EC9604: Advanced Macroeconomics[1]

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Contents

[1] These notes are intended to summarize the main concepts, definitions and results covered in the core macroeconomics course of the PhD program at the University of Western Ontario, EC9604. These notes only include selected sections of books or articles relevant to the course used here only in part for reference and teaching purposes. Please let me know of any errors that persist in the document.

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1 Durable Goods

Consider a single agent problem where each period, w total output is produced and can be divided into consumption of a perishable good, c_t and investment in a durable good, d_{xt}. The durable depreciates like a capital good, but is not directly productive. The stock of durables at any date, d_t, produces a flow of services that enters the utility function. Thus, the problem faced by the household with initial stock d_0 is:

$$\begin{aligned}
&\max_{c_t, d_t, d_{xt}} \sum_t \beta^t \left\{ u_1(c_t) + u_2(d_t) \right\} \\
&\text{s.t. } c_t + d_{xt} \le w \\
&\quad d_{t+1} \le (1-\delta) d_t + d_{xt} \\
&\quad c_t, d_t, d_{xt} \ge 0 \\
&\quad d_0 \text{ given}
\end{aligned}$$

where both u_1 and u_2 are strictly increasing and continuous. Note: you can ignore non-negativity constraints on investment, d_{xt} in this problem.

i. State a condition on either u_1 or u_2 (or both) such that you can write an equivalent problem in the following form:

$$\begin{aligned}
&\max_{\{d_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(d_t, d_{t+1}) \\
&\text{s.t.} \\
&d_{t+1} \in \Gamma(d_t) \\
&d_0 \text{given}
\end{aligned}$$

where $\Gamma(d) \in \mathbb{R}_+$. What is F? What is the correspondence Γ?

ii. Write the Bellman equation for this problem.

iii. State additional conditions on u_1 and u_2 such that the value function $v(d)$ is both strictly increasing and strictly concave. Prove these two properties.

iv. For the remaining questions, assume that both u_1 and u_2 satisfy the Inada conditions and are continuously differentiable. State the envelope and the FOC for the functional equation problem in (b)

v. Show that there is a unique steady state value of the stock, $d^\star$, such that if $d_0 = d^\star$, then $d_t = d^\star$ for all t. Show that $d^\star > 0$.

vi. Show that the policy functions for the solution, $c^\star(d)$ and $d' = g^\star(d)$ are increasing.

vii. Show that the system is globally stable. You can assume that the policy functions are differentiable for this part. [Hint: Check Chapter 6 of SLP and use the Euler equation to show that g, the policy function, is increasing and concave. Use the fact that v is strictly concave when showing this.]

2 Aggregation

Consider a heterogeneous agents economy.

Firms: The economy has two kinds of firms: investment firms and consumption firms. Both use capital and labor to produce, respectively, investment specific goods for the formation of capital or general consumption goods. The production technologies are potentially firm- and time-specific:

$$\underbrace{x_{h,t} = F^x_{h,t}\left(k^x_{h,t}, n^x_{h,t}\right);}_{\text{Investment Firm } h} \qquad \underbrace{y_{j,t} = F^y_{j,t}\left(k^y_{j,t}, n^y_{j,t}\right)}_{\text{Consumption Firm } j}.$$

Households: There are also households indexed by i. They are infinitely lived and supply labor and consume every period. Household i owns a share $\theta^x_{i,h}$ of investment firm h and a share $\theta^y_{i,j}$ of consumption firm j. These shares are constant (but we can allow for a market in which they are traded). The entirety of the firms are owned by households so that

$$\theta^x_{i,h}, \theta^y_{i,j} \geq 0; \qquad \sum_{i=1}^{I} \theta^x_{i,h} = 1; \qquad \sum_{i=1}^{I} \theta^y_{i,j} = 1.$$

We assume that households make investment decisions. The initial endowment of households (that determines their lifetime wealth) is an amount of initial capital $k_{i,0}$ and a sequence of potential labor supplies (or labor productivities) $\overline{n}_{i,t}$. The payoff of a household is the lifetime value of their consumption and labor choices captured by a utility $U^i(\cdot)$, a function of the sequence $(c_{i,t}, n_{i,t})_{t=0}^{\infty}$. Capital accumulates according to $k_{i,t+1} \leq (1-\delta)\, k_{i,t} + x_{i,t}$.

i. Define a competitive equilibrium for this economy.

ii. Express the budget constraint of the households in terms of their present value of their lifetime resources depending only on their endowments.

