Computing Longitudinal Moments for Heterogeneous Agent Models*

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Abstract

Computing population moments for heterogeneous agent models is a necessary step for their estimation and evaluation. Computation based on Monte Carlo methods is usually time- and resource-consuming because it involves simulating a large sample of agents and potentially tracking them over time. We argue in favor of an alternative non-stochastic method for computing cross-sectional and longitudinal moments that exploits the endogenous Markov transition function that defines the stationary distribution of agents in the model. The method relies on following the distribution of populations of interest by iterating forward the Markov transition function rather than focusing on a simulated sample of agents. Approximations of this function are readily available from standard solution methods of dynamic programming problems. The method provides precise estimates of moments like top-wealth shares, auto-correlations, transition rates, age-profiles, or coefficients of population regressions at lower time- and resource-costs compared to Monte Carlo based methods. The method is particularly useful for moments of small groups of agents or involving rare events.

JEL: C6, E2

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Computing cross-sectional and longitudinal moments is integral to the estimation and use of heterogeneous agents models that are common in the study of a wide variety of economic phenomena (e.g., Heathcote, Storesletten, and Violante 2009; De Nardi, French, and Jones 2016; De Nardi and Fella 2017). However, calculating these moments frequently poses computational challenges that arise from the repeated simulation of the models. These challenges limit how researchers can use these models and the features they are able to include in them, even as computational power continues to improve.

One key challenge is the cost of calculating longitudinal moments that require following individuals over time (e.g., mobility rates across occupations or the wealth distribution, income persistence, or inter-generational correlations), or population regressions used in the models' estimation. Standard Monte Carlo based methods used to compute these moments rely on a simulated panel of agents that can fail to be representative of small sub-populations like the "very rich", or of the effects of rare events (e.g., health shocks, tail risks). So, in order to obtain accurate moments, these panels must be simulated with a large number of agents, often millions of them, which is computationally costly.

We argue in favor of an alternative non-stochastic method for computing longitudinal moments that directly follows the distribution of any sub-population over time. We do this by iterating forward the Markov kernel that characterizes how agents transition between states. This method comes at minimal cost because the Markov kernel is already approximated as part of most solution methods (e.g., Young 2010; Heer and Maußner 2005, Ch. 7). Moreover, it generates moments that avoid the impreciseness and inefficiencies of Monte Carlo simulation that result from the use of random number generators and costly simulation initialization.

We take as given the model's solution in the form of policy functions for agents that, together with the stochastic processes of exogenous states, imply the evolution of the distribution of agents in the economy. This evolution is captured by a Markov kernel, $T\left(s^{\prime}|s\right)$, that maps the transition of a mass of agents from a current state s into a future state s' in the state space $\mathcal{S}$. The stationary distribution, λ, is the solution to

$$\lambda\left(s'\right) = \int_{s\in\mathcal{S}} T\left(s'|s\right)\lambda\left(s\right) ds. \tag{1}$$

We describe how to use λ and T to directly compute cross-sectional and longitudinal moments, rather than using a simulated panel of agents.

Cross-sectional moments. These moments involve taking expectations over some outcome of interest, $x(s)$, for some sub-population characterized by states $s \in S \subseteq \mathcal{S}$,

$$E\left[x|s\in S\right] = \int_{s\in S} x\left(s\right)\lambda_S\left(s\right) ds, \tag{2}$$

where $\lambda_S \equiv \mathbb{I}_{s\in S}\lambda(s)/\int \mathbb{I}_{s\in S}\lambda(s)\,ds$ is the marginal distribution of the sub-population in S, and where $\mathbb{I}_{s\in S}$ is an indicator variable for whether or not $s \in S$. Equation (2) applies to a wide range of moments. For example, the skewness or kurtosis of the endogenous wealth distribution for the whole population (when $S = \mathcal{S}$) or for a subgroup (say top income earners).[1] These moments can be computed immediately from the solution of the model's stationary distribution (λ), either by approximating the integral (Judd 1998, Ch. 7) or by calculating the moment from a discrete approximation of the distribution (Young 2010).

Longitudinal moments. Many other moments require knowing either the collective outcomes of a group of agents over time (e.g., for computing transition rates across occupations) or the outcomes of individual agents (e.g., for computing the auto-correlation of their wealth).[2] Calculating these moments is difficult because of the stochastic nature of the individuals' time-paths. However, we show that it is possible to extend the approach described above for cross-sectional moments to the calculation of longitudinal moments at low computational

[1] Equation (2) can also be used to define percentiles or other descriptors of the distribution. These expectations can also characterize the population value of coefficients in cross-sectional regressions.

[2] Moments that require collective outcomes include mobility rates across the income or wealth distribution, or inter-generational mobility in life cycle models. Moments that require individual outcomes include the distribution of growth rates of income or wealth for individual agents, or the distribution of lifetime earnings.

cost. This is accomplished by focusing on the transition of the distribution of agents, taking into account all possible paths an individual can take rather than relying on a sample of realized paths from a Monte Carlo simulation.

In general, we consider an outcome of interest $x\left(s, s^{'}\right)$ that depends on some initial and final state of an agent. This outcome could be any function of the agent's initial or final states. However, the procedure for computing the expectation of interest depends on whether we focus on the outcomes of a group of agents (as in transition rates) or of individual agents (as in the auto-correlation of wealth). We tackle these cases separately.

In the first case, only group outcomes matter. This is the case for the transition rate in or out of a particular wealth quantile, say the share of individuals in the top 1% of the wealth distribution who remain in the top 1% at some future period. To compute this, we must follow the group of individuals in the top 1% at a given time, characterized by the subset of the state space $S \subseteq \mathcal{S}$, in order to obtain

$$E\left[x|s \in S\right] = \int_{s \in S} \int_{s^{'} \in \mathcal{S}} x\left(s, s^{'}\right) \lambda_{S}^{'}\left(s^{'}\right) ds^{'} \, \lambda_{S}\left(s\right) ds, \tag{3}$$

where $x(s, s')$ is a binary variable equal to one if both s and s' place an individual in the top 1% of the wealth distribution, λ_S is the distribution of individuals in the top 1%, those with $s \in S$, and $\lambda_{S}^{'}$ is the future distribution of agents conditional on the initial distribution λ_S. We directly compute the distribution $\lambda_{S}^{'}$ as:

$$\lambda_{S}^{'}\left(s^{'}\right) = \int_{s \in \mathcal{S}} T\left(s^{'}|s\right) \lambda_{S}\left(s\right) ds \tag{4}$$

For any initial and final state, computing this integral merely employs the same Markov kernel T as in equation (1).

