} = -3\mathbf{i} + 6\mathbf{j}.$$

Solution

The cross product $\mathbf{u} \times \mathbf{v}$, as shown in Figure 10.18, is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -4 & 1 \\ -3 & 6 & 0 \end{vmatrix} = -6\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$$

Because

$$\begin{aligned} \|\mathbf{u} \times \mathbf{v}\| &= \sqrt{(-6)^2 + (-3)^2 + 6^2} \\ &= \sqrt{81} \\ &= 9 \end{aligned}$$

a unit vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ is

$$\frac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = -\frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}.$$

CHECKPOINT Now try Exercise 31.

Figure 10.18

What's Wrong?

You use a graphing utility to find $\mathbf{u} \times \mathbf{v}$ for

$$\mathbf{u} = -\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$$

and

$$\mathbf{v} = \mathbf{i} - 2\mathbf{k}$$

as shown in the figure. You conclude that

$$\mathbf{w} = -8\mathbf{i} + \mathbf{j} - 4\mathbf{k}$$

is a unit vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. What's wrong?

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In Example 2, note that you could have used the cross product $\mathbf{v} \times \mathbf{u}$ to form a unit vector that is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. With that choice, you would have obtained the *negative* of the unit vector found in the example.

The fourth geometric property of the cross product states that $\|\mathbf{u} \times \mathbf{v}\|$ is the area of the parallelogram that has $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides. A simple example of this is given by the unit square with adjacent sides of $\mathbf{i}$ and $\mathbf{j}$. Because

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}$$

and $\|\mathbf{k}\| = 1$, it follows that the square has an area of 1. This geometric property of the cross product is illustrated further in the next example.

Example 3 Geometric Application of the Cross Product

Show that the quadrilateral with vertices at the following points is a parallelogram. Then find the area of the parallelogram. Is the parallelogram a rectangle?

$$A(5, 2, 0), \qquad B(2, 6, 1), \qquad C(2, 4, 7), \qquad D(5, 0, 6)$$

Solution

From Figure 10.19 you can see that the sides of the quadrilateral correspond to the following four vectors.

$$\overrightarrow{AB} = -3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$$

$$\overrightarrow{CD} = 3\mathbf{i} - 4\mathbf{j} - \mathbf{k} = -\overrightarrow{AB}$$

$$\overrightarrow{AD} = 0\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$$

$$\overrightarrow{CB} = 0\mathbf{i} + 2\mathbf{j} - 6\mathbf{k} = -\overrightarrow{AD}$$

Figure 10.19

Because $\overrightarrow{CD} = -\overrightarrow{AB}$ and $\overrightarrow{CB} = -\overrightarrow{AD}$, you can conclude that $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$ and $\overrightarrow{AD}$ is parallel to $\overrightarrow{CB}$. It follows that the quadrilateral is a parallelogram with $\overrightarrow{AB}$ and $\overrightarrow{AD}$ as adjacent sides. Moreover, because

$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 4 & 1 \\ 0 & -2 & 6 \end{vmatrix} = 26\mathbf{i} + 18\mathbf{j} + 6\mathbf{k}$$

the area of the parallelogram is

$$\begin{aligned} \|\overrightarrow{AB} \times \overrightarrow{AD}\| &= \sqrt{26^2 + 18^2 + 6^2} \\ &= \sqrt{1036} \\ &\approx 32.19. \end{aligned}$$

You can tell whether the parallelogram is a rectangle by finding the angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$.

$$\sin\theta = \frac{\|\overrightarrow{AB} \times \overrightarrow{AD}\|}{\|\overrightarrow{AB}\|\,\|\overrightarrow{AD}\|}$$

$$\sin\theta = \frac{\sqrt{1036}}{\sqrt{26}\sqrt{40}}$$

$$\sin\theta \approx 0.998$$

$$\theta \approx \arcsin 0.998$$

$$\theta \approx 86.4^\circ$$

Because $\theta \neq 90^\circ$, the parallelogram is not a rectangle.

CHECKPOINT Now try Exercise 43.

Explore the Concept

When you connect the terminal points of two vectors $\mathbf{u}$ and $\mathbf{v}$ that have the same initial points, a triangle is formed. Is it possible to use the cross product $\mathbf{u} \times \mathbf{v}$ to determine the area of the triangle? Explain. Verify your conclusion using two vectors from Example 3.

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The Triple Scalar Product

For the vectors **u**, **v**, and **w** in space, the dot product of **u** and $\mathbf{v} \times \mathbf{w}$ is called the **triple scalar product** of **u**, **v**, and **w**.

The Triple Scalar Product

For $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, and $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$, the triple scalar product is given by

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

When the vectors **u**, **v**, and **w** do not lie in the same plane, the triple scalar product

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$$

can be used to determine the volume of the parallelepiped (a polyhedron, all of whose faces are parallelograms) with **u**, **v**, and **w** as adjacent edges, as shown in Figure 10.20.

Area of base $= \|\mathbf{v} \times \mathbf{w}\|$
Volume of parallelepiped $= |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$

Figure 10.20

Geometric Property of Triple Scalar Product

The volume V of a parallelepiped with vectors **u**, **v**, and **w** as adjacent edges is given by

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|.$$

Example 4 Volume by the Triple Scalar Product

Find the volume of the parallelepiped having

$$\mathbf{u} = 3\mathbf{i} - 5\mathbf{j} + \mathbf{k}, \quad \mathbf{v} = 2\mathbf{j} - 2\mathbf{k}, \quad \text{and} \quad \mathbf{w} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$$

as adjacent edges, as shown in Figure 10.21.

Figure 10.21

Solution

The value of the triple scalar product is

$$\begin{aligned} \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) &= \begin{vmatrix} 3 & -5 & 1 \\ 0 & 2 & -2 \\ 3 & 1 & 1 \end{vmatrix} \\ &= 3\begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} - (-5)\begin{vmatrix} 0 & -2 \\ 3 & 1 \end{vmatrix} + 1\begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} \\ &= 3(4) + 5(6) + 1(-6) \\ &= 36. \end{aligned}$$

So, the volume of the parallelepiped is

$$|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = |36| = 36.$$

You can confirm this result using a graphing utility, as shown in Figure 10.22.

Figure 10.22

CHECKPOINT Now try Exercise 55.

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10.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1–4, fill in the blank.

1. To find a vector in space that is orthogonal to two given vectors, find the _____ of the two vectors.
2. $\mathbf{u} \times \mathbf{u} =$ _____
3. $\|\mathbf{u} \times \mathbf{v}\| =$ _____
4. The dot product of $\mathbf{u}$ and $\mathbf{v} \times \mathbf{w}$ is called the _____ of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.
5. Given $\mathbf{u} \times \mathbf{v} = \langle 1, 0, -2 \rangle$, what is $\mathbf{v} \times \mathbf{u}$?
6. Given $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in space, when is $\mathbf{u} \times \mathbf{v} = \mathbf{0}$?

Procedures and Problem Solving

Finding the Cross Product **In Exercises 7–22, find $\mathbf{u} \times \mathbf{v}$ and show that it is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.**

7. $\mathbf{u} = \langle 1, -1, 0 \rangle$
   $\mathbf{v} = \langle 0, 1, -1 \rangle$
8. $\mathbf{u} = \langle -1, 1, 0 \rangle$
   $\mathbf{v} = \langle 1, 0, -1 \rangle$
9. $\mathbf{u} = \langle 3, -2, 5 \rangle$
   $\mathbf{v} = \langle 0, -1, 1 \rangle$
10. $\mathbf{u} = \langle 6, 8, 3 \rangle$
    $\mathbf{v} = \langle 4, -1, -4 \rangle$
11. $\mathbf{u} = \langle -10, 0, 6 \rangle$
    $\mathbf{v} = \langle 7, 0, 0 \rangle$
12. $\mathbf{u} = \langle -5, 5, 11 \rangle$
    $\mathbf{v} = \langle 2, 2, 3 \rangle$
13. $\mathbf{u} = 6\mathbf{i} + 2\mathbf{j} + \mathbf{k}$
    $\mathbf{v} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$
14. $\mathbf{u} = \mathbf{i} + \frac{3}{2}\mathbf{j} - \frac{5}{2}\mathbf{k}$
    $\mathbf{v} = \frac{1}{2}\mathbf{i} - \frac{3}{4}\mathbf{j} + \frac{1}{4}\mathbf{k}$
15. $\mathbf{u} = 2\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$
    $\mathbf{v} = -\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$
16. $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$
    $\mathbf{v} = -2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$
17. $\mathbf{u} = \frac{1}{2}\mathbf{i} - \frac{2}{3}\mathbf{j} + \mathbf{k}$
    $\mathbf{v} = -\frac{3}{4}\mathbf{i} + \mathbf{j} + \frac{1}{4}\mathbf{k}$
18. $\mathbf{u} = \frac{2}{5}\mathbf{i} - \frac{1}{4}\mathbf{j} + \frac{1}{2}\mathbf{k}$
    $\mathbf{v} = -\frac{3}{5}\mathbf{i} + \mathbf{j} + \frac{1}{5}\mathbf{k}$
19. $\mathbf{u} = 6\mathbf{k}$
    $\mathbf{v} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k}$
20. $\mathbf{u} = \frac{2}{3}\mathbf{i}$
    $\mathbf{v} = \frac{1}{3}\mathbf{j} - 3\mathbf{k}$
21. $\mathbf{u} = -\mathbf{i} + \mathbf{k}$
    $\mathbf{v} = \mathbf{j} - 2\mathbf{k}$
22. $\mathbf{u} = \mathbf{i} - \mathbf{k}$
    $\mathbf{v} = -\mathbf{j} + \mathbf{k}$

Using a Graphing Utility to Find the Cross Product **In Exercises 23–28, use a graphing utility to find $\mathbf{u} \times \mathbf{v}$.**

23. $\mathbf{u} = \langle 2, 4, 3 \rangle$
    $\mathbf{v} = \langle 0, -2, 1 \rangle$
24. $\mathbf{u} = \langle 4, -2, 6 \rangle$
    $\mathbf{v} = \langle -1, 5, 7 \rangle$
25. $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$
    $\mathbf{v} = -4\mathbf{i} + 2\mathbf{j} - \mathbf{k}$
26. $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$
    $\mathbf{v} = -\mathbf{i} + \mathbf{j} - 4\mathbf{k}$
27. $\mathbf{u} = 6\mathbf{i} - 5\mathbf{j} + \mathbf{k}$
    $\mathbf{v} = \frac{1}{2}\mathbf{i} - \frac{3}{4}\mathbf{j} + \frac{2}{10}\mathbf{k}$
28. $\mathbf{u} = 8\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$
    $\mathbf{v} = \frac{1}{2}\mathbf{i} + \frac{3}{4}\mathbf{j} - \frac{1}{4}\mathbf{k}$

Using the Cross Product **In Exercises 29–36, find a unit vector orthogonal to $\mathbf{u}$ and $\mathbf{v}$.**

29. $\mathbf{u} = \langle 2, -3, 4 \rangle$
    $\mathbf{v} = \langle 0, -1, 1 \rangle$
30. $\mathbf{u} = \langle 2, -1, 3 \rangle$
    $\mathbf{v} = \langle 1, 0, -2 \rangle$
31. $\mathbf{u} = 3\mathbf{i} + \mathbf{j}$
    $\mathbf{v} = \mathbf{j} + \mathbf{k}$
32. $\mathbf{u} = \mathbf{i} + 2\mathbf{j}$
    $\mathbf{v} = \mathbf{i} - 3\mathbf{k}$
33. $\mathbf{u} = -3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$
    $\mathbf{v} = \frac{1}{2}\mathbf{i} - \frac{3}{4}\mathbf{j} + \frac{1}{10}\mathbf{k}$
34. $\mathbf{u} = 7\mathbf{i} - 14\mathbf{j} + 5\mathbf{k}$
    $\mathbf{v} = 14\mathbf{i} + 28\mathbf{j} - 15\mathbf{k}$
35. $\mathbf{u} = \mathbf{i} + \mathbf{j} - \mathbf{k}$
    $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$
36. $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$
    $\mathbf{v} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$

Finding the Area of a Parallelogram **In Exercises 37–42, find the area of the parallelogram that has the vectors as adjacent sides.**

37. $\mathbf{u} = \mathbf{k}$
    $\mathbf{v} = \mathbf{i} + \mathbf{k}$
38. $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$
    $\mathbf{v} = \mathbf{i} + \mathbf{k}$
39. $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$
    $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$
40. $\mathbf{u} = -2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$
    $\mathbf{v} = \mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$
41. $\mathbf{u} = \langle 4, 4, -6 \rangle$
    $\mathbf{v} = \langle 0, 4, 6 \rangle$
42. $\mathbf{u} = \langle 4, -3, 2 \rangle$
    $\mathbf{v} = \langle 5, 0, 1 \rangle$

Geometric Application of the Cross Product **In Exercises 43 and 44, (a) verify that the points are the vertices of a parallelogram, (b) find its area, and (c) determine whether the parallelogram is a rectangle.**

43. $A(2, -1, 4)$, $B(3, 1, 2)$, $C(0, 5, 6)$, $D(-1, 3, 8)$
44. $A(2, 1, 1)$, $B(2, 3, 1)$, $C(-2, 4, 1)$, $D(-2, 6, 1)$

Finding the Area of a Triangle **In Exercises 45–48, find the area of the triangle with the given vertices. (The area A of the triangle having $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides is given by $A = \frac{1}{2}\|\mathbf{u} \times \mathbf{v}\|$.)**

45. $(0, 0, 0)$, $(1, 2, 3)$, $(-3, 0, 0)$
46. $(1, -4, 3)$, $(2, 0, 2)$, $(-2, 2, 0)$
47. $(2, 3, -5)$, $(-2, -2, 0)$, $(3, 0, 6)$
48. $(2, 4, 0)$, $(-2, -4, 0)$, $(0, 0, 4)$

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Finding the Triple Scalar Product **In Exercises 49–52, find the triple scalar product.**

49. $\mathbf{u} = \langle 2, 3, 3\rangle$, $\mathbf{v} = \langle 4, 4, 0\rangle$, $\mathbf{w} = \langle 0, 0, 4\rangle$

50. $\mathbf{u} = \langle 4, 0, 1\rangle$, $\mathbf{v} = \langle 0, 5, 0\rangle$, $\mathbf{w} = \langle 0, 0, 1\rangle$

51. $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$, $\mathbf{v} = \mathbf{i} - \mathbf{j}$, $\mathbf{w} = 4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$

52. $\mathbf{u} = \mathbf{i} + 4\mathbf{j} - 7\mathbf{k}$, $\mathbf{v} = 2\mathbf{i} + 4\mathbf{k}$, $\mathbf{w} = -3\mathbf{j} + 6\mathbf{k}$

Volume by the Triple Scalar Product **In Exercises 53–56, use the triple scalar product to find the volume of the parallelepiped having adjacent edges u, v, and w.**

53. $\mathbf{u} = \mathbf{i} + \mathbf{j}$
$\mathbf{v} = \mathbf{j} + \mathbf{k}$
$\mathbf{w} = \mathbf{i} + \mathbf{k}$

54. $\mathbf{u} = \mathbf{i} + \mathbf{j} + 3\mathbf{k}$
$\mathbf{v} = 3\mathbf{j} + 3\mathbf{k}$
$\mathbf{w} = 3\mathbf{i} + 3\mathbf{k}$

✓ **55.** $\mathbf{u} = \langle 0, 2, 2\rangle$
$\mathbf{v} = \langle 0, 0, -2\rangle$
$\mathbf{w} = \langle 3, 0, 2\rangle$

56. $\mathbf{u} = \langle 1, 2, -1\rangle$
$\mathbf{v} = \langle -1, 2, 2\rangle$
$\mathbf{w} = \langle 2, 0, 1\rangle$

Finding the Volume of a Parallelepiped **In Exercises 57 and 58, find the volume of the parallelepiped with the given vertices.**

57. $A(0, 0, 0)$, $B(4, 0, 0)$, $C(4, -2, 3)$, $D(0, -2, 3)$, $E(4, 5, 3)$, $F(0, 5, 3)$, $G(0, 3, 6)$, $H(4, 3, 6)$

58. $A(0, 0, 0)$, $B(1, 1, 0)$, $C(1, 0, 2)$, $D(0, 1, 1)$, $E(2, 1, 2)$, $F(1, 1, 3)$, $G(1, 2, 1)$, $H(2, 2, 3)$

59. ***Why you should learn it*** *(p. 726)* The brakes on a bicycle are applied by using a downward force of p pounds on the pedal when the six-inch crank makes a 40° angle with the horizontal (see figure). Vectors representing the position of the crank and the force are $\mathbf{V} = \frac{1}{2}(\cos 40°\mathbf{j} + \sin 40°\mathbf{k})$ and $\mathbf{F} = -p\mathbf{k}$, respectively.

(a) The magnitude of the torque on the crank is given by $\|\mathbf{V} \times \mathbf{F}\|$. Using the given information, write the torque T on the crank as a function of p.

(b) Use the function from part (a) to complete the table.

p	15	20	25	30	35	40	45
T							

Figure for 59

Figure for 60

60. MODELING DATA

Both the magnitude and direction of the force on a crankshaft change as the crankshaft rotates. Use the technique given in Exercise 59 to find the magnitude of the torque on the crankshaft using the position and data shown in the figure.

Conclusions

True or False? **In Exercises 61 and 62, determine whether the statement is true or false. Justify your answer.**

61. The cross product is not defined for vectors in the plane.

62. If $\mathbf{u}$ and $\mathbf{v}$ are vectors in space that are nonzero and not parallel, then $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{u}$.

63. **Proof** Prove that $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\|$ if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

64. **Think About It** When the magnitudes of two vectors are doubled, how will the magnitude of the cross product of the vectors change?

65. **Proof** Consider the vectors $\mathbf{u} = \langle \cos\alpha, \sin\alpha, 0\rangle$ and $\mathbf{v} = \langle \cos\beta, \sin\beta, 0\rangle$, where $\alpha > \beta$. Find the cross product of the vectors and use the result to prove the identity $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$.

66. **CAPSTONE** Define the cross product of two vectors in space, $\mathbf{u}$ and $\mathbf{v}$, where $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Explain, in your own words, what the cross product $\mathbf{u} \times \mathbf{v}$ represents. What does it mean when $\mathbf{u} \times \mathbf{v} = \mathbf{0}$?

Cumulative Mixed Review

Evaluating a Trigonometric Expression **In Exercises 67 and 68, evaluate the expression without using a calculator.**

67. cos 480°

68. sin 690°

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10.4 Lines and Planes in Space

Lines in Space

In the plane, *slope* is used to determine an equation of a line. In space, it is more convenient to use *vectors* to determine the equation of a line.

In Figure 10.23, consider the line L through the point

$P(x_1, y_1, z_1)$

and parallel to the vector

$\mathbf{v} = \langle a, b, c \rangle.$ Direction vector for L

Figure 10.23

The vector $\mathbf{v}$ is the **direction vector** for the line L, and a, b, and c are the **direction numbers.** One way of describing the line L is to say that it consists of all points $Q(x, y, z)$ for which the vector $\overrightarrow{PQ}$ is parallel to $\mathbf{v}$. This means that $\overrightarrow{PQ}$ is a scalar multiple of $\mathbf{v}$, and you can write $\overrightarrow{PQ} = t\mathbf{v}$, where t is a scalar.

$$\begin{aligned}\overrightarrow{PQ} &= \langle x - x_1, y - y_1, z - z_1 \rangle \\ &= \langle at, bt, ct \rangle \\ &= t\mathbf{v}\end{aligned}$$

By equating corresponding components, you can obtain the **parametric equations of a line in space.**

Parametric Equations of a Line in Space

A line L parallel to the nonzero vector

$\mathbf{v} = \langle a, b, c \rangle$

and passing through the point

$P(x_1, y_1, z_1)$

is represented by the parametric equations

$x = x_1 + at, \quad y = y_1 + bt, \quad \text{and} \quad z = z_1 + ct.$

When the direction numbers a, b, and c are all nonzero, you can eliminate the parameter t to obtain the **symmetric equations** of a line.

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$ Symmetric equations

What you should learn

- Find parametric and symmetric equations of lines in space.
- Find equations of planes in space.
- Sketch planes in space.
- Find distances between points and planes in space.

Why you should learn it

Normal vectors to a plane are important in modeling and solving real-life problems. For instance, in Exercise 66 on page 741, normal vectors are used to find the angle between two adjacent sides of a tapered bread pan.

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Example 1 Finding Parametric and Symmetric Equations

Find parametric and symmetric equations of the line L that passes through the point $(1, -2, 4)$ and is parallel to $\mathbf{v} = \langle 2, 4, -4 \rangle$.

Figure 10.24

Solution

To find a set of parametric equations of the line, use the coordinates

$$x_1 = 1, \quad y_1 = -2, \quad \text{and} \quad z_1 = 4$$ See Figure 10.24.

and direction numbers

$$a = 2, \quad b = 4, \quad \text{and} \quad c = -4.$$ See Figure 10.24.

$$x = 1 + 2t, \quad y = -2 + 4t, \quad z = 4 - 4t$$ Parametric equations

Because a, b, and c are all nonzero, a set of symmetric equations is

$$\frac{x-1}{2} = \frac{y+2}{4} = \frac{z-4}{-4}.$$ Symmetric equations

CHECKPOINT Now try Exercise 7.

Neither the parametric equations nor the symmetric equations of a given line are unique. For instance, in Example 1, by letting $t = 1$ in the parametric equations you would obtain the point $(3, 2, 0)$. Using this point with the direction numbers

$$a = 2, \quad b = 4, \quad \text{and} \quad c = -4$$

produces the parametric equations

$$x = 3 + 2t, \quad y = 2 + 4t, \quad \text{and} \quad z = -4t.$$

Example 2 Parametric and Symmetric Equations of a Line Through Two Points

Find a set of parametric and symmetric equations of the line that passes through the points $(-2, 1, 0)$ and $(1, 3, 5)$.

Solution

Begin by using the points $P = (-2, 1, 0)$ and $Q = (1, 3, 5)$. Then a direction vector for the line passing through P and Q is

$$\begin{aligned} \mathbf{v} &= \overrightarrow{PQ} \\ &= \langle 1 - (-2), 3 - 1, 5 - 0 \rangle \\ &= \langle 3, 2, 5 \rangle \\ &= \langle a, b, c \rangle. \end{aligned}$$

Using the direction numbers

$$a = 3, \quad b = 2, \quad \text{and} \quad c = 5$$

with the point $P(-2, 1, 0)$, you can obtain the parametric equations

$$x = -2 + 3t, \quad y = 1 + 2t, \quad \text{and} \quad z = 5t.$$ Parametric equations

Because a, b, and c are all nonzero, a set of symmetric equations is

$$\frac{x+2}{3} = \frac{y-1}{2} = \frac{z}{5}.$$ Symmetric equations

CHECKPOINT Now try Exercise 13.

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Planes in Space

You have seen how an equation of a line in space can be obtained from a point on the line and a vector *parallel* to it. You will now see that an equation of a plane in space can be obtained from a point in the plane and a vector *normal* (perpendicular) to the plane.

Figure 10.25

Consider the plane containing the point $P(x_1, y_1, z_1)$ having a nonzero normal vector $\mathbf{n} = \langle a, b, c \rangle$, as shown in Figure 10.25. This plane consists of all points $Q(x, y, z)$ for which the vector $\overrightarrow{PQ}$ is orthogonal to $\mathbf{n}$. Using the dot product, you can write the following.

$$\mathbf{n} \cdot \overrightarrow{PQ} = 0$$

$$\langle a, b, c \rangle \cdot \langle x - x_1, y - y_1, z - z_1 \rangle = 0$$

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

The third equation of the plane is said to be in **standard form.**

Standard Equation of a Plane in Space

The plane containing the point (x_1, y_1, z_1) and having nonzero normal vector $\mathbf{n} = \langle a, b, c \rangle$ can be represented by the **standard form of the equation of a plane**

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0. \qquad \text{Standard form of equation of plane}$$

Regrouping terms yields the **general form of the equation of a plane** in space

$$ax + by + cz + d = 0. \qquad \text{General form of equation of plane}$$

Given the general form of the equation of a plane, it is easy to find a normal vector to the plane. Use the coefficients of x, y, and z to write $\mathbf{n} = \langle a, b, c \rangle$.

Explore the Concept

Consider the following four planes.

$$2x + 3y - z = 2 \qquad 4x + 6y - 2z = 5$$

$$-2x - 3y + z = -2 \qquad -6x - 9y + 3z = 11$$

What are the normal vectors to each plane? What can you say about the relative positions of these planes in space?

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Example 3 Finding an Equation of a Plane in Three-Space

Find the general equation of the plane passing through the points $(2, 1, 1)$, $(0, 4, 1)$, and $(-2, 1, 4)$.

Solution

To find the equation of the plane, you need a point in the plane and a vector that is normal to the plane. There are three choices for the point, but no normal vector is given. To obtain a normal vector, use the cross product of vectors $\mathbf{u}$ and $\mathbf{v}$ extending from the point $(2, 1, 1)$ to the points $(0, 4, 1)$ and $(-2, 1, 4)$, as shown in Figure 10.26. The component forms of $\mathbf{u}$ and $\mathbf{v}$ are

$$\mathbf{u} = \langle 0 - 2, 4 - 1, 1 - 1\rangle = \langle -2, 3, 0\rangle$$

$$\mathbf{v} = \langle -2 - 2, 1 - 1, 4 - 1\rangle = \langle -4, 0, 3\rangle$$

Figure 10.26

and it follows that

$$\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 3 & 0 \\ -4 & 0 & 3 \end{vmatrix}$$

$$= 9\mathbf{i} + 6\mathbf{j} + 12\mathbf{k}$$

$$= \langle a, b, c\rangle$$

is normal to the given plane. Using the direction numbers for $\mathbf{n}$ and the point $(x_1, y_1, z_1) = (2, 1, 1)$, you can determine an equation of the plane to be

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

$$9(x - 2) + 6(y - 1) + 12(z - 1) = 0 \qquad \text{Standard form}$$

$$9x + 6y + 12z - 36 = 0$$

$$3x + 2y + 4z - 12 = 0. \qquad \text{General form}$$

Check that each of the three points satisfies the equation $3x + 2y + 4z - 12 = 0$.

CHECKPOINT Now try Exercise 31.

Two distinct planes in three-space either are parallel or intersect in a line. When they intersect, you can determine the angle θ $(0 \le \theta \le 90°)$ between them from the angle between their normal vectors, as shown in Figure 10.27. Specifically, if vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are normal to two intersecting planes, then the angle θ between the normal vectors is equal to the **angle between the two planes** and is given by

Figure 10.27

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|}. \qquad \text{Angle between two planes}$$

Consequently, two planes with normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are

1. *perpendicular* when $\mathbf{n}_1 \cdot \mathbf{n}_2 = 0$.
2. *parallel* when $\mathbf{n}_1$ is a scalar multiple of $\mathbf{n}_2$.

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Example 4 Finding the Line of Intersection of Two Planes

Find the angle between the two planes given by

$$x - 2y + z = 0 \qquad \text{Equation for plane 1}$$

$$2x + 3y - 2z = 0 \qquad \text{Equation for plane 2}$$

and find parametric equations of their line of intersection (see Figure 10.28).

Figure 10.28

Solution

The normal vectors to the planes are $\mathbf{n}_1 = \langle 1, -2, 1\rangle$ and $\mathbf{n}_2 = \langle 2, 3, -2\rangle$. Consequently, the angle between the two planes is determined as follows.

$$\begin{aligned}\cos\theta &= \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|}\\ &= \frac{|-6|}{\sqrt{6}\sqrt{17}}\\ &= \frac{6}{\sqrt{102}}\\ &\approx 0.59409\end{aligned}$$

This implies that the angle between the two planes is $\theta \approx 53.55°$. You can find the line of intersection of the two planes by simultaneously solving the two linear equations representing the planes. One way to do this is to multiply the first equation by -2 and add the result to the second equation.

$$\begin{aligned} x - 2y + z &= 0 \\ 2x + 3y - 2z &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} -2x + 4y - 2z &= 0 \\ \underline{2x + 3y - 2z} &\underline{= 0} \\ 7y - 4z &= 0 \end{aligned} \quad \Rightarrow \quad y = \frac{4z}{7}$$

Substituting $y = 4z/7$ back into one of the original equations, you can determine that $x = z/7$. Finally, by letting $t = z/7$, you obtain the parametric equations

$$x = t = x_1 + at, \quad y = 4t = y_1 + bt, \quad z = 7t = z_1 + ct.$$

Because $(x_1, y_1, z_1) = (0, 0, 0)$ lies in both planes, you can substitute for x_1, y_1, and z_1 in these parametric equations, which indicates that $a = 1, b = 4$, and $c = 7$ are direction numbers for the line of intersection.

CHECKPOINT Now try Exercise 51.

Note that the direction numbers in Example 4 can be obtained from the cross product of the two normal vectors as follows.

$$\begin{aligned}\mathbf{n}_1 \times \mathbf{n}_2 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ 2 & 3 & -2 \end{vmatrix}\\ &= \begin{vmatrix} -2 & 1 \\ 3 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 1 \\ 2 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix}\mathbf{k}\\ &= \mathbf{i} + 4\mathbf{j} + 7\mathbf{k}\end{aligned}$$

This means that the *line of intersection of the two planes is parallel to the cross product of their normal vectors.*

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Sketching Planes in Space

As discussed in Section 10.1, when a plane in space intersects one of the coordinate planes, the line of intersection is called the *trace* of the given plane in the coordinate plane. To sketch a plane in space, it is helpful to find its points of intersection with the coordinate axes and its traces in the coordinate planes. For example, consider the plane

$$3x + 2y + 4z = 12. \qquad \text{Equation of plane}$$

You can find the xy-trace by letting $z = 0$ and sketching the line

$$3x + 2y = 12 \qquad xy\text{-trace}$$

in the xy-plane. This line intersects the x-axis at $(4, 0, 0)$ and the y-axis at $(0, 6, 0)$. In Figure 10.29, this process is continued by finding the yz-trace and the xz-trace and then shading the triangular region lying in the first octant.

(a) ***xy-trace*** **($z = 0$):** $3x + 2y = 12$

(b) ***yz-trace*** **($x = 0$):** $2y + 4z = 12$

(c) ***xz-trace*** **($y = 0$):** $3x + 4z = 12$

Figure 10.29

If the equation of a plane has a missing variable, such as

$$2x + z = 1$$

then the plane must be *parallel to the axis* represented by the missing variable, as shown in Figure 10.30.

If two variables are missing from the equation of a plane, then it is *parallel to the coordinate plane* represented by the missing variables, as shown in Figure 10.31. For instance, the graph of

$$z + 2 = 0$$

is parallel to the xy-plane. Try confirming this using a three-dimensional graphing utility.

Plane is parallel to y-axis.

Figure 10.30

(a) ***Plane $ax + d = 0$ is parallel to yz-plane.***

(b) ***Plane $by + d = 0$ is parallel to xz-plane.***

(c) ***Plane $cz + d = 0$ is parallel to xy-plane.***

Figure 10.31

Technology Tip

Most three-dimensional graphing utilities and computer algebra systems can graph a plane in space. Consult the user's guide for your utility for specific instructions.

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Distance Between a Point and a Plane

The distance D between a point Q and a plane is the length of the shortest line segment connecting Q to the plane, as shown in Figure 10.32. When P is *any* point in the plane, you can find this distance by projecting the vector

$$\overrightarrow{PQ}$$

onto the normal vector $\mathbf{n}$. The length of this projection is the desired distance.

$D = \|\text{proj}_{\mathbf{n}}\overrightarrow{PQ}\|$

Figure 10.32

Distance Between a Point and a Plane

The **distance between a plane and a point** Q (not in the plane) is

$$D = \|\text{proj}_{\mathbf{n}}\overrightarrow{PQ}\| = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|}$$

where P is a point in the plane and $\mathbf{n}$ is normal to the plane.

To find a point in the plane given by $ax + by + cz + d = 0$, where $a \neq 0$, let $y = 0$ and $z = 0$. Then, from the equation $ax + d = 0$, you can conclude that the point

$$(-d/a, 0, 0)$$

lies in the plane.

Example 5 Finding the Distance Between a Point and a Plane

Find the distance between the point $Q(1, 5, -4)$ and the plane $3x - y + 2z = 6$.

Solution

You know that $\mathbf{n} = \langle 3, -1, 2\rangle$ is normal to the given plane. To find a point in the plane, let $y = 0$ and $z = 0$, and obtain the point $P(2, 0, 0)$. The vector from P to Q is

$$\overrightarrow{PQ} = \langle 1 - 2, 5 - 0, -4 - 0\rangle = \langle -1, 5, -4\rangle.$$

The formula for the distance between a point and a plane produces

$$\begin{aligned} D &= \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} \\ &= \frac{|\langle -1, 5, -4\rangle \cdot \langle 3, -1, 2\rangle|}{\sqrt{9 + 1 + 4}} \\ &= \frac{|-3 - 5 - 8|}{\sqrt{14}} \\ &= \frac{16}{\sqrt{14}}. \end{aligned}$$

CHECKPOINT Now try Exercise 63.

The choice of the point P in Example 5 is arbitrary. Try choosing a different point to verify that you obtain the same distance.

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10.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1 and 2, fill in the blank.

1. The _____ vector for a line L is parallel to line L.
2. A vector that is perpendicular to a plane is said to be _____ to the plane.
3. For two distinct planes in three-space with normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$, $\mathbf{n}_1 \cdot \mathbf{n}_2 = 0$. Are the planes parallel, perpendicular, or neither?
4. Write the component form of a normal vector to the plane $ax + by + cz + d = 0$.
5. Which coordinate plane is parallel to the plane $x = 2$?
6. Which coordinate axis is parallel to the plane $y + z = 3$?

Procedures and Problem Solving

Finding Parametric and Symmetric Equations **In Exercises 7–12, find (a) a set of parametric equations and (b) if possible, a set of symmetric equations for the line through the point and parallel to the specified vector or line. (There are many correct answers.)**

	Point	*Parallel to*
✓ 7.	$(0, 0, 0)$	$\mathbf{v} = \langle 1, 2, 3 \rangle$
8.	$(3, -5, 1)$	$\mathbf{v} = \langle 3, -7, -10 \rangle$
9.	$(-4, 1, 0)$	$\mathbf{v} = \frac{1}{2}\mathbf{i} + \frac{4}{3}\mathbf{j} - \mathbf{k}$
10.	$(5, 0, 10)$	$\mathbf{v} = 4\mathbf{i} + 3\mathbf{k}$
11.	$(2, -3, 5)$	$x = 5 + 2t, y = 7 - 3t, z = -2 + t$
12.	$(1, 0, 1)$	$x = 3 + 3t, y = 5 - 2t, z = -7 + t$

Parametric and Symmetric Equations of a Line Through Two Points **In Exercises 13–20, find (a) a set of parametric equations and (b) if possible, a set of symmetric equations of the line that passes through the given points. (There are many correct answers.)**

✓ 13. $(2, 0, 2), (1, 4, -3)$
14. $(2, 3, 0), (10, 8, 12)$
15. $(-3, 8, 15), (1, -2, 16)$
16. $(2, 3, -1), (1, -5, 3)$
17. $(3, 1, 2), (-1, 1, 5)$
18. $(2, -1, 5), (2, 1, -3)$
19. $\left(-\frac{1}{2}, 2, \frac{1}{2}\right), \left(1, -\frac{1}{2}, 0\right)$
20. $\left(-\frac{3}{2}, \frac{3}{2}, 2\right), (3, -5, -4)$

Using Parametric Equations **In Exercises 21 and 22, sketch a graph of the line.**

21. $x = 2t, y = 2 + t, z = 1 + \frac{1}{2}t$
22. $x = 5 - 2t, y = 1 + t, z = 5 - \frac{1}{2}t$

Finding an Equation of a Plane in Three-Space **In Exercises 23–28, find the general form of the equation of the plane passing through the point and perpendicular to the specified vector or line.**

	Point	*Perpendicular to*
23.	$(2, 1, 2)$	$\mathbf{n} = \mathbf{i}$
24.	$(1, 0, -3)$	$\mathbf{n} = \mathbf{k}$
25.	$(5, 6, 3)$	$\mathbf{n} = -2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$
26.	$(0, 0, 0)$	$\mathbf{n} = -3\mathbf{j} + 5\mathbf{k}$
27.	$(2, 0, 0)$	$x = 3 - t, y = 2 - 2t, z = 4 + t$
28.	$(0, 0, 6)$	$x = 1 - t, y = 2 + t, z = 4 - 2t$

Finding an Equation of a Plane in Three-Space **In Exercises 29–32, find the general form of the equation of the plane passing through the three points.**

29. $(0, 0, 0), (1, 2, 3), (-2, 3, 3)$
30. $(4, -1, 3), (2, 5, 1), (-1, 2, 1)$
✓ 31. $(2, 3, -2), (3, 4, 2), (1, -1, 0)$
32. $(5, -1, 4), (1, -1, 2), (2, 1, -3)$

Finding an Equation of a Plane in Three-Space **In Exercises 33–38, find the general form of the equation of the plane with the given characteristics.**

33. Passes through $(2, 5, 3)$ and is parallel to the xz-plane
34. Passes through $(1, 2, 3)$ and is parallel to the yz-plane
35. Passes through $(0, 2, 4)$ and $(-1, -2, 0)$ and is perpendicular to the yz-plane
36. Passes through $(1, -2, 4)$ and $(4, 0, -1)$ and is perpendicular to the xz-plane
37. Passes through $(2, 2, 1)$ and $(-1, 1, -1)$ and is perpendicular to $2x - 3y + z = 3$
38. Passes through $(1, 2, 0)$ and $(-1, -1, 2)$ and is perpendicular to $2x - 3y + z = 6$

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Finding Parametric Equations of a Line **In Exercises 39–46, find a set of parametric equations of the line. (There are many correct answers.)**

39. Passes through $(2, 3, 4)$ and is parallel to the xz-plane and the yz-plane

40. Passes through $(-4, 5, 2)$ and is parallel to the xy-plane and the yz-plane

41. Passes through $(2, 3, 4)$ and is perpendicular to the plane given by $3x + 2y - z = 6$

42. Passes through $(-4, 5, 2)$ and is perpendicular to the plane given by $-x + 2y + z = 5$

43. Passes through $(5, -3, -4)$ and is parallel to $\mathbf{v} = \langle 2, -1, 3 \rangle$

44. Passes through $(-1, 4, -3)$ and is parallel to $\mathbf{v} = 5\mathbf{i} - \mathbf{j}$

45. Passes through $(2, 1, 2)$ and is parallel to $x = -t$, $y = 1 + t$, $z = -2 + t$

46. Passes through $(-6, 0, 8)$ and is parallel to $x = 5 - 2t$, $y = -4 + 2t$, $z = 0$

Parallel or Orthogonal Planes **In Exercises 47–50, determine whether the planes are parallel, orthogonal, or neither. If they are neither parallel nor orthogonal, find the angle of intersection.**

47. $5x - 3y + z = 4$
$x + 4y + 7z = 1$

48. $3x + y - 4z = 3$
$-9x - 3y + 12z = 4$

49. $2x - z = 1$
$4x + y + 8z = 10$

50. $x - 5y - z = 1$
$5x - 25y - 5z = -3$

Finding the Line of Intersection of Two Planes **In Exercises 51–54, (a) find the angle between the two planes and (b) find parametric equations of their line of intersection.**

✓ **51.** $3x - 4y + 5z = 6$
$x + y - z = 2$

52. $x - 3y + z = -2$
$2x + 5z + 3 = 0$

53. $x + y - z = 0$
$2x - 5y - z = 1$

54. $2x + 4y - 2z = 1$
$-3x - 6y + 3z = 10$

Sketching a Plane in Space **In Exercises 55–60, plot the intercepts and sketch a graph of the plane.**

55. $x + 2y + 3z = 6$

56. $2x - y + 4z = 4$

57. $x + 2y = 4$

58. $y + z = 5$

59. $3x + 2y - z = 6$

60. $x - 3z = 6$

Finding the Distance Between a Point and a Plane **In Exercises 61–64, find the distance between the point and the plane.**

61. $(0, 0, 0)$
$8x - 4y + z = 8$

62. $(3, 2, 1)$
$x - y + 2z = 4$

✓ **63.** $(4, -2, -2)$
$2x - y + z = 4$

64. $(-1, 2, 5)$
$2x + 3y + z = 12$

65. **Mechanical Engineering** A company that produces after-market machine parts is designing a reproduction of a fuel tank with the shape and dimensions shown in the figure. Find the angle between two adjacent sides of the tank.

66. ***Why you should learn it*** *(p. 733)* A bread pan is tapered so that a loaf of bread can be easily removed. The shape and dimensions of the pan are shown in the figure. Find the angle between two adjacent sides of the pan.

Conclusions

True or False? **In Exercises 67 and 68, determine whether the statement is true or false. Justify your answer.**

67. Two lines in space are either parallel or they intersect.

68. Two nonparallel planes in space will always intersect.

69. **Error Analysis** Find the error in writing symmetric equations of the line that passes through $(1, 2, 6)$ and is parallel to $\mathbf{v} = \langle 3, 5, 4 \rangle$.

$$\cancel{\frac{x-3}{1} = \frac{y-5}{2} = \frac{z-4}{6}}$$

70. **CAPSTONE**

(a) Explain why you use a vector that is *parallel* to a line L to find parametric equations of L, whereas you use a vector that is *normal* to a plane P to find an equation of P.

(b) Explain when you use each formula. Why does the formula in part (ii) involve absolute value?

(i) $\cos\theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}$ (ii) $\cos\theta = \dfrac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|}$

71. **Think About It** The direction numbers of two distinct lines in space are $10, -18, 20$, and $-15, 27, -30$. What is the relationship between the lines? Explain.

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10 Chapter Summary

	What did you learn?	Explanation and Examples	Review Exercises
10.1	**Plot points in the three-dimensional coordinate system *(p. 712)*.**		1–4
	Find distances between points in space and find midpoints of line segments joining points in space *(p. 713)*.	The distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) given by the Distance Formula in Space is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$. The midpoint of the line segment joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) given by the Midpoint Formula in Space is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$.	5–12
	Write equations of spheres in standard form and find traces of surfaces in space *(p. 714)*.	**Standard Equation of a Sphere** The standard equation of a sphere with center (h, k, j) and radius r is given by $(x - h)^2 + (y - k)^2 + (z - j)^2 = r^2$.	13–20
10.2	**Find the component forms of, the unit vectors in the same direction of, the magnitudes of, the dot products of, and the angles between vectors in space *(p. 719)*.**	**Vectors in Space** **1.** Two vectors are equal if and only if their corresponding components are equal. **2.** Magnitude of $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$: $\lVert \mathbf{u} \rVert = \sqrt{u_1^2 + u_2^2 + u_3^2}$ **3.** A unit vector $\mathbf{u}$ in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{\mathbf{v}}{\lVert \mathbf{v} \rVert}$, $\mathbf{v} \neq \mathbf{0}$. **4.** The sum of $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$. **5.** The scalar multiple of the real number c and $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ is $c\mathbf{u} = \langle cu_1, cu_2, cu_3 \rangle$. **6.** The dot product of $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$. **Angle Between Two Vectors** If θ is the angle between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, then $\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\lVert \mathbf{u} \rVert \lVert \mathbf{v} \rVert}$.	21–32
	Determine whether vectors in space are parallel or orthogonal *(p. 721)*.	Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel when there is some scalar c such that $\mathbf{u} = c\mathbf{v}$.	33–40
	Use vectors in space to solve real-life problems *(p. 723)*.	Vectors can be used to solve equilibrium problems in space. (See Example 7.)	41, 42

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	What did you learn?	Explanation and Examples	Review Exercises
10.3	**Find cross products of vectors in space *(p. 726)*.**	**Definition of Cross Product of Two Vectors in Space** Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ be vectors in space. The cross product of $\mathbf{u}$ and $\mathbf{v}$, $\mathbf{u} \times \mathbf{v}$, is the vector $(u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.$	43, 44
	Use geometric properties of cross products of vectors in space *(p. 728)*.	**Geometric Properties of the Cross Product** Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in space, and let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. **1.** $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. **2.** $\lVert\mathbf{u} \times \mathbf{v}\rVert = \lVert\mathbf{u}\rVert\,\lVert\mathbf{v}\rVert \sin\theta$ **3.** $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are scalar multiples of each other. **4.** $\lVert\mathbf{u} \times \mathbf{v}\rVert =$ area of parallelogram having $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides.	45–48
	Use triple scalar products to find volumes of parallelepipeds *(p. 730)*.	**The Triple Scalar Product** For $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, and $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$, the triple scalar product is given by $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$ **Geometric Property of Triple Scalar Product** The volume V of a parallelepiped with vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ as adjacent edges is given by $V = \lvert\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})\rvert.$	49, 50
10.4	**Find parametric and symmetric equations of lines in space *(p. 733)*.**	**Parametric Equations of a Line in Space** A line L parallel to the nonzero vector $\mathbf{v} = \langle a, b, c\rangle$ and passing through the point $P(x_1, y_1, z_1)$ is represented by the parametric equations $x = x_1 + at, \quad y = y_1 + bt, \quad$ and $\quad z = z_1 + ct.$	51–56
	Find equations of planes in space *(p. 735)*.	**Standard Equation of a Plane in Space** The plane containing the point (x_1, y_1, z_1) and having nonzero normal vector $\mathbf{n} = \langle a, b, c\rangle$ can be represented by the standard form of the equation of a plane $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$	57–60
	Sketch planes in space *(p. 738)*.	See Figure 10.29, which shows how to sketch the plane $3x + 2y + 4z = 12$.	61–64
	Find distances between points and planes in space *(p. 739)*.	**Distance Between a Point and a Plane** The distance between a plane and a point Q (not in the plane) is $D = \lVert\text{proj}_{\mathbf{n}}\overrightarrow{PQ}\rVert = \dfrac{\lvert\overrightarrow{PQ} \cdot \mathbf{n}\rvert}{\lVert\mathbf{n}\rVert}$ where P is a point in the plane and $\mathbf{n}$ is normal to the plane.	65–68

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10 Review Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

10.1

Plotting Points in Space **In Exercises 1 and 2, plot both points in the same three-dimensional coordinate system.**

1. (a) $(5, -1, 3)$ (b) $(-3, 3, 0)$

2. (a) $(2, 4, -2)$ (b) $(0, 0, 5)$

Finding the Coordinates of a Point **In Exercises 3 and 4, find the coordinates of the point.**

3. The point is located in the xy-plane, three units to the right of the xz-plane, and five units behind the yz-plane.

4. The point is located on the y-axis and six units to the left of the xz-plane.

Finding the Distance Between Two Points in Space **In Exercises 5 and 6, find the distance between the points.**

5. $(4, 0, 6), (5, 2, 1)$

6. $(2, 3, -4), (-1, -2, 0)$

Using the Pythagorean Theorem **In Exercises 7 and 8, find the lengths of the sides of the right triangle. Show that these lengths satisfy the Pythagorean Theorem.**

7.

8.

Using the Midpoint Formula in Space **In Exercises 9–12, find the midpoint of the line segment joining the points.**

9. $(-2, 3, 2), (2, -5, -4)$

10. $(7, 1, -4), (3, -1, 2)$

11. $(10, 6, -12), (-8, -4, -6)$

12. $(-5, -3, 1), (-9, -9, -5)$

Finding the Equation of a Sphere **In Exercises 13–16, find the standard form of the equation of the sphere with the given characteristics.**

13. Center: $(2, 3, 5)$; radius: 1

14. Center: $(3, -2, 4)$; radius: 4

15. Center: $(1, 5, 2)$; diameter: 12

16. Center: $(0, 4, -1)$; diameter: 15

Finding the Center and Radius of a Sphere **In Exercises 17 and 18, find the center and radius of the sphere.**

17. $x^2 + y^2 + z^2 - 4x - 6y + 4 = 0$

18. $x^2 + y^2 + z^2 - 10x + 6y - 4z + 34 = 0$

Finding a Trace of a Surface **In Exercises 19 and 20, sketch the graph of the equation and sketch the specified trace.**

19. $x^2 + (y - 3)^2 + z^2 = 16$

(a) xz-trace (b) yz-trace

20. $(x + 2)^2 + (y - 1)^2 + z^2 = 9$

(a) xy-trace (b) yz-trace

10.2

Finding the Component Form of a Vector **In Exercises 21–24, (a) write the component form of the vector v, (b) find the magnitude of v, and (c) find a unit vector in the direction of v.**

	Initial point	*Terminal point*
21.	$(2, -1, 3)$	$(3, 3, 0)$
22.	$(2, -1, 2)$	$(-3, 2, 3)$
23.	$(7, -4, 3)$	$(-3, 2, 10)$
24.	$(0, 3, -1)$	$(5, -6, 6)$

Finding the Dot Product of Two Vectors **In Exercises 25–28, find the dot product of u and v.**

25. $\mathbf{u} = \langle -1, 4, 3\rangle$
$\mathbf{v} = \langle 0, -6, 5\rangle$

26. $\mathbf{u} = \langle 8, -4, 2\rangle$
$\mathbf{v} = \langle 2, 5, 2\rangle$

27. $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$
$\mathbf{v} = \mathbf{i} - \mathbf{k}$

28. $\mathbf{u} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$
$\mathbf{v} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$

Finding the Angle Between Two Vectors **In Exercises 29–32, find the angle θ between the vectors.**

29. $\mathbf{u} = \langle 2, -1, 0\rangle$
$\mathbf{v} = \langle 1, 2, 1\rangle$

30. $\mathbf{u} = \langle 10, -5, 15\rangle$
$\mathbf{v} = \langle -2, 1, -3\rangle$

31. $\mathbf{u} = \langle 2\sqrt{2}, -4, 4\rangle$
$\mathbf{v} = \langle -\sqrt{2}, 1, 2\rangle$

32. $\mathbf{u} = \langle 3, 1, -1\rangle$
$\mathbf{v} = \langle 4, 5, 2\rangle$

The Relationship of Two Vectors **In Exercises 33–36, determine whether u and v are orthogonal, parallel, or neither.**

33. $\mathbf{u} = \langle 7, -2, 3\rangle$
$\mathbf{v} = \langle -1, 4, 5\rangle$

34. $\mathbf{u} = \langle -4, 3, -6\rangle$
$\mathbf{v} = \langle 16, -12, 24\rangle$

35. $\mathbf{u} = 6\mathbf{i} + 5\mathbf{j} + 9\mathbf{k}$
$\mathbf{v} = 5\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$

36. $\mathbf{u} = 3\mathbf{j} + 2\mathbf{k}$
$\mathbf{v} = 12\mathbf{i} - 18\mathbf{k}$

Using Vectors to Determine Collinear Points **In Exercises 37–40, use vectors to determine whether the points are collinear.**

37. $(5, 2, 0), (2, 6, 1), (2, 4, 7)$

38. $(6, 3, -1), (5, 8, 3), (7, -2, -5)$

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39. $(3, 4, -1), (-1, 6, 9), (5, 3, -6)$

40. $(5, -4, 7), (8, -5, 5), (11, 6, 3)$

41. **Forces in Equilibrium** A load of 300 pounds is supported by three cables, as shown in the figure. Find the tension in each of the supporting cables.