   (a) For this you must relate the price of investment to the rate of return on capital in equilibrium. Notice that the capital accumulation equation has constant returns to scale in initial capital and investment.

   Assume that the production functions are constant returns to scale and common across firms and that the utility function is homothetic and common across households. All other differences stay.

iii. Define the competitive equilibrium of an economy populated by a representative household and representative investment and consumption firms.

iv. Show that if a sequence of prices $\{p_t^c, p_t^x, r_t, w_t\}_{t=0}^{\infty}$ a sequence of input demands and output from firms $\left\{x_{h,t}, k_{h,t}^x, n_{h,t}^x, y_{j,t}, k_{j,t}^y, n_{j,t}^y\right\}_{t=0;h=1;j=1}^{t=\infty;h=H;j=J}$ and a sequence of household actions $\{c_{i,t}, n_{i,t}, k_{i,t+1}, x_{i,t}\}_{t=0;i=1}^{t=\infty;i=I}$ are an equilibrium for the heterogeneous agent economy, then prices $\{p_t^c, p_t^x, r_t, w_t\}_{t=0}^{\infty}$ a sequence of input demands and output from the representative firms $\left\{\sum_h x_{h,t}, \sum_h k_{h,t}^x, \sum_h n_{h,t}^x, \sum_j y_{j,t}, \sum_j k_{j,t}^y, \sum_j n_{j,t}^y\right\}_{t=0}^{t=\infty}$ and a sequence of household actions $\{\sum_i c_{i,t}, \sum_i n_{i,t}, \sum_i k_{i,t+1}, \sum_i x_{i,t}\}_{t=0}^{t=\infty}$ are an equilibrium for the representative agent economy

v. Show that the solution to the planner's problem constitutes an equilibrium for the representative agent economy and use it construct the equilibrium of the heterogeneous agent economy given some initial endowments.

3 Gorman Aggregation

Consider an economy with goods $j = 1, \ldots, J$ and households $h \in \mathcal{H}$. Households differ in their income $\{Y_h\}$ and face common prices $p = \left[p_1, \ldots, p_{j,\ldots}, p_J\right]^T$. We are going to work directly with these households' Marshallian demands and indirect utility functions. The Marshallian demand of household h is $x_h\left(p, Y_h\right)$, their indirect utility function is $v_h\left(p, Y_h\right)$.

The aggregate demand function is defined as

$$X\left(p, \{Y_h\}\right) \equiv \int_{\mathcal{H}} x_h\left(p, Y_h\right) dh.$$

A representative agent exists if there is a demand function $\overline{x}\left(p, Y\right)$ such that

$$\overline{x}\left(p, \int_{\mathcal{H}} Y_h di\right) = X\left(p, \{Y_h\}\right).$$

i. Show that if the indirect utility functions of households are of the polar Gorman form, $v_h\left(p, Y_h\right) = \varphi_h\left(p\right) + \eta\left(p\right) Y_h$, then a representative agent exists. Show also the Marshallian demand and indirect utility function of the representative agent.

ii. Show that any feasible allocation that maximizes the utility of the representative agent is Pareto optimal for the decentralized economy.

iii. Show that if $\varphi_h\left(p\right) = \psi_h + \overline{\psi}\left(p\right)$ then any Pareto optimal allocation maximizes the utility of the representative agent.

iv. Show that if the households have common homothetic preferences then their indirect utility function is of the polar Gorman form. Also show that the representative agent has the same preferences as the households.

4 Lifetime Income Taxes (Krueger & Wu, 2025)

Environment Consider an economy that operates for two periods where households face uncertainty about their second period income.

There is a continuum of measure 1 of ex ante identical households that work and consume in each period.. They can also save or borrow at the risk-free interest rate $r = 0$. There is no time discounting.

In the first period, all households earn wage w_1, but the wage in the second period $w_2(s)$ depends on a realization of an exogenous state $s \in S$ that is uncertain.[2] The economy operates in partial equilibrium with exogenous prices w_1, w_2, and r. The price w_2 is, of course, a function.

