Computing Longitudinal Moments for Heterogeneous Agent Models*

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May 4, 2022

Abstract

Computing population moments for heterogeneous agent models is a necessary step for their estimation and evaluation. Computation based on Monte Carlo methods is usually time- and resource-consuming because it involves simulating a large sample of agents and (potentially) tracking them over time. We argue in favor of an alternative method for computing both cross-sectional and longitudinal moments that exploits the (endogenous) Markov transition function that defines the model's stationary distribution. The method relies on iterating forward an approximation of the transition function which is readily available from the computation of the model's solution. This provides precise estimates of the moments at lower time- and resource-costs compared to Monte Carlo based methods.

JEL: C6, E2

Keywords: Computational Methods, Heterogeneous Agents, Simulation.

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Increasingly available micro economic data and substantial improvements in computational power have led to growing use of heterogeneous agent models to analyze a wide variety of economic phenomena (Heathcote, Storesletten, and Violante, 2009; De Nardi, French, and Jones, 2016). Estimating and using these models typically requires computing cross-sectional and longitudinal moments that characterize the distribution of the agents and their behavior. Longitudinal moments involve following sub-populations or individuals over time (e.g., mobility rates across the wealth distribution, income persistence, or inter-generational correlations). However, standard Monte Carlo based methods used to compute these moments rely on costly and imprecise simulations of the population. This makes the estimation and evaluation of models difficult, limiting their use.

We argue in favor of an alternative method for computing longitudinal moments in heterogeneous agent models that relies on following the distribution of any sub-population over time, by iterating forward the Markov kernel that characterizes the model's stationary distribution of agents. This method comes at minimal additional cost because the Markov kernel is already approximated as part of most solution methods (e.g., Young, 2010; Heer and Maußner, 2005, Ch. 7) and avoids the impreciseness and inefficiencies of Monte Carlo simulation by directly approximating the distribution of agents of interest.

We take as given the model's solution in the form of policy functions for agents that, together with the stochastic processes of exogenous states, implies an evolution for the agents in the economy. This evolution is captured by a Markov kernel, $T\left(s'|s\right)$, that maps the transition of agents from a current state s into a future state s' in the state space $\mathcal{S}$. The stationary distribution, λ, is the solution to

$$\lambda\left(s'\right) = \int_{s\in\mathcal{S}} T\left(s'|s\right)\lambda\left(s\right)ds, \tag{1}$$

We describe how to use λ and T to directly compute cross-sectional and longitudinal moments, focusing on the distribution of agents rather than a simulated sample of them.

Cross-sectional moments. These moments involve taking expectations over some variable of interest (x) for some sub-population characterized by states $s \in S \subseteq \mathcal{S}$,

$$E[x|s \in S] = \int_{s \in S} x(s) \lambda_S(s) \, ds, \tag{2}$$

where $\lambda_S \equiv {}^{\mathbb{I}_{s\in S}\lambda(s)}/_{\int \mathbb{I}_{s\in S}\lambda(s)ds}$ is the marginal distribution of the sub-population in S, with $\mathbb{I}_{s\in S}$ and indicator variable for whether or not $s \in S$. Equation (2) applies to a wide range of moments. For example, the moments of the wealth distribution (an endogenous state) for the whole population (when $S = \mathcal{S}$) or a subgroup (say among the top income earners).[1] As is well understood, these moments can be computed immediately from the solution of the model's stationary distribution (λ), either by approximating the integral (Judd, 1998, Ch. 7) or by calculating the moment from the discrete approximation of the distribution itself.

Longitudinal moments. Many other moments require knowing either the collective outcomes of a group of agents over time, for instance to compute the transition rates across different occupations, or the outcomes of individual agents, for example to compute the auto-correlation of their income.[2] Direct computation of these moments is hard because of the stochastic nature of the time-paths of individuals. However, we demonstrate that it is both possible and computationally efficient to extend the approach used in computing cross-sectional moments to longitudinal moments by focusing on the transition of the distribution of agents, rather than relying on a sample of realized paths as in Monte Carlo based methods.

Consider an outcome of interest $x\left(s, s'\right)$ that depends on the initial and final state of an agent. This outcome could be any function of the initial, final, or intervening states of the

[1] Equation (2) can also be used to define percentiles or other descriptors of the distribution. It also applies to moments that depend on both endogenous and exogenous states, for example by making x total income.

[2] Other moments that require collective outcomes include mobility rates across the income or wealth distribution, or inter-generational mobility in life cycle models. Other moments that require individual outcomes include the distribution of growth rates of income or wealth for individual agents, or the distribution of lifetime earnings.

agent. For example, it could be an indicator function for whether the agent satisfies some condition in the future as being a top earner or having a certain occupation, or the agent's income. The expectation of interest depends on whether we focus on the behavior of the group of agents (as in transition rates) or of individual agents (as in the auto-correlation of income). In the first case, we must follow the group (S) as a whole and compute

$$E\left[x|s\in S\right]=\int_{s\in S}\int_{s^{'}\in\mathcal{S}}x\left(s,s^{'}\right)\lambda_{S}^{'}\left(s^{'}\right)ds^{'}\,\lambda_{S}\left(s\right)ds, \tag{3}$$

where $\lambda_S^{'}$ is the future distribution of agents conditional on the initial distribution λ_S. In the second case, we must follow the possible paths of each individual and compute

$$E\left[x|s\in S\right]=\int_{s\in S}\int_{s^{'}\in\mathcal{S}}x\left(s,s^{'}\right)\lambda_{\{s\}}^{'}\left(s^{'}\right)ds^{'}\,\lambda_{S}\left(s\right)ds, \tag{4}$$

where $\lambda_{\{s\}}^{'}$ is the future distribution of the mass of agents that starts in state $s\in S$ (i.e., given an initial distribution $\delta_{\{s\}}$).

The difficulty in evaluating the expectations in (3) and (4) resides in obtaining $\lambda_S^{'}$ and $\lambda_{\{s\}}^{'}$ because this requires accounting for the variation in individual paths between the initial and final period. We directly compute $\lambda_S^{'}$ and $\lambda_{\{s\}}^{'}$ by iterating forward the initial distribution of agents using the Markov kernel T,

$$\lambda_{S}^{'}\left(s^{'}\right)=\int_{s\in\mathcal{S}}T\left(s^{'}|s\right)\lambda_{S}\left(s\right)ds;\quad \lambda_{\{s\}}^{'}\left(s^{'}\right)=\int_{s\in\mathcal{S}}T\left(s^{'}|s\right)\delta_{\{s\}}\left(s\right)ds. \tag{5}$$

Performing the iteration in (5) is relatively costless, as similar iterations are involved in finding the stationary distribution (see 1) and the number of iterations required to compute $\lambda_S^{'}$ are finite (and known).[3] Once $\lambda_S^{'}$ and $\lambda_{\{s\}}^{'}$ are obtained, the moments can be computed.

In practice, the solution of these models is typically computed using a discrete

[3]The integral in the computation of $\lambda_{\{s\}}^{'}$ is of course superfluous. Nevertheless, integration becomes necessary when iterating more than one period into the future, as the initial (degenerate) distribution $\delta_{\{s\}}$ generically distributes mass across the state space $\mathcal{S}$.

approximation of the Markov kernel $\hat{T}$ that operates over a discrete state space and induces a discrete distribution $\hat{\lambda}$ in the form of a histogram (see, Young, 2010; Tan, 2020). In this case the formulas in (1)-(4) replace integrals for sums over the discretized state space. We show that this approach provides fast and precise estimates of the moments of interest without involving the computation of new objects, relative to those already involved in the solution of the model.

We apply our method to two versions of the standard heterogeneous agent model based on Aiyagari (1994), one with infinitely lived agents and the other with overlapping generations, both in partial equilibrium with fixed interest rates (a common assumption in life-cycle models and models of small open economies). We focus on the computation of moments from the stationary distribution of agents. We show how to use our method to efficiently compute cross-sectional and longitudinal moments. The method applies for unconditional moments (following the whole population) and conditional (or sub-population) moments.

We compare our method with a Monte Carlo simulation method common in the literature (e.g., Judd, 1998, Ch. 8). We find that our method is more precise and efficient than the simulation method. Sentence about exact moments (how large must simulation be to get the same moments). Sentence about speed and memory demands across methods (efficiency).

1 Baseline heterogeneous agent models

We illustrate our method in the context of the baseline Bewley-Hugget-Aiyagari-Imrohoroglu model that is the workhorse heterogeneous agent model. The model economy is populated by a continuum of agents indexed by $i \in [0, 1]$ that differ only on their age (h), their exogenous labor productivity (ε), and their endogenous asset holdings (a). Labor productivity follows a Markov process with transition function H. Agents are price takers and receive income from a return (r) on their savings and from wages (w) paid for their supply of efficiency units of labor (ℓ) that depend on their age and labor productivity.

The dynamic programming problem of an agent is

$$V_h(\varepsilon, a) = \max_{a',c} \; u(c) + \beta E\left[V_{h+1}\left(\varepsilon', a'\right) |\varepsilon\right] \tag{6}$$
$$\text{s.t.} \;\; (1+r)\,a + w\ell(h,\varepsilon) = c + a'; \; a' \geq \underline{a}.$$

The solution to (6) is a savings function $(a_h^\star)$, such that $a_h^\star(\varepsilon, a) \geq \underline{a}$ for all (ε, a) and

$$V_h(\varepsilon, a) = u\left(y\,(1+r)\,a + w\ell(h,\varepsilon) - a_h^\star(\varepsilon, a)\right) + \beta E\left[V_{h+1}\left(\varepsilon', a_h^\star(\varepsilon, a)\right) |\varepsilon\right]. \tag{7}$$

We will focus on a stationary equilibrium with a (time-invariant) distribution of agents $\mathcal{S}$ is the state space with typical element $s = (h, \varepsilon, a)$. Given a birth/death process for agents, the transition function of labor productivity, and the savings functions, the stationary distribution is a solution to (1), where the Markov kernel $T\left(s'|s\right)$ is constructed using the policy functions and evolution of exogenous states.

We close the model by specifying the markets for labor and capital. Factor demand comes from competitive final good producers that produce the consumption good by operating a constant-returns-to-scale production function that is twice continuously differentiable, $Y = F(K, L)$. We assume that ours is a small open economy, so that the rental rate of capital (the rate of return on assets) is exogenously given by the international interest rate $r^\star$. The total supply of labor, $L = \int \ell(h(s), \varepsilon(s))\, \lambda(s)\, ds$, is exogenous. It depends only on the birth/death process and the transition function for ε and can be computed independently from the agent's dynamic programming problem. The labor market clears when $w = F_L\left({}^{L}\!/_{K}\right)$, where the ratio ${}^{L}\!/_{K}$ is pinned down by the international interest rate from $r^\star = F_K\left({}^{L}\!/_{K}\right)$.[4]

We solve for two versions of the model that differ in the birth/death process of agents. In both models, we adopt the following functional forms for the utility and the production

[4]Euler's theorem guarantees that the partial derivatives of F are homogeneous of degree 0. The differentiability assumption is enough to guarantee their invertibility to get ${}^{L}\!/_{K}$ in terms of $r^\star$.

functions:

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}; \quad F(K, L) = ZK^{\alpha}L^{1-\alpha}. \tag{8}$$

We set α equal to 0.36, as in Aiyagari (1994), and set σ equal to 2 which is in the range of values used in that paper. We take $r^{\star}$ to be 3.2 to match the average (wealth-weighted) real return on net-worth found in Fagereng, Guiso, Malacrino, and Pistaferri (2020) of 3.79 percent. We set Z so that labour income in our model matches the US value for 2019, which is \$53,624.[5] We set $\underline{a} = 0$, preventing borrowing. Below, we outline the differences between the two different versions of this model and their parametrization.

Infinitely lived heterogeneous agent model We consider a version of the model where agents are infinitely lived and their labor efficiency depends only on their labor productivity, without loss, $\ell(h, \varepsilon) = \exp(\varepsilon)$. Then, we focus on the age-invariant solutions to (6) and (7), a value function $V(\varepsilon, a)$ and a savings function $a^{\star}(\varepsilon, a)$. Accordingly, we drop age from the state vector when referring to the infinitely lived agents model.

We assume that labor productivity follows a discrete Markov process with $n_{\varepsilon} = 15$ states and transition matrix $P^{\varepsilon}_{n_{\varepsilon} \times n_{\varepsilon}}$. We obtain P^{ε} by discretizing an AR(1) process with persistence $\rho_{\varepsilon} = 0.963$ and innovation variance $\sigma^2_{\varepsilon} = 0.162$ using Rouwenhorst (1995)'s method. The values of ρ_{ε} and σ_{ε} come from Storesletten et al. (2004).

We further modify the model by allowing for heterogeneous returns on savings, a key ingredient for generating high levels of wealth inequality (Benhabib, Bisin, and Zhu, 2011; Stachurski and Toda, 2019). We add a state to the agent's dynamic problem that captures the evolution of their returns, so that $r_i = r \exp(\zeta_i)$ is the return of agent i, where r is the market's rate and the state ζ follows a discrete Markov process with $n_{\zeta} = 7$ states and transition matrix $P^{\zeta}_{n_{\zeta} \times n_{\zeta}}$. As before, we obtain P^{ζ} by discretizing an AR(1) process with

[5] We construct this value from FRED Data (U.S. Bureau of Economic Analysis, 2022) as Total Wages and Salaries (BA06RC1A027NBEA) divided by the 12-month average of Civilian Labor Force Level (CLF16OV).

persistence ρ_ζ and innovation variance σ_ζ^2 using Tauchen (1986)'s method. We set $\rho_\zeta = 0.70$ and $\sigma_\zeta^2 = 1.3$.

Overlapping generations heterogeneous agent model In the second version of the model, agents live for $H > 0$ periods. We add mortality risk to the model, so that an agent conditional survival probability going into age h is ϕ_h. We set $\phi_{H+1} = 1$ so that agents die with certainty after age H. We set the survival probabilities following Bell and Miller (2002) projections for the U.S., with each model period corresponding to a single year and $H = 100$. Upon death, agents are replaced by a newborn (age 20) agent who inherits a bequest $a_h^\star$ from the previous generation. Accordingly, we modify the model by introducing a value for bequests. The problem for an agent of age h is

$$V_h(\varepsilon, a) = \max_{a', c} \; u(c) + \phi_{h+1}\beta E\left[V_{h+1}\left(\varepsilon', a'\right)|\varepsilon\right] + (1 - \phi_{h+1})\, v\left(a'\right) \tag{9}$$

$$\text{s.t.} \;\; (1 + r)\, a + w\ell(h, \varepsilon) = c + a'; \; a' \geq 0.$$

We set $v\left(a'\right) = \chi \frac{\left(a'\right)^{1-\gamma} - 1}{1-\gamma}$ as in De Nardi and Yang (2016), with $\chi_1 = 1.27$ and $\gamma = 2$. These parameters are chosen to replicate two features of De Nardi and Yang (2016). First, a marginal propensity to bequeath out of an additional dollar of 56%, and second, a threshold above which an individual wants to start making a bequest of 19.2 times average income.

Efficiency units of labor are $\ell(h, \varepsilon) = \exp(\xi(h) + \varepsilon)$, where $\xi(h)$ is a quadratic polynomial that generates a 50 percent rise in average labor income from age 21 to its peak at age 51 as in Guvenen, Kambourov, Kuruscu, Ocampo, and Chen (2019).[6] Finally, we use the same process for labor productivity (ε) as in the infinitely lived agent model.

[6] $\xi(h) = \left(60(h-1) - (h-1)^2\right)/1800$.

1.1 Solving the model

We solve for the policy functions of agents using readily available solution methods that exploit the optimality conditions of the savings choice (i.e., Carroll, 2006). Having computed the policy functions, we approximate the Markov kernel, T, of the distribution of agents by discretizing it over assets on a grid $\vec{a}_{n_a}$ following Young (2010) and Tan (2020). The result is a transition matrix $\hat{T}$, whose elements $\hat{T}\left(s, s^{'}\right)$ gives the probability that an agent with current state s transitions to state $s^{'}$. This probability depends on the birth/death process (for instance, agents of age $h < 1$ transition to age $h+1$ with certainty), the transition matrix of the labor productivity process (H), and the approximation of the transition of assets on the fixed grid $\vec{a}_{n_a}$.[7] Finally, we compute the stationary distribution of agents on the discrete grid by iterating over

$$\hat{\lambda}^{n+1}\left(s^{'}\right) = \sum_{s} \hat{T}\left(s, s^{'}\right) \hat{\lambda}^{n}\left(s\right), \tag{10}$$

for some initial $\hat{\lambda}^{0}$. The stationary distribution $\left(\hat{\lambda}\right)$ is the limit of $\hat{\lambda}^{n}$ as n grows large.

The approximated Markov kernel, $\hat{T}$, plays an important role in computing moments because it describes the evolution of states given any initial distribution. This is crucial for computing longitudinal moments where it is necessary to know the joint distribution of agents across time. This is the case for computing inter-temporal (or inter-generational) transition rates across the income or wealth distribution, or for computing the persistence or auto-correlation of individual variables (e.g., income, consumption, savings). Cross-sectional moments are obtained directly from the stationary distribution, $\hat{\lambda}$ as they do not require knowing the behavior of agents over time.

[7] An agent with state s transitions with certainty to having assets $a^{'} = a^{\star}_{h(s)}\left(\varepsilon\left(s\right), a\left(s\right)\right) \in \left[\vec{a}_{j}, \vec{a}_{j+1}\right]$, for some j. In the discrete approximation the agent transitions to either $\vec{a}_{j}$ with probability ${a^{'}-\vec{a}_{j}}/{\vec{a}_{j+1}-\vec{a}_{j}}$ or $\vec{a}_{j+1}$ with probability ${\vec{a}_{j+1}-a^{'}}/{\vec{a}_{j+1}-\vec{a}_{j}}$.

2 Computing population moments

The objective is to compute population (and sub-population) moments that depend on agents' behavior both within and across periods. These moments can describe the dispersion of the (cross-sectional) distribution of agents, like the ratio between the 90^{th} and 10^{th} percentiles of wealth, or the concentration of income or wealth among given groups, say the top 1 percent wealth-share. However, it is often necessary to follow individuals over time in order to compute longitudinal moments like transition rates

Cross-sectional moments can be readily computed by integrating with respect to the stationary distributional of agents (λ) or its discrete approximation $\left(\hat{\lambda}\right)$. In contrast, longitudinal moments involve keeping track of individuals over time. This is often achieved through Monte Carlo based simulations of a sample of individuals. Our method relies on tracking the distribution of the relevant group of individuals (sub-population) instead of tracking the individuals themselves.

The histogram iteration method Consider a moment describing the expectation over some outcome in some future period $x\left(s^{'}, s\right)$ for a group of individuals satisfying some condition, say having a certain level of wealth or income. For cross-sectional moments the outcome x depends only on the current state. It is possible to determine a subset of the state space $S \subseteq \mathcal{S}$ such that any agent with state $s \in S$ belongs to the sub-population of interest. The objective is to compute the expected value as in (3) or (4). We obtain the sub-population's initial distribution, λ_S, from the stationary distribution λ by restricting its domain to S and normalizing. Tracking the distribution of the sub-population involves iterating over λ_S with the Markov kernel T (or its approximation) as in (10).

The expectation of interest is then

$$E\left[x|s \in S\right] = \sum_{s} \sum_{s^{'}} x\left(s^{'}, x\right) \hat{\lambda}_S^{'}\left(s^{'}\right) \hat{\lambda}_S\left(s\right) \tag{11}$$

when the moment requires tracking only the outcomes of the group in S, and

$$E\left[x|s\in S\right]=\sum_{s}\sum_{s^{'}}x\left(s^{'},x\right)\hat{\lambda}_{\{s\}}^{'}\left(s^{'}\right)\hat{\lambda}_{S}\left(s\right) \tag{12}$$

when the moment requires tracking the future outcomes of individuals. In this case $\hat{\lambda}_{\{s\}}^{'}$ is the future distribution starting from the degenerate distribution $\delta_{\{s\}}$ that places full probability on the state $s\in S$. Below, we apply this method to computing cross-sectional and longitudinal moments for the models described in Section 1.

2.1 Moments for the infinitely lived agents model

2.2 Moments for the overlapping generations model

- Example of moments (some need histogram simulation some do not)
  1. Cross-Sectional moments
     - Lorenz curve and Pareto tail (perhaps cite Bounenfant & Toda)
     - 90-10 ratio of wealth
     - First 4 moments of wealth distribution
  2. Longitudinal moments
     - Transition rates in wealth distribution (Top 1% in 10 years). These are equivalent to occupational transition rates in discrete choice models.
     - Auto-correlation of consumption (5 years), for Aiyagari model These are equivalent to auto-correlation of income when there are endogenous choices involved (say entrepreneurial income, or occupational choices).
     - Distribution of lifetime earnings, for OLG model (Useful to compare to new evidence from administrative data on evolution of inequality in the U.S. Guvenen, Kaplan, Song, and Weidner (2021))
     - Choose a cohort, say 50 year olds, compute 90-10 wealth ratio in each

subsequent year to get time series (for newborns compute it directly from histogram with no simulation using marginal distributions by age)

  - Repeat previous moment with sub-cohort (say those born in the top/bottom decile of wealth distribution)

- Two methods: Monte Carlo simulation or Histogram simulation (non-stochastic)
- Monte Carlo is well understood, provide cite to textbooks (Canova? Heer + Maussner? Judd?) and provide algorithm in appendix
- Histogram simulation in words: Follow population over time and compute conditional expectations. Use Histogram Kernel.

3 Discussion

- Discuss generalization of method
- It applies almost without change to continuous time cite Achdou, Han, Lasry, Lions, and Moll (2021). Relation between finite difference method kernel and histogram method. Describe in appendix.
- It applies to continuously distributed states by taking the histogram as the value of the distribution at certain points (normalized so that it integrates to 1).
- Applies to non-stationary distributions. No change, just let time index actual time, not just cohort.
- Applies with more endogenous choices. Policy functions solved with extensions of EGM, Barillas and Fernández-Villaverde (2007) and Fella (2014). Then integrate over choices using histogram.
- Discussion of the trade-offs in constructing T. Or how to construct P, how is the transition matrix constructed.
- Take the true policy function. Can we approximate it to get closed form solutions to moments? Perhaps we know how to take convolutions of normal (for the shocks).

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Appendices for Online Publication

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B Continuous time models