42. **Forces in Equilibrium** Determine the tension in each of the supporting cables in Exercise 41 when the load is 200 pounds.

10.3

Finding the Cross Product **In Exercises 43 and 44, find $\mathbf{u} \times \mathbf{v}$.**

43. $\mathbf{u} = \langle -2, 8, 2 \rangle$
$\mathbf{v} = \langle 1, 1, -1 \rangle$

44. $\mathbf{u} = \langle 10, 15, 5 \rangle$
$\mathbf{v} = \langle 5, -3, 0 \rangle$

Using the Cross Product **In Exercises 45 and 46, find a unit vector orthogonal to u and v.**

45. $\mathbf{u} = -3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$
$\mathbf{v} = 10\mathbf{i} - 15\mathbf{j} + 2\mathbf{k}$

46. $\mathbf{u} = 4\mathbf{k}$
$\mathbf{v} = \mathbf{i} + 12\mathbf{k}$

Geometric Application of the Cross Product **In Exercises 47 and 48, (a) verify that the points are the vertices of a parallelogram, (b) find its area, and (c) determine whether the parallelogram is a rectangle.**

47. $A(2, -1, 1), B(5, 1, 4), C(0, 1, 1), D(3, 3, 4)$

48. $A(0, 4, 0), B(1, 4, 1), C(0, 6, 0), D(1, 6, 1)$

Finding the Volume of a Parallelepiped **In Exercises 49 and 50, find the volume of the parallelepiped with the given vertices.**

49. $A(0, 0, 0), B(3, 0, 0), C(0, 5, 1), D(3, 5, 1),$
$E(2, 0, 5), F(5, 0, 5), G(2, 5, 6), H(5, 5, 6)$

50. $A(0, 0, 0), B(2, 0, 0), C(2, 4, 0), D(0, 4, 0),$
$E(0, 0, 6), F(2, 0, 6), G(2, 4, 6), H(0, 4, 6)$

10.4

Parametric and Symmetric Equations of a Line Through Two Points **In Exercises 51 and 52, find a set of (a) parametric equations and (b) symmetric equations for the line that passes through the given points. (There are many correct answers.)**

51. $(3, 0, 2), (9, 11, 6)$

52. $(-1, 4, 3), (8, 10, 5)$

Finding Parametric and Symmetric Equations **In Exercises 53–56, find a set of (a) parametric equations and (b) symmetric equations for the specified line. (There are many correct answers.)**

53. Passes through $(-1, 3, 5)$ and $(3, 6, -1)$

54. Passes through $(0, -10, 3)$ and $(5, 10, 0)$

55. Passes through $(0, 0, 0)$ and is parallel to $\mathbf{v} = \langle -2, \frac{5}{2}, 1 \rangle$

56. Passes through $(3, 2, 1)$ and is parallel to $x = y = z$

Finding the General Form of the Equation of a Plane **In Exercises 57–60, find the general form of the equation of the plane with the given characteristics.**

57. Passes through $(0, 0, 0), (5, 0, 2)$, and $(2, 3, 8)$

58. Passes through $(-1, 3, 4), (4, -2, 2)$, and $(2, 8, 6)$

59. Passes through $(5, 3, 2)$ and is parallel to the xy-plane

60. Passes through $(0, 0, 6)$ and is perpendicular to $x = 1 - t, y = 2 + t$, and $z = 4 - 2t$

Sketching the Graph of a Plane **In Exercises 61–64, plot the intercepts and sketch a graph of the plane.**

61. $3x - 2y + 3z = 6$

62. $5x - y - 5z = 5$

63. $2x - 3z = 6$

64. $4y - 3z = 12$

Finding the Distance Between a Point and a Plane **In Exercises 65–68, find the distance between the point and the plane.**

65. $(2, 3, 10)$
$2x - 20y + 6z = 6$

66. $(1, 2, 3)$
$2x - y + z = 4$

67. $(0, 0, 0)$
$x - 10y + 3z = 2$

68. $(0, 0, 0)$
$2x + 3y + z = 12$

Conclusions

True or False? **In Exercises 69 and 70, determine whether the statement is true or false. Justify your answer.**

69. The cross product is commutative.

70. The triple scalar product of three vectors in space is a scalar.

Verifying Properties **In Exercises 71–74, let $\mathbf{u} = \langle 3, -2, 1 \rangle$, $\mathbf{v} = \langle 2, -4, -3 \rangle$, and $\mathbf{w} = \langle -1, 2, 2 \rangle$.**

71. Show that $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$.

72. Show that $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$.

73. Show that $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$.

74. Show that $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$.

75. **Writing** Define the cross product of vectors $\mathbf{u}$ and $\mathbf{v}$.

76. **Writing** The vertices of a triangle in space are (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3). Explain how to find a vector perpendicular to the triangle.

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10 Chapter Test

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Take this test as you would take a test in class. After you are finished, check your work against the answers in the back of the book.

1. Plot each point in the same three-dimensional coordinate system.
(a) $(5, -2, 3)$ (b) $(-2, -2, 3)$ (c) $(-1, 4, 1)$

In Exercises 2–4, use the points $A(8, -2, 5)$, $B(6, 4, -1)$, and $C(-4, 3, 0)$ to solve the problem.

2. Consider the triangle with vertices A, B, and C. Is it a right triangle? Explain.

3. Find the coordinates of the midpoint of the line segment joining points A and B.

4. Find the standard form of the equation of the sphere for which A and B are the endpoints of a diameter. Sketch the sphere and its xz-trace.

In Exercises 5 and 6, find the component form and the magnitude of the vector v.

5. Initial point: $(2, -1, 3)$
Terminal point: $(4, 4, -7)$

6. Initial point: $(6, 2, 0)$
Terminal point: $(3, -3, 8)$

In Exercises 7–10, let u and v be the vectors from $A(8, -2, 5)$ to $B(6, 4, -1)$ and from A to $C(-4, 3, 0)$, respectively.

7. Write **u** and **v** in component form.

8. Find (a) $\|\mathbf{v}\|$, (b) $\mathbf{u} \cdot \mathbf{v}$, and (c) $\mathbf{u} \times \mathbf{v}$.

9. Find the angle between **u** and **v**.

10. Find a set of (a) parametric equations and (b) symmetric equations for the line through points A and B.

In Exercises 11 and 12, determine whether u and v are orthogonal, parallel, or neither.

11. $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - \mathbf{k}$
$\mathbf{v} = \mathbf{j} + 6\mathbf{k}$

12. $\mathbf{u} = -3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$
$\mathbf{v} = \mathbf{i} + \mathbf{j} - \mathbf{k}$

13. Verify that the points $A(2, -3, 1)$, $B(6, 5, -1)$, $C(3, -6, 4)$, and $D(7, 2, 2)$ are the vertices of a parallelogram, and find its area.

14. Find the general form of the equation of the plane passing through the points $(-3, -4, 2)$, $(-3, 4, 1)$, and $(1, 1, -2)$.

15. Find the volume of the parallelepiped at the right with the given vertices.
$A(0, 0, 5)$, $B(0, 10, 5)$, $C(4, 10, 5)$, $D(4, 0, 5)$,
$E(0, 1, 0)$, $F(0, 11, 0)$, $G(4, 11, 0)$, $H(4, 1, 0)$

Figure for 15

In Exercises 16 and 17, label the intercepts and sketch a graph of the plane.

16. $3x + 6y + 2z = 18$

17. $5x - y - 2z = 10$

18. Find the distance between the point $(2, -1, 6)$ and the plane $3x - 2y + z = 6$.

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Proofs in Mathematics

Algebraic Properties of the Cross Product (p. 727)

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in space and let c be a scalar.

1. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$
3. $c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$
4. $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$
5. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$
6. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$

Proof

Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$, $\mathbf{0} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$, and let c be a scalar.

1. $\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$

$\mathbf{v} \times \mathbf{u} = (v_2u_3 - v_3u_2)\mathbf{i} - (v_1u_3 - v_3u_1)\mathbf{j} + (v_1u_2 - v_2u_1)\mathbf{k}$

So, this implies $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$.

2.
$$\begin{aligned}\mathbf{u} \times (\mathbf{v} + \mathbf{w}) &= [u_2(v_3 + w_3) - u_3(v_2 + w_2)]\mathbf{i} - [u_1(v_3 + w_3) - \\ &\qquad u_3(v_1 + w_1)]\mathbf{j} + [u_1(v_2 + w_2) - u_2(v_1 + w_1)]\mathbf{k} \\ &= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k} + \\ &\qquad (u_2w_3 - u_3w_2)\mathbf{i} - (u_1w_3 - u_3w_1)\mathbf{j} + (u_1w_2 - u_2w_1)\mathbf{k} \\ &= (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})\end{aligned}$$

3.
$$\begin{aligned}(c\mathbf{u}) \times \mathbf{v} &= (cu_2v_3 - cu_3v_2)\mathbf{i} - (cu_1v_3 - cu_3v_1)\mathbf{j} + (cu_1v_2 - cu_2v_1)\mathbf{k} \\ &= c[(u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}] \\ &= c(\mathbf{u} \times \mathbf{v})\end{aligned}$$

4.
$$\begin{aligned}\mathbf{u} \times \mathbf{0} &= (u_2 \cdot 0 - u_3 \cdot 0)\mathbf{i} - (u_1 \cdot 0 - u_3 \cdot 0)\mathbf{j} + (u_1 \cdot 0 - u_2 \cdot 0)\mathbf{k} \\ &= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}\end{aligned}$$
$$\begin{aligned}\mathbf{0} \times \mathbf{u} &= (0 \cdot u_3 - 0 \cdot u_2)\mathbf{i} - (0 \cdot u_3 - 0 \cdot u_1)\mathbf{j} + (0 \cdot u_2 - 0 \cdot u_1)\mathbf{k} \\ &= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}\end{aligned}$$

So, this implies $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$.

5. $\mathbf{u} \times \mathbf{u} = (u_2u_3 - u_3u_2)\mathbf{i} - (u_1u_3 - u_3u_1)\mathbf{j} + (u_1u_2 - u_2u_1)\mathbf{k} = \mathbf{0}$

6. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$ and $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

$$\begin{aligned}\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) &= u_1(v_2w_3 - w_2v_3) - u_2(v_1w_3 - w_1v_3) + u_3(v_1w_2 - w_1v_2) \\ &= u_1v_2w_3 - u_1w_2v_3 - u_2v_1w_3 + u_2w_1v_3 + u_3v_1w_2 - u_3w_1v_2 \\ &= u_2w_1v_3 - u_3w_1v_2 - u_1w_2v_3 + u_3v_1w_2 + u_1v_2w_3 - u_2v_1w_3 \\ &= w_1(u_2v_3 - v_2u_3) - w_2(u_1v_3 - v_1u_3) + w_3(u_1v_2 - v_1u_2) \\ &= (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}\end{aligned}$$

Notation for Dot and Cross Products

The notation for the dot products and cross products of vectors was first introduced by the American physicist Josiah Willard Gibbs (1839–1903). In the early 1880s, Gibbs built a system to represent physical quantities called *vector analysis*. The system was a departure from William Hamilton's theory of quaternions.

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Geometric Properties of the Cross Product (p. 728)

Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in space, and let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$.

1. $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.
2. $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\| \sin\theta$
3. $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are scalar multiples of each other.
4. $\|\mathbf{u} \times \mathbf{v}\|$ = area of parallelogram having $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides.

Proof

Let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, and $\mathbf{0} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$.

1. $$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

$$\begin{aligned}(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} &= (u_2v_3 - u_3v_2)u_1 - (u_1v_3 - u_3v_1)u_2 + (u_1v_2 - u_2v_1)u_3 \\ &= u_1u_2v_3 - u_1u_3v_2 - u_1u_2v_3 + u_2u_3v_1 + u_1u_3v_2 - u_2u_3v_1 \\ &= 0\end{aligned}$$

$$\begin{aligned}(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} &= (u_2v_3 - u_3v_2)v_1 - (u_1v_3 - u_3v_1)v_2 + (u_1v_2 - u_2v_1)v_3 \\ &= u_2v_1v_3 - u_3v_1v_2 - u_1v_2v_3 + u_3v_1v_2 + u_1v_2v_3 - u_2v_1v_3 \\ &= 0\end{aligned}$$

Because two vectors are orthogonal when their dot product is zero, it follows that $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

2. Note that $\cos\theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}$. Therefore,

$$\begin{aligned}\|\mathbf{u}\|\,\|\mathbf{v}\| \sin\theta &= \|\mathbf{u}\|\,\|\mathbf{v}\| \sqrt{1 - \cos^2\theta} \\ &= \|\mathbf{u}\|\,\|\mathbf{v}\| \sqrt{1 - \frac{(\mathbf{u} \cdot \mathbf{v})^2}{\|\mathbf{u}\|^2\,\|\mathbf{v}\|^2}} \\ &= \sqrt{\|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2} \\ &= \sqrt{(u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2} \\ &= \sqrt{(u_2v_3 - u_3v_2)^2 + (u_1v_3 - u_3v_1)^2 + (u_1v_2 - u_2v_1)^2} \\ &= \|\mathbf{u} \times \mathbf{v}\|.\end{aligned}$$

3. If $\mathbf{u}$ and $\mathbf{v}$ are scalar multiples of each other, then $\mathbf{u} = c\mathbf{v}$ for some scalar c.

$$\mathbf{u} \times \mathbf{v} = (c\mathbf{v}) \times \mathbf{v} = c(\mathbf{v} \times \mathbf{v}) = c(\mathbf{0}) = \mathbf{0}$$

If $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, then $\|\mathbf{u}\|\,\|\mathbf{v}\| \sin\theta = 0$. (Assume $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{v} \neq \mathbf{0}$.) So, $\sin\theta = 0$, and $\theta = 0$ or $\theta = \pi$. In either case, because θ is the angle between the vectors, $\mathbf{u}$ and $\mathbf{v}$ are parallel. Therefore, $\mathbf{u} = c\mathbf{v}$ for some scalar c.

4. The figure at the right is a parallelogram having $\mathbf{v}$ and $\mathbf{u}$ as adjacent sides. Because the height of the parallelogram is $\|\mathbf{v}\| \sin\theta$, the area is

$$\begin{aligned}\text{Area} &= (\text{base})(\text{height}) \\ &= \|\mathbf{u}\|\,\|\mathbf{v}\| \sin\theta \\ &= \|\mathbf{u} \times \mathbf{v}\|.\end{aligned}$$

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11 Limits and an Introduction to Calculus

Section 11.4, Example 3
Average Cost

11.1 Introduction to Limits
11.2 Techniques for Evaluating Limits
11.3 The Tangent Line Problem
11.4 Limits at Infinity and Limits of Sequences
11.5 The Area Problem

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11.1 Introduction to Limits

The Limit Concept

The notion of a limit is a *fundamental* concept of calculus. In this chapter, you will learn how to evaluate limits and how they are used in the two basic problems of calculus: the tangent line problem and the area problem.

Example 1 Finding a Rectangle of Maximum Area

You are given 24 inches of wire and are asked to form a rectangle whose area is as large as possible. What dimensions should the rectangle have?

Solution

Let w represent the width of the rectangle and let l represent the length of the rectangle. Because

$$2w + 2l = 24 \qquad \text{Perimeter is 24.}$$

it follows that

$$l = 12 - w$$

as shown in Figure 11.1. So, the area of the rectangle is

$$\begin{aligned} A &= lw && \text{Formula for area} \\ &= (12 - w)w && \text{Substitute } 12 - w \text{ for } l. \\ &= 12w - w^2. && \text{Simplify.} \end{aligned}$$

Figure 11.1

Using this model for area, you can experiment with different values of w to see how to obtain the maximum area. After trying several values, it appears that the maximum area occurs when

$$w = 6$$

as shown in the table.

Width, w	5.0	5.5	5.9	6.0	6.1	6.5	7.0
Area, A	35.00	35.75	35.99	36.00	35.99	35.75	35.00

In limit terminology, you can say that "the limit of A as w approaches 6 is 36." This is written as

$$\lim_{w \to 6} A = \lim_{w \to 6} (12w - w^2) = 36.$$

CHECKPOINT Now try Exercise 5.

What you should learn

- Understand the limit concept.
- Use the definition of a limit to estimate limits.
- Determine whether limits of functions exist.
- Use properties of limits and direct substitution to evaluate limits.

Why you should learn it

The concept of a limit is useful in applications involving maximization. For instance, in Exercise 5 on page 757, the concept of a limit is used to verify the maximum volume of an open box.

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Definition of Limit

Definition of Limit

If $f(x)$ becomes arbitrarily close to a unique number L as x approaches c from either side, then the **limit** of $f(x)$ as x approaches c is L. This is written as

$$\lim_{x \to c} f(x) = L.$$

Example 2 Estimating a Limit Numerically

Use a table to estimate the limit numerically.

$$\lim_{x \to 2} (3x - 2)$$

Solution

Let $f(x) = 3x - 2$. Then construct a table that shows values of $f(x)$ for two sets of x-values—one set that approaches 2 from the left and one that approaches 2 from the right.

x	1.9	1.99	1.999	2.0	2.001	2.01	2.1
$f(x)$	3.700	3.970	3.997	?	4.003	4.030	4.300

Figure 11.2

From the table, it appears that the closer x gets to 2, the closer $f(x)$ gets to 4. So, you can estimate the limit to be 4. Figure 11.2 adds further support to this conclusion.

CHECKPOINT Now try Exercise 7.

In Figure 11.2, note that the graph of

$$f(x) = 3x - 2$$

is continuous. For graphs that are not continuous, finding a limit can be more difficult.

Example 3 Estimating a Limit Numerically

Use a table to estimate the limit numerically.

$$\lim_{x \to 0} \frac{x}{\sqrt{x + 1} - 1}$$

Solution

Let $f(x) = x/\left(\sqrt{x + 1} - 1\right)$. Then construct a table that shows values of $f(x)$ for two sets of x-values—one set that approaches 0 from the left and one that approaches 0 from the right.

x	-0.01	-0.001	-0.0001	0	0.0001	0.001	0.01
$f(x)$	1.99499	1.99949	1.99995	?	2.00005	2.00050	2.00499

Figure 11.3

From the table, it appears that the limit is 2. This limit is reinforced by the graph of f (see Figure 11.3).

CHECKPOINT Now try Exercise 9.

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In Example 3, note that $f(x)$ has a limit as $x \to 0$ even though the function is not defined at $x = 0$. This often happens, and it is important to realize that *the existence or nonexistence of $f(x)$ at $x = c$ has no bearing on the existence of the limit of $f(x)$ as x approaches c.*

Example 4 Using a Graphing Utility to Estimate a Limit

Estimate the limit.

$$\lim_{x \to 1} \frac{x^3 - x^2 + x - 1}{x - 1}$$

Numerical Solution

Let $f(x) = (x^3 - x^2 + x - 1)/(x - 1)$.

Create a table that shows values of $f(x)$ for several x-values near 1.

Figure 11.4

From Figure 11.4, it appears that the closer x gets to 1, the closer $f(x)$ gets to 2. So, you can estimate the limit to be 2.

CHECKPOINT Now try Exercise 15.

Graphical Solution

Use a graphing utility to graph

$$f(x) = (x^3 - x^2 + x - 1)/(x - 1)$$

using a *decimal* setting, as shown in Figure 11.5.

Use the *trace* feature to determine that as x gets closer and closer to 1, $f(x)$ gets closer and closer to 2 from the left and from the right.

Figure 11.5

From Figure 11.5, you can estimate the limit to be 2. As you use the *trace* feature, notice that there is no value given for y when $x = 1$, and that there is a hole or break in the graph at $x = 1$.

Example 5 Using a Graph to Find a Limit

Find the limit of $f(x)$ as x approaches 3, where f is defined as

$$f(x) = \begin{cases} 2, & x \neq 3 \\ 0, & x = 3 \end{cases}.$$

Solution

Because $f(x) = 2$ for all x other than $x = 3$ and because the value of $f(3)$ is immaterial, it follows that the limit is 2 (see Figure 11.6). So, you can write

$$\lim_{x \to 3} f(x) = 2.$$

The fact that $f(3) = 0$ has no bearing on the existence or value of the limit as x approaches 3. For instance, if the function were defined as

$$f(x) = \begin{cases} 2, & x \neq 3 \\ 4, & x = 3 \end{cases}$$

then the limit as x approaches 3 would be the same.

Figure 11.6

CHECKPOINT Now try Exercise 29.

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Limits That Fail to Exist

In the next three examples, you will examine some limits that fail to exist.

Example 6 Comparing Left and Right Behavior

Show that the limit does not exist.

$$\lim_{x \to 0} \frac{|x|}{x}$$

Solution

Consider the graph of the function given by $f(x) = |x|/x$. In Figure 11.7, you can see that for positive x-values

$$\frac{|x|}{x} = 1, \quad x > 0$$

and for negative x-values

$$\frac{|x|}{x} = -1, \quad x < 0.$$

This means that no matter how close x gets to 0, there will be both positive and negative x-values that yield

$$f(x) = 1$$

and

$$f(x) = -1.$$

This implies that the limit does not exist.

Figure 11.7

CHECKPOINT Now try Exercise 35.

What's Wrong?

You use a graphing utility to graph

$$y_1 = \frac{x^3 - 1}{x - 1}$$

using a *decimal* setting, as shown in the figure. You use the *trace* feature to conclude that the limit

$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$$

does not exist. What's wrong?

Example 7 Unbounded Behavior

Discuss the existence of the limit.

$$\lim_{x \to 0} \frac{1}{x^2}$$

Solution

Let $f(x) = 1/x^2$. In Figure 11.8, note that as x approaches 0 from either the right or the left, $f(x)$ increases without bound. This means that by choosing x close enough to 0, you can force $f(x)$ to be as large as you want. For instance, $f(x)$ will be larger than 100 when you choose x that is within $\frac{1}{10}$ of 0. That is,

$$0 < |x| < \frac{1}{10} \quad \Rightarrow \quad f(x) = \frac{1}{x^2} > 100.$$

Similarly, you can force $f(x)$ to be larger than 1,000,000, as follows.

$$0 < |x| < \frac{1}{1000} \quad \Rightarrow \quad f(x) = \frac{1}{x^2} > 1{,}000{,}000$$

Because $f(x)$ is not approaching a unique real number L as x approaches 0, you can conclude that the limit does not exist.

Figure 11.8

CHECKPOINT Now try Exercise 37.

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Example 8 Oscillating Behavior

Discuss the existence of the limit.

$$\lim_{x \to 0} \sin \frac{1}{x}$$

Solution

Let $f(x) = \sin(1/x)$. In Figure 11.9, you can see that as x approaches 0, $f(x)$ oscillates between -1 and 1.

Figure 11.9

So, the limit does not exist because no matter how close you are to 0, it is possible to choose values of x_1 and x_2 such that

$$\sin \frac{1}{x_1} = 1 \quad \text{and} \quad \sin \frac{1}{x_2} = -1$$

as indicated in the table.

x	$\frac{2}{\pi}$	$\frac{2}{3\pi}$	$\frac{2}{5\pi}$	$\frac{2}{7\pi}$	$\frac{2}{9\pi}$	$\frac{2}{11\pi}$	$x \to 0$
$\sin \frac{1}{x}$	1	-1	1	-1	1	-1	Limit does not exist.

CHECKPOINT Now try Exercise 39.

Examples 6, 7, and 8 show three of the most common types of behavior associated with the *nonexistence* of a limit.

Conditions Under Which Limits Do Not Exist

The limit of $f(x)$ as $x \to c$ does not exist under any of the following conditions.

1. $f(x)$ approaches a different number from the right side of c than it approaches from the left side of c. — Example 6
2. $f(x)$ increases or decreases without bound as x approaches c. — Example 7
3. $f(x)$ oscillates between two fixed values as x approaches c. — Example 8

Technology Tip

When using a graphing utility to investigate the behavior of a function near the x-value at which you are trying to evaluate a limit, remember that you cannot always trust the graphs that the graphing utility displays. For instance, consider the incorrect graph shown in Figure 11.10. The graphing utility can't show the correct graph because $f(x) = \sin(1/x)$ has infinitely many oscillations over any interval that contains 0.

Figure 11.10

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Properties of Limits and Direct Substitution

You have seen that sometimes the limit of $f(x)$ as $x \to c$ is simply $f(c)$. In such cases, it is said that the limit can be evaluated by **direct substitution.** That is,

$$\lim_{x \to c} f(x) = f(c). \qquad \text{Substitute } c \text{ for } x.$$

There are many "well-behaved" functions, such as polynomial functions and rational functions with nonzero denominators, that have this property. Some of the basic ones are included in the following list.

Basic Limits

Let b and c be real numbers and let n be a positive integer.

1. $\lim_{x \to c} b = b$
2. $\lim_{x \to c} x = c$
3. $\lim_{x \to c} x^n = c^n$ (See the proof on page 804.)
4. $\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$, for n even and $c > 0$

Trigonometric functions can also be included in this list. For instance,

$$\lim_{x \to \pi} \sin x = \sin \pi = 0 \quad \text{and} \quad \lim_{x \to 0} \cos x = \cos 0 = 1.$$

By combining the basic limits with the following operations, you can find limits for a wide variety of functions.

Properties of Limits

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits.

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = K$$

1. Scalar multiple: $\lim_{x \to c} [b f(x)] = bL$
2. Sum or difference: $\lim_{x \to c} [f(x) \pm g(x)] = L \pm K$
3. Product: $\lim_{x \to c} [f(x) g(x)] = LK$
4. Quotient: $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{K}$, provided $K \neq 0$
5. Power: $\lim_{x \to c} [f(x)]^n = L^n$

Explore the Concept

Use a graphing utility to graph the tangent function. What are $\lim_{x \to 0} \tan x$ and $\lim_{x \to \pi/4} \tan x$? What can you say about the existence of the limit $\lim_{x \to \pi/2} \tan x$?

Technology Tip

When evaluating limits, remember that there are several ways to solve most problems. Often, a problem can be solved *numerically*, *graphically*, or *algebraically*. You can use a graphing utility to confirm the limits in the examples and in the exercise set numerically using the *table* feature or graphically using the *zoom* and *trace* features.

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Example 9 Direct Substitution and Properties of Limits

a. $\lim_{x\to 4} x^2 = (4)^2 = 16$ — Direct Substitution

b. $\lim_{x\to 4} 5x = 5\lim_{x\to 4} x = 5(4) = 20$ — Scalar Multiple Property

c. $\lim_{x\to \pi} \frac{\tan x}{x} = \frac{\lim_{x\to\pi} \tan x}{\lim_{x\to\pi} x} = \frac{0}{\pi} = 0$ — Quotient Property

d. $\lim_{x\to 9} \sqrt{x} = \sqrt{9} = 3$ — Direct Substitution

e. $\lim_{x\to \pi} (x\cos x) = (\lim_{x\to\pi} x)(\lim_{x\to\pi} \cos x)$ — Product Property

$= \pi(\cos \pi)$

$= -\pi$

f. $\lim_{x\to 3} (x+4)^2 = \left[\left(\lim_{x\to 3} x\right) + \left(\lim_{x\to 3} 4\right)\right]^2$ — Sum and Power Properties

$= (3+4)^2$

$= 7^2 = 49$

CHECKPOINT Now try Exercise 51.

Explore the Concept

Sketch the graph of each function. Then find the limits of each function as x approaches 1 and as x approaches 2. What conclusions can you make?

a. $f(x) = x + 1$

b. $g(x) = \frac{x^2 - 1}{x - 1}$

c. $h(x) = \frac{x^3 - 2x^2 - x + 2}{x^2 - 3x + 2}$

Use a graphing utility to graph each function above. Does the graphing utility distinguish among the three graphs? Write a short explanation of your findings.

The results of using direct substitution to evaluate limits of polynomial and rational functions are summarized as follows.

Limits of Polynomial and Rational Functions

1. If p is a polynomial function and c is a real number, then

$$\lim_{x\to c} p(x) = p(c).$$ (See the proof on page 804.)

2. If r is a rational function given by $r(x) = p(x)/q(x)$, and c is a real number such that $q(c) \neq 0$, then

$$\lim_{x\to c} r(x) = r(c) = \frac{p(c)}{q(c)}.$$

Example 10 Evaluating Limits by Direct Substitution

Find each limit.

a. $\lim_{x\to -1} (x^2 + x - 6)$ **b.** $\lim_{x\to -1} \frac{x^2 + x - 6}{x + 3}$

Solution

a. To evaluate the limit of a polynomial function, use direct substitution.

$$\lim_{x\to -1} (x^2 + x - 6) = (-1)^2 + (-1) - 6 = -6$$

b. The denominator is not 0 when $x = -1$, so you can evaluate the limit of the rational function using direct substitution.

$$\lim_{x\to -1} \frac{x^2 + x - 6}{x + 3} = \frac{(-1)^2 + (-1) - 6}{-1 + 3} = -\frac{6}{2} = -3$$

CHECKPOINT Now try Exercise 55.

Explore the Concept

Use a graphing utility to graph the function

$$f(x) = \frac{x^2 - 3x - 10}{x - 5}.$$

Use the *trace* feature to approximate $\lim_{x\to 4} f(x)$. What do you think $\lim_{x\to 5} f(x)$ equals? Is f defined at $x = 5$? Does this affect the existence of the limit as x approaches 5?

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11.1 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1 and 2, fill in the blank.

1. If $f(x)$ becomes arbitrarily close to a unique number L as x approaches c from either side, then the _____ of $f(x)$ as x approaches c is L.

2. To find a limit of a polynomial function, use _____ .

3. Find the limit: $\lim_{x \to 0} 3$.

4. List three conditions under which limits do not exist.

Procedures and Problem Solving

5. *Why you should learn it* *(p. 750)* You create an open box from a square piece of material, 24 centimeters on a side. You cut equal squares from the corners and turn up the sides.

(a) Draw and label a diagram that represents the box.

(b) Verify that the volume V of the box is given by $V = 4x(12 - x)^2$.

(c) The box has a maximum volume when $x = 4$. Use a graphing utility to complete the table and observe the behavior of the function as x approaches 4. Use the table to find $\lim_{x \to 4} V$.

x	3	3.5	3.9	4	4.1	4.5	5
V							

(d) Use the graphing utility to graph the volume function. Verify that the volume is maximum when $x = 4$.

6. **Landscape Design** A landscaper arranges bricks to enclose a region shaped like a right triangle with a hypotenuse of $\sqrt{18}$ meters whose area is as large as possible.

(a) Draw and label a diagram that shows the base x and height y of the triangle.

(b) Verify that the area A of the triangle is given by

$$A = \tfrac{1}{2}x\sqrt{18 - x^2}.$$

(c) The triangle has a maximum area when $x = 3$ meters. Use a graphing utility to complete the table and observe the behavior of the function as x approaches 3. Use the table to find $\lim_{x \to 3} A$.

x	2	2.5	2.9	3	3.1	3.5	4
A							

(d) Use the graphing utility to graph the area function. Verify that the area is maximum when $x = 3$ meters.

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Estimating a Limit Numerically **In Exercises 7–12, complete the table and use the result to estimate the limit numerically. Determine whether the limit can be reached.**

7. $\lim_{x \to 2} (5x + 4)$

x	1.9	1.99	1.999	2	2.001	2.01	2.1
$f(x)$				?			

8. $\lim_{x \to 1} (2x^2 + x - 4)$

x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$				?			

9. $\lim_{x \to -1} \dfrac{x + 1}{x^2 - x - 2}$

x	-1.1	-1.01	-1.001	-1	-0.999
$f(x)$				?	

x	-0.99	-0.9
$f(x)$		

10. $\lim_{x \to 0} \dfrac{\sin 2x}{x}$

x	-0.1	-0.01	-0.001	0	0.001
$f(x)$				?	

x	0.01	0.1
$f(x)$		

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11. $\lim_{x\to 0} \dfrac{\tan x}{2x}$

x	-0.1	-0.01	-0.001	0	0.001
$f(x)$				?	

x	0.01	0.1
$f(x)$		

12. $\lim_{x\to 1} \dfrac{\ln x}{x-1}$

x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$				?			

Using a Graphing Utility to Estimate a Limit **In Exercises 13–28, use the *table* feature of a graphing utility to create a table for the function and use the result to estimate the limit numerically. Use the graphing utility to graph the corresponding function to confirm your result graphically.**

13. $\lim_{x\to -1} \dfrac{x^2-1}{x+1}$

14. $\lim_{x\to 2} \dfrac{4-x^2}{x-2}$

15. $\lim_{x\to 1} \dfrac{x-1}{x^2+2x-3}$

16. $\lim_{x\to -2} \dfrac{x+2}{x^2+5x+6}$

17. $\lim_{x\to 0} \dfrac{\sqrt{x+5}-\sqrt{5}}{x}$

18. $\lim_{x\to -3} \dfrac{\sqrt{1-x}-2}{x+3}$

19. $\lim_{x\to -4} \dfrac{\dfrac{x}{x+2}-2}{x+4}$

20. $\lim_{x\to 2} \dfrac{\dfrac{1}{x+2}-\dfrac{1}{4}}{x-2}$

21. $\lim_{x\to 0} \dfrac{\sin x}{x}$

22. $\lim_{x\to 0} \dfrac{\cos x - 1}{x}$

23. $\lim_{x\to 0} \dfrac{\sin^2 x}{x}$

24. $\lim_{x\to 0} \dfrac{2x}{\tan 4x}$

25. $\lim_{x\to 0} \dfrac{e^{2x}-1}{2x}$

26. $\lim_{x\to 0} \dfrac{1-e^{-4x}}{x}$

27. $\lim_{x\to 2} \dfrac{\ln(2x-3)}{x-2}$

28. $\lim_{x\to 1} \dfrac{\ln(x^2)}{x-1}$

Using a Graph to Find a Limit **In Exercises 29–32, graph the function and find the limit (if it exists) as x approaches 2.**

29. $f(x) = \begin{cases} 3, & x \neq 2 \\ 1, & x = 2 \end{cases}$

30. $f(x) = \begin{cases} x, & x \neq 2 \\ -4, & x = 2 \end{cases}$

31. $f(x) = \begin{cases} 2x+1, & x < 2 \\ x+3, & x \geq 2 \end{cases}$

32. $f(x) = \begin{cases} -2x, & x \leq 2 \\ x^2-4x+1, & x > 2 \end{cases}$

Using a Graph to Find a Limit **In Exercises 33–40, use the graph to find the limit (if it exists). If the limit does not exist, explain why.**

33. $\lim_{x\to 0} (2-x^2)$

34. $\lim_{x\to -2} \dfrac{x^2-4}{x+2}$

35. $\lim_{x\to -2} \dfrac{|x+2|}{x+2}$

36. $\lim_{x\to -1} \sin \dfrac{\pi x}{2}$

37. $\lim_{x\to 1} \dfrac{1}{x-1}$

38. $\lim_{x\to \pi/2} \tan x$

39. $\lim_{x\to 0} 2\cos \dfrac{\pi}{x}$

40. $\lim_{x\to \pi/2} \sec x$

Determining Whether a Limit Exists **In Exercises 41–48, use a graphing utility to graph the function and use the graph to determine whether the limit exists. If the limit does not exist, explain why.**

41. $f(x) = \dfrac{5}{2+e^{1/x}}, \quad \lim_{x\to 0} f(x)$

42. $f(x) = \dfrac{e^x-1}{x}, \quad \lim_{x\to 0} f(x)$

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43. $f(x) = \cos \dfrac{1}{x}, \quad \lim\limits_{x \to 0} f(x)$

44. $f(x) = \sin \pi x, \quad \lim\limits_{x \to -1} f(x)$

45. $f(x) = \dfrac{\sqrt{x+3} - 1}{x - 4}, \quad \lim\limits_{x \to 4} f(x)$

46. $f(x) = \dfrac{\sqrt{x+5} - 4}{x - 2}, \quad \lim\limits_{x \to 2} f(x)$

47. $f(x) = \ln(x + 3), \quad \lim\limits_{x \to 4} f(x)$

48. $f(x) = \ln(7 - x), \quad \lim\limits_{x \to -1} f(x)$

Evaluating Limits **In Exercises 49 and 50, use the given information to evaluate each limit.**

49. $\lim\limits_{x \to c} f(x) = 4, \quad \lim\limits_{x \to c} g(x) = 8$

(a) $\lim\limits_{x \to c} [-2g(x)]$ (b) $\lim\limits_{x \to c} [f(x) + g(x)]$

(c) $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)}$ (d) $\lim\limits_{x \to c} \sqrt{f(x)}$

50. $\lim\limits_{x \to c} f(x) = 3, \quad \lim\limits_{x \to c} g(x) = -1$

(a) $\lim\limits_{x \to c} [f(x) + g(x)]^2$ (b) $\lim\limits_{x \to c} [6f(x)g(x)]$

(c) $\lim\limits_{x \to c} \dfrac{5g(x)}{4f(x)}$ (d) $\lim\limits_{x \to c} \dfrac{1}{\sqrt{f(x)}}$

Evaluating Limits **In Exercises 51 and 52, find (a) $\lim\limits_{x \to 2} f(x)$, (b) $\lim\limits_{x \to 2} g(x)$, (c) $\lim\limits_{x \to 2} [f(x)g(x)]$, and (d) $\lim\limits_{x \to 2} [g(x) - f(x)]$.**

✓ **51.** $f(x) = x^3, \quad g(x) = \dfrac{\sqrt{x^2 + 5}}{2x^2}$

52. $f(x) = \dfrac{x}{3 - x}, \quad g(x) = \sin \pi x$

Evaluating a Limit by Direct Substitution **In Exercises 53–68, find the limit by direct substitution.**

53. $\lim\limits_{x \to 5} (10 - x^2)$

54. $\lim\limits_{x \to -2} \left(\frac{1}{2}x^3 - 5x\right)$

✓ **55.** $\lim\limits_{x \to -3} (2x^2 + 4x + 1)$

56. $\lim\limits_{x \to -2} (x^3 - 6x + 5)$

57. $\lim\limits_{x \to -3} \dfrac{3x}{x^2 + 1}$

58. $\lim\limits_{x \to 4} \dfrac{x - 1}{x^2 + 2x + 3}$

59. $\lim\limits_{x \to -2} \dfrac{5x + 3}{2x - 9}$

60. $\lim\limits_{x \to 3} \dfrac{x^2 + 1}{x}$

61. $\lim\limits_{x \to -1} \sqrt{x + 2}$

62. $\lim\limits_{x \to 3} \sqrt[3]{x^2 - 1}$

63. $\lim\limits_{x \to 3} e^x$

64. $\lim\limits_{x \to e} \ln x$

65. $\lim\limits_{x \to \pi} \sin 2x$

66. $\lim\limits_{x \to \pi} \tan x$

67. $\lim\limits_{x \to 1/2} \arcsin x$

68. $\lim\limits_{x \to 1} \arccos \dfrac{x}{2}$

Conclusions

True or False? **In Exercises 69 and 70, determine whether the statement is true or false. Justify your answer.**

69. The limit of a function as x approaches c does not exist when the function approaches -3 from the left of c and 3 from the right of c.

70. If f is a rational function, then the limit of $f(x)$ as x approaches c is $f(c)$.

71. **Think About It** From Exercises 7 to 12, select a limit that can be reached and one that cannot be reached.

(a) Use a graphing utility to graph the corresponding functions using a standard viewing window. Do the graphs reveal whether the limit can be reached? Explain.

(b) Use the graphing utility to graph the corresponding functions using a *decimal* setting. Do the graphs reveal whether the limit can be reached? Explain.

72. **Think About It** Use the results of Exercise 71 to draw a conclusion as to whether you can use the graph generated by a graphing utility to determine reliably when a limit can be reached.

73. **Think About It**

(a) Given $f(2) = 4$, can you conclude anything about $\lim\limits_{x \to 2} f(x)$? Explain your reasoning.

(b) Given $\lim\limits_{x \to 2} f(x) = 4$, can you conclude anything about $f(2)$? Explain your reasoning.

74. **CAPSTONE** Use the graph of the function f to decide whether the value of the given quantity exists. If it does, find it. If not, explain why.

(a) $f(0)$

(b) $\lim\limits_{x \to 0} f(x)$

(c) $f(2)$

(d) $\lim\limits_{x \to 2} f(x)$

Cumulative Mixed Review

Simplifying Rational Expressions **In Exercises 75–80, simplify the rational expression.**

75. $\dfrac{5 - x}{3x - 15}$

76. $\dfrac{x^2 - 81}{9 - x}$

77. $\dfrac{15x^2 + 7x - 4}{15x^2 + x - 2}$

78. $\dfrac{x^2 - 12x + 36}{x^2 - 7x + 6}$

79. $\dfrac{x^3 + 27}{x^2 + x - 6}$

80. $\dfrac{x^3 - 8}{x^2 - 4}$

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11.2 Techniques for Evaluating Limits

Dividing Out Technique

In Section 11.1, you studied several types of functions whose limits can be evaluated by direct substitution. In this section, you will study several techniques for evaluating limits of functions for which direct substitution fails.

Suppose you were asked to find the following limit.

$$\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3}$$

Direct substitution fails because -3 is a zero of the denominator. By using a table, however, it appears that the limit of the function as x approaches -3 is -5.

x	-3.01	-3.001	-3.0001	-3	-2.9999	-2.999	-2.99
$\dfrac{x^2 + x - 6}{x + 3}$	-5.01	-5.001	-5.0001	?	-4.9999	-4.999	-4.99

Another way to find the limit of this function is shown in Example 1.

What you should learn

- Use the dividing out technique to evaluate limits of functions.
- Use the rationalizing technique to evaluate limits of functions.
- Use technology to approximate limits of functions graphically and numerically.
- Evaluate one-sided limits of functions.
- Evaluate limits of difference quotients from calculus.

Why you should learn it

Many definitions in calculus involve the limit of a function. For instance, in Exercises 69 and 70 on page 768, the definition of the velocity of a free-falling object at any instant in time involves finding the limit of a position function.

Example 1 Dividing Out Technique

Find the limit.

$$\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3}$$

Solution

Begin by factoring the numerator and dividing out any common factors.

$$\begin{aligned}
\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3} &= \lim_{x \to -3} \frac{(x - 2)(x + 3)}{x + 3} && \text{Factor numerator.} \\
&= \lim_{x \to -3} \frac{(x - 2)\cancel{(x + 3)}}{\cancel{x + 3}} && \text{Divide out common factor.} \\
&= \lim_{x \to -3} (x - 2) && \text{Simplify.} \\
&= -3 - 2 && \text{Direct substitution} \\
&= -5 && \text{Simplify.}
\end{aligned}$$

CHECKPOINT Now try Exercise 11.

This procedure for evaluating a limit is called the **dividing out technique.** The validity of this technique stems from the fact that when two functions agree at all but a single number c, they must have identical limit behavior at $x = c$. In Example 1, the functions given by

$$f(x) = \frac{x^2 + x - 6}{x + 3} \quad \text{and} \quad g(x) = x - 2$$

agree at all values of x other than

$$x = -3.$$

So, you can use $g(x)$ to find the limit of $f(x)$.

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The dividing out technique should be applied only when direct substitution produces 0 in both the numerator *and* the denominator. An expression such as $\frac{0}{0}$ has no meaning as a real number. It is called an **indeterminate form** because you cannot, from the form alone, determine the limit. When you try to evaluate a limit of a rational function by direct substitution and encounter this form, you can conclude that the numerator and denominator must have a common factor. After factoring and dividing out, you should try direct substitution again.

Example 2 Dividing Out Technique

Find the limit.

$$\lim_{x \to 1} \frac{x - 1}{x^3 - x^2 + x - 1}$$

Solution

Begin by substituting $x = 1$ into the numerator and denominator.

$$\frac{x - 1}{x^3 - x^2 + x - 1}$$

$1 - 1 = 0$ Numerator is 0 when $x = 1$.

$1^3 - 1^2 + 1 - 1 = 0$ Denominator is 0 when $x = 1$.

Because both the numerator and denominator are zero when $x = 1$, direct substitution will not yield the limit. To find the limit, you should factor the numerator and denominator, divide out any common factors, and then try direct substitution again.

$$\lim_{x \to 1} \frac{x - 1}{x^3 - x^2 + x - 1} = \lim_{x \to 1} \frac{x - 1}{(x - 1)(x^2 + 1)} \quad \text{Factor denominator.}$$

$$= \lim_{x \to 1} \frac{\cancel{x - 1}}{\cancel{(x - 1)}(x^2 + 1)} \quad \text{Divide out common factor.}$$

$$= \lim_{x \to 1} \frac{1}{x^2 + 1} \quad \text{Simplify.}$$

$$= \frac{1}{1^2 + 1} \quad \text{Direct substitution}$$

$$= \frac{1}{2} \quad \text{Simplify.}$$

This result is shown graphically in Figure 11.11.

Figure 11.11

Study Tip

In Example 2, the factorization of the denominator can be obtained by dividing by $(x - 1)$ or by grouping as follows.

$$x^3 - x^2 + x - 1$$
$$= x^2(x - 1) + (x - 1)$$
$$= (x - 1)(x^2 + 1)$$

CHECKPOINT Now try Exercise 15.

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Rationalizing Technique

Another way to find the limits of some functions is first to rationalize the numerator of the function. This is called the **rationalizing technique.** Recall that rationalizing the numerator means multiplying the numerator and denominator by the conjugate of the numerator. For instance, the conjugate of $\sqrt{x} + 4$ is

$$\sqrt{x} - 4.$$

Example 3 Rationalizing Technique

Find the limit.

$$\lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x}$$

Solution

By direct substitution, you obtain the indeterminate form $\frac{0}{0}$.

$$\lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x} = \frac{\sqrt{0+1} - 1}{0} = \frac{0}{0} \qquad \text{Indeterminate form}$$

In this case, you can rewrite the fraction by rationalizing the numerator.

$$\begin{aligned}
\frac{\sqrt{x+1} - 1}{x} &= \left(\frac{\sqrt{x+1} - 1}{x}\right)\left(\frac{\sqrt{x+1} + 1}{\sqrt{x+1} + 1}\right) \\
&= \frac{(x+1) - 1}{x\left(\sqrt{x+1} + 1\right)} && \text{Multiply.} \\
&= \frac{x}{x\left(\sqrt{x+1} + 1\right)} && \text{Simplify.} \\
&= \frac{\cancel{x}}{\cancel{x}\left(\sqrt{x+1} + 1\right)} && \text{Divide out common factor.} \\
&= \frac{1}{\sqrt{x+1} + 1}, \quad x \neq 0 && \text{Simplify.}
\end{aligned}$$

Now you can evaluate the limit by direct substitution.

$$\lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x} = \lim_{x \to 0} \frac{1}{\sqrt{x+1} + 1} = \frac{1}{\sqrt{0+1} + 1} = \frac{1}{1+1} = \frac{1}{2}$$

You can reinforce your conclusion that the limit is $\frac{1}{2}$ by constructing a table, as shown below, or by sketching a graph, as shown in Figure 11.12.

Figure 11.12

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	0.5132	0.5013	0.5001	?	0.4999	0.4988	0.4881

CHECKPOINT Now try Exercise 23.

The rationalizing technique for evaluating limits is based on multiplication by a convenient form of 1. In Example 3, the convenient form is

$$1 = \frac{\sqrt{x+1} + 1}{\sqrt{x+1} + 1}.$$

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Using Technology

The dividing out and rationalizing techniques may not work well for finding limits of nonalgebraic functions. You often need to use more sophisticated analytic techniques to find limits of these types of functions.

Technology Tip

In Example 4, a graph of $f(x) = (1 + x)^{1/x}$ on a graphing utility would appear continuous at $x = 0$ (see below). But when you try to use the *trace* feature of a graphing utility to determine $f(0)$, no value is given. Some graphing utilities can show breaks or holes in a graph when an appropriate viewing window is used. Because the hole in the graph of f occurs on the y-axis, the hole is not visible.

Example 4 Approximating a Limit Numerically

Approximate the limit: $\lim_{x \to 0} (1 + x)^{1/x}$.

Solution

Let $f(x) = (1 + x)^{1/x}$.

Create a table that shows values of $f(x)$ for several x-values near 0.

Figure 11.13

Because 0 is halfway between -0.001 and 0.001 (see Figure 11.13), use the average of the values of f at these two x-coordinates to estimate the limit.

$$\lim_{x \to 0} (1 + x)^{1/x} \approx \frac{2.7196 + 2.7169}{2} = 2.71825$$

The actual limit can be found algebraically to be

$$e \approx 2.71828.$$

CHECKPOINT Now try Exercise 37.

Example 5 Approximating a Limit Graphically

Approximate the limit: $\lim_{x \to 0} \frac{\sin x}{x}$.

Solution

Direct substitution produces the indeterminate form $\frac{0}{0}$. To approximate the limit, begin by using a graphing utility to graph $f(x) = (\sin x)/x$, as shown in Figure 11.14. Then use the *zoom* and *trace* features of the graphing utility to choose a point on each side of 0, such as $(-0.0012467, 0.9999997)$ and $(0.0012467, 0.9999997)$. Because 0 is halfway between -0.0012467 and 0.0012467, use the average of the values of f at these two x-coordinates to estimate the limit.