Households value consumption and dislike labor, their lifetime preferences are represented by

$$U(c_1, h_1) + E_s [U(c_2(s), h_2(s))]$$

Taxes The government levies (potentially negative) taxes on labor income. There is no government spending on goods and services. As private households, the government has access to an inter-temporal technology that turns one unit of consumption today into $1 + r$ units of consumption tomorrow.

Taxes in the first period are determined by a (potentially non-linear) tax function $T(y)$ that determines the tax bill for a household with labor income $y = w_1 h_1$.

In general, second period taxes depend on the income in both periods, so that the tax bill is $\tilde{T}(y_1, y_2)$, where y_t is period's t realized labor income.

We will study two possible regimes for the second period taxes:

i. Annual Income Tax: In this case taxes depend only on current period income and so the tax function is $\tilde{T}(y_1, y_2) = T(y_2)$.

ii. Lifetime Income Tax: In this case taxes depend on lifetime income, so that $\tilde{T}(y_1, y_2) = T(y_1 + y_2) - T(y_1)$.

In the second case we subtract the taxes paid in the first period so that lifetime taxes are only a function of lifetime income:

$$T(y_1) + E_s\left[\tilde{T}(y_1, y_2(s))\right] = E_s[T(y_1 + y_2(s))]$$

- Recall that $r = 0$ and so there is no need to discount payoffs across periods.

[2] You can denote by f and F the probability density function (pdf) and cumulative density function (cdf) for the state s. Shocks are idiosyncratic, so that f and F are also the pdf's and cdf's of the population distributions across households in the second period.

Household Problem

$$\max_{\{c_1,h_1,c_2(s),h_2(s)\}} U(c_1,h_1) + E_s\left[U(c_2(s),h_2(s))\right]$$
$$\text{s.t. } c_1 + a = w_1 h_1 - T(w_1 h_1)$$
$$\forall_s \; c_1(s) = a + w_2(s) h_2(s) - \tilde{T}(w_1 h_1, w_2(s) h_2(s))$$

Questions:

i. Pose the social planner that characterizes the efficient allocation in this economy (this planner maximizes the expected utility of the ex-ante identical households subject to feasibility)

ii. Derive the Euler conditions (optimality conditions for marginal rates of substitution) that characterize:

  (a) Consumption allocation between the first and second period (states)

  (b) Intra-temporal labor choice

  (c) Inter-temporal labor choice between first period h_1 and second period $h_2(s)$

  (d) Allocation of labor across states in the second period $\left(h_2(s), h_2\left(s'\right)\right)$

iii. Suppose that the utility function takes the form

$$U(c,h) = \frac{c^{1-\gamma}}{1-\gamma} - \psi \frac{h^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}}$$

Show that the efficient allocation demands perfect consumption insurance ($c_1 = c_2(s)$).

iv. Consider the equilibrium allocation without government intervention (set $T(y) = 0$).

  (a) Obtain the consumption Euler equation

  (b) Obtain the labor intra-temporal Euler equations

  (c) Combine the Euler equations to characterize the inter-temporal condition that determines the labor allocation

  (d) Is the allocation efficient? If not, explain how the market is failing.

v. Consider the equilibrium allocation with taxes.

  (a) Show that annual taxes distort the inter-temporal allocation of labor (that is that they show up as a wedge in the Euler equation that links the first and second period labor choice).

(b) Show that the lifetime taxes do not distort the inter-temporal allocation of labor.

vi. Explain how lifetime taxes can be used to improve over the laissez-faire allocation (the one without taxes).

5 Recursive Competitive Equilibrium with Two Households

Households:

- Consider an economy with two types of households indexed by $i \in \{a, b\}$ (arguers and bores).
- There is a continuum of each type of equal measure and they are both infinitely lived.
- Preferences for both households are given by the expected discounted value (both types have the same discount factor, β).
- The utility of bores is $u^b(c, h)$. The function is increasing in the first argument and decreasing in the second.
- Type a households like it more when other agents of their type are not working so that they can spend time together, and, you guessed it, argue. Their utility is $u^a(c, H^b, h)$, it is increasing in the first argument and decreasing in the other two arguments.

Production:

- There is a continuum of competitive firms.
- The labor from each type of household are not perfect substitutes in production.
- There is an aggregate production function $f(Z, K, H^a, H^b)$, where z is productivity, K is aggregate capital, and H^a and H^b are total hours worked by each type of household.
- The production function f has constant returns to scale in K, H^a, and H^b.
- Productivity shocks, Z, follow a Markov chain with transition $\Pi_{zz'}$
- Capital is rented from households at a rate R. There is no depreciation.
- Labor is hired at wage rates W^a and W^b.

Setup:

- Use the convention that big letters represent aggregate variables and small letters represent individual variables.
- The only shock in the economy is the productivity shock.

i. Firm problem:

   (a) Describe the demand for capital and the two types of labor.

(b) What does the demand tell you about market clearing in these markets.

ii. Define Recursive Competitive Equilibrium. Make sure to include the goods market in your definition. Make sure that you not only define the required objects but also state the conditions that such objects must satisfy.

iii. Stochastic Processes.

(a) Is the stochastic process for output implied by the equilibrium a Markov process? Explain.

(b) What variables form the state vector of the economy? Choose the minimal state vector.

(c) A Markov process is characterized by its transition function $\mathcal{Q}$. What are the domain and the codomain of the Transition function for the state of this economy.

(d) Construct the stochastic process for output implied by the equilibrium.

iv. Planner's problem:

(a) Pose the Planner's problem.

(b) What are the optimality conditions that characterize the solution to the Planner's problem? List them all.

v. Optimality of the equilibrium:

(a) Is the equilibrium optimal? Explain mathematically and intuitively why or why not.

(b) How do the equilibrium and the planner's labor allocations differ?

vi. Now, assume that there are no productivity shocks, so Z is constant over time, and that households are subject to preference shocks that affect the disutility from labor. The utility functions are now

$$u^b\left(c_i, \gamma_i h_i\right); \text{ and } \qquad u^a\left(c_j, H^b, \nu_j h_j\right);$$

When γ or ν are high the disutility from working is higher. These variables follow Markov processes that are independent across households and are characterized by transition functions Q and P, respectively.

Define a Stationary Recursive Competitive Equilibrium for this Economy. Mace sure to include all market clearing conditions and the dynamic programming problem of each household type.

6 Recursive Competitive Equilibrium with Public Goods

There is an economy with many identical agents with preferences given by

$$\sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\sigma}}{1-\sigma} + \alpha \left(1 - n_t\right)^{\frac{1}{2}} + \gamma P_t^{\frac{1}{2}} \right]$$

where c_t is their own consumption at time t, n_t is the fraction of their own time worked at time t,and P_t are public parks. Their initial wealth is A . The technology to produce output uses capital (that depreciates at rate δ) and labor: $Y_t = F\left(K_t, N_t\right)$.

i. What conditions would be satisfied in a Pareto Optimum in steady state? [Hint: Use the planner's problem to obtain these conditions.]

   Imagine now that the government levies income taxes and issues debt to pay for the parks. Its initial debt is B.

ii. Define an RCE for this economy. That is, define (recursively) the set of private and government policies that constitute an equilibrium together with all the necessary elements (like prices).

   Imagine now that this is a small open economy and borrowing and lending can occur a and sell at the international rate $\bar{r}$.

iii. Define Recursive competitive equilibrium for this case and for the appropriate policies.

iv. Give an expression for the wage, and for the stock of capital in this equilibrium.

7 Recursive Competitive Equilibrium

Consider an economy populated by a representative firm and a representative household.

The firm produces using a constant-returns-to-scale technology that combines capital and labor. The firm chooses capital and labor to maximize its profits every period taking as given its current productivity (z_t) the period's prices: the rental rate of capital (r_t) and the wage rate (w_t). In this formulation, the problem of the firm is static:

$$\pi_t = \max_{\left\{k_t, \ell_t^d\right\}} f\left(z_t, k_t, \ell_t^d\right) - (r_t + \delta)\, k_t - w_t \ell_t^d.$$

The household chooses contingent plans for consumption and labor (or equivalently leisure) taking as given the return on their assets (r_t) and the wage rate (w_t). The household owns the firms and hence receives the profits the firm generates (π_t), which are also taken as given by the household.

$$\max_{\{c_t, \ell_t\}_{t=0}^{\infty}} E_0 \left[\sum_{t=0}^{\infty} \beta^t u\,(c, \ell)\right] \qquad \text{s.t. } c_t + a_{t+1} = (1 + r_t)\, a_t + w_t \ell_t + \pi_t,$$

The variable z_t is exogenous and random. In particular $\{z_t\}$ follows a Markov Process with transition function Q and initial value z_0.

i. What are the conditions that characterize the solution to the firm's problem? Explain how you obtain them and their implications.

ii. Pose the recursive problem of the household. Indicate clearly what constitutes the state vector.

iii. Define a recursive competitive equilibrium for this economy.

Assume now that households do not choose labor (so the utility is now $u\,(c_t)$) and that ℓ_t is instead exogenous, random, and that it varies across households. In particular, the labor of a household $\{\ell_{i,t}\}$ follows a Markov process with transition function P. You can assume that this process is discrete. Labor realizations are independent across households but the process itself is common.

4. Define a recursive competitive equilibrium for this economy.

Assume now that z does not vary.

5. Define a stationary recursive competitive equilibrium for this economy.

6. What can you say about the stock the capital (K) and the interest rate $(1 + r)$ with respect to the levels they would have if ℓ were to be constant (and equal for all households)? Explain your claims.

8 Defining equilibrium with heterogeneity

Consider an economy populated by a continuum of households who receive a stochastic endowment of consumption goods (say, sushi rolls) every period. Denote by ϵ the amount of sushi rolls received by a household. The amount of sushi that a household receives follows a discrete Markov process with transition matrix Π. The households receive sushi independently of each others.

Sadly, sushi goes bad very quickly if not eaten and refrigerators have not been invented in this economy. This is a problem for the households because some days they might have too much sushi, and some others not enough. Nevertheless, the households can smooth their sushi consumption by contracting with one another.

The contracts that households can sign are limited in two ways. First, contracts take the form of saving/debt contracts, where the household exchanges an amount $a^{'}$ of sushi rolls today in exchange for $(1+r)\,a^{'}$ rolls in the future. When $a^{'} > 0$ the household is saving and will receive payment in the future. When $a^{'} < 0$ the household is in debt and receives sushi rolls today in exchange for a payment in the future. Second, the credit balances are constrained so that $a^{'} \geq \underline{a}$, with $\underline{a} \leq 0$.

Households live forever, discount the future at a constant rate β, and derive utility only from consumption, $u\,(c)$.

i. Pose the dynamic programming problem of the household. Denote by ε the endowment of sushi rolls. Indicate clear what constitutes the state of the problem and what are the controls.

ii. The distribution of households (with respect to their states) evolves depending on the stochastic process for sushi endowments and the households' decisions. Write down an evolution equation for the distribution of households.

iii. Define an equilibrium for this economy. Make sure to write down market clearing conditions for all markets.

9 Heterogeneous Agent Economy

The economy is populated by a continuum of households who receive a stochastic endowment ϵ of consumption goods very period. The endowment ϵ that a household receives follows independent Markov processes with identical transition function Q.

Sadly, consumption goods cannot be stored and thus have to be consumed the same period they are received. Households can smooth consumption by contracting with one another, that is by exchanging goods among them.

The contracts that households can sign are limited in two ways. First, contracts can only specify a fixed amount of goods being transferred today in exchange for a fixed amount in the future, regardless of the endowment the household actually has. Second, credit balances (receiving goods today in exchange for payment of jelly beans in the future) are restricted, so that there is a limit to how many goods can be obtained by a household in the market.

Therefore, the contracts take the form of saving/debt contracts, where the household exchanges an amount $a^{'}$ today in exchange for $(1+r)\,a^{'}$ in the future. When $a^{'} > 0$ the household is saving and will receive payment in the future. When $a^{'} < 0$ the household is in debt and receives goods today in exchange for a payment in the future.

Households live forever and there is no aggregate risk. The problem of the households is then

$$V(\epsilon, a) = \max_{\{c, a^{'}\}} u(c) + \beta E\left[V\left(\epsilon^{'}, a^{'}\right) | \epsilon\right]$$
$$\text{s.t. } c + a^{'} = (1+r)\,a + \epsilon \qquad a^{'} \geq \underline{a}$$

i. What is the natural borrowing limit $\underline{a}$ in this economy? Explain.

ii. State the market clearing condition for this economy. Explain.

iii. Define a stationary recursive competitive equilibrium for this economy.

10 Income fluctuations and the cost of inflation İmrohoroğlu (1992)

Consider an economy with a continuum of infinitely-lived agents who seek to maximize the expected discounted value of their utility,

$$E_0 \left[\sum_{t=0}^{\infty} \beta^t u\left(c_t\right) \right].$$

These agents have (common) preferences for consumption and supply labor inelastically. Their income in consumption units, ϵ, fluctuates stochastically following a Markov chain process with a Markov matrix Π. These are idiosyncratic fluctuations independent across agents and can take only N values $\{\epsilon_1, \ldots, \epsilon_N\}$. These correspond to the endowments in this economy.

The consumption good cannot be stored. But the agents have access to paper money, printed by the government. Money cannot be consumed, but it can be stored. People in this economy are extremely distrustful of one another, and so they cannot sign contracts with one another (financial or otherwise). They are worried that if they sign a contract that stipulates future payments their counterpart will default on them. The outcome of this is that there are no financial markets in this economy.

New money is printed by the government at a rate g, so $M_t = (1+g)\, M_{t-1}$ is the amount of money in period t. The government transfers all the new money to the agents lump-sum.

The price of consumption goods in terms of money units is p_t. Let $1 + \pi_t = {p_t}/{p_{t-1}}$ denote the inflation rate (the growth rate of the price).

i. What is the nominal budget constraint of an agent? (nominal refers to the constraint being written in terms of money units). What is the real budget constraint? (real refers to the constraint being written in terms of consumption units, this requires you to introduce notation for "real money balances")

ii. Write down the agents' dynamic programming problem using the real budget constraint.

iii. Write down the agents' Euler equation. Interpret in terms of the saving motives of the agents. What do they depend on? In particular, how are they affected by inflation?

iv. Define the law of motion of the distribution of agents using the policy functions from the agents' dynamic problem and explain how to obtain the stationary distribution of agents.

v. Define a stationary recursive competitive equilibrium for this economy.

vi. Why did we have to use the real (and not the nominal) budget to define the S-RCE?

vii. In the S-RCE the real money balances are constant because there is no aggregate risk. Explain. Then, use this fact (together with aggregation and market clearing) to show that the equilibrium inflation rate constant and is equal to the growth rate of money, $\pi_t = g$.

viii. Pose the problem of a utilitarian planner that weights all agents equally. Characterize the planners' allocation (of consumption) across agents and over time.

11 Cash-in-Advanced and the Friedman Rule

Consider the following cash-in-advance model. Time is discrete and goes on forever. There is a representative household, a representative firm, and a government.

The government follows a monetary policy rule so that it prints money at a rate α_{t+1} every period

$$M_{t+1}^{s} = (1 + \alpha_{t+1}) M_t^s,$$

where M_t^s is the money supply in period t. and a fiscal rule that balances the budget every period. There is no government spending so that

$$M_{t+1} - M_t = T_t,$$

where T_t are (nominal) transfers to the household.

The firm produces using only labor with a linear technology

$$Y_t = z_t N_t,$$

hires labor in a perfectly competitive labor market at a nominal wage W_t and sells output at a price P_t in a perfectly competitive market.

The household chooses consumption, leisure, and savings in nominal bonds to maximize the present discounted value of their utility $\left(\sum_{t=0}^{\infty} \beta^t \left(U\left(C_t\right) - V\left(N_t\right)\right)\right)$ subject to a sequence of budget and cash-in-advance constraints

$$\begin{aligned} P_t C_t + q_t B_{t+1} + M_{t+1} &= M_t + B_t + W_t N_t + P_t \pi_t + T_t \\ P_t C_t + q_t B_{t+1} &= M_t + B_t + T_t \end{aligned}$$

Inflation is defined as $1 + i_{t+1} = \frac{P_{t+1}}{P_t}$.

i. Express the consumer's budget constraint, the cash in advance constraint, and the government's budget below in real terms in relation to the price level.

ii. Define an equilibrium for this economy (it can be a sequential or a recursive competitive equilibrium). For this, take as given the exogenous sequences of productivity $\{z_t\}_{t=0}^{\infty}$ and monetary policy $\{\alpha_{t+1}\}_{t=0}^{\infty}$.

iii. Pose the planner's problem for this economy and obtain a condition that characterizes the efficient allocation of labor.

iv. In the market economy, the optimal level of labor and consumption in the model solve the following equation that relates the marginal benefit of supplying more labor (in terms of the consumption it generates) with its marginal cost (in terms of the disutility of labor):

$$\frac{\beta}{1+i_{t+1}} U^{'}\left(C_{t+1}\right) z_t \quad = \quad V^{'}\left(N_t\right);$$

(a) Show how this condition is obtained from the household problem

(b) How does inflation affect the optimal labor choice of individuals? Explain what is the economic intuition behind your result.

v. What is the optimal inflation rate in this economy? Explain using the solution to the Planner's problem and the market equilibrium. Provide the economic intuition behind your result.

(a) What is the price of nominal bonds under the optimal inflation rate when $z_t = z$?

vi. Now assume that $U\left(C\right) = \frac{c^{1-\sigma}}{1-\sigma}$ and that $V\left(N\right) = \frac{N^{1+\frac{1}{\eta}}}{1+\frac{1}{\eta}}$. What is the optimal monetary policy? that is, what is the optimal growth rate of money $\{\alpha_{t+t}\}_{t=0}^{\infty}$?

vii. What is the value of the Lagrange multiplier of the cash in advanced constraint under the optimal monetary policy? You can do this in steady state.

viii. What is the value of the rate of return on nominal bonds and on currency under the optimal monetary policy? You can do this in steady state.

12 Entrepreneurial productivity and aggregate productivity (Moll, 2014)

Entrepreneurial production problem:

Consider an entrepreneur with idea quality z and business capital k producing output y using a production technology:

$$y(z,k,n) = (zk)^{\alpha} n^{1-\alpha},$$

with parameters $\alpha \in [0,1)$, and n denotes the number of workers the firm employs. The firm produces the numeraire good, with a normalized price of one. A higher z means a higher quality idea.

i. Formulate the entrepreneur's profit maximization problem. Take the amount of capital as given so that profits depend on productivity and capital, $\pi(z,k)$.

ii. Characterize the solution to the entrepreneur's profit maximization problem in closed-form. The solution should include a characterization of the firm's indirect profit function and for the demand for labor of an entrepreneur.

Occupational Choice

Next, suppose instead that the entrepreneur is endowed with idea quality z and wealth level a, but not with business capital k.

The entrepreneur can borrow up to $\bar{\lambda} a$ to invest in entrepreneurial capital $k \geq 0$, where $\bar{\lambda} > 0$. Borrowing takes place at a rate $r > 0$. The entrepreneur can thus invest $\lambda \in [0, \bar{\lambda}]$ of their wealth in business capital. If $\lambda < 1$, the complementary share of wealth is invested in an asset that yields a net return $r > 0$, if $1 < \lambda < \overline{\lambda}$ the entrepreneur pays an interest rate r on the excess funds.

The choice of how much to invest in their businesses and how much to invest in the market captures the (continuous) occupational choice of the entrepreneur. The entrepreneur will choose their occupation to maximize their income.

i. Formulate the entrepreneur's income maximization problem and characterize the optimal factor of wealth λ that entrepreneurs invest in business capital..

   (a) Does wealth level a affect the share of wealth λ invested in business capital? Explain.

   (b) Show that returns per unit of wealth $r(z,a)$ are independent of wealth, $r(z,a) = r(z)$.

Dynamic Problem

For the remainder of this question, consider an infinite-horizon specification of the entrepreneur's problem (entrepreneurs live forever). The entrepreneur has logarithmic preferences over consumption in each period, or $u(c) = \log c$, a discount factor $\beta \in (0,1)$ and the budget constraint is:

$$c + a' = R(z)a,$$

where $R(z)$ is the gross returns per unit of wealth corresponding to the net return $r(z)$ from above, or $R(z) = 1 + r(z)$.

The idea quality of entrepreneurs takