Precalculus with Limits

A Graphing Approach

Sixth Edition

Ron Larson

FORMULAS FROM GEOMETRY

Triangle

$h = a \sin \theta$

$\text{Area} = \frac{1}{2}bh$

Laws of Cosines:
$c^2 = a^2 + b^2 - 2ab \cos \theta$

Circular Ring

$\text{Area} = \pi(R^2 - r^2)$
$= 2\pi pw$

(p = average radius,
w = width of ring)

Right Triangle

Pythagorean Theorem:
$c^2 = a^2 + b^2$

Ellipse

$\text{Area} = \pi ab$

$\text{Circumference} \approx 2\pi\sqrt{\frac{a^2 + b^2}{2}}$

Equilateral Triangle

$h = \frac{\sqrt{3}s}{2}$

$\text{Area} = \frac{\sqrt{3}s^2}{4}$

Cone

(A = area of base)

$\text{Volume} = \frac{Ah}{3}$

Parallelogram

$\text{Area} = bh$

Right Circular Cone

$\text{Volume} = \frac{\pi r^2 h}{3}$

$\text{Lateral Surface Area} = \pi r\sqrt{r^2 + h^2}$

Trapezoid

$\text{Area} = \frac{h}{2}(a + b)$

Frustum of Right Circular Cone

$\text{Volume} = \frac{\pi(r^2 + rR + R^2)h}{3}$

$\text{Lateral Surface Area} = \pi s(R + r)$

Circle

$\text{Area} = \pi r^2$
$\text{Circumference} = 2\pi r$

Right Circular Cylinder

$\text{Volume} = \pi r^2 h$
$\text{Lateral Surface Area} = 2\pi rh$

Sector of Circle

$\text{Area} = \frac{\theta r^2}{2}$

$s = r\theta$

(θ in radians)

Sphere

$\text{Volume} = \frac{4}{3}\pi r^3$

$\text{Surface Area} = 4\pi r^2$

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Definition of the Six Trigonometric Functions

Right triangle definitions, where $0 < \theta < \pi/2$

$$\sin\theta = \frac{\text{opp.}}{\text{hyp.}} \qquad \csc\theta = \frac{\text{hyp.}}{\text{opp.}}$$

$$\cos\theta = \frac{\text{adj.}}{\text{hyp.}} \qquad \sec\theta = \frac{\text{hyp.}}{\text{adj.}}$$

$$\tan\theta = \frac{\text{opp.}}{\text{adj.}} \qquad \cot\theta = \frac{\text{adj.}}{\text{opp.}}$$

Circular function definitions, where θ *is any angle*

$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

Reciprocal Identities

$$\sin u = \frac{1}{\csc u} \qquad \cos u = \frac{1}{\sec u} \qquad \tan u = \frac{1}{\cot u}$$

$$\csc u = \frac{1}{\sin u} \qquad \sec u = \frac{1}{\cos u} \qquad \cot u = \frac{1}{\tan u}$$

Quotient Identities

$$\tan u = \frac{\sin u}{\cos u} \qquad \cot u = \frac{\cos u}{\sin u}$$

Pythagorean Identities

$$\sin^2 u + \cos^2 u = 1$$

$$1 + \tan^2 u = \sec^2 u \qquad 1 + \cot^2 u = \csc^2 u$$

Cofunction Identities

$$\sin\left(\frac{\pi}{2} - u\right) = \cos u \qquad \cot\left(\frac{\pi}{2} - u\right) = \tan u$$

$$\cos\left(\frac{\pi}{2} - u\right) = \sin u \qquad \sec\left(\frac{\pi}{2} - u\right) = \csc u$$

$$\tan\left(\frac{\pi}{2} - u\right) = \cot u \qquad \csc\left(\frac{\pi}{2} - u\right) = \sec u$$

Even/Odd Identities

$$\sin(-u) = -\sin u \qquad \cot(-u) = -\cot u$$

$$\cos(-u) = \cos u \qquad \sec(-u) = \sec u$$

$$\tan(-u) = -\tan u \qquad \csc(-u) = -\csc u$$

Sum and Difference Formulas

$$\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$$

$$\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$$

$$\tan(u \pm v) = \frac{\tan u \pm \tan v}{1 \mp \tan u \tan v}$$

Double-Angle Formulas

$$\sin 2u = 2 \sin u \cos u$$

$$\cos 2u = \cos^2 u - \sin^2 u = 2\cos^2 u - 1 = 1 - 2\sin^2 u$$

$$\tan 2u = \frac{2 \tan u}{1 - \tan^2 u}$$

Power-Reducing Formulas

$$\sin^2 u = \frac{1 - \cos 2u}{2}$$

$$\cos^2 u = \frac{1 + \cos 2u}{2}$$

$$\tan^2 u = \frac{1 - \cos 2u}{1 + \cos 2u}$$

Sum-to-Product Formulas

$$\sin u + \sin v = 2 \sin\left(\frac{u+v}{2}\right) \cos\left(\frac{u-v}{2}\right)$$

$$\sin u - \sin v = 2 \cos\left(\frac{u+v}{2}\right) \sin\left(\frac{u-v}{2}\right)$$

$$\cos u + \cos v = 2 \cos\left(\frac{u+v}{2}\right) \cos\left(\frac{u-v}{2}\right)$$

$$\cos u - \cos v = -2 \sin\left(\frac{u+v}{2}\right) \sin\left(\frac{u-v}{2}\right)$$

Product-to-Sum Formulas

$$\sin u \sin v = \frac{1}{2}[\cos(u - v) - \cos(u + v)]$$

$$\cos u \cos v = \frac{1}{2}[\cos(u - v) + \cos(u + v)]$$

$$\sin u \cos v = \frac{1}{2}[\sin(u + v) + \sin(u - v)]$$

$$\cos u \sin v = \frac{1}{2}[\sin(u + v) - \sin(u - v)]$$

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Precalculus with Limits
A Graphing Approach
Sixth Edition

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$$f(x) = \frac{2x}{x-3}$$

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Preface

We all know that *Precalculus* is the mathematics needed to master calculus. But, *Precalculus* is so much more than that. It is not simply preliminary material for calculus. It stands alone as "real mathematics" in itself. It was developed to model dozens of different technical applications in real life. In this book, you will see how *Precalculus* can be used by real people to solve real-life problems and make real-life decisions.

New To This Edition

NEW *Chapter Openers*

Each *Chapter Opener* highlights a real modeling data problem, showing a graph of the data, a section reference, and a short description of the data.

NEW *Explore the Concept*

Each *Explore the Concept* engages the student in active discovery of mathematical concepts, strengthens critical thinking skills, and helps build intuition.

NEW *What's Wrong?*

Each *What's Wrong?* points out common errors made using graphing utilities.

NEW *Vocabulary and Concept Check*

The *Vocabulary and Concept Check* appears at the beginning of the exercise set for each section. Each of these checks asks fill-in-the-blank, matching, and non-computational questions designedto help students learn mathematical terminology and to test basic understanding of that section's concepts.

NEW *Organization of Section Exercises*

The exercise sets are grouped into four categories: (1) *Vocabulary and Concept Check*, (2) *Procedures and Problem Solving*, (3) *Conclusions*, and (4) *Cumulative Mixed Review*. Many of the exercises are titled for easy reference.

NEW *Algebraic-Graphical-Numerical Exercises*

These exercises allow students to solve a problem using multiple approaches—algebraic, graphical, and numerical. This helps students to see that a problem can be solved in more than one way and to see that different methods yield the same result.

NEW *Modeling Data Exercises*

These multi-part applications that involve real-life data offer students the opportunity to generate and analyze mathematical models.

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NEW *Capstone Exercise*

The *Capstone* is a conceptual problem that synthesizes key topics and provides students with a better understanding of the concepts in a section. This exercise is excellent for classroom discussion or test preparation. I recommend it.

130. CAPSTONE Decide whether the two functions shown in each graph appear to be inverse functions of each other. Explain your reasoning.

(a) (b) (c) (d)

NEW *Why You Should Learn It Exercise*

An engaging real-life application of the concepts of the section. This application exercise is typically described in the section opener as a motivator for the section.

NEW *Error Analysis Exercise*

This exercise presents a sample solution that contains a common error which the students are asked to identify.

REVISED *Library of Parent Functions*

To facilitate familiarity with the basic functions, I have compiled several elementary and nonelementary functions as a *Library of Parent Functions*. Each function is introduced at the first point of use in the text with a definition and description of basic characteristics. New to this edition are the *Library of Parent Functions Examples*, which are identified in the title of the example, and the *Review of Library of Parent Functions* after Chapter 4. A summary of the functions is presented on the inside cover of this text.

REVISED *Side-by-Side Examples*

Many examples present side-by-side solutions with multiple approaches—algebraic, graphical, and numerical. I have revised the graphical solutions to be more visual. Often, the algebraic solution is formal, with step-by-step work. As a change in this edition, the graphical solution is often more of a visual check to ensure the reasonableness of the solution obtained algebraically.

REVISED *Exercise Sets*

The exercise sets have been carefully and extensively examined to ensure they are rigorous, relevant, and cover all topics suggested by our users. Many new skill building and challenging exercises have been added.

REVISED *Chapter Summaries*

The Chapter Summary now includes explanations and examples of the objectives taught in the chapter.

Calc Chat

For the past several years, we have maintained an independent website—**CalcChat.com**—that provides free solutions to all odd-numbered exercises in the text. Thousands of students using our textbooks have visited the site for practice and help with their homework. For the Sixth Edition, I was able to use information from *CalcChat.com*, including which solutions students accessed most often, to help guide the revision of the exercises.

Table of Contents Changes

In Chapter 8 (Sequences, Series, and Probability), old Section 8.4 (Mathematical Induction) is now Appendix G and has been moved to *CengageBrain.com*. In Chapter 9 (Topics in Analytical Geometry), we have combined old Section 9.3 (Hyperbolas) and old Section 9.4 (Rotation and Systems of Quadratic Equations) into a new combined Section 9.3 (Hyperbolas and Rotation of Conics). Added Appendix B (Concepts in Statistics). We have moved Appendix C (Review of Graphs, Equations, and Inequalities), Appendix D (Variation), Appendix E (Solving Linear Equations and Inequalities), and Appendix F (Systems of Inequalities) to *CengageBrain.com*.

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Trusted Features

What you should learn/Why you should learn it

These summarize important topics in the section and why they are important in math and life.

Study Tip

These hints and tips can be used to reinforce concepts, help students learn how to study mathematics, caution students about common errors, or show alternative or additional steps to a solution of an example.

Technology Tip

Technology Tips provide graphing calculator tips or provide alternative methods of solving a problem using a graphing utility.

Checkpoint

Checkpoints guide students to an odd-numbered exercise that is similar to the example they just read.

Algebra of Calculus

Throughout the text, special emphasis is given to the algebraic techniques used in calculus. *Algebra of Calculus* examples and exercises are integrated throughout the text and are identified by the symbol $\int$.

Instructor Resources

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ISBN-10: 1-111-57402-2; ISBN-13: 978-1-111-57402-4

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Media

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- Practice Another Version feature (activated at the instructor's discretion) allows students to attempt the questions with new sets of values until they feel confident enough to work the original problem
- Graphing utility tool enables students to graph lines, segments, parabolas, and circles as they answer questions
- MathPad simplifies the input of mathematical symbols

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I hope that you enjoy learning the mathematics presented in this text. More than that, I hope you gain a new appreciation for the relevance of mathematics to careers in science, technology, business, and medicine.

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If you have suggestions for improving this text, please feel free to write me. Over the past two decades I have received many useful comments from both instructors and students, and I value these very much.

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1 Functions and Their Graphs

Section 1.7, Example 4
Alternative-Fueled Vehicles

1.1 Lines in the Plane
1.2 Functions
1.3 Graphs of Functions
1.4 Shifting, Reflecting, and Stretching Graphs
1.5 Combinations of Functions
1.6 Inverse Functions
1.7 Linear Models and Scatter Plots

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Introduction to Library of Parent Functions

In Chapter 1, you will be introduced to the concept of a *function*. As you proceed through the text, you will see that functions play a primary role in modeling real-life situations.

There are three basic types of functions that have proven to be the most important in modeling real-life situations. These functions are algebraic functions, exponential and logarithmic functions, and trigonometric and inverse trigonometric functions. These three types of functions are referred to as the *elementary functions*, though they are often placed in the two categories of *algebraic functions* and *transcendental functions*. Each time a new type of function is studied in detail in this text, it will be highlighted in a box similar to those shown below. The graphs of these functions are shown on the inside covers of this text.

ALGEBRAIC FUNCTIONS

These functions are formed by applying algebraic operations to the linear function $f(x) = x$.

Name	*Function*	*Location*
Linear	$f(x) = x$	Section 1.1
Quadratic	$f(x) = x^2$	Section 2.1
Cubic	$f(x) = x^3$	Section 2.2
Rational	$f(x) = \dfrac{1}{x}$	Section 2.7
Square root	$f(x) = \sqrt{x}$	Section 1.2

TRANSCENDENTAL FUNCTIONS

These functions cannot be formed from the linear function by using algebraic operations.

Name	*Function*	*Location*
Exponential	$f(x) = a^x, a > 0, a \neq 1$	Section 3.1
Logarithmic	$f(x) = \log_a x, x > 0, a > 0, a \neq 1$	Section 3.2
Trigonometric	$f(x) = \sin x$	Section 4.5
	$f(x) = \cos x$	Section 4.5
	$f(x) = \tan x$	Section 4.6
	$f(x) = \csc x$	Section 4.6
	$f(x) = \sec x$	Section 4.6
	$f(x) = \cot x$	Section 4.6
Inverse trigonometric	$f(x) = \arcsin x$	Section 4.7
	$f(x) = \arccos x$	Section 4.7
	$f(x) = \arctan x$	Section 4.7

NONELEMENTARY FUNCTIONS

Some useful nonelementary functions include the following.

Name	*Function*	*Location*
Absolute value	$f(x) = \lvert x \rvert$	Section 1.2
Greatest integer	$f(x) = [\![x]\!]$	Section 1.3

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1.1 Lines in the Plane

The Slope of a Line

In this section, you will study lines and their equations. The **slope** of a nonvertical line represents the number of units the line rises or falls vertically for each unit of horizontal change from left to right. For instance, consider the two points

$$(x_1, y_1) \qquad \text{and} \qquad (x_2, y_2)$$

on the line shown in Figure 1.1.

Figure 1.1

As you move from left to right along this line, a change of $(y_2 - y_1)$ units in the vertical direction corresponds to a change of $(x_2 - x_1)$ units in the horizontal direction. That is,

$$y_2 - y_1 = \text{the change in } y$$

and

$$x_2 - x_1 = \text{the change in } x.$$

The slope of the line is given by the ratio of these two changes.

Definition of the Slope of a Line

The **slope** m of the nonvertical line through (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{change in } y}{\text{change in } x}$$

where $x_1 \neq x_2$.

When this formula for slope is used, the *order of subtraction* is important. Given two points on a line, you are free to label either one of them as (x_1, y_1) and the other as (x_2, y_2). Once you have done this, however, you must form the numerator and denominator using the same order of subtraction.

$$\underbrace{m = \frac{y_2 - y_1}{x_2 - x_1}}_{\text{Correct}} \qquad \underbrace{m = \frac{y_1 - y_2}{x_1 - x_2}}_{\text{Correct}} \qquad \underbrace{\cancel{m = \frac{y_2 - y_1}{x_1 - x_2}}}_{\text{Incorrect}}$$

Throughout this text, the term *line* always means a *straight* line.

What you should learn

- Find the slopes of lines.
- Write linear equations given points on lines and their slopes.
- Use slope-intercept forms of linear equations to sketch lines.
- Use slope to identify parallel and perpendicular lines.

Why you should learn it

The slope of a line can be used to solve real-life problems. For instance, in Exercise 97 on page 14, you will use a linear equation to model student enrollment at Penn State University.

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Example 1 Finding the Slope of a Line

Find the slope of the line passing through each pair of points.

a. $(-2, 0)$ and $(3, 1)$

b. $(-1, 2)$ and $(2, 2)$

c. $(0, 4)$ and $(1, -1)$

Solution

a. $$m = \frac{\overbrace{y_2 - y_1}^{\text{Difference in } y\text{-values}}}{\underbrace{x_2 - x_1}_{\text{Difference in } x\text{-values}}} = \frac{1 - 0}{3 - (-2)} = \frac{1}{3 + 2} = \frac{1}{5}$$

b. $$m = \frac{2 - 2}{2 - (-1)} = \frac{0}{3} = 0$$

c. $$m = \frac{-1 - 4}{1 - 0} = \frac{-5}{1} = -5$$

The graphs of the three lines are shown in Figure 1.2. Note that the *square setting* gives the correct "steepness" of the lines.

(a) **(b)** **(c)**

Figure 1.2

CHECKPOINT Now try Exercise 15.

The definition of slope does not apply to vertical lines. For instance, consider the points $(3, 4)$ and $(3, 1)$ on the vertical line shown in Figure 1.3. Applying the formula for slope, you obtain

$$m = \frac{4 - 1}{3 - 3} = \frac{3}{0}. \qquad \text{Undefined}$$

Figure 1.3

Because division by zero is undefined, the slope of a vertical line is undefined.

From the slopes of the lines shown in Figures 1.2 and 1.3, you can make the following generalizations about the slope of a line.

The Slope of a Line

1. A line with positive slope $(m > 0)$ *rises* from left to right.
2. A line with negative slope $(m < 0)$ *falls* from left to right.
3. A line with zero slope $(m = 0)$ is *horizontal*.
4. A line with undefined slope is *vertical*.

Explore the Concept

Use a graphing utility to compare the slopes of the lines $y = 0.5x$, $y = x$, $y = 2x$, and $y = 4x$. What do you observe about these lines? Compare the slopes of the lines $y = -0.5x$, $y = -x$, $y = -2x$, and $y = -4x$. What do you observe about these lines? (*Hint:* Use a *square setting* to obtain a true geometric perspective.)

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The Point-Slope Form of the Equation of a Line

When you know the slope of a line *and* you also know the coordinates of one point on the line, you can find an equation of the line. For instance, in Figure 1.4, let (x_1, y_1) be a point on the line whose slope is m. When (x, y) is any *other* point on the line, it follows that

$$\frac{y - y_1}{x - x_1} = m.$$

This equation in the variables x and y can be rewritten in the **point-slope form** of the equation of a line.

Figure 1.4

Point-Slope Form of the Equation of a Line

The **point-slope form** of the equation of the line that passes through the point (x_1, y_1) and has a slope of m is

$$y - y_1 = m(x - x_1).$$

Example 2 The Point-Slope Form of the Equation of a Line

Find an equation of the line that passes through the point

$$(1, -2)$$

and has a slope of 3.

Solution

$y - y_1 = m(x - x_1)$	Point-slope form
$y - (-2) = 3(x - 1)$	Substitute for y_1, m, and x_1.
$y + 2 = 3x - 3$	Simplify.
$y = 3x - 5$	Solve for y.

The line is shown in Figure 1.5.

Figure 1.5

CHECKPOINT Now try Exercise 25.

The point-slope form can be used to find an equation of a nonvertical line passing through two points

$$(x_1, y_1) \quad \text{and} \quad (x_2, y_2).$$

First, find the slope of the line.

$$m = \frac{y_2 - y_1}{x_2 - x_1}, \quad x_1 \neq x_2$$

Then use the point-slope form to obtain the equation

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

This is sometimes called the **two-point form** of the equation of a line.

Study Tip

When you find an equation of the line that passes through two given points, you need to substitute the coordinates of only one of the points into the point-slope form. It does not matter which point you choose because both points will yield the same result.

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Example 3 A Linear Model for Profits Prediction

During 2006, Research In Motion's net profits were \$631.6 million, and in 2007 net profits were \$1293.9 million. Write a linear equation giving the net profits y in terms of the year x. Then use the equation to predict the net profits for 2008. (Source: Research In Motion Limited)

Solution

Let $x = 0$ represent 2000. In Figure 1.6, let (6, 631.6) and (7, 1293.9) be two points on the line representing the net profits. The slope of this line is

$$m = \frac{1293.9 - 631.6}{7 - 6} = 662.3.$$

By the point-slope form, the equation of the line is as follows.

$$y - 631.6 = 662.3(x - 6)$$

$$y = 662.3x - 3342.2$$

Figure 1.6

Now, using this equation, you can predict the 2008 net profits ($x = 8$) to be

$$y = 662.3(8) - 3342.2 = 5298.4 - 3342.2 = \$1956.2 \text{ million}.$$

(In this case, the prediction is quite good—the actual net profits in 2008 were \$1968.8 million.)

CHECKPOINT Now try Exercise 33.

Library of Parent Functions: Linear Function

In the next section, you will be introduced to the precise meaning of the term *function*. The simplest type of function is the *parent linear function*

$$f(x) = x.$$

As its name implies, the graph of the parent linear function is a line. The basic characteristics of the parent linear function are summarized below and on the inside cover of this text. (Note that some of the terms below will be defined later in the text.)

Graph of $f(x) = x$

Domain: $(-\infty, \infty)$

Range: $(-\infty, \infty)$

Intercept: $(0, 0)$

Increasing

The function $f(x) = x$ is also referred to as the *identity function*. Later in this text, you will learn that the graph of the linear function $f(x) = mx + b$ is a line with slope m and y-intercept $(0, b)$. When $m = 0$, $f(x) = b$ is called a *constant function* and its graph is a horizontal line.

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Sketching Graphs of Lines

Many problems in coordinate geometry can be classified as follows.

1. Given a graph (or parts of it), find its equation.

2. Given an equation, sketch its graph.

For lines, the first problem is solved easily by using the point-slope form. This formula, however, is not particularly useful for solving the second type of problem. The form that is better suited to graphing linear equations is the **slope-intercept form** of the equation of a line, $y = mx + b$.

Slope-Intercept Form of the Equation of a Line

The graph of the equation

$$y = mx + b$$

is a line whose slope is m and whose y-intercept is $(0, b)$.

Example 4 Using the Slope-Intercept Form

Determine the slope and y-intercept of each linear equation. Then describe its graph.

a. $x + y = 2$

b. $y = 2$

Algebraic Solution

a. Begin by writing the equation in slope-intercept form.

$x + y = 2$	Write original equation.
$y = 2 - x$	Subtract x from each side.
$y = -x + 2$	Write in slope-intercept form.

From the slope-intercept form of the equation, the slope is -1 and the y-intercept is

$$(0, 2).$$

Because the slope is negative, you know that the graph of the equation is a line that falls one unit for every unit it moves to the right.

b. By writing the equation $y = 2$ in slope-intercept form

$$y = (0)x + 2$$

you can see that the slope is 0 and the y-intercept is

$$(0, 2).$$

A zero slope implies that the line is horizontal.

CHECKPOINT Now try Exercise 35.

Graphical Solution

a.

b.

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From the slope-intercept form of the equation of a line, you can see that a horizontal line ($m = 0$) has an equation of the form

$$y = b. \qquad \text{Horizontal line}$$

This is consistent with the fact that each point on a horizontal line through $(0, b)$ has a y-coordinate of b. Similarly, each point on a vertical line through $(a, 0)$ has an x-coordinate of a. So, a vertical line has an equation of the form

$$x = a. \qquad \text{Vertical line}$$

This equation cannot be written in slope-intercept form because the slope of a vertical line is undefined. However, every line has an equation that can be written in the general form

$$Ax + By + C = 0 \qquad \text{General form of the equation of a line}$$

where A and B are not *both* zero.

Summary of Equations of Lines

1. General form: $Ax + By + C = 0$
2. Vertical line: $x = a$
3. Horizontal line: $y = b$
4. Slope-intercept form: $y = mx + b$
5. Point-slope form: $y - y_1 = m(x - x_1)$

Example 5 Different Viewing Windows

When a graphing utility is used to graph a line, it is important to realize that the graph of the line may not visually appear to have the slope indicated by its equation. This occurs because of the viewing window used for the graph. For instance, Figure 1.7 shows graphs of $y = 2x + 1$ produced on a graphing utility using three different viewing windows. Notice that the slopes in Figures 1.7(a) and (b) do not visually appear to be equal to 2. When you use a *square setting*, as in Figure 1.7(c), the slope visually appears to be 2.

(a) ***Nonsquare setting*** **(b)** ***Nonsquare setting***

(c) ***Square setting***

Figure 1.7

CHECKPOINT Now try Exercise 61.

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Parallel and Perpendicular Lines

The slope of a line is a convenient tool for determining whether two lines are parallel or perpendicular.

Parallel Lines

Two distinct nonvertical lines are **parallel** if and only if their slopes are equal. That is,

$$m_1 = m_2.$$

Explore the Concept

Graph the lines $y_1 = \frac{1}{2}x + 1$ and $y_2 = -2x + 1$ in the same viewing window. What do you observe?

Graph the lines $y_1 = 2x + 1$, $y_2 = 2x$, and $y_3 = 2x - 1$ in the same viewing window. What do you observe?

Example 6 Equations of Parallel Lines

Find the slope-intercept form of the equation of the line that passes through the point $(2, -1)$ and is parallel to the line

$$2x - 3y = 5.$$

Solution

Begin by writing the equation of the given line in slope-intercept form.

$2x - 3y = 5$	Write original equation.
$-2x + 3y = -5$	Multiply by -1.
$3y = 2x - 5$	Add $2x$ to each side.
$y = \frac{2}{3}x - \frac{5}{3}$	Write in slope-intercept form.

Therefore, the given line has a slope of

$$m = \tfrac{2}{3}.$$

Any line parallel to the given line must also have a slope of $\frac{2}{3}$. So, the line through $(2, -1)$ has the following equation.

$y - y_1 = m(x - x_1)$	Point-slope form
$y - (-1) = \frac{2}{3}(x - 2)$	Substitute for y_1, m, and x_1.
$y + 1 = \frac{2}{3}x - \frac{4}{3}$	Simplify.
$y = \frac{2}{3}x - \frac{7}{3}$	Write in slope-intercept form.

Notice the similarity between the slope-intercept form of the original equation and the slope-intercept form of the parallel equation. The graphs of both equations are shown in Figure 1.8.

Figure 1.8

CHECKPOINT Now try Exercise 67(a).

Perpendicular Lines

Two nonvertical lines are **perpendicular** if and only if their slopes are negative reciprocals of each other. That is,

$$m_1 = -\frac{1}{m_2}.$$

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Example 7 Equations of Perpendicular Lines

Find the slope-intercept form of the equation of the line that passes through the point $(2, -1)$ and is perpendicular to the line

$$2x - 3y = 5.$$

Solution

From Example 6, you know that the equation can be written in the slope-intercept form

$$y = \tfrac{2}{3}x - \tfrac{5}{3}.$$

You can see that the line has a slope of $\frac{2}{3}$. So, any line perpendicular to this line must have a slope of $-\frac{3}{2}$ $\left(\text{because } -\frac{3}{2} \text{ is the negative reciprocal of } \frac{2}{3}\right)$. So, the line through the point $(2, -1)$ has the following equation.

$$y - (-1) = -\tfrac{3}{2}(x - 2) \qquad \text{Write in point-slope form.}$$

$$y + 1 = -\tfrac{3}{2}x + 3 \qquad \text{Simplify.}$$

$$y = -\tfrac{3}{2}x + 2 \qquad \text{Write in slope-intercept form.}$$

The graphs of both equations are shown in Figure 1.9.

Figure 1.9

CHECKPOINT Now try Exercise 67(b).

Example 8 Graphs of Perpendicular Lines

Use a graphing utility to graph the lines $y = x + 1$ and $y = -x + 3$ in the same viewing window. The lines are supposed to be perpendicular (they have slopes of $m_1 = 1$ and $m_2 = -1$). Do they appear to be perpendicular on the display?

Solution

When the viewing window is nonsquare, as in Figure 1.10, the two lines will not appear perpendicular. When, however, the viewing window is square, as in Figure 1.11, the lines will appear perpendicular.

Figure 1.10

Figure 1.11

CHECKPOINT Now try Exercise 81.

What's Wrong?

You use a graphing utility to graph $y_1 = 1.5x$ and $y_2 = -1.5x + 5$, as shown in the figure. You use the graph to conclude that the lines are perpendicular. What's wrong?

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1.1 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

1. Match each equation with its form.

(a) $Ax + By + C = 0$ (i) vertical line
(b) $x = a$ (ii) slope-intercept form
(c) $y = b$ (iii) general form
(d) $y = mx + b$ (iv) point-slope form
(e) $y - y_1 = m(x - x_1)$ (v) horizontal line

In Exercises 2 and 3, fill in the blank.

2. For a line, the ratio of the change in y to the change in x is called the _____ of the line.

3. Two lines are _____ if and only if their slopes are equal.

4. What is the relationship between two lines whose slopes are -3 and $\frac{1}{3}$?

5. What is the slope of a line that is perpendicular to the line represented by $x = 3$?

6. Give the coordinates of a point on the line whose equation in point-slope form is $y - (-2) = \frac{1}{2}(x - 5)$.

Procedures and Problem Solving

Using Slope **In Exercises 7 and 8, identify the line that has the indicated slope.**

7. (a) $m = \frac{2}{3}$ (b) m is undefined. (c) $m = -2$

8. (a) $m = 0$ (b) $m = -\frac{3}{4}$ (c) $m = 1$

Estimating Slope **In Exercises 9 and 10, estimate the slope of the line.**

9.

10.

Sketching Lines **In Exercises 11 and 12, sketch the lines through the point with the indicated slopes on the same set of coordinate axes.**

	Point	*Slopes*			
11.	$(2, 3)$	(a) 0	(b) 1	(c) 2	(d) -3
12.	$(-4, 1)$	(a) 3	(b) -3	(c) $\frac{1}{2}$	(d) Undefined

Finding the Slope of a Line **In Exercises 13–16, find the slope of the line passing through the pair of points. Then use a graphing utility to plot the points and use the *draw* feature to graph the line segment connecting the two points. (Use a *square setting*.)**

13. $(0, -10), (-4, 0)$ **14.** $(2, 4), (4, -4)$
✓ **15.** $(-6, -1), (-6, 4)$ **16.** $(-3, -2), (1, 6)$

Using Slope **In Exercises 17–24, use the point on the line and the slope of the line to find three additional points through which the line passes. (There are many correct answers.)**

	Point	*Slope*
17.	$(2, 1)$	$m = 0$
18.	$(3, -2)$	$m = 0$
19.	$(1, 5)$	m is undefined.
20.	$(-4, 1)$	m is undefined.
21.	$(0, -9)$	$m = -2$
22.	$(-5, 4)$	$m = 2$
23.	$(7, -2)$	$m = \frac{1}{2}$
24.	$(-1, -6)$	$m = -\frac{1}{2}$

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The Point-Slope Form of the Equation of a Line **In Exercises 25–32, find an equation of the line that passes through the given point and has the indicated slope. Sketch the line by hand. Use a graphing utility to verify your sketch, if possible.**

25. $(0, -2)$, $m = 3$ **26.** $(-3, 6)$, $m = -2$

27. $(2, -3)$, $m = -\frac{1}{2}$ **28.** $(-2, -5)$, $m = \frac{3}{4}$

29. $(6, -1)$, m is undefined

30. $(-10, 4)$, m is undefined

31. $\left(-\frac{1}{2}, \frac{3}{2}\right)$, $m = 0$ **32.** $(2.3, -8.5)$, $m = 0$

33. **Finance** The median player salary for the New York Yankees was \$1.6 million in 2001 and \$5.2 million in 2009. Write a linear equation giving the median salary y in terms of the year x. Then use the equation to predict the median salary in 2017.

34. **Finance** The median player salary for the Dallas Cowboys was \$441,300 in 2000 and \$1,326,720 in 2008. Write a linear equation giving the median salary y in terms of the year x. Then use the equation to predict the median salary in 2016.

Using the Slope-Intercept Form **In Exercises 35–42, determine the slope and y-intercept (if possible) of the linear equation. Then describe its graph.**

35. $x - 2y = 4$ **36.** $3x + 4y = 1$

37. $2x - 5y + 10 = 0$ **38.** $4x - 3y - 9 = 0$

39. $x = -6$ **40.** $y = 12$

41. $3y + 2 = 0$ **42.** $2x - 5 = 0$

Using the Slope-Intercept Form **In Exercises 43–48, (a) find the slope and y-intercept (if possible) of the equation of the line algebraically, and (b) sketch the line by hand. Use a graphing utility to verify your answers to parts (a) and (b).**

43. $5x - y + 3 = 0$ **44.** $2x + 3y - 9 = 0$

45. $5x - 2 = 0$ **46.** $3x + 7 = 0$

47. $3y + 5 = 0$ **48.** $-11 - 8y = 0$

Finding the Slope-Intercept Form **In Exercises 49 and 50, find the slope-intercept form of the equation of the line shown.**

49.

50.

Finding the Slope-Intercept Form **In Exercises 51–60, write an equation of the line that passes through the points. Use the slope-intercept form (if possible). If not possible, explain why and use the general form. Use a graphing utility to graph the line (if possible).**

51. $(5, -1), (-5, 5)$ **52.** $(4, 3), (-4, -4)$

53. $(-8, 1), (-8, 7)$ **54.** $(-1, 4), (6, 4)$

55. $\left(2, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{5}{4}\right)$ **56.** $(1, 1), \left(6, -\frac{2}{3}\right)$

57. $\left(-\frac{1}{10}, -\frac{3}{5}\right), \left(\frac{9}{10}, -\frac{9}{5}\right)$ **58.** $\left(\frac{3}{4}, \frac{3}{2}\right), \left(-\frac{4}{3}, \frac{7}{4}\right)$

59. $(1, 0.6), (-2, -0.6)$ **60.** $(-8, 0.6), (2, -2.4)$

Different Viewing Windows **In Exercises 61 and 62, use a graphing utility to graph the equation using each viewing window. Describe the differences in the graphs.**

61. $y = 0.5x - 3$

Xmin = -5	Xmin = -2	Xmin = -5
Xmax = 10	Xmax = 10	Xmax = 10
Xscl = 1	Xscl = 1	Xscl = 1
Ymin = -1	Ymin = -4	Ymin = -7
Ymax = 10	Ymax = 1	Ymax = 3
Yscl = 1	Yscl = 1	Yscl = 1

62. $y = -8x + 5$

Xmin = -5	Xmin = -5	Xmin = -5
Xmax = 5	Xmax = 10	Xmax = 13
Xscl = 1	Xscl = 1	Xscl = 1
Ymin = -10	Ymin = -80	Ymin = -2
Ymax = 10	Ymax = 80	Ymax = 10
Yscl = 1	Yscl = 20	Yscl = 1

Parallel and Perpendicular Lines **In Exercises 63–66, determine whether the lines L_1 and L_2 passing through the pairs of points are parallel, perpendicular, or neither.**

63. L_1: $(0, -1), (5, 9)$
L_2: $(0, 3), (4, 1)$

64. L_1: $(-2, -1), (1, 5)$
L_2: $(1, 3), (5, -5)$

65. L_1: $(3, 6), (-6, 0)$
L_2: $(0, -1), \left(5, \frac{7}{3}\right)$

66. L_1: $(4, 8), (-4, 2)$
L_2: $(3, -5), \left(-1, \frac{1}{3}\right)$

Equations of Parallel and Perpendicular Lines **In Exercises 67–76, write the slope-intercept forms of the equations of the lines through the given point (a) parallel to the given line and (b) perpendicular to the given line.**

67. $(2, 1)$, $4x - 2y = 3$ **68.** $(-3, 2)$, $x + y = 7$

69. $\left(-\frac{2}{3}, \frac{7}{8}\right)$, $3x + 4y = 7$ **70.** $\left(\frac{2}{5}, -1\right)$, $3x - 2y = 6$

71. $(-3.9, -1.4)$, $6x + 2y = 9$

72. $(-1.2, 2.4)$, $5x + 4y = 1$

73. $(3, -2)$, $x - 4 = 0$ **74.** $(3, -1)$, $y - 2 = 0$

75. $(-4, 1)$, $y + 2 = 0$ **76.** $(-2, 4)$, $x + 5 = 0$

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Equations of Parallel Lines **In Exercises 77 and 78, the lines are parallel. Find the slope-intercept form of the equation of line y_2.**

77.

78.

Equations of Perpendicular Lines **In Exercises 79 and 80, the lines are perpendicular. Find the slope-intercept form of the equation of line y_2.**

79.

80.

Graphs of Parallel and Perpendicular Lines **In Exercises 81–84, identify any relationships that exist among the lines, and then use a graphing utility to graph the three equations in the same viewing window. Adjust the viewing window so that each slope appears visually correct. Use the slopes of the lines to verify your results.**

✓ **81.** (a) $y = 2x$ (b) $y = -2x$ (c) $y = \frac{1}{2}x$

82. (a) $y = \frac{2}{3}x$ (b) $y = -\frac{3}{2}x$ (c) $y = \frac{2}{3}x + 2$

83. (a) $y = -\frac{1}{2}x$ (b) $y = -\frac{1}{2}x + 3$ (c) $y = 2x - 4$

84. (a) $y = x - 8$ (b) $y = x + 1$ (c) $y = -x + 3$

85. **Architectural Design** The "rise to run" ratio of the roof of a house determines the steepness of the roof. The rise to run ratio of the roof in the figure is 3 to 4. Determine the maximum height in the attic of the house if the house is 32 feet wide.

86. **Highway Engineering** When driving down a mountain road, you notice warning signs indicating that it is a "12% grade." This means that the slope of the road is $-\frac{12}{100}$. Approximate the amount of horizontal change in your position if you note from elevation markers that you have descended 2000 feet vertically.

87. **MODELING DATA**

The graph shows the sales y (in millions of dollars) of the Coca-Cola Bottling Company each year x from 2000 through 2008, where $x = 0$ represents 2000. (Source: Coca-Cola Bottling Company)

(a) Use the slopes to determine the years in which the sales showed the greatest increase and greatest decrease.

(b) Find the equation of the line between the years 2000 and 2008.

(c) Interpret the meaning of the slope of the line from part (b) in the context of the problem.

(d) Use the equation from part (b) to estimate the sales of the Coca-Cola Bottling Company in 2010. Do you think this is an accurate estimate? Explain.

88. **MODELING DATA**

The table shows the profits y (in millions of dollars) for Buffalo Wild Wings for each year x from 2002 through 2008, where $x = 2$ represents 2002. (Source: Buffalo Wild Wings Inc.)

Year, x	Profits, y
2	3.1
3	3.9
4	7.2
5	8.9
6	16.3
7	19.7
8	24.4

(a) Sketch a graph of the data.

(b) Use the slopes to determine the years in which the profits showed the greatest and least increases.

(c) Find the equation of the line between the years 2002 and 2008.

(d) Interpret the meaning of the slope of the line from part (c) in the context of the problem.

(e) Use the equation from part (c) to estimate the profit for Buffalo Wild Wings in 2010. Do you think this is an accurate estimate? Explain.

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Using a Rate of Change to Write an Equation **In Exercises 89–92, you are given the dollar value of a product in 2009 and the rate at which the value of the product is expected to change during the next 5 years. Write a linear equation that gives the dollar value V of the product in terms of the year t. (Let $t = 9$ represent 2009.)**

	2009 Value	*Rate*
89.	\$2540	\$125 increase per year
90.	\$156	\$4.50 increase per year
91.	\$20,400	\$2000 decrease per year
92.	\$245,000	\$5600 decrease per year

93. Accounting A school district purchases a high-volume printer, copier, and scanner for \$25,000. After 10 years, the equipment will have to be replaced. Its value at that time is expected to be \$2000.

(a) Write a linear equation giving the value V of the equipment for each year t during its 10 years of use.

(b) Use a graphing utility to graph the linear equation representing the depreciation of the equipment, and use the value or trace feature to complete the table. Verify your answers algebraically by using the equation you found in part (a).

t	0	1	2	3	4	5	6	7	8	9	10
V											

94. Meterology Recall that water freezes at 0°C (32°F) and boils at 100°C (212°F).

(a) Find an equation of the line that shows the relationship between the temperature in degrees Celsius C and degrees Fahrenheit F.

(b) Use the result of part (a) to complete the table.

C		$-10°$	$10°$			$177°$
F	$0°$			$68°$	$90°$	

95. Business A contractor purchases a bulldozer for \$36,500. The bulldozer requires an average expenditure of \$9.25 per hour for fuel and maintenance, and the operator is paid \$18.50 per hour.

(a) Write a linear equation giving the total cost C of operating the bulldozer for t hours. (Include the purchase cost of the bulldozer.)

(b) Assuming that customers are charged \$65 per hour of bulldozer use, write an equation for the revenue R derived from t hours of use.

(c) Use the profit formula ($P = R - C$) to write an equation for the profit gained from t hours of use.

(d) Use the result of part (c) to find the break-even point (the number of hours the bulldozer must be used to gain a profit of 0 dollars).

96. Real Estate A real estate office handles an apartment complex with 50 units. When the rent per unit is \$580 per month, all 50 units are occupied. However, when the rent is \$625 per month, the average number of occupied units drops to 47. Assume that the relationship between the monthly rent p and the demand x is linear.

(a) Write the equation of the line giving the demand x in terms of the rent p.

(b) Use a graphing utility to graph the demand equation and use the *trace* feature to estimate the number of units occupied when the rent is \$655. Verify your answer algebraically.

(c) Use the demand equation to predict the number of units occupied when the rent is lowered to \$595. Verify your answer graphically.

97. ***Why you should learn it*** *(p. 3)* In 1990, Penn State University had an enrollment of 75,365 students. By 2009, the enrollment had increased to 87,163. (Source: Penn State Fact Book)

(a) What was the average annual change in enrollment from 1990 to 2009?

(b) Use the average annual change in enrollment to estimate the enrollments in 1995, 2000, and 2005.

(c) Write the equation of a line that represents the given data. What is its slope? Interpret the slope in the context of the problem.

98. Writing Using the results of Exercise 97, write a short paragraph discussing the concepts of *slope* and *average rate of change*.

Conclusions

True or False? **In Exercises 99 and 100, determine whether the statement is true or false. Justify your answer.**

99. The line through $(-8, 2)$ and $(-1, 4)$ and the line through $(0, -4)$ and $(-7, 7)$ are parallel.

100. If the points $(10, -3)$ and $(2, -9)$ lie on the same line, then the point $\left(-12, -\frac{37}{2}\right)$ also lies on that line.

Exploration **In Exercises 101–104, use a graphing utility to graph the equation of the line in the form**

$$\frac{x}{a} + \frac{y}{b} = 1, \quad a \neq 0, b \neq 0.$$

Use the graphs to make a conjecture about what a and b represent. Verify your conjecture.

101. $\frac{x}{5} + \frac{y}{-3} = 1$

102. $\frac{x}{-6} + \frac{y}{2} = 1$

103. $\frac{x}{4} + \frac{y}{-\frac{2}{3}} = 1$

104. $\frac{x}{\frac{1}{2}} + \frac{y}{5} = 1$

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Using Intercepts **In Exercises 105–108, use the results of Exercises 101–104 to write an equation of the line that passes through the points.**

105. x-intercept: $(2, 0)$
y-intercept: $(0, 3)$

106. x-intercept: $(-5, 0)$
y-intercept: $(0, -4)$

107. x-intercept: $\left(-\frac{1}{6}, 0\right)$
y-intercept: $\left(0, -\frac{2}{3}\right)$

108. x-intercept: $\left(\frac{3}{4}, 0\right)$
y-intercept: $\left(0, \frac{4}{5}\right)$

Think About It **In Exercises 109 and 110, determine which equation(s) may be represented by the graphs shown. (There may be more than one correct answer.)**

109.

(a) $2x - y = -10$
(b) $2x + y = 10$
(c) $x - 2y = 10$
(d) $x + 2y = 10$

110.

(a) $2x + y = 5$
(b) $2x + y = -5$
(c) $x - 2y = 5$
(d) $x - 2y = -5$

Think About It **In Exercises 111 and 112, determine which pair of equations may be represented by the graphs shown.**

111.

(a) $2x - y = 5$
$2x - y = 1$
(b) $2x + y = -5$
$2x + y = 1$
(c) $2x - y = -5$
$2x - y = 1$
(d) $x - 2y = -5$
$x - 2y = -1$

112.

(a) $2x - y = 2$
$x + 2y = 12$
(b) $x - y = 1$
$x + y = 6$
(c) $2x + y = 2$
$x - 2y = 12$
(d) $x - 2y = 2$
$x + 2y = 12$

113. **Think About It** Does every line have both an x-intercept and a y-intercept? Explain.

114. **Think About It** Can every line be written in slope-intercept form? Explain.

115. **Think About It** Does every line have an infinite number of lines that are parallel to it? Explain.

116. **CAPSTONE** Match the description with its graph. Determine the slope of each graph and how it is interpreted in the given context. [The graphs are labeled (i), (ii), (iii), and (iv).]

(a) You are paying \$10 per week to repay a \$100 loan.
(b) An employee is paid \$12.50 per hour plus \$1.50 for each unit produced per hour.
(c) A sales representative receives \$30 per day for food plus \$0.35 for each mile traveled.
(d) A computer that was purchased for \$600 depreciates \$100 per year.

Cumulative Mixed Review

Identifying Polynomials **In Exercises 117–122, determine whether the expression is a polynomial. If it is, write the polynomial in standard form.**

117. $x + 20$

118. $3x - 10x^2 + 1$

119. $4x^2 + x^{-1} - 3$

120. $2x^2 - 2x^4 - x^3 + 2$

121. $\dfrac{x^2 + 3x + 4}{x^2 - 9}$

122. $\sqrt{x^2 + 7x + 6}$

Factoring Trinomials **In Exercises 123–126, factor the trinomial.**

123. $x^2 - 6x - 27$

124. $x^2 - 11x + 28$

125. $2x^2 + 11x - 40$

126. $3x^2 - 16x + 5$

127. ***Make a Decision*** To work an extended application analyzing the numbers of bachelor's degrees earned by women in the United States from 1985 through 2007, visit this textbook's *Companion Website*. (Source: National Center for Education Statistics)

The *Make a Decision* exercise indicates a multipart exercise using large data sets. Go to this textbook's *Companion Website* to view these exercises.

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1.2 Functions

Introduction to Functions

Many everyday phenomena involve two quantities that are related to each other by some rule of correspondence. The mathematical term for such a rule of correspondence is a **relation.** Here are two examples.

1. The simple interest I earned on an investment of \$1000 for 1 year is related to the annual interest rate r by the formula $I = 1000r$.
2. The area A of a circle is related to its radius r by the formula $A = \pi r^2$.

Not all relations have simple mathematical formulas. For instance, people commonly match up NFL starting quarterbacks with touchdown passes, and time of day with temperature. In each of these cases, there is some relation that matches each item from one set with exactly one item from a different set. Such a relation is called a **function.**

Definition of a Function

A **function** f from a set A to a set B is a relation that assigns to each element x in the set A exactly one element y in the set B. The set A is the **domain** (or set of inputs) of the function f, and the set B contains the **range** (or set of outputs).

To help understand this definition, look at the function that relates the time of day to the temperature in Figure 1.12.

Figure 1.12

This function can be represented by the ordered pairs

$$\{(1, 9°), (2, 13°), (3, 15°), (4, 15°), (5, 12°), (6, 10°)\}.$$

In each ordered pair, the first coordinate (x-value) is the **input** and the second coordinate (y-value) is the **output.**

Characteristics of a Function from Set A to Set B

1. Each element of A must be matched with an element of B.
2. Some elements of B may not be matched with any element of A.
3. Two or more elements of A may be matched with the same element of B.
4. An element of A (the domain) cannot be matched with two different elements of B.

What you should learn

- Decide whether a relation between two variables represents a function.
- Use function notation and evaluate functions.
- Find the domains of functions.
- Use functions to model and solve real-life problems.
- Evaluate difference quotients.

Why you should learn it

Many natural phenomena can be modeled by functions, such as the force of water against the face of a dam, explored in Exercise 78 on page 27.

Study Tip

Be sure you see that the *range* of a function is not the same as the use of *range* relating to the viewing window of a graphing utility.

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To determine whether or not a relation is a function, you must decide whether each input value is matched with exactly one output value. When any input value is matched with two or more output values, the relation is not a function.

Example 1 Testing for Functions

Decide whether the relation represents y as a function of x.

a.

Input, x	2	2	3	4	5
Output, y	11	10	8	5	1

b.

Figure 1.13

Solution

a. This table *does not* describe y as a function of x. The input value 2 is matched with two different y-values.

b. The graph in Figure 1.13 *does* describe y as a function of x. Each input value is matched with exactly one output value.

CHECKPOINT Now try Exercise 11.

In algebra, it is common to represent functions by equations or formulas involving two variables. For instance, the equation $y = x^2$ represents the variable y as a function of the variable x. In this equation, x is the **independent variable** and y is the **dependent variable.** The domain of the function is the set of all values taken on by the independent variable x, and the range of the function is the set of all values taken on by the dependent variable y.

Example 2 Testing for Functions Represented Algebraically

Determine whether the equation represents y as a function of x.

a. $x^2 + y = 1$ **b.** $-x + y^2 = 1$

Solution

To determine whether y is a function of x, try to solve for y in terms of x.

a. $x^2 + y = 1$ Write original equation.

$y = 1 - x^2$ Solve for y.

Each value of x corresponds to exactly one value of y. So, y is a function of x.

b. $-x + y^2 = 1$ Write original equation.

$y^2 = 1 + x$ Add x to each side.

$y = \pm\sqrt{1 + x}$ Solve for y.

The $\pm$ indicates that for a given value of x there correspond two values of y. For instance, when $x = 3$, $y = 2$ or $y = -2$. So, y is not a function of x.

CHECKPOINT Now try Exercise 23.

Explore the Concept

Use a graphing utility to graph $x^2 + y = 1$. Then use the graph to write a convincing argument that each x-value corresponds to at most one y-value.

Use a graphing utility to graph $-x + y^2 = 1$. (*Hint:* You will need to use two equations.) Does the graph represent y as a function of x? Explain.

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Function Notation

When an equation is used to represent a function, it is convenient to name the function so that it can be referenced easily. For example, you know that the equation $y = 1 - x^2$ describes y as a function of x. Suppose you give this function the name "f." Then you can use the following **function notation.**

Input	*Output*	*Equation*
x	$f(x)$	$f(x) = 1 - x^2$

The symbol $f(x)$ is read as the *value of f at x* or simply *f of x*. The symbol $f(x)$ corresponds to the y-value for a given x. So, you can write $y = f(x)$. Keep in mind that f is the *name* of the function, whereas $f(x)$ is the *output value* of the function at the *input value* x. In function notation, the *input* is the independent variable and the *output* is the dependent variable. For instance, the function $f(x) = 3 - 2x$ has *function values* denoted by $f(-1)$, $f(0)$, and so on. To find these values, substitute the specified input values into the given equation.

For $x = -1$, $\quad f(-1) = 3 - 2(-1) = 3 + 2 = 5.$

For $x = 0$, $\quad f(0) = 3 - 2(0) = 3 - 0 = 3.$

Although f is often used as a convenient function name and x is often used as the independent variable, you can use other letters. For instance,

$$f(x) = x^2 - 4x + 7, \quad f(t) = t^2 - 4t + 7, \quad \text{and} \quad g(s) = s^2 - 4s + 7$$

all define the same function. In fact, the role of the independent variable is that of a "placeholder." Consequently, the function could be written as

$$f(\quad) = (\quad)^2 - 4(\quad) + 7.$$

Example 3 Evaluating a Function

Let $g(x) = -x^2 + 4x + 1$. Find each value of the function.

a. $g(2)$

b. $g(t)$

c. $g(x + 2)$

Solution

a. Replacing x with 2 in $g(x) = -x^2 + 4x + 1$ yields the following.

$$g(2) = -(2)^2 + 4(2) + 1 = -4 + 8 + 1 = 5$$

b. Replacing x with t yields the following.

$$g(t) = -(t)^2 + 4(t) + 1 = -t^2 + 4t + 1$$

c. Replacing x with $x + 2$ yields the following.

$$\begin{aligned} g(x + 2) &= -(x + 2)^2 + 4(x + 2) + 1 && \text{Substitute } x + 2 \text{ for } x. \\ &= -(x^2 + 4x + 4) + 4x + 8 + 1 && \text{Multiply.} \\ &= -x^2 - 4x - 4 + 4x + 8 + 1 && \text{Distributive Property} \\ &= -x^2 + 5 && \text{Simplify.} \end{aligned}$$

CHECKPOINT Now try Exercise 31.

In Example 3, note that $g(x + 2)$ is not equal to $g(x) + g(2)$. In general,

$$g(u + v) \neq g(u) + g(v).$$

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Library of Parent Functions: Absolute Value Function

The *parent absolute value function* given by $f(x) = |x|$ can be written as a piecewise-defined function. The basic characteristics of the parent absolute value function are summarized below and on the inside cover of this text.

Graph of $f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

Domain: $(-\infty, \infty)$
Range: $[0, \infty)$
Intercept: $(0, 0)$
Decreasing on $(-\infty, 0)$
Increasing on $(0, \infty)$

A function defined by two or more equations over a specified domain is called a **piecewise-defined function.**

Example 4 A Piecewise-Defined Function

Evaluate the function when $x = -1$ and $x = 0$.

$$f(x) = \begin{cases} x^2 + 1, & x < 0 \\ x - 1, & x \geq 0 \end{cases}$$

Solution

Because $x = -1$ is less than 0, use $f(x) = x^2 + 1$ to obtain

$f(-1) = (-1)^2 + 1$ Substitute -1 for x.

$= 2.$ Simplify.

For $x = 0$, use $f(x) = x - 1$ to obtain

$f(0) = 0 - 1$ Substitute 0 for x.

$= -1.$ Simplify.

The graph of f is shown in Figure 1.14.

Figure 1.14

CHECKPOINT Now try Exercise 39.

Technology Tip

Most graphing utilities can graph piecewise-defined functions. For instructions on how to enter a piecewise-defined function into your graphing utility, consult your user's manual. You may find it helpful to set your graphing utility to *dot mode* before graphing such functions.

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The Domain of a Function

The domain of a function can be described explicitly or it can be *implied* by the expression used to define the function. The **implied domain** is the set of all real numbers for which the expression is defined. For instance, the function

$$f(x) = \frac{1}{x^2 - 4}$$

Domain excludes x-values that result in division by zero.

has an implied domain that consists of all real numbers x other than $x = \pm 2$. These two values are excluded from the domain because division by zero is undefined. Another common type of implied domain is that used to avoid even roots of negative numbers. For example, the function

$$f(x) = \sqrt{x}$$

Domain excludes x-values that result in even roots of negative numbers.

is defined only for $x \geq 0$. So, its implied domain is the interval $[0, \infty)$. In general, the domain of a function *excludes* values that would cause division by zero *or* result in the even root of a negative number.

Explore the Concept

Use a graphing utility to graph $y = \sqrt{4 - x^2}$. What is the domain of this function? Then graph $y = \sqrt{x^2 - 4}$. What is the domain of this function? Do the domains of these two functions overlap? If so, for what values?

Library of Parent Functions: Square Root Function

Radical functions arise from the use of rational exponents. The most common radical function is the *parent square root* function given by $f(x) = \sqrt{x}$. The basic characteristics of the parent square root function are summarized below and on the inside cover of this text.

Graph of $f(x) = \sqrt{x}$

Domain: $[0, \infty)$
Range: $[0, \infty)$
Intercept: $(0, 0)$
Increasing on $(0, \infty)$

Study Tip

Because the square root function is not defined for $x < 0$, you must be careful when analyzing the domains of complicated functions involving the square root symbol.

Example 5 Finding the Domain of a Function

Find the domain of each function.

a. $f: \{(-3, 0), (-1, 4), (0, 2), (2, 2), (4, -1)\}$

b. $g(x) = -3x^2 + 4x + 5$

c. $h(x) = \dfrac{1}{x + 5}$

Solution

a. The domain of f consists of all first coordinates in the set of ordered pairs.

$$\text{Domain} = \{-3, -1, 0, 2, 4\}$$

b. The domain of g is the set of all *real* numbers.

c. Excluding x-values that yield zero in the denominator, the domain of h is the set of all real numbers x except $x = -5$.

CHECKPOINT Now try Exercise 55.

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Example 6 Finding the Domain of a Function

Find the domain of each function.

a. Volume of a sphere: $V = \frac{4}{3}\pi r^3$ **b.** $k(x) = \sqrt{4 - 3x}$

Solution

a. Because this function represents the volume of a sphere, the values of the radius r must be positive (see Figure 1.15). So, the domain is the set of all real numbers r such that $r > 0$.

Figure 1.15

b. This function is defined only for x-values for which $4 - 3x \geq 0$. By solving this inequality, you will find that the domain of k is all real numbers that are less than or equal to $\frac{4}{3}$.

CHECKPOINT Now try Exercise 61.

In Example 6(a), note that the *domain of a function may be implied by the physical context*. For instance, from the equation $V = \frac{4}{3}\pi r^3$ you would have no reason to restrict r to positive values, but the physical context implies that a sphere cannot have a negative or zero radius.

For some functions, it may be easier to find the domain and range of the function by examining its graph.

Example 7 Finding the Domain and Range of a Function

Use a graphing utility to find the domain and range of the function $f(x) = \sqrt{9 - x^2}$.

Solution

Graph the function as $y = \sqrt{9 - x^2}$, as shown in Figure 1.16. Using the *trace* feature of a graphing utility, you can determine that the x-values extend from -3 to 3 and the y-values extend from 0 to 3. So, the domain of the function f is all real numbers such that

$$-3 \leq x \leq 3 \qquad \text{Domain of } f$$

and the range of f is all real numbers such that

$$0 \leq y \leq 3. \qquad \text{Range of } f$$

Figure 1.16

CHECKPOINT Now try Exercise 65.

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Applications

Example 8 Construction Employees

The number N (in millions) of employees in the construction industry in the United States increased in a linear pattern from 2003 through 2006 (see Figure 1.17). In 2007, the number dropped, then decreased through 2008 in a different linear pattern. These two patterns can be approximated by the function

$$N(t) = \begin{cases} 0.32t + 5.7, & 3 \le t \le 6 \\ -0.42t + 10.5, & 7 \le t \le 8 \end{cases}$$

where t represents the year, with $t = 3$ corresponding to 2003. Use this function to approximate the number of employees for each year from 2003 to 2008.
(Source: U.S. Bureau of Labor Statistics)

Figure 1.17

Solution

From 2003 to 2006, use $N(t) = 0.32t + 5.7$.

$$\underbrace{6.66}_{2003},\quad \underbrace{6.98}_{2004},\quad \underbrace{7.3}_{2005},\quad \underbrace{7.62}_{2006}$$

From 2007 to 2008, use $N(t) = -0.42t + 10.5$.

$$\underbrace{7.56}_{2007},\quad \underbrace{7.14}_{2008}$$

CHECKPOINT Now try Exercise 77.

Example 9 The Path of a Baseball

A baseball is hit at a point 3 feet above the ground at a velocity of 100 feet per second and an angle of 45°. The path of the baseball is given by the function

$$f(x) = -0.0032x^2 + x + 3$$

where $f(x)$ is the height of the baseball (in feet) and x is the horizontal distance from home plate (in feet). Will the baseball clear a 10-foot fence located 300 feet from home plate?

Algebraic Solution

The height of the baseball is a function of the horizontal distance from home plate. When $x = 300$, you can find the height of the baseball as follows.

$f(x) = -0.0032x^2 + x + 3$	Write original function.
$f(300) = -0.0032(300)^2 + 300 + 3$	Substitute 300 for x.
$= 15$	Simplify.

When $x = 300$, the height of the baseball is 15 feet. So, the baseball will clear a 10-foot fence.

CHECKPOINT Now try Exercise 79.

Graphical Solution

When $x = 300$, $y = 15$. So, the ball will clear a 10-foot fence.

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Difference Quotients

One of the basic definitions in calculus employs the ratio

$$\frac{f(x+h)-f(x)}{h}, \quad h \neq 0.$$

This ratio is called a **difference quotient,** as illustrated in Example 10.

Example 10 Evaluating a Difference Quotient

For $f(x) = x^2 - 4x + 7$, find $\dfrac{f(x+h)-f(x)}{h}$.

Solution

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= \frac{[(x+h)^2 - 4(x+h) + 7] - (x^2 - 4x + 7)}{h} \\
&= \frac{x^2 + 2xh + h^2 - 4x - 4h + 7 - x^2 + 4x - 7}{h} \\
&= \frac{2xh + h^2 - 4h}{h} \\
&= \frac{h(2x + h - 4)}{h} \\
&= 2x + h - 4, \quad h \neq 0
\end{aligned}$$

CHECKPOINT Now try Exercise 83.

Study Tip

Notice in Example 10 that h cannot be zero in the original expression. Therefore, you must restrict the domain of the simplified expression by listing $h \neq 0$ so that the simplified expression is equivalent to the original expression.

Summary of Function Terminology

Function: A **function** is a relationship between two variables such that to each value of the independent variable there corresponds exactly one value of the dependent variable.

Function Notation: $y = f(x)$

f is the *name* of the function.
y is the **dependent variable,** or output value.
x is the **independent variable,** or input value.
$f(x)$ is the *value of the function at x.*

Domain: The **domain** of a function is the set of all values (inputs) of the independent variable for which the function is defined. If x is in the domain of f, then f is said to be *defined* at x. If x is not in the domain of f, then f is said to be *undefined* at x.

Range: The **range** of a function is the set of all values (outputs) assumed by the dependent variable (that is, the set of all function values).

Implied Domain: If f is defined by an algebraic expression and the domain is not specified, then the **implied domain** consists of all real numbers for which the expression is defined.

The symbol ∫ indicates an example or exercise that highlights algebraic techniques specifically used in calculus.

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1.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1 and 2, fill in the blanks.

1. A relation that assigns to each element x from a set of inputs, or _____ , exactly one element y in a set of outputs, or _____ , is called a _____ .
2. For an equation that represents y as a function of x, the _____ variable is the set of all x in the domain, and the _____ variable is the set of all y in the range.
3. Can the ordered pairs (3, 0) and (3, 5) represent a function?
4. To find $g(x + 1)$, what do you substitute for x in the function $g(x) = 3x - 2$?
5. Does the domain of the function $f(x) = \sqrt{1 + x}$ include $x = -2$?
6. Is the domain of a piecewise-defined function *implied* or *explicitly described*?

Procedures and Problem Solving

Testing for Functions **In Exercises 7–10, does the relation describe a function? Explain your reasoning.**

7. *Domain* *Range*

8. *Domain* *Range*

9. *Domain* *Range*

10. *Domain* (State) *Range* (Electoral votes 2000–2010)

Testing for Functions **In Exercises 11 and 12, decide whether the relation represents y as a function of x. Explain your reasoning.**

11.

Input, x	-3	-1	0	1	3
Output, y	-9	-1	0	1	9

12.

Input, x	0	1	2	1	0
Output, y	-4	-2	0	2	4

Testing for Functions **In Exercises 13 and 14, which sets of ordered pairs represent functions from A to B? Explain.**

13. $A = \{0, 1, 2, 3\}$ and $B = \{-2, -1, 0, 1, 2\}$
    (a) $\{(0, 1), (1, -2), (2, 0), (3, 2)\}$
    (b) $\{(0, -1), (2, 2), (1, -2), (3, 0), (1, 1)\}$
    (c) $\{(0, 0), (1, 0), (2, 0), (3, 0)\}$
14. $A = \{a, b, c\}$ and $B = \{0, 1, 2, 3\}$
    (a) $\{(a, 1), (c, 2), (c, 3), (b, 3)\}$
    (b) $\{(a, 1), (b, 2), (c, 3)\}$
    (c) $\{(1, a), (0, a), (2, c), (3, b)\}$

Pharmacology **In Exercises 15 and 16, use the graph, which shows the average prices of name brand and generic drug prescriptions in the United States.** (Source: National Association of Chain Drug Stores)

15. Is the average price of a name brand prescription a function of the year? Is the average price of a generic prescription a function of the year? Explain.
16. Let $b(t)$ and $g(t)$ represent the average prices of name brand and generic prescriptions, respectively, in year t. Find $b(2007)$ and $g(2000)$.

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Testing for Functions Represented Algebraically **In Exercises 17–28, determine whether the equation represents y as a function of x.**

17. $x^2 + y^2 = 4$

18. $x = y^2 + 1$

19. $y = \sqrt{x^2 - 1}$

20. $y = \sqrt{x + 5}$

21. $2x + 3y = 4$

22. $x = -y + 5$

23. $y^2 = x^2 - 1$

24. $x + y^2 = 3$

25. $y = |4 - x|$

26. $|y| = 4 - x$

27. $x = -7$

28. $y = 8$

Evaluating a Function **In Exercises 29–44, evaluate the function at each specified value of the independent variable and simplify.**

29. $f(t) = 3t + 1$

(a) $f(2)$ (b) $f(-4)$ (c) $f(t + 2)$

30. $g(y) = 7 - 3y$

(a) $g(0)$ (b) $g\left(\frac{7}{3}\right)$ (c) $g(s + 2)$

31. $h(t) = t^2 - 2t$

(a) $h(2)$ (b) $h(1.5)$ (c) $h(x + 2)$

32. $V(r) = \frac{4}{3}\pi r^3$

(a) $V(3)$ (b) $V\left(\frac{3}{2}\right)$ (c) $V(2r)$

33. $f(y) = 3 - \sqrt{y}$

(a) $f(4)$ (b) $f(0.25)$ (c) $f(4x^2)$

34. $f(x) = \sqrt{x + 8} + 2$

(a) $f(-4)$ (b) $f(8)$ (c) $f(x - 8)$

35. $q(x) = \dfrac{1}{x^2 - 9}$

(a) $q(-3)$ (b) $q(2)$ (c) $q(y + 3)$

36. $q(t) = \dfrac{2t^2 + 3}{t^2}$

(a) $q(2)$ (b) $q(0)$ (c) $q(-x)$

37. $f(x) = \dfrac{|x|}{x}$

(a) $f(9)$ (b) $f(-9)$ (c) $f(t)$

38. $f(x) = |x| + 4$

(a) $f(5)$ (b) $f(-5)$ (c) $f(t)$

39. $f(x) = \begin{cases} 2x + 1, & x < 0 \\ 2x + 2, & x \geq 0 \end{cases}$

(a) $f(-1)$ (b) $f(0)$ (c) $f(2)$

40. $f(x) = \begin{cases} 2x + 5, & x \leq 0 \\ 2 - x^2, & x > 0 \end{cases}$

(a) $f(-2)$ (b) $f(0)$ (c) $f(1)$

41. $f(x) = \begin{cases} x^2 + 2, & x \leq 1 \\ 2x^2 + 2, & x > 1 \end{cases}$

(a) $f(-2)$ (b) $f(1)$ (c) $f(2)$

42. $f(x) = \begin{cases} x^2 - 4, & x \leq 0 \\ 1 - 2x^2, & x > 0 \end{cases}$

(a) $f(-2)$ (b) $f(0)$ (c) $f(1)$

43. $f(x) = \begin{cases} x + 2, & x < 0 \\ 4, & 0 \leq x < 2 \\ x^2 + 1, & x \geq 2 \end{cases}$

(a) $f(-2)$ (b) $f(1)$ (c) $f(4)$

44. $f(x) = \begin{cases} 5 - 2x, & x < 0 \\ 5, & 0 \leq x < 1 \\ 4x + 1, & x \geq 1 \end{cases}$

(a) $f(-2)$ (b) $f\left(\frac{1}{2}\right)$ (c) $f(1)$

Evaluating a Function **In Exercises 45–48, assume that the domain of f is the set $A = \{-2, -1, 0, 1, 2\}$. Determine the set of ordered pairs representing the function f.**

45. $f(x) = x^2$

46. $f(x) = x^2 - 3$

47. $f(x) = |x| + 2$

48. $f(x) = |x + 1|$

Evaluating a Function **In Exercises 49 and 50, complete the table.**

49. $h(t) = \frac{1}{2}|t + 3|$

t	-5	-4	-3	-2	-1
$h(t)$					

50. $f(s) = \dfrac{|s - 2|}{s - 2}$

s	0	1	$\frac{3}{2}$	$\frac{5}{2}$	4
$f(s)$					

Finding the Inputs That Have Outputs of Zero **In Exercises 51–54, find all values of x such that $f(x) = 0$.**

51. $f(x) = 15 - 3x$

52. $f(x) = 5x + 1$

53. $f(x) = \dfrac{3x - 4}{5}$

54. $f(x) = \dfrac{2x - 3}{7}$

Finding the Domain of a Function **In Exercises 55–64, find the domain of the function.**

55. $f(x) = 5x^2 + 2x - 1$

56. $g(x) = 1 - 2x^2$

57. $h(t) = \dfrac{4}{t}$

58. $s(y) = \dfrac{3y}{y + 5}$

59. $f(x) = \sqrt[3]{x - 4}$

60. $f(x) = \sqrt[4]{x^2 + 3x}$

61. $g(x) = \dfrac{1}{x} - \dfrac{3}{x + 2}$

62. $h(x) = \dfrac{10}{x^2 - 2x}$

63. $g(y) = \dfrac{y + 2}{\sqrt{y - 10}}$

64. $f(x) = \dfrac{\sqrt{x + 6}}{6 + x}$

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Finding the Domain and Range of a Function **In Exercises 65–68, use a graphing utility to graph the function. Find the domain and range of the function.**

65. $f(x) = \sqrt{4 - x^2}$

66. $f(x) = \sqrt{x^2 + 1}$

67. $g(x) = |2x + 3|$

68. $g(x) = |x - 5|$

69. Geometry Write the area A of a circle as a function of its circumference C.

70. Geometry Write the area A of an equilateral triangle as a function of the length s of its sides.

71. Exploration An open box of maximum volume is to be made from a square piece of material, 24 centimeters on a side, by cutting equal squares from the corners and turning up the sides (see figure).

(a) The table shows the volume V (in cubic centimeters) of the box for various heights x (in centimeters). Use the table to estimate the maximum volume.

Height, x	Volume, V
1	484
2	800
3	972
4	1024
5	980
6	864

(b) Plot the points (x, V) from the table in part (a). Does the relation defined by the ordered pairs represent V as a function of x?

(c) If V is a function of x, write the function and determine its domain.

(d) Use a graphing utility to plot the points from the table in part (a) with the function from part (c). How closely does the function represent the data? Explain.

72. Geometry A right triangle is formed in the first quadrant by the x- and y-axes and a line through the point $(2, 1)$ (see figure). Write the area A of the triangle as a function of x, and determine the domain of the function.

73. Geometry A rectangle is bounded by the x-axis and the semicircle $y = \sqrt{36 - x^2}$ (see figure). Write the area A of the rectangle as a function of x, and determine the domain of the function.

74. Geometry A rectangular package to be sent by the U.S. Postal Service can have a maximum combined length and girth (perimeter of a cross section) of 108 inches (see figure).

(a) Write the volume V of the package as a function of x. What is the domain of the function?

(b) Use a graphing utility to graph the function. Be sure to use an appropriate viewing window.

(c) What dimensions will maximize the volume of the package? Explain.

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75. Business A company produces a handheld game console for which the variable cost is \$68.20 per unit and the fixed costs are \$248,000. The game console sells for \$98.98. Let x be the number of units produced and sold.

(a) The total cost for a business is the sum of the variable cost and the fixed costs. Write the total cost C as a function of the number of units produced.

(b) Write the revenue R as a function of the number of units sold.

(c) Write the profit P as a function of the number of units sold. (*Note:* $P = R - C$.)

76. MODELING DATA

The table shows the revenue y (in thousands of dollars) of a landscaping business for each month of 2010, with $x = 1$ representing January.

Month, x	Revenue, y
1	5.2
2	5.6
3	6.6
4	8.3
5	11.5
6	15.8
7	12.8
8	10.1
9	8.6
10	6.9
11	4.5
12	2.7

The mathematical model below represents the data.

$$f(x) = \begin{cases} -1.97x + 26.3 \\ 0.505x^2 - 1.47x + 6.3 \end{cases}$$

(a) Identify the independent and dependent variables and explain what they represent in the context of the problem.

(b) What is the domain of each part of the piecewise-defined function? Explain your reasoning.

(c) Use the mathematical model to find $f(5)$. Interpret your result in the context of the problem.

(d) Use the mathematical model to find $f(11)$. Interpret your result in the context of the problem.

(e) How do the values obtained from the models in parts (c) and (d) compare with the actual data values?

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✓ **77. Civil Engineering** The numbers n (in billions) of miles traveled by vans, pickup trucks, and sport utility vehicles in the United States from 1990 through 2007 can be approximated by the model

$$n(t) = \begin{cases} -5.24t^2 + 69.5t + 581, & 0 \le t \le 6 \\ 25.7t + 664, & 6 < t \le 17 \end{cases}$$

where t represents the year, with $t = 0$ corresponding to 1990. The actual numbers are shown in the bar graph. (Source: U.S. Federal Highway Administration)

(a) Identify the independent and dependent variables and explain what they represent in the context of the problem.

(b) Use the *table* feature of a graphing utility to approximate the number of miles traveled by vans, pickup trucks, and sport utility vehicles each year from 1990 through 2007.

(c) Compare the values in part (b) with the actual values shown in the bar graph. How well does the model fit the data?

78. ***Why you should learn it*** *(p. 16)* The force F (in tons) of water against the face of a dam is estimated by the function

$$F(y) = 149.76\sqrt{10}y^{5/2}$$

where y is the depth of the water (in feet).

(a) Complete the table. What can you conclude from it?

y	5	10	20	30	40
$F(y)$					

(b) Use a graphing utility to graph the function. Describe your viewing window.

(c) Use the table to approximate the depth at which the force against the dam is 1,000,000 tons. Verify your answer graphically. How could you find a better estimate?

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79. Projectile Motion A second baseman throws a baseball toward the first baseman 60 feet away. The path of the ball is given by

$$y = -0.004x^2 + 0.3x + 6$$

where y is the height (in feet) and x is the horizontal distance (in feet) from the second baseman. The first baseman can reach 8 feet high. Can the first baseman catch the ball without jumping? Explain.

80. Business The graph shows the sales (in millions of dollars) of Peet's Coffee & Tea from 2000 through 2008. Let $f(x)$ represent the sales in year x. (Source: Peet's Coffee & Tea, Inc.)

(a) Find $\dfrac{f(2008) - f(2000)}{2008 - 2000}$ and interpret the result in the context of the problem.

(b) An approximate model for the function is

$$S(t) = 2.484t^2 + 5.71t + 84.0, \quad 0 \le t \le 8$$

where S is the sales (in millions of dollars) and $t = 0$ represents 2000. Complete the table and compare the results with the data in the graph.

t	0	1	2	3	4	5	6	7	8
$S(t)$									

Evaluating a Difference Quotient **In Exercises 81–86, find the difference quotient and simplify your answer.**

81. $f(x) = 2x, \quad \dfrac{f(x + c) - f(x)}{c}, \quad c \neq 0$

82. $g(x) = 3x - 1, \quad \dfrac{g(x + h) - g(x)}{h}, \quad h \neq 0$

83. $f(x) = x^2 - x + 1, \quad \dfrac{f(2 + h) - f(2)}{h}, \quad h \neq 0$

84. $f(x) = x^3 + x, \quad \dfrac{f(x + h) - f(x)}{h}, \quad h \neq 0$

85. $f(t) = \dfrac{1}{t}, \quad \dfrac{f(t) - f(1)}{t - 1}, \quad t \neq 1$

86. $f(x) = \dfrac{4}{x + 1}, \quad \dfrac{f(x) - f(7)}{x - 7}, \quad x \neq 7$

Conclusions

True or False? In Exercises 87 and 88, determine whether the statement is true or false. Justify your answer.

87. The domain of the function $f(x) = x^4 - 1$ is $(-\infty, \infty)$, and the range of $f(x)$ is $(0, \infty)$.

88. The set of ordered pairs $\{(-8, -2), (-6, 0), (-4, 0), (-2, 2), (0, 4), (2, -2)\}$ represents a function.

Think About It **In Exercises 89 and 90, write a square root function for the graph shown. Then, identify the domain and range of the function.**

89.

90.

91. Think About It Given $f(x) = x^2$, is f the independent variable? Why or why not?

92. CAPSTONE

(a) Describe any differences between a *relation* and a *function*.

(b) In your own words, explain the meanings of *domain* and *range*.

Cumulative Mixed Review

Operations with Rational Expressions **In Exercises 93–96, perform the operation and simplify.**

93. $12 - \dfrac{4}{x + 2}$

94. $\dfrac{3}{x^2 + x - 20} + \dfrac{x}{x^2 + 4x - 5}$

95. $\dfrac{2x^3 + 11x^2 - 6x}{5x} \cdot \dfrac{x + 10}{2x^2 + 5x - 3}$

96. $\dfrac{x + 7}{2(x - 9)} \div \dfrac{x - 7}{2(x - 9)}$

The symbol ∫ indicates an example or exercise that highlights algebraic techniques specifically used in calculus.

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1.3 Graphs of Functions

The Graph of a Function

In Section 1.2, some functions were represented graphically by points on a graph in a coordinate plane in which the input values are represented by the horizontal axis and the output values are represented by the vertical axis. The **graph of a function** f is the collection of ordered pairs $(x, f(x))$ such that x is in the domain of f. As you study this section, remember the geometric interpretations of x and $f(x)$.

$$x = \text{the directed distance from the } y\text{-axis}$$

$$f(x) = \text{the directed distance from the } x\text{-axis}$$

Example 1 shows how to use the graph of a function to find the domain and range of the function.

Example 1 Finding the Domain and Range of a Function

Use the graph of the function f shown in Figure 1.18 to find (a) the domain of f, (b) the function values $f(-1)$ and $f(2)$, and (c) the range of f.

Figure 1.18

Solution

a. The closed dot at $(-1, -5)$ indicates that $x = -1$ is in the domain of f, whereas the open dot at $(4, 0)$ indicates that $x = 4$ is not in the domain. So, the domain of f is all x in the interval $[-1, 4)$.

b. Because $(-1, -5)$ is a point on the graph of f, it follows that

$$f(-1) = -5.$$

Similarly, because $(2, 4)$ is a point on the graph of f, it follows that

$$f(2) = 4.$$

c. Because the graph does not extend below $f(-1) = -5$ or above $f(2) = 4$, the range of f is the interval $[-5, 4]$.

CHECKPOINT Now try Exercise 9.

The use of dots (open or closed) at the extreme left and right points of a graph indicates that the graph does not extend beyond these points. When no such dots are shown, assume that the graph extends beyond these points.

What you should learn

- Find the domains and ranges of functions and use the Vertical Line Test for functions.
- Determine intervals on which functions are increasing, decreasing, or constant.
- Determine relative maximum and relative minimum values of functions.
- Identify and graph step functions and other piecewise-defined functions.
- Identify even and odd functions.

Why you should learn it

Graphs of functions provide visual relationships between two variables. For example, in Exercise 92 on page 39, you will use the graph of a step function to model the cost of sending a package.

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Example 2 Finding the Domain and Range of a Function

Find the domain and range of

$$f(x) = \sqrt{x - 4}.$$

Algebraic Solution

Because the expression under a radical cannot be negative, the domain of $f(x) = \sqrt{x - 4}$ is the set of all real numbers such that

$$x - 4 \geq 0.$$

Solve this linear inequality for x as follows. (For help with solving linear inequalities, see Appendix E at this textbook's *Companion Website*.)

$x - 4 \geq 0$ Write original inequality.

$x \geq 4$ Add 4 to each side.

So, the domain is the set of all real numbers greater than or equal to 4. Because the value of a radical expression is never negative, the range of $f(x) = \sqrt{x - 4}$ is the set of all nonnegative real numbers.

CHECKPOINT Now try Exercise 13.

Graphical Solution

The x-coordinates of points on the graph extend from 4 to the right. So, the domain is the set of all real numbers greater than or equal to 4.

The y-coordinates of points on the graph extend from 0 upwards. So, the range is the set of all nonnegative real numbers.

By the definition of a function, at most one y-value corresponds to a given x-value. It follows, then, that a vertical line can intersect the graph of a function at most once. This leads to the **Vertical Line Test** for functions.

Vertical Line Test for Functions

A set of points in a coordinate plane is the graph of y as a function of x if and only if no vertical line intersects the graph at more than one point.

Example 3 Vertical Line Test for Functions

Use the Vertical Line Test to decide whether the graphs in Figure 1.19 represent y as a function of x.

(a) **(b)**

Figure 1.19

Solution

a. This is *not* a graph of y as a function of x because you can find a vertical line that intersects the graph twice.

b. This *is* a graph of y as a function of x because every vertical line intersects the graph at most once.

CHECKPOINT Now try Exercise 21.

Technology Tip

Most graphing utilities are designed to graph functions of x more easily than other types of equations. For instance, the graph shown in Figure 1.19(a) represents the equation $x - (y - 1)^2 = 0$. To use a graphing utility to duplicate this graph you must first solve the equation for y to obtain $y = 1 \pm \sqrt{x}$, and then graph the two equations $y_1 = 1 + \sqrt{x}$ and $y_2 = 1 - \sqrt{x}$ in the same viewing window.

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Increasing and Decreasing Functions

The more you know about the graph of a function, the more you know about the function itself. Consider the graph shown in Figure 1.20. Moving from *left to right*, this graph falls from $x = -2$ to $x = 0$, is constant from $x = 0$ to $x = 2$, and rises from $x = 2$ to $x = 4$.

Figure 1.20

Increasing, Decreasing, and Constant Functions

A function f is **increasing** on an interval when, for any x_1 and x_2 in the interval,

$$x_1 < x_2 \text{ implies } f(x_1) < f(x_2).$$

A function f is **decreasing** on an interval when, for any x_1 and x_2 in the interval,

$$x_1 < x_2 \text{ implies } f(x_1) > f(x_2).$$

A function f is **constant** on an interval when, for any x_1 and x_2 in the interval,

$$f(x_1) = f(x_2).$$

Example 4 Increasing and Decreasing Functions

In Figure 1.21, determine the open intervals on which each function is increasing, decreasing, or constant.

(a) **(b)** **(c)**

Figure 1.21

Solution

a. Although it might appear that there is an interval in which this function is constant, you can see that if $x_1 < x_2$, then $(x_1)^3 < (x_2)^3$, which implies that $f(x_1) < f(x_2)$. So, the function is increasing over the entire real line.

b. This function is increasing on the interval $(-\infty, -1)$, decreasing on the interval $(-1, 1)$, and increasing on the interval $(1, \infty)$.

c. This function is increasing on the interval $(-\infty, 0)$, constant on the interval $(0, 2)$, and decreasing on the interval $(2, \infty)$.

CHECKPOINT Now try Exercise 25.

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Relative Minimum and Maximum Values

The points at which a function changes its increasing, decreasing, or constant behavior are helpful in determining the relative maximum or relative minimum values of the function.

Definition of Relative Minimum and Relative Maximum

A function value $f(a)$ is called a **relative minimum** of f when there exists an interval (x_1, x_2) that contains a such that

$$x_1 < x < x_2 \quad \text{implies} \quad f(a) \le f(x).$$

A function value $f(a)$ is called a **relative maximum** of f when there exists an interval (x_1, x_2) that contains a such that

$$x_1 < x < x_2 \quad \text{implies} \quad f(a) \ge f(x).$$

Figure 1.22

Figure 1.22 shows several different examples of relative minima and relative maxima. In Section 2.1, you will study a technique for finding the *exact points* at which a second-degree polynomial function has a relative minimum or relative maximum. For the time being, however, you can use a graphing utility to find reasonable approximations of these points.

Example 5 Approximating a Relative Minimum

Use a graphing utility to approximate the relative minimum of the function given by $f(x) = 3x^2 - 4x - 2$.

Solution

The graph of f is shown in Figure 1.23. By using the *zoom* and *trace* features of a graphing utility, you can estimate that the function has a relative minimum at the point

$$(0.67, -3.33).$$ See Figure 1.24.

Later, in Section 2.1, you will be able to determine that the exact point at which the relative minimum occurs is $\left(\frac{2}{3}, -\frac{10}{3}\right)$.

Figure 1.23

Figure 1.24

CHECKPOINT Now try Exercise 35.

Technology Tip

When you use a graphing utility to estimate the x- and y-values of a relative minimum or relative maximum, the *zoom* feature will often produce graphs that are nearly flat, as shown in Figure 1.24. To overcome this problem, you can manually change the vertical setting of the viewing window. The graph will stretch vertically when the values of Ymin and Ymax are closer together.

Technology Tip

Some graphing utilities have built-in programs that will find minimum or maximum values. These features are demonstrated in Example 6.

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Example 6 Approximating Relative Minima and Maxima

Use a graphing utility to approximate the relative minimum and relative maximum of the function given by $f(x) = -x^3 + x$.

Solution

By using the *minimum* and *maximum* features of the graphing utility, you can estimate that the function has a relative minimum at the point

$(-0.58, -0.38)$ See Figure 1.25.

and a relative maximum at the point

$(0.58, 0.38)$. See Figure 1.26.

If you take a course in calculus, you will learn a technique for finding the exact points at which this function has a relative minimum and a relative maximum.

Figure 1.25

Figure 1.26

CHECKPOINT Now try Exercise 37.

Example 7 Temperature

During a 24-hour period, the temperature y (in degrees Fahrenheit) of a certain city can be approximated by the model

$$y = 0.026x^3 - 1.03x^2 + 10.2x + 34, \quad 0 \le x \le 24$$

where x represents the time of day, with $x = 0$ corresponding to 6 A.M. Approximate the maximum and minimum temperatures during this 24-hour period.

Solution

Using the *maximum* feature of a graphing utility, you can determine that the maximum temperature during the 24-hour period was approximately 64°F. This temperature occurred at about 12:36 P.M. ($x \approx 6.6$), as shown in Figure 1.27. Using the *minimum* feature, you can determine that the minimum temperature during the 24-hour period was approximately 34°F, which occurred at about 1:48 A.M. ($x \approx 19.8$), as shown in Figure 1.28.

Figure 1.27

Figure 1.28

CHECKPOINT Now try Exercise 95.

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Step Functions and Piecewise-Defined Functions

Library of Parent Functions: Greatest Integer Function

The *greatest integer function*, denoted by $[\![x]\!]$ and defined as the greatest integer less than or equal to x, has an infinite number of breaks or steps—one at each integer value in its domain. The basic characteristics of the greatest integer function are summarized below.

Graph of $f(x) = [\![x]\!]$

Domain: $(-\infty, \infty)$

Range: the set of integers

x-intercepts: in the interval $[0, 1)$

y-intercept: $(0, 0)$

Constant between each pair of consecutive integers

Jumps vertically one unit at each integer value

Technology Tip

Most graphing utilities display graphs in *connected mode*, which works well for graphs that do not have breaks. For graphs that do have breaks, such as the greatest integer function, it is better to use *dot mode*. Graph the greatest integer function [often called Int (x)] in *connected* and *dot modes*, and compare the two results.

Because of the vertical jumps described above, the greatest integer function is an example of a **step function** whose graph resembles a set of stairsteps. Some values of the greatest integer function are as follows.

$$[\![-1]\!] = (\text{greatest integer} \le -1) = -1$$

$$[\![-\tfrac{1}{2}]\!] = \left(\text{greatest integer} \le -\tfrac{1}{2}\right) = -1$$

$$[\![\tfrac{1}{10}]\!] = \left(\text{greatest integer} \le \tfrac{1}{10}\right) = 0$$

$$[\![1.5]\!] = (\text{greatest integer} \le 1.5) = 1$$

In Section 1.2, you learned that a piecewise-defined function is a function that is defined by two or more equations over a specified domain. To sketch the graph of a piecewise-defined function, you need to sketch the graph of each equation on the appropriate portion of the domain.

What's Wrong?

You use a graphing utility to graph

$$f(x) = \begin{cases} x^2 + 1, & x \le 0 \\ 4 - x, & x > 0 \end{cases}$$

by letting $y_1 = x^2 + 1$ and $y_2 = 4 - x$, as shown in the figure. You conclude that this is the graph of f. What's wrong?

Example 8 Sketching a Piecewise-Defined Function

Sketch the graph of

$$f(x) = \begin{cases} 2x + 3, & x \le 1 \\ -x + 4, & x > 1 \end{cases}$$

by hand.

Solution

This piecewise-defined function is composed of two linear functions. At and to the left of $x = 1$, the graph is the line given by

$$y = 2x + 3.$$

To the right of $x = 1$, the graph is the line given by

$$y = -x + 4$$

as shown in Figure 1.29. Notice that the point $(1, 5)$ is a solid dot and the point $(1, 3)$ is an open dot. This is because $f(1) = 5$.

Figure 1.29

CHECKPOINT Now try Exercise 55.

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Even and Odd Functions

A graph has *symmetry with respect to the y-axis* if whenever (x, y) is on the graph, then so is the point $(-x, y)$. A graph has *symmetry with respect to the origin* if whenever (x, y) is on the graph, then so is the point $(-x, -y)$. A graph has *symmetry with respect to the x-axis* if whenever (x, y) is on the graph, then so is the point $(x, -y)$. A function whose graph is symmetric with respect to the y-axis is an **even function.** A function whose graph is symmetric with respect to the origin is an **odd function.** A graph that is symmetric with respect to the x-axis is not the graph of a function (except for the graph of $y = 0$). These three types of symmetry are illustrated in Figure 1.30.

Symmetric to y-axis
Even function

Symmetric to origin
Odd function

Symmetric to x-axis
Not a function

Figure 1.30

Explore the Concept

Graph each function with a graphing utility. Determine whether the function is *even*, *odd*, or *neither*.

$f(x) = x^2 - x^4$

$g(x) = 2x^3 + 1$

$h(x) = x^5 - 2x^3 + x$

$j(x) = 2 - x^6 - x^8$

$k(x) = x^5 - 2x^4 + x - 2$

$p(x) = x^9 + 3x^5 - x^3 + x$

What do you notice about the equations of functions that are (a) odd and (b) even? Describe a way to identify a function as (c) odd, (d) even, or (e) neither odd nor even by inspecting the equation.

Example 9 Even and Odd Functions

Use the figure to determine whether the function is even, odd, or neither.

Solution

a. The graph is symmetric with respect to the y-axis. So, the function is even.

b. The graph is symmetric with respect to the origin. So, the function is odd.

c. The graph is neither symmetric with respect to the origin nor with respect to the y-axis. So, the function is neither even nor odd.

d. The graph is symmetric with respect to the y-axis. So, the function is even.

CHECKPOINT Now try Exercise 67.

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Test for Even and Odd Functions

A function f is **even** when, for each x in the domain of f, $f(-x) = f(x)$.
A function f is **odd** when, for each x in the domain of f, $f(-x) = -f(x)$.

Example 10 Even and Odd Functions

Determine whether each function is even, odd, or neither.

a. $g(x) = x^3 - x$
b. $h(x) = x^2 + 1$
c. $f(x) = x^3 - 1$

Algebraic Solution

a. This function is odd because

$$\begin{aligned} g(-x) &= (-x)^3 - (-x) \\ &= -x^3 + x \\ &= -(x^3 - x) \\ &= -g(x). \end{aligned}$$

b. This function is even because

$$\begin{aligned} h(-x) &= (-x)^2 + 1 \\ &= x^2 + 1 \\ &= h(x). \end{aligned}$$

c. Substituting $-x$ for x produces

$$\begin{aligned} f(-x) &= (-x)^3 - 1 \\ &= -x^3 - 1. \end{aligned}$$

Because

$$f(x) = x^3 - 1$$

and

$$-f(x) = -x^3 + 1$$

you can conclude that

$$f(-x) \neq f(x)$$

and

$$f(-x) \neq -f(x).$$

So, the function is neither even nor odd.

CHECKPOINT Now try Exercise 81.

Graphical Solution

a. In Figure 1.31, the graph is symmetric with respect to the origin. So, this function is odd.

Figure 1.31

b. In Figure 1.32, the graph is symmetric with respect to the y-axis. So, this function is even.

Figure 1.32

c. In Figure 1.33, the graph is neither symmetric with respect to the origin nor with respect to the y-axis. So, this function is neither even nor odd.

Figure 1.33

To help visualize symmetry with respect to the origin, place a pin at the origin of a graph and rotate the graph 180°. If the result after rotation coincides with the original graph, then the graph is symmetric with respect to the origin.

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1.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1 and 2, fill in the blank.

1. A function f is _____ on an interval when, for any x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) > f(x_2)$.
2. A function f is _____ when, for each x in the domain of f, $f(-x) = f(x)$.
3. The graph of a function f is the segment from $(1, 2)$ to $(4, 5)$, including the endpoints. What is the domain of f?
4. A vertical line intersects a graph twice. Does the graph represent a function?
5. Let f be a function such that $f(2) \geq f(x)$ for all values of x in the interval $(0, 3)$. Does $f(2)$ represent a relative minimum or a relative maximum?
6. Given $f(x) = [\![x]\!]$, in what interval does $f(x) = 5$?

Procedures and Problem Solving

Finding the Domain and Range of a Function **In Exercises 7–10, use the graph of the function to find the domain and range of f. Then find $f(0)$.**

7.

8.

✓ 9.

10.

Finding the Domain and Range of a Function **In Exercises 11–16, use a graphing utility to graph the function and estimate its domain and range. Then find the domain and range algebraically.**

11. $f(x) = 2x^2 + 3$
12. $f(x) = -x^2 - 1$
✓ 13. $f(x) = \sqrt{x - 1}$
14. $h(t) = \sqrt{4 - t^2}$
15. $f(x) = |x + 3|$
16. $f(x) = -\frac{1}{4}|x - 5|$

Analyzing a Graph **In Exercises 17 and 18, use the graph of the function to answer the questions.**

(a) Determine the domain of the function.

(b) Determine the range of the function.

(c) Find the value(s) of x for which $f(x) = 0$.

(d) What are the values of x from part (c) referred to graphically?

(e) Find $f(0)$, if possible.

(f) What is the value from part (e) referred to graphically?

(g) What is the value of f at $x = 1$? What are the coordinates of the point?

(h) What is the value of f at $x = -1$? What are the coordinates of the point?

(i) The coordinates of the point on the graph of f at which $x = -3$ can be labeled $(-3, f(-3))$, or $(-3,$).

17.

18.

Vertical Line Test for Functions **In Exercises 19–22, use the Vertical Line Test to determine whether y is a function of x. Describe how you can use a graphing utility to produce the given graph.**

19. $y = \frac{1}{2}x^2$

20. $x - y^2 = 1$

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✓ **21.** $x^2 + y^2 = 25$

22. $x^2 = 2xy - 1$

Increasing and Decreasing Functions **In Exercises 23–26, determine the open intervals on which the function is increasing, decreasing, or constant.**

23. $f(x) = \frac{3}{2}x$

24. $f(x) = x^2 - 4x$

✓ **25.** $f(x) = x^3 - 3x^2 + 2$

26. $f(x) = \sqrt{x^2 - 1}$

Increasing and Decreasing Functions **In Exercises 27–34, (a) use a graphing utility to graph the function and (b) determine the open intervals on which the function is increasing, decreasing, or constant.**

27. $f(x) = 3$

28. $f(x) = x$

29. $f(x) = x^{2/3}$

30. $f(x) = -x^{3/4}$

31. $f(x) = x\sqrt{x + 3}$

32. $f(x) = \sqrt{1 - x}$

33. $f(x) = |x + 1| + |x - 1|$

34. $f(x) = -|x + 4| - |x + 1|$

Approximating Relative Minima and Maxima **In Exercises 35–46, use a graphing utility to graph the function and to approximate any relative minimum or relative maximum values of the function.**

✓ **35.** $f(x) = x^2 - 6x$

36. $f(x) = 3x^2 - 2x - 5$

✓ **37.** $y = 2x^3 + 3x^2 - 12x$

38. $y = x^3 - 6x^2 + 15$

39. $h(x) = (x - 1)\sqrt{x}$

40. $g(x) = x\sqrt{4 - x}$

41. $f(x) = x^2 - 4x - 5$

42. $f(x) = 3x^2 - 12x$

43. $f(x) = x^3 - 3x$

44. $f(x) = -x^3 + 3x^2$

45. $f(x) = 3x^2 - 6x + 1$

46. $f(x) = 8x - 4x^2$

Library of Parent Functions **In Exercises 47–52, sketch the graph of the function by hand. Then use a graphing utility to verify the graph.**

47. $f(x) = [\![x]\!] + 2$

48. $f(x) = [\![x]\!] - 3$

49. $f(x) = [\![x - 1]\!] + 2$

50. $f(x) = [\![x - 2]\!] + 1$

51. $f(x) = [\![2x]\!]$

52. $f(x) = [\![4x]\!]$

Describing a Step Function **In Exercises 53 and 54, use a graphing utility to graph the function. State the domain and range of the function. Describe the pattern of the graph.**

53. $s(x) = 2\left(\frac{1}{4}x - \left[\!\left[\frac{1}{4}x\right]\!\right]\right)$

54. $g(x) = 2\left(\frac{1}{4}x - \left[\!\left[\frac{1}{4}x\right]\!\right]\right)^2$

Sketching a Piecewise-Defined Function **In Exercises 55–62, sketch the graph of the piecewise-defined function by hand.**

✓ **55.** $f(x) = \begin{cases} 2x + 3, & x < 0 \\ 3 - x, & x \geq 0 \end{cases}$

56. $f(x) = \begin{cases} x + 6, & x \leq -4 \\ 2x - 4, & x > -4 \end{cases}$

57. $f(x) = \begin{cases} \sqrt{4 + x}, & x < 0 \\ \sqrt{4 - x}, & x \geq 0 \end{cases}$

58. $f(x) = \begin{cases} 1 - (x - 1)^2, & x \leq 2 \\ \sqrt{x - 2}, & x > 2 \end{cases}$

59. $f(x) = \begin{cases} x + 3, & x \leq 0 \\ 3, & 0 < x \leq 2 \\ 2x - 1, & x > 2 \end{cases}$

60. $g(x) = \begin{cases} x + 5, & x \leq -3 \\ -2, & -3 < x < 1 \\ 5x - 4, & x \geq 1 \end{cases}$

61. $f(x) = \begin{cases} 2x + 1, & x \leq -1 \\ x^2 - 2, & x > -1 \end{cases}$

62. $h(x) = \begin{cases} 3 + x, & x < 0 \\ x^2 + 1, & x \geq 0 \end{cases}$

Even and Odd Functions **In Exercises 63–72, use a graphing utility to graph the function and determine whether it is even, odd, or neither.**

63. $f(x) = 5$

64. $f(x) = -9$

65. $f(x) = 3x - 2$

66. $f(x) = 5 - 3x$

✓ **67.** $h(x) = x^2 - 4$

68. $f(x) = -x^2 - 8$

69. $f(x) = \sqrt{1 - x}$

70. $g(t) = \sqrt[3]{t - 1}$

71. $f(x) = |x + 2|$

72. $f(x) = -|x - 5|$

Think About It **In Exercises 73–78, find the coordinates of a second point on the graph of a function *f* if the given point is on the graph and the function is (a) even and (b) odd.**

73. $\left(-\frac{3}{2}, 4\right)$

74. $\left(-\frac{5}{3}, -7\right)$

75. $(4, 9)$

76. $(5, -1)$

77. $(x, -y)$

78. $(2a, 2c)$

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Algebraic-Graphical-Numerical **In Exercises 79–86, determine whether the function is even, odd, or neither (a) algebraically, (b) graphically by using a graphing utility to graph the function, and (c) numerically by using the *table* feature of the graphing utility to compare $f(x)$ and $f(-x)$ for several values of x.**

79. $f(t) = t^2 + 2t - 3$

80. $f(x) = x^6 - 2x^2 + 3$

81. $g(x) = x^3 - 5x$

82. $h(x) = x^3 - 5$

83. $f(x) = x\sqrt{1 - x^2}$

84. $f(x) = x\sqrt{x + 5}$

85. $g(s) = 4s^{2/3}$

86. $f(s) = 4s^{3/2}$

Finding the Intervals Where a Function is Positive **In Exercises 87–90, graph the function and determine the interval(s) (if any) on the real axis for which $f(x) \geq 0$. Use a graphing utility to verify your results.**

87. $f(x) = 4 - x$

88. $f(x) = 4x + 2$

89. $f(x) = x^2 - 9$

90. $f(x) = x^2 - 4x$

91. **Business** The cost of using a telephone calling card is \$1.05 for the first minute and \$0.08 for each additional minute or portion of a minute.

(a) A customer needs a model for the cost C of using the calling card for a call lasting t minutes. Which of the following is the appropriate model?

$$C_1(t) = 1.05 + 0.08[\![t - 1]\!]$$

$$C_2(t) = 1.05 - 0.08[\![-(t - 1)]\!]$$

(b) Use a graphing utility to graph the appropriate model. Estimate the cost of a call lasting 18 minutes and 45 seconds.

92. ***Why you should learn it*** *(p. 29)* The cost of sending an overnight package from New York to Atlanta is \$18.80 for a package weighing up to but not including 1 pound and \$3.50 for each additional pound or portion of a pound. Use the greatest integer function to create a model for the cost C of overnight delivery of a package weighing x pounds, where $x > 0$. Sketch the graph of the function.

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Using the Graph of a Function **In Exercises 93 and 94, write the height h of the rectangle as a function of x.**

93.

94.

95. **MODELING DATA**

The number N (in thousands) of existing condominiums and cooperative homes sold each year from 2000 through 2008 in the United States is approximated by the model

$$N = 0.4825t^4 - 11.293t^3 + 65.26t^2 - 48.8t + 578,$$
$$0 \leq t \leq 8$$

where t represents the year, with $t = 0$ corresponding to 2000. (Source: National Association of Realtors)

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(a) Use a graphing utility to graph the model over the appropriate domain.

(b) Use the graph from part (a) to determine during which years the number of cooperative homes and condos was increasing. During which years was the number decreasing?

(c) Approximate the maximum number of cooperative homes and condos sold from 2000 through 2008.

96. **Mechanical Engineering** The intake pipe of a 100-gallon tank has a flow rate of 10 gallons per minute, and two drain pipes have a flow rate of 5 gallons per minute each. The graph shows the volume V of fluid in the tank as a function of time t. Determine in which pipes the fluid is flowing in specific subintervals of the one-hour interval of time shown on the graph. (There are many correct answers.)

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Conclusions

True or False? In Exercises 97 and 98, determine whether the statement is true or false. Justify your answer.

97. A function with a square root cannot have a domain that is the set of all real numbers.

98. It is possible for an odd function to have the interval $[0, \infty)$ as its domain.

Think About It In Exercises 99–104, match the graph of the function with the description that best fits the situation.

(a) The air temperature at a beach on a sunny day

(b) The height of a football kicked in a field goal attempt

(c) The number of children in a family over time

(d) The population of California as a function of time

(e) The depth of the tide at a beach over a 24-hour period

(f) The number of cupcakes on a tray at a party

99.

100.

101.

102.

103.

104.

105. Think About It Does the graph in Exercise 20 represent x as a function of y? Explain.

106. Think About It Does the graph in Exercise 21 represent x as a function of y? Explain.

107. Think About It Can you represent the greatest integer function using a piecewise-defined function?

108. Think About It How does the graph of the greatest integer function differ from the graph of a line with a slope of zero?

109. Let f be an even function. Determine whether g is even, odd, or neither. Explain.

(a) $g(x) = -f(x)$ (b) $g(x) = f(-x)$

(c) $g(x) = f(x) - 2$ (d) $g(x) = -f(x - 2)$

110. CAPSTONE Half of the graph of an odd function is shown.

(a) Sketch a complete graph of the function.

(b) Find the domain and range of the function.

(c) Identify the open intervals on which the function is increasing, decreasing, or constant.

(d) Find any relative minimum and relative maximum values of the function.

111. Proof Prove that a function of the following form is odd.

$$y = a_{2n+1}x^{2n+1} + a_{2n-1}x^{2n-1} + \cdot \cdot \cdot + a_3x^3 + a_1x$$

112. Proof Prove that a function of the following form is even.

$$y = a_{2n}x^{2n} + a_{2n-2}x^{2n-2} + \cdot \cdot \cdot + a_2x^2 + a_0$$

Cumulative Mixed Review

Identifying Terms and Coefficients In Exercises 113–116, identify the terms. Then identify the coefficients of the variable terms of the expression.

113. $-2x^2 + 8x$ **114.** $10 + 3x$

115. $\frac{x}{3} - 5x^2 + x^3$ **116.** $7x^4 + \sqrt{2}x^2$

Evaluating a Function In Exercises 117 and 118, evaluate the function at each specified value of the independent variable and simplify.

117. $f(x) = -x^2 - x + 3$

(a) $f(4)$ (b) $f(-2)$ (c) $f(x - 2)$

118. $f(x) = x\sqrt{x - 3}$

(a) $f(3)$ (b) $f(12)$ (c) $f(6)$

Evaluating a Difference Quotient In Exercises 119 and 120, find the difference quotient and simplify your answer.

119. $f(x) = x^2 - 2x + 9, \dfrac{f(3 + h) - f(3)}{h}, h \neq 0$

120. $f(x) = 5 + 6x - x^2, \dfrac{f(6 + h) - f(6)}{h}, h \neq 0$

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1.4 Shifting, Reflecting, and Stretching Graphs

Summary of Graphs of Parent Functions

One of the goals of this text is to enable you to build your intuition for the basic shapes of the graphs of different types of functions. For instance, from your study of lines in Section 1.1, you can determine the basic shape of the graph of the parent linear function

$$f(x) = x.$$

Specifically, you know that the graph of this function is a line whose slope is 1 and whose y-intercept is $(0, 0)$.

The six graphs shown in Figure 1.34 represent the most commonly used types of functions in algebra. Familiarity with the basic characteristics of these simple parent graphs will help you analyze the shapes of more complicated graphs.

What you should learn

- Recognize graphs of parent functions.
- Use vertical and horizontal shifts and reflections to graph functions.
- Use nonrigid transformations to graph functions.

Why you should learn it

Recognizing the graphs of parent functions and knowing how to shift, reflect, and stretch graphs of functions can help you sketch or describe the graphs of a wide variety of simple functions. For example, in Exercise 66 on page 49, you are asked to describe a transformation that produces the graph of a model for the sales of the WD-40 Company.

Library of Parent Functions: Commonly Used Functions

(a) *Linear Function* (b) *Absolute Value Function*

(c) *Square Root Function* (d) *Quadratic Function*

(e) *Cubic Function* (f) *Rational Function*

Figure 1.34

Throughout this section, you will discover how many complicated graphs are derived by shifting, stretching, shrinking, or reflecting the parent graphs shown above. Shifts, stretches, shrinks, and reflections are called *transformations*. Many graphs of functions can be created from combinations of these transformations.

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Vertical and Horizontal Shifts

Many functions have graphs that are simple transformations of the graphs of parent functions summarized in Figure 1.34. For example, you can obtain the graph of

$$h(x) = x^2 + 2$$

by shifting the graph of $f(x) = x^2$ two units *upward*, as shown in Figure 1.35. In function notation, h and f are related as follows.

$$h(x) = x^2 + 2$$
$$= f(x) + 2 \qquad \text{Upward shift of two units}$$

Similarly, you can obtain the graph of

$$g(x) = (x - 2)^2$$

by shifting the graph of $f(x) = x^2$ two units to the *right*, as shown in Figure 1.36. In this case, the functions g and f have the following relationship.

$$g(x) = (x - 2)^2$$
$$= f(x - 2) \qquad \text{Right shift of two units}$$

Explore the Concept

Use a graphing utility to display (in the same viewing window) the graphs of $y = x^2 + c$, where $c = -2, 0, 2$, and 4. Use the results to describe the effect that c has on the graph.

Use a graphing utility to display (in the same viewing window) the graphs of $y = (x + c)^2$, where $c = -2, 0, 2$, and 4. Use the results to describe the effect that c has on the graph.

Vertical shift upward: two units
Figure 1.35

Horizontal shift to the right: two units
Figure 1.36

The following list summarizes vertical and horizontal shifts.

Vertical and Horizontal Shifts

Let c be a positive real number. **Vertical and horizontal shifts** in the graph of $y = f(x)$ are represented as follows.

1. Vertical shift c units *upward*: $h(x) = f(x) + c$
2. Vertical shift c units *downward*: $h(x) = f(x) - c$
3. Horizontal shift c units to the *right*: $h(x) = f(x - c)$
4. Horizontal shift c units to the *left*: $h(x) = f(x + c)$

In items 3 and 4, be sure you see that

$$h(x) = f(x - c)$$

corresponds to a *right* shift and

$$h(x) = f(x + c)$$

corresponds to a *left* shift for $c > 0$.

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Example 1 Shifts in the Graph of a Function

Compare the graph of each function with the graph of $f(x) = x^3$.

a. $g(x) = x^3 - 1$ **b.** $h(x) = (x - 1)^3$ **c.** $k(x) = (x + 2)^3 + 1$

Solution

a. You obtain the graph of g by shifting the graph of f one unit downward.

b. You obtain the graph of h by shifting the graph of f one unit to the right.

c. You obtain the graph of k by shifting the graph of f two units to the left and then one unit upward.

(a) ***Vertical shift: one unit downward*** **(b)** ***Horizontal shift: one unit right*** **(c)** ***Two units left and one unit upward***

Figure 1.37

CHECKPOINT Now try Exercise 23.

Example 2 Finding Equations from Graphs

The graph of $f(x) = x^2$ is shown in Figure 1.38. Each of the graphs in Figure 1.39 is a transformation of the graph of f. Find an equation for each function.

Figure 1.38

(a)

(b)

Figure 1.39

Solution

a. The graph of g is a vertical shift of four units upward of the graph of $f(x) = x^2$. So, the equation for g is $g(x) = x^2 + 4$.

b. The graph of h is a horizontal shift of two units to the left, and a vertical shift of one unit downward, of the graph of $f(x) = x^2$. So, the equation for h is $h(x) = (x + 2)^2 - 1$.

CHECKPOINT Now try Exercise 31.

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Reflecting Graphs

Another common type of transformation is called a **reflection.** For instance, when you consider the x-axis to be a mirror, the graph of $h(x) = -x^2$ is the mirror image (or reflection) of the graph of $f(x) = x^2$ (see Figure 1.40).

Figure 1.40

Reflections in the Coordinate Axes

Reflections in the coordinate axes of the graph of $y = f(x)$ are represented as follows.

1. Reflection in the x-axis: $h(x) = -f(x)$
2. Reflection in the y-axis: $h(x) = f(-x)$

Explore the Concept

Compare the graph of each function with the graph of $f(x) = x^2$ by using a graphing utility to graph the function and f in the same viewing window. Describe the transformation.

a. $g(x) = -x^2$

b. $h(x) = (-x)^2$

Example 3 Finding Equations from Graphs

The graph of $f(x) = x^2$ is shown in Figure 1.40. Each of the graphs in Figure 1.41 is a transformation of the graph of f. Find an equation for each function.

(a)

(b)

Figure 1.41

Solution

a. The graph of g is a reflection in the x-axis *followed by* an upward shift of two units of the graph of $f(x) = x^2$. So, the equation for g is $g(x) = -x^2 + 2$.

b. The graph of h is a horizontal shift of three units to the right *followed by* a reflection in the x-axis of the graph of $f(x) = x^2$. So, the equation for h is $h(x) = -(x - 3)^2$.

CHECKPOINT Now try Exercise 33.

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Example 4 Reflections and Shifts

Compare the graph of each function with the graph of

$$f(x) = \sqrt{x}.$$

a. $g(x) = -\sqrt{x}$

b. $h(x) = \sqrt{-x}$

c. $k(x) = -\sqrt{x+2}$

Algebraic Solution

a. Relative to the graph of $f(x) = \sqrt{x}$, the graph of g is a reflection in the x-axis because

$$\begin{aligned} g(x) &= -\sqrt{x} \\ &= -f(x). \end{aligned}$$

b. The graph of h is a reflection of the graph of $f(x) = \sqrt{x}$ in the y-axis because

$$\begin{aligned} h(x) &= \sqrt{-x} \\ &= f(-x). \end{aligned}$$

c. From the equation

$$\begin{aligned} k(x) &= -\sqrt{x+2} \\ &= -f(x+2) \end{aligned}$$

you can conclude that the graph of k is a left shift of two units, followed by a reflection in the x-axis, of the graph of $f(x) = \sqrt{x}$.

CHECKPOINT Now try Exercise 35.

Graphical Solution

a. From the graph in Figure 1.42, you can see that the graph of g is a reflection of the graph of f in the x-axis. Note that the domain of g is $x \geq 0$.

Figure 1.42

b. From the graph in Figure 1.43, you can see that the graph of h is a reflection of the graph of f in the y-axis. Note that the domain of h is $x \leq 0$.

Figure 1.43

c. From the graph in Figure 1.44, you can see that the graph of k is a left shift of two units of the graph of f, followed by a reflection in the x-axis. Note that the domain of k is $x \geq -2$.

Figure 1.44

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Nonrigid Transformations

Horizontal shifts, vertical shifts, and reflections are called **rigid transformations** because the basic shape of the graph is unchanged. These transformations change only the *position* of the graph in the coordinate plane. **Nonrigid transformations** are those that cause a *distortion*—a change in the shape of the original graph. For instance, a nonrigid transformation of the graph of $y = f(x)$ is represented by $g(x) = cf(x)$, where the transformation is a **vertical stretch** when $c > 1$ and a **vertical shrink** when $0 < c < 1$. Another nonrigid transformation of the graph of $y = f(x)$ is represented by $h(x) = f(cx)$, where the transformation is a **horizontal shrink** when $c > 1$ and a **horizontal stretch** when $0 < c < 1$.

Example 5 Nonrigid Transformations

Compare the graph of each function with the graph of $f(x) = |x|$.

a. $h(x) = 3|x|$

b. $g(x) = \frac{1}{3}|x|$

Solution

a. Relative to the graph of $f(x) = |x|$, the graph of

$$h(x) = 3|x| = 3f(x)$$

is a vertical stretch (each y-value is multiplied by 3) of the graph of f. (See Figure 1.45.)

b. Similarly, the graph of

$$g(x) = \tfrac{1}{3}|x| = \tfrac{1}{3}f(x)$$

is a vertical shrink $\left(\text{each } y\text{-value is multiplied by } \frac{1}{3}\right)$ of the graph of f. (See Figure 1.46.)

Figure 1.45

Figure 1.46

CHECKPOINT Now try Exercise 41.

Example 6 Nonrigid Transformations

Compare the graph of $h(x) = f\left(\frac{1}{2}x\right)$ with the graph of $f(x) = 2 - x^3$.

Solution

Relative to the graph of $f(x) = 2 - x^3$, the graph of

$$h(x) = f\left(\tfrac{1}{2}x\right) = 2 - \left(\tfrac{1}{2}x\right)^3 = 2 - \tfrac{1}{8}x^3$$

is a horizontal stretch (each x-value is multiplied by 2) of the graph of f. (See Figure 1.47.)

CHECKPOINT Now try Exercise 49.

Figure 1.47

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1.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

1. Name three types of rigid transformations.

2. Match the rigid transformation of $y = f(x)$ with the correct representation, where $c > 0$.

(a) $h(x) = f(x) + c$ (i) horizontal shift c units to the left

(b) $h(x) = f(x) - c$ (ii) vertical shift c units upward

(c) $h(x) = f(x - c)$ (iii) horizontal shift c units to the right

(d) $h(x) = f(x + c)$ (iv) vertical shift c units downward

In Exercises 3 and 4, fill in the blanks.

3. A reflection in the x-axis of $y = f(x)$ is represented by $h(x) =$ _____ , while a reflection in the y-axis of $y = f(x)$ is represented by $h(x) =$ _____ .

4. A nonrigid transformation of $y = f(x)$ represented by $cf(x)$ is a vertical stretch when _____ and a vertical shrink when _____ .

Procedures and Problem Solving

Sketching Transformations **In Exercises 5–18, sketch the graphs of the three functions by hand on the same rectangular coordinate system. Verify your results with a graphing utility.**

5. $f(x) = x$
$g(x) = x - 4$
$h(x) = 3x$

6. $f(x) = \frac{1}{2}x$
$g(x) = \frac{1}{2}x + 2$
$h(x) = \frac{1}{2}(x - 2)$

7. $f(x) = x^2$
$g(x) = x^2 + 2$
$h(x) = (x - 2)^2$

8. $f(x) = x^2$
$g(x) = x^2 - 4$
$h(x) = (x + 2)^2 + 1$

9. $f(x) = -x^2$
$g(x) = -x^2 + 1$
$h(x) = -(x - 2)^2$

10. $f(x) = (x - 2)^2$
$g(x) = (x + 2)^2 + 2$
$h(x) = -(x - 2)^2 - 1$

11. $f(x) = x^2$
$g(x) = \frac{1}{2}x^2$
$h(x) = (2x)^2$

12. $f(x) = x^2$
$g(x) = \frac{1}{4}x^2 + 2$
$h(x) = -\frac{1}{4}x^2$

13. $f(x) = |x|$
$g(x) = |x| - 1$
$h(x) = |x - 3|$

14. $f(x) = |x|$
$g(x) = |x + 3|$
$h(x) = -2|x + 2| - 1$

15. $f(x) = \sqrt{x}$
$g(x) = \sqrt{x + 1}$
$h(x) = \sqrt{x - 2} + 1$

16. $f(x) = \sqrt{x}$
$g(x) = \frac{1}{2}\sqrt{x}$
$h(x) = -\sqrt{x + 4}$

17. $f(x) = \dfrac{1}{x}$
$g(x) = \dfrac{1}{x} + 2$
$h(x) = \dfrac{1}{x - 1} + 2$

18. $f(x) = \dfrac{1}{x}$
$g(x) = \dfrac{1}{x} - 4$
$h(x) = \dfrac{1}{x + 3} - 1$

Sketching Transformations **In Exercises 19 and 20, use the graph of *f* to sketch each graph. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

19. (a) $y = f(x) + 2$
(b) $y = -f(x)$
(c) $y = f(x - 2)$
(d) $y = f(x + 3)$
(e) $y = 2f(x)$
(f) $y = f(-x)$
(g) $y = f(\frac{1}{2}x)$

20. (a) $y = f(x) - 1$
(b) $y = f(x + 1)$
(c) $y = f(x - 1)$
(d) $y = -f(x - 2)$
(e) $y = f(-x)$
(f) $y = \frac{1}{2}f(x)$
(g) $y = f(2x)$

Error Analysis **In Exercises 21 and 22, describe the error in graphing the function.**

21. $f(x) = (x + 1)^2$

22. $f(x) = (x - 1)^2$

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Library of Parent Functions **In Exercises 23–28, compare the graph of the function with the graph of its parent function.**

23. $y = \sqrt{x} + 2$ **24.** $y = \dfrac{1}{x} - 5$

25. $y = (x - 4)^3$ **26.** $y = |x + 5|$

27. $y = x^2 - 2$ **28.** $y = \sqrt{x - 2}$

Library of Parent Functions **In Exercises 29–34, identify the parent function and describe the transformation shown in the graph. Write an equation for the graphed function.**

29.

30.

31.

32.

33.

34.

Rigid and Nonrigid Transformations **In Exercises 35–46, compare the graph of the function with the graph of its parent function.**

35. $y = -|x|$ **36.** $y = |-x|$

37. $y = (-x)^2$ **38.** $y = -x^3$

39. $y = \dfrac{1}{-x}$ **40.** $y = -\dfrac{1}{x}$

41. $h(x) = 4|x|$ **42.** $p(x) = \frac{1}{2}x^2$

43. $g(x) = \frac{1}{4}x^3$ **44.** $y = 2\sqrt{x}$

45. $f(x) = \sqrt{4x}$ **46.** $y = |\frac{1}{2}x|$

Rigid and Nonrigid Transformations **In Exercises 47–50, use a graphing utility to graph the three functions in the same viewing window. Describe the graphs of *g* and *h* relative to the graph of *f*.**

47. $f(x) = x^3 - 3x^2$
$g(x) = f(x + 2)$
$h(x) = \frac{1}{2}f(x)$

48. $f(x) = x^3 - 3x^2 + 2$
$g(x) = f(x - 1)$
$h(x) = f(3x)$

49. $f(x) = x^3 - 3x^2$
$g(x) = -\frac{1}{3}f(x)$
$h(x) = f(-x)$

50. $f(x) = x^3 - 3x^2 + 2$
$g(x) = -f(x)$
$h(x) = f(2x)$

Describing Transformations **In Exercises 51–64, *g* is related to one of the six parent functions on page 41. (a) Identify the parent function *f*. (b) Describe the sequence of transformations from *f* to *g*. (c) Sketch the graph of *g* by hand. (d) Use function notation to write *g* in terms of the parent function *f*.**

51. $g(x) = 2 - (x + 5)^2$ **52.** $g(x) = -(x + 10)^2 + 5$

53. $g(x) = 3 + 2(x - 4)^2$ **54.** $g(x) = -\frac{1}{4}(x + 2)^2 - 2$

55. $g(x) = 3(x - 2)^3$ **56.** $g(x) = -\frac{1}{2}(x + 1)^3$

57. $g(x) = (x - 1)^3 + 2$

58. $g(x) = -(x + 3)^3 - 10$

59. $g(x) = \dfrac{1}{x + 8} - 9$ **60.** $g(x) = \dfrac{1}{x - 7} + 4$

61. $g(x) = -2|x - 1| - 4$ **62.** $g(x) = \frac{1}{2}|x - 2| - 3$

63. $g(x) = -\frac{1}{2}\sqrt{x + 3} - 1$ **64.** $g(x) = -\sqrt{x + 1} - 6$

65. MODELING DATA

The amounts of fuel F (in billions of gallons) used by motor vehicles from 1991 through 2007 are given by the ordered pairs of the form $(t, F(t))$, where $t = 1$ represents 1991. A model for the data is

$$F(t) = -0.099(t - 24.7)^2 + 183.4.$$

(Source: U.S. Federal Highway Administration)

(1, 128.6)
(2, 132.9)
(3, 137.3)
(4, 140.8)
(5, 143.8)
(6, 147.4)
(7, 150.4)
(8, 155.4)
(9, 161.4)
(10, 162.5)
(11, 163.5)
(12, 168.7)
(13, 170.0)
(14, 173.5)
(15, 174.8)
(16, 175.0)
(17, 176.1)

(a) Describe the transformation of the parent function $f(t) = t^2$.

(b) Use a graphing utility to graph the model and the data in the same viewing window.

(c) Rewrite the function so that $t = 0$ represents 2000. Explain how you got your answer.

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66. ***Why you should learn it*** *(p. 41)* The sales S (in millions of dollars) of the WD-40 Company from 2000 through 2008 can be approximated by the function

$$S(t) = 99\sqrt{t + 2.37}$$

where $t = 0$ represents 2000. (Source: WD-40 Company)

(a) Describe the transformation of the parent function $f(t) = \sqrt{t}$.

(b) Use a graphing utility to graph the model over the interval $0 \le t \le 8$.

(c) According to the model, in what year will the sales of WD-40 be approximately 400 million dollars?

(d) Rewrite the function so that $t = 0$ represents 2005. Explain how you got your answer.

Conclusions

True or False? In Exercises 67 and 68, determine whether the statement is true or false. Justify your answer.

67. The graph of $y = f(-x)$ is a reflection of the graph of $y = f(x)$ in the x-axis.

68. The graphs of $f(x) = |x| + 6$ and $f(x) = |-x| + 6$ are identical.

Exploration In Exercises 69–72, use the fact that the graph of $y = f(x)$ has x-intercepts at $x = 2$ and $x = -3$ to find the x-intercepts of the given graph. If not possible, state the reason.

69. $y = f(-x)$

70. $y = 2f(x)$

71. $y = f(x) + 2$

72. $y = f(x - 3)$

Library of Parent Functions In Exercises 73–76, determine which equation(s) may be represented by the graph shown. There may be more than one correct answer.

73.

(a) $f(x) = |x + 2| + 1$

(b) $f(x) = |x - 1| + 2$

(c) $f(x) = |x - 2| + 1$

(d) $f(x) = 2 + |x - 2|$

(e) $f(x) = |(x - 2) + 1|$

(f) $f(x) = 1 - |x - 2|$

74.

(a) $f(x) = -\sqrt{x} - 4$

(b) $f(x) = -4 - \sqrt{x}$

(c) $f(x) = -4 - \sqrt{-x}$

(d) $f(x) = \sqrt{-x} - 4$

(e) $f(x) = \sqrt{-x} + 4$

(f) $f(x) = \sqrt{x} - 4$

75.

(a) $f(x) = (x - 2)^2 - 2$

(b) $f(x) = (x + 4)^2 - 4$

(c) $f(x) = (x - 2)^2 - 4$

(d) $f(x) = (x + 2)^2 - 4$

(e) $f(x) = 4 - (x - 2)^2$

(f) $f(x) = 4 - (x + 2)^2$

76.

(a) $f(x) = -(x - 4)^3 + 2$

(b) $f(x) = -(x + 4)^3 + 2$

(c) $f(x) = -(x - 2)^3 + 4$

(d) $f(x) = (-x - 4)^3 + 2$

(e) $f(x) = (x + 4)^3 + 2$

(f) $f(x) = (-x + 4)^3 + 2$

77. Think About It You can use either of two methods to graph a function: plotting points, or translating a parent function as shown in this section. Which method do you prefer to use for each function? Explain.

(a) $f(x) = 3x^2 - 4x + 1$ (b) $f(x) = 2(x - 1)^2 - 6$

78. Think About It The graph of $y = f(x)$ passes through the points $(0, 1)$, $(1, 2)$, and $(2, 3)$. Find the corresponding points on the graph of $y = f(x + 2) - 1$.

79. Think About It Compare the graph of $g(x) = ax^2$ with the graph of $f(x) = x^2$ when (a) $0 < a < 1$ and (b) $a > 1$.

80. CAPSTONE Use the fact that the graph of $y = f(x)$ is increasing on the interval $(-\infty, 2)$ and decreasing on the interval $(2, \infty)$ to find the intervals on which the graph is increasing and decreasing. If not possible, state the reason.

(a) $y = f(-x)$ (b) $y = -f(x)$ (c) $y = 2f(x)$

(d) $y = f(x) - 3$ (e) $y = f(x + 1)$

Cumulative Mixed Review

Parallel and Perpendicular Lines In Exercises 81 and 82, determine whether the lines L_1 and L_2 passing through the pairs of points are parallel, perpendicular, or neither.

81. L_1: $(-2, -2)$, $(2, 10)$

L_2: $(-1, 3)$, $(3, 9)$

82. L_1: $(-1, -7)$, $(4, 3)$

L_2: $(1, 5)$, $(-2, -7)$

Finding the Domain of a Function In Exercises 83–86, find the domain of the function.

83. $f(x) = \dfrac{4}{9 - x}$

84. $f(x) = \dfrac{\sqrt{x - 5}}{x - 7}$

85. $f(x) = \sqrt{100 - x^2}$

86. $f(x) = \sqrt[3]{16 - x^2}$

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1.5 Combinations of Functions

Arithmetic Combinations of Functions

Just as two real numbers can be combined by the operations of addition, subtraction, multiplication, and division to form other real numbers, two *functions* can be combined to create new functions. When

$$f(x) = 2x - 3 \qquad \text{and} \qquad g(x) = x^2 - 1$$

you can form the sum, difference, product, and quotient of f and g as follows.

$$\begin{aligned} f(x) + g(x) &= (2x - 3) + (x^2 - 1) \\ &= x^2 + 2x - 4 && \text{Sum} \\ f(x) - g(x) &= (2x - 3) - (x^2 - 1) \\ &= -x^2 + 2x - 2 && \text{Difference} \\ f(x) \cdot g(x) &= (2x - 3)(x^2 - 1) \\ &= 2x^3 - 3x^2 - 2x + 3 && \text{Product} \\ \frac{f(x)}{g(x)} &= \frac{2x - 3}{x^2 - 1}, \quad x \neq \pm 1 && \text{Quotient} \end{aligned}$$

The domain of an **arithmetic combination** of functions f and g consists of all real numbers that are common to the domains of f and g. In the case of the quotient

$$\frac{f(x)}{g(x)}$$

there is the further restriction that $g(x) \neq 0$.

What you should learn

- Add, subtract, multiply, and divide functions.
- Find compositions of one function with another function.
- Use combinations of functions to model and solve real-life problems.

Why you should learn it

You can model some situations by combining functions. For instance, in Exercise 79 on page 57, you will model the stopping distance of a car by combining the driver's reaction time with the car's braking distance.

Sum, Difference, Product, and Quotient of Functions

Let f and g be two functions with overlapping domains. Then, for all x common to both domains, the sum, difference, product, and quotient of f and g are defined as follows.

1. Sum: $(f + g)(x) = f(x) + g(x)$
2. Difference: $(f - g)(x) = f(x) - g(x)$
3. Product: $(fg)(x) = f(x) \cdot g(x)$
4. Quotient: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad g(x) \neq 0$

Example 1 Finding the Sum of Two Functions

Given $f(x) = 2x + 1$ and $g(x) = x^2 + 2x - 1$, find $(f + g)(x)$. Then evaluate the sum when $x = 2$.

Solution

$$\begin{aligned} (f + g)(x) &= f(x) + g(x) \\ &= (2x + 1) + (x^2 + 2x - 1) \\ &= x^2 + 4x \end{aligned}$$

When $x = 2$, the value of this sum is $(f + g)(2) = 2^2 + 4(2) = 12$.

CHECKPOINT Now try Exercise 13(a).

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Example 2 Finding the Difference of Two Functions

Given $f(x) = 2x + 1$ and $g(x) = x^2 + 2x - 1$, find $(f - g)(x)$. Then evaluate the difference when $x = 2$.

Algebraic Solution

The difference of the functions f and g is

$$\begin{aligned}(f - g)(x) &= f(x) - g(x)\\ &= (2x + 1) - (x^2 + 2x - 1)\\ &= -x^2 + 2.\end{aligned}$$

When $x = 2$, the value of this difference is

$$\begin{aligned}(f - g)(2) &= -(2)^2 + 2\\ &= -2.\end{aligned}$$

CHECKPOINT Now try Exercise 13(b).

Graphical Solution

Enter the functions in a graphing utility (see Figure 1.48). Then graph the difference of the two functions, y_3, as shown in Figure 1.49.

Figure 1.48

Figure 1.49

Example 3 Finding the Product of Two Functions

Given $f(x) = x^2$ and $g(x) = x - 3$, find $(fg)(x)$. Then evaluate the product when $x = 4$.

Solution

$$\begin{aligned}(fg)(x) &= f(x)g(x)\\ &= (x^2)(x - 3)\\ &= x^3 - 3x^2\end{aligned}$$

When $x = 4$, the value of this product is

$$(fg)(4) = 4^3 - 3(4)^2 = 16.$$

CHECKPOINT Now try Exercise 13(c).

In Examples 1–3, both f and g have domains that consist of all real numbers. So, the domain of both $(f + g)$ and $(f - g)$ is also the set of all real numbers. Remember that any restrictions on the domains of f or g must be considered when forming the sum, difference, product, or quotient of f and g. For instance, the domain of $f(x) = 1/x$ is all $x \neq 0$, and the domain of $g(x) = \sqrt{x}$ is $[0, \infty)$. This implies that the domain of $(f + g)$ is $(0, \infty)$.

Example 4 Finding the Quotient of Two Functions

Given $f(x) = \sqrt{x}$ and $g(x) = \sqrt{4 - x^2}$, find $(f/g)(x)$. Then find the domain of f/g.

Solution

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x}}{\sqrt{4 - x^2}}$$

The domain of f is $[0, \infty)$ and the domain of g is $[-2, 2]$. The intersection of these domains is $[0, 2]$. So, the domain of f/g is $[0, 2)$.

CHECKPOINT Now try Exercise 13(d).

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Compositions of Functions

Another way of combining two functions is to form the **composition** of one with the other. For instance, when $f(x) = x^2$ and $g(x) = x + 1$, the composition of f with g is

$$\begin{aligned} f(g(x)) &= f(x + 1) \\ &= (x + 1)^2. \end{aligned}$$

This composition is denoted as $f \circ g$ and is read as "f composed with g."

Definition of Composition of Two Functions

The **composition** of the function f with the function g is

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f. (See Figure 1.50.)

Figure 1.50

Example 5 Forming the Composition of f with g

Find $(f \circ g)(x)$ for

$$f(x) = \sqrt{x}, x \geq 0, \qquad \text{and} \qquad g(x) = x - 1, x \geq 1.$$

If possible, find $(f \circ g)(2)$ and $(f \circ g)(0)$.

Solution

The composition of f with g is

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) && \text{Definition of } f \circ g \\ &= f(x - 1) && \text{Definition of } g(x) \\ &= \sqrt{x - 1}, \quad x \geq 1. && \text{Definition of } f(x) \end{aligned}$$

The domain of $f \circ g$ is $[1, \infty)$. (See Figure 1.51). So,

$$(f \circ g)(2) = \sqrt{2 - 1} = 1$$

is defined, but $(f \circ g)(0)$ is not defined because 0 is not in the domain of $f \circ g$.

Figure 1.51

CHECKPOINT Now try Exercise 41.

Explore the Concept

Let $f(x) = x + 2$ and $g(x) = 4 - x^2$. Are the compositions $f \circ g$ and $g \circ f$ equal? You can use your graphing utility to answer this question by entering and graphing the following functions.

$$y_1 = (4 - x^2) + 2$$

$$y_2 = 4 - (x + 2)^2$$

What do you observe? Which function represents $f \circ g$ and which represents $g \circ f$?

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The composition of f with g is generally not the same as the composition of g with f. This is illustrated in Example 6.

Example 6 Compositions of Functions

Given $f(x) = x + 2$ and $g(x) = 4 - x^2$, evaluate

(a) $(f \circ g)(x)$ and (b) $(g \circ f)(x)$

when $x = 0$ and 1.

Algebraic Solution

a. $(f \circ g)(x) = f(g(x))$ Definition of $f \circ g$

$= f(4 - x^2)$ Definition of $g(x)$

$= (4 - x^2) + 2$ Definition of $f(x)$

$= -x^2 + 6$

$(f \circ g)(0) = -0^2 + 6 = 6$

$(f \circ g)(1) = -1^2 + 6 = 5$

b. $(g \circ f)(x) = g(f(x))$ Definition of $g \circ f$

$= g(x + 2)$ Definition of $f(x)$

$= 4 - (x + 2)^2$ Definition of $g(x)$

$= 4 - (x^2 + 4x + 4)$

$= -x^2 - 4x$

$(g \circ f)(0) = -0^2 - 4(0) = 0$

$(g \circ f)(1) = -1^2 - 4(1) = -5$

Note that $f \circ g \neq g \circ f$.

CHECKPOINT Now try Exercise 43.

Graphical Solution

a. and b. Enter $y_1 = f(x)$, $y_2 = g(x)$, $y_3 = (f \circ g)(x)$, and $y_4 = (g \circ f)(x)$, as shown in Figure 1.52. Then use the *table* feature to find the desired function values (see Figure 1.53).

Figure 1.52

Figure 1.53

From the table you can see that $f \circ g \neq g \circ f$.

Example 7 Finding the Domain of a Composite Function

Find the domain of $f \circ g$ for the functions given by

$f(x) = x^2 - 9$ and $g(x) = \sqrt{9 - x^2}$.

Algebraic Solution

The composition of the functions is as follows.

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f\left(\sqrt{9 - x^2}\right) \\ &= \left(\sqrt{9 - x^2}\right)^2 - 9 \\ &= 9 - x^2 - 9 \\ &= -x^2\end{aligned}$$

From this, it might appear that the domain of the composition is the set of all real numbers. This, however, is not true. Because the domain of f is the set of all real numbers and the domain of g is $[-3, 3]$, the domain of $f \circ g$ is $[-3, 3]$.

CHECKPOINT Now try Exercise 45.

Graphical Solution

The x-coordinates of points on the graph extend from -3 to 3. So, the domain of $f \circ g$ is $[-3, 3]$.

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Example 8 A Case in Which $f \circ g = g \circ f$

Given

$$f(x) = 2x + 3 \text{ and } g(x) = \tfrac{1}{2}(x - 3)$$

find each composition.

a. $(f \circ g)(x)$

b. $(g \circ f)(x)$

Solution

a.
$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f\left(\frac{1}{2}(x - 3)\right) \\ &= 2\left[\frac{1}{2}(x - 3)\right] + 3 \\ &= x - 3 + 3 \\ &= x\end{aligned}$$

b.
$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(2x + 3) \\ &= \frac{1}{2}\left[(2x + 3) - 3\right] \\ &= \frac{1}{2}(2x) \\ &= x\end{aligned}$$

CHECKPOINT Now try Exercise 57.

In Examples 5–8, you formed the composition of two given functions. In calculus, it is also important to be able to identify two functions that make up a given composite function. Basically, to "decompose" a composite function, look for an "inner" and an "outer" function.

Example 9 Identifying a Composite Function

algebra of calculus

Write the function

$$h(x) = \frac{1}{(x - 2)^2}$$

as a composition of two functions.

Solution

One way to write h as a composition of two functions is to take the inner function to be $g(x) = x - 2$ and the outer function to be

$$f(x) = \frac{1}{x^2} = x^{-2}.$$

Then you can write

$$h(x) = \frac{1}{(x - 2)^2} = (x - 2)^{-2} = f(x - 2) = f(g(x)).$$

CHECKPOINT Now try Exercise 75.

Study Tip

In Example 8, note that the two composite functions $f \circ g$ and $g \circ f$ are equal, and both represent the identity function. That is, $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$. You will study this special case in the next section.

Explore the Concept

Write each function as a composition of two functions.

a. $h(x) = |x^3 - 2|$

b. $r(x) = |x^3| - 2$

What do you notice about the inner and outer functions?

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Application

Example 10 Bacteria Count

The number N of bacteria in a refrigerated petri dish is given by

$$N(T) = 20T^2 - 80T + 500, \quad 2 \le T \le 14$$

where T is the temperature of the petri dish (in degrees Celsius). When the petri dish is removed from refrigeration, the temperature of the petri dish is given by

$$T(t) = 4t + 2, \quad 0 \le t \le 3$$

where t is the time (in hours).

a. Find the composition $N(T(t))$ and interpret its meaning in context.

b. Find the number of bacteria in the petri dish when $t = 2$ hours.

c. Find the time when the bacteria count reaches 2000.

Microbiologist

Solution

a.
$$\begin{aligned} N(T(t)) &= 20(4t + 2)^2 - 80(4t + 2) + 500 \\ &= 20(16t^2 + 16t + 4) - 320t - 160 + 500 \\ &= 320t^2 + 320t + 80 - 320t - 160 + 500 \\ &= 320t^2 + 420 \end{aligned}$$

The composite function $N(T(t))$ represents the number of bacteria as a function of the amount of time the petri dish has been out of refrigeration.

b. When $t = 2$, the number of bacteria is

$$N = 320(2)^2 + 420 = 1280 + 420 = 1700.$$

c. The bacteria count will reach $N = 2000$ when $320t^2 + 420 = 2000$. You can solve this equation for t algebraically as follows.

$$\begin{aligned} 320t^2 + 420 &= 2000 \\ 320t^2 &= 1580 \\ t^2 &= \frac{79}{16} \\ t &= \frac{\sqrt{79}}{4} \quad \Rightarrow \quad t \approx 2.22 \text{ hours} \end{aligned}$$

So, the count will reach 2000 when $t \approx 2.22$ hours. Note that the negative value is rejected because it is not in the domain of the composite function. To confirm your solution, graph the equation $N = 320t^2 + 420$, as shown in Figure 1.54. Then use the *zoom* and *trace* features to approximate $N = 2000$ when $t \approx 2.22$, as shown in Figure 1.55.

Figure 1.54

Figure 1.55

CHECKPOINT Now try Exercise 85.

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1.5 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1–4, fill in the blank(s).

1. Two functions f and g can be combined by the arithmetic operations of _____ , _____ , _____ , and _____ to create new functions.
2. The _____ of the function f with the function g is $(f \circ g)(x) = f(g(x))$.
3. The domain of $f \circ g$ is the set of all x in the domain of g such that _____ is in the domain of f.
4. To decompose a composite function, look for an _____ and an _____ function.
5. Given $f(x) = x^2 + 1$ and $(fg)(x) = 2x(x^2 + 1)$, what is $g(x)$?
6. Given $(f \circ g)(x) = f(x^2 + 1)$, what is $g(x)$?

Procedures and Problem Solving

Finding the Sum of Two Functions **In Exercises 7–10, use the graphs of f and g to graph $h(x) = (f + g)(x)$. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

7.

8.

9.

10.

Finding Arithmetic Combinations of Functions **In Exercises 11–18, find (a) $(f + g)(x)$, (b) $(f - g)(x)$, (c) $(fg)(x)$, and (d) $(f/g)(x)$. What is the domain of f/g?**

11. $f(x) = x + 3, \quad g(x) = x - 3$
12. $f(x) = 2x - 5, \quad g(x) = 1 - x$
13. $f(x) = x^2, \quad g(x) = 1 - x$
14. $f(x) = 2x - 5, \quad g(x) = 5$
15. $f(x) = x^2 + 5, \quad g(x) = \sqrt{1 - x}$
16. $f(x) = \sqrt{x^2 - 4}, \quad g(x) = \dfrac{x^2}{x^2 + 1}$
17. $f(x) = \dfrac{1}{x}, \quad g(x) = \dfrac{1}{x^2}$
18. $f(x) = \dfrac{x}{x + 1}, \quad g(x) = x^3$

Evaluating an Arithmetic Combination of Functions **In Exercises 19–32, evaluate the indicated function for $f(x) = x^2 - 1$ and $g(x) = x - 2$ algebraically. If possible, use a graphing utility to verify your answer.**

19. $(f + g)(3)$
20. $(f - g)(-2)$
21. $(f - g)(0)$
22. $(f + g)(1)$
23. $(fg)(6)$
24. $(fg)(-4)$
25. $(f/g)(-5)$
26. $(f/g)(0)$
27. $(f - g)(2t)$
28. $(f + g)(t - 4)$
29. $(fg)(-5t)$
30. $(fg)(3t^2)$
31. $(f/g)(-t)$
32. $(f/g)(t + 2)$

Graphing an Arithmetic Combination of Functions **In Exercises 33–36, use a graphing utility to graph the functions f, g, and h in the same viewing window.**

33. $f(x) = \frac{1}{2}x, \quad g(x) = x - 1, \quad h(x) = f(x) + g(x)$
34. $f(x) = \frac{1}{3}x, \quad g(x) = -x + 4, \quad h(x) = f(x) - g(x)$
35. $f(x) = x^2, \quad g(x) = -2x, \quad h(x) = f(x) \cdot g(x)$
36. $f(x) = 4 - x^2, \quad g(x) = x, \quad h(x) = f(x)/g(x)$

Graphing a Sum of Functions **In Exercises 37–40, use a graphing utility to graph f, g, and $f + g$ in the same viewing window. Which function contributes most to the magnitude of the sum when $0 \le x \le 2$? Which function contributes most to the magnitude of the sum when $x > 6$?**

37. $f(x) = 3x, \quad g(x) = -\dfrac{x^3}{10}$
38. $f(x) = \dfrac{x}{2}, \quad g(x) = \sqrt{x}$
39. $f(x) = 3x + 2, \quad g(x) = -\sqrt{x + 5}$
40. $f(x) = x^2 - \frac{1}{2}, \quad g(x) = -3x^2 - 1$

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Compositions of Functions **In Exercises 41–44, find (a) $f \circ g$, (b) $g \circ f$, and, if possible, (c) $(f \circ g)(0)$.**

41. $f(x) = x^2, \quad g(x) = x - 1$

42. $f(x) = \sqrt[3]{x - 1}, \quad g(x) = x^3 + 1$

43. $f(x) = 3x + 5, \quad g(x) = 5 - x$

44. $f(x) = x^3, \quad g(x) = \dfrac{1}{x}$

Finding the Domain of a Composite Function **In Exercises 45–54, determine the domains of (a) f, (b) g, and (c) $f \circ g$. Use a graphing utility to verify your results.**

45. $f(x) = \sqrt{x + 4}, \quad g(x) = x^2$

46. $f(x) = \sqrt{x + 3}, \quad g(x) = \dfrac{x}{2}$

47. $f(x) = x^2 + 1, \quad g(x) = \sqrt{x}$

48. $f(x) = x^{1/4}, \quad g(x) = x^4$

49. $f(x) = \dfrac{1}{x}, \quad g(x) = x + 3$

50. $f(x) = \dfrac{1}{x}, \quad g(x) = \dfrac{1}{2x}$

51. $f(x) = |x - 4|, \quad g(x) = 3 - x$

52. $f(x) = \dfrac{2}{|x|}, \quad g(x) = x - 1$

53. $f(x) = x + 2, \quad g(x) = \dfrac{1}{x^2 - 4}$

54. $f(x) = \dfrac{3}{x^2 - 1}, \quad g(x) = x + 1$

Determining Whether $f \circ g = g \circ f$ **In Exercises 55–60, (a) find $f \circ g$, $g \circ f$, and the domain of $f \circ g$. (b) Use a graphing utility to graph $f \circ g$ and $g \circ f$. Determine whether $f \circ g = g \circ f$.**

55. $f(x) = \sqrt{x + 4}, \quad g(x) = x^2$

56. $f(x) = \sqrt[3]{x + 1}, \quad g(x) = x^3 - 1$

57. $f(x) = \frac{1}{3}x - 3, \quad g(x) = 3x + 9$

58. $f(x) = \sqrt{x}, \quad g(x) = \sqrt{x}$

59. $f(x) = x^{2/3}, \quad g(x) = x^6$

60. $f(x) = |x|, \quad g(x) = -x^2 + 1$

Determining Whether $f \circ g = g \circ f$ **In Exercises 61–66, (a) find $(f \circ g)(x)$ and $(g \circ f)(x)$, (b) determine algebraically whether $(f \circ g)(x) = (g \circ f)(x)$, and (c) use a graphing utility to complete a table of values for the two compositions to confirm your answer to part (b).**

61. $f(x) = 5x + 4, \quad g(x) = 4 - x$

62. $f(x) = \frac{1}{4}(x - 1), \quad g(x) = 4x + 1$

63. $f(x) = \sqrt{x + 6}, \quad g(x) = x^2 - 5$

64. $f(x) = x^3 - 4, \quad g(x) = \sqrt[3]{x + 10}$

65. $f(x) = |x|, \quad g(x) = 2x^3$

66. $f(x) = \dfrac{6}{3x - 5}, \quad g(x) = -x$

Evaluating Combinations of Functions **In Exercises 67–70, use the graphs of f and g to evaluate the functions.**

67. (a) $(f + g)(3)$ (b) $(f/g)(2)$

68. (a) $(f - g)(1)$ (b) $(fg)(4)$

69. (a) $(f \circ g)(2)$ (b) $(g \circ f)(2)$

70. (a) $(f \circ g)(1)$ (b) $(g \circ f)(3)$

Identifying a Composite Function **In Exercises 71–78, find two functions f and g such that $(f \circ g)(x) = h(x)$. (There are many correct answers.)**

71. $h(x) = (2x + 1)^2$

72. $h(x) = (1 - x)^3$

73. $h(x) = \sqrt[3]{x^2 - 4}$

74. $h(x) = \sqrt{9 - x}$

75. $h(x) = \dfrac{1}{x + 2}$

76. $h(x) = \dfrac{4}{(5x + 2)^2}$

77. $h(x) = (x + 4)^2 + 2(x + 4)$

78. $h(x) = (x + 3)^{3/2} + 4(x + 3)^{1/2}$

79. ***Why you should learn it*** *(p. 50)* The research and development department of an automobile manufacturer has determined that when required to stop quickly to avoid an accident, the distance (in feet) a car travels during the driver's reaction time is given by

$$R(x) = \tfrac{3}{4}x$$

where x is the speed of the car in miles per hour. The distance (in feet) traveled while the driver is braking is given by

$$B(x) = \tfrac{1}{15}x^2.$$

(a) Find the function that represents the total stopping distance T.

(b) Use a graphing utility to graph the functions R, B, and T in the same viewing window for $0 \le x \le 60$.

(c) Which function contributes most to the magnitude of the sum at higher speeds? Explain.

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80. MODELING DATA

The table shows the total amounts (in billions of dollars) of private expenditures on health services and supplies in the United States (including Puerto Rico) for the years 1997 through 2007. The variables y_1, y_2, and y_3 represent out-of-pocket payments, insurance premiums, and other types of payments, respectively. (Source: U.S. Centers for Medicare and Medicaid Services)

Year	y_1	y_2	y_3
1997	162	359	52
1998	175	385	56
1999	184	417	59
2000	193	455	58
2001	200	498	58
2002	211	551	59
2003	225	604	65
2004	235	646	66
2005	247	690	70
2006	255	731	75
2007	269	775	80

The data are approximated by the following models, where t represents the year, with $t = 7$ corresponding to 1997.

$y_1 = 10.5t + 88$

$y_2 = 0.66t^2 + 27.6t + 123$

$y_3 = 0.23t^2 - 3.0t + 64$

(a) Use the models and the *table* feature of a graphing utility to create a table showing the values of y_1, y_2, and y_3 for each year from 1997 through 2007. Compare these models with the original data. Are the models a good fit? Explain.

(b) Use the graphing utility to graph y_1, y_2, y_3, and $y_T = y_1 + y_2 + y_3$ in the same viewing window. What does the function y_T represent?

81. Geometry A square concrete foundation was prepared as a base for a large cylindrical gasoline tank (see figure).

(a) Write the radius r of the tank as a function of the length x of the sides of the square.

(b) Write the area A of the circular base of the tank as a function of the radius r.

(c) Find and interpret $(A \circ r)(x)$.

82. Geometry A pebble is dropped into a calm pond, causing ripples in the form of concentric circles. The radius (in feet) of the outermost ripple is given by $r(t) = 0.6t$, where t is the time (in seconds) after the pebble strikes the water. The area of the circle is given by $A(r) = \pi r^2$. Find and interpret $(A \circ r)(t)$.

83. Business A company owns two retail stores. The annual sales (in thousands of dollars) of the stores each year from 2004 through 2010 can be approximated by the models

$$S_1 = 830 + 1.2t^2 \quad \text{and} \quad S_2 = 390 + 75.4t$$

where t is the year, with $t = 4$ corresponding to 2004.

(a) Write a function T that represents the total annual sales of the two stores.

(b) Use a graphing utility to graph S_1, S_2, and T in the same viewing window.

84. Business The annual cost C (in thousands of dollars) and revenue R (in thousands of dollars) for a company each year from 2004 through 2010 can be approximated by the models

$$C = 260 - 8t + 1.6t^2 \quad \text{and} \quad R = 320 + 2.8t$$

where t is the year, with $t = 4$ corresponding to 2004.

(a) Write a function P that represents the annual profits of the company.

(b) Use a graphing utility to graph C, R, and P in the same viewing window.

✓ **85. Biology** The number of bacteria in a refrigerated food product is given by

$$N(T) = 10T^2 - 20T + 600, \quad 1 \le T \le 20$$

where T is the temperature of the food in degrees Celsius. When the food is removed from the refrigerator, the temperature of the food is given by

$$T(t) = 2t + 1$$

where t is the time in hours.

(a) Find the composite function $N(T(t))$ or $(N \circ T)(t)$ and interpret its meaning in the context of the situation.

(b) Find $(N \circ T)(6)$ and interpret its meaning.

(c) Find the time when the bacteria count reaches 800.

86. Environmental Science The spread of a contaminant is increasing in a circular pattern on the surface of a lake. The radius of the contaminant can be modeled by $r(t) = 5.25\sqrt{t}$, where r is the radius in meters and t is time in hours since contamination.

(a) Find a function that gives the area A of the circular leak in terms of the time t since the spread began.

(b) Find the size of the contaminated area after 36 hours.

(c) Find when the size of the contaminated area is 6250 square meters.

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87. **Air Traffic Control** An air traffic controller spots two planes flying at the same altitude. Their flight paths form a right angle at point P. One plane is 150 miles from point P and is moving at 450 miles per hour. The other plane is 200 miles from point P and is moving at 450 miles per hour. Write the distance s between the planes as a function of time t.

88. **Marketing** The suggested retail price of a new car is p dollars. The dealership advertised a factory rebate of \$1200 and an 8% discount.

(a) Write a function R in terms of p giving the cost of the car after receiving the rebate from the factory.

(b) Write a function S in terms of p giving the cost of the car after receiving the dealership discount.

(c) Form the composite functions $(R \circ S)(p)$ and $(S \circ R)(p)$ and interpret each.

(d) Find $(R \circ S)(18{,}400)$ and $(S \circ R)(18{,}400)$. Which yields the lower cost for the car? Explain.

Conclusions

True or False? **In Exercises 89 and 90, determine whether the statement is true or false. Justify your answer.**

89. A function that represents the graph of $f(x) = x^2$ shifted three units to the right is $f(g(x))$, where $g(x) = x + 3$.

90. Given two functions $f(x)$ and $g(x)$, you can calculate $(f \circ g)(x)$ if and only if the range of g is a subset of the domain of f.

Exploration **In Exercises 91 and 92, three siblings are of three different ages. The oldest is twice the age of the middle sibling, and the middle sibling is six years older than one-half the age of the youngest.**

91. (a) Write a composite function that gives the oldest sibling's age in terms of the youngest. Explain how you arrived at your answer.

(b) The oldest sibling is 16 years old. Find the ages of the other two siblings.

92. (a) Write a composite function that gives the youngest sibling's age in terms of the oldest. Explain how you arrived at your answer.

(b) The youngest sibling is two years old. Find the ages of the other two siblings.

93. **Proof** Prove that the product of two odd functions is an even function, and that the product of two even functions is an even function.

94. **Proof** Use examples to hypothesize whether the product of an odd function and an even function is even or odd. Then prove your hypothesis.

95. **Proof** Given a function f, prove that $g(x)$ is even and $h(x)$ is odd, where $g(x) = \frac{1}{2}[f(x) + f(-x)]$ and $h(x) = \frac{1}{2}[f(x) - f(-x)]$.

96. (a) Use the result of Exercise 95 to prove that any function can be written as a sum of even and odd functions. (*Hint:* Add the two equations in Exercise 95.)

(b) Use the result of part (a) to write each function as a sum of even and odd functions.

$$f(x) = x^2 - 2x + 1, \quad g(x) = \frac{1}{x+1}$$

97. **Exploration** The function in Example 9 can be decomposed in other ways. For which of the following pairs of functions is $h(x) = \dfrac{1}{(x-2)^2}$ equal to $f(g(x))$?

(a) $g(x) = \dfrac{1}{x-2}$ and $f(x) = x^2$

(b) $g(x) = x^2$ and $f(x) = \dfrac{1}{x-2}$

(c) $g(x) = (x-2)^2$ and $f(x) = \dfrac{1}{x}$

98. **CAPSTONE** Consider the functions $f(x) = x^2$ and $g(x) = \sqrt{x}$. Describe the restrictions that need to be made on the domains of f and g so that $f(g(x)) = g(f(x))$.

Cumulative Mixed Review

Evaluating an Equation **In Exercises 99–102, find three points that lie on the graph of the equation. (There are many correct answers.)**

99. $y = -x^2 + x - 5$

100. $y = \frac{1}{5}x^3 - 4x^2 + 1$

101. $x^2 + y^2 = 24$

102. $y = \dfrac{x}{x^2 - 5}$

Finding the Slope-Intercept Form **In Exercises 103–106, find the slope-intercept form of the equation of the line that passes through the two points.**

103. $(-4, -2), (-3, 8)$

104. $(1, 5), (-8, 2)$

105. $\left(\frac{3}{2}, -1\right), \left(-\frac{1}{3}, 4\right)$

106. $(0, 1.1), (-4, 3.1)$

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1.6 Inverse Functions

Inverse Functions

Recall from Section 1.2 that a function can be represented by a set of ordered pairs. For instance, the function $f(x) = x + 4$ from the set $A = \{1, 2, 3, 4\}$ to the set $B = \{5, 6, 7, 8\}$ can be written as follows.

$$f(x) = x + 4\text{: } \{(1, 5), (2, 6), (3, 7), (4, 8)\}$$

In this case, by interchanging the first and second coordinates of each of these ordered pairs, you can form the **inverse function** of f, which is denoted by f^{-1}. It is a function from the set B to the set A, and can be written as follows.

$$f^{-1}(x) = x - 4\text{: } \{(5, 1), (6, 2), (7, 3), (8, 4)\}$$

Note that the domain of f is equal to the range of f^{-1}, and vice versa, as shown in Figure 1.56. Also note that the functions f and f^{-1} have the effect of "undoing" each other. In other words, when you form the composition of f with f^{-1} or the composition of f^{-1} with f, you obtain the identity function.

$$f(f^{-1}(x)) = f(x - 4) = (x - 4) + 4 = x$$

$$f^{-1}(f(x)) = f^{-1}(x + 4) = (x + 4) - 4 = x$$

Figure 1.56

What you should learn

- Find inverse functions informally and verify that two functions are inverse functions of each other.
- Use graphs of functions to decide whether functions have inverse functions.
- Determine whether functions are one-to-one.
- Find inverse functions algebraically.

Why you should learn it

Inverse functions can be helpful in further exploring how two variables relate to each other. For example, in Exercise 115 on page 69, you will use inverse functions to find the European shoe sizes from the corresponding U.S. shoe sizes.

Example 1 Finding Inverse Functions Informally

Find the inverse function of

$$f(x) = 4x.$$

Then verify that both $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to the identity function.

Solution

The function f *multiplies* each input by 4. To "undo" this function, you need to *divide* each input by 4. So, the inverse function of $f(x) = 4x$ is given by

$$f^{-1}(x) = \frac{x}{4}.$$

You can verify that both $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to the identity function as follows.

$$f(f^{-1}(x)) = f\left(\frac{x}{4}\right) = 4\left(\frac{x}{4}\right) = x$$

$$f^{-1}(f(x)) = f^{-1}(4x) = \frac{4x}{4} = x$$

CHECKPOINT Now try Exercise 7.

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Don't be confused by the use of the exponent -1 to denote the inverse function f^{-1}. In this text, whenever f^{-1} is written, it always refers to the inverse function of the function f and not to the reciprocal of $f(x)$, which is given by

$$\frac{1}{f(x)}.$$

Example 2 Finding Inverse Functions Informally

Find the inverse function of $f(x) = x - 6$. Then verify that both $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to the identity function.

Solution

The function f *subtracts* 6 from each input. To "undo" this function, you need to *add* 6 to each input. So, the inverse function of $f(x) = x - 6$ is given by

$$f^{-1}(x) = x + 6.$$

You can verify that both $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to the identity function as follows.

$$f(f^{-1}(x)) = f(x + 6) = (x + 6) - 6 = x$$

$$f^{-1}(f(x)) = f^{-1}(x - 6) = (x - 6) + 6 = x$$

CHECKPOINT Now try Exercise 9.

A table of values can help you understand inverse functions. For instance, the first table below shows several values of the function in Example 2. Interchange the rows of this table to obtain values of the inverse function.

x	-2	-1	0	1	2
$f(x)$	-8	-7	-6	-5	-4

x	-8	-7	-6	-5	-4
$f^{-1}(x)$	-2	-1	0	1	2

In the table at the left, each output is 6 less than the input, and in the table at the right, each output is 6 more than the input.

The formal definition of an inverse function is as follows.

Definition of Inverse Function

Let f and g be two functions such that

$$f(g(x)) = x \qquad \text{for every } x \text{ in the domain of } g$$

and

$$g(f(x)) = x \qquad \text{for every } x \text{ in the domain of } f.$$

Under these conditions, the function g is the **inverse function** of the function f. The function g is denoted by f^{-1} (read "f-inverse"). So,

$$f(f^{-1}(x)) = x \qquad \text{and} \qquad f^{-1}(f(x)) = x.$$

The domain of f must be equal to the range of f^{-1}, and the range of f must be equal to the domain of f^{-1}.

If the function g is the inverse function of the function f, then it must also be true that the function f is the inverse function of the function g. For this reason, you can say that the functions f and g are *inverse functions of each other*.

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Example 3 Verifying Inverse Functions Algebraically

Show that the functions are inverse functions of each other.

$$f(x) = 2x^3 - 1 \qquad \text{and} \qquad g(x) = \sqrt[3]{\frac{x+1}{2}}$$

Solution

$$\begin{aligned} f(g(x)) &= f\left(\sqrt[3]{\frac{x+1}{2}}\right) \\ &= 2\left(\sqrt[3]{\frac{x+1}{2}}\right)^3 - 1 \\ &= 2\left(\frac{x+1}{2}\right) - 1 \\ &= x + 1 - 1 \\ &= x \end{aligned}$$

$$\begin{aligned} g(f(x)) &= g(2x^3 - 1) \\ &= \sqrt[3]{\frac{(2x^3 - 1) + 1}{2}} \\ &= \sqrt[3]{\frac{2x^3}{2}} \\ &= \sqrt[3]{x^3} \\ &= x \end{aligned}$$

CHECKPOINT Now try Exercise 19.

Technology Tip

Most graphing utilities can graph $y = x^{1/3}$ in two ways:

$y_1 = x \wedge (1/3)$ or

$y_1 = \sqrt[3]{x}$.

On some graphing utilities, you may not be able to obtain the complete graph of $y = x^{2/3}$ by entering $y_1 = x \wedge (2/3)$. If not, you should use

$y_1 = (x \wedge (1/3))^2$ or

$y_1 = \sqrt[3]{x^2}$.

Example 4 Verifying Inverse Functions Algebraically

Which of the functions is the inverse function of $f(x) = \dfrac{5}{x-2}$?

$$g(x) = \frac{x-2}{5} \qquad \text{or} \qquad h(x) = \frac{5}{x} + 2$$

Solution

By forming the composition of f with g, you have

$$f(g(x)) = f\left(\frac{x-2}{5}\right) = \frac{5}{\left(\dfrac{x-2}{5}\right) - 2} = \frac{25}{x - 12} \neq x.$$

Because this composition is not equal to the identity function x, it follows that g *is not* the inverse function of f. By forming the composition of f with h, you have

$$f(h(x)) = f\left(\frac{5}{x} + 2\right) = \frac{5}{\left(\dfrac{5}{x} + 2\right) - 2} = \frac{5}{\left(\dfrac{5}{x}\right)} = x.$$

So, it appears that h is the inverse function of f. You can confirm this by showing that the composition of h with f is also equal to the identity function.

CHECKPOINT Now try Exercise 23.

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The Graph of an Inverse Function

The graphs of a function f and its inverse function f^{-1} are related to each other in the following way. If the point

$$(a, b)$$

lies on the graph of f, then the point

$$(b, a)$$

must lie on the graph of f^{-1}, and vice versa. This means that the graph of f^{-1} is a reflection of the graph of f in the line $y = x$, as shown in Figure 1.57.

Figure 1.57

Technology Tip

Many graphing utilities have a built-in feature for drawing an inverse function. For instructions on how to use the *draw inverse* feature, see Appendix A; for specific keystrokes, go to this texbook's *Companion Website*.

Example 5 Verifying Inverse Functions Graphically

Verify that the functions f and g from Example 3 are inverse functions of each other graphically.

Solution

From Figure 1.58, you can conclude that f and g are inverse functions of each other.

Figure 1.58

CHECKPOINT Now try Exercise 33(b).

Example 6 Verifying Inverse Functions Numerically

Verify that the functions $f(x) = \dfrac{x - 5}{2}$ and $g(x) = 2x + 5$ are inverse functions of each other numerically.

Solution

You can verify that f and g are inverse functions of each other *numerically* by using a graphing utility. Enter $y_1 = f(x)$, $y_2 = g(x)$, $y_3 = f(g(x))$, and $y_4 = g(f(x))$, as shown in Figure 1.59. Then use the *table* feature to create a table (see Figure 1.60).

Figure 1.59

Figure 1.60

Note that the entries for x, y_3, and y_4 are the same. So, $f(g(x)) = x$ and $g(f(x)) = x$. You can conclude that f and g are inverse functions of each other.

CHECKPOINT Now try Exercise 33(c).

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The Existence of an Inverse Function

To have an inverse function, a function must be **one-to-one,** which means that no two elements in the domain of f correspond to the same element in the range of f.

Definition of a One-to-One Function

A function f is **one-to-one** when, for a and b in its domain, $f(a) = f(b)$ implies that $a = b$.

Existence of an Inverse Function

A function f has an inverse function f^{-1} if and only if f is one-to-one.

From its graph, it is easy to tell whether a function of x is one-to-one. Simply check to see that every horizontal line intersects the graph of the function at most once. This is called the **Horizontal Line Test.** For instance, Figure 1.61 shows the graph of $y = x^2$. On the graph, you can find a horizontal line that intersects the graph twice.

Figure 1.61 ***$f(x) = x^2$ is not one-to-one.***

Two special types of functions that pass the Horizontal Line Test are those that are increasing or decreasing on their entire domains.

1. If f is *increasing* on its entire domain, then f is one-to-one.

2. If f is *decreasing* on its entire domain, then f is one-to-one.

Example 7 Testing for One-to-One Functions

Is the function $f(x) = \sqrt{x} + 1$ one-to-one?

Algebraic Solution

Let a and b be nonnegative real numbers with $f(a) = f(b)$.

$$\sqrt{a} + 1 = \sqrt{b} + 1 \qquad \text{Set } f(a) = f(b).$$

$$\sqrt{a} = \sqrt{b}$$

$$a = b$$

So, $f(a) = f(b)$ implies that $a = b$. You can conclude that f *is* one-to-one and *does* have an inverse function.

CHECKPOINT Now try Exercise 67.

Graphical Solution

Figure 1.62

From Figure 1.62, you can conclude that f is one-to-one and *does* have an inverse function.

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Finding Inverse Functions Algebraically

For simple functions, you can find inverse functions by inspection. For more complicated functions, however, it is best to use the following guidelines.

Finding an Inverse Function

1. Use the Horizontal Line Test to decide whether f has an inverse function.
2. In the equation for $f(x)$, replace $f(x)$ by y.
3. Interchange the roles of x and y, and solve for y.
4. Replace y by $f^{-1}(x)$ in the new equation.
5. Verify that f and f^{-1} are inverse functions of each other by showing that the domain of f is equal to the range of f^{-1}, the range of f is equal to the domain of f^{-1}, and $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

What's Wrong?

You use a graphing utility to graph $y_1 = x^2$ and then use the *draw inverse* feature to conclude that $f(x) = x^2$ has an inverse function (see figure). What's wrong?

Example 8 Finding an Inverse Function Algebraically

Find the inverse function of

$$f(x) = \frac{5 - 3x}{2}.$$

Solution

The graph of f in Figure 1.63 passes the Horizontal Line Test. So, you know that f is one-to-one and has an inverse function.

Figure 1.63

$f(x) = \dfrac{5 - 3x}{2}$	Write original function.
$y = \dfrac{5 - 3x}{2}$	Replace $f(x)$ by y.
$x = \dfrac{5 - 3y}{2}$	Interchange x and y.
$2x = 5 - 3y$	Multiply each side by 2.
$3y = 5 - 2x$	Isolate the y-term.
$y = \dfrac{5 - 2x}{3}$	Solve for y.
$f^{-1}(x) = \dfrac{5 - 2x}{3}$	Replace y by $f^{-1}(x)$.

The domains and ranges of f and f^{-1} consist of all real numbers. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

CHECKPOINT Now try Exercise 71.

A function f with an implied domain of all real numbers may not pass the Horizontal Line Test. In this case, the domain of f may be restricted so that f does have an inverse function. For instance, when the domain of $f(x) = x^2$ is restricted to the nonnegative real numbers, then f does have an inverse function.

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Example 9 Finding an Inverse Function Algebraically

Find the inverse function of

$$f(x) = x^3 - 4.$$

Solution

The graph of f in Figure 1.64 passes the Horizontal Line Test. So, you know that f is one-to-one and has an inverse function.

Figure 1.64

$f(x) = x^3 - 4$	Write original function.
$y = x^3 - 4$	Replace $f(x)$ by y.
$x = y^3 - 4$	Interchange x and y.
$y^3 = x + 4$	Isolate y.
$y = \sqrt[3]{x + 4}$	Solve for y.
$f^{-1}(x) = \sqrt[3]{x + 4}$	Replace y by $f^{-1}(x)$.

The domains and ranges of f and f^{-1} consist of all real numbers. You can verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$ as follows.

$$\begin{aligned} f(f^{-1}(x)) &= f\left(\sqrt[3]{x + 4}\right) \\ &= \left(\sqrt[3]{x + 4}\right)^3 - 4 \\ &= x + 4 - 4 \\ &= x \end{aligned} \qquad \begin{aligned} f^{-1}(f(x)) &= f^{-1}(x^3 - 4) \\ &= \sqrt[3]{(x^3 - 4) + 4} \\ &= \sqrt[3]{x^3} \\ &= x \end{aligned}$$

CHECKPOINT Now try Exercise 73.

Example 10 Finding an Inverse Function Algebraically

Find the inverse function of

$$f(x) = \sqrt{2x - 3}.$$

Solution

The graph of f in Figure 1.65 passes the Horizontal Line Test. So, you know that f is one-to-one and has an inverse function.

Figure 1.65

$f(x) = \sqrt{2x - 3}$	Write original function.
$y = \sqrt{2x - 3}$	Replace $f(x)$ by y.
$x = \sqrt{2y - 3}$	Interchange x and y.
$x^2 = 2y - 3$	Square each side.
$2y = x^2 + 3$	Isolate y.
$y = \dfrac{x^2 + 3}{2}$	Solve for y.
$f^{-1}(x) = \dfrac{x^2 + 3}{2}, \quad x \geq 0$	Replace y by $f^{-1}(x)$.

Note that the range of f is the interval $[0, \infty)$, which implies that the domain of f^{-1} is the interval $[0, \infty)$. Moreover, the domain of f is the interval $\left[\frac{3}{2}, \infty\right)$, which implies that the range of f^{-1} is the interval $\left[\frac{3}{2}, \infty\right)$. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

CHECKPOINT Now try Exercise 77.

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1.6 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1–4, fill in the blank(s).

1. If f and g are functions such that $f(g(x)) = x$ and $g(f(x)) = x$, then the function g is the _____ function of f, and is denoted by _____ .
2. The domain of f is the _____ of f^{-1}, and the _____ of f^{-1} is the range of f.
3. The graphs of f and f^{-1} are reflections of each other in the line _____ .
4. To have an inverse function, a function f must be _____ ; that is, $f(a) = f(b)$ implies $a = b$.
5. How many times can a horizontal line intersect the graph of a function that is one-to-one?
6. Can $(1, 4)$ and $(2, 4)$ be two ordered pairs of a one-to-one function?

Procedures and Problem Solving

Finding Inverse Functions Informally **In Exercises 7–14, find the inverse function of f informally. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.**

7. $f(x) = 6x$
8. $f(x) = \frac{1}{3}x$
9. $f(x) = x + 7$
10. $f(x) = x - 3$
11. $f(x) = 2x + 1$
12. $f(x) = (x - 1)/4$
13. $f(x) = \sqrt[3]{x}$
14. $f(x) = x^5$

Identifying Graphs of Inverse Functions **In Exercises 15–18, match the graph of the function with the graph of its inverse function. [The graphs of the inverse functions are labeled (a), (b), (c), and (d).]**

(a)

(b)

(c)

(d)

15.

16.

17.

18.

Verifying Inverse Functions Algebraically **In Exercises 19–24, show that f and g are inverse functions algebraically. Use a graphing utility to graph f and g in the same viewing window. Describe the relationship between the graphs.**

19. $f(x) = x^3, \quad g(x) = \sqrt[3]{x}$
20. $f(x) = \dfrac{1}{x}, \quad g(x) = \dfrac{1}{x}$
21. $f(x) = \sqrt{x - 4}; \quad g(x) = x^2 + 4, \ x \geq 0$
22. $f(x) = 9 - x^2, \quad x \geq 0; \quad g(x) = \sqrt{9 - x}$
23. $f(x) = 1 - x^3, \quad g(x) = \sqrt[3]{1 - x}$
24. $f(x) = \dfrac{1}{1 + x}, \ x \geq 0; \quad g(x) = \dfrac{1 - x}{x}, \ 0 < x \leq 1$

Algebraic-Graphical-Numerical **In Exercises 25–34, show that f and g are inverse functions (a) algebraically, (b) graphically, and (c) numerically.**

25. $f(x) = -\dfrac{7}{2}x - 3, \quad g(x) = -\dfrac{2x + 6}{7}$
26. $f(x) = \dfrac{x - 9}{4}, \quad g(x) = 4x + 9$
27. $f(x) = x^3 + 5, \quad g(x) = \sqrt[3]{x - 5}$
28. $f(x) = \dfrac{x^3}{2}, \quad g(x) = \sqrt[3]{2x}$
29. $f(x) = -\sqrt{x - 8}, \quad g(x) = 8 + x^2, \quad x \leq 0$
30. $f(x) = \sqrt[3]{3x - 10}, \quad g(x) = \dfrac{x^3 + 10}{3}$
31. $f(x) = 2x, \quad g(x) = \dfrac{x}{2}$
32. $f(x) = x - 5, \quad g(x) = x + 5$
33. $f(x) = \dfrac{x - 1}{x + 5}, \quad g(x) = -\dfrac{5x + 1}{x - 1}$
34. $f(x) = \dfrac{x + 3}{x - 2}, \quad g(x) = \dfrac{2x + 3}{x - 1}$

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Identifying Whether Functions Have Inverses **In Exercises 35–38, does the function have an inverse? Explain.**

35.

Domain	*Range*
1 can	\$1
6 cans	\$5
12 cans	\$9
24 cans	\$16

36.

Domain	*Range*
1/2 hour	\$40
1 hour	\$70
2 hours	\$120
4 hours	

37. $\{(-3, 6), (-1, 5), (0, 6)\}$

38. $\{(2, 4), (3, 7), (7, 2)\}$

Recognizing One-to-One Functions **In Exercises 39–44, determine whether the graph is that of a function. If so, determine whether the function is one-to-one.**

39.

40.

41.

42.

43.

44.

Using the Horizontal Line Test **In Exercises 45–56, use a graphing utility to graph the function and use the Horizontal Line Test to determine whether the function is one-to-one and thus has an inverse function.**

45. $f(x) = 3 - \frac{1}{2}x$

46. $f(x) = \frac{1}{4}(x + 2)^2 - 1$

47. $h(x) = \dfrac{x^2}{x^2 + 1}$

48. $g(x) = \dfrac{4 - x}{6x^2}$

49. $h(x) = \sqrt{16 - x^2}$

50. $f(x) = -2x\sqrt{16 - x^2}$

51. $f(x) = 10$

52. $f(x) = -0.65$

53. $g(x) = (x + 5)^3$

54. $f(x) = x^5 - 7$

55. $h(x) = |x + 4| - |x - 4|$

56. $f(x) = -\dfrac{|x - 6|}{|x + 6|}$

Analyzing a Piecewise-Defined Function **In Exercises 57 and 58, sketch the graph of the piecewise-defined function by hand and use the graph to determine whether an inverse function exists.**

57. $f(x) = \begin{cases} x^2, & 0 \le x \le 1 \\ x, & x > 1 \end{cases}$

58. $f(x) = \begin{cases} (x - 2)^3, & x < 3 \\ (x - 4)^2, & x \ge 3 \end{cases}$

Testing for One-to-One Functions **In Exercises 59–70, determine algebraically whether the function is one-to-one. Verify your answer graphically. If the function is one-to-one, find its inverse.**

59. $f(x) = x^4$

60. $g(x) = x^2 - x^4$

61. $f(x) = \dfrac{3x + 4}{5}$

62. $f(x) = 3x + 5$

63. $f(x) = \dfrac{1}{x^2}$

64. $h(x) = \dfrac{4}{x^2}$

65. $f(x) = (x + 3)^2, \quad x \ge -3$

66. $q(x) = (x - 5)^2, \quad x \le 5$

✓ **67.** $f(x) = \sqrt{2x + 3}$

68. $f(x) = \sqrt{x - 2}$

69. $f(x) = |x - 2|, \quad x \le 2$

70. $f(x) = \dfrac{x^2}{x^2 + 1}$

Finding an Inverse Function Algebraically **In Exercises 71–80, find the inverse function of f algebraically. Use a graphing utility to graph both f and f^{-1} in the same viewing window. Describe the relationship between the graphs.**

✓ **71.** $f(x) = 2x - 3$

72. $f(x) = 3x$

✓ **73.** $f(x) = x^5$

74. $f(x) = x^3 + 1$

75. $f(x) = x^{3/5}$

76. $f(x) = x^2, \quad x \ge 0$

✓ **77.** $f(x) = \sqrt{4 - x^2}, \quad 0 \le x \le 2$

78. $f(x) = \sqrt{16 - x^2}, \quad -4 \le x \le 0$

79. $f(x) = \dfrac{4}{x}$

80. $f(x) = \dfrac{6}{\sqrt{x}}$

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Think About It **In Exercises 81–90, restrict the domain of the function f so that the function is one-to-one and has an inverse function. Then find the inverse function f^{-1}. State the domains and ranges of f and f^{-1}. Explain your results. (There are many correct answers.)**

81. $f(x) = (x - 2)^2$

82. $f(x) = 1 - x^4$

83. $f(x) = |x + 2|$

84. $f(x) = |x - 2|$

85. $f(x) = (x + 3)^2$

86. $f(x) = (x - 4)^2$

87. $f(x) = -2x^2 + 5$

88. $f(x) = \frac{1}{2}x^2 - 1$

89. $f(x) = |x - 4| + 1$

90. $f(x) = -|x - 1| - 2$

Using the Properties of Inverse Functions **In Exercises 91 and 92, use the graph of the function f to complete the table and sketch the graph of f^{-1}.**

91.

x	$f^{-1}(x)$
-4	
-2	
2	
3	

92.

x	$f^{-1}(x)$
-3	
-2	
0	
6	

Using Graphs to Evaluate a Function **In Exercises 93–100, use the graphs of $y = f(x)$ and $y = g(x)$ to evaluate the function.**

93. $f^{-1}(0)$

94. $g^{-1}(0)$

95. $(f \circ g)(2)$

96. $g(f(-4))$

97. $f^{-1}(g(0))$

98. $(g^{-1} \circ f)(3)$

99. $(g \circ f^{-1})(2)$

100. $(f^{-1} \circ g^{-1})(6)$

Using the *Draw Inverse* Feature **In Exercises 101–104, (a) use a graphing utility to graph the function f, (b) use the *draw inverse* feature of the graphing utility to draw the inverse relation of the function, and (c) determine whether the inverse relation is an inverse function. Explain your reasoning.**

101. $f(x) = x^3 + x + 1$

102. $f(x) = x\sqrt{4 - x^2}$

103. $f(x) = \dfrac{3x^2}{x^2 + 1}$

104. $f(x) = \dfrac{4x}{\sqrt{x^2 + 15}}$

Evaluating a Composition of Functions **In Exercises 105–110, use the functions $f(x) = \frac{1}{8}x - 3$ and $g(x) = x^3$ to find the indicated value or function.**

105. $(f^{-1} \circ g^{-1})(1)$

106. $(g^{-1} \circ f^{-1})(-3)$

107. $(f^{-1} \circ f^{-1})(6)$

108. $(g^{-1} \circ g^{-1})(-4)$

109. $(f \circ g)^{-1}$

110. $g^{-1} \circ f^{-1}$

Finding a Composition of Functions **In Exercises 111–114, use the functions $f(x) = x + 4$ and $g(x) = 2x - 5$ to find the specified function.**

111. $g^{-1} \circ f^{-1}$

112. $f^{-1} \circ g^{-1}$

113. $(f \circ g)^{-1}$

114. $(g \circ f)^{-1}$

115. ***Why you should learn it*** *(p. 60)* The table shows men's shoe sizes in the United States and the corresponding European shoe sizes. Let $y = f(x)$ represent the function that gives the men's European shoe size in terms of x, the men's U.S. size.

Men's U.S. shoe size	Men's European shoe size
8	41
9	42
10	43
11	45
12	46
13	47

(a) Is f one-to-one? Explain.

(b) Find $f(11)$.

(c) Find $f^{-1}(43)$, if possible.

(d) Find $f(f^{-1}(41))$.

(e) Find $f^{-1}(f(13))$.

116. **Fashion Design** Let $y = g(x)$ represent the function that gives the women's European shoe size in terms of x, the women's U.S. size. A women's U.S. size 6 shoe corresponds to a European size 38. Find $g^{-1}(g(6))$.

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117. **Military Science** You can encode and decode messages using functions and their inverses. To code a message, first translate the letters to numbers using 1 for "A," 2 for "B," and so on. Use 0 for a space. So, "A ball" becomes

1 0 2 1 12 12.

Then, use a one-to-one function to convert to coded numbers. Using $f(x) = 2x - 1$, "A ball" becomes

1 −1 3 1 23 23.

(a) Encode "Call me later" using the function $f(x) = 5x + 4$.

(b) Find the inverse function of $f(x) = 5x + 4$ and use it to decode 119 44 9 104 4 104 49 69 29.

118. **Production Management** Your wage is \$10.00 per hour plus \$0.75 for each unit produced per hour. So, your hourly wage y in terms of the number of units produced x is $y = 10 + 0.75x$.

(a) Find the inverse function. What does each variable in the inverse function represent?

(b) Use a graphing utility to graph the function and its inverse function.

(c) Use the *trace* feature of the graphing utility to find the hourly wage when 10 units are produced per hour.

(d) Use the *trace* feature of the graphing utility to find the number of units produced per hour when your hourly wage is \$21.25.

Conclusions

True or False? **In Exercises 119 and 120, determine whether the statement is true or false. Justify your answer.**

119. If f is an even function, then f^{-1} exists.

120. If the inverse function of f exists, and the graph of f has a y-intercept, then the y-intercept of f is an x-intercept of f^{-1}.

Think About It **In Exercises 121–124, determine whether the situation could be represented by a one-to-one function. If so, write a statement that describes the inverse function.**

121. The number of miles n a marathon runner has completed in terms of the time t in hours

122. The population p of a town in terms of the year t from 1990 through 2010 given that the population was greatest in 2000

123. The depth of the tide d at a beach in terms of the time t over a 24-hour period

124. The height h in inches of a human child from age 2 to age 14 in terms of his or her age n in years

125. **Writing** Describe the relationship between the graph of a function f and the graph of its inverse function f^{-1}.

126. **Think About It** The domain of a one-to-one function f is $[0, 9]$ and the range is $[-3, 3]$. Find the domain and range of f^{-1}.

127. **Think About It** The function $f(x) = \frac{9}{5}x + 32$ can be used to convert a temperature of x degrees Celsius to its corresponding temperature in degrees Fahrenheit.

(a) Using the expression for f, make a conceptual argument to show that f has an inverse function.

(b) What does $f^{-1}(50)$ represent?

128. **Think About It** A function f is increasing over its entire domain. Does f have an inverse function? Explain.

129. **Think About It** Describe a type of function that is *not* one-to-one on any interval of its domain.

130. **CAPSTONE** Decide whether the two functions shown in each graph appear to be inverse functions of each other. Explain your reasoning.

131. **Proof** Prove that if f and g are one-to-one functions, then $(f \circ g)^{-1}(x) = (g^{-1} \circ f^{-1})(x)$.

132. **Proof** Prove that if f is a one-to-one odd function, then f^{-1} is an odd function.

Cumulative Mixed Review

Simplifying a Rational Expression **In Exercises 133–136, write the rational expression in simplest form.**

133. $\dfrac{27x^3}{3x^2}$

134. $\dfrac{5x^2y}{xy + 5x}$

135. $\dfrac{x^2 - 36}{6 - x}$

136. $\dfrac{x^2 + 3x - 40}{x^2 - 3x - 10}$

Testing for Functions **In Exercises 137–140, determine whether the equation represents y as a function of x.**

137. $x = 5$

138. $y = \sqrt{x + 2}$

139. $x^2 + y^2 = 9$

140. $x - y^2 = 0$

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1.7 Linear Models and Scatter Plots

What you should learn

- Construct scatter plots and interpret correlation.
- Use scatter plots and a graphing utility to find linear models for data.

Why you should learn it

Real-life data often follow a linear pattern. For instance, in Exercise 25 on page 79, you will find a linear model for the winning times in the women's 400-meter freestyle Olympic swimming event.

Scatter Plots and Correlation

Many real-life situations involve finding relationships between two variables, such as the year and the number of employees in the cellular telecommunications industry. In a typical situation, data are collected and written as a set of ordered pairs. The graph of such a set is called a *scatter plot*. (For a brief discussion of scatter plots, see Appendix B.1.)

Example 1 Constructing a Scatter Plot

The data in the table show the numbers E (in thousands) of employees in the cellular telecommunications industry in the United States from 2002 through 2007. Construct a scatter plot of the data. (Source: CTIA–The Wireless Association)

Year	Employees in the cellular telecommunications industry, E (in thousands)
2002	192
2003	206
2004	226
2005	233
2006	254
2007	267

Solution

Begin by representing the data with a set of ordered pairs. Let t represent the year, with $t = 2$ corresponding to 2002.

$$(2, 192), (3, 206), (4, 226),$$

$$(5, 233), (6, 254), (7, 267)$$

Then plot each point in a coordinate plane, as shown in Figure 1.66.

Figure 1.66

CHECKPOINT Now try Exercise 5.

From the scatter plot in Figure 1.66, it appears that the points describe a relationship that is nearly linear. The relationship is not *exactly* linear because the number of employees did not increase by precisely the same amount each year.

A mathematical equation that approximates the relationship between t and E is a *mathematical model*. When developing a mathematical model to describe a set of data, you strive for two (often conflicting) goals—accuracy and simplicity. For the data above, a linear model of the form

$$E = at + b$$

(where a and b are constants) appears to be best. It is simple and relatively accurate.

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Consider a collection of ordered pairs of the form (x, y). If y tends to increase as x increases, then the collection is said to have a **positive correlation.** If y tends to decrease as x increases, then the collection is said to have a **negative correlation.** Figure 1.67 shows three examples: one with a positive correlation, one with a negative correlation, and one with no (discernible) correlation.

Positive Correlation ***Negative Correlation*** ***No Correlation***

Figure 1.67

Example 2 Interpreting Correlation

On a Friday, 22 students in a class were asked to record the numbers of hours they spent studying for a test on Monday and the numbers of hours they spent watching television. The results are shown below. (The first coordinate is the number of hours and the second coordinate is the score obtained on the test.)

Study Hours: (0, 40), (1, 41), (2, 51), (3, 58), (3, 49), (4, 48), (4, 64), (5, 55), (5, 69), (5, 58), (5, 75), (6, 68), (6, 63), (6, 93), (7, 84), (7, 67), (8, 90), (8, 76), (9, 95), (9, 72), (9, 85), (10, 98)

TV Hours: (0, 98), (1, 85), (2, 72), (2, 90), (3, 67), (3, 93), (3, 95), (4, 68), (4, 84), (5, 76), (7, 75), (7, 58), (9, 63), (9, 69), (11, 55), (12, 58), (14, 64), (16, 48), (17, 51), (18, 41), (19, 49), (20, 40)

a. Construct a scatter plot for each set of data.

b. Determine whether the points are positively correlated, are negatively correlated, or have no discernible correlation. What can you conclude?

Solution

a. Scatter plots for the two sets of data are shown in Figure 1.68.

b. The scatter plot relating study hours and test scores has a positive correlation. This means that the more a student studied, the higher his or her score tended to be. The scatter plot relating television hours and test scores has a negative correlation. This means that the more time a student spent watching television, the lower his or her score tended to be.

Figure 1.68

CHECKPOINT Now try Exercise 7.

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Fitting a Line to Data

Finding a linear model to represent the relationship described by a scatter plot is called **fitting a line to data.** You can do this graphically by simply sketching the line that appears to fit the points, finding two points on the line, and then finding the equation of the line that passes through the two points.

Example 3 Fitting a Line to Data

Find a linear model that relates the year to the number of employees in the cellular telecommunications industry in the United States. (See Example 1.)

Year	Employees in the cellular telecommunications industry, E (in thousands)
2002	192
2003	206
2004	226
2005	233
2006	254
2007	267

Figure 1.69

Solution

Let t represent the year, with $t = 2$ corresponding to 2002. After plotting the data in the table, draw the line that you think best represents the data, as shown in Figure 1.69. Two points that lie on this line are

$$(3, 206) \quad \text{and} \quad (6, 254).$$

Using the point-slope form, you can find the equation of the line to be

$$E = 16(t - 3) + 206$$
$$= 16t + 158. \qquad \text{Linear model}$$

CHECKPOINT Now try Exercise 11.

Once you have found a model, you can measure how well the model fits the data by comparing the actual values with the values given by the model, as shown in the following table.

	t	2	3	4	5	6	7
Actual	E	192	206	226	233	254	267
Model	E	190	206	222	238	254	270

The sum of the squares of the differences between the actual values and the model values is called the **sum of the squared differences.** The model that has the least sum is called the **least squares regression line** for the data. For the model in Example 3, the sum of the squared differences is 54. The least squares regression line for the data is

$$E = 15.0t + 162. \qquad \text{Best-fitting linear model}$$

Its sum of squared differences is 37. For more on the least squares regression line, see Appendix C.2 at this textbook's *Companion Website*.

Study Tip

The model in Example 3 is based on the two data points chosen. When different points are chosen, the model may change somewhat. For instance, when you choose (5, 233) and (7, 267), the new model is

$$E = 17(t - 5) + 233$$
$$= 17t + 148.$$

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Another way to find a linear model to represent the relationship described by a scatter plot is to enter the data points into a graphing utility and use the *linear regression* feature. This method is demonstrated in Example 4.

Technology Tip

For instructions on how to use the *linear regression* feature, see Appendix A; for specific keystrokes, go to this textbook's *Companion Website*.

Example 4 A Mathematical Model

The data in the table show the estimated numbers V (in thousands) of alternative-fueled vehicles in use in the United States from 2001 through 2007. (Source: Energy Information Administration)

Year	Alternative-fueled vehicles in use, V (in thousands)
2001	425
2002	471
2003	534
2004	565
2005	592
2006	635
2007	696

a. Use the *regression* feature of a graphing utility to find a linear model for the data. Let t represent the year, with $t = 1$ corresponding to 2001.

b. How closely does the model represent the data?

Graphical Solution

a. Use the *linear regression* feature of a graphing utility to obtain the model shown in Figure 1.70.

Figure 1.70

You can approximate the model to be $V = 42.8t + 388$.

b. Graph the actual data and the model. From Figure 1.71, it appears that the model is a good fit for the actual data.

Figure 1.71

CHECKPOINT Now try Exercise 15.

Numerical Solution

a. Using the *linear regression* feature of a graphing utility, you can find that a linear model for the data is $V = 42.8t + 388$.

b. You can see how well the model fits the data by comparing the actual values of V with the values of V given by the model, which are labeled V^* in the table below. From the table, you can see that the model appears to be a good fit for the actual data.

Year	V	V^*
2001	425	431
2002	471	474
2003	534	516
2004	565	559
2005	592	602
2006	635	645
2007	696	688

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When you use the *regression* feature of a graphing calculator or computer program to find a linear model for data, you will notice that the program may also output an "r-value." For instance, the r-value from Example 4 was $r \approx 0.994$. This r-value is the **correlation coefficient** of the data and gives a measure of how well the model fits the data. The correlation coefficient r varies between -1 and 1. Basically, the closer $|r|$ is to 1, the better the points can be described by a line. Three examples are shown in Figure 1.72.

$r = 0.972$ $r = -0.856$ $r = 0.190$

Figure 1.72

Technology Tip

For some calculators, the *diagnostics on* feature must be selected before the *regression* feature is used in order to see the value of the correlation coefficient r. To learn how to use this feature, consult your user's manual.

Example 5 Finding a Least Squares Regression Line

The following ordered pairs (w, h) represent the shoe sizes w and the heights h (in inches) of 25 men. Use the *regression* feature of a graphing utility to find the least squares regression line for the data.

(10.0, 70.5)	(10.5, 71.0)	(9.5, 69.0)	(11.0, 72.0)	(12.0, 74.0)
(8.5, 67.0)	(9.0, 68.5)	(13.0, 76.0)	(10.5, 71.5)	(10.5, 70.5)
(10.0, 71.0)	(9.5, 70.0)	(10.0, 71.0)	(10.5, 71.0)	(11.0, 71.5)
(12.0, 73.5)	(12.5, 75.0)	(11.0, 72.0)	(9.0, 68.0)	(10.0, 70.0)
(13.0, 75.5)	(10.5, 72.0)	(10.5, 71.0)	(11.0, 73.0)	(8.5, 67.5)

Solution

After entering the data into a graphing utility (see Figure 1.73), you obtain the model shown in Figure 1.74. So, the least squares regression line for the data is

$$h = 1.84w + 51.9.$$

In Figure 1.75, this line is plotted with the data. Note that the plot does not have 25 points because some of the ordered pairs graph as the same point. The correlation coefficient for this model is $r \approx 0.981$, which implies that the model is a good fit for the data.

Figure 1.73

Figure 1.74

Figure 1.75

CHECKPOINT Now try Exercise 25.

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1.7 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1 and 2, fill in the blank.

1. Consider a collection of ordered pairs of the form (x, y). If y tends to increase as x increases, then the collection is said to have a _____ correlation.

2. To find the least squares regression line for data, you can use the _____ feature of a graphing utility.

3. In a collection of ordered pairs (x, y), y tends to decrease as x increases. Does the collection have a positive correlation or a negative correlation?

4. You find the least squares regression line for a set of data. The correlation coefficient is 0.114. Is the model a good fit?

Procedures and Problem Solving

✓ **5. Constructing a Scatter Plot** The following ordered pairs give the years of experience x for 15 sales representatives and the monthly sales y (in thousands of dollars).

(1.5, 41.7), (1.0, 32.4), (0.3, 19.2), (3.0, 48.4), (4.0, 51.2), (0.5, 28.5), (2.5, 50.4), (1.8, 35.5), (2.0, 36.0), (1.5, 40.0), (3.5, 50.3), (4.0, 55.2), (0.5, 29.1), (2.2, 43.2), (2.0, 41.6)

(a) Create a scatter plot of the data.

(b) Does the relationship between x and y appear to be approximately linear? Explain.

6. Constructing a Scatter Plot The following ordered pairs give the scores on two consecutive 15-point quizzes for a class of 18 students.

(7, 13), (9, 7), (14, 14), (15, 15), (10, 15), (9, 7), (14, 11), (14, 15), (8, 10), (9, 10), (15, 9), (10, 11), (11, 14), (7, 14), (11, 10), (14, 11), (10, 15), (9, 6)

(a) Create a scatter plot of the data.

(b) Does the relationship between consecutive quiz scores appear to be approximately linear? If not, give some possible explanations.

Interpreting Correlation **In Exercises 7–10, the scatter plot of a set of data is shown. Determine whether the points are positively correlated, are negatively correlated, or have no discernible correlation.**

✓ **7.**

8.

9.

10.

Fitting a Line to Data **In Exercises 11–14, (a) create a scatter plot of the data, (b) draw a line of best fit that passes through two of the points, and (c) use the two points to find an equation of the line.**

✓ **11.** $(-3, -3)$, $(3, 4)$, $(1, 1)$, $(3, 2)$, $(4, 4)$, $(-1, -1)$

12. $(-2, 3)$, $(-2, 4)$, $(-1, 2)$, $(1, -2)$, $(0, 0)$, $(0, 1)$

13. $(0, 2)$, $(-2, 1)$, $(3, 3)$, $(1, 3)$, $(4, 4)$

14. $(3, 2)$, $(2, 3)$, $(1, 5)$, $(4, 0)$, $(5, 0)$

A Mathematical Model **In Exercises 15 and 16, use the *regression* feature of a graphing utility to find a linear model for the data. Then use the graphing utility to decide how closely the model fits the data (a) graphically and (b) numerically. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

✓ **15.**

16.

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17. MODELING DATA

Hooke's Law states that the force F required to compress or stretch a spring (within its elastic limits) is proportional to the distance d that the spring is compressed or stretched from its original length. That is, $F = kd$, where k is the measure of the stiffness of the spring and is called the *spring constant*. The table shows the elongation d in centimeters of a spring when a force of F kilograms is applied.

Force, F	Elongation, d
20	1.4
40	2.5
60	4.0
80	5.3
100	6.6

(a) Sketch a scatter plot of the data.

(b) Find the equation of the line that seems to best fit the data.

(c) Use the *regression* feature of a graphing utility to find a linear model for the data.

(d) Use the model from part (c) to estimate the elongation of the spring when a force of 55 kilograms is applied.

18. MODELING DATA

The numbers of subscribers S (in millions) to wireless networks from 2002 through 2008 are shown in the table. (Source: CTIA–The Wireless Association)

Year	Subscribers, S (in millions)
2002	140.8
2003	158.7
2004	182.1
2005	207.9
2006	233.0
2007	255.4
2008	270.3

(a) Use a graphing utility to create a scatter plot of the data, with $t = 2$ corresponding to 2002.

(b) Use the *regression* feature of the graphing utility to find a linear model for the data.

(c) Use the graphing utility to plot the data and graph the model in the same viewing window. Is the model a good fit? Explain.

(d) Use the model to predict the number of subscribers in 2015. Is your answer reasonable? Explain.

19. MODELING DATA

The total player payrolls T (in millions of dollars) for the Pittsburgh Steelers from 2004 through 2008 are shown in the table. (Source: USA Today)

Year	Total player payroll, T (in millions of dollars)
2004	78.0
2005	84.2
2006	94.0
2007	106.3
2008	128.8

(a) Use a graphing utility to create a scatter plot of the data, with $t = 4$ corresponding to 2004.

(b) Use the *regression* feature of the graphing utility to find a linear model for the data.

(c) Use the graphing utility to plot the data and graph the model in the same viewing window. Is the model a good fit? Explain.

(d) Use the model to predict the payrolls in 2010 and 2015. Do the results seem reasonable? Explain.

(e) What is the slope of your model? What does it tell you about the player payroll?

20. MODELING DATA

The mean salaries S (in thousands of dollars) of public school teachers in the United States from 2002 through 2008 are shown in the table. (Source: National Education Association)

Year	Mean salary, S (in thousands of dollars)
2002	44.7
2003	45.7
2004	46.6
2005	47.7
2006	49.0
2007	50.8
2008	52.3

(a) Use a graphing utility to create a scatter plot of the data, with $t = 2$ corresponding to 2002.

(b) Use the *regression* feature of the graphing utility to find a linear model for the data.

(c) Use the graphing utility to plot the data and graph the model in the same viewing window. Is the model a good fit? Explain.

(d) Use the model to predict the mean salaries in 2016 and 2018. Do the results seem reasonable? Explain.

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21. MODELING DATA

The projected populations P (in thousands) of New Jersey for selected years, based on the 2000 census, are shown in the table. (Source: U.S. Census Bureau)

Year	Population, P (in thousands)
2010	9018
2015	9256
2020	9462
2025	9637
2030	9802

(a) Use a graphing utility to create a scatter plot of the data, with $t = 10$ corresponding to 2010.

(b) Use the *regression* feature of the graphing utility to find a linear model for the data.

(c) Use the graphing utility to plot the data and graph the model in the same viewing window.

(d) Create a table showing the actual values of P and the values of P given by the model. How closely does the model fit the data?

(e) Use the model to predict the population of New Jersey in 2050. Does the result seem reasonable? Explain.

22. MODELING DATA

The projected populations P (in thousands) of Wyoming for selected years, based on the 2000 census, are shown in the table. (Source: U.S. Census Bureau)

Year	Population, P (in thousands)
2010	520
2015	528
2020	531
2025	529
2030	523

(a) Use a graphing utility to create a scatter plot of the data, with $t = 10$ corresponding to 2010.

(b) Use the *regression* feature of the graphing utility to find a linear model for the data.

(c) Use the graphing utility to plot the data and graph the model in the same viewing window.

(d) Create a table showing the actual values of P and the values of P given by the model. How closely does the model fit the data?

(e) Use the model to predict the population of Wyoming in 2050. Does the result seem reasonable? Explain.

23. MODELING DATA

The table shows the advertising expenditures x and sales volumes y for a company for seven randomly selected months. Both are measured in thousands of dollars.

Month	Advertising expenditures, x	Sales volume, y
1	2.4	202
2	1.6	184
3	2.0	220
4	2.6	240
5	1.4	180
6	1.6	164
7	2.0	186

(a) Use the *regression* feature of a graphing utility to find a linear model for the data.

(b) Use the graphing utility to plot the data and graph the model in the same viewing window.

(c) Interpret the slope of the model in the context of the problem.

(d) Use the model to estimate sales for advertising expenditures of $1500.

24. MODELING DATA

The table shows the numbers T of stores owned by the Target Corporation from 2000 through 2008. (Source: Target Corp.)

Year	Number of stores, T
2000	1307
2001	1381
2002	1475
2003	1553
2004	1308
2005	1397
2006	1488
2007	1591
2008	1682

(a) Use a graphing utility to make a scatter plot of the data, with $t = 0$ corresponding to 2000. Identify two sets of points in the scatter plot that are approximately linear.

(b) Use the *regression* feature of the graphing utility to find a linear model for each set of points.

(c) Write a piecewise-defined model for the data. Use the graphing utility to graph the piecewise-defined model.

(d) Describe a scenario that could be the cause of the break in the data.

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25. ***Why you should learn it*** *(p. 71)* The following ordered pairs (t, T) represent the Olympic year t and the winning time T (in minutes) in the women's 400-meter freestyle swimming event. (Source: International Olympic Committee)

(1952, 5.20)	(1972, 4.32)	(1992, 4.12)
(1956, 4.91)	(1976, 4.16)	(1996, 4.12)
(1960, 4.84)	(1980, 4.15)	(2000, 4.10)
(1964, 4.72)	(1984, 4.12)	(2004, 4.09)
(1968, 4.53)	(1988, 4.06)	(2008, 4.05)

(a) Use the *regression* feature of a graphing utility to find a linear model for the data and to identify the correlation coefficient. Let t represent the year, with $t = 0$ corresponding to 1950.

(b) What information is given by the sign of the slope of the model?

(c) Use the graphing utility to plot the data and graph the model in the same viewing window.

(d) Create a table showing the actual values of y and the values of y given by the model. How closely does the model fit the data?

(e) How can you use the value of the correlation coefficient to help answer the question in part (d)?

(f) Would you use the model to predict the winning times in the future? Explain.

26. MODELING DATA

In a study, 60 colts were measured every 14 days from birth. The ordered pairs (d, l) represent the average length l (in centimeters) of the 60 colts d days after birth: (14, 81.2), (28, 87.1), (42, 93.7), (56, 98.3), (70, 102.4), (84, 106.2), and (98, 110.0). (Source: American Society of Animal Science)

(a) Use the *regression* feature of a graphing utility to find a linear model for the data and to identify the correlation coefficient.

(b) According to the correlation coefficient, does the model represent the data well? Explain.

(c) Use the graphing utility to plot the data and graph the model in the same viewing window. How closely does the model fit the data?

(d) Use the model to predict the average length of a colt 112 days after birth.

Conclusions

True or False? In Exercises 27 and 28, determine whether the statement is true or false. Justify your answer.

27. A linear regression model with a positive correlation will have a slope that is greater than 0.

28. When the correlation coefficient for a linear regression model is close to -1, the regression line is a poor fit for the data.

29. Writing Use your school's library, the Internet, or some other reference source to locate data that you think describes a linear relationship. Create a scatter plot of the data and find the least squares regression line that represents the points. Interpret the slope and y-intercept in the context of the data. Write a summary of your findings.

30. CAPSTONE Each graphing utility screen below shows the least squares regression line for a set of data. The equations and r-values for the models are given.

$y = 0.68x + 2.7$
$y = 0.41x + 2.7$
$y = -0.62x + 10.0$

$r = 0.973$
$r = -0.986$
$r = 0.624$

(i)

(ii)

(iii)

(a) Determine the equation and correlation coefficient (r-value) that represents each graph. Explain how you found your answers.

(b) According to the correlation coefficients, which model is the best fit for its data? Explain.

Cumulative Mixed Review

Evaluating a Function In Exercises 31 and 32, evaluate the function at each value of the independent variable and simplify.

31. $f(x) = 2x^2 - 3x + 5$

(a) $f(-1)$

(b) $f(w + 2)$

32. $g(x) = 5x^2 - 6x + 1$

(a) $g(-2)$

(b) $g(z - 2)$

Solving Equations In Exercises 33–36, solve the equation algebraically. Check your solution graphically.

33. $6x + 1 = -9x - 8$

34. $3(x - 3) = 7x + 2$

35. $8x^2 - 10x - 3 = 0$

36. $10x^2 - 23x - 5 = 0$

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1 Chapter Summary

	What did you learn?	Explanation and Examples	Review Exercises
1.1	**Find the slopes of lines *(p. 3)*.**	The slope m of the nonvertical line through (x_1, y_1) and (x_2, y_2) is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, $x_1 \neq x_2$.	1–8
	Write linear equations given points on lines and their slopes *(p. 5)*.	The point-slope form of the equation of the line that passes through the point (x_1, y_1) and has a slope of m is $y - y_1 = m(x - x_1)$.	9–16
	Use slope-intercept forms of linear equations to sketch lines *(p. 7)*.	The graph of the equation $y = mx + b$ is a line whose slope is m and whose y-intercept is $(0, b)$.	17–30
	Use slope to identify parallel and perpendicular lines *(p. 9)*.	**Parallel lines:** Slopes are equal. **Perpendicular lines:** Slopes are negative reciprocals of each other.	31, 32
1.2	**Decide whether a relation between two variables represents a function *(p. 16)*.**	A function f from a set A to a set B is a relation that assigns to each element x in the set A exactly one element y in the set B. The set A is the domain (or set of inputs) of the function f, and the set B contains the range (or set of outputs).	33–42
	Use function notation and evaluate functions *(p. 18)*, and find the domains of functions *(p. 20)*.	**Equation:** $f(x) = 5 - x^2$ $\boldsymbol{f(2)}$**:** $f(2) = 5 - 2^2 = 1$ **Domain of** $\boldsymbol{f(x) = 5 - x^2}$**:** All real numbers x	43–50
	Use functions to model and solve real-life problems *(p. 22)*.	A function can be used to model the number of construction employees in the United States. (See Example 8.)	51, 52
	Evaluate difference quotients *(p. 23)*.	**Difference quotient:** $\dfrac{f(x + h) - f(x)}{h}$, $h \neq 0$	53, 54
1.3	**Find the domains and ranges of functions *(p. 29)*.**		55–62
	Use the Vertical Line Test for functions *(p. 30)*.	A set of points in a coordinate plane is the graph of y as a function of x if and only if no *vertical* line intersects the graph at more than one point.	63–66
	Determine intervals on which functions are increasing, decreasing, or constant *(p. 31)*.	A function f is increasing on an interval when, for any x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) < f(x_2)$. A function f is decreasing on an interval when, for any x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) > f(x_2)$. A function f is constant on an interval when, for any x_1 and x_2 in the interval, $f(x_1) = f(x_2)$.	67–70

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	What did you learn?	Explanation and Examples	Review Exercises
1.3	**Determine relative maximum and relative minimum values of functions *(p. 32)*.**	A function value $f(a)$ is called a relative minimum of f when there exists an interval (x_1, x_2) that contains a such that $x_1 < x < x_2$ implies $f(a) \leq f(x)$. A function value $f(a)$ is called a relative maximum of f when there exists an interval (x_1, x_2) that contains a such that $x_1 < x < x_2$ implies $f(a) \geq f(x)$.	71–74
	Identify and graph step functions and other piecewise-defined functions *(p. 34)*.	**Greatest integer:** $f(x) = [\![x]\!]$	75–78
	Identify even and odd functions *(p. 35)*.	**Even:** For each x in the domain of f, $f(-x) = f(x)$. **Odd:** For each x in the domain of f, $f(-x) = -f(x)$.	79–86
1.4	**Recognize graphs of parent functions *(p. 41)*.**	**Linear:** $f(x) = x$; **Quadratic:** $f(x) = x^2$; **Cubic:** $f(x) = x^3$; **Absolute value:** $f(x) = \|x\|$; **Square root:** $f(x) = \sqrt{x}$; **Rational:** $f(x) = 1/x$ (See Figure 1.34, page 41.)	87–92
	Use vertical and horizontal shifts *(p. 42)*, reflections *(p. 44)*, and nonrigid transformations *(p. 46)* to graph functions.	**Vertical shifts:** $h(x) = f(x) + c$ or $h(x) = f(x) - c$ **Horizontal shifts:** $h(x) = f(x - c)$ or $h(x) = f(x + c)$ **Reflection in the *x*-axis:** $h(x) = -f(x)$ **Reflection in the *y*-axis:** $h(x) = f(-x)$ **Nonrigid transformations:** $h(x) = cf(x)$ or $h(x) = f(cx)$	93–106
1.5	**Add, subtract, multiply, and divide functions *(p. 50)*, find the compositions of functions *(p. 52)*, and write a function as a composition of two functions *(p. 54)*.**	$(f + g)(x) = f(x) + g(x)$ $\quad (f - g)(x) = f(x) - g(x)$ $(fg)(x) = f(x) \cdot g(x)$ $\quad (f/g)(x) = f(x)/g(x), g(x) \neq 0$ **Composition of functions:** $(f \circ g)(x) = f(g(x))$	107–122
	Use combinations of functions to model and solve real-life problems *(p. 55)*.	A composite function can be used to represent the number of bacteria in a petri dish as a function of the amount of time the petri dish has been out of refrigeration. (See Example 10.)	123, 124
1.6	**Find inverse functions informally and verify that two functions are inverse functions of each other *(p. 60)*.**	Let f and g be two functions such that $f(g(x)) = x$ for every x in the domain of g and $g(f(x)) = x$ for every x in the domain of f. Under these conditions, the function g is the inverse function of the function f.	125–128
	Use graphs of functions to decide whether functions have inverse functions *(p. 63)*.	If the point (a, b) lies on the graph of f, then the point (b, a) must lie on the graph of f^{-1}, and vice versa. In short, f^{-1} is a reflection of f in the line $y = x$.	129, 130
	Determine whether functions are one-to-one *(p. 64)*.	A function f is one-to-one when, for a and b in its domain, $f(a) = f(b)$ implies $a = b$.	131–134
	Find inverse functions algebraically *(p. 65)*.	To find inverse functions, replace $f(x)$ by y, interchange the roles of x and y, and solve for y. Replace y by $f^{-1}(x)$.	135–142
1.7	**Construct scatter plots *(p. 71)* and interpret correlation *(p. 72)*.**	A scatter plot is a graphical representation of data written as a set of ordered pairs.	143–146
	Use scatter plots *(p. 73)* and a graphing utility *(p. 74)* to find linear models for data.	The best-fitting linear model can be found using the *linear regression* feature of a graphing utility or a computer program.	147, 148

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1 Review Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

1.1

Finding the Slope of a Line **In Exercises 1–8, plot the two points and find the slope of the line passing through the points.**

1. $(-3, 2), (8, 2)$
2. $(3, -1), (-3, -1)$
3. $(7, -1), (7, 12)$
4. $(8, -1), (8, 2)$
5. $\left(\frac{3}{2}, 1\right), \left(5, \frac{5}{2}\right)$
6. $\left(-\frac{3}{4}, \frac{5}{6}\right), \left(\frac{1}{2}, -\frac{5}{2}\right)$
7. $(-4.5, 6), (2.1, 3)$
8. $(-2.7, -6.3), (0, 1.8)$

The Point-Slope Form of the Equation of a Line **In Exercises 9–16, (a) use the point on the line and the slope of the line to find an equation of the line, and (b) find three additional points through which the line passes. (There are many correct answers.)**

	Point	*Slope*
9.	$(2, -1)$	$m = \frac{1}{4}$
10.	$(-3, 5)$	$m = -\frac{3}{2}$
11.	$(0, -5)$	$m = \frac{3}{2}$
12.	$\left(0, \frac{7}{8}\right)$	$m = -\frac{4}{5}$
13.	$(-2, 6)$	$m = 0$
14.	$(-8, 8)$	$m = 0$
15.	$(10, -6)$	m is undefined.
16.	$(5, 4)$	m is undefined.

Finding the Slope-Intercept Form **In Exercises 17–24, write an equation of the line that passes through the points. Use the slope-intercept form, if possible. If not possible, explain why. Use a graphing utility to graph the line (if possible).**

17. $(2, -1), (4, -1)$
18. $(0, 0), (0, 10)$
19. $\left(7, \frac{11}{3}\right), \left(9, \frac{11}{3}\right)$
20. $\left(\frac{5}{8}, 4\right), \left(\frac{5}{8}, -6\right)$
21. $(-1, 0), (6, 2)$
22. $(1, 6), (4, 2)$
23. $(3, -1), (-3, 2)$
24. $\left(-\frac{5}{2}, 1\right), \left(-4, \frac{2}{9}\right)$

Using a Rate of Change to Write an Equation **In Exercises 25–28, you are given the dollar value of a product in 2010 *and* the rate at which the value of the item is expected to change during the next 5 years. Use this information to write a linear equation that gives the dollar value V of the product in terms of the year t. (Let $t = 0$ represent 2010.)**

	2010 Value	*Rate*
25.	\$12,500	\$850 increase per year
26.	\$3795	\$115 decrease per year
27.	\$625.50	\$42.70 increase per year
28.	\$72.95	\$5.15 decrease per year

29. **Business** During the second and third quarters of the year, an e-commerce business had sales of \$160,000 and \$185,000, respectively. The growth of sales follows a linear pattern. Estimate sales during the fourth quarter.

30. **Accounting** The dollar value of a DVD player in 2010 is \$134. The product will decrease in value at an expected rate of \$26.80 per year.

(a) Write a linear equation that gives the dollar value V of the DVD player in terms of the year t. (Let $t = 0$ represent 2010.)

(b) Use a graphing utility to graph the equation found in part (a). Be sure to choose an appropriate viewing window. State the dimensions of your viewing window, and explain why you chose the values that you did.

(c) Use the *value* or *trace* feature of the graphing utility to estimate the dollar value of the DVD player in 2014. Confirm your answer algebraically.

(d) According to the model, when will the DVD player have no value?

Equations of Parallel and Perpendicular Lines **In Exercises 31 and 32, write the slope-intercept forms of the equations of the lines through the given point (a) parallel to the given line and (b) perpendicular to the given line. Verify your result with a graphing utility (use a *square setting*).**

	Point	*Line*
31.	$(3, -2)$	$5x - 4y = 8$
32.	$(-8, 3)$	$2x + 3y = 5$

1.2

Testing for Functions **In Exercises 33 and 34, which set of ordered pairs represents a function from A to B? Explain.**

33. $A = \{10, 20, 30, 40\}$ and $B = \{0, 2, 4, 6\}$

(a) $\{(20, 4), (40, 0), (20, 6), (30, 2)\}$

(b) $\{(10, 4), (20, 4), (30, 4), (40, 4)\}$

34. $A = \{u, v, w\}$ and $B = \{-2, -1, 0, 1, 2\}$

(a) $\{(u, -2), (v, 2), (w, 1)\}$

(b) $\{(w, -2), (v, 0), (w, 2)\}$

Testing for Functions Represented Algebraically **In Exercises 35–42, determine whether the equation represents y as a function of x.**

35. $16x^2 - y^2 = 0$
36. $x^3 + y^2 = 64$
37. $2x - y - 3 = 0$
38. $2x + y = 10$
39. $y = \sqrt{1 - x}$
40. $y = \sqrt{x^2 + 4}$
41. $|y| = x + 2$
42. $16 - |y| - 4x = 0$

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Evaluating a Function **In Exercises 43–46, evaluate the function at each specified value of the independent variable, and simplify.**

43. $f(x) = x^2 + 1$

(a) $f(1)$ (b) $f(-3)$

(c) $f(b^3)$ (d) $f(x - 1)$

44. $g(x) = \sqrt{x^2 + 1}$

(a) $g(-1)$ (b) $g(3)$

(c) $g(3x)$ (d) $g(x + 2)$

45. $h(x) = \begin{cases} 2x + 1, & x \le -1 \\ x^2 + 2, & x > -1 \end{cases}$

(a) $h(-2)$ (b) $h(-1)$

(c) $h(0)$ (d) $h(2)$

46. $f(x) = \dfrac{3}{2x - 5}$

(a) $f(1)$ (b) $f(-2)$

(c) $f(t)$ (d) $f(10)$

Finding the Domain of a Function **In Exercises 47–50, find the domain of the function.**

47. $f(x) = \dfrac{x - 1}{x + 2}$ **48.** $f(x) = \dfrac{x^2}{x^2 + 1}$

49. $f(x) = \sqrt{25 - x^2}$ **50.** $f(x) = \sqrt{x^2 - 16}$

51. **Industrial Engineering** A hand tool manufacturer produces a product for which the variable cost is \$5.35 per unit and the fixed costs are \$16,000. The company sells the product for \$8.20 and can sell all that it produces.

(a) Write the total cost C as a function of x, the number of units produced.

(b) Write the profit P as a function of x.

52. **Education** The numbers n (in millions) of students enrolled in public schools in the United States from 2000 through 2008 can be approximated by

$$n(t) = \begin{cases} 0.76t + 61.4, & 0 \le t \le 4 \\ -0.3333t^3 + 6.6t^2 - 42.37t + 152.7, & 4 < t \le 8 \end{cases}$$

where t is the year, with $t = 0$ corresponding to 2000. (Source: U.S. Census Bureau)

(a) Use the *table* feature of a graphing utility to approximate the enrollment from 2000 through 2008.

(b) Use the graphing utility to graph the model and estimate the enrollment for the years 2009 through 2012. Do the values seem reasonable? Explain.

Evaluating a Difference Quotient **In Exercises 53 and 54, find the difference quotient $\dfrac{f(x + h) - f(x)}{h}$ for the given function and simplify your answer.**

53. $f(x) = 2x^2 + 3x - 1$ **54.** $f(x) = x^2 - 3x + 5$

1.3

Finding the Domain and Range of a Function **In Exercises 55–62, use a graphing utility to graph the function and estimate its domain and range. Then find the domain and range algebraically.**

55. $f(x) = 3 - 2x^2$ **56.** $f(x) = 2x^2 + 5$

57. $f(x) = \sqrt{x + 3} + 4$ **58.** $f(x) = 2 - \sqrt{x - 5}$

59. $h(x) = \sqrt{36 - x^2}$ **60.** $f(x) = \sqrt{x^2 - 9}$

61. $f(x) = |x + 5| + 2$ **62.** $f(x) = |x + 1| - 3$

Vertical Line Test for Functions **In Exercises 63–66, use the Vertical Line Test to determine whether y is a function of x. Describe how to enter the equation into a graphing utility to produce the given graph.**

63. $y - 4x = x^2$

64. $|x + 5| - 2y = 0$

65. $3x + y^2 - 2 = 0$

66. $x^2 + y^2 - 49 = 0$

Increasing and Decreasing Functions **In Exercises 67–70, (a) use a graphing utility to graph the function and (b) determine the open intervals on which the function is increasing, decreasing, or constant.**

67. $f(x) = x^3 - 3x$ **68.** $f(x) = \sqrt{x^2 - 9}$

69. $f(x) = x\sqrt{x - 6}$ **70.** $f(x) = \dfrac{|x + 8|}{2}$

Approximating Relative Minima and Maxima **In Exercises 71–74, use a graphing utility to approximate (to two decimal places) any relative minimum or relative maximum values of the function.**

71. $f(x) = (x^2 - 4)^2$ **72.** $f(x) = x^2 - x - 1$

73. $h(x) = 4x^3 - x^4$ **74.** $f(x) = x^3 - 4x^2 - 1$

Sketching Graphs **In Exercises 75–78, sketch the graph of the function by hand.**

75. $f(x) = \begin{cases} 3x + 5, & x < 0 \\ x - 4, & x \ge 0 \end{cases}$ **76.** $f(x) = \begin{cases} \frac{1}{2}x + 3, & x < 0 \\ 4 - x^2, & x \ge 0 \end{cases}$

77. $f(x) = [\![x]\!] + 3$ **78.** $f(x) = [\![x + 2]\!]$

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Even and Odd Functions **In Exercises 79–86, determine algebraically whether the function is even, odd, or neither. Verify your answer using a graphing utility.**

79. $f(x) = x^2 + 6$

80. $f(x) = x^2 - x - 1$

81. $f(x) = (x^2 - 8)^2$

82. $f(x) = 2x^3 - x^2$

83. $f(x) = 3x^{5/2}$

84. $f(x) = 3x^{2/5}$

85. $f(x) = \sqrt{4 - x^2}$

86. $f(x) = x\sqrt{x^2 - 1}$

1.4

Library of Parent Functions **In Exercises 87–92, identify the parent function and describe the transformation shown in the graph. Write an equation for the graphed function.**

87.

88.

89.

90.

91.

92.

Sketching Transformations **In Exercises 93–96, use the graph of $y = f(x)$ to graph the function.**

93. $y = f(-x)$

94. $y = -f(x)$

95. $y = f(x) - 2$

96. $y = f(x - 1)$

Describing Transformations **In Exercises 97–106, h is related to one of the six parent functions on page 41. (a) Identify the parent function f. (b) Describe the sequence of transformations from f to h. (c) Sketch the graph of h by hand. (d) Use function notation to write h in terms of the parent function f.**

97. $h(x) = \dfrac{1}{x} - 6$

98. $h(x) = -\dfrac{1}{x} - 3$

99. $h(x) = (x - 2)^3 + 5$

100. $h(x) = -(x - 2)^2 - 8$

101. $h(x) = -\sqrt{x} + 6$

102. $h(x) = \sqrt{x - 1} + 4$

103. $h(x) = |x| + 9$

104. $h(x) = |x + 8| - 1$

105. $h(x) = \dfrac{-2}{x + 1} - 3$

106. $h(x) = \dfrac{1}{x + 2} - 4$

1.5

Evaluating a Combination of Functions **In Exercises 107–116, let $f(x) = 3 - 2x$, $g(x) = \sqrt{x}$, and $h(x) = 3x^2 + 2$, and find the indicated values.**

107. $(f - g)(4)$

108. $(f + h)(5)$

109. $(f + g)(25)$

110. $(g - h)(1)$

111. $(fh)(1)$

112. $\left(\dfrac{g}{h}\right)(1)$

113. $(h \circ g)(5)$

114. $(g \circ f)(-3)$

115. $(f \circ h)(-4)$

116. $(g \circ h)(6)$

Identifying a Composite Function **In Exercises 117–122, find two functions f and g such that $(f \circ g)(x) = h(x)$. (There are many correct answers.)**

117. $h(x) = (x + 3)^2$

118. $h(x) = (1 - 2x)^3$

119. $h(x) = \sqrt{4x + 2}$

120. $h(x) = \sqrt[3]{(x + 2)^2}$

121. $h(x) = \dfrac{4}{x + 2}$

122. $h(x) = \dfrac{6}{(3x + 1)^3}$

Education **In Exercises 123 and 124, the numbers (in thousands) of students taking the SAT (y_1) and the ACT (y_2) for the years 2000 through 2009 can be modeled by**

$$y_1 = -2.61t^2 + 55.0t + 1244 \quad \text{and}$$

$$y_2 = 0.949t^3 - 8.02t^2 + 44.4t + 1056$$

where t represents the year, with $t = 0$ corresponding to 2000. (Source: College Entrance Examination Board and ACT, Inc.)

123. Use a graphing utility to graph y_1, y_2, and $y_1 + y_2$ in the same viewing window.

124. Use the model $y_1 + y_2$ to estimate the total number of students taking the SAT and the ACT in 2010.

1.6

Finding Inverse Functions Informally **In Exercises 125–128, find the inverse function of f informally. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.**

125. $f(x) = 6x$

126. $f(x) = x + 5$

127. $f(x) = \dfrac{1}{2}x + 3$

128. $f(x) = \dfrac{x - 4}{5}$

Algebraic-Graphical-Numerical **In Exercises 129 and 130, show that f and g are inverse functions (a) algebraically, (b) graphically, and (c) numerically.**

129. $f(x) = 3 - 4x, \quad g(x) = \dfrac{3 - x}{4}$

130. $f(x) = \sqrt{x + 1}, \quad g(x) = x^2 - 1, x \geq 0$

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Using the Horizontal Line Test **In Exercises 131–134, use a graphing utility to graph the function and use the Horizontal Line Test to determine whether the function is one-to-one and an inverse function exists.**

131. $f(x) = \frac{1}{2}x - 3$

132. $f(x) = (x - 1)^2$

133. $h(t) = \dfrac{2}{t - 3}$

134. $g(x) = \sqrt{x + 6}$

Finding an Inverse Function Algebraically **In Exercises 135–142, find the inverse function of f algebraically.**

135. $f(x) = \dfrac{1}{2}x - 5$

136. $f(x) = \dfrac{7x + 3}{8}$

137. $f(x) = 4x^3 - 3$

138. $f(x) = 5x^3 + 2$

139. $f(x) = \sqrt{x + 10}$

140. $f(x) = 4\sqrt{6 - x}$

141. $f(x) = \frac{1}{4}x^2 + 1, \quad x \geq 0$

142. $f(x) = 5 - \frac{1}{9}x^2, \quad x \geq 0$

1.7

Interpreting Correlation **In Exercises 143 and 144, the scatter plot of a set of data is shown. Determine whether the points are positively correlated, are negatively correlated, or have no discernible correlation.**

143.

144.

145. **Education** The following ordered pairs give the entrance exam scores x and the grade-point averages y after 1 year of college for 10 students.

(75, 2.3), (82, 3.0), (90, 3.6), (65, 2.0), (70, 2.1), (88, 3.5), (93, 3.9), (69, 2.0), (80, 2.8), (85, 3.3)

(a) Create a scatter plot of the data.

(b) Does the relationship between x and y appear to be approximately linear? Explain.

146. **Industrial Engineering** A machine part was tested by bending it x centimeters 10 times per minute until it failed (y equals the time to failure in hours). The results are given as the following ordered pairs.

(3, 61), (6, 56), (9, 53), (12, 55), (15, 48), (18, 35), (21, 36), (24, 33), (27, 44), (30, 23)

(a) Create a scatter plot of the data.

(b) Does the relationship between x and y appear to be approximately linear? If not, give some possible explanations.

147. **MODELING DATA**

In an experiment, students measured the speed s (in meters per second) of a ball t seconds after it was released. The results are shown in the table.

Time, t	Speed, s
0	0
1	11.0
2	19.4
3	29.2
4	39.4

(a) Sketch a scatter plot of the data.

(b) Find the equation of the line that seems to fit the data best.

(c) Use the *regression* feature of a graphing utility to find a linear model for the data and identify the correlation coefficient.

(d) Use the model from part (c) to estimate the speed of the ball after 2.5 seconds.

148. **MODELING DATA**

The following ordered pairs (x, y) represent the Olympic year x and the winning time y (in minutes) in the men's 400-meter freestyle swimming event. (Source: International Olympic Committee)

(1964, 4.203)	(1980, 3.855)	(1996, 3.800)
(1968, 4.150)	(1984, 3.854)	(2000, 3.677)
(1972, 4.005)	(1988, 3.783)	(2004, 3.718)
(1976, 3.866)	(1992, 3.750)	(2008, 3.698)

(a) Use the *regression* feature of a graphing utility to find a linear model for the data. Let x represent the year, with $x = 4$ corresponding to 1964.

(b) Use the graphing utility to create a scatter plot of the data. Graph the model in the same viewing window.

(c) Is the model a good fit for the data? Explain.

(d) Is this model appropriate for predicting the winning times in future Olympics? Explain.

Conclusions

True or False? **In Exercises 149–151, determine whether the statement is true or false. Justify your answer.**

149. If the graph of the parent function $f(x) = x^2$ is moved six units to the right, moved three units upward, and reflected in the x-axis, then the point $(-1, 28)$ will lie on the graph of the transformation.

150. If $f(x) = x^n$ where n is odd, then f^{-1} exists.

151. There exists no function f such that $f = f^{-1}$.

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1 Chapter Test

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Take this test as you would take a test in class. After you are finished, check your work against the answers in the back of the book.

Figure for 3

1. Find the equations of the lines that pass through the point $(0, 4)$ and are (a) parallel to and (b) perpendicular to the line $5x + 2y = 3$.
2. Find the slope-intercept form of the equation of the line that passes through the points $(2, -1)$ and $(-3, 4)$.
3. Does the graph at the right represent y as a function of x? Explain.
4. Evaluate $f(x) = |x + 2| - 15$ at each value of the independent variable and simplify.
   (a) $f(-8)$ (b) $f(14)$ (c) $f(t - 6)$
5. Find the domain of $f(x) = 10 - \sqrt{3 - x}$.
6. An electronics company produces a car stereo for which the variable cost is \$25.60 per unit and the fixed costs are \$24,000. The product sells for \$99.50. Write the total cost C as a function of the number of units produced and sold, x. Write the profit P as a function of the number of units produced and sold, x.

In Exercises 7 and 8, determine algebraically whether the function is even, odd, or neither.

7. $f(x) = 2x^3 - 3x$
8. $f(x) = 3x^4 + 5x^2$

In Exercises 9 and 10, determine the open intervals on which the function is increasing, decreasing, or constant.

9. $h(x) = \frac{1}{4}x^4 - 2x^2$
10. $g(t) = |t + 2| - |t - 2|$

In Exercises 11 and 12, use a graphing utility to graph the functions and to approximate (to two decimal places) any relative minimum or relative maximum values of the function.

11. $f(x) = -x^3 - 5x^2 + 12$
12. $f(x) = x^5 - x^3 + 2$

In Exercises 13–15, (a) identify the parent function *f*, (b) describe the sequence of transformations from *f* to *g*, and (c) sketch the graph of *g*.

13. $g(x) = -2(x - 5)^3 + 3$
14. $g(x) = \sqrt{-x - 7}$
15. $g(x) = 4|-x| - 7$

16. Use the functions $f(x) = x^2$ and $g(x) = \sqrt{2 - x}$ to find the specified function and its domain.
   (a) $(f - g)(x)$ (b) $\left(\dfrac{f}{g}\right)(x)$ (c) $(f \circ g)(x)$ (d) $(g \circ f)(x)$

In Exercises 17–19, determine whether the function has an inverse function, and if so, find the inverse function.

17. $f(x) = x^3 + 8$
18. $f(x) = x^2 + 6$
19. $f(x) = \dfrac{3x\sqrt{x}}{8}$

20. The table shows the average monthly cost C of basic cable television from 2000 through 2008, where t represents the year, with $t = 0$ corresponding to 2000. Use the *regression* feature of a graphing utility to find a linear model for the data. Use the model to estimate the year in which the average monthly cost reached \$50. (Source: SNL Kagan)

Year, t	Average monthly cost, C (in dollars)
0	30.37
1	32.87
2	34.71
3	36.59
4	38.14
5	39.63
6	41.17
7	42.72
8	44.28

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Proofs in Mathematics

Conditional Statements

Many theorems are written in the **if-then form** "if p, then q," which is denoted by

$p \rightarrow q$ Conditional statement

where p is the **hypothesis** and q is the **conclusion.** Here are some other ways to express the conditional statement $p \rightarrow q$.

p implies q. p, only if q. p is sufficient for q.

Conditional statements can be either true or false. The conditional statement $p \rightarrow q$ is false only when p is true and q is false. To show that a conditional statement is true, you must prove that the conclusion follows for all cases that fulfill the hypothesis. To show that a conditional statement is false, you need to describe only a single **counterexample** that shows that the statement is not always true.

For instance, $x = -4$ is a counterexample that shows that the following statement is false.

If $x^2 = 16$, then $x = 4$.

The hypothesis "$x^2 = 16$" is true because $(-4)^2 = 16$. However, the conclusion "$x = 4$" is false. This implies that the given conditional statement is false.

For the conditional statement $p \rightarrow q$, there are three important associated conditional statements.

1. The **converse** of $p \rightarrow q$: $q \rightarrow p$
2. The **inverse** of $p \rightarrow q$: $\sim p \rightarrow \sim q$
3. The **contrapositive** of $p \rightarrow q$: $\sim q \rightarrow \sim p$

The symbol $\sim$ means the **negation** of a statement. For instance, the negation of "The engine is running" is "The engine is not running."

Example 1 Writing the Converse, Inverse, and Contrapositive

Write the converse, inverse, and contrapositive of the conditional statement "If I get a B on my test, then I will pass the course."

Solution

a. *Converse:* If I pass the course, then I got a B on my test.

b. *Inverse:* If I do not get a B on my test, then I will not pass the course.

c. *Contrapositive:* If I do not pass the course, then I did not get a B on my test.

In the example above, notice that neither the converse nor the inverse is logically equivalent to the original conditional statement. On the other hand, the contrapositive is logically equivalent to the original conditional statement.

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Biconditional Statements

Recall that a conditional statement is a statement of the form "if p, then q." A statement of the form "p if and only if q" is called a **biconditional statement.** A biconditional statement, denoted by

$$p \leftrightarrow q \qquad \text{Biconditional statement}$$

is the conjunction of the conditional statement $p \rightarrow q$ and its converse $q \rightarrow p$.

A biconditional statement can be either true or false. To be true, *both* the conditional statement and its converse must be true.

Example 2 Analyzing a Biconditional Statement

Consider the statement $x = 3$ if and only if $x^2 = 9$.

a. Is the statement a biconditional statement? **b.** Is the statement true?

Solution

a. The statement is a biconditional statement because it is of the form "p if and only if q."

b. The statement can be rewritten as the following conditional statement and its converse.

Conditional statement: If $x = 3$, then $x^2 = 9$.

Converse: If $x^2 = 9$, then $x = 3$.

The first of these statements is true, but the second is false because x could also equal -3. So, the biconditional statement is false.

Knowing how to use biconditional statements is an important tool for reasoning in mathematics.

Example 3 Analyzing a Biconditional Statement

Determine whether the biconditional statement is true or false. If it is false, provide a counterexample.

A number is divisible by 5 if and only if it ends in 0.

Solution

The biconditional statement can be rewritten as the following conditional statement and its converse.

Conditional statement: If a number is divisible by 5, then it ends in 0.

Converse: If a number ends in 0, then it is divisible by 5.

The conditional statement is false. A counterexample is the number 15, which is divisible by 5 but does not end in 0. So, the biconditional statement is false.

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2 Polynomial and Rational Functions

Section 2.8, Example 4
Regular Soft Drinks Consumed

2.1 Quadratic Functions
2.2 Polynomial Functions of Higher Degree
2.3 Real Zeros of Polynomial Functions
2.4 Complex Numbers
2.5 The Fundamental Theorem of Algebra
2.6 Rational Functions and Asymptotes
2.7 Graphs of Rational Functions
2.8 Quadratic Models

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2.1 Quadratic Functions

The Graph of a Quadratic Function

In this and the next section, you will study the graphs of polynomial functions.

Definition of Polynomial Function

Let n be a nonnegative integer and let $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ be real numbers with $a_n \neq 0$. The function given by

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$$

is called a **polynomial function of x of degree n.**

Polynomial functions are classified by degree. For instance, the polynomial function

$$f(x) = a, \quad a \neq 0 \qquad \text{Constant function}$$

has degree 0 and is called a **constant function.** In Chapter 1, you learned that the graph of this type of function is a horizontal line. The polynomial function

$$f(x) = mx + b, \quad m \neq 0 \qquad \text{Linear function}$$

has degree 1 and is called a **linear function.** You learned in Chapter 1 that the graph of $f(x) = mx + b$ is a line whose slope is m and whose y-intercept is $(0, b)$. In this section, you will study second-degree polynomial functions, which are called **quadratic functions.**

Definition of Quadratic Function

Let a, b, and c be real numbers with $a \neq 0$. The function given by

$$f(x) = ax^2 + bx + c \qquad \text{Quadratic function}$$

is called a **quadratic function.**

Often real-life data can be modeled by quadratic functions. For instance, the table at the right shows the height h (in feet) of a projectile fired from a height of 6 feet with an initial velocity of 256 feet per second at any time t (in seconds). A quadratic model for the data in the table is

$$h(t) = -16t^2 + 256t + 6 \quad \text{for} \quad 0 \leq t \leq 16.$$

t	h
0	6
2	454
4	774
6	966
8	1030
10	966
12	774
14	454
16	6

The graph of a quadratic function is a special type of U-shaped curve called a **parabola.** Parabolas occur in many real-life applications, especially those involving reflective properties, such as satellite dishes or flashlight reflectors. You will study these properties in a later chapter.

All parabolas are symmetric with respect to a line called the **axis of symmetry,** or simply the **axis** of the parabola. The point where the axis intersects the parabola is called the **vertex** of the parabola.

What you should learn

- Analyze graphs of quadratic functions.
- Write quadratic functions in standard form and use the results to sketch graphs of functions.
- Find minimum and maximum values of quadratic functions in real-life applications.

Why you should learn it

Quadratic functions can be used to model the design of a room. For instance, Exercise 63 on page 97 shows how the size of an indoor fitness room with a running track can be modeled.

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Basic Characteristics of Quadratic Functions

Graph of $f(x) = ax^2,\ a > 0$
Domain: $(-\infty, \infty)$
Range: $[0, \infty)$
Intercept: $(0, 0)$
Decreasing on $(-\infty, 0)$
Increasing on $(0, \infty)$
Even function
Axis of symmetry: $x = 0$
Relative minimum or vertex: $(0, 0)$

Graph of $f(x) = ax^2,\ a < 0$
Domain: $(-\infty, \infty)$
Range: $(-\infty, 0]$
Intercept: $(0, 0)$
Increasing on $(-\infty, 0)$
Decreasing on $(0, \infty)$
Even function
Axis of symmetry: $x = 0$
Relative maximum or vertex: $(0, 0)$

For the general quadratic form $f(x) = ax^2 + bx + c$, when the leading coefficient a is positive, the parabola opens upward; and when the leading coefficient a is negative, the parabola opens downward. Later in this section you will learn ways to find the coordinates of the vertex of a parabola.

When sketching the graph of $f(x) = ax^2$, it is helpful to use the graph of $y = x^2$ as a reference, as discussed in Section 1.4. There you saw that when $a > 1$, the graph of $y = af(x)$ is a vertical stretch of the graph of $y = f(x)$. When $0 < a < 1$, the graph of $y = af(x)$ is a vertical shrink of the graph of $y = f(x)$. Notice in Figure 2.1 that the coefficient a determines how widely the parabola given by $f(x) = ax^2$ opens. When $|a|$ is small, the parabola opens more widely than when $|a|$ is large.

Vertical shrink

Vertical stretch

Figure 2.1

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Library of Parent Functions: Quadratic Function

The *parent quadratic function* is $f(x) = x^2$, also known as the *squaring function*. The basic characteristics of the parent quadratic function are summarized below and on the inside cover of this text.

Graph of $f(x) = x^2$
Domain: $(-\infty, \infty)$
Range: $[0, \infty)$
Intercept: $(0, 0)$
Decreasing on $(-\infty, 0)$
Increasing on $(0, \infty)$
Even function
Axis of symmetry: $x = 0$
Relative minimum or vertex: $(0, 0)$

Example 1 Library of Parent Functions: $f(x) = x^2$

Sketch the graph of the function and describe how the graph is related to the graph of $f(x) = x^2$.

a. $g(x) = -x^2 + 1$

b. $h(x) = (x + 2)^2 - 3$

Solution

a. With respect to the graph of $f(x) = x^2$, the graph of g is obtained by a *reflection* in the x-axis and a vertical shift one unit *upward*, as shown in Figure 2.2. Confirm this with a graphing utility.

b. With respect to the graph of $f(x) = x^2$, the graph of h is obtained by a horizontal shift two units *to the left* and a vertical shift three units *downward*, as shown in Figure 2.3. Confirm this with a graphing utility.

Figure 2.2

Figure 2.3

CHECKPOINT Now try Exercise 11.

Recall from Section 1.4 that the graphs of $y = f(x \pm c)$, $y = f(x) \pm c$, $y = -f(x)$, and $y = f(-x)$ are rigid transformations of the graph of $y = f(x)$.

$y = f(x \pm c)$	Horizontal shift	$y = -f(x)$	Reflection in x-axis
$y = f(x) \pm c$	Vertical shift	$y = f(-x)$	Reflection in y-axis

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The Standard Form of a Quadratic Function

The equation in Example 1(b) is written in the **standard form**

$$f(x) = a(x - h)^2 + k.$$

This form is especially convenient for sketching a parabola because it identifies the vertex of the parabola as (h, k).

Standard Form of a Quadratic Function

The quadratic function given by

$$f(x) = a(x - h)^2 + k, \quad a \neq 0$$

is in **standard form.** The graph of f is a parabola whose axis is the vertical line $x = h$ and whose vertex is the point (h, k). When $a > 0$, the parabola opens upward, and when $a < 0$, the parabola opens downward.

Explore the Concept

Use a graphing utility to graph $y = ax^2$ with $a = -2, -1, -0.5, 0.5, 1,$ and 2. How does changing the value of a affect the graph?

Use a graphing utility to graph $y = (x - h)^2$ with $h = -4, -2, 2,$ and 4. How does changing the value of h affect the graph?

Use a graphing utility to graph $y = x^2 + k$ with $k = -4, -2, 2,$ and 4. How does changing the value of k affect the graph?

Example 2 Identifying the Vertex of a Quadratic Function

Describe the graph of

$$f(x) = 2x^2 + 8x + 7$$

and identify the vertex.

Solution

Write the quadratic function in standard form by completing the square. Recall that the first step is to factor out any coefficient of x^2 that is not 1.

$f(x) = 2x^2 + 8x + 7$	Write original function.
$= (2x^2 + 8x) + 7$	Group x-terms.
$= 2(x^2 + 4x) + 7$	Factor 2 out of x-terms.
$= 2(x^2 + 4x + \underbrace{4}_{\left(\frac{4}{2}\right)^2} - 4) + 7$	Add and subtract $(4/2)^2 = 4$ within parentheses to complete the square.
$= 2(x^2 + 4x + 4) - 2(4) + 7$	Regroup terms.
$= 2(x + 2)^2 - 1$	Write in standard form.

From the standard form, you can see that the graph of f is a parabola that opens upward with vertex

$$(-2, -1)$$

as shown in Figure 2.4. This corresponds to a left shift of two units and a downward shift of one unit relative to the graph of

$$y = 2x^2.$$

Figure 2.4

CHECKPOINT Now try Exercise 23.

To find the x-intercepts of the graph of $f(x) = ax^2 + bx + c$, solve the equation $ax^2 + bx + c = 0$. When $ax^2 + bx + c$ does not factor, you can use the Quadratic Formula to find the x-intercepts, or a graphing utility to approximate the x-intercepts. Remember, however, that a parabola may not have x-intercepts.

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Example 3 Identifying x-Intercepts of a Quadratic Function

Describe the graph of $f(x) = -x^2 + 6x - 8$ and identify any x-intercepts.

Solution

$f(x) = -x^2 + 6x - 8$	Write original function.
$= -(x^2 - 6x) - 8$	Factor -1 out of x-terms.
$= -(x^2 - 6x + 9 - 9) - 8$	Because $b = 6$, add and subtract $(6/2)^2 = 9$ within parentheses.
$\left(\frac{6}{2}\right)^2$	
$= -(x^2 - 6x + 9) - (-9) - 8$	Regroup terms.
$= -(x - 3)^2 + 1$	Write in standard form.

Figure 2.5

The graph of f is a parabola that opens downward with vertex $(3, 1)$, as shown in Figure 2.5. The x-intercepts are determined as follows.

$-(x^2 - 6x + 8) = 0$	Factor out -1.
$-(x - 2)(x - 4) = 0$	Factor.
$x - 2 = 0 \Rightarrow x = 2$	Set 1st factor equal to 0.
$x - 4 = 0 \Rightarrow x = 4$	Set 2nd factor equal to 0.

So, the x-intercepts are $(2, 0)$ and $(4, 0)$, as shown in Figure 2.5.

CHECKPOINT Now try Exercise 31.

Example 4 Writing the Equation of a Parabola in Standard Form

Write the standard form of the equation of the parabola whose vertex is $(1, 2)$ and that passes through the point $(3, -6)$.

Solution

Because the vertex of the parabola is $(h, k) = (1, 2)$, the equation has the form

$f(x) = a(x - 1)^2 + 2.$	Substitute for h and k in standard form.

Because the parabola passes through the point $(3, -6)$, it follows that $f(3) = -6$. So, you obtain

$f(x) = a(x - 1)^2 + 2$	Write in standard form.
$-6 = a(3 - 1)^2 + 2$	Substitute -6 for $f(x)$ and 3 for x.
$-6 = 4a + 2$	Simplify.
$-8 = 4a$	Subtract 2 from each side.
$-2 = a.$	Divide each side by 4.

The equation in standard form is

$$f(x) = -2(x - 1)^2 + 2.$$

You can confirm this answer by graphing $f(x) = -2(x - 1)^2 + 2$ with a graphing utility, as shown in Figure 2.6. Use the *zoom* and *trace* features or the *maximum* and *value* features to confirm that its vertex is $(1, 2)$ and that it passes through the point $(3, -6)$.

Figure 2.6

CHECKPOINT Now try Exercise 39.

Study Tip

In Example 4, there are infinitely many different parabolas that have a vertex at $(1, 2)$. Of these, however, the only one that passes through the point $(3, -6)$ is the one given by

$$f(x) = -2(x - 1)^2 + 2.$$

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Finding Minimum and Maximum Values

Many applications involve finding the maximum or minimum value of a quadratic function. By completing the square of the quadratic function $f(x) = ax^2 + bx + c$, you can rewrite the function in standard form.

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right) \qquad \text{Standard form}$$

So, the vertex of the graph of f is $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$, which implies the following.

Minimum and Maximum Values of Quadratic Functions

Consider the function $f(x) = ax^2 + bx + c$ with vertex $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

1. If $a > 0$, then f has a *minimum* at $x = -\frac{b}{2a}$.

   The minimum value is $f\left(-\frac{b}{2a}\right)$.

2. If $a < 0$, then f has a *maximum* at $x = -\frac{b}{2a}$.

   The maximum value is $f\left(-\frac{b}{2a}\right)$.

Example 5 The Maximum Height of a Projectile

A baseball is hit at a point 3 feet above the ground at a velocity of 100 feet per second and at an angle of 45° with respect to the ground. The path of the baseball is given by the function $f(x) = -0.0032x^2 + x + 3$, where $f(x)$ is the height of the baseball (in feet) and x is the horizontal distance from home plate (in feet). What is the maximum height reached by the baseball?

Algebraic Solution

For this quadratic function, you have

$$f(x) = ax^2 + bx + c = -0.0032x^2 + x + 3$$

which implies that $a = -0.0032$ and $b = 1$. Because the function has a maximum when $x = -b/(2a)$, you can conclude that the baseball reaches its maximum height when it is x feet from home plate, where x is

$$\begin{aligned} x &= -\frac{b}{2a} \\ &= -\frac{1}{2(-0.0032)} \\ &= 156.25 \text{ feet.} \end{aligned}$$

At this distance, the maximum height is

$$\begin{aligned} f(156.25) &= -0.0032(156.25)^2 + 156.25 + 3 \\ &= 81.125 \text{ feet.} \end{aligned}$$

CHECKPOINT Now try Exercise 65.

Graphical Solution

The maximum height is $y \approx 81.125$ feet at $x \approx 156.25$ feet.

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2.1 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1 and 2, fill in the blanks.

1. A polynomial function of degree n and leading coefficient a_n is a function of the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$, $a_n \neq 0$, where n is a _____ and $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ are _____ numbers.
2. A _____ function is a second-degree polynomial function, and its graph is called a _____ .
3. Is the quadratic function $f(x) = (x - 2)^2 + 3$ written in standard form? Identify the vertex of the graph of f.
4. Does the graph of the quadratic function $f(x) = -3x^2 + 5x + 2$ have a relative minimum value at its vertex?

Procedures and Problem Solving

Graphs of Quadratic Functions **In Exercises 5–8, match the quadratic function with its graph. [The graphs are labeled (a), (b), (c), and (d).]**

(a)

(b)

(c)

(d)

5. $f(x) = (x - 2)^2$
6. $f(x) = 3 - x^2$
7. $f(x) = x^2 + 3$
8. $f(x) = -(x - 4)^2$

Library of Parent Functions **In Exercises 9–16, sketch the graph of the function and describe how the graph is related to the graph of $y = x^2$.**

9. $y = -x^2$
10. $y = x^2 - 1$
11. $y = (x + 3)^2$
12. $y = -(x + 3)^2 - 1$
13. $y = (x + 1)^2$
14. $y = -x^2 + 2$
15. $y = (x - 3)^2$
16. $y = -(x - 3)^2 + 1$

Identifying the Vertex of a Quadratic Function **In Exercises 17–30, describe the graph of the function and identify the vertex. Use a graphing utility to verify your results.**

17. $f(x) = 25 - x^2$
18. $f(x) = x^2 - 7$
19. $f(x) = \frac{1}{2}x^2 - 4$
20. $f(x) = 16 - \frac{1}{4}x^2$
21. $f(x) = (x + 4)^2 - 3$
22. $f(x) = (x - 6)^2 + 3$
23. $h(x) = x^2 - 8x + 16$
24. $g(x) = x^2 + 2x + 1$
25. $f(x) = x^2 - x + \frac{5}{4}$
26. $f(x) = x^2 + 3x + \frac{1}{4}$
27. $f(x) = -x^2 + 2x + 5$
28. $f(x) = -x^2 - 4x + 1$
29. $h(x) = 4x^2 - 4x + 21$
30. $f(x) = 2x^2 - x + 1$

Identifying x-Intercepts of a Quadratic Function **In Exercises 31–36, describe the graph of the quadratic function. Identify the vertex and x-intercept(s). Use a graphing utility to verify your results.**

31. $f(x) = -(x^2 + 2x - 3)$
32. $f(x) = -(x^2 + x - 30)$
33. $g(x) = x^2 + 8x + 11$
34. $f(x) = x^2 + 10x + 14$
35. $f(x) = -2x^2 + 16x - 31$
36. $f(x) = -4x^2 + 24x - 41$

Writing the Equation of a Parabola in Standard Form **In Exercises 37 and 38, write an equation of the parabola in standard form. Use a graphing utility to graph the equation and verify your result.**

37.

38.

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Writing the Equation of a Parabola in Standard Form **In Exercises 39–44, write the standard form of the quadratic function that has the indicated vertex and whose graph passes through the given point. Use a graphing utility to verify your result.**

39. Vertex: $(-2, 5)$; Point: $(0, 9)$

40. Vertex: $(4, 1)$; Point: $(6, -7)$

41. Vertex: $(1, -2)$; Point: $(-1, 14)$

42. Vertex: $(-4, -1)$; Point: $(-2, 4)$

43. Vertex: $\left(\frac{1}{2}, 1\right)$; Point: $\left(-2, -\frac{21}{5}\right)$

44. Vertex: $\left(-\frac{1}{4}, -1\right)$; Point: $\left(0, -\frac{17}{16}\right)$

Using a Graph to Identify *x*-Intercepts **In Exercises 45–48, determine the *x*-intercept(s) of the graph visually. Then find the *x*-intercept(s) algebraically to verify your answer.**

45.

46.

47.

48.

Graphing to Identify *x*-Intercepts **In Exercises 49–54, use a graphing utility to graph the quadratic function and find the *x*-intercepts of the graph. Then find the *x*-intercepts algebraically to verify your answer.**

49. $y = x^2 - 4x$

50. $y = -2x^2 + 10x$

51. $y = 2x^2 - 7x - 30$

52. $y = 4x^2 + 25x - 21$

53. $y = -\frac{1}{2}(x^2 - 6x - 7)$

54. $y = \frac{7}{10}(x^2 + 12x - 45)$

Using the *x*-Intercepts to Write Equations **In Exercises 55–58, find two quadratic functions, one that opens upward and one that opens downward, whose graphs have the given *x*-intercepts. (There are many correct answers.)**

55. $(-1, 0), (3, 0)$

56. $(0, 0), (10, 0)$

57. $(-3, 0), \left(-\frac{1}{2}, 0\right)$

58. $\left(-\frac{5}{2}, 0\right), (2, 0)$

Maximizing a Product of Two Numbers **In Exercises 59–62, find the two positive real numbers with the given sum whose product is a maximum.**

59. The sum is 110.

60. The sum is 66.

61. The sum of the first and twice the second is 24.

62. The sum of the first and three times the second is 42.

63. ***Why you should learn it*** *(p. 90)* An indoor physical fitness room consists of a rectangular region with a semicircle on each end. The perimeter of the room is to be a 200-meter single-lane running track.

(a) Draw a diagram that illustrates the problem. Let x and y represent the length and width of the rectangular region, respectively.

(b) Determine the radius of the semicircular ends of the track. Determine the distance, in terms of y, around the inside edge of the two semicircular parts of the track.

(c) Use the result of part (b) to write an equation, in terms of x and y, for the distance traveled in one lap around the track. Solve for y.

(d) Use the result of part (c) to write the area A of the rectangular region as a function of x.

(e) Use a graphing utility to graph the area function from part (d). Use the graph to approximate the dimensions that will produce a rectangle of maximum area.

64. **Algebraic-Graphical-Numerical** A child care center has 200 feet of fencing to enclose two adjacent rectangular safe play areas (see figure). Use the following methods to determine the dimensions that will produce a maximum enclosed area.

(a) Write the total area A of the play areas as a function of x.

(b) Use the *table* feature of a graphing utility to create a table showing possible values of x and the corresponding total area A of the play areas. Use the table to estimate the dimensions that will produce the maximum enclosed area.

(c) Use the graphing utility to graph the area function. Use the graph to approximate the dimensions that will produce the maximum enclosed area.

(d) Write the area function in standard form to find algebraically the dimensions that will produce the maximum enclosed area.

(e) Compare your results from parts (b), (c), and (d).

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65. Height of a Projectile The height y (in feet) of a punted football is approximated by

$$y = -\frac{16}{2025}x^2 + \frac{9}{5}x + \frac{3}{2}$$

where x is the horizontal distance (in feet) from where the football is punted.

(a) Use a graphing utility to graph the path of the football.

(b) How high is the football when it is punted? (*Hint:* Find y when $x = 0$.)

(c) What is the maximum height of the football?

(d) How far from the punter does the football strike the ground?

66. Physics The path of a diver is approximated by

$$y = -\frac{4}{9}x^2 + \frac{24}{9}x + 12$$

where y is the height (in feet) and x is the horizontal distance (in feet) from the end of the diving board (see figure). What is the maximum height of the diver? Verify your answer using a graphing utility.

67. Geometry To make a sign holder, you bend a 100-inch long steel wire x inches from each end to form two right angles. To use the sign holder, you insert each end 6 inches into the ground.

(a) Write a function for the rectangular area A enclosed by the sign holder in terms of x.

(b) Use the *table* feature of a graphing utility to determine the value of x that maximizes the rectangular area enclosed by the sign holder.

68. Aerodynamic Engineering The number of horsepower H required to overcome wind drag on a certain automobile is approximated by

$$H(s) = 0.002s^2 + 0.05s - 0.029, \quad 0 \le s \le 100$$

where s is the speed of the car (in miles per hour).

(a) Use a graphing utility to graph the function.

(b) Graphically estimate the maximum speed of the car given that the power required to overcome wind drag is not to exceed 10 horsepower. Verify your result algebraically.

69. Economics The monthly revenue R (in thousands of dollars) from the sales of a digital picture frame is approximated by $R(p) = -10p^2 + 1580p$, where p is the price per unit (in dollars).

(a) Find the monthly revenues for unit prices of \$50, \$70, and \$90.

(b) Find the unit price that will yield a maximum monthly revenue.

(c) What is the maximum monthly revenue?

(d) Explain your results.

70. Economics The weekly revenue R (in dollars) earned by a computer repair service is given by

$$R(p) = -12p^2 + 372p$$

where p is the price charged per service hour (in dollars).

(a) Find the weekly revenues for prices per service hour of \$12, \$16, and \$20.

(b) Find the price that will yield a maximum weekly revenue.

(c) What is the maximum weekly revenue?

(d) Explain your results.

71. Public Health From 1955 through 2000, the annual per capita consumption C of cigarettes by Americans (age 18 and older) can be modeled by

$$C(t) = -2.10t^2 + 70.9t + 3557, \quad 5 \le t \le 50$$

where t is the year, with $t = 5$ corresponding to 1955. (Source: U.S. Department of Agriculture)

(a) Use a graphing utility to graph the model.

(b) Use the graph of the model to approximate the year when the maximum annual consumption of cigarettes occurred. Approximate the maximum average annual consumption. Beginning in 1966, all cigarette packages were required by law to carry a health warning. Do you think the warning had any effect? Explain.

(c) In 2000, the U.S. population (age 18 and older) was 209,117,000. Of those, about 48,306,000 were smokers. What was the average annual cigarette consumption *per smoker* in 2000? What was the average daily cigarette consumption *per smoker*?

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72. **Demography** The population P of Germany (in thousands) from 1999 through 2009 can be modeled by

$$P(t) = -8.87t^2 + 271.4t + 80{,}362, \quad 9 \le t \le 19$$

where t is the year, with $t = 9$ corresponding to 1999. (Source: U.S. Census Bureau)

(a) According to the model, in what year did Germany have its greatest population? What was the population?

(b) According to the model, what will Germany's population be in the year 2100? Is this result reasonable? Explain.

Conclusions

True or False? **In Exercises 73 and 74, determine whether the statement is true or false. Justify your answer.**

73. The function $f(x) = -12x^2 - 1$ has no x-intercepts.

74. The graphs of $f(x) = -4x^2 - 10x + 7$ and $g(x) = 12x^2 + 30x + 1$ have the same axis of symmetry.

Library of Parent Functions **In Exercises 75 and 76, determine which equation(s) may be represented by the graph shown. (There may be more than one correct answer.)**

75. (a) $f(x) = -(x - 4)^2 + 2$
(b) $f(x) = -(x + 2)^2 + 4$
(c) $f(x) = -(x + 2)^2 - 4$
(d) $f(x) = -x^2 - 4x - 8$
(e) $f(x) = -(x - 2)^2 - 4$
(f) $f(x) = -x^2 + 4x - 8$

76. (a) $f(x) = (x - 1)^2 + 3$
(b) $f(x) = (x + 1)^2 + 3$
(c) $f(x) = (x - 3)^2 + 1$
(d) $f(x) = x^2 + 2x + 4$
(e) $f(x) = (x + 3)^2 + 1$
(f) $f(x) = x^2 + 6x + 10$

Describing Parabolas **In Exercises 77–80, let z represent a positive real number. Describe how the family of parabolas represented by the given function compares with the graph of $g(x) = x^2$.**

77. $f(x) = (x - z)^2$

78. $f(x) = x^2 - z$

79. $f(x) = z(x - 3)^2$

80. $f(x) = zx^2 + 4$

Think About It **In Exercises 81–84, find the value of b such that the function has the given maximum or minimum value.**

81. $f(x) = -x^2 + bx - 75$; Maximum value: 25

82. $f(x) = -x^2 + bx - 16$; Maximum value: 48

83. $f(x) = x^2 + bx + 26$; Minimum value: 10

84. $f(x) = x^2 + bx - 25$; Minimum value: -50

85. **Proof** Let x and y be two positive real numbers whose sum is S. Show that the maximum product of x and y occurs when x and y are both equal to $S/2$.

86. **Proof** Assume that the function given by $f(x) = ax^2 + bx + c$, $a \neq 0$, has two real zeros. Show that the x-coordinate of the vertex of the graph is the average of the zeros of f. (*Hint:* Use the Quadratic Formula.)

87. **Writing** The parabola in the figure below has an equation of the form $y = ax^2 + bx - 4$. Find the equation of this parabola in two different ways, by hand and with technology (graphing utility or computer software). Write a paragraph describing the methods you used and comparing the results of the two methods.

88. **CAPSTONE** The annual profit P (in dollars) of a company is modeled by a function of the form $P = at^2 + bt + c$, where t represents the year. Discuss which of the following models the company might prefer.

(a) a is positive and $t \ge -b/(2a)$.
(b) a is positive and $t \le -b/(2a)$.
(c) a is negative and $t \ge -b/(2a)$.
(d) a is negative and $t \le -b/(2a)$.

Cumulative Mixed Review

Finding Points of Intersection **In Exercises 89–92, determine algebraically any point(s) of intersection of the graphs of the equations. Verify your results using the *intersect* feature of a graphing utility.**

89. $x + y = 8$
$-\frac{2}{3}x + y = 6$

90. $y = 3x - 10$
$y = \frac{1}{4}x + 1$

91. $y = 9 - x^2$
$y = x + 3$

92. $y = x^3 + 2x - 1$
$y = -2x + 15$

93. ***Make a Decision*** To work an extended application analyzing the height of a basketball after it has been dropped, visit this textbook's *Companion Website*.

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2.2 Polynomial Functions of Higher Degree

Graphs of Polynomial Functions

At this point, you should be able to sketch accurate graphs of polynomial functions of degrees 0, 1, and 2.

Function	*Graph*
$f(x) = a$	Horizontal line
$f(x) = ax + b$	Line of slope a
$f(x) = ax^2 + bx + c$	Parabola

The graphs of polynomial functions of degree greater than 2 are more difficult to sketch by hand. However, in this section you will learn how to recognize some of the basic features of the graphs of polynomial functions. Using these features along with point plotting, intercepts, and symmetry, you should be able to make reasonably accurate sketches *by hand*.

The graph of a polynomial function is **continuous.** Essentially, this means that the graph of a polynomial function has no breaks, holes, or gaps, as shown in Figure 2.7. Informally, you can say that a function is continuous when its graph can be drawn with a pencil without lifting the pencil from the paper.

What you should learn

- Use transformations to sketch graphs of polynomial functions.
- Use the Leading Coefficient Test to determine the end behavior of graphs of polynomial functions.
- Find and use zeros of polynomial functions as sketching aids.
- Use the Intermediate Value Theorem to help locate zeros of polynomial functions.

Why you should learn it

You can use polynomial functions to model various aspects of nature, such as the growth of a red oak tree, as shown in Exercise 112 on page 111.

(a) Polynomial functions have continuous graphs.

(b) Functions with graphs that are not continuous are not polynomial functions.

Figure 2.7

Another feature of the graph of a polynomial function is that it has only smooth, rounded turns, as shown in Figure 2.8(a). It cannot have a sharp turn such as the one shown in Figure 2.8(b).

(a) Polynomial functions have graphs with smooth, rounded turns.

(b) Functions with graphs that have sharp turns are not polynomial functions.

Figure 2.8

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The graphs of polynomial functions of degree 1 are lines, and those of functions of degree 2 are parabolas. The graphs of all polynomial functions are smooth and continuous. A polynomial function of degree n has the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2x^2 + a_1x + a_0$$

where n is a positive integer and $a_n \neq 0$.

The polynomial functions that have the simplest graphs are monomials of the form $f(x) = x^n$, where n is an integer greater than zero. The greater the value of n, the flatter the graph near the origin. When n is even, the graph is similar to the graph of $f(x) = x^2$ and touches the x-axis at the x-intercept. When n is odd, the graph is similar to the graph of $f(x) = x^3$ and crosses the x-axis at the x-intercept. Polynomial functions of the form $f(x) = x^n$ are often referred to as **power functions.**

Explore the Concept

Use a graphing utility to graph $y = x^n$ for $n = 2, 4,$ and 8. (Use the viewing window $-1.5 \leq x \leq 1.5$ and $-1 \leq y \leq 6$.) Compare the graphs. In the interval $(-1, 1)$, which graph is on the bottom? Outside the interval $(-1, 1)$, which graph is on the bottom?

Use a graphing utility to graph $y = x^n$ for $n = 3, 5,$ and 7. (Use the viewing window $-1.5 \leq x \leq 1.5$ and $-4 \leq y \leq 4$.) Compare the graphs. In the intervals $(-\infty, -1)$ and $(0, 1)$, which graph is on the bottom? In the intervals $(-1, 0)$ and $(1, \infty)$, which graph is on the bottom?

Library of Parent Functions: Cubic Function

The basic characteristics of the *parent cubic function* $f(x) = x^3$ are summarized below and on the inside cover of this text.

Graph of $f(x) = x^3$
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$
Intercept: $(0, 0)$
Increasing on $(-\infty, \infty)$
Odd function
Origin symmetry

Example 1 Library of Parent Functions: $f(x) = x^3$

Sketch the graphs of (a) $g(x) = -x^3$, (b) $h(x) = x^3 + 1$, and (c) $k(x) = (x - 1)^3$.

Solution

a. With respect to the graph of $f(x) = x^3$, the graph of g is obtained by a *reflection* in the x-axis, as shown in Figure 2.9.

b. With respect to the graph of $f(x) = x^3$, the graph of h is obtained by a vertical shift one unit *upward*, as shown in Figure 2.10.

c. With respect to the graph of $f(x) = x^3$, the graph of k is obtained by a horizontal shift one unit *to the right*, as shown in Figure 2.11.

Figure 2.9

Figure 2.10

Figure 2.11

CHECKPOINT Now try Exercise 17.

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The Leading Coefficient Test

In Example 1, note that all three graphs eventually rise or fall without bound as x moves to the right. Whether the graph of a polynomial eventually rises or falls can be determined by the polynomial function's degree (even or odd) and by its leading coefficient, as indicated in the **Leading Coefficient Test.**

Leading Coefficient Test

As x moves without bound to the left or to the right, the graph of the polynomial function

$$f(x) = a_n x^n + \cdot \cdot \cdot + a_1 x + a_0, \quad a_n \neq 0$$

eventually rises or falls in the following manner.

1. When n is odd:

If the leading coefficient is positive ($a_n > 0$), then the graph falls to the left and rises to the right.

If the leading coefficient is negative ($a_n < 0$), then the graph rises to the left and falls to the right.

2. When n is even:

If the leading coefficient is positive ($a_n > 0$), then the graph rises to the left and right.

If the leading coefficient is negative ($a_n < 0$), then the graph falls to the left and right.

Note that the dashed portions of the graphs indicate that the test determines only the right-hand and left-hand behavior of the graph.

As you continue to study polynomial functions and their graphs, you will notice that the degree of a polynomial plays an important role in determining other characteristics of the polynomial function and its graph.

Explore the Concept

For each function, identify the degree of the function and whether the degree of the function is even or odd. Identify the leading coefficient and whether the leading coefficient is positive or negative. Use a graphing utility to graph each function. Describe the relationship between the degree and sign of the leading coefficient of the function, and the right- and left-hand behavior of the graph of the function.

a. $y = x^3 - 2x^2 - x + 1$
b. $y = 2x^5 + 2x^2 - 5x + 1$
c. $y = -2x^5 - x^2 + 5x + 3$
d. $y = -x^3 + 5x - 2$
e. $y = 2x^2 + 3x - 4$
f. $y = x^4 - 3x^2 + 2x - 1$
g. $y = -x^2 + 3x + 2$
h. $y = -x^6 - x^2 - 5x + 4$

Study Tip

The notation "$f(x) \to -\infty$ as $x \to -\infty$" indicates that the graph falls to the left. The notation "$f(x) \to \infty$ as $x \to \infty$" indicates that the graph rises to the right.

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Example 2 Applying the Leading Coefficient Test

Use the Leading Coefficient Test to describe the right-hand and left-hand behavior of the graph of

$$f(x) = -x^3 + 4x.$$

Solution

Because the degree is odd and the leading coefficient is negative, the graph rises to the left and falls to the right, as shown in Figure 2.12.

Figure 2.12

CHECKPOINT Now try Exercise 29.

Explore the Concept

For each of the graphs in Examples 2 and 3, count the number of zeros of the polynomial function and the number of relative extrema, and compare these numbers with the degree of the polynomial. What do you observe?

Example 3 Applying the Leading Coefficient Test

Use the Leading Coefficient Test to describe the right-hand and left-hand behavior of the graph of each polynomial function.

a. $f(x) = x^4 - 5x^2 + 4$

b. $f(x) = x^5 - x$

Solution

a. Because the degree is even and the leading coefficient is positive, the graph rises to the left and right, as shown in Figure 2.13.

Figure 2.13

b. Because the degree is odd and the leading coefficient is positive, the graph falls to the left and rises to the right, as shown in Figure 2.14.

Figure 2.14

CHECKPOINT Now try Exercise 31.

In Examples 2 and 3, note that the Leading Coefficient Test tells you only whether the graph *eventually* rises or falls to the right or left. Other characteristics of the graph, such as intercepts and minimum and maximum points, must be determined by other tests.

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Zeros of Polynomial Functions

It can be shown that for a polynomial function f of degree n, the following statements are true.

1. The function f has at most n real zeros. (You will study this result in detail in Section 2.5 on the Fundamental Theorem of Algebra.)
2. The graph of f has at most $n - 1$ relative **extrema** (relative minima or maxima).

Recall that a zero of a function f is a number x for which $f(x) = 0$. Finding the zeros of polynomial functions is one of the most important problems in algebra. You have already seen that there is a strong interplay between graphical and algebraic approaches to this problem. Sometimes you can use information about the graph of a function to help find its zeros. In other cases, you can use information about the zeros of a function to find a good viewing window.

Real Zeros of Polynomial Functions

If f is a polynomial function and a is a real number, then the following statements are equivalent.

1. $x = a$ is a *zero* of the function f.
2. $x = a$ is a *solution* of the polynomial equation $f(x) = 0$.
3. $(x - a)$ is a *factor* of the polynomial $f(x)$.
4. $(a, 0)$ is an x-intercept of the graph of f.

Finding zeros of polynomial functions is closely related to factoring and finding x-intercepts, as demonstrated in Examples 4, 5, and 6.

Example 4 Finding Zeros of a Polynomial Function

Find all real zeros of $f(x) = x^3 - x^2 - 2x$.

Algebraic Solution

$f(x) = x^3 - x^2 - 2x$	Write original function.
$0 = x^3 - x^2 - 2x$	Substitute 0 for $f(x)$.
$0 = x(x^2 - x - 2)$	Remove common monomial factor.
$0 = x(x - 2)(x + 1)$	Factor completely.

So, the real zeros are

$$x = 0, \quad x = 2, \quad \text{and} \quad x = -1$$

and the corresponding x-intercepts are

$$(0, 0), \quad (2, 0), \quad \text{and} \quad (-1, 0).$$

Check

$(0)^3 - (0)^2 - 2(0) = 0$	$x = 0$ is a zero. ✓
$(2)^3 - (2)^2 - 2(2) = 0$	$x = 2$ is a zero. ✓
$(-1)^3 - (-1)^2 - 2(-1) = 0$	$x = -1$ is a zero. ✓

CHECKPOINT Now try Exercise 37.

Graphical Solution

The graph of f has the x-intercepts

$$(0, 0), \quad (2, 0), \quad \text{and} \quad (-1, 0)$$

as shown in Figure 2.15. So, the real zeros of f are

$$x = 0, \quad x = 2, \quad \text{and} \quad x = -1.$$

Use the *zero* or *root* feature of a graphing utility to verify these zeros.

Figure 2.15

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Example 5 Analyzing a Polynomial Function

Find all real zeros and relative extrema of $f(x) = -2x^4 + 2x^2$.

Solution

$$0 = -2x^4 + 2x^2 \qquad \text{Substitute 0 for } f(x).$$

$$0 = -2x^2(x^2 - 1) \qquad \text{Remove common monomial factor.}$$

$$0 = -2x^2(x - 1)(x + 1) \qquad \text{Factor completely.}$$

So, the real zeros are $x = 0$, $x = 1$, and $x = -1$, and the corresponding x-intercepts are $(0, 0)$, $(1, 0)$, and $(-1, 0)$, as shown in Figure 2.16. Using the *minimum* and *maximum* features of a graphing utility, you can approximate the three relative extrema to be $(-0.71, 0.5)$, $(0, 0)$, and $(0.71, 0.5)$.

Figure 2.16

CHECKPOINT Now try Exercise 59.

Repeated Zeros

For a polynomial function, a factor of $(x - a)^k$, $k > 1$, yields a **repeated zero** $x = a$ of **multiplicity** k.

1. If k is odd, then the graph *crosses* the x-axis at $x = a$.
2. If k is even, then the graph *touches* the x-axis (but does not cross the x-axis) at $x = a$.

Study Tip

In Example 5, note that because k is even, the factor $-2x^2$ yields the repeated zero $x = 0$. The graph touches (but does not cross) the x-axis at $x = 0$, as shown in Figure 2.16.

Example 6 Analyzing a Polynomial Function

Find all real zeros of $f(x) = x^5 - 3x^3 - x^2 - 4x - 1$.

Solution

From Figure 2.17, you can see that there are three zeros. Using the *zero* feature of a graphing utility, you can determine that the zeros are approximately $x \approx -1.86$, $x \approx -0.25$, and $x \approx 2.11$. It should be noted that this fifth-degree polynomial factors as

$$f(x) = x^5 - 3x^3 - x^2 - 4x - 1 = (x^2 + 1)(x^3 - 4x - 1).$$

The three zeros obtained above are the zeros of the cubic factor $x^3 - 4x - 1$. The quadratic factor $x^2 + 1$ has no real zeros, but does have two *complex* zeros. You will learn more about complex zeros in Section 2.5.

Figure 2.17

CHECKPOINT Now try Exercise 61.

Example 7 Finding a Polynomial Function with Given Zeros

Find a polynomial function with zeros $-\frac{1}{2}$, 3, and 3. (There are many correct solutions.)

Solution

Note that the zero $x = -\frac{1}{2}$ corresponds to either $\left(x + \frac{1}{2}\right)$ or $(2x + 1)$. To avoid fractions, choose the second factor and write

$$\begin{aligned} f(x) &= (2x + 1)(x - 3)^2 \\ &= (2x + 1)(x^2 - 6x + 9) \\ &= 2x^3 - 11x^2 + 12x + 9. \end{aligned}$$

CHECKPOINT Now try Exercise 67.

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Note in Example 7 that there are many polynomial functions with the indicated zeros. In fact, multiplying the function by any real number does not change the zeros of the function. For instance, multiply the function from Example 7 by $\frac{1}{2}$ to obtain

$$f(x) = x^3 - \tfrac{11}{2}x^2 + 6x + \tfrac{9}{2}.$$

Then find the zeros of the function. You will obtain the zeros $-\frac{1}{2}$, 3, and 3, as given in Example 7.

Example 8 Sketching the Graph of a Polynomial Function

Sketch the graph of

$$f(x) = 3x^4 - 4x^3$$

by hand.

Solution

1. *Apply the Leading Coefficient Test.* Because the leading coefficient is positive and the degree is even, you know that the graph eventually rises to the left and to the right (see Figure 2.18).
2. *Find the Real Zeros of the Polynomial.* By factoring

$$f(x) = 3x^4 - 4x^3 = x^3(3x - 4)$$

you can see that the real zeros of f are $x = 0$ (of odd multiplicity 3) and $x = \frac{4}{3}$ (of odd multiplicity 1). So, the x-intercepts occur at $(0, 0)$ and $\left(\frac{4}{3}, 0\right)$. Add these points to your graph, as shown in Figure 2.18.

3. *Plot a Few Additional Points.* To sketch the graph by hand, find a few additional points, as shown in the table. Be sure to choose points between the zeros and to the left and right of the zeros. Then plot the points (see Figure 2.19).

x	-1	0.5	1	1.5
$f(x)$	7	-0.31	-1	1.69

4. *Draw the Graph.* Draw a continuous curve through the points, as shown in Figure 2.19. Because both zeros are of odd multiplicity, you know that the graph should cross the x-axis at $x = 0$ and $x = \frac{4}{3}$. When you are unsure of the shape of a portion of the graph, plot some additional points.

Figure 2.18

Figure 2.19

CHECKPOINT Now try Exercise 87.

Technology Tip

Because it is easy to make mistakes when entering functions into a graphing utility, it is important to understand the basic shapes of graphs and to be able to graph simple polynomials *by hand*. For instance, suppose you had entered the function in Example 8 as $y = 3x^5 - 4x^3$. From the graph, what mathematical principles would alert you to the fact that you had made a mistake?

Explore the Concept

Partner Activity
Multiply three, four, or five distinct linear factors to obtain the equation of a polynomial function of degree 3, 4, or 5. Exchange equations with your partner and sketch, *by hand*, the graph of the equation that your partner wrote. When you are finished, use a graphing utility to check each other's work.

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Example 9 Sketching the Graph of a Polynomial Function

Sketch the graph of

$$f(x) = -2x^3 + 6x^2 - \tfrac{9}{2}x.$$

Solution

1. *Apply the Leading Coefficient Test.* Because the leading coefficient is negative and the degree is odd, you know that the graph eventually rises to the left and falls to the right (see Figure 2.20).

2. *Find the Real Zeros of the Polynomial.* By factoring

$$\begin{aligned} f(x) &= -2x^3 + 6x^2 - \tfrac{9}{2}x \\ &= -\tfrac{1}{2}x(4x^2 - 12x + 9) \\ &= -\tfrac{1}{2}x(2x - 3)^2 \end{aligned}$$

you can see that the real zeros of f are $x = 0$ (of odd multiplicity 1) and $x = \frac{3}{2}$ (of even multiplicity 2). So, the x-intercepts occur at $(0, 0)$ and $\left(\frac{3}{2}, 0\right)$. Add these points to your graph, as shown in Figure 2.20.

3. *Plot a Few Additional Points.* To sketch the graph by hand, find a few additional points, as shown in the table. Then plot the points (see Figure 2.21).

x	-0.5	0.5	1	2
$f(x)$	4	-1	-0.5	-1

4. *Draw the Graph.* Draw a continuous curve through the points, as shown in Figure 2.21. As indicated by the multiplicities of the zeros, the graph crosses the x-axis at $(0, 0)$ and touches (but does not cross) the x-axis at $\left(\frac{3}{2}, 0\right)$.

Figure 2.20

Figure 2.21

CHECKPOINT Now try Exercise 89.

Study Tip

Observe in Example 9 that the sign of $f(x)$ is positive to the left of and negative to the right of the zero $x = 0$. Similarly, the sign of $f(x)$ is negative to the left and to the right of the zero $x = \frac{3}{2}$. This suggests that if a zero of a polynomial function is of *odd* multiplicity, then the sign of $f(x)$ changes from one side of the zero to the other side. If a zero is of *even* multiplicity, then the sign of $f(x)$ does not change from one side of the zero to the other side. The following table helps to illustrate this result.

x	-0.5	0	0.5
$f(x)$	4	0	-1
Sign	$+$		$-$

x	1	$\frac{3}{2}$	2
$f(x)$	-0.5	0	-1
Sign	$-$		$-$

This sign analysis may be helpful in graphing polynomial functions.

Technology Tip

Remember that when using a graphing utility to verify your graphs, you may need to adjust your viewing window in order to see all the features of the graph.

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The Intermediate Value Theorem

The **Intermediate Value Theorem** concerns the existence of real zeros of polynomial functions. The theorem states that if

$$(a, f(a)) \quad \text{and} \quad (b, f(b))$$

are two points on the graph of a polynomial function such that $f(a) \neq f(b)$, then for any number d between $f(a)$ and $f(b)$ there must be a number c between a and b such that $f(c) = d$. (See Figure 2.22.)

Figure 2.22

Intermediate Value Theorem

Let a and b be real numbers such that $a < b$. If f is a polynomial function such that $f(a) \neq f(b)$, then in the interval $[a, b]$, f takes on every value between $f(a)$ and $f(b)$.

This theorem helps you locate the real zeros of a polynomial function in the following way. If you can find a value $x = a$ at which a polynomial function is positive, and another value $x = b$ at which it is negative, then you can conclude that the function has at least one real zero between these two values. For example, the function $f(x) = x^3 + x^2 + 1$ is negative when $x = -2$ and positive when $x = -1$. Therefore, it follows from the Intermediate Value Theorem that f must have a real zero somewhere between -2 and -1.

Example 10 Approximating the Zeros of a Function

Find three intervals of length 1 in which the polynomial

$$f(x) = 12x^3 - 32x^2 + 3x + 5$$

is guaranteed to have a zero.

Graphical Solution

From Figure 2.23, you can see that the graph of f crosses the x-axis three times—between -1 and 0, between 0 and 1, and between 2 and 3. So, you can conclude that the function has zeros in the intervals $(-1, 0)$, $(0, 1)$, and $(2, 3)$.

Figure 2.23

CHECKPOINT Now try Exercise 95.

Numerical Solution

From the table in Figure 2.24, you can see that $f(-1)$ and $f(0)$ differ in sign. So, you can conclude from the Intermediate Value Theorem that the function has a zero between -1 and 0. Similarly, $f(0)$ and $f(1)$ differ in sign, so the function has a zero between 0 and 1. Likewise, $f(2)$ and $f(3)$ differ in sign, so the function has a zero between 2 and 3. So, you can conclude that the function has zeros in the intervals $(-1, 0)$, $(0, 1)$, and $(2, 3)$.

Figure 2.24

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2.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1–4, fill in the blank(s).

1. The graph of a polynomial function is _____ , so it has no breaks, holes, or gaps.

2. A polynomial function of degree n has at most _____ real zeros and at most _____ relative extrema.

3. If $x = a$ is a zero of a polynomial function f, then the following statements are true.

(a) $x = a$ is a _____ of the polynomial equation $f(x) = 0$.

(b) _____ is a factor of the polynomial $f(x)$.

(c) The point _____ is an x-intercept of the graph of f.

4. If a zero of a polynomial function f is of even multiplicity, then the graph of f _____ the x-axis, and if the zero is of odd multiplicity, then the graph of f _____ the x-axis.

For Exercises 5–8, the graph shows the right-hand and left-hand behavior of a polynomial function *f*.

5. Can f be a fourth-degree polynomial function?

6. Can the leading coefficient of f be negative?

7. The graph shows that $f(x_1) < 0$. What other information shown in the graph allows you to apply the Intermediate Value Theorem to guarantee that f has a zero in the interval $[x_1, x_2]$?

8. Is the repeated zero of f in the interval $[x_3, x_4]$ of even or odd multiplicity?

Procedures and Problem Solving

Identifying Graphs of Polynomial Functions **In Exercises 9–16, match the polynomial function with its graph. [The graphs are labeled (a) through (h).]**

9. $f(x) = -2x + 3$

10. $f(x) = x^2 - 4x$

11. $f(x) = -2x^2 - 5x$

12. $f(x) = 2x^3 - 3x + 1$

13. $f(x) = -\frac{1}{4}x^4 + 3x^2$

14. $f(x) = -\frac{1}{3}x^3 + x^2 - \frac{4}{3}$

15. $f(x) = x^4 + 2x^3$

16. $f(x) = \frac{1}{5}x^5 - 2x^3 + \frac{9}{5}x$

Library of Parent Functions **In Exercises 17–22, sketch the graph of $y = x^3$ and the graph of the function f. Describe the transformation from y to f.**

17. $f(x) = (x - 2)^3$

18. $f(x) = x^3 - 2$

19. $f(x) = -x^3 + 1$

20. $f(x) = (x - 2)^3 - 2$

21. $f(x) = -(x - 2)^3$

22. $f(x) = -x^3 + 3$

Comparing End Behavior **In Exercises 23–28, use a graphing utility to graph the functions f and g in the same viewing window. Zoom out far enough to see the right-hand and left-hand behavior of each graph. Do the graphs of f and g have the same right-hand and left-hand behavior? Explain why or why not.**

23. $f(x) = 3x^3 - 9x + 1, \quad g(x) = 3x^3$

24. $f(x) = -\frac{1}{3}(x^3 - 3x + 2), \quad g(x) = -\frac{1}{3}x^3$

25. $f(x) = -(x^4 - 4x^3 + 16x), \quad g(x) = -x^4$

26. $f(x) = 3x^4 - 6x^2, \quad g(x) = 3x^4$

27. $f(x) = -2x^3 + 4x^2 - 1, \quad g(x) = 2x^3$

28. $f(x) = -(x^4 - 6x^2 - x + 10), \quad g(x) = x^4$

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Applying the Leading Coefficient Test **In Exercises 29–36, use the Leading Coefficient Test to describe the right-hand and left-hand behavior of the graph of the polynomial function. Use a graphing utility to verify your result.**

29. $f(x) = 2x^4 - 3x + 1$ **30.** $h(x) = 1 - x^6$

31. $g(x) = 5 - \frac{7}{2}x - 3x^2$ **32.** $f(x) = \frac{1}{3}x^3 + 5x$

33. $f(x) = \dfrac{6x^5 - 2x^4 + 4x^2 - 5x}{3}$

34. $f(x) = \dfrac{3x^7 - 2x^5 + 5x^3 + 6x^2}{4}$

35. $h(t) = -\frac{2}{3}(t^2 - 5t + 3)$

36. $f(s) = -\frac{7}{8}(s^3 + 5s^2 - 7s + 1)$

Finding Zeros of a Polynomial Function **In Exercises 37–48, (a) find the zeros algebraically, (b) use a graphing utility to graph the function, and (c) use the graph to approximate any zeros and compare them with those from part (a).**

37. $f(x) = 3x^2 - 12x + 3$ **38.** $g(x) = 5x^2 - 10x - 5$

39. $g(t) = \frac{1}{2}t^4 - \frac{1}{2}$ **40.** $y = \frac{1}{4}x^3(x^2 - 9)$

41. $f(x) = x^5 + x^3 - 6x$ **42.** $g(t) = t^5 - 6t^3 + 9t$

43. $f(x) = 2x^4 - 2x^2 - 40$

44. $f(x) = 5x^4 + 15x^2 + 10$

45. $f(x) = x^3 - 4x^2 - 25x + 100$

46. $y = 4x^3 + 4x^2 - 7x + 2$

47. $y = 4x^3 - 20x^2 + 25x$

48. $y = x^5 - 5x^3 + 4x$

Finding Zeros and Their Multiplicities **In Exercises 49–58, find all the real zeros of the polynomial function. Determine the multiplicity of each zero. Use a graphing utility to verify your result.**

49. $f(x) = x^2 - 25$ **50.** $f(x) = 49 - x^2$

51. $h(t) = t^2 - 6t + 9$ **52.** $f(x) = x^2 + 10x + 25$

53. $f(x) = x^2 + x - 2$ **54.** $f(x) = 2x^2 - 14x + 24$

55. $f(t) = t^3 - 4t^2 + 4t$ **56.** $f(x) = x^4 - x^3 - 20x^2$

57. $f(x) = \frac{1}{2}x^2 + \frac{5}{2}x - \frac{3}{2}$ **58.** $f(x) = \frac{5}{3}x^2 + \frac{8}{3}x - \frac{4}{3}$

Analyzing a Polynomial Function **In Exercises 59–64, use a graphing utility to graph the function and approximate (accurate to three decimal places) any real zeros and relative extrema.**

59. $f(x) = 2x^4 - 6x^2 + 1$

60. $f(x) = -\frac{3}{8}x^4 - x^3 + 2x^2 + 5$

61. $f(x) = x^5 + 3x^3 - x + 6$

62. $f(x) = -3x^3 - 4x^2 + x - 3$

63. $f(x) = -2x^4 + 5x^2 - x - 1$

64. $f(x) = 3x^5 - 2x^2 - x + 1$

Finding a Polynomial Function with Given Zeros **In Exercises 65–74, find a polynomial function that has the given zeros. (There are many correct answers.)**

65. 0, 4 **66.** $-7, 2$

67. $0, -2, -3$ **68.** 0, 2, 5

69. $4, -3, 3, 0$ **70.** $-2, -1, 0, 1, 2$

71. $1 + \sqrt{3}, 1 - \sqrt{3}$ **72.** $6 + \sqrt{3}, 6 - \sqrt{3}$

73. $2, 4 + \sqrt{5}, 4 - \sqrt{5}$ **74.** $4, 2 + \sqrt{7}, 2 - \sqrt{7}$

Finding a Polynomial Function with Given Zeros **In Exercises 75–80, find a polynomial function with the given zeros, multiplicities, and degree. (There are many correct answers.)**

75. Zero: -2, multiplicity: 2
Zero: -1, multiplicity: 1
Degree: 3

76. Zero: 3, multiplicity: 1
Zero: 2, multiplicity: 3
Degree: 4

77. Zero: -4, multiplicity: 2
Zero: 3, multiplicity: 2
Degree: 4

78. Zero: 5, multiplicity: 3
Zero: 0, multiplicity: 2
Degree: 5

79. Zero: -1, multiplicity: 2
Zero: -2, multiplicity: 1
Degree: 3
Rises to the left,
Falls to the right

80. Zero: 1, multiplicity: 2
Zero: 4, multiplicity: 2
Degree: 4
Falls to the left,
Falls to the right

Sketching a Polynomial with Given Conditions **In Exercises 81–84, sketch the graph of a polynomial function that satisfies the given conditions. If not possible, explain your reasoning. (There are many correct answers.)**

81. Third-degree polynomial with two real zeros and a negative leading coefficient

82. Fourth-degree polynomial with three real zeros and a positive leading coefficient

83. Fifth-degree polynomial with three real zeros and a positive leading coefficient

84. Fourth-degree polynomial with two real zeros and a negative leading coefficient

Sketching the Graph of a Polynomial Function **In Exercises 85–94, sketch the graph of the function by (a) applying the Leading Coefficient Test, (b) finding the zeros of the polynomial, (c) plotting sufficient solution points, and (d) drawing a continuous curve through the points.**

85. $f(x) = x^3 - 9x$ **86.** $g(x) = x^4 - 4x^2$

87. $f(x) = x^3 - 3x^2$ **88.** $f(x) = 3x^3 - 24x^2$

89. $f(x) = -x^4 + 9x^2 - 20$ **90.** $f(x) = -x^6 + 7x^3 + 8$

91. $f(x) = x^3 + 3x^2 - 9x - 27$

92. $h(x) = x^5 - 4x^3 + 8x^2 - 32$

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93. $g(t) = -\frac{1}{4}t^4 + 2t^2 - 4$

94. $g(x) = \frac{1}{10}(x^4 - 4x^3 - 2x^2 + 12x + 9)$

Approximating the Zeros of a Function **In Exercises 95–100, (a) use the Intermediate Value Theorem and a graphing utility to find graphically any intervals of length 1 in which the polynomial function is guaranteed to have a zero, and (b) use the *zero* or *root* feature of the graphing utility to approximate the real zeros of the function. Verify your answers in part (a) by using the *table* feature of the graphing utility.**

95. $f(x) = x^3 - 3x^2 + 3$ **96.** $f(x) = -2x^3 - 6x^2 + 3$

97. $g(x) = 3x^4 + 4x^3 - 3$ **98.** $h(x) = x^4 - 10x^2 + 2$

99. $f(x) = x^4 - 3x^3 - 4x - 3$

100. $f(x) = x^3 - 4x^2 - 2x + 10$

Identifying Symmetry and *x*-Intercepts **In Exercises 101–108, use a graphing utility to graph the function. Identify any symmetry with respect to the *x*-axis, *y*-axis, or origin. Determine the number of *x*-intercepts of the graph.**

101. $f(x) = x^2(x + 6)$ **102.** $h(x) = x^3(x - 4)^2$

103. $g(t) = -\frac{1}{2}(t - 4)^2(t + 4)^2$

104. $g(x) = \frac{1}{8}(x + 1)^2(x - 3)^3$

105. $f(x) = x^3 - 4x$ **106.** $f(x) = x^4 - 2x^2$

107. $g(x) = \frac{1}{5}(x + 1)^2(x - 3)(2x - 9)$

108. $h(x) = \frac{1}{5}(x + 2)^2(3x - 5)^2$

109. **Geometry** An open box is to be made from a square piece of material 36 centimeters on a side by cutting equal squares with sides of length x from the corners and turning up the sides (see figure).

(a) Verify that the volume of the box is given by the function $V(x) = x(36 - 2x)^2$.

(b) Determine the domain of the function V.

(c) Use the *table* feature of a graphing utility to create a table that shows various box heights x and the corresponding volumes V. Use the table to estimate a range of dimensions within which the maximum volume is produced.

(d) Use the graphing utility to graph V and use the range of dimensions from part (c) to find the x-value for which $V(x)$ is maximum.

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110. **Geometry** An open box with locking tabs is to be made from a square piece of material 24 inches on a side. This is done by cutting equal squares from the corners and folding along the dashed lines, as shown in the figure.

(a) Verify that the volume of the box is given by the function $V(x) = 8x(6 - x)(12 - x)$.

(b) Determine the domain of the function V.

(c) Sketch the graph of the function and estimate the value of x for which $V(x)$ is maximum.

111. **Marketing** The total revenue R (in millions of dollars) for a company is related to its advertising expense by the function

$$R = 0.00001(-x^3 + 600x^2), \quad 0 \le x \le 400$$

where x is the amount spent on advertising (in tens of thousands of dollars). Use the graph of the function shown in the figure to estimate the point on the graph at which the function is increasing most rapidly. This point is called the **point of diminishing returns** because any expense above this amount will yield less return per dollar invested in advertising.

112. ***Why you should learn it*** *(p. 100)* The growth of a red oak tree is approximated by the function

$$G = -0.003t^3 + 0.137t^2 + 0.458t - 0.839$$

where G is the height of the tree (in feet) and t $(2 \le t \le 34)$ is its age (in years). Use a graphing utility to graph the function and estimate the age of the tree when it is growing most rapidly. This point is called the **point of diminishing returns** because the increase in growth will be less with each additional year. (*Hint:* Use a viewing window in which $0 \le x \le 35$ and $0 \le y \le 60$.)

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113. MODELING DATA

The U.S. production of crude oil y_1 (in quadrillions of British thermal units) and of solar and photovoltaic energy y_2 (in trillions of British thermal units) are shown in the table for the years 1999 through 2008, where t represents the year, with $t = 9$ corresponding to 1999. These data can be approximated by the models

$y_1 = 7.204t^3 - 301.60t^2 + 3854.2t - 3130$ and

$y_2 = 0.077t^3 - 2.31t^2 + 21.3t + 8.$

(Source: Energy Information Administration)

Year, t	y_1	y_2
9	12,451	69
10	12,358	66
11	12,282	65
12	12,163	64
13	12,026	64
14	11,503	65
15	10,963	66
16	10,801	72
17	10,721	81
18	10,519	91

(a) Use a graphing utility to plot the data and graph the model for y_1 in the same viewing window. How closely does the model represent the data?

(b) Extend the viewing window of the graphing utility to show the right-hand behavior of the model y_1. Would you use the model to estimate the production of crude oil in 2010? in 2020? Explain.

(c) Repeat parts (a) and (b) for y_2.

Conclusions

True or False? **In Exercises 114–118, determine whether the statement is true or false. Justify your answer.**

114. It is possible for a sixth-degree polynomial to have only one zero.

115. The graph of the function

$$f(x) = 2 + x - x^2 + x^3 - x^4 + x^5 + x^6 - x^7$$

rises to the left and falls to the right.

116. The graph of the function $f(x) = 2x(x - 1)^2(x + 3)^3$ crosses the x-axis at $x = 1$.

117. The graph of the function $f(x) = 2x(x - 1)^2(x + 3)^3$ touches, but does not cross, the x-axis.

118. The graph of the function $f(x) = 2x(x - 1)^2(x + 3)^3$ rises to the left and falls to the right.

119. Exploration Use a graphing utility to graph

$y_1 = x + 2$ and $y_2 = (x + 2)(x - 1).$

Predict the shape of the graph of

$y_3 = (x + 2)(x - 1)(x - 3).$

Use the graphing utility to verify your answer.

120. CAPSTONE For each graph, describe a polynomial function that could represent the graph. (Indicate the degree of the function and the sign of its leading coefficient.)

Cumulative Mixed Review

Evaluating Combinations of Functions **In Exercises 121–126, let $f(x) = 14x - 3$ and $g(x) = 8x^2$. Find the indicated value.**

121. $(f + g)(-4)$

122. $(g - f)(3)$

123. $(fg)\left(-\frac{4}{7}\right)$

124. $\left(\frac{f}{g}\right)(-1.5)$

125. $(f \circ g)(-1)$

126. $(g \circ f)(0)$

Solving Inequalities **In Exercises 127–130, solve the inequality and sketch the solution on the real number line. Use a graphing utility to verify your solution graphically.**

127. $3(x - 5) < 4x - 7$

128. $2x^2 - x \geq 1$

129. $\frac{5x - 2}{x - 7} \leq 4$

130. $|x + 8| - 1 \geq 15$

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2.3 Real Zeros of Polynomial Functions

Long Division of Polynomials

Consider the graph of

$$f(x) = 6x^3 - 19x^2 + 16x - 4.$$

Notice in Figure 2.25 that $x = 2$ appears to be a zero of f. Because $f(2) = 0$, you know that $x = 2$ is a zero of the polynomial function f, and that $(x - 2)$ is a factor of $f(x)$. This means that there exists a second-degree polynomial $q(x)$ such that $f(x) = (x - 2) \cdot q(x)$. To find $q(x)$, you can use **long division of polynomials.**

Figure 2.25

What you should learn

- Use long division to divide polynomials by other polynomials.
- Use synthetic division to divide polynomials by binomials of the form $(x - k)$.
- Use the Remainder and Factor Theorems.
- Use the Rational Zero Test to determine possible rational zeros of polynomial functions.
- Use Descartes's Rule of Signs and the Upper and Lower Bound Rules to find zeros of polynomials.

Why you should learn it

The Remainder Theorem can be used to determine the number of employees in education and health services in the United States in a given year based on a polynomial model, as shown in Exercise 104 on page 127.

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Example 1 Long Division of Polynomials

Divide $6x^3 - 19x^2 + 16x - 4$ by $x - 2$, and use the result to factor the polynomial completely.

Solution

Partial quotients

$$
\begin{array}{rrrrrl}
 & & 6x^2 & - & 7x & + \quad 2 \\
x - 2 \,\big) & 6x^3 - & 19x^2 & + & 16x & - \quad 4 \\
 & 6x^3 - & 12x^2 & & & \\
\hline
 & & -7x^2 & + & 16x & \\
 & & -7x^2 & + & 14x & \\
\hline
 & & & & 2x & - \quad 4 \\
 & & & & 2x & - \quad 4 \\
\hline
 & & & & & \quad 0
\end{array}
$$

Multiply: $6x^2(x - 2)$.
Subtract.
Multiply: $-7x(x - 2)$.
Subtract.
Multiply: $2(x - 2)$.
Subtract.

You can see that

$$
\begin{aligned}
6x^3 - 19x^2 + 16x - 4 &= (x - 2)(6x^2 - 7x + 2) \\
&= (x - 2)(2x - 1)(3x - 2).
\end{aligned}
$$

Note that this factorization agrees with the graph of f (see Figure 2.25) in that the three x-intercepts occur at $x = 2$, $x = \frac{1}{2}$, and $x = \frac{2}{3}$.

CHECKPOINT Now try Exercise 9.

Note that in Example 1, the division process requires $-7x^2 + 14x$ to be subtracted from $-7x^2 + 16x$. Therefore, it is implied that

$$
\begin{array}{r}
-7x^2 + 16x \\
-(-7x^2 + 14x) \\
\hline
\end{array}
\quad = \quad
\begin{array}{r}
-7x^2 + 16x \\
7x^2 - 14x \\
\hline
\end{array}
$$

and instead is written simply as

$$
\begin{array}{r}
-7x^2 + 16x \\
-7x^2 + 14x. \\
\hline
2x \qquad
\end{array}
$$

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In Example 1, $x - 2$ is a factor of the polynomial

$$6x^3 - 19x^2 + 16x - 4$$

and the long division process produces a remainder of zero. Often, long division will produce a nonzero remainder. For instance, when you divide $x^2 + 3x + 5$ by $x + 1$, you obtain the following.

$$\begin{array}{rl} & \;x + 2 \\ \text{Divisor} \Rightarrow x + 1 & \overline{)\,x^2 + 3x + 5} \;\Leftarrow \text{Quotient, Dividend} \\ & \underline{x^2 + x} \\ & \phantom{x^2 +{}} 2x + 5 \\ & \phantom{x^2 +{}} \underline{2x + 2} \\ & \phantom{x^2 + 2x +{}} 3 \;\Leftarrow \text{Remainder} \end{array}$$

In fractional form, you can write this result as follows.

$$\frac{\overbrace{x^2 + 3x + 5}^{\text{Dividend}}}{\underbrace{x + 1}_{\text{Divisor}}} = \overbrace{x + 2}^{\text{Quotient}} + \frac{\overset{\text{Remainder}}{3}}{\underbrace{x + 1}_{\text{Divisor}}}$$

This implies that

$$x^2 + 3x + 5 = (x + 1)(x + 2) + 3 \qquad \text{Multiply each side by } (x + 1).$$

which illustrates the following theorem, called the **Division Algorithm.**

The Division Algorithm

If $f(x)$ and $d(x)$ are polynomials such that $d(x) \neq 0$, and the degree of $d(x)$ is less than or equal to the degree of $f(x)$, then there exist unique polynomials $q(x)$ and $r(x)$ such that

$$\underset{\text{Dividend}}{f(x)} = \underset{\text{Divisor}}{d(x)}\,\underset{\text{Quotient}}{q(x)} + \underset{\text{Remainder}}{r(x)}$$

where $r(x) = 0$ *or* the degree of $r(x)$ is less than the degree of $d(x)$. If the remainder $r(x)$ is zero, then $d(x)$ *divides evenly* into $f(x)$.

The Division Algorithm can also be written as

$$\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}.$$

In the Division Algorithm, the rational expression $f(x)/d(x)$ is **improper** because the degree of $f(x)$ is greater than or equal to the degree of $d(x)$. On the other hand, the rational expression $r(x)/d(x)$ is **proper** because the degree of $r(x)$ is less than the degree of $d(x)$.

Before you apply the Division Algorithm, follow these steps.

1. Write the dividend and divisor in descending powers of the variable.

2. Insert placeholders with zero coefficients for missing powers of the variable.

Note how these steps are applied in the next two examples.

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Example 2 Long Division of Polynomials

Divide $8x^3 - 1$ by $2x - 1$.

Solution

Because there is no x^2-term or x-term in the dividend, you need to line up the subtraction by using zero coefficients (or leaving spaces) for the missing terms.

$$
\begin{array}{r}
4x^2 + 2x + 1 \\
2x - 1 \overline{\big)\, 8x^3 + 0x^2 + 0x - 1} \\
\underline{8x^3 - 4x^2} \qquad\qquad \\
4x^2 + 0x \qquad \\
\underline{4x^2 - 2x} \qquad \\
2x - 1 \\
\underline{2x - 1} \\
0
\end{array}
$$

So, $2x - 1$ divides evenly into $8x^3 - 1$, and you can write

$$\frac{8x^3 - 1}{2x - 1} = 4x^2 + 2x + 1, \quad x \neq \frac{1}{2}.$$

CHECKPOINT Now try Exercise 15.

You can check the result of Example 2 by multiplying.

$$
\begin{aligned}
(2x - 1)(4x^2 + 2x + 1) &= 8x^3 + 4x^2 + 2x - 4x^2 - 2x - 1 \\
&= 8x^3 - 1
\end{aligned}
$$

In each of the long division examples presented so far, the divisor has been a first-degree polynomial. The long division algorithm works just as well with polynomial divisors of degree two or more, as shown in Example 3.

Example 3 Long Division of Polynomials

Divide $-2 + 3x - 5x^2 + 4x^3 + 2x^4$ by $x^2 + 2x - 3$.

Solution

Begin by writing the dividend in descending powers of x.

$$
\begin{array}{r}
2x^2 \qquad + 1 \\
x^2 + 2x - 3 \overline{\big)\, 2x^4 + 4x^3 - 5x^2 + 3x - 2} \\
\underline{2x^4 + 4x^3 - 6x^2 \qquad\qquad\qquad} \\
x^2 + 3x - 2 \\
\underline{x^2 + 2x - 3} \\
x + 1
\end{array}
$$

Note that the first subtraction eliminated two terms from the dividend. When this happens, the quotient skips a term. You can write the result as

$$\frac{2x^4 + 4x^3 - 5x^2 + 3x - 2}{x^2 + 2x - 3} = 2x^2 + 1 + \frac{x + 1}{x^2 + 2x - 3}.$$

CHECKPOINT Now try Exercise 17.

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Synthetic Division

There is a nice shortcut for long division of polynomials when dividing by divisors of the form

$$x - k.$$

The shortcut is called **synthetic division.** The pattern for synthetic division of a cubic polynomial is summarized as follows. (The pattern for higher-degree polynomials is similar.)

Synthetic Division (of a Cubic Polynomial)

To divide $ax^3 + bx^2 + cx + d$ by $x - k$, use the following pattern.

Vertical pattern: Add terms.
Diagonal pattern: Multiply by k.

This algorithm for synthetic division works *only* for divisors of the form $x - k$. Remember that

$$x + k = x - (-k).$$

Example 4 Using Synthetic Division

Use synthetic division to divide

$$x^4 - 10x^2 - 2x + 4 \quad \text{by} \quad x + 3.$$

Solution

You should set up the array as follows. Note that a zero is included for each missing term in the dividend.

Then, use the synthetic division pattern by adding terms in columns and multiplying the results by -3.

Divisor: $x + 3$ Dividend: $x^4 - 10x^2 - 2x + 4$

$$\begin{array}{r|rrrrr} -3 & 1 & 0 & -10 & -2 & 4 \\ & & -3 & 9 & 3 & -3 \\ \hline & 1 & -3 & -1 & 1 & 1 \end{array}$$

Remainder: 1

Quotient: $x^3 - 3x^2 - x + 1$

So, you have

$$\frac{x^4 - 10x^2 - 2x + 4}{x + 3} = x^3 - 3x^2 - x + 1 + \frac{1}{x + 3}.$$

CHECKPOINT Now try Exercise 23.

Explore the Concept

Evaluate the polynomial $x^4 - 10x^2 - 2x + 4$ at $x = -3$. What do you observe?

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The Remainder and Factor Theorems

The remainder obtained in the synthetic division process has an important interpretation, as described in the **Remainder Theorem.**

The Remainder Theorem (See the proof on page 176.)

If a polynomial $f(x)$ is divided by $x - k$, then the remainder is

$$r = f(k).$$

The Remainder Theorem tells you that synthetic division can be used to evaluate a polynomial function. That is, to evaluate a polynomial function $f(x)$ when $x = k$, divide $f(x)$ by $x - k$. The remainder will be $f(k)$.

Example 5 Using the Remainder Theorem

Use the Remainder Theorem to evaluate the following function at $x = -2$.

$$f(x) = 3x^3 + 8x^2 + 5x - 7$$

Solution

Using synthetic division, you obtain the following.

$$\begin{array}{r|rrrr} -2 & 3 & 8 & 5 & -7 \\ & & -6 & -4 & -2 \\ \hline & 3 & 2 & 1 & -9 \end{array}$$

Because the remainder is $r = -9$, you can conclude that

$$f(-2) = -9. \qquad r = f(k)$$

This means that $(-2, -9)$ is a point on the graph of f. You can check this by substituting $x = -2$ in the original function.

Check

$$\begin{aligned} f(-2) &= 3(-2)^3 + 8(-2)^2 + 5(-2) - 7 \\ &= 3(-8) + 8(4) - 10 - 7 \\ &= -24 + 32 - 10 - 7 \\ &= -9 \end{aligned}$$

CHECKPOINT Now try Exercise 43.

Another important theorem is the **Factor Theorem.** This theorem states that you can test whether a polynomial has $(x - k)$ as a factor by evaluating the polynomial at $x = k$. If the result is 0, then $(x - k)$ is a factor.

The Factor Theorem (See the proof on page 176.)

A polynomial $f(x)$ has a factor

$$(x - k)$$

if and only if

$$f(k) = 0.$$

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Example 6 Factoring a Polynomial: Repeated Division

Show that $(x - 2)$ and $(x + 3)$ are factors of

$$f(x) = 2x^4 + 7x^3 - 4x^2 - 27x - 18.$$

Then find the remaining factors of $f(x)$.

Algebraic Solution

Using synthetic division with the factor $(x - 2)$, you obtain the following.

$$\begin{array}{r|rrrrr} 2 & 2 & 7 & -4 & -27 & -18 \\ & & 4 & 22 & 36 & 18 \\ \hline & 2 & 11 & 18 & 9 & 0 \end{array} \longrightarrow$$

0 remainder; $(x - 2)$ is a factor.

Take the result of this division and perform synthetic division again using the factor $(x + 3)$.

$$\begin{array}{r|rrrr} -3 & 2 & 11 & 18 & 9 \\ & & -6 & -15 & -9 \\ \hline & \underbrace{2 \quad 5 \quad 3}_{2x^2 + 5x + 3} & & & 0 \end{array} \longrightarrow$$

0 remainder; $(x + 3)$ is a factor.

Because the resulting quadratic factors as

$$2x^2 + 5x + 3 = (2x + 3)(x + 1)$$

the complete factorization of $f(x)$ is

$$f(x) = (x - 2)(x + 3)(2x + 3)(x + 1).$$

CHECKPOINT Now try Exercise 53.

Graphical Solution

From the graph of

$$f(x) = 2x^4 + 7x^3 - 4x^2 - 27x - 18$$

you can see that there are four x-intercepts (see Figure 2.26). These occur at $x = -3$, $x = -\frac{3}{2}$, $x = -1$, and $x = 2$. (Check this algebraically.) This implies that $(x + 3)$, $\left(x + \frac{3}{2}\right)$, $(x + 1)$, and $(x - 2)$ are factors of $f(x)$. [Note that $\left(x + \frac{3}{2}\right)$ and $(2x + 3)$ are equivalent factors because they both yield the same zero, $x = -\frac{3}{2}$.]

Figure 2.26

Note in Example 6 that the complete factorization of $f(x)$ implies that f has four real zeros:

$$x = 2, \quad x = -3, \quad x = -\tfrac{3}{2}, \quad \text{and} \quad x = -1.$$

This is confirmed by the graph of f, which is shown in Figure 2.26.

Using the Remainder in Synthetic Division

In summary, the remainder r, obtained in the synthetic division of $f(x)$ by $x - k$, provides the following information.

1. The remainder r gives the value of f at $x = k$. That is, $r = f(k)$.
2. If $r = 0$, then $(x - k)$ is a factor of $f(x)$.
3. If $r = 0$, then $(k, 0)$ is an x-intercept of the graph of f.

Throughout this text, the importance of developing several problem-solving strategies is emphasized. In the exercises for this section, try using more than one strategy to solve several of the exercises. For instance, when you find that $x - k$ divides evenly into $f(x)$, try sketching the graph of f. You should find that $(k, 0)$ is an x-intercept of the graph.

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The Rational Zero Test

The **Rational Zero Test** relates the possible rational zeros of a polynomial (having integer coefficients) to the leading coefficient and to the constant term of the polynomial.

The Rational Zero Test

If the polynomial

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2 x^2 + a_1 x + a_0$$

has integer coefficients, then every rational zero of f has the form

$$\text{Rational zero} = \frac{p}{q}$$

where p and q have no common factors other than 1, p is a factor of the constant term a_0, and q is a factor of the leading coefficient a_n.

To use the Rational Zero Test, first list all rational numbers whose numerators are factors of the constant term and whose denominators are factors of the leading coefficient.

$$\text{Possible rational zeros} = \frac{\text{factors of constant term}}{\text{factors of leading coefficient}}$$

Now that you have formed this list of *possible rational zeros*, use a trial-and-error method to determine which, if any, are actual zeros of the polynomial. Note that when the leading coefficient is 1, the possible rational zeros are simply the factors of the constant term. This case is illustrated in Example 7.

Study Tip

Use a graphing utility to graph the polynomial

$$y = x^3 - 53x^2 + 103x - 51$$

in a standard viewing window. From the graph alone, it appears that there is only one zero. From the Leading Coefficient Test, you know that because the degree of the polynomial is odd and the leading coefficient is positive, the graph falls to the left and rises to the right. So, the function must have another zero. From the Rational Zero Test, you know that ± 51 might be zeros of the function. When you zoom out several times, you will see a more complete picture of the graph. Your graph should confirm that $x = 51$ is a zero of f.

Example 7 Rational Zero Test with Leading Coefficient of 1

Find the rational zeros of $f(x) = x^3 + x + 1$.

Solution

Because the leading coefficient is 1, the possible rational zeros are simply the factors of the constant term.

Possible rational zeros: ± 1

By testing these possible zeros, you can see that neither works.

$$f(1) = (1)^3 + 1 + 1 = 3$$

$$f(-1) = (-1)^3 + (-1) + 1 = -1$$

So, you can conclude that the polynomial has *no* rational zeros. Note from the graph of f in Figure 2.27 that f does have one real zero between -1 and 0. However, by the Rational Zero Test, you know that this real zero is *not* a rational number.

Figure 2.27

CHECKPOINT Now try Exercise 57.

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When the leading coefficient of a polynomial is not 1, the list of possible rational zeros can increase dramatically. In such cases, the search can be shortened in several ways.

1. A programmable calculator can be used to speed up the calculations.
2. A graphing utility can give a good estimate of the locations of the zeros.
3. The Intermediate Value Theorem, along with a table generated by a graphing utility, can give approximations of zeros.
4. The Factor Theorem and synthetic division can be used to test the possible rational zeros.

Finding the first zero is often the most difficult part. After that, the search is simplified by working with the lower-degree polynomial obtained in synthetic division, as shown in Example 8.

Example 8 Using the Rational Zero Test

Find the rational zeros of

$$f(x) = 2x^3 + 3x^2 - 8x + 3.$$

Solution

The leading coefficient is 2 and the constant term is 3.

Possible rational zeros:

$$\frac{\text{Factors of 3}}{\text{Factors of 2}} = \frac{\pm 1, \pm 3}{\pm 1, \pm 2} = \pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}$$

By synthetic division, you can determine that $x = 1$ is a rational zero.

$$\begin{array}{r|rrrr} 1 & 2 & 3 & -8 & 3 \\ & & 2 & 5 & -3 \\ \hline & 2 & 5 & -3 & 0 \end{array}$$

So, $f(x)$ factors as

$$\begin{aligned} f(x) &= (x - 1)(2x^2 + 5x - 3) \\ &= (x - 1)(2x - 1)(x + 3) \end{aligned}$$

and you can conclude that the rational zeros of f are $x = 1$, $x = \frac{1}{2}$, and $x = -3$, as shown in Figure 2.28.

Figure 2.28

CHECKPOINT Now try Exercise 59.

Remember that when you try to find the rational zeros of a polynomial function with many possible rational zeros, as in Example 8, you must use trial and error. There is no quick algebraic method to determine which of the possibilities is an actual zero; however, sketching a graph may be helpful.

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Other Tests for Zeros of Polynomials

You know that an nth-degree polynomial function can have *at most* n real zeros. Of course, many nth-degree polynomials do not have that many real zeros. For instance, $f(x) = x^2 + 1$ has no real zeros, and $f(x) = x^3 + 1$ has only one real zero. The following theorem, called **Descartes's Rule of Signs,** sheds more light on the number of real zeros of a polynomial.

Decartes's Rule of Signs

Let $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2x^2 + a_1x + a_0$ be a polynomial with real coefficients and $a_0 \neq 0$.

1. The number of *positive real zeros* of f is either equal to the number of variations in sign of $f(x)$ or less than that number by an even integer.
2. The number of *negative real zeros* of f is either equal to the number of variations in sign of $f(-x)$ or less than that number by an even integer.

A **variation in sign** means that two consecutive (nonzero) coefficients have opposite signs.

When using Descartes's Rule of Signs, a zero of multiplicity k should be counted as k zeros. For instance, the polynomial $x^3 - 3x + 2$ has two variations in sign, and so has either two positive or no positive real zeros. Because

$$x^3 - 3x + 2 = (x - 1)(x - 1)(x + 2)$$

you can see that the two positive real zeros are $x = 1$ of multiplicity 2.

Example 9 Using Descartes's Rule of Signs

Describe the possible real zeros of $f(x) = 3x^3 - 5x^2 + 6x - 4$.

Solution

The original polynomial has *three* variations in sign.

$$f(x) = 3x^3 \overset{+ \text{ to } -}{-} 5x^2 \overset{+ \text{ to } -}{+} 6x - 4$$

$$\underset{- \text{ to } +}{}$$

The polynomial

$$\begin{aligned} f(-x) &= 3(-x)^3 - 5(-x)^2 + 6(-x) - 4 \\ &= -3x^3 - 5x^2 - 6x - 4 \end{aligned}$$

has no variations in sign. So, from Descartes's Rule of Signs, the polynomial $f(x) = 3x^3 - 5x^2 + 6x - 4$ has either three positive real zeros or one positive real zero, and has no negative real zeros. By using the *trace* feature of a graphing utility, you can see that the function has only one real zero (it is a positive number near $x = 1$), as shown in Figure 2.29.

Figure 2.29

CHECKPOINT Now try Exercise 61.

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Another test for zeros of a polynomial function is related to the sign pattern in the last row of the synthetic division array. This test can give you an upper or lower bound of the real zeros of f, which can help you eliminate possible real zeros. A real number c is an **upper bound** for the real zeros of f when no zeros are greater than c. Similarly, c is a **lower bound** when no real zeros of f are less than c.

Upper and Lower Bound Rules

Let $f(x)$ be a polynomial with real coefficients and a positive leading coefficient. Suppose $f(x)$ is divided by $x - c$, using synthetic division.

1. If $c > 0$ and each number in the last row is either positive or zero, then c is an **upper bound** for the real zeros of f.
2. If $c < 0$ and the numbers in the last row are alternately positive and negative (zero entries count as positive or negative), then c is a **lower bound** for the real zeros of f.

Example 10 Finding the Zeros of a Polynomial Function

Find the real zeros of $f(x) = 6x^3 - 4x^2 + 3x - 2$.

Solution

The possible real zeros are as follows.

$$\frac{\text{Factors of 2}}{\text{Factors of 6}} = \frac{\pm 1, \pm 2}{\pm 1, \pm 2, \pm 3, \pm 6} = \pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}, \pm\frac{2}{3}, \pm 2$$

The original polynomial $f(x)$ has three variations in sign. The polynomial

$$\begin{aligned} f(-x) &= 6(-x)^3 - 4(-x)^2 + 3(-x) - 2 \\ &= -6x^3 - 4x^2 - 3x - 2 \end{aligned}$$

has no variations in sign. As a result of these two findings, you can apply Descartes's Rule of Signs to conclude that there are three positive real zeros or one positive real zero, and no negative real zeros. Trying $x = 1$ produces the following.

$$\begin{array}{r|rrrr} 1 & 6 & -4 & 3 & -2 \\ & & 6 & 2 & 5 \\ \hline & 6 & 2 & 5 & 3 \end{array}$$

So, $x = 1$ is not a zero, but because the last row has all positive entries, you know that $x = 1$ is an upper bound for the real zeros. Therefore, you can restrict the search to zeros between 0 and 1. By trial and error, you can determine that $x = \frac{2}{3}$ is a zero. So,

$$f(x) = \left(x - \frac{2}{3}\right)(6x^2 + 3).$$

Because $6x^2 + 3$ has no real zeros, it follows that $x = \frac{2}{3}$ is the only real zero, as shown in Figure 2.30.

Figure 2.30

CHECKPOINT Now try Exercise 71.

Explore the Concept

Use a graphing utility to graph the polynomial

$$y_1 = 6x^3 - 4x^2 + 3x - 2.$$

Notice that the graph intersects the x-axis at the point $\left(\frac{2}{3}, 0\right)$. How does this information relate to the real zero found in Example 10? Use the graphing utility to graph

$$y_2 = x^4 - 5x^3 + 3x^2 + x.$$

How many times does the graph intersect the x-axis? How many real zeros does y_2 have?

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Here are two additional hints that can help you find the real zeros of a polynomial.

1. When the terms of $f(x)$ have a common monomial factor, it should be factored out before applying the tests in this section. For instance, by writing

$$f(x) = x^4 - 5x^3 + 3x^2 + x = x(x^3 - 5x^2 + 3x + 1)$$

you can see that $x = 0$ is a zero of f and that the remaining zeros can be obtained by analyzing the cubic factor.

2. When you are able to find all but two zeros of $f(x)$, you can always use the Quadratic Formula on the remaining quadratic factor. For instance, after writing

$$f(x) = x^4 - 5x^3 + 3x^2 + x = x(x - 1)(x^2 - 4x - 1)$$

you can apply the Quadratic Formula to $x^2 - 4x - 1$ to conclude that the two remaining zeros are $x = 2 + \sqrt{5}$ and $x = 2 - \sqrt{5}$.

Note how these hints are applied in the next example.

Explore the Concept

Use a graphing utility to graph the polynomial

$$y = x^3 + 4.8x^2 - 127x + 309$$

in a standard viewing window. From the graph, what do the real zeros appear to be? Discuss how the mathematical tools of this section might help you realize that the graph does not show all the important features of the polynomial function. Now use the *zoom* feature to find all the zeros of this function.

Example 11 Finding the Zeros of a Polynomial Function

Find all the real zeros of $f(x) = 10x^4 - 15x^3 - 16x^2 + 12x$.

Solution

Remove the common monomial factor x to write

$$f(x) = 10x^4 - 15x^3 - 16x^2 + 12x = x(10x^3 - 15x^2 - 16x + 12).$$

So, $x = 0$ is a zero of f. You can find the remaining zeros of f by analyzing the cubic factor. Because the leading coefficient is 10 and the constant term is 12, there is a long list of possible rational zeros.

Possible rational zeros:

$$\frac{\text{Factors of } 12}{\text{Factors of } 10} = \frac{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12}{\pm 1, \pm 2, \pm 5, \pm 10}$$

With so many possibilities (32, in fact), it is worth your time to use a graphing utility to focus on just a few. By using the *trace* feature of a graphing utility, it looks like three reasonable choices are $x = -\frac{6}{5}$, $x = \frac{1}{2}$, and $x = 2$ (see Figure 2.31). Synthetic division shows that only $x = 2$ works. (You could also use the Factor Theorem to test these choices.)

Figure 2.31

$$\begin{array}{r|rrrr} 2 & 10 & -15 & -16 & 12 \\ & & 20 & 10 & -12 \\ \hline & 10 & 5 & -6 & 0 \end{array}$$

So, $x = 2$ is one zero and you have

$$f(x) = x(x - 2)(10x^2 + 5x - 6).$$

Using the Quadratic Formula, you find that the two additional zeros are irrational numbers.

$$x = \frac{-5 + \sqrt{265}}{20} \approx 0.56 \quad \text{and} \quad x = \frac{-5 - \sqrt{265}}{20} \approx -1.06$$

CHECKPOINT Now try Exercise 87.

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2.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

1. Two forms of the Division Algorithm are shown below. Identify and label each part.

$$f(x) = d(x)q(x) + r(x) \qquad \frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$$

In Exercises 2–5, fill in the blank(s).

2. The rational expression $p(x)/q(x)$ is called _____ when the degree of the numerator is greater than or equal to that of the denominator.

3. Every rational zero of a polynomial function with integer coefficients has the form p/q, where p is a factor of the _____ and q is a factor of the _____.

4. The theorem that can be used to determine the possible numbers of positive real zeros and negative real zeros of a function is called _____ of _____ .

5. A real number c is a(n) _____ bound for the real zeros of f when no zeros are greater than c, and is a(n) _____ bound when no real zeros of f are less than c.

6. How many negative real zeros are possible for a polynomial function $f(x)$, given that $f(-x)$ has 5 variations in sign?

7. You divide the polynomial $f(x)$ by $(x - 4)$ and obtain a remainder of 7. What is $f(4)$?

8. What value should you write in the circle to check whether $(x - 3)$ is a factor of $f(x) = x^3 - 2x^2 + 3x + 4$?

$$\bigcirc \;\big|\; \begin{array}{rrrr} 1 & -2 & 3 & 4 \\ \hline \end{array}$$

Procedures and Problem Solving

Long Division of Polynomials **In Exercises 9–22, use long division to divide.**

9. Divide $2x^2 + 10x + 12$ by $x + 3$.

10. Divide $5x^2 - 17x - 12$ by $x - 4$.

11. Divide $x^4 + 5x^3 + 6x^2 - x - 2$ by $x + 2$.

12. Divide $x^3 - 4x^2 - 17x + 6$ by $x - 3$.

13. Divide $4x^3 - 7x^2 - 11x + 5$ by $4x + 5$.

14. Divide $2x^3 - 3x^2 - 50x + 75$ by $2x - 3$.

15. Divide $7x^3 + 3$ by $x + 2$.

16. Divide $8x^4 - 5$ by $2x + 1$.

17. $(x + 8 + 6x^3 + 10x^2) \div (2x^2 + 1)$

18. $(1 + 3x^2 + x^4) \div (3 - 2x + x^2)$

19. $(x^3 - 9) \div (x^2 + 1)$

20. $(x^5 + 7) \div (x^3 - 1)$

21. $\dfrac{2x^3 - 4x^2 - 15x + 5}{(x - 1)^2}$

22. $\dfrac{x^4}{(x - 1)^3}$

Using Synthetic Division **In Exercises 23–32, use synthetic division to divide.**

23. $(3x^3 - 17x^2 + 15x - 25) \div (x - 5)$

24. $(5x^3 + 18x^2 + 7x - 6) \div (x + 3)$

25. $(6x^3 + 7x^2 - x + 26) \div (x - 3)$

26. $(2x^3 + 14x^2 - 20x + 7) \div (x + 6)$

27. $(9x^3 - 18x^2 - 16x + 32) \div (x - 2)$

28. $(5x^3 + 6x + 8) \div (x + 2)$

29. $(x^3 + 512) \div (x + 8)$

30. $(x^3 - 729) \div (x - 9)$

31. $\dfrac{4x^3 + 16x^2 - 23x - 15}{x + \frac{1}{2}}$

32. $\dfrac{3x^3 - 4x^2 + 5}{x - \frac{3}{2}}$

Verifying Quotients **In Exercises 33–36, use a graphing utility to graph the two equations in the same viewing window. Use the graphs to verify that the expressions are equivalent. Verify the results algebraically.**

33. $y_1 = \dfrac{x^2}{x + 2}, \quad y_2 = x - 2 + \dfrac{4}{x + 2}$

34. $y_1 = \dfrac{x^2 + 2x - 1}{x + 3}, \quad y_2 = x - 1 + \dfrac{2}{x + 3}$

35. $y_1 = \dfrac{x^4 - 3x^2 - 1}{x^2 + 5}, \quad y_2 = x^2 - 8 + \dfrac{39}{x^2 + 5}$

36. $y_1 = \dfrac{x^4 + x^2 - 1}{x^2 + 1}, \quad y_2 = x^2 - \dfrac{1}{x^2 + 1}$

Verifying the Remainder Theorem **In Exercises 37–42, write the function in the form $f(x) = (x - k)q(x) + r(x)$ for the given value of k. Use a graphing utility to demonstrate that $f(k) = r$.**

	Function	*Value of k*
37.	$f(x) = x^3 - x^2 - 14x + 11$	$k = 4$
38.	$f(x) = 15x^4 + 10x^3 - 6x^2 + 14$	$k = -\frac{2}{3}$

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Function	*Value of k*
39. $f(x) = x^3 + 3x^2 - 2x - 14$	$k = \sqrt{2}$
40. $f(x) = x^3 + 2x^2 - 5x - 4$	$k = -\sqrt{5}$
41. $f(x) = 4x^3 - 6x^2 - 12x - 4$	$k = 1 - \sqrt{3}$
42. $f(x) = -3x^3 + 8x^2 + 10x - 8$	$k = 2 + \sqrt{2}$

Using the Remainder Theorem **In Exercises 43–46, use the Remainder Theorem and synthetic division to evaluate the function at each given value. Use a graphing utility to verify your results.**

✓ **43.** $f(x) = 2x^3 - 7x + 3$
(a) $f(1)$ (b) $f(-2)$ (c) $f\left(\frac{1}{2}\right)$ (d) $f(2)$

44. $g(x) = 2x^6 + 3x^4 - x^2 + 3$
(a) $g(2)$ (b) $g(1)$ (c) $g(3)$ (d) $g(-1)$

45. $h(x) = x^3 - 5x^2 - 7x + 4$
(a) $h(3)$ (b) $h(2)$ (c) $h(-2)$ (d) $h(-5)$

46. $f(x) = 4x^4 - 16x^3 + 7x^2 + 20$
(a) $f(1)$ (b) $f(-2)$ (c) $f(5)$ (d) $f(-10)$

Using the Factor Theorem **In Exercises 47–50, use synthetic division to show that *x* is a solution of the third-degree polynomial equation, and use the result to factor the polynomial completely. List all the real solutions of the equation.**

Polynomial Equation	*Value of x*
47. $x^3 - 7x + 6 = 0$	$x = 2$
48. $x^3 - 28x - 48 = 0$	$x = -4$
49. $2x^3 - 15x^2 + 27x - 10 = 0$	$x = \frac{1}{2}$
50. $48x^3 - 80x^2 + 41x - 6 = 0$	$x = \frac{2}{3}$

Factoring a Polynomial **In Exercises 51–56, (a) verify the given factor(s) of the function *f*, (b) find the remaining factors of *f*, (c) use your results to write the complete factorization of *f*, and (d) list all real zeros of *f*. Confirm your results by using a graphing utility to graph the function.**

Function	*Factor(s)*
51. $f(x) = 2x^3 + x^2 - 5x + 2$	$(x + 2)$
52. $f(x) = 3x^3 + 2x^2 - 19x + 6$	$(x + 3)$
✓ **53.** $f(x) = x^4 - 4x^3 - 15x^2 + 58x - 40$	$(x - 5), (x + 4)$
54. $f(x) = 8x^4 - 14x^3 - 71x^2 - 10x + 24$	$(x + 2), (x - 4)$
55. $f(x) = 6x^3 + 41x^2 - 9x - 14$	$(2x + 1)$
56. $f(x) = 2x^3 - x^2 - 10x + 5$	$(2x - 1)$

Using the Rational Zero Test **In Exercises 57–60, use the Rational Zero Test to list all possible rational zeros of *f*. Then find the rational zeros.**

✓ **57.** $f(x) = x^3 + 3x^2 - x - 3$

58. $f(x) = x^3 - 4x^2 - 4x + 16$

✓ **59.** $f(x) = 2x^4 - 17x^3 + 35x^2 + 9x - 45$

60. $f(x) = 4x^5 - 8x^4 - 5x^3 + 10x^2 + x - 2$

Using Descartes's Rule of Signs **In Exercises 61–64, use Descartes's Rule of Signs to determine the possible numbers of positive and negative real zeros of the function.**

✓ **61.** $f(x) = 2x^4 - x^3 + 6x^2 - x + 5$

62. $f(x) = 3x^4 + 5x^3 - 6x^2 + 8x - 3$

63. $g(x) = 4x^3 - 5x + 8$

64. $g(x) = 2x^3 - 4x^2 - 5$

Finding the Zeros of a Polynomial Function **In Exercises 65–70, (a) use Descartes's Rule of Signs to determine the possible numbers of positive and negative real zeros of *f*, (b) list the possible rational zeros of *f*, (c) use a graphing utility to graph *f* so that some of the possible zeros in parts (a) and (b) can be disregarded, and (d) determine all the real zeros of *f*.**

65. $f(x) = x^3 + x^2 - 4x - 4$

66. $f(x) = -3x^3 + 20x^2 - 36x + 16$

67. $f(x) = -2x^4 + 13x^3 - 21x^2 + 2x + 8$

68. $f(x) = 4x^4 - 17x^2 + 4$

69. $f(x) = 32x^3 - 52x^2 + 17x + 3$

70. $f(x) = x^4 - x^3 - 29x^2 - x - 30$

Finding the Zeros of a Polynomial Function **In Exercises 71–74, use synthetic division to verify the upper and lower bounds of the real zeros of *f*. Then find the real zeros of the function.**

✓ **71.** $f(x) = x^4 - 4x^3 + 15$
Upper bound: $x = 4$
Lower bound: $x = -1$

72. $f(x) = 2x^3 - 3x^2 - 12x + 8$
Upper bound: $x = 4$
Lower bound: $x = -3$

73. $f(x) = x^4 - 4x^3 + 16x - 16$
Upper bound: $x = 5$
Lower bound: $x = -3$

74. $f(x) = 2x^4 - 8x + 3$
Upper bound: $x = 3$
Lower bound: $x = -4$

Occasionally, throughout this text, you will be asked to round to a place value rather than to a number of decimal places.

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Rewriting to Use the Rational Zero Test **In Exercises 75–78, find the rational zeros of the polynomial function.**

75. $P(x) = x^4 - \frac{25}{4}x^2 + 9 = \frac{1}{4}(4x^4 - 25x^2 + 36)$

76. $f(x) = x^3 - \frac{3}{2}x^2 - \frac{23}{2}x + 6 = \frac{1}{2}(2x^3 - 3x^2 - 23x + 12)$

77. $f(x) = x^3 - \frac{1}{4}x^2 - x + \frac{1}{4} = \frac{1}{4}(4x^3 - x^2 - 4x + 1)$

78. $f(z) = z^3 + \frac{11}{6}z^2 - \frac{1}{2}z - \frac{1}{3} = \frac{1}{6}(6z^3 + 11z^2 - 3z - 2)$

A Cubic Polynomial with Two Terms **In Exercises 79–82, match the cubic function with the correct number of rational and irrational zeros.**

(a) Rational zeros: 0; Irrational zeros: 1

(b) Rational zeros: 3; Irrational zeros: 0

(c) Rational zeros: 1; Irrational zeros: 2

(d) Rational zeros: 1; Irrational zeros: 0

79. $f(x) = x^3 - 1$

80. $f(x) = x^3 - 2$

81. $f(x) = x^3 - x$

82. $f(x) = x^3 - 2x$

Using a Graph to Help Find Zeros **In Exercises 83–86, the graph of $y = f(x)$ is shown. Use the graph as an aid to find all the real zeros of the function.**

83. $y = 2x^4 - 9x^3 + 5x^2 + 3x - 1$

84. $y = x^4 - 5x^3 - 7x^2 + 13x - 2$

85. $y = -2x^4 + 17x^3 - 3x^2 - 25x - 3$

86. $y = -x^4 + 5x^3 - 10x - 4$

Finding the Zeros of a Polynomial Function **In Exercises 87–98, find all real zeros of the polynomial function.**

87. $f(x) = 3x^4 - 14x^2 - 4x$

88. $g(x) = 4x^4 - 11x^3 - 22x^2 + 8x$

89. $f(z) = z^4 - z^3 - 2z - 4$

90. $f(x) = 4x^3 + 7x^2 - 11x - 18$

91. $g(y) = 2y^4 + 7y^3 - 26y^2 + 23y - 6$

92. $h(x) = x^5 - x^4 - 3x^3 + 5x^2 - 2x$

93. $f(x) = 4x^4 - 55x^2 - 45x + 36$

94. $z(x) = 4x^4 - 43x^2 - 9x + 90$

95. $g(x) = 8x^4 + 28x^3 + 9x^2 - 9x$

96. $h(x) = x^5 + 5x^4 - 5x^3 - 15x^2 - 6x$

97. $f(x) = 4x^5 + 12x^4 - 11x^3 - 42x^2 + 7x + 30$

98. $g(x) = 4x^5 + 8x^4 - 15x^3 - 23x^2 + 11x + 15$

Using a Rational Zero **In Exercises 99–102, (a) use the *zero* or *root* feature of a graphing utility to approximate (accurate to the nearest thousandth) the zeros of the function, (b) determine one of the exact zeros and use synthetic division to verify your result, and (c) factor the polynomial completely.**

99. $h(t) = t^3 - 2t^2 - 7t + 2$

100. $f(s) = s^3 - 12s^2 + 40s - 24$

101. $h(x) = x^5 - 7x^4 + 10x^3 + 14x^2 - 24x$

102. $g(x) = 6x^4 - 11x^3 - 51x^2 + 99x - 27$

103. MODELING DATA

The table shows the numbers S of cellular phone subscriptions per 100 people in the United States from 1991 through 2008. (Source: U.S. International Telecommunications Union)

Year	Subscriptions per 100 people, S
1991	3.0
1992	4.3
1993	6.2
1994	9.2
1995	12.7
1996	16.4
1997	20.3
1998	25.1
1999	30.8
2000	38.9
2001	45.1
2002	49.2
2003	55.2
2004	62.9
2005	71.5
2006	77.4
2007	85.2
2008	86.8

The data can be approximated by the model

$$S = -0.0135t^3 + 0.545t^2 - 0.71t + 3.6, \quad 1 \le t \le 18$$

where t represents the year, with $t = 1$ corresponding to 1991.

(a) Use a graphing utility to graph the data and the model in the same viewing window.

(b) How well does the model fit the data?

(c) Use the Remainder Theorem to evaluate the model for the year 2015. Is the value reasonable? Explain.

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104. ***Why you should learn it*** *(p. 113)* The numbers of employees E (in thousands) in education and health services in the United States from 1960 through 2008 are approximated by $E = -0.084t^3 + 10.32t^2 - 23.5t + 3167$, $0 \le t \le 48$, where t is the year, with $t = 0$ corresponding to 1960. (Source: U.S. Bureau of Labor Statistics)

(a) Use a graphing utility to graph the model over the domain.

(b) Estimate the number of employees in education and health services in 1960. Use the Remainder Theorem to estimate the number in 2000.

(c) Is this a good model for making predictions in future years? Explain.

105. **Geometry** A rectangular package sent by a delivery service can have a maximum combined length and girth (perimeter of a cross section) of 120 inches (see figure).

(a) Show that the volume of the package is given by the function $V(x) = 4x^2(30 - x)$.

(b) Use a graphing utility to graph the function and approximate the dimensions of the package that yield a maximum volume.

(c) Find values of x such that $V = 13{,}500$. Which of these values is a physical impossibility in the construction of the package? Explain.

106. **Environmental Science** The number of parts per million of nitric oxide emissions y from a car engine is approximated by $y = -5.05x^3 + 3857x - 38{,}411.25$, $13 \le x \le 18$, where x is the air-fuel ratio.

(a) Use a graphing utility to graph the model.

(b) There are two air-fuel ratios that produce 2400 parts per million of nitric oxide. One is $x = 15$. Use the graph to approximate the other.

(c) Find the second air-fuel ratio from part (b) algebraically. (*Hint:* Use the known value of $x = 15$ and synthetic division.)

Conclusions

True or False? **In Exercises 107 and 108, determine whether the statement is true or false. Justify your answer.**

107. If $(7x + 4)$ is a factor of some polynomial function f, then $\frac{4}{7}$ is a zero of f.

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108. The value $x = \frac{1}{7}$ is a zero of the polynomial function

$$f(x) = 3x^5 - 2x^4 + x^3 - 16x^2 + 3x - 8.$$

Think About It **In Exercises 109 and 110, the graph of a cubic polynomial function $y = f(x)$ with integer zeros is shown. Find the factored form of f.**

109.

110.

111. **Think About It** Let $y = f(x)$ be a fourth-degree polynomial with leading coefficient $a = -1$ and $f(\pm 1) = f(\pm 2) = 0$. Find the factored form of f.

112. **Think About It** Find the value of k such that $x - 3$ is a factor of $x^3 - kx^2 + 2kx - 12$.

113. **Writing** Complete each polynomial division. Write a brief description of the pattern that you obtain, and use your result to find a formula for the polynomial division $(x^n - 1)/(x - 1)$. Create a numerical example to test your formula.

(a) $\dfrac{x^2 - 1}{x - 1} =$ ____ (b) $\dfrac{x^3 - 1}{x - 1} =$ ____

(c) $\dfrac{x^4 - 1}{x - 1} =$ ____

114. **CAPSTONE** A graph of $f(x)$ is shown, where

$$f(x) = 2x^5 - 3x^4 + x^3 - 8x^2 + 5x + 3 \text{ and}$$
$$f(-x) = -2x^5 - 3x^4 - x^3 - 8x^2 - 5x + 3.$$

(a) How many negative real zeros does f have? Explain.

(b) How many positive real zeros are *possible* for f? Explain. What does this tell you about the eventual right-hand behavior of the graph?

(c) Is $x = -\frac{1}{3}$ a possible rational zero of f? Explain.

(d) Explain how to check whether $\left(x - \frac{3}{2}\right)$ is a factor of f and whether $x = \frac{3}{2}$ is an upper bound for the real zeros of f.

Cumulative Mixed Review

Solving a Quadratic Equation **In Exercises 115–118, use any convenient method to solve the quadratic equation.**

115. $9x^2 - 25 = 0$

116. $16x^2 - 21 = 0$

117. $2x^2 + 6x + 3 = 0$

118. $8x^2 - 22x + 15 = 0$

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2.4 Complex Numbers

The Imaginary Unit *i*

Some quadratic equations have no real solutions. For instance, the quadratic equation $x^2 + 1 = 0$ has no real solution because there is no real number x that can be squared to produce -1. To overcome this deficiency, mathematicians created an expanded system of numbers using the **imaginary unit *i*,** defined as

$$i = \sqrt{-1} \qquad \text{Imaginary unit}$$

where $i^2 = -1$. By adding real numbers to real multiples of this imaginary unit, you obtain the set of **complex numbers.** Each complex number can be written in the **standard form** $a + bi$. For instance, the standard form of the complex number $\sqrt{-9} - 5$ is $-5 + 3i$ because

$$\begin{aligned}\sqrt{-9} - 5 &= \sqrt{3^2(-1)} - 5\\ &= 3\sqrt{-1} - 5\\ &= 3i - 5\\ &= -5 + 3i.\end{aligned}$$

In the standard form $a + bi$, the real number a is called the **real part** of the **complex number** $a + bi$, and the number bi (where b is a real number) is called the **imaginary part** of the complex number.

What you should learn

- Use the imaginary unit i to write complex numbers.
- Add, subtract, and multiply complex numbers.
- Use complex conjugates to write the quotient of two complex numbers in standard form.
- Find complex solutions of quadratic equations.

Why you should learn it

Complex numbers are used to model numerous aspects of the natural world, such as the impedance of an electrical circuit, as shown in Exercise 94 on page 134.

Definition of a Complex Number

If a and b are real numbers, then the number $a + bi$ is a **complex number,** and it is said to be written in **standard form.** If $b = 0$, then the number $a + bi = a$ is a real number. If $b \neq 0$, then the number $a + bi$ is called an **imaginary number.** A number of the form bi, where $b \neq 0$, is called a **pure imaginary number.**

The set of real numbers is a subset of the set of complex numbers, as shown in Figure 2.32. This is true because every real number a can be written as a complex number using $b = 0$. That is, for every real number a, you can write $a = a + 0i$.

Figure 2.32

Equality of Complex Numbers

Two complex numbers $a + bi$ and $c + di$, written in standard form, are equal to each other

$$a + bi = c + di \qquad \text{Equality of two complex numbers}$$

if and only if $a = c$ and $b = d$.

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Operations with Complex Numbers

To add (or subtract) two complex numbers, you add (or subtract) the real and imaginary parts of the numbers separately.

Addition and Subtraction of Complex Numbers

If $a + bi$ and $c + di$ are two complex numbers written in standard form, then their sum and difference are defined as follows.

Sum: $(a + bi) + (c + di) = (a + c) + (b + d)i$

Difference: $(a + bi) - (c + di) = (a - c) + (b - d)i$

The **additive identity** in the complex number system is zero (the same as in the real number system). Furthermore, the **additive inverse** of the complex number $a + bi$ is

$$-(a + bi) = -a - bi. \qquad \text{Additive inverse}$$

So, you have

$$(a + bi) + (-a - bi) = 0 + 0i = 0.$$

Example 1 Adding and Subtracting Complex Numbers

Perform the operation(s) and write the result in standard form.

a. $(3 - i) + (2 + 3i)$ **b.** $(1 + 2i) - (4 + 2i)$

c. $3 - (-2 + 3i) + (-5 + i)$ **d.** $(3 + 2i) + (4 - i) - (7 + i)$

Solution

a. $(3 - i) + (2 + 3i) = 3 - i + 2 + 3i$ Remove parentheses.

$= (3 + 2) + (-i + 3i)$ Group like terms.

$= 5 + 2i$ Write in standard form.

b. $(1 + 2i) - (4 + 2i) = 1 + 2i - 4 - 2i$ Remove parentheses.

$= (1 - 4) + (2i - 2i)$ Group like terms.

$= -3 + 0i$ Simplify.

$= -3$ Write in standard form.

c. $3 - (-2 + 3i) + (-5 + i) = 3 + 2 - 3i - 5 + i$

$= (3 + 2 - 5) + (-3i + i)$

$= 0 - 2i$

$= -2i$

d. $(3 + 2i) + (4 - i) - (7 + i) = 3 + 2i + 4 - i - 7 - i$

$= (3 + 4 - 7) + (2i - i - i)$

$= 0 + 0i$

$= 0$

CHECKPOINT Now try Exercise 25.

In Examples 1(b) and 1(d), note that the sum of complex numbers can be a real number.

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Many of the properties of real numbers are valid for complex numbers as well. Here are some examples.

Associative Properties of Addition and Multiplication
Commutative Properties of Addition and Multiplication
Distributive Property of Multiplication over Addition

Notice how these properties are used when two complex numbers are multiplied.

$$
\begin{aligned}
(a + bi)(c + di) &= a(c + di) + bi(c + di) && \text{Distributive Property}\\
&= ac + (ad)i + (bc)i + (bd)i^2 && \text{Distributive Property}\\
&= ac + (ad)i + (bc)i + (bd)(-1) && i^2 = -1\\
&= ac - bd + (ad)i + (bc)i && \text{Commutative Property}\\
&= (ac - bd) + (ad + bc)i && \text{Associative Property}
\end{aligned}
$$

The procedure above is similar to multiplying two polynomials and combining like terms, as in the FOIL Method.

Explore the Concept

Complete the following:

$i^1 = i$	$i^7 =$
$i^2 = -1$	$i^8 =$
$i^3 = -i$	$i^9 =$
$i^4 = 1$	$i^{10} =$
$i^5 =$	$i^{11} =$
$i^6 =$	$i^{12} =$

What pattern do you see? Write a brief description of how you would find i raised to any positive integer power.

Example 2 Multiplying Complex Numbers

Perform the operation(s) and write the result in standard form.

a. $5(-2 + 3i)$
b. $(2 - i)(4 + 3i)$
c. $(3 + 2i)(3 - 2i)$
d. $4i(-1 + 5i)$
e. $(3 + 2i)^2$

Solution

a.
$$
\begin{aligned}
5(-2 + 3i) &= 5(-2) + 5(3i) && \text{Distributive Property}\\
&= -10 + 15i && \text{Simplify.}
\end{aligned}
$$

b.
$$
\begin{aligned}
(2 - i)(4 + 3i) &= 2(4 + 3i) - i(4 + 3i) && \text{Distributive Property}\\
&= 8 + 6i - 4i - 3i^2 && \text{Product of binomials}\\
&= 8 + 6i - 4i - 3(-1) && i^2 = -1\\
&= 8 + 3 + 6i - 4i && \text{Group like terms.}\\
&= 11 + 2i && \text{Write in standard form.}
\end{aligned}
$$

c.
$$
\begin{aligned}
(3 + 2i)(3 - 2i) &= 3(3 - 2i) + 2i(3 - 2i) && \text{Distributive Property}\\
&= 9 - 6i + 6i - 4i^2 && \text{Product of binomials}\\
&= 9 - 4(-1) && i^2 = -1\\
&= 9 + 4 && \text{Simplify.}\\
&= 13 && \text{Write in standard form.}
\end{aligned}
$$

d.
$$
\begin{aligned}
4i(-1 + 5i) &= 4i(-1) + 4i(5i) && \text{Distributive Property}\\
&= -4i + 20i^2 && \text{Simplify.}\\
&= -4i + 20(-1) && i^2 = -1\\
&= -20 - 4i && \text{Write in standard form.}
\end{aligned}
$$

e.
$$
\begin{aligned}
(3 + 2i)^2 &= 9 + 6i + 6i + 4i^2 && \text{Product of binomials}\\
&= 9 + 12i + 4(-1) && i^2 = -1\\
&= 9 - 4 + 12i && \text{Group like terms.}\\
&= 5 + 12i && \text{Write in standard form.}
\end{aligned}
$$

CHECKPOINT Now try Exercise 33.

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Complex Conjugates

Notice in Example 2(c) that the product of two complex numbers can be a real number. This occurs with pairs of complex numbers of the forms $a + bi$ and $a - bi$, called **complex conjugates.**

$$\begin{aligned}(a + bi)(a - bi) &= a^2 - abi + abi - b^2i^2\\ &= a^2 - b^2(-1)\\ &= a^2 + b^2\end{aligned}$$

Example 3 Multiplying Conjugates

Multiply each complex number by its complex conjugate.

a. $1 + i$ **b.** $3 - 5i$

Solution

a. The complex conjugate of $1 + i$ is $1 - i$.

$$(1 + i)(1 - i) = 1^2 - i^2 = 1 - (-1) = 2$$

b. The complex conjugate of $3 - 5i$ is $3 + 5i$.

$$(3 - 5i)(3 + 5i) = 3^2 - (5i)^2 = 9 - 25i^2 = 9 - 25(-1) = 34$$

CHECKPOINT Now try Exercise 43.

To write the quotient of $a + bi$ and $c + di$ in standard form, where c and d are not both zero, multiply the numerator and denominator by the complex conjugate of the *denominator* to obtain

$$\frac{a + bi}{c + di} = \frac{a + bi}{c + di}\left(\frac{c - di}{c - di}\right) \quad \text{Multiply numerator and denominator by complex conjugate of denominator.}$$

$$= \frac{ac + bd}{c^2 + d^2} + \left(\frac{bc - ad}{c^2 + d^2}\right)i. \quad \text{Standard form}$$

Technology Tip

Some graphing utilities can perform operations with complex numbers. For instance, on some graphing utilities, to divide $2 + 3i$ by $4 - 2i$, use the following keystrokes.

(2 + 3 i) ÷

(4 − 2 i) ENTER

Example 4 Writing a Quotient of Complex Numbers in Standard Form

Write the quotient $\dfrac{2 + 3i}{4 - 2i}$ in standard form.

Solution

$$\frac{2 + 3i}{4 - 2i} = \frac{2 + 3i}{4 - 2i}\left(\frac{4 + 2i}{4 + 2i}\right) \quad \text{Multiply numerator and denominator by complex conjugate of denominator.}$$

$$= \frac{8 + 4i + 12i + 6i^2}{16 - 4i^2} \quad \text{Expand.}$$

$$= \frac{8 - 6 + 16i}{16 + 4} \quad i^2 = -1$$

$$= \frac{2 + 16i}{20} \quad \text{Simplify.}$$

$$= \frac{1}{10} + \frac{4}{5}i \quad \text{Write in standard form.}$$

CHECKPOINT Now try Exercise 55.

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Complex Solutions of Quadratic Equations

When using the Quadratic Formula to solve a quadratic equation, you often obtain a result such as $\sqrt{-3}$, which you know is not a real number. By factoring out $i = \sqrt{-1}$, you can write this number in standard form.

$$\sqrt{-3} = \sqrt{3(-1)} = \sqrt{3}\sqrt{-1} = \sqrt{3}i$$

The number $\sqrt{3}i$ is called the *principal square root* of -3.

Principal Square Root of a Negative Number

If a is a positive number, then the **principal square root** of the negative number $-a$ is defined as

$$\sqrt{-a} = \sqrt{a}i.$$

Example 5 Writing Complex Numbers in Standard Form

a. $\sqrt{-3}\sqrt{-12} = \sqrt{3}i\sqrt{12}i$
$= \sqrt{36}i^2$
$= 6(-1)$
$= -6$

b. $\sqrt{-48} - \sqrt{-27} = \sqrt{48}i - \sqrt{27}i$
$= 4\sqrt{3}i - 3\sqrt{3}i$
$= \sqrt{3}i$

c. $(-1 + \sqrt{-3})^2 = (-1 + \sqrt{3}i)^2$
$= (-1)^2 - 2\sqrt{3}i + (\sqrt{3})^2(i^2)$
$= 1 - 2\sqrt{3}i + 3(-1)$
$= -2 - 2\sqrt{3}i$

CHECKPOINT Now try Exercise 63.

Example 6 Complex Solutions of a Quadratic Equation

Solve (a) $x^2 + 4 = 0$ and (b) $3x^2 - 2x + 5 = 0$.

Solution

a. $x^2 + 4 = 0$ — Write original equation.

$x^2 = -4$ — Subtract 4 from each side.

$x = \pm 2i$ — Extract square roots.

b. $3x^2 - 2x + 5 = 0$ — Write original equation.

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(5)}}{2(3)}$$ — Quadratic Formula

$$= \frac{2 \pm \sqrt{-56}}{6}$$ — Simplify.

$$= \frac{2 \pm 2\sqrt{14}i}{6}$$ — Write $\sqrt{-56}$ in standard form.

$$= \frac{1}{3} \pm \frac{\sqrt{14}}{3}i$$ — Write in standard form.

CHECKPOINT Now try Exercise 73.

Study Tip

The definition of principal square root uses the rule

$$\sqrt{ab} = \sqrt{a}\sqrt{b}$$

for $a > 0$ and $b < 0$. This rule is not valid when *both* a and b are negative. For example,

$$\sqrt{-5}\sqrt{-5} = \sqrt{5(-1)}\sqrt{5(-1)}$$
$$= \sqrt{5}i\sqrt{5}i$$
$$= \sqrt{25}i^2$$
$$= 5i^2 = -5$$

whereas

$$\sqrt{(-5)(-5)} = \sqrt{25} = 5.$$

To avoid problems with square roots of negative numbers, be sure to convert complex numbers to standard form *before* multiplying.

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2.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

1. Match the type of complex number with its definition.

(a) real number (i) $a + bi, a = 0, b \neq 0$

(b) imaginary number (ii) $a + bi, b = 0$

(c) pure imaginary number (iii) $a + bi, a \neq 0, b \neq 0$

In Exercises 2 and 3, fill in the blanks.

2. The imaginary unit i is defined as $i =$ _____ , where $i^2 =$ _____ .

3. When you add $(7 + 6i)$ and $(8 + 5i)$, the real part of the sum is _____ and the imaginary part of the sum is _____.

4. What method for multiplying two polynomials can you use when multiplying two complex numbers?

5. What is the additive inverse of the complex number $2 - 4i$?

6. What is the complex conjugate of the complex number $2 - 4i$?

Procedures and Problem Solving

Equality of Complex Numbers **In Exercises 7–10, find real numbers *a* and *b* such that the equation is true.**

7. $a + bi = -9 + 4i$ **8.** $a + bi = 12 + 5i$

9. $3a + (b + 3)i = 9 + 8i$ **10.** $(a + 6) + 2bi = 6 - i$

Writing a Complex Number in Standard Form **In Exercises 11–20, write the complex number in standard form.**

11. $5 + \sqrt{-16}$ **12.** $2 - \sqrt{-9}$

13. -6 **14.** 8

15. $-5i + i^2$ **16.** $-3i^2 + i$

17. $\left(\sqrt{-75}\right)^2$ **18.** $\left(\sqrt{-4}\right)^2 - 7$

19. $\sqrt{-0.09}$ **20.** $\sqrt{-0.0004}$

Adding and Subtracting Complex Numbers **In Exercises 21–30, perform the addition or subtraction and write the result in standard form.**

21. $(4 + i) - (7 - 2i)$ **22.** $(11 - 2i) - (-3 + 6i)$

23. $\left(-1 + 8i\right) + \left(8 - 5i\right)$ **24.** $\left(7 + 6i\right) + \left(3 + 12i\right)$

✓ **25.** $13i - (14 - 7i)$ **26.** $22 + (-5 + 8i) - 9i$

27. $\left(\frac{3}{2} + \frac{5}{2}i\right) + \left(\frac{5}{3} + \frac{11}{3}i\right)$ **28.** $\left(\frac{3}{4} + \frac{7}{5}i\right) - \left(\frac{5}{6} - \frac{1}{6}i\right)$

29. $(1.6 + 3.2i) + (-5.8 + 4.3i)$

30. $-(-3.7 - 12.8i) - \left(6.1 - 16.3i\right)$

Multiplying Complex Numbers **In Exercises 31–42, perform the operation and write the result in standard form.**

31. $4(3 + 5i)$ **32.** $-6(5 - 3i)$

✓ **33.** $(1 + i)(3 - 2i)$ **34.** $(6 - 2i)(2 - 3i)$

35. $4i(8 + 5i)$ **36.** $-3i(6 - i)$

37. $\left(\sqrt{14} + \sqrt{10}i\right)\left(\sqrt{14} - \sqrt{10}i\right)$

38. $\left(\sqrt{3} + \sqrt{15}i\right)\left(\sqrt{3} - \sqrt{15}i\right)$

39. $(6 + 7i)^2$ **40.** $(5 - 4i)^2$

41. $(4 + 5i)^2 - (4 - 5i)^2$ **42.** $(1 - 2i)^2 - (1 + 2i)^2$

Multiplying Conjugates **In Exercises 43–50, write the complex conjugate of the complex number. Then multiply the number by its complex conjugate.**

✓ **43.** $4 + 3i$ **44.** $7 - 5i$

45. $-6 - \sqrt{5}i$ **46.** $-3 + \sqrt{2}i$

47. $\sqrt{-20}$ **48.** $\sqrt{-13}$

49. $3 - \sqrt{-2}$ **50.** $1 + \sqrt{-8}$

Writing a Quotient of Complex Numbers in Standard Form **In Exercises 51–58, write the quotient in standard form.**

51. $\dfrac{6}{i}$ **52.** $-\dfrac{5}{2i}$

53. $\dfrac{2}{4 - 5i}$ **54.** $\dfrac{3}{1 - i}$

✓ **55.** $\dfrac{2 + i}{2 - i}$ **56.** $\dfrac{8 - 7i}{1 - 2i}$

57. $i/(4 - 5i)^2$ **58.** $5i/(2 + 3i)^2$

Adding or Subtracting Quotients of Complex Numbers **In Exercises 59–62, perform the operation and write the result in standard form.**

59. $\dfrac{2}{1 + i} - \dfrac{3}{1 - i}$ **60.** $\dfrac{2i}{2 + i} + \dfrac{5}{2 - i}$

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61. $\dfrac{i}{3-2i} + \dfrac{2i}{3+8i}$ **62.** $\dfrac{1+i}{i} - \dfrac{3}{4-i}$

Writing Complex Numbers in Standard Form **In Exercises 63–72, perform the operation and write the result in standard form.**

63. $\sqrt{-18} - \sqrt{-54}$ **64.** $\sqrt{-50} + \sqrt{-275}$

65. $(-3 + \sqrt{-24}) + (7 - \sqrt{-44})$

66. $(-12 - \sqrt{-72}) + (9 + \sqrt{-108})$

67. $\sqrt{-6} \cdot \sqrt{-2}$ **68.** $\sqrt{-5} \cdot \sqrt{-10}$

69. $(\sqrt{-10})^2$ **70.** $(\sqrt{-75})^2$

71. $(2 - \sqrt{-6})^2$ **72.** $(3 + \sqrt{-5})(7 - \sqrt{-10})$

Complex Solutions of a Quadratic Equation **In Exercises 73–84, solve the quadratic equation.**

73. $x^2 + 25 = 0$ **74.** $x^2 + 32 = 0$

75. $x^2 - 2x + 2 = 0$ **76.** $x^2 + 6x + 10 = 0$

77. $4x^2 + 16x + 17 = 0$ **78.** $9x^2 - 6x + 37 = 0$

79. $16t^2 - 4t + 3 = 0$ **80.** $4x^2 + 16x + 15 = 0$

81. $\frac{3}{2}x^2 - 6x + 9 = 0$ **82.** $\frac{7}{8}x^2 - \frac{3}{4}x + \frac{5}{16} = 0$

83. $1.4x^2 - 2x - 10 = 0$ **84.** $4.5x^2 - 3x + 12 = 0$

Expressions Involving Powers of *i* **In Exercises 85–90, simplify the complex number and write it in standard form.**

85. $-6i^3 + i^2$ **86.** $4i^2 - 2i^3$

87. $(\sqrt{-75})^3$ **88.** $(\sqrt{-2})^6$

89. $\dfrac{1}{i^3}$ **90.** $\dfrac{1}{(2i)^3}$

91. Cube each complex number. What do you notice?

(a) 2 (b) $-1 + \sqrt{3}i$ (c) $-1 - \sqrt{3}i$

92. Raise each complex number to the fourth power and simplify.

(a) 2 (b) -2 (c) $2i$ (d) $-2i$

93. Use the results of the Explore the Concept on page 130 to find each power of i.

(a) i^{20} (b) i^{45} (c) i^{67} (d) i^{114}

94. ***Why you should learn it*** *(p. 128)* The opposition to current in an electrical circuit is called its impedance. The impedance z in a parallel circuit with two pathways satisfies the equation $1/z = 1/z_1 + 1/z_2$, where z_1 is the impedance (in ohms) of pathway 1, and z_2 is the impedance (in ohms) of pathway 2. Use the table to determine the impedance of each parallel circuit. (*Hint:* You can find the impedance of each pathway in a parallel circuit by adding the impedances of all components in the pathway.)

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	Resistor	Inductor	Capacitor
Symbol	$a\ \Omega$	$b\ \Omega$	$c\ \Omega$
Impedance	a	bi	$-ci$

(a)

(b)

Conclusions

True or False? **In Exercises 95–100, determine whether the statement is true or false. Justify your answer.**

95. No complex number is equal to its complex conjugate.

96. $i^{44} + i^{150} - i^{74} - i^{109} + i^{61} = -1$

97. The sum of two imaginary numbers is always an imaginary number.

98. The product of two imaginary numbers is always an imaginary number.

99. The conjugate of the product of two complex numbers is equal to the product of the conjugates of the two complex numbers.

100. The conjugate of the sum of two complex numbers is equal to the sum of the conjugates of the two complex numbers.

101. **Error Analysis** Describe the error.

$$\cancel{\sqrt{-6}\sqrt{-6} = \sqrt{(-6)(-6)} = \sqrt{36} = 6}$$

102. **CAPSTONE** Consider the functions

$f(x) = 2(x-3)^2 - 4$ and $g(x) = -2(x-3)^2 - 4$.

(a) Without graphing either function, determine whether the graph of f and the graph of g have x-intercepts. Explain your reasoning.

(b) Solve $f(x) = 0$ and $g(x) = 0$.

(c) Explain how the zeros of f and g are related to whether their graphs have x-intercepts.

(d) For the function $f(x) = a(x-h)^2 + k$, make a general statement about how a, h, and k affect whether the graph of f has x-intercepts, and whether the zeros of f are real or complex.

Cumulative Mixed Review

Multiplying Polynomials **In Exercises 103–106, perform the operation and write the result in standard form.**

103. $(4x - 5)(4x + 5)$ **104.** $(x + 2)^3$

105. $(3x - \frac{1}{2})(x + 4)$ **106.** $(2x - 5)^2$

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2.5 The Fundamental Theorem of Algebra

The Fundamental Theorem of Algebra

You know that an nth-degree polynomial can have at most n real zeros. In the complex number system, this statement can be improved. That is, in the complex number system, every nth-degree polynomial function has *precisely n zeros*. This important result is derived from the **Fundamental Theorem of Algebra,** first proved by the German mathematician Carl Friedrich Gauss (1777–1855).

The Fundamental Theorem of Algebra

If $f(x)$ is a polynomial of degree n, where $n > 0$, then f has at least one zero in the complex number system.

Using the Fundamental Theorem of Algebra and the equivalence of zeros and factors, you obtain the **Linear Factorization Theorem.**

Linear Factorization Theorem (See the proof on page 177.)

If $f(x)$ is a polynomial of degree n, where $n > 0$, then f has precisely n linear factors

$$f(x) = a_n(x - c_1)(x - c_2) \cdot \cdot \cdot (x - c_n)$$

where $c_1, c_2, \ldots, c_n$ are complex numbers.

Note that neither the Fundamental Theorem of Algebra nor the Linear Factorization Theorem tells you *how* to find the zeros or factors of a polynomial. Such theorems are called *existence theorems*. To find the zeros of a polynomial function, you still must rely on other techniques.

What you should learn

- Use the Fundamental Theorem of Algebra to determine the number of zeros of a polynomial function.
- Find all zeros of polynomial functions, including complex zeros.
- Find conjugate pairs of complex zeros.
- Find zeros of polynomials by factoring.

Why you should learn it

Being able to find zeros of polynomial functions is an important part of modeling real-life problems. For instance, Exercise 71 on page 141 shows how to determine whether a football kicked with a given velocity can reach a certain height.

Example 1 Zeros of Polynomial Functions

a. The first-degree polynomial $f(x) = x - 2$ has exactly *one* zero: $x = 2$.

b. Counting multiplicity, the second-degree polynomial function

$$\begin{aligned} f(x) &= x^2 - 6x + 9 \\ &= (x - 3)(x - 3) \end{aligned}$$

has exactly *two* zeros: $x = 3$ and $x = 3$. (This is called a *repeated zero*.)

c. The third-degree polynomial function

$$f(x) = x^3 + 4x = x(x^2 + 4) = x(x - 2i)(x + 2i)$$

has exactly *three* zeros: $x = 0$, $x = 2i$, and $x = -2i$.

d. The fourth-degree polynomial function

$$\begin{aligned} f(x) &= x^4 - 1 \\ &= (x - 1)(x + 1)(x - i)(x + i) \end{aligned}$$

has exactly *four* zeros: $x = 1$, $x = -1$, $x = i$, and $x = -i$.

CHECKPOINT Now try Exercise 5.

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Finding Zeros of a Polynomial Function

Remember that the n zeros of a polynomial function can be real or complex, and they may be repeated. Examples 2 and 3 illustrate several cases.

Example 2 Real and Complex Zeros of a Polynomial Function

Confirm that the third-degree polynomial function

$$f(x) = x^3 + 4x$$

has exactly three zeros: $x = 0$, $x = 2i$, and $x = -2i$.

Solution

Factor the polynomial completely as $x(x - 2i)(x + 2i)$. So, the zeros are

$$\begin{aligned} x(x - 2i)(x + 2i) &= 0 \\ x &= 0 \\ x - 2i &= 0 \quad \Rightarrow \quad x = 2i \\ x + 2i &= 0 \quad \Rightarrow \quad x = -2i. \end{aligned}$$

In the graph in Figure 2.33, only the real zero $x = 0$ appears as an x-intercept.

Figure 2.33

CHECKPOINT Now try Exercise 11.

Example 3 shows how to use the methods described in Sections 2.2 and 2.3 (the Rational Zero Test, synthetic division, and factoring) to find all the zeros of a polynomial function, including complex zeros.

Example 3 Finding the Zeros of a Polynomial Function

Write $f(x) = x^5 + x^3 + 2x^2 - 12x + 8$ as the product of linear factors, and list all the zeros of f.

Solution

The possible rational zeros are $\pm 1, \pm 2, \pm 4$, and ± 8. The graph shown in Figure 2.34 indicates that 1 and -2 are likely zeros, and that 1 is possibly a repeated zero because it appears that the graph touches (but does not cross) the x-axis at this point. Using synthetic division, you can determine that -2 is a zero and 1 is a repeated zero of f. So, you have

$$\begin{aligned} f(x) &= x^5 + x^3 + 2x^2 - 12x + 8 \\ &= (x - 1)(x - 1)(x + 2)(x^2 + 4). \end{aligned}$$

Figure 2.34

By factoring $x^2 + 4$ as

$$\begin{aligned} x^2 - (-4) &= \left(x - \sqrt{-4}\right)\left(x + \sqrt{-4}\right) \\ &= (x - 2i)(x + 2i) \end{aligned}$$

you obtain

$$f(x) = (x - 1)(x - 1)(x + 2)(x - 2i)(x + 2i)$$

which gives the following five zeros of f.

$$x = 1,\ x = 1,\ x = -2,\ x = 2i, \text{ and } x = -2i$$

Note from the graph of f shown in Figure 2.34 that the *real* zeros are the only ones that appear as x-intercepts.

CHECKPOINT Now try Exercise 35.

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Conjugate Pairs

In Example 3, note that the two complex zeros are **conjugates.** That is, they are of the forms $a + bi$ and $a - bi$.

Complex Zeros Occur in Conjugate Pairs

Let $f(x)$ be a polynomial function that has *real coefficients*. If $a + bi$, where $b \neq 0$, is a zero of the function, then the conjugate $a - bi$ is also a zero of the function.

Be sure you see that this result is true only when the polynomial function has *real coefficients*. For instance, the result applies to the function $f(x) = x^2 + 1$, but not to the function $g(x) = x - i$.

Example 4 Finding a Polynomial with Given Zeros

Find a *fourth-degree* polynomial function with real coefficients that has -1, -1, and $3i$ as zeros.

Solution

Because $3i$ is a zero *and* the polynomial is stated to have real coefficients, you know that the conjugate $-3i$ must also be a zero. So, from the Linear Factorization Theorem, $f(x)$ can be written as

$$f(x) = a(x + 1)(x + 1)(x - 3i)(x + 3i).$$

For simplicity, let $a = 1$ to obtain

$$\begin{aligned} f(x) &= (x^2 + 2x + 1)(x^2 + 9) \\ &= x^4 + 2x^3 + 10x^2 + 18x + 9. \end{aligned}$$

CHECKPOINT Now try Exercise 47.

Example 5 Finding a Polynomial with Given Zeros

Find a *cubic* polynomial function f with real coefficients that has 2 and $1 - i$ as zeros, and $f(1) = 3$.

Solution

Because $1 - i$ is a zero of f, the conjugate $1 + i$ must also be a zero.

$$\begin{aligned} f(x) &= a(x - 2)[x - (1 - i)][x - (1 + i)] \\ &= a(x - 2)[x^2 - x(1 + i) - x(1 - i) + 1 - i^2] \\ &= a(x - 2)(x^2 - 2x + 2) \\ &= a(x^3 - 4x^2 + 6x - 4) \end{aligned}$$

To find the value of a, use the fact that $f(1) = 3$ to obtain

$$a[(1)^3 - 4(1)^2 + 6(1) - 4)] = 3.$$

Thus, $a = -3$ and you can conclude that

$$\begin{aligned} f(x) &= -3(x^3 - 4x^2 + 6x - 4) \\ &= -3x^3 + 12x^2 - 18x + 12. \end{aligned}$$

CHECKPOINT Now try Exercise 51.

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Factoring a Polynomial

The Linear Factorization Theorem states that you can write any nth-degree polynomial as the product of n linear factors.

$$f(x) = a_n(x - c_1)(x - c_2)(x - c_3) \cdot \cdot \cdot (x - c_n)$$

This result, however, includes the possibility that some of the values of c_i are complex. The following theorem states that even when you do not want to get involved with "complex factors," you can still write $f(x)$ as the product of linear and/or quadratic factors.

Factors of a Polynomial (See the proof on page 177.)

Every polynomial of degree $n > 0$ with real coefficients can be written as the product of linear and quadratic factors with real coefficients, where the quadratic factors have no real zeros.

A quadratic factor with no real zeros is said to be **prime** or **irreducible over the reals.** Be sure you see that this is not the same as being *irreducible over the rationals*. For example, the quadratic

$$x^2 + 1 = (x - i)(x + i)$$

is irreducible over the reals (and therefore over the rationals). On the other hand, the quadratic

$$x^2 - 2 = \left(x - \sqrt{2}\right)\left(x + \sqrt{2}\right)$$

is irreducible over the rationals, but *reducible* over the reals.

Study Tip

Recall that irrational and rational numbers are subsets of the set of real numbers, and the real numbers are a subset of the set of complex numbers.

Example 6 Factoring a Polynomial

Write the polynomial $f(x) = x^4 - x^2 - 20$

a. as the product of factors that are irreducible over the *rationals*,

b. as the product of linear factors and quadratic factors that are irreducible over the *reals*, and

c. in completely factored form.

Solution

a. Begin by factoring the polynomial into the product of two quadratic polynomials.

$$x^4 - x^2 - 20 = (x^2 - 5)(x^2 + 4)$$

Both of these factors are irreducible over the rationals.

b. By factoring over the reals, you have

$$x^4 - x^2 - 20 = \left(x + \sqrt{5}\right)\left(x - \sqrt{5}\right)(x^2 + 4)$$

where the quadratic factor is irreducible over the reals.

c. In completely factored form, you have

$$x^4 - x^2 - 20 = \left(x + \sqrt{5}\right)\left(x - \sqrt{5}\right)(x - 2i)(x + 2i).$$

CHECKPOINT Now try Exercise 55.

In Example 6, notice from the completely factored form that the *fourth*-degree polynomial has *four* zeros.

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Throughout this chapter, the results and theorems have been stated in terms of zeros of polynomial functions. Be sure you see that the same results could have been stated in terms of solutions of polynomial equations. This is true because the zeros of the polynomial function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2x^2 + a_1x + a_0$$

are precisely the solutions of the polynomial equation

$$a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2x^2 + a_1x + a_0 = 0.$$

Example 7 Finding the Zeros of a Polynomial Function

Find all the zeros of

$$f(x) = x^4 - 3x^3 + 6x^2 + 2x - 60$$

given that $1 + 3i$ is a zero of f.

Algebraic Solution

Because complex zeros occur in conjugate pairs, you know that $1 - 3i$ is also a zero of f. This means that both

$$x - (1 + 3i) \qquad \text{and} \qquad x - (1 - 3i)$$

are factors of f. Multiplying these two factors produces

$$\begin{aligned}[x - (1 + 3i)][x - (1 - 3i)] &= [(x - 1) - 3i][(x - 1) + 3i] \\ &= (x - 1)^2 - 9i^2 \\ &= x^2 - 2x + 10.\end{aligned}$$

Using long division, you can divide $x^2 - 2x + 10$ into f to obtain the following.

$$\begin{array}{r}
x^2 - x - 6 \\
500x^2 - 2x + 10 \overline{\big) x^4 - 3x^3 + 6x^2 + 2x - 60} \\
\underline{x^4 - 2x^3 + 10x^2 \qquad\qquad} \\
-x^3 - 4x^2 + 2x \qquad \\
\underline{-x^3 + 2x^2 - 10x \qquad} \\
-6x^2 + 12x - 60 \\
\underline{-6x^2 + 12x - 60} \\
0
\end{array}$$

So, you have

$$\begin{aligned}f(x) &= (x^2 - 2x + 10)(x^2 - x - 6) \\ &= (x^2 - 2x + 10)(x - 3)(x + 2)\end{aligned}$$

and you can conclude that the zeros of f are

$$x = 1 + 3i,\ x = 1 - 3i,\ x = 3, \text{ and } x = -2.$$

CHECKPOINT Now try Exercise 61.

Graphical Solution

You can use a graphing utility to determine that $x = -2$ and $x = 3$ are x-intercepts of the graph of f (see Figure 2.35).

Figure 2.35

Because $1 + 3i$ is a zero of f, you know that the conjugate $1 - 3i$ must also be a zero. So, you can conclude that the zeros of f are

$$x = 1 + 3i,\ x = 1 - 3i,\ x = 3, \text{ and } x = -2.$$

In Example 7, if you were not told that $1 + 3i$ is a zero of f, you could still find all the zeros of the function by using synthetic division to find the real zeros -2 and 3. Then, you could factor the polynomial as $(x + 2)(x - 3)(x^2 - 2x + 10)$. Finally, by using the Quadratic Formula, you could determine that the zeros are $x = 1 + 3i$, $x = 1 - 3i$, $x = 3$, and $x = -2$.

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2.5 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1 and 2, fill in the blanks.

1. The _____ of _____ states that if $f(x)$ is a polynomial function of degree n $(n > 0)$, then f has at least one zero in the complex number system.
2. A quadratic factor that cannot be factored as a product of linear factors containing real numbers is said to be _____ over the _____ .
3. How many linear factors does a polynomial function f of degree n have, where $n > 0$?
4. Three of the zeros of a fourth-degree polynomial function f are -1, 3, and $2i$. What is the other zero of f?

Procedures and Problem Solving

Zeros of a Polynomial Function **In Exercises 5–8, match the function with its exact number of zeros.**

5. $f(x) = x^3 + x$ (a) 1 zero
6. $f(x) = -x + 7$ (b) 2 zeros
7. $f(x) = x^5 + 9x^3$ (c) 3 zeros
8. $f(x) = x^2 - 14x + 49$ (d) 5 zeros

Real and Complex Zeros of a Polynomial Function **In Exercises 9–12, confirm that the function has the indicated zeros.**

9. $f(x) = x^2 + 25$; $-5i$, $5i$
10. $f(x) = x^2 + 2$; $-\sqrt{2}i$, $\sqrt{2}i$
11. $f(x) = x^3 + 9x$; 0, $-3i$, $3i$
12. $f(x) = x^3 + 49x$; 0, $-7i$, $7i$

Comparing the Zeros and the x-Intercepts **In Exercises 13–16, find all the zeros of the function. Is there a relationship between the number of real zeros and the number of x-intercepts of the graph? Explain.**

13. $f(x) = x^3 - 4x^2 + x - 4$

14. $f(x) = x^3 - 4x^2 - 4x + 16$

15. $f(x) = x^4 + 4x^2 + 4$

16. $f(x) = x^4 - 3x^2 - 4$

Finding the Zeros of a Polynomial Function **In Exercises 17–36, find all the zeros of the function and write the polynomial as a product of linear factors. Use a graphing utility to verify your results graphically. (If possible, use the graphing utility to verify the complex zeros.)**

17. $h(x) = x^2 - 4x + 1$
18. $g(x) = x^2 + 10x + 23$
19. $f(x) = x^2 - 12x + 26$
20. $f(x) = x^2 + 6x - 2$
21. $f(x) = x^2 + 25$
22. $f(x) = x^2 + 36$
23. $f(x) = 16x^4 - 81$
24. $f(y) = 81y^4 - 625$
25. $f(z) = z^2 - z + 56$
26. $h(x) = x^2 - 4x - 3$
27. $f(x) = x^4 + 10x^2 + 9$
28. $f(x) = x^4 + 29x^2 + 100$
29. $f(x) = 3x^3 - 5x^2 + 48x - 80$
30. $f(x) = 3x^3 - 2x^2 + 75x - 50$
31. $f(t) = t^3 - 3t^2 - 15t + 125$
32. $f(x) = x^3 + 11x^2 + 39x + 29$
33. $f(x) = 5x^3 - 9x^2 + 28x + 6$
34. $f(s) = 3s^3 - 4s^2 + 8s + 8$
35. $g(x) = x^4 - 4x^3 + 8x^2 - 16x + 16$
36. $h(x) = x^4 + 6x^3 + 10x^2 + 6x + 9$

Using the Zeros to Find x-Intercepts **In Exercises 37–44, (a) find all zeros of the function, (b) write the polynomial as a product of linear factors, and (c) use your factorization to determine the x-intercepts of the graph of the function. Use a graphing utility to verify that the real zeros are the only x-intercepts.**

37. $f(x) = x^2 - 14x + 46$
38. $f(x) = x^2 - 12x + 34$
39. $f(x) = 2x^3 - 3x^2 + 8x - 12$
40. $f(x) = 2x^3 - 5x^2 + 18x - 45$
41. $f(x) = x^3 - 11x + 150$
42. $f(x) = x^3 + 10x^2 + 33x + 34$
43. $f(x) = x^4 + 25x^2 + 144$
44. $f(x) = x^4 - 8x^3 + 17x^2 - 8x + 16$

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Finding a Polynomial with Given Zeros **In Exercises 45–50, find a polynomial function with real coefficients that has the given zeros. (There are many correct answers.)**

45. $2, i, -i$

46. $3, 4i, -4i$

47. $2, 2, 4 - i$

48. $-1, -1, 2 + 5i$

49. $0, -5, 1 + \sqrt{2}i$

50. $0, 4, 1 + \sqrt{2}i$

Finding a Polynomial with Given Zeros **In Exercises 51–54, a polynomial function f with real coefficients has the given degree, zeros, and solution point. Write the function (a) in completely factored form and (b) in polynomial form.**

	Degree	*Zeros*	*Solution Point*
51.	4	$1, -2, 2i$	$f(-1) = 10$
52.	4	$-1, 2, i$	$f(1) = 8$
53.	3	$-1, 2 + \sqrt{5}i$	$f(-2) = 42$
54.	3	$-2, 2 + 2\sqrt{2}i$	$f(-1) = -34$

Factoring a Polynomial **In Exercises 55–58, write the polynomial (a) as the product of factors that are irreducible over the *rationals*, (b) as the product of linear and quadratic factors that are irreducible over the *reals*, and (c) in completely factored form.**

55. $f(x) = x^4 - 6x^2 - 7$

56. $f(x) = x^4 + 6x^2 - 27$

57. $f(x) = x^4 - 2x^3 - 3x^2 + 12x - 18$
(*Hint:* One factor is $x^2 - 6$.)

58. $f(x) = x^4 - 3x^3 - x^2 - 12x - 20$
(*Hint:* One factor is $x^2 + 4$.)

Finding the Zeros of a Polynomial Function **In Exercises 59–66, use the given zero to find all the zeros of the function.**

	Function	*Zero*
59.	$f(x) = 2x^3 + 3x^2 + 50x + 75$	$5i$
60.	$f(x) = x^3 + x^2 + 9x + 9$	$3i$
61.	$g(x) = x^3 - 7x^2 - x + 87$	$5 + 2i$
62.	$g(x) = 4x^3 + 23x^2 + 34x - 10$	$-3 + i$
63.	$h(x) = 3x^3 - 4x^2 + 8x + 8$	$1 - \sqrt{3}i$
64.	$f(x) = x^3 + 4x^2 + 14x + 20$	$-1 - 3i$
65.	$h(x) = 8x^3 - 14x^2 + 18x - 9$	$\frac{1}{2}(1 - \sqrt{5}i)$
66.	$f(x) = 25x^3 - 55x^2 - 54x - 18$	$\frac{1}{5}(-2 + \sqrt{2}i)$

Using a Graph to Locate the Real Zeros **In Exercises 67–70, (a) use a graphing utility to find the real zeros of the function, and then (b) use the real zeros to find the exact values of the complex zeros.**

67. $f(x) = x^4 + 3x^3 - 5x^2 - 21x + 22$

68. $f(x) = x^3 + 4x^2 + 14x + 20$

69. $h(x) = 8x^3 - 14x^2 + 18x - 9$

70. $f(x) = 25x^3 - 55x^2 - 54x - 18$

71. ***Why you should learn it*** *(p. 135)* A football is kicked off the ground with an initial upward velocity of 48 feet per second. The football's height h (in feet) is given by

$$h(t) = -16t^2 + 48t, \quad 0 \le t \le 3$$

where t is the time (in seconds). Does the football reach a height of 50 feet? Explain.

72. **Marketing** The demand equation for a microwave is $p = 140 - 0.001x$, where p is the unit price (in dollars) of the microwave and x is the number of units produced and sold. The cost equation for the microwave is $C = 40x + 150{,}000$, where C is the total cost (in dollars) and x is the number of units produced. The total profit P obtained by producing and selling x units is given by $P = R - C = xp - C$. Is there a price p that yields a profit of \$3 million? Explain.

Conclusions

True or False? **In Exercises 73 and 74, decide whether the statement is true or false. Justify your answer.**

73. It is possible for a third-degree polynomial function with integer coefficients to have no real zeros.

74. If $[x + (4 + 3i)]$ is a factor of a polynomial function f with real coefficients, then $[x - (4 + 3i)]$ is also a factor of f.

75. **Writing** Compile a list of all the various techniques for factoring a polynomial that have been covered so far in the text. Give an example illustrating each technique, and write a paragraph discussing when the use of each technique is appropriate.

76. **CAPSTONE** Use a graphing utility to graph the function given by $f(x) = x^4 - 4x^2 + k$ for different values of k. Find the values of k such that the zeros of f satisfy the specified characteristics.

(a) Four real zeros

(b) Two real zeros, each of multiplicity 2

(c) Two real zeros and two complex zeros

(d) Four complex zeros

Cumulative Mixed Review

Identifying the Vertex of a Quadratic Function **In Exercises 77–80, describe the graph of the function and identify the vertex.**

77. $f(x) = x^2 - 7x - 8$

78. $f(x) = -x^2 + x + 6$

79. $f(x) = 6x^2 + 5x - 6$

80. $f(x) = 4x^2 + 2x - 12$

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2.6 Rational Functions and Asymptotes

Introduction to Rational Functions

A **rational function** can be written in the form

$$f(x) = \frac{N(x)}{D(x)}$$

where $N(x)$ and $D(x)$ are polynomials and $D(x)$ is not the zero polynomial.

In general, the *domain* of a rational function of x includes all real numbers except x-values that make the denominator zero. Much of the discussion of rational functions will focus on their graphical behavior near these x-values.

What you should learn

- Find the domains of rational functions.
- Find vertical and horizontal asymptotes of graphs of rational functions.
- Use rational functions to model and solve real-life problems.

Why you should learn it

Rational functions are convenient in modeling a wide variety of real-life problems, such as environmental scenarios. For instance, Exercise 45 on page 148 shows how to determine the cost of removing pollutants from a river.

Example 1 Finding the Domain of a Rational Function

Find the domain of $f(x) = 1/x$ and discuss the behavior of f near any excluded x-values.

Solution

Because the denominator is zero when $x = 0$, the domain of f is all real numbers except $x = 0$. To determine the behavior of f near this excluded value, evaluate $f(x)$ to the left and right of $x = 0$, as indicated in the following tables.

x	-1	-0.5	-0.1	-0.01	-0.001	$\to 0$
$f(x)$	-1	-2	-10	-100	-1000	$\to -\infty$

x	$0 \leftarrow$	0.001	0.01	0.1	0.5	1
$f(x)$	$\infty \leftarrow$	1000	100	10	2	1

From the table, note that as x approaches 0 *from the left*, $f(x)$ decreases without bound. In contrast, as x approaches 0 *from the right*, $f(x)$ increases without bound. Because $f(x)$ decreases without bound from the left and increases without bound from the right, you can conclude that f is not continuous. The graph of f is shown in Figure 2.36.

Figure 2.36

CHECKPOINT Now try Exercise 5.

Explore the Concept

Use the *table* and *trace* features of a graphing utility to verify that the function $f(x) = 1/x$ in Example 1 is not continuous.

Technology Tip

The graphing utility graphs in this section and the next section were created using the *dot* mode. A blue curve is placed behind the graphing utility's display to indicate where the graph should appear. You will learn more about how graphing utilities graph rational functions in the next section.

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Vertical and Horizontal Asymptotes

In Example 1, the behavior of f near $x = 0$ is denoted as follows.

$$\underbrace{f(x) \to -\infty \text{ as } x \to 0^-}_{\substack{f(x) \text{ decreases without bound as} \\ x \text{ approaches 0 from the left.}}} \qquad \underbrace{f(x) \to \infty \text{ as } x \to 0^+}_{\substack{f(x) \text{ increases without bound as} \\ x \text{ approaches 0 from the right.}}}$$

The line $x = 0$ is a **vertical asymptote** of the graph of f, as shown in Figure 2.37. From this figure you can see that the graph of f also has a **horizontal asymptote**—the line $y = 0$. This means the values of

$$f(x) = \frac{1}{x}$$

approach zero as x increases or decreases without bound.

$$\underbrace{f(x) \to 0 \text{ as } x \to -\infty}_{\substack{f(x) \text{ approaches 0 as } x \\ \text{decreases without bound.}}} \qquad \underbrace{f(x) \to 0 \text{ as } x \to \infty}_{\substack{f(x) \text{ approaches 0 as } x \\ \text{increases without bound.}}}$$

Figure 2.37

Definition of Vertical and Horizontal Asymptotes

1. The line $x = a$ is a **vertical asymptote** of the graph of f when

$$f(x) \to \infty \quad \text{or} \quad f(x) \to -\infty$$

as $x \to a$, either from the right or from the left.

2. The line $y = b$ is a **horizontal asymptote** of the graph of f when

$$f(x) \to b$$

as $x \to \infty$ or $x \to -\infty$.

Explore the Concept

Use a table of values to determine whether the functions in Figure 2.38 are continuous. When the graph of a function has an asymptote, can you conclude that the function is not continuous? Explain.

Figure 2.38 shows the vertical and horizontal asymptotes of the graphs of three rational functions. Note in Figure 2.38 that eventually (as $x \to \infty$ or $x \to -\infty$) the distance between the horizontal asymptote and the points on the graph must approach zero.

(a) (b) (c)

Figure 2.38

The graphs of $f(x) = 1/x$ in Figure 2.37 and $f(x) = (2x + 1)/(x + 1)$ in Figure 2.38 (a) are **hyperbolas.** You will study hyperbolas in Section 9.3.

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Vertical and Horizontal Asymptotes of a Rational Function

Let f be the rational function

$$f(x) = \frac{N(x)}{D(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdot \cdot \cdot + b_1 x + b_0}$$

where $N(x)$ and $D(x)$ have no common factors.

1. The graph of f has *vertical* asymptotes at the zeros of $D(x)$.
2. The graph of f has at most one *horizontal* asymptote determined by comparing the degrees of $N(x)$ and $D(x)$.
   - **a.** If $n < m$, then the graph of f has the line $y = 0$ (the x-axis) as a horizontal asymptote.
   - **b.** If $n = m$, then the graph of f has the line
     $$y = \frac{a_n}{b_m}$$
     as a horizontal asymptote, where a_n is the leading coefficient of the numerator and b_m is the leading coefficient of the denominator.
   - **c.** If $n > m$, then the graph of f has no horizontal asymptote.

What's Wrong?

You use a graphing utility to graph

$$y_1 = \frac{2x^3 + 1000x^2 + x}{x^3 + 1000x^2 + x + 1000}$$

as shown in the figure. You use the graph to conclude that the graph of y_1 has the line $y = 1$ as a horizontal asymptote. What's wrong?

Example 2 Finding Vertical and Horizontal Asymptotes

Find all asymptotes of the graph of each rational function.

a. $f(x) = \dfrac{2x}{3x^2 + 1}$ **b.** $f(x) = \dfrac{2x^2}{x^2 - 1}$

Solution

a. For this rational function, the degree of the numerator is *less than* the degree of the denominator, so the graph has the line $y = 0$ as a horizontal asymptote. To find any vertical asymptotes, set the denominator equal to zero and solve the resulting equation for x.

$$3x^2 + 1 = 0 \qquad \text{Set denominator equal to zero.}$$

Because this equation has no real solutions, you can conclude that the graph has no vertical asymptote. The graph of the function is shown in Figure 2.39.

Figure 2.39

b. For this rational function, the degree of the numerator is *equal* to the degree of the denominator. The leading coefficient of the numerator is 2 and the leading coefficient of the denominator is 1, so the graph has the line $y = 2$ as a horizontal asymptote. To find any vertical asymptotes, set the denominator equal to zero and solve the resulting equation for x.

$$x^2 - 1 = 0 \qquad \text{Set denominator equal to zero.}$$

$$(x + 1)(x - 1) = 0 \qquad \text{Factor.}$$

$$x + 1 = 0 \Rightarrow x = -1 \qquad \text{Set 1st factor equal to 0.}$$

$$x - 1 = 0 \Rightarrow x = 1 \qquad \text{Set 2nd factor equal to 0.}$$

This equation has two real solutions, $x = -1$ and $x = 1$, so the graph has the lines $x = -1$ and $x = 1$ as vertical asymptotes, as shown in Figure 2.40.

Figure 2.40

CHECKPOINT Now try Exercise 17.

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Values for which a rational function is undefined (the denominator is zero) result in a vertical asymptote or a hole in the graph, as shown in Example 3.

Example 3 Finding Asymptotes and Holes

Find all asymptotes and holes in the graph of

$$f(x) = \frac{x^2 + x - 2}{x^2 - x - 6}.$$

Solution

For this rational function the degree of the numerator is *equal to* the degree of the denominator. The leading coefficients of the numerator and denominator are both 1, so the graph has the line $y = 1$ as a horizontal asymptote. To find any vertical asymptotes, first factor the numerator and denominator as follows.

$$f(x) = \frac{x^2 + x - 2}{x^2 - x - 6} = \frac{(x - 1)\cancel{(x + 2)}}{\cancel{(x + 2)}(x - 3)} = \frac{x - 1}{x - 3}, \quad x \neq -2$$

By setting the denominator $x - 3$ (of the simplified function) equal to zero, you can determine that the graph has the line $x = 3$ as a vertical asymptote, as shown in Figure 2.41. To find any holes in the graph, note that the function is undefined at $x = -2$ and $x = 3$. Because $x = -2$ is not a vertical asymptote of the function, there is a hole in the graph at $x = -2$. To find the y-coordinate of the hole, substitute $x = -2$ into the simplified form of the function.

Figure 2.41

$$y = \frac{x - 1}{x - 3} = \frac{-2 - 1}{-2 - 3} = \frac{3}{5}$$

So, the graph of the rational function has a hole at $\left(-2, \frac{3}{5}\right)$.

CHECKPOINT Now try Exercise 23.

Technology Tip

Graphing utilities are limited in their resolution and therefore may not show a break or hole in the graph. You can use the *table* feature of a graphing utility to verify the values of x at which a function is not defined. Try doing this for the function in Example 3.

Example 4 Finding a Function's Domain and Asymptotes

For the function f, find (a) the domain of f, (b) the vertical asymptote of f, and (c) the horizontal asymptote of f.

$$f(x) = \frac{3x^3 + 7x^2 + 2}{-2x^3 + 16}$$

Solution

a. Because the denominator is zero when $-2x^3 + 16 = 0$, solve this equation to determine that the domain of f is all real numbers except $x = 2$.

b. Because the denominator of f has a zero at $x = 2$, and 2 is not a zero of the numerator, the graph of f has the vertical asymptote $x = 2$.

c. Because the degrees of the numerator and denominator are the same, and the leading coefficient of the numerator is 3 and the leading coefficient of the denominator is -2, the horizontal asymptote of f is $y = -\frac{3}{2}$.

CHECKPOINT Now try Exercise 25.

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Application

There are many examples of asymptotic behavior in real life. For instance, Example 5 shows how a vertical asymptote can be used to analyze the cost of removing pollutants from smokestack emissions.

Example 5 Cost-Benefit Model

A utility company burns coal to generate electricity. The cost C (in dollars) of removing p% of the smokestack pollutants is given by

$$C = \frac{80{,}000p}{100 - p}$$

for $0 \le p < 100$. Use a graphing utility to graph this function. You are a member of a state legislature that is considering a law that would require utility companies to remove 90% of the pollutants from their smokestack emissions. The current law requires 85% removal. How much additional cost would the utility company incur as a result of the new law?

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Solution

The graph of this function is shown in Figure 2.42. Note that the graph has a vertical asymptote at

$$p = 100.$$

Because the current law requires 85% removal, the current cost to the utility company is

$$C = \frac{80{,}000p}{100 - p} \qquad \text{Write original function.}$$

$$C = \frac{80{,}000(85)}{100 - 85} \qquad \text{Substitute 85 for } p.$$

$$\approx \$453{,}333. \qquad \text{Simplify.}$$

If the new law increases the percent removal to 90%, the cost will be

$$C = \frac{80{,}000p}{100 - p} \qquad \text{Write original function.}$$

$$C = \frac{80{,}000(90)}{100 - 90} \qquad \text{Substitute 90 for } p.$$

$$= \$720{,}000. \qquad \text{Simplify.}$$

So, the new law would require the utility company to spend an additional

$$720{,}000 - 453{,}333 = \$266{,}667. \qquad \text{Subtract 85\% removal cost from 90\% removal cost.}$$

Figure 2.42

CHECKPOINT Now try Exercise 45.

Explore the Concept

The *table* feature of a graphing utility can be used to estimate vertical and horizontal asymptotes of rational functions. Use the *table* feature to find any vertical or horizontal asymptotes of

$$f(x) = \frac{2x}{x + 1}.$$

Write a statement explaining how you found the asymptote(s) using the table.

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2.6 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1 and 2, fill in the blank.

1. Functions of the form $f(x) = N(x)/D(x)$, where $N(x)$ and $D(x)$ are polynomials and $D(x)$ is not the zero polynomial, are called _____ .

2. If $f(x) \to \pm\infty$ as $x \to a$ from the left (or right), then $x = a$ is a _____ of the graph of f.

3. What feature of the graph of $y = \dfrac{9}{x-3}$ can you find by solving $x - 3 = 0$?

4. Is $y = \dfrac{2}{3}$ a horizontal asymptote of the function $y = \dfrac{2x}{3x^2 - 5}$?

Procedures and Problem Solving

Finding the Domain of a Rational Function **In Exercises 5–10, (a) find the domain of the function, (b) complete each table, and (c) discuss the behavior of f near any excluded x-values.**

x	$f(x)$
0.5	
0.9	
0.99	
0.999	

x	$f(x)$
1.5	
1.1	
1.01	
1.001	

x	$f(x)$
5	
10	
100	
1000	

x	$f(x)$
-5	
-10	
-100	
-1000	

✓ **5.** $f(x) = \dfrac{1}{x-1}$

6. $f(x) = \dfrac{5x}{x-1}$

7. $f(x) = \dfrac{3x}{|x-1|}$

8. $f(x) = \dfrac{3}{|x-1|}$

9. $f(x) = \dfrac{3x^2}{x^2-1}$

10. $f(x) = \dfrac{4x}{x^2-1}$

Identifying Graphs of Rational Functions **In Exercises 11–16, match the function with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]**

(a)

(b)

(c)

(d)

(e)

(f)

11. $f(x) = \dfrac{2}{x+2}$

12. $f(x) = \dfrac{1}{x-3}$

13. $f(x) = \dfrac{4x+1}{x}$

14. $f(x) = \dfrac{1-x}{x}$

15. $f(x) = \dfrac{x-2}{x-4}$

16. $f(x) = -\dfrac{x+2}{x+4}$

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Finding Vertical and Horizontal Asymptotes **In Exercises 17–20, find any asymptotes of the graph of the rational function. Verify your answers by using a graphing utility to graph the function.**

✓ **17.** $f(x) = \dfrac{1}{x^2}$

18. $f(x) = \dfrac{3}{(x-2)^3}$

19. $f(x) = \dfrac{2x^2}{x^2 + x - 6}$

20. $f(x) = \dfrac{x^2 - 4x}{x^2 - 4}$

Finding Asymptotes and Holes **In Exercises 21–24, find any asymptotes and holes in the graph of the rational function. Verify your answers by using a graphing utility.**

21. $f(x) = \dfrac{x(2+x)}{2x - x^2}$

22. $f(x) = \dfrac{x^2 + 2x + 1}{2x^2 - x - 3}$

✓ **23.** $f(x) = \dfrac{x^2 - 25}{x^2 + 5x}$

24. $f(x) = \dfrac{3 - 14x - 5x^2}{3 + 7x + 2x^2}$

Finding a Function's Domain and Asymptotes **In Exercises 25–28, (a) find the domain of the function, (b) decide whether the function is continuous, and (c) identify any horizontal and vertical asymptotes. Verify your answer to part (a) both graphically by using a graphing utility and numerically by creating a table of values.**

✓ **25.** $f(x) = \dfrac{3x^2 + x - 5}{x^2 + 1}$

26. $f(x) = \dfrac{3x^2 + 1}{x^2 + x + 9}$

27. $f(x) = \dfrac{x^2 + 3x - 4}{-x^3 + 27}$

28. $f(x) = \dfrac{4x^3 - x^2 + 3}{3x^3 + 24}$

Algebraic-Graphical-Numerical **In Exercises 29–32, (a) determine the domains of *f* and *g*, (b) find any vertical asymptotes and holes in the graphs of *f* and *g*, (c) compare *f* and *g* by completing the table, (d) use a graphing utility to graph *f* and *g*, and (e) explain why the differences in the domains of *f* and *g* are not shown in their graphs.**

29. $f(x) = \dfrac{x^2 - 16}{x - 4}, \quad g(x) = x + 4$

x	1	2	3	4	5	6	7
$f(x)$							
$g(x)$							

30. $f(x) = \dfrac{x^2 - 9}{x - 3}, \quad g(x) = x + 3$

x	0	1	2	3	4	5	6
$f(x)$							
$g(x)$							

31. $f(x) = \dfrac{x^2 - 1}{x^2 - 2x - 3}, \quad g(x) = \dfrac{x - 1}{x - 3}$

x	-2	-1	0	1	2	3	4
$f(x)$							
$g(x)$							

32. $f(x) = \dfrac{x^2 - 4}{x^2 - 3x + 2}, \quad g(x) = \dfrac{x + 2}{x - 1}$

x	-3	-2	-1	0	1	2	3
$f(x)$							
$g(x)$							

Exploration **In Exercises 33–36, determine the value that the function *f* approaches as the magnitude of *x* increases. Is $f(x)$ greater than or less than this value when *x* is positive and large in magnitude? What about when *x* is negative and large in magnitude?**

33. $f(x) = 4 - \dfrac{1}{x}$

34. $f(x) = 2 + \dfrac{1}{x - 3}$

35. $f(x) = \dfrac{2x - 1}{x - 3}$

36. $f(x) = \dfrac{2x - 1}{x^2 + 1}$

Finding the Zeros of a Rational Function **In Exercises 37–44, find the zeros (if any) of the rational function. Use a graphing utility to verify your answer.**

37. $g(x) = \dfrac{x^2 - 4}{x + 3}$

38. $g(x) = \dfrac{x^3 - 8}{x^2 + 4}$

39. $f(x) = 1 - \dfrac{2}{x - 5}$

40. $h(x) = 5 + \dfrac{3}{x^2 + 1}$

41. $g(x) = \dfrac{x^2 - 2x - 3}{x^2 + 1}$

42. $g(x) = \dfrac{x^2 - 5x + 6}{x^2 + 4}$

43. $f(x) = \dfrac{2x^2 - 5x + 2}{2x^2 - 7x + 3}$

44. $f(x) = \dfrac{2x^2 + 3x - 2}{x^2 + x - 2}$

✓ **45.** ***Why you should learn it*** *(p. 142)* The cost C (in millions of dollars) of removing p% of the industrial and municipal pollutants discharged into a river is given by

$$C = \frac{255p}{100 - p}, \quad 0 \le p < 100.$$

(a) Find the costs of removing 10%, 40%, and 75% of the pollutants.

(b) Use a graphing utility to graph the cost function. Explain why you chose the values that you used in your viewing window.

(c) According to this model, is it possible to remove 100% of the pollutants? Explain.

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46. MODELING DATA

The endpoints of the interval over which distinct vision is possible are called the *near point* and *far point* of the eye (see figure). With increasing age these points normally change. The table shows the approximate near points y (in inches) for various ages x (in years).

Age, x	Near point, y
16	3.0
32	4.7
44	9.8
50	19.7
60	39.4

(a) Find a rational model for the data. Take the reciprocals of the near points to generate the points $(x, 1/y)$. Use the *regression* feature of a graphing utility to find a linear model for the data. The resulting line has the form

$$\frac{1}{y} = ax + b.$$

Solve for y.

(b) Use the *table* feature of the graphing utility to create a table showing the predicted near point based on the model for each of the ages in the original table.

(c) Do you think the model can be used to predict the near point for a person who is 70 years old? Explain.

47. Physics Consider a physics laboratory experiment designed to determine an unknown mass. A flexible metal meter stick is clamped to a table with 50 centimeters overhanging the edge (see figure). Known masses M ranging from 200 grams to 2000 grams are attached to the end of the meter stick. For each mass, the meter stick is displaced vertically and then allowed to oscillate. The average time t (in seconds) of one oscillation for each mass is recorded in the table.

Mass, M	Time, t
200	0.450
400	0.597
600	0.712
800	0.831
1000	0.906
1200	1.003
1400	1.088
1600	1.126
1800	1.218
2000	1.338

A model for the data is given by

$$t = \frac{38M + 16{,}965}{10(M + 5000)}.$$

(a) Use the *table* feature of a graphing utility to create a table showing the estimated time based on the model for each of the masses shown in the table. What can you conclude?

(b) Use the model to approximate the mass of an object when the average time for one oscillation is 1.056 seconds.

48. Biology The game commission introduces 100 deer into newly acquired state game lands. The population N of the herd is given by

$$N = \frac{20(5 + 3t)}{1 + 0.04t}, \quad t \geq 0$$

where t is the time in years.

(a) Use a graphing utility to graph the model.

(b) Find the populations when $t = 5$, $t = 10$, and $t = 25$.

(c) What is the limiting size of the herd as time increases? Explain.

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49. **Military Science** The table shows the numbers N (in thousands) of Department of Defense personnel from 1990 through 2008. The data can be modeled by

$$N = \frac{77.095t^2 - 216.04t + 2050}{0.052t^2 - 0.08t + 1}, \quad 0 \le t \le 18$$

where t is the year, with $t = 0$ corresponding to 1990. (Source: U.S. Department of Defense)

Year	Number, N (in thousands)
1990	2044
1991	1986
1992	1807
1993	1705
1994	1610
1995	1518
1996	1472
1997	1439
1998	1407
1999	1386
2000	1384
2001	1385
2002	1414
2003	1434
2004	1427
2005	1389
2006	1385
2007	1380
2008	1402

(a) Use a graphing utility to plot the data and graph the model in the same viewing window. How well does the model represent the data?

(b) Use the model to estimate the numbers of Department of Defense personnel in the years 2009, 2010, and 2011. Are the estimates reasonable?

(c) Find the horizontal asymptote of the graph of the model. Does it seem realistic that the numbers of Department of Defense personnel will keep getting closer to the value of the asymptote? Explain.

Conclusions

True or False? **In Exercises 50 and 51, determine whether the statement is true or false. Justify your answer.**

50. A rational function can have infinitely many vertical asymptotes.

51. A rational function must have at least one vertical asymptote.

52. **Think About It** Describe the possible features of the graph of a rational function f at $x = c$, when c is not in the domain of f.

53. **Think About It** A real zero of the numerator of a rational function f is $x = c$. Must $x = c$ also be a zero of f? Explain.

54. **Think About It** When the graph of a rational function f has a vertical asymptote at $x = 4$, can f have a common factor of $(x - 4)$ in the numerator and denominator? Explain.

55. **Exploration** Use a graphing utility to compare the graphs of y_1 and y_2.

$$y_1 = \frac{3x^3 - 5x^2 + 4x - 5}{2x^2 - 6x + 7}, \quad y_2 = \frac{3x^3}{2x^2}$$

Start with a viewing window of $-5 \le x \le 5$ and $-10 \le y \le 10$, and then zoom out. Make a conjecture about how the graph of a rational function f is related to the graph of $y = a_n x^n / b_m x^m$, where $a_n x^n$ is the leading term of the numerator of f and $b_m x^m$ is the leading term of the denominator of f.

56. **CAPSTONE** Write a rational function f that has the specified characteristics. (There are many correct answers.)

(a) Vertical asymptote: $x = 2$
Horizontal asymptote: $y = 0$
Zero: $x = 1$

(b) Vertical asymptote: $x = -1$
Horizontal asymptote: $y = 0$
Zero: $x = 2$

(c) Vertical asymptotes: $x = -2, x = 1$
Horizontal asymptote: $y = 2$
Zeros: $x = 3, x = -3$

(d) Vertical asymptotes: $x = -1, x = 2$
Horizontal asymptote: $y = -2$
Zeros: $x = -2, x = 3$

Cumulative Mixed Review

Finding the Equation of a Line **In Exercises 57–60, write the general form of the equation of the line that passes through the points.**

57. $(3, 2), (0, -1)$

58. $(-6, 1), (4, -5)$

59. $(2, 7), (3, 10)$

60. $(0, 0), (-9, 4)$

Long Division of Polynomials **In Exercises 61–64, divide using long division.**

61. $(x^2 + 5x + 6) \div (x - 4)$

62. $(x^2 - 10x + 15) \div (x - 3)$

63. $(2x^4 + x^2 - 11) \div (x^2 + 5)$

64. $(4x^5 + 3x^3 - 10) \div (2x + 3)$

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2.7 Graphs of Rational Functions

The Graph of a Rational Function

To sketch the graph of a rational function, use the following guidelines.

Guidelines for Graphing Rational Functions

Let

$$f(x) = N(x)/D(x)$$

where $N(x)$ and $D(x)$ are polynomials.

1. Simplify f, if possible. Any restrictions on the domain of f not in the simplified function should be listed.
2. Find and plot the y-intercept (if any) by evaluating $f(0)$.
3. Find the zeros of the numerator (if any) by setting the numerator equal to zero. Then plot the corresponding x-intercepts.
4. Find the zeros of the denominator (if any) by setting the denominator equal to zero. Then sketch the corresponding vertical asymptotes using dashed vertical lines and plot the corresponding holes using open circles.
5. Find and sketch any other asymptotes of the graph using dashed lines.
6. Plot at least one point *between* and one point *beyond* each x-intercept and vertical asymptote.
7. Use smooth curves to complete the graph between and beyond the vertical asymptotes, excluding any points where f is not defined.

What you should learn

- Analyze and sketch graphs of rational functions.
- Sketch graphs of rational functions that have slant asymptotes.
- Use graphs of rational functions to model and solve real-life problems.

Why you should learn it

The graph of a rational function provides a good indication of the behavior of a mathematical model. Exercise 89 on page 159 models the concentration of a chemical in the bloodstream after injection.

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Technology Tip

Some graphing utilities have difficulty graphing rational functions that have vertical asymptotes. Often, the utility will connect parts of the graph that are not supposed to be connected. Notice that the graph in Figure 2.43(a) should consist of two *unconnected* portions—one to the left of $x = 2$ and the other to the right of $x = 2$. To eliminate this problem, you can try changing the *mode* of the graphing utility to *dot mode*. The problem with this mode is that the graph is then represented as a collection of dots rather than as a smooth curve, as shown in Figure 2.43(b). In this text, a blue curve is placed behind the graphing utility's display to indicate where the graph should appear. [See Figure 2.43(b).]

(a) Connected mode **(b) Dot mode**

Figure 2.43

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Library of Parent Functions: Rational Function

The simplest type of rational function is the *parent rational function* $f(x) = 1/x$, also known as the *reciprocal function*. The basic characteristics of the parent rational function are summarized below and on the inside cover of this text.

Graph of $f(x) = \dfrac{1}{x}$

Domain: $(-\infty, 0) \cup (0, \infty)$

Range: $(-\infty, 0) \cup (0, \infty)$

No intercepts

Decreasing on $(-\infty, 0)$ and $(0, \infty)$

Odd function

Origin symmetry

Vertical asymptote: y-axis

Horizontal asymptote: x-axis

Explore the Concept

Use a graphing utility to graph

$$f(x) = 1 + \frac{1}{x - \frac{1}{x}}.$$

Set the graphing utility to *dot* mode and use a decimal viewing window. Use the *trace* feature to find three "holes" or "breaks" in the graph. Do all three holes or breaks represent zeros of the denominator

$$x - \frac{1}{x}?$$

Explain.

Example 1 Library of Parent Functions: $f(x) = 1/x$

Sketch the graph of the function and describe how the graph is related to the graph of $f(x) = 1/x$.

a. $g(x) = \dfrac{-1}{x + 2}$

b. $h(x) = \dfrac{1}{x - 1} + 3$

Solution

a. With respect to the graph of $f(x) = 1/x$, the graph of g is obtained by a *reflection* in the y-axis and a horizontal shift two units *to the left*, as shown in Figure 2.44. Confirm this with a graphing utility.

b. With respect to the graph of $f(x) = 1/x$, the graph of h is obtained by a horizontal shift one unit *to the right* and a vertical shift three units *upward*, as shown in Figure 2.45. Confirm this with a graphing utility.

Figure 2.44

Figure 2.45

CHECKPOINT Now try Exercise 5.

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Example 2 Sketching the Graph of a Rational Function

Sketch the graph of

$$g(x) = \frac{3}{x-2}$$

by hand.

Solution

y-intercept: $\left(0, -\frac{3}{2}\right)$, because $g(0) = -\frac{3}{2}$

x-intercept: None, because $3 \neq 0$

Vertical asymptote: $x = 2$, zero of denominator

Horizontal asymptote: $y = 0$, because degree of $N(x) <$ degree of $D(x)$

Additional points:

x	-4	1	2	3	5
$g(x)$	-0.5	-3	Undefined	3	1

By plotting the intercept, asymptotes, and a few additional points, you can obtain the graph shown in Figure 2.46. Confirm this with a graphing utility.

Figure 2.46

CHECKPOINT Now try Exercise 17.

Note that the graph of g in Example 2 is a vertical stretch and a right shift of the graph of

$$f(x) = \frac{1}{x}$$

because

$$g(x) = \frac{3}{x-2} = 3\left(\frac{1}{x-2}\right) = 3f(x-2).$$

Study Tip

Note in Examples 2–6 that the vertical asymptotes are included in the tables of additional points. This is done to emphasize numerically the behavior of the graph of the function.

Example 3 Sketching the Graph of a Rational Function

Sketch the graph of

$$f(x) = \frac{2x-1}{x}$$

by hand.

Solution

y-intercept: None, because $x = 0$ is not in the domain

x-intercept: $\left(\frac{1}{2}, 0\right)$, because $2x - 1 = 0$ when $x = \frac{1}{2}$

Vertical asymptote: $x = 0$, zero of denominator

Horizontal asymptote: $y = 2$, because degree of $N(x) =$ degree of $D(x)$

Additional points:

x	-4	-1	0	$\frac{1}{4}$	4
$f(x)$	2.25	3	Undefined	-2	1.75

By plotting the intercept, asymptotes, and a few additional points, you can obtain the graph shown in Figure 2.47. Confirm this with a graphing utility.

Figure 2.47

CHECKPOINT Now try Exercise 21.

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Example 4 Sketching the Graph of a Rational Function

Sketch the graph of $f(x) = \dfrac{x}{x^2 - x - 2}$.

Solution

Factor the denominator to determine more easily the zeros of the denominator.

$$f(x) = \frac{x}{x^2 - x - 2}$$

$$= \frac{x}{(x + 1)(x - 2)}$$

y-intercept: $(0, 0)$, because $f(0) = 0$

x-intercept: $(0, 0)$

Vertical asymptotes: $x = -1$, $x = 2$, zeros of denominator

Horizontal asymptote: $y = 0$, because degree of $N(x) <$ degree of $D(x)$

Additional points:

x	-3	-1	-0.5	1	2	3
$f(x)$	-0.3	Undefined	0.4	-0.5	Undefined	0.75

The graph is shown in Figure 2.48.

Figure 2.48

CHECKPOINT Now try Exercise 27.

Example 5 Sketching the Graph of a Rational Function

Sketch the graph of $f(x) = \dfrac{x^2 - 9}{x^2 - 2x - 3}$.

Solution

By factoring the numerator and denominator, you have

$$f(x) = \frac{x^2 - 9}{x^2 - 2x - 3}$$

$$= \frac{\cancel{(x - 3)}(x + 3)}{\cancel{(x - 3)}(x + 1)}$$

$$= \frac{x + 3}{x + 1}, \quad x \neq 3.$$

y-intercept: $(0, 3)$, because $f(0) = 3$

x-intercept: $(-3, 0)$, because $x + 3 = 0$ when $x = -3$

Vertical asymptote: $x = -1$, zero of (simplified) denominator

Hole: $\left(3, \frac{3}{2}\right)$, f is not defined at $x = 3$

Horizontal asymptote: $y = 1$, because degree of $N(x) =$ degree of $D(x)$

Additional points:

x	-5	-2	-1	-0.5	1	3	4
$f(x)$	0.5	-1	Undefined	5	2	Undefined	1.4

The graph is shown in Figure 2.49.

Figure 2.49 ***Hole at $x = 3$***

CHECKPOINT Now try Exercise 29.

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Slant Asymptotes

Consider a rational function whose denominator is of degree 1 or greater. If the degree of the numerator is exactly *one more* than the degree of the denominator, then the graph of the function has a **slant** (or **oblique**) **asymptote.** For example, the graph of

$$f(x) = \frac{x^2 - x}{x + 1}$$

has a slant asymptote, as shown in Figure 2.50. To find the equation of a slant asymptote, use long division. For instance, by dividing $x + 1$ into $x^2 - x$, you have

$$f(x) = \frac{x^2 - x}{x + 1} = \underbrace{x - 2}_{\substack{\text{Slant asymptote} \\ (y = x - 2)}} + \frac{2}{x + 1}.$$

Figure 2.50

As x increases or decreases without bound, the remainder term

$$\frac{2}{x + 1}$$

approaches 0, so the graph of f approaches the line $y = x - 2$, as shown in Figure 2.50.

Explore the Concept

Do you think it is possible for the graph of a rational function to cross its horizontal asymptote or its slant asymptote? Use the graphs of the following functions to investigate this question. Write a summary of your conclusion. Explain your reasoning.

$$f(x) = \frac{x}{x^2 + 1}$$

$$g(x) = \frac{2x}{3x^2 - 2x + 1}$$

$$h(x) = \frac{x^3}{x^2 + 1}$$

Example 6 A Rational Function with a Slant Asymptote

Sketch the graph of $f(x) = \dfrac{x^2 - x - 2}{x - 1}$.

Solution

First write $f(x)$ in two different ways. Factoring the numerator

$$f(x) = \frac{x^2 - x - 2}{x - 1} = \frac{(x - 2)(x + 1)}{x - 1}$$

enables you to recognize the x-intercepts. Long division

$$f(x) = \frac{x^2 - x - 2}{x - 1} = x - \frac{2}{x - 1}$$

enables you to recognize that the line $y = x$ is a slant asymptote of the graph.

y-intercept: $(0, 2)$, because $f(0) = 2$

x-intercepts: $(-1, 0)$ and $(2, 0)$

Vertical asymptote: $x = 1$, zero of denominator

Horizontal asymptote: None, because degree of $N(x) >$ degree of $D(x)$

Slant asymptote: $y = x$

Additional points:

x	-2	0.5	1	1.5	3
$f(x)$	-1.33	4.5	Undefined	-2.5	2

The graph is shown in Figure 2.51.

Figure 2.51

CHECKPOINT Now try Exercise 51.

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Application

Example 7 Publishing

A rectangular page is designed to contain 48 square inches of print. The margins on each side of the page are $1\frac{1}{2}$ inches wide. The margins at the top and bottom are each 1 inch deep. What should the dimensions of the page be so that the minimum amount of paper is used?

Figure 2.52

Graphical Solution

Let A be the area to be minimized. From Figure 2.52, you can write

$$A = (x + 3)(y + 2).$$

The printed area inside the margins is modeled by $48 = xy$ or $y = 48/x$. To find the minimum area, rewrite the equation for A in terms of just one variable by substituting $48/x$ for y.

$$A = (x + 3)\left(\frac{48}{x} + 2\right) = \frac{(x + 3)(48 + 2x)}{x}, \quad x > 0$$

The graph of this rational function is shown in Figure 2.53. Because x represents the width of the printed area, you need consider only the portion of the graph for which x is positive. Using the *minimum* feature of a graphing utility, you can approximate the minimum value of A to occur when $x \approx 8.5$ inches. The corresponding value of y is $48/8.5 \approx 5.6$ inches. So, the dimensions should be

$$x + 3 \approx 11.5 \text{ inches by } y + 2 \approx 7.6 \text{ inches.}$$

Figure 2.53

CHECKPOINT Now try Exercise 85.

Numerical Solution

Let A be the area to be minimized. From Figure 2.52, you can write

$$A = (x + 3)(y + 2).$$

The printed area inside the margins is modeled by $48 = xy$ or $y = 48/x$. To find the minimum area, rewrite the equation for A in terms of just one variable by substituting $48/x$ for y.

$$A = (x + 3)\left(\frac{48}{x} + 2\right) = \frac{(x + 3)(48 + 2x)}{x}, \quad x > 0$$

Use the *table* feature of a graphing utility to create a table of values for the function

$$y_1 = \frac{(x + 3)(48 + 2x)}{x}$$

beginning at $x = 1$. From the table, you can see that the minimum value of y_1 occurs when x is somewhere between 8 and 9, as shown in Figure 2.54. To approximate the minimum value of y_1 to one decimal place, change the table to begin at $x = 8$ and set the table step to 0.1. The minimum value of y_1 occurs when $x \approx 8.5$, as shown in Figure 2.55. The corresponding value of y is $48/8.5 \approx 5.6$ inches. So, the dimensions should be

$$x + 3 \approx 11.5 \text{ inches by } y + 2 \approx 7.6 \text{ inches.}$$

Figure 2.54

Figure 2.55

If you go on to take a course in calculus, you will learn an analytic technique for finding the exact value of x that produces a minimum area in Example 7. In this case, that value is

$$x = 6\sqrt{2} \approx 8.485.$$

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2.7 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1 and 2, fill in the blank(s).

1. For the rational function $f(x) = N(x)/D(x)$, if the degree of $N(x)$ is exactly one more than the degree of $D(x)$, then the graph of f has a _____ (or oblique) _____ .
2. The graph of $f(x) = 1/x$ has a _____ asymptote at $x = 0$.
3. Does the graph of $f(x) = \dfrac{x^3 - 1}{x^2 + 2}$ have a slant asymptote?
4. Using long division, you find that $f(x) = \dfrac{x^2 + 1}{x + 1} = x - 1 + \dfrac{2}{x + 1}$. What is the slant asymptote of the graph of f?

Procedures and Problem Solving

Library of Parent Functions: $f(x) = 1/x$ **In Exercises 5–8, sketch the graph of the function g and describe how the graph is related to the graph of $f(x) = 1/x$.**

5. $g(x) = \dfrac{1}{x - 4}$
6. $g(x) = \dfrac{-1}{x} - 5$
7. $g(x) = \dfrac{-1}{x + 3}$
8. $g(x) = \dfrac{1}{x + 6} - 2$

Describing a Transformation of $f(x) = 2/x$ **In Exercises 9–12, use a graphing utility to graph $f(x) = 2/x$ and the function g in the same viewing window. Describe the relationship between the two graphs.**

9. $g(x) = f(x) + 1$
10. $g(x) = f(x - 1)$
11. $g(x) = -f(x)$
12. $g(x) = \frac{1}{2}f(x + 2)$

Describing a Transformation of $f(x) = 2/x^2$ **In Exercises 13–16, use a graphing utility to graph $f(x) = 2/x^2$ and the function g in the same viewing window. Describe the relationship between the two graphs.**

13. $g(x) = f(x) - 2$
14. $g(x) = -f(x)$
15. $g(x) = f(x - 2)$
16. $g(x) = \frac{1}{4}f(x)$

Sketching the Graph of a Rational Function **In Exercises 17–32, sketch the graph of the rational function by hand. As sketching aids, check for intercepts, vertical asymptotes, horizontal asymptotes, and holes. Use a graphing utility to verify your graph.**

17. $f(x) = \dfrac{1}{x + 2}$
18. $f(x) = \dfrac{1}{x - 6}$
19. $C(x) = \dfrac{5 + 2x}{1 + x}$
20. $P(x) = \dfrac{1 - 3x}{1 - x}$
21. $f(t) = \dfrac{1 - 2t}{t}$
22. $g(x) = \dfrac{1}{x + 2} + 2$
23. $f(x) = \dfrac{x^2}{x^2 - 4}$
24. $g(x) = \dfrac{x}{x^2 - 9}$
25. $g(x) = \dfrac{4(x + 1)}{x(x - 4)}$
26. $h(x) = \dfrac{2}{x^2(x - 3)}$
27. $f(x) = \dfrac{3x}{x^2 - x - 2}$
28. $f(x) = \dfrac{2x}{x^2 + x - 2}$
29. $f(x) = \dfrac{x^2 + 3x}{x^2 + x - 6}$
30. $g(x) = \dfrac{5(x + 4)}{x^2 + x - 12}$
31. $f(x) = \dfrac{x^2 - 1}{x + 1}$
32. $f(x) = \dfrac{x^2 - 16}{x - 4}$

Finding the Domain and Asymptotes **In Exercises 33–42, use a graphing utility to graph the function. Determine its domain and identify any vertical or horizontal asymptotes.**

33. $f(x) = \dfrac{2 + x}{1 - x}$
34. $f(x) = \dfrac{3 - x}{2 - x}$
35. $f(t) = \dfrac{3t + 1}{t}$
36. $h(x) = \dfrac{x - 2}{x - 3}$
37. $h(t) = \dfrac{4}{t^2 + 1}$
38. $g(x) = -\dfrac{x}{(x - 2)^2}$
39. $f(x) = \dfrac{x + 1}{x^2 - x - 6}$
40. $f(x) = \dfrac{x + 4}{x^2 + x - 6}$
41. $f(x) = \dfrac{20x}{x^2 + 1} - \dfrac{1}{x}$
42. $f(x) = 5\left(\dfrac{1}{x - 4} - \dfrac{1}{x + 2}\right)$

Exploration **In Exercises 43–48, use a graphing utility to graph the function. What do you observe about its asymptotes?**

43. $h(x) = \dfrac{6x}{\sqrt{x^2 + 1}}$
44. $f(x) = -\dfrac{x}{\sqrt{9 + x^2}}$
45. $g(x) = \dfrac{4|x - 2|}{x + 1}$
46. $f(x) = -\dfrac{8|3 + x|}{x - 2}$
47. $f(x) = \dfrac{4(x - 1)^2}{x^2 - 4x + 5}$
48. $g(x) = \dfrac{3x^4 - 5x + 3}{x^4 + 1}$

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A Rational Function with a Slant Asymptote **In Exercises 49–56, sketch the graph of the rational function by hand. As sketching aids, check for intercepts, vertical asymptotes, and slant asymptotes.**

49. $f(x) = \dfrac{2x^2 + 1}{x}$

50. $g(x) = \dfrac{1 - x^2}{x}$

51. $h(x) = \dfrac{x^2}{x - 1}$

52. $f(x) = \dfrac{x^3}{x^2 - 1}$

53. $g(x) = \dfrac{x^3}{2x^2 - 8}$

54. $f(x) = \dfrac{x^3}{x^2 + 4}$

55. $f(x) = \dfrac{x^3 + 2x^2 + 4}{2x^2 + 1}$

56. $f(x) = \dfrac{2x^2 - 5x + 5}{x - 2}$

Finding the *x*-Intercepts **In Exercises 57–60, use the graph to estimate any *x*-intercepts of the rational function. Set $y = 0$ and solve the resulting equation to confirm your result.**

57. $y = \dfrac{x + 1}{x - 3}$

58. $y = \dfrac{2x}{x - 3}$

59. $y = \dfrac{1}{x} - x$

60. $y = x - 3 + \dfrac{2}{x}$

Finding the Domain and Asymptotes **In Exercises 61–64, use a graphing utility to graph the rational function. Determine the domain of the function and identify any asymptotes.**

61. $y = \dfrac{2x^2 + x}{x + 1}$

62. $y = \dfrac{x^2 + 5x + 8}{x + 3}$

63. $y = \dfrac{1 + 3x^2 - x^3}{x^2}$

64. $y = \dfrac{12 - 2x - x^2}{2(4 + x)}$

Finding Asymptotes and Holes **In Exercises 65–70, find all vertical asymptotes, horizontal asymptotes, slant asymptotes, and holes in the graph of the function. Then use a graphing utility to verify your result.**

65. $f(x) = \dfrac{x^2 - 5x + 4}{x^2 - 4}$

66. $f(x) = \dfrac{x^2 - 2x - 8}{x^2 - 9}$

67. $f(x) = \dfrac{2x^2 - 5x + 2}{2x^2 - x - 6}$

68. $f(x) = \dfrac{3x^2 - 8x + 4}{2x^2 - 3x - 2}$

69. $f(x) = \dfrac{2x^3 - x^2 - 2x + 1}{x^2 + 3x + 2}$

70. $f(x) = \dfrac{2x^3 + x^2 - 8x - 4}{x^2 - 3x + 2}$

Finding *x*-Intercepts Graphically **In Exercises 71–82, use a graphing utility to graph the function and determine any *x*-intercepts. Set $y = 0$ and solve the resulting equation to confirm your result.**

71. $y = \dfrac{1}{x + 5} + \dfrac{4}{x}$

72. $y = \dfrac{2}{x + 1} - \dfrac{3}{x}$

73. $y = \dfrac{1}{x + 2} + \dfrac{2}{x + 4}$

74. $y = \dfrac{2}{x + 2} - \dfrac{3}{x - 1}$

75. $y = x - \dfrac{6}{x - 1}$

76. $y = x - \dfrac{9}{x}$

77. $y = x + 2 - \dfrac{1}{x + 1}$

78. $y = 2x - 1 + \dfrac{1}{x - 2}$

79. $y = x + 1 + \dfrac{2}{x - 1}$

80. $y = x + 2 + \dfrac{2}{x + 2}$

81. $y = x + 3 - \dfrac{2}{2x - 1}$

82. $y = x - 1 - \dfrac{2}{2x - 3}$

83. **Chemistry** A 1000-liter tank contains 50 liters of a 25% brine solution. You add x liters of a 75% brine solution to the tank.

(a) Show that the concentration C (the ratio of brine to the total solution) of the final mixture is given by

$$C = \frac{3x + 50}{4(x + 50)}.$$

(b) Determine the domain of the function based on the physical constraints of the problem.

(c) Use a graphing utility to graph the function. As the tank is filled, what happens to the rate at which the concentration of brine increases? What percent does the concentration of brine appear to approach?

84. **Geometry** A rectangular region of length x and width y has an area of 500 square meters.

(a) Write the width y as a function of x.

(b) Determine the domain of the function based on the physical constraints of the problem.

(c) Sketch a graph of the function and determine the width of the rectangle when $x = 30$ meters.

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85. **Publishing** A page that is x inches wide and y inches high contains 30 square inches of print. The margins at the top and bottom are 2 inches deep and the margins on each side are 1 inch wide (see figure).

(a) Show that the total area A of the page is given by

$$A = \frac{2x(2x + 11)}{x - 2}.$$

(b) Determine the domain of the function based on the physical constraints of the problem.

(c) Use a graphing utility to graph the area function and approximate the page size such that the minimum amount of paper will be used. Verify your answer numerically using the *table* feature of the graphing utility.

86. **Geometry** A right triangle is formed in the first quadrant by the x-axis, the y-axis, and a line segment through the point $(3, 2)$ (see figure).

(a) Show that an equation of the line segment is given by

$$y = \frac{2(a - x)}{a - 3}, \quad 0 \le x \le a.$$

(b) Show that the area of the triangle is given by

$$A = \frac{a^2}{a - 3}.$$

(c) Use a graphing utility to graph the area function and estimate the value of a that yields a minimum area. Estimate the minimum area. Verify your answer numerically using the *table* feature of the graphing utility.

87. **Cost Management** The ordering and transportation cost C (in thousands of dollars) for the components used in manufacturing a product is given by

$$C = 100\left(\frac{200}{x^2} + \frac{x}{x + 30}\right), \quad x \ge 1$$

where x is the order size (in hundreds). Use a graphing utility to graph the cost function. From the graph, estimate the order size that minimizes cost.

88. **Cost Management** The cost C of producing x units of a product is given by $C = 0.2x^2 + 10x + 5$, and the average cost per unit is given by

$$\overline{C} = \frac{C}{x} = \frac{0.2x^2 + 10x + 5}{x}, \quad x > 0.$$

Sketch the graph of the average cost function, and estimate the number of units that should be produced to minimize the average cost per unit.

89. ***Why you should learn it*** *(p. 151)* The concentration C of a chemical in the bloodstream t hours after injection into muscle tissue is given by

$$C = \frac{3t^2 + t}{t^3 + 50}, \quad t \ge 0.$$

(a) Determine the horizontal asymptote of the function and interpret its meaning in the context of the problem.

(b) Use a graphing utility to graph the function and approximate the time when the bloodstream concentration is greatest.

(c) Use the graphing utility to determine when the concentration is less than 0.345.

90. **Algebraic-Graphical-Numerical** A driver averaged 50 miles per hour on the round trip between Baltimore, Maryland and Philadelphia, Pennsylvania, 100 miles away. The average speeds for going and returning were x and y miles per hour, respectively.

(a) Show that $y = \dfrac{25x}{x - 25}$.

(b) Determine the vertical and horizontal asymptotes of the function.

(c) Use a graphing utility to complete the table. What do you observe?

x	30	35	40	45	50	55	60
y							

(d) Use the graphing utility to graph the function.

(e) Is it possible to average 20 miles per hour in one direction and still average 50 miles per hour on the round trip? Explain.

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91. MODELING DATA

Data are recorded at 124 monitoring sites throughout the United States to study national trends in air quality. The table shows the mean amount A of carbon monoxide (in parts per million) recorded at these sites in each year from 1999 through 2008. (Source: Environmental Protection Agency)

Year	Amount, A (in parts per million)
1999	3.9
2000	3.5
2001	3.3
2002	2.9
2003	2.7
2004	2.5
2005	2.3
2006	2.2
2007	2.0
2008	1.9

(a) Use the *regression* feature of a graphing utility to find a linear model for the data. Let $t = 9$ represent 1999. Use the graphing utility to plot the data and graph the model in the same viewing window.

(b) Find a rational model for the data. Take the reciprocal of A to generate the points $(t, 1/A)$. Use the *regression* feature of the graphing utility to find a linear model for these data. The resulting line has the form $1/A = at + b$. Solve for A. Use the graphing utility to plot the data and graph the rational model in the same viewing window.

(c) Use the *table* feature of the graphing utility to create a table showing the mean amounts of carbon monoxide generated by each model for the years in the original table. Which model do you prefer? Why?

92. Biology A herd of elk is released onto state game lands. The expected population P of the herd can be modeled by the equation $P = (10 + 2.7t)/(1 + 0.1t)$, where t is the time in years since the initial number of elk were released.

(a) State the domain of the model. Explain your answer.

(b) Find the initial number of elk in the herd.

(c) Find the populations of elk after 25, 50, and 100 years.

(d) Is there a limit to the size of the herd? If so, what is the expected population?

Use a graphing utility to confirm your results for parts (a) through (d).

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Conclusions

True or False? In Exercises 93 and 94, determine whether the statement is true or false. Justify your answer.

93. The graph of a rational function is continuous only when the denominator is a constant polynomial.

94. The graph of a rational function can never cross one of its asymptotes.

Think About It In Exercises 95 and 96, use a graphing utility to graph the function. Explain why there is no vertical asymptote when a superficial examination of the function might indicate that there should be one.

95. $h(x) = \dfrac{6 - 2x}{3 - x}$

96. $g(x) = \dfrac{x^2 + x - 2}{x - 1}$

97. Writing Write a set of guidelines for finding all the asymptotes of a rational function given that the degree of the numerator is not more than 1 greater than the degree of the denominator.

98. CAPSTONE Write a rational function that has the specified characteristics. (There are many correct answers.)

(a) Vertical asymptote: $x = -2$
Slant asymptote: $y = x + 1$
Zero of the function: $x = 2$

(b) Vertical asymptote: $x = -4$
Slant asymptote: $y = x - 2$
Zero of the function: $x = 3$

Cumulative Mixed Review

Simplifying Exponential Expressions In Exercises 99–102, simplify the expression.

99. $\left(\dfrac{x}{8}\right)^{-3}$

100. $(4x^2)^{-2}$

101. $\dfrac{3^{7/6}}{3^{1/6}}$

102. $\dfrac{(x^{-2})(x^{1/2})}{(x^{-1})(x^{5/2})}$

Finding the Domain and Range of a Function In Exercises 103–106, use a graphing utility to graph the function and find its domain and range.

103. $f(x) = \sqrt{6 + x^2}$

104. $f(x) = \sqrt{121 - x^2}$

105. $f(x) = -|x + 9|$

106. $f(x) = -x^2 + 9$

107. *Make a Decision* To work an extended application analyzing the median sales prices of existing one-family homes, visit this textbook's *Companion Website*. (Data Source: National Association of Realtors)

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2.8 Quadratic Models

Classifying Scatter Plots

In real life, many relationships between two variables are parabolic, as in Section 2.1, Example 5. A scatter plot can be used to give you an idea of which type of model will best fit a set of data.

What you should learn

- Classify scatter plots.
- Use scatter plots and a graphing utility to find quadratic models for data.
- Choose a model that best fits a set of data.

Why you should learn it

Many real-life situations can be modeled by quadratic equations. For instance, in Exercise 17 on page 165, a quadratic equation is used to model the monthly precipitation for San Francisco, California.

Example 1 Classifying Scatter Plots

Decide whether each set of data could be better modeled by a linear model,

$$y = ax + b$$

a quadratic model,

$$y = ax^2 + bx + c$$

or neither.

a. (0.9, 1.7), (1.2, 2.0), (1.3, 1.9), (1.4, 2.1), (1.6, 2.5), (1.8, 2.8), (2.1, 3.0), (2.5, 3.4), (2.9, 3.7), (3.2, 3.9), (3.3, 4.1), (3.6, 4.4), (4.0, 4.7), (4.2, 4.8), (4.3, 5.0)

b. (0.9, 3.2), (1.2, 4.0), (1.3, 4.1), (1.4, 4.4), (1.6, 5.1), (1.8, 6.0), (2.1, 7.6), (2.5, 9.8), (2.9, 12.4), (3.2, 14.3), (3.3, 15.2), (3.6, 18.1), (4.0, 22.7), (4.2, 24.9), (4.3, 27.2)

c. (0.9, 1.2), (1.2, 6.5), (1.3, 9.3), (1.4, 11.6), (1.6, 15.2), (1.8, 16.9), (2.1, 14.7), (2.5, 8.1), (2.9, 3.7), (3.2, 5.8), (3.3, 7.1), (3.6, 11.5), (4.0, 20.2), (4.2, 23.7), (4.3, 26.9)

Solution

a. Begin by entering the data into a graphing utility. Then display the scatter plot, as shown in Figure 2.56. From the scatter plot, it appears the data follow a linear pattern. So, the data can be better modeled by a linear function.

Figure 2.56

b. Enter the data into a graphing utility and then display the scatter plot (see Figure 2.57). From the scatter plot, it appears the data follow a parabolic pattern. So, the data can be better modeled by a quadratic function.

Figure 2.57

c. Enter the data into a graphing utility and then display the scatter plot (see Figure 2.58). From the scatter plot, it appears the data do not follow either a linear or a parabolic pattern. So, the data cannot be modeled by either a linear function or a quadratic function.

Figure 2.58

CHECKPOINT Now try Exercise 5.

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Fitting a Quadratic Model to Data

In Section 1.7, you created scatter plots of data and used a graphing utility to find the least squares regression lines for the data. You can use a similar procedure to find a model for nonlinear data. Once you have used a scatter plot to determine the type of model that would best fit a set of data, there are several ways that you can actually find the model. Each method is best used with a computer or calculator, rather than with hand calculations.

Example 2 Fitting a Quadratic Model to Data

A study was done to compare the speed x (in miles per hour) with the mileage y (in miles per gallon) of an automobile. The results are shown in the table.

a. Use a graphing utility to create a scatter plot of the data.

b. Use the *regression* feature of the graphing utility to find a model that best fits the data.

c. Approximate the speed at which the mileage is the greatest.

Speed, x	Mileage, y
15	22.3
20	25.5
25	27.5
30	29.0
35	28.7
40	29.9
45	30.4
50	30.2
55	30.0
60	28.8
65	27.4
70	25.3
75	23.3

Solution

a. Begin by entering the data into a graphing utility and displaying the scatter plot, as shown in Figure 2.59. From the scatter plot, you can see that the data appear to follow a parabolic pattern.

Figure 2.59

b. Using the *regression* feature of the graphing utility, you can find the quadratic model, as shown in Figure 2.60. So, the quadratic equation that best fits the data is given by

$$y = -0.0082x^2 + 0.75x + 13.5. \qquad \text{Quadratic model}$$

Figure 2.60

c. Graph the data and the model in the same viewing window, as shown in Figure 2.61. Use the *maximum* feature or the *zoom* and *trace* features of the graphing utility to approximate the speed at which the mileage is greatest. You should obtain a maximum of approximately (46, 31), as shown in Figure 2.61. So, the speed at which the mileage is greatest is about 46 miles per hour.

Figure 2.61

CHECKPOINT Now try Exercise 17.

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Example 3 Fitting a Quadratic Model to Data

A basketball is dropped from a height of about 5.25 feet. The height of the basketball is recorded 23 times at intervals of about 0.02 second. The results are shown in the table. Use a graphing utility to find a model that best fits the data. Then use the model to predict the time when the basketball will hit the ground.

Time, x	Height, y
0.0	5.23594
0.02	5.20353
0.04	5.16031
0.06	5.09910
0.08	5.02707
0.099996	4.95146
0.119996	4.85062
0.139992	4.74979
0.159988	4.63096
0.179988	4.50132
0.199984	4.35728
0.219984	4.19523
0.23998	4.02958
0.25993	3.84593
0.27998	3.65507
0.299976	3.44981
0.319972	3.23375
0.339961	3.01048
0.359961	2.76921
0.379951	2.52074
0.399941	2.25786
0.419941	1.98058
0.439941	1.63488

Solution

Begin by entering the data into a graphing utility and displaying the scatter plot, as shown in Figure 2.62.

Figure 2.62

From the scatter plot, you can see that the data show a parabolic trend. So, using the *regression* feature of the graphing utility, you can find the quadratic model, as shown in Figure 2.63. The quadratic model that best fits the data is given by $y = -15.449x^2 - 1.30x + 5.2$.

Figure 2.63

You can graph the data and the model in the same viewing window to see that the model fits the data well, as shown in Figure 2.64.

Figure 2.64

Using this model, you can predict the time when the basketball will hit the ground by substituting 0 for y and solving the resulting equation for x.

$$y = -15.449x^2 - 1.30x + 5.2 \qquad \text{Write original model.}$$

$$0 = -15.449x^2 - 1.30x + 5.2 \qquad \text{Substitute 0 for } y.$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{Quadratic Formula}$$

$$= \frac{-(-1.30) \pm \sqrt{(-1.30)^2 - 4(-15.449)(5.2)}}{2(-15.449)} \qquad \text{Substitute for } a, b, \text{ and } c.$$

$$\approx 0.54 \qquad \text{Choose positive solution.}$$

So, the solution is about 0.54 second. In other words, the basketball will continue to fall for about $0.54 - 0.44 = 0.1$ second more before hitting the ground.

CHECKPOINT Now try Exercise 19.

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Choosing a Model

Sometimes it is not easy to distinguish from a scatter plot which type of model will best fit the data. You should first find several models for the data, using the *Library of Parent Functions*, and then choose the model that best fits the data by comparing the y-values of each model with the actual y-values.

Example 4 Choosing a Model

The table shows the amounts y (in gallons per person) of regular soft drinks consumed in the United States in the years 2000 through 2007. Use the *regression* feature of a graphing utility to find a linear model and a quadratic model for the data. Determine which model better fits the data. (Source: United States Department of Agriculture)

Year	Amounts, y
2000	39.4
2001	39.0
2002	38.5
2003	37.5
2004	37.0
2005	36.3
2006	35.4
2007	33.9

Solution

Let x represent the year, with $x = 0$ corresponding to 2000. Begin by entering the data into a graphing utility. Using the *regression* feature of the graphing utility, a linear model for the data is

$$y = -0.76x + 39.8 \qquad \text{Linear model}$$

and a quadratic model for the data is

$$y = -0.056x^2 - 0.37x + 39.4. \qquad \text{Quadratic model}$$

Plot the data and the linear model in the same viewing window, as shown in Figure 2.65. Then plot the data and the quadratic model in the same viewing window, as shown in Figure 2.66. To determine which model fits the data better, compare the y-values given by each model with the actual y-values. The model whose y-values are closest to the actual values is the better fit. In this case, the better-fitting model is the quadratic model.

Figure 2.65

Figure 2.66

CHECKPOINT Now try Exercise 21.

Technology Tip

When you use the regression feature of a graphing utility, the program may output an "r^2-value." This r^2-value is the **coefficient of determination** of the data and gives a measure of how well the model fits the data. The coefficient of determination for the linear model in Example 4 is

$$r^2 \approx 0.97105$$

and the coefficient of determination for the quadratic model is

$$r^2 \approx 0.99226.$$

Because the coefficient of determination for the quadratic model is closer to 1, the quadratic model better fits the data.

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2.8 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

1. What type of model best represents data that follow a parabolic pattern?
2. Which coefficient of determination indicates a better model for a set of data, $r^2 = 0.0365$, or $r^2 = 0.9688$?

Procedures and Problem Solving

Classifying Scatter Plots **In Exercises 3–8, determine whether the scatter plot could best be modeled by a linear model, a quadratic model, or neither.**

3.

4.

5.

6.

7.

8.

Using a Scatter Plot **In Exercises 9–16, (a) use a graphing utility to create a scatter plot of the data, (b) determine whether the data could be better modeled by a linear model or a quadratic model, (c) use the *regression* feature of the graphing utility to find a model for the data, (d) use the graphing utility to graph the model with the scatter plot from part (a), and (e) create a table comparing the original data with the data given by the model.**

9. (0, 2.1), (1, 2.4), (2, 2.5), (3, 2.8), (4, 2.9), (5, 3.0), (6, 3.0), (7, 3.2), (8, 3.4), (9, 3.5), (10, 3.6)
10. (−2, 11.0), (−1, 10.7), (0, 10.4), (1, 10.3), (2, 10.1), (3, 9.9), (4, 9.6), (5, 9.4), (6, 9.4), (7, 9.2), (8, 9.0)
11. (0, 3480), (5, 2235), (10, 1250), (15, 565), (20, 150), (25, 12), (30, 145), (35, 575), (40, 1275), (45, 2225), (50, 3500), (55, 5010)
12. (0, 6140), (2, 6815), (4, 7335), (6, 7710), (8, 7915), (10, 7590), (12, 7975), (14, 7700), (16, 7325), (18, 6820), (20, 6125), (22, 5325)
13. (1, 4.0), (2, 6.5), (3, 8.8), (4, 10.6), (5, 13.9), (6, 15.0), (7, 17.5), (8, 20.1), (9, 24.0), (10, 27.1)
14. (−6, 10.7), (−4, 9.0), (−2, 7.0), (0, 5.4), (2, 3.5), (4, 1.7), (6, −0.1), (8, −1.8), (10, −3.6), (12, −5.3)
15. (0, 587), (5, 551), (10, 512), (15, 478), (20, 436), (25, 430) (30, 424), (35, 420), (40, 423), (45, 429), (50, 444)
16. (2, 34.3), (3, 33.8), (4, 32.6), (5, 30.1), (6, 27.8), (7, 22.5), (8, 19.1), (9, 14.8), (10, 9.4). (11, 3.7), (12, −1.6)
17. ***Why you should learn it*** *(p. 161)* The table shows the monthly normal precipitation P (in inches) for San Francisco, California. (Source: U.S. National Oceanic and Atmospheric Administration)

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Month	Precipitation, P
January	4.45
February	4.01
March	3.26
April	1.17
May	0.38
June	0.11
July	0.03
August	0.07
September	0.20
October	1.40
November	2.49
December	2.89

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the month, with $t = 1$ corresponding to January.

(b) Use the *regression* feature of the graphing utility to find a quadratic model for the data.

(c) Use the graphing utility to graph the model with the scatter plot from part (a).

(d) Use the graph from part (c) to determine in which month the normal precipitation in San Francisco is the least.

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18. MODELING DATA

The table shows the annual sales S (in billions of dollars) of department stores in the United States from 2003 through 2008. (Source: U.S. Census Bureau)

Year	Sales, S (in billions of dollars)
2003	221.0
2004	222.0
2005	220.7
2006	219.0
2007	215.9
2008	206.1

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 3$ corresponding to 2003.

(b) Use the *regression* feature of the graphing utility to find a quadratic model for the data.

(c) Use the graphing utility to graph the model with the scatter plot from part (a).

(d) Use the model to estimate the first year when the annual sales of department stores will be below \$180 billion. Is this a good model for predicting future sales? Explain.

19. MODELING DATA

The table shows the percents P of the U.S. population who used the Internet from 2003 through 2008. (Source: U.S. Census Bureau)

Year	Percent, P
2003	55.58
2004	63.00
2005	66.33
2006	69.83
2007	71.83
2008	74.00

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 3$ corresponding to 2003.

(b) Use the *regression* feature of the graphing utility to find a quadratic model for the data.

(c) Use the graphing utility to graph the model with the scatter plot from part (a).

(d) According to the model, in what year will the percent of the U.S. population who use the Internet fall below 60%? Is this a good model for making future predictions? Explain.

20. MODELING DATA

The table shows the average numbers of hours H that adults in the United States spent reading newspapers each year from 2002 through 2007. (Source: Veronis Suhler Stevenson)

Year	Hours, H
2002	188
2003	195
2004	192
2005	187
2006	178
2007	171

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 2$ corresponding to 2002.

(b) A cubic model for the data is

$$H = 0.500t^3 - 8.43t^2 + 38.9t + 140$$

which has an r^2-value of 0.9965. Use the graphing utility to graph the model with the scatter plot from part (a). Is the cubic model a good fit for the data? Explain.

(c) Use the *regression* feature of the graphing utility to find a quadratic model for the data and identify the coefficient of determination.

(d) Use the graphing utility to graph the quadratic model with the scatter plot from part (a). Is the quadratic model a good fit for the data? Explain.

(e) Which model is a better fit for the data? Explain.

(f) The projected average numbers of hours H^* that adults spent reading newspapers each year from 2008 through 2010 are shown in the table. Use the models from parts (b) and (c) to *predict* the average numbers of hours for 2008 through 2010. Explain why your values may differ from those in the table.

Year	2008	2009	2010
H^*	164	159	155

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21. MODELING DATA

The table shows the numbers of televisions T (in millions) in homes in the United States from 1997 through 2008. (Source: The Nielsen Company)

Year	Televisions, T (in millions)
1997	229
1998	235
1999	240
2000	245
2001	248
2002	254
2003	260
2004	268
2005	287
2006	301
2007	311
2008	310

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 7$ corresponding to 1997.

(b) Use the *regression* feature of the graphing utility to find a linear model for the data and identify the coefficient of determination.

(c) Use the graphing utility to graph the linear model with the scatter plot from part (a).

(d) Use the *regression* feature of the graphing utility to find a quadratic model for the data and identify the coefficient of determination.

(e) Use the graphing utility to graph the quadratic model with the scatter plot from part (a).

(f) Which model is a better fit for the data? Explain.

(g) Use each model to approximate the year when the number of televisions in homes will reach 350 million.

Conclusions

True or False? **In Exercises 22–24, determine whether the statement is true or false. Justify your answer.**

22. The graph of a quadratic model with a negative leading coefficient will have a maximum value at its vertex.

23. The graph of a quadratic model with a positive leading coefficient will have a minimum value at its vertex.

24. Data that are positively correlated are always better modeled by a linear equation than by a quadratic equation.

25. Writing Explain why the parabola shown in the figure is not a good fit for the data.

26. CAPSTONE The r^2-values representing the coefficients of determination for the least squares linear model and the least squares quadratic model for the data shown are given below. Which is which? Explain your reasoning.

$r^2 \approx 0.9995$

$r^2 \approx 0.9782$

Cumulative Mixed Review

Compositions of Functions **In Exercises 27–30, find (a) $f \circ g$ and (b) $g \circ f$.**

27. $f(x) = 2x - 1, \quad g(x) = x^2 + 3$

28. $f(x) = 5x + 8, \quad g(x) = 2x^2 - 1$

29. $f(x) = x^3 - 1, \quad g(x) = \sqrt[3]{x + 1}$

30. $f(x) = \sqrt[3]{x + 5}, \quad g(x) = x^3 - 5$

Testing for One-to-One Functions **In Exercises 31–34, determine algebraically whether the function is one-to-one. If it is, find its inverse function. Verify your answer graphically.**

31. $f(x) = 2x + 5$

32. $f(x) = \dfrac{x - 4}{5}$

33. $f(x) = x^2 + 5, x \geq 0$

34. $f(x) = 2x^2 - 3, x \geq 0$

Multiplying Complex Conjugates **In Exercises 35–38, write the complex conjugate of the complex number. Then multiply the number by its complex conjugate.**

35. $1 - 3i$

36. $-2 + 4i$

37. $-5i$

38. $8i$

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2 Chapter Summary

	What did you learn?	Explanation and Examples	Review Exercises
2.1	**Analyze graphs of quadratic functions *(p. 90)*.**	Let a, b, and c be real numbers with $a \neq 0$. The function $f(x) = ax^2 + bx + c$ is called a quadratic function. Its graph is a "U-shaped" curve called a parabola.	1–6
	Write quadratic functions in standard form and use the results to sketch graphs of functions *(p. 93)*.	The quadratic function $f(x) = a(x - h)^2 + k$, $a \neq 0$, is in standard form. The graph of f is a parabola whose axis is the vertical line $x = h$ and whose vertex is the point (h, k). The parabola opens upward when $a > 0$ and opens downward when $a < 0$.	7–12
	Find minimum and maximum values of quadratic functions in real-life applications *(p. 95)*.	Consider $f(x) = ax^2 + bx + c$ with vertex $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$. If $a > 0$, then f has a *minimum* at $x = -b/(2a)$. If $a < 0$, then f has a *maximum* at $x = -b/(2a)$.	13, 14
2.2	**Use transformations to sketch graphs of polynomial functions *(p. 100)*.**	The graph of a polynomial function is continuous (no breaks, holes, or gaps) and has only smooth, rounded turns.	15–20
	Use the Leading Coefficient Test to determine the end behavior of graphs of polynomial functions *(p. 102)*.	Consider $f(x) = a_nx^n + \cdots + a_1x + a_0$, $a_n \neq 0$. ***n* is odd:** If $a_n > 0$, then the graph falls to the left and rises to the right. If $a_n < 0$, then the graph rises to the left and falls to the right. ***n* is even:** If $a_n > 0$, then the graph rises to the left and right. If $a_n < 0$, then the graph falls to the left and right.	21–26
	Find and use zeros of polynomial functions as sketching aids *(p. 104)*.	If f is a polynomial function and a is a real number, the following are equivalent: (1) $x = a$ is a *zero* of the function f, (2) $x = a$ is a *solution* of the polynomial equation $f(x) = 0$, (3) $(x - a)$ is a *factor* of the polynomial $f(x)$, and (4) $(a, 0)$ is an *x-intercept* of the graph of f.	27–38
	Use the Intermediate Value Theorem to help locate zeros of polynomial functions *(p. 108)*.	Let a and b be real numbers such that $a < b$. If f is a polynomial function such that $f(a) \neq f(b)$, then, in $[a, b]$, f takes on every value between $f(a)$ and $f(b)$.	39–42
2.3	**Use long division to divide polynomials by other polynomials *(p. 113)*.**	Dividend, Quotient, Remainder, Divisor $\dfrac{x^2 + 3x + 5}{x + 1} = \underbrace{x + 2}_{\text{Quotient}} + \dfrac{3}{x + 1}$	43–50
	Use synthetic division to divide polynomials by binomials of the form $(x - k)$ *(p. 116)*.	Divisor: $x + 3$ Dividend: $x^4 - 10x^2 - 2x + 4$ $\begin{array}{r\|rrrrr} -3 & 1 & 0 & -10 & -2 & 4 \\ & & -3 & 9 & 3 & -3 \\ \hline & 1 & -3 & -1 & 1 & 1 \end{array}$ Remainder: 1 Quotient: $x^3 - 3x^2 - x + 1$	51–56
	Use the Remainder Theorem and the Factor Theorem *(p. 117)*.	**The Remainder Theorem:** If a polynomial $f(x)$ is divided by $x - k$, then the remainder is $r = f(k)$. **The Factor Theorem:** A polynomial $f(x)$ has a factor $(x - k)$ if and only if $f(k) = 0$.	57–62
	Use the Rational Zero Test to determine possible rational zeros of polynomial functions *(p. 119)*.	The Rational Zero Test relates the possible rational zeros of a polynomial to the leading coefficient and to the constant term of the polynomial.	63, 64

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	What did you learn?	Explanation and Examples	Review Exercises
2.3	**Use Descartes's Rule of Signs *(p. 121)* and the Upper and Lower Bound Rules *(p. 122)* to find zeros of polynomials.**	Example 9 shows how to use Descartes's Rule of Signs. Example 10 uses Descartes's Rule of Signs and the Upper and Lower Bound Rules.	65–72
2.4	**Use the imaginary unit *i* to write complex numbers *(p. 128)*.**	The imaginary unit i is defined as $i = \sqrt{-1}$. If a and b are real numbers, then the number $a + bi$ is a complex number, and it is written in standard form.	73–76
	Add, subtract, and multiply complex numbers *(p. 129)*.	**Sum:** $(a + bi) + (c + di) = (a + c) + (b + d)i$ **Difference:** $(a + bi) - (c + di) = (a - c) + (b - d)i$	77–88
	Use complex conjugates to write the quotient of two complex numbers in standard form *(p. 131)*.	Complex numbers of the forms $a + bi$ and $a - bi$ are complex conjugates. To write $(a + bi)/(c + di)$ in standard form, multiply by $(c - di)/(c - di)$.	89–92
	Find complex solutions of quadratic equations *(p. 132)*.	If a is a positive number, then the principal square root of the negative number $-a$ is defined as $\sqrt{-a} = \sqrt{a}i$.	93–98
2.5	**Use the Fundamental Theorem of Algebra to determine the number of zeros of a polynomial function *(p. 135)*.**	**The Fundamental Theorem of Algebra** If $f(x)$ is a polynomial of degree n, where $n > 0$, then f has at least one zero in the complex number system.	99–102
	Find all zeros of polynomial functions, including complex zeros *(p. 136)*, and find conjugate pairs of complex zeros *(p. 137)*.	**Complex Zeros Occur in Conjugate Pairs** Let $f(x)$ be a polynomial function that has real coefficients. If $a + bi$ $(b \neq 0)$ is a zero of the function, the conjugate $a - bi$ is also a zero of the function.	103–120
	Find zeros of polynomials by factoring *(p. 138)*.	Every polynomial of degree $n > 0$ with real coefficients can be written as the product of linear and quadratic factors with real coefficients, where the quadratic factors have no real zeros.	121–124
2.6	**Find the domains *(p. 142)* and vertical and horizontal asymptotes *(p. 143)* of rational functions.**	The domain of a rational function of x includes all real numbers except x-values that make the denominator zero.	125–136
	Use rational functions to model and solve real-life problems *(p. 146)*.	A rational function can be used to model the cost of removing a given percent of smokestack pollutants at a utility company that burns coal. (See Example 5.)	137, 138
2.7	**Analyze and sketch graphs of rational functions *(p. 151)*, including functions with slant asymptotes *(p. 155)*.**	Consider a rational function whose denominator is of degree 1 or greater. If the degree of the numerator is exactly *one more* than the degree of the denominator, then the graph of the function has a slant asymptote.	139–152
	Use rational functions to model and solve real-life problems *(p. 156)*.	A rational function can be used to model the area of a page. The model can be used to determine the dimensions of the page that use the minimum amount of paper. (See Example 7.)	153, 154
2.8	**Classify scatter plots *(p. 161)*, find quadratic models for data *(p. 162)*, and choose a model that best fits a set of data *(p. 164)*.**	Sometimes it is not easy to distinguish from a scatter plot which type of model will best fit the data. You should first find several models for the data and then choose the model that best fits the data by comparing the y-values of each model with the actual y-values.	155–160

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2 Review Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

2.1

Library of Parent Functions **In Exercises 1–6, sketch the graph of each function and describe how the graph is related to the graph of $y = x^2$.**

1. $y = x^2 - 2$ **2.** $y = x^2 + 4$

3. $y = (x - 2)^2$ **4.** $y = -(x + 4)^2$

5. $y = (x + 5)^2 - 2$ **6.** $y = -(x - 4)^2 + 1$

Identifying the Vertex of a Quadratic Function **In Exercises 7–10, describe the graph of the function and identify the vertex. Then, sketch the graph of the function. Identify any x-intercepts.**

7. $f(x) = \left(x + \frac{3}{2}\right)^2 + 1$ **8.** $f(x) = (x - 4)^2 - 4$

9. $f(x) = \frac{1}{3}(x^2 + 5x - 4)$ **10.** $f(x) = 3x^2 - 12x + 11$

Writing the Equation of a Parabola in Standard Form **In Exercises 11 and 12, write the standard form of the quadratic function that has the indicated vertex and whose graph passes through the given point. Use a graphing utility to verify your result.**

11. Vertex: $(1, -4)$; Point: $(2, -3)$

12. Vertex: $(2, 3)$; Point: $(0, 2)$

13. **Geometry** A rectangle is inscribed in the region bounded by the x-axis, the y-axis, and the graph of $x + 2y - 8 = 0$, as shown in the figure.

(a) Write the area A of the rectangle as a function of x. Determine the domain of the function in the context of the problem.

(b) Use a graphing utility to graph the area function. Use the graph to approximate the dimensions that will produce a maximum area.

(c) Write the area function in standard form to find algebraically the dimensions that will produce a maximum area. Compare your results with your answer from part (b).

14. **Physical Education** A college has 1500 feet of portable rink boards to form three adjacent outdoor ice rinks, as shown in the figure. Determine the dimensions that will produce the maximum total area of ice surface.

2.2

Library of Parent Functions **In Exercises 15–20, sketch the graph of $y = x^3$ and the graph of the function f. Describe the transformation from y to f.**

15. $f(x) = (x + 4)^3$ **16.** $f(x) = x^3 - 4$

17. $f(x) = -x^3 + 2$ **18.** $f(x) = (x + 3)^3 - 2$

19. $f(x) = -(x + 7)^3 - 2$ **20.** $f(x) = -(x - 1)^3 + 3$

Comparing End Behavior **In Exercises 21 and 22, use a graphing utility to graph the functions f and g in the same viewing window. Zoom out far enough to see the right-hand and left-hand behavior of each graph. Do the graphs of f and g have the same right-hand and left-hand behavior? Explain why or why not.**

21. $f(x) = \frac{1}{2}x^3 - 2x + 1, \quad g(x) = \frac{1}{2}x^3$

22. $f(x) = -x^4 + 2x^3, \quad g(x) = -x^4$

Applying the Leading Coefficient Test **In Exercises 23–26, use the Leading Coefficient Test to describe the right-hand and left-hand behavior of the graph of the polynomial function.**

23. $f(x) = -x^2 + 6x + 9$

24. $f(x) = \frac{1}{2}x^3 + 2x$

25. $g(x) = \frac{3}{4}(x^4 + 3x^2 + 2)$

26. $h(x) = -x^5 - 7x^2 + 10x$

Finding Zeros of a Polynomial Function **In Exercises 27–32, (a) find the zeros algebraically, (b) use a graphing utility to graph the function, and (c) use the graph to approximate any zeros and compare them with those in part (a).**

27. $g(x) = x^4 - x^3 - 2x^2$ **28.** $h(x) = -2x^3 - x^2 + x$

29. $f(t) = t^3 - 3t$ **30.** $f(x) = -(x + 6)^3 - 8$

31. $f(x) = x(x + 3)^2$ **32.** $f(t) = t^4 - 4t^2$

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Finding a Polynomial Function with Given Zeros **In Exercises 33–36, find a polynomial function that has the given zeros. (There are many correct answers.)**

33. $-2, 1, 1, 5$

34. $-3, 0, 1, 4$

35. $3, 2 - \sqrt{3}, 2 + \sqrt{3}$

36. $-7, 4 - \sqrt{6}, 4 + \sqrt{6}$

Sketching the Graph of a Polynomial Function **In Exercises 37 and 38, sketch the graph of the function by (a) applying the Leading Coefficient Test, (b) finding the zeros of the polynomial, (c) plotting sufficient solution points, and (d) drawing a continuous curve through the points.**

37. $f(x) = x^4 - 2x^3 - 12x^2 + 18x + 27$

38. $f(x) = 18 + 27x - 2x^2 - 3x^3$

Approximating the Zeros of a Function **In Exercises 39–42, (a) use the Intermediate Value Theorem and a graphing utility to find graphically any intervals of length 1 in which the polynomial function is guaranteed to have a zero and (b) use the *zero* or *root* feature of the graphing utility to approximate the real zeros of the function. Verify your results in part (a) by using the *table* feature of the graphing utility.**

39. $f(x) = x^3 + 2x^2 - x - 1$

40. $f(x) = 0.24x^3 - 2.6x - 1.4$

41. $f(x) = x^4 - 6x^2 - 4$

42. $f(x) = 2x^4 + \frac{7}{2}x^3 - 2$

2.3

Long Division of Polynomials **In Exercises 43–50, use long division to divide.**

43. $\dfrac{24x^2 - x - 8}{3x - 2}$

44. $\dfrac{4x^2 + 7}{3x - 2}$

45. $\dfrac{x^4 - 3x^2 + 2}{x^2 - 1}$

46. $\dfrac{3x^4 + x^2 - 1}{x^2 - 1}$

47. $(5x^3 - 13x^2 - x + 2) \div (x^2 - 3x + 1)$

48. $(x^4 + x^3 - x^2 + 2x) \div (x^2 + 2x)$

49. $\dfrac{6x^4 + 10x^3 + 13x^2 - 5x + 2}{2x^2 - 1}$

50. $\dfrac{x^4 - 3x^3 + 4x^2 - 6x + 3}{x^2 + 2}$

Using Synthetic Division **In Exercises 51–56, use synthetic division to divide.**

51. $(0.25x^4 - 4x^3) \div (x + 2)$

52. $(0.1x^3 + 0.3x^2 - 0.5) \div (x - 5)$

53. $(6x^4 - 4x^3 - 27x^2 + 18x) \div \left(x - \frac{2}{3}\right)$

54. $(2x^3 + 2x^2 - x + 2) \div \left(x - \frac{1}{2}\right)$

55. $(3x^3 - 10x^2 + 12x - 22) \div (x - 4)$

56. $(2x^3 + 6x^2 - 14x + 9) \div (x - 1)$

Using the Remainder Theorem **In Exercises 57 and 58, use the Remainder Theorem and synthetic division to evaluate the function at each given value. Use a graphing utility to verify your results.**

57. $f(x) = x^4 + 10x^3 - 24x^2 + 20x + 44$

(a) $f(-3)$ (b) $f(-2)$

58. $g(t) = 2t^5 - 5t^4 - 8t + 20$

(a) $g(-4)$ (b) $g(\sqrt{2})$

Factoring a Polynomial **In Exercises 59–62, (a) verify the given factor(s) of the function *f*, (b) find the remaining factors of *f*, (c) use your results to write the complete factorization of *f*, and (d) list all real zeros of *f*. Confirm your results by using a graphing utility to graph the function.**

	Function	*Factor(s)*
59.	$f(x) = x^3 + 4x^2 - 25x - 28$	$(x - 4)$
60.	$f(x) = 2x^3 + 11x^2 - 21x - 90$	$(x + 6)$
61.	$f(x) = x^4 - 4x^3 - 7x^2 + 22x + 24$	$(x + 2)$, $(x - 3)$
62.	$f(x) = x^4 - 11x^3 + 41x^2 - 61x + 30$	$(x - 2)$, $(x - 5)$

Using the Rational Zero Test **In Exercises 63 and 64, use the Rational Zero Test to list all possible rational zeros of *f*. Use a graphing utility to verify that all the zeros of *f* are contained in the list.**

63. $f(x) = 4x^3 - 11x^2 + 10x - 3$

64. $f(x) = 10x^3 + 21x^2 - x - 6$

Using Descartes's Rule of Signs **In Exercises 65 and 66, use Descartes's Rule of Signs to determine the possible numbers of positive and negative real zeros of the function.**

65. $g(x) = 5x^3 - 6x + 9$

66. $f(x) = 2x^5 - 3x^2 + 2x - 1$

Finding the Zeros of a Polynomial Function **In Exercises 67 and 68, use synthetic division to verify the upper and lower bounds of the real zeros of *f*. Then find the real zeros of the function.**

67. $f(x) = 4x^3 - 3x^2 + 4x - 3$

Upper bound: $x = 1$; Lower bound: $x = -\frac{1}{4}$

68. $f(x) = 2x^3 - 5x^2 - 14x + 8$

Upper bound: $x = 8$; Lower bound: $x = -4$

Finding the Zeros of a Polynomial Function **In Exercises 69–72, find all the real zeros of the polynomial function.**

69. $f(x) = 6x^3 + 31x^2 - 18x - 10$

70. $f(x) = x^3 - 1.3x^2 - 1.7x + 0.6$

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71. $f(x) = 6x^4 - 25x^3 + 14x^2 + 27x - 18$

72. $f(x) = 5x^4 + 126x^2 + 25$

2.4

Writing a Complex Number in Standard Form **In Exercises 73–76, write the complex number in standard form.**

73. $6 + \sqrt{-25}$

74. $-\sqrt{-12} + 3$

75. $-2i^2 + 7i$

76. $-i^2 - 4i$

Operations with Complex Numbers **In Exercises 77–88, perform the operations and write the result in standard form.**

77. $(7 + 5i) + (-4 + 2i)$

78. $\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) - \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$

79. $5i(13 - 8i)$

80. $(1 + 6i)(5 - 2i)$

81. $(10 - 8i)(2 - 3i)$

82. $i(6 + i)(3 - 2i)$

83. $(3 + 7i)^2 + (3 - 7i)^2$

84. $(4 - i)^2 - (4 + i)^2$

85. $\left(\sqrt{-16} + 3\right)\left(\sqrt{-25} - 2\right)$

86. $\left(5 - \sqrt{-4}\right)\left(5 + \sqrt{-4}\right)$

87. $\sqrt{-9} + 3 + \sqrt{-36}$

88. $7 - \sqrt{-81} + \sqrt{-49}$

Writing a Quotient of Complex Numbers in Standard Form **In Exercises 89–92, write the quotient in standard form.**

89. $\frac{6 + i}{i}$

90. $\frac{4}{-3i}$

91. $\frac{3 + 2i}{5 + i}$

92. $\frac{1 - 7i}{2 + 3i}$

Complex Solutions of a Quadratic Equation **In Exercises 93–98, solve the quadratic equation.**

93. $x^2 + 16 = 0$

94. $x^2 + 48 = 0$

95. $x^2 + 3x + 6 = 0$

96. $x^2 + 4x + 8 = 0$

97. $3x^2 - 5x + 6 = 0$

98. $5x^2 - 2x + 4 = 0$

2.5

Zeros of a Polynomial Function **In Exercises 99–102, confirm that the function has the indicated zero(s).**

99. $f(x) = x^2 + 6x + 9$; Repeated zero: -3

100. $f(x) = x^2 - 10x + 25$; Repeated zero: 5

101. $f(x) = x^3 + 16x$; $0, -4i, 4i$

102. $f(x) = x^3 + 144x$; $0, -12i, 12i$

Using the Factored Form of a Function **In Exercises 103 and 104, find all the zeros of the function.**

103. $f(x) = 3x(x - 2)^2$

104. $f(x) = (x - 4)(x + 9)^2$

Finding the Zeros of a Polynomial Function **In Exercises 105–110, find all the zeros of the function and write the polynomial as a product of linear factors. Verify your results by using a graphing utility to graph the function.**

105. $h(x) = x^3 - 7x^2 + 18x - 24$

106. $f(x) = 2x^3 - 5x^2 - 9x + 40$

107. $f(x) = 2x^4 - 5x^3 + 10x - 12$

108. $g(x) = 3x^4 - 4x^3 + 7x^2 + 10x - 4$

109. $f(x) = x^5 + x^4 + 5x^3 + 5x^2$

110. $f(x) = x^5 - 5x^3 + 4x$

Using the Zeros to Find the *x*-Intercepts **In Exercises 111–116, (a) find all the zeros of the function, (b) write the polynomial as a product of linear factors, and (c) use your factorization to determine the *x*-intercepts of the graph of the function. Use a graphing utility to verify that the real zeros are the only *x*-intercepts.**

111. $f(x) = x^3 - 4x^2 + 6x - 4$

112. $f(x) = x^3 - 5x^2 - 7x + 51$

113. $f(x) = -3x^3 - 19x^2 - 4x + 12$

114. $f(x) = 2x^3 - 9x^2 + 22x - 30$

115. $f(x) = x^4 + 34x^2 + 225$

116. $f(x) = x^4 + 10x^3 + 26x^2 + 10x + 25$

Finding a Polynomial with Given Zeros **In Exercises 117–120, find a polynomial function with real coefficients that has the given zeros. (There are many correct answers.)**

117. $4, -2, 5i$

118. $2, -2, 2i$

119. $1, -4, -3 + 5i$

120. $-4, -4, 1 + \sqrt{3}i$

Factoring a Polynomial **In Exercises 121 and 122, write the polynomial (a) as the product of factors that are irreducible over the *rationals*, (b) as the product of linear and quadratic factors that are irreducible over the *reals*, and (c) in completely factored form.**

121. $f(x) = x^4 - 2x^3 + 8x^2 - 18x - 9$
(*Hint:* One factor is $x^2 + 9$.)

122. $f(x) = x^4 - 4x^3 + 3x^2 + 8x - 16$
(*Hint:* One factor is $x^2 - x - 4$.)

Finding the Zeros of a Polynomial Function **In Exercises 123 and 124, use the given zero to find all the zeros of the function.**

	Function	*Zero*
123.	$f(x) = x^3 + 3x^2 + 4x + 12$	$-2i$
124.	$f(x) = 2x^3 - 7x^2 + 14x + 9$	$2 + \sqrt{5}i$

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2.6

Finding a Function's Domain and Asymptotes **In Exercises 125–136, (a) find the domain of the function, (b) decide whether the function is continuous, and (c) identify any horizontal and vertical asymptotes.**

125. $f(x) = \dfrac{2 - x}{x + 3}$

126. $f(x) = \dfrac{4x}{x - 8}$

127. $f(x) = \dfrac{2}{x^2 - 3x - 18}$

128. $f(x) = \dfrac{2x^2 + 3}{x^2 + x + 3}$

129. $f(x) = \dfrac{7 + x}{7 - x}$

130. $f(x) = \dfrac{6x}{x^2 - 1}$

131. $f(x) = \dfrac{4x^2}{2x^2 - 3}$

132. $f(x) = \dfrac{3x^2 - 11x - 4}{x^2 + 2}$

133. $f(x) = \dfrac{2x - 10}{x^2 - 2x - 15}$

134. $f(x) = \dfrac{x^3 - 4x^2}{x^2 + 3x + 2}$

135. $f(x) = \dfrac{x - 2}{|x| + 2}$

136. $f(x) = \dfrac{2x}{|2x - 1|}$

137. **Criminology** The cost C (in millions of dollars) for the U.S. government to seize p% of an illegal drug as it enters the country is given by

$$C = \frac{528p}{100 - p}, \quad 0 \le p < 100.$$

(a) Find the costs of seizing 25%, 50%, and 75% of the illegal drug.

(b) Use a graphing utility to graph the function. Be sure to choose an appropriate viewing window. Explain why you chose the values you used in your viewing window.

(c) According to this model, would it be possible to seize 100% of the drug? Explain.

138. **Biology** A biology class performs an experiment comparing the quantity of food consumed by a certain kind of moth with the quantity supplied. The model for the experimental data is given by

$$y = \frac{1.568x - 0.001}{6.360x + 1}, \quad x > 0$$

where x is the quantity (in milligrams) of food supplied and y is the quantity (in milligrams) eaten (see figure). At what level of consumption will the moth become satiated?

2.7

Finding Asymptotes and Holes **In Exercises 139–142, find all of the vertical, horizontal, and slant asymptotes, and any holes in the graph of the function. Then use a graphing utility to verify your result.**

139. $f(x) = \dfrac{x^2 - 5x + 4}{x^2 - 1}$

140. $f(x) = \dfrac{2x^2 - 7x + 3}{2x^2 - 3x - 9}$

141. $f(x) = \dfrac{3x^2 + 5x - 2}{x + 1}$

142. $f(x) = \dfrac{2x^2 + 5x + 3}{x - 2}$

Sketching the Graph of a Rational Function **In Exercises 143–152, sketch the graph of the rational function by hand. As sketching aids, check for intercepts, vertical asymptotes, horizontal asymptotes, slant asymptotes, and holes.**

143. $f(x) = \dfrac{2x - 1}{x - 5}$

144. $f(x) = \dfrac{x - 3}{x - 2}$

145. $f(x) = \dfrac{2x^2}{x^2 - 4}$

146. $f(x) = \dfrac{5x}{x^2 + 1}$

147. $f(x) = \dfrac{2}{(x + 1)^2}$

148. $f(x) = \dfrac{4}{(x - 1)^2}$

149. $f(x) = \dfrac{2x^3}{x^2 + 1}$

150. $f(x) = \dfrac{x^3}{3x^2 - 6}$

151. $f(x) = \dfrac{x^2 - x + 1}{x - 3}$

152. $f(x) = \dfrac{2x^2 + 7x + 3}{x + 1}$

153. **Biology** A Parks and Wildlife Commission releases 80,000 fish into a lake. After t years, the population N of the fish (in thousands) is given by

$$N = \frac{20(4 + 3t)}{1 + 0.05t}, \quad t \ge 0.$$

(a) Use a graphing utility to graph the function and find the populations when $t = 5$, $t = 10$, and $t = 25$.

(b) What is the maximum number of fish in the lake as time passes? Explain your reasoning.

154. **Publishing** A page that is x inches wide and y inches high contains 30 square inches of print. The top and bottom margins are 2 inches deep and the margins on each side are 2 inches wide.

(a) Draw a diagram that illustrates the problem.

(b) Show that the total area A of the page is given by

$$A = \frac{2x(2x + 7)}{x - 4}.$$

(c) Determine the domain of the function based on the physical constraints of the problem.

(d) Use a graphing utility to graph the area function and approximate the page size such that the minimum amount of paper will be used.

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2.8

Classifying Scatter Plots **In Exercises 155–158, determine whether the scatter plot could best be modeled by a linear model, a quadratic model, or neither.**

155.

156.

157.

158.

159. MODELING DATA

The table shows the numbers of FM radio stations S in the United States from 2000 through 2009. (Source: Federal Communication Commission)

Year	FM stations, S
2000	5892
2001	6051
2002	6161
2003	6207
2004	6217
2005	6215
2006	6252
2007	6290
2008	6309
2009	6427

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 0$ corresponding to 2000.

(b) A cubic model for the data is

$$S = 2.520t^3 - 37.51t^2 + 192.4t + 5895.$$

Use the graphing utility to graph this model with the scatter plot from part (a).

(c) Use the *regression* feature of the graphing utility to find a quadratic model for the data and identify the coefficient of determination.

(d) Use the graphing utility to graph the quadratic model with the scatter plot from part (a).

(e) Which model is a better fit for the data? Explain.

(f) Use the model you chose in part (e) to predict the number of FM radio stations in 2012.

160. MODELING DATA

The table shows the sales S (in millions of dollars) of Office Depot for each of the years from 2002 through 2008. (Source: Office Depot)

Year	Sales, S (in millions of dollars)
2002	11,357
2003	12,359
2004	13,565
2005	14,279
2006	15,011
2007	15,528
2008	14,496

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 2$ corresponding to 2002.

(b) Use the *regression* feature of the graphing utility to find a quadratic model for the data.

(c) Use the graphing utility to graph the quadratic model with the scatter plot from part (a). Is the quadratic model a good fit for the data?

(d) According to the model, what is the first year when Office Depot will have sales of less than \$11 billion?

(e) Is this a good model for predicting the sales of Office Depot in future years? Explain.

Conclusions

True or False? **In Exercises 161–163, determine whether the statement is true or false. Justify your answer.**

161. The graph of $f(x) = \dfrac{2x^3}{x + 1}$ has a slant asymptote.

162. A fourth-degree polynomial with real coefficients can have -5, $-8i$, $4i$, and 5 as its zeros.

163. The sum of two complex numbers cannot be a real number.

164. **Think About It** Describe the domain restrictions of a rational function when the denominator divides evenly into the numerator.

165. **Writing** Write a paragraph discussing whether every rational function has a vertical asymptote.

166. **Error Analysis** Describe the error.

$$\cancel{\sqrt{-6}\,\sqrt{-6} = \sqrt{(-6)(-6)} = \sqrt{36} = 6}$$

167. **Error Analysis** Describe the error.

$$\cancel{-i\left(\sqrt{-4} - 1\right) = -i(4i - 1) = -4i^2 + i = 4 + i}$$

168. Write each of the powers of i as i, $-i$, 1, or -1.

(a) i^{40} (b) i^{25} (c) i^{50} (d) i^{67}

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2 Chapter Test

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Take this test as you would take a test in class. After you are finished, check your work against the answers given in the back of the book.

Figure for 2

1. Identify the vertex and intercepts of the graph of $y = x^2 + 4x + 3$.

2. Write an equation of the parabola shown at the right.

3. Find all the real zeros of $f(x) = 4x^3 + 4x^2 + x$. Determine the multiplicity of each zero.

4. Sketch the graph of the function $f(x) = -x^3 + 7x + 6$.

5. Divide using long division: $(3x^3 + 4x - 1) \div (x^2 + 1)$.

6. Divide using synthetic division: $(2x^4 - 5x^2 - 3) \div (x - 2)$.

7. Use synthetic division to evaluate $f(-2)$ for $f(x) = 3x^4 - 6x^2 + 5x - 1$.

In Exercises 8 and 9, list all the possible rational zeros of the function. Use a graphing utility to graph the function and find all the rational zeros.

8. $g(t) = 2t^4 - 3t^3 + 16t - 24$

9. $h(x) = 3x^5 + 2x^4 - 3x - 2$

10. Find all the zeros of the function $f(x) = x^3 - 7x^2 + 11x + 19$ and write the polynomial as a product of linear factors.

In Exercises 11–14, perform the operations and write the result in standard form.

11. $(-8 - 3i) + (-1 - 15i)$

12. $\left(10 + \sqrt{-20}\right) - \left(4 - \sqrt{-14}\right)$

13. $(2 + i)(6 - i)$

14. $(4 + 3i)^2 - (5 + i)^2$

In Exercises 15–17, write the quotient in standard form.

15. $\dfrac{8 + 5i}{6 - i}$

16. $\dfrac{5i}{2 + i}$

17. $(2i - 1) \div (3i + 2)$

In Exercises 18 and 19, solve the quadratic equation.

18. $x^2 + 75 = 0$

19. $x^2 - 2x + 8 = 0$

In Exercises 20–22, sketch the graph of the rational function. As sketching aids, check for intercepts, vertical asymptotes, horizontal asymptotes, and slant asymptotes.

20. $h(x) = \dfrac{4}{x^2} - 1$

21. $g(x) = \dfrac{x^2 + 2}{x - 1}$

22. $f(x) = \dfrac{2x^2 + 9}{5x^2 + 2}$

23. The table shows the amounts A (in billions of dollars) spent on military procurement by the Department of Defense for the years 2002 through 2008. (Source: U.S. Office of Management and Budget)

Year	Military procurement, A (in billions of dollars)
2002	62.5
2003	67.9
2004	76.2
2005	82.3
2006	89.8
2007	99.6
2008	117.4

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 2$ corresponding to 2002.

(b) Use the *regression* feature of the graphing utility to find a quadratic model for the data.

(c) Use the graphing utility to graph the quadratic model with the scatter plot from part (a). Is the quadratic model a good fit for the data?

(d) Use the model to estimate the amounts spent on military procurement in 2010 and 2012.

(e) Do you believe the model is useful for predicting the amounts spent on military procurement for years beyond 2008? Explain.

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Proofs in Mathematics

These two pages contain proofs of four important theorems about polynomial functions. The first two theorems are from Section 2.3, and the second two theorems are from Section 2.5.

The Remainder Theorem (p. 117)

If a polynomial $f(x)$ is divided by $x - k$, then the remainder is

$$r = f(k).$$

Proof

From the Division Algorithm, you have

$$f(x) = (x - k)q(x) + r(x)$$

and because either $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $x - k$, you know that $r(x)$ must be a constant. That is, $r(x) = r$. Now, by evaluating $f(x)$ at $x = k$, you have

$$\begin{aligned} f(k) &= (k - k)q(k) + r \\ &= (0)q(k) + r \\ &= r. \end{aligned}$$

To be successful in algebra, it is important that you understand the connection among the *factors* of a polynomial, the *zeros* of a polynomial function, and the *solutions* or *roots* of a polynomial equation. The Factor Theorem is the basis for this connection.

The Factor Theorem (p. 117)

A polynomial $f(x)$ has a factor $(x - k)$ if and only if $f(k) = 0$.

Proof

Using the Division Algorithm with the factor $(x - k)$, you have

$$f(x) = (x - k)q(x) + r(x).$$

By the Remainder Theorem, $r(x) = r = f(k)$, and you have

$$f(x) = (x - k)q(x) + f(k)$$

where $q(x)$ is a polynomial of lesser degree than $f(x)$. If $f(k) = 0$, then

$$f(x) = (x - k)q(x)$$

and you see that $(x - k)$ is a factor of $f(x)$. Conversely, if $(x - k)$ is a factor of $f(x)$, then division of $f(x)$ by $(x - k)$ yields a remainder of 0. So, by the Remainder Theorem, you have $f(k) = 0$.

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Linear Factorization Theorem (p. 135)

If $f(x)$ is a polynomial of degree n, where $n > 0$, then f has precisely n linear factors

$$f(x) = a_n(x - c_1)(x - c_2) \cdot \cdot \cdot (x - c_n)$$

where $c_1, c_2, \ldots, c_n$ are complex numbers.

Proof

Using the Fundamental Theorem of Algebra, you know that f must have at least one zero, c_1. Consequently, $(x - c_1)$ is a factor of $f(x)$, and you have

$$f(x) = (x - c_1)f_1(x).$$

If the degree of $f_1(x)$ is greater than zero, then apply the Fundamental Theorem again to conclude that f_1 must have a zero c_2, which implies that

$$f(x) = (x - c_1)(x - c_2)f_2(x).$$

It is clear that the degree of $f_1(x)$ is $n - 1$, that the degree of $f_2(x)$ is $n - 2$, and that you can repeatedly apply the Fundamental Theorem n times until you obtain

$$f(x) = a_n(x - c_1)(x - c_2) \cdot \cdot \cdot (x - c_n)$$

where a_n is the leading coefficient of the polynomial $f(x)$.

Factors of a Polynomial (p. 138)

Every polynomial of degree $n > 0$ with real coefficients can be written as the product of linear and quadratic factors with real coefficients, where the quadratic factors have no real zeros.

Proof

To begin, you use the Linear Factorization Theorem to conclude that $f(x)$ can be *completely* factored in the form

$$f(x) = d(x - c_1)(x - c_2)(x - c_3) \cdot \cdot \cdot (x - c_n).$$

If each c_i is real, then there is nothing more to prove. If any c_i is complex ($c_i = a + bi$, $b \neq 0$), then, because the coefficients of $f(x)$ are real, you know that the conjugate $c_j = a - bi$ is also a zero. By multiplying the corresponding factors, you obtain

$$\begin{aligned}(x - c_i)(x - c_j) &= [x - (a + bi)][x - (a - bi)] \\ &= x^2 - 2ax + (a^2 + b^2)\end{aligned}$$

where each coefficient is real.

The Fundamental Theorem of Algebra

The Linear Factorization Theorem is closely related to the Fundamental Theorem of Algebra. The Fundamental Theorem of Algebra has a long and interesting history. In the early work with polynomial equations, the Fundamental Theorem of Algebra was thought to have been not true, because imaginary solutions were not considered. In fact, in the very early work by mathematicians such as Abu al-Khwarizmi (c. 800 A.D.), negative solutions were also not considered.

Once imaginary numbers were accepted, several mathematicians attempted to give a general proof of the Fundamental Theorem of Algebra. These included Gottfried von Leibniz (1702), Jean d'Alembert (1746), Leonhard Euler (1749), Joseph-Louis Lagrange (1772), and Pierre Simon Laplace (1795). The mathematician usually credited with the first correct proof of the Fundamental Theorem of Algebra is Carl Friedrich Gauss, who published the proof in his doctoral thesis in 1799.

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Progressive Summary (Chapters 1–2)

This chart outlines the topics that have been covered so far in this text. Progressive Summary charts appear after Chapters 2, 3, 6, and 9. In each Progressive Summary, new topics encountered for the first time appear in red.

ALGEBRAIC FUNCTIONS

Polynomial, Rational, Radical

Rewriting

Polynomial form ↔ Factored form
Operations with polynomials
Rationalize denominators
Simplify rational expressions
Operations with complex numbers

Solving

Equation	*Strategy*
Linear	Isolate variable
Quadratic	Factor, set to zero Extract square roots Complete the square Quadratic Formula
Polynomial	Factor, set to zero Rational Zero Test
Rational	Multiply by LCD
Radical	Isolate, raise to power
Absolute value	Isolate, form two equations

Analyzing

Graphically	*Algebraically*
Intercepts	Domain, Range
Symmetry	Transformations
Slope	Composition
Asymptotes	Standard forms of equations
End behavior	Leading Coefficient Test
Minimum values	Synthetic division
Maximum values	Descartes's Rule of Signs

Numerically

Table of values

TRANSCENDENTAL FUNCTIONS

Rewriting

Solving

Analyzing

OTHER TOPICS

Rewriting

Solving

Analyzing

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3 Exponential and Logarithmic Functions

Section 3.6, Example 4
Internal Revenue Service

3.1 Exponential Functions and Their Graphs
3.2 Logarithmic Functions and Their Graphs
3.3 Properties of Logarithms
3.4 Solving Exponential and Logarithmic Equations
3.5 Exponential and Logarithmic Models
3.6 Nonlinear Models

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3.1 Exponential Functions and Their Graphs

Exponential Functions

So far, this text has dealt mainly with **algebraic functions,** which include polynomial functions and rational functions. In this chapter you will study two types of nonalgebraic functions—*exponential functions* and *logarithmic functions*. These functions are examples of **transcendental functions.**

Definition of Exponential Function

The **exponential function *f* with base *a*** is denoted by

$$f(x) = a^x$$

where $a > 0$, $a \neq 1$, and x is any real number.

Note that in the definition of an exponential function, the base $a = 1$ is excluded because it yields

$$f(x) = 1^x = 1. \qquad \text{Constant function}$$

This is a constant function, not an exponential function.

You have already evaluated a^x for integer and rational values of x. For example, you know that

$$4^3 = 64 \quad \text{and} \quad 4^{1/2} = 2.$$

However, to evaluate 4^x for any real number x, you need to interpret forms with *irrational* exponents. For the purposes of this text, it is sufficient to think of

$$a^{\sqrt{2}} \left(\text{where } \sqrt{2} \approx 1.41421356\right)$$

as the number that has the successively closer approximations

$$a^{1.4}, a^{1.41}, a^{1.414}, a^{1.4142}, a^{1.41421}, \ldots.$$

Example 1 shows how to use a calculator to evaluate exponential functions.

What you should learn

- Recognize and evaluate exponential functions with base *a*.
- Graph exponential functions with base *a*.
- Recognize, evaluate, and graph exponential functions with base *e*.
- Use exponential functions to model and solve real-life problems.

Why you should learn it

Exponential functions are useful in modeling data that represent quantities that increase or decrease quickly. For instance, Exercise 74 on page 191 shows how an exponential function is used to model the depreciation of a new vehicle.

Example 1 Evaluating Exponential Functions

Use a calculator to evaluate each function at the indicated value of x.

	Function	*Value*
a.	$f(x) = 2^x$	$x = -3.1$
b.	$f(x) = 2^{-x}$	$x = \pi$
c.	$f(x) = 0.6^x$	$x = \frac{3}{2}$
d.	$f(x) = 1.05^{2x}$	$x = 12$

Solution

	Function Value	*Graphing Calculator Keystrokes*	*Display*
a.	$f(-3.1) = 2^{-3.1}$	2 ^ (–) 3.1 ENTER	0.1166291
b.	$f(\pi) = 2^{-\pi}$	2 ^ (–) π ENTER	0.1133147
c.	$f\left(\frac{3}{2}\right) = (0.6)^{3/2}$	.6 ^ (3 ÷ 2) ENTER	0.4647580
d.	$f(12) = (1.05)^{2(12)}$	1.05 ^ (2 × 12) ENTER	3.2250999

CHECKPOINT Now try Exercise 7.

Technology Tip

When evaluating exponential functions with a calculator, remember to enclose fractional exponents in parentheses. Because the calculator follows the order of operations, parentheses are crucial in order to obtain the correct result.

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Graphs of Exponential Functions

The graphs of all exponential functions have similar characteristics, as shown in Examples 2, 3, and 4.

Example 2 Graphs of $y = a^x$

In the same coordinate plane, sketch the graph of each function by hand.

a. $f(x) = 2^x$

b. $g(x) = 4^x$

Solution

The table below lists some values for each function. By plotting these points and connecting them with smooth curves, you obtain the graphs shown in Figure 3.1. Note that both graphs are increasing. Moreover, the graph of $g(x) = 4^x$ is increasing more rapidly than the graph of $f(x) = 2^x$.

Figure 3.1

x	-2	-1	0	1	2	3
2^x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
4^x	$\frac{1}{16}$	$\frac{1}{4}$	1	4	16	64

CHECKPOINT Now try Exercise 9.

Example 3 Graphs of $y = a^{-x}$

In the same coordinate plane, sketch the graph of each function by hand.

a. $F(x) = 2^{-x}$

b. $G(x) = 4^{-x}$

Solution

The table below lists some values for each function. By plotting these points and connecting them with smooth curves, you obtain the graphs shown in Figure 3.2. Note that both graphs are decreasing. Moreover, the graph of $G(x) = 4^{-x}$ is decreasing more rapidly than the graph of $F(x) = 2^{-x}$.

Figure 3.2

x	-3	-2	-1	0	1	2
2^{-x}	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$
4^{-x}	64	16	4	1	$\frac{1}{4}$	$\frac{1}{16}$

CHECKPOINT Now try Exercise 11.

The properties of exponents can also be applied to real-number exponents. For review, these properties are listed below.

1. $a^x a^y = a^{x+y}$ **2.** $\dfrac{a^x}{a^y} = a^{x-y}$ **3.** $a^{-x} = \dfrac{1}{a^x} = \left(\dfrac{1}{a}\right)^x$ **4.** $a^0 = 1$

5. $(ab)^x = a^x b^x$ **6.** $(a^x)^y = a^{xy}$ **7.** $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$ **8.** $|a^2| = |a|^2 = a^2$

Study Tip

In Example 3, note that the functions $F(x) = 2^{-x}$ and $G(x) = 4^{-x}$ can be rewritten with positive exponents.

$$F(x) = 2^{-x} = \left(\frac{1}{2}\right)^x \quad \text{and}$$

$$G(x) = 4^{-x} = \left(\frac{1}{4}\right)^x$$

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Comparing the functions in Examples 2 and 3, observe that

$$F(x) = 2^{-x} = f(-x) \qquad \text{and} \qquad G(x) = 4^{-x} = g(-x).$$

Consequently, the graph of F is a reflection (in the y-axis) of the graph of f, as shown in Figure 3.3. The graphs of G and g have the same relationship, as shown in Figure 3.4.

Figure 3.3

Figure 3.4

The graphs in Figures 3.3 and 3.4 are typical of the graphs of the exponential functions

$$f(x) = a^x \qquad \text{and} \qquad f(x) = a^{-x}.$$

They have one y-intercept and one horizontal asymptote (the x-axis), and they are continuous. The basic characteristics of these exponential functions are summarized below.

Library of Parent Functions: Exponential Function

The *parent exponential function*

$$f(x) = a^x, \ a > 0, \ a \neq 1$$

is different from all the functions you have studied so far because the variable x is an *exponent*. A distinguishing characteristic of an exponential function is its rapid increase as x increases (for $a > 1$). Many real-life phenomena with patterns of rapid growth (or decline) can be modeled by exponential functions. The basic characteristics of the exponential function are summarized below and on the inside cover of this text.

Graph of $f(x) = a^x, \ a > 1$

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$

Intercept: $(0, 1)$

Increasing on $(-\infty, \infty)$

x-axis is a horizontal asymptote

$(a^x \to 0 \text{ as } x \to -\infty)$

Continuous

Graph of $f(x) = a^{-x}, \ a > 1$

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$

Intercept: $(0, 1)$

Decreasing on $(-\infty, \infty)$

x-axis is a horizontal asymptote

$(a^{-x} \to 0 \text{ as } x \to \infty)$

Continuous

Explore the Concept

Use a graphing utility to graph $y = a^x$ for $a = 3$, 5, and 7 in the same viewing window. (Use a viewing window in which $-2 \le x \le 1$ and $0 \le y \le 2$.) How do the graphs compare with each other? Which graph is on the top in the interval $(-\infty, 0)$? Which is on the bottom? Which graph is on the top in the interval $(0, \infty)$? Which is on the bottom? Repeat this experiment with the graphs of $y = b^x$ for $b = \frac{1}{3}, \frac{1}{5}$, and $\frac{1}{7}$. (Use a viewing window in which $-1 \le x \le 2$ and $0 \le y \le 2$.) What can you conclude about the shape of the graph of $y = b^x$ and the value of b?

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In the following example, the graph of

$$y = a^x$$

is used to graph functions of the form

$$f(x) = b \pm a^{x+c}$$

where b and c are any real numbers.

Example 4 Library of Parent Functions: $f(x) = a^x$

Each of the following graphs is a transformation of the graph of $f(x) = 3^x$.

a. Because $g(x) = 3^{x+1} = f(x + 1)$, the graph of g can be obtained by shifting the graph of f one unit to the *left*, as shown in Figure 3.5.

b. Because $h(x) = 3^x - 2 = f(x) - 2$, the graph of h can be obtained by shifting the graph of f *downward* two units, as shown in Figure 3.6.

c. Because $k(x) = -3^x = -f(x)$, the graph of k can be obtained by *reflecting* the graph of f in the x-axis, as shown in Figure 3.7.

d. Because $j(x) = 3^{-x} = f(-x)$, the graph of j can be obtained by *reflecting* the graph of f in the y-axis, as shown in Figure 3.8.

Figure 3.5

Figure 3.6

Figure 3.7

Figure 3.8

CHECKPOINT Now try Exercise 21.

Notice that the transformations in Figures 3.5, 3.7, and 3.8 keep the x-axis ($y = 0$) as a horizontal asymptote, but the transformation in Figure 3.6 yields a new horizontal asymptote of $y = -2$. Also, be sure to note how the y-intercept is affected by each transformation.

What's Wrong?

You use a graphing utility to graph $f(x) = 3^x$ and $g(x) = 3^{x+2}$, as shown in the figure. You use the graph to conclude that the graph of g can be obtained by shifting the graph of f upward two units. What's wrong?

Explore the Concept

The following table shows some points on the graphs in Figure 3.5. The functions $f(x)$ and $g(x)$ are represented by Y_1 and Y_2, respectively. Explain how you can use the table to describe the transformation.

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The Natural Base e

For many applications, the convenient choice for a base is the irrational number

$$e = 2.718281828\ldots.$$

This number is called the **natural base.** The function

$$f(x) = e^x$$

is called the **natural exponential function** and its graph is shown in Figure 3.9. The graph of the natural exponential function has the same basic characteristics as the graph of the function $f(x) = a^x$ (see page 182). Be sure you see that for the natural exponential function $f(x) = e^x$, e is the constant $2.718281828\ldots$, whereas x is the variable.

Figure 3.9 *The Natural Exponential Function*

In Example 5, you will see that the number e can be approximated by the expression

$$\left(1 + \frac{1}{x}\right)^x \text{ for large values of } x.$$

Explore the Concept

Use your graphing utility to graph the functions

$$y_1 = 2^x$$
$$y_2 = e^x$$
$$y_3 = 3^x$$

in the same viewing window. From the relative positions of these graphs, make a guess as to the value of the real number e. Then try to find a number a such that the graphs of $y_2 = e^x$ and $y_4 = a^x$ are as close to each other as possible.

Example 5 Approximation of the Number e

Evaluate the expression

$$\left(1 + \frac{1}{x}\right)^x$$

for several large values of x to see that the values approach

$$e \approx 2.718281828$$

as x increases without bound.

Graphical Solution

As x increases, the graph of y_1 gets closer and closer to the graph of the line $y_2 = e$.

Figure 3.10

CHECKPOINT Now try Exercise 27.

Numerical Solution

Enter $y_1 = [1 + (1/x)]^x$.

Use the *table* feature (in *ask* mode) to evaluate y_1 for increasing values of x.

Figure 3.11

From Figure 3.11, it seems reasonable to conclude that

$$\left(1 + \frac{1}{x}\right)^x \to e \text{ as } x \to \infty.$$

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Example 6 Evaluating the Natural Exponential Function

Use a calculator to evaluate the function

$$f(x) = e^x$$

at each indicated value of x.

a. $x = -2$
b. $x = 0.25$
c. $x = -0.4$
d. $x = \frac{2}{3}$

Solution

	Function Value	*Graphing Calculator Keystrokes*	*Display*
a.	$f(-2) = e^{-2}$	[e^x] [(−)] 2 [ENTER]	0.1353353
b.	$f(0.25) = e^{0.25}$	[e^x] .25 [ENTER]	1.2840254
c.	$f(-0.4) = e^{-0.4}$	[e^x] [(−)] .4 [ENTER]	0.6703200
d.	$f(\frac{2}{3}) = e^{2/3}$	[e^x] [(] 2 [÷] 3 [)] [ENTER]	1.9477340

CHECKPOINT Now try Exercise 29.

Explore the Concept

Use a graphing utility to graph $y = (1 + x)^{1/x}$. Describe the behavior of the graph near $x = 0$. Is there a y-intercept? How does the behavior of the graph near $x = 0$ relate to the result of Example 5? Use the *table* feature of the graphing utility to create a table that shows values of y for values of x near $x = 0$ to help you describe the behavior of the graph near this point.

Example 7 Graphing Natural Exponential Functions

Sketch the graph of each natural exponential function.

a. $f(x) = 2e^{0.24x}$
b. $g(x) = \frac{1}{2}e^{-0.58x}$

Solution

To sketch these two graphs, you can use a calculator to construct a table of values, as shown below.

x	-3	-2	-1	0	1	2	3
$f(x)$	0.974	1.238	1.573	2.000	2.542	3.232	4.109
$g(x)$	2.849	1.595	0.893	0.500	0.280	0.157	0.088

After constructing the table, plot the points and connect them with smooth curves. Note that the graph in Figure 3.12 is increasing, whereas the graph in Figure 3.13 is decreasing. Use a graphing calculator to verify these graphs.

Figure 3.12

Figure 3.13

CHECKPOINT Now try Exercise 47.

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Applications

One of the most familiar examples of exponential growth is an investment earning *continuously compounded interest*. Suppose a principal P is invested at an annual interest rate r, compounded once a year. If the interest is added to the principal at the end of the year, then the new balance P_1 is

$$P_1 = P + Pr = P(1 + r).$$

This pattern of multiplying the previous principal by $1 + r$ is then repeated each successive year, as shown in the table.

Time in years	Balance after each compounding
0	$P = P$
1	$P_1 = P(1 + r)$
2	$P_2 = P_1(1 + r) = P(1 + r)(1 + r) = P(1 + r)^2$
$\vdots$	$\vdots$
t	$P_t = P(1 + r)^t$

To accommodate more frequent (quarterly, monthly, or daily) compounding of interest, let n be the number of compoundings per year and let t be the number of years. (The product nt represents the total number of times the interest will be compounded.) Then the interest rate per compounding period is r/n, and the account balance after t years is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}. \qquad \text{Amount (balance) with } n \text{ compoundings per year}$$

When the number of compoundings n increases without bound, the process approaches what is called **continuous compounding.** In the formula for n compoundings per year, let $m = n/r$. This produces

$$A = P\left(1 + \frac{r}{n}\right)^{nt} = P\left(1 + \frac{1}{m}\right)^{mrt} = P\left[\left(1 + \frac{1}{m}\right)^{m}\right]^{rt}.$$

As m increases without bound, you know from Example 5 that

$$\left(1 + \frac{1}{m}\right)^{m}$$

approaches e. So, for continuous compounding, it follows that

$$P\left[\left(1 + \frac{1}{m}\right)^{m}\right]^{rt} \Rightarrow P[e]^{rt}$$

and you can write $A = Pe^{rt}$. This result is part of the reason that e is the "natural" choice for a base of an exponential function.

Formulas for Compound Interest

After t years, the balance A in an account with principal P and annual interest rate r (in decimal form) is given by the following formulas.

1. For n compoundings per year: $A = P\left(1 + \frac{r}{n}\right)^{nt}$
2. For continuous compounding: $A = Pe^{rt}$

Explore the Concept

Use the formula

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

to calculate the amount in an account when $P = \$3000$, $r = 6\%$, $t = 10$ years, and the interest is compounded (a) by the day, (b) by the hour, (c) by the minute, and (d) by the second. Does increasing the number of compoundings per year result in unlimited growth of the amount in the account? Explain.

Study Tip

The interest rate r in the formula for compound interest should be written as a decimal. For example, an interest rate of 7% would be written as $r = 0.07$.

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Example 8 Finding the Balance for Compound Interest

A total of \$9000 is invested at an annual interest rate of 2.5%, compounded annually. Find the balance in the account after 5 years.

Algebraic Solution

In this case,

$$P = 9000,\ r = 2.5\% = 0.025,\ n = 1,\ t = 5.$$

Using the formula for compound interest with n compoundings per year, you have

$$A = P\left(1 + \frac{r}{n}\right)^{nt} \qquad \text{Formula for compound interest}$$

$$= 9000\left(1 + \frac{0.025}{1}\right)^{1(5)} \qquad \text{Substitute for } P, r, n, \text{ and } t.$$

$$= 9000(1.025)^5 \qquad \text{Simplify.}$$

$$\approx \$10{,}182.67. \qquad \text{Use a calculator.}$$

So, the balance in the account after 5 years will be about \$10,182.67.

CHECKPOINT Now try Exercise 57.

Graphical Solution

Substitute the values for P, r, and n into the formula for compound interest with n compoundings per year and simplify to obtain

$$A = 9000(1.025)^t.$$

Use a graphing utility to graph $A = 9000(1.025)^t$. Then use the *value* feature to approximate the value of A when $t = 5$, as shown in Figure 3.14.

Figure 3.14

Example 9 Finding Compound Interest

A total of \$12,000 is invested at an annual interest rate of 3%. Find the balance after 4 years for each type of compounding.

a. Quarterly

b. Continuous

Solution

a. For quarterly compoundings, $n = 4$. So, after 4 years at 3%, the balance is

$$A = P\left(1 + \frac{r}{n}\right)^{nt} \qquad \text{Formula for compound interest}$$

$$= 12{,}000\left(1 + \frac{0.03}{4}\right)^{4(4)} \qquad \text{Substitute for } P, r, n, \text{ and } t.$$

$$\approx \$13{,}523.91. \qquad \text{Use a calculator.}$$

b. For continuous compounding, the balance is

$$A = Pe^{rt} \qquad \text{Formula for continuous compounding}$$

$$= 12{,}000e^{0.03(4)} \qquad \text{Substitute for } P, r, \text{ and } t.$$

$$\approx \$13{,}529.96. \qquad \text{Use a calculator.}$$

Note that a continuous-compounding account yields more than a quarterly-compounding account.

CHECKPOINT Now try Exercise 59.

Financial Analyst

Example 9 illustrates the following general rule. For a given principal, interest rate, and time, the more often the interest is compounded per year, the greater the balance will be. Moreover, the balance obtained by continuous compounding is greater than the balance obtained by compounding n times per year.

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Example 10 Radioactive Decay

Let y represent a mass, in grams, of radioactive strontium (^{90}Sr), whose half-life is 29 years. The quantity of strontium present after t years is

$$y = 10\left(\tfrac{1}{2}\right)^{t/29}.$$

a. What is the initial mass (when $t = 0$)?

b. How much of the initial mass is present after 80 years?

Algebraic Solution

a. $y = 10\left(\frac{1}{2}\right)^{t/29}$ Write original equation.

$= 10\left(\frac{1}{2}\right)^{0/29}$ Substitute 0 for t.

$= 10$ Simplify.

So, the initial mass is 10 grams.

b. $y = 10\left(\frac{1}{2}\right)^{t/29}$ Write original equation.

$= 10\left(\frac{1}{2}\right)^{80/29}$ Substitute 80 for t.

$\approx 10\left(\frac{1}{2}\right)^{2.759}$ Simplify.

≈ 1.48 Use a calculator.

So, about 1.48 grams are present after 80 years.

CHECKPOINT Now try Exercise 71.

Graphical Solution

a. When $t = 0$, $y = 10$. So, the initial mass is 10 grams.

b. When $t = 80$, $y \approx 1.48$. So, about 1.48 grams are present after 80 years.

Example 11 Population Growth

The approximate number of fruit flies in an experimental population after t hours is given by

$$Q(t) = 20e^{0.03t}$$

where $t \geq 0$.

a. Find the initial number of fruit flies in the population.

b. How large is the population of fruit flies after 72 hours?

c. Graph Q.

Solution

a. To find the initial population, evaluate $Q(t)$ when $t = 0$.

$$Q(0) = 20e^{0.03(0)} = 20e^0 = 20(1) = 20 \text{ flies}$$

b. After 72 hours, the population size is

$$Q(72) = 20e^{0.03(72)} = 20e^{2.16} \approx 173 \text{ flies.}$$

c. The graph of Q is shown in Figure 3.15.

Figure 3.15

CHECKPOINT Now try Exercise 73.

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3.1 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1 and 2, fill in the blank(s).

1. Exponential and logarithmic functions are examples of nonalgebraic functions, also called _____ functions.
2. The exponential function $f(x) = e^x$ is called the _____ function, and the base e is called the _____ base.
3. What type of transformation of the graph of $f(x) = 5^x$ is the graph of $f(x + 1)$?
4. The formula $A = Pe^{rt}$ gives the balance A of an account earning what type of interest?

Procedures and Problem Solving

Evaluating Exponential Functions **In Exercises 5–8, use a calculator to evaluate the function at the indicated value of x. Round your result to three decimal places.**

	Function	*Value*
5.	$f(x) = 3.4^x$	$x = 6.8$
6.	$f(x) = 1.2^x$	$x = \frac{1}{3}$
✓ 7.	$g(x) = 5^x$	$x = -\pi$
8.	$h(x) = 8.6^{-3x}$	$x = -\sqrt{2}$

Graphs of $y = a^x$ and $y = a^{-x}$ **In Exercises 9–16, graph the exponential function by hand. Identify any asymptotes and intercepts and determine whether the graph of the function is increasing or decreasing.**

✓ 9. $g(x) = 5^x$
✓ 11. $f(x) = 5^{-x}$
13. $h(x) = 3^x$
15. $g(x) = 3^{-x}$
10. $f(x) = \left(\frac{3}{2}\right)^x$
12. $h(x) = \left(\frac{3}{2}\right)^{-x}$
14. $g(x) = 10^x$
16. $f(x) = 10^{-x}$

Library of Parent Functions **In Exercises 17–20, use the graph of $y = 2^x$ to match the function with its graph. [The graphs are labeled (a), (b), (c), and (d).]**

(a)

(b)

(c)

(d)

17. $f(x) = 2^{x-2}$
19. $f(x) = 2^x - 4$
18. $f(x) = 2^{-x}$
20. $f(x) = 2^x + 1$

Library of Parent Functions **In Exercises 21–26, use the graph of f to describe the transformation that yields the graph of g. Then sketch the graphs of f and g by hand.**

✓ 21. $f(x) = 3^x,\ g(x) = 3^{x-5}$
22. $f(x) = -2^x,\ g(x) = 5 - 2^x$
23. $f(x) = \left(\frac{3}{5}\right)^x,\ g(x) = -\left(\frac{3}{5}\right)^{x+4}$
24. $f(x) = 0.3^x,\ g(x) = -0.3^x + 5$
25. $f(x) = 4^x,\ g(x) = 4^{x-2} - 3$
26. $f(x) = \left(\frac{1}{2}\right)^x,\ g(x) = \left(\frac{1}{2}\right)^{-(x+4)}$

Approximation of a Power with Base e **In Exercises 27 and 28, show that the value of $f(x)$ approaches the value of $g(x)$ as x increases without bound (a) graphically and (b) numerically.**

✓ 27. $f(x) = [1 + (2/x)]^x,\quad g(x) = e^2$
28. $f(x) = [1 + (3/x)]^x,\quad g(x) = e^3$

Evaluating the Natural Exponential Function **In Exercises 29–32, use a calculator to evaluate the function at the indicated value of x. Round your result to the nearest thousandth.**

	Function	*Value*
✓ 29.	$f(x) = e^x$	$x = 9.2$
30.	$f(x) = e^{-x}$	$x = -\frac{3}{4}$
31.	$g(x) = 50e^{4x}$	$x = 0.02$
32.	$h(x) = -5.5e^{-x}$	$x = 200$

Graphing an Exponential Function **In Exercises 33–48, use a graphing utility to construct a table of values for the function. Then sketch the graph of the function. Identify any asymptotes of the graph.**

33. $f(x) = \left(\frac{5}{2}\right)^x$
35. $f(x) = 6^x$
34. $f(x) = \left(\frac{5}{2}\right)^{-x}$
36. $f(x) = 2^{x-1}$

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37. $f(x) = 3^{x+2}$

38. $y = 2^{-x^2}$

39. $y = 3^{x-2} + 1$

40. $y = 4^{x+1} - 2$

41. $f(x) = e^{-x}$

42. $s(t) = 3e^{-0.2t}$

43. $f(x) = 3e^{x+4}$

44. $f(x) = 2e^{-0.5x}$

45. $f(x) = 2 + e^{x-5}$

46. $g(x) = 2 - e^{-x}$

47. $s(t) = 2e^{0.12t}$

48. $g(x) = 1 + e^{-x}$

Finding Asymptotes **In Exercises 49–52, use a graphing utility to (a) graph the function and (b) find any asymptotes numerically by creating a table of values for the function.**

49. $f(x) = \dfrac{8}{1 + e^{-0.5x}}$

50. $g(x) = \dfrac{8}{1 + e^{-0.5/x}}$

51. $f(x) = -\dfrac{6}{2 - e^{0.2x}}$

52. $f(x) = \dfrac{6}{2 - e^{0.2/x}}$

Finding Points of Intersection **In Exercises 53 and 54, use a graphing utility to find the point(s) of intersection, if any, of the graphs of the functions. Round your result to three decimal places.**

53. $y = 20e^{0.05x}$
$y = 1500$

54. $y = 100e^{0.01x}$
$y = 12{,}500$

Approximating Relative Extrema **In Exercises 55 and 56, (a) use a graphing utility to graph the function, (b) use the graph to find the open intervals on which the function is increasing and decreasing, and (c) approximate any relative maximum or minimum values.**

55. $f(x) = x^2e^{-x}$

56. $f(x) = 2x^2e^{x+1}$

Finding the Balance for Compound Interest **In Exercises 57–60, complete the table to determine the balance A for \$2500 invested at rate r for t years and compounded n times per year.**

n	1	2	4	12	365	Continuous
A						

57. $r = 2\%$, $t = 10$ years

58. $r = 6\%$, $t = 10$ years

59. $r = 4\%$, $t = 20$ years

60. $r = 3\%$, $t = 40$ years

Finding the Balance for Compound Interest **In Exercises 61–64, complete the table to determine the balance A for \$12,000 invested at rate r for t years, compounded continuously.**

t	1	10	20	30	40	50
A						

61. $r = 4\%$

62. $r = 6\%$

63. $r = 3.5\%$

64. $r = 2.5\%$

Finding the Amount of an Annuity **In Exercises 65–68, you build an annuity by investing P dollars every month at interest rate r, compounded monthly. Find the amount A accrued after n months using the formula**

$$A = P\left[\frac{(1 + r/12)^n - 1}{r/12}\right]$$

where r is in decimal form.

65. $P = \$25$, $r = 0.12$, $n = 48$ months

66. $P = \$100$, $r = 0.09$, $n = 60$ months

67. $P = \$200$, $r = 0.06$, $n = 72$ months

68. $P = \$75$, $r = 0.03$, $n = 24$ months

69. MODELING DATA

There are three options for investing \$500. The first earns 7% compounded annually, the second earns 7% compounded quarterly, and the third earns 7% compounded continuously.

(a) Find equations that model the growth of each investment and use a graphing utility to graph each model in the same viewing window over a 20-year period.

(b) Use the graph from part (a) to determine which investment yields the highest return after 20 years. What are the differences in earnings among the three investments?

70. Radioactive Decay Let Q represent a mass, in grams, of radioactive radium (^{226}Ra), whose half-life is 1599 years. The quantity of radium present after t years is given by

$$Q = 25\left(\tfrac{1}{2}\right)^{t/1599}.$$

(a) Determine the initial quantity (when $t = 0$).

(b) Determine the quantity present after 1000 years.

(c) Use a graphing utility to graph the function over the interval $t = 0$ to $t = 5000$.

(d) When will the quantity of radium be 0 grams? Explain.

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✓ **71. Radioactive Decay** Let Q represent a mass, in grams, of carbon 14 (^{14}C), whose half-life is 5715 years. The quantity present after t years is given by $Q = 10\left(\frac{1}{2}\right)^{t/5715}$.

(a) Determine the initial quantity (when $t = 0$).

(b) Determine the quantity present after 2000 years.

(c) Sketch the graph of the function over the interval $t = 0$ to $t = 10{,}000$.

72. Algebraic-Graphical-Numerical Suppose the annual rate of inflation is 4% for the next 10 years. The approximate cost C of goods or services during these years is $C(t) = P(1.04)^t$, where t is the time (in years) and P is the present cost. An oil change for your car presently costs \$23.95. Use the following methods to approximate the cost 10 years from now.

(a) Use a graphing utility to graph the function and then use the *value* feature.

(b) Use the *table* feature of the graphing utility to find a numerical approximation.

(c) Use a calculator to evaluate the cost function algebraically.

✓ **73. Population Growth** The projected populations of California for the years 2015 through 2030 can be modeled by $P = 34.706e^{0.0097t}$, where P is the population (in millions) and t is the time (in years), with $t = 15$ corresponding to 2015. (Source: U.S. Census Bureau)

(a) Use a graphing utility to graph the function for the years 2015 through 2030.

(b) Use the *table* feature of the graphing utility to create a table of values for the same time period as in part (a).

(c) According to the model, in what year will the population of California exceed 50 million?

74. ***Why you should learn it*** *(p. 180)* In early 2010, a new sedan had a manufacturer's suggested retail price of \$31,915. After t years, the sedan's value is given by

$$V(t) = 31{,}915\left(\tfrac{4}{5}\right)^t.$$

(a) Use a graphing utility to graph the function.

(b) Use the graphing utility to create a table of values that shows the value V for $t = 1$ to $t = 10$ years.

(c) According to the model, when will the sedan have no value?

Conclusions

True or False? In Exercises 75 and 76, determine whether the statement is true or false. Justify your answer.

75. $f(x) = 1^x$ is not an exponential function.

76. $e = \dfrac{271{,}801}{99{,}990}$

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77. Library of Parent Functions Determine which equation(s) may be represented by the graph shown. (There may be more than one correct answer.)

(a) $y = e^x + 1$

(b) $y = -e^{-x} + 1$

(c) $y = e^{-x} - 1$

(d) $y = e^{-x} + 1$

78. Exploration Use a graphing utility to graph $y_1 = e^x$ and each of the functions $y_2 = x^2$, $y_3 = x^3$, $y_4 = \sqrt{x}$, and $y_5 = |x|$ in the same viewing window.

(a) Which function increases at the fastest rate for "large" values of x?

(b) Use the result of part (a) to make a conjecture about the rates of growth of $y_1 = e^x$ and $y = x^n$, where n is a natural number and x is "large."

(c) Use the results of parts (a) and (b) to describe what is implied when it is stated that a quantity is growing exponentially.

79. Think About It Graph $y = 3^x$ and $y = 4^x$. Use the graph to solve the inequality $3^x < 4^x$.

80. CAPSTONE The figure shows the graphs of $y = 2^x$, $y = e^x$, $y = 10^x$, $y = 2^{-x}$, $y = e^{-x}$, and $y = 10^{-x}$. Match each function with its graph. [The graphs are labeled (a) through (f).] Explain your reasoning.

Think About It In Exercises 81–84, place the correct symbol (< or >) between the two of numbers.

81. e^{π} ☐ π^e

82. 2^{10} ☐ 10^2

83. 5^{-3} ☐ 3^{-5}

84. $4^{1/2}$ ☐ $\left(\frac{1}{2}\right)^4$

Cumulative Mixed Review

Inverse Functions In Exercises 85–88, determine whether the function has an inverse function. If it does, find f^{-1}.

85. $f(x) = 5x - 7$

86. $f(x) = -\frac{2}{3}x + \frac{5}{2}$

87. $f(x) = \sqrt[3]{x + 8}$

88. $f(x) = \sqrt{x^2 + 6}$

89. ***Make a Decision*** To work an extended application analyzing the population per square mile in the United States, visit this textbook's *Companion Website*. (Data Source: U.S. Census Bureau)

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3.2 Logarithmic Functions and Their Graphs

Logarithmic Functions

In Section 1.6, you studied the concept of an inverse function. There, you learned that when a function is one-to-one—that is, when the function has the property that no horizontal line intersects its graph more than once—the function must have an inverse function. By looking back at the graphs of the exponential functions introduced in Section 3.1, you will see that every function of the form

$$f(x) = a^x, \quad a > 0, a \neq 1$$

passes the Horizontal Line Test and therefore must have an inverse function. This inverse function is called the **logarithmic function with base *a*.**

Definition of Logarithmic Function

For $x > 0$, $a > 0$, and $a \neq 1$,

$$y = \log_a x \quad \text{if and only if} \quad x = a^y.$$

The function given by

$$f(x) = \log_a x \qquad \text{Read as "log base } a \text{ of } x\text{."}$$

is called the **logarithmic function with base *a*.**

From the definition above, you can see that every logarithmic equation can be written in an equivalent exponential form and every exponential equation can be written in logarithmic form. The equations

$$y = \log_a x \quad \text{and} \quad x = a^y$$

are equivalent.

When evaluating logarithms, remember that *a logarithm is an exponent*. This means that $\log_a x$ is the exponent to which a must be raised to obtain x. For instance, $\log_2 8 = 3$ because 2 must be raised to the third power to get 8.

Example 1 Evaluating Logarithms

Use the definition of logarithmic function to evaluate each logarithm at the indicated value of x.

	Function	*Value*
a.	$f(x) = \log_2 x$	$x = 32$
b.	$f(x) = \log_3 x$	$x = 1$
c.	$f(x) = \log_4 x$	$x = 2$
d.	$f(x) = \log_{10} x$	$x = \frac{1}{100}$

Solution

a. $f(32) = \log_2 32 = 5$ because $2^5 = 32$.

b. $f(1) = \log_3 1 = 0$ because $3^0 = 1$.

c. $f(2) = \log_4 2 = \frac{1}{2}$ because $4^{1/2} = \sqrt{4} = 2$.

d. $f\left(\frac{1}{100}\right) = \log_{10} \frac{1}{100} = -2$ because $10^{-2} = \frac{1}{10^2} = \frac{1}{100}$.

CHECKPOINT Now try Exercise 23.

What you should learn

- Recognize and evaluate logarithmic functions with base *a*.
- Graph logarithmic functions with base *a*.
- Recognize, evaluate, and graph natural logarithmic functions.
- Use logarithmic functions to model and solve real-life problems.

Why you should learn it

Logarithmic functions are useful in modeling data that represent quantities that increase or decrease slowly. For instance, Exercise 114 on page 201 shows how to use a logarithmic function to model the minimum required ventilation rates in public school classrooms.

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Study Tip

In this text, the parentheses in $\log_a(u)$ are sometimes omitted when u is an expression involving exponents, radicals, products, or quotients. For instance, $\log_{10}(2x)$ can be written as $\log_{10} 2x$. To evaluate $\log_{10} 2x$, find the logarithm of the product $2x$.

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The logarithmic function with base 10 is called the **common logarithmic function.** On most calculators, this function is denoted by [LOG]. Example 2 shows how to use a calculator to evaluate common logarithmic functions. You will learn how to use a calculator to calculate logarithms to any base in the next section.

Example 2 Evaluating Common Logarithms on a Calculator

Use a calculator to evaluate the function

$$f(x) = \log_{10} x$$

at each value of x.

a. $x = 10$ **b.** $x = 2.5$

c. $x = -2$ **d.** $x = \frac{1}{4}$

Solution

	Function Value	*Graphing Calculator Keystrokes*	*Display*
a.	$f(10) = \log_{10} 10$	[LOG] 10 [ENTER]	1
b.	$f(2.5) = \log_{10} 2.5$	[LOG] 2.5 [ENTER]	0.3979400
c.	$f(-2) = \log_{10}(-2)$	[LOG] [(−)] 2 [ENTER]	ERROR
d.	$f(\frac{1}{4}) = \log_{10} \frac{1}{4}$	[LOG] [(] 1 [÷] 4 [)] [ENTER]	−0.6020600

Note that the calculator displays an error message when you try to evaluate $\log_{10}(-2)$. In this case, there is no *real* power to which 10 can be raised to obtain -2.

CHECKPOINT Now try Exercise 27.

Technology Tip

Some graphing utilities do not give an error message for $\log_{10}(-2)$. Instead, the graphing utility will display a complex number. For the purpose of this text, however, it will be said that the domain of a logarithmic function is the set of positive *real* numbers.

The following properties follow directly from the definition of the logarithmic function with base a.

Properties of Logarithms

1. $\log_a 1 = 0$ because $a^0 = 1$.
2. $\log_a a = 1$ because $a^1 = a$.
3. $\log_a a^x = x$ and $a^{\log_a x} = x$. Inverse Properties
4. If $\log_a x = \log_a y$, then $x = y$. One-to-One Property

Example 3 Using Properties of Logarithms

a. Solve for x: $\log_2 x = \log_2 3$

b. Solve for x: $\log_4 4 = x$

c. Simplify: $\log_5 5^x$

d. Simplify: $7^{\log_7 14}$

Solution

a. Using the One-to-One Property (Property 4), you can conclude that $x = 3$.

b. Using Property 2, you can conclude that $x = 1$.

c. Using the Inverse Property (Property 3), it follows that $\log_5 5^x = x$.

d. Using the Inverse Property (Property 3), it follows that $7^{\log_7 14} = 14$.

CHECKPOINT Now try Exercise 31.

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Graphs of Logarithmic Functions

To sketch the graph of

$$y = \log_a x$$

you can use the fact that the graphs of inverse functions are reflections of each other in the line $y = x$.

Example 4 Graphs of Exponential and Logarithmic Functions

In the same coordinate plane, sketch the graph of each function by hand.

a. $f(x) = 2^x$

b. $g(x) = \log_2 x$

Solution

a. For $f(x) = 2^x$, construct a table of values. By plotting these points and connecting them with a smooth curve, you obtain the graph of f shown in Figure 3.16.

x	-2	-1	0	1	2	3
$f(x) = 2^x$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

b. Because $g(x) = \log_2 x$ is the inverse function of $f(x) = 2^x$, the graph of g is obtained by plotting the points $(f(x), x)$ and connecting them with a smooth curve. The graph of g is a reflection of the graph of f in the line $y = x$, as shown in Figure 3.16.

Figure 3.16

CHECKPOINT Now try Exercise 41.

Before you can confirm the result of Example 4 using a graphing utility, you need to know how to enter $\log_2 x$. You will learn how to do this using the *change-of-base formula* discussed in Section 3.3.

Example 5 Sketching the Graph of a Logarithmic Function

Sketch the graph of the common logarithmic function $f(x) = \log_{10} x$ by hand.

Solution

Begin by constructing a table of values. Note that some of the values can be obtained without a calculator by using the Inverse Property of Logarithms. Others require a calculator. Next, plot the points and connect them with a smooth curve, as shown in Figure 3.17.

	Without calculator				With calculator		
x	$\frac{1}{100}$	$\frac{1}{10}$	1	10	2	5	8
$f(x) = \log_{10} x$	-2	-1	0	1	0.301	0.699	0.903

Figure 3.17

CHECKPOINT Now try Exercise 45.

The nature of the graph in Figure 3.17 is typical of functions of the form $f(x) = \log_a x$, $a > 1$. They have one x-intercept and one vertical asymptote. Notice how slowly the graph rises for $x > 1$.

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Library of Parent Functions: Logarithmic Function

The *parent logarithmic function*

$$f(x) = \log_a x, \quad a > 0, \ a \neq 1$$

is the inverse function of the exponential function. Its domain is the set of positive real numbers and its range is the set of all real numbers. This is the opposite of the exponential function. Moreover, the logarithmic function has the y-axis as a vertical asymptote, whereas the exponential function has the x-axis as a horizontal asymptote. Many real-life phenomena with slow rates of growth can be modeled by logarithmic functions. The basic characteristics of the logarithmic function are summarized below and on the inside cover of this text.

Graph of $f(x) = \log_a x, \ a > 1$

Domain: $(0, \infty)$

Range: $(-\infty, \infty)$

Intercept: $(1, 0)$

Increasing on $(0, \infty)$

y-axis is a vertical asymptote ($\log_a x \to -\infty$ as $x \to 0^+$)

Continuous

Reflection of graph of $f(x) = a^x$ in the line $y = x$

Explore the Concept

Use a graphing utility to graph $y = \log_{10} x$ and $y = 8$ in the same viewing window. Find a viewing window that shows the point of intersection. What is the point of intersection? Use the point of intersection to complete the equation $\log_{10} \square = 8$.

Example 6 Library of Parent Functions $f(x) = \log_a x$

Each of the following functions is a transformation of the graph of

$$f(x) = \log_{10} x.$$

a. Because $g(x) = \log_{10}(x - 1) = f(x - 1)$, the graph of g can be obtained by shifting the graph of f one unit to the *right*, as shown in Figure 3.18.

b. Because $h(x) = 2 + \log_{10} x = 2 + f(x)$, the graph of h can be obtained by shifting the graph of f two units *upward*, as shown in Figure 3.19.

Figure 3.18

Figure 3.19

Notice that the transformation in Figure 3.19 keeps the y-axis as a vertical asymptote, but the transformation in Figure 3.18 yields the new vertical asymptote $x = 1$.

CHECKPOINT Now try Exercise 55.

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The Natural Logarithmic Function

By looking back at the graph of the natural exponential function introduced in Section 3.1, you will see that $f(x) = e^x$ is one-to-one and so has an inverse function. This inverse function is called the **natural logarithmic function** and is denoted by the special symbol $\ln x$, read as "the natural log of x" or "el en of x."

The Natural Logarithmic Function

For $x > 0$,

$$y = \ln x \quad \text{if and only if} \quad x = e^y.$$

The function given by

$$f(x) = \log_e x = \ln x$$

is called the **natural logarithmic function.**

The equations $y = \ln x$ and $x = e^y$ are equivalent. Note that the natural logarithm $\ln x$ is written without a base. The base is understood to be e.

Because the functions

$$f(x) = e^x \quad \text{and} \quad g(x) = \ln x$$

are inverse functions of each other, their graphs are reflections of each other in the line $y = x$. This reflective property is illustrated in Figure 3.20.

Reflection of graph of $f(x) = e^x$ in the line $y = x$
Figure 3.20

Example 7 Evaluating the Natural Logarithmic Function

Use a calculator to evaluate the function

$$f(x) = \ln x$$

at each indicated value of x.

a. $x = 2$
b. $x = 0.3$
c. $x = -1$

Solution

	Function Value	*Graphing Calculator Keystrokes*	*Display*
a.	$f(2) = \ln 2$	LN 2 ENTER	0.6931472
b.	$f(0.3) = \ln 0.3$	LN .3 ENTER	−1.2039728
c.	$f(-1) = \ln(-1)$	LN (−) 1 ENTER	ERROR

CHECKPOINT Now try Exercise 77.

The four properties of logarithms listed on page 193 are also valid for natural logarithms.

Properties of Natural Logarithms

1. $\ln 1 = 0$ because $e^0 = 1$.
2. $\ln e = 1$ because $e^1 = e$.
3. $\ln e^x = x$ and $e^{\ln x} = x$. Inverse Properties
4. If $\ln x = \ln y$, then $x = y$. One-to-One Property

Technology Tip

On most calculators, the natural logarithm is denoted by LN, as illustrated in Example 7.

Study Tip

In Example 7(c), be sure you see that $\ln(-1)$ gives an error message on most calculators. This occurs because the domain of $\ln x$ is the set of *positive* real numbers (see Figure 3.20). So, $\ln(-1)$ is undefined.

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Example 8 Using Properties of Natural Logarithms

Use the properties of natural logarithms to rewrite each expression.

a. $\ln \frac{1}{e}$ **b.** $e^{\ln 5}$ **c.** $4 \ln 1$ **d.** $2 \ln e$

Solution

a. $\ln \frac{1}{e} = \ln e^{-1} = -1$ Inverse Property **b.** $e^{\ln 5} = 5$ Inverse Property

c. $4 \ln 1 = 4(0) = 0$ Property 1 **d.** $2 \ln e = 2(1) = 2$ Property 2

CHECKPOINT Now try Exercise 81.

Example 9 Finding the Domains of Logarithmic Functions

Find the domain of each function.

a. $f(x) = \ln(x - 2)$ **b.** $g(x) = \ln(2 - x)$ **c.** $h(x) = \ln x^2$

Algebraic Solution

a. Because $\ln(x - 2)$ is defined only when

$$x - 2 > 0$$

it follows that the domain of f is $(2, \infty)$.

b. Because $\ln(2 - x)$ is defined only when

$$2 - x > 0$$

it follows that the domain of g is $(-\infty, 2)$.

c. Because $\ln x^2$ is defined only when

$$x^2 > 0$$

it follows that the domain of h is all real numbers except $x = 0$.

CHECKPOINT Now try Exercise 89.

Graphical Solution

a.

The x-coordinates of points on the graph appear to extend from the right of 2 to ∞. So, you can estimate the domain to be $(2, \infty)$.

b.

The x-coordinates of points on the graph appear to extend from $-\infty$ to the left of 2. So, you can estimate the domain to be $(-\infty, 2)$.

c.

The x-coordinates of points on the graph appear to include all real numbers except 0. So, you can estimate the domain to be all real numbers except $x = 0$.

In Example 9, suppose you had been asked to analyze the function $h(x) = \ln|x - 2|$. How would the domain of this function compare with the domains of the functions given in parts (a) and (b) of the example?

Technology Tip

When a graphing utility graphs a logarithmic function, it may appear that the graph has an endpoint. This is because some graphing utilities have a limited resolution. So, in this text, a blue curve is placed behind the graphing utility's display to indicate where the graph should appear.

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Application

Logarithmic functions are used to model many situations in real life, as shown in the next example.

Psychologist

Example 10 Psychology

Students participating in a psychology experiment attended several lectures on a subject and were given an exam. Every month for a year after the exam, the students were retested to see how much of the material they remembered. The average scores for the group are given by the *human memory model*

$$f(t) = 75 - 6\ln(t + 1), \quad 0 \le t \le 12$$

where t is the time in months. The graph of f is shown in Figure 3.21.

a. What was the average score on the original exam ($t = 0$)?

b. What was the average score at the end of $t = 2$ months?

c. What was the average score at the end of $t = 6$ months?

Figure 3.21

Algebraic Solution

a. The original average score was

$$\begin{aligned} f(0) &= 75 - 6\ln(0 + 1) \\ &= 75 - 6\ln 1 \\ &= 75 - 6(0) \\ &= 75. \end{aligned}$$

b. After 2 months, the average score was

$$\begin{aligned} f(2) &= 75 - 6\ln(2 + 1) \\ &= 75 - 6\ln 3 \\ &\approx 75 - 6(1.0986) \\ &\approx 68.41. \end{aligned}$$

c. After 6 months, the average score was

$$\begin{aligned} f(6) &= 75 - 6\ln(6 + 1) \\ &= 75 - 6\ln 7 \\ &\approx 75 - 6(1.9459) \\ &\approx 63.32. \end{aligned}$$

CHECKPOINT Now try Exercise 109.

Graphical Solution

a. When $t = 0$, $f(0) = 75$. So, the original average score was 75.

b. When $t = 2$, $f(2) \approx 68.41$. So, the average score after 2 months was about 68.41.

c. When $t = 6$, $f(6) \approx 63.32$. So, the average score after 6 months was about 63.32.

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3.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1–4, fill in the blank(s).

1. The inverse function of the exponential function $f(x) = a^x$ is called the _____ with base a.
2. The base of the _____ logarithmic function is 10, and the base of the _____ logarithmic function is e.
3. The inverse properties of logarithms are $\log_a a^x = x$ and _____ .
4. If $x = e^y$, then $y =$ _____.
5. What exponential equation is equivalent to the logarithmic equation $\log_a b = c$?
6. For what value(s) of x is $\ln x = \ln 7$?

Procedures and Problem Solving

Rewriting Logarithmic Equations **In Exercises 7–14, write the logarithmic equation in exponential form. For example, the exponential form of $\log_5 25 = 2$ is $5^2 = 25$.**

7. $\log_4 64 = 3$
8. $\log_3 81 = 4$
9. $\log_7 \frac{1}{49} = -2$
10. $\log_{10} \frac{1}{1000} = -3$
11. $\log_{32} 4 = \frac{2}{5}$
12. $\log_{16} 8 = \frac{3}{4}$
13. $\log_2 \sqrt{2} = \frac{1}{2}$
14. $\log_5 \sqrt[3]{25} = \frac{2}{3}$

Rewriting Exponential Equations **In Exercises 15–22, write the exponential equation in logarithmic form. For example, the logarithmic form of $2^3 = 8$ is $\log_2 8 = 3$.**

15. $5^3 = 125$
16. $8^2 = 64$
17. $81^{1/4} = 3$
18. $9^{3/2} = 27$
19. $6^{-2} = \frac{1}{36}$
20. $10^{-3} = 0.001$
21. $g^a = 4$
22. $n^t = 10$

Evaluating Logarithms **In Exercises 23–26, use the definition of logarithmic function to evaluate the function at the indicated value of x without using a calculator.**

	Function	*Value*
23.	$f(x) = \log_2 x$	$x = 16$
24.	$f(x) = \log_{16} x$	$x = \frac{1}{4}$
25.	$g(x) = \log_{10} x$	$x = \frac{1}{1000}$
26.	$g(x) = \log_{10} x$	$x = 10{,}000$

Evaluating Common Logarithms on a Calculator **In Exercises 27–30, use a calculator to evaluate the function at the indicated value of x. Round your result to three decimal places.**

	Function	*Value*
27.	$f(x) = \log_{10} x$	$x = 345$
28.	$f(x) = \log_{10} x$	$x = \frac{4}{5}$
29.	$h(x) = 6 \log_{10} x$	$x = 14.8$
30.	$h(x) = 1.9 \log_{10} x$	$x = 4.3$

Using Properties of Logarithms **In Exercises 31–36, solve the equation for x.**

31. $\log_7 x = \log_7 9$
32. $\log_5 5 = x$
33. $\log_4 4^2 = x$
34. $\log_3 3^{-5} = x$
35. $\log_8 x = \log_8 10^{-1}$
36. $\log_4 4^3 = x$

Using Properties of Logarithms **In Exercises 37–40, use the properties of logarithms to simplify the expression.**

37. $\log_4 4^{3x}$
38. $6^{\log_6 36}$
39. $3 \log_2 \frac{1}{2}$
40. $\frac{1}{4} \log_4 16$

Graphs of Exponential and Logarithmic Functions **In Exercises 41–44, sketch the graph of f. Then use the graph of f to sketch the graph of g.**

41. $f(x) = 3^x$, $g(x) = \log_3 x$
42. $f(x) = 5^x$, $g(x) = \log_5 x$
43. $f(x) = 15^x$, $g(x) = \log_{15} x$
44. $f(x) = 4^x$, $g(x) = \log_4 x$

Sketching the Graph of a Logarithmic Function **In Exercises 45–50, find the domain, vertical asymptote, and x-intercept of the logarithmic function, and sketch its graph by hand.**

45. $y = \log_2(x + 2)$
46. $y = \log_2(x - 1)$
47. $y = 1 + \log_2 x$
48. $y = 2 - \log_2 x$
49. $y = 1 + \log_2(x - 2)$
50. $y = 2 + \log_2(x + 1)$

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Library of Parent Functions **In Exercises 51–54, use the graph of $y = \log_3 x$ to match the function with its graph. [The graphs are labeled (a), (b), (c), and (d).]**

(a)

(b)

(c)

(d)

51. $f(x) = \log_3 x + 2$

52. $f(x) = -\log_3 x$

53. $f(x) = -\log_3(x + 2)$

54. $f(x) = \log_3(1 - x)$

Library of Parent Functions **In Exercises 55–60, describe the transformation of the graph of f that yields the graph of g.**

55. $f(x) = \log_{10} x, \quad g(x) = -\log_{10} x$

56. $f(x) = \log_{10} x, \quad g(x) = \log_{10}(x + 7)$

57. $f(x) = \log_2 x, \quad g(x) = 4 - \log_2 x$

58. $f(x) = \log_2 x, \quad g(x) = 3 + \log_2 x$

59. $f(x) = \log_8 x, \quad g(x) = -2 + \log_8(x + 3)$

60. $f(x) = \log_8 x, \quad g(x) = 4 + \log_8(x - 1)$

Rewriting Logarithmic Equations **In Exercises 61–68, write the logarithmic equation in exponential form. For example, the exponential form of $\ln 5 = 1.6094\ldots$ is $e^{1.6094\ldots} = 5$.**

61. $\ln 1 = 0$

62. $\ln 4 = 1.3862\ldots$

63. $\ln e = 1$

64. $\ln e^3 = 3$

65. $\ln \sqrt{e} = \dfrac{1}{2}$

66. $\ln \dfrac{1}{e^2} = -2$

67. $\ln 9 = 2.1972\ldots$

68. $\ln \sqrt[3]{e} = \frac{1}{3}$

Rewriting Exponential Equations **In Exercises 69–76, write the exponential equation in logarithmic form. For example, the logarithmic form of $e^2 = 7.3890\ldots$ is $\ln 7.3890\ldots = 2$.**

69. $e^3 = 20.0855\ldots$

70. $e^4 = 54.5981\ldots$

71. $e^{1.3} = 3.6692\ldots$

72. $e^{2.5} = 12.1824\ldots$

73. $\sqrt[3]{e} = 1.3956\ldots$

74. $\dfrac{1}{e^4} = 0.0183\ldots$

75. $\sqrt{e^3} = 4.4816\ldots$

76. $e^{3/4} = 2.1170\ldots$

Evaluating the Natural Logarithmic Function **In Exercises 77–80, use a calculator to evaluate the function at the indicated value of x. Round your result to three decimal places.**

	Function	*Value*
77.	$f(x) = \ln x$	$x = \sqrt{42}$
78.	$f(x) = \ln x$	$x = 18.31$
79.	$f(x) = -\ln x$	$x = \frac{1}{2}$
80.	$f(x) = 3 \ln x$	$x = 0.75$

Using Properties of Natural Logarithms **In Exercises 81–88, use the properties of natural logarithms to rewrite the expression.**

81. $\ln e^2$

82. $-\ln e$

83. $e^{\ln 1.8}$

84. $7 \ln e^0$

85. $e \ln 1$

86. $e^{\ln 22}$

87. $\ln e^{\ln e}$

88. $\ln \dfrac{1}{e^4}$

Library of Parent Functions **In Exercises 89–92, find the domain, vertical asymptote, and x-intercept of the logarithmic function, and sketch its graph by hand. Verify using a graphing utility.**

89. $f(x) = \ln(x - 1)$

90. $h(x) = \ln(x + 1)$

91. $g(x) = \ln(-x)$

92. $f(x) = \ln(3 - x)$

Library of Parent Functions **In Exercises 93–98, use the graph of $f(x) = \ln x$ to describe the transformation that yields the graph of g.**

93. $g(x) = \ln(x + 3)$

94. $g(x) = \ln(x - 4)$

95. $g(x) = \ln x - 5$

96. $g(x) = \ln x + 4$

97. $g(x) = \ln(x - 1) + 2$

98. $g(x) = \ln(x + 2) - 5$

Analyzing Graphs of Functions **In Exercises 99–108, (a) use a graphing utility to graph the function, (b) find the domain, (c) use the graph to find the open intervals on which the function is increasing and decreasing, and (d) approximate any relative maximum or minimum values of the function. Round your result to three decimal places.**

99. $f(x) = \dfrac{x}{2} - \ln \dfrac{x}{4}$

100. $g(x) = \dfrac{12 \ln x}{x}$

101. $h(x) = 4x \ln x$

102. $f(x) = \dfrac{x}{\ln x}$

103. $f(x) = \ln \dfrac{x + 2}{x - 1}$

104. $f(x) = \ln \dfrac{2x}{x + 2}$

105. $f(x) = \ln \dfrac{x^2}{10}$

106. $f(x) = \ln \dfrac{x}{x^2 + 1}$

107. $f(x) = \sqrt{\ln x}$

108. $f(x) = (\ln x)^2$

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109. **Psychology** Students in a mathematics class were given an exam and then tested monthly with an equivalent exam. The average scores for the class are given by the human memory model

$$f(t) = 80 - 17 \log_{10}(t + 1), \quad 0 \le t \le 12$$

where t is the time in months.

(a) What was the average score on the original exam ($t = 0$)?

(b) What was the average score after 4 months?

(c) What was the average score after 10 months?

Verify your answers in parts (a), (b), and (c) using a graphing utility.

110. **MODELING DATA**

The table shows the temperatures T (in degrees Fahrenheit) at which water boils at selected pressures p (in pounds per square inch). (Source: Standard Handbook of Mechanical Engineers)

Pressure, p	Temperature, T
5	162.24°
10	193.21°
14.696 (1 atm)	212.00°
20	227.96°
30	250.33°
40	267.25°
60	292.71°
80	312.03°
100	327.81°

A model that approximates the data is

$$T = 87.97 + 34.96 \ln p + 7.91\sqrt{p}.$$

(a) Use a graphing utility to plot the data and graph the model in the same viewing window. How well does the model fit the data?

(b) Use the graph to estimate the pressure at which the boiling point of water is 300°F.

(c) Calculate T when the pressure is 74 pounds per square inch. Verify your answer graphically.

111. **Finance** A principal P, invested at $5\frac{1}{2}\%$ and compounded continuously, increases to an amount K times the original principal after t years, where $t = (\ln K)/0.055$.

(a) Complete the table and interpret your results.

K	1	2	4	6	8	10	12
t							

(b) Use a graphing utility to graph the function.

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112. **Science** The relationship between the number of decibels β and the intensity of a sound I in watts per square meter is given by

$$\beta = 10 \log_{10}\left(\frac{I}{10^{-12}}\right).$$

(a) Determine the number of decibels of a sound with an intensity of 1 watt per square meter.

(b) Determine the number of decibels of a sound with an intensity of 10^{-2} watt per square meter.

(c) The intensity of the sound in part (a) is 100 times as great as that in part (b). Is the number of decibels 100 times as great? Explain.

113. **Real Estate** The model

$$t = 16.625 \ln \frac{x}{x - 750}, \quad x > 750$$

approximates the length of a home mortgage of \$150,000 at 6% in terms of the monthly payment. In the model, t is the length of the mortgage in years and x is the monthly payment in dollars.

(a) Use the model to approximate the lengths of a \$150,000 mortgage at 6% when the monthly payment is \$897.72 and when the monthly payment is \$1659.24.

(b) Approximate the total amounts paid over the term of the mortgage with a monthly payment of \$897.72 and with a monthly payment of \$1659.24. What amount of the total is interest costs for each payment?

114. ***Why you should learn it*** *(p. 192)* The rate of ventilation required in a public school classroom depends on the volume of air space per child. The model

$$y = 80.4 - 11 \ln x, \quad 100 \le x \le 1500$$

approximates the minimum required rate of ventilation y (in cubic feet per minute per child) in a classroom with x cubic feet of air space per child.

(a) Use a graphing utility to graph the function and approximate the required rate of ventilation in a room with 300 cubic feet of air space per child.

(b) A classroom of 30 students has an air conditioning system that moves 450 cubic feet of air per minute. Determine the rate of ventilation per child.

(c) Use the graph in part (a) to estimate the minimum required air space per child for the classroom in part (b).

(d) The classroom in part (b) has 960 square feet of floor space and a ceiling that is 12 feet high. Is the rate of ventilation for this classroom adequate? Explain.

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Conclusions

True or False? **In Exercises 115 and 116, determine whether the statement is true or false. Justify your answer.**

115. You can determine the graph of $f(x) = \log_6 x$ by graphing $g(x) = 6^x$ and reflecting it about the x-axis.

116. The graph of $f(x) = \log_3 x$ contains the point $(27, 3)$.

Think About It **In Exercises 117–120, find the value of the base b so that the graph of $f(x) = \log_b x$ contains the given point.**

117. $(32, 5)$ **118.** $(81, 4)$

119. $\left(\frac{1}{16}, 2\right)$ **120.** $\left(\frac{1}{27}, 3\right)$

Library of Parent Functions **In Exercises 121 and 122, determine which equation(s) may be represented by the graph shown. (There may be more than one correct answer.)**

121.

(a) $y = \log_2(x + 1) + 2$
(b) $y = \log_2(x - 1) + 2$
(c) $y = 2 - \log_2(x - 1)$
(d) $y = \log_2(x + 2) + 1$

122.

(a) $y = \ln(x - 1) + 2$
(b) $y = \ln(x + 2) - 1$
(c) $y = 2 - \ln(x - 1)$
(d) $y = \ln(x - 2) + 1$

123. **Writing** Explain why $\log_a x$ is defined only for $0 < a < 1$ and $a > 1$.

124. **Exploration** Let $f(x) = \ln x$ and $g(x) = x^{1/n}$.

(a) Use a graphing utility to graph g (for $n = 2$) and f in the same viewing window.

(b) Determine which function is increasing at a greater rate as x approaches infinity.

(c) Repeat parts (a) and (b) for $n = 3$, 4, and 5. What do you notice?

125. **Exploration**

(a) Use a graphing utility to compare the graph of the function $y = \ln x$ with the graph of each function.

$y_1 = x - 1, y_2 = (x - 1) - \frac{1}{2}(x - 1)^2,$

$y_3 = (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3$

(b) Identify the pattern of successive polynomials given in part (a). Extend the pattern one more term and compare the graph of the resulting polynomial function with the graph of $y = \ln x$. What do you think the pattern implies?

126. **CAPSTONE** The following table of values was obtained by evaluating a function. Determine which of the statements may be true and which must be false. Explain your reasoning.

x	1	2	8
y	0	1	3

(a) y is an exponential function of x.

(b) y is a logarithmic function of x.

(c) x is an exponential function of y.

(d) y is a linear function of x.

127. **Exploration**

(a) Use a graphing utility to complete the table for the function

$$f(x) = \frac{\ln x}{x}.$$

x	1	5	10	10^2	10^4	10^6
$f(x)$						

(b) Use the table in part (a) to determine what value $f(x)$ approaches as x increases without bound. Use the graphing utility to confirm your result.

128. **Writing** Use a graphing utility to determine how many months it would take for the average score in Example 10 to decrease to 60. Explain your method of solving the problem. Describe another way that you can use the graphing utility to determine the answer. Also, based on the shape of the graph, does the rate at which a student forgets information *increase* or *decrease* with time? Explain.

Cumulative Mixed Review

Factoring a Polynomial **In Exercises 129–136, factor the polynomial.**

129. $x^2 + 2x - 3$ **130.** $2x^2 + 3x - 5$

131. $12x^2 + 5x - 3$ **132.** $16x^2 + 16x + 7$

133. $16x^2 - 25$ **134.** $36x^2 - 49$

135. $2x^3 + x^2 - 45x$ **136.** $3x^3 - 5x^2 - 12x$

Evaluating an Arithmetic Combination of Functions **In Exercises 137 and 138, evaluate the function for $f(x) = 3x + 2$ and $g(x) = x^3 - 1$.**

137. $(f + g)(2)$ **138.** $(f - g)(-1)$

Using Graphs **In Exercises 139–142, solve the equation graphically.**

139. $5x - 7 = x + 4$ **140.** $-2x + 3 = 8x$

141. $\sqrt{3x - 2} = 9$ **142.** $\sqrt{x - 11} = x + 2$

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3.3 Properties of Logarithms

Change of Base

Most calculators have only two types of log keys, one for common logarithms (base 10) and one for natural logarithms (base e). Although common logs and natural logs are the most frequently used, you may occasionally need to evaluate logarithms to other bases. To do this, you can use the following **change-of-base formula.**

Change-of-Base Formula

Let a, b, and x be positive real numbers such that $a \neq 1$ and $b \neq 1$. Then $\log_a x$ can be converted to a different base using any of the following formulas.

Base b	*Base 10*	*Base e*
$\log_a x = \dfrac{\log_b x}{\log_b a}$	$\log_a x = \dfrac{\log_{10} x}{\log_{10} a}$	$\log_a x = \dfrac{\ln x}{\ln a}$

One way to look at the change-of-base formula is that logarithms to base a are simply *constant multiples* of logarithms to base b. The constant multiplier is

$$\frac{1}{\log_b a}.$$

What you should learn

- Rewrite logarithms with different bases.
- Use properties of logarithms to evaluate or rewrite logarithmic expressions.
- Use properties of logarithms to expand or condense logarithmic expressions.
- Use logarithmic functions to model and solve real-life problems.

Why you should learn it

Logarithmic functions can be used to model and solve real-life problems, such as the model for the number of decibels of a sound in Exercise 107 on page 208.

Example 1 Changing Bases Using Common Logarithms

a. $\log_4 25 = \dfrac{\log_{10} 25}{\log_{10} 4}$ $\qquad \log_a x = \dfrac{\log_{10} x}{\log_{10} a}$

$\approx \dfrac{1.39794}{0.60206}$ $\qquad$ Use a calculator.

≈ 2.32 $\qquad$ Simplify.

b. $\log_2 12 = \dfrac{\log_{10} 12}{\log_{10} 2} \approx \dfrac{1.07918}{0.30103} \approx 3.58$

CHECKPOINT Now try Exercise 13.

Example 2 Changing Bases Using Natural Logarithms

a. $\log_4 25 = \dfrac{\ln 25}{\ln 4}$ $\qquad \log_a x = \dfrac{\ln x}{\ln a}$

$\approx \dfrac{3.21888}{1.38629}$ $\qquad$ Use a calculator.

≈ 2.32 $\qquad$ Simplify.

b. $\log_2 12 = \dfrac{\ln 12}{\ln 2} \approx \dfrac{2.48491}{0.69315} \approx 3.58$

CHECKPOINT Now try Exercise 19.

Notice in Examples 1 and 2 that the result is the same whether common logarithms or natural logarithms are used in the change-of-base formula.

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Properties of Logarithms

You know from the previous section that the logarithmic function with base a is the *inverse function* of the exponential function with base a. So, it makes sense that the properties of exponents (see Section 3.1) should have corresponding properties involving logarithms. For instance, the exponential property

$$a^0 = 1$$

has the corresponding logarithmic property

$$\log_a 1 = 0.$$

Properties of Logarithms (See the proof on page 251.)

Let a be a positive real number such that $a \neq 1$, and let n be a real number. If u and v are positive real numbers, then the following properties are true.

	Logarithm with Base a	*Natural Logarithm*
1. Product Property:	$\log_a(uv) = \log_a u + \log_a v$	$\ln(uv) = \ln u + \ln v$
2. Quotient Property:	$\log_a \frac{u}{v} = \log_a u - \log_a v$	$\ln \frac{u}{v} = \ln u - \ln v$
3. Power Property:	$\log_a u^n = n \log_a u$	$\ln u^n = n \ln u$

Study Tip

There is no general property that can be used to rewrite $\log_a(u \pm v)$. Specifically, $\log_a(x + y)$ is *not* equal to $\log_a x + \log_a y$.

Example 3 Using Properties of Logarithms

Write each logarithm in terms of ln 2 and ln 3.

a. $\ln 6$

b. $\ln \frac{2}{27}$

Solution

a. $\ln 6 = \ln(2 \cdot 3)$ — Rewrite 6 as $2 \cdot 3$.

$= \ln 2 + \ln 3$ — Product Property

b. $\ln \frac{2}{27} = \ln 2 - \ln 27$ — Quotient Property

$= \ln 2 - \ln 3^3$ — Rewrite 27 as 3^3.

$= \ln 2 - 3 \ln 3$ — Power Property

CHECKPOINT Now try Exercise 21.

Example 4 Using Properties of Logarithms

Use the properties of logarithms to verify that $-\log_{10} \frac{1}{100} = \log_{10} 100$.

Solution

$-\log_{10} \frac{1}{100} = -\log_{10}(100^{-1})$ — Rewrite $\frac{1}{100}$ as 100^{-1}.

$= -(-1) \log_{10} 100$ — Power Property

$= \log_{10} 100$ — Simplify.

CHECKPOINT Now try Exercise 45.

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Rewriting Logarithmic Expressions

The properties of logarithms are useful for rewriting logarithmic expressions in forms that simplify the operations of algebra. This is true because they convert complicated products, quotients, and exponential forms into simpler sums, differences, and products, respectively.

Example 5 Expanding Logarithmic Expressions

Use the properties of logarithms to expand each expression.

a. $\log_4 5x^3y$

b. $\ln \dfrac{\sqrt{3x-5}}{7}$

Solution

a. $\log_4 5x^3y = \log_4 5 + \log_4 x^3 + \log_4 y$ — Product Property

$= \log_4 5 + 3\log_4 x + \log_4 y$ — Power Property

b. $\ln \dfrac{\sqrt{3x-5}}{7} = \ln \dfrac{(3x-5)^{1/2}}{7}$ — Rewrite radical using rational exponent.

$= \ln(3x-5)^{1/2} - \ln 7$ — Quotient Property

$= \dfrac{1}{2}\ln(3x-5) - \ln 7$ — Power Property

CHECKPOINT Now try Exercise 63.

In Example 5, the properties of logarithms were used to *expand* logarithmic expressions. In Example 6, this procedure is reversed and the properties of logarithms are used to *condense* logarithmic expressions.

Explore the Concept

Use a graphing utility to graph the functions

$$y = \ln x - \ln(x-3)$$

and

$$y = \ln \frac{x}{x-3}$$

in the same viewing window. Does the graphing utility show the functions with the same domain? Should it? Explain your reasoning.

Example 6 Condensing Logarithmic Expressions

Use the properties of logarithms to condense each expression.

a. $\frac{1}{2}\log_{10} x + 3\log_{10}(x+1)$

b. $2\ln(x+2) - \ln x$

c. $\frac{1}{3}[\log_2 x + \log_2(x-4)]$

Solution

a. $\frac{1}{2}\log_{10} x + 3\log_{10}(x+1) = \log_{10} x^{1/2} + \log_{10}(x+1)^3$ — Power Property

$= \log_{10}\left[\sqrt{x}(x+1)^3\right]$ — Product Property

b. $2\ln(x+2) - \ln x = \ln(x+2)^2 - \ln x$ — Power Property

$= \ln \dfrac{(x+2)^2}{x}$ — Quotient Property

c. $\frac{1}{3}[\log_2 x + \log_2(x-4)] = \frac{1}{3}\{\log_2[x(x-4)]\}$ — Product Property

$= \log_2[x(x-4)]^{1/3}$ — Power Property

$= \log_2 \sqrt[3]{x(x-4)}$ — Rewrite with a radical.

CHECKPOINT Now try Exercise 81.

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Application

Example 7 Finding a Mathematical Model

The table shows the mean distance x from the sun and the period y (the time it takes a planet to orbit the sun) for each of the six planets that are closest to the sun. In the table, the mean distance is given in astronomical units (where the Earth's mean distance is defined as 1.0), and the period is given in years. The points in the table are plotted in Figure 3.22. Find an equation that relates y and x.

Figure 3.22

Planet	Mercury	Venus	Earth	Mars	Jupiter	Saturn
Mean distance, x	0.387	0.723	1.000	1.524	5.203	9.555
Period, y	0.241	0.615	1.000	1.881	11.860	29.420

Solution

From Figure 3.22, it is not clear how to find an equation that relates y and x. To solve this problem, take the natural log of each of the x- and y-values in the table. This produces the following results.

Planet	Mercury	Venus	Earth	Mars	Jupiter	Saturn
$\ln x = X$	-0.949	-0.324	0.000	0.421	1.649	2.257
$\ln y = Y$	-1.423	-0.486	0.000	0.632	2.473	3.382

Now, by plotting the points in the table, you can see that all six of the points appear to lie in a line, as shown in Figure 3.23. To find an equation of the line through these points, you can use one of the following methods.

Figure 3.23

Method 1: Algebraic

Choose any two points to determine the slope of the line. Using the two points (0.421, 0.632) and (0, 0), you can determine that the slope of the line is

$$m = \frac{0.632 - 0}{0.421 - 0} \approx 1.5 = \frac{3}{2}.$$

By the point-slope form, the equation of the line is

$$Y = \tfrac{3}{2}X$$

where $Y = \ln y$ and $X = \ln x$. You can therefore conclude that

$$\ln y = \frac{3}{2} \ln x.$$

CHECKPOINT Now try Exercise 109.

Method 2: Graphical

Using the *linear regression* feature of a graphing utility, you can find a linear model for the data, as shown in Figure 3.24. You can approximate this model to be $Y = 1.5X$, where $Y = \ln y$ and $X = \ln x$. From the model, you can see that the slope of the line is $\frac{3}{2}$. So, you can conclude that

$$\ln y = \tfrac{3}{2} \ln x.$$

Figure 3.24

In Example 7, try to convert the final equation to $y = f(x)$ form. You will get a function of the form $y = ax^b$, which is called a *power model*.

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3.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1 and 2, fill in the blank(s).

1. You can evaluate logarithms to any base using the _____ formula.
2. Two properties of logarithms are _____ $= n \log_a u$ and $\ln(uv) =$ _____.
3. Is $\log_3 24 = \dfrac{\ln 3}{\ln 24}$ or $\log_3 24 = \dfrac{\ln 24}{\ln 3}$ correct?
4. Which property of logarithms can you use to condense the expression $\ln x - \ln 2$?

Procedures and Problem Solving

Changing the Base **In Exercises 5–12, rewrite the logarithm as a ratio of (a) common logarithms and (b) natural logarithms.**

5. $\log_5 x$
6. $\log_3 x$
7. $\log_{1/5} x$
8. $\log_{1/3} x$
9. $\log_a \frac{3}{10}$
10. $\log_a \frac{4}{5}$
11. $\log_{2.6} x$
12. $\log_{7.1} x$

Changing the Base **In Exercises 13–20, evaluate the logarithm using the change-of-base formula. Round your result to three decimal places.**

13. $\log_3 7$
14. $\log_7 4$
15. $\log_{1/2} 4$
16. $\log_{1/8} 64$
17. $\log_6 0.9$
18. $\log_4 0.045$
19. $\log_{15} 1460$
20. $\log_{20} 175$

Using Properties of Logarithms **In Exercises 21–24, rewrite the expression in terms of ln 4 and ln 5.**

21. $\ln 20$
22. $\ln 500$
23. $\ln \frac{25}{4}$
24. $\ln \frac{5}{2}$

Using Properties to Evaluate Logarithms **In Exercises 25–28, approximate the logarithm using the properties of logarithms, given the values $\log_b 2 \approx 0.3562$, $\log_b 3 \approx 0.5646$, and $\log_b 5 \approx 0.8271$. Round your result to four decimal places.**

25. $\log_b 25$
26. $\log_b 30$
27. $\log_b \sqrt{3}$
28. $\log_b \frac{16}{25}$

Graphing a Logarithm **In Exercises 29–36, use the change-of-base formula $\log_a x = (\ln x)/(\ln a)$ and a graphing utility to graph the function.**

29. $f(x) = \log_3(x + 2)$
30. $f(x) = \log_2(x - 1)$
31. $f(x) = \log_{1/2}(x - 2)$
32. $f(x) = \log_{1/3}(x + 1)$
33. $f(x) = \log_{1/4} x^2$
34. $f(x) = \log_3 \sqrt{x}$
35. $f(x) = \log_{1/2}\left(\dfrac{x}{2}\right)$
36. $\log_5\left(\dfrac{x}{3}\right)$

Simplifying a Logarithm **In Exercises 37–44, use the properties of logarithms to rewrite and simplify the logarithmic expression.**

37. $\log_4 8$
38. $\log_9 243$
39. $\log_2 4^2 \cdot 3^4$
40. $\log_3 9^2 \cdot 2^4$
41. $\ln 5e^6$
42. $\ln 8e^3$
43. $\ln \dfrac{6}{e^2}$
44. $\ln \dfrac{e^5}{7}$

Using Properties of Logarithms **In Exercises 45 and 46, use the properties of logarithms to verify the equation.**

45. $\log_5 \frac{1}{250} = -3 - \log_5 2$
46. $-\ln 24 = -(3 \ln 2 + \ln 3)$

Expanding Logarithmic Expressions **In Exercises 47–64, use the properties of logarithms to expand the expression as a sum, difference, and/or constant multiple of logarithms. (Assume all variables are positive.)**

47. $\log_{10} 5x$
48. $\log_{10} 10z$
49. $\log_{10} \dfrac{t}{8}$
50. $\log_{10} \dfrac{7}{z}$
51. $\log_8 x^4$
52. $\log_6 z^{-3}$
53. $\ln \sqrt{z}$
54. $\ln \sqrt[3]{t}$
55. $\ln xyz$
56. $\ln \dfrac{xy}{z}$
57. $\log_6 ab^3c^2$
58. $\log_4 xy^6z^4$
59. $\ln \sqrt[3]{\dfrac{x}{y}}$
60. $\ln \sqrt{\dfrac{x^2}{y^3}}$
61. $\ln \dfrac{x^2 - 1}{x^3}, \quad x > 1$
62. $\ln \dfrac{x}{\sqrt{x^2 + 1}}$
63. $\ln \dfrac{x^4\sqrt{y}}{z^5}$
64. $\log_b \dfrac{\sqrt{x}y^4}{z^4}$

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Algebraic-Graphical-Numerical **In Exercises 65–68, (a) use a graphing utility to graph the two equations in the same viewing window and (b) use the *table* feature of the graphing utility to create a table of values for each equation. (c) What do the graphs and tables suggest? Verify your conclusion algebraically.**

65. $y_1 = \ln[x^3(x + 4)], \quad y_2 = 3 \ln x + \ln(x + 4)$

66. $y_1 = \ln\left(\dfrac{\sqrt{x}}{x - 2}\right), \quad y_2 = \dfrac{1}{2} \ln x - \ln(x - 2)$

67. $y_1 = \ln\left(\dfrac{x^4}{x - 2}\right), \quad y_2 = 4 \ln x - \ln(x - 2)$

68. $y_1 = \ln 4x^3, \quad y_2 = \ln 4 + 3 \ln x$

Condensing Logarithmic Expressions **In Exercises 69–84, condense the expression to the logarithm of a single quantity.**

69. $\ln x + \ln 4$

70. $\ln y + \ln z$

71. $\log_4 z - \log_4 y$

72. $\log_5 8 - \log_5 t$

73. $2 \log_2(x + 3)$

74. $\frac{5}{2} \log_7(z - 4)$

75. $\frac{1}{2} \ln(x^2 + 4)$

76. $2 \ln x + \ln(x + 1)$

77. $\ln x - 3 \ln(x + 1)$

78. $\ln x - 2 \ln(x + 2)$

79. $\ln(x - 2) - \ln(x + 2)$

80. $3 \ln x + 2 \ln y - 4 \ln z$

81. $\ln x - 2[\ln(x + 2) + \ln(x - 2)]$

82. $4[\ln z + \ln(z + 5)] - 2 \ln(z - 5)$

83. $\frac{1}{3}[2 \ln(x + 3) + \ln x - \ln(x^2 - 1)]$

84. $2[\ln x - \ln(x + 1) - \ln(x - 1)]$

Algebraic-Graphical-Numerical **In Exercises 85–88, (a) use a graphing utility to graph the two equations in the same viewing window and (b) use the *table* feature of the graphing utility to create a table of values for each equation. (c) What do the graphs and tables suggest? Verify your conclusion algebraically.**

85. $y_1 = 2[\ln 8 - \ln(x^2 + 1)], \quad y_2 = \ln\left[\dfrac{64}{(x^2 + 1)^2}\right]$

86. $y_1 = 2[\ln 6 + \ln(x^2 + 1)], \quad y_2 = \ln[36(x^2 + 1)^2]$

87. $y_1 = \ln x + \frac{1}{2} \ln(x + 1), \quad y_2 = \ln\left(x\sqrt{x + 1}\right)$

88. $y_1 = \dfrac{1}{2} \ln x - \ln(x + 2), \quad y_2 = \ln\left(\dfrac{\sqrt{x}}{x + 2}\right)$

Algebraic-Graphical-Numerical **In Exercises 89–92, (a) use a graphing utility to graph the two equations in the same viewing window and (b) use the *table* feature of the graphing utility to create a table of values for each equation. (c) Are the expressions equivalent? Explain. Verify your conclusion algebraically.**

89. $y_1 = \ln x^2, \quad y_2 = 2 \ln x$

90. $y_1 = 2(\ln 2 + \ln x), \quad y_2 = \ln 4x^2$

91. $y_1 = \ln(x - 2) + \ln(x + 2), \quad y_2 = \ln(x^2 - 4)$

92. $y_1 = \frac{1}{4} \ln[x^4(x^2 + 1)], \quad y_2 = \ln x + \frac{1}{4} \ln(x^2 + 1)$

Using Properties to Evaluate Logarithms **In Exercises 93–106, find the exact value of the logarithm without using a calculator. If this is not possible, state the reason.**

93. $\log_3 9$

94. $\log_6 \sqrt[3]{6}$

95. $\log_4 16^{3.4}$

96. $\log_5\left(\frac{1}{125}\right)$

97. $\log_2(-4)$

98. $\log_4(-16)$

99. $\log_5 75 - \log_5 3$

100. $\log_4 2 + \log_4 32$

101. $\ln e^3 - \ln e^7$

102. $\ln e^6 - 2 \ln e^5$

103. $2 \ln e^4$

104. $\ln e^{4.5}$

105. $\ln \dfrac{1}{\sqrt{e}}$

106. $\ln \sqrt[5]{e^3}$

107. ***Why you should learn it*** *(p. 203)* The relationship between the number of decibels β and the intensity of a sound I in watts per square meter is given by

$$\beta = 10 \log_{10}\left(\frac{I}{10^{-12}}\right).$$

(a) Use the properties of logarithms to write the formula in a simpler form.

(b) Use a graphing utility to complete the table. Verify your answers algebraically.

I	10^{-4}	10^{-6}	10^{-8}	10^{-10}	10^{-12}	10^{-14}
β						

108. **Psychology** Students participating in a psychology experiment attended several lectures and were given an exam. Every month for the next year, the students were retested to see how much of the material they remembered. The average scores for the group are given by the human memory model

$$f(t) = 90 - 15 \log_{10}(t + 1), \quad 0 \le t \le 12$$

where t is the time (in months).

(a) Use a graphing utility to graph the function over the specified domain.

(b) What was the average score on the original exam ($t = 0$)?

(c) What was the average score after 6 months?

(d) What was the average score after 12 months?

(e) When did the average score decrease to 75?

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109. MODELING DATA

A beaker of liquid at an initial temperature of 78°C is placed in a room at a constant temperature of 21°C. The temperature of the liquid is measured every 5 minutes during a half-hour period. The results are recorded as ordered pairs of the form (t, T), where t is the time (in minutes) and T is the temperature (in degrees Celsius).

(0, 78.0°), (5, 66.0°), (10, 57.5°), (15, 51.2°), (20, 46.3°), (25, 42.5°), (30, 39.6°)

(a) The graph of the temperature of the room should be an asymptote of the graph of the model for the data. Subtract the room temperature from each of the temperatures in the ordered pairs. Use a graphing utility to plot the data points (t, T) and $(t, T - 21)$.

(b) An exponential model for the data $(t, T - 21)$ is given by

$$T - 21 = 54.4(0.964)^t.$$

Solve for T and graph the model. Compare the result with the plot of the original data.

(c) Take the natural logarithms of the revised temperatures. Use the graphing utility to plot the points $(t, \ln(T - 21))$ and observe that the points appear linear. Use the *regression* feature of the graphing utility to fit a line to the data. The resulting line has the form

$$\ln(T - 21) = at + b.$$

Use the properties of logarithms to solve for T. Verify that the result is equivalent to the model in part (b).

(d) Fit a rational model to the data. Take the reciprocals of the y-coordinates of the revised data points to generate the points

$$\left(t, \frac{1}{T - 21}\right).$$

Use the graphing utility to plot these points and observe that they appear linear. Use the *regression* feature of the graphing utility to fit a line to the data. The resulting line has the form

$$\frac{1}{T - 21} = at + b.$$

Solve for T, and use the graphing utility to graph the rational function and the original data points.

110. Writing Write a short paragraph explaining why the transformations of the data in Exercise 109 were necessary to obtain the models. Why did taking the logarithms of the temperatures lead to a linear scatter plot? Why did taking the reciprocals of the temperatures lead to a linear scatter plot?

Conclusions

True or False? **In Exercises 111–116, determine whether the statement is true or false given that $f(x) = \ln x$, where $x > 0$. Justify your answer.**

111. $f(ax) = f(a) + f(x),\ a > 0$

112. $f(x - a) = f(x) - f(a),\ x > a$

113. $\sqrt{f(x)} = \frac{1}{2}f(x)$

114. $[f(x)]^n = nf(x)$

115. If $f(x) < 0$, then $0 < x < e$.

116. If $f(x) > 0$, then $x > e$.

117. Error Analysis Describe the error.

$$\cancel{\ln\left(\frac{x^2}{\sqrt{x^2 + 4}}\right) = \frac{\ln x^2}{\ln \sqrt{x^2 + 4}}}$$

118. Think About It Consider the functions below.

$$f(x) = \ln \frac{x}{2}, \quad g(x) = \frac{\ln x}{\ln 2}, \quad h(x) = \ln x - \ln 2$$

Which two functions have identical graphs? Verify your answer by using a graphing utility to graph all three functions in the same viewing window.

119. Exploration For how many integers between 1 and 20 can the natural logarithms be approximated given that $\ln 2 \approx 0.6931$, $\ln 3 \approx 1.0986$, and $\ln 5 \approx 1.6094$? Approximate these logarithms. (Do not use a calculator.)

120. CAPSTONE Show that each expression is equivalent to ln 8. Then write three more expressions that are equivalent to ln 8.

(a) $3 \ln 2$ (b) $-\ln \frac{1}{8}$ (c) $-3 \ln \frac{1}{2}$

(d) $\ln 16 - \ln 2$ (e) $(\log_{10} 2 + \log_{10} 4) \div \log_{10} e$

121. Think About It Does $y_1 = \ln[x(x - 2)]$ have the same domain as $y_2 = \ln x + \ln(x - 2)$? Explain.

122. Proof Prove that $\dfrac{\log_a x}{\log_{a/b} x} = 1 + \log_a \dfrac{1}{b}$.

Cumulative Mixed Review

Using Rules of Exponents **In Exercises 123–126, simplify the expression.**

123. $\dfrac{24xy^{-2}}{16x^{-3}y}$

124. $\left(\dfrac{2x^3}{3y}\right)^{-3}$

125. $(18x^3y^4)^{-3}(18x^3y^4)^4$

126. $xy(x^{-1} + y^{-1})^{-1}$

Solving Polynomial Equations **In Exercises 127–130, find all solutions of the equation.**

127. $x^2 - 6x + 2 = 0$

128. $2x^3 + 20x^2 + 50x = 0$

129. $x^4 - 19x^2 + 48 = 0$

130. $9x^4 - 37x^2 + 4 = 0$

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3.4 Solving Exponential and Logarithmic Equations

Introduction

So far in this chapter, you have studied the definitions, graphs, and properties of exponential and logarithmic functions. In this section, you will study procedures for *solving equations* involving exponential and logarithmic functions.

There are two basic strategies for solving exponential or logarithmic equations. The first is based on the One-to-One Properties and the second is based on the Inverse Properties. For $a > 0$ and $a \neq 1$, the following properties are true for all x and y for which

$$\log_a x \quad \text{and} \quad \log_a y$$

are defined.

One-to-One Properties

$a^x = a^y$ if and only if $x = y$.

$\log_a x = \log_a y$ if and only if $x = y$.

Inverse Properties

$a^{\log_a x} = x$

$\log_a a^x = x$

What you should learn

- Solve simple exponential and logarithmic equations.
- Solve more complicated exponential equations.
- Solve more complicated logarithmic equations.
- Use exponential and logarithmic equations to model and solve real-life problems.

Why you should learn it

Exponential and logarithmic equations can be used to model and solve real-life problems. For instance, Exercise 148 on page 219 shows how to use an exponential function to model the average heights of men and women.

Example 1 Solving Simple Exponential and Logarithmic Equations

	Original Equation	*Rewritten Equation*	*Solution*	*Property*
a.	$2^x = 32$	$2^x = 2^5$	$x = 5$	One-to-One
b.	$\log_4 x - \log_4 8 = 0$	$\log_4 x = \log_4 8$	$x = 8$	One-to-One
c.	$\ln x - \ln 3 = 0$	$\ln x = \ln 3$	$x = 3$	One-to-One
d.	$\left(\frac{1}{3}\right)^x = 9$	$3^{-x} = 3^2$	$x = -2$	One-to-One
e.	$e^x = 7$	$\ln e^x = \ln 7$	$x = \ln 7$	Inverse
f.	$\ln x = -3$	$e^{\ln x} = e^{-3}$	$x = e^{-3}$	Inverse
g.	$\log_{10} x = -1$	$10^{\log_{10} x} = 10^{-1}$	$x = 10^{-1} = \frac{1}{10}$	Inverse
h.	$\log_3 x = 4$	$3^{\log_3 x} = 3^4$	$x = 81$	Inverse

CHECKPOINT Now try Exercise 27.

The strategies used in Example 1 are summarized as follows.

Strategies for Solving Exponential and Logarithmic Equations

1. Rewrite the original equation in a form that allows the use of the One-to-One Properties of exponential or logarithmic functions.
2. Rewrite an *exponential* equation in logarithmic form and apply the Inverse Property of logarithmic functions.
3. Rewrite a *logarithmic* equation in exponential form and apply the Inverse Property of exponential functions.

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Solving Exponential Equations

Example 2 Solving Exponential Equations

Solve each equation.

a. $e^x = 72$

b. $3(2^x) = 42$

Algebraic Solution

a.

$e^x = 72$	Write original equation.
$\ln e^x = \ln 72$	Take natural log of each side.
$x = \ln 72$	Inverse Property
$x \approx 4.28$	Use a calculator.

The solution is $x = \ln 72 \approx 4.28$. Check this in the original equation.

b.

$3(2^x) = 42$	Write original equation.
$2^x = 14$	Divide each side by 3.
$\log_2 2^x = \log_2 14$	Take log (base 2) of each side.
$x = \log_2 14$	Inverse Property
$x = \dfrac{\ln 14}{\ln 2}$	Change-of-base formula
$x \approx 3.81$	Use a calculator.

The solution is $x = \log_2 14 \approx 3.81$. Check this in the original equation.

CHECKPOINT Now try Exercise 33.

Graphical Solution

To solve an equation using a graphing utility, you can graph the left- and right-hand sides of the equation and use the *intersect* feature.

a.

The intersection point is about (4.28, 72). So, the solution is $x \approx 4.28$.

b.

The intersection point is about (3.81, 42). So, the solution is $x \approx 3.81$.

Example 3 Solving an Exponential Equation

Solve $4e^{2x} - 3 = 2$.

Algebraic Solution

$4e^{2x} - 3 = 2$	Write original equation.
$4e^{2x} = 5$	Add 3 to each side.
$e^{2x} = \frac{5}{4}$	Divide each side by 4.
$\ln e^{2x} = \ln \frac{5}{4}$	Take natural log of each side.
$2x = \ln \frac{5}{4}$	Inverse Property
$x = \frac{1}{2} \ln \frac{5}{4}$	Divide each side by 2.
$x \approx 0.11$	Use a calculator.

The solution is

$$x = \tfrac{1}{2} \ln \tfrac{5}{4} \approx 0.11.$$

Check this in the original equation.

CHECKPOINT Now try Exercise 61.

Graphical Solution

Rather than using the procedure in Example 2, another way to solve the equation graphically is first to rewrite the equation as

$$4e^{2x} - 5 = 0$$

and then use a graphing utility to graph

$$y = 4e^{2x} - 5.$$

Use the *zero* or *root* feature of the graphing utility to approximate the value of x for which $y = 0$, as shown in Figure 3.25.

The zero occurs at $x \approx 0.11$. So, the solution of the original equation is $x \approx 0.11$.

Figure 3.25

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Example 4 Solving an Exponential Equation

Solve $2(3^{2t-5}) - 4 = 11$.

Solution

$2(3^{2t-5}) - 4 = 11$	Write original equation.
$2(3^{2t-5}) = 15$	Add 4 to each side.
$3^{2t-5} = \frac{15}{2}$	Divide each side by 2.
$\log_3 3^{2t-5} = \log_3 \frac{15}{2}$	Take log (base 3) of each side.
$2t - 5 = \log_3 \frac{15}{2}$	Inverse Property
$2t = 5 + \log_3 7.5$	Add 5 to each side.
$t = \frac{5}{2} + \frac{1}{2}\log_3 7.5$	Divide each side by 2.
$t \approx 3.42$	Use a calculator.

The solution is $t = \frac{5}{2} + \frac{1}{2}\log_3 7.5 \approx 3.42$. Check this in the original equation.

CHECKPOINT Now try Exercise 65.

Study Tip

Remember that to evaluate a logarithm such as $\log_3 7.5$, you need to use the change-of-base formula.

When an equation involves two or more exponential expressions, you can still use a procedure similar to that demonstrated in the previous three examples. However, the algebra is a bit more complicated.

Example 5 Solving an Exponential Equation in Quadratic Form

Solve $e^{2x} - 3e^x + 2 = 0$.

Algebraic Solution

$e^{2x} - 3e^x + 2 = 0$	Write original equation.
$(e^x)^2 - 3e^x + 2 = 0$	Write in quadratic form.
$(e^x - 2)(e^x - 1) = 0$	Factor.
$e^x - 2 = 0$	Set 1st factor equal to 0.
$e^x = 2$	Add 2 to each side.
$x = \ln 2$	Solution
$e^x - 1 = 0$	Set 2nd factor equal to 0.
$e^x = 1$	Add 1 to each side.
$x = \ln 1$	Inverse Property
$x = 0$	Solution

The solutions are

$$x = \ln 2 \approx 0.69 \text{ and } x = 0.$$

Check these in the original equation.

CHECKPOINT Now try Exercise 71.

Graphical Solution

Use a graphing utility to graph $y = e^{2x} - 3e^x + 2$ and then find the zeros.

Figure 3.26

Figure 3.27

From Figures 3.26 and 3.27, you can conclude that the solutions are $x = 0$ and $x \approx 0.69$.

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Solving Logarithmic Equations

To solve a logarithmic equation, you can write it in exponential form.

$\ln x = 3$	Logarithmic form
$e^{\ln x} = e^3$	Exponentiate each side.
$x = e^3$	Exponential form

This procedure is called *exponentiating* each side of an equation. It is applied after the logarithmic expression has been isolated.

Example 6 Solving Logarithmic Equations

Solve each logarithmic equation.

a. $\ln 3x = 2$

b. $\log_3(5x - 1) = \log_3(x + 7)$

Solution

a.

$\ln 3x = 2$	Write original equation.
$e^{\ln 3x} = e^2$	Exponentiate each side.
$3x = e^2$	Inverse Property
$x = \frac{1}{3}e^2$	Multiply each side by $\frac{1}{3}$.
$x \approx 2.46$	Use a calculator.

The solution is $x = \frac{1}{3}e^2 \approx 2.46$. Check this in the original equation.

b.

$\log_3(5x - 1) = \log_3(x + 7)$	Write original equation.
$5x - 1 = x + 7$	One-to-One Property
$x = 2$	Solve for x.

The solution is $x = 2$. Check this in the original equation.

CHECKPOINT Now try Exercise 93.

Example 7 Solving a Logarithmic Equation

Solve $5 + 2 \ln x = 4$.

Algebraic Solution

$5 + 2 \ln x = 4$	Write original equation.
$2 \ln x = -1$	Subtract 5 from each side.
$\ln x = -\frac{1}{2}$	Divide each side by 2.
$e^{\ln x} = e^{-1/2}$	Exponentiate each side.
$x = e^{-1/2}$	Inverse Property
$x \approx 0.61$	Use a calculator.

The solution is $x = e^{-1/2} \approx 0.61$. Check this in the original equation.

CHECKPOINT Now try Exercise 97.

Graphical Solution

The intersection point is about (0.61, 4). So, the solution is $x \approx 0.61$.

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Example 8 Solving a Logarithmic Equation

Solve $2 \log_5 3x = 4$.

Solution

$2 \log_5 3x = 4$	Write original equation.
$\log_5 3x = 2$	Divide each side by 2.
$5^{\log_5 3x} = 5^2$	Exponentiate each side (base 5).
$3x = 25$	Inverse Property
$x = \frac{25}{3}$	Divide each side by 3.

The solution is $x = \frac{25}{3}$. Check this in the original equation. Or, perform a graphical check by graphing

$$y_1 = 2 \log_5 3x = 2\left(\frac{\log_{10} 3x}{\log_{10} 5}\right) \quad \text{and} \quad y_2 = 4$$

in the same viewing window. The two graphs should intersect at

$$x = \frac{25}{3} \approx 8.33$$

and

$$y = 4$$

as shown in Figure 3.28.

Figure 3.28

CHECKPOINT Now try Exercise 99.

Because the domain of a logarithmic function generally does not include all real numbers, you should be sure to check for extraneous solutions of logarithmic equations, as shown in the next example.

Example 9 Checking for Extraneous Solutions

Solve $\ln(x - 2) + \ln(2x - 3) = 2 \ln x$.

Algebraic Solution

$\ln(x - 2) + \ln(2x - 3) = 2 \ln x$	Write original equation.
$\ln[(x - 2)(2x - 3)] = \ln x^2$	Use properties of logarithms.
$\ln(2x^2 - 7x + 6) = \ln x^2$	Multiply binomials.
$2x^2 - 7x + 6 = x^2$	One-to-One Property
$x^2 - 7x + 6 = 0$	Write in general form.
$(x - 6)(x - 1) = 0$	Factor.
$x - 6 = 0 \Rightarrow x = 6$	Set 1st factor equal to 0.
$x - 1 = 0 \Rightarrow x = 1$	Set 2nd factor equal to 0.

Finally, by checking these two "solutions" in the original equation, you can conclude that $x = 1$ is not valid. This is because when $x = 1$,

$$\ln(x - 2) + \ln(2x - 3) = \ln(-1) + \ln(-1)$$

which is invalid because -1 is not in the domain of the natural logarithmic function. So, the only solution is $x = 6$.

CHECKPOINT Now try Exercise 109.

Graphical Solution

First rewrite the original equation as

$$\ln(x - 2) + \ln(2x - 3) - 2 \ln x = 0.$$

Then use a graphing utility to graph the equation

$$y = \ln(x - 2) + \ln(2x - 3) - 2 \ln x$$

and find the zeros (see Figure 3.29).

Figure 3.29

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Example 10 The Change-of-Base Formula

Prove the change-of-base formula: $\log_a x = \dfrac{\log_b x}{\log_b a}$.

Solution

Begin by letting

$$y = \log_a x$$

and writing the equivalent exponential form

$$a^y = x.$$

Now, taking the logarithms *with base b* of each side produces the following.

$$\log_b a^y = \log_b x$$

$$y \log_b a = \log_b x \qquad \text{Power Property}$$

$$y = \frac{\log_b x}{\log_b a} \qquad \text{Divide each side by } \log_b a.$$

$$\log_a x = \frac{\log_b x}{\log_b a} \qquad \text{Replace } y \text{ with } \log_a x.$$

CHECKPOINT Now try Exercise 113.

Equations that involve combinations of algebraic functions, exponential functions, and/or logarithmic functions can be very difficult to solve by algebraic procedures. Here again, you can take advantage of a graphing utility.

Example 11 Approximating the Solution of an Equation

Approximate (to three decimal places) the solution of $\ln x = x^2 - 2$.

Solution

First, rewrite the equation as

$$\ln x - x^2 + 2 = 0.$$

Then use a graphing utility to graph

$$y = -x^2 + 2 + \ln x$$

as shown in Figure 3.30. From this graph, you can see that the equation has two solutions. Next, using the *zero* or *root* feature, you can approximate the two solutions to be $x \approx 0.138$ and $x \approx 1.564$.

Figure 3.30

Check

$$\ln x = x^2 - 2 \qquad \text{Write original equation.}$$

$$\ln(0.138) \overset{?}{\approx} (0.138)^2 - 2 \qquad \text{Substitute 0.138 for } x.$$

$$-1.9805 \approx -1.9810 \qquad \text{Solution checks. } \checkmark$$

$$\ln(1.564) \overset{?}{\approx} (1.564)^2 - 2 \qquad \text{Substitute 1.564 for } x.$$

$$0.4472 \approx 0.4461 \qquad \text{Solution checks. } \checkmark$$

So, the two solutions $x \approx 0.138$ and $x \approx 1.564$ seem reasonable.

CHECKPOINT Now try Exercise 119.

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Applications

Example 12 Doubling an Investment

You have deposited \$500 in an account that pays 6.75% interest, compounded continuously. How long will it take your money to double?

Solution

Using the formula for continuous compounding, you can find that the balance in the account is

$$A = Pe^{rt} = 500e^{0.0675t}.$$

To find the time required for the balance to double, let $A = 1000$, and solve the resulting equation for t.

$500e^{0.0675t} = 1000$	Substitute 1000 for A.
$e^{0.0675t} = 2$	Divide each side by 500.
$\ln e^{0.0675t} = \ln 2$	Take natural log of each side.
$0.0675t = \ln 2$	Inverse Property
$t = \dfrac{\ln 2}{0.0675}$	Divide each side by 0.0675.
$t \approx 10.27$	Use a calculator.

The balance in the account will double after approximately 10.27 years. This result is demonstrated graphically in Figure 3.31.

Figure 3.31

CHECKPOINT Now try Exercise 143.

Example 13 Average Salary for Public School Teachers

From 1985 through 2007, the average salary y (in thousands of dollars) for public school teachers for the year t can be modeled by the equation

$$y = -2.983 + 15.206 \ln t, \quad 5 \le t \le 27$$

where $t = 5$ represents 1985. During which year did the average salary for public school teachers reach \$45,000? (Source: National Education Association)

Solution

$-2.983 + 15.206 \ln t = y$	Write original equation.
$-2.983 + 15.206 \ln t = 45$	Substitute 45 for y.
$15.206 \ln t = 47.983$	Add 2.983 to each side.
$\ln t = \dfrac{47.983}{15.206}$	Divide each side by 15.206.
$e^{\ln t} = e^{47.983/15.206}$	Exponentiate each side.
$t = e^{47.983/15.206}$	Inverse Property
$t \approx 23.47$	Use a calculator.

The solution is $t \approx 23.47$ years. Because $t = 5$ represents 1985, it follows that the average salary for public school teachers reached \$45,000 in 2003.

CHECKPOINT Now try Exercise 149.

Teacher

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3.4 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercise 1 and 2, fill in the blank.

1. To solve exponential and logarithmic equations, you can use the following One-to-One and Inverse Properties.

(a) $a^x = a^y$ if and only if _____ .

(b) $\log_a x = \log_a y$ if and only if _____ .

(c) $a^{\log_a x} =$ _____

(d) $\log_a a^x =$ _____

2. An _____ solution does not satisfy the original equation.

3. What is the value of $\ln e^7$?

4. Can you solve $5^x = 125$ using a One-to-One Property?

5. What is the first step in solving the equation $3 + \ln x = 10$?

6. Do you solve $\log_4 x = 2$ by using a One-to-One Property or an Inverse Property?

Procedures and Problem Solving

Checking Solutions **In Exercises 7–14, determine whether each x-value is a solution of the equation.**

7. $4^{2x-7} = 64$

(a) $x = 5$

(b) $x = 2$

8. $2^{3x+1} = 32$

(a) $x = -1$

(b) $x = 2$

9. $3e^{x+2} = 75$

(a) $x = -2 + e^{25}$

(b) $x = -2 + \ln 25$

(c) $x \approx 1.2189$

10. $4e^{x-1} = 60$

(a) $x = 1 + \ln 15$

(b) $x \approx 3.7081$

(c) $x = \ln 16$

11. $\log_4(3x) = 3$

(a) $x \approx 21.3560$

(b) $x = -4$

(c) $x = \frac{64}{3}$

12. $\log_6\left(\frac{5}{3}x\right) = 2$

(a) $x \approx 20.2882$

(b) $x = \frac{108}{5}$

(c) $x = 7.2$

13. $\ln(x - 1) = 3.8$

(a) $x = 1 + e^{3.8}$

(b) $x \approx 45.7012$

(c) $x = 1 + \ln 3.8$

14. $\ln(2 + x) = 2.5$

(a) $x = e^{2.5} - 2$

(b) $x \approx \frac{4073}{400}$

(c) $x = \frac{1}{2}$

Solving Equations Graphically **In Exercises 15–22, use a graphing utility to graph f and g in the same viewing window. Approximate the point of intersection of the graphs of f and g. Then solve the equation $f(x) = g(x)$ algebraically.**

15. $f(x) = 2^x$
$g(x) = 8$

16. $f(x) = 27^x$
$g(x) = 9$

17. $f(x) = 5^{x-2} - 15$
$g(x) = 10$

18. $f(x) = 2^{-x+1} - 3$
$g(x) = 13$

19. $f(x) = 4\log_3 x$
$g(x) = 20$

20. $f(x) = 3\log_5 x$
$g(x) = 6$

21. $f(x) = \ln e^{x+1}$
$g(x) = 2x + 5$

22. $f(x) = \ln e^{x-2}$
$g(x) = 3x + 2$

Solving an Exponential Equation **In Exercises 23–36, solve the exponential equation.**

23. $4^x = 16$

24. $3^x = 243$

25. $5^x = \frac{1}{625}$

26. $7^x = \frac{1}{49}$

✓ **27.** $\left(\frac{1}{8}\right)^x = 64$

28. $\left(\frac{1}{2}\right)^x = 32$

29. $\left(\frac{2}{3}\right)^x = \frac{81}{16}$

30. $\left(\frac{3}{4}\right)^x = \frac{27}{64}$

31. $e^x = 14$

32. $e^x = 66$

✓ **33.** $6(10^x) = 216$

34. $5(8^x) = 325$

35. $2^{x+3} = 256$

36. $3^{x-1} = \frac{1}{81}$

Solving a Logarithmic Equation **In Exercises 37–46, solve the logarithmic equation.**

37. $\ln x - \ln 5 = 0$

38. $\ln x - \ln 2 = 0$

39. $\ln x = -9$

40. $\ln x = -14$

41. $\log_x 625 = 4$

42. $\log_x 25 = 2$

43. $\log_{10} x = -1$

44. $\log_{10} x = -\frac{1}{2}$

45. $\ln(2x - 1) = 5$

46. $\ln(3x + 5) = 8$

Using Inverse Properties **In Exercises 47–54, simplify the expression.**

47. $\ln e^{x^2}$

48. $\ln e^{2x-1}$

49. $e^{\ln x^2}$

50. $e^{\ln(x^2+2)}$

51. $-1 + \ln e^{2x}$

52. $-4 + e^{\ln x^4}$

53. $5 + e^{\ln(x^2+1)}$

54. $3 - \ln(e^{x^2+2})$

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Solving an Exponential Equation **In Exercises 55–80, solve the exponential equation algebraically. Round your result to three decimal places. Use a graphing utility to verify your answer.**

55. $8^{3x} = 360$

56. $6^{5x} = 3000$

57. $5^{-t/2} = 0.20$

58. $4^{-3t} = 0.10$

59. $250e^{0.02x} = 10{,}000$

60. $100e^{0.005x} = 125{,}000$

✓ **61.** $500e^{-x} = 300$

62. $1000e^{-4x} = 75$

63. $7 - 2e^{x} = 5$

64. $-14 + 3e^{x} = 11$

✓ **65.** $5(2^{3-x}) - 13 = 100$

66. $6(8^{-2-x}) + 15 = 2601$

67. $\left(1 + \dfrac{0.10}{12}\right)^{12t} = 2$

68. $\left(16 + \dfrac{0.878}{26}\right)^{3t} = 30$

69. $5000\left[\dfrac{(1 + 0.005)^{x}}{0.005}\right] = 250{,}000$

70. $250\left[\dfrac{(1 + 0.01)^{x}}{0.01}\right] = 150{,}000$

✓ **71.** $e^{2x} - 4e^{x} - 5 = 0$

72. $e^{2x} - 5e^{x} + 6 = 0$

73. $e^{x} = e^{x^2-2}$

74. $e^{2x} = e^{x^2-8}$

75. $e^{x^2-3x} = e^{x-2}$

76. $e^{-x^2} = e^{x^2-2x}$

77. $\dfrac{400}{1 + e^{-x}} = 350$

78. $\dfrac{525}{1 + e^{-x}} = 275$

79. $\dfrac{40}{1 - 5e^{-0.01x}} = 200$

80. $\dfrac{50}{1 - 2e^{-0.001x}} = 1000$

Algebraic-Graphical-Numerical **In Exercises 81 and 82, (a) complete the table to find an interval containing the solution of the equation, (b) use a graphing utility to graph both sides of the equation to estimate the solution, and (c) solve the equation algebraically. Round your results to three decimal places.**

81. $e^{3x} = 12$

x	0.6	0.7	0.8	0.9	1.0
e^{3x}					

82. $20(100 - e^{x/2}) = 500$

x	5	6	7	8	9
$20(100 - e^{x/2})$					

Solving an Exponential Equation Graphically **In Exercises 83–86, use the *zero* or *root* feature or the *zoom* and *trace* features of a graphing utility to approximate the solution of the exponential equation accurate to three decimal places.**

83. $\left(1 + \dfrac{0.065}{365}\right)^{365t} = 4$

84. $\left(4 - \dfrac{2.471}{40}\right)^{9t} = 21$

85. $\dfrac{3000}{2 + e^{2x}} = 2$

86. $\dfrac{119}{e^{6x} - 14} = 7$

Finding the Zero of a Function **In Exercises 87–90, use a graphing utility to graph the function and approximate its zero accurate to three decimal places.**

87. $g(x) = 6e^{1-x} - 25$

88. $f(x) = 3e^{3x/2} - 962$

89. $g(t) = e^{0.09t} - 3$

90. $h(t) = e^{0.125t} - 8$

Solving a Logarithmic Equation **In Exercises 91–112, solve the logarithmic equation algebraically. Round the result to three decimal places. Verify your answer using a graphing utility.**

91. $\ln x = -3$

92. $\ln x = -4$

✓ **93.** $\ln 4x = 2.1$

94. $\ln 2x = 1.5$

95. $\log_5(3x + 2) = \log_5(6 - x)$

96. $\log_9(4 + x) = \log_9(2x - 1)$

✓ **97.** $-2 + 2\ln 3x = 17$

98. $3 + 2\ln x = 10$

✓ **99.** $7\log_4(0.6x) = 12$

100. $4\log_{10}(x - 6) = 11$

101. $\log_{10}(z - 3) = 2$

102. $\log_{10} x^2 = 6$

103. $\ln\sqrt{x + 2} = 1$

104. $\ln\sqrt{x - 8} = 5$

105. $\ln(x + 1)^2 = 2$

106. $\ln(x^2 + 1) = 8$

107. $\log_4 x - \log_4(x - 1) = \frac{1}{2}$

108. $\log_3 x + \log_3(x - 8) = 2$

✓ **109.** $\ln(x + 5) = \ln(x - 1) - \ln(x + 1)$

110. $\ln(x + 1) - \ln(x - 2) = \ln x$

111. $\log_{10} 8x - \log_{10}\left(1 + \sqrt{x}\right) = 2$

112. $\log_{10} 4x - \log_{10}\left(12 + \sqrt{x}\right) = 2$

The Change-of-Base Formula **In Exercises 113 and 114, use the method of Example 10 to prove the change-of-base formula for the indicated base.**

✓ **113.** $\log_a x = \dfrac{\log_e x}{\log_e a}$

114. $\log_a x = \dfrac{\log_{10} x}{\log_{10} a}$

Algebraic-Graphical-Numerical **In Exercises 115–118, (a) complete the table to find an interval containing the solution of the equation, (b) use a graphing utility to graph both sides of the equation to estimate the solution, and (c) solve the equation algebraically. Round your results to three decimal places.**

115. $\ln 2x = 2.4$

x	2	3	4	5	6
$\ln 2x$					

116. $3\ln 5x = 10$

x	4	5	6	7	8
$3\ln 5x$					

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117. $6 \log_3(0.5x) = 11$

x	12	13	14	15	16
$6 \log_3(0.5x)$					

118. $5 \log_{10}(x - 2) = 11$

x	150	155	160	165	170
$5 \log_{10}(x - 2)$					

Approximating the Solution of an Equation **In Exercises 119–124, use the *zero* or *root* feature of a graphing utility to approximate the solution of the logarithmic equation.**

119. $\log_{10} x = x^3 - 3$ **120.** $\log_{10} x^2 = 4$

121. $\ln x + \ln(x - 2) = 1$ **122.** $\ln x + \ln(x + 1) = 2$

123. $\ln(x - 3) + \ln(x + 3) = 1$

124. $\ln x + \ln(x^2 + 4) = 10$

Finding the Point of Intersection **In Exercises 125–130, use a graphing utility to approximate the point of intersection of the graphs. Round your result to three decimal places.**

125. $y_1 = 7$, $y_2 = 2^{x-1} - 5$ **126.** $y_1 = 4$, $y_2 = 3^{x+1} - 2$

127. $y_1 = 80$, $y_2 = 4e^{-0.2x}$ **128.** $y_1 = 500$, $y_2 = 1500e^{-x/2}$

129. $y_1 = 3.25$, $y_2 = \frac{1}{2}\ln(x + 2)$ **130.** $y_1 = 1.05$, $y_2 = \ln\sqrt{x - 2}$

Solving Exponential and Logarithmic Equations **In Exercises 131–138, solve the equation algebraically. Round the result to three decimal places. Verify your answer using a graphing utility.**

131. $2x^2e^{2x} + 2xe^{2x} = 0$ **132.** $-x^2e^{-x} + 2xe^{-x} = 0$

133. $-xe^{-x} + e^{-x} = 0$ **134.** $e^{-2x} - 2xe^{-2x} = 0$

135. $2x \ln x + x = 0$ **136.** $\dfrac{1 - \ln x}{x^2} = 0$

137. $\dfrac{1 + \ln x}{2} = 0$ **138.** $2x \ln\left(\dfrac{1}{x}\right) - x = 0$

Solving a Model for x **In Exercises 139–142, the equation represents the given type of model, which you will use in Section 3.5. Solve the equation for x.**

	Model type	*Equation*
139.	Exponential growth	$y = ae^{bx}$
140.	Exponential decay	$y = ae^{-bx}$
141.	Gaussian	$y = ae^{-(x-b)^2/c}$
142.	Logarithmic	$y = a + b \ln x$

Doubling and Tripling an Investment **In Exercises 143–146, find the time required for a \$1000 investment to (a) double at interest rate r, compounded continuously, and (b) triple at interest rate r, compounded continuously. Round your results to two decimal places.**

143. $r = 7.5\%$ **144.** $r = 6\%$

145. $r = 2.5\%$ **146.** $r = 3.75\%$

147. **Economics** The demand x for a handheld electronic organizer is given by

$$p = 5000\left(1 - \frac{4}{4 + e^{-0.002x}}\right)$$

where p is the price in dollars. Find the demands x for prices of (a) $p = \$300$ and (b) $p = \$250$.

148. ***Why you should learn it*** *(p. 210)* The percent m of American males between the ages of 18 and 24 who are no more than x inches tall is modeled by

$$m(x) = \frac{100}{1 + e^{-0.6114(x-69.71)}}$$

and the percent f of American females between the ages of 18 and 24 who are no more than x inches tall is modeled by

$$f(x) = \frac{100}{1 + e^{-0.66607(x-64.51)}}.$$

(Source: U.S. National Center for Health Statistics)

(a) Use a graphing utility to graph the two functions in the same viewing window.

(b) Use the graphs in part (a) to determine the horizontal asymptotes of the functions. Interpret their meanings in the context of the problem.

(c) What is the average height for each sex?

149. **Finance** The numbers y of commercial banks in the United States from 1999 through 2009 can be modeled by

$$y = 13{,}107 - 2077.6 \ln t, \quad 9 \le t \le 19$$

where t represents the year, with $t = 9$ corresponding to 1999. In what year were there about 7100 commercial banks? (Source: Federal Deposit Insurance Corp.)

150. **Forestry** The yield V (in millions of cubic feet per acre) for a forest at age t years is given by $V = 6.7e^{-48.1/t}$.

(a) Use a graphing utility to graph the function.

(b) Determine the horizontal asymptote of the function. Interpret its meaning in the context of the problem.

(c) Find the time necessary to obtain a yield of 1.3 million cubic feet.

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151. **Science** An object at a temperature of 160°C was removed from a furnace and placed in a room at 20°C. The temperature T of the object was measured after each hour h and recorded in the table. A model for the data is given by $T = 20[1 + 7(2^{-h})]$.

Hour, h	Temperature
0	160°
1	90°
2	56°
3	38°
4	29°
5	24°

(a) Use a graphing utility to plot the data and graph the model in the same viewing window.

(b) Identify the horizontal asymptote of the graph. Interpret its meaning in the context of the problem.

(c) Approximate the time when the temperature of the object is 100°C.

152. **MODELING DATA**

The table shows the numbers N of college-bound seniors intending to major in computer or information sciences who took the SAT exam from 2001 through 2009. The data can be modeled by the logarithmic function $N = 77{,}010 - 21{,}554.3 \ln t$, where t represents the year, with $t = 1$ corresponding to 2001. (Source: The College Board)

Year	Number, N
2001	73,466
2002	68,051
2003	53,449
2004	45,879
2005	42,890
2006	37,943
2007	33,965
2008	30,495
2009	31,022

(a) According to the model, in what year would 25,325 seniors intending to major in computer or information sciences take the SAT exam?

(b) Use a graphing utility to graph the model with the data, and use the graph to verify your answer in part (a).

(c) Do you think this is a good model for predicting future values? Explain.

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Conclusions

True or False? **In Exercises 153 and 154, determine whether the statement is true or false. Justify your answer.**

153. An exponential equation must have at least one solution.

154. A logarithmic equation can have at most one extraneous solution.

155. **Error Analysis** Describe the error.

$$\begin{aligned} 2e^x &= 10 \\ \ln(2e^x) &= \ln 10 \\ 2x &= \ln 10 \\ x &= \tfrac{1}{2}\ln 10 \end{aligned}$$

156. **CAPSTONE** Write two or three sentences stating the general guidelines that you follow when you solve (a) exponential equations and (b) logarithmic equations.

157. **Think About It** Would you use a One-to-One Property or an Inverse Property to solve $5^x = 34$? Explain.

158. **Exploration** Let $f(x) = \log_a x$ and $g(x) = a^x$, where $a > 1$.

(a) Let $a = 1.2$ and use a graphing utility to graph the two functions in the same viewing window. What do you observe? Approximate any points of intersection of the two graphs.

(b) Determine the value(s) of a for which the two graphs have one point of intersection.

(c) Determine the value(s) of a for which the two graphs have two points of intersection.

159. **Think About It** Is the time required for a continuously compounded investment to quadruple twice as long as the time required for it to double? Give a reason for your answer and verify your answer algebraically.

160. **Writing** Write a paragraph explaining whether or not the time required for a continuously compounded investment to double is dependent on the size of the investment.

Cumulative Mixed Review

Sketching Graphs **In Exercises 161–166, sketch the graph of the function.**

161. $f(x) = 3x^3 - 4$

162. $f(x) = -(x + 1)^3 + 2$

163. $f(x) = |x| + 9$

164. $f(x) = |x + 2| - 8$

165. $f(x) = \begin{cases} 2x, & x < 0 \\ -x^2 + 4, & x \geq 0 \end{cases}$

166. $f(x) = \begin{cases} x - 9, & x \leq -1 \\ x^2 + 1, & x > -1 \end{cases}$

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3.5 Exponential and Logarithmic Models

Introduction

There are many examples of exponential and logarithmic models in real life. In Section 3.1, you used the formula

$$A = Pe^{rt} \qquad \text{Exponential model}$$

to find the balance in an account when the interest was compounded continuously. In Section 3.2, Example 10, you used the human memory model

$$f(t) = 75 - 6\ln(t + 1). \qquad \text{Logarithmic model}$$

The five most common types of mathematical models involving exponential functions or logarithmic functions are as follows.

1. **Exponential growth model:** $y = ae^{bx}, \quad b > 0$
2. **Exponential decay model:** $y = ae^{-bx}, \quad b > 0$
3. **Gaussian model:** $y = ae^{-(x-b)^2/c}$
4. **Logistic growth model:** $y = \dfrac{a}{1 + be^{-rx}}$
5. **Logarithmic models:** $y = a + b\ln x, \quad y = a + b\log_{10} x$

The basic shapes of these graphs are shown in Figure 3.32.

Exponential Growth Model *Exponential Decay Model* *Gaussian Model*

Logistic Growth Model *Natural Logarithmic Model* *Common Logarithmic Model*

Figure 3.32

You can often gain quite a bit of insight into a situation modeled by an exponential or logarithmic function by identifying and interpreting the function's asymptotes.

What you should learn

- Recognize the five most common types of models involving exponential or logarithmic functions.
- Use exponential growth and decay functions to model and solve real-life problems.
- Use Gaussian functions to model and solve real-life problems.
- Use logistic growth functions to model and solve real-life problems.
- Use logarithmic functions to model and solve real-life problems.

Why you should learn it

Exponential decay models are used in carbon dating. For instance, in Exercise 37 on page 230, you will use an exponential decay model to estimate the age of a piece of ancient charcoal.

Archaeologist

Explore the Concept

Use a graphing utility to graph each model shown in Figure 3.32. Use the *table* and *trace* features of the graphing utility to identify the asymptotes of the graph of each function.

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Exponential Growth and Decay

Example 1 Demography

Estimates of the world population (in millions) from 2003 through 2009 are shown in the table. A scatter plot of the data is shown in Figure 3.33. (Source: U.S. Census Bureau)

Year	Population, P
2003	6313
2004	6387
2005	6462
2006	6538
2007	6615
2008	6691
2009	6768

Figure 3.33

An exponential growth model that approximates these data is given by

$$P = 6097e^{0.0116t}, \quad 3 \le t \le 9$$

where P is the population (in millions) and $t = 3$ represents 2003. Compare the values given by the model with the estimates shown in the table. According to this model, when will the world population reach 7.1 billion?

Algebraic Solution

The following table compares the two sets of population figures.

Year	2003	2004	2005	2006	2007	2008	2009
Population	6313	6387	6462	6538	6615	6691	6768
Model	6313	6387	6461	6536	6613	6690	6768

From the table, it appears that the model is a good fit for the data. To find when the world population will reach 7.1 billion, let

$$P = 7100$$

in the model and solve for t.

$6097e^{0.0116t} = P$	Write original equation.
$6097e^{0.0116t} = 7100$	Substitute 7100 for P.
$e^{0.0116t} \approx 1.16451$	Divide each side by 6097.
$\ln e^{0.0116t} \approx \ln 1.16451$	Take natural log of each side.
$0.0116t \approx 0.15230$	Inverse Property
$t \approx 13.1$	Divide each side by 0.0116.

According to the model, the world population will reach 7.1 billion in 2013.

CHECKPOINT Now try Exercise 33.

Graphical Solution

The model appears to fit the data closely.

The intersection point of the model and the line $y = 7100$ is about (13.1, 7100). So, according to the model, the world population will reach 7.1 billion in 2013.

An exponential model increases (or decreases) by the same percent each year. What is the annual percent increase for the model in Example 1?

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In Example 1, you were given the exponential growth model. Sometimes you must find such a model. One technique for doing this is shown in Example 2.

Example 2 Modeling Population Growth

In a research experiment, a population of fruit flies is increasing according to the law of exponential growth. After 2 days there are 100 flies, and after 4 days there are 300 flies. How many flies will there be after 5 days?

Solution

Let y be the number of flies at time t (in days). From the given information, you know that $y = 100$ when $t = 2$ and $y = 300$ when $t = 4$. Substituting this information into the model $y = ae^{bt}$ produces

$$100 = ae^{2b} \quad \text{and} \quad 300 = ae^{4b}.$$

To solve for b, solve for a in the first equation.

$$100 = ae^{2b} \quad \Rightarrow \quad a = \frac{100}{e^{2b}} \qquad \text{Solve for } a \text{ in the first equation.}$$

Then substitute the result into the second equation.

$$300 = ae^{4b} \qquad \text{Write second equation.}$$

$$300 = \left(\frac{100}{e^{2b}}\right)e^{4b} \qquad \text{Substitute } \frac{100}{e^{2b}} \text{ for } a.$$

$$300 = 100e^{2b} \qquad \text{Simplify.}$$

$$3 = e^{2b} \qquad \text{Divide each side by 100.}$$

$$\ln 3 = \ln e^{2b} \qquad \text{Take natural log of each side.}$$

$$\ln 3 = 2b \qquad \text{Inverse Property}$$

$$\frac{1}{2}\ln 3 = b \qquad \text{Solve for } b.$$

Using $b = \frac{1}{2}\ln 3$ and the equation you found for a, you can determine that

$$a = \frac{100}{e^{2[(1/2)\ln 3]}} \qquad \text{Substitute } \tfrac{1}{2}\ln 3 \text{ for } b.$$

$$= \frac{100}{e^{\ln 3}} \qquad \text{Simplify.}$$

$$= \frac{100}{3} \qquad \text{Inverse Property}$$

$$\approx 33.33. \qquad \text{Simplify.}$$

So, with $a \approx 33.33$ and

$$b = \tfrac{1}{2}\ln 3 \approx 0.5493$$

the exponential growth model is

$$y = 33.33e^{0.5493t},$$

as shown in Figure 3.34. This implies that after 5 days, the population will be

$$y = 33.33e^{0.5493(5)} \approx 520 \text{ flies.}$$

Figure 3.34

CHECKPOINT Now try Exercise 35.

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In living organic material, the ratio of the content of radioactive carbon isotopes (carbon 14) to the content of nonradioactive carbon isotopes (carbon 12) is about 1 to 10^{12}. When organic material dies, its carbon 12 content remains fixed, whereas its radioactive carbon 14 begins to decay with a half-life of 5700 years. To estimate the age of dead organic material, scientists use the following formula, which denotes the ratio of carbon 14 to carbon 12 present at any time t (in years).

Figure 3.35

$$R = \frac{1}{10^{12}} e^{-t/8223} \qquad \text{Carbon dating model}$$

The graph of R is shown in Figure 3.35. Note that R decreases as t increases.

Example 3 Carbon Dating

The ratio of carbon 14 to carbon 12 in a newly discovered fossil is

$$R = \frac{1}{10^{13}}.$$

Estimate the age of the fossil.

Algebraic Solution

In the carbon dating model, substitute the given value of R to obtain the following.

$$\frac{1}{10^{12}} e^{-t/8223} = R \qquad \text{Write original model.}$$

$$\frac{e^{-t/8223}}{10^{12}} = \frac{1}{10^{13}} \qquad \text{Substitute } \frac{1}{10^{13}} \text{ for } R.$$

$$e^{-t/8223} = \frac{1}{10} \qquad \text{Multiply each side by } 10^{12}.$$

$$\ln e^{-t/8223} = \ln \frac{1}{10} \qquad \text{Take natural log of each side.}$$

$$-\frac{t}{8223} \approx -2.3026 \qquad \text{Inverse Property}$$

$$t \approx 18{,}934 \qquad \text{Multiply each side by } -8223.$$

So, to the nearest thousand years, you can estimate the age of the fossil to be 19,000 years.

CHECKPOINT Now try Exercise 37.

Graphical Solution

Use a graphing utility to graph the formula for the ratio of carbon 14 to carbon 12 at any time t as

$$y_1 = \frac{1}{10^{12}} e^{-x/8223}.$$

In the same viewing window, graph $y_2 = 1/(10^{13})$, as shown in Figure 3.36.

Use the *intersect* feature to estimate that $x \approx 18{,}934$ when $y = 1/(10^{13})$.

Figure 3.36

So, to the nearest thousand years, you can estimate the age of the fossil to be 19,000 years.

The carbon dating model in Example 3 assumed that the carbon 14 to carbon 12 ratio was one part in 10,000,000,000,000. Suppose an error in measurement occurred and the actual ratio was only one part in 8,000,000,000,000. The fossil age corresponding to the actual ratio would then be approximately 17,000 years. Try checking this result.

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Gaussian Models

As mentioned at the beginning of this section, Gaussian models are of the form

$$y = ae^{-(x-b)^2/c}.$$

This type of model is commonly used in probability and statistics to represent populations that are **normally distributed.** For *standard* normal distributions, the model takes the form

$$y = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}.$$

The graph of a Gaussian model is called a **bell-shaped curve.** Try graphing the normal distribution curve with a graphing utility. Can you see why it is called a bell-shaped curve?

The average value for a population can be found from the bell-shaped curve by observing where the maximum y-value of the function occurs. The x-value corresponding to the maximum y-value of the function represents the average value of the independent variable—in this case, x.

Example 4 SAT Scores

In 2009, the Scholastic Aptitude Test (SAT) mathematics scores for college-bound seniors roughly followed the normal distribution

$$y = 0.0034e^{-(x-515)^2/26{,}912}, \quad 200 \le x \le 800$$

where x is the SAT score for mathematics. Use a graphing utility to graph this function and estimate the average SAT score. (Source: College Board)

Solution

The graph of the function is shown in Figure 3.37. On this bell-shaped curve, the maximum value of the curve represents the average score. Using the *maximum* feature of the graphing utility, you can see that the average mathematics score for college-bound seniors in 2009 was 515.

Figure 3.37

CHECKPOINT Now try Exercise 41.

In Example 4, note that 50% of the seniors who took the test received scores lower than 515 (see Figure 3.38).

Figure 3.38

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Logistic Growth Models

Some populations initially have rapid growth, followed by a declining rate of growth, as indicated by the graph in Figure 3.39. One model for describing this type of growth pattern is the **logistic curve** given by the function

$$y = \frac{a}{1 + be^{-rx}}$$

where y is the population size and x is the time. An example is a bacteria culture that is initially allowed to grow under ideal conditions, and then under less favorable conditions that inhibit growth. A logistic growth curve is also called a **sigmoidal curve.**

Figure 3.39 *Logistic Curve*

Example 5 Spread of a Virus

On a college campus of 5000 students, one student returns from vacation with a contagious flu virus. The spread of the virus is modeled by

$$y = \frac{5000}{1 + 4999e^{-0.8t}}, \quad t \geq 0$$

where y is the total number of students infected after t days. The college will cancel classes when 40% or more of the students are infected.

a. How many students are infected after 5 days?

b. After how many days will the college cancel classes?

Algebraic Solution

a. After 5 days, the number of students infected is

$$\begin{aligned} y &= \frac{5000}{1 + 4999e^{-0.8(5)}} \\ &= \frac{5000}{1 + 4999e^{-4}} \\ &\approx 54. \end{aligned}$$

b. Classes are canceled when the number of infected students is $(0.40)(5000) = 2000$.

$$\begin{aligned} 2000 &= \frac{5000}{1 + 4999e^{-0.8t}} \\ 1 + 4999e^{-0.8t} &= 2.5 \\ e^{-0.8t} &= \frac{1.5}{4999} \\ \ln e^{-0.8t} &= \ln \frac{1.5}{4999} \\ -0.8t &= \ln \frac{1.5}{4999} \\ t &= -\frac{1}{0.8} \ln \frac{1.5}{4999} \\ t &\approx 10.14 \end{aligned}$$

So, after about 10 days, at least 40% of the students will be infected, and classes will be canceled.

CHECKPOINT Now try Exercise 43.

Graphical Solution

a. Use the *value* feature to estimate that $y \approx 54$ when $x = 5$. So, after 5 days, about 54 students will be infected.

b. Classes are canceled when the number of infected students is $(0.40)(5000) = 2000$. Use a graphing utility to graph

$$y_1 = \frac{5000}{1 + 4999e^{-0.8x}} \quad \text{and} \quad y_2 = 2000$$

in the same viewing window. Use the *intersect* feature of the graphing utility to find the point of intersection of the graphs, as shown in Figure 3.40.

The intersection point occurs near $x \approx 10.14$. So, after about 10 days, at least 40% of the students will be infected, and classes will be canceled.

Figure 3.40

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Logarithmic Models

On the Richter scale, the magnitude R of an earthquake of intensity I is given by

$$R = \log_{10} \frac{I}{I_0}$$

where $I_0 = 1$ is the minimum intensity used for comparison. Intensity is a measure of the wave energy of an earthquake.

Earthquake Relief Worker

Example 6 Magnitudes of Earthquakes

In 2009, Crete, Greece experienced an earthquake that measured 6.4 on the Richter scale. Also in 2009, the north coast of Indonesia experienced an earthquake that measured 7.6 on the Richter scale. Find the intensity of each earthquake and compare the two intensities.

Solution

Because $I_0 = 1$ and $R = 6.4$, you have

$$6.4 = \log_{10} \frac{I}{1}$$

$$10^{6.4} = 10^{\log_{10} I} \quad \Rightarrow \quad 10^{6.4} = I.$$

For $R = 7.6$, you have

$$7.6 = \log_{10} \frac{I}{1}$$

$$10^{7.6} = 10^{\log_{10} I} \quad \Rightarrow \quad 10^{7.6} = I.$$

Note that an increase of 1.2 units on the Richter scale (from 6.4 to 7.6) represents an increase in intensity by a factor of

$$\frac{10^{7.6}}{10^{6.4}} = 10^{1.2} \approx 16.$$

In other words, the intensity of the earthquake near the north coast of Indonesia was about 16 times as great as the intensity of the earthquake in Greece.

CHECKPOINT Now try Exercise 45.

Example 7 pH Levels

Acidity, or pH level, is a measure of the hydrogen ion concentration $[H^+]$ (measured in moles of hydrogen per liter) of a solution. Use the model given by $pH = -\log_{10}[H^+]$ to determine the hydrogen ion concentration of milk of magnesia, which has a pH of 10.5.

Solution

$pH = -\log_{10}[H^+]$	Write original model.
$10.5 = -\log_{10}[H^+]$	Substitute 10.5 for pH.
$-10.5 = \log_{10}[H^+]$	Multiply each side by -1.
$10^{-10.5} = 10^{\log_{10}[H^+]}$	Exponentiate each side (base 10).
$3.16 \times 10^{-11} = [H^+]$	Simplify.

So, the hydrogen ion concentration of milk of magnesia is 3.16×10^{-11} mole of hydrogen per liter.

CHECKPOINT Now try Exercise 51.

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3.5 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

1. Match the equation with its model.

(a) Exponential growth model — (i) $y = ae^{-bx},\ b > 0$

(b) Exponential decay model — (ii) $y = a + b \ln x$

(c) Logistic growth model — (iii) $y = \dfrac{a}{1 + be^{-rx}}$

(d) Gaussian model — (iv) $y = ae^{bx},\ b > 0$

(e) Natural logarithmic model — (v) $y = a + b \log_{10} x$

(f) Common logarithmic model — (vi) $y = ae^{-(x-b)^2/c}$

In Exercises 2 and 3, fill in the blank.

2. Gaussian models are commonly used in probability and statistics to represent populations that are _____ distributed.

3. Logistic growth curves are also called _____ curves.

4. Which model in Exercise 1 has a graph called a bell-shaped curve?

5. Does the model $y = 120e^{-0.25x}$ represent exponential growth or exponential decay?

6. Which model in Exercise 1 has a graph with two horizontal asymptotes?

Procedures and Problem Solving

Identifying Graphs of Models **In Exercises 7–12, match the function with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]**

(a)

(b)

(c)

(d)

(e)

(f)

7. $y = 2e^{x/4}$ **8.** $y = 6e^{-x/4}$

9. $y = 6 + \log_{10}(x + 2)$ **10.** $y = 3e^{-(x-2)^2/5}$

11. $y = \ln(x + 1)$ **12.** $y = \dfrac{4}{1 + e^{-2x}}$

Using a Compound Interest Formula **In Exercises 13–20, complete the table for a savings account in which interest is compounded continuously.**

	Initial Investment	*Annual % Rate*	*Time to Double*	*Amount After 10 Years*
13.	\$10,000	3.5%		
14.	\$2000	1.5%		
15.	\$7500		21 years	
16.	\$1000		12 years	
17.	\$5000			\$5665.74
18.	\$300			\$385.21
19.		4.5%		\$100,000.00
20.		2%		\$2500.00

21. **Tripling an Investment** Complete the table for the time t (in years) necessary for P dollars to triple when interest is compounded continuously at rate r. Create a scatter plot of the data.

r	2%	4%	6%	8%	10%	12%
t						

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22. Tripling an Investment Complete the table for the time t (in years) necessary for P dollars to triple when interest is compounded annually at rate r. Create a scatter plot of the data.

r	2%	4%	6%	8%	10%	12%
t						

23. Finance When \$1 is invested in an account over a 10-year period, the amount A in the account after t years is given by

$$A = 1 + 0.075[\![t]\!] \quad \text{or} \quad A = e^{0.07t}$$

depending on whether the account pays simple interest at $7\frac{1}{2}$% or continuous compound interest at 7%. Use a graphing utility to graph each function in the same viewing window. Which grows at a greater rate? (Remember that $[\![t]\!]$ is the greatest integer function discussed in Section 1.3.)

24. Finance When \$1 is invested in an account over a 10-year period, the amount A in the account after t years is given by

$$A = 1 + 0.06[\![t]\!] \quad \text{or} \quad A = \left(1 + \frac{0.055}{365}\right)^{[\![365t]\!]}$$

depending on whether the account pays simple interest at 6% or compound interest at $5\frac{1}{2}$% compounded daily. Use a graphing utility to graph each function in the same viewing window. Which grows at a greater rate?

Radioactive Decay **In Exercises 25–28, complete the table for the radioactive isotope.**

	Isotope	*Half-Life (years)*	*Initial Quantity*	*Amount After 1000 Years*
25.	^{226}Ra	1599	10 g	
26.	^{226}Ra	1599		1.5 g
27.	^{14}C	5700	3 g	
28.	^{239}Pu	24,100		0.4 g

Identifying a Model **In Exercises 29–32, find the exponential model $y = ae^{bx}$ that fits the points shown in the graph or table.**

29.

30.

31.

x	0	5
y	4	1

32.

x	0	3
y	1	$\frac{1}{4}$

33. Demography The populations P (in thousands) of Pittsburgh, Pennsylvania from 2000 through 2008 can be modeled by $P = 333.68e^{-0.0099t}$, where t is the year, with $t = 0$ corresponding to 2000. (Source: U.S. Census Bureau)

(a) According to the model, was the population of Pittsburgh increasing or decreasing from 2000 through 2008? Explain your reasoning.

(b) What were the populations of Pittsburgh in 2000, 2005, and 2008?

(c) According to the model, when will the population of Pittsburgh be approximately 290,000?

34. MODELING DATA

The table shows the populations (in millions) of five countries in 2005 and the projected populations (in millions) for 2015. (Source: U.S. Census Bureau)

Country	2005	2015
Australia	20.2	22.8
Canada	32.4	35.1
Hungary	10.0	9.7
Philippines	90.4	109.6
Turkey	72.7	82.5

(a) Find the exponential growth or decay model, $y = ae^{bt}$ or $y = ae^{-bt}$, for the population of each country, where t is the year, with $t = 5$ corresponding to 2005. Use the model to predict the population of each country in 2030.

(b) You can see that the populations of Canada and the Philippines are growing at different rates. What constant in the equation $y = ae^{bt}$ is determined by these different growth rates? Discuss the relationship between the different growth rates and the magnitude of the constant.

(c) The population of Turkey is increasing while the population of Hungary is decreasing. What constant in the equation $y = ae^{bt}$ reflects this difference? Explain.

35. Demography The populations P (in thousands) of San Antonio, Texas from 2000 through 2008 can be modeled by

$$P = 1155.4e^{kt}$$

where t is the year, with $t = 0$ corresponding to 2000. In 2002, the population was 1,200,000. (Source: U.S. Census Bureau)

(a) Find the value of k for the model. Round your result to four decimal places.

(b) Use your model to predict the population in 2015.

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36. Demography The populations P (in thousands) of Raleigh, North Carolina from 2000 through 2008 can be modeled by $P = 289.81e^{kt}$, where t is the year, with $t = 0$ corresponding to 2000. In 2006, the population was 363,000. (Source: U.S. Census Bureau)

(a) Find the value of k for the model. Round your result to four decimal places.

(b) Use your model to predict the population in 2015.

37. ***Why you should learn it*** *(p. 221)* Carbon 14 (^{14}C) dating assumes that the carbon dioxide on Earth today has the same radioactive content as it did centuries ago. If this is true, then the amount of ^{14}C absorbed by a tree that grew several centuries ago should be the same as the amount of ^{14}C absorbed by a tree growing today. A piece of ancient charcoal contains only 15% as much radioactive carbon as a piece of modern charcoal. How long ago was the tree burned to make the ancient charcoal given that the half-life of ^{14}C is 5700 years?

38. Radioactive Decay The half-life of radioactive radium (^{226}Ra) is 1599 years. What percent of a present amount of radioactive radium will remain after 100 years?

39. MODELING DATA

A new 2009 luxury sedan that sold for \$49,200 has a book value V of \$32,590 after 2 years.

(a) Find a linear model for the value V of the sedan.

(b) Find an exponential model for the value V of the sedan. Round the numbers in the model to four decimal places.

(c) Use a graphing utility to graph the two models in the same viewing window.

(d) Which model represents a greater depreciation rate in the first year?

(e) For what years is the value of the sedan greater using the linear model? the exponential model?

40. MODELING DATA

A new laptop computer that sold for \$935 in 2009 has a book value V of \$385 after 2 years.

(a) Find a linear model for the value V of the laptop.

(b) Find an exponential model for the value V of the laptop. Round the numbers in the model to four decimal places.

(c) Use a graphing utility to graph the two models in the same viewing window.

(d) Which model represents a greater depreciation rate in the first year?

(e) For what years is the value of the laptop greater using the linear model? the exponential model?

41. Psychology The IQ scores for adults roughly follow the normal distribution $y = 0.0266e^{-(x-100)^2/450}$, $70 \le x \le 115$, where x is the IQ score.

(a) Use a graphing utility to graph the function.

(b) Use the graph in part (a) to estimate the average IQ score.

42. Marketing The sales S (in thousands of units) of a cleaning solution after x hundred dollars is spent on advertising are given by $S = 10(1 - e^{kx})$. When \$500 is spent on advertising, 2500 units are sold.

(a) Complete the model by solving for k.

(b) Estimate the number of units that will be sold when advertising expenditures are raised to \$700.

43. Forestry A conservation organization releases 100 animals of an endangered species into a game preserve. The organization believes that the preserve has a carrying capacity of 1000 animals and that the growth of the herd will follow the logistic curve

$$p(t) = \frac{1000}{1 + 9e^{-0.1656t}}$$

where t is measured in months.

(a) What is the population after 5 months?

(b) After how many months will the population reach 500?

(c) Use a graphing utility to graph the function. Use the graph to determine the values of p at which the horizontal asymptotes occur. Identify the asymptote that is most relevant in the context of the problem and interpret its meaning.

44. Biology The number Y of yeast organisms in a culture is given by the model

$$Y = \frac{663}{1 + 72e^{-0.547t}}, \quad 0 \le t \le 18$$

where t represents the time (in hours).

(a) Use a graphing utility to graph the model.

(b) Use the model to predict the populations for the 19th hour and the 30th hour.

(c) According to this model, what is the limiting value of the population?

(d) Why do you think this population of yeast follows a logistic growth model instead of an exponential growth model?

Geology **In Exercises 45 and 46, use the Richter scale (see page 227) for measuring the magnitudes of earthquakes.**

45. Find the intensities I of the following earthquakes measuring R on the Richter scale (let $I_0 = 1$). (Source: U.S. Geological Survey)

(a) Haiti in 2010, $R = 7.0$

(b) Samoa Islands in 2009, $R = 8.1$

(c) Virgin Islands in 2008, $R = 6.1$

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46. Find the magnitudes R of the following earthquakes of intensity I (let $I_0 = 1$).

(a) $I = 39{,}811{,}000$

(b) $I = 12{,}589{,}000$

(c) $I = 251{,}200$

Audiology In Exercises 47–50, use the following information for determining sound intensity. The level of sound β (in decibels) with an intensity I is

$$\beta = 10 \log_{10} \frac{I}{I_0}$$

where I_0 is an intensity of 10^{-12} watt per square meter, corresponding roughly to the faintest sound that can be heard by the human ear. In Exercises 47 and 48, find the level of each sound β.

47. (a) $I = 10^{-10}$ watt per m^2 (quiet room)

(b) $I = 10^{-5}$ watt per m^2 (busy street corner)

(c) $I \approx 10^0$ watt per m^2 (threshold of pain)

48. (a) $I = 10^{-4}$ watt per m^2 (door slamming)

(b) $I = 10^{-3}$ watt per m^2 (loud car horn)

(c) $I = 10^{-2}$ watt per m^2 (siren at 30 meters)

49. As a result of the installation of a muffler, the noise level of an engine was reduced from 88 to 72 decibels. Find the percent decrease in the intensity level of the noise due to the installation of the muffler.

50. As a result of the installation of noise suppression materials, the noise level in an auditorium was reduced from 93 to 80 decibels. Find the percent decrease in the intensity level of the noise due to the installation of these materials.

Chemistry In Exercises 51–54, use the acidity model given in Example 7.

51. Find the pH when $[H^+] = 2.3 \times 10^{-5}$.

52. Compute $[H^+]$ for a solution for which pH $= 5.8$.

53. A grape has a pH of 3.5, and baking soda has a pH of 8.0. The hydrogen ion concentration of the grape is how many times that of the baking soda?

54. The pH of a solution is decreased by one unit. The hydrogen ion concentration is increased by what factor?

55. Finance The total interest u paid on a home mortgage of P dollars at interest rate r for t years is given by

$$u = P\left[\frac{rt}{1 - \left(\dfrac{1}{1 + r/12}\right)^{12t}} - 1\right].$$

Consider a \$120,000 home mortgage at $7\frac{1}{2}$%.

(a) Use a graphing utility to graph the total interest function.

(b) Approximate the length of the mortgage when the total interest paid is the same as the amount of the mortgage. Is it possible that a person could pay twice as much in interest charges as the amount of his or her mortgage?

56. Finance A \$120,000 home mortgage for 30 years at $7\frac{1}{2}$% has a monthly payment of \$839.06. Part of the monthly payment goes toward the interest charge on the unpaid balance, and the remainder of the payment is used to reduce the principal. The amount that goes toward the interest is given by

$$u = M - \left(M - \frac{Pr}{12}\right)\left(1 + \frac{r}{12}\right)^{12t}$$

and the amount that goes toward reduction of the principal is given by

$$v = \left(M - \frac{Pr}{12}\right)\left(1 + \frac{r}{12}\right)^{12t}.$$

In these formulas, P is the size of the mortgage, r is the interest rate, M is the monthly payment, and t is the time (in years).

(a) Use a graphing utility to graph each function in the same viewing window. (The viewing window should show all 30 years of mortgage payments.)

(b) In the early years of the mortgage, the larger part of the monthly payment goes for what purpose? Approximate the time when the monthly payment is evenly divided between interest and principal reduction.

(c) Repeat parts (a) and (b) for a repayment period of 20 years ($M = \$966.71$). What can you conclude?

57. Forensics At 8:30 A.M., a coroner was called to the home of a person who had died during the night. In order to estimate the time of death, the coroner took the person's temperature twice. At 9:00 A.M. the temperature was 85.7°F, and at 11:00 A.M. the temperature was 82.8°F. From these two temperatures the coroner was able to determine that the time elapsed since death and the body temperature were related by the formula

$$t = -10 \ln \frac{T - 70}{98.6 - 70}$$

where t is the time (in hours elapsed since the person died) and T is the temperature (in degrees Fahrenheit) of the person's body. Assume that the person had a normal body temperature of 98.6°F at death and that the room temperature was a constant 70°F. Use the formula to estimate the time of death of the person. (This formula is derived from a general cooling principle called Newton's Law of Cooling.)

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58. **Culinary Arts** You take a five-pound package of steaks out of a freezer at 11 A.M. and place it in a refrigerator. Will the steaks be thawed in time to be grilled at 6 P.M.? Assume that the refrigerator temperature is 40°F and the freezer temperature is 0°F. Use the formula for Newton's Law of Cooling

$$t = -5.05 \ln \frac{T - 40}{0 - 40}$$

where t is the time in hours (with $t = 0$ corresponding to 11 A.M.) and T is the temperature of the package of steaks (in degrees Fahrenheit).

Conclusions

True or False? In Exercises 59 and 60, determine whether the statement is true or false. Justify your answer.

59. The domain of a logistic growth function cannot be the set of real numbers.

60. The graph of a logistic growth function will always have an x-intercept.

61. **Think About It** Can the graph of a Gaussian model ever have an x-intercept? Explain.

62. **CAPSTONE** For each graph, state whether an exponential, Gaussian, logarithmic, logistic, or quadratic model will fit the data best. Explain your reasoning. Then describe a real-life situation that could be represented by the data.

Identifying Models In Exercises 63–66, identify the type of model you studied in this section that has the given characteristic.

63. The maximum value of the function occurs at the average value of the independent variable.

64. A horizontal asymptote of its graph represents the limiting value of a population.

65. Its graph shows a steadily increasing rate of growth.

66. The only asymptote of its graph is a vertical asymptote.

Cumulative Mixed Review

Identifying Graphs of Linear Equations In Exercises 67–70, match the equation with its graph, and identify any intercepts. [The graphs are labeled (a), (b), (c), and (d).]

67. $4x - 3y - 9 = 0$

68. $2x + 5y - 10 = 0$

69. $y = 25 - 2.25x$

70. $\frac{x}{2} + \frac{y}{4} = 1$

Applying the Leading Coefficient Test In Exercises 71–74, use the Leading Coefficient Test to determine the right-hand and left-hand behavior of the graph of the polynomial function.

71. $f(x) = 2x^3 - 3x^2 + x - 1$

72. $f(x) = 5 - x^2 - 4x^4$

73. $g(x) = -1.6x^5 + 4x^2 - 2$

74. $g(x) = 7x^6 + 9.1x^5 - 3.2x^4 + 25x^3$

Using Synthetic Division In Exercises 75 and 76, divide using synthetic division.

75. $(2x^3 - 8x^2 + 3x - 9) \div (x - 4)$

76. $(x^4 - 3x + 1) \div (x + 5)$

77. ***Make a Decision*** To work an extended application analyzing the net sales for Kohl's Corporation from 1992 through 2008, visit this textbook's *Companion Website*. (Data Source: Kohl's Illinois, Inc.)

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3.6 Nonlinear Models

Classifying Scatter Plots

In Section 1.7, you saw how to fit linear models to data, and in Section 2.8, you saw how to fit quadratic models to data. In real life, many relationships between two variables are represented by different types of growth patterns. A scatter plot can be used to give you an idea of which type of model will best fit a set of data.

What you should learn

- Classify scatter plots.
- Use scatter plots and a graphing utility to find models for data and choose the model that best fits a set of data.
- Use a graphing utility to find exponential and logistic models for data.

Why you should learn it

Many real-life applications can be modeled by nonlinear equations. For instance, in Exercise 34 on page 240, you are asked to find a nonlinear model that relates air pressure to altitude.

Example 1 Classifying Scatter Plots

Decide whether each set of data could best be modeled by a linear model, $y = ax + b$, an exponential model, $y = ab^x$, or a logarithmic model, $y = a + b \ln x$.

a. (2, 1), (2.5, 1.2), (3, 1.3), (3.5, 1.5), (4, 1.8), (4.5, 2), (5, 2.4), (5.5, 2.5), (6, 3.1), (6.5, 3.8), (7, 4.5), (7.5, 5), (8, 6.5), (8.5, 7.8), (9, 9), (9.5, 10)

b. (2, 2), (2.5, 3.1), (3, 3.8), (3.5, 4.3), (4, 4.6), (4.5, 5.3), (5, 5.6), (5.5, 5.9), (6, 6.2), (6.5, 6.4), (7, 6.9), (7.5, 7.2), (8, 7.6), (8.5, 7.9), (9, 8), (9.5, 8.2)

c. (2, 1.9), (2.5, 2.5), (3, 3.2), (3.5, 3.6), (4, 4.3), (4.5, 4.7), (5, 5.2), (5.5, 5.7), (6, 6.4), (6.5, 6.8), (7, 7.2), (7.5, 7.9), (8, 8.6), (8.5, 8.9), (9, 9.5), (9.5, 9.9)

Solution

a. From Figure 3.41, it appears that the data can best be modeled by an exponential function.

Figure 3.41

b. From Figure 3.42, it appears that the data can best be modeled by a logarithmic function.

Figure 3.42

c. From Figure 3.43, it appears that the data can best be modeled by a linear function.

Figure 3.43

CHECKPOINT Now try Exercise 13.

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Fitting Nonlinear Models to Data

Once you have used a scatter plot to determine the type of model that would best fit a set of data, there are several ways that you can actually find the model. Each method is best used with a computer or calculator, rather than with hand calculations.

Example 2 Fitting a Model to Data

Fit the following data from Example 1(a) to an exponential model and a power model. Identify the coefficient of determination and determine which model fits the data better.

(2, 1), (2.5, 1.2), (3, 1.3), (3.5, 1.5),

(4, 1.8), (4.5, 2), (5, 2.4), (5.5, 2.5),

(6, 3.1), (6.5, 3.8), (7, 4.5), (7.5, 5),

(8, 6.5), (8.5, 7.8), (9, 9), (9.5, 10)

Solution

Begin by entering the data into a graphing utility. Then use the *regression* feature of the graphing utility to find exponential and power models for the data, as shown in Figure 3.44.

Exponential Model

Power Model

Figure 3.44

So, an exponential model for the data is $y = 0.507(1.368)^x$, and a power model for the data is $y = 0.249x^{1.518}$. Plot the data and each model in the same viewing window, as shown in Figure 3.45. To determine which model fits the data better, compare the coefficients of determination for each model. The model whose r^2-value is closest to 1 is the model that better fits the data. In this case, the better-fitting model is the exponential model.

Exponential Model

Power Model

Figure 3.45

CHECKPOINT Now try Exercise 31.

Deciding which model best fits a set of data is a question that is studied in detail in statistics. Recall from Section 1.7 that the model that best fits a set of data is the one whose *sum of squared differences* is the least. In Example 2, the sums of squared differences are 0.90 for the exponential model and 14.30 for the power model.

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Example 3 Fitting a Model to Data

The table shows the yield y (in milligrams) of a chemical reaction after x minutes. Use a graphing utility to find a logarithmic model and a linear model for the data and identify the coefficient of determination for each model. Determine which model fits the data better.

Minutes, x	Yield, y
1	1.5
2	7.4
3	10.2
4	13.4
5	15.8
6	16.3
7	18.2
8	18.3

Solution

Begin by entering the data into a graphing utility. Then use the *regression* feature of the graphing utility to find logarithmic and linear models for the data, as shown in Figure 3.46.

Logarithmic Model

Linear Model

Figure 3.46

So, a logarithmic model for the data is $y = 1.538 + 8.373 \ln x$ and a linear model for the data is $y = 2.29x + 2.3$. Plot the data and each model in the same viewing window, as shown in Figure 3.47. To determine which model fits the data better, compare the coefficients of determination for each model. The model whose coefficient of determination is closer to 1 is the model that better fits the data. In this case, the better-fitting model is the logarithmic model.

Logarithmic Model

Linear Model

Figure 3.47

CHECKPOINT Now try Exercise 33.

In Example 3, the sum of the squared differences for the logarithmic model is 1.59 and the sum of the squared differences for the linear model is 24.31.

Explore the Concept

Use a graphing utility to find a quadratic model for the data in Example 3. Do you think this model fits the data better than the logarithmic model in Example 3? Explain your reasoning.

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Modeling With Exponential and Logistic Functions

Example 4 Fitting an Exponential Model to Data

The table at the right shows the amounts of revenue R (in billions of dollars) collected by the Internal Revenue Service (IRS) for selected years from 1963 through 2008. Use a graphing utility to find a model for the data. Then use the model to estimate the revenue collected in 2013. (Source: IRS Data Book)

Year	Revenue, R
1963	105.9
1968	153.6
1973	237.8
1978	399.8
1983	627.2
1988	935.1
1993	1176.7
1998	1769.4
2003	1952.9
2008	2745.0

Solution

Let x represent the year, with $x = 3$ corresponding to 1963. Begin by entering the data into a graphing utility and displaying the scatter plot, as shown in Figure 3.48.

Figure 3.48

Figure 3.49

From the scatter plot, it appears that an exponential model is a good fit. Use the *regression* feature of the graphing utility to find the exponential model, as shown in Figure 3.49. Change the model to a natural exponential model, as follows.

$$R = 96.56(1.076)^x \qquad \text{Write original model.}$$

$$= 96.56e^{(\ln 1.076)x} \qquad b = e^{\ln b}$$

$$\approx 96.56e^{0.073x} \qquad \text{Simplify.}$$

Graph the data and the natural exponential model

$$R = 96.56e^{0.073x}$$

in the same viewing window, as shown in Figure 3.50. From the model, you can see that the revenue collected by the IRS from 1963 through 2008 had an average annual increase of about 7%. From this model, you can estimate the 2013 revenue to be

$$R = 96.56e^{0.073x} \qquad \text{Write natural exponential model.}$$

$$= 96.56e^{0.073(53)} \qquad \text{Substitute 53 for } x.$$

$$\approx \$4624.7 \text{ billion} \qquad \text{Use a calculator.}$$

which is more than twice the amount collected in 2003. You can also use the *value* feature of the graphing utility to approximate the revenue in 2013 to be \$4624.7 billion, as shown in Figure 3.50.

Figure 3.50

CHECKPOINT Now try Exercise 35.

Study Tip

You can change an exponential model of the form

$$y = ab^x$$

to one of the form

$$y = ae^{cx}$$

by rewriting b in the form

$$b = e^{\ln b}.$$

For instance,

$$y = 3(2^x)$$

can be written as

$$y = 3(2^x) = 3e^{(\ln 2)x} \approx 3e^{0.693x}.$$

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The next example demonstrates how to use a graphing utility to fit a logistic model to data.

Forester

Example 5 Fitting a Logistic Model to Data

To estimate the amount of defoliation caused by the gypsy moth during a given year, a forester counts the number x of egg masses on $\frac{1}{40}$ of an acre (circle of radius 18.6 feet) in the fall. The percent of defoliation y the next spring is shown in the table. (Source: USDA, Forest Service)

Egg masses, x	Percent of defoliation, y
0	12
25	44
50	81
75	96
100	99

a. Use the *regression* feature of a graphing utility to find a logistic model for the data.

b. How closely does the model represent the data?

Graphical Solution

a. Enter the data into a graphing utility. Using the *regression* feature of the graphing utility, you can find the logistic model, as shown in Figure 3.51.

Figure 3.51

You can approximate this model to be

$$y = \frac{100}{1 + 7e^{-0.069x}}.$$

b. You can use the graphing utility to graph the actual data and the model in the same viewing window. In Figure 3.52, it appears that the model is a good fit for the actual data.

Figure 3.52

CHECKPOINT Now try Exercise 37.

Numerical Solution

a. Enter the data into a graphing utility. Using the *regression* feature of the graphing utility, you can approximate the logistic model to be

$$y = \frac{100}{1 + 7e^{-0.069x}}.$$

b. You can see how well the model fits the data by comparing the actual values of y with the values of y given by the model, which are labeled y^* in the table below.

x	0	25	50	75	100
y	12	44	81	96	99
y^*	12.5	44.5	81.8	96.2	99.3

In the table, you can see that the model appears to be a good fit for the actual data.

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3.6 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1 and 2, fill in the blank.

1. A power model has the form _____ .
2. An exponential model of the form $y = ab^x$ can be rewritten as a natural exponential model of the form _____.
3. What type of visual display can you create to get an idea of which type of model will best fit the data set?
4. A power model for a set of data has a coefficient of determination of $r^2 \approx 0.901$ and an exponential model for the data has a coefficient of determination of $r^2 \approx 0.967$. Which model fits the data better?

Procedures and Problem Solving

Classifying Scatter Plots **In Exercises 5–12, determine whether the scatter plot could best be modeled by a linear model, a quadratic model, an exponential model, a logarithmic model, or a logistic model.**

5.
6.
7.
8.
9.
10.
11.
12.

Classifying Scatter Plots **In Exercises 13–18, use a graphing utility to create a scatter plot of the data. Decide whether the data could best be modeled by a linear model, an exponential model, or a logarithmic model.**

13. (1, 2.0), (1.5, 3.5), (2, 4.0), (4, 5.8), (6, 7.0), (8, 7.8)
14. (1, 5.8), (1.5, 6.0), (2, 6.5), (4, 7.6), (6, 8.9), (8, 10.0)
15. (1, 4.4), (1.5, 4.7), (2, 5.5), (4, 9.9), (6, 18.1), (8, 33.0)
16. (1, 11.0), (1.5, 9.6), (2, 8.2), (4, 4.5), (6, 2.5), (8, 1.4)
17. (1, 7.5), (1.5, 7.0), (2, 6.8), (4, 5.0), (6, 3.5), (8, 2.0)
18. (1, 5.0), (1.5, 6.0), (2, 6.4), (4, 7.8), (6, 8.6), (8, 9.0)

Finding an Exponential Model **In Exercises 19–22, use the *regression* feature of a graphing utility to find an exponential model $y = ab^x$ for the data and identify the coefficient of determination. Use the graphing utility to plot the data and graph the model in the same viewing window.**

19. (0, 5), (1, 6), (2, 7), (3, 9), (4, 13)
20. (0, 4.0), (2, 6.9), (4, 18.0), (6, 32.3), (8, 59.1), (10, 118.5)
21. (0, 10.0), (1, 6.1), (2, 4.2), (3, 3.8), (4, 3.6)
22. (−3, 120.2), (0, 80.5), (3, 64.8), (6, 58.2), (10, 55.0)

Finding a Logarithmic Model **In Exercises 23–26, use the *regression* feature of a graphing utility to find a logarithmic model $y = a + b \ln x$ for the data and identify the coefficient of determination. Use the graphing utility to plot the data and graph the model in the same viewing window.**

23. (1, 2.0), (2, 3.0), (3, 3.5), (4, 4.0), (5, 4.1), (6, 4.2), (7, 4.5)
24. (1, 8.5), (2, 11.4), (4, 12.8), (6, 13.6), (8, 14.2), (10, 14.6)
25. (1, 10), (2, 6), (3, 6), (4, 5), (5, 3), (6, 2)
26. (3, 14.6), (6, 11.0), (9, 9.0), (12, 7.6), (15, 6.5)

Finding a Power Model **In Exercises 27–30, use the *regression* feature of a graphing utility to find a power model $y = ax^b$ for the data and identify the coefficient of determination. Use the graphing utility to plot the data and graph the model in the same viewing window.**

27. (1, 2.0), (2, 3.4), (5, 6.7), (6, 7.3), (10, 12.0)
28. (0.5, 1.0), (2, 12.5), (4, 33.2), (6, 65.7), (8, 98.5), (10, 150.0)
29. (1, 10.0), (2, 4.0), (3, 0.7), (4, 0.1)
30. (2, 450), (4, 385), (6, 345), (8, 332), (10, 312)

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31. MODELING DATA

The table shows the yearly sales S (in millions of dollars) of Whole Foods Market for the years 2001 through 2008. (Source: Whole Foods Market)

Year	Sales, S
2001	2272.2
2002	2690.5
2003	3148.6
2004	3865.0
2005	4701.3
2006	5607.4
2007	6591.8
2008	7953.9

(a) Use the *regression* feature of a graphing utility to find an exponential model and a power model for the data and identify the coefficient of determination for each model. Let t represent the year, with $t = 1$ corresponding to 2001.

(b) Use the graphing utility to graph each model with the data.

(c) Use the coefficients of determination to determine which model best fits the data.

32. MODELING DATA

The table shows the annual amounts A (in billions of dollars) spent in the U.S. by the cruise lines and passengers of the North American cruise industry from 2003 through 2008. (Source: Cruise Lines International Association)

Year	Amount, A
2003	12.92
2004	14.70
2005	16.18
2006	17.64
2007	18.70
2008	19.07

(a) Use the *regression* feature of a graphing utility to find a linear model, an exponential model, and a logarithmic model for the data. Let t represent the year, with $t = 3$ corresponding to 2003.

(b) Use the graphing utility to graph each model with the data. Use the graphs to determine which model best fits the data.

(c) Use the model you chose in part (b) to predict the amount spent in 2009. Is the amount reasonable?

33. MODELING DATA

The populations P (in millions) of the United States for the years 1995 through 2008 are shown in the table, where t represents the year, with $t = 5$ corresponding to 1995. (Source: U.S. Census Bureau)

Year	Population, P
1995	266.6
1996	269.7
1997	272.9
1998	276.1
1999	279.3
2000	282.4
2001	285.3
2002	288.0
2003	290.7
2004	293.3
2005	296.0
2006	298.8
2007	301.7
2008	304.5

(a) Use the *regression* feature of a graphing utility to find a linear model for the data and to identify the coefficient of determination. Plot the model and the data in the same viewing window.

(b) Use the *regression* feature of the graphing utility to find a power model for the data and to identify the coefficient of determination. Plot the model and the data in the same viewing window.

(c) Use the *regression* feature of the graphing utility to find an exponential model for the data and to identify the coefficient of determination. Plot the model and the data in the same viewing window.

(d) Use the *regression* feature of the graphing utility to find a logarithmic model for the data and to identify the coefficient of determination. Plot the model and the data in the same viewing window.

(e) Which model is the best fit for the data? Explain.

(f) Use each model to predict the populations of the United States for the years 2009 through 2014.

(g) Which model is the best choice for predicting the future population of the United States? Explain.

(h) Were your choices of models the same for parts (e) and (g)? If not, explain why your choices were different.

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34. ***Why you should learn it*** *(p. 233)* The atmospheric pressure decreases with increasing altitude. At sea level, the average air pressure is approximately 1.03323 kilograms per square centimeter, and this pressure is called one atmosphere. Variations in weather conditions cause changes in the atmospheric pressure of up to ± 5 percent. The ordered pairs (h, p) give the pressures p (in atmospheres) for various altitudes h (in kilometers).

$(0, 1), (10, 0.25), (20, 0.06), (5, 0.55),$
$(15, 0.12), (25, 0.02)$

(a) Use the *regression* feature of a graphing utility to attempt to find the logarithmic model $p = a + b \ln h$ for the data. Explain why the result is an error message.

(b) Use the *regression* feature of the graphing utility to find the logarithmic model $h = a + b \ln p$ for the data.

(c) Use the graphing utility to plot the data and graph the logarithmic model in the same viewing window.

(d) Use the model to estimate the altitude at which the pressure is 0.75 atmosphere.

(e) Use the graph in part (c) to estimate the pressure at an altitude of 13 kilometers.

35. MODELING DATA

The table shows the numbers N of office supply stores operated by Staples from 2001 through 2008. (Source: Staples, Inc.)

Year	Number, N
2001	1436
2002	1488
2003	1559
2004	1680
2005	1780
2006	1884
2007	2038
2008	2218

(a) Use the *regression* feature of a graphing utility to find an exponential model for the data. Let t represent the year, with $t = 1$ corresponding to 2001.

(b) Rewrite the model as a natural exponential model.

(c) Use the natural exponential model to predict the number of Staples stores in 2009. Is the number reasonable?

36. MODELING DATA

A beaker of liquid at an initial temperature of 78°C is placed in a room at a constant temperature of 21°C. The temperature of the liquid is measured every 5 minutes for a period of $\frac{1}{2}$ hour. The results are recorded in the table, where t is the time (in minutes) and T is the temperature (in degrees Celsius).

Time, t	Temperature, T
0	78.0°
5	66.0°
10	57.5°
15	51.2°
20	46.3°
25	42.5°
30	39.6°

(a) Use the *regression* feature of a graphing utility to find a linear model for the data. Use the graphing utility to plot the data and graph the model in the same viewing window. Do the data appear linear? Explain.

(b) Use the *regression* feature of the graphing utility to find a quadratic model for the data. Use the graphing utility to plot the data and graph the model in the same viewing window. Do the data appear quadratic? Even though the quadratic model appears to be a good fit, explain why it might not be a good model for predicting the temperature of the liquid when $t = 60$.

(c) The graph of the temperature of the room should be an asymptote of the graph of the model. Subtract the room temperature from each of the temperatures in the table. Use the *regression* feature of the graphing utility to find an exponential model for the revised data. Add the room temperature to this model. Use the graphing utility to plot the original data and graph the model in the same viewing window.

(d) Explain why the procedure in part (c) was necessary for finding the exponential model.

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37. MODELING DATA

The table shows the percents P of women in different age groups (in years) who have been married at least once. (Source: U.S. Census Bureau)

Age group	Percent, P
20–24	21.1
25–29	54.5
30–34	73.9
35–39	84.0
40–44	86.5
45–54	89.7
55–64	93.1
65–74	95.8

(a) Use the *regression* feature of a graphing utility to find a logistic model for the data. Let x represent the midpoint of the age group.

(b) Use the graphing utility to graph the model with the original data. How closely does the model represent the data?

38. MODELING DATA

The table shows the annual sales S (in millions of dollars) of AutoZone for the years from 2002 through 2009. (Source: AutoZone, Inc.)

Year	Sales, S
2002	5325.5
2003	5457.1
2004	5637.0
2005	5710.9
2006	5948.4
2007	6169.8
2008	6522.7
2009	6816.8

(a) Use the *regression* feature of a graphing utility to find a logarithmic model, an exponential model, and a power model for the data. Let t represent the year, with $t = 2$ corresponding to 2002.

(b) Use each of the following methods to choose the model that best fits the data. Compare your results.

(i) Create a table of values for each model.

(ii) Use the graphing utility to graph each model with the data.

(iii) Find and compare the coefficients of determination for the models.

Conclusions

True or False? In Exercises 39 and 40, determine whether the statement is true or false. Justify your answer.

39. The exponential model $y = ae^{bx}$ represents a growth model when $b > 0$.

40. To change an exponential model of the form $y = ab^x$ to one of the form $y = ae^{cx}$, rewrite b as $b = \ln e^b$.

41. Writing In your own words, explain how to fit a model to a set of data using a graphing utility.

42. CAPSTONE You use a graphing utility to create the scatter plot of a set of data.

(a) What types of models are likely to fit the data well? Explain.

(b) Discuss the methods you can use to find the model of best fit for the data. Which method would you prefer? Explain.

Cumulative Mixed Review

Using the Slope-Intercept Form In Exercises 43–46, find the slope and y-intercept of the equation of the line. Then sketch the line by hand.

43. $2x + 5y = 10$

44. $3x - 2y = 9$

45. $1.2x + 3.5y = 10.5$

46. $0.4x - 2.5y = 12.0$

Writing the Equation of a Parabola in Standard Form In Exercises 47–50, write an equation of the parabola in standard form.

47.

48.

49.

50.

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3 Chapter Summary

	What did you learn?	Explanation and Examples	Review Exercises
3.1	**Recognize and evaluate exponential functions with base *a* *(p. 180)*.**	The exponential function f with base a is denoted by $f(x) = a^x$, where $a > 0$, $a \neq 1$, and x is any real number.	1–4
	Graph exponential functions with base *a* *(p. 181)*.		5–12
	Recognize, evaluate, and graph exponential functions with base *e* *(p. 184)*.	The function $f(x) = e^x$ is called the natural exponential function.	13–18
	Use exponential functions to model and solve real-life problems *(p. 186)*.	Exponential functions are used in compound interest formulas (see Example 8) and in radioactive decay models (see Example 10).	19–22
3.2	**Recognize and evaluate logarithmic functions with base *a* *(p. 192)*.**	For $x > 0$, $a > 0$, and $a \neq 1$, $y = \log_a x$ if and only if $x = a^y$. The function $f(x) = \log_a x$ is called the logarithmic function with base a.	23–36
	Graph logarithmic functions with base *a* *(p. 194)*, and recognize, evaluate, and graph natural logarithmic functions *(p. 196)*.	The graphs of $g(x) = \log_a x$ and $f(x) = a^x$ are reflections of each other in the line $y = x$. The function defined by $g(x) = \ln x$, $x > 0$, is called the natural logarithmic function.	37–48
	Use logarithmic functions to model and solve real-life problems *(p. 198)*.	A logarithmic function is used in the human memory model. (See Example 10.)	49, 50
3.3	**Rewrite logarithms with different bases *(p. 203)*.**	Let a, b, and x be positive real numbers such that $a \neq 1$ and $b \neq 1$. Then $\log_a x$ can be converted to a different base as follows. *Base b*: $\log_a x = \dfrac{\log_b x}{\log_b a}$ *Base 10*: $\log_a x = \dfrac{\log_{10} x}{\log_{10} a}$ *Base e*: $\log_a x = \dfrac{\ln x}{\ln a}$	51–58

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	What did you learn?	Explanation and Examples	Review Exercises
3.3	**Use properties of logarithms to evaluate, rewrite, expand, or condense logarithmic expressions *(p. 204)*.**	Let a be a positive number ($a \neq 1$), n be a real number, and u and v be positive real numbers. **1. Product Property:** $\log_a(uv) = \log_a u + \log_a v$ $\ln(uv) = \ln u + \ln v$ **2. Quotient Property:** $\log_a(u/v) = \log_a u - \log_a v$ $\ln(u/v) = \ln u - \ln v$ **3. Power Property:** $\log_a u^n = n \log_a u, \quad \ln u^n = n \ln u$	59–78
	Use logarithmic functions to model and solve real-life problems *(p. 206)*.	Logarithmic functions can be used to find an equation that relates the periods of several planets and their distances from the sun. (See Example 7.)	79, 80
3.4	**Solve simple exponential and logarithmic equations *(p. 210)*.**	Solve simple exponential or logarithmic equations using the One-to-One Properties and Inverse Properties of exponential and logarithmic functions.	81–94
	Solve more complicated exponential *(p. 211)* and logarithmic *(p. 213)* equations.	To solve more complicated equations, rewrite the equations so that the One-to-One Properties and Inverse Properties of exponential and logarithmic functions can be used. (See Examples 2–8.)	95–118
	Use exponential and logarithmic equations to model and solve real-life problems *(p. 216)*.	Exponential and logarithmic equations can be used to find how long it will take to double an investment (see Example 12) and to find the year in which the average salary for public school teachers reached $45,000 (see Example 13).	119, 120
3.5	**Recognize the five most common types of models involving exponential or logarithmic functions *(p. 221)*.**	**1. Exponential growth model:** $y = ae^{bx}, \quad b > 0$ **2. Exponential decay model:** $y = ae^{-bx}, \quad b > 0$ **3. Gaussian model:** $y = ae^{-(x-b)^2/c}$ **4. Logistic growth model:** $y = \dfrac{a}{1 + be^{-rx}}$ **5. Logarithmic models:** $y = a + b \ln x$ $y = a + b \log_{10} x$	121–126
	Use exponential growth and decay functions to model and solve real-life problems *(p. 222)*.	An exponential growth function can be used to model the world population (see Example 1) and an exponential decay function can be used to estimate the age of a fossil (see Example 3).	127
	Use Gaussian functions *(p. 225)*, logistic growth functions *(p. 226)*, and logarithmic functions *(p. 227)* to model and solve real-life problems.	A Gaussian function can be used to model SAT math scores for college-bound seniors (see Example 4). A logistic growth function can be used to model the spread of a flu virus (see Example 5). A logarithmic function can be used to find the intensity of an earthquake using its magnitude (see Example 6).	128–130
3.6	**Classify scatter plots *(p. 233)*, and use scatter plots and a graphing utility to find models for data and choose the model that best fits a set of data *(p. 234)*.**	You can use a scatter plot and a graphing utility to choose a model that best fits a set of data that represents the yield of a chemical reaction. (See Example 3.)	131, 132
	Use a graphing utility to find exponential and logistic models for data *(p. 236)*.	An exponential model can be used to estimate the amount of revenue collected by the Internal Revenue Service for a given year (see Example 4) and a logistic model can be used to estimate the percent of defoliation caused by the gypsy moth (see Example 5).	133, 134

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3 Review Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

3.1

Evaluating Exponential Functions **In Exercises 1–4, use a calculator to evaluate the function at the indicated value of x. Round your result to four decimal places.**

1. $f(x) = 1.45^x, \quad x = 2\pi$ **2.** $f(x) = 7^x, \quad x = -\sqrt{11}$

3. $g(x) = 60^{2x}, \quad x = -1.1$ **4.** $g(x) = 25^{-3x}, \quad x = \frac{3}{2}$

Library of Parent Functions **In Exercises 5–8, match the function with its graph. [The graphs are labeled (a), (b), (c), and (d).]**

(a)

(b)

(c)

(d)

5. $f(x) = 4^x$ **6.** $f(x) = 4^{-x}$

7. $f(x) = -4^x$ **8.** $f(x) = 4^x + 1$

Graphs of $y = a^x$ and $y = a^{-x}$ **In Exercises 9–12, graph the exponential function by hand. Identify any asymptotes and intercepts and determine whether the graph of the function is increasing or decreasing.**

9. $f(x) = 6^x$ **10.** $f(x) = 0.3^x$

11. $g(x) = 6^{-x}$ **12.** $g(x) = 0.3^{-x}$

Graphing an Exponential Function **In Exercises 13–18, use a graphing utility to construct a table of values for the function. Then sketch the graph of the function. Identify any asymptotes of the graph.**

13. $h(x) = e^{x-1}$ **14.** $f(x) = e^{x+2}$

15. $h(x) = -e^x$ **16.** $f(x) = 3 - e^{-x}$

17. $f(x) = 4e^{-0.5x}$ **18.** $f(x) = 2 + e^{x+3}$

Finding the Balance for Compound Interest **In Exercises 19 and 20, complete the table to determine the balance A for \$10,000 invested at rate r for t years, compounded continuously.**

t	1	10	20	30	40	50
A						

19. $r = 8\%$ **20.** $r = 3\%$

21. Economics A new SUV costs \$32,000. The value V of the SUV after t years is modeled by $V(t) = 32{,}000\left(\frac{3}{4}\right)^t$.

(a) Use a graphing utility to graph the function.

(b) Find the value of the SUV after 2 years.

(c) According to the model, when does the SUV depreciate most rapidly? Is this realistic? Explain.

22. Radioactive Decay Let Q represent the mass, in grams, of a quantity of plutonium 241 (^{241}Pu), whose half-life is 14 years. The quantity of plutonium present after t years is given by $Q = 50\left(\frac{1}{2}\right)^{t/14}$.

(a) Determine the initial quantity (when $t = 0$).

(b) Determine the quantity present after 10 years.

(c) Use a graphing utility to graph the function over the interval $t = 0$ to $t = 50$.

3.2

Rewriting Equations **In Exercises 23–32, write the logarithmic equation in exponential form or write the exponential equation in logarithmic form.**

23. $\log_5 125 = 3$ **24.** $\log_9 81 = 2$

25. $\log_{64} 2 = \frac{1}{6}$ **26.** $\log_{10}\left(\frac{1}{100}\right) = -2$

27. $4^3 = 64$ **28.** $3^5 = 243$

29. $125^{2/3} = 25$ **30.** $12^{-1} = \frac{1}{12}$

31. $\left(\frac{1}{2}\right)^{-3} = 8$ **32.** $\left(\frac{2}{3}\right)^{-2} = \frac{9}{4}$

Evaluating Logarithms **In Exercises 33–36, use the definition of logarithmic function to evaluate the function at the indicated value of x without using a calculator.**

	Function	*Value*
33.	$f(x) = \log_6 x$	$x = 216$
34.	$f(x) = \log_7 x$	$x = 1$
35.	$f(x) = \log_4 x$	$x = \frac{1}{4}$
36.	$f(x) = \log_{10} x$	$x = 0.00001$

Sketching the Graph of a Logarithmic Function **In Exercises 37–40, find the domain, vertical asymptote, and x-intercept of the logarithmic function, and sketch its graph by hand.**

37. $g(x) = -\log_2 x + 5$ **38.** $g(x) = \log_5(x - 3)$

39. $f(x) = \log_2(x - 1) + 6$ **40.** $f(x) = \log_5(x + 2) - 3$

Evaluating the Natural Logarithmic Function **In Exercises 41–44, use a calculator to evaluate the function $f(x) = \ln x$ at the indicated value of x. Round your result to three decimal places, if necessary.**

41. $x = 21.5$ **42.** $x = 0.46$

43. $x = \sqrt{6}$ **44.** $x = \frac{5}{6}$

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Analyzing Graphs of Functions **In Exercises 45–48, use a graphing utility to graph the logarithmic function. Determine the domain and identify any vertical asymptote and x-intercept.**

45. $f(x) = \ln x + 3$

46. $f(x) = \ln(x - 3)$

47. $h(x) = \frac{1}{2}\ln x$

48. $f(x) = \frac{1}{4}\ln x$

49. **Aeronautics** The time t (in minutes) for a small plane to climb to an altitude of h feet is given by

$$t = 50\log_{10}[18{,}000/(18{,}000 - h)]$$

where 18,000 feet is the plane's absolute ceiling.

(a) Determine the domain of the function appropriate for the context of the problem.

(b) Use a graphing utility to graph the function and identify any asymptotes.

(c) As the plane approaches its absolute ceiling, what can be said about the time required to further increase its altitude?

(d) Find the amount of time it will take for the plane to climb to an altitude of 4000 feet.

50. **Real Estate** The model

$$t = 12.542\ln[x/(x - 1000)], \quad x > 1000$$

approximates the length of a home mortgage of \$150,000 at 8% in terms of the monthly payment. In the model, t is the length of the mortgage in years and x is the monthly payment in dollars.

(a) Use the model to approximate the length of a \$150,000 mortgage at 8% when the monthly payment is \$1254.68.

(b) Approximate the total amount paid over the term of the mortgage with a monthly payment of \$1254.68. What amount of the total is interest costs?

3.3

Changing the Base **In Exercises 51–54, evaluate the logarithm using the change-of-base formula. Do each problem twice, once with common logarithms and once with natural logarithms. Round your results to three decimal places.**

51. $\log_4 9$

52. $\log_{1/2} 9$

53. $\log_{14} 364$

54. $\log_3 0.28$

Graphing a Logarithm with Any Base **In Exercises 55–58, use the change-of-base formula and a graphing utility to graph the function.**

55. $f(x) = \log_2(x - 1)$

56. $f(x) = 2 - \log_3 x$

57. $f(x) = -\log_{1/2}(x + 2)$

58. $f(x) = \log_{1/3}(x - 1) + 1$

Using Properties to Evaluate Logarithms **In Exercises 59–62, approximate the logarithm using the properties of logarithms, given the values $\log_b 2 \approx 0.3562$, $\log_b 3 \approx 0.5646$, and $\log_b 5 \approx 0.8271$.**

59. $\log_b 9$

60. $\log_b \frac{4}{9}$

61. $\log_b \sqrt{5}$

62. $\log_b 50$

Simplifying a Logarithm **In Exercises 63–66, use the properties of logarithms to rewrite and simplify the logarithmic expression.**

63. $\ln(5e^{-2})$

64. $\ln \sqrt{e^5}$

65. $\log_{10} 200$

66. $\log_{10} 0.002$

Expanding Logarithmic Expressions **In Exercises 67–72, use the properties of logarithms to expand the expression as a sum, difference, and/or constant multiple of logarithms. (Assume all variables are positive.)**

67. $\log_5 5x^2$

68. $\log_4 16xy^2$

69. $\log_{10} \dfrac{5\sqrt{y}}{x^2}$

70. $\ln \dfrac{\sqrt{x}}{4}$

71. $\ln \dfrac{x + 3}{xy}$

72. $\ln \dfrac{xy^5}{\sqrt{z}}$

Condensing Logarithmic Expressions **In Exercises 73–78, condense the expression to the logarithm of a single quantity.**

73. $\log_2 9 + \log_2 x$

74. $\log_6 y - 2\log_6 z$

75. $\frac{1}{2}\ln(2x - 1) - 2\ln(x + 1)$

76. $5\ln(x - 2) - \ln(x + 2) - 3\ln x$

77. $\ln 3 + \frac{1}{3}\ln(4 - x^2) - \ln x$

78. $3[\ln x - 2\ln(x^2 + 1)] + 2\ln 5$

79. **Public Service** The number of miles s of roads cleared of snow in 1 hour is approximated by the model

$$s = 25 - \frac{13\ln(h/12)}{\ln 3}, \quad 2 \le h \le 15$$

where h is the depth of the snow (in inches).

(a) Use a graphing utility to graph the function.

(b) Complete the table.

h	4	6	8	10	12	14
s						

(c) Using the graph of the function and the table, what conclusion can you make about the number of miles of roads cleared as the depth of the snow increases?

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80. **Psychology** Students in a sociology class were given an exam and then retested monthly with an equivalent exam. The average scores for the class are given by the human memory model $f(t) = 85 - 17\log_{10}(t + 1)$, where t is the time in months and $0 \le t \le 10$. When will the average score decrease to 68?

3.4

Solving an Exponential or Logarithmic Equation **In Exercises 81–94, solve the equation for x without using a calculator.**

81. $10^x = 10{,}000$
82. $7^x = 343$
83. $6^x = \frac{1}{216}$
84. $6^{x-2} = 1296$
85. $2^{x+1} = \frac{1}{16}$
86. $4^{x/2} = 64$
87. $\log_8 x = 4$
88. $\log_x 729 = 6$
89. $\log_2(x - 1) = 3$
90. $\log_5(2x + 1) = 2$
91. $\ln x = 4$
92. $\ln x = -3$
93. $\ln(x - 1) = 2$
94. $\ln(2x + 1) = -4$

Solving an Exponential Equation **In Exercises 95–104, solve the exponential equation algebraically. Round your result to three decimal places.**

95. $3e^{-5x} = 132$
96. $14e^{3x+2} = 560$
97. $2^x + 13 = 35$
98. $6^x - 28 = -8$
99. $-4(5^x) = -68$
100. $2(12^x) = 190$
101. $2e^{x-3} - 1 = 4$
102. $-e^{x/2} + 1 = \frac{1}{2}$
103. $e^{2x} - 7e^x + 10 = 0$
104. $e^{2x} - 6e^x + 8 = 0$

Solving a Logarithmic Equation **In Exercises 105–114, solve the logarithmic equation algebraically. Round your result to three decimal places.**

105. $\ln 3x = 6.4$
106. $\ln 5x = 4.5$
107. $\ln x - \ln 5 = 2$
108. $\ln x - \ln 3 = 4$
109. $\ln \sqrt{x + 1} = 2$
110. $\ln \sqrt{x + 40} = 3$
111. $\log_4(x - 1) = \log_4(x - 2) - \log_4(x + 2)$
112. $\log_5(x + 2) - \log_5 x = \log_5(x + 5)$
113. $\log_{10}(1 - x) = -1$
114. $\log_{10}(-x - 4) = 2$

Solving an Exponential or Logarithmic Equation **In Exercises 115–118, solve the equation algebraically. Round your result to three decimal places.**

115. $xe^x + e^x = 0$
116. $2xe^{2x} + e^{2x} = 0$
117. $x\ln x + x = 0$
118. $\dfrac{1 - \ln x}{x^2} = 0$

119. **Finance** You deposit \$7550 in an account that pays 6.9% interest, compounded continuously. How long will it take for the money to double?

120. **Economics** The demand x for a 32-inch plasma television is modeled by

$$p = 5000\left(1 - \frac{4}{4 + e^{-0.0005x}}\right).$$

Find the demands x for prices of (a) $p = \$450$ and (b) $p = \$400$.

3.5

Identifying Graphs of Models **In Exercises 121–126, match the function with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]**

(a)

(b)

(c)

(d)

(e)

(f)

121. $y = 3e^{-2x/3}$
122. $y = 4e^{2x/3}$
123. $y = \ln(x + 3)$
124. $y = 7 - \log_{10}(x + 3)$
125. $y = 2e^{-(x+4)^2/3}$
126. $y = \dfrac{6}{1 + 2e^{-2x}}$

127. **Demography** The populations P (in thousands) of North Carolina from 1990 through 2008 can be modeled by $P = 6707.7e^{kt}$, where t is the year, with $t = 0$ corresponding to 1990. In 2008, the population was about 9,222,000. Find the value of k and use the result to predict the population in the year 2020. (Source: U.S. Census Bureau)

128. **Education** The scores for a biology test follow a normal distribution modeled by $y = 0.0499e^{-(x-74)^2/128}$, where x is the test score and $40 \le x \le 100$.

(a) Use a graphing utility to graph the function.

(b) Use the graph to estimate the average test score.

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129. Education The average number N of words per minute that the students in a first grade class could read orally after t weeks of school is modeled by

$$N = \frac{62}{1 + 5.4e^{-0.24t}}.$$

Find the numbers of weeks it took the class to read at average rates of (a) 40 words per minute and (b) 60 words per minute.

130. Geology On the Richter scale, the magnitude R of an earthquake of intensity I is modeled by

$$R = \log_{10} \frac{I}{I_0}$$

where $I_0 = 1$ is the minimum intensity used for comparison. Find the intensities I of the following earthquakes measuring R on the Richter scale.

(a) $R = 7.1$ (b) $R = 8.4$ (c) $R = 5.5$

3.6

Classifying Scatter Plots **In Exercises 131 and 132, determine whether the scatter plot could best be modeled by a linear model, an exponential model, a logarithmic model, or a logistic model.**

131.

132.

133. MODELING DATA

Each ordered pair (t, N) represents the year t and the number N (in thousands) of female participants in high school athletic programs during nine school years, with $t = 1$ corresponding to the 2000–2001 school year. (Source: National Federation of State High School Associations)

(1, 2784), (2, 2807), (3, 2856), (4, 2865), (5, 2908), (6, 2953), (7, 3022), (8, 3057), (9, 3114)

(a) Use the *regression* feature of a graphing utility to find a linear model, an exponential model, and a power model for the data and identify the coefficient of determination for each model.

(b) Use the graphing utility to graph each model with the original data.

(c) Determine which model best fits the data. Explain.

(d) Use the model you chose in part (c) to predict the number of participants during the 2009–2010 school year.

(e) Use the model you chose in part (c) to predict the school year in which about 3,720,000 girls will participate.

134. MODELING DATA

You plant a tree when it is 1 meter tall and check its height h (in meters) every 10 years, as shown in the table.

Year	Height, h
0	1
10	3
20	7.5
30	14.5
40	19
50	20.5
60	21

(a) Use the *regression* feature of a graphing utility to find a logistic model for the data. Let x represent the year.

(b) Use the graphing utility to graph the model with the original data.

(c) How closely does the model represent the data?

(d) What is the limiting height of the tree?

Conclusions

True or False? **In Exercises 135–138, determine whether the equation or statement is true or false. Justify your answer.**

135. $e^{x-1} = \dfrac{e^x}{e}$

136. $\ln(x + y) = \ln(xy)$

137. The domain of the function $f(x) = \ln x$ is the set of all real numbers.

138. The logarithm of the quotient of two numbers is equal to the difference of the logarithms of the numbers.

139. Think About It Without using a calculator, explain why you know that $2^{\sqrt{2}}$ is greater than 2, but less than 4.

140. Exploration

(a) Use a graphing utility to compare the graph of the function $y = e^x$ with the graph of each function below. [$n!$ (read as "n factorial") is defined as $n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$.]

$$y_1 = 1 + \frac{x}{1!},\ y_2 = 1 + \frac{x}{1!} + \frac{x^2}{2!},$$

$$y_3 = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}$$

(b) Identify the pattern of successive polynomials given in part (a). Extend the pattern one more term and compare the graph of the resulting polynomial function with the graph of $y = e^x$. What do you think this pattern implies?

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3 Chapter Test

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Take this test as you would take a test in class. After you are finished, check your work against the answers given in the back of the book.

In Exercises 1–3, use a graphing utility to construct a table of values for the function. Then sketch a graph of the function. Identify any asymptotes and intercepts.

1. $f(x) = 10^{-x}$ **2.** $f(x) = -6^{x-2}$ **3.** $f(x) = 1 - e^{2x}$

In Exercises 4–6, evaluate the expression.

4. $\log_7 7^{-0.89}$ **5.** $4.6 \ln e^2$ **6.** $5 - \log_{10} 1000$

In Exercises 7–9, find the domain, vertical asymptote, and x-intercept of the logarithmic function, and sketch its graph by hand.

7. $f(x) = -\log_{10} x - 6$ **8.** $f(x) = \ln(x - 4)$ **9.** $f(x) = 1 + \ln(x + 6)$

In Exercises 10–12, evaluate the logarithm using the change-of-base formula. Round your result to three decimal places.

10. $\log_7 44$ **11.** $\log_{2/5} 0.9$ **12.** $\log_{12} 64$

In Exercises 13–15, use the properties of logarithms to expand the expression as a sum, difference, and/or multiple of logarithms.

13. $\log_2 3a^4$ **14.** $\ln \dfrac{5\sqrt{x}}{6}$ **15.** $\ln \dfrac{x\sqrt{x+1}}{2e^4}$

In Exercises 16–18, condense the expression to the logarithm of a single quantity.

16. $\log_3 13 + \log_3 y$ **17.** $4 \ln x - 4 \ln y$

18. $\ln x - \ln(x + 2) + \ln(2x - 3)$

In Exercises 19–22, solve the equation for x.

19. $3^x = 81$ **20.** $5^{2x} = 2500$

21. $\log_7 x = 3$ **22.** $\log_{10}(x - 4) = 5$

In Exercises 23–26, solve the equation algebraically. Round your result to three decimal places.

23. $\dfrac{1025}{8 + e^{4x}} = 5$ **24.** $-xe^{-x} + e^{-x} = 0$

25. $\log_{10} x - \log_{10}(8 - 5x) = 2$ **26.** $2x \ln x - x = 0$

27. The half-life of radioactive actinium (^{227}Ac) is 22 years. What percent of a present amount of radioactive actinium will remain after 19 years?

28. The table shows the annual revenues R (in millions of dollars) for Daktronics from 2001 through 2008. (Source: Daktronics, Inc.)

(a) Use the *regression* feature of a graphing utility to find a logarithmic model, an exponential model, and a power model for the data. Let t represent the year, with $t = 1$ corresponding to 2001.

(b) Use the graphing utility to graph each model with the original data.

(c) Determine which model best fits the data. Use the model to predict the revenue of Daktronics in 2015.

Year	Revenue, R
2001	148.8
2002	177.8
2003	209.9
2004	230.3
2005	309.4
2006	433.2
2007	499.7
2008	581.9

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1–3 Cumulative Test

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Take this test to review the material in Chapters 1–3. After you are finished, check your work against the answers given in the back of the book.

In Exercises 1–3, (a) write the slope-intercept form of the equation of the line that satisfies the given conditions and (b) find three additional points through which the line passes.

1. The line contains the points $(-5, 8)$ and $(-1, 4)$.

2. The line contains the point $\left(-\frac{1}{2}, 1\right)$ and has a slope of -2.

3. The line has an undefined slope and contains the point $\left(-\frac{3}{7}, \frac{1}{8}\right)$.

In Exercises 4 and 5, evaluate the function at each value of the independent variable and simplify.

4. $f(x) = \dfrac{x}{x-2}$

(a) $f(5)$ (b) $f(2)$ (c) $f(5 + 4s)$

5. $f(x) = \begin{cases} 3x - 8, & x < 0 \\ x^2 + 4, & x \geq 0 \end{cases}$

(a) $f(-8)$ (b) $f(0)$ (c) $f(4)$

Figure for 6

6. Does the graph at the right represent y as a function of x? Explain.

7. Use a graphing utility to graph the function $f(x) = 2|x - 5| - |x + 5|$. Then determine the open intervals over which the function is increasing, decreasing, or constant.

8. Compare the graph of each function with the graph of $f(x) = \sqrt{x}$.

(a) $r(x) = \dfrac{1}{4}\sqrt{x}$ (b) $h(x) = \sqrt{x} - 3$ (c) $g(x) = -\sqrt{x + 3}$

In Exercises 9–12, evaluate the indicated function for

$$f(x) = -x^2 + 3x - 10 \quad \text{and} \quad g(x) = 4x + 1.$$

9. $(f + g)(-4)$

10. $(g - f)\left(\frac{3}{4}\right)$

11. $(g \circ f)(-2)$

12. $(fg)(-1)$

13. Determine whether $h(x) = 5x - 2$ has an inverse function. If so, find it.

In Exercises 14–16, sketch the graph of the function. Use a graphing utility to verify the graph.

14. $f(x) = -(x - 2)^2 + 5$

15. $f(x) = x^2 - 6x + 5$

16. $f(x) = x^3 + 2x^2 - 9x - 18$

17. Find all the zeros of $f(x) = x^3 + 2x^2 + 4x + 8$.

18. Use a graphing utility to approximate any real zeros of $g(x) = x^3 + 4x^2 - 11$ accurate to three decimal places.

19. Divide $(4x^2 + 14x - 9)$ by $(x + 3)$ using long division.

20. Divide $(2x^3 - 5x^2 + 6x - 20)$ by $(x - 6)$ using synthetic division.

21. Multiply the complex number $-5 + 4i$ by its complex conjugate.

22. Find a polynomial function with real coefficients that has the zeros 0, -3, and $1 + \sqrt{5}i$.

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In Exercises 23–25, sketch the graph of the rational function. Identify any asymptotes. Use a graphing utility to verify your graph.

23. $f(x) = \dfrac{2x}{x - 3}$ **24.** $f(x) = \dfrac{5x}{x^2 + x - 6}$ **25.** $f(x) = \dfrac{x^2 - 3x + 8}{x - 2}$

In Exercises 26–29, use a calculator to evaluate the expression. Round your answer to three decimal places.

26. $(1.85)^{3.1}$ **27.** $58^{\sqrt{5}}$ **28.** $e^{-8/5}$ **29.** $4e^{2.56}$

In Exercises 30–33, sketch the graph of the function by hand. Use a graphing utility to verify your graph.

30. $f(x) = -3^{x+4} - 5$ **31.** $f(x) = -\left(\frac{1}{2}\right)^{-x} - 3$

32. $f(x) = 4 + \log_{10}(x - 3)$ **33.** $f(x) = \ln(4 - x)$

In Exercises 34–36, evaluate the logarithm using the change-of-base formula. Round your result to three decimal places.

34. $\log_5 16$ **35.** $\log_9 6.8$ **36.** $\log_2\left(\frac{3}{2}\right)$

37. Use the properties of logarithms to expand $\ln \dfrac{x^2 - 4}{x^2 + 1}$.

38. Write $2 \ln x - \ln(x - 1) + \ln(x + 1)$ as a logarithm of a single quantity.

In Exercises 39–44, solve the equation algebraically. Round your result to three decimal places and verify your result graphically.

39. $6e^{2x} = 72$

40. $4^{x-5} + 21 = 30$

41. $\log_2 x + \log_2 5 = 6$

42. $250e^{0.05x} = 500{,}000$

43. $2x^2e^{2x} - 2xe^{2x} = 0$

44. $\ln(2x - 5) - \ln x = 1$

45. A rectangular plot of land with a perimeter of 546 feet has a width of x.

(a) Write the area A of the plot as a function of x.

(b) Use a graphing utility to graph the area function. What is the domain of the function?

(c) Approximate the dimensions of the plot when the area is 15,000 square feet.

46. The table shows the average prices y (in dollars) received by commercial trout producers per pound of trout in the United States from 2001 to 2008. (Source: U.S. Department of Agriculture)

Year	Average price, y (in dollars)
2001	1.13
2002	1.08
2003	1.04
2004	1.03
2005	1.05
2006	1.11
2007	1.19
2008	1.38

(a) Use the *regression* feature of a graphing utility to find a quadratic model, an exponential model, and a power model for the data and identify the coefficient of determination for each model. Let t represent the year, with $t = 1$ corresponding to 2001.

(b) Use the graphing utility to graph each model with the original data.

(c) Determine which model best fits the data. Explain.

(d) Use the model you chose in part (c) to predict the average price of one pound of trout in 2010. Is your answer reasonable? Explain.

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Proofs in Mathematics

Each of the following three properties of logarithms can be proved by using properties of exponential functions.

Properties of Logarithms (p. 204)

Let a be a positive real number such that $a \neq 1$, and let n be a real number. If u and v are positive real numbers, then the following properties are true.

	Logarithm with Base a	*Natural Logarithm*
1. Product Property:	$\log_a(uv) = \log_a u + \log_a v$	$\ln(uv) = \ln u + \ln v$
2. Quotient Property:	$\log_a \frac{u}{v} = \log_a u - \log_a v$	$\ln \frac{u}{v} = \ln u - \ln v$
3. Power Property:	$\log_a u^n = n \log_a u$	$\ln u^n = n \ln u$

Proof

Let

$$x = \log_a u \quad \text{and} \quad y = \log_a v.$$

The corresponding exponential forms of these two equations are

$$a^x = u \quad \text{and} \quad a^y = v.$$

To prove the Product Property, multiply u and v to obtain

$$uv = a^x a^y = a^{x+y}.$$

The corresponding logarithmic form of $uv = a^{x+y}$ is

$$\log_a(uv) = x + y.$$

So,

$$\log_a(uv) = \log_a u + \log_a v.$$

To prove the Quotient Property, divide u by v to obtain

$$\frac{u}{v} = \frac{a^x}{a^y} = a^{x-y}.$$

The corresponding logarithmic form of $u/v = a^{x-y}$ is

$$\log_a \frac{u}{v} = x - y.$$

So,

$$\log_a \frac{u}{v} = \log_a u - \log_a v.$$

To prove the Power Property, substitute a^x for u in the expression $\log_a u^n$, as follows.

$$\begin{aligned} \log_a u^n &= \log_a (a^x)^n && \text{Substitute } a^x \text{ for } u. \\ &= \log_a a^{nx} && \text{Property of exponents} \\ &= nx && \text{Inverse Property of logarithms} \\ &= n \log_a u && \text{Substitute } \log_a u \text{ for } x. \end{aligned}$$

So, $\log_a u^n = n \log_a u$.

Slide Rules

The slide rule was invented by William Oughtred (1574–1660) in 1625. The slide rule is a computational device with a sliding portion and a fixed portion. A slide rule enables you to perform multiplication by using the Product Property of logarithms. There are other slide rules that allow for the calculation of roots and trigonometric functions. Slide rules were used by mathematicians and engineers until the invention of the handheld calculator in 1972.

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Progressive Summary (Chapters 1–3)

This chart outlines the topics that have been covered so far in this text. Progressive Summary charts appear after Chapters 2, 3, 6, and 9. In each Progressive Summary, new topics encountered for the first time appear in red.

ALGEBRAIC FUNCTIONS

Polynomial, Rational, Radical

Rewriting

Polynomial form ↔ Factored form
Operations with polynomials
Rationalize denominators
Simplify rational expressions
Operations with complex numbers

Solving

Equation	*Strategy*
Linear	Isolate variable
Quadratic	Factor, set to zero Extract square roots Complete the square Quadratic Formula
Polynomial	Factor, set to zero Rational Zero Test
Rational	Multiply by LCD
Radical	Isolate, raise to power
Absolute value	Isolate, form two equations

Analyzing

Graphically	*Algebraically*
Intercepts	Domain, Range
Symmetry	Transformations
Slope	Composition
Asymptotes	Standard forms of equations
End behavior	Leading Coefficient Test
Minimum values	Synthetic division
Maximum values	Descartes's Rule of Signs

Numerically

Table of values

TRANSCENDENTAL FUNCTIONS

Exponential, Logarithmic

Rewriting

Exponential form ↔ Logarithmic form
Condense/expand logarithmic expressions

Solving

Equation	*Strategy*
Exponential	Take logarithm of each side
Logarithmic	Exponentiate each side

Analyzing

Graphically	*Algebraically*
Intercepts	Domain, Range
Asymptotes	Transformations
	Composition
	Inverse Properties

Numerically

Table of values

OTHER TOPICS

Rewriting

Solving

Analyzing

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4 Trigonometric Functions

Section 4.5, Example 8
A Depth of at Least 10 Feet

4.1 Radian and Degree Measure
4.2 Trigonometric Functions: The Unit Circle
4.3 Right Triangle Trigonometry
4.4 Trigonometric Functions of Any Angle
4.5 Graphs of Sine and Cosine Functions
4.6 Graphs of Other Trigonometric Functions
4.7 Inverse Trigonometric Functions
4.8 Applications and Models

Chris Schmidt/iStockphoto.com

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4.1 Radian and Degree Measure

Angles

As derived from the Greek language, the word **trigonometry** means "measurement of triangles." Initially, trigonometry dealt with relationships among the sides and angles of triangles and was used in the development of astronomy, navigation, and surveying. With the development of calculus and the physical sciences in the 17th century, a different perspective arose—one that viewed the classic trigonometric relationships as *functions* with the set of real numbers as their domains. Consequently, the applications of trigonometry expanded to include a vast number of physical phenomena involving rotations and vibrations, including the following.

- sound waves
- light rays
- planetary orbits
- vibrating strings
- pendulums
- orbits of atomic particles

The approach in this text incorporates *both* perspectives, starting with angles and their measure.

What you should learn

- Describe angles.
- Use radian measure.
- Use degree measure and convert between degree and radian measure.
- Use angles to model and solve real-life problems.

Why you should learn it

Radian measures of angles are involved in numerous aspects of our daily lives. For instance, in Exercise 106 on page 263, you are asked to determine the measure of the angle generated as a skater performs an axel jump.

Figure 4.1

Figure 4.2

An **angle** is determined by rotating a ray (half-line) about its endpoint. The starting position of the ray is the **initial side** of the angle, and the position after rotation is the **terminal side,** as shown in Figure 4.1. The endpoint of the ray is the **vertex** of the angle. This perception of an angle fits a coordinate system in which the origin is the vertex and the initial side coincides with the positive x-axis. Such an angle is in **standard position,** as shown in Figure 4.2. **Positive angles** are generated by counterclockwise rotation, and **negative angles** by clockwise rotation, as shown in Figure 4.3. Angles are labeled with Greek letters such as α (alpha), β (beta), and θ (theta), as well as uppercase letters such as A, B, and C. In Figure 4.4, note that angles α and β have the same initial and terminal sides. Such angles are **coterminal.**

Figure 4.3

Figure 4.4

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Radian Measure

The **measure of an angle** is determined by the amount of rotation from the initial side to the terminal side. One way to measure angles is in *radians*. This type of measure is especially useful in calculus. To define a radian, you can use a **central angle** of a circle, one whose vertex is the center of the circle, as shown in Figure 4.5.

Arc length = radius when θ = 1 radian.
Figure 4.5

Definition of Radian

One **radian** (rad) is the measure of a central angle θ that intercepts an arc s equal in length to the radius r of the circle. (See Figure 4.5.) Algebraically this means that

$$\theta = \frac{s}{r}$$

where θ is measured in radians.

Because the circumference of a circle is $2\pi r$ units, it follows that a central angle of one full revolution (counterclockwise) corresponds to an arc length of

$$s = 2\pi r.$$

Moreover, because

$$2\pi \approx 6.28$$

there are just over six radius lengths in a full circle, as shown in Figure 4.6. Because the units of measure for s and r are the same, the ratio

$$\frac{s}{r}$$

has no units—it is simply a real number.

Figure 4.6

Because the radian measure of an angle of one full revolution is 2π, you can obtain the following.

$$\frac{1}{2}\text{ revolution} = \frac{2\pi}{2} = \pi \text{ radians}$$

$$\frac{1}{4}\text{ revolution} = \frac{2\pi}{4} = \frac{\pi}{2} \text{ radians}$$

$$\frac{1}{6}\text{ revolution} = \frac{2\pi}{6} = \frac{\pi}{3} \text{ radians}$$

These and other common angles are shown in Figure 4.7.

Figure 4.7

Recall that the four quadrants in a coordinate system are numbered I, II, III, and IV. Figure 4.8 shows which angles between 0 and 2π lie in each of the four quadrants. Note that angles between 0 and $\pi/2$ are **acute** and that angles between $\pi/2$ and π are **obtuse.**

Figure 4.8

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Two angles are coterminal when they have the same initial and terminal sides. For instance, the angles 0 and 2π are coterminal, as are the angles $\pi/6$ and $13\pi/6$. You can find an angle that is coterminal to a given angle θ by adding or subtracting 2π (one revolution), as demonstrated in Example 1. A given angle θ has infinitely many coterminal angles. For instance, $\theta = \pi/6$ is coterminal with

$$\frac{\pi}{6} + 2n\pi, \text{ where } n \text{ is an integer.}$$

Study Tip

The phrase "the terminal side of θ lies in a quadrant" is often abbreviated by simply saying that "θ lies in a quadrant." The terminal sides of the "quadrant angles" 0, $\pi/2$, π, and $3\pi/2$ do not lie within quadrants.

Example 1 Sketching and Finding Coterminal Angles

a. For the positive angle $\theta = \dfrac{13\pi}{6}$, subtract 2π to obtain a coterminal angle.

$$\frac{13\pi}{6} - 2\pi = \frac{\pi}{6}$$

See Figure 4.9.

Figure 4.9

b. For the positive angle $\theta = \dfrac{3\pi}{4}$, subtract 2π to obtain a coterminal angle.

$$\frac{3\pi}{4} - 2\pi = -\frac{5\pi}{4}$$

See Figure 4.10.

Figure 4.10

c. For the negative angle $\theta = -\dfrac{2\pi}{3}$, add 2π to obtain a coterminal angle.

$$-\frac{2\pi}{3} + 2\pi = \frac{4\pi}{3}$$

See Figure 4.11.

Figure 4.11

CHECKPOINT Now try Exercise 25.

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Degree Measure

A second way to measure angles is in terms of **degrees,** denoted by the symbol °. A measure of one degree (1°) is equivalent to a rotation of $\frac{1}{360}$ of a complete revolution about the vertex. To measure angles, it is convenient to mark degrees on the circumference of a circle, as shown in Figure 4.12. So, a full revolution (counterclockwise) corresponds to 360°, a half revolution to 180°, a quarter revolution to 90°, and so on.

Because 2π radians corresponds to one complete revolution, degrees and radians are related by the equations

$$360° = 2\pi \text{ rad} \qquad \text{and} \qquad 180° = \pi \text{ rad}.$$

From the second equation, you obtain

$$1° = \frac{\pi}{180} \text{ rad} \qquad \text{and} \qquad 1 \text{ rad} = \frac{180°}{\pi}$$

which lead to the following conversion rules.

Figure 4.12

Conversions Between Degrees and Radians

1. To convert degrees to radians, multiply degrees by $\dfrac{\pi \text{ rad}}{180°}$.
2. To convert radians to degrees, multiply radians by $\dfrac{180°}{\pi \text{ rad}}$.

To apply these two conversion rules, use the basic relationship π rad $= 180°$. (See Figure 4.13.)

Figure 4.13

When no units of angle measure are specified, *radian measure is implied*. For instance, $\theta = \pi$ or $\theta = 2$ implies that $\theta = \pi$ radians or $\theta = 2$ radians.

Example 2 Converting from Degrees to Radians

a. $135° = (135 \cancel{\text{deg}})\left(\dfrac{\pi \text{ rad}}{180 \cancel{\text{deg}}}\right) = \dfrac{3\pi}{4}$ radians — Multiply by $\frac{\pi}{180}$.

b. $540° = (540 \cancel{\text{deg}})\left(\dfrac{\pi \text{ rad}}{180 \cancel{\text{deg}}}\right) = 3\pi$ radians — Multiply by $\frac{\pi}{180}$.

c. $-270° = (-270 \cancel{\text{deg}})\left(\dfrac{\pi \text{ rad}}{180 \cancel{\text{deg}}}\right) = -\dfrac{3\pi}{2}$ radians — Multiply by $\frac{\pi}{180}$.

CHECKPOINT Now try Exercise 47.

Technology Tip

Historically, fractional parts of degrees were expressed in *minutes* and *seconds*, using the prime (′) and double prime (″) notations, respectively. That is,

$$1' = \text{one minute} = \tfrac{1}{60}(1°)$$

$$1'' = \text{one second} = \tfrac{1}{3600}(1°).$$

Many calculators have special keys for converting angles in degrees, minutes, and seconds (D° M′ S″) to decimal degree form, and vice versa.

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Example 3 Converting from Radians to Degrees

a. $-\dfrac{\pi}{2}\text{ rad} = \left(-\dfrac{\pi}{2}\cancel{\text{rad}}\right)\left(\dfrac{180\text{ deg}}{\pi\cancel{\text{rad}}}\right) = -90°$ Multiply by $\dfrac{180}{\pi}$.

b. $2\text{ rad} = (2\cancel{\text{rad}})\left(\dfrac{180\text{ deg}}{\pi\cancel{\text{rad}}}\right) = \dfrac{360}{\pi} \approx 114.59°$ Multiply by $\dfrac{180}{\pi}$.

c. $\dfrac{9\pi}{2}\text{ rad} = \left(\dfrac{9\pi}{2}\cancel{\text{rad}}\right)\left(\dfrac{180\text{ deg}}{\pi\cancel{\text{rad}}}\right) = 810°$ Multiply by $\dfrac{180}{\pi}$.

CHECKPOINT Now try Exercise 51.

Two positive angles α and β are **complementary** (complements of each other) when their sum is $90°$ (or $\pi/2$). Two positive angles are **supplementary** (supplements of each other) when their sum is $180°$ (or π). (See Figure 4.14.)

Complementary angles *Supplementary angles*

Figure 4.14

Example 4 Complementary and Supplementary Angles

If possible, find the complement and supplement of each angle.

a. $72°$ **b.** $148°$ **c.** $\dfrac{2\pi}{5}$ **d.** $\dfrac{4\pi}{5}$

Solution

a. The complement is

$$90° - 72° = 18°.$$

The supplement is

$$180° - 72° = 108°.$$

b. Because $148°$ is greater than $90°$, it has no complement. (Remember that complements are *positive* angles.) The supplement is

$$180° - 148° = 32°.$$

c. The complement is

$$\frac{\pi}{2} - \frac{2\pi}{5} = \frac{5\pi}{10} - \frac{4\pi}{10} = \frac{\pi}{10}.$$

The supplement is

$$\pi - \frac{2\pi}{5} = \frac{5\pi}{5} - \frac{2\pi}{5} = \frac{3\pi}{5}.$$

d. Because $4\pi/5$ is greater than $\pi/2$, it has no complement. The supplement is

$$\pi - \frac{4\pi}{5} = \frac{5\pi}{5} - \frac{4\pi}{5} = \frac{\pi}{5}.$$

CHECKPOINT Now try Exercise 79.

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Linear and Angular Speed

The *radian measure* formula

$$\theta = \frac{s}{r}$$

can be used to measure arc length along a circle.

Arc Length

For a circle of radius r, a central angle θ intercepts an arc of length s given by

$$s = r\theta \qquad \text{Length of circular arc}$$

where θ is measured in radians. Note that if $r = 1$, then $s = \theta$, and the radian measure of θ equals the arc length.

Example 5 Finding Arc Length

A circle has a radius of 4 inches. Find the length of the arc intercepted by a central angle of 240°, as shown in Figure 4.15.

Figure 4.15

Solution

To use the formula

$$s = r\theta$$

first convert 240° to radian measure.

$$240° = (240\ \cancel{\text{deg}})\left(\frac{\pi \text{ rad}}{180\ \cancel{\text{deg}}}\right) = \frac{4\pi}{3} \text{ radians}$$

Then, using a radius of $r = 4$ inches, you can find the arc length to be

$$s = r\theta = 4\left(\frac{4\pi}{3}\right) = \frac{16\pi}{3} \approx 16.76 \text{ inches.}$$

Note that the units for $r\theta$ are determined by the units for r because θ is given in radian measure and therefore has no units.

CHECKPOINT Now try Exercise 93.

The formula for the length of a circular arc can be used to analyze the motion of a particle moving at a *constant speed* along a circular path.

Linear and Angular Speed

Consider a particle moving at a constant speed along a circular arc of radius r. If s is the length of the arc traveled in time t, then the **linear speed** of the particle is

$$\text{Linear speed} = \frac{\text{arc length}}{\text{time}} = \frac{s}{t}.$$

Moreover, if θ is the angle (in radian measure) corresponding to the arc length s, then the **angular speed** of the particle is

$$\text{Angular speed} = \frac{\text{central angle}}{\text{time}} = \frac{\theta}{t}.$$

Linear speed measures how fast the particle moves, and angular speed measures how fast the angle changes.

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Example 6 Finding Linear Speed

The second hand of a clock is 10.2 centimeters long, as shown in Figure 4.16. Find the linear speed of the tip of this second hand.

Figure 4.16

Solution

In one revolution, the arc length traveled is

$$\begin{aligned} s &= 2\pi r \\ &= 2\pi(10.2) \qquad \text{Substitute for } r. \\ &= 20.4\pi \text{ centimeters.} \end{aligned}$$

The time required for the second hand to travel this distance is

$$t = 1 \text{ minute} = 60 \text{ seconds.}$$

So, the linear speed of the tip of the second hand is

$$\begin{aligned} \text{Linear speed} &= \frac{s}{t} \\ &= \frac{20.4\pi \text{ centimeters}}{60 \text{ seconds}} \\ &\approx 1.07 \text{ centimeters per second.} \end{aligned}$$

CHECKPOINT Now try Exercise 107.

Example 7 Finding Angular and Linear Speed

A 15-inch diameter tire on a car makes 9.3 revolutions per second (see Figure 4.17).

a. Find the angular speed of the tire in radians per second.

b. Find the linear speed of the car.

Figure 4.17

Solution

a. Because each revolution generates 2π radians, it follows that the tire turns

$$(9.3)(2\pi) = 18.6\pi \text{ radians per second.}$$

In other words, the angular speed is

$$\begin{aligned} \text{Angular speed} &= \frac{\theta}{t} \\ &= \frac{18.6\pi \text{ radians}}{1 \text{ second}} \\ &= 18.6\pi \text{ radians per second.} \end{aligned}$$

b. The linear speed of the tire and car is

$$\begin{aligned} \text{Linear speed} &= \frac{s}{t} \\ &= \frac{r\theta}{t} \\ &= \frac{\left(\frac{1}{2}\right)15(18.6\pi) \text{ inches}}{1 \text{ second}} \\ &\approx 438.25 \text{ inches per second.} \end{aligned}$$

CHECKPOINT Now try Exercise 109.

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4.1 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1–6, fill in the blank.

1. _____ means "measurement of triangles."
2. An _____ is determined by rotating a ray about its endpoint.
3. An angle with its initial side coinciding with the positive x-axis and the origin as its vertex is said to be in _____ .
4. Two angles that have the same initial and terminal sides are _____ .
5. One _____ is the measure of a central angle that intercepts an arc equal in length to the radius of the circle.
6. The _____ speed of a particle is a ratio of the change in the central angle to the time.

7. Is one-half revolution of a circle equal to 90° or 180°?
8. What is the sum of two complementary angles?
9. Are the angles 315° and $-225°$ coterminal?
10. Is the angle $\frac{2\pi}{3}$ acute or obtuse?

Procedures and Problem Solving

Estimating an Angle **In Exercises 11 and 12, estimate the angle to the nearest one-half radian.**

11.

12.

Determining Quadrants **In Exercises 13–18, determine the quadrant in which each angle lies. (The angle measure is given in radians.)**

13. (a) $\frac{\pi}{6}$ (b) $\frac{5\pi}{4}$
14. (a) $\frac{5\pi}{6}$ (b) $-\frac{5\pi}{3}$
15. (a) $\frac{7\pi}{4}$ (b) $\frac{11\pi}{4}$
16. (a) $-\frac{5\pi}{12}$ (b) $-\frac{13\pi}{9}$
17. (a) -1 (b) -2
18. (a) 3.5 (b) 2.25

Sketching Angles **In Exercises 19–24, sketch each angle in standard position.**

19. (a) $\frac{3\pi}{2}$ (b) $-\frac{7\pi}{2}$
20. (a) $\frac{3\pi}{4}$ (b) $\frac{4\pi}{3}$
21. (a) $-\frac{7\pi}{4}$ (b) $-\frac{5\pi}{2}$
22. (a) $\frac{11\pi}{6}$ (b) $-\frac{2\pi}{3}$
23. (a) 4 (b) -3
24. (a) 2 (b) 5π

Finding Coterminal Angles **In Exercises 25–28, determine two coterminal angles in radian measure (one positive and one negative) for each angle. (There are many correct answers).**

25. (a)

(b)

26. (a)

(b)

27. (a) $-\frac{9\pi}{4}$ (b) $-\frac{2\pi}{15}$
28. (a) $\frac{7\pi}{8}$ (b) $\frac{\pi}{12}$

Estimating an Angle **In Exercises 29 and 30, estimate the number of degrees in the angle.**

29.

30.

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Determining Quadrants **In Exercises 31–36, determine the quadrant in which each angle lies.**

31. (a) $55°$ (b) $215°$
32. (a) $121°$ (b) $181°$
33. (a) $150°$ (b) $282°$
34. (a) $87.9°$ (b) $8.5°$
35. (a) $-132° \, 50'$ (b) $-336° \, 30'$
36. (a) $-245.25°$ (b) $-12.35°$

Sketching Angles **In Exercises 37–42, sketch each angle in standard position.**

37. (a) $45°$ (b) $90°$ **38.** (a) $60°$ (b) $180°$
39. (a) $30°$ (b) $150°$ **40.** (a) $-270°$ (b) $-120°$
41. (a) $405°$ (b) $780°$ **42.** (a) $-450°$ (b) $-600°$

Finding Coterminal Angles **In Exercises 43–46, determine two coterminal angles in degree measure (one positive and one negative) for each angle. (There are many correct answers.)**

43. (a)

(b)

44. (a)

(b)

45. (a) $300°$ (b) $230°$
46. (a) $-445°$ (b) $-740°$

Converting From Degrees to Radians **In Exercises 47–50, rewrite each angle in radian measure as a multiple of π. (Do not use a calculator.)**

47. (a) $30°$ (b) $150°$
48. (a) $315°$ (b) $120°$
49. (a) $-20°$ (b) $-240°$
50. (a) $-270°$ (b) $144°$

Converting From Radians to Degrees **In Exercises 51–54, rewrite each angle in degree measure. (Do not use a calculator.)**

51. (a) $\frac{3\pi}{2}$ (b) $-\frac{7\pi}{6}$
52. (a) -4π (b) 3π
53. (a) $\frac{7\pi}{3}$ (b) $-\frac{13\pi}{60}$
54. (a) $-\frac{15\pi}{6}$ (b) $\frac{28\pi}{15}$

Converting From Degrees to Radians **In Exercises 55–60, convert the angle measure from degrees to radians. Round your answer to three decimal places.**

55. $115°$ **56.** $83.7°$
57. $-216.35°$ **58.** $-46.52°$
59. $-0.78°$ **60.** $395°$

Converting From Radians to Degrees **In Exercises 61–66, convert the angle measure from radians to degrees. Round your answer to three decimal places.**

61. $\frac{\pi}{7}$ **62.** $\frac{5\pi}{11}$
63. 6.5π **64.** -4.2π
65. -2 **66.** -0.57

Converting to Decimal Degree Form **In Exercises 67–72, use the angle-conversion capabilities of a graphing utility to convert the angle measure to decimal degree form. Round your answer to three decimal places if necessary.**

67. $64° \, 45'$ **68.** $-124° \, 30'$
69. $85° \, 18' \, 30''$ **70.** $-408° \, 16' \, 25''$
71. $-125° \, 36''$ **72.** $330° \, 25''$

Converting to $D°M'S''$ Form **In Exercises 73–78, use the angle-conversion capabilities of a graphing utility to convert the angle measure to $D°M'S''$ form.**

73. $280.6°$ **74.** $-115.8°$
75. $-345.12°$
76. $310.75°$
77. -0.355
78. 0.7865

Complementary and Supplementary Angles **In Exercises 79–86, find (if possible) the complement and supplement of the angle.**

79. $24°$ **80.** $129°$
81. $87°$ **82.** $167°$
83. $\frac{\pi}{3}$ **84.** $\frac{3\pi}{4}$
85. $\frac{\pi}{6}$ **86.** $\frac{2\pi}{3}$

Finding an Angle **In Exercises 87 and 88, find the angle in radians.**

87.

88.

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Finding the Central Angle **In Exercises 89–92, find the radian measure of the central angle of a circle of radius *r* that intercepts an arc of length *s*.**

	Radius r	*Arc Length s*
89.	15 inches	8 inches
90.	22 feet	10 feet
91.	14.5 centimeters	35 centimeters
92.	80 kilometers	160 kilometers

Finding Arc Length **In Exercises 93–96, find the length of the arc on a circle of radius *r* intercepted by a central angle *θ*.**

	Radius r	*Central Angle θ*
✓ **93.**	14 inches	180°
94.	9 feet	60°
95.	27 meters	$\frac{2\pi}{3}$ radians
96.	12 centimeters	$\frac{3\pi}{4}$ radians

Finding the Radius **In Exercises 97–100, find the radius *r* of a circle with an arc length *s* and a central angle *θ*.**

	Arc Length s	*Central Angle θ*
97.	36 feet	$\frac{\pi}{2}$ radians
98.	3 meters	$\frac{4\pi}{3}$ radians
99.	82 miles	135°
100.	8 inches	330°

Earth-Space Science **In Exercises 101 and 102, find the distance between the cities. Assume that Earth is a sphere of radius 4000 miles and the cities are on the same longitude (one city is due north of the other).**

	City	*Latitude*
101.	Dallas, Texas	32° 47′ 39″ N
	Omaha, Nebraska	41° 15′ 50″ N
102.	San Francisco, California	37° 47′ 36″ N
	Seattle, Washington	47° 37′ 18″ N

103. **Earth-Space Science** Assuming that Earth is a sphere of radius 6378 kilometers, what is the difference in the latitudes of Syracuse, New York and Annapolis, Maryland, where Syracuse is 450 kilometers due north of Annapolis?

104. **Electrical Engineering** A voltmeter's pointer is 6 centimeters in length (see figure). Find the number of degrees through which it rotates when it moves 2.5 centimeters on the scale.

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Figure for 104

Figure for 105

105. **Mechanical Engineering** An electric hoist is used to lift a piece of equipment 2 feet (see figure). The diameter of the drum on the hoist is 10 inches. Find the number of degrees through which the drum must rotate.

106. ***Why you should learn it*** *(p. 254)* The number of revolutions made by a figure skater for each type of axel jump is given. Determine the measure of the angle generated as the skater performs each jump. Give the answer in both degrees and radians.

(a) Single axel: $1\frac{1}{2}$ revolutions

(b) Double axel: $2\frac{1}{2}$ revolutions

(c) Triple axel: $3\frac{1}{2}$ revolutions

✓ **107.** **Linear Speed** A satellite in a circular orbit 1250 kilometers above Earth makes one complete revolution every 110 minutes. What is its linear speed? Assume that Earth is a sphere of radius 6400 kilometers.

108. **Mechanical Engineering** The circular blade on a saw has a diameter of 7.25 inches and rotates at 4800 revolutions per minute.

(a) Find the angular speed of the blade in radians per second.

(b) Find the linear speed of the saw teeth (in feet per second) as they contact the wood being cut.

✓ **109.** **Mechanical Engineering** The circular blade on a saw has a diameter of 7.5 inches (see figure) and rotates at 2400 revolutions per minute.

(a) Find the angular speed in radians per second.

(b) Find the linear speed of the saw teeth (in feet per second) as they contact the wood being cut.

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110. Angular Speed A computerized spin balance machine rotates a 25-inch diameter tire at 480 revolutions per minute.

(a) Find the road speed (in miles per hour) at which the tire is being balanced.

(b) At what rate should the spin balance machine be set so that the tire is being tested for 70 miles per hour?

111. Mechanical Engineering A DVD is approximately 12 centimeters in diameter. The drive motor of the DVD player is controlled to rotate precisely between 200 and 500 revolutions per minute, depending on what track is being read.

(a) Find an interval for the angular speed of a disc as it rotates.

(b) Find the linear speed of a point on the outermost track as the disc rotates.

112. MODELING DATA

The radii of the pedal sprocket, the wheel sprocket, and the wheel of the bicycle in the figure are 4 inches, 2 inches, and 14 inches, respectively. A cyclist is pedaling at a rate of 1 revolution per second.

(a) Find the speed of the bicycle in feet per second and miles per hour.

(b) Use your result from part (a) to write a function for the distance d (in miles) a cyclist travels in terms of the number n of revolutions of the pedal sprocket.

(c) Write a function for the distance d (in miles) a cyclist travels in terms of time t (in seconds). Compare this function with the function from part (b).

Conclusions

True or False? In Exercises 113–115, determine whether the statement is true or false. Justify your answer.

113. A degree is a larger unit of measure than a radian.

114. An angle that measures $-1260°$ lies in Quadrant III.

115. The angles of a triangle can have radian measures of $2\pi/3$, $\pi/4$, and $\pi/12$.

116. Geometry Show that the area of a circular sector of radius r with central angle θ is $A = \frac{1}{2}r^2\theta$, where θ is measured in radians.

Proof In Exercises 117 and 118, use the result of Exercise 116 to find the area of the sector.

117.

118.

119. Geometry The formulas for the area of a circular sector and arc length are $A = \frac{1}{2}r^2\theta$ and $s = r\theta$, respectively. (r is the radius and θ is the angle measured in radians.)

(a) Let $\theta = 0.8$. Write the area and arc length as functions of r. What is the domain of each function? Use a graphing utility to graph the functions. Use the graphs to determine which function changes more rapidly as r increases. Explain.

(b) Let $r = 10$ centimeters. Write the area and arc length as functions of θ. What is the domain of each function? Use a graphing utility to graph and identify the functions.

120. CAPSTONE

(a) Find each angle (in radians) shown on the unit circle.

(b) Find each angle (in degrees) shown on the unit circle.

121. Writing In your own words, write a definition of 1 radian.

122. Writing In your own words, explain the difference between 1 radian and 1 degree.

Cumulative Mixed Review

Library of Parent Functions In Exercises 123–128, sketch the graph of $f(x) = x^3$ and the graph of the function g. Describe the transformation from f to g.

123. $g(x) = (x - 1)^3$

124. $g(x) = x^3 - 4$

125. $g(x) = 2 - x^3$

126. $g(x) = -(x + 3)^3$

127. $g(x) = (x + 1)^3 - 3$

128. $g(x) = (x - 5)^3 + 1$

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4.2 Trigonometric Functions: The Unit Circle

The Unit Circle

The two historical perspectives of trigonometry incorporate different methods of introducing the trigonometric functions. Our first introduction to these functions is based on the unit circle.

Consider the **unit circle** given by

$$x^2 + y^2 = 1 \qquad \text{Unit circle}$$

as shown in Figure 4.18.

Figure 4.18

Imagine that the real number line is wrapped around this circle, with positive numbers corresponding to a counterclockwise wrapping and negative numbers corresponding to a clockwise wrapping, as shown in Figure 4.19.

Figure 4.19

As the real number line is wrapped around the unit circle, each real number t corresponds to a point

$$(x, y)$$

on the circle. For example, the real number 0 corresponds to the point $(1, 0)$. Moreover, because the unit circle has a circumference of 2π, the real number 2π also corresponds to the point $(1, 0)$.

In general, each real number t also corresponds to a central angle θ (in standard position) whose radian measure is t. With this interpretation of t, the arc length formula

$$s = r\theta$$

(with $r = 1$) indicates that the real number t is the (directional) length of the arc intercepted by the angle θ, given in radians.

What you should learn

- Identify a unit circle and describe its relationship to real numbers.
- Evaluate trigonometric functions using the unit circle.
- Use domain and period to evaluate sine and cosine functions and use a calculator to evaluate trigonometric functions.

Why you should learn it

Trigonometric functions are used to model the movement of an oscillating weight. For instance, in Exercise 77 on page 271, the displacement from equilibrium of an oscillating weight suspended by a spring is modeled as a function of time.

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The Trigonometric Functions

From the preceding discussion, it follows that the coordinates x and y are two functions of the real variable t. You can use these coordinates to define the six trigonometric functions of t.

sine	**cosine**	**tangent**
cosecant	**secant**	**cotangent**

These six functions are normally abbreviated sin, cos, tan, csc, sec, and cot, respectively.

Definitions of Trigonometric Functions

Let t be a real number and let (x, y) be the point on the unit circle corresponding to t.

$$\sin t = y \qquad \cos t = x \qquad \tan t = \frac{y}{x}, \quad x \neq 0$$

$$\csc t = \frac{1}{y}, \quad y \neq 0 \qquad \sec t = \frac{1}{x}, \quad x \neq 0 \qquad \cot t = \frac{x}{y}, \quad y \neq 0$$

Study Tip

Note in the definition at the left that the functions in the second row are the *reciprocals* of the corresponding functions in the first row.

In the definitions of the trigonometric functions, note that the tangent and secant are not defined when $x = 0$. For instance, because $t = \pi/2$ corresponds to $(x, y) = (0, 1)$, it follows that $\tan(\pi/2)$ and $\sec(\pi/2)$ are *undefined*. Similarly, the cotangent and cosecant are not defined when $y = 0$. For instance, because $t = 0$ corresponds to $(x, y) = (1, 0)$, cot 0 and csc 0 are *undefined*.

In Figure 4.20, the unit circle has been divided into eight equal arcs, corresponding to t-values of

$$0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, \text{ and } 2\pi.$$

Similarly, in Figure 4.21, the unit circle has been divided into 12 equal arcs, corresponding to t-values of

$$0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}, \frac{11\pi}{6}, \text{ and } 2\pi.$$

Using the (x, y) coordinates in Figures 4.20 and 4.21, you can easily evaluate the exact values of trigonometric functions for common t-values. This procedure is demonstrated in Example 1. You should study and learn these exact values for common t-values because they will help you to perform calculations in later sections.

Figure 4.20

Figure 4.21

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Example 1 Evaluating Trigonometric Functions

Evaluate the six trigonometric functions at each real number.

a. $t = \dfrac{\pi}{6}$ **b.** $t = \dfrac{5\pi}{4}$ **c.** $t = \pi$ **d.** $t = -\dfrac{\pi}{3}$

Solution

For each t-value, begin by finding the corresponding point (x, y) on the unit circle. Then use the definitions of trigonometric functions listed on page 266.

a. $t = \pi/6$ corresponds to the point $(x, y) = \left(\sqrt{3}/2, 1/2\right)$.

$$\sin\frac{\pi}{6} = y = \frac{1}{2} \qquad \csc\frac{\pi}{6} = \frac{1}{y} = \frac{1}{1/2} = 2$$

$$\cos\frac{\pi}{6} = x = \frac{\sqrt{3}}{2} \qquad \sec\frac{\pi}{6} = \frac{1}{x} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\tan\frac{\pi}{6} = \frac{y}{x} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \qquad \cot\frac{\pi}{6} = \frac{x}{y} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

b. $t = 5\pi/4$ corresponds to the point $(x, y) = \left(-\sqrt{2}/2, -\sqrt{2}/2\right)$.

$$\sin\frac{5\pi}{4} = y = -\frac{\sqrt{2}}{2} \qquad \csc\frac{5\pi}{4} = \frac{1}{y} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$$

$$\cos\frac{5\pi}{4} = x = -\frac{\sqrt{2}}{2} \qquad \sec\frac{5\pi}{4} = \frac{1}{x} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$$

$$\tan\frac{5\pi}{4} = \frac{y}{x} = \frac{-\sqrt{2}/2}{-\sqrt{2}/2} = 1 \qquad \cot\frac{5\pi}{4} = \frac{x}{y} = \frac{-\sqrt{2}/2}{-\sqrt{2}/2} = 1$$

c. $t = \pi$ corresponds to the point $(x, y) = (-1, 0)$.

$$\sin\pi = y = 0 \qquad \csc\pi = \frac{1}{y} \text{ is undefined.}$$

$$\cos\pi = x = -1 \qquad \sec\pi = \frac{1}{x} = \frac{1}{-1} = -1$$

$$\tan\pi = \frac{y}{x} = \frac{0}{-1} = 0 \qquad \cot\pi = \frac{x}{y} \text{ is undefined.}$$

d. Moving *clockwise* around the unit circle, it follows that $t = -\pi/3$ corresponds to the point $(x, y) = \left(1/2, -\sqrt{3}/2\right)$.

$$\sin\left(-\frac{\pi}{3}\right) = y = -\frac{\sqrt{3}}{2} \qquad \csc\left(-\frac{\pi}{3}\right) = \frac{1}{y} = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$$

$$\cos\left(-\frac{\pi}{3}\right) = x = \frac{1}{2} \qquad \sec\left(-\frac{\pi}{3}\right) = \frac{1}{x} = \frac{1}{1/2} = 2$$

$$\tan\left(-\frac{\pi}{3}\right) = \frac{y}{x} = \frac{-\sqrt{3}/2}{1/2} = -\sqrt{3}$$

$$\cot\left(-\frac{\pi}{3}\right) = \frac{x}{y} = \frac{1/2}{-\sqrt{3}/2} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

CHECKPOINT Now try Exercise 33.

Explore the Concept

With your graphing utility in *radian* and *parametric* modes, enter X1T = cos T and Y1T = sin T and use the following settings.

Tmin = 0, Tmax = 6.3, Tstep = 0.1
Xmin = −1.5, Xmax = 1.5, Xscl = 1
Ymin = −1, Ymax = 1, Yscl = 1

1. Graph the entered equations and describe the graph.
2. Use the *trace* feature to move the cursor around the graph. What do the t-values represent? What do the x- and y-values represent?
3. What are the least and greatest values of x and y?

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Domain and Period of Sine and Cosine

The *domain* of the sine and cosine functions is the set of all real numbers. To determine the *range* of these two functions, consider the unit circle shown in Figure 4.22. Because $r = 1$, it follows that $\sin t = y$ and $\cos t = x$. Moreover, because (x, y) is on the unit circle, you know that $-1 \le y \le 1$ and $-1 \le x \le 1$. So, the values of sine and cosine also range between -1 and 1.

$$-1 \le y \le 1 \qquad \text{and} \qquad -1 \le x \le 1$$
$$-1 \le \sin t \le 1 \qquad\qquad -1 \le \cos t \le 1$$

Figure 4.22

Adding 2π to each value of t in the interval $[0, 2\pi]$ completes a second revolution around the unit circle, as shown in Figure 4.23. The values of $\sin(t + 2\pi)$ and $\cos(t + 2\pi)$ correspond to those of $\sin t$ and $\cos t$. Similar results can be obtained for repeated revolutions (positive or negative) around the unit circle. This leads to the general result

$$\sin(t + 2\pi n) = \sin t \quad \text{and} \quad \cos(t + 2\pi n) = \cos t$$

for any integer n and real number t. Functions that behave in such a repetitive (or cyclic) manner are called **periodic.**

Figure 4.23

Definition of Periodic Function

A function f is **periodic** when there exists a positive real number c such that

$$f(t + c) = f(t)$$

for all t in the domain of f. The least number c for which f is periodic is called the **period** of f.

It follows from the definition of periodic function that the sine and cosine functions are periodic and have a period of 2π. The other four trigonometric functions are also periodic, and will be discussed further in Section 4.6.

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Recall from Section 1.3 that a function f is *even* when

$$f(-t) = f(t)$$

and is *odd* when

$$f(-t) = -f(t).$$

Of the six trigonometric functions, two are even and four are odd.

Even and Odd Trigonometric Functions

The cosine and secant functions are *even*.

$\cos(-t) = \cos t$ $\qquad$ $\sec(-t) = \sec t$

The sine, cosecant, tangent, and cotangent functions are *odd*.

$\sin(-t) = -\sin t$ $\qquad$ $\csc(-t) = -\csc t$

$\tan(-t) = -\tan t$ $\qquad$ $\cot(-t) = -\cot t$

Study Tip

In this text, the parentheses in $\sin(u)$ are sometimes omitted when u is an expression involving exponents, radicals, products, or quotients. For instance, $\sin(3\pi)$ can be written as $\sin 3\pi$. To evaluate $\sin 3\pi$, find the sine of the product 3π. The other trigonometric functions can be written and evaluated in the same manner.

Example 2 Using the Period to Evaluate Sine and Cosine

a. Because $\dfrac{13\pi}{6} = 2\pi + \dfrac{\pi}{6}$, you have

$$\sin\frac{13\pi}{6} = \sin\left(2\pi + \frac{\pi}{6}\right) = \sin\frac{\pi}{6} = \frac{1}{2}.$$

b. Because $-\dfrac{7\pi}{2} = -4\pi + \dfrac{\pi}{2}$, you have

$$\cos\left(-\frac{7\pi}{2}\right) = \cos\left(-4\pi + \frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0.$$

c. For $\sin t = \dfrac{4}{5}$, $\sin(-t) = -\dfrac{4}{5}$ because the function is odd.

CHECKPOINT Now try Exercise 41.

You can use a calculator to evaluate trigonometric functions. When evaluating a trigonometric function with a calculator, you need to set the calculator to the desired *mode* of measurement (degrees or radians). Most calculators do not have keys for the cosecant, secant, and cotangent functions. To evaluate these functions, you can use the [x^{-1}] key with their respective reciprocal functions sine, cosine, and tangent. For example, to evaluate $\csc(\pi/8)$, use the fact that

$$\csc\frac{\pi}{8} = \frac{1}{\sin(\pi/8)}$$

and enter the following keystroke sequence in *radian* mode.

[(] [SIN] [(] [π] [÷] 8 [)] [)] [x^{-1}] [ENTER] $\qquad$ Display 2.6131259

Technology Tip

When evaluating trigonometric functions with a calculator, remember to enclose all fractional angle measures in parentheses. For instance, to evaluate $\sin\theta$ for $\theta = \pi/6$, you should enter

[SIN] [(] [π] [÷] 6 [)] [ENTER].

Example 3 Using a Calculator

	Function	*Mode*	*Graphing Calculator Keystrokes*	*Display*
a.	$\sin\dfrac{2\pi}{3}$	Radian	[SIN] [(] 2 [π] [÷] 3 [)] [ENTER]	0.8660254
b.	$\cot 1.5$	Radian	[(] [TAN] [(] 1.5 [)] [)] [x^{-1}] [ENTER]	0.0709148

CHECKPOINT Now try Exercise 57.

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4.2 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1–3, fill in the blank(s).

1. Each real number t corresponds to a point (x, y) on the _____ .
2. A function f is _____ if there exists a positive real number c such that $f(t + c) = f(t)$ for all t in the domain of f.
3. A function f is _____ if $f(-t) = -f(t)$ and _____ if $f(-t) = f(t)$.
4. List the names and abbreviations of the six trigonometric functions.
5. Which trigonometric functions are even? odd?
6. Which trigonometric functions are undefined when $x = 0$?
7. How many equal arcs can you divide the unit circle into using t-values that are multiples of $\pi/4$?
8. How many times is $\tan t = y/x$ undefined on the unit circle?

Procedures and Problem Solving

Determining Values of Trigonometric Functions **In Exercises 9–12, determine the exact values of the six trigonometric functions of the angle θ.**

9.

10.

11.

12.

Finding a Point on the Unit Circle **In Exercises 13–22, find the point (x, y) on the unit circle that corresponds to the real number t.**

13. $t = \dfrac{\pi}{4}$
14. $t = \dfrac{\pi}{3}$
15. $t = \dfrac{7\pi}{6}$
16. $t = \dfrac{5\pi}{4}$
17. $t = \dfrac{2\pi}{3}$
18. $t = \dfrac{5\pi}{3}$
19. $t = \dfrac{3\pi}{2}$
20. $t = \pi$
21. $t = -\dfrac{7\pi}{4}$
22. $t = -\dfrac{4\pi}{3}$

Evaluating Sine, Cosine, and Tangent **In Exercises 23–32, evaluate (if possible) the sine, cosine, and tangent of the real number.**

23. $t = \dfrac{\pi}{4}$
24. $t = \dfrac{\pi}{3}$
25. $t = -\dfrac{7\pi}{4}$
26. $t = -\dfrac{5\pi}{4}$
27. $t = \dfrac{2\pi}{3}$
28. $t = \dfrac{5\pi}{3}$
29. $t = -\dfrac{5\pi}{3}$
30. $t = \dfrac{11\pi}{6}$
31. $t = -\dfrac{\pi}{6}$
32. $t = -\dfrac{\pi}{4}$

Evaluating Trigonometric Functions **In Exercises 33–38, evaluate (if possible) the six trigonometric functions of the real number.**

33. $t = 3\pi/4$
34. $t = 5\pi/6$
35. $t = \pi/2$
36. $t = 3\pi/2$
37. $t = -4\pi/3$
38. $t = 7\pi/4$

Using the Period to Evaluate Sine and Cosine **In Exercises 39–46, evaluate the trigonometric function using its period as an aid.**

39. $\sin 5\pi$
40. $\cos 5\pi$
41. $\cos \dfrac{8\pi}{3}$
42. $\sin \dfrac{9\pi}{4}$
43. $\cos\left(-\dfrac{13\pi}{6}\right)$
44. $\sin\left(-\dfrac{19\pi}{6}\right)$
45. $\sin\left(-\dfrac{9\pi}{4}\right)$
46. $\cos\left(-\dfrac{8\pi}{3}\right)$

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Using the Value of a Trigonometric Function **In Exercises 47–52, use the value of the trigonometric function to evaluate the indicated functions.**

47. $\sin t = \frac{1}{3}$
(a) $\sin(-t)$
(b) $\csc(-t)$

48. $\cos t = -\frac{3}{4}$
(a) $\cos(-t)$
(b) $\sec(-t)$

49. $\cos(-t) = -\frac{1}{5}$
(a) $\cos t$
(b) $\sec(-t)$

50. $\sin(-t) = \frac{3}{8}$
(a) $\sin t$
(b) $\csc t$

51. $\sin t = \frac{4}{5}$
(a) $\sin(\pi - t)$
(b) $\sin(t + \pi)$

52. $\cos t = \frac{4}{5}$
(a) $\cos(\pi - t)$
(b) $\cos(t + \pi)$

Using a Calculator **In Exercises 53–70, use a calculator to evaluate the trigonometric expression. Round your answer to four decimal places.**

53. $\sin \frac{\pi}{7}$

54. $\tan \frac{3\pi}{5}$

55. $\cos \frac{11\pi}{5}$

56. $\sin \frac{11\pi}{9}$

57. $\csc 1.3$

58. $\cot 3.7$

59. $\cos(-1.7)$

60. $\cos(-2.5)$

61. $\csc 0.8$

62. $\sec 1.8$

63. $\sec 22.8$

64. $\sin(-13.4)$

65. $\cot 2.5$

66. $\tan 1.75$

67. $\csc(-1.5)$

68. $\tan(-2.25)$

69. $\sec(-4.3)$

70. $\csc(-5.2)$

Approximating the Value of a Trigonometric Function **In Exercises 71 and 72, use the figure and a straightedge to approximate the value of each trigonometric function. Check your approximation using a graphing utility. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

71. (a) $\sin 5$ (b) $\cos 2$

72. (a) $\sin 0.75$ (b) $\cos 2.5$

Figure for 71–74

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Approximating the Solution of an Equation **In Exercises 73 and 74, use the figure and a straightedge to approximate the solution of each equation, where $0 \le t \le 2\pi$. Check your approximation using a graphing utility. To print an enlarged copy of the graph, go to the website *www.mathgraphs.com*.**

73. (a) $\sin t = 0.25$ (b) $\cos t = -0.25$

74. (a) $\sin t = -0.75$ (b) $\cos t = 0.75$

75. **Electrical Engineering** The initial current and charge in an electrical circuit are zero. The current when 100 volts is applied to the circuit is given by

$$I = 5e^{-2t} \sin t$$

where the resistance, inductance, and capacitance are 80 ohms, 20 henrys, and 0.01 farad, respectively. Approximate the current (in amperes) $t = 0.7$ second after the voltage is applied.

76. **Electrical Engineering** Approximate the current (in amperes) in the electrical circuit in Exercise 75 $t = 2.1$ seconds after the voltage is applied.

77. ***Why you should learn it*** *(p. 265)* The displacement from equilibrium of an oscillating weight suspended by a spring is given by

$$y(t) = \frac{1}{4} \cos 6t$$

where y is the displacement (in feet) and t is the time (in seconds) (see figure). Find the displacement when (a) $t = 0$, (b) $t = \frac{1}{4}$, and (c) $t = \frac{1}{2}$.

Equilibrium *Maximum negative displacement* *Maximum positive displacement*

Simple Harmonic Motion

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78. MODELING DATA

The displacement from equilibrium of an oscillating weight suspended by a spring and subject to the damping effect of friction is given by

$y(t) = \frac{1}{4}e^{-t} \cos 6t$

where y is the displacement (in feet) and t is the time (in seconds).

(a) What is the initial displacement ($t = 0$)?

(b) Use a graphing utility to complete the table.

t	0.50	1.02	1.54	2.07	2.59
y					

(c) The approximate times when the weight is at its maximum distance from equilibrium are shown in the table in part (b). Explain why the magnitude of the maximum displacement is decreasing. What causes this decrease in maximum displacement in the physical system? What factor in the model measures this decrease?

(d) Find the first two times when the weight is at the equilibrium point ($y = 0$).

Conclusions

True or False? **In Exercises 79–83, determine whether the statement is true or false. Justify your answer.**

79. Because $\sin(-t) = -\sin t$, it can be said that the sine of a negative angle is a negative number.

80. $\sin a = \sin(a - 6\pi)$

81. $\tan a = \tan(a - 6\pi)$

82. The real number 0 corresponds to the point (0, 1) on the unit circle.

83. $\cos\left(-\frac{7\pi}{2}\right) = \cos\left(\pi + \frac{\pi}{2}\right)$

84. Exploration Let (x_1, y_1) and (x_2, y_2) be points on the unit circle corresponding to $t = t_1$ and $t = \pi - t_1$, respectively.

(a) Identify the symmetry of the points (x_1, y_1) and (x_2, y_2).

(b) Make a conjecture about any relationship between $\sin t_1$ and $\sin(\pi - t_1)$.

(c) Make a conjecture about any relationship between $\cos t_1$ and $\cos(\pi - t_1)$.

85. Exploration Let (x_1, y_1) and (x_2, y_2) be points on the unit circle corresponding to $t = t_1$ and $t = t_1 + \pi$, respectively.

(a) Identify the symmetry of the points (x_1, y_1) and (x_2, y_2).

(b) Make a conjecture about any relationship between $\sin t_1$ and $\sin(t_1 + \pi)$.

(c) Make a conjecture about any relationship between $\cos t_1$ and $\cos(t_1 + \pi)$.

86. Verify that $\cos 2t \neq 2 \cos t$ by approximating $\cos 1.5$ and $2 \cos 0.75$.

87. Use the unit circle to verify that the cosine and secant functions are even and that the sine, cosecant, tangent, and cotangent functions are odd.

88. Think About It Because $f(t) = \sin t$ is an odd function and $g(t) = \cos t$ is an even function, what can be said about the function $h(t) = f(t)g(t)$?

89. Think About It Because $f(t) = \sin t$ and $g(t) = \tan t$ are odd functions, what can be said about the function $h(t) = f(t)g(t)$?

90. CAPSTONE A student you are tutoring has used a unit circle divided into 8 equal parts to complete the table for selected values of t. What is wrong?

t	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
x	1	$\frac{\sqrt{2}}{2}$	0	$-\frac{\sqrt{2}}{2}$	-1
y	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{2}}{2}$	0
$\sin t$	1	$\frac{\sqrt{2}}{2}$	0	$-\frac{\sqrt{2}}{2}$	-1
$\cos t$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{2}}{2}$	0
$\tan t$	Undef.	1	0	-1	Undef.

Cumulative Mixed Review

Finding Vertical and Horizontal Asymptotes **In Exercises 91–94, find any asymptotes of the graph of the rational function. Verify your results by using a graphing utility to graph the function.**

91. $f(x) = \dfrac{2x}{x - 3}$

92. $f(x) = \dfrac{5x}{x^2 + x - 6}$

93. $f(x) = \dfrac{x^2 + 3x - 10}{2x^2 - 8}$

94. $f(x) = \dfrac{x^3 - 6x^2 + x - 1}{2x^2 - 5x - 8}$

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4.3 Right Triangle Trigonometry

The Six Trigonometric Functions

Our second look at the trigonometric functions is from a *right triangle* perspective. Consider a right triangle, with one acute angle labeled θ, as shown in Figure 4.24. Relative to the angle θ, the three sides of the triangle are the **hypotenuse,** the **opposite side** (the side opposite the angle θ), and the **adjacent side** (the side adjacent to the angle θ).

Figure 4.24

Using the lengths of these three sides, you can form six ratios that define the six trigonometric functions of the acute angle θ.

sine	**cosecant**
cosine	**secant**
tangent	**cotangent**

In the following definitions it is important to see that

$0° < \theta < 90°$ θ lies in the first quadrant.

and that for such angles the value of each trigonometric function is *positive*.

Right Triangle Definitions of Trigonometric Functions

Let θ be an *acute* angle of a right triangle. Then the six trigonometric functions *of the angle* θ are defined as follows. (Note that the functions in the second column are the *reciprocals* of the corresponding functions in the first column.)

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

The abbreviations

"opp," "adj," and "hyp"

represent the lengths of the three sides of a right triangle.

opp = the length of the side *opposite* θ

adj = the length of the side *adjacent* to θ

hyp = the length of the *hypotenuse*

What you should learn

- Evaluate trigonometric functions of acute angles and use a calculator to evaluate trigonometric functions.
- Use the fundamental trigonometric identities.
- Use trigonometric functions to model and solve real-life problems.

Why you should learn it

You can use trigonometry to analyze all aspects of a geometric figure. For instance, Exercise 79 on page 282 shows you how trigonometric functions can be used to approximate the angle of elevation of a zip-line.

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Example 1 Evaluating Trigonometric Functions

Use the triangle in Figure 4.25 to find the exact values of the six trigonometric functions of θ.

Figure 4.25

Solution

By the Pythagorean Theorem, $(\text{hyp})^2 = (\text{opp})^2 + (\text{adj})^2$, it follows that

$$\begin{aligned} \text{hyp} &= \sqrt{4^2 + 3^2} \\ &= \sqrt{25} \\ &= 5. \end{aligned}$$

So, the six trigonometric functions of θ are

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{4}{5} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{5}{4}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{3}{5} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{5}{3}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{4}{3} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{3}{4}$$

CHECKPOINT Now try Exercise 9.

In Example 1, you were given the lengths of two sides of the right triangle, but not the angle θ. Often you will be asked to find the trigonometric functions for a *given* acute angle θ. To do this, you can construct a right triangle having θ as one of its angles.

Example 2 Evaluating Trigonometric Functions of 45°

Find the exact values of sin 45°, cos 45°, and tan 45°.

Solution

Construct a right triangle having 45° as one of its acute angles, as shown in Figure 4.26. Choose 1 as the length of the adjacent side. From geometry, you know that the other acute angle is also 45°. So, the triangle is isosceles and the length of the opposite side is also 1. Using the Pythagorean Theorem, you can find the length of the hypotenuse to be $\sqrt{2}$.

Figure 4.26

$$\sin 45° = \frac{\text{opp}}{\text{hyp}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cos 45° = \frac{\text{adj}}{\text{hyp}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan 45° = \frac{\text{opp}}{\text{adj}} = \frac{1}{1} = 1$$

CHECKPOINT Now try Exercise 21.

Technology Tip

You can use a calculator to convert the answers in Example 2 to decimals. However, the radical form is the exact value and, in most cases, the exact value is preferred.

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Example 3 Evaluating Trigonometric Functions of 30° and 60°

Use the equilateral triangle shown in Figure 4.27 to find the exact values of sin 60°, cos 60°, sin 30°, and cos 30°.

Figure 4.27

Solution

Use the Pythagorean Theorem and Figure 4.27 to verify the lengths of the sides. For $\theta = 60°$, you have adj $= 1$, opp $= \sqrt{3}$, and hyp $= 2$. So,

$$\sin 60° = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{3}}{2} \quad \text{and} \quad \cos 60° = \frac{\text{adj}}{\text{hyp}} = \frac{1}{2}.$$

For $\theta = 30°$, adj $= \sqrt{3}$, opp $= 1$, and hyp $= 2$. So,

$$\sin 30° = \frac{\text{opp}}{\text{hyp}} = \frac{1}{2} \quad \text{and} \quad \cos 30° = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{3}}{2}.$$

CHECKPOINT Now try Exercise 23.

Study Tip

Because the angles 30°, 45°, and 60° ($\pi/6$, $\pi/4$, and $\pi/3$) occur frequently in trigonometry, you should learn to construct the triangles shown in Figures 4.26 and 4.27.

Sines, Cosines, and Tangents of Special Angles

$$\sin 30° = \sin\frac{\pi}{6} = \frac{1}{2} \qquad \cos 30° = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad \tan 30° = \tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}$$

$$\sin 45° = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} \qquad \cos 45° = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} \qquad \tan 45° = \tan\frac{\pi}{4} = 1$$

$$\sin 60° = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \qquad \cos 60° = \cos\frac{\pi}{3} = \frac{1}{2} \qquad \tan 60° = \tan\frac{\pi}{3} = \sqrt{3}$$

In the box, note that $\sin 30° = \frac{1}{2} = \cos 60°$. This occurs because 30° and 60° are complementary angles, and, in general, it can be shown from the right triangle definitions that *cofunctions of complementary angles are equal*. That is, if θ is an acute angle, then the following relationships are true.

$$\sin(90° - \theta) = \cos\theta \qquad \cos(90° - \theta) = \sin\theta$$

$$\tan(90° - \theta) = \cot\theta \qquad \cot(90° - \theta) = \tan\theta$$

$$\sec(90° - \theta) = \csc\theta \qquad \csc(90° - \theta) = \sec\theta$$

To use a calculator to evaluate trigonometric functions of angles measured in degrees, set the calculator to *degree* mode and then proceed as demonstrated in Section 4.2.

Study Tip

Remember that throughout this text, it is assumed that angles are measured in radians unless noted otherwise. For instance, sin 1 means the sine of 1 radian and sin 1° means the sine of 1 degree.

Example 4 Using a Calculator

Use a calculator to evaluate $\cos(5° 40' 12'')$.

Solution

Begin by converting to decimal degree form.

$$5° 40' 12'' = 5° + \left(\frac{40}{60}\right)^{\circ} + \left(\frac{12}{3600}\right)^{\circ} = 5.67°$$

Then use a calculator in *degree* mode to evaluate cos 5.67°.

Function	*Graphing Calculator Keystrokes*	*Display*
$\cos(5°40'12'') = \cos 5.67°$	COS (5.67) ENTER	0.9951074

CHECKPOINT Now try Exercise 33.

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Trigonometric Identities

In trigonometry, a great deal of time is spent studying relationships between trigonometric functions (identities).

Fundamental Trigonometric Identities

Reciprocal Identities

$$\sin\theta = \frac{1}{\csc\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\cos\theta = \frac{1}{\sec\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{1}{\cot\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

Note that $\sin^2\theta$ represents $(\sin\theta)^2$, $\cos^2\theta$ represents $(\cos\theta)^2$, and so on.

Explore the Concept

Select a number t and use your graphing utility to calculate

$$(\sin t)^2 + (\cos t)^2.$$

Repeat this experiment for other values of t and explain why the answer is always the same. Is the result true in both *radian* and *degree* modes?

Example 5 Applying Trigonometric Identities

Let θ be an acute angle such that $\cos\theta = 0.8$. Find the values of (a) $\sin\theta$ and (b) $\tan\theta$ using trigonometric identities.

Solution

a. To find the value of $\sin\theta$, use the Pythagorean identity

$$\sin^2\theta + \cos^2\theta = 1.$$

So, you have

$\sin^2\theta + (0.8)^2 = 1$	Substitute 0.8 for $\cos\theta$.
$\sin^2\theta = 1 - (0.8)^2$	Subtract $(0.8)^2$ from each side.
$\sin^2\theta = 0.36$	Simplify.
$\sin\theta = \sqrt{0.36}$	Extract positive square root.
$\sin\theta = 0.6$	Simplify.

b. Now, knowing the sine and cosine of θ, you can find the tangent of θ to be

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{0.6}{0.8} = 0.75.$$

Figure 4.28

Use the definitions of $\sin\theta$ and $\tan\theta$ and the triangle shown in Figure 4.28 to check these results.

CHECKPOINT Now try Exercise 55.

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Example 6 Applying Trigonometric Identities

Let θ be an acute angle such that $\tan\theta = 3$. Find the values of (a) $\cot\theta$ and (b) $\sec\theta$ using trigonometric identities.

Solution

a. $\cot\theta = \dfrac{1}{\tan\theta}$ — Reciprocal identity

$\cot\theta = \dfrac{1}{3}$

b. $\sec^2\theta = 1 + \tan^2\theta$ — Pythagorean identity

$\sec^2\theta = 1 + 3^2$ — Substitute 3 for $\tan\theta$.

$\sec^2\theta = 10$ — Simplify.

$\sec\theta = \sqrt{10}$ — Extract positive square root.

Use the definitions of $\cot\theta$ and $\sec\theta$, and the triangle shown in Figure 4.29, to check these results.

Figure 4.29

CHECKPOINT Now try Exercise 57.

Example 7 Using Trigonometric Identities

Use trigonometric identities to transform one side of the equation into the other $(0 < \theta < \pi/2)$.

a. $\cos\theta\sec\theta = 1$ **b.** $(\sec\theta + \tan\theta)(\sec\theta - \tan\theta) = 1$

Solution

Simplify the expression on the left-hand side of the equation until you obtain the right-hand side.

a. $\cos\theta\sec\theta = \left(\dfrac{1}{\cancel{\sec\theta}}\right)\cancel{\sec\theta}$ — Reciprocal identity

$= 1$ — Divide out common factor.

b. $(\sec\theta + \tan\theta)(\sec\theta - \tan\theta)$

$= \sec^2\theta - \sec\theta\tan\theta + \sec\theta\tan\theta - \tan^2\theta$ — Distributive Property

$= \sec^2\theta - \tan^2\theta$ — Simplify.

$= 1$ — Pythagorean identity

CHECKPOINT Now try Exercise 65.

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Applications

Many applications of trigonometry involve a process called **solving right triangles.** In this type of application, you are usually given one side of a right triangle and one of the acute angles and are asked to find one of the other sides, *or* you are given two sides and are asked to find one of the acute angles.

In Example 8, the angle you are given is the **angle of elevation,** which represents the angle from the horizontal upward to the object. In other applications you may be given the **angle of depression,** which represents the angle from the horizontal downward to the object. (See Figure 4.30.)

Figure 4.30

Example 8 Using Trigonometry to Solve a Right Triangle

A surveyor is standing 50 feet from the base of a large tree, as shown in Figure 4.31. The surveyor measures the angle of elevation to the top of the tree as 71.5°. How tall is the tree?

Figure 4.31

Surveyor

Solution

From Figure 4.31, you can see that

$$\tan 71.5° = \frac{\text{opp}}{\text{adj}} = \frac{y}{x}$$

where $x = 50$ and y is the height of the tree. So, the height of the tree is

$$\begin{aligned} y &= x \tan 71.5° \\ &= 50 \tan 71.5° \\ &\approx 149.43 \text{ feet.} \end{aligned}$$

CHECKPOINT Now try Exercise 77.

Example 9 Using Trigonometry to Solve a Right Triangle

You are 200 yards from a river. Rather than walking directly to the river, you walk 400 yards along a straight path to the river's edge. Find the acute angle θ between this path and the river's edge, as illustrated in Figure 4.32.

Solution

From Figure 4.32, you can see that the sine of the angle θ is

$$\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{200}{400} = \frac{1}{2}.$$

Now you should recognize that $\theta = 30°$.

CHECKPOINT Now try Exercise 79.

Figure 4.32

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In Example 9, you were able to recognize that $\theta = 30°$ is the acute angle that satisfies the equation $\sin\theta = \frac{1}{2}$. Suppose, however, that you were given the equation $\sin\theta = 0.6$ and were asked to find the acute angle θ. Because

$$\sin 30° = \frac{1}{2} = 0.5000$$

and

$$\sin 45° = \frac{1}{\sqrt{2}} \approx 0.7071$$

you might guess that θ lies somewhere between 30° and 45°. In a later section, you will study a method by which a more precise value of θ can be determined.

Example 10 Solving a Right Triangle

Find the length c of the skateboard ramp shown in Figure 4.33.

Figure 4.33

Solution

From Figure 4.33, you can see that

$$\sin 18.4° = \frac{\text{opp}}{\text{hyp}} = \frac{4}{c}.$$

So, the length of the ramp is

$$c = \frac{4}{\sin 18.4°} \approx \frac{4}{0.3156} \approx 12.67 \text{ feet.}$$

CHECKPOINT Now try Exercise 81.

What's Wrong?

You use a graphing utility to check the answer to Example 10 and obtain the screen shown below. What's wrong?

Technology Tip

Calculators and graphing utilities have both *degree* and *radian* modes. As you progress through this chapter, be sure you use the correct mode.

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4.3 Exercises

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

1. Match the trigonometric function with its right triangle definition.

(a) sine (b) cosine (c) tangent (d) cosecant (e) secant (f) cotangent

(i) $\frac{\text{hyp}}{\text{adj}}$ (ii) $\frac{\text{opp}}{\text{adj}}$ (iii) $\frac{\text{opp}}{\text{hyp}}$ (iv) $\frac{\text{adj}}{\text{opp}}$ (v) $\frac{\text{hyp}}{\text{opp}}$ (vi) $\frac{\text{adj}}{\text{hyp}}$

In Exercises 2 and 3, fill in the blanks.

2. Relative to the acute angle θ, the three sides of a right triangle are the _____ , the _____ side, and the _____ side.

3. An angle that measures from the horizontal upward to an object is called the angle of _____ , whereas an angle that measures from the horizontal downward to an object is called the angle of _____ .

In Exercises 4–6, use the figure to answer the question.

4. What is the length of the side opposite the angle θ?

5. What is the length of the side adjacent to the angle θ?

6. What is the length of the hypotenuse?

Figure for Exercises 4–6

Procedures and Problem Solving

Evaluating Trigonometric Functions **In Exercises 7–10, find the exact values of the six trigonometric functions of the angle θ shown in the figure. (Use the Pythagorean Theorem to find the third side of the triangle.)**

7.

8.

✓ **9.**

10.

Evaluating Trigonometric Functions **In Exercises 11 and 12, find the exact values of the six trigonometric functions of the angle θ for each of the triangles. Explain why the function values are the same.**

11.

12.

Evaluating Trigonometric Functions **In Exercises 13–20, sketch a right triangle corresponding to the trigonometric function of the acute angle θ. Use the Pythagorean Theorem to determine the third side of the triangle and then find the other five trigonometric functions of θ.**

13. $\sin\theta = \frac{5}{6}$

14. $\cot\theta = 5$

15. $\sec\theta = 4$

16. $\cos\theta = \frac{3}{7}$

17. $\tan\theta = 3$

18. $\csc\theta = \frac{17}{4}$

19. $\cot\theta = \frac{3}{2}$

20. $\sin\theta = \frac{3}{8}$

Evaluating Trigonometric Functions of 30°, 45°, and 60° **In Exercises 21–30, construct an appropriate triangle to complete the table. ($0° \le \theta \le 90°$, $0 \le \theta \le \pi/2$)**

	Function	θ (deg)	θ (rad)	*Function Value*
✓ **21.**	sin	30°		
22.	cos	60°		
✓ **23.**	tan		$\frac{\pi}{3}$	
24.	sec		$\frac{\pi}{4}$	
25.	cot			$\frac{\sqrt{3}}{3}$
26.	csc			$\sqrt{2}$

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	Function	θ (deg)	θ (rad)	*Function Value*
27.	cos		$\frac{\pi}{6}$	
28.	sin		$\frac{\pi}{4}$	
29.	cot			1
30.	tan			$\frac{1}{\sqrt{3}}$

Using a Calculator **In Exercises 31–36, use a calculator to evaluate each function. Round your answers to four decimal places. (Be sure the calculator is in the correct angle mode.)**

31. (a) $\sin 10°$ (b) $\cos 80°$

32. (a) $\tan 18.5°$ (b) $\cot 71.5°$

✓ **33.** (a) $\sec 42° 12'$ (b) $\csc 48° 7'$

34. (a) $\cos 8° 50' 25''$ (b) $\sec 56° 8' 10''$

35. (a) $\cot \frac{\pi}{16}$ (b) $\tan \frac{\pi}{8}$

36. (a) $\sec 1.54$ (b) $\cos 1.25$

Applying Trigonometric Identities **In Exercises 37–52, complete the identity.**

37. $\sin \theta = \frac{1}{\square}$

38. $\cos \theta = \frac{1}{\square}$

39. $\tan \theta = \frac{1}{\square}$

40. $\csc \theta = \frac{1}{\square}$

41. $\sec \theta = \frac{1}{\square}$

42. $\cot \theta = \frac{1}{\square}$

43. $\tan \theta = \frac{\square}{\square}$

44. $\cot \theta = \frac{\square}{\square}$

45. $\sin^2 \theta + \cos^2 \theta = \square$

46. $1 + \tan^2 \theta = \square$

47. $\sin(90° - \theta) = \square$

48. $\cos(90° - \theta) = \square$

49. $\tan(90° - \theta) = \square$

50. $\cot(90° - \theta) = \square$

51. $\sec(90° - \theta) = \square$

52. $\csc(90° - \theta) = \square$

Applying Trigonometric Identities **In Exercises 53–58, use the given function value(s) and the trigonometric identities to find the indicated trigonometric functions.**

53. $\sin 60° = \frac{\sqrt{3}}{2}, \quad \cos 60° = \frac{1}{2}$

(a) $\tan 60°$ (b) $\sin 30°$

(c) $\cos 30°$ (d) $\cot 60°$

54. $\sin 30° = \frac{1}{2}, \quad \tan 30° = \frac{\sqrt{3}}{3}$

(a) $\csc 30°$ (b) $\cot 60°$

(c) $\cos 30°$ (d) $\cot 30°$

✓ **55.** $\csc \theta = 3, \quad \sec \theta = \frac{3\sqrt{2}}{4}$

(a) $\sin \theta$ (b) $\cos \theta$

(c) $\tan \theta$ (d) $\sec(90° - \theta)$

56. $\sec \theta = 5, \quad \tan \theta = 2\sqrt{6}$

(a) $\cos \theta$ (b) $\cot \theta$

(c) $\cot(90° - \theta)$ (d) $\sin \theta$

✓ **57.** $\cos \alpha = \frac{1}{3}$

(a) $\sec \alpha$ (b) $\sin \alpha$

(c) $\cot \alpha$ (d) $\sin(90° - \alpha)$

58. $\tan \beta = 5$

(a) $\cot \beta$ (b) $\cos \beta$

(c) $\tan(90° - \beta)$ (d) $\csc \beta$

Using Trigonometric Identities **In Exercises 59–66, use trigonometric identities to transform one side of the equation into the other ($0 < \theta < \pi/2$).**

59. $\tan \theta \cot \theta = 1$

60. $\csc \theta \tan \theta = \sec \theta$

61. $\tan \theta \cos \theta = \sin \theta$

62. $\cot \theta \sin \theta = \cos \theta$

63. $(1 + \cos \theta)(1 - \cos \theta) = \sin^2 \theta$

64. $(\csc \theta + \cot \theta)(\csc \theta - \cot \theta) = 1$

✓ **65.** $\frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} = \csc \theta \sec \theta$

66. $\frac{\tan \theta + \cot \theta}{\tan \theta} = \csc^2 \theta$

Evaluating Trigonometric Functions **In Exercises 67–72, find each value of θ in degrees ($0° < \theta < 90°$) and radians ($0 < \theta < \pi/2$) without using a calculator.**

67. (a) $\sin \theta = \frac{1}{2}$ (b) $\csc \theta = 2$

68. (a) $\cos \theta = \frac{\sqrt{2}}{2}$ (b) $\tan \theta = 1$

69. (a) $\sec \theta = 2$ (b) $\cot \theta = 1$

70. (a) $\tan \theta = \sqrt{3}$ (b) $\cos \theta = \frac{1}{2}$

71. (a) $\csc \theta = \frac{2\sqrt{3}}{3}$ (b) $\sin \theta = \frac{\sqrt{2}}{2}$

72. (a) $\cot \theta = \frac{\sqrt{3}}{3}$ (b) $\sec \theta = \sqrt{2}$

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Finding Side Lengths of a Triangle **In Exercises 73–76, find the exact values of the indicated variables (selected from *x*, *y*, and *r*).**

73. Find y and r.

74. Find x and y.

75. Find x and y.

76. Find x and r.

✓ **77. Geometry** A six-foot person walks from the base of a streetlight directly toward the tip of the shadow cast by the streetlight. When the person is 16 feet from the streetlight and 5 feet from the tip of the streetlight's shadow, the person's shadow starts to appear beyond the streetlight's shadow.

(a) Draw a right triangle that gives a visual representation of the problem. Show the known quantities and use a variable to indicate the height of the streetlight.

(b) Use a trigonometric function to write an equation involving the unknown quantity.

(c) What is the height of the streetlight?

78. Geometry A 25-meter line is used to tether a helium-filled balloon. Because of a breeze, the line makes an angle of approximately 75° with the ground.

(a) Draw a right triangle that gives a visual representation of the problem. Show the known quantities and use a variable to indicate the height of the balloon.

(b) Use a trigonometric function to write an equation involving the unknown quantity.

(c) What is the height of the balloon?

✓ **79.** ***Why you should learn it*** *(p. 273)* A zip-line steel cable is being constructed for a reality television competition show. The high end of the zip-line is attached to the top of a 50-foot pole while the lower end is anchored at ground level to a stake 50 feet from the base of the pole.

(a) Find the angle of elevation of the zip-line.

(b) Find the number of feet of steel cable needed for the zip-line.

(c) A contestant takes 6 seconds to reach the ground from the top of the zip-line. At what rate is the contestant moving down the line? At what rate is the contestant dropping vertically?

80. Geomatics In traveling across flat land you notice a mountain directly in front of you. Its angle of elevation (to the peak) is 3.5°. After you drive 13 miles closer to the mountain, the angle of elevation is 9° (see figure). Approximate the height of the mountain.

✓ **81. Biology** A biologist wants to know the width w of a river (see figure) in order to properly set instruments for studying the pollutants in the water. From point A, the biologist walks upstream 100 feet and sights to point C, From this sighting, it is determined that $\theta = 58°$. How wide is the river? Verify your result numerically.

82. Mechanical Engineering A steel plate has the form of one-fourth of a circle with a radius of 60 centimeters. Two 2-centimeter holes are to be drilled in the plate, positioned as shown in the figure. Find the coordinates of the center of each hole.

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83. MODELING DATA

A 20-meter line is used to tether a helium-filled balloon. Because of a breeze, the line makes an angle of approximately 85° with the ground.

(a) Draw a right triangle that gives a visual representation of the problem. Show the known quantities of the triangle and use a variable to indicate the height of the balloon.

(b) Use a trigonometric function to write an equation involving the unknown quantity.

(c) What is the height of the balloon?

(d) The breeze becomes stronger and the angle the balloon makes with the ground decreases. How does this affect the triangle you drew in part (a)?

(e) Complete the table, which shows the heights (in meters) of the balloon for decreasing angle measures θ.

Angle, θ	80°	70°	60°	50°
Height				

Angle, θ	40°	30°	20°	10°
Height				

(f) As the angle the balloon makes with the ground approaches 0°, how does this affect the height of the balloon? Draw a right triangle to explain your reasoning.

Conclusions

True or False? **In Exercises 84–86, determine whether the statement is true or false. Justify your answer.**

84. $\sin 60° \csc 60° = 1$

85. $\sin 45° + \cos 45° = 1$

86. $\cot^2 10° - \csc^2 10° = -1$

87. Think About It You are given the value of $\tan \theta$. Is it possible to find the value of $\sec \theta$ without finding the measure of θ? Explain.

88. Exploration Use a graphing utility to complete the table and make a conjecture about the relationship between $\cos \theta$ and $\sin(90° - \theta)$. What are the angles θ and $90° - \theta$ called?

θ	0°	20°	40°	60°	80°
$\cos \theta$					
$\sin(90° - \theta)$					

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89. Exploration

(a) Use a graphing utility to complete the table. Round your results to four decimal places.

θ	0°	20°	40°	60°	80°
$\sin \theta$					
$\cos \theta$					
$\tan \theta$					

(b) Classify each of the three trigonometric functions as increasing or decreasing for the table values.

(c) From the values in the table, verify that the tangent function is the quotient of the sine and cosine functions.

90. CAPSTONE The Johnstown Inclined Plane in Pennsylvania is one of the longest and steepest hoists in the world. The railway cars travel a distance of 896.5 feet at an angle of approximately 35.4°, rising to a height of 1693.5 feet above sea level.

(a) Find the vertical rise of the inclined plane.

(b) Find the elevation of the lower end of the inclined plane.

(c) The cars move up the mountain at a rate of 300 feet per minute. Find the rate at which they rise vertically.

Cumulative Mixed Review

Graphing an Exponential Function **In Exercises 91–94, use a graphing utility to construct a table of values for the function. Then sketch the graph of the function. Identify any asymptote of the graph.**

91. $f(x) = e^{3x}$

92. $f(x) = -e^{3x}$

93. $f(x) = 2 + e^{3x}$

94. $f(x) = -4 + e^{3x}$

Graphing a Logarithmic Function **In Exercises 95–98, use a graphing utility to graph the logarithmic function. Find the domain, vertical asymptote, and x-intercept of the logarithmic function.**

95. $f(x) = \log_3 x$

96. $f(x) = \log_3 x + 1$

97. $f(x) = -\log_3 x$

98. $f(x) = \log_3(x - 4)$

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4.4 Trigonometric Functions of Any Angle

Introduction

In Section 4.3, the definitions of trigonometric functions were restricted to acute angles. In this section, the definitions are extended to cover *any* angle. When θ is an *acute* angle, the definitions here coincide with those given in the preceding section.

Definition of Trigonometric Functions of Any Angle

Let θ be an angle in standard position with (x, y) a point on the terminal side of θ and $r = \sqrt{x^2 + y^2} \neq 0$.

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r}$$

$$\tan\theta = \frac{y}{x}, \quad x \neq 0 \qquad \cot\theta = \frac{x}{y}, \quad y \neq 0$$

$$\sec\theta = \frac{r}{x}, \quad x \neq 0 \qquad \csc\theta = \frac{r}{y}, \quad y \neq 0$$

What you should learn

- Evaluate trigonometric functions of any angle.
- Find reference angles.
- Evaluate trigonometric functions of real numbers.

Why you should learn it

You can use trigonometric functions to model and solve real-life problems. For instance, Exercise 124 on page 291 shows you how trigonometric functions can be used to model the monthly sales of a seasonal product, such as wakeboards.

Because $r = \sqrt{x^2 + y^2}$ *cannot* be zero, it follows that the sine and cosine functions are defined for any real value of θ. However, when $x = 0$, the tangent and secant of θ are undefined. For example, the tangent of 90° is undefined. Similarly, when $y = 0$, the cotangent and cosecant of θ are undefined.

Example 1 Evaluating Trigonometric Functions

Let $(-3, 4)$ be a point on the terminal side of θ (see Figure 4.34). Find the sine, cosine, and tangent of θ.

Figure 4.34

Solution

Referring to Figure 4.34, you can see that $x = -3$, $y = 4$, and

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{(-3)^2 + 4^2} \\ &= \sqrt{25} \\ &= 5. \end{aligned}$$

So, you have $\sin\theta = \dfrac{y}{r} = \dfrac{4}{5}$, $\cos\theta = \dfrac{x}{r} = -\dfrac{3}{5}$, and $\tan\theta = \dfrac{y}{x} = -\dfrac{4}{3}$.

CHECKPOINT Now try Exercise 11.

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