Figure 11.14

$$\lim_{x \to 0} \frac{\sin x}{x} \approx \frac{0.9999997 + 0.9999997}{2} = 0.9999997$$

It can be shown algebraically that this limit is exactly 1.

CHECKPOINT Now try Exercise 43.

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One-Sided Limits

In Section 11.1, you saw that one way in which a limit can fail to exist is when a function approaches a different value from the left side of c than it approaches from the right side of c. This type of behavior can be described more concisely with the concept of a **one-sided limit.**

$$\lim_{x \to c^-} f(x) = L_1 \text{ or } f(x) \to L_1 \text{ as } x \to c^- \qquad \text{Limit from the left}$$

$$\lim_{x \to c^+} f(x) = L_2 \text{ or } f(x) \to L_2 \text{ as } x \to c^+ \qquad \text{Limit from the right}$$

Example 6 Evaluating One-Sided Limits

Find the limit as $x \to 0$ from the left and the limit as $x \to 0$ from the right for

$$f(x) = \frac{|2x|}{x}.$$

Solution

From the graph of f, shown in Figure 11.15, you can see that $f(x) = -2$ for all $x < 0$.

Figure 11.15

So, the limit from the left is

$$\lim_{x \to 0^-} \frac{|2x|}{x} = -2. \qquad \text{Limit from the left}$$

Because $f(x) = 2$ for all $x > 0$, the limit from the right is

$$\lim_{x \to 0^+} \frac{|2x|}{x} = 2. \qquad \text{Limit from the right}$$

CHECKPOINT Now try Exercise 49.

In Example 6, note that the function approaches different limits from the left and from the right. In such cases, the limit of $f(x)$ as $x \to c$ does not exist. For the limit of a function to exist as $x \to c$, it must be true that both one-sided limits exist and are equal.

Existence of a Limit

If f is a function and c and L are real numbers, then

$$\lim_{x \to c} f(x) = L$$

if and only if both the left and right limits *exist* and are *equal* to L.

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Example 7 Finding One-Sided Limits

Find the limit of $f(x)$ as x approaches 1.

$$f(x) = \begin{cases} 4 - x, & x < 1 \\ 4x - x^2, & x > 1 \end{cases}$$

Solution

Remember that you are concerned about the value of f *near* $x = 1$ rather than *at* $x = 1$. So, for $x < 1$, $f(x)$ is given by $4 - x$, and you can use direct substitution to obtain

$$\begin{aligned} \lim_{x \to 1^-} f(x) &= \lim_{x \to 1^-} (4 - x) \\ &= 4 - 1 \\ &= 3. \end{aligned}$$

For $x > 1$, $f(x)$ is given by $4x - x^2$, and you can use direct substitution to obtain

$$\begin{aligned} \lim_{x \to 1^+} f(x) &= \lim_{x \to 1^+} (4x - x^2) \\ &= 4(1) - 1^2 \\ &= 3. \end{aligned}$$

Because the one-sided limits both exist and are equal to 3, it follows that

$$\lim_{x \to 1} f(x) = 3.$$

The graph in Figure 11.16 confirms this conclusion.

Figure 11.16

CHECKPOINT Now try Exercise 53.

Example 8 Comparing Limits from the Left and Right

To ship a package overnight, a delivery service charges \$17.80 for the first pound and \$1.40 for each additional pound or portion of a pound. Let x represent the weight of a package and let $f(x)$ represent the shipping cost. Show that the limit of $f(x)$ as $x \to 2$ does not exist.

$$f(x) = \begin{cases} 17.80, & 0 < x \le 1 \\ 19.20, & 1 < x \le 2 \\ 20.60, & 2 < x \le 3 \end{cases}$$

Solution

The graph of f is shown in Figure 11.17. The limit of $f(x)$ as x approaches 2 from the left is

$$\lim_{x \to 2^-} f(x) = 19.20$$

whereas the limit of $f(x)$ as x approaches 2 from the right is

$$\lim_{x \to 2^+} f(x) = 20.60.$$

Because these one-sided limits are not equal, the limit of $f(x)$ as $x \to 2$ does not exist.

Figure 11.17

CHECKPOINT Now try Exercise 71.

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A Limit from Calculus

In the next section, you will study an important type of limit from calculus—the limit of a *difference quotient.*

Example 9 Evaluating a Limit from Calculus

algebra of calculus

For the function given by $f(x) = x^2 - 1$, find

$$\lim_{h \to 0} \frac{f(3 + h) - f(3)}{h}.$$

Solution

Direct substitution produces an indeterminate form.

$$\begin{aligned}
\lim_{h \to 0} \frac{f(3 + h) - f(3)}{h} &= \lim_{h \to 0} \frac{[(3 + h)^2 - 1] - [(3)^2 - 1]}{h} \\
&= \lim_{h \to 0} \frac{9 + 6h + h^2 - 1 - 9 + 1}{h} \\
&= \lim_{h \to 0} \frac{6h + h^2}{h} \\
&= \frac{0}{0}
\end{aligned}$$

By factoring and dividing out, you obtain the following.

$$\begin{aligned}
\lim_{h \to 0} \frac{f(3 + h) - f(3)}{h} &= \lim_{h \to 0} \frac{6h + h^2}{h} \\
&= \lim_{h \to 0} \frac{h(6 + h)}{h} \\
&= \lim_{h \to 0} (6 + h) \\
&= 6 + 0 \\
&= 6
\end{aligned}$$

So, the limit is 6.

CHECKPOINT Now try Exercise 79.

Note that for any x-value, the limit of a difference quotient is an expression of the form

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.$$

Direct substitution into the difference quotient always produces the indeterminate form $\frac{0}{0}$. For instance,

$$\begin{aligned}
\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} &= \frac{f(x + 0) - f(x)}{0} \\
&= \frac{f(x) - f(x)}{0} \\
&= \frac{0}{0}.
\end{aligned}$$

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11.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1 and 2, fill in the blank.

1. To find a limit of a rational function that has common factors in its numerator and denominator, use the _____ .

2. The expression $\frac{0}{0}$ has no meaning as a real number and is called an _____ because you cannot, from the form alone, determine the limit.

3. Which algebraic technique can you use to find $\lim_{x \to 0} \dfrac{\sqrt{x+4}-2}{x}$?

4. Describe in words what is meant by $\lim_{x \to 0^+} f(x) = -2$.

Procedures and Problem Solving

Using a Graph to Determine Limits **In Exercises 5–8, use the graph to determine each limit (if it exists). Then identify another function that agrees with the given function at all but one point.**

5. $g(x) = \dfrac{-2x^2 + x}{x}$

(a) $\lim_{x \to 0} g(x)$

(b) $\lim_{x \to -1} g(x)$

(c) $\lim_{x \to -2} g(x)$

6. $h(x) = \dfrac{x^2 - 3x}{x}$

(a) $\lim_{x \to -2} h(x)$

(b) $\lim_{x \to 0} h(x)$

(c) $\lim_{x \to 3} h(x)$

7. $g(x) = \dfrac{x^3 - x}{x - 1}$

(a) $\lim_{x \to 1} g(x)$

(b) $\lim_{x \to -1} g(x)$

(c) $\lim_{x \to 0} g(x)$

8. $f(x) = \dfrac{x^2 - 1}{x + 1}$

(a) $\lim_{x \to 1} f(x)$

(b) $\lim_{x \to 2} f(x)$

(c) $\lim_{x \to -1} f(x)$

Finding a Limit **In Exercises 9–36, find the limit (if it exists). Use a graphing utility to confirm your result graphically.**

9. $\lim_{x \to 6} \dfrac{x - 6}{x^2 - 36}$

10. $\lim_{x \to 9} \dfrac{9 - x}{x^2 - 81}$

✓ **11.** $\lim_{x \to 2} \dfrac{x^2 - x - 2}{x - 2}$

12. $\lim_{x \to -1} \dfrac{x^2 + 6x + 5}{x + 1}$

13. $\lim_{x \to -1} \dfrac{1 - 2x - 3x^2}{1 + x}$

14. $\lim_{x \to -4} \dfrac{2x^2 + 7x - 4}{x + 4}$

✓ **15.** $\lim_{t \to 2} \dfrac{t^3 - 8}{t - 2}$

16. $\lim_{a \to -4} \dfrac{a^3 + 64}{a + 4}$

17. $\lim_{x \to 2} \dfrac{x^5 - 32}{x - 2}$

18. $\lim_{x \to 1} \dfrac{x^4 - 1}{x - 1}$

19. $\lim_{x \to -4} \dfrac{x^2 + x - 12}{x^2 + 6x + 8}$

20. $\lim_{x \to 3} \dfrac{x^2 - 8x + 15}{x^2 - 2x - 3}$

21. $\lim_{x \to -1} \dfrac{x^3 + 2x^2 - x - 2}{x^3 + 4x^2 - x - 4}$

22. $\lim_{x \to -3} \dfrac{x^3 + 2x^2 - 9x - 18}{x^3 + x^2 - 9x - 9}$

✓ **23.** $\lim_{y \to 0} \dfrac{\sqrt{5 + y} - \sqrt{5}}{y}$

24. $\lim_{z \to 0} \dfrac{\sqrt{7 - z} - \sqrt{7}}{z}$

25. $\lim_{x \to -3} \dfrac{\sqrt{x + 7} - 2}{x + 3}$

26. $\lim_{x \to 2} \dfrac{4 - \sqrt{18 - x}}{x - 2}$

27. $\lim_{x \to 0} \dfrac{\dfrac{1}{x + 1} - 1}{x}$

28. $\lim_{x \to 0} \dfrac{\dfrac{1}{x - 8} + \dfrac{1}{8}}{x}$

29. $\lim_{x \to 0} \dfrac{\dfrac{1}{x + 4} - \dfrac{1}{4}}{x}$

30. $\lim_{x \to 0} \dfrac{\dfrac{1}{2 + x} - \dfrac{1}{2}}{x}$

31. $\lim_{x \to \pi/2} \dfrac{1 - \sin x}{\cos x}$

32. $\lim_{x \to 0} \dfrac{\cos x - 1}{\sin x}$

33. $\lim_{x \to 0} \dfrac{\cos 2x}{\cot 2x}$

34. $\lim_{x \to \pi} \dfrac{\sin x}{\csc x}$

35. $\lim_{x \to \pi/2} \dfrac{\sin x - 1}{x}$

36. $\lim_{x \to \pi} \dfrac{1 + \cos x}{x}$

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Approximating a Limit Numerically **In Exercises 37–42, use the *table* feature of a graphing utility to create a table for the function and use the result to approximate the limit numerically. Write an approximation that is accurate to three decimal places.**

37. $\lim_{x\to 0} \frac{e^{2x} - 1}{x}$ **38.** $\lim_{x\to 0} \frac{1 - e^{-x}}{x}$

39. $\lim_{x\to 0} \frac{\sqrt{2x + 1} - 1}{x}$ **40.** $\lim_{x\to 9} \frac{3 - \sqrt{x}}{x - 9}$

41. $\lim_{x\to 0} (1 - x)^{2/x}$ **42.** $\lim_{x\to 0} (1 + 2x)^{1/x}$

Approximating a Limit Graphically **In Exercises 43–48, use a graphing utility to graph the function and approximate the limit. Write an approximation that is accurate to three decimal places.**

43. $\lim_{x\to 0} \frac{\sin 2x}{x}$ **44.** $\lim_{x\to 0} \frac{\sin 3x}{x}$

45. $\lim_{x\to 0} \frac{\tan x}{x}$ **46.** $\lim_{x\to 0} \frac{1 - \cos 2x}{x}$

47. $\lim_{x\to 1} \frac{1 - \sqrt[3]{x}}{1 - x}$ **48.** $\lim_{x\to 1} \frac{\sqrt[3]{x} - x}{x - 1}$

Evaluating One-Sided Limits **In Exercises 49–56, graph the function. Determine the limit (if it exists) by evaluating the corresponding one-sided limits.**

49. $\lim_{x\to 6} \frac{|x - 6|}{x - 6}$ **50.** $\lim_{x\to 2} \frac{|x - 2|}{x - 2}$

51. $\lim_{x\to 1} \frac{1}{x^2 + 1}$ **52.** $\lim_{x\to 1} \frac{1}{x^2 - 1}$

53. $\lim_{x\to 2} f(x)$ where $f(x) = \begin{cases} x - 1, & x \le 2 \\ 2x - 3, & x > 2 \end{cases}$

54. $\lim_{x\to 1} f(x)$ where $f(x) = \begin{cases} 2x + 1, & x < 1 \\ 4 - x^2, & x \ge 1 \end{cases}$

55. $\lim_{x\to 1} f(x)$ where $f(x) = \begin{cases} 4 - x^2, & x \le 1 \\ 3 - x, & x > 1 \end{cases}$

56. $\lim_{x\to 0} f(x)$ where $f(x) = \begin{cases} 4 - x^2, & x \le 0 \\ x + 4, & x > 0 \end{cases}$

Algebraic-Graphical-Numerical **In Exercises 57–60, (a) graphically approximate the limit (if it exists) by using a graphing utility to graph the function, (b) numerically approximate the limit (if it exists) by using the *table* feature of the graphing utility to create a table, and (c) algebraically evaluate the limit (if it exists) by the appropriate technique(s).**

57. $\lim_{x\to 1^-} \frac{x - 1}{x^2 - 1}$ **58.** $\lim_{x\to 5^+} \frac{5 - x}{25 - x^2}$

59. $\lim_{x\to 16^+} \frac{4 - \sqrt{x}}{x - 16}$ **60.** $\lim_{x\to 0^-} \frac{\sqrt{x + 2} - \sqrt{2}}{x}$

Finding a Limit **In Exercises 61–66, use a graphing utility to graph the function and the equations $y = x$ and $y = -x$ in the same viewing window. Use the graph to find $\lim_{x\to 0} f(x)$.**

61. $f(x) = x \cos x$ **62.** $f(x) = |x \sin x|$

63. $f(x) = |x| \sin x$ **64.** $f(x) = |x| \cos x$

65. $f(x) = x \sin \frac{1}{x}$ **66.** $f(x) = x \cos \frac{1}{x}$

Finding Limits **In Exercises 67 and 68, state which limit can be evaluated by using direct substitution. Then evaluate or approximate each limit.**

67. (a) $\lim_{x\to 0} x^2 \sin x^2$ (b) $\lim_{x\to 0} \frac{\sin x^2}{x^2}$

68. (a) $\lim_{x\to 0} \frac{x}{\cos x}$ (b) $\lim_{x\to 0} \frac{1 - \cos x}{x}$

Why you should learn it *(p. 760)* **In Exercises 69 and 70, use the position function**

$$s(t) = -16t^2 + 128$$

which gives the height (in feet) of a free-falling object. The velocity at time $t = a$ seconds is given by

$$\lim_{t\to a} \frac{s(a) - s(t)}{a - t}.$$

69. Find the velocity when $t = 1$ second.

70. Find the velocity when $t = 2$ seconds.

71. **Human Resources** A union contract guarantees an 8% salary increase yearly for 3 years. For a current salary of \$30,000, the salaries $f(t)$ (in thousands of dollars) for the next 3 years are given by

$$f(t) = \begin{cases} 30.000, & 0 < t \le 1 \\ 32.400, & 1 < t \le 2 \\ 34.992, & 2 < t \le 3 \end{cases}$$

where t represents the time in years. Show that the limit of f as $t \to 2$ does not exist.

72. **Business** The cost of sending a package overnight is \$15 for the first pound and \$1.30 for each additional pound or portion of a pound. A plastic mailing bag can hold up to 3 pounds. The cost $f(x)$ of sending a package in a plastic mailing bag is given by

$$f(x) = \begin{cases} 15.00, & 0 < x \le 1 \\ 16.30, & 1 < x \le 2 \\ 17.60, & 2 < x \le 3 \end{cases}$$

where x represents the weight of the package (in pounds). Show that the limit of f as $x \to 1$ does not exist.

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73. MODELING DATA

The cost of hooking up and towing a car is \$85 for the first mile and \$5 for each additional mile or portion of a mile. A model for the cost C (in dollars) is $C(x) = 85 - 5[\![-(x-1)]\!]$, where x is the distance in miles. (Recall from Section 1.3 that $f(x) = [\![x]\!]$ = the greatest integer less than or equal to x.)

(a) Use a graphing utility to graph C for $0 < x \leq 10$.

(b) Complete the table and observe the behavior of C as x approaches 5.5. Use the graph from part (a) and the table to find $\lim_{x \to 5.5} C(x)$.

x	5	5.3	5.4	5.5	5.6	5.7	6
$C(x)$				?			

(c) Complete the table and observe the behavior of C as x approaches 5. Does the limit of $C(x)$ as x approaches 5 exist? Explain.

x	4	4.5	4.9	5	5.1	5.5	6
$C(x)$				?			

74. MODELING DATA

The cost C (in dollars) of making x photocopies at a copy shop is given by the function

$$C(x) = \begin{cases} 0.15x, & 0 < x \leq 25 \\ 0.10x, & 25 < x \leq 100 \\ 0.07x, & 100 < x \leq 500 \\ 0.05x, & x > 500 \end{cases}.$$

(a) Sketch a graph of the function.

(b) Find each limit and interpret your result in the context of the situation.

(i) $\lim_{x \to 15} C(x)$ (ii) $\lim_{x \to 99} C(x)$ (iii) $\lim_{x \to 305} C(x)$

(c) Create a table of values to show numerically that each limit does not exist.

(i) $\lim_{x \to 25} C(x)$ (ii) $\lim_{x \to 100} C(x)$ (iii) $\lim_{x \to 500} C(x)$

(d) Explain how you can use the graph in part (a) to verify that the limits in part (c) do not exist.

Evaluating a Limit from Calculus **In Exercises 75–82, find**

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

75. $f(x) = 3x - 1$

76. $f(x) = 5 - 6x$

77. $f(x) = \sqrt{x}$

78. $f(x) = \sqrt{x - 2}$

79. $f(x) = x^2 - 3x$

80. $f(x) = 4 - 2x - x^2$

81. $f(x) = \dfrac{1}{x+2}$

82. $f(x) = \dfrac{1}{x-1}$

Conclusions

True or False? **In Exercises 83 and 84, determine whether the statement is true or false. Justify your answer.**

83. When your attempt to find the limit of a rational function yields the indeterminate form $\frac{0}{0}$, the rational function's numerator and denominator have a common factor.

84. If $f(c) = L$, then $\lim_{x \to c} f(x) = L$.

85. Think About It Sketch the graph of a function for which $f(2)$ is defined but the limit of $f(x)$ as x approaches 2 does not exist.

86. Think About It Sketch the graph of a function for which the limit of $f(x)$ as x approaches 1 is 4 but $f(1) \neq 4$.

87. Writing Consider the limit of the rational function $p(x)/q(x)$. What conclusion can you make when direct substitution produces each expression? Write a short paragraph explaining your reasoning.

(a) $\lim_{x \to c} \dfrac{p(x)}{q(x)} = \dfrac{0}{1}$ (b) $\lim_{x \to c} \dfrac{p(x)}{q(x)} = \dfrac{1}{1}$

(c) $\lim_{x \to c} \dfrac{p(x)}{q(x)} = \dfrac{1}{0}$ (d) $\lim_{x \to c} \dfrac{p(x)}{q(x)} = \dfrac{0}{0}$

88. CAPSTONE Given

$$f(x) = \begin{cases} 2x, & x \leq 0 \\ x^2 + 1, & x > 0 \end{cases},$$

find each of the following limits. If the limit does not exist, explain why.

(a) $\lim_{x \to 0^-} f(x)$ (b) $\lim_{x \to 0^+} f(x)$ (c) $\lim_{x \to 0} f(x)$

Cumulative Mixed Review

Identifying a Conic from Its Equation **In Exercises 89–92, identify the type of conic represented by the equation. Use a graphing utility to confirm your result.**

89. $r = \dfrac{3}{1 + \cos\theta}$

90. $r = \dfrac{12}{3 + 2\sin\theta}$

91. $r = \dfrac{9}{2 + 3\cos\theta}$

92. $r = \dfrac{4}{4 + \cos\theta}$

A Relationship of Two Vectors **In Exercises 93–96, determine whether the vectors are orthogonal, parallel, or neither.**

93. $\langle 7, -2, 3\rangle, \langle -1, 4, 5\rangle$

94. $\langle 5, 5, 0\rangle, \langle 0, 5, 1\rangle$

95. $\langle 2, -3, 1\rangle, \langle -2, 2, 2\rangle$

96. $\langle -1, 3, 1\rangle, \langle 3, -9, -3\rangle$

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11.3 The Tangent Line Problem

Tangent Line to a Graph

Calculus is a branch of mathematics that studies rates of change of functions. If you go on to take a course in calculus, you will learn that rates of change have many applications in real life.

Earlier in the text, you learned how the slope of a line indicates the rate at which a line rises or falls. For a line, this rate (or slope) is the same at every point on the line. For graphs other than lines, the rate at which the graph rises or falls changes from point to point. For instance, in Figure 11.18, the parabola is rising more quickly at the point (x_1, y_1) than it is at the point (x_2, y_2). At the vertex (x_3, y_3), the graph levels off, and at the point (x_4, y_4), the graph is falling.

Figure 11.18

What you should learn

- Understand the tangent line problem.
- Use a tangent line to approximate the slope of a graph at a point.
- Use the limit definition of slope to find exact slopes of graphs.
- Find derivatives of functions and use derivatives to find slopes of graphs.

Why you should learn it

The derivative, or the slope of the tangent line to the graph of a function at a point, can be used to analyze rates of change. For instance, in Exercise 69 on page 779, the derivative is used to analyze the rate of change of the volume of a spherical balloon.

To determine the rate at which a graph rises or falls at a *single point*, you can find the slope of the tangent line at that point. In simple terms, the **tangent line** to the graph of a function f at a point

$$P(x_1, y_1)$$

is the line that best approximates the slope of the graph at the point. Figure 11.19 shows other examples of tangent lines.

Figure 11.19

From geometry, you know that a line is tangent to a circle when the line intersects the circle at only one point (see Figure 11.20). Tangent lines to noncircular graphs, however, can intersect the graph at more than one point. For instance, in the first graph in Figure 11.19, if the tangent line were extended, then it would intersect the graph at a point other than the point of tangency.

Figure 11.20

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Slope of a Graph

Because a tangent line approximates the slope of a graph at a point, the problem of finding the slope of a graph at a point is the same as finding the slope of the tangent line at the point.

Example 1 Visually Approximating the Slope of a Graph

Use the graph in Figure 11.21 to approximate the slope of the graph of

$$f(x) = x^2$$

at the point $(1, 1)$.

Figure 11.21

Solution

From the graph of $f(x) = x^2$, you can see that the tangent line at $(1, 1)$ rises approximately two units for each unit change in x. So, you can estimate the slope of the tangent line at $(1, 1)$ to be

$$\text{Slope} = \frac{\text{change in } y}{\text{change in } x}$$

$$\approx \frac{2}{1}$$

$$= 2.$$

Because the tangent line at the point $(1, 1)$ has a slope of about 2, you can conclude that the graph of f has a slope of about 2 at the point $(1, 1)$.

CHECKPOINT Now try Exercise 7.

When you are visually approximating the slope of a graph, remember that the scales on the horizontal and vertical axes may differ. When this happens (as it frequently does in applications), the slope of the tangent line is distorted, and you must be careful to account for the difference in the scales.

Example 2 Approximating the Slope of a Graph

Figure 11.22 graphically depicts the monthly normal temperatures (in degrees Fahrenheit) for Dallas, Texas. Approximate the slope of this graph at the indicated point and give a physical interpretation of the result. (Source: National Climatic Data Center)

Figure 11.22

Solution

From the graph, you can see that the tangent line at the given point falls approximately 16 units for each two-unit change in x. So, you can estimate the slope at the given point to be

$$\text{Slope} = \frac{\text{change in } y}{\text{change in } x}$$

$$\approx \frac{-16}{2}$$

$$= -8 \text{ degrees per month.}$$

This means that you can expect the monthly normal temperature in November to be about 8 degrees lower than the normal temperature in October.

CHECKPOINT Now try Exercise 9.

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Slope and the Limit Process

In Examples 1 and 2, you approximated the slope of a graph at a point by creating a graph and then "eyeballing" the tangent line at the point of tangency. A more systematic method of approximating tangent lines makes use of a **secant line** through the point of tangency and a second point on the graph, as shown in Figure 11.23. If $(x, f(x))$ is the point of tangency and

$$(x + h, f(x + h))$$

is a second point on the graph of f, then the slope of the secant line through the two points is given by

Figure 11.23

$$m_{\text{sec}} = \frac{f(x + h) - f(x)}{h}. \qquad \text{Slope of secant line}$$

The right side of this equation is called the **difference quotient.** The denominator h is the *change in x*, and the numerator is the *change in y*. The beauty of this procedure is that you obtain more and more accurate approximations of the slope of the tangent line by choosing points closer and closer to the point of tangency, as shown in Figure 11.24.

As h approaches 0, the secant line approaches the tangent line.
Figure 11.24

Using the limit process, you can find the *exact* slope of the tangent line at $(x, f(x))$.

Definition of the Slope of a Graph

The **slope** m of the graph of f at the point $(x, f(x))$ is equal to the slope of its tangent line at $(x, f(x))$, and is given by

$$m = \lim_{h \to 0} m_{\text{sec}}$$
$$= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

provided this limit exists.

From the definition above and from Section 11.2, you can see that the difference quotient is used frequently in calculus. Using the difference quotient to find the slope of a tangent line to a graph is a major concept of calculus.

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Example 3 Finding the Slope of a Graph

Find the slope of the graph of $f(x) = x^2$ at the point $(-2, 4)$.

Solution

Find an expression that represents the slope of a secant line at $(-2, 4)$.

$$m_{\text{sec}} = \frac{f(-2+h) - f(-2)}{h} \qquad \text{Set up difference quotient.}$$

$$= \frac{(-2+h)^2 - (-2)^2}{h} \qquad \text{Substitute into } f(x) = x^2.$$

$$= \frac{4 - 4h + h^2 - 4}{h} \qquad \text{Expand terms.}$$

$$= \frac{-4h + h^2}{h} \qquad \text{Simplify.}$$

$$= \frac{\cancel{h}(-4+h)}{\cancel{h}} \qquad \text{Factor and divide out.}$$

$$= -4 + h, \quad h \neq 0 \qquad \text{Simplify.}$$

Next, take the limit of m_{sec} as h approaches 0.

$$m = \lim_{h \to 0} m_{\text{sec}}$$
$$= \lim_{h \to 0} (-4 + h)$$
$$= -4 + 0$$
$$= -4$$

The graph has a slope of -4 at the point $(-2, 4)$, as shown in Figure 11.25.

Figure 11.25

CHECKPOINT Now try Exercise 11.

Example 4 Finding the Slope of a Graph

Find the slope of $f(x) = -2x + 4$.

Solution

$$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad \text{Set up difference quotient.}$$

$$= \lim_{h \to 0} \frac{[-2(x+h) + 4] - (-2x + 4)}{h} \qquad \text{Substitute into } f(x) = -2x + 4.$$

$$= \lim_{h \to 0} \frac{-2x - 2h + 4 + 2x - 4}{h} \qquad \text{Expand terms.}$$

$$= \lim_{h \to 0} \frac{-2\cancel{h}}{\cancel{h}} \qquad \text{Divide out.}$$

$$= -2 \qquad \text{Simplify.}$$

You know from your study of linear functions that the line given by

$$f(x) = -2x + 4$$

has a slope of -2, as shown in Figure 11.26. This conclusion is consistent with that obtained by the limit definition of slope, as shown above.

Figure 11.26

CHECKPOINT Now try Exercise 13.

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It is important that you see the difference between the ways the difference quotients were set up in Examples 3 and 4. In Example 3, you were finding the slope of a graph at a specific point $(c, f(c))$. To find the slope in such a case, you can use the following form of the difference quotient.

$$m = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h} \qquad \text{Slope at specific point}$$

In Example 4, however, you were finding a *formula* for the slope at *any* point on the graph. In such cases, you should use x, rather than c, in the difference quotient.

$$m = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \qquad \text{Formula for slope}$$

Example 5 Finding a Formula for the Slope of a Graph

Find a formula for the slope of the graph of

$$f(x) = x^2 + 1.$$

What are the slopes at the points $(-1, 2)$ and $(2, 5)$?

Solution

$$m_{\text{sec}} = \frac{f(x + h) - f(x)}{h} \qquad \text{Set up difference quotient.}$$

$$= \frac{[(x + h)^2 + 1] - (x^2 + 1)}{h} \qquad \text{Substitute into } f(x) = x^2 + 1.$$

$$= \frac{x^2 + 2xh + h^2 + 1 - x^2 - 1}{h} \qquad \text{Expand terms.}$$

$$= \frac{2xh + h^2}{h} \qquad \text{Simplify.}$$

$$= \frac{\cancel{h}(2x + h)}{\cancel{h}} \qquad \text{Factor and divide out.}$$

$$= 2x + h, \quad h \neq 0 \qquad \text{Simplify.}$$

Next, take the limit of m_{sec} as h approaches 0.

$$\begin{aligned} m &= \lim_{h \to 0} m_{\text{sec}} \\ &= \lim_{h \to 0} (2x + h) \\ &= 2x + 0 \\ &= 2x \end{aligned}$$

Using the formula $m = 2x$ for the slope at $(x, f(x))$, you can find the slope at the specified points. At $(-1, 2)$, the slope is

$$m = 2(-1) = -2$$

and at $(2, 5)$, the slope is

$$m = 2(2) = 4.$$

The graph of f is shown in Figure 11.27.

Figure 11.27

CHECKPOINT Now try Exercise 19.

Technology Tip

Try verifying the result in Example 5 by using a graphing utility to graph the function and the tangent lines at $(-1, 2)$ and $(2, 5)$ as

$$y_1 = x^2 + 1$$
$$y_2 = -2x$$
$$y_3 = 4x - 3$$

in the same viewing window. You can also verify the result using the *tangent* feature. For instructions on how to use the *tangent* feature, see Appendix A; for specific keystrokes, go to this textbook's *Companion Website.*

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The Derivative of a Function

In Example 5, you started with the function $f(x) = x^2 + 1$ and used the limit process to derive another function, $m = 2x$, that represents the slope of the graph of f at the point $(x, f(x))$. This derived function is called the **derivative** of f at x. It is denoted by $f'(x)$, which is read as "f prime of x."

Definition of the Derivative

The **derivative** of f at x is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided this limit exists.

Remember that the derivative $f'(x)$ is a formula for the slope of the tangent line to the graph of f at the point $(x, f(x))$.

Example 6 Finding a Derivative

Find the derivative of

$$f(x) = 3x^2 - 2x.$$

Solution

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{[3(x+h)^2 - 2(x+h)] - (3x^2 - 2x)}{h} \\
&= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 2x - 2h - 3x^2 + 2x}{h} \\
&= \lim_{h \to 0} \frac{6xh + 3h^2 - 2h}{h} \\
&= \lim_{h \to 0} \frac{\cancel{h}(6x + 3h - 2)}{\cancel{h}} \\
&= \lim_{h \to 0} (6x + 3h - 2) \\
&= 6x + 3(0) - 2 \\
&= 6x - 2
\end{aligned}$$

So, the derivative of $f(x) = 3x^2 - 2x$ is

$$f'(x) = 6x - 2. \qquad \text{Derivative of } f \text{ at } x$$

CHECKPOINT Now try Exercise 31.

Note that in addition to $f'(x)$, other notations can be used to denote the derivative of $y = f(x)$. The most common are

$$\frac{dy}{dx}, \quad y', \quad \frac{d}{dx}[f(x)], \quad \text{and} \quad D_x[y].$$

Study Tip

In Section 1.1, you studied the slope of a line, which represents the *average rate of change* over an interval. The derivative of a function is a formula which represents the *instantaneous rate of change* at a point.

Explore the Concept

Use a graphing utility to graph the function $f(x) = 3x^2 - 2x$. Use the *trace* feature to approximate the coordinates of the vertex of this parabola. Then use the derivative of $f(x) = 3x^2 - 2x$ to find the slope of the tangent line at the vertex. Make a conjecture about the slope of the tangent line at the vertex of an arbitrary parabola.

Hasan Kursad Ergan/iStockphoto.com

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Example 7 Using the Derivative

Find $f'(x)$ for

$$f(x) = \sqrt{x}.$$

Then find the slopes of the graph of f at the points $(1, 1)$ and $(4, 2)$ and equations of the tangent lines to the graph at the points.

Solution

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

Because direct substitution yields the indeterminate form $\frac{0}{0}$, you should use the rationalizing technique discussed in Section 11.2 to find the limit.

$$f'(x) = \lim_{h \to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h}\right)\left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}\right)$$

$$= \lim_{h \to 0} \frac{(x+h) - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x+0} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

At the point $(1, 1)$, the slope is

$$f'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2}.$$

An equation of the tangent line at the point $(1, 1)$ is

$y - y_1 = m(x - x_1)$	Point-slope form
$y - 1 = \frac{1}{2}(x - 1)$	Substitute $\frac{1}{2}$ for m, 1 for x_1, and 1 for y_1.
$y = \frac{1}{2}x + \frac{1}{2}.$	Tangent line

At the point $(4, 2)$, the slope is

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

An equation of the tangent line at the point $(4, 2)$ is

$y - y_1 = m(x - x_1)$	Point-slope form
$y - 2 = \frac{1}{4}(x - 4)$	Substitute $\frac{1}{4}$ for m, 4 for x_1, and 2 for y_1.
$y = \frac{1}{4}x + 1.$	Tangent line

The graphs of f and the tangent lines at the points $(1, 1)$ and $(4, 2)$ are shown in Figure 11.28.

Figure 11.28

CHECKPOINT Now try Exercise 43.

Study Tip

Remember that in order to rationalize the numerator of an expression, you must multiply the numerator and denominator by the conjugate of the numerator.

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11.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1–4, fill in the blank.

1. _____ is the study of the rates of change of functions.
2. The _____ to the graph of a function at a point is the line that best approximates the slope of the graph at the point.
3. To approximate a tangent line to a graph, you can make use of a _____ through the point of tangency and a second point on the graph.
4. The _____ of a function f at x represents the slope of the graph of f at the point $(x, f(x))$.
5. The slope of the tangent line to the graph of f at the point $(1, 5)$ is 2. What is the slope of the graph of f at the point $(1, 5)$?
6. Given $f(1) = 2$ and $f'(1) = -4$, what is the slope of the graph of f at the point $(1, 2)$?

Procedures and Problem Solving

Approximating the Slope of a Graph **In Exercises 7–10, use the figure to approximate the slope of the curve at the point (x, y).**

7.

8.

9.

10.

Finding the Slope of a Graph **In Exercises 11–18, use the limit process to find the slope of the graph of the function at the specified point. Use a graphing utility to confirm your result.**

11. $g(x) = x^2 - 4x$, $(3, -3)$
12. $f(x) = 10x - 2x^2$, $(3, 12)$
13. $g(x) = 5 - 2x$, $(1, 3)$
14. $h(x) = 2x + 5$, $(-1, -3)$
15. $g(x) = \dfrac{4}{x}$, $(2, 2)$
16. $g(x) = \dfrac{1}{x - 2}$, $\left(4, \dfrac{1}{2}\right)$
17. $h(x) = \sqrt{x}$, $(9, 3)$
18. $h(x) = \sqrt{x + 10}$, $(-1, 3)$

Finding a Formula for the Slope of a Graph **In Exercises 19–24, find a formula for the slope of the graph of f at the point $(x, f(x))$. Then use it to find the slopes at the two specified points.**

19. $f(x) = 4 - x^2$
    (a) $(0, 4)$
    (b) $(-2, 0)$
20. $f(x) = x^3$
    (a) $(1, 1)$
    (b) $(2, 8)$
21. $f(x) = \dfrac{1}{x + 4}$
    (a) $\left(0, \frac{1}{4}\right)$
    (b) $\left(-2, \frac{1}{2}\right)$
22. $f(x) = \dfrac{1}{x + 2}$
    (a) $\left(0, \frac{1}{2}\right)$
    (b) $(-3, -1)$
23. $f(x) = \sqrt{x - 1}$
    (a) $(2, 1)$
    (b) $(10, 3)$
24. $f(x) = \sqrt{x - 4}$
    (a) $(5, 1)$
    (b) $(8, 2)$

Approximating the Slope of a Tangent Line **In Exercises 25–30, use a graphing utility to graph the function and the tangent line at the point $(1, f(1))$. Use the graph to approximate the slope of the tangent line.**

25. $f(x) = x^2 - 3$
26. $f(x) = x^2 - 2x + 1$
27. $f(x) = \sqrt{2 - x}$
28. $f(x) = \sqrt{x + 3}$
29. $f(x) = \dfrac{4}{x + 1}$
30. $f(x) = \dfrac{3}{2 - x}$

Finding a Derivative **In Exercises 31–42, find the derivative of the function.**

31. $f(x) = 4 - 3x^2$
32. $f(x) = x^2 - 3x + 4$
33. $f(x) = 5$
34. $f(x) = -1$
35. $f(x) = 9 - \frac{1}{3}x$
36. $f(x) = -5x + 2$
37. $f(x) = \dfrac{1}{x^2}$
38. $f(x) = \dfrac{1}{x^3}$
39. $f(x) = \sqrt{x - 4}$
40. $f(x) = \sqrt{x + 1}$

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41. $f(x) = \dfrac{1}{\sqrt{x-9}}$ **42.** $f(x) = \dfrac{1}{\sqrt{x+1}}$

Using the Derivative **In Exercises 43–50, (a) find the slope of the graph of f at the given point, (b) find an equation of the tangent line to the graph at the point, and (c) graph the function and the tangent line.**

43. $f(x) = x^2 - 1, (2, 3)$ **44.** $f(x) = 4 - x^2, (1, 3)$

45. $f(x) = x^3 - 2x, (1, -1)$

46. $f(x) = x^2 + 2x + 1, (-3, 4)$

47. $f(x) = \sqrt{x+1}, (3, 2)$ **48.** $f(x) = \sqrt{x-2}, (3, 1)$

49. $f(x) = \dfrac{1}{x+5}, (-4, 1)$ **50.** $f(x) = \dfrac{1}{x-3}, (4, 1)$

Graphing a Function Over an Interval **In Exercises 51–54, use a graphing utility to graph f over the interval $[-2, 2]$ and complete the table. Compare the value of the first derivative with a visual approximation of the slope of the graph.**

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
$f(x)$									
$f'(x)$									

51. $f(x) = \frac{1}{2}x^2$ **52.** $f(x) = \frac{1}{4}x^3$

53. $f(x) = \sqrt{x+3}$ **54.** $f(x) = \dfrac{x^2-4}{x+4}$

Using the Derivative **In Exercises 55–58, find the derivative of f. Use the derivative to determine any points on the graph of f at which the tangent line is horizontal. Use a graphing utility to verify your results.**

55. $f(x) = x^2 - 4x + 3$ **56.** $f(x) = x^2 - 6x + 4$

57. $f(x) = 3x^3 - 9x$ **58.** $f(x) = x^3 + 3x$

Using the Derivative **In Exercises 59–66, use the function and its derivative to determine any points on the graph of f at which the tangent line is horizontal. Use a graphing utility to verify your results.**

59. $f(x) = x^4 - 2x^2, f'(x) = 4x^3 - 4x$

60. $f(x) = 3x^4 + 4x^3, f'(x) = 12x^3 + 12x^2$

61. $f(x) = 2\cos x + x, f'(x) = -2\sin x + 1,$ over the interval $(0, 2\pi)$

62. $f(x) = x - 2\sin x, f'(x) = 1 - 2\cos x,$ over the interval $(0, 2\pi)$

63. $f(x) = x^2e^x, f'(x) = x^2e^x + 2xe^x$

64. $f(x) = xe^{-x}, f'(x) = e^{-x} - xe^{-x}$

65. $f(x) = x\ln x, f'(x) = \ln x + 1$

66. $f(x) = \dfrac{\ln x}{x}, f'(x) = \dfrac{1 - \ln x}{x^2}$

67. MODELING DATA

The projected populations y (in thousands) of New Jersey for selected years from 2015 to 2030 are shown in the table. (Source: U.S. Census Bureau)

Year	Population, y (in thousands)
2015	9256
2020	9462
2025	9637
2030	9802

(a) Use the *regression* feature of a graphing utility to find a quadratic model for the data. Let t represent the year, with $t = 15$ corresponding to 2015.

(b) Use the graphing utility to graph the model found in part (a). Estimate the slope of the graph when $t = 20$, and interpret the result.

(c) Find the derivative of the model in part (a). Then evaluate the derivative for $t = 20$.

(d) Write a brief statement regarding your results for parts (a) through (c).

68. MODELING DATA

The data in the table show the number N (in thousands) of books sold when the price per book is p (in dollars).

Price, p	Number of books, N (in thousands)
\$10	900
\$15	630
\$20	396
\$25	227
\$30	102
\$35	36

(a) Use the *regression* feature of a graphing utility to find a quadratic model for the data.

(b) Use the graphing utility to graph the model found in part (a). Estimate the slopes of the graph when $p = \$15$ and $p = \$30$.

(c) Use the graphing utility to graph the tangent lines to the model when $p = \$15$ and $p = \$30$. Compare the slopes given by the graphing utility with your estimates in part (b).

(d) The slopes of the tangent lines at $p = \$15$ and $p = \$30$ are not the same. Explain what this means to the company selling the books.

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69. *Why you should learn it* *(p. 770)* A spherical balloon is inflated. The volume V is approximated by the formula $V(r) = \frac{4}{3}\pi r^3$, where r is the radius.

(a) Find the derivative of V with respect to r.

(b) Evaluate the derivative when the radius is 4 inches.

(c) What type of unit would be applied to your answer in part (b)? Explain.

70. **Rate of Change** An approximately spherical benign tumor is reducing in size. The surface area S is given by the formula $S(r) = 4\pi r^2$, where r is the radius.

(a) Find the derivative of S with respect to r.

(b) Evaluate the derivative when the radius is 3 millimeters.

(c) What type of unit would be applied to your answer in part (b)? Explain.

71. **Vertical Motion** A water balloon is thrown upward from the top of an 80-foot building with a velocity of 64 feet per second. The height or displacement s (in feet) of the balloon can be modeled by the position function $s(t) = -16t^2 + 64t + 80$, where t is the time in seconds from when it was thrown.

(a) Find a formula for the instantaneous rate of change of the balloon.

(b) Find the average rate of change of the balloon after the first three seconds of flight. Explain your results.

(c) Find the time at which the balloon reaches its maximum height. Explain your method.

(d) Velocity is given by the derivative of the position function. Find the velocity of the balloon as it impacts the ground.

(e) Use a graphing utility to graph the model and verify your results for parts (a)–(d).

72. **Vertical Motion** A coin is dropped from the top of a 120-foot building. The height or displacement s (in feet) of the coin can be modeled by the position function $s(t) = -16t^2 + 120$, where t is the time in seconds from when it was dropped.

(a) Find a formula for the instantaneous rate of change of the coin.

(b) Find the average rate of change of the coin after the first two seconds of free fall. Explain your results.

(c) Velocity is given by the derivative of the position function. Find the velocity of the coin as it impacts the ground.

(d) Find the time when the coin's velocity is -70 feet per second.

(e) Use a graphing utility to graph the model and verify your results for parts (a)–(d).

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Conclusions

True or False? **In Exercises 73 and 74, determine whether the statement is true or false. Justify your answer.**

73. The slope of the graph of $y = x^2$ is different at every point on the graph of f.

74. A tangent line to a graph can intersect the graph only at the point of tangency.

Graphing the Derivative of a Function **In Exercises 75–78, match the function with the graph of its *derivative*. It is not necessary to find the derivative of the function. [The graphs are labeled (a), (b), (c), and (d).]**

(a)

(b)

(c)

(d)

75. $f(x) = \sqrt{x}$

76. $f(x) = \dfrac{1}{x}$

77. $f(x) = |x|$

78. $f(x) = x^3$

79. **Think About It** Sketch the graph of a function whose derivative is always positive.

80. **CAPSTONE** Consider the graph of a function f.

(a) Explain how you can use a secant line to approximate the tangent line at $(x, f(x))$.

(b) Explain how you can use the limit process to find the exact slope of the tangent line at $(x, f(x))$.

81. **Think About It** Sketch the graph of a function for which $f'(x) < 0$ for $x < 1$, $f'(x) \geq 0$ for $x > 1$, and $f'(1) = 0$.

Cumulative Mixed Review

Sketching the Graph of a Rational Function **In Exercises 82 and 83, sketch the graph of the rational function.**

82. $f(x) = \dfrac{1}{x^2 - x - 2}$

83. $f(x) = \dfrac{x - 2}{x^2 - 4x + 3}$

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11.4 Limits at Infinity and Limits of Sequences

Limits at Infinity and Horizontal Asymptotes

As pointed out at the beginning of this chapter, there are two basic problems in calculus: finding **tangent lines** and finding the **area** of a region. In Section 11.3, you saw how limits can be used to solve the tangent line problem. In this section and the next, you will see how a different type of limit, a *limit at infinity*, can be used to solve the area problem. To get an idea of what is meant by a limit at infinity, consider the function

$$f(x) = (x + 1)/(2x).$$

The graph of f is shown in Figure 11.29. From earlier work, you know that $y = \frac{1}{2}$ is a horizontal asymptote of the graph of this function. Using limit notation, this can be written as follows.

$$\lim_{x \to -\infty} f(x) = \frac{1}{2} \qquad \text{Horizontal asymptote to the left}$$

$$\lim_{x \to \infty} f(x) = \frac{1}{2} \qquad \text{Horizontal asymptote to the right}$$

These limits mean that the value of $f(x)$ gets arbitrarily close to $\frac{1}{2}$ as x decreases or increases without bound.

Figure 11.29

What you should learn

- Evaluate limits of functions at infinity.
- Find limits of sequences

Why you should learn it

Finding limits at infinity is useful in analyzing functions that model real-life situations. For instance, in Exercise 60 on page 788, you are asked to find a limit at infinity to decide whether you can use a given model to predict the payroll of the legislative branch of the United States government.

Definition of Limits at Infinity

If f is a function and L_1 and L_2 are real numbers, then the statements

$$\lim_{x \to -\infty} f(x) = L_1 \qquad \text{Limit as } x \text{ approaches } -\infty$$

and

$$\lim_{x \to \infty} f(x) = L_2 \qquad \text{Limit as } x \text{ approaches } \infty$$

denote the **limits at infinity.** The first statement is read "*the limit of $f(x)$ as x approaches $-\infty$ is L_1,*" and the second is read "*the limit of $f(x)$ as x approaches ∞ is L_2.*"

Technology Tip

Recall from Section 2.7 that some graphing utilities have difficulty graphing rational functions. In this text, rational functions are graphed using the *dot* mode of a graphing utility, and a blue curve is placed behind the graphing utility's display to indicate where the graph should appear.

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To help evaluate limits at infinity, you can use the following definition.

Limits at Infinity

If r is a positive real number, then

$$\lim_{x\to\infty} \frac{1}{x^r} = 0.$$ Limit toward the right

Furthermore, if x^r is defined when $x < 0$, then

$$\lim_{x\to-\infty} \frac{1}{x^r} = 0.$$ Limit toward the left

Explore the Concept

Use a graphing utility to graph the two functions given by

$$y_1 = \frac{1}{\sqrt{x}} \quad \text{and} \quad y_2 = \frac{1}{\sqrt[3]{x}}$$

in the same viewing window. Why doesn't y_1 appear to the left of the y-axis? How does this relate to the statement at the left about the infinite limit

$$\lim_{x\to-\infty} \frac{1}{x^r}?$$

Limits at infinity share many of the properties of limits listed in Section 11.1. Some of these properties are demonstrated in the next example.

Example 1 Evaluating a Limit at Infinity

Find the limit.

$$\lim_{x\to\infty} \left(4 - \frac{3}{x^2}\right)$$

Algebraic Solution

Use the properties of limits listed in Section 11.1.

$$\begin{aligned}\lim_{x\to\infty} \left(4 - \frac{3}{x^2}\right) &= \lim_{x\to\infty} 4 - \lim_{x\to\infty} \frac{3}{x^2}\\ &= \lim_{x\to\infty} 4 - 3\left(\lim_{x\to\infty} \frac{1}{x^2}\right)\\ &= 4 - 3(0)\\ &= 4\end{aligned}$$

So, the limit of

$$f(x) = 4 - \frac{3}{x^2}$$

as x approaches ∞ is 4.

CHECKPOINT Now try Exercise 13.

Graphical Solution

Use a graphing utility to graph

$$y = 4 - \frac{3}{x^2}.$$

Figure 11.30

From Figure 11.30, you can estimate the limit to be 4. Note in the figure that the line $y = 4$ is a horizontal asymptote to the right.

In Figure 11.30, it appears that the line

$$y = 4$$

is also a horizontal asymptote *to the left*. You can verify this by showing that

$$\lim_{x\to-\infty} \left(4 - \frac{3}{x^2}\right) = 4.$$

The graph of a rational function need not have a horizontal asymptote. When it does, however, its left and right asymptotes must be the same.

When evaluating limits at infinity for more complicated rational functions, divide the numerator and denominator by the *highest-powered term* in the denominator. This enables you to evaluate each limit using the limits at infinity at the top of this page.

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Example 2 Comparing Limits at Infinity

Find the limit as x approaches ∞ for each function.

a. $f(x) = \dfrac{-2x + 3}{3x^2 + 1}$

b. $f(x) = \dfrac{-2x^2 + 3}{3x^2 + 1}$

c. $f(x) = \dfrac{-2x^3 + 3}{3x^2 + 1}$

Solution

In each case, begin by dividing both the numerator and denominator by

$$x^2$$

the highest-powered term in the denominator.

a. $$\lim_{x\to\infty} \frac{-2x + 3}{3x^2 + 1} = \lim_{x\to\infty} \frac{-\dfrac{2}{x} + \dfrac{3}{x^2}}{3 + \dfrac{1}{x^2}}$$

$$= \frac{-0 + 0}{3 + 0}$$

$$= 0$$

b. $$\lim_{x\to\infty} \frac{-2x^2 + 3}{3x^2 + 1} = \lim_{x\to\infty} \frac{-2 + \dfrac{3}{x^2}}{3 + \dfrac{1}{x^2}}$$

$$= \frac{-2 + 0}{3 + 0}$$

$$= -\frac{2}{3}$$

c. $$\lim_{x\to\infty} \frac{-2x^3 + 3}{3x^2 + 1} = \lim_{x\to\infty} \frac{-2x + \dfrac{3}{x^2}}{3 + \dfrac{1}{x^2}}$$

In this case, you can conclude that the limit does not exist because the numerator decreases without bound as the denominator approaches 3.

CHECKPOINT Now try Exercise 19.

In Example 2, observe that when the degree of the numerator is less than the degree of the denominator, as in part (a), the limit is 0. When the degrees of the numerator and denominator are equal, as in part (b), the limit is the ratio of the coefficients of the highest-powered terms. When the degree of the numerator is greater than the degree of the denominator, as in part (c), the limit does not exist.

This result seems reasonable when you realize that for large values of x, the highest-powered term of a polynomial is the most "influential" term. That is, a polynomial tends to behave as its highest-powered term behaves as x approaches positive or negative infinity.

Explore the Concept

Use a graphing utility to complete the table below to verify that

$$\lim_{x\to\infty} \frac{1}{x} = 0.$$

x	10^0	10^1	10^2
$\frac{1}{x}$			

x	10^3	10^4	10^5
$\frac{1}{x}$			

Make a conjecture about

$$\lim_{x\to 0} \frac{1}{x}.$$

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Limits at Infinity for Rational Functions

Consider the rational function

$$f(x) = \frac{N(x)}{D(x)}$$

where

$$N(x) = a_n x^n + \cdot \cdot \cdot + a_0 \qquad \text{and} \qquad D(x) = b_m x^m + \cdot \cdot \cdot + b_0.$$

The limit of $f(x)$ as x approaches positive or negative infinity is as follows.

$$\lim_{x \to \pm\infty} f(x) = \begin{cases} 0, & n < m \\ \dfrac{a_n}{b_m}, & n = m \end{cases}$$

If $n > m$, then the limit does not exist.

Example 3 Finding the Average Cost

You are manufacturing greeting cards that cost \$0.50 per card to produce. Your initial investment is \$5000, which implies that the total cost C of producing x cards is given by

$$C = 0.50x + 5000.$$

The average cost $\overline{C}$ per card is given by

$$\overline{C} = \frac{C}{x} = \frac{0.50x + 5000}{x}.$$

Find the average cost per card when (a) $x = 1000$, (b) $x = 10{,}000$, and (c) $x = 100{,}000$. (d) What is the limit of $\overline{C}$ as x approaches infinity?

Solution

a. When $x = 1000$, the average cost per card is

$$\overline{C} = \frac{0.50(1000) + 5000}{1000} \qquad x = 1000$$

$$= \$5.50.$$

b. When $x = 10{,}000$, the average cost per card is

$$\overline{C} = \frac{0.50(10{,}000) + 5000}{10{,}000} \qquad x = 10{,}000$$

$$= \$1.00.$$

c. When $x = 100{,}000$, the average cost per card is

$$\overline{C} = \frac{0.50(100{,}000) + 5000}{100{,}000} \qquad x = 100{,}000$$

$$= \$0.55.$$

d. As x approaches infinity, the limit of $\overline{C}$ is

$$\lim_{x \to \infty} \frac{0.50x + 5000}{x} = \$0.50. \qquad x \to \infty$$

The graph of $\overline{C}$ is shown in Figure 11.31.

As $x \to \infty$, the average cost per card approaches \$0.50.
Figure 11.31

CHECKPOINT Now try Exercise 57.

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Limits of Sequences

Limits of sequences have many of the same properties as limits of functions. For instance, consider the sequence whose nth term is $a_n = 1/2^n$.

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$$

As n increases without bound, the terms of this sequence get closer and closer to 0, and the sequence is said to **converge** to 0. Using limit notation, you can write

$$\lim_{n\to\infty} \frac{1}{2^n} = 0.$$

The following relationship shows how limits of functions of x can be used to evaluate the limit of a sequence.

Limit of a Sequence

Let L be a real number. Let f be a function of a real variable such that

$$\lim_{x\to\infty} f(x) = L.$$

If $\{a_n\}$ is a sequence such that

$$f(n) = a_n$$

for every positive integer n, then

$$\lim_{n\to\infty} a_n = L.$$

A sequence that does not converge is said to **diverge.** For instance, the sequence

$$1, -1, 1, -1, 1, \ldots$$

diverges because it does not approach a unique number.

Example 4 Finding the Limit of a Sequence

Find the limit of each sequence. (Assume n begins with 1.)

a. $a_n = \dfrac{2n+1}{n+4}$

b. $b_n = \dfrac{2n+1}{n^2+4}$

c. $c_n = \dfrac{2n^2+1}{4n^2}$

Solution

a. $\displaystyle\lim_{n\to\infty} \frac{2n+1}{n+4} = 2$ $\qquad \frac{3}{5}, \frac{5}{6}, \frac{7}{7}, \frac{9}{8}, \frac{11}{9}, \frac{13}{10}, \ldots \to 2$

b. $\displaystyle\lim_{n\to\infty} \frac{2n+1}{n^2+4} = 0$ $\qquad \frac{3}{5}, \frac{5}{8}, \frac{7}{13}, \frac{9}{20}, \frac{11}{29}, \frac{13}{40}, \ldots \to 0$

c. $\displaystyle\lim_{n\to\infty} \frac{2n^2+1}{4n^2} = \frac{1}{2}$ $\qquad \frac{3}{4}, \frac{9}{16}, \frac{19}{36}, \frac{33}{64}, \frac{51}{100}, \frac{73}{144}, \ldots \to \frac{1}{2}$

CHECKPOINT Now try Exercise 43.

Study Tip

You can use the definition of limits at infinity for rational functions on page 783 to verify the limits of the sequences in Example 4.

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In the next section, you will encounter limits of sequences such as that shown in Example 5. A strategy for evaluating such limits is to begin by writing the nth term in standard rational function form. Then you can determine the limit by comparing the degrees of the numerator and denominator, as shown on page 783.

Example 5 Finding the Limit of a Sequence

Find the limit of the sequence whose nth term is

$$a_n = \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right].$$

Algebraic Solution

Begin by writing the nth term in standard rational function form—as the ratio of two polynomials.

$$a_n = \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] \qquad \text{Write original } n\text{th term.}$$

$$= \frac{8(n)(n+1)(2n+1)}{6n^3} \qquad \text{Multiply fractions.}$$

$$= \frac{8n^3 + 12n^2 + 4n}{3n^3} \qquad \text{Write in standard rational form.}$$

From this form, you can see that the degree of the numerator is equal to the degree of the denominator. So, the limit of the sequence is the ratio of the coefficients of the highest-powered terms.

$$\lim_{n\to\infty} \frac{8n^3 + 12n^2 + 4n}{3n^3} = \frac{8}{3}$$

CHECKPOINT Now try Exercise 53.

Numerical Solution

Enter the sequence

$$a_n = \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$$

into a graphing utility and create a table. (Be sure the graphing utility is set to *sequence* mode.)

As n increases, a_n approaches 2.667.

Figure 11.32

From Figure 11.32, you can estimate that as n approaches ∞, a_n gets closer and closer to $2.667 \approx \frac{8}{3}$.

The result of Example 5 is supported by Figure 11.33, which shows the graph of the sequence a_n and $y = 8/3$.

Figure 11.33

Explore the Concept

In Figure 11.32, the value of a_n approaches its limit of $\frac{8}{3}$ rather slowly. (The first term to be accurate to three decimal places is $a_{4801} \approx 2.667$.) Each of the following sequences converges to 0. Which converges the quickest? Which converges the slowest? Why? Write a short paragraph discussing your conclusions.

a. $a_n = \dfrac{1}{n}$ **b.** $b_n = \dfrac{1}{n^2}$ **c.** $c_n = \dfrac{1}{2^n}$ **d.** $d_n = \dfrac{1}{n!}$ **e.** $h_n = \dfrac{2^n}{n!}$

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11.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

1. The line $y = 5$ is a horizontal asymptote to the right of the graph of a function f. What is the limit of f as x approaches infinity?
2. Given $\lim_{x\to\infty} f(x) = 2$ for a rational function f, how does the degree of the numerator compare with the degree of the denominator?

In Exercises 3 and 4, fill in the blank.

3. A sequence that has a limit is said to _____ .
4. A sequence that does not have a limit is said to _____ .

Procedures and Problem Solving

Identifying the Graph of an Equation **In Exercises 5–12, match the function with its graph, using horizontal asymptotes as aids. [The graphs are labeled (a), (b), (c), (d), (e), (f), (g), and (h).]**

(a)

(b)

(c)

(d)

(e)

(f)

(g)

(h)

5. $f(x) = \dfrac{4x^2}{x^2 + 1}$
6. $f(x) = \dfrac{x^2}{x^2 + 1}$
7. $f(x) = 4 - \dfrac{1}{x^2}$
8. $f(x) = x + \dfrac{1}{x}$
9. $f(x) = \dfrac{x^2}{x^2 - 1}$
10. $f(x) = \dfrac{2x + 1}{x - 2}$
11. $f(x) = \dfrac{1 - 2x}{x - 2}$
12. $f(x) = \dfrac{1 - 4x^2}{x^2 - 4}$

Evaluating a Limit at Infinity **In Exercises 13–32, find the limit (if it exists). If the limit does not exist, explain why. Use a graphing utility to verify your result graphically.**

13. $\lim_{x\to\infty} \dfrac{3}{x^2}$
14. $\lim_{x\to\infty} \dfrac{5}{2x}$
15. $\lim_{x\to\infty} \dfrac{3 + x}{3 - x}$
16. $\lim_{x\to\infty} \dfrac{2 - 7x}{2 + 3x}$
17. $\lim_{x\to-\infty} \dfrac{5x - 2}{6x + 1}$
18. $\lim_{x\to-\infty} \dfrac{5 - 3x}{x + 4}$
19. $\lim_{x\to-\infty} \dfrac{4x^2 - 3}{2 - x^2}$
20. $\lim_{x\to-\infty} \dfrac{x^2 + 3}{5x^2 - 4}$
21. $\lim_{t\to\infty} \dfrac{t^2}{t + 3}$
22. $\lim_{y\to\infty} \dfrac{4y^4}{y^2 + 3}$
23. $\lim_{t\to\infty} \dfrac{4t^2 + 3t - 1}{3t^2 + 2t - 5}$
24. $\lim_{x\to\infty} \dfrac{5 - 6x - 3x^2}{2x^2 + x + 4}$
25. $\lim_{y\to-\infty} \dfrac{3 + 8y - 4y^2}{3 - y - 2y^2}$
26. $\lim_{t\to-\infty} \dfrac{t^2 + 9t - 10}{2 + 4t - 3t^2}$
27. $\lim_{x\to-\infty} \dfrac{-(x^2 + 3)}{(2 - x)^2}$
28. $\lim_{x\to\infty} \dfrac{2x^2 - 6}{(x - 1)^2}$
29. $\lim_{x\to-\infty} \left[\dfrac{x}{(x + 1)^2} - 4\right]$
30. $\lim_{x\to\infty} \left[7 + \dfrac{2x^2}{(x + 3)^2}\right]$
31. $\lim_{t\to\infty} \left(\dfrac{1}{3t^2} - \dfrac{5t}{t + 2}\right)$
32. $\lim_{x\to\infty} \left[\dfrac{x}{2x + 1} + \dfrac{3x^2}{(x - 3)^2}\right]$

Algebraic-Graphical-Numerical **In Exercises 33–38, (a) complete the table and numerically estimate the limit as x approaches infinity, (b) use a graphing utility to graph the function and estimate the limit graphically, and (c) find the limit algebraically.**

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$							

33. $f(x) = \dfrac{3x}{1 - x}$
34. $f(x) = \dfrac{x^2}{x^2 + 4}$

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35. $f(x) = \dfrac{2x}{1 - x^2}$

36. $f(x) = \dfrac{2x + 1}{x^2 - 1}$

37. $f(x) = 1 - \dfrac{3}{x^2}$

38. $f(x) = 2 + \dfrac{1}{x}$

Estimating the Limit at Infinity **In Exercises 39–42, (a) complete the table and numerically estimate the limit as x approaches infinity and (b) use a graphing utility to graph the function and estimate the limit graphically.**

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$							

39. $f(x) = x - \sqrt{x^2 + 2}$

40. $f(x) = 3x - \sqrt{9x^2 + 1}$

41. $f(x) = 3\left(2x - \sqrt{4x^2 + x}\right)$

42. $f(x) = 4\left(4x - \sqrt{16x^2 - x}\right)$

Finding the Limit of a Sequence **In Exercises 43–52, write the first five terms of the sequence and find the limit of the sequence (if it exists). If the limit does not exist, explain why. Assume n begins with 1.**

43. $a_n = \dfrac{n + 1}{n^2 + 1}$

44. $a_n = \dfrac{n}{n^2 + 1}$

45. $a_n = \dfrac{n}{2n + 1}$

46. $a_n = \dfrac{4n - 1}{n + 3}$

47. $a_n = \dfrac{n^2}{3n + 2}$

48. $a_n = \dfrac{4n^2 + 1}{2n}$

49. $a_n = \dfrac{(n + 1)!}{n!}$

50. $a_n = \dfrac{(3n - 1)!}{(3n + 1)!}$

51. $a_n = \dfrac{(-1)^n}{n}$

52. $a_n = \dfrac{(-1)^{n+1}}{n^2}$

Finding the Limit of a Sequence **In Exercises 53–56, use a graphing utility to complete the table and estimate the limit of the sequence as n approaches infinity. Then find the limit algebraically.**

n	10^0	10^1	10^2	10^3
a_n				

53. $a_n = \dfrac{1}{n}\left(n + \dfrac{1}{n}\left[\dfrac{n(n + 1)}{2}\right]\right)$

54. $a_n = \dfrac{4}{n}\left(n + \dfrac{4}{n}\left[\dfrac{n(n + 1)}{2}\right]\right)$

55. $a_n = \dfrac{10}{n^3}\left[\dfrac{n(n + 1)(3n + 1)}{6}\right]$

56. $a_n = \dfrac{3n(n + 1)}{n^2} - \dfrac{4}{n^4}\left[\dfrac{n(n + 1)}{2}\right]^2$

57. **Business** The cost function for a certain model of a personal digital assistant (PDA) is given by $C = 13.50x + 45{,}750$, where C is the cost (in dollars) and x is the number of PDAs produced.

(a) Write a model for the average cost per unit produced.

(b) Find the average costs per unit when $x = 100$ and $x = 1000$.

(c) Determine the limit of the average cost function as x approaches infinity. Explain the meaning of the limit in the context of the problem.

58. **Business** The cost function for a new supermarket to recycle x tons of organic material is given by $C = 60x + 1650$, where C is the cost (in dollars).

(a) Write a model for the average cost per ton of organic material recycled.

(b) Find the average costs of recycling 100 tons and 1000 tons of organic material.

(c) Determine the limit of the average cost function as x approaches infinity. Explain the meaning of the limit in the context of the problem.

59. MODELING DATA

The table shows the numbers R (in thousands) of United States military reserve personnel in the years 2002 through 2008. (Source: U.S. Dept. of Defense)

Year	Reserves, R (in thousands)
2002	1222
2003	1189
2004	1167
2005	1136
2006	1120
2007	1110
2008	1100

A model for the data is given by

$$R(t) = \frac{61.018t^2 + 1260.64}{0.0578t^2 + 1}, \quad 2 \le t \le 8$$

where t represents the year, with $t = 2$ corresponding to 2002.

(a) Use a graphing utility to create a scatter plot of the data and graph the model in the same viewing window. How do they compare?

(b) Use the model to predict the numbers of reserves in 2009 and 2010.

(c) Find the limit of the model as $t \to \infty$ and interpret its meaning in the context of the situation.

(d) Is this a good model for predicting future numbers of reserves? Explain.

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60. ***Why you should learn it*** *(p. 780)* The table shows the annual payrolls P (in millions of dollars) of the legislative branch of the United States government for the years 2001 through 2008. (Source: U.S. Office of Personnel Management)

Year	Payroll, P (in millions of dollars)
2001	1682
2002	1781
2003	1908
2004	1977
2005	2048
2006	2109
2007	2119
2008	2162

A model for the data is given by

$$P(t) = \frac{182.4312t^2 + 1634.39}{0.0808t^2 + 1}, \quad 1 \le t \le 8$$

where t represents the year, with $t = 1$ corresponding to 2001.

(a) Use a graphing utility to create a scatter plot of the data and graph the model in the same viewing window. How do they compare?

(b) Use the model to predict the payrolls in 2009 and 2010.

(c) Find the limit of the model as $t \to \infty$ and interpret its meaning in the context of the situation.

(d) Is this a good model for predicting the annual payrolls in future years? Explain.

Conclusions

True or False? **In Exercises 61–64, determine whether the statement is true or false. Justify your answer.**

61. Every rational function has a horizontal asymptote.

62. If a rational function f has a horizontal asymptote to the right, then the limit of $f(x)$ as x approaches $-\infty$ exists.

63. If a sequence converges, then it has a limit.

64. When the degrees of the numerator and denominator of a rational function are equal, the limit as x goes to infinity does not exist.

65. **Think About It** Find functions f and g such that both $f(x)$ and $g(x)$ increase without bound as x approaches ∞, but $\lim_{x\to\infty} [f(x) - g(x)] \neq \infty$.

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66. **Think About It** Use a graphing utility to graph the function $f(x) = x/\sqrt{x^2 + 1}$. How many horizontal asymptotes does the function appear to have? What are the horizontal asymptotes?

Exploration **In Exercises 67–70, use a graphing utility to create a scatter plot of the terms of the sequence. Determine whether the sequence converges or diverges. If it converges, estimate its limit.**

67. $a_n = 4\left(\frac{2}{3}\right)^n$

68. $a_n = 3\left(\frac{3}{2}\right)^n$

69. $a_n = \dfrac{3[1 - (1.5)^n]}{1 - 1.5}$

70. $a_n = \dfrac{3[1 - (0.5)^n]}{1 - 0.5}$

71. **Error Analysis** Describe the error in finding the limit.

$$\cancel{\lim_{x\to\infty} \frac{1 - 2x - x^2}{4x^2 + 1} = 0}$$

72. **CAPSTONE** Let f be a rational function whose graph has the line $y = 3$ as a horizontal asymptote to the right.

(a) Find $\lim_{x\to\infty} f(x)$.

(b) Does the graph of f have a horizontal asymptote to the left? Explain your reasoning.

(c) Find $\lim_{x\to-\infty} f(x)$.

(d) Let $3x^3 - x + 4$ be the numerator of f. Which of the following expressions are possible denominators of f?

(i) $x^2 + 1$ (ii) $x^3 + 1$ (iii) $x^4 + 1$

Cumulative Mixed Review

Sketching Transformations **In Exercises 73 and 74, sketch the graphs of y and each transformation on the same rectangular coordinate system.**

73. $y = x^4$

(a) $f(x) = (x + 3)^4$ (b) $f(x) = x^4 - 1$

(c) $f(x) = -2 + x^4$ (d) $f(x) = \frac{1}{2}(x - 4)^4$

74. $y = x^3$

(a) $f(x) = (x + 2)^3$ (b) $f(x) = 3 + x^3$

(c) $f(x) = 2 - \frac{1}{4}x^3$ (d) $f(x) = 3(x + 1)^3$

Using Sigma Notation **In Exercises 75–78, find the sum.**

75. $\sum_{i=1}^{6} (2i + 3)$

76. $\sum_{i=0}^{4} 5i^2$

77. $\sum_{k=1}^{10} 15$

78. $\sum_{k=0}^{8} \dfrac{3}{k^2 + 1}$

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11.5 The Area Problem

Limits of Summations

Earlier in the text, you used the concept of a limit to obtain a formula for the sum S of an infinite geometric series

$$S = a_1 + a_1r + a_1r^2 + \cdots = \sum_{i=1}^{\infty} a_1r^{i-1} = \frac{a_1}{1-r}, \quad |r| < 1.$$

Using limit notation, this sum can be written as

$$\begin{aligned} S &= \lim_{n\to\infty} \sum_{i=1}^{n} a_1r^{i-1} \\ &= \lim_{n\to\infty} \frac{a_1(1-r^n)}{1-r} \\ &= \frac{a_1}{1-r}. \end{aligned}$$

$\lim_{n\to\infty} r^n = 0$ for $0 < |r| < 1$

The following summation formulas and properties are used to evaluate finite and infinite summations.

Summation Formulas and Properties

1. $\sum_{i=1}^{n} c = cn$, c is a constant.
2. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
3. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
4. $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$
5. $\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$
6. $\sum_{i=1}^{n} ka_i = k\sum_{i=1}^{n} a_i$, k is a constant.

What you should learn

- Find limits of summations.
- Use rectangles to approximate and limits of summations to find areas of plane regions.

Why you should learn it

Limits of summations are useful in determining areas of plane regions. For instance, in Exercise 46 on page 795, you are asked to find the limit of a summation to determine the area of a parcel of land bounded by a stream and two roads.

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Example 1 Evaluating a Summation

Evaluate the summation.

$$\sum_{i=1}^{200} i = 1 + 2 + 3 + 4 + \cdots + 200$$

Solution

Using Formula 2 with $n = 200$, you can write

$$\begin{aligned} \sum_{i=1}^{n} i &= \frac{n(n+1)}{2} \\ \sum_{i=1}^{200} i &= \frac{200(200+1)}{2} \\ &= \frac{40{,}200}{2} \\ &= 20{,}100. \end{aligned}$$

CHECKPOINT Now try Exercise 5.

Study Tip

Recall from Section 8.3 that the sum of a finite geometric sequence is given by

$$\sum_{i=1}^{n} a_1r^{i-1} = a_1\left(\frac{1-r^n}{1-r}\right).$$

Furthermore, if $0 < |r| < 1$, then $r^n \to 0$ as $n \to \infty$.

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Example 2 Evaluating a Summation

Evaluate the summation

$$S = \sum_{i=1}^{n} \frac{i+2}{n^2}$$
$$= \frac{3}{n^2} + \frac{4}{n^2} + \frac{5}{n^2} + \cdots + \frac{n+2}{n^2}$$

for $n = 10$, 100, 1000, and 10,000.

Solution

Begin by applying summation formulas and properties to simplify S. In the second line of the solution, note that

$$\frac{1}{n^2}$$

can be factored out of the sum because n is considered to be constant. You could not factor i out of the summation because i is the (variable) index of summation.

$$S = \sum_{i=1}^{n} \frac{i+2}{n^2} \qquad \text{Write original form of summation.}$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}(i+2) \qquad \text{Factor constant } 1/n^2 \text{ out of sum.}$$

$$= \frac{1}{n^2}\left(\sum_{i=1}^{n} i + \sum_{i=1}^{n} 2\right) \qquad \text{Write as two sums.}$$

$$= \frac{1}{n^2}\left[\frac{n(n+1)}{2} + 2n\right] \qquad \text{Apply Formulas 1 and 2.}$$

$$= \frac{1}{n^2}\left(\frac{n^2+5n}{2}\right) \qquad \text{Add fractions.}$$

$$= \frac{n+5}{2n} \qquad \text{Simplify.}$$

Now you can evaluate the sum by substituting the appropriate values of n, as shown in the following table.

n	10	100	1000	10,000
$\sum_{i=1}^{n} \frac{i+2}{n^2} = \frac{n+5}{2n}$	0.75	0.525	0.5025	0.50025

CHECKPOINT Now try Exercise 15(a) and (b).

In Example 2, note that the sum appears to approach a limit as n increases. To find the limit of

$$\frac{n+5}{2n}$$

as n approaches infinity, you can use the techniques from Section 11.4 to write

$$\lim_{n\to\infty} \frac{n+5}{2n} = \frac{1}{2}.$$

Technology Tip

Some graphing utilities have a *sum sequence* feature that is useful for computing summations. For instructions on how to use the *sum sequence* feature, see Appendix A; for specific keystrokes, go to this textbook's *Companion Website.*

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Be sure you notice the strategy used in Example 2. Rather than separately evaluating the sums

$$\sum_{i=1}^{10} \frac{i+2}{n^2}, \qquad \sum_{i=1}^{100} \frac{i+2}{n^2}, \qquad \sum_{i=1}^{1000} \frac{i+2}{n^2}, \qquad \sum_{i=1}^{10{,}000} \frac{i+2}{n^2}$$

it was more efficient first to convert to rational form using the summation formulas and properties listed on page 789.

$$S = \underbrace{\sum_{i=1}^{n} \frac{i+2}{n^2}}_{\text{Summation form}} = \underbrace{\frac{n+5}{2n}}_{\text{Rational form}}$$

With this rational form, each sum can be evaluated by simply substituting appropriate values of n.

Example 3 Finding the Limit of a Summation

Find the limit of $S(n)$ as $n \to \infty$.

$$S(n) = \sum_{i=1}^{n} \left(1 + \frac{i}{n}\right)^2 \left(\frac{1}{n}\right)$$

Solution

Begin by rewriting the summation in rational form.

$$S(n) = \sum_{i=1}^{n} \left(1 + \frac{i}{n}\right)^2 \left(\frac{1}{n}\right)$$ Write original form of summation.

$$= \sum_{i=1}^{n} \left(\frac{n^2 + 2ni + i^2}{n^2}\right)\left(\frac{1}{n}\right)$$ Square $(1 + i/n)$ and write as a single fraction.

$$= \frac{1}{n^3} \sum_{i=1}^{n} (n^2 + 2ni + i^2)$$ Factor constant $1/n^3$ out of the sum.

$$= \frac{1}{n^3}\left(\sum_{i=1}^{n} n^2 + \sum_{i=1}^{n} 2ni + \sum_{i=1}^{n} i^2\right)$$ Write as three sums.

$$= \frac{1}{n^3}\left\{n^3 + 2n\left[\frac{n(n+1)}{2}\right] + \frac{n(n+1)(2n+1)}{6}\right\}$$ Use summation formulas.

$$= \frac{14n^3 + 9n^2 + n}{6n^3}$$ Simplify.

In this rational form, you can now find the limit as $n \to \infty$.

$$\lim_{n\to\infty} S(n) = \lim_{n\to\infty} \frac{14n^3 + 9n^2 + n}{6n^3} = \frac{14}{6} = \frac{7}{3}$$

CHECKPOINT Now try Exercise 15(c).

Study Tip

As you can see from Example 3, there is a lot of algebra involved in rewriting a summation in rational form. You may want to review simplifying rational expressions if you are having difficulty with this procedure.

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The Area Problem

You now have the tools needed to solve the second basic problem of calculus: the area problem. The problem is to find the *area* of the region R bounded by the graph of a nonnegative, continuous function f, the x-axis, and the vertical lines $x = a$ and $x = b$, as shown in Figure 11.34.

When the region R is a square, a triangle, a trapezoid, or a semicircle, you can find its area by using a geometric formula. For more general regions, however, you must use a different approach—one that involves the limit of a summation. The basic strategy is to use a collection of rectangles of equal width that approximates the region R, as illustrated in Example 4.

Figure 11.34

Example 4 Approximating the Area of a Region

Use the five rectangles in Figure 11.35 to approximate the area of the region bounded by the graph of

$$f(x) = 6 - x^2$$

the x-axis, and the lines $x = 0$ and $x = 2$.

Figure 11.35

Solution

Because the length of the interval along the x-axis is 2 and there are five rectangles, the width of each rectangle is $\frac{2}{5}$. The height of each rectangle can be obtained by evaluating f at the right endpoint of each interval. The five intervals are as follows.

$$\left[0, \frac{2}{5}\right], \quad \left[\frac{2}{5}, \frac{4}{5}\right], \quad \left[\frac{4}{5}, \frac{6}{5}\right], \quad \left[\frac{6}{5}, \frac{8}{5}\right], \quad \left[\frac{8}{5}, \frac{10}{5}\right]$$

Notice that the right endpoint of each interval is $\frac{2}{5}i$ for $i = 1, 2, 3, 4$, and 5. The sum of the areas of the five rectangles is

$$\begin{aligned}
\sum_{i=1}^{5} \overbrace{f\left(\frac{2i}{5}\right)}^{\text{Height}}\overbrace{\left(\frac{2}{5}\right)}^{\text{Width}} &= \sum_{i=1}^{5}\left[6 - \left(\frac{2i}{5}\right)^2\right]\left(\frac{2}{5}\right) \\
&= \frac{2}{5}\left(\sum_{i=1}^{5} 6 - \frac{4}{25}\sum_{i=1}^{5} i^2\right) \\
&= \frac{2}{5}\left(30 - \frac{44}{5}\right) \\
&= \frac{212}{25} \\
&= 8.48.
\end{aligned}$$

So, you can approximate the area of R as 8.48 square units.

CHECKPOINT Now try Exercise 21.

By increasing the number of rectangles used in Example 4, you can obtain closer and closer approximations of the area of the region. For instance, using 25 rectangles of width $\frac{2}{25}$ each, you can approximate the area to be

$$A \approx 9.17 \text{ square units.}$$

The following table shows even better approximations.

n	5	25	100	1000	5000
Approximate area	8.48	9.17	9.29	9.33	9.33

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Based on the procedure illustrated in Example 4, the *exact* **area of a plane region** R is given by the limit of the sum of n rectangles as n approaches ∞.

Area of a Plane Region

Let f be continuous and nonnegative on the interval $[a, b]$. The **area** A of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{f\left(a + \frac{(b-a)i}{n}\right)}_{\text{Height}} \underbrace{\left(\frac{b-a}{n}\right)}_{\text{Width}}.$$

Example 5 Finding the Area of a Region

Find the area of the region bounded by the graph of

$$f(x) = x^2$$

and the x-axis between

$$x = 0 \quad \text{and} \quad x = 1$$

as shown in Figure 11.36.

Figure 11.36

Solution

Begin by finding the dimensions of the rectangles.

$$\textit{Width:}\ \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

$$\textit{Height:}\ f\left(a + \frac{(b-a)i}{n}\right) = f\left(0 + \frac{(1-0)i}{n}\right) = f\left(\frac{i}{n}\right) = \frac{i^2}{n^2}$$

Next, approximate the area as the sum of the areas of n rectangles.

$$\begin{aligned}
A &\approx \sum_{i=1}^{n} f\left(a + \frac{(b-a)i}{n}\right)\left(\frac{b-a}{n}\right) \\
&= \sum_{i=1}^{n} \left(\frac{i^2}{n^2}\right)\left(\frac{1}{n}\right) \\
&= \sum_{i=1}^{n} \frac{i^2}{n^3} \\
&= \frac{1}{n^3}\sum_{i=1}^{n} i^2 \\
&= \frac{1}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] \\
&= \frac{2n^3 + 3n^2 + n}{6n^3}
\end{aligned}$$

Finally, find the exact area by taking the limit as n approaches ∞.

$$A = \lim_{n\to\infty} \frac{2n^3 + 3n^2 + n}{6n^3} = \frac{1}{3}$$

CHECKPOINT Now try Exercise 33.

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11.5 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1 and 2, fill in the blank.

1. $\sum_{i=1}^{n} i =$ _____ **2.** $\sum_{i=1}^{n} i^3 =$ _____

3. Can you obtain a better approximation of the area of the shaded region using 10 rectangles of equal width or 100 rectangles of equal width?

4. Does the limit of the sum of n rectangles as n approaches infinity represent the *exact* area of a plane region or an *approximation* of the area?

Procedures and Problem Solving

Evaluating a Summation **In Exercises 5–12, evaluate the sum using the summation formulas and properties.**

5. $\sum_{i=1}^{60} 7$ **6.** $\sum_{i=1}^{45} 3$

7. $\sum_{i=1}^{20} i^3$ **8.** $\sum_{i=1}^{30} i^2$

9. $\sum_{k=1}^{20} (k^3 + 2)$ **10.** $\sum_{k=1}^{50} (2k + 1)$

11. $\sum_{j=1}^{25} (j^2 + j)$ **12.** $\sum_{j=1}^{10} (j^3 - 3j^2)$

Finding the Limit of a Summation **In Exercises 13–18, (a) rewrite the sum as a rational function $S(n)$. (b) Use $S(n)$ to complete the table. (c) Find $\lim_{n\to\infty} S(n)$.**

n	10^0	10^1	10^2	10^3	10^4
$S(n)$					

13. $\sum_{i=1}^{n} \frac{i^3}{n^4}$ **14.** $\sum_{i=1}^{n} \frac{i}{n^2}$

15. $\sum_{i=1}^{n} \frac{3}{n^3}(1 + i^2)$ **16.** $\sum_{i=1}^{n} \frac{2i + 3}{n^2}$

17. $\sum_{i=1}^{n} \left(\frac{i^2}{n^3} + \frac{2}{n}\right)\left(\frac{1}{n}\right)$ **18.** $\sum_{i=1}^{n} \left[3 - 2\left(\frac{i}{n}\right)\right]\left(\frac{1}{n}\right)$

Approximating the Area of a Region **In Exercises 19–22, approximate the area of the region using the indicated number of rectangles of equal width.**

19. $f(x) = x + 4$

20. $f(x) = 2 - x^2$

21. $f(x) = \frac{1}{4}x^3$ (8 rectangles)

22. $f(x) = \frac{1}{2}(x - 1)^3$ (4 rectangles)

Approximating the Area of a Region **In Exercises 23–26, complete the table to show the approximate area of the region using the indicated numbers n of rectangles of equal width.**

n	4	8	20	50
Approximate area				

23. $f(x) = -\frac{1}{3}x + 4$

24. $f(x) = 9 - x^2$

25. $f(x) = \frac{1}{9}x^3$

26. $f(x) = 3 - \frac{1}{4}x^3$

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The Area of a Region **In Exercises 27–32, use the given expression for the sum of the areas of n rectangles. For each finite value of n in the table, approximate the area of the region bounded by the graph of f and the x-axis over the specified interval. Then find the exact area as $n \to \infty$.**

n	4	8	20	50	100	∞
Area						

	Function	*Interval*	*Sum of areas of n rectangles*
27.	$f(x) = 2x + 5$	$[0, 4]$	$\dfrac{36n^2 + 16n}{n^2}$
28.	$f(x) = 3x + 1$	$[0, 4]$	$\dfrac{28n^2 + 24n}{n^2}$
29.	$f(x) = 9 - x^2$	$[0, 2]$	$\dfrac{46n^3 - 12n^2 - 4n}{3n^3}$
30.	$f(x) = x^2 + 1$	$[4, 6]$	$\dfrac{158n^3 + 60n^2 + 4n}{3n^3}$
31.	$f(x) = \frac{1}{2}x + 4$	$[-1, 3]$	$\dfrac{18n^2 + 4n}{n^2}$
32.	$f(x) = \frac{1}{2}x + 1$	$[-2, 2]$	$\dfrac{4n^2 + 4n}{n^2}$

Finding the Area of a Region **In Exercises 33–44, use the limit process to find the area of the region bounded by the graph of the function and the x-axis over the specified interval.**

	Function	*Interval*
33.	$f(x) = 4x + 1$	$[0, 1]$
34.	$f(x) = 3x + 2$	$[0, 2]$
35.	$f(x) = -x + 4$	$[0, 3]$
36.	$f(x) = 3x - 6$	$[2, 5]$
37.	$f(x) = 16 - x^2$	$[0, 4]$
38.	$f(x) = x^2 + 2$	$[0, 1]$
39.	$g(x) = 1 - x^3$	$[0, 1]$
40.	$g(x) = 64 - x^3$	$[0, 3]$
41.	$g(x) = 2x - x^3$	$[0, 1]$
42.	$g(x) = 4x - x^3$	$[0, 2]$
43.	$f(x) = x^2 + 4x$	$[0, 6]$
44.	$f(x) = x^2 - x^3$	$[0, 1]$

45. **Geometry** The boundaries of a parcel of land are two edges modeled by the coordinate axes and a stream modeled by the equation

$$y = (-3.0 \times 10^{-6})x^3 + 0.002x^2 - 1.05x + 400.$$

Use a graphing utility to graph the equation. Find the area of the property. (All distances are measured in feet.)

Lukasz Laska/iStockphoto.com

46. ***Why you should learn it*** *(p. 789)* The table shows the measurements (in feet) of a lot bounded by a stream and two straight roads that meet at right angles (see figure).

x	0	50	100	150
y	450	362	305	268

x	200	250	300
y	245	156	0

(a) Use the *regression* feature of a graphing utility to find a model of the form $y = ax^3 + bx^2 + cx + d$.

(b) Use the graphing utility to plot the data and graph the model in the same viewing window.

(c) Use the model in part (a) to estimate the area of the lot.

Conclusions

True or False? **In Exercises 47 and 48, determine whether the statement is true or false. Justify your answer.**

47. The sum of the first n positive integers is $n(n + 1)/2$.

48. The exact area of a region is given by the limit of the sum of n rectangles as n approaches 0.

49. **Think About It** Determine which value best approximates the area of the region shown in the graph. (Make your selection on the basis of the sketch of the region and *not* by performing any calculations.)

(a) -2

(b) 1

(c) 4

(d) 6

(e) 9

50. **CAPSTONE** Describe the process of finding the areas of a region bounded by the graph of a nonnegative, continuous function f, the x-axis, and the vertical lines $x = a$ and $x = b$.

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11 Chapter Summary

	What did you learn?	Explanation and Examples	Review Exercises
11.1	**Understand the limit concept *(p. 750)* and use the definition of a limit to estimate limits *(p. 751)*.**	If $f(x)$ becomes arbitrarily close to a unique number L as x approaches c from either side, then the limit of $f(x)$ as x approaches c is L. This is written as $\lim_{x \to c} f(x) = L$.	1–4
	Determine whether limits of functions exist *(p. 753)*.	**Conditions Under Which Limits Do Not Exist** The limit of $f(x)$ as $x \to c$ does not exist under any of the following conditions. **1.** $f(x)$ approaches a different number from the right side of c than it approaches from the left side of c. **2.** $f(x)$ increases or decreases without bound as x approaches c. **3.** $f(x)$ oscillates between two fixed values as x approaches c.	5–8
	Use properties of limits and direct substitution to evaluate limits *(p. 755)*.	Let b and c be real numbers and let n be a positive integer. **1.** $\lim_{x \to c} b = b$ **2.** $\lim_{x \to c} x = c$ **3.** $\lim_{x \to c} x^n = c^n$ **4.** $\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$, for n even and $c > 0$ **Properties of Limits** Let b and c be real numbers, let n be a positive integer, and let f and g be functions where $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = K$. **1.** $\lim_{x \to c} [bf(x)] = bL$ **2.** $\lim_{x \to c} [f(x) \pm g(x)] = L \pm K$ **3.** $\lim_{x \to c} [f(x)g(x)] = LK$ **4.** $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{K}, \quad K \neq 0$ **5.** $\lim_{x \to c} [f(x)]^n = L^n$	9–22
11.2	**Use the dividing out technique to evaluate limits of functions *(p. 760)*.**	When evaluating a limit of a rational function by direct substitution, you may encounter the indeterminate form $0/0$. In this case, factor and divide out any common factors, then try direct substitution again. (See Examples 1 and 2.)	23–30
	Use the rationalizing technique to evaluate limits of functions *(p. 762)*.	The rationalizing technique involves rationalizing the numerator of the function when finding a limit. (See Example 3.)	31, 32
	Use technology to approximate limits of functions graphically and numerically *(p. 763)*.	The *table* feature or *zoom* and *trace* features of a graphing utility can be used to approximate limits. (See Examples 4 and 5.)	33–40
	Evaluate one-sided limits of functions *(p. 764)*.	**Limit from left:** $\lim_{x \to c^-} f(x) = L_1$ or $f(x) \to L_1$ as $x \to c^-$ **Limit from right:** $\lim_{x \to c^+} f(x) = L_2$ or $f(x) \to L_2$ as $x \to c^+$	41–48
	Evaluate limits of difference quotients from calculus *(p. 766)*.	For any x-value, the limit of a difference quotient is an expression of the form $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.	49, 50

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	What did you learn?	Explanation and Examples	Review Exercises
11.3	**Understand the tangent line problem *(p. 770)* and use a tangent line to approximate the slope of a graph at a point *(p. 771)*.**	The tangent line to the graph of a function f at a point $P(x_1, y_1)$ is the line that best approximates the slope of the graph at the point.	51–58
	Use the limit definition of slope to find exact slopes of graphs *(p. 772)*.	**Definition of the Slope of a Graph** The slope m of the graph of f at the point $(x, f(x))$ is equal to the slope of its tangent line at $(x, f(x))$, and is given by $m = \lim_{h \to 0} m_{\text{sec}} = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$ provided this limit exists.	59–62
	Find derivatives of functions and use derivatives to find slopes of graphs *(p. 775)*.	The derivative of f at x is given by $f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$ provided this limit exists. The derivative $f'(x)$ is a formula for the slope of the tangent line to the graph of f at the point $(x, f(x))$.	63–78
11.4	**Evaluate limits of functions at infinity *(p. 780)*.**	If f is a function and L_1 and L_2 are real numbers, then the statements $\lim_{x \to -\infty} f(x) = L_1$ and $\lim_{x \to \infty} f(x) = L_2$ denote the limits at infinity.	79–86
	Find limits of sequences *(p. 784)*.	**Limit of a Sequence** Let L be a real number. Let f be a function of a real variable such that $\lim_{x \to \infty} f(x) = L$. If $\{a_n\}$ is a sequence such that $f(n) = a_n$ for every positive integer n, then $\lim_{n \to \infty} a_n = L$.	87–92
11.5	**Find limits of summations *(p. 789)*.**	**Summation Formulas and Properties** **1.** $\sum_{i=1}^{n} c = cn$, c is a constant. **2.** $\sum_{i=1}^{n} i = \frac{n(n + 1)}{2}$ **3.** $\sum_{i=1}^{n} i^2 = \frac{n(n + 1)(2n + 1)}{6}$ **4.** $\sum_{i=1}^{n} i^3 = \frac{n^2(n + 1)^2}{4}$ **5.** $\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$ **6.** $\sum_{i=1}^{n} ka_i = k\sum_{i=1}^{n} a_i$, k is a constant.	93, 94
	Use rectangles to approximate and limits of summations to find areas of plane regions *(p. 792)*.	A collection of rectangles of equal width can be used to approximate the area of a region. Increasing the number of rectangles gives a closer approximation. (See Example 4.) **Area of a Plane Region** Let f be continuous and nonnegative on $[a, b]$. The area A of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by $A = \lim_{n \to \infty} \sum_{i=1}^{n} f\left(a + \frac{(b - a)i}{n}\right)\left(\frac{b - a}{n}\right).$	95–105

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11 Review Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

11.1

Estimating a Limit Numerically **In Exercises 1–4, complete the table and use the result to estimate the limit numerically. Determine whether the limit can be reached.**

1. $\lim_{x\to 3}(6x - 1)$

x	2.9	2.99	2.999	3	3.001	3.01	3.1
$f(x)$				?			

2. $\lim_{x\to 2}\dfrac{x-2}{3x^2-4x-4}$

x	1.9	1.99	1.999	2	2.001	2.01	2.1
$f(x)$				?			

3. $\lim_{x\to 0}\dfrac{1-e^{-x}}{x}$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$				?			

4. $\lim_{x\to 0}\dfrac{\ln(1-x)}{x}$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$				?			

Using a Graph to Find a Limit **In Exercises 5–8, use the graph to find the limit (if it exists). If the limit does not exist, explain why.**

5. $\lim_{x\to 1}(3 - x)$

6. $\lim_{x\to 2}\dfrac{1}{x-2}$

7. $\lim_{x\to -3}\dfrac{|x+3|}{x+3}$

8. $\lim_{x\to 1}\dfrac{x^2-1}{x-1}$

Evaluating Limits **In Exercises 9 and 10, use the given information to evaluate each limit.**

9. $\lim_{x\to c} f(x) = 2,\ \lim_{x\to c} g(x) = 5$

(a) $\lim_{x\to c}[f(x)]^3$ (b) $\lim_{x\to c}[3f(x) - g(x)]$

(c) $\lim_{x\to c}[f(x)g(x)]$ (d) $\lim_{x\to c}\dfrac{f(x)}{g(x)}$

10. $\lim_{x\to c} f(x) = 8,\ \lim_{x\to c} g(x) = 3$

(a) $\lim_{x\to c}\sqrt[3]{f(x)}$ (b) $\lim_{x\to c}\dfrac{f(x)}{18}$

(c) $\lim_{x\to c}[f(x)g(x)]$ (d) $\lim_{x\to c}[f(x) - 2g(x)]$

Evaluating Limits by Direct Substitution **In Exercises 11–22, find the limit by direct substitution.**

11. $\lim_{x\to 4}\left(\frac{1}{2}x + 3\right)$

12. $\lim_{x\to 3}(5x - 4)$

13. $\lim_{t\to 3}\dfrac{t^2+1}{t}$

14. $\lim_{x\to 2}\dfrac{3x+5}{5x-3}$

15. $\lim_{x\to -2}\sqrt[3]{4x}$

16. $\lim_{x\to -1}\sqrt{5-x}$

17. $\lim_{x\to \pi}\sin 3x$

18. $\lim_{x\to 0}\tan x$

19. $\lim_{x\to -1} e^{-x}$

20. $\lim_{x\to 3} 2\ln x$

21. $\lim_{x\to -1/2}\arcsin x$

22. $\lim_{x\to 0}\arctan x$

11.2

Finding Limits **In Exercises 23–32, find the limit (if it exists). Use a graphing utility to confirm your result graphically.**

23. $\lim_{t\to -2}\dfrac{t+2}{t^2-4}$

24. $\lim_{t\to 3}\dfrac{t^2-9}{t-3}$

25. $\lim_{x\to 5}\dfrac{x-5}{x^2+5x-50}$

26. $\lim_{x\to -1}\dfrac{x+1}{x^2-5x-6}$

27. $\lim_{x\to 1}\dfrac{x^2+7x-8}{x^2-3x+2}$

28. $\lim_{x\to -2}\dfrac{x^2+8x+12}{x^2-3x-10}$

29. $\lim_{x\to -1}\dfrac{\dfrac{1}{x+2}-1}{x+1}$

30. $\lim_{x\to 0}\dfrac{\dfrac{1}{x+1}-1}{x}$

31. $\lim_{u\to 0}\dfrac{\sqrt{4+u}-2}{u}$

32. $\lim_{v\to 0}\dfrac{\sqrt{v+9}-3}{v}$

Approximating a Limit **In Exercises 33–40, (a) graphically approximate the limit (if it exists) by using a graphing utility to graph the function and (b) numerically approximate the limit (if it exists) by using the *table* feature of the graphing utility to create a table.**

33. $\lim_{x\to 3}\dfrac{x-3}{x^2-9}$

34. $\lim_{x\to 4}\dfrac{4-x}{16-x^2}$

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35. $\lim_{x\to 0} e^{-2/x}$

36. $\lim_{x\to 0} e^{-4/x^2}$

37. $\lim_{x\to 0} \dfrac{\sin 4x}{2x}$

38. $\lim_{x\to 0} \dfrac{\tan 2x}{x}$

39. $\lim_{x\to 1^+} \dfrac{\sqrt{2x+1}-\sqrt{3}}{x-1}$

40. $\lim_{x\to 1^+} \dfrac{1-\sqrt{x}}{x-1}$

Evaluating One-Sided Limits **In Exercises 41–48, graph the function. Determine the limit (if it exists) by evaluating the corresponding one-sided limits.**

41. $\lim_{x\to 3} \dfrac{|x-3|}{x-3}$

42. $\lim_{x\to 8} \dfrac{|8-x|}{8-x}$

43. $\lim_{x\to 2} \dfrac{2}{x^2-4}$

44. $\lim_{x\to -3} \dfrac{1}{x^2+9}$

45. $\lim_{x\to 5} \dfrac{|x-5|}{x-5}$

46. $\lim_{x\to -2} \dfrac{|x+2|}{x+2}$

47. $\lim_{x\to 2} f(x)$ where $f(x) = \begin{cases} 5-x, & x \le 2 \\ x^2-3, & x > 2 \end{cases}$

48. $\lim_{x\to 0} f(x)$ where $f(x) = \begin{cases} x-6, & x \ge 0 \\ x^2-4, & x < 0 \end{cases}$

Evaluating a Limit from Calculus **In Exercises 49 and 50, find $\lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$.**

49. $f(x) = 3x - x^2$

50. $f(x) = x^2 - 5x - 2$

11.3

Approximating the Slope of a Graph **In Exercises 51 and 52, use the figure to approximate the slope of the curve at the point (x, y).**

51.

52.

Approximating the Slope of a Tangent Line **In Exercises 53–58, use a graphing utility to graph the function and the tangent line at the point $(2, f(2))$. Use the graph to approximate the slope of the tangent line.**

53. $f(x) = x^2 - 2x$

54. $f(x) = 6 - x^2$

55. $f(x) = \sqrt{x+2}$

56. $f(x) = \sqrt{x^2+5}$

57. $f(x) = \dfrac{6}{x-4}$

58. $f(x) = \dfrac{1}{3-x}$

Finding a Formula for the Slope of a Graph **In Exercises 59–62, find a formula for the slope of the graph of f at the point $(x, f(x))$. Then use it to find the slopes at the two specified points.**

59. $f(x) = x^2 - 4x$
(a) $(0, 0)$
(b) $(5, 5)$

60. $f(x) = \frac{1}{4}x^4$
(a) $(-2, 4)$
(b) $\left(1, \frac{1}{4}\right)$

61. $f(x) = \dfrac{4}{x-6}$
(a) $(7, 4)$
(b) $(8, 2)$

62. $f(x) = \sqrt{x}$
(a) $(1, 1)$
(b) $(4, 2)$

Finding a Derivative **In Exercises 63–74, find the derivative of the function.**

63. $f(x) = 5$

64. $g(x) = -3$

65. $h(x) = 5 - \frac{1}{2}x$

66. $f(x) = 3x$

67. $g(x) = 2x^2 - 1$

68. $f(x) = -x^3 + 4x$

69. $f(t) = \sqrt{t+5}$

70. $g(t) = \sqrt{t-3}$

71. $g(s) = \dfrac{4}{s+5}$

72. $g(t) = \dfrac{6}{5-t}$

73. $g(x) = \dfrac{1}{\sqrt{x+4}}$

74. $f(x) = \dfrac{1}{\sqrt{12-x}}$

Using the Derivative **In Exercises 75–78, (a) find the slope of the graph of f at the given point, (b) find an equation of the tangent line to the graph at the point, and (c) graph the function and the tangent line.**

75. $f(x) = 2x^2 - 1$, $(0, -1)$

76. $f(x) = x^2 + 10$, $(2, 14)$

77. $f(x) = x^3 + 1$, $(-1, 0)$

78. $f(x) = x^3 - x$, $(2, 6)$

11.4

Evaluating a Limit at Infinity **In Exercises 79–86, find the limit (if it exists). If the limit does not exist, explain why. Use a graphing utility to verify your result graphically.**

79. $\lim_{x\to\infty} \dfrac{4x}{2x-3}$

80. $\lim_{x\to\infty} \dfrac{7x}{14x+2}$

81. $\lim_{x\to-\infty} \dfrac{2x}{x^2-25}$

82. $\lim_{x\to-\infty} \dfrac{3x}{(1-x)^3}$

83. $\lim_{x\to\infty} \dfrac{x^2}{2x+3}$

84. $\lim_{y\to\infty} \dfrac{3y^4}{y^2+1}$

85. $\lim_{x\to\infty} \left[\dfrac{x}{(x-2)^2} + 3\right]$

86. $\lim_{x\to\infty} \left[2 - \dfrac{2x^2}{(x+1)^2}\right]$

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Finding the Limit of a Sequence **In Exercises 87–92, write the first five terms of the sequence and find the limit of the sequence (if it exists). If the limit does not exist, explain why. Assume n begins with 1.**

87. $a_n = \dfrac{2n - 3}{5n + 4}$

88. $a_n = \dfrac{2n}{n^2 + 1}$

89. $a_n = \dfrac{(-1)^n}{n^3}$

90. $a_n = \dfrac{(-1)^{n+1}}{n}$

91. $a_n = \dfrac{1}{2n^2}[3 - 2n(n + 1)]$

92. $a_n = \left(\dfrac{2}{n}\right)\left\{n + \dfrac{2}{n}\left[\dfrac{n(n - 1)}{2} - n\right]\right\}$

11.5

Finding the Limit of a Summation **In Exercises 93 and 94, (a) use the summation formulas and properties to rewrite the sum as a rational function $S(n)$. (b) Use $S(n)$ to complete the table. (c) Find $\lim_{n\to\infty} S(n)$.**

n	10^0	10^1	10^2	10^3	10^4
$S(n)$					

93. $\displaystyle\sum_{i=1}^{n}\left(\frac{4i^2}{n^2} - \frac{i}{n}\right)\left(\frac{1}{n}\right)$

94. $\displaystyle\sum_{i=1}^{n}\left[4 - \left(\frac{3i}{n}\right)^2\right]\left(\frac{3i}{n^2}\right)$

Approximating the Area of a Region **In Exercises 95 and 96, approximate the area of the region using the indicated number of rectangles of equal width.**

95. $f(x) = 4 - x$

96. $f(x) = 4 - x^2$

Approximating the Area of a Region **In Exercises 97 and 98, complete the table to show the approximate area of the region using the indicated numbers n of rectangles of equal width.**

n	4	8	20	50
Approximate area				

97. $f(x) = \frac{1}{4}x^2$

98. $f(x) = 4x - x^2$

Finding the Area of a Region **In Exercises 99–104, use the limit process to find the area of the region bounded by the graph of the function and the x-axis over the specified interval.**

	Function	*Interval*
99.	$f(x) = 10 - x$	$[0, 10]$
100.	$f(x) = 2x - 6$	$[3, 6]$
101.	$f(x) = x^2 + 4$	$[0, 3]$
102.	$f(x) = 6(x - x^2)$	$[0, 1]$
103.	$f(x) = x^3 + 1$	$[0, 4]$
104.	$f(x) = 8 - x^3$	$[0, 2]$

105. **Civil Engineering** The table shows the measurements (in feet) of a lot bounded by a stream and two straight roads that meet at right angles (see figure).

x	0	100	200	300	400	500
y	125	125	120	112	90	90

x	600	700	800	900	1000
y	95	88	75	35	0

(a) Use the *regression* feature of a graphing utility to find a model of the form

$$y = ax^3 + bx^2 + cx + d.$$

(b) Use the graphing utility to plot the data and graph the model in the same viewing window.

(c) Use the model in part (a) to estimate the area of the lot.

Conclusions

True or False? **In Exercises 106 and 107, determine whether the statement is true or false. Justify your answer.**

106. If the degree of the numerator $N(x)$ of a rational function $f(x) = N(x)/D(x)$ is greater than the degree of its denominator $D(x)$, then the limit of the rational function as x approaches ∞ is 0.

107. The expression $f'(z)$ gives the slope of the tangent line to the graph of f at the point $(z, f(z))$.

108. **Writing** Write a short paragraph explaining several reasons why the limit of a function may not exist.

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11 Chapter Test

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Take this test as you would take a test in class. After you are finished, check your work against the answers in the back of the book.

In Exercises 1–3, use a graphing utility to graph the function and approximate the limit (if it exists). Then find the limit (if it exists) algebraically by using appropriate techniques.

1. $\lim_{x \to -2} \dfrac{x^2 - 1}{2x}$ **2.** $\lim_{x \to 1} \dfrac{-x^2 + 5x - 3}{1 - x}$ **3.** $\lim_{x \to 5} \dfrac{\sqrt{x} - 2}{x - 5}$

In Exercises 4 and 5, use a graphing utility to graph the function and approximate the limit. Write an approximation that is accurate to four decimal places. Then create a table to verify your limit numerically.

4. $\lim_{x \to 0} \dfrac{\sin 3x}{x}$ **5.** $\lim_{x \to 0} \dfrac{e^{2x} - 1}{x}$

6. Find a formula for the slope of the graph of f at the point $(x, f(x))$. Then use it to find the slope at the specified point.

(a) $f(x) = 3x^2 - 5x - 2,\ (2, 0)$ (b) $f(x) = 2x^3 + 6x,\ (-1, -8)$

In Exercises 7–9, find the derivative of the function.

7. $f(x) = 3 - \dfrac{2}{5}x$ **8.** $f(x) = 2x^2 + 4x - 1$ **9.** $f(x) = \dfrac{1}{x + 1}$

In Exercises 10–12, find the limit (if it exists). If the limit does not exist, explain why. Use a graphing utility to verify your result graphically.

10. $\lim_{x \to \infty} \dfrac{6}{5x - 1}$ **11.** $\lim_{x \to \infty} \dfrac{1 - 3x^2}{x^2 - 5}$ **12.** $\lim_{x \to -\infty} \dfrac{x^2}{3x + 2}$

In Exercises 13 and 14, write the first five terms of the sequence and find the limit of the sequence (if it exists). If the limit does not exist, explain why. Assume n begins with 1.

13. $a_n = \dfrac{n^2 + 3n - 4}{2n^2 + n - 2}$ **14.** $a_n = \dfrac{1 + (-1)^n}{n}$

15. Approximate the area of the region bounded by the graph of $f(x) = 8 - 2x^2$ shown at the right using the indicated number of rectangles of equal width.

Figure for 15

In Exercises 16 and 17, use the limit process to find the area of the region bounded by the graph of the function and the x-axis over the specified interval.

16. $f(x) = x + 2$; interval: $[-2, 2]$

17. $f(x) = 7 - x^2$; interval: $[0, 2]$

18. The table shows the height of a space shuttle during its first 5 seconds of motion.

(a) Use the *regression* feature of a graphing utility to find a quadratic model $y = ax^2 + bx + c$ for the data.

(b) The value of the derivative of the model is the rate of change of height with respect to time, or the velocity, at that instant. Find the velocity of the shuttle after 5 seconds.

Time (seconds), x	Height (feet), y
0	0
1	1
2	23
3	60
4	115
5	188

Table for 18

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10–11 Cumulative Test

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Take this test to review the material in Chapters 10 and 11. After you are finished, check your work against the answers in the back of the book.

In Exercises 1 and 2, find the coordinates of the point.

1. The point is located six units behind the *yz*-plane, one unit to the right of the *xz*-plane, and two units above the *xy*-plane.

2. The point is located on the *y*-axis, five units to the left of the *xz*-plane.

3. Find the distance between the points $(-2, 3, -6)$ and $(4, -5, 1)$.

4. Find the lengths of the sides of the right triangle at the right. Show that these lengths satisfy the Pythagorean Theorem.

Figure for 4

5. Find the coordinates of the midpoint of the line segment joining $(3, 4, -1)$ and $(-5, 0, 2)$.

6. Find an equation of the sphere for which the endpoints of a diameter are $(0, 0, 0)$ and $(4, 4, 8)$.

7. Sketch the graph of the equation $(x - 2)^2 + (y + 1)^2 + z^2 = 4$, and then sketch the *xy*-trace and the *yz*-trace.

8. For the vectors $\mathbf{u} = \langle 2, -6, 0\rangle$ and $\mathbf{v} = \langle -4, 5, 3\rangle$, find $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{u} \times \mathbf{v}$.

In Exercises 9–11, determine whether u and v are orthogonal, parallel, or neither.

9. $\mathbf{u} = \langle 4, 4, 0\rangle$
$\mathbf{v} = \langle 0, -8, 6\rangle$

10. $\mathbf{u} = \langle 4, -2, 10\rangle$
$\mathbf{v} = \langle -2, 6, 2\rangle$

11. $\mathbf{u} = \langle -1, 6, -3\rangle$
$\mathbf{v} = \langle 3, -18, 9\rangle$

12. Find the volume of the parallelepiped with the vertices $A(1, 3, 2)$, $B(3, 4, 2)$, $C(3, 2, 2)$, $D(1, 1, 2)$, $E(1, 3, 5)$, $F(3, 4, 5)$, $G(3, 2, 5)$, and $H(1, 1, 5)$.

13. Find sets of (a) parametric equations and (b) symmetric equations for the line passing through the points $(-2, 3, 0)$ and $(5, 8, 25)$.

14. Find the parametric form of the equation of the line passing through the point $(-1, 2, 0)$ and perpendicular to $2x - 4y + z = 8$.

15. Find an equation of the plane passing through the points $(0, 0, 0)$, $(-2, 3, 0)$, and $(5, 8, 25)$.

16. Label the intercepts and sketch the graph of the plane given by $3x - 6y - 12z = 24$.

17. Find the distance between the point $(0, 0, 25)$ and the plane $2x - 5y + z = 10$.

18. A plastic wastebasket has the shape and dimensions shown in the figure. In fabricating a mold for making the wastebasket, it is necessary to know the angle between two adjacent sides. Find the angle.

Figure for 18

In Exercises 19–27, find the limit (if it exists). If the limit does not exist, explain why. Use a graphing utility to verify your result graphically.

19. $\lim_{x\to 4}(5x - x^2)$

20. $\lim_{x\to -2^+} \dfrac{x + 2}{x^2 + x - 2}$

21. $\lim_{x\to 7} \dfrac{x - 7}{x^2 - 49}$

22. $\lim_{x\to 0} \dfrac{\sqrt{x + 4} - 2}{x}$

23. $\lim_{x\to 4^-} \dfrac{|x - 4|}{x - 4}$

24. $\lim_{x\to 0} \sin\left(\dfrac{\pi}{x}\right)$

25. $\lim_{x\to 0} \dfrac{\dfrac{1}{x + 3} - \dfrac{1}{3}}{x}$

26. $\lim_{x\to 0} \dfrac{\sqrt{x + 16} - 4}{x}$

27. $\lim_{x\to 2^-} \dfrac{x - 2}{x^2 - 4}$

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In Exercises 28–31, find a formula for the slope of the graph of f at the point $(x, f(x))$. Then use it to find the slope at the specified point.

28. $f(x) = 4 - x^2, \quad (-2, 0)$

29. $f(x) = \sqrt{x + 3}, \quad (-2, 1)$

30. $f(x) = \dfrac{1}{x + 3}, \quad \left(1, \dfrac{1}{4}\right)$

31. $f(x) = x^2 - x, \quad (1, 0)$

In Exercises 32–37, find the limit (if it exists). If the limit does not exist, explain why. Use a graphing utility to verify your result graphically.

32. $\displaystyle\lim_{x\to\infty} \frac{x^3}{x^2 - 9}$

33. $\displaystyle\lim_{x\to\infty} \frac{3 - 7x}{x + 4}$

34. $\displaystyle\lim_{x\to\infty} \frac{3x^2 + 1}{x^2 + 4}$

35. $\displaystyle\lim_{x\to\infty} \frac{2x}{x^2 + 3x - 2}$

36. $\displaystyle\lim_{x\to\infty} \frac{3 - x}{x^2 + 1}$

37. $\displaystyle\lim_{x\to\infty} \frac{3 + 4x - x^3}{2x^2 + 3}$

In Exercises 38–40, evaluate the sum using the summation formulas and properties.

38. $\displaystyle\sum_{i=1}^{50} (1 - i^2)$

39. $\displaystyle\sum_{k=1}^{20} (3k^2 - 2k)$

40. $\displaystyle\sum_{i=1}^{40} (12 + i^3)$

In Exercises 41–44, approximate the area of the region using the indicated number of rectangles of equal width.

41.

42.

43.

44.

In Exercises 45–48, use the limit process to find the area of the region bounded by the graph of the function and the x-axis over the specified interval.

45. $f(x) = x + 2$
Interval: $[0, 1]$

46. $f(x) = x^2 + 1$
Interval: $[0, 4]$

47. $f(x) = 4 - x^2$
Interval: $[0, 2]$

48. $f(x) = 1 - x^3$
Interval: $[0, 1]$

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Proofs in Mathematics

Many of the proofs of the definitions and properties presented in this chapter are beyond the scope of this text. Included below are simple proofs for the limit of a power function and the limit of a polynomial function.

Limit of a Power Function (p. 755)

$\lim_{x \to c} x^n = c^n$, c is a real number and n is a positive integer.

Proof

$$\lim_{x \to c} x^n = \lim_{x \to c} (\underbrace{x \cdot x \cdot x \cdot \cdots \cdot x}_{n \text{ factors}})$$

$$= \underbrace{\lim_{x \to c} x \cdot \lim_{x \to c} x \cdot \lim_{x \to c} x \cdot \cdots \cdot \lim_{x \to c} x}_{n \text{ factors}} \qquad \text{Product Property of Limits}$$

$$= \underbrace{c \cdot c \cdot c \cdot \cdots \cdot c}_{n \text{ factors}} \qquad \text{Limit of the identity function}$$

$$= c^n \qquad \text{Exponential form}$$

Limit of a Polynomial Function (p. 756)

If p is a polynomial function and c is a real number, then

$$\lim_{x \to c} p(x) = p(c).$$

Proof

Let p be a polynomial function such that

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0.$$

Because a polynomial function is the sum of monomial functions, you can write the following.

$$\lim_{x \to c} p(x) = \lim_{x \to c} (a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0)$$

$$= \lim_{x \to c} a_n x^n + \lim_{x \to c} a_{n-1} x^{n-1} + \cdots + \lim_{x \to c} a_2 x^2 + \lim_{x \to c} a_1 x + \lim_{x \to c} a_0$$

$$= a_n c^n + a_{n-1} c^{n-1} + \cdots + a_2 c^2 + a_1 c + a_0 \qquad \text{Scalar Multiple Property of Limits and limit of a power function}$$

$$= p(c) \qquad p \text{ evaluated at } c$$

Proving Limits

To prove most of the definitions and properties from this chapter, you must use the *formal* definition of a limit. This definition is called the *epsilon-delta definition* and was first introduced by Karl Weierstrass (1815–1897). If you go on to take a course in calculus, you will use this definition of a limit extensively.

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Progressive Summary (Chapters 3–11)

This chart outlines the topics that have been covered so far in this text. Progressive Summary charts appear after Chapters 2, 3, 6, 9, and 11. In each Progressive Summary, new topics encountered for the first time appear in red.

TRANSCENDENTAL FUNCTIONS

Exponential, Logarithmic, Trigonometric, Inverse Trigonometric

Rewriting

Exponential form ↔ Logarithmic form
Condense/expand logarithmic expressions
Simplify trigonometric expressions
Prove trigonometric identities
Use conversion formulas
Operations with vectors
Powers and roots of complex numbers

Solving

Equation	*Strategy*
Exponential	Take logarithm of each side
Logarithmic	Exponentiate each side
Trigonometric	Isolate function Factor, use inverse function
Multiple angle or high powers	Use trigonometric identities

Analyzing

Graphically	*Algebraically*
Intercepts	Domain, Range
Asymptotes	Transformations
Minimum values	Composition
Maximum values	Inverse Properties
	Amplitude, period
	Reference angles

Numerically

Table of values

SYSTEMS AND SERIES

Systems, Sequences, Series

Rewriting

Row operations for systems of equations
Partial fraction decomposition
Operations with matrices
Matrix form of a system of equations
*n*th term of a sequence
Summation form of a series

Solving

Equation	*Strategy*
System of linear equations	Substitution Elimination Gaussian Gauss-Jordan Inverse matrices Cramer's Rule

Analyzing

Systems:
Intersecting, parallel, and coincident lines, determinants
Sequences:
Graphing utility in *dot* mode, *n*th term, partial sums, summation formulas

OTHER TOPICS

Conics, Parametric and Polar Equations
3-space, Limits

Rewriting

Standard forms of conics
Eliminate parameters
Rectangular form ↔ Parametric form
Rectangular form ↔ Polar form
Rationalize numerator
Difference quotient

Solving

Equation	*Strategy*
Conics	Convert to standard form Convert to polar form
3-space	Equation of plane Equation of line Cross-product of vectors
Limits	Direct substitution One-sided limits Limits at infinity

Analyzing

Conics:
Table of values, vertices, foci, axes, symmetry, asymptotes, translations, eccentricity
Parametric forms:
Point plotting, eliminate parameters
Polar forms:
Point plotting, special equations, symmetry, zeros, eccentricity, maximum *r*-values, directrix
3-space:
Point plotting, intercepts, traces, vectors

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Appendix A: Technology Support

Introduction

Graphing utilities such as graphing calculators and computers with graphing software are very valuable tools for visualizing mathematical principles, verifying solutions of equations, exploring mathematical ideas, and developing mathematical models. Although graphing utilities are extremely helpful in learning mathematics, their use does not mean that learning algebra is any less important. In fact, the combination of knowledge of mathematics and the use of graphing utilities enables you to explore mathematics more easily and to a greater depth. If you are using a graphing utility in this course, it is up to you to learn its capabilities and to practice using this tool to enhance your mathematical learning.

In this text, there are many opportunities to use a graphing utility, some of which are described below.

Uses of a Graphing Utility

1. Check or validate answers to problems obtained using algebraic methods.
2. Discover and explore algebraic properties, rules, and concepts.
3. Graph functions, and approximate solutions of equations involving functions.
4. Efficiently perform complicated mathematical procedures such as those found in many real-life applications.
5. Find mathematical models for sets of data.

In this appendix, the features of graphing utilities are discussed from a generic perspective and are listed in alphabetical order. To learn how to use the features of a specific graphing utility, consult your user's manual or go to this textbook's *Companion Website*. Additional keystroke guides are available for most graphing utilities, and your college library may have a resource on how to use your graphing utility.

Many graphing utilities are designed to act as "function graphers." In this course, functions and their graphs are studied in detail. You may recall from previous courses that a function can be thought of as a rule that describes the relationship between two variables. These rules are frequently written in terms of x and y. For instance, the equation

$$y = 3x + 5$$

represents y as a function of x.

Many graphing utilities have an *equation editor* feature that requires that an equation be written in "$y =$" form in order to be entered, as shown in Figure A.1. (You should note that your *equation editor* screen may not look like the screen shown in Figure A.1.)

Figure A.1

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Cumulative Sum Feature

The *cumulative sum* feature finds partial sums of a series. For instance, to find the first four partial sums of the series

$$\sum_{k=1}^{4} 2(0.1)^k$$

choose the *cumulative sum* feature, which is found in the *operations* menu of the *list* feature (see Figure A.2). To use this feature, you will also have to use the *sequence* feature (see Figure A.2 and page A15). You must enter an expression for the sequence, a variable, the lower limit of summation, and the upper limit of summation, as shown in Figure A.3. After pressing ENTER, you can see that the first four partial sums are 0.2, 0.22, 0.222, and 0.2222. You may have to scroll to the right in order to see all the partial sums.

Figure A.2

Figure A.3

Technology Tip

As you use your graphing utility, be aware of how parentheses are inserted in an expression. Some graphing utilities automatically insert the left parenthesis when certain calculator buttons are pressed. The placement of parentheses can make a difference between a correct answer and an incorrect answer.

Determinant Feature

The *determinant* feature evaluates the determinant of a square matrix. For instance, to evaluate the determinant of

$$A = \begin{bmatrix} 7 & -1 & 0 \\ 2 & 2 & 3 \\ -6 & 4 & 1 \end{bmatrix}$$

enter the 3×3 matrix in the graphing utility using the *matrix editor*, as shown in Figure A.4. Then choose the *determinant* feature from the *math* menu of the *matrix* feature, as shown in Figure A.5. Once you choose the matrix name, A, press ENTER and you should obtain a determinant of -50, as shown in Figure A.6.

Figure A.4

Figure A.5

Figure A.6

Draw Inverse Feature

The *draw inverse* feature graphs the inverse function of a *one-to-one* function. For instance, to graph the inverse function of $f(x) = x^3 + 4$, first enter the function in the *equation editor* (see Figure A.7) and graph the function (using a square viewing window), as shown in Figure A.8. Then choose the *draw inverse* feature from the *draw* feature menu, as shown in Figure A.9. You must enter the function you want to graph the inverse function of, as shown in Figure A.10. Finally, press ENTER to obtain the graph of the inverse function of $f(x) = x^3 + 4$, as shown in Figure A.11. This feature can be used only when the graphing utility is in *function* mode.

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Figure A.7

Figure A.8

Figure A.9

Figure A.10

Figure A.11

Elementary Row Operations Features

Most graphing utilities can perform elementary row operations on matrices.

Row Swap Feature

The *row swap* feature interchanges two rows of a matrix. To interchange rows 1 and 3 of the matrix

$$A = \begin{bmatrix} -1 & -2 & 1 & 2 \\ 2 & -4 & 6 & -2 \\ 1 & 3 & -3 & 0 \end{bmatrix}$$

first enter the matrix in the graphing utility using the *matrix editor*, as shown in Figure A.12. Then choose the *row swap* feature from the *math* menu of the *matrix* feature, as shown in Figure A.13. When using this feature, you must enter the name of the matrix and the two rows that are to be interchanged. After pressing ENTER, you should obtain the matrix shown in Figure A.14. Because the resulting matrix will be used to demonstrate the other elementary row operation features, use the *store* feature to copy the resulting matrix to [A], as shown in Figure A.15.

Figure A.12

Figure A.13

Figure A.14

Figure A.15

Technology Tip

The *store* feature of a graphing utility is used to store a value in a variable or to copy one matrix to another matrix. For instance, as shown at the left, after performing a row operation on a matrix, you can copy the answer to another matrix (see Figure A.15). You can then perform another row operation on the copied matrix. To continue performing row operations to obtain a matrix in row-echelon form or reduced row-echelon form, you must copy the resulting matrix to a new matrix before each operation.

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Row Addition and Row Multiplication and Addition Features

The *row addition* and *row multiplication and addition* features add a row or a multiple of a row of a matrix to another row of the same matrix. To add row 1 to row 3 of the matrix stored in [A], choose the *row addition* feature from the *math* menu of the *matrix* feature, as shown in Figure A.16. When using this feature, you must enter the name of the matrix and the two rows that are to be added. After pressing ENTER, you should obtain the matrix shown in Figure A.17. Copy the resulting matrix to [A].

Figure A.16

Figure A.17

To add -2 times row 1 to row 2 of the matrix stored in [A], choose the *row multiplication and addition* feature from the *math* menu of the *matrix* feature, as shown in Figure A.18. When using this feature, you must enter the constant, the name of the matrix, the row the constant is multiplied by, and the row to be added to. After pressing ENTER, you should obtain the matrix shown in Figure A.19. Copy the resulting matrix to [A].

Figure A.18

Figure A.19

Row Multiplication Feature

The *row multiplication* feature multiplies a row of a matrix by a nonzero constant. To multiply row 2 of the matrix stored in [A] by

$$-\frac{1}{10}$$

choose the *row multiplication* feature from the *math* menu of the *matrix* feature, as shown in Figure A.20. When using this feature, you must enter the constant, the name of the matrix, and the row to be multiplied. After pressing ENTER, you should obtain the matrix shown in Figure A.21.

Figure A.20

Figure A.21

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Intersect Feature

The *intersect* feature finds the point(s) of intersection of two graphs. The *intersect* feature is found in the *calculate* menu (see Figure A.22). To find the point(s) of intersection of the graphs of $y_1 = -x + 2$ and $y_2 = x + 4$, first enter the equations in the *equation editor*, as shown in Figure A.23. Then graph the equations, as shown in Figure A.24. Next, use the *intersect* feature to find the point of intersection. Trace the cursor along the graph of y_1 near the intersection and press ENTER (see Figure A.25). Then trace the cursor along the graph of y_2 near the intersection and press ENTER (see Figure A.26). Marks are then placed on the graph at these points (see Figure A.27). Finally, move the cursor near the point of intersection and press ENTER. In Figure A.28, you can see that the coordinates of the point of intersection are displayed at the bottom of the window. So, the point of intersection is $(-1, 3)$.

Figure A.22

Figure A.23

Figure A.24

Figure A.25

Figure A.26

Figure A.27

Figure A.28

List Editor

Most graphing utilities can store data in lists. The *list editor* can be used to create tables and to store statistical data. The *list editor* can be found in the *edit* menu of the *statistics* feature, as shown in Figure A.29. To enter the numbers 1 through 10 in a list, first choose a list (L_1) and then begin entering the data into each row, as shown in Figure A.30.

Figure A.29

Figure A.30

You can also attach a formula to a list. For instance, you can multiply each of the data values in L_1 by 3. First, display the *list editor* and move the cursor to the top line.

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Then move the cursor onto the list to which you want to attach the formula (L_2). Finally, enter the formula 3* L_1 (see Figure A.31) and then press ENTER. You should obtain the list shown in Figure A.32.

Figure A.31

Figure A.32

Matrix Feature

The *matrix* feature of a graphing utility has many uses, such as evaluating a determinant and performing row operations. (For instructions on how to perform row operations, see Elementary Row Operations Features on page A3.)

Matrix Editor

You can define, display, and edit matrices using the *matrix editor*. The *matrix editor* can be found in the *edit* menu of the *matrix* feature. For instance, to enter the matrix

$$A = \begin{bmatrix} 6 & -3 & 4 \\ 9 & 0 & -1 \end{bmatrix}$$

first choose the matrix name [A], as shown in Figure A.33. Then enter the dimension of the matrix (in this case, the dimension is 2×3) and enter the entries of the matrix, as shown in Figure A.34. To display the matrix on the home screen, choose the *name* menu of the *matrix* feature and select the matrix [A] (see Figure A.35), then press ENTER. The matrix A should now appear on the home screen, as shown in Figure A.36.

Figure A.33

Figure A.34

Figure A.35

Figure A.36

Matrix Operations

Most graphing utilities can perform matrix operations. To find the sum $A + B$ of the matrices shown at the right, first enter the matrices in the *matrix editor* as [A] and [B]. Then find the sum, as shown in Figure A.37. Scalar multiplication can be performed in a similar manner. For instance, you can evaluate $7A$, where A is the matrix at the right, as shown in Figure A.38. To find the product AB of the matrices A and B at the right, first be sure that the product is defined. Because the number of columns of A (2 columns) equals the number of rows of B (2 rows), you can find the product AB, as shown in Figure A.39.

$$A = \begin{bmatrix} -3 & 5 \\ 0 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 7 & -2 \\ -1 & 2 \end{bmatrix}$$

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Figure A.37

Figure A.38

Figure A.39

Inverse Matrix

Some graphing utilities may not have an *inverse matrix* feature. However, you can find the inverse of a square matrix by using the inverse key x^{-1}. To find the inverse of the matrix

$$A = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 3 & 0 \\ 2 & 4 & 5 \end{bmatrix}$$

enter the matrix in the *matrix editor* as [A]. Then find the inverse, as shown in Figure A.40.

Figure A.40

Maximum and Minimum Features

The *maximum* and *minimum* features find relative extrema of a function. For instance, the graph of

$$y = x^3 - 3x$$

is shown in Figure A.41. In the figure, the graph appears to have a relative maximum at $x = -1$ and a relative minimum at $x = 1$. To find the exact values of the relative extrema, you can use the *maximum* and *minimum* features found in the *calculate* menu (see Figure A.42). First, to find the relative maximum, choose the *maximum* feature and trace the cursor along the graph to a point left of the maximum and press ENTER (see Figure A.43). Then trace the cursor along the graph to a point right of the maximum and press ENTER (see Figure A.44). Note the two arrows near the top of the display marking the left and right bounds, as shown in Figure A.45. Next, trace the cursor along the graph between the two bounds and as close to the maximum as you can (see Figure A.45) and press ENTER. In Figure A.46, you can see that the coordinates of the maximum point are displayed at the bottom of the window. So, the relative maximum is $(-1, 2)$.

Figure A.41

Figure A.42

Figure A.43

Figure A.44

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Figure A.45

Figure A.46

Figure A.47

You can find the relative minimum in a similar manner. In Figure A.47, you can see that the relative minimum is $(1, -2)$.

Mean and Median Features

In real-life applications, you often encounter large data sets and want to calculate statistical values. The *mean* and *median* features calculate the mean and median of a data set. For instance, in a survey, 100 people were asked how much money (in dollars) per week they withdraw from an automatic teller machine (ATM). The results are shown in the table below. The frequency represents the number of responses.

Amount	10	20	30	40	50	60	70	80	90	100
Frequency	3	8	10	19	24	13	13	7	2	1

To find the mean and median of the data set, first enter the data in the *list editor*, as shown in Figure A.48. Enter the amount in L_1 and the frequency in L_2. Then choose the *mean* feature from the *math* menu of the *list* feature, as shown in Figure A.49. When using this feature, you must enter a list and a frequency list (if applicable). In this case, the list is L_1 and the frequency list is L_2. After pressing [ENTER], you should obtain a mean of

$49.80

as shown in Figure A.50. You can follow the same steps (except choose the *median* feature) to find the median of the data. You should obtain a median of

$50

as shown in Figure A.51.

Figure A.48

Figure A.49

Figure A.50

Figure A.51

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Mode Settings

Mode settings of a graphing utility control how the utility displays and interprets numbers and graphs. The default mode settings are shown in Figure A.52.

Figure A.52

Radian and Degree Modes

The trigonometric functions can be applied to angles measured in either radians or degrees. When your graphing utility is in *radian* mode, it interprets angle values as radians and displays answers in radians. When your graphing utility is in *degree* mode, it interprets angle values as degrees and displays answers in degrees. For instance, to calculate $\sin(\pi/6)$, make sure the calculator is in *radian* mode. You should obtain an answer of 0.5, as shown in Figure A.53. To calculate $\sin 45°$, make sure the calculator is in *degree* mode, as shown in Figure A.54. You should obtain an approximate answer of 0.7071, as shown in Figure A.55. Without changing the mode of the calculator before evaluating $\sin 45°$, you would obtain an answer of approximately 0.8509, which is the sine of 45 radians.

Figure A.53

Figure A.54

Figure A.55

Function, Parametric, Polar, and Sequence Modes

Most graphing utilities can graph using four different modes.

Function Mode The *function* mode is used to graph standard algebraic and trigonometric functions. For instance, to graph $y = 2x^2$, use the *function* mode, as shown in Figure A.52. Then enter the equation in the *equation editor*, as shown in Figure A.56. Using a standard viewing window (see Figure A.57), you obtain the graph shown in Figure A.58.

Figure A.56

Figure A.57

Figure A.58

Parametric Mode To graph parametric equations such as

$$x = t + 1 \quad \text{and} \quad y = t^2$$

use the *parametric* mode, as shown in Figure A.59. Then enter the equations in the *equation editor*, as shown in Figure A.60. Using the viewing window shown in Figure A.61, you obtain the graph shown in Figure A.62.

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Figure A.59

Figure A.60

Figure A.61

Figure A.62

Polar Mode To graph polar equations of the form $r = f(\theta)$, you can use the *polar* mode of a graphing utility. For instance, to graph the polar equation

$$r = 2 \cos \theta$$

use the *polar* mode (and *radian* mode), as shown in Figure A.63. Then enter the equation in the *equation editor*, as shown in Figure A.64. Using the viewing window shown in Figure A.65, you obtain the graph shown in Figure A.66.

Figure A.63

Figure A.64

Figure A.65

Figure A.66

Sequence Mode To graph the first five terms of a sequence such as

$$a_n = 4n - 5$$

use the *sequence* mode, as shown in Figure A.67. Then enter the sequence in the *equation editor*, as shown in Figure A.68 (assume that n begins with 1). Using the viewing window shown in Figure A.69, you obtain the graph shown in Figure A.70.

Figure A.67

Figure A.68

Technology Tip

Note that when using the different graphing modes of a graphing utility, the utility uses different variables. When the utility is in *function* mode, it uses the variables x and y. In *parametric* mode, the utility uses the variables x, y, and t. In *polar* mode, the utility uses the variables r and θ. In *sequence* mode, the utility uses the variables u (instead of a) and n.

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Figure A.69

Figure A.70

Connected and Dot Modes

Graphing utilities use the point-plotting method to graph functions. When a graphing utility is in *connected* mode, the utility connects the points that are plotted. When the utility is in *dot* mode, it does not connect the points that are plotted. For instance, the graph of

$$y = x^3$$

in *connected* mode is shown in Figure A.71. To graph this function using *dot* mode, first change the mode to *dot* mode (see Figure A.72) and then graph the equation, as shown in Figure A.73. As you can see in Figure A.73, the graph is a collection of dots.

Figure A.71

Figure A.72

Figure A.73

A problem arises in some graphing utilities when the *connected* mode is used. Graphs with vertical asymptotes, such as rational functions and tangent functions, appear to be connected. For instance, the graph of

$$y = \frac{1}{x + 3}$$

is shown in Figure A.74. Notice how the two portions of the graph appear to be connected with a vertical line at $x = -3$. From your study of rational functions, you know that the graph has a vertical asymptote at $x = -3$ and therefore is undefined when $x = -3$. When using a graphing utility to graph rational functions and other functions that have vertical asymptotes, you should use the *dot* mode to eliminate extraneous vertical lines. Because the *dot* mode of a graphing utility displays a graph as a collection of dots rather than as a smooth curve, in this text, a blue curve is placed behind the graphing utility's display to indicate where the graph should appear, as shown in Figure A.75.

Figure A.74

Figure A.75

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$_nC_r$ Feature

The $_nC_r$ feature calculates binomial coefficients and the number of combinations of n elements taken r at a time. For instance, to find the number of combinations of eight elements taken five at a time, enter 8 (the n-value) on the home screen and choose the $_nC_r$ feature from the *probability* menu of the *math* feature (see Figure A.76). Next, enter 5 (the r-value) on the home screen and press ENTER. You should obtain 56, as shown in Figure A.77.

Figure A.76

Figure A.77

$_nP_r$ Feature

The $_nP_r$ feature calculates the number of permutations of n elements taken r at a time. For instance, to find the number of permutations of six elements taken four at a time, enter 6 (the n-value) on the home screen and choose the $_nP_r$ feature from the *probability* menu of the *math* feature (see Figure A.78). Next enter 4 (the r-value) on the home screen and press ENTER. You should obtain 360, as shown in Figure A.79.

Figure A.78

Figure A.79

One-Variable Statistics Feature

Graphing utilities are useful in calculating statistical values for a set of data. The *one-variable statistics* feature analyzes data with one measured variable. This feature outputs the mean of the data, the sum of the data, the sum of the data squared, the sample standard deviation of the data, the population standard deviation of the data, the number of data points, the minimum data value, the maximum data value, the first quartile of the data, the median of the data, and the third quartile of the data. Consider the following data, which show the hourly earnings (in dollars) of 12 retail sales associates.

8.05, 10.25, 8.45, 9.15, 8.90, 8.20, 9.25, 10.30, 8.60, 9.60, 10.05, 11.35

You can use the *one-variable statistics* feature to determine the mean and standard deviation of the data. First, enter the data in the *list editor*, as shown in Figure A.80. Then choose the *one-variable statistics* feature from the *calculate* menu of the *statistics* feature, as shown in Figure A.81. When using this feature, you must enter a list. In this case, the list is L_1. In Figure A.82, you can see that the mean of the data is

$$\bar{x} \approx 9.35$$

and the standard deviation of the data is

$$\sigma x \approx 0.95.$$

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Figure A.80

Figure A.81

Figure A.82

Regression Feature

Throughout the text, you are asked to use the *regression* feature of a graphing utility to find models for sets of data. Most graphing utilities have built-in regression programs for the following.

Regression	*Form of Model*
Linear	$y = ax + b$ or $y = a + bx$
Quadratic	$y = ax^2 + bx + c$
Cubic	$y = ax^3 + bx^2 + cx + d$
Quartic	$y = ax^4 + bx^3 + cx^2 + dx + e$
Logarithmic	$y = a + b \ln x$
Exponential	$y = ab^x$
Power	$y = ax^b$
Logistic	$y = \dfrac{c}{1 + ae^{-bx}}$
Sine	$y = a \sin(bx + c) + d$

For instance, you can find a linear model for the data in the table.

x	y
6	222.8
7	228.7
8	235.0
9	240.3
10	245.0
11	248.2
12	254.4
13	260.2
14	268.3
15	287.0

Enter the data in the *list editor*, as shown in Figure A.83. Note that L_1 contains the x-values, and L_2 contains the y-values. Now choose the *linear regression* feature from the *calculate* menu of the *statistics* feature, as shown in Figure A.84. In Figure A.85, you can see that a linear model for the data is given by

$$y = 6.22x + 183.7.$$

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Figure A.83

Figure A.84

Figure A.85

When you use the *regression* feature of a graphing utility, you will notice that the program may also output an "*r*-value." (For some calculators, make sure you select the *diagnostics on* feature before you use the *regression* feature. Otherwise, the calculator will not output an *r*-value.) The *r*-value or *correlation coefficient* measures how well the linear model fits the data. The closer the value of $|r|$ is to 1, the better the fit. For the data above,

$$r \approx 0.97514$$

which implies that the model is a good fit for the data.

Row-Echelon and Reduced Row-Echelon Features

Some graphing utilities have features that can automatically transform a matrix to row-echelon form and reduced row-echelon form. These features can be used to check your solutions to systems of equations.

Row-Echelon Feature

Consider the system of equations and the corresponding augmented matrix shown below.

Linear System

$$\begin{cases} 2x + 5y - 3z = 4 \\ 4x + y = 2 \\ -x + 3y - 2z = -1 \end{cases}$$

Augmented Matrix

$$\left[\begin{array}{ccc|c} 2 & 5 & -3 & 4 \\ 4 & 1 & 0 & 2 \\ -1 & 3 & -2 & -1 \end{array}\right]$$

You can use the *row-echelon* feature of a graphing utility to write the augmented matrix in row-echelon form. First, enter the matrix in the graphing utility using the *matrix editor*, as shown in Figure A.86. Next, choose the *row-echelon* feature from the *math* menu of the *matrix* feature, as shown in Figure A.87. When using this feature, you must enter the name of the matrix. In this case, the name of the matrix is [A]. You should obtain the matrix shown in Figure A.88. You may have to scroll to the right in order to see all the entries of the matrix.

Figure A.86

Figure A.87

Figure A.88

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Reduced Row-Echelon Feature

To write the augmented matrix in reduced row-echelon form, follow the same steps used to write a matrix in row-echelon form except choose the *reduced row-echelon* feature, as shown in Figure A.89. You should obtain the matrix shown in Figure A.90. From Figure A.90, you can conclude that the solution to the system is $x = 3$, $y = -10$, and $z = -16$.

Figure A.89

Figure A.90

Sequence Feature

The *sequence* feature is used to display the terms of sequences. For instance, to determine the first five terms of the arithmetic sequence

$$a_n = 3n + 5 \qquad \text{Assume } n \text{ begins with 1.}$$

set the graphing utility to *sequence* mode. Then choose the *sequence* feature from the *operations* menu of the *list* feature, as shown in Figure A.91. When using this feature, you must enter the sequence, the variable (in this case n), the beginning value (in this case 1), and the end value (in this case 5). The first five terms of the sequence are 8, 11, 14, 17, and 20, as shown in Figure A.92. You may have to scroll to the right in order to see all the terms of the sequence.

Figure A.91

Figure A.92

Shade Feature

Most graphing utilities have a *shade* feature that can be used to graph inequalities. For instance, to graph the inequality

$$y \le 2x - 3$$

first enter the equation $y = 2x - 3$ in the *equation editor*, as shown in Figure A.93. Next, using a standard viewing window (see Figure A.94), graph the equation, as shown in Figure A.95.

Figure A.93

Figure A.94

Figure A.95

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Because the inequality sign is $\leq$, you want to shade the region below the line $y = 2x - 3$. Choose the *shade* feature from the *draw* feature menu, as shown in Figure A.96. You must enter a lower function and an upper function. In this case, the lower function is -10 (this is the least y-value in the viewing window) and the upper function is Y_1 ($y = 2x - 3$), as shown in Figure A.97. Then press ENTER to obtain the graph shown in Figure A.98.

Figure A.96

Figure A.97

Figure A.98

To graph the inequality

$$y \geq 2x - 3$$

(using a standard viewing window), you would enter the lower function as Y_1 ($y = 2x - 3$) and the upper function as 10 (the greatest y-value in the viewing window).

Statistical Plotting Feature

The *statistical plotting* feature plots data that are stored in lists. Most graphing utilities can display the following types of plots.

Plot Type	*Variables*
Scatter plot	x-list, y-list
xy line graph	x-list, y-list
Histogram	x-list, frequency
Box-and-whisker plot	x-list, frequency
Normal probability plot	data list, data axis

For instance, use a box-and-whisker plot to represent the following set of data. Then use the graphing utility plot to find the least number, the lower quartile, the median, the upper quartile, and the greatest number.

17, 19, 21, 27, 29, 30, 37, 27, 15, 23, 19, 16

First, use the *list editor* to enter the values in a list, as shown in Figure A.99. Then go to the *statistical plotting editor*. In this editor you will turn the plot on, select the box-and-whisker plot, select the list you entered in the *list editor*, and enter the frequency of each item in the list, as shown in Figure A.100. Now use the *zoom* feature and choose the *zoom stat* option to set the viewing window and plot the graph, as shown in Figure A.101. Use the *trace* feature to find that the least number is 15, the lower quartile is 18, the median is 22, the upper quartile is 28, and the greatest number is 37.

Figure A.99

Figure A.100

Figure A.101

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Sum Feature

The *sum* feature finds the sum of a list of data. For instance, the data below represent a student's quiz scores on 10 quizzes throughout an algebra course.

22, 23, 19, 24, 20, 15, 25, 21, 18, 24

To find the total quiz points the student earned, enter the data in the *list editor*, as shown in Figure A.102. To find the sum, choose the *sum* feature from the *math* menu of the *list* feature, as shown in Figure A.103. You must enter a list. In this case the list is L_1. You should obtain a sum of 211, as shown in Figure A.104.

Figure A.102

Figure A.103

Figure A.104

Sum Sequence Feature

The *sum* feature and the *sequence* feature can be combined to find the sum of a sequence or series. For instance, to find the sum

$$\sum_{k=0}^{10} 5^{k+1}$$

first choose the *sum* feature from the *math* menu of the *list* feature, as shown in Figure A.105. Then choose the *sequence* feature from the *operations* menu of the *list* feature, as shown in Figure A.106. You must enter an expression for the sequence, a variable, the lower limit of summation, and the upper limit of summation. After pressing [ENTER], you should obtain the sum 61,035,155, as shown in Figure A.107.

Figure A.105

Figure A.106

Figure A.107

Table Feature

Most graphing utilities are capable of displaying a table of values with *x*-values and one or more corresponding *y*-values. These tables can be used to check solutions of an equation and to generate ordered pairs to assist in graphing an equation by hand.

To use the *table* feature, enter an equation in the *equation editor*. The table may have a setup screen, which allows you to select the starting *x*-value and the table step or *x*-increment. You may then have the option of automatically generating values for *x* and *y* or building your own table using the *ask* mode (see Figure A.108).

Figure A.108

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For instance, enter the equation

$$y = \frac{3x}{x + 2}$$

in the *equation editor*, as shown in Figure A.109. In the table setup screen, set the table to start at $x = -4$ and set the table step to 1, as shown in Figure A.110. When you view the table, notice that the first x-value is -4 and that each value after it increases by 1. Also notice that the Y_1 column gives the resulting y-value for each x-value, as shown in Figure A.111. The table shows that the y-value for $x = -2$ is ERROR. This means that the equation is undefined when $x = -2$.

Figure A.109

Figure A.110

Figure A.111

With the same equation in the *equation editor*, set the independent variable in the table to *ask* mode, as shown in Figure A.112. In this mode, you do not need to set the starting x-value or the table step because you are entering any value you choose for x. You may enter any real value for x—integers, fractions, decimals, irrational numbers, and so forth. When you enter

$$x = 1 + \sqrt{3}$$

the graphing utility may rewrite the number as a decimal approximation, as shown in Figure A.113. You can continue to build your own table by entering additional x-values in order to generate y-values, as shown in Figure A.114.

Figure A.112

Figure A.113

Figure A.114

When you have several equations in the *equation editor*, the table may generate y-values for each equation.

Tangent Feature

Some graphing utilities have the capability of drawing a tangent line to a graph at a given point. For instance, consider the equation

$$y = -x^3 + x + 2.$$

To draw the line tangent to the point (1, 2) on the graph of y, enter the equation in the *equation editor*, as shown in Figure A.115. Using the viewing window shown in Figure A.116, graph the equation, as shown in Figure A.117. Next, choose the *tangent* feature from the *draw* feature menu, as shown in Figure A.118. You can either move the cursor to select a point or enter the x-value at which you want the tangent line to be drawn.

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Because you want the tangent line to the point

(1, 2)

enter 1 (see Figure A.119) and then press ENTER. The x-value you entered and the equation of the tangent line are displayed at the bottom of the window, as shown in Figure A.120.

Figure A.115

Figure A.116

Figure A.117

Figure A.118

Figure A.119

Figure A.120

Trace Feature

For instructions on how to use the *trace* feature, see Zoom and Trace Features on page A23.

Value Feature

The *value* feature finds the value of a function y for a given x-value. To find the value of a function such as

$$f(x) = 0.5x^2 - 1.5x$$

at $x = 1.8$, first enter the function in the *equation editor* (see Figure A.121) and then graph the function (using a standard viewing window), as shown in Figure A.122. Next, choose the *value* feature from the *calculate* menu, as shown in Figure A.123. You will see "X= " displayed at the bottom of the window. Enter the x-value, in this case $x = 1.8$, as shown in Figure A.124. When entering an x-value, be sure it is between the Xmin and Xmax values you entered for the viewing window. Then press ENTER. In Figure A.125, you can see that when $x = 1.8$,

$$y = -1.08.$$

Figure A.121

Figure A.122

Figure A.123

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Figure A.124

Figure A.125

Viewing Window

A viewing window for a graph is a rectangular portion of the coordinate plane. A viewing window is determined by the following six values (see Figure A.126).

Xmin = the minimum value of x
Xmax = the maximum value of x
Xscl = the number of units per tick mark on the x-axis
Ymin = the minimum value of y
Ymax = the maximum value of y
Yscl = the number of units per tick mark on the y-axis

Figure A.126

When you enter these six values in a graphing utility, you are setting the viewing window. On some graphing utilities there is a seventh value for the viewing window labeled Xres. This sets the pixel resolution (1 through 8). For instance, when

Xres = 1

functions are evaluated and graphed at each pixel on the x-axis. Some graphing utilities have a standard viewing window, as shown in Figure A.127. To initialize the standard viewing window quickly, choose the *standard viewing window* feature from the *zoom* feature menu (see page A23), as shown in Figure A.128.

Figure A.127

Figure A.128

By choosing different viewing windows for a graph, it is possible to obtain different impressions of the graph's shape. For instance, Figure A.129 shows four different viewing windows for the graph of

$y = 0.1x^4 - x^3 + 2x^2.$

Of these viewing windows, the one shown in part (a) is the most complete.

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Figure A.129

On most graphing utilities, the display screen is two-thirds as high as it is wide. On such screens, you can obtain a graph with a true geometric perspective by using a square setting—one in which

$$\frac{\text{Ymax} - \text{Ymin}}{\text{Xmax} - \text{Xmin}} = \frac{2}{3}.$$

One such setting is shown in Figure A.130. Notice that the x and y tick marks are equally spaced on a square setting, but not on a standard setting (see Figure A.127). To initialize the square viewing window quickly, choose the *square viewing window* feature from the *zoom* feature menu (see page A23), as shown in Figure A.131.

Figure A.130

Figure A.131

To see how the viewing window affects the geometric perspective, graph the semicircles

$$y_1 = \sqrt{9 - x^2}$$

and

$$y_2 = -\sqrt{9 - x^2}$$

using a standard viewing window, as shown in Figure A.132. Notice how the circle appears elliptical rather than circular. Now graph y_1 and y_2 using a square viewing window, as shown in Figure A.133. Notice how the circle appears circular. (Note that when you graph the two semicircles, your graphing utility may not connect them. This is because some graphing utilities are limited in their resolution. So, in this text, a blue curve is placed behind the graphing utility's display to indicate where the graph should appear.)

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Figure A.132

Figure A.133

Zero or Root Feature

The *zero* or *root* feature finds the real zeros of the various types of functions studied in this text. To find the zeros of a function such as

$$f(x) = 2x^3 - 4x$$

first enter the function in the *equation editor*, as shown in Figure A.134. Now graph the equation (using a standard viewing window), as shown in Figure A.135. From the graph you can see that the graph of the function crosses the x-axis three times, so the function has three zeros.

Figure A.134

Figure A.135

To find these zeros, choose the *zero* feature found in the *calculate* menu (see Figure A.136). Next, trace the cursor along the graph to a point left of one of the zeros and press ENTER (see Figure A.137). Then trace the cursor along the graph to a point right of the zero and press ENTER (see Figure A.138). Note the two arrows near the top of the display marking the left and right bounds, as shown in Figure A.139. Now trace the cursor along the graph between the two bounds and as close to the zero as you can (see Figure A.140) and press ENTER. In Figure A.141, you can see that one zero of the function is $x \approx -1.414214$.

Figure A.136

Figure A.137

Figure A.138

Figure A.139

Figure A.140

Figure A.141

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Repeat this process to determine that the other two zeros of the function are $x = 0$ (see Figure A.142) and $x \approx 1.414214$ (see Figure A.143).

Figure A.142

Figure A.143

Zoom and Trace Features

The *zoom* feature enables you to adjust the viewing window of a graph quickly (see Figure A.144). For example, the *zoom box* feature allows you to create a new viewing window by drawing a box around any part of the graph.

Figure A.144

The *trace* feature moves from point to point along a graph. For instance, enter the equation

$$y = 2x^3 - 3x + 2$$

in the *equation editor* (see Figure A.145) and graph the equation, as shown in Figure A.146. To activate the *trace* feature, press TRACE; then use the arrow keys to move the cursor along the graph. As you trace the graph, the coordinates of each point are displayed, as shown in Figure A.147.

Figure A.145

Figure A.146

Figure A.147

The *trace* feature combined with the *zoom* feature enables you to obtain better and better approximations of desired points on a graph. For instance, you can use the *zoom* feature to approximate the x-intercept of the graph of

$$y = 2x^3 - 3x + 2.$$

From the viewing window shown in Figure A.146, the graph appears to have only one x-intercept. This intercept lies between -2 and -1. To zoom in on the x-intercept, choose the *zoom-in* feature from the *zoom* feature menu, as shown in Figure A.148. Next, trace the cursor to the point you want to zoom in on, in this case the x-intercept (see Figure A.149). Then press ENTER. You should obtain the graph shown in Figure A.150.

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Now, using the *trace* feature, you can approximate the x-intercept to be

$$x \approx -1.468085$$

as shown in Figure A.151. Use the *zoom-in* feature again to obtain the graph shown in Figure A.152. Using the *trace* feature, you can approximate the x-intercept to be

$$x \approx -1.476064$$

as shown in Figure A.153.

Figure A.148

Figure A.149

Figure A.150

Figure A.151

Figure A.152

Figure A.153

Here are some suggestions for using the *zoom* feature.

1. With each successive zoom-in, adjust the x-scale so that the viewing window shows at least one tick mark on each side of the x-intercept.
2. The error in your approximation will be less than the distance between two scale marks.
3. The *trace* feature can usually be used to add one more decimal place of accuracy without changing the viewing window.

You can adjust the scale in Figure A.153 to obtain a better approximation of the x-intercept. Using the suggestions above, change the viewing window settings so that the viewing window shows at least one tick mark on each side of the x-intercept, as shown in Figure A.154. From Figure A.154, you can determine that the error in your approximation will be less than 0.001 (the Xscl value). Then, using the *trace* feature, you can improve the approximation, as shown in Figure A.155. To three decimal places, the x-intercept is

$$x \approx -1.476.$$

Figure A.154

Figure A.155

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Appendix B: Concepts in Statistics

B.1 Representing Data Graphically

What you should learn

- Use line plots to order and analyze data.
- Use histograms to represent frequency distributions.
- Use bar graphs to represent and analyze data.
- Use scatter plots to represent and analyze data.
- Use line graphs to represent and analyze data.

Why you should learn it

Double bar graphs allow you to compare visually two sets of data over time. For example, in Exercise 15 on page A32, you will create a double bar graph showing the yearly numbers of college enrollments for women and men.

Line Plots

Statistics is the branch of mathematics that studies techniques for collecting, organizing, and interpreting data. In this section, you will study several ways to organize data. The first is a **line plot,** which uses a portion of a real number line to order numbers. Line plots are especially useful for ordering small sets of numbers (about 50 or less) by hand.

Many statistical measures can be obtained from a line plot. Two such measures are the *frequency* and *range* of the data. The **frequency** measures the number of times a value occurs in a data set. The **range** is the difference between the greatest and least data values. For example, consider the data values

20, 21, 21, 25, 32.

The frequency of 21 in the data set is 2 because 21 occurs twice. The range is 12 because the difference between the greatest and least data values is $32 - 20 = 12$.

Example 1 Constructing a Line Plot

The scores from an economics class of 30 students are listed below. The scores are for a 100-point exam.

93, 70, 76, 67, 86, 93, 82, 78, 83, 86, 64, 78, 76, 66, 83
83, 96, 74, 69, 76, 64, 74, 79, 76, 88, 76, 81, 82, 74, 70

a. Use a line plot to organize the scores.

b. Which score occurs with the greatest frequency?

c. What is the range of the scores?

Solution

a. Begin by scanning the data to find the least and greatest numbers. For the data, the least number is 64 and the greatest is 96. Next, draw a portion of a real number line that includes the interval [64, 96]. To create the line plot, start with the first number, 93, and record an × above 93 on the number line. Continue recording an × for each number in the list until you obtain the line plot shown in Figure B.1.

Figure B.1

b. From the line plot, you can see that 76 occurs with the greatest frequency.

c. Because the range is the difference between the greatest and least data values, the range of scores is

$$96 - 64 = 32.$$

CHECKPOINT Now try Exercise 7.

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Histograms and Frequency Distributions

When you want to organize large sets of data, it is useful to group the data into intervals and plot the frequency of the data in each interval. A **frequency distribution** can be used to construct a **histogram.** A histogram uses a portion of a real number line as its horizontal axis. The bars of a histogram are not separated by spaces.

Example 2 Constructing a Histogram

The table at the right shows the percent of the resident population of each state and the District of Columbia who were at least 65 years old in 2008. Construct a frequency distribution and a histogram for the data. (Source: U.S. Census Bureau)

AK	7.3	MT	14.2
AL	13.8	NC	12.4
AR	14.3	ND	14.7
AZ	13.3	NE	13.5
CA	11.2	NH	12.9
CO	10.4	NJ	13.3
CT	13.7	NM	13.1
DC	11.9	NV	11.4
DE	13.9	NY	13.4
FL	17.4	OH	13.7
GA	10.1	OK	13.5
HI	14.8	OR	13.3
IA	14.8	PA	15.4
ID	12.0	RI	14.1
IL	12.2	SC	13.3
IN	12.8	SD	14.4
KS	13.1	TN	13.2
KY	13.3	TX	10.2
LA	12.3	UT	9.0
MA	13.4	VA	12.1
MD	12.1	VT	14.0
ME	15.1	WA	12.0
MI	13.0	WI	13.3
MN	12.5	WV	15.7
MO	13.6	WY	12.3
MS	12.7		

Solution

To begin constructing a frequency distribution, you must first decide on the number of intervals. There are several ways to group the data. However, because the least number is 7.3 and the greatest is 17.4, it seems that six intervals would be appropriate. The first would be the interval

[7, 9)

the second would be

[9, 11)

and so on. By tallying the data into the six intervals, you obtain the frequency distribution shown below.

Interval	*Tally*
[7, 9)	|
[9, 11)	||||
[11, 13)	~~||||~~ ~~||||~~ ~~||||~~
[13, 15)	~~||||~~ ~~||||~~ ~~||||~~ ~~||||~~ ~~||||~~ ||
[15, 17)	|||
[17, 19)	|

You can construct the histogram by drawing a vertical axis to represent the number of states and a horizontal axis to represent the percent of the population 65 and older. Then, for each interval, draw a vertical bar whose height is the total tally, as shown in Figure B.2. Note that the break in the horizontal axis indicates that the numbers 0 through 6 have been omitted.

Figure B.2

CHECKPOINT Now try Exercise 11.

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Bar Graphs

A **bar graph** is similar to a histogram, except that the bars can be either horizontal or vertical and the labels of the bars are not necessarily numbers. For instance, the labels of the bars can be the months of a year, names of cities, or types of vehicles. Another difference between a bar graph and a histogram is that the bars in a bar graph are usually separated by spaces.

Example 3 Constructing a Bar Graph

The data below show the normal monthly precipitation (in inches) in Houston, Texas. Construct a bar graph for the data. What can you conclude? (Source: U.S. National Oceanic and Atmospheric Administration)

Month	Precipitation
January	3.7
February	3.0
March	3.4
April	3.6
May	5.2
June	5.4
July	3.2
August	3.8
September	4.3
October	4.5
November	4.2
December	3.7

Solution

To create a bar graph, begin by drawing a vertical axis to represent the precipitation and a horizontal axis to represent the month. The bar graph is shown in Figure B.3.

Figure B.3

From the graph, you can see that Houston receives a fairly consistent amount of rain throughout the year—the driest month tends to be February and the wettest month tends to be June.

CHECKPOINT Now try Exercise 13.

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Example 4 Constructing a Double Bar Graph

The table shows the percents of associate degrees awarded to males and females for selected fields of study in the United States in 2007. Construct a double bar graph for the data. (Source: U.S. National Center for Education Statistics)

Field of Study	% Female	% Male
Agriculture and Natural Resources	37.9	62.1
Biological Sciences/Life Sciences	67.2	32.8
Accounting	77.5	22.5
Business	60.3	39.7
Communications	49.8	50.2
Education	86.4	13.6
Engineering	14.1	85.9
Family and Consumer Sciences	96.4	3.6
Emergency Medical Technician	32.4	67.6
Law and Legal Studies	89.9	10.1
Liberal/General Studies	62.5	37.5
Mathematics	33.7	66.3
Physical Sciences	43.0	57.0
Precision Production Trades	6.3	93.7
Psychology	76.6	23.4
Social Sciences	66.3	33.7

Solution

For the data, a horizontal bar graph seems to be appropriate. This makes it easier to label and read the bars. Such a graph is shown in Figure B.4.

Figure B.4

CHECKPOINT Now try Exercise 15.

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Scatter Plots

Data can be represented graphically by points plotted in a rectangular coordinate system. This type of graph is called a **scatter plot.** (To review the rectangular coordinate system, see Appendix C.1 at this textbook's *Companion Website*.)

Example 5 Sketching a Scatter Plot

The numbers of employees E (in thousands) in dentist offices in the United States from 1999 through 2009 are shown in the table, where t represents the year. Sketch a scatter plot of the data by hand. (Source: U.S. Bureau of Labor Statistics)

Year, t	Employees, E
1999	667
2000	688
2001	705
2002	725
2003	744
2004	760
2005	774
2006	786
2007	808
2008	818
2009	818

Solution

Before you sketch the scatter plot, it is helpful to represent each pair of values by an ordered pair (t, E) as follows.

(1999, 667), (2000, 688), (2001, 705), (2002, 725), (2003, 744), (2004, 760), (2005, 774), (2006, 786), (2007, 808), (2008, 818), (2009, 818)

To sketch a scatter plot of the data shown in the table, first draw a vertical axis to represent the number of employees (in thousands) and a horizontal axis to represent the year. Then plot the resulting points, as shown in Figure B.5. Note that the break in the t-axis indicates that the numbers 0 through 1998 have been omitted.

Figure B.5

CHECKPOINT Now try Exercise 17.

In Example 5, you could have let $t = 1$ represent the year 1999. In that case, the horizontal axis of the graph would not have been broken, and the tick marks would have been labeled 1 through 11 (instead of 1999 through 2009).

Technology Tip

You can use a graphing utility to graph the scatter plot in Example 5. For instructions on how to use the *list editor* and the *statistical plotting* feature, see Appendix A; for specific keystrokes, go to this textbook's *Companion Website.*

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Line Graphs

A **line graph** is similar to a standard coordinate graph. Line graphs are usually used to show trends over periods of time.

Example 6 Constructing a Line Graph

The table at the right shows the number of immigrants (in thousands) entering the United States for each decade from 1901 through 2000. Construct a line graph for the data. What can you conclude? (Source: U.S. Immigration and Naturalization Service)

Decade	Number (in thousands)
1901–1910	8795
1911–1920	5736
1921–1930	4107
1931–1940	528
1941–1950	1035
1951–1960	2515
1961–1970	3322
1971–1980	4493
1981–1990	7338
1991–2000	9095

Solution

Begin by drawing a vertical axis to represent the number of immigrants in thousands. Then label the horizontal axis with decades and plot the points shown in the table. Finally, connect the points with line segments, as shown in Figure B.6. From the line graph, you can see that the number of immigrants hit a low point during the depression of the 1930s. Since then the number has steadily increased.

Figure B.6

CHECKPOINT Now try Exercise 25.

You can use a graphing utility to check your sketch, as shown in Figure B.7.

Figure B.7

Technology Tip

You can use the *statistical plotting* feature of a graphing utility to create different types of graphs, such as line graphs. For instructions on how to use the *statistical plotting* feature, see Appendix A; for specific keystrokes, go to this textbook's *Companion Website.*

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B.1 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1–6, fill in the blank.

1. _____ are useful for ordering small sets of numbers by hand.
2. A _____ uses a portion of a real number line as its horizontal axis, and the bars are not separated by spaces.
3. You can use a _____ to construct a histogram.
4. The bars in a _____ can be either vertical or horizontal.
5. A _____ represents data graphically as points plotted in a rectangular coordinate system.
6. _____ show trends over periods of time.

Procedures and Problem Solving

✓ **7. Economics** The line plot shows a sample of prices of unleaded regular gasoline in 25 different cities.

(a) What price occurred with the greatest frequency?

(b) What is the range of prices?

8. Environmental Science The line plot shows the weights (to the nearest hundred pounds) of municipal waste hauled by a garbage truck in 30 trips to a landfill.

(a) What weight occurred with the greatest frequency?

(b) What is the range of weights?

Education In Exercises 9 and 10, use the following scores from an algebra class of 30 students. The scores are for one 25-point quiz and one 100-point exam.

Quiz **20, 15, 14, 20, 16, 19, 10, 21, 24, 15, 15, 14, 15, 21, 19, 15, 20, 18, 18, 22, 18, 16, 18, 19, 21, 19, 16, 20, 14, 12**

Exam **77, 100, 77, 70, 83, 89, 87, 85, 81, 84, 81, 78, 89, 78, 88, 85, 90, 92, 75, 81, 85, 100, 98, 81, 78, 75, 85, 89, 82, 75**

9. Construct a line plot for the quiz. Which score(s) occurred with the greatest frequency?
10. Construct a line plot for the exam. Which score(s) occurred with the greatest frequency?

✓ **11. Demographics** The list shows the percents of individuals living below the poverty level in the 50 states in 2008. Use a frequency distribution and a histogram to organize the data. (Source: U.S. Census Bureau)

AK 9.5	AL 16.3	AR 17.6	AZ 14.3
CA 12.9	CO 11.9	CT 8.5	DE 10.4
FL 12.6	GA 14.5	HI 8.8	IA 11.2
ID 12.4	IL 12.1	IN 12.7	KS 11.7
KY 17.2	LA 18.5	MA 10.0	MD 8.0
ME 12.6	MI 14.0	MN 9.7	MO 13.3
MS 21.0	MT 14.3	NC 14.6	ND 12.1
NE 11.1	NH 7.6	NJ 8.7	NM 17.9
NV 10.8	NY 13.8	OH 13.2	OK 16.2
OR 13.4	PA 11.9	RI 11.6	SC 15.5
SD 13.2	TN 15.7	TX 16.3	UT 10.0
VA 9.9	VT 10.5	WA 11.6	WI 10.7
WV 17.2	WY 8.9		

12. Education The list shows the numbers of public high school graduates (in thousands) in the 50 states and the District of Columbia in 2007. Use a frequency distribution and a histogram to organize the data. (Source: U.S. National Center for Education Statistics)

AK 7.7	AL 38.9	AR 27.2	AZ 56.0
CA 356.6	CO 45.6	CT 37.5	DC 2.9
DE 7.2	FL 142.3	GA 77.8	HI 11.1
IA 34.1	ID 16.2	IL 130.2	IN 59.9
KS 30.1	KY 39.1	LA 34.3	MA 63.9
MD 57.6	ME 13.2	MI 111.8	MN 59.5
MO 60.3	MS 24.2	MT 10.1	NC 76.0
ND 7.2	NE 19.9	NH 14.5	NJ 93.0
NM 16.1	NV 16.5	NY 168.3	OH 117.7
OK 37.1	OR 33.4	PA 128.6	RI 10.4
SC 35.1	SD 8.3	TN 54.5	TX 241.2
UT 28.3	VA 74.0	VT 7.3	WA 62.8
WI 64.0	WV 17.4	WY 5.4	

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13. Business The table shows the numbers of Wal-Mart stores from 1995 through 2008. Construct a bar graph for the data. Write a brief statement regarding the number of Wal-Mart stores over time. (Source: Wal-Mart Stores, Inc.)

Year	Number of stores
1995	2943
1996	3054
1997	3406
1998	3599
1999	3985
2000	4189
2001	4414
2002	4688
2003	4906
2004	5289
2005	6141
2006	6779
2007	7262
2008	7720

14. Business The table shows the revenues (in billions of dollars) for Costco Wholesale stores in the years from 1995 through 2009. Construct a bar graph for the data. Write a brief statement regarding the revenue of Costco Wholesale stores over time. (Source: Costco Wholesale Corporation)

Year	Revenue (in billions of dollars)
1995	18.247
1996	19.566
1997	21.874
1998	24.270
1999	27.456
2000	32.164
2001	34.797
2002	38.762
2003	42.546
2004	48.107
2005	52.935
2006	60.151
2007	64.909
2008	72.483
2009	71.422

15. ***Why you should learn it*** *(p. A25)* The table shows the total college enrollments (in thousands) for women and men in the United States from 2002 through 2008. Construct a double bar graph for the data. (Source: U.S. National Center for Education Statistics)

Year	Women (in thousands)	Men (in thousands)
2002	8998	6875
2003	9258	7240
2004	9319	7318
2005	9885	7387
2006	10,032	7456
2007	10,184	7575
2008	10,432	7816

16. Demographics The table shows the populations (in millions) in the coastal regions of the United States in 1990 and 2008. Construct a double bar graph for the data. (Source: U.S. Census Bureau)

Region	1990 population (in millions)	2008 population (in millions)
Atlantic	59.0	69.2
Gulf of Mexico	15.2	20.1
Great Lakes	25.9	27.4
Pacific	33.2	40.8

Sketching a Scatter Plot **In Exercises 17 and 18, sketch a scatter plot of the data shown in the table.**

17. Accounting The table shows the sales y (in millions of dollars) for Apple Inc. for the years 2000 through 2009. (Source: Apple Inc.)

Year	Sales, y (in millions of dollars)
2000	7983
2001	5363
2002	5742
2003	6207
2004	8279
2005	13,931
2006	19,315
2007	24,006
2008	32,479
2009	36,537

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18. **Meteorology** The table shows the lowest temperature on record y (in degrees Fahrenheit) in Flagstaff, Arizona for each month x, where $x = 1$ represents January. (Source: U.S. National Oceanic and Atmospheric Administration)

Month, x	Temperature, y
1	−22
2	−23
3	−16
4	−2
5	14
6	22
7	32
8	24
9	23
10	−2
11	−13
12	−23

Economics **In Exercises 19–24, use the line graph, which shows the average prices of a gallon of premium unleaded gasoline from 2000 through 2008.** (Source: U.S. Energy Information Administration)

19. Describe the trend in the price of premium unleaded gasoline from 2000 to 2002.

20. Describe the trend in the price of premium unleaded gasoline from 2002 to 2008.

21. Determine when the price of premium unleaded gasoline showed the greatest rate of decrease from one year to the next.

22. Determine when the price of premium unleaded gasoline showed the greatest rate of increase from one year to the next.

23. Approximate the percent increase in the price per gallon of premium unleaded gasoline from 2005 to 2008.

24. Approximate the percent increase in the price per gallon of premium unleaded gasoline from 2002 to 2008.

25. **Human Resources** The table shows the total numbers of women in the work force (in thousands) in the United States from 1998 through 2009. Construct a line graph for the data. Write a brief statement describing what the graph reveals. (Source: U.S. Bureau of Labor Statistics)

Year	Women in the work force (in thousands)
1998	63,714
1999	64,855
2000	66,303
2001	66,848
2002	67,363
2003	68,272
2004	68,421
2005	69,288
2006	70,173
2007	70,988
2008	71,767
2009	72,019

26. **Economics** The table shows the total amounts (in trillions of dollars) of private fixed assets held by noncorporate entities in the United States from 2000 through 2008. Construct a line graph for the data. Write a brief statement describing what the graph reveals. (Source: U.S. Bureau of Economic Analysis)

Year	Fixed assets (in trillions of dollars)
2000	13.04
2001	13.95
2002	14.79
2003	15.94
2004	17.77
2005	19.74
2006	21.15
2007	21.56
2008	20.96

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Education **In Exercises 27–30, refer to the scatter plot, which shows the mathematics entrance test scores *x* and the final examination scores *y* in an algebra course for a sample of 10 students.**

27. Find the entrance test score of any student with a final exam score in the 80s.

28. Find the final exam score of any student with an entrance test score in the 50s.

29. Did the student with the highest entrance test score have the highest final exam score?

30. Does a higher entrance test score necessarily imply a higher final exam score? Explain.

31. **Networking** The list shows the percent of households in each of the 50 states and the District of Columbia with Internet access in 2009. Use a graphing utility to organize the data in the graph of your choice. Explain your choice of graph. (Source: National Telecommunications and Information Administration)

AK 77.3	AL 56.1	AR 55.9
AZ 72.1	CA 73.1	CO 72.9
CT 74.8	DC 71.8	DE 71.1
FL 71.5	GA 67.6	HI 73.3
IA 67.8	ID 72.2	IL 68.3
IN 62.5	KS 70.5	KY 59.5
LA 60.6	MA 75.9	MD 73.4
ME 70.2	MI 68.0	MN 72.4
MO 63.7	MS 51.8	MT 64.1
NC 63.9	ND 67.1	NE 70.0
NH 78.7	NJ 76.8	NM 61.7
NV 72.4	NY 70.7	OH 66.9
OK 60.7	OR 76.0	PA 67.3
RI 71.6	SC 58.3	SD 65.5
TN 62.3	TX 63.9	UT 77.9
VA 71.0	VT 70.8	WA 77.9
WI 71.8	WV 59.7	WY 69.8

32. **Athletics** The table shows the numbers of participants (in thousands) in high school athletic programs in the United States from 1998 through 2009. Organize the data in an appropriate display. Explain your choice of graph. (Source: National Federation of State High School Associations)

Year	Female athletes (in thousands)	Male athletes (in thousands)
1998	2570	3763
1999	2653	3832
2000	2676	3862
2001	2784	3921
2002	2807	3961
2003	2856	3989
2004	2865	4038
2005	2908	4110
2006	2953	4207
2007	3022	4321
2008	3057	4372
2009	3114	4423

Conclusions

33. **Writing** Describe the differences between a bar graph and a histogram.

34. **Writing** Describe the differences between a line plot and a scatter plot.

35. **Think About It** How can you decide which type of graph to use when you are organizing data?

36. **Think About It** The graphs shown below represent the same data points.

(a) Which of the two graphs is misleading, and why?

(b) Discuss other ways in which graphs can be misleading.

(c) Why would it be beneficial for someone to use a misleading graph?

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B.2 Measures of Central Tendency and Dispersion

Mean, Median, and Mode

In many real-life situations, it is helpful to describe data by a single number that is most representative of the entire collection of numbers. Such a number is called a **measure of central tendency.** The most commonly used measures are as follows.

1. The **mean,** or **average,** of n numbers is the sum of the numbers divided by n.
2. The **median** of n numbers is the middle number when the numbers are written in numerical order. If n is even, then the median is the average of the two middle numbers.
3. The **mode** of n numbers is the number that occurs most frequently. If two numbers tie for most frequent occurrence, then the collection has two modes and is called **bimodal.**

Example 1 Comparing Measures of Central Tendency

On an interview for a job, the interviewer tells you that the average annual income of the company's 25 employees is $60,849. The actual annual incomes of the 25 employees are shown below. What are the mean, median, and mode of the incomes?

$17,305	$478,320	$45,678	$18,980	$17,408
$25,676	$28,906	$12,500	$24,540	$33,450
$12,500	$33,855	$37,450	$20,432	$28,956
$34,983	$36,540	$250,921	$36,853	$16,430
$32,654	$98,213	$48,980	$94,024	$35,671

Solution

The mean of the incomes is

$$\text{Mean} = \frac{17{,}305 + 478{,}320 + 45{,}678 + 18{,}980 + \cdots + 35{,}671}{25}$$

$$= \frac{1{,}521{,}225}{25}$$

$$= \$60{,}849.$$

To find the median, order the incomes as follows.

$12,500	$12,500	$16,430	$17,305	$17,408
$18,980	$20,432	$24,540	$25,676	$28,906
$28,956	$32,654	$33,450	$33,855	$34,983
$35,671	$36,540	$36,853	$37,450	$45,678
$48,980	$94,024	$98,213	$250,921	$478,320

From this list, you can see that the median income is $33,450. You can also see that $12,500 is the only income that occurs more than once. So, the mode is $12,500.

CHECKPOINT Now try Exercise 5.

In Example 1, was the interviewer telling you the truth about the annual incomes? Technically, the person was telling the truth because the average is (generally) defined to be the mean. However, of the three measures of central tendency—*mean:* $60,849, *median:* $33,450, *mode:* $12,500—it seems clear that the median is most representative. The mean is inflated by the two highest salaries.

What you should learn

- Find and interpret the mean, median, and mode of a set of data.
- Determine the measure of central tendency that best represents a set of data.
- Find the standard deviation of a set of data.
- Use box-and-whisker plots.

Why you should learn it

Measures of central tendency and dispersion provide a convenient way to describe and compare sets of data. For instance, in Exercise 38 on page A43, the mean and standard deviation are used to analyze the prices of gold for the years 1985 through 2008.

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Choosing a Measure of Central Tendency

Which of the three measures of central tendency is most representative of a particular data set? The answer is that it depends on the distribution of the data *and* the way in which you plan to use the data.

For instance, in Example 1, the mean salary of $60,849 does not seem very representative to a potential employee. To a city income tax collector who wants to estimate 1% of the total income of the 25 employees, however, the mean is precisely the right measure.

Example 2 Choosing a Measure of Central Tendency

Which measure of central tendency is most representative of the data given in each frequency distribution?

a.

Number	1	2	3	4	5	6	7	8	9
Frequency	7	20	15	11	8	3	2	0	15

b.

Number	1	2	3	4	5	6	7	8	9
Frequency	9	8	7	6	5	6	7	8	9

c.

Number	1	2	3	4	5	6	7	8	9
Frequency	6	1	2	3	5	5	8	3	0

Solution

a. For this data set, the mean is 4.23, the median is 3, and the mode is 2. Of these, the median or mode is probably the most representative measure.

b. For this data set, the mean and median are each 5 and the modes are 1 and 9 (the distribution is bimodal). Of these, the mean or median is the most representative measure.

c. For this data set, the mean is 4.88, the median is 5, and the mode is 7. Of these, the mean or median is the most representative measure.

CHECKPOINT Now try Exercise 19.

Technology Tip

Calculating the mean and median of a large data set can become time-consuming. Most graphing utilities have *mean* and *median* features that can be used to find the means and medians of data sets. Enter the data from Example 2(a) in the *list editor* of a graphing utility. Then use the *mean* and *median* features to verify the solution to Example 2(a), as shown below.

For instructions on how to use the *list* feature, the *mean* feature, and the *median* feature, see Appendix A; for specific keystrokes, go to this textbook's *Companion Website.*

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Variance and Standard Deviation

Very different sets of numbers can have the same mean. You will now study two **measures of dispersion,** which give you an idea of how much the numbers in a set differ from the mean of the set. These two measures are called the *variance* of the set and the *standard deviation* of the set.

Definitions of Variance and Standard Deviation

Consider a set of numbers $\{x_1, x_2, \ldots, x_n\}$ with a mean of $\bar{x}$. The **variance** of the set is

$$v = \frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \cdots + (x_n - \bar{x})^2}{n}$$

and the **standard deviation** of the set is $\sigma = \sqrt{v}$ (σ is the lowercase Greek letter sigma).

The standard deviation of a set is a measure of how much a typical number in the set differs from the mean. The greater the standard deviation, the more the numbers in the set vary from the mean. For instance, each of the following sets has a mean of 5.

$$\{5, 5, 5, 5\}, \quad \{4, 4, 6, 6\}, \quad \text{and} \quad \{3, 3, 7, 7\}$$

The standard deviations of the sets are 0, 1, and 2.

$$\sigma_1 = \sqrt{\frac{(5-5)^2 + (5-5)^2 + (5-5)^2 + (5-5)^2}{4}} = 0$$

$$\sigma_2 = \sqrt{\frac{(4-5)^2 + (4-5)^2 + (6-5)^2 + (6-5)^2}{4}} = 1$$

$$\sigma_3 = \sqrt{\frac{(3-5)^2 + (3-5)^2 + (7-5)^2 + (7-5)^2}{4}} = 2$$

Example 3 Estimations of Standard Deviation

Consider the three frequency distributions represented by the bar graphs in Figure B.8. Which set has the smallest standard deviation? Which has the largest?

Figure B.8

Solution

Of the three sets, the numbers in set A are grouped most closely to the center and the numbers in set C are the most dispersed. So, set A has the smallest standard deviation and set C has the largest standard deviation.

CHECKPOINT Now try Exercise 21.

Study Tip

In Example 3, you may find it helpful to write each set numerically. For instance, set A is

$$\{1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 4, 5, 5, 5, 6, 6, 7\}.$$

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Example 4 Finding a Standard Deviation

Find the standard deviation of each set shown in Example 3.

Solution

Because of the symmetry of each bar graph, you can conclude that each has a mean of

$$\bar{x} = 4.$$

The standard deviation of set A is

$$\sigma = \sqrt{\frac{(-3)^2 + 2(-2)^2 + 3(-1)^2 + 5(0)^2 + 3(1)^2 + 2(2)^2 + (3)^2}{17}}$$

$$\approx 1.53.$$

The standard deviation of set B is

$$\sigma = \sqrt{\frac{2(-3)^2 + 2(-2)^2 + 2(-1)^2 + 2(0)^2 + 2(1)^2 + 2(2)^2 + 2(3)^2}{14}}$$

$$= 2.$$

The standard deviation of set C is

$$\sigma = \sqrt{\frac{5(-3)^2 + 4(-2)^2 + 3(-1)^2 + 2(0)^2 + 3(1)^2 + 4(2)^2 + 5(3)^2}{26}}$$

$$\approx 2.22.$$

These values confirm the results of Example 3. That is, set A has the smallest standard deviation and set C has the largest.

CHECKPOINT Now try Exercise 25.

The following alternative formula provides a more efficient way to compute the standard deviation.

Alternative Formula for Standard Deviation

The standard deviation of $\{x_1, x_2, \ldots, x_n\}$ is given by

$$\sigma = \sqrt{\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n} - \bar{x}^2}.$$

Because of lengthy computations, this formula is difficult to verify. Conceptually, however, the process is straightforward. It consists of showing that the expressions

$$\sqrt{\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \cdots + (x_n - \bar{x})^2}{n}}$$

and

$$\sqrt{\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n} - \bar{x}^2}$$

are equivalent. Try verifying this equivalence for the set $\{x_1, x_2, x_3\}$ with

$$\bar{x} = \frac{x_1 + x_2 + x_3}{3}.$$

Technology Tip

Calculating the standard deviation of a large data set can become time-consuming. Most graphing utilities have *statistical* features that can be used to find different statistical values of data sets. Enter the data from set A in Example 3 in the *list editor* of a graphing utility. Then use the *one-variable statistics* feature to verify the solution in Example 4, as shown below.

In the figure above, the standard deviation is represented as σx, which is about 1.53. For instructions on how to use the *one-variable statistics* feature, see Appendix A; for specific keystrokes, go to this textbook's *Companion Website.*

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Example 5 Using the Alternative Formula

Use the alternative formula for standard deviation to find the standard deviation of the following set of numbers.

5, 6, 6, 7, 7, 8, 8, 8, 9, 10

Solution

Begin by finding the mean of the set, which is 7.4. So, the standard deviation is

$$\sigma = \sqrt{\frac{5^2 + 2(6^2) + 2(7^2) + 3(8^2) + 9^2 + 10^2}{10} - (7.4)^2}$$

$$= \sqrt{\frac{568}{10} - 54.76}$$

$$= \sqrt{2.04}$$

$$\approx 1.43.$$

You can use the *one-variable statistics* feature of a graphing utility to check this result.

CHECKPOINT Now try Exercise 31.

A well-known theorem in statistics, called *Chebychev's Theorem*, states that at least

$$1 - \frac{1}{k^2}$$

of the numbers in a distribution must lie within k standard deviations of the mean. So, at least 75% of the numbers in a collection must lie within two standard deviations of the mean, and at least 88.9% of the numbers must lie within three standard deviations of the mean. For most distributions, these percents are low. For instance, in all three distributions shown in Example 3, 100% of the numbers lie within two standard deviations of the mean.

Example 6 Describing a Distribution

The table at the right shows the estimated numbers of registered nurses (in thousands) in each state and the District of Columbia in 2009. Find the mean and standard deviation of the numbers. What percent of the numbers lie within two standard deviations of the mean? (Source: U.S. Bureau of Labor Statistics)

AK	5	MT	8
AL	43	NC	88
AR	23	ND	6
AZ	39	NE	19
CA	233	NH	13
CO	42	NJ	75
CT	36	NM	12
DC	9	NV	16
DE	10	NY	166
FL	151	OH	118
GA	65	OK	27
HI	9	OR	31
IA	31	PA	130
ID	11	RI	12
IL	116	SC	38
IN	58	SD	11
KS	26	TN	62
KY	43	TX	168
LA	40	UT	18
MA	83	VA	60
MD	52	VT	6
ME	14	WA	54
MI	85	WI	54
MN	58	WV	17
MO	62	WY	5
MS	28		

Solution

Begin by entering the numbers in a graphing utility. Then use the *one-variable statistics* feature to obtain

$$\bar{x} \approx 50.7 \quad \text{and} \quad \sigma \approx 49.6.$$

The interval that contains all numbers that lie within two standard deviations of the mean is

$$[50.7 - 2(49.6), 50.7 + 2(49.6)] \quad \text{or} \quad [-48.5, 149.9].$$

From the table you can see that all but four of the numbers (about 92%) lie in this interval—all but the numbers that correspond to the numbers of registered nurses in California, Florida, New York, and Texas.

CHECKPOINT Now try Exercise 39.

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Box-and-Whisker Plots

Standard deviation is the measure of dispersion that is associated with the mean. **Quartiles** measure dispersion associated with the median.

Definition of Quartiles

Consider an ordered set of numbers whose median is m. The **lower quartile** is the median of the numbers that occur on or before m. The **upper quartile** is the median of the numbers that occur on or after m.

Example 7 Finding Quartiles of a Set

Find the lower and upper quartiles of the following set.

34, 14, 24, 16, 12, 18, 20, 24, 16, 26, 13, 27

Solution

Begin by ordering the set.

$$\underbrace{12, 13, 14,}_{\text{1st 25\%}} \quad \underbrace{16, 16, 18,}_{\text{2nd 25\%}} \quad \underbrace{20, 24, 24,}_{\text{3rd 25\%}} \quad \underbrace{26, 27, 34}_{\text{4th 25\%}}$$

The median of the entire set is 19. The median of the six numbers that are less than 19 is 15. So, the lower quartile is 15. The median of the six numbers that are greater than 19 is 25. So, the upper quartile is 25.

CHECKPOINT Now try Exercise 41(a).

Quartiles are represented graphically by a **box-and-whisker plot,** as shown in Figure B.9. In the plot, notice that five numbers are listed: the smallest number, the lower quartile, the median, the upper quartile, and the largest number. Also notice that the numbers are spaced proportionally, as though they were on a real number line.

Figure B.9

Figure B.10

Figure B.11

Figure B.12

Technology Tip

You can use a graphing utility to graph the box-and-whisker plot in Figure B.9. First enter the data in the graphing utility's *list editor*, as shown in Figure B.10. Then use the *statistical plotting* feature to set up the box-and-whisker plot, as shown in Figure B.11. Finally, display the box-and-whisker plot (using the *ZoomStat* feature), as shown in Figure B.12. For instructions on how to use the *list editor* and *statistical plotting* features, see Appendix A; for specific keystrokes, go to this textbook's *Companion Website.*

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The next example shows how to find quartiles when the number of elements in a set is not divisible by 4.

Example 8 Sketching Box-and-Whisker Plots

Sketch a box-and-whisker plot for each data set.

a. 82, 82, 83, 85, 87, 89, 90, 94, 95, 95, 96, 98, 99

b. 11, 13, 13, 15, 17, 17, 20, 24, 24, 27

Solution

a. This set has 13 numbers. The median is 90 (the seventh number). The lower quartile is 84 (the median of the first six numbers). The upper quartile is 95.5 (the median of the last six numbers). See Figure B.13.

Figure B.13

b. This set has 10 numbers. The median is 17 (the average of the fifth and sixth numbers). The lower quartile is 13 (the median of the first five numbers). The upper quartile is 24 (the median of the last five numbers). See Figure B.14.

Figure B.14

CHECKPOINT Now try Exercise 41(b).

B.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1–4, fill in the blank(s).

1. A single number that is the most representative of a data set is called a _____ of _____ .

2. If two numbers are tied for the most frequent occurrence, then the collection has two _____ and is called _____ .

3. Two measures of dispersion are called the _____ and the _____ of a data set.

4. _____ measure dispersion associated with the median.

Procedures and Problem Solving

Comparing Measures of Central Tendency **In Exercises 5–10, find the mean, median, and mode of the data set.**

✓ **5.** 5, 12, 7, 14, 8, 9, 7

6. 30, 37, 32, 39, 33, 34, 32

7. 5, 12, 7, 24, 8, 9, 7

8. 20, 37, 32, 39, 33, 34, 32

9. 5, 12, 7, 14, 9, 7

10. 30, 37, 32, 39, 34, 32

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11. Reasoning

(a) Compare your answers in Exercises 5 and 7 with those in Exercises 6 and 8. Which of the measures of central tendency is sensitive to extreme measurements? Explain your reasoning.

(b) Add 6 to each measurement in Exercise 5 and calculate the mean, median, and mode of the revised measurements. How are the measures of central tendency changed?

(c) When a constant k is added to each measurement in a set of data, how will the measures of central tendency change?

12. Finance A person had the following monthly bills for electricity. What are the mean and median of the collection of bills?

January	\$67.92	February	\$59.84
March	\$52.00	April	\$52.50
May	\$57.99	June	\$65.35
July	\$81.76	August	\$74.98
September	\$87.82	October	\$83.18
November	\$65.35	December	\$57.00

13. Sales A car rental company kept the following record of the numbers of miles a rental car was driven. What are the mean, median, and mode of the data?

Monday	410	Tuesday	260
Wednesday	320	Thursday	320
Friday	460	Saturday	150

14. Demographics A study was done on families having six children. The table shows the numbers of families in the study with the indicated numbers of girls. Determine the mean, median, and mode of the data.

Number of girls	0	1	2	3	4	5	6
Frequency	1	24	45	54	50	19	7

15. Athletics The table shows the bowling scores of a three-member team for a three-game series.

Team member	Game 1	Game 2	Game 3
Jay	181	222	196
Hank	199	195	205
Buck	202	251	235

(a) Find the mean for each team member.

(b) Find the mean of the nine scores.

(c) Find the median of the nine scores.

16. Sales The selling prices of 12 new homes built in one subdivision are listed.

\$525,000	\$375,000	\$425,000	\$550,000
\$385,000	\$500,000	\$550,000	\$425,000
\$475,000	\$500,000	\$350,000	\$450,000

(a) Find the mean, mode, and median of the selling prices.

(b) Which measure of central tendency best describes the prices? Explain.

17. Think About It Construct a collection of numbers that has the following properties. If this is not possible, explain why.

Mean = 6, median = 4, mode = 4

18. Think About It Construct a collection of numbers that has the following properties. If this is not possible, explain why.

Mean = 6, median = 6, mode = 4

✓ **19. Education** An English professor records the following scores for a 100-point exam.

99, 64, 80, 77, 59, 72, 87, 79, 92, 88, 90, 42, 20, 89, 42, 100, 98, 84, 78, 91

Which measure of central tendency best describes these test scores?

20. Sales A salesman sold eight pairs of men's brown dress shoes. The sizes of the eight pairs were as follows: $10\frac{1}{2}$, 8, 12, $10\frac{1}{2}$, 10, $9\frac{1}{2}$, 11, and $10\frac{1}{2}$. Which measure (or measures) of central tendency best describes (describe) the typical shoe size for these data?

Finding Mean and Standard Deviation **In Exercises 21 and 22, line plots of data sets are given. Determine the mean and standard deviation of each set.**

✓ **21.** (a)

(b)

(c)

(d)

22. (a)

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(b)

(c)

(d)

Finding Mean, Variance, and Standard Deviation **In Exercises 23–30, find the mean ($\bar{x}$), variance (v), and standard deviation (σ) of the set.**

23. 4, 10, 8, 2

24. 3, 15, 6, 9, 2

25. 0, 1, 1, 2, 2, 2, 3, 3, 4

26. 2, 2, 2, 2, 2, 2

27. 1, 2, 3, 4, 5, 6, 7

28. 1, 1, 1, 5, 5, 5

29. 49, 62, 40, 29, 32, 70

30. 1.5, 0.4, 2.1, 0.7, 0.8

Using the Alternative Formula **In Exercises 31–34, use the alternative formula to find the standard deviation of the set.**

31. 2, 4, 6, 6, 13, 5

32. 246, 336, 473, 167, 219, 359

33. 8.1, 6.9, 3.7, 4.2, 6.1

34. 9.0, 7.5, 3.3, 7.4, 6.0

35. **Reasoning** Without calculating the standard deviation, explain why the set $\{4, 4, 20, 20\}$ has a standard deviation of 8.

36. **Reasoning** When the standard deviation of a set of numbers is 0, what does this imply about the set?

37. **Education** An instructor adds five points to each student's exam score. Will this change the mean or standard deviation of the exam scores? Explain.

38. ***Why you should learn it*** *(p. A35)* The following data represent the average prices of gold (in dollars per ounce) for the years 1985 through 2008. Use a graphing utility to find the mean, variance, and standard deviation of the data. What percent of the data lie within two standard deviations of the mean? (Sources: U.S. Bureau of Mines; U.S. Geological Survey)

318,	368,	478,	438,	383,	385,
363,	345,	361,	385,	386,	389,
332,	295,	280,	280,	272,	311,
365,	411,	446,	606,	699,	900

39. **Education** The scores on a mathematics exam given to 600 science and engineering students at a college had a mean and standard deviation of 235 and 28, respectively. Use Chebychev's Theorem to determine the intervals containing at least $\frac{3}{4}$ and at least $\frac{8}{9}$ of the scores. How do the intervals change when the standard deviation is 16?

40. **Think About It** The histograms represent the test scores of two classes of a college course in mathematics. Which histogram has the smaller standard deviation?

(a)

(b)

Sketching a Box-and-Whisker Plot **In Exercises 41–44, (a) find the lower and upper quartiles of the data and (b) sketch a box-and-whisker plot for the data without using a graphing utility.**

41. 23, 15, 14, 23, 13, 14, 13, 20, 12

42. 11, 10, 11, 14, 17, 16, 14, 11, 8, 14, 20

43. 46, 48, 48, 50, 52, 47, 51, 47, 49, 53

44. 25, 20, 22, 28, 24, 28, 25, 19, 27, 29, 28, 21

Creating a Box-and-Whisker Plot **In Exercises 45–48, use a graphing utility to create a box-and-whisker plot for the data.**

45. 19, 12, 14, 9, 14, 15, 17, 13, 19, 11, 10, 19

46. 9, 5, 5, 5, 6, 5, 4, 12, 7, 10, 7, 11, 8, 9, 9

47. 20.1, 43.4, 34.9, 23.9, 33.5, 24.1, 22.5, 42.4, 25.7, 17.4, 23.8, 33.3, 17.3, 36.4, 21.8

48. 78.4, 76.3, 107.5, 78.5, 93.2, 90.3, 77.8, 37.1, 97.1, 75.5, 58.8, 65.6

49. **Manufacturing** A company has redesigned a product in an attempt to increase the lifetime of the product. The two sets of data list the lifetimes (in months) of 20 units with the original design and 20 units with the new design. Create a box-and-whisker plot for each set of data, and then comment on the differences between the plots.

Original Design

15.1	78.3	56.3	68.9	30.6
27.2	12.5	42.7	72.7	20.2
53.0	13.5	11.0	18.4	85.2
10.8	38.3	85.1	10.0	12.6

New Design

55.8	71.5	25.6	19.0	23.1
37.2	60.0	35.3	18.9	80.5
46.7	31.1	67.9	23.5	99.5
54.0	23.2	45.5	24.8	87.8

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B.3 Least Squares Regression

What you should learn

- Use the sum of squared differences to determine a least squares regression line.
- Find a least squares regression line for a set of data.
- Find a least squares regression parabola for a set of data.

Why you should learn it

The method of least squares provides a way of creating a mathematical model for a set of data, which can then be analyzed.

In many of the examples and exercises in this text, you have been asked to use the *regression* feature of a graphing utility to find mathematical models for sets of data. The *regression* feature of a graphing utility uses the **method of least squares** to find a mathematical model for a set of data. As a measure of how well a model fits a set of data points

$$\{(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_n, y_n)\}$$

you can add the squares of the differences between the actual y-values and the values given by the model to obtain the **sum of the squared differences.** For instance, the table shows the heights x (in feet) and the diameters y (in inches) of eight trees. The table also shows the values of a linear model $y^* = 0.54x - 29.5$ for each x-value. The sum of squared differences for the model is 51.7.

x	70	72	75	76	85	78	77	80
y	8.3	10.5	11.0	11.4	12.9	14.0	16.3	18.0
y^*	8.3	9.38	11.0	11.54	16.4	12.62	12.08	13.7
$(y - y^*)^2$	0	1.2544	0	0.0196	12.25	1.9044	17.8084	18.49

The model that has the *least* sum of squared differences is the **least squares regression** line for the data. The least squares regression line for the data in the table is $y \approx 0.43x - 20.3$. The sum of squared differences is 43.3.

To find the least squares regression line $y = ax + b$ for the points $\{(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_n, y_n)\}$ algebraically, you need to solve the following system for a and b.

$$\begin{cases} nb + \left(\sum_{i=1}^{n} x_i\right)a = \sum_{i=1}^{n} y_i \\ \left(\sum_{i=1}^{n} x_i\right)b + \left(\sum_{i=1}^{n} x_i^2\right)a = \sum_{i=1}^{n} x_i y_i \end{cases}$$

In the system,

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots + x_n$$

$$\sum_{i=1}^{n} y_i = y_1 + y_2 + \cdots + y_n$$

$$\sum_{i=1}^{n} x_i^2 = x_1^2 + x_2^2 + \cdots + x_n^2$$

$$\sum_{i=1}^{n} x_i y_i = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

Technology Tip

Recall from Section 1.7 that when you use the *regression* feature of a graphing utility, the program may output a correlation coefficient, r. When $|r|$ is close to 1, the model is a good fit for the data.

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Example 1 Finding a Least Squares Regression Line

Find the least squares regression line for $(-3, 0)$, $(-1, 1)$, $(0, 2)$, and $(2, 3)$.

Solution

Begin by constructing a table, as shown below.

x	y	xy	x^2
-3	0	0	9
-1	1	-1	1
0	2	0	0
2	3	6	4
$\sum_{i=1}^{n} x_i = -2$	$\sum_{i=1}^{n} y_i = 6$	$\sum_{i=1}^{n} x_i y_i = 5$	$\sum_{i=1}^{n} x_i^2 = 14$

Applying the system for the least squares regression line with $n = 4$ produces

$$\begin{cases} nb + \left(\sum_{i=1}^{n} x_i\right)a = \sum_{i=1}^{n} y_i \\ \left(\sum_{i=1}^{n} x_i\right)b + \left(\sum_{i=1}^{n} x_i^2\right)a = \sum_{i=1}^{n} x_i y_i \end{cases} \Rightarrow \begin{cases} 4b - 2a = 6 \\ -2b + 14a = 5 \end{cases}.$$

Solving this system of equations produces $a = \frac{8}{13}$ and $b = \frac{47}{26}$. So, the least squares regression line is $y = \frac{8}{13}x + \frac{47}{26}$, as shown in Figure B.15.

Figure B.15

CHECKPOINT Now try Exercise 1.

The least squares regression parabola $y = ax^2 + bx + c$ for the points

$\{(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_n, y_n)\}$

is obtained in a similar manner by solving the following system of three equations in three unknowns for a, b, and c.

$$\begin{cases} nc + \left(\sum_{i=1}^{n} x_i\right) b + \left(\sum_{i=1}^{n} x_i^2\right)a = \sum_{i=1}^{n} y_i \\ \left(\sum_{i=1}^{n} x_i\right)c + \left(\sum_{i=1}^{n} x_i^2\right)b + \left(\sum_{i=1}^{n} x_i^3\right)a = \sum_{i=1}^{n} x_i y_i \\ \left(\sum_{i=1}^{n} x_i^2\right)c + \left(\sum_{i=1}^{n} x_i^3\right)b + \left(\sum_{i=1}^{n} x_i^4\right)a = \sum_{i=1}^{n} x_i^2 y_i \end{cases}$$

B.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Finding a Least Squares Regression Line **In Exercises 1–4, find the least squares regression line for the points. Verify your answer with a graphing utility.**

✓ **1.** $(-4, 1), (-3, 3), (-2, 4), (-1, 6)$

2. $(0, -1), (2, 0), (4, 3), (6, 5)$

3. $(-3, 1), (-1, 2), (1, 2), (4, 3)$

4. $(0, -1), (2, 1), (3, 2), (5, 3)$

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Answers to Odd-Numbered Exercises and Tests

Chapter 1

Section 1.1 (page 11)

1. (a) iii (b) i (c) v (d) ii (e) iv
3. parallel **5.** 0 **7.** (a) L_2 (b) L_3 (c) L_1
9. $\frac{3}{2}$
11.

13. $m = -\frac{5}{2}$

15. m is undefined.

17. (0, 1), (3, 1), (−1, 1) **19.** (1, 4), (1, 6), (1, 9)
21. (−1, −7), (−2, −5), (−5, 1)
23. (3, −4), (5, −3), (9, −1)
25. $y = 3x - 2$

27. $y = -\frac{1}{2}x - 2$

29. $x = 6$

31. $y = \frac{3}{2}$

33. $y = 0.45x + 1.15$; \$8.8 million
35. $m = \frac{1}{2}$; y-intercept: $(0, -2)$; a line that rises from left to right
37. $m = \frac{2}{5}$; y-intercept: $(0, 2)$; a line that rises from left to right
39. Slope is undefined; no y-intercept; a vertical line at $x = -6$
41. $m = 0$; y-intercept: $\left(0, -\frac{2}{3}\right)$; a horizontal line at $y = -\frac{2}{3}$
43. (a) $m = 5$; y-intercept: $(0, 3)$
(b)

45. (a) Slope is undefined; there is no y-intercept.
(b)

47. (a) $m = 0$; y-intercept: $\left(0, -\frac{5}{3}\right)$
(b)

49. $y = 2x - 5$
51. $y = -\frac{3}{5}x + 2$

53. $x + 8 = 0$

55. $y = -\frac{1}{2}x + \frac{3}{2}$

57. $y = -\frac{6}{5}x - \frac{18}{25}$

59. $y = \frac{2}{5}x + \frac{1}{5}$

61.

The first graph does not show both intercepts. The third graph is best because it shows both intercepts and gives the most accurate view of the slope by using a square setting.

CHAPTER 1

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63. Perpendicular **65.** Parallel

67. (a) $y = 2x - 3$ (b) $y = -\frac{1}{2}x + 2$

69. (a) $y = -\frac{3}{4}x + \frac{3}{8}$ (b) $y = \frac{4}{3}x + \frac{127}{72}$

71. (a) $y = -3x - 13.1$ (b) $y = \frac{1}{3}x - \frac{1}{10}$

73. (a) $x = 3$ (b) $y = -2$

75. (a) $y = 1$ (b) $x = -4$

77. $y = 2x + 1$ **79.** $y = -\frac{1}{2}x + 1$

81. The lines $y = \frac{1}{2}x$ and $y = -2x$ are perpendicular.

83. The lines $y = -\frac{1}{2}x$ and $y = -\frac{1}{2}x + 3$ are parallel. Both are perpendicular to $y = 2x - 4$.

85. 12 ft

87. (a) The greatest increase was from 2001 to 2002, and the greatest decrease was from 2002 to 2003.
(b) $y = 58.625x + 995$
(c) There is an increase of about \$58.625 million per year.
(d) \$1581.25 million; Answers will vary.

89. $V = 125t + 1415$ **91.** $V = -2000t + 38{,}400$

93. (a) $V = 25{,}000 - 2300t$
(b)

t	0	1	2	3	4
V	25,000	22,700	20,400	18,100	15,800

t	5	6	7	8	9	10
V	13,500	11,200	8900	6600	4300	2000

95. (a) $C = 27.75t + 36{,}500$ (b) $R = 65t$
(c) $P = 37.25t - 36{,}500$ (d) $t \approx 979.9$ h

97. (a) Increase of about 621 students per year
(b) 78,470; 81,575; 84,680
(c) $y = 621x + 75{,}365$, where $x = 0$ corresponds to 1990; $m = 621$; The slope determines the average increase in enrollment.

99. False. The slopes $\left(\frac{2}{7}\text{ and } -\frac{11}{7}\right)$ are not equal.

101.

a and b represent the x- and y-intercepts.

103.

a and b represent the x- and y-intercepts.

105. $3x + 2y - 6 = 0$ **107.** $12x + 3y + 2 = 0$

109. a **111.** c

113. No. Answers will vary. Sample answer: The line $y = 2$ does not have an x-intercept.

115. Yes. Once a parallel line is established to the given line, there are an infinite number of distances away from that line, and thus an infinite number of parallel lines.

117. Yes; $x + 20$ **119.** No **121.** No

123. $(x - 9)(x + 3)$ **125.** $(2x - 5)(x + 8)$

127. Answers will vary.

Section 1.2 (page 24)

1. domain, range, function **3.** No **5.** No

7. Yes. Each element of the domain is assigned to exactly one element of the range.

9. No. The National League, an element in the domain, is assigned to three items in the range, the Cubs, the Pirates, and the Dodgers; the American League, an element in the domain, is also assigned to three items in the range, the Orioles, the Yankees, and the Twins.

11. Yes. Each input value is matched with one output value.

13. (a) Function
(b) Not a function because the element 1 in A corresponds to two elements, -2 and 1, in B.
(c) Function

15. Yes; yes; Each input value (year) is matched with exactly one output value (average price) for both name brand and generic drug prescriptions.

17. Not a function **19.** Function **21.** Function

23. Not a function **25.** Function **27.** Not a function

29. (a) 7 (b) -11 (c) $3t + 7$

31. (a) 0 (b) -0.75 (c) $x^2 + 2x$

33. (a) 1 (b) 2.5 (c) $3 - 2|x|$

35. (a) Undefined (b) $-\dfrac{1}{5}$ (c) $\dfrac{1}{y^2 + 6y}$

37. (a) 1 (b) -1 (c) $\dfrac{|t|}{t}$

39. (a) -1 (b) 2 (c) 6

41. (a) 6 (b) 3 (c) 10

43. (a) 0 (b) 4 (c) 17

45. $\{(-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4)\}$

47. $\{(-2, 4), (-1, 3), (0, 2), (1, 3), (2, 4)\}$

49.

t	-5	-4	-3	-2	-1
$h(t)$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1

51. 5 **53.** $\frac{4}{3}$ **55.** All real numbers x

57. All real numbers t except $t = 0$ **59.** All real numbers x

61. All real numbers x except $x = 0, -2$

63. All real numbers y such that $y > 10$

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65.

Domain: $[-2, 2]$
Range: $[0, 2]$

67.

Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

69. $A = \dfrac{C^2}{4\pi}$

71. (a) 1024 cm^3

(b)

Yes, it is a function.

(c) $V = x(24 - 2x)^2,\ 0 < x < 12$

(d)

The function fits the data points; Answers will vary.

73. $A = 2xy = 2x\sqrt{36 - x^2},\ 0 < x < 6$

75. (a) $C = 68.20x + 248{,}000$ (b) $R = 98.98x$
(c) $P = 30.78x - 248{,}000$

77. (a) The independent variable is t and represents the year. The dependent variable is n and represents the numbers of miles traveled.

(b)

t	0	1	2	3	4	5	6
$n(t)$	581	645	699	742	775	798	809

t	7	8	9	10	11	12	13
$n(t)$	844	870	895	921	947	972	998

t	14	15	16	17
$n(t)$	1024	1050	1075	1101

(c) The model fits the data well.

79. No; The ball is 9.6 feet high when it reaches the first baseman.

81. $2, c \neq 0$ **83.** $3 + h, h \neq 0$

85. $-\dfrac{1}{t}, t \neq 1$ **87.** False. The range is $[-1, \infty)$.

89. $f(x) = \sqrt{x} + 2$
Domain: $x \geq 0$
Range: $[2, \infty)$

91. No, f is not the independent variable because the value of f depends on the value of x.

93. $\dfrac{12x + 20}{x + 2}$ **95.** $\dfrac{(x + 6)(x + 10)}{5(x + 3)}, x \neq 0, \dfrac{1}{2}$

Section 1.3 (page 37)

1. decreasing **3.** $[1, 4]$ **5.** Relative maximum

7. Domain: $(-\infty, \infty)$
Range: $(-\infty, 1]$
$f(0) = 1$

9. Domain: $[-4, 4]$
Range: $[0, 4]$
$f(0) = 4$

11.

Domain: $(-\infty, \infty)$
Range: $[3, \infty)$

13.

Domain: $[1, \infty)$
Range: $[0, \infty)$

15.

Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

17. (a) $(-\infty, \infty)$ (b) $[-2, \infty)$ (c) $-1, 3$
(d) x-intercepts (e) -1 (f) y-intercept
(g) $-2; (1, -2)$ (h) $0; (-1, 0)$ (i) 2

19. Function. Graph the given function over the window shown in the figure.

21. Not a function. Solve for y and graph the two resulting functions.

23. Increasing on $(-\infty, \infty)$

25. Increasing on $(-\infty, 0), (2, \infty)$
Decreasing on $(0, 2)$

27. (a)

(b) Constant: $(-\infty, \infty)$

29. (a)

(b) Decreasing on $(-\infty, 0)$
Increasing on $(0, \infty)$

31. (a)

(b) Increasing on $(-2, \infty)$
Decreasing on $(-3, -2)$

33. (a)

(b) Decreasing on $(-\infty, -1)$
Constant on $(-1, 1)$
Increasing on $(1, \infty)$

35.

Relative minimum: $(3, -9)$

CHAPTER 1

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37.

Relative minimum: $(1, -7)$
Relative maximum: $(-2, 20)$

39.

Relative minimum: $(0.33, -0.38)$

41.

Relative minimum: $(2, -9)$

43.

Relative minimum: $(1, -2)$
Relative maximum: $(-1, 2)$

45.

Relative minimum: $(1, -2)$

47.

49.

51.

53.

Domain: $(-\infty, \infty)$
Range: $[0, 2)$
Sawtooth pattern

55.

57.

59.

61.

63. Even

65. Neither

67. Even

69. Neither

71. Neither

73. (a) $\left(\frac{3}{2}, 4\right)$ (b) $\left(\frac{3}{2}, -4\right)$
75. (a) $(-4, 9)$ (b) $(-4, -9)$
77. (a) $(-x, -y)$ (b) $(-x, y)$
79. (a)–(c) Neither **81.** (a)–(c) Odd
83. (a)–(c) Odd **85.** (a)–(c) Even

87.

$(-\infty, 4]$

89.

$(-\infty, -3], [3, \infty)$

91. (a) C_2 is the appropriate model. The cost of the first minute is \$1.05 and the cost increases \$0.08 when the next minute begins, and so on.

(b)

\$2.49

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93. $h = -x^2 + 4x - 3,\ 1 \le x \le 3$

95. (a)

(b) Increasing from 2000 to 2005; decreasing from 2005 to 2008

(c) About 850

97. False. Counterexample: $f(x) = \sqrt{1 + x^2}$

99. c **100.** d **101.** b **102.** e **103.** a **104.** f

105. Yes. To check whether x is a function of y, determine whether any horizontal line can be drawn to intersect the graph more than once.

107. Yes

109. (a) Even. g is a reflection in the x-axis.

(b) Even. g is a reflection in the y-axis.

(c) Even. g is a vertical shift downward.

(d) Neither. g is shifted to the right and reflected in the x-axis.

111. Proof **113.** Terms: $-2x^2$, $8x$; coefficients: -2, 8

115. Terms: $\frac{x}{3}$, $-5x^2$, x^3; coefficients: $\frac{1}{3}$, -5, 1

117. (a) -17 (b) 1 (c) $-x^2 + 3x + 1$

119. $h + 4,\ h \neq 0$

Section 1.4 (page 47)

1. Horizontal shifts, vertical shifts, reflections

3. $-f(x),\ f(-x)$

5.

7.

9.

11.

13.

15.

17.

19. (a)

(b)

(c)

(d)

(e)

(f)

(g)

21. The graph of $f(x) = x^2$ should have been shifted one unit to the left instead of one unit to the right.

23. Vertical shift two units upward

25. Horizontal shift four units to the right

27. Vertical shift two units downward

29. Vertical shift of $y = x$; $y = x + 3$

31. Vertical shift of $y = x^2$; $y = x^2 - 1$

33. Reflection in the x-axis and a vertical shift one unit upward of $y = \sqrt{x}$; $y = 1 - \sqrt{x}$

35. Reflection in the x-axis

37. Reflection in the y-axis (identical)

39. Reflection in the x-axis

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41. Vertical stretch **43.** Vertical shrink

45. Horizontal shrink

47.

g is a horizontal shift and h is a vertical shrink.

49.

g is a vertical shrink and a reflection in the x-axis and h is a reflection in the y-axis.

51. (a) $f(x) = x^2$

(b) Horizontal shift five units to the left, reflection in the x-axis, and vertical shift two units upward

(c)

(d) $g(x) = 2 - f(x + 5)$

53. (a) $f(x) = x^2$

(b) Horizontal shift four units to the right, vertical stretch, and vertical shift three units upward

(c)

(d) $g(x) = 3 + 2f(x - 4)$

55. (a) $f(x) = x^3$

(b) Horizontal shift two units to the right and vertical stretch

(c)

(d) $g(x) = 3f(x - 2)$

57. (a) $f(x) = x^3$

(b) Horizontal shift one unit to the right and vertical shift two units upward

(c)

(d) $g(x) = f(x - 1) + 2$

59. (a) $f(x) = \dfrac{1}{x}$

(b) Horizontal shift eight units to the left and vertical shift nine units downward

(c)

(d) $g(x) = f(x + 8) - 9$

61. (a) $f(x) = |x|$

(b) Horizontal shift one unit to the right, reflection in the x-axis, vertical stretch, and vertical shift four units downward

(c)

(d) $g(x) = -2f(x - 1) - 4$

63. (a) $f(x) = \sqrt{x}$

(b) Horizontal shift three units to the left, reflection in the x-axis, vertical shrink, and vertical shift one unit downward

(c)

(d) $g(x) = -\frac{1}{2}f(x + 3) - 1$

65. (a) Horizontal shift 24.7 units to the right, vertical shift 183.4 units upward, reflection in the x-axis, and a vertical shrink

(b)

(c) $G(t) = F(t + 10) = -0.099[(t + 10) - 24.7]^2 + 183.4$
$= -0.099(t - 14.7)^2 + 183.4$

To make a horizontal shift 10 years backward (10 units left), add 10 to t.

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67. False. When $f(x) = x^2$, $f(-x) = (-x)^2 = x^2$. Because $f(x) = f(-x)$ in this case, $y = f(-x)$ is not a reflection of $y = f(x)$ across the x-axis in all cases.

69. $x = -2$ and $x = 3$

71. Cannot be determined because it is a vertical shift

73. c **75.** c **77.** Answers will vary.

79. (a) The graph of g is a vertical shrink of the graph of f.
(b) The graph of g is a vertical stretch of the graph of f.

81. Neither

83. All real numbers x except $x = 9$

85. All real numbers x such that $-10 \le x \le 10$

Section 1.5 (page 56)

1. addition, subtraction, multiplication, division

3. $g(x)$ **5.** $2x$

7.

9.

11. (a) $2x$ (b) 6 (c) $x^2 - 9$ (d) $\dfrac{x+3}{x-3}$

All real numbers x, except $x = 3$

13. (a) $x^2 - x + 1$ (b) $x^2 + x - 1$

(c) $x^2 - x^3$ (d) $\dfrac{x^2}{1-x}$

All real numbers x, except $x = 1$

15. (a) $x^2 + 5 + \sqrt{1-x}$ (b) $x^2 + 5 - \sqrt{1-x}$

(c) $(x^2 + 5)\sqrt{1-x}$ (d) $\dfrac{x^2+5}{\sqrt{1-x}}$, $x < 1$

17. (a) $\dfrac{x+1}{x^2}$ (b) $\dfrac{x-1}{x^2}$ (c) $\dfrac{1}{x^3}$ (d) x, $x \ne 0$

19. 9 **21.** 1 **23.** 140 **25.** $-\frac{24}{7}$ **27.** $4t^2 - 2t + 1$

29. $-125t^3 - 50t^2 + 5t + 2$ **31.** $\dfrac{t^2-1}{-t-2}$

33.

35.

37.

$f(x)$, $0 \le x \le 2$;
$g(x)$, $x > 6$

39.

$f(x)$, $0 \le x \le 2$;
$f(x)$, $x > 6$

41. (a) $(x-1)^2$ (b) $x^2 - 1$ (c) 1

43. (a) $20 - 3x$ (b) $-3x$ (c) 20

45. (a) All real numbers x such that $x \ge -4$
(b) All real numbers x
(c) All real numbers x

47. (a) All real numbers x
(b) All real numbers x such that $x \ge 0$
(c) All real numbers x such that $x \ge 0$

49. (a) All real numbers x except $x = 0$
(b) All real numbers x
(c) All real numbers x except $x = -3$

51. (a) All real numbers x
(b) All real numbers x
(c) All real numbers x

53. (a) All real numbers x
(b) All real numbers x except $x = \pm 2$
(c) All real numbers x except $x = \pm 2$

55. (a) $(f \circ g)(x) = \sqrt{x^2 + 4}$
$(g \circ f)(x) = x + 4$, $x \ge -4$;
Domain of $f \circ g$: all real numbers x
(b) $f \circ g \ne g \circ f$

57. (a) $(f \circ g)(x) = x$; $(g \circ f)(x) = x$;
Domain of $f \circ g$: all real numbers x
(b) $f \circ g = g \circ f$

59. (a) $(f \circ g)(x) = x^4$; $(g \circ f)(x) = x^4$;
Domain of $f \circ g$: all real numbers x
(b) $f \circ g = g \circ f$

61. (a) $(f \circ g)(x) = 24 - 5x$; $(g \circ f)(x) = -5x$
(b) $24 - 5x \ne -5x$
(c)

x	0	1	2	3
$g(x)$	4	3	2	1
$(f \circ g)(x)$	24	19	14	9

x	0	1	2	3
$f(x)$	4	9	14	19
$(g \circ f)(x)$	0	-5	-10	-15

63. (a) $(f \circ g)(x) = \sqrt{x^2 + 1}$; $(g \circ f)(x) = x + 1$, $x \ge -6$
(b) $x + 1 \ne \sqrt{x^2 + 1}$

CHAPTER 1

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(c)

x	0	1	2	3
$g(x)$	-5	-4	-1	4
$(f \circ g)(x)$	1	$\sqrt{2}$	$\sqrt{5}$	$\sqrt{10}$

x	0	1	2	3
$f(x)$	$\sqrt{6}$	$\sqrt{7}$	$\sqrt{8}$	3
$(g \circ f)(x)$	1	2	3	4

65. (a) $(f \circ g)(x) = |2x^3|$; $(g \circ f)(x) = 2|x|^3$
(b) $|2x^3| = 2|x|^3$
(c)

x	-2	-1	0	1	2
$g(x)$	-16	-2	0	2	16
$(f \circ g)(x)$	16	2	0	2	16

x	-2	-1	0	1	2
$f(x)$	2	1	0	1	2
$(g \circ f)(x)$	16	2	0	2	16

67. (a) 3 (b) 0 **69.** (a) 0 (b) 4
71. $f(x) = x^2$, $g(x) = 2x + 1$
73. $f(x) = \sqrt[3]{x}$, $g(x) = x^2 - 4$
75. $f(x) = \dfrac{1}{x}$, $g(x) = x + 2$
77. $f(x) = x^2 + 2x$, $g(x) = x + 4$
79. (a) $T = \frac{3}{4}x + \frac{1}{15}x^2$
(b)

(c) B. For example, $B(60) = 240$, whereas $R(60)$ is only 45.
81. (a) $r(x) = \dfrac{x}{2}$ (b) $A(r) = \pi r^2$ (c) $(A \circ r)(x) = \pi\left(\dfrac{x}{2}\right)^2$
$(A \circ r)(x)$ represents the area of the circular base of the tank with radius $x/2$.
83. (a) $T = 1.2t^2 + 75.4t + 1220$
(b)

85. (a) $N(T(t))$ or $(N \circ T)(t) = 40t^2 + 590$; $N(T(t))$ or $(N \circ T)(t)$ represents the number of bacteria after t hours outside the refrigerator.
(b) $(N \circ T)(6) = 2030$; There are 2030 bacteria in a refrigerated food product after 6 hours outside the refrigerator.
(c) About 2.3 h
87. $s(t) = \sqrt{(150 - 450t)^2 + (200 - 450t)^2}$
$= 50\sqrt{162t^2 - 126t + 25}$

89. False. $g(x) = x - 3$
91. (a) $O(M(Y)) = 2\left(6 + \frac{1}{2}Y\right) = 12 + Y$; Answers will vary.
(b) Middle child is 8 years old, youngest child is 4 years old.
93. Proof **95.** Proof **97.** a, c
99. $(0, -5)$, $(1, -5)$, $(2, -7)$
101. $\left(0, 2\sqrt{6}\right)$, $\left(1, \sqrt{23}\right)$, $\left(2, 2\sqrt{5}\right)$
103. $y = 10x + 38$ **105.** $y = -\frac{30}{11}x + \frac{34}{11}$

Section 1.6 (page 67)

1. inverse, f^{-1} **3.** $y = x$ **5.** At most once
7. $f^{-1}(x) = \dfrac{x}{6}$ **9.** $f^{-1}(x) = x - 7$
11. $f^{-1}(x) = \frac{1}{2}(x - 1)$ **13.** $f^{-1}(x) = x^3$
15. c **16.** b **17.** a **18.** d
19. $f(g(x)) = f\left(\sqrt[3]{x}\right) = \left(\sqrt[3]{x}\right)^3 = x$
$g(f(x)) = g(x^3) = \sqrt[3]{x^3} = x$

Reflections in the line $y = x$
21. $f(g(x)) = f(x^2 + 4), \quad x \geq 0$
$= \sqrt{(x^2 + 4) - 4} = x$
$g(f(x)) = g\left(\sqrt{x - 4}\right)$
$= \left(\sqrt{x - 4}\right)^2 + 4 = x$

Reflections in the line $y = x$
23. $f(g(x)) = f\left(\sqrt[3]{1 - x}\right) = 1 - \left(\sqrt[3]{1 - x}\right)^3 = x$
$g(f(x)) = g(1 - x^3) = \sqrt[3]{1 - (1 - x^3)} = x$

Reflections in the line $y = x$
25. (a) $f(g(x)) = f\left(-\dfrac{2x + 6}{7}\right)$
$= -\dfrac{7}{2}\left(-\dfrac{2x + 6}{7}\right) - 3 = x$
$g(f(x)) = g\left(-\dfrac{7}{2}x - 3\right)$
$= -\dfrac{2\left(-\frac{7}{2}x - 3\right) + 6}{7} = x$
(b)

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(c)

x	0	2	-2	6
$f(x)$	-3	-10	4	-24

x	-3	-10	4	-24
$g(x)$	0	2	-2	6

27. (a) $f(g(x)) = f\left(\sqrt[3]{x-5}\right) = \left(\sqrt[3]{x-5}\right)^3 + 5 = x$

$g(f(x)) = g(x^3+5) = \sqrt[3]{(x^3+5)-5} = x$

(b)

(c)

x	0	1	-1	-2	4
$f(x)$	5	6	4	-3	69

x	5	6	4	-3	69
$g(x)$	0	1	-1	-2	4

29. (a) $f(g(x)) = f(8+x^2)$

$= -\sqrt{(8+x^2)-8}$

$= -\sqrt{x^2} = -(-x) = x,\ x \le 0$

$g(f(x)) = g\left(-\sqrt{x-8}\right)$

$= 8 + \left(-\sqrt{x-8}\right)^2$

$= 8 + (x-8) = x,\ x \ge 8$

(b)

(c)

x	8	9	12	15
$f(x)$	0	-1	-2	$-\sqrt{7}$

x	0	-1	-2	$-\sqrt{7}$
$g(x)$	8	9	12	15

31. (a) $f(g(x)) = f\left(\dfrac{x}{2}\right)$

$= 2\left(\dfrac{x}{2}\right) = x$

$g(f(x)) = g(2x)$

$= \dfrac{2x}{2} = x$

(b)

(c)

x	-4	-2	0	2	4
$f(x)$	-8	-4	0	4	8

x	-8	-4	0	4	8
$g(x)$	-4	-2	0	2	4

33. (a) $f(g(x)) = f\left(-\dfrac{5x+1}{x-1}\right)$

$$= \frac{\left(-\dfrac{5x+1}{x-1}\right) - 1}{\left(-\dfrac{5x+1}{x-1}\right) + 5} = \frac{\dfrac{6x}{x-1}}{\dfrac{6}{x-1}} = x, \quad x \ne 1$$

$g(f(x)) = g\left(\dfrac{x-1}{x+5}\right)$

$$= -\frac{5\left(\dfrac{x-1}{x+5}\right) + 1}{\left(\dfrac{x-1}{x+5}\right) - 1} = \frac{\dfrac{6x}{x+5}}{\dfrac{6}{x+5}} = x,\ x \ne -5$$

(b)

(c)

x	-3	-2	-1	0	2	3	4
$f(x)$	-2	-1	$-\frac{1}{2}$	$-\frac{1}{5}$	$\frac{1}{7}$	$\frac{1}{4}$	$\frac{1}{3}$

x	-2	-1	$-\frac{1}{2}$	$-\frac{1}{5}$	$\frac{1}{7}$	$\frac{1}{4}$	$\frac{1}{3}$
$g(x)$	-3	-2	-1	0	2	3	4

35. Yes. No two elements in the domain of f correspond to the same element in the range of f.

37. No. -3 and 0 both correspond to 6, so f is not one-to-one.

39. Not a function **41.** Function; one-to-one

43. Function; one-to-one

45.

One-to-one

47.

Not one-to-one

49.

Not one-to-one

51.

Not one-to-one

53.

One-to-one

55.

Not one-to-one

CHAPTER 1

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57.

An inverse function exists.

59. Not one-to-one **61.** $f^{-1}(x) = \dfrac{5x - 4}{3}$

63. Not one-to-one **65.** $f^{-1}(x) = \sqrt{x} - 3$

67. $f^{-1}(x) = \dfrac{x^2 - 3}{2}, x \geq 0$ **69.** $f^{-1}(x) = 2 - x, x \geq 0$

71. $f^{-1}(x) = \dfrac{x + 3}{2}$ **73.** $f^{-1}(x) = \sqrt[5]{x}$

Reflections in the line $y = x$

Reflections in the line $y = x$

75. $f^{-1}(x) = x^{5/3}$

Reflections in the line $y = x$

77. $f^{-1}(x) = \sqrt{4 - x^2}$, $0 \leq x \leq 2$

The graphs are the same.

79. $f^{-1}(x) = \dfrac{4}{x}$

The graphs are the same.

81. $f^{-1}(x) = \sqrt{x} + 2$
Domain of f: all real numbers x such that $x \geq 2$
Range of f: all real numbers y such that $y \geq 0$
Domain of f^{-1}: all real numbers x such that $x \geq 0$
Range of f^{-1}: all real numbers y such that $y \geq 2$

83. $f^{-1}(x) = x - 2$
Domain of f: all real numbers x such that $x \geq -2$
Range of f: all real numbers y such that $y \geq 0$
Domain of f^{-1}: all real numbers x such that $x \geq 0$
Range of f^{-1}: all real numbers y such that $y \geq -2$

85. $f^{-1}(x) = \sqrt{x} - 3$
Domain of f: all real numbers x such that $x \geq -3$
Range of f: all real numbers y such that $y \geq 0$
Domain of f^{-1}: all real numbers x such that $x \geq 0$
Range of f^{-1}: all real numbers y such that $y \geq -3$

87. $f^{-1}(x) = \dfrac{\sqrt{-2(x - 5)}}{2}$
Domain of f: all real numbers x such that $x \geq 0$
Range of f: all real numbers y such that $y \leq 5$
Domain of f^{-1}: all real numbers x such that $x \leq 5$
Range of f^{-1}: all real numbers y such that $y \geq 0$

89. $f^{-1}(x) = x + 3$
Domain of f: all real numbers x such that $x \geq 4$
Range of f: all real numbers y such that $y \geq 1$
Domain of f^{-1}: all real numbers x such that $x \geq 1$
Range of f^{-1}: all real numbers y such that $y \geq 4$

91.

x	-4	-2	2	3
$f^{-1}(x)$	-2	-1	1	3

93. $\frac{1}{2}$ **95.** -2 **97.** 0 **99.** 2

101. (a) and (b)

(c) Inverse function because it satisfies the Vertical Line Test.

103. (a) and (b)

(c) Not an inverse function because it does not satisfy the Vertical Line Test.

105. 32 **107.** 600 **109.** $2\sqrt[3]{x + 3}$

111. $\dfrac{x + 1}{2}$ **113.** $\dfrac{x + 1}{2}$

115. (a) f is one-to-one because no two elements in the domain (men's U.S. shoe sizes) correspond to the same element in the range (men's European shoe sizes).
(b) 45 (c) 10 (d) 41 (e) 13

117. (a) 19 9 64 64 4 69 29 4 64 9 104 29 94
(b) $f^{-1}(x) = \dfrac{x - 4}{5}$; What time

119. False. For example, $y = x^2$ is even, but does not have an inverse.

121. This situation could be represented by a one-to-one function. The inverse function would represent the number of miles completed in terms of time in hours.

123. This function could not be represented by a one-to-one function because it oscillates.

125. The graph of f^{-1} is a reflection of the graph of f in the line $y = x$.

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127. (a) The function will be one-to-one because no two values of x will produce the same value for $f(x)$.

(b) $f^{-1}(50)$ represents the value of 50 degrees Fahrenheit in degrees Celsius.

129. Constant function **131.** Proof **133.** $9x,\ x \neq 0$

135. $-(x + 6),\ x \neq 6$ **137.** Not a function

139. Not a function

Section 1.7 (page 76)

1. positive **3.** Negative correlation

5. (a)

(b) Yes, the data appear somewhat linear. More experience, x, corresponds to higher sales, y.

7. Negative correlation **9.** No correlation

11. (a) and (b)

(c) $y = x$

13. (a) and (b)

(c) $y = \frac{1}{2}x + 2$

15. $y = 0.46x + 1.6$

(a)

(b)

x	-3	-1	0	2	4
Linear equation	0.22	1.14	1.6	2.52	3.44
Given data	0	1	2	3	3

The model fits the data well.

17. (a)

(b) $d = 0.07F - 0.3$

(c) $d = 0.066F$

(d) 3.63 cm

19. (a) and (c)

The model fits the data well.

(b) $T = 12.37t + 24.04$

(d) 2010: \$147.74 million; 2015: \$209.59 million; Answers will vary.

(e) 12.37; The slope represents the average annual increase in salaries (in millions of dollars).

21. (a) and (c)

(b) $P = 38.98t + 8655.4$

(d)

Year	2010	2015	2020	2025	2030
Actual	9018	9256	9462	9637	9802
Model	9045.2	9240.1	9435	9629.9	9824.8

The model fits the data well.

(e) 10,604,400 people; Answers will vary.

23. (a) $y = 47.77x + 103.8$

(b)

(c) The slope represents the increase in sales due to increased advertising.

(d) \$175,455

25. (a) $T = -0.019t + 4.92$

$r \approx -0.886$

(b) The negative slope means that the winning times are generally decreasing over time.

(c)

CHAPTER 1

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(d)

Year	1952	1956	1960	1964	1968
Actual	5.20	4.91	4.84	4.72	4.53
Model	4.88	4.81	4.73	4.65	4.58

Year	1972	1976	1980	1984	1988
Actual	4.32	4.16	4.15	4.12	4.06
Model	4.50	4.43	4.35	4.27	4.20

Year	1992	1996	2000	2004	2008
Actual	4.12	4.12	4.10	4.09	4.05
Model	4.12	4.05	3.97	3.89	3.82

The model does not fit the data well.

(e) The closer $|r|$ is to 1, the better the model fits the data.

(f) No; The winning times have leveled off in recent years, but the model values continue to decrease to unrealistic times.

27. True. To have positive correlation, the y-values tend to increase as x increases.

29. Answers will vary.

31. (a) 10 (b) $2w^2 + 5w + 7$ **33.** $-\frac{3}{5}$ **35.** $-\frac{1}{4}, \frac{3}{2}$

Review Exercises (page 82)

1.

$m = 0$

3.

m is undefined.

5.

$m = \frac{3}{7}$

7.

$m = -\frac{5}{11}$

9. $x - 4y - 6 = 0$; $(6, 0), (10, 1), (-2, -2)$

11. $3x - 2y - 10 = 0$; $(4, 1), (2, -2), (-2, -8)$

13. $y - 6 = 0$; $(0, 6), (1, 6), (-1, 6)$

15. $x - 10 = 0$; $(10, 1), (10, 3), (10, -2)$

17. $y = -1$

19. $y = \frac{11}{3}$

21. $y = \frac{2}{7}x + \frac{2}{7}$

23. $y = -\frac{1}{2}x + \frac{1}{2}$

25. $V = 850t + 12{,}500$ **27.** $V = 42.70t + 625.5$

29. \$210,000 **31.** (a) $y = \frac{5}{4}x - \frac{23}{4}$ (b) $y = -\frac{4}{5}x + \frac{2}{5}$

33. (a) Not a function because element 20 in A corresponds to two elements, 4 and 6, in B.

(b) Function

35. Not a function **37.** Function

39. Function **41.** Not a function

43. (a) 2 (b) 10 (c) $b^6 + 1$ (d) $x^2 - 2x + 2$

45. (a) -3 (b) -1 (c) 2 (d) 6

47. All real numbers x except $x = -2$

49. All real numbers x such that $-5 \le x \le 5$

51. (a) $C = 5.35x + 16{,}000$ (b) $P = 2.85x - 16{,}000$

53. $2h + 4x + 3,\ h \neq 0$

55.

Domain: all real numbers x
Range: $(-\infty, 3]$

57.

Domain: $[-3, \infty)$
Range: $[4, \infty)$

59.

Domain: $[-6, 6]$
Range: $[0, 6]$

61.

Domain: all real numbers x
Range: $[2, \infty)$

63. Function. Solve for y and graph the resulting function.

65. Not a function. Solve for y and graph the two resulting functions.

67. (a)

(b) Increasing on $(-\infty, -1), (1, \infty)$
Decreasing on $(-1, 1)$

69. (a)

(b) Increasing on $(6, \infty)$

71. Relative maximum: $(0, 16)$
Relative minima: $(-2, 0), (2, 0)$

73. Relative maximum: $(3, 27)$

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75.

77.

79. Even **81.** Even **83.** Neither **85.** Even

87. Rational function $f(x) = \dfrac{1}{x}$

Horizontal shift three units right

$g(x) = \dfrac{1}{x - 3}$

89. Quadratic function $f(x) = x^2$

Vertical shift one unit upward, horizontal shift two units right

$g(x) = (x - 2)^2 + 1$

91. Absolute value function $f(x) = |x|$

Vertical shift three units upward

$g(x) = |x| + 3$

93.

95.

97. (a) Rational function $f(x) = \dfrac{1}{x}$

(b) Vertical shift six units downward

(c)

(d) $h(x) = f(x) - 6$

99. (a) Cubic function $f(x) = x^3$

(b) Horizontal shift two units right, vertical shift five units upward

(c)

(d) $h(x) = f(x - 2) + 5$

101. (a) Square root function $f(x) = \sqrt{x}$

(b) Reflection in the x-axis, vertical shift six units upward

(c)

(d) $h(x) = -f(x) + 6$

103. (a) Absolute value function $f(x) = |x|$

(b) Vertical shift nine units upward

(c)

(d) $h(x) = f(x) + 9$

105. (a) Rational function $f(x) = \dfrac{1}{x}$

(b) Horizontal shift one unit left, reflection in the x-axis, vertical stretch, vertical shift three units downward

(c)

(d) $h(x) = -2f(x + 1) - 3$

107. -7 **109.** -42 **111.** 5 **113.** 17 **115.** -97

117. $f(x) = x^2$, $g(x) = x + 3$

119. $f(x) = \sqrt{x}$, $g(x) = 4x + 2$

121. $f(x) = \dfrac{4}{x}$, $g(x) = x + 2$

123.

125. $f^{-1}(x) = \dfrac{x}{6}$ **127.** $f^{-1}(x) = 2x - 6$

129. Answers will vary.

131.

One-to-one

133.

One-to-one

135. $f^{-1}(x) = 2x + 10$ **137.** $f^{-1}(x) = \sqrt[3]{\dfrac{x + 3}{4}}$

139. $f^{-1}(x) = x^2 - 10$, $x \geq 0$

141. $f^{-1}(x) = \sqrt{4x - 4}$ **143.** Negative correlation

CHAPTER 1

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145. (a)

(b) Yes. Answers will vary.

147. (a)

(b) Answers will vary.
Sample answer: $s = 10t - 0.4$

(c) $s = 9.7t + 0.4$; $r \approx 0.99933$

(d) 24.65 m/sec

149. False. The point $(-1, 28)$ does not lie on the graph of the function $g(x) = -(x - 6)^2 - 3$.

151. False. For example, $f(x) = 4 - x = f^{-1}(x)$.

Chapter Test (page 86)

1. (a) $5x + 2y - 8 = 0$ (b) $2x - 5y + 20 = 0$

2. $y = -x + 1$

3. No. To some x there correspond two values of y.

4. (a) -9 (b) 1 (c) $|t - 4| - 15$ **5.** $(-\infty, 3]$

6. $C = 25.60x + 24{,}000$ **7.** Odd **8.** Even
$P = 73.9x - 24{,}000$

9. Increasing: $(-2, 0), (2, \infty)$; Decreasing: $(-\infty, -2), (0, 2)$

10. Increasing: $(-2, 2)$; Constant: $(-\infty, -2), (2, \infty)$

11.

Relative minimum: $(-3.33, -6.52)$
Relative maximum: $(0, 12)$

12.

Relative minimum: $(0.77, 1.81)$
Relative maximum: $(-0.77, 2.19)$

13. (a) Cubic function $f(x) = x^3$

(b) Horizontal shift five units to the right, reflection in the x-axis, vertical stretch, and vertical shift three units upward

(c)

14. (a) Square root function $f(x) = \sqrt{x}$

(b) Reflection in the y-axis and horizontal shift seven units to the left

(c)

15. (a) Absolute value function $f(x) = |x|$

(b) Reflection in the y-axis (no effect), vertical stretch, and vertical shift seven units downward

(c)

16. (a) $x^2 - \sqrt{2 - x}$, $(-\infty, 2]$ (b) $\dfrac{x^2}{\sqrt{2 - x}}$, $(-\infty, 2)$

(c) $2 - x$, $(-\infty, 2]$ (d) $\sqrt{2 - x^2}$, $\left[-\sqrt{2}, \sqrt{2}\right]$

17. $f^{-1}(x) = \sqrt[3]{x - 8}$ **18.** No inverse

19. $f^{-1}(x) = \left(\frac{8}{3}x\right)^{2/3}$, $x \geq 0$ **20.** $C = 1.686t + 31.09$; 2012

Chapter 2

Section 2.1 (page 96)

1. nonnegative integer, real **3.** Yes; (2, 3)

5. c **6.** d **7.** b **8.** a

9.

Reflection in the x-axis

11.

Horizontal shift three units to the left

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13.

Horizontal shift one unit to the left

15.

Horizontal shift three units to the right

17. Parabola opening downward
Vertex: $(0, 25)$

19. Parabola opening upward
Vertex: $(0, -4)$

21. Parabola opening upward
Vertex: $(-4, -3)$

23. Parabola opening upward
Vertex: $(4, 0)$

25. Parabola opening upward
Vertex: $\left(\frac{1}{2}, 1\right)$

27. Parabola opening downward
Vertex: $(1, 6)$

29. Parabola opening upward
Vertex: $\left(\frac{1}{2}, 20\right)$

31. Parabola opening downward
Vertex: $(-1, 4)$
x-intercepts: $(1, 0), (-3, 0)$

33. Parabola opening upward
Vertex: $(-4, -5)$
x-intercepts: $\left(-4 \pm \sqrt{5}, 0\right)$

35. Parabola opening downward
Vertex: $(4, 1)$
x-intercepts: $\left(4 \pm \frac{1}{2}\sqrt{2}, 0\right)$

37. $y = -(x + 1)^2 + 4$ **39.** $f(x) = (x + 2)^2 + 5$

41. $y = 4(x - 1)^2 - 2$ **43.** $y = -\frac{104}{125}\left(x - \frac{1}{2}\right)^2 + 1$

45. $(5, 0), (-1, 0)$ **47.** $(-4, 0)$

49.

$(0, 0), (4, 0)$

51.

$\left(-\frac{5}{2}, 0\right), (6, 0)$

53.

$(7, 0), (-1, 0)$

55. $f(x) = x^2 - 2x - 3$
$g(x) = -x^2 + 2x + 3$

57. $f(x) = 2x^2 + 7x + 3$
$g(x) = -2x^2 - 7x - 3$

59. 55, 55 **61.** 12, 6

63. (a)

(b) $r = \frac{1}{2}y;\ d = y\pi$ (c) $y = \dfrac{200 - 2x}{\pi}$

(d) $A = x\left(\dfrac{200 - 2x}{\pi}\right)$

(e)

$x = 50$ m, $y = \dfrac{100}{\pi}$ m

65. (a)

(b) $\frac{3}{2}$ ft

(c) About 104 ft (d) About 228.6 ft

67. (a) $A = -2x^2 + 112x - 600$ (b) $x = 28$ in.

69. (a) \$54,000; \$61,600; \$61,200 (b) \$79
(c) \$62,410 (d) Answers will vary.

71. (a)

(b) 1966; 4155 cigarettes; Yes, the warning had an effect because the maximum consumption occurred in 1966.
(c) 1852 cigarettes per year; 5 cigarettes per day

73. True. The vertex is $(0, -1)$ and the parabola opens down.

75. c, d **77.** Horizontal shift z units to the right

79. Vertical stretch $(z > 1)$ or shrink $(0 < z < 1)$ and horizontal shift three units to the right

81. $b = \pm 20$ **83.** $b = \pm 8$ **85.** Proof

87. $y = -x^2 + 5x - 4$; Answers will vary.

89. $(1.2, 6.8)$ **91.** $(2, 5), (-3, 0)$

93. Answers will vary.

Section 2.2 (page 109)

1. continuous

3. (a) solution (b) $(x - a)$ (c) $(a, 0)$

5. No **7.** $f(x_2) > 0$

9. f **10.** h **11.** c **12.** a **13.** e

14. d **15.** g **16.** b

17.

Horizontal shift two units to the right

CHAPTER 2

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19.

Reflection in the x-axis and vertical shift one unit upward

21.

Reflection in the x-axis and horizontal shift two units to the right

23.

Yes, because both graphs have the same leading coefficient.

25.

Yes, because both graphs have the same leading coefficient.

27.

No, because the graphs have different leading coefficients.

29. Rises to the left, rises to the right

31. Falls to the left, falls to the right

33. Falls to the left, rises to the right

35. Falls to the left, falls to the right

37. (a) $\left(2 \pm \sqrt{3}, 0\right)$

(b)

(c) $(0.27, 0), (3.73, 0)$; Answers are approximately the same.

39. (a) $(-1, 0), (1, 0)$

(b)

(c) $(-1, 0), (1, 0)$; Answers are the same.

41. (a) $(0, 0), \left(\pm\sqrt{2}, 0\right)$

(b)

(c) $(-1.41, 0), (0, 0), (1.41, 0)$; Answers are approximately the same.

43. (a) $\left(\pm\sqrt{5}, 0\right)$

(b)

(c) $(-2.236, 0), (2.236, 0)$; Answers are approximately the same.

45. (a) $(4, 0), (\pm 5, 0)$

(b)

(c) $(-5, 0), (4, 0), (5, 0)$; Answers are the same.

47. (a) $(0, 0), \left(\frac{5}{2}, 0\right)$

(b)

(c) $(0, 0), (2.5, 0)$ Answers are the same.

49. ± 5 (multiplicity 1)

51. 3 (multiplicity 2)

53. $1, -2$ (multiplicity 1)

55. 2 (multiplicity 2), 0 (multiplicity 1)

57. $\dfrac{-5 \pm \sqrt{37}}{2}$ (multiplicity 1)

59.

Zeros: $\pm 1.680, \pm 0.421$

Relative minima: $(\pm 1.225, -3.500)$

Relative maximum: $(0, 1)$

61.

Zero: -1.178

Relative minimum: $(0.324, 5.782)$

Relative maximum: $(-0.324, 6.218)$

63.

Zeros: $-1.618, -0.366, 0.618, 1.366$

Relative minimum: $(0.101, -1.050)$

Relative maxima: $(-1.165, 3.267), (1.064, 1.033)$

65. $f(x) = x^2 - 4x$

67. $f(x) = x^3 + 5x^2 + 6x$

69. $f(x) = x^4 - 4x^3 - 9x^2 + 36x$

71. $f(x) = x^2 - 2x - 2$

73. $f(x) = x^3 - 10x^2 + 27x - 22$

75. $f(x) = x^3 + 5x^2 + 8x + 4$

77. $f(x) = x^4 + 2x^3 - 23x^2 - 24x + 144$

79. $f(x) = -x^3 - 4x^2 - 5x - 2$

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81.

83.

85. (a) Falls to the left, rises to the right
(b) $(0, 0), (3, 0), (-3, 0)$
(c) and (d)

87. (a) Falls to the left, rises to the right
(b) $(0, 0), (3, 0)$
(c) and (d)

89. (a) Falls to the left, falls to the right
(b) $(\pm 2, 0), (\pm\sqrt{5}, 0)$
(c) and (d)

91. (a) Falls to the left, rises to the right
(b) $(\pm 3, 0)$
(c) and (d)

93. (a) Falls to the left, falls to the right
(b) $(-2, 0), (2, 0)$
(c) and (d)

95. (a)

$(-1, 0), (1, 2), (2, 3)$
(b) $-0.879, 1.347, 2.532$

97. (a)

$(-2, -1), (0, 1)$
(b) $-1.585, 0.779$

99. (a)

$(-1, 0), (3, 4)$
(b) $-0.578, 3.418$

101.

Two x-intercepts

103.

y-axis symmetry
Two x-intercepts

105.

Origin symmetry
Three x-intercepts

107.

Three x-intercepts

109. (a) Answers will vary.
(b) Domain: $0 < x < 18$
(c)

Height, x	Volume, V
1	1156
2	2048
3	2700
4	3136
5	3380
6	3456
7	3388

(d)

$x = 6$

$5 < x < 7$

111. $(200, 160)$

CHAPTER 2

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113. (a)

The model fits the data well.

(b)

Answers will vary. Sample answer: You could use the model to estimate production in 2010 because the result is somewhat reasonable, but you would not use the model to estimate the 2020 production because the result is unreasonably high.

(c)

The model fits the data well.

Answers will vary. Sample answer: You could use the model to estimate production in 2010 because the result is somewhat reasonable, but you would not use the model to estimate the 2020 production because the result is unreasonably high.

115. True. The degree is odd and the leading coefficient is -1.

117. False. The graph crosses the x-axis at $x = -3$ and $x = 0$.

119.

121. 69 **123.** $-\frac{1408}{49} \approx -28.73$ **125.** 109

127. $x > -8$

129. $-26 \le x < 7$

Section 2.3 (page 124)

1. $f(x)$ is the dividend, $d(x)$ is the divisor, $q(x)$ is the quotient, and $r(x)$ is the remainder.

3. constant term, leading coefficient

5. upper, lower **7.** 7 **9.** $2x + 4,\ x \neq -3$

11. $x^3 + 3x^2 - 1,\ x \neq -2$ **13.** $x^2 - 3x + 1,\ x \neq -\frac{5}{4}$

15. $7x^2 - 14x + 28 - \dfrac{53}{x+2}$ **17.** $3x + 5 - \dfrac{2x-3}{2x^2+1}$

19. $x - \dfrac{x+9}{x^2+1}$ **21.** $2x - \dfrac{17x-5}{x^2-2x+1}$

23. $3x^2 - 2x + 5,\ x \neq 5$ **25.** $6x^2 + 25x + 74 + \dfrac{248}{x-3}$

27. $9x^2 - 16,\ x \neq 2$ **29.** $x^2 - 8x + 64,\ x \neq -8$

31. $4x^2 + 14x - 30,\ x \neq -\frac{1}{2}$

33.

35.

37. $f(x) = (x-4)(x^2+3x-2) + 3,\ f(4) = 3$

39. $f(x) = \left(x - \sqrt{2}\right)\left[x^2 + \left(3 + \sqrt{2}\right)x + 3\sqrt{2}\right] - 8,$
$f\left(\sqrt{2}\right) = -8$

41. $f(x) = \left(x - 1 + \sqrt{3}\right)\left[4x^2 - \left(2 + 4\sqrt{3}\right)x - \left(2 + 2\sqrt{3}\right)\right],$
$f\left(1 - \sqrt{3}\right) = 0$

43. (a) -2 (b) 1 (c) $-\frac{1}{4}$ (d) 5

45. (a) -35 (b) -22 (c) -10 (d) -211

47. $(x-2)(x+3)(x-1)$
Zeros: $2, -3, 1$

49. $(2x-1)(x-5)(x-2)$
Zeros: $\frac{1}{2}, 5, 2$

51. (a) Answers will vary. (b) $(2x-1), (x-1)$
(c) $(x+2)(x-1)(2x-1)$ (d) $-2, 1, \frac{1}{2}$

53. (a) Answers will vary. (b) $(x-1), (x-2)$
(c) $(x-5)(x+4)(x-1)(x-2)$ (d) $-4, 1, 2, 5$

55. (a) Answers will vary. (b) $(x+7), (3x-2)$
(c) $(2x+1)(3x-2)(x+7)$ (d) $-7, -\frac{1}{2}, \frac{2}{3}$

57. $\pm 1, \pm 3; \pm 1, -3$

59. $\pm 1, \pm 3, \pm 5, \pm 9, \pm 15, \pm 45, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \pm\frac{9}{2}, \pm\frac{15}{2}, \pm\frac{45}{2};$
$-1, \frac{3}{2}, 3, 5$

61. 4, 2, or 0 positive real zeros, no negative real zeros

63. 2 or 0 positive real zeros, 1 negative real zero

65. (a) 1 positive real zero, 2 or 0 negative real zeros
(b) $\pm 1, \pm 2, \pm 4$
(c)

(d) $-2, -1, 2$

67. (a) 3 or 1 positive real zeros, 1 negative real zero
(b) $\pm 1, \pm 2, \pm 4, \pm 8, \pm\frac{1}{2}$
(c)

(d) $-\frac{1}{2}, 1, 2, 4$

69. (a) 2 or 0 positive real zeros, 1 negative real zero
(b) $\pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{1}{4}, \pm\frac{3}{4}, \pm\frac{1}{8}, \pm\frac{3}{8}, \pm\frac{1}{16}, \pm\frac{3}{16}, \pm\frac{1}{32}, \pm\frac{3}{32}$
(c)

(d) $-\frac{1}{8}, \frac{3}{4}, 1$

71. Answers will vary; 1.937, 3.705

73. Answers will vary; ± 2 **75.** $\pm 2, \pm\frac{3}{2}$ **77.** $\pm 1, \frac{1}{4}$

79. d **80.** a **81.** b **82.** c **83.** $-\frac{1}{2}, 2 \pm \sqrt{3}, 1$

85. $-1, \dfrac{3}{2}, 4 \pm \sqrt{17}$ **87.** $-2, 0, 1 \pm \dfrac{\sqrt{15}}{3}$ **89.** $-1, 2$

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91. $-6, \frac{1}{2}, 1$ **93.** $-3, -\frac{3}{2}, \frac{1}{2}, 4$

95. $-3, 0, \dfrac{-1 \pm \sqrt{7}}{4}$ **97.** $-\dfrac{5}{2}, -2, \pm 1, \dfrac{3}{2}$

99. (a) $-2, 0.268, 3.732$ (b) -2
(c) $h(t) = (t + 2)\left(t - 2 + \sqrt{3}\right)\left(t - 2 - \sqrt{3}\right)$

101. (a) $0, 3, 4, -1.414, 1.414$ (b) $0, 3, 4$
(c) $h(x) = x(x - 3)(x - 4)\left(x + \sqrt{2}\right)\left(x - \sqrt{2}\right)$

103. (a)

(b) The model fits the data well.
(c) About 116 subscriptions; No, because you cannot have more subscriptions than people.

105. (a) Answers will vary.
(b)

$20 \times 20 \times 40$
(c) $15, \dfrac{15 \pm 15\sqrt{5}}{2}$;
$\dfrac{15 - 15\sqrt{5}}{2}$ represents a negative volume.

107. False. If $(7x + 4)$ is a factor of f, then $-\frac{4}{7}$ is a zero of f.

109. $-2(x - 1)^2(x + 2)$

111. $-(x - 2)(x + 2)(x + 1)(x - 1)$

113. (a) $x + 1, \quad x \neq 1$ (b) $x^2 + x + 1, \quad x \neq 1$
(c) $x^3 + x^2 + x + 1, \quad x \neq 1$
$\dfrac{x^n - 1}{x - 1} = x^{n-1} + x^{n-2} + \cdot \cdot \cdot + x^2 + x + 1, \; x \neq 1$

115. $\pm\dfrac{5}{3}$ **117.** $\dfrac{-3 \pm \sqrt{3}}{2}$

Section 2.4 (page 133)

1. (a) ii (b) iii (c) i **3.** $15; 11i$ **5.** $-2 + 4i$
7. $a = -9, b = 4$ **9.** $a = 3, b = 5$ **11.** $5 + 4i$
13. -6 **15.** $-1 - 5i$ **17.** -75 **19.** $0.3i$
21. $-3 + 3i$ **23.** $7 + 3i$ **25.** $-14 + 20i$
27. $\frac{19}{6} + \frac{37}{6}i$ **29.** $-4.2 + 7.5i$ **31.** $12 + 20i$
33. $5 + i$ **35.** $-20 + 32i$ **37.** 24 **39.** $-13 + 84i$
41. $80i$ **43.** $4 - 3i; 25$ **45.** $-6 + \sqrt{5}i; 41$
47. $-\sqrt{20}i; 20$ **49.** $3 + \sqrt{-2}; 11$ **51.** $-6i$
53. $\frac{8}{41} + \frac{10}{41}i$ **55.** $\frac{3}{5} + \frac{4}{5}i$ **57.** $-\frac{40}{1681} - \frac{9}{1681}i$
59. $-\frac{1}{2} - \frac{5}{2}i$ **61.** $\frac{62}{949} + \frac{297}{949}i$ **63.** $3\left(\sqrt{2} - \sqrt{6}\right)i$
65. $4 + 2\left(\sqrt{6} - \sqrt{11}\right)i$ **67.** $-2\sqrt{3}$ **69.** -10
71. $-2 - 4\sqrt{6}i$ **73.** $\pm 5i$ **75.** $1 \pm i$
77. $-2 \pm \dfrac{i}{2}$ **79.** $\dfrac{1}{8} \pm \dfrac{\sqrt{11}}{8}i$ **81.** $2 \pm \sqrt{2}i$
83. $\dfrac{5}{7} \pm \dfrac{5\sqrt{13}}{7}i$ **85.** $-1 + 6i$ **87.** $-375\sqrt{3}i$
89. i **91.** (a) 8 (b) 8 (c) 8; Answers will vary.
93. (a) 1 (b) i (c) $-i$ (d) -1
95. False. Any real number is equal to its conjugate.
97. False. Example: $(1 + i) + (1 - i) = 2$, which is not an imaginary number.
99. True. Answers will vary.
101. $\sqrt{-6}\sqrt{-6} = \left(\sqrt{6}i\right)\left(\sqrt{6}i\right) = 6i^2 = -6$
103. $16x^2 - 25$ **105.** $3x^2 + \frac{23}{2}x - 2$

Section 2.5 (page 140)

1. Fundamental Theorem, Algebra **3.** n linear factors
5. c **6.** a **7.** d **8.** b
9 and 11. Answers will vary.
13. Zeros: $4, -i, i$. One real zero; they are the same.
15. Zeros: $\sqrt{2}i, \sqrt{2}i, -\sqrt{2}i, -\sqrt{2}i$. No real zeros; they are the same.
17. $2 \pm \sqrt{3}$
$\left(x - 2 - \sqrt{3}\right)\left(x - 2 + \sqrt{3}\right)$
19. $6 \pm \sqrt{10}$
$\left(x - 6 - \sqrt{10}\right)\left(x - 6 + \sqrt{10}\right)$
21. $\pm 5i$
$(x + 5i)(x - 5i)$
23. $\pm\frac{3}{2}, \pm\frac{3}{2}i$
$(2x - 3)(2x + 3)(2x - 3i)(2x + 3i)$
25. $\dfrac{1 \pm \sqrt{223}i}{2}$
$\left(z - \dfrac{1 - \sqrt{223}i}{2}\right)\left(z - \dfrac{1 + \sqrt{223}i}{2}\right)$
27. $\pm i, \pm 3i$
$(x + i)(x - i)(x + 3i)(x - 3i)$
29. $\frac{5}{3}, \pm 4i$
$(3x - 5)(x - 4i)(x + 4i)$
31. $-5, 4 \pm 3i$
$(t + 5)(t - 4 + 3i)(t - 4 - 3i)$
33. $1 \pm \sqrt{5}i, -\frac{1}{5}$
$(5x + 1)\left(x - 1 + \sqrt{5}i\right)\left(x - 1 - \sqrt{5}i\right)$
35. $2, 2, \pm 2i$
$(x - 2)^2(x + 2i)(x - 2i)$
37. (a) $7 \pm \sqrt{3}$ (b) $\left(x - 7 - \sqrt{3}\right)\left(x - 7 + \sqrt{3}\right)$
(c) $\left(7 \pm \sqrt{3}, 0\right)$
39. (a) $\frac{3}{2}, \pm 2i$ (b) $(2x - 3)(x - 2i)(x + 2i)$ (c) $\left(\frac{3}{2}, 0\right)$
41. (a) $-6, 3 \pm 4i$ (b) $(x + 6)(x - 3 - 4i)(x - 3 + 4i)$
(c) $(-6, 0)$
43. (a) $\pm 4i, \pm 3i$ (b) $(x + 4i)(x - 4i)(x + 3i)(x - 3i)$
(c) None
45. $f(x) = x^3 - 2x^2 + x - 2$
47. $f(x) = x^4 - 12x^3 + 53x^2 - 100x + 68$
49. $f(x) = x^4 + 3x^3 - 7x^2 + 15x$
51. (a) $-(x - 1)(x + 2)(x - 2i)(x + 2i)$
(b) $f(x) = -(x^4 + x^3 + 2x^2 + 4x - 8)$
53. (a) $-2(x + 1)\left(x - 2 + \sqrt{5}i\right)\left(x - 2 - \sqrt{5}i\right)$
(b) $f(x) = -2x^3 + 6x^2 - 10x - 18$
55. (a) $(x^2 + 1)(x^2 - 7)$ (b) $(x^2 + 1)\left(x + \sqrt{7}\right)\left(x - \sqrt{7}\right)$
(c) $(x + i)(x - i)\left(x + \sqrt{7}\right)\left(x - \sqrt{7}\right)$
57. (a) $(x^2 - 6)(x^2 - 2x + 3)$
(b) $\left(x + \sqrt{6}\right)\left(x - \sqrt{6}\right)(x^2 - 2x + 3)$
(c) $\left(x + \sqrt{6}\right)\left(x - \sqrt{6}\right)\left(x - 1 - \sqrt{2}i\right)\left(x - 1 + \sqrt{2}i\right)$
59. $-\frac{3}{2}, \pm 5i$ **61.** $-3, 5 \pm 2i$ **63.** $-\frac{2}{3}, 1 \pm \sqrt{3}i$
65. $\frac{3}{4}, \frac{1}{2}\left(1 \pm \sqrt{5}i\right)$ **67.** (a) 1.000, 2.000 (b) $-3 \pm \sqrt{2}i$

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69. (a) 0.750 (b) $\frac{1}{2} \pm \frac{\sqrt{5}}{2}i$

71. No. Setting $h = 50$ and solving the resulting equation yields imaginary roots.

73. False. A third-degree polynomial must have at least one real zero.

75. Answers will vary.

77. Parabola opening upward
Vertex: $\left(\frac{7}{2}, -\frac{81}{4}\right)$

79. Parabola opening upward
Vertex: $\left(-\frac{5}{12}, -\frac{169}{24}\right)$

Section 2.6 (page 147)

1. rational functions **3.** vertical asymptote

5. (a) Domain: all real numbers x except $x = 1$

(b)

x	$f(x)$	x	$f(x)$
0.5	-2	1.5	2
0.9	-10	1.1	10
0.99	-100	1.01	100
0.999	-1000	1.001	1000

x	$f(x)$	x	$f(x)$
5	0.25	-5	$-0.\overline{16}$
10	$0.\overline{1}$	-10	$-0.\overline{09}$
100	$0.\overline{01}$	-100	$-0.\overline{0099}$
1000	$0.\overline{001}$	-1000	$-0.\overline{000999}$

(c) f approaches $-\infty$ from the left and ∞ from the right of $x = 1$.

7. (a) Domain: all real numbers x except $x = 1$

(b)

x	$f(x)$	x	$f(x)$
0.5	3	1.5	9
0.9	27	1.1	33
0.99	297	1.01	303
0.999	2997	1.001	3003

x	$f(x)$	x	$f(x)$
5	3.75	-5	-2.5
10	$3.\overline{33}$	-10	-2.727
100	$3.\overline{03}$	-100	-2.97
1000	$3.\overline{003}$	-1000	-2.997

(c) f approaches ∞ from both the left and the right of $x = 1$.

9. (a) Domain: all real numbers x except $x = \pm 1$

(b)

x	$f(x)$	x	$f(x)$
0.5	-1	1.5	5.4
0.9	-12.79	1.1	17.29
0.99	-147.8	1.01	152.3
0.999	-1498	1.001	1502.3

x	$f(x)$	x	$f(x)$
5	3.125	-5	3.125
10	$3.\overline{03}$	-10	$3.\overline{03}$
100	$3.\overline{0003}$	-100	$3.\overline{0003}$
1000	3	-1000	3.000003

(c) f approaches ∞ from the left and $-\infty$ from the right of $x = -1$. f approaches $-\infty$ from the left and ∞ from the right of $x = 1$.

11. a **12.** d **13.** c **14.** e **15.** b **16.** f

17. Vertical asymptote: $x = 0$
Horizontal asymptote: $y = 0$

19. Vertical asymptote: $x = -3, 2$
Horizontal asymptote: $y = 2$

21. Vertical asymptote: $x = 2$
Horizontal asymptote: $y = -1$
Hole at $x = 0$

23. Vertical asymptote: $x = 0$
Horizontal asymptote: $y = 1$
Hole at $x = -5$

25. (a) Domain: all real numbers x
(b) Continuous
(c) Horizontal asymptote: $y = 3$

27. (a) Domain: all real numbers x except $x = 3$
(b) Not continuous
(c) Vertical asymptote: $x = 3$
Horizontal asymptote: $y = 0$

29. (a) Domain of f: all real numbers x except $x = 4$
Domain of g: all real numbers x
(b) Vertical asymptote: none; Hole in f at $x = 4$
(c)

x	1	2	3	4	5	6	7
$f(x)$	5	6	7	Undef.	9	10	11
$g(x)$	5	6	7	8	9	10	11

(d)

(e) Graphing utilities are limited in their resolution and therefore may not show a hole in a graph.

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31. (a) Domain of f: all real numbers x except $x = -1, 3$
Domain of g: all real numbers x except $x = 3$

(b) Vertical asymptote: $x = 3$; Hole in f at $x = -1$

(c)

x	-2	-1	0	1	2	3	4
$f(x)$	$\frac{3}{5}$	Undef.	$\frac{1}{3}$	0	-1	Undef.	3
$g(x)$	$\frac{3}{5}$	$\frac{1}{2}$	$\frac{1}{3}$	0	-1	Undef.	3

(d)

33. 4; less than; greater than **35.** 2; greater than; less than

37. ± 2 **39.** 7 **41.** $-1, 3$ **43.** 2

45. (a) \$28.33 million; \$170 million; \$765 million

(b)

Answers will vary.

(c) No. The function is undefined at the 100% level.

47. (a)

M	200	400	600	800	1000
t	0.472	0.596	0.710	0.817	0.916

M	1200	1400	1600	1800	2000
t	1.009	1.096	1.178	1.255	1.328

The greater the mass, the more time required per oscillation.

(b) $M \approx 1306$ g

49. (a)

The model fits the data well.

(b) 1412 thousand; 1414 thousand; 1416 thousand; Answers will vary.

(c) $y = 1482.6$; Answers will vary.

51. False. $\dfrac{1}{x^2 + 1}$ has no vertical asymptote.

53. No. If $x = c$ is also a zero in the denominator, then f is undefined at $x = c$.

55.

Both graphs have the same slope.

57. $x - y - 1 = 0$ **59.** $3x - y + 1 = 0$

61. $x + 9 + \dfrac{42}{x - 4}$ **63.** $2x^2 - 9 + \dfrac{34}{x^2 + 5}$

Section 2.7 (page 157)

1. slant, asymptote **3.** Yes

5.

Horizontal shift four units to the right

7.

Reflection in the x-axis, horizontal shift three units to the left

9.

Vertical shift

11.

Reflection in the x-axis

13.

Vertical shift

15.

Horizontal shift

17.

19.

21.

23.

25.

27.

CHAPTER 2

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29.

There is a hole at $x = -3$.

31.

There is a hole at $x = -1$.

33.

Domain: $(-\infty, 1), (1, \infty)$
Vertical asymptote: $x = 1$
Horizontal asymptote: $y = -1$

35.

Domain: $(-\infty, 0), (0, \infty)$
Vertical asymptote: $t = 0$
Horizontal asymptote: $y = 3$

37.

Domain: $(-\infty, \infty)$
Horizontal asymptote: $y = 0$

39.

Domain:
$(-\infty, -2), (-2, 3), (3, \infty)$
Vertical asymptotes:
$x = -2, x = 3$
Horizontal asymptote: $y = 0$

41.

Domain: $(-\infty, 0), (0, \infty)$
Vertical asymptote: $x = 0$
Horizontal asymptote: $y = 0$

43.

There are two horizontal asymptotes, $y = \pm 6$.

45.

There are two horizontal asymptotes, $y = \pm 4$, and one vertical asymptote, $x = -1$.

47.

The graph crosses the horizontal asymptote, $y = 4$.

49.

51.

53.

55.

57. $(-1, 0)$ **59.** $(1, 0), (-1, 0)$

61.

Domain: $(-\infty, -1), (-1, \infty)$
Vertical asymptote: $x = -1$
Slant asymptote: $y = 2x - 1$

63.

Domain: $(-\infty, 0), (0, \infty)$
Vertical asymptote: $x = 0$
Slant asymptote: $y = -x + 3$

65. Vertical asymptotes: $x = \pm 2$; horizontal asymptote: $y = 1$; slant asymptote: none; holes: none

67. Vertical asymptote: $x = -\frac{3}{2}$; horizontal asymptote: $y = 1$; slant asymptote: none; hole at $x = 2$

69. Vertical asymptote: $x = -2$; horizontal asymptote: none; slant asymptote: $y = 2x - 7$; hole at $x = -1$

71.

$(-4, 0)$

73.

$\left(-\frac{8}{3}, 0\right)$

75.

$(3, 0), (-2, 0)$

77.

$\left(\frac{-3 \pm \sqrt{5}}{2}, 0\right)$

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79.

None

81.

$\left(\dfrac{-5 \pm \sqrt{65}}{4}, 0\right)$

83. (a) Answers will vary.

(b) $[0, 950]$

(c)

The concentration increases more slowly; the concentration approaches 75%.

85. (a) Answers will vary.

(b) $(2, \infty)$

(c)

5.9 in. $\times$ 11.8 in.

87.

$x \approx 40$

89. (a) $C = 0$. The chemical will eventually dissipate.

(b)

$t \approx 4.5$ h

(c) Before about 2.6 hours and after about 8.3 hours

91. (a) $A = -0.2182t + 5.665$

(b) $A = \dfrac{1}{0.0302t - 0.020}$

(c)

Year	1999	2000	2001	2002
Original data, A	3.9	3.5	3.3	2.9
Model from (a), A	3.7	3.5	3.3	3.0
Model from (b), A	4.0	3.5	3.2	2.9

Year	2003	2004	2005	2006
Original data, A	2.7	2.5	2.3	2.2
Model from (a), A	2.8	2.6	2.4	2.2
Model from (b), A	2.7	2.5	2.3	2.2

Year	2007	2008
Original data, A	2.0	1.9
Model from (a), A	2.0	1.7
Model from (b), A	2.0	1.9

Answers will vary.

93. False. The graph of a rational function is continuous when the polynomial in the denominator has no real zeros.

95.

The denominator is a factor of the numerator.

97. *Horizontal asymptotes:*

If the degree of the numerator is greater than the degree of the denominator, then there is no horizontal asymptote.

If the degree of the numerator is less than the degree of the denominator, then there is a horizontal asymptote at $y = 0$.

If the degree of the numerator is equal to the degree of the denominator, then there is a horizontal asymptote at the line given by the ratio of the leading coefficients.

Vertical asymptotes:

Set the denominator equal to zero and solve.

Slant asymptotes:

If there is no horizontal asymptote and the degree of the numerator is exactly one greater than the degree of the denominator, then divide the numerator by the denominator. The slant asymptote is the result, not including the remainder.

99. $\dfrac{512}{x^3}$ **101.** 3

103.

Domain: $(-\infty, \infty)$

Range: $[\sqrt{6}, \infty)$

105.

Domain: $(-\infty, \infty)$

Range: $(-\infty, 0]$

107. Answers will vary.

CHAPTER 2

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Section 2.8 (page 165)

1. Quadratic **3.** Quadratic **5.** Linear **7.** Neither

9. (a)

(b) Linear

(c) $y = 0.14x + 2.2$

(d)

(e)

x	0	1	2	3	4	5
Actual, y	2.1	2.4	2.5	2.8	2.9	3.0
Model, y	2.2	2.3	2.5	2.6	2.8	2.9

x	6	7	8	9	10
Actual, y	3.0	3.2	3.4	3.5	3.6
Model, y	3.0	3.2	3.3	3.5	3.6

11. (a)

(b) Quadratic

(c) $y = 5.55x^2 - 277.5x + 3478$

(d)

(e)

x	0	5	10	15	20	25
Actual, y	3480	2235	1250	565	150	12
Model, y	3478	2229	1258	564	148	9

x	30	35	40	45	50	55
Actual, y	145	575	1275	2225	3500	5010
Model, y	148	564	1258	2229	3478	5004

13. (a)

(b) Linear

(c) $y = 2.48x + 1.1$

(d)

(e)

x	1	2	3	4	5
Actual, y	4.0	6.5	8.8	10.6	13.9
Model, y	3.6	6.1	8.5	11.0	13.5

x	6	7	8	9	10
Actual, y	15.0	17.5	20.1	24.0	27.1
Model, y	16.0	18.5	20.9	23.4	25.9

15. (a)

(b) Quadratic

(c) $y = 0.14x^2 - 9.9x + 591$

(d)

(e)

x	0	5	10	15	20	25
Actual, y	587	551	512	478	436	430
Model, y	591	545	506	474	449	431

x	30	35	40	45	50
Actual, y	424	420	423	429	444
Model, y	420	416	419	429	446

17. (a)

(b) $P = 0.1323t^2 - 1.893t + 6.85$

(c)

(d) July

19. (a)

(b) $P = -0.5638t^2 + 9.690t + 32.17$

(c)

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(d) 2014; No. By 2021, the model gives negative values for the number of Internet users.

21. (a)

(b) $T = 7.97t + 166.1$
$r^2 \approx 0.9469$

(c)

(d) $T = 0.459t^2 - 3.51t + 232.4$
$r^2 \approx 0.9763$

(e)

(f) Quadratic

(g) Linear: 2014
Quadratic: 2011

23. True. See "Basic Characteristics of Quadratic Functions" on page 91.

25. The model is consistently above the data points.

27. (a) $(f \circ g)(x) = 2x^2 + 5$ (b) $(g \circ f)(x) = 4(x^2 - x + 1)$

29. (a) $(f \circ g)(x) = x$ (b) $(g \circ f)(x) = x$

31. $f^{-1}(x) = \dfrac{x - 5}{2}$ **33.** $f^{-1}(x) = \sqrt{x - 5}$

35. $1 + 3i$; 10 **37.** $5i$; 25

Review Exercises (page 170)

1.

Vertical shift two units downward

3.

Horizontal shift two units to the right

5.

Horizontal shift five units to the left, vertical shift two units downward

7. Parabola opening upward
Vertex: $\left(-\frac{3}{2}, 1\right)$

x-intercept: none

9. Parabola opening upward
Vertex: $\left(-\dfrac{5}{2}, -\dfrac{41}{12}\right)$

x-intercepts: $\left(\dfrac{-5 \pm \sqrt{41}}{2}, 0\right)$

11. $f(x) = (x - 1)^2 - 4$

13. (a) $A = x\left(\dfrac{8 - x}{2}\right), 0 < x < 8$

(b)

$x = 4, y = 2$

(c) $A = -\frac{1}{2}(x - 4)^2 + 8$; $x = 4, y = 2$;
They are the same.

15.

Horizontal shift four units to the left

17.

Reflection in the x-axis, vertical shift two units upward

19.

Reflection in the x-axis, horizontal shift seven units to the left, vertical shift two units downward

CHAPTER 2

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21.

Yes. Both functions are of the same degree and have positive leading coefficients.

23. Falls to the left, falls to the right

25. Rises to the left, rises to the right

27. (a) $x = -1, 0, 0, 2$
(b)

(c) $x = -1, 0, 0, 2$; They are the same.

29. (a) $t = 0, \pm\sqrt{3}$
(b)

(c) $t = 0, \pm 1.73$; They are the same.

31. (a) $x = -3, -3, 0$
(b)

(c) $x = -3, -3, 0$; They are the same.

33. $f(x) = x^4 - 5x^3 - 3x^2 + 17x - 10$

35. $f(x) = x^3 - 7x^2 + 13x - 3$

37. (a) Rises to the left, rises to the right
(b) $x = -3, -1, 3, 3$
(c) and (d)

39. (a) $(-3, -2), (-1, 0), (0, 1)$
(b) $x = -2.25, -0.56, 0.80$

41. (a) $(-3, -2), (2, 3)$ (b) $x = -2.57, 2.57$

43. $8x + 5 + \dfrac{2}{3x - 2}$

45. $x^2 - 2,\ x \neq \pm 1$

47. $5x + 2,\ x \neq \dfrac{3 \pm \sqrt{5}}{2}$

49. $3x^2 + 5x + 8 + \dfrac{10}{2x^2 - 1}$

51. $0.25x^3 - 4.5x^2 + 9x - 18 + \dfrac{36}{x + 2}$

53. $6x^3 - 27x,\ x \neq \dfrac{2}{3}$

55. $3x^2 + 2x + 20 + \dfrac{58}{x - 4}$

57. (a) -421 (b) -156

59. (a) Answers will vary. (b) $(x + 1)(x + 7)$
(c) $f(x) = (x - 4)(x + 1)(x + 7)$ (d) $x = 4, -1, -7$

61. (a) Answers will vary. (b) $(x + 1)(x - 4)$
(c) $f(x) = (x + 2)(x - 3)(x + 1)(x - 4)$
(d) $x = -2, 3, -1, 4$

63. $\pm 1, \pm 3, \pm\frac{3}{2}, \pm\frac{3}{4}, \pm\frac{1}{2}, \pm\frac{1}{4}$

65. 2 or 0 positive real zeros
1 negative real zero

67. Answers will vary.
$x = \frac{3}{4}$

69. $x = \frac{5}{6}, -3 \pm \sqrt{7}$

71. $x = -1, \frac{3}{2}, 3, \frac{2}{3}$

73. $6 + 5i$

75. $2 + 7i$

77. $3 + 7i$

79. $40 + 65i$

81. $-4 - 46i$

83. -80

85. $-26 + 7i$

87. $3 + 9i$

89. $1 - 6i$

91. $\frac{17}{26} + \frac{7}{26}i$

93. $x = \pm 4i$

95. $x = -\dfrac{3}{2} \pm \dfrac{\sqrt{15}}{2}i$

97. $x = \dfrac{5}{6} \pm \dfrac{\sqrt{47}}{6}i$

99 and 101. Answers will vary.

103. $x = 0, 2, 2$

105. $x = 4, \dfrac{3 \pm \sqrt{15}i}{2}$;
$(x - 4)\left(x - \dfrac{3 + \sqrt{15}i}{2}\right)\left(x - \dfrac{3 - \sqrt{15}i}{2}\right)$

107. $x = 2, -\frac{3}{2}, 1 \pm i$;
$(x - 2)(2x + 3)(x - 1 + i)(x - 1 - i)$

109. $x = 0, -1, \pm\sqrt{5}i$; $x^2(x + 1)(x + \sqrt{5}i)(x - \sqrt{5}i)$

111. (a) $x = 2, 1 \pm i$
(b) $(x - 2)(x - 1 - i)(x - 1 + i)$ (c) $(2, 0)$

113. (a) $x = -6, -1, \frac{2}{3}$ (b) $-(x + 1)(x + 6)(3x - 2)$
(c) $(-6, 0), (-1, 0), \left(\frac{2}{3}, 0\right)$

115. (a) $\pm 3i, \pm 5i$
(b) $(x - 3i)(x + 3i)(x - 5i)(x + 5i)$ (c) None

117. $f(x) = x^4 - 2x^3 + 17x^2 - 50x - 200$

119. $f(x) = x^4 + 9x^3 + 48x^2 + 78x - 136$

121. (a) $(x^2 + 9)(x^2 - 2x - 1)$
(b) $(x^2 + 9)(x - 1 + \sqrt{2})(x - 1 - \sqrt{2})$
(c) $(x + 3i)(x - 3i)(x - 1 + \sqrt{2})(x - 1 - \sqrt{2})$

123. $x = -3, \pm 2i$

125. (a) Domain: all real numbers x except $x = -3$
(b) Not continuous
(c) Vertical asymptote: $x = -3$
Horizontal asymptote: $y = -1$

127. (a) Domain: all real numbers x except $x = 6, -3$
(b) Not continuous
(c) Vertical asymptotes: $x = 6, x = -3$
Horizontal asymptote: $y = 0$

129. (a) Domain: all real numbers x except $x = 7$
(b) Not continuous
(c) Vertical asymptote: $x = 7$
Horizontal asymptote: $y = -1$

131. (a) Domain: all real numbers x except $x = \pm\dfrac{\sqrt{6}}{2}$
(b) Not continuous
(c) Vertical asymptotes: $x = \pm\dfrac{\sqrt{6}}{2}$
Horizontal asymptote: $y = 2$

133. (a) Domain: all real numbers x except $x = 5, -3$
(b) Not continuous
(c) Vertical asymptote: $x = -3$
Horizontal asymptote: $y = 0$

135. (a) Domain: all real numbers x
(b) Continuous
(c) Horizontal asymptotes: $y = \pm 1$

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137. (a) \$176 million; \$528 million; \$1584 million

(b)

Answers will vary.

(c) No. As $p \to 100$, the cost approaches ∞.

139. Vertical asymptote: $x = -1$
Horizontal asymptote: $y = 1$
Hole at $x = 1$

141. Vertical asymptote: $x = -1$
Slant asymptote: $y = 3x + 2$

143.

145.

147.

149.

151.

153. (a)

304,000; 453,333; 702,222

(b) 1,200,000, because N has a horizontal asymptote at $y = 1200$.

155. Quadratic **157.** Linear

159. (a)

(b)

(c) $S = -3.49t^2 + 76.3t + 5958$; $r^2 \approx 0.8915$

(d)

(e) Cubic; The cubic model more closely follows the pattern of the data.

(f) 7157 stations

161. False. For the graph of a rational function to have a slant asymptote, the degree of its numerator must be exactly one more than the degree of its denominator.

163. False. Example: $(1 + 2i) + (1 - 2i) = 2$

165. Answers will vary.

167. The first step is completed incorrectly: $\sqrt{-4} = 2i \neq 4i$

Chapter Test (page 175)

1. Vertex: $(-2, -1)$
Intercepts: $(0, 3), (-3, 0), (-1, 0)$

2. $y = (x - 3)^2 - 6$

3. 0, multiplicity 1; $-\frac{1}{2}$, multiplicity 2

4.

5. $3x + \dfrac{x - 1}{x^2 + 1}$ **6.** $2x^3 + 4x^2 + 3x + 6 + \dfrac{9}{x - 2}$

7. 13

8. $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24, \pm\frac{1}{2}, \pm\frac{3}{2}$

$t = -2, \frac{3}{2}$

9. $\pm 1, \pm 2, \pm\frac{1}{3}, \pm\frac{2}{3}$

$x = \pm 1, -\frac{2}{3}$

10. $x = -1, 4 \pm \sqrt{3}i$
$(x + 1)\left(x - 4 + \sqrt{3}i\right)\left(x - 4 - \sqrt{3}i\right)$

11. $-9 - 18i$ **12.** $6 + \left(2\sqrt{5} + \sqrt{14}\right)i$

13. $13 + 4i$ **14.** $-17 + 14i$ **15.** $\frac{43}{37} + \frac{38}{37}i$

16. $1 + 2i$ **17.** $\frac{4}{13} + \frac{7}{13}i$ **18.** $x = \pm 5\sqrt{3}i$

19. $x = 1 \pm \sqrt{7}i$

20.

21.

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22.

23. (a)

(b) $A = 0.861t^2 + 0.03t + 60.0$

(c)

The model fits the data well.

(d) \$146.4 billion; \$184.3 billion

(e) Answers will vary.

Chapter 3

Section 3.1 (page 189)

1. transcendental **3.** Horizontal shift one unit to the left

5. 4112.033 **7.** 0.006

9.

$y = 0$, $(0, 1)$, increasing

11.

$y = 0$, $(0, 1)$, decreasing

13.

$y = 0$, $(0, 1)$, increasing

15.

$y = 0$, $(0, 1)$, decreasing

17. d **18.** a **19.** c **20.** b

21. Right shift of five units

23. Left shift of four units and reflection in the x-axis

25. Right shift of two units and downward shift of three units

27. (a)

(b) $e^2 \approx 7.3891$

29. 9897.129 **31.** 54.164

33.

x	-2	-1	0	1	2
$f(x)$	0.16	0.4	1	2.5	6.25

Asymptote: $y = 0$

35.

x	-2	-1	0	1	2
$f(x)$	0.03	0.17	1	6	36

Asymptote: $y = 0$

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37.

x	-3	-2	-1	0	1
$f(x)$	0.33	1	3	9	27

Asymptote: $y = 0$

39.

x	-1	0	1	2	3	4
y	1.04	1.11	1.33	2	4	10

Asymptote: $y = 1$

41.

x	-2	-1	0	1	2
$f(x)$	7.39	2.72	1	0.37	0.14

Asymptote: $y = 0$

43.

x	-6	-5	-4	-3	-2
$f(x)$	0.41	1.10	3	8.15	22.17

Asymptote: $y = 0$

45.

x	3	4	5	6	7
$f(x)$	2.14	2.37	3	4.72	9.39

Asymptote: $y = 2$

47.

t	-2	-1	0	1	2
$s(t)$	1.57	1.77	2	2.26	2.54

Asymptote: $y = 0$

49. (a)

(b) $y = 0, y = 8$

51. (a)

(b) $y = -3, y = 0, x \approx 3.47$

53. (86.350, 1500)

55. (a)

(b) Decreasing on $(-\infty, 0), (2, \infty)$
Increasing on $(0, 2)$

(c) Relative minimum: $(0, 0)$
Relative maximum: $(2, 0.54)$

CHAPTER 3

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57.

n	1	2	4	12
A	\$3047.49	\$3050.48	\$3051.99	\$3053.00

n	365	Continuous
A	\$3053.49	\$3053.51

59.

n	1	2	4	12
A	\$5477.81	\$5520.10	\$5541.79	\$5556.46

n	365	Continuous
A	\$5563.61	\$5563.85

61.

t	1	10	20
A	\$12,489.73	\$17,901.90	\$26,706.49

t	30	40	50
A	\$39,841.40	\$59,436.39	\$88,668.67

63.

t	1	10	20
A	\$12,427.44	\$17,028.81	\$24,165.03

t	30	40	50
A	\$34,291.81	\$48,662.40	\$69,055.23

65. \$1530.57 **67.** \$17,281.77

69. (a) $y_1 = 500(1 + 0.07)^t$

$y_2 = 500\left(1 + \dfrac{0.07}{4}\right)^{4t}$

$y_3 = 500e^{0.07t}$

(b) y_3 yields the highest return after 20 years.

$y_2 - y_1 = \$68.36$

$y_3 - y_2 = \$24.40$

$y_3 - y_1 = \$92.76$

71. (a) 10 g (b) 7.85 g

(c)

73. (a)

(b)

t	15	16	17	18	19
P	40.14	40.53	40.93	41.33	41.73

t	20	21	22	23	24
P	42.14	42.55	42.96	43.38	43.80

t	25	26	27	28	29	30
P	44.23	44.66	45.10	45.54	45.98	46.43

(c) 2037

75. True. The definition of an exponential function is $f(x) = a^x$, $a > 0$, $a \neq 1$.

77. d

79.

$3^x < 4^x$ when $x > 0$

81. $>$ **83.** $>$ **85.** $f^{-1}(x) = \dfrac{x + 7}{5}$

87. $f^{-1}(x) = x^3 - 8$ **89.** Answers will vary.

Section 3.2 (page 199)

1. logarithmic function **3.** $a^{\log_a x} = x$ **5.** $b = a^c$

7. $4^3 = 64$ **9.** $7^{-2} = \frac{1}{49}$ **11.** $32^{2/5} = 4$

13. $2^{1/2} = \sqrt{2}$ **15.** $\log_5 125 = 3$ **17.** $\log_{81} 3 = \frac{1}{4}$

19. $\log_6 \frac{1}{36} = -2$ **21.** $\log_g 4 = a$ **23.** 4 **25.** -3

27. 2.538 **29.** 7.022 **31.** 9 **33.** 2 **35.** $\frac{1}{10}$

37. $3x$ **39.** -3

41.

43.

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45.

Domain: $(-2, \infty)$
Vertical asymptote: $x = -2$
x-intercept: $(-1, 0)$

47.

Domain: $(0, \infty)$
Vertical asymptote: $x = 0$
x-intercept: $\left(\frac{1}{2}, 0\right)$

49.

Domain: $(2, \infty)$
Vertical asymptote: $x = 2$
x-intercept: $\left(\frac{5}{2}, 0\right)$

51. b **52.** c **53.** d **54.** a

55. Reflection in the x-axis

57. Reflection in the x-axis, vertical shift four units upward

59. Horizontal shift three units to the left and vertical shift two units downward

61. $e^0 = 1$ **63.** $e^1 = e$ **65.** $e^{1/2} = \sqrt{e}$

67. $e^{2.1972\ldots} = 9$ **69.** $\ln 20.0855\ldots = 3$

71. $\ln 3.6692\ldots = 1.3$ **73.** $\ln 1.3956\ldots = \frac{1}{3}$

75. $\ln 4.4816\ldots = \frac{3}{2}$ **77.** 1.869 **79.** 0.693

81. 2 **83.** 1.8 **85.** 0 **87.** 1

89.

Domain: $(1, \infty)$
Vertical asymptote: $x = 1$
x-intercept: $(2, 0)$

91.

Domain: $(-\infty, 0)$
Vertical asymptote: $x = 0$
x-intercept: $(-1, 0)$

93. Horizontal shift three units to the left

95. Vertical shift five units downward

97. Horizontal shift one unit to the right and vertical shift two units upward

99. (a)

(b) Domain: $(0, \infty)$
(c) Decreasing on $(0, 2)$; increasing on $(2, \infty)$
(d) Relative minimum: $(2, 1.693)$

101. (a)

(b) Domain: $(0, \infty)$
(c) Decreasing on $(0, 0.368)$; increasing on $(0.368, \infty)$
(d) Relative minimum: $(0.368, -1.472)$

103. (a)

(b) Domain: $(-\infty, -2), (1, \infty)$
(c) Decreasing on $(-\infty, -2); (1, \infty)$
(d) No relative maxima or minima

105. (a)

(b) Domain: $(-\infty, 0), (0, \infty)$
(c) Decreasing on $(-\infty, 0)$; increasing on $(0, \infty)$
(d) No relative maxima or minima

107. (a)

(b) Domain: $[1, \infty)$
(c) Increasing on $(1, \infty)$ (d) Relative minimum: $(1, 0)$

109. (a) 80 (b) 68.12 (c) 62.30

111. (a)

K	1	2	4	6
t	0	12.60	25.21	32.57

K	8	10	12
t	37.81	41.87	45.18

It takes 12.60 years for the principal to double.

(b)

113. (a) 30 yr, 10 yr
(b) Total amount: \$323,179.20; interest: \$173,179.20
Total amount: \$199,108.80; interest: \$49,108.80

115. False. Reflect $g(x)$ about the line $y = x$.

117. 2 **119.** $\frac{1}{4}$ **121.** b

123. $\log_a x$ is the inverse of a^x only if $0 < a < 1$ and $a > 1$, so $\log_a x$ is defined only for $0 < a < 1$ and $a > 1$.

CHAPTER 3

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125. (a)

(b)

$y_4 = (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3 - \frac{1}{4}(x - 1)^4$

Answers will vary.

127. (a)

x	1	5	10	10^2
$f(x)$	0	0.32	0.23	0.046

x	10^4	10^6
$f(x)$	0.00092	0.0000138

(b) 0

129. $(x + 3)(x - 1)$ **131.** $(4x + 3)(3x - 1)$

133. $(4x + 5)(4x - 5)$ **135.** $x(2x - 9)(x + 5)$

137. 15 **139.** 2.75 **141.** 27.67

Section 3.3 (page 207)

1. change-of-base **3.** $\log_3 24 = \dfrac{\ln 24}{\ln 3}$

5. (a) $\dfrac{\log_{10} x}{\log_{10} 5}$ (b) $\dfrac{\ln x}{\ln 5}$ **7.** (a) $\dfrac{\log_{10} x}{\log_{10} \frac{1}{5}}$ (b) $\dfrac{\ln x}{\ln \frac{1}{5}}$

9. (a) $\dfrac{\log_{10} \frac{3}{10}}{\log_{10} a}$ (b) $\dfrac{\ln \frac{3}{10}}{\ln a}$ **11.** (a) $\dfrac{\log_{10} x}{\log_{10} 2.6}$ (b) $\dfrac{\ln x}{\ln 2.6}$

13. 1.771 **15.** -2 **17.** -0.059 **19.** 2.691

21. $\ln 5 + \ln 4$ **23.** $2 \ln 5 - \ln 4$ **25.** 1.6542

27. 0.2823

29.

31.

33.

35.

37. $\frac{3}{2}$ **39.** $4 + 4 \log_2 3$ **41.** $6 + \ln 5$ **43.** $\ln 6 - 2$

45. Answers will vary. **47.** $\log_{10} 5 + \log_{10} x$

49. $\log_{10} t - \log_{10} 8$ **51.** $4 \log_8 x$ **53.** $\frac{1}{2} \ln z$

55. $\ln x + \ln y + \ln z$ **57.** $\log_6 a + 3 \log_6 b + 2 \log_6 c$

59. $\frac{1}{3} \ln x - \frac{1}{3} \ln y$

61. $\ln(x + 1) + \ln(x - 1) - 3 \ln x,\ x > 1$

63. $4 \ln x + \frac{1}{2} \ln y - 5 \ln z$

65. (a)

(b)

x	1	2	3	4	5	6
y_1	1.61	3.87	5.24	6.24	7.03	7.68
y_2	1.61	3.87	5.24	6.24	7.03	7.68

x	7	8	9	10	11
y_1	8.24	8.72	9.16	9.55	9.90
y_2	8.24	8.72	9.16	9.55	9.90

(c) $y_1 = y_2$

67. (a)

(b)

x	0	3	4	5
y_1	Error	4.39	4.85	5.34
y_2	Error	4.39	4.85	5.34

x	6	7	8
y_1	5.78	6.17	6.53
y_2	5.78	6.17	6.53

(c) $y_1 = y_2$

69. $\ln 4x$ **71.** $\log_4 \dfrac{z}{y}$ **73.** $\log_2 (x + 3)^2$

75. $\ln \sqrt{x^2 + 4}$ **77.** $\ln \dfrac{x}{(x + 1)^3}$ **79.** $\ln \dfrac{x - 2}{x + 2}$

81. $\ln \dfrac{x}{(x^2 - 4)^2}$ **83.** $\ln \sqrt[3]{\dfrac{x(x + 3)^2}{x^2 - 1}}$

85. (a)

(b)

x	-5	-4	-3	-2	-1
y_1	-2.36	-1.51	-0.45	0.94	2.77
y_2	-2.36	-1.51	-0.45	0.94	2.77

x	0	1	2	3	4	5
y_1	4.16	2.77	0.94	-0.45	-1.51	-2.36
y_2	4.16	2.77	0.94	-0.45	-1.51	-2.36

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(c) $y_1 = y_2$

87. (a)

(b)

x	-1	0	1	2
y_1	Error	Error	0.35	1.24
y_2	Error	Error	0.35	1.24

x	3	4	5
y_1	1.79	2.19	2.51
y_2	1.79	2.19	2.51

(c) $y_1 = y_2$

89. (a)

(b)

x	-5	-4	-3	-2	-1
y_1	3.22	2.77	2.20	1.39	0
y_2	Error	Error	Error	Error	Error

x	0	1	2	3	4	5
y_1	Error	0	1.39	2.20	2.77	3.22
y_2	Error	0	1.39	2.20	2.77	3.22

(c) No. The domains differ.

91. (a)

(b)

x	-4	-3	0	3
y_1	Error	Error	Error	1.61
y_2	2.48	1.61	Error	1.61

x	4	5	6
y_1	2.48	3.04	3.47
y_2	2.48	3.04	3.47

(c) No. The domains differ.

93. 2 **95.** 6.8

97. Not possible; -4 is not in the domain of $\log_2 x$.

99. 2 **101.** -4 **103.** 8 **105.** $-\frac{1}{2}$

107. (a) $\beta = 120 + 10 \log_{10} I$

(b)

I	10^{-4}	10^{-6}	10^{-8}	10^{-10}	10^{-12}	10^{-14}
β	80	60	40	20	0	-20

109. (a)

(b) $T = 54.4(0.964)^t + 21$

The model fits the data.

(c)

$$\ln(T - 21) = -0.037t + 3.997$$
$$T = e^{(-0.037t + 3.997)} + 21$$

(d)

$$\frac{1}{T - 21} = 0.0012t + 0.0162$$
$$T = \frac{1}{0.0012t + 0.0162} + 21$$

111. True **113.** False. $f(\sqrt{x}) = \frac{1}{2} f(x)$

115. True. When $f(x) = 0$, $x = 1 < e$.

117. The error is an improper use of the Quotient Property of logarithms.

$$\ln \frac{x^2}{\sqrt{x^2 + 4}} = \ln x^2 - \ln \sqrt{x^2 + 4}$$
$$= 2 \ln x - \frac{1}{2} \ln(x^2 + 4)$$

119.

$\ln 1 = 0$	$\ln 9 \approx 2.1972$
$\ln 2 \approx 0.6931$	$\ln 10 \approx 2.3025$
$\ln 3 \approx 1.0986$	$\ln 12 \approx 2.4848$
$\ln 4 \approx 1.3862$	$\ln 15 \approx 2.7080$
$\ln 5 \approx 1.6094$	$\ln 16 \approx 2.7724$
$\ln 6 \approx 1.7917$	$\ln 18 \approx 2.8903$
$\ln 8 \approx 2.0793$	$\ln 20 \approx 2.9956$

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121. No.
Domain of y_1: $(-\infty, 0), (2, \infty)$
Domain of y_2: $(2, \infty)$

123. $\dfrac{3x^4}{2y^3}$ **125.** $18x^3y^4$ **127.** $3 \pm \sqrt{7}$ **129.** $\pm 4, \pm\sqrt{3}$

Section 3.4 (page 217)

1. (a) $x = y$ (b) $x = y$ (c) x (d) x **3.** 7
5. Subtract 3 from both sides. **7.** (a) Yes (b) No
9. (a) No (b) Yes (c) Yes, approximate
11. (a) Yes, approximate (b) No (c) Yes
13. (a) Yes (b) Yes, approximate (c) No

15.

(3, 8); 3

17.

(4, 10); 4

19.

(243, 20); 243

21.

$(-4, -3)$; -4

23. 2 **25.** -4 **27.** -2 **29.** -4 **31.** ln 14
33. $\log_{10} 36$ **35.** 5 **37.** 5 **39.** e^{-9} **41.** 5
43. 0.1 **45.** $\dfrac{e^5 + 1}{2}$ **47.** x^2 **49.** x^2 **51.** $2x - 1$
53. $x^2 + 6$ **55.** 0.944 **57.** 2 **59.** 184.444
61. 0.511 **63.** 0 **65.** -1.498 **67.** 6.960
69. -277.951 **71.** 1.609 **73.** $-1, 2$
75. 0.586, 3.414 **77.** 1.946 **79.** 183.258

81. (a)

x	0.6	0.7	0.8	0.9	1.0
e^{3x}	6.05	8.17	11.02	14.88	20.09

(0.8, 0.9)

(b)

(c) 0.828

83. 21.330 **85.** 3.656

87.

$x = -0.427$

89.

$x = 12.207$

91. 0.050 **93.** 2.042 **95.** 1 **97.** 4453.242
99. 17.945 **101.** 103 **103.** 5.389
105. 1.718, -3.718 **107.** 2 **109.** No real solution
111. 180.384

113. $y = \log_a x, \quad a^y = x$

$$\log_e a^y = \log_e x$$
$$y \log_e a = \log_e x$$
$$y = \frac{\log_e x}{\log_e a}$$
$$\log_a x = \frac{\log_e x}{\log_e a} = \frac{\ln x}{\ln a}$$

115. (a)

x	2	3	4	5	6
$\ln 2x$	1.39	1.79	2.08	2.30	2.48

(5, 6)

(b)

(c) 5.512

5.512

117. (a)

x	12	13	14	15	16
$6 \log_3(0.5x)$	9.79	10.22	10.63	11.00	11.36

(14, 15)

(b)

(c) 14.988

14.988

119. 1.469, 0.001 **121.** 2.928 **123.** 3.423

125.

(4.585, 7)

127.

$(-14.979, 80)$

129.

(663.142, 3.25)

131. $-1, 0$ **133.** 1 **135.** $e^{-1/2} \approx 0.607$
137. $e^{-1} \approx 0.368$ **139.** $\dfrac{\ln y - \ln a}{b}$
141. $b \pm \sqrt{c(\ln y - \ln a)}$ **143.** (a) 9.24 yr (b) 14.65 yr
145. (a) 27.73 yr (b) 43.94 yr
147. (a) 682 units (b) 779 units **149.** 2008
151. (a)

(b) $y = 20$. The object's temperature cannot cool below the room's temperature.
(c) 0.81 hr

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