In the second case, individual outcomes matter. This is the case for the auto-correlation of individual wealth. To compute this, we must follow all the possible paths of each individual and compute

$$E\left[x|s\in S\right]=\int_{s\in S}\int_{s^{'}\in\mathcal{S}}x\left(s,s^{'}\right)\lambda^{'}_{\{s\}}\left(s^{'}\right)ds^{'}\,\lambda_{S}\left(s\right)ds. \tag{5}$$

Here, the outcome of interest is $x(s,s^{'})=(w(s)-\bar{w})(w^{'}(s^{'})-\bar{w}^{'})$, where we denote the wealth of an agent by $w(s)$. The average wealth can change across periods, $\bar{w}\neq\bar{w}^{'}$, reflecting potential non-stationary environments, like in transition paths. Finally, $\lambda^{'}_{\{s\}}$ is the future distribution of the mass of agents that starts in state $s\in S$, which we again compute using the Markov kernel T as:

$$\lambda^{'}_{\{s\}}\left(s^{'}\right)=\int_{s\in\mathcal{S}}T\left(s^{'}|s\right)\delta_{\{s\}}\left(s\right)ds, \tag{6}$$

where $\delta_{\{s\}}$ is the (degenerate) distribution concentrated in state s.[3]

In practice, the stationary distribution of heterogeneous agents models is typically computed using a discrete approximation of the Markov kernel $\hat{T}$ that operates over a discrete state space and induces a discrete distribution $\hat{\lambda}$ in the form of a histogram (see, Young 2010; Tan 2020; Gouin-Bonenfant and Toda 2023).[4] In this case, the formulas in (1)-(5) replace integrals for sums over the discretized state space. Accordingly, we call our method the histogram iteration method. We show that it provides fast and precise estimates of moments of interest without involving the computation of new objects, relative to those involved in the model's solution.

We apply our method to two partial equilibrium versions of the standard heterogeneous agent model based on Aiyagari (1994), one with infinitely lived agents and one with overlapping generations. We approximate the stationary distribution and its associated Markov kernel following Young (2010). We calculate moments characterizing the right tail of the wealth

[3] The integral in the computation of $\lambda^{'}_{\{s\}}$ is of course initially superfluous. Nevertheless, it becomes necessary when iterating more than one period, as $\delta_{\{s\}}$ generically distributes mass across the state space $\mathcal{S}$.

[4] Alternatively, projection methods can also be used to represent the distribution of heterogeneous agents. See Algan, Allais, and Den Haan (2008) for a demonstration for how projection methods can minimize cross-sectional sampling variation in heterogeneous agent models.

distribution and the persistence of consumption and wealth in the infinitely lived agents model. In the overlapping-generations model, we calculate the age-profile of wealth and five- and fifteen-year auto-correlation of wealth. In this way, we use our method to calculate cross-sectional and longitudinal moments for both the entire population of agents and specific sub-populations, like the wealthiest and poorest agents. We compare the results with those from a Monte Carlo simulation, which is common in the literature (e.g., Judd 1998, Ch. 8).

We find that our histogram iteration method is at least as precise as using large simulated panels while significantly reducing computational time. Time savings come from avoiding the simulation of a large enough panel of agents, in favor of computing the histogram that approximates the distribution of agents. In most cases, these time savings more than compensate for the time it takes to compute longitudinal moments by iterating on the histogram, which is somewhat greater than the time it takes to compute them from a simulated panel. There are further gains when computing cross-sectional moments because no iteration or simulation is needed when using the histogram.

1. Baseline heterogeneous agent models

We illustrate our method in the context of the baseline Bewley-Hugget-Aiyagari-Imrohoroglu model. The economy is populated by a continuum of agents indexed by $i \in [0,1]$ that differ on their age (h), labor productivity (ε), rate of return (ζ), and endogenous asset holdings (a). Labor productivity and rates of return follow discrete Markov processes with transition matrices P^{ε} and P^{ζ}. Agents are price takers. They receive income from the return on their savings, $r(\zeta)$, and from wages, w, paid for their supply of efficiency units of labor, $\ell(h,\varepsilon)$, which depends on their age and labor productivity.

The dynamic programming problem of an agent of age h is

$$V_h\left(\varepsilon,\zeta,a\right) = \max_{a',c}\ u\left(c\right) + \beta\phi_h E\left[V_{h+1}\left(\varepsilon',\zeta',a'\right)|\varepsilon,\zeta\right] \tag{7}$$

$$\text{s.t. } [1 + r(\zeta)]\, a + w\ell\,(h, \varepsilon) = c + a^{'};\ a^{'} \geq \underline{a}.$$

The solution to (7) is a savings function $(a_h^{\star})$, such that $a_h^{\star}(\varepsilon, \zeta, a) \geq \underline{a}$ for all (ε, ζ, a) and

$$V_h\,(\varepsilon, \zeta, a) = u\left([1 + r(\zeta)]\, a + w\ell\,(h, \varepsilon) - a_h^{\star}(\varepsilon, \zeta, a)\right) + \beta E\left[V_{h+1}\left(\varepsilon^{'}, \zeta^{'}, a_h^{\star}(\varepsilon, a)\right) |\varepsilon, \zeta\right]. \tag{8}$$

We will focus on a stationary equilibrium with a time-invariant distribution of agents. $\mathcal{S}$ is the state space with typical element $s = (h, \varepsilon, \zeta, a)$. Given a birth and death process for agents, the transition function of labor productivity, and the savings functions, the stationary distribution is a solution to (1), where the Markov kernel $T\left(s^{'}|s\right)$ is constructed using the policy functions and the evolution of exogenous states.

We solve the model in partial equilibrium taking the wage rate, w, and the average return on savings, $\bar{r}$, as exogenous. We do this to focus on the computation of moments for any given solution of the agents' problem. Our results apply in a general equilibrium setting when computing the moments after finding the market clearing prices.

We solve for two versions of the model that differ in the birth and death process of agents. In both models, we adopt the following functional form for agents' utility:

$$u\,(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}. \tag{9}$$

We set σ equal to 2 which is in the range of values used in Aiyagari (1994). We take $\bar{r}$ to be 3.2 in line with historical values for the U.S. and we set w so that labour income in our model matches average labor income for the U.S. in 2019, which is \$53,624.[5] We set $\underline{a} = 0$, preventing borrowing. Below, we outline the differences between the two different versions of the model and their parametrization.

[5]We construct this value from FRED Data (U.S. Bureau of Economic Analysis 2022) as Total Wages and Salaries (BA06RC1A027NBEA) divided by the 12-month average of Civilian Labor Force Level (CLF16OV).

Infinitely lived heterogeneous agent model. We consider a version of the model where agents are infinitely lived and their labor efficiency depends only on their labor productivity. In particular, $\ell(h,\varepsilon) = \exp(\varepsilon)$. We focus on the age-invariant solutions to (7) and (8), a value function $V(\varepsilon,\zeta,a)$ and a savings function $a^{\star}(\varepsilon,\zeta,a)$. Accordingly, we drop age from the state vector when referring to the infinitely lived agents model.

Labor productivity follows a discrete Markov process with $n_\varepsilon = 11$. We obtain P^ε by discretizing an AR(1) process using Rouwenhorst (1995)'s method and use persistence $\rho_\varepsilon = 0.963$ and innovation variance $\sigma_\varepsilon^2 = 0.162$ from Storesletten, Telmer, and Yaron (2004).

We include heterogeneous returns on savings, a key ingredient for generating high levels of wealth inequality (Benhabib, Bisin, and Zhu 2011; Stachurski and Toda 2019), by setting an agent's returns to be $r(\zeta) = \bar{r}\exp(\zeta)$. The state ζ follows a discrete Markov process with $n_\zeta = 7$ states. We obtain P^ζ by discretizing an AR(1) process with persistence $\rho_\zeta = 0.70$ and innovation variance $\sigma_\zeta^2 = 1.3$ using Tauchen (1986)'s method.

Overlapping generations heterogeneous agent model. In the second version of the model agents live for $H > 0$ periods and have a terminal value of $V_{H+1} = 0$. Agents face mortality risk and have a survival probability ϕ_h of surviving into age h, conditional on surviving to age $h-1$. We set the survival probabilities following Bell and Miller (2002) projections for the U.S., with each model period corresponding to a single year. Agents are born at age 20 ($h = 1$) and can live to a maximum age of 100 ($H = 81$), when $\phi_{H+1} = 0$. Upon death, agents are replaced by a newborn who starts life with $a_1^\star = \$1,000$ of assets.

Efficiency units of labor are $\ell(h,\varepsilon) = \exp(\xi(h) + \varepsilon)$, where $\xi(h)$ is a quadratic polynomial that generates a 50 percent rise in average labor income from age 21 to its peak at age 51 as in Guvenen, Kambourov, Kuruscu, Ocampo, and Chen (2023).[6] We use the same process for labor productivity (ε) as in the infinitely lived agent model. Finally, we eliminate rate of return heterogeneity, so that all agents earn $r_i = \bar{r}$. Accordingly, we drop ζ from the state vector when referring to the overlapping generations model.

[6] $\xi(h) = \left(60(h-1) - (h-1)^2\right)/1800$.

2. Solving the models

We solve for the policy functions in (7) using readily available solution methods that exploit the optimality conditions of the savings choice (i.e., Carroll 2006). Having computed the policy functions, we approximate the Markov kernel, T, of the distribution of agents by discretizing it over assets on a grid $\vec{a}_{n_a}$ following Young (2010).[7] The result is a transition matrix $\hat{T}$, whose elements $\hat{T}\left(s, s^{'}\right)$ gives the probability that an agent with current state s transitions to state $s^{'}$.[8] This probability depends on the birth and death process (for instance, agents of age $h = H$ transition to age $h = 1$ with certainty), the transition matrix of the labor productivity process (ε), the transition matrix of the return heterogeneity process (ζ), and the approximation of the transition of assets on the fixed grid $\vec{a}_{n_a}$.[9] Finally, we compute the stationary distribution of agents on the discrete grid by iterating over

$$\hat{\lambda}^{n+1}\left(s^{'}\right) = \sum_{s} \hat{T}\left(s, s^{'}\right) \hat{\lambda}^{n}\left(s\right), \tag{10}$$

for some initial $\hat{\lambda}^{0}$. The stationary distribution, $\hat{\lambda}$, is the limit of $\hat{\lambda}^{n}$ as n grows large.

In the next section, we use the approximated distribution, $\hat{\lambda}$, and Markov kernel, $\hat{T}$, to compute moments for both models. The Markov kernel plays an important role in computing moments because it describes the evolution of states given any initial distribution. This is crucial for computing longitudinal moments where it is necessary to know how agents transition between states over time. We explore results with grids of different sizes for the approximation of the distribution and the Markov kernel. All grids are curved so that they

[7] Young (2010)'s method can lead to sizeable approximation error for models with fat-tailed distributions. Methods based on Pareto extrapolation are more accurate, as shown by Gouin-Bonenfant and Toda (2023). The transition probability matrix that these method generate can be used instead of the one from Young (2010)'s method when calculating moments as we describe below.

[8] The transition matrix $\hat{T}$ can be further exploited to speed up the computation of the model's solution as shown by Rendahl (2022).

[9] An agent with state s transitions with certainty to having assets $a^{'} = a^{\star}_{h(s)}\left(\varepsilon\left(s\right), \zeta\left(s\right), a\left(s\right)\right) \in \left[\vec{a}_{j}, \vec{a}_{j+1}\right]$, for some j. In the discrete approximation the agent transitions to either $\vec{a}_{j}$ with probability $\vec{a}_{j+1}-a^{'}/\vec{a}_{j+1}-\vec{a}_{j}$ or $\vec{a}_{j+1}$ with probability $a^{'}-\vec{a}_{j}/\vec{a}_{j+1}-\vec{a}_{j}$.

are denser for low wealth values. In particular, the n^{th} node of an asset grid with N satisfies $\vec{a}_n = \underline{a} + (\overline{a} - \underline{a}) \left(n-1/N-1\right)^{\theta_a}$, where $\theta_a > 1$ measures the curvature. We use a curvature of $\theta_a = 3.5$ and solve for the policy functions on a grid with 250 nodes before approximating the Markov kernel and the stationary distribution.

3. Computing moments

We now compute cross-sectional and longitudinal moments for both the entire population and sub-populations of interest, like agents at the top or bottom of the wealth distribution. Cross-sectional moments, like the share of wealth owned by the wealthiest 1% of individuals, can be readily computed from the stationary distribution (λ) or its approximation $\left(\hat{\lambda}\right)$. However, it is often necessary to follow individuals over time when computing longitudinal moments like the auto-correlation of wealth. This is often achieved through costly Monte Carlo based simulations of a sample of individuals. Our histogram iteration method relies instead on tracking the distribution of the relevant group of individuals (the sub-population), following its evolution as described by the Markov kernel T. We now describe the method.

The histogram iteration method. Consider a moment describing the expectation over some outcome in some future period $x\left(s^{'}, s\right)$ for a sub-population satisfying some condition, say having a certain level of wealth or income. It is possible to determine a subset of the state space $S \subseteq \mathcal{S}$ such that any agent with state $s \in S$ belongs to the sub-population of interest. These moments take the form of the expectations in equations (3) and (5). The objective is to approximate the value of these expectations. We obtain the sub-population's initial distribution, $\hat{\lambda}_S$, from the stationary distribution $\hat{\lambda}$ by restricting its domain to S and normalizing. Tracking the distribution of the sub-population over time involves iterating over $\hat{\lambda}_S$ with the Markov kernel $\hat{T}$ as in (10).

When the moment requires tracking only the outcomes of the group in S, the expectation of interest is

$$\hat{E}\left[x|s\in S\right]=\sum_{s}\sum_{s'}x\left(s',s\right)\hat{\lambda}'_{S}\left(s'\right)\hat{\lambda}_{S}\left(s\right) \tag{11}$$

and when the moment requires tracking the future outcomes of individuals, the expectation of interest is

$$\hat{E}\left[x|s\in S\right]=\sum_{s}\sum_{s'}x\left(s',s\right)\hat{\lambda}'_{\{s\}}\left(s'\right)\hat{\lambda}_{S}\left(s\right) \tag{12}$$

In this case, $\hat{\lambda}'_{\{s\}}$ is the future distribution of agents that started in state $s\in S$. Below, we apply this method in the models described in Section 1.

3.1. Moments for the infinitely lived agents model

We now compute several moments for the infinitely lived agents model and compare the performance of the histogram iteration method relative to a traditional Monte Carlo simulation. We focus on moments characterizing the wealth distribution and the behaviour of consumption, which are the endogenous outcomes in our setting. In particular, we present results for the tail of the wealth distribution, top wealth shares, the persistence of consumption and wealth, and the ten-year transition rates across wealth deciles.

In terms of the accuracy, we find that both methods provide similar estimates for the moments, except for those regarding top-wealth holders: the shape of the tail of the wealth distribution and top wealth shares. The challenge for the Monte Carlo simulation method comes from the large number of agents needed in order to obtain a representative sample of top-wealth holders. The histogram iteration method provides more consistent values of these moments when varying the number of grid nodes in the approximation.

In terms of the computational cost of calculating moments, cross-sectional moments come almost for free after solving for the histogram or simulating a panel of agents. Longitudinal moments are significantly more expensive to calculate with the histogram iteration method than from a simulated panel of agents. This is because the histogram iteration method requires iterating forward for all initial states, as all the possible histories of agents are mapped for each initial condition. This makes the computation more expensive than computing the same moment using an already existing panel of agents containing realized histories of consumption and wealth. However, the time required to solve for the histogram is substantially less than the time required to simulate the Monte Carlo panel of agents. As a result, the total time it takes to calculate longitudinal moments is less when using the histogram iteration method than when using the simulated panel.[10]

Pareto Tail. One characteristic of the cross-sectional distribution of wealth that is often difficult to capture in heterogeneous agent models is the behavior of its right tail and the level of wealth concentration. These statistics are crucial when studying inequality, particularly because of their implications for taxation. We report the right tail of the wealth distribution (above ten million dollars) and the corresponding Pareto coefficient for simulations with sample sizes between two hundred and fifty thousand and one million agents in Figure 1 and contrast them with the tail of the stationary distribution of wealth approximated with a histogram with 500 grid points. We find that simulation-based results require a large number of agents to correctly represent the properties of the right tail of the wealth distribution,[11] and that, by contrast, the histogram provides a more stable picture of the distribution at lower computational cost.[12]

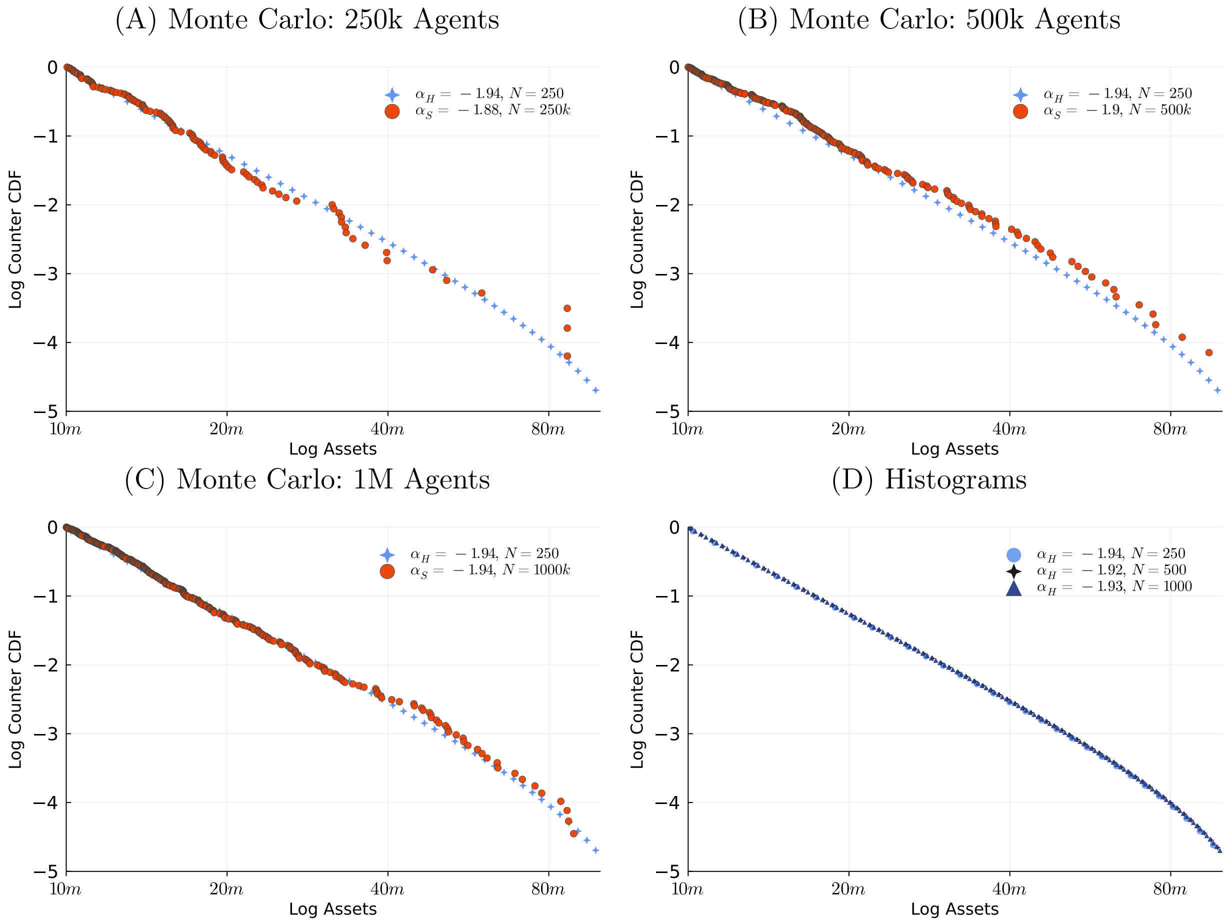
The Monte Carlo simulation captures the general shape of the tail, but has issues populating the top end, even with one million agents. This is apparent in the discrepancies between the tail indexes (α) and the wealth shares of the richest agents across simulation samples, as shown in Table 1. Figure 1D shows that the histogram provides more stable

[10] All times are for a Mac Mini with an M1 processor running Julia v1.7.

[11] The same logic applies when studying the effects of rare events and tail risks, like extreme health shocks.

[12] This is similar to Gouin-Bonenfant and Toda (2023), who propose replacing the grid at the right end of the distribution with an approximation of the continuous distribution using limit results.

Figure 1. Pareto Tail - Monte Carlo Simulation and Histogram Method



Notes: The figures plot the log counter CDF of the conditional distribution of wealth above $10 million. Panels 1A to 1C approximate the CDF using samples of agents from a Monte Carlo simulation and differ in the number of agents being simulated. The blue diamonds correspond to the approximation of the counter CDF using the histogram method with 500 grid nodes. The final panel approximates the CDF using the histogram method with 250, 500, and 1000 grid nodes.

outcomes across grid sizes for both the shape of the distribution and the tail index.[13]

The sensitivity of the right tail to the number of agents being simulated becomes an issue in models that aim to capture the extent of wealth inequality in the data. For instance, Guvenen et al. (2023) pose a model capable of reproducing the tail of the wealth distribution in the U.S., including the presence of multi-billionaires. In order to generate these very wealthy agents, they use a Monte Carlo simulation with twenty million agents.

[13] This stability makes moments computed via the histogram iteration method more likely to be continuous in the model parameters than those computed via Monte Carlo simulation. This smoothness can benefit estimation or calibration routines as it smooths the objective function.

Table 1. Cross-sectional and Longitudinal Moments: Infinitely Lived Agents

	Percentage Point Deviations from Reference Value						Ref. Value
	Monte Carlo: Sample Size			Histogram: Grid Size			
	250k	500k	1M	250	500	1000	
Top Wealth Shares							
Top 0.1%	0.12	0.14	−0.52	0.08	−0.04	−0.10	6.29
Top 1%	0.08	0.02	−0.76	0.12	−0.06	0.03	19.00
Pareto Coefficient	−0.04	−0.01	0.03	0.03	0.01	0.02	1.91
Auto-Correlations							
$\rho(c_t, c_{t+2})$	0.24	0.00	0.05	0.13	0.01	0.04	82.52
$\rho(a_t, a_{t+2})$	0.20	0.97	0.08	1.04	0.07	0.43	49.73
Transition Rates							
$\Pr\left(a_i^{\prime} \in D_1 \vert a_i \in D_1\right)$	0.17	−0.06	0.05	0.59	0.12	0.31	50.04
$\Pr\left(a_i^{\prime} \in D_2 \vert a_i \in D_1\right)$	−0.10	0.24	−0.06	−0.19	0.02	−0.03	34.16
Computational Time							
Simulation	689.3	1386.1	2744.3	—	—	—	—
Distribution $\hat{\lambda}$	—	—	—	478.9	881.4	1827.0	—
Top Inequality	0.01	0.02	0.04	1E-4	4E-4	2E-4	—
Auto-Correlation	0.05	0.08	0.18	9.81	21.47	54.76	—
Transition Rates	0.39	0.83	1.58	13.48	26.04	50.48	—

Notes: The table reports the deviation of calculated moments and computational time in seconds for the infinitely lived agents model. The first block computes the moments approximating the distribution with Monte Carlo simulation on three different samples of 250k, 500k, and 1M agents. The second block computes the moments approximating the stationary distribution with histograms on three different grids with 250, 500, and 1000 nodes. The reference value is obtained from a histogram grid with 5000 nodes.

Top Wealth Shares. We compute the share of wealth owned by the top 1% and top 0.1% of individuals in our model and report them in Table 1. Just as with the shape of the right tail, these measures of top wealth concentration are difficult to measure with the Monte Carlo simulation because a small number of "very rich" agents play a large role in determining the value of the moments. As a consequence, the top wealth shares are still varying even when the number of simulated agents is increased to one million. The time required to compute the moments is negligible next to the time required to either obtain the stationary distribution of the model or to simulate the agents.

Persistence of Consumption and Wealth. We continue by computing the two-year auto-correlations of consumption and wealth, which are informative about the ability of individuals to insure themselves against temporary income fluctuations. These are longitudinal moments that require comparing the level of consumption and wealth for individuals across time. Both the histogram iteration method and Monte Carlo simulation give very similar results for the moments, but they differ markedly on the time it takes to compute the moments. While it is faster to compute moments from an existing panel of agents, this does not take into account the time it takes to generate the panel.

Mobility. Finally, we calculate the ten-year transition rates across deciles of the wealth distribution. These rates are commonly used to study the persistence of wealth inequality and mobility. Unlike the auto-correlation of wealth, computing transition rates does not require following the full path of individuals, rather, it is enough to follow a subset of the population, e.g., those in a given decile. The histogram iteration method takes advantage of this by iterating from the conditional distribution of agents of each decile to obtain their final distribution as in (11). The transition rates are calculated directly as the mass of the final distribution in each decile. As with the auto-correlations, these transition rates take longer to calculate via the histogram method than using an existing panel of agents. However, simulating that panel of agents is costly relative to solving for the histogram.

3.2. Moments for the overlapping generations model

We now conduct similar exercises on the overlapping generations model. We focus on the behavior of agents along their life-cycle. In particular, we present age-profiles of the wealth distribution for agents with above median income at age 45 and the auto-correlation of wealth between the ages of 35 and 40, and the ages of 35 and 50. We compute the moments using the histogram iteration method to iterate over the evolution of a cohort and contrast the

Table 2. Cross-sectional and Longitudinal Moments: Overlapping Generations

	Monte Carlo: Sample Size			Histogram: Grid Size		
	100k	250k	500k	250	500	1000
Wealth Auto-Correlation						
Age 35-40	89.43	89.44	89.46	89.39	89.59	89.66
Age 35-50	62.94	62.85	62.69	63.70	63.84	63.96
Computational Time						
Population Simulation	300.5	750.7	1501.1	—	—	—
Cohort Simulation	17.57	43.71	87.33	—	—	—
Distribution $\hat{\lambda}$	—	—	—	113.1	223.9	459.0
Wealth Profiles	0.19	0.71	1.33	0.51	0.99	1.97
Auto-Correlation 35-40	5E-4	4E-3	2E-3	12.45	42.13	145.5
Auto-Correlation 35-50	5E-4	1E-3	2E-3	97.97	355.1	1450.8

Notes: The table reports the auto-correlation of wealth between the ages of 35 and 55. The first block computes the moments approximating the distribution with a Monte Carlo simulation. The second block computes the moments approximating the stationary distribution with histograms on three different grids with 250, 500, and 1000 nodes. The auto-correlation of wealth is computed from the simulation of cohorts between the ages of 35 and 50 of 100k, 250k, and 500k agents, without attrition. The initial distribution is obtained from the histogram with 500 nodes. All times are in seconds.

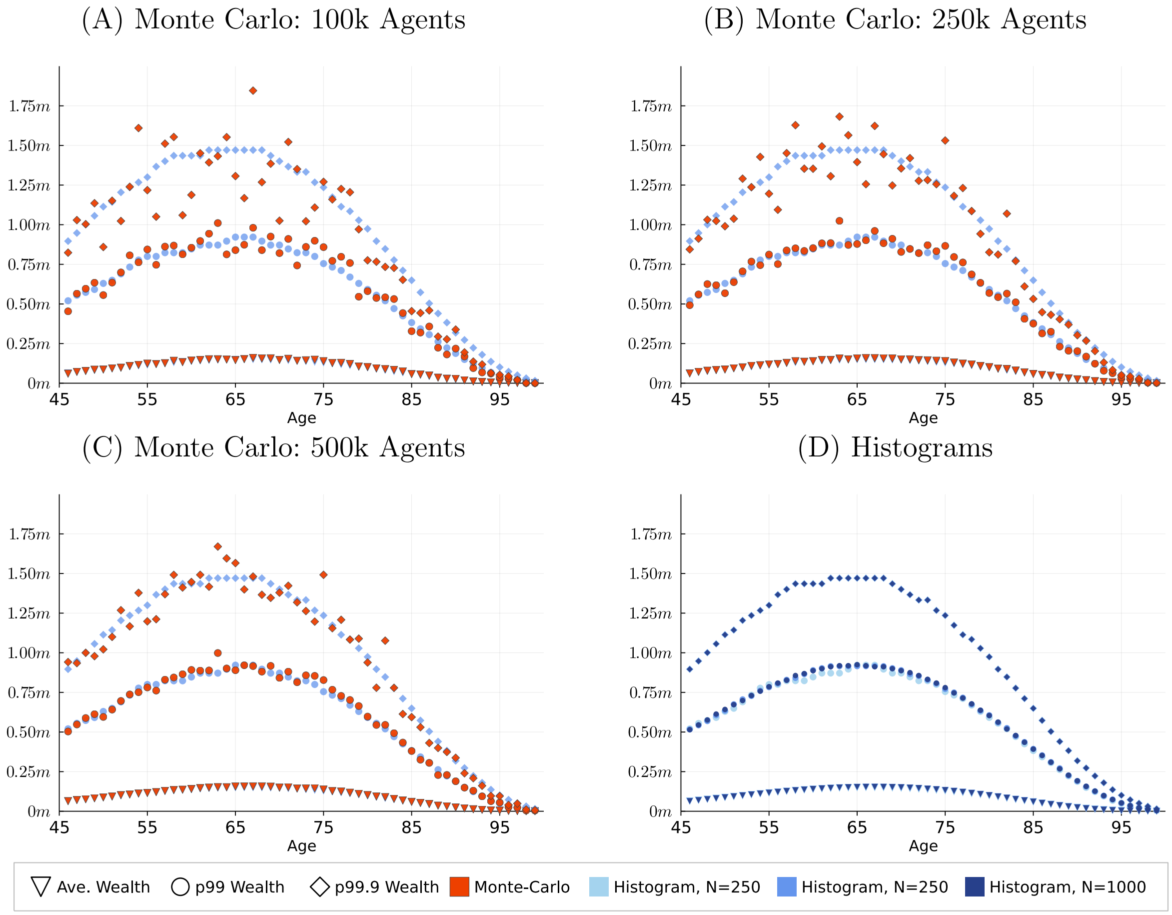
results with those of Monte Carlo simulations of up to five hundred thousand agents.[14]

The two methods produce similar moments, with the exception of moments characterizing the top of the wealth distribution. The reason is again that a large number of agents must be simulated in order to have a representative sample of wealthy agents. This is especially the case for life-cycle moments because the sample is also conditioned by age, making it more difficult to ensure large sample sizes. In terms of the computational cost, the simulation time is again the main factor making Monte Carlo simulation-based moments more costly.

Wealth Age Profiles. We report the age profile of wealth for agents with above median income at age 45 in Figure 2. The figures show the average, and the 99th and 99.9th percentile of wealth for every age. It is clear that the results obtained from Monte Carlo simulations

[14]For the cross-sectional moments the sample size refers to the total sample, including agents of all ages. For the auto-correlation we simulate a single cohort of individuals.

FIGURE 2. Wealth-Age Profiles - Monte Carlo Simulation and Histogram Method



Notes: The figures plot the age profile of wealth starting at age 45. Panels 2A to 2C compute the moments using samples of agents from Monte Carlo simulation and differ in the number of agents being simulated. Triangles correspond to the average wealth at every age, circles to the 99^{th} percentile of the wealth distribution, and diamonds to the 99.9^{th} percentile. Markers in blue correspond to the age profiles using the histogram method. The final panel computes the moments from the conditional distribution of wealth by age using the histogram method with 250, 500, and 1000 grid nodes.

struggle to capture the top percentiles of the wealth distribution, even though they do successfully capture the average wealth profile. As before, this is because there are only a small number of "very rich" agents in the Monte Carlo simulation, producing volatile age profiles. This is in contrast with the results obtained from the histogram that provides stable results even for relatively coarse grids as shown in Figure 2D.

The time required to compute the distribution or simulate the agents follows the same pattern described above. As we show in Table 2, the bulk of the computational time is accounted for by computing the histogram $\hat{\lambda}$ or performing the Monte Carlo simulation, with the calculation of the wealth profiles taking just a few seconds at most.

Auto-correlation of Wealth. Finally, we compute the five- and fifteen-year auto-correlation of wealth starting at age 35. Both the histogram iteration method and Monte Carlo simulation produce similar results, see Table 2. However, the time required to iterate the histogram increases markedly with the time horizon, making the Monte Carlo simulation faster when computing the fifteen-year auto-correlation.

This result is instructive about the practical limitations of the histogram iteration method. When using Monte Carlo methods, calculating the auto-correlation requires simulating a single cohort of agents, generating a representative sample of paths. This cohort simulation takes less time than a full simulation of the whole population and can take advantage of the histogram by using it to obtain the initial distribution of agents at age 35. By contrast, computing the auto-correlation with the histogram iteration method requires solving for the conditional distribution of agents at age 50 $\left(\lambda_s^{'}\right)$ starting from each initial state s at age 35, see (12). $\lambda_s^{'}$ describes all the possible paths that a 35 year old can take in their next fifteen years. Computing $\lambda_s^{'}$ requires iterating forward as in equation (6) multiple times. The complexity of this step increases with the time horizon as the initial mass of agents fans out across the state space.

4. Discussion

We have shown how to use a histogram approximation of the stationary distribution of agents and its associated Markov kernel to efficiently compute cross-sectional and longitudinal moments without having to simulate large samples of agents through Monte Carlo methods. We illustrated the workings of the method in the context of baseline models that abstract from many of the characteristics of applied work. However, the method we propose can also be used in other scenarios. We therefore end with a short discussion of some of the natural extensions of the histogram iteration method and its main limitations.

Extensions. The histogram iteration method can be easily applied to models that allow for additional endogenous choices (e.g., labor supply). In this case the policy functions can be solved with extensions of the endogenous grid method like those in Barillas and Fernández-Villaverde (2007) and Fella (2014). Once the policy functions are obtained, the construction of the Markov kernel and the histogram that approximates the distribution follow as above. At this point it is also possible to complement the solution with methods that provide better approximation for the dynamics of wealthy agents (Gouin-Bonenfant and Toda 2023), or that speed up the computation by taking advantage of the sparseness of the Markov kernel (Tan 2020; Rendahl 2022).

Similarly, the method applies to non-stationary problems where the distribution of agents changes over time, or the agents' choices change (therefore making the Markov kernel time-varying). This can happen, for example, in the transition path to a new steady state after changes in policy variables. The histogram iteration method is already built to capture changes in the distribution, as the iteration in equation (6) shows. The only change comes in by indexing the Markov kernel by time when iterating over an initial distribution of agents.

Continuous-time methods. The histogram iteration method can also be applied with only minor changes to continuous-time heterogeneous agent models (see for instance Herreño and Ocampo 2023). In particular, the solution of these models by means of the Finite Difference method is constructed from a (sparse) matrix A that characterizes the approximation to the Hamilton-Jacobi-Bellman equation (see, Achdou, Han, Lasry, Lions, and Moll 2021). The adjoint of this matrix plays the same role as the Markov kernel T described above and characterizes the solution to the Kolmogorov Forward equation that describes the evolution of the distribution of agents. In this way, the solution of the model generates a value function, policy functions, a distribution over states, and an operator to iterate the distribution just as in Section 2.

The changes in the implementation of the histogram iteration method come from how the method can take advantage of the sparseness of matrix A, the stipulation of a time step (Δt), and the need to integrate with respect to the distribution of continuous states. Alternative solution methods for continuous time models as those in Phelan and Eslami (2022) also allow for a direct implementation of the histogram iteration method.

Limitations. The histogram iteration method is generally an efficient way for calculating cross-sectional and longitudinal moments. However, longitudinal moments that involve individual outcomes of a large subset of the population, or that involve long periods of time, can be expensive to calculate. As we discussed in Section 3, this is because the full history of individuals' paths must be mapped in order to compare the individuals' initial and final outcomes, unlike for other moments that focus on group outcomes like transition rates. This leads to cases where Monte Carlo methods can be more efficient, as was the case with the computation of the fifteen-year auto-correlation of wealth discussed in Section 3.2.[15]

The histogram iteration method takes advantage of the histogram approximation of the distribution of agents and the associated Markov kernel, which are often already computed as part of solving the model. Because of this, the histogram method will usually generate time-savings even when the computation of specific moments is costlier than the computation from a simulated Monte Carlo panel, as the simulation has to be conducted on top of the model solution. This makes the key computational trade-off for computing moments clear: the complexity of the moment is weighed against the complexity of simulating a representative sample of agents. We have shown that this trade-off will usually land in favor of using the model's own stationary distribution and Markov kernel, allowing researchers to avoid both coding and running computationally-costly Monte Carlo simulations.

[15] The same principle applies to moments that involve the outcomes of agents in intervening periods, rather than just the initial and final outcomes. For example, computing the distribution of lifetime earnings in our OLG model proves to be unfeasible. Doing so would require us to compute the time paths of each possible income realization over the 81 year lifespan of agents. With 11 income states, there are $11^{81} \approx 6.8 \times 10^{17}$ possible histories of lifetime income.

References

Achdou, Yves, Jiequn Han, Jean-Michel Lasry, Pierre-Louis Lions, and Benjamin Moll. 2021. "Income and Wealth Distribution in Macroeconomics: A Continuous-Time Approach." *The Review of Economic Studies* 89 (1):45–86. URL https://doi.org/10.1093/restud/rdab002.

Aiyagari, S. Rao. 1994. "Uninsured Idiosyncratic Risk and Aggregate Saving." *The Quarterly Journal of Economics* 109 (3):659–684. URL http://www.jstor.org/stable/2118417.

Algan, Yann, Olivier Allais, and Wouter J Den Haan. 2008. "Solving Heterogeneous-Agent Models with Parameterized Cross-Sectional Distributions." *Journal of Economic Dynamics and Control* 32 (3):875–908. URL https://doi.org/10.1016/j.jedc.2007.03.007.

Barillas, Francisco and Jesús Fernández-Villaverde. 2007. "A generalization of the endogenous grid method." *Journal of Economic Dynamics and Control* 31 (8):2698–2712. URL https://doi.org/10.1016/j.jedc.2006.08.005.

Bell, Felicite C. and Michael L. Miller. 2002. "Life Tables for the United States Social Security Area: 1900-2100"." Actuarial Study 116, Office of the Actuary, Social Security Administration. URL https://www.ssa.gov/oact/NOTES/as120/LifeTables_Body.html.

Benhabib, Jess, Alberto Bisin, and Shenghao Zhu. 2011. "The Distribution of Wealth and Fiscal Policy in Economies With Finitely Lived Agents." *Econometrica* 79 (1):123–157. URL https://doi.org/10.3982/ECTA8416.

Carroll, Christopher D. 2006. "The method of endogenous gridpoints for solving dynamic stochastic optimization problems." *Economics Letters* 91 (3):312–320. URL https://doi.org/10.1016/j.econlet.2005.09.013.

De Nardi, Mariacristina and Giulio Fella. 2017. "Saving and wealth inequality." *Review of Economic Dynamics* 26:280–300. URL https://doi.org/10.1016/j.red.2017.06.002.

De Nardi, Mariacristina, Eric French, and John Bailey Jones. 2016. "Savings After Retirement: A Survey." *Annual Review of Economics* 8 (1):177–204. URL https://doi.org/10.1146/annurev-economics-080315-015127.

Fella, Giulio. 2014. "A generalized endogenous grid method for non-smooth and non-concave problems." *Review of Economic Dynamics* 17 (2):329–344. URL https://doi.org/10.1016/j.red.2013.07.001.

Gouin-Bonenfant, Émilien and Alexis Akira Toda. 2023. "Pareto Extrapolation: An Analytical Framework for Studying Tail Inequality." *Quantitative Economics* URL https://doi.org/10.3982/QE1817.

Guvenen, Fatih, Gueorgui Kambourov, Burhan Kuruscu, Sergio Ocampo, and Daphne Chen. 2023. "Use It or Lose It: Efficiency and Redistributional Effects of Wealth Taxation." *The Quarterly Journal of Economics* URL https://doi.org/10.1093/qje/qjac047.

Heathcote, Jonathan, Kjetil Storesletten, and Giovanni L. Violante. 2009. "Quantitative Macroeconomics with Heterogeneous Households." *Annual Review of Economics* 1 (1):319–354. URL https://doi.org/10.1146/annurev.economics.050708.142922.

Heer, Burkhard and Alfred Maußner. 2005. *Dynamic General Equilibrium Modelling*. Springer.

Herreño, Juan and Sergio Ocampo. 2023. "The Macroeconomic Consequences of Subsistence Self-Employment." *Journal of Monetary Economics* URL https://doi.org/10.1016/j.jmoneco.2023.02.002.

Judd, Kenneth L. 1998. *Numerical Methods in Economics*. MIT Press.

Phelan, Thomas and Keyvan Eslami. 2022. "Applications of Markov chain approximation methods to optimal control problems in economics." *Journal of Economic Dynamics and Control* 143:104437. URL https://doi.org/10.1016/j.jedc.2022.104437.

Rendahl, Pontus. 2022. "Continuous vs. Discrete Time: Some Computational Insights." *Journal of Economic Dynamics and Control* :104522URL https://doi.org/10.1016/j.jedc.2022.104522.

Rouwenhorst, K. Geert. 1995. "Asset Pricing Implications of Equilibrium Business Cycle Models." In *Frontiers of Business Cycle Research*, edited by Thomas Cooley, chap. 10. Princeton University Press, 394–330.

Stachurski, John and Alexis Akira Toda. 2019. "An impossibility theorem for wealth in heterogeneous-agent models with limited heterogeneity." *Journal of Economic Theory* 182:1–24. URL https://doi.org/10.1016/j.jet.2019.04.001.

Storesletten, Kjetil, Chris I. Telmer, and Amir Yaron. 2004. "Cyclical Dynamics in Idiosyncratic Labor Market Risk." *Journal of Political Economy* 112 (3):695–717. URL https://doi.org/10.1086/383105.

Tan, Eugene. 2020. "A fast and low computational memory algorithm for non-stochastic simulations in heterogeneous agent models." *Economics Letters* 193:109285. URL https://doi.org/10.1016/j.econlet.2020.109285.

Tauchen, George. 1986. "Finite state markov-chain approximations to univariate and vector autoregressions." *Economics Letters* 20 (2):177–181. URL https://doi.org/10.1016/0165-1765(86)90168-0.

U.S. Bureau of Economic Analysis. 2022. "Economic Data." FRED, Federal Reserve Bank of St. Louis; April 9, 2022., https://fred.stlouisfed.org.

Young, Eric R. 2010. "Solving the incomplete markets model with aggregate uncertainty using the Krusell–Smith algorithm and non-stochastic simulations." *Journal of Economic Dynamics and Control* 34 (1):36–41. URL https://doi.org/10.1016/j.jedc.2008.11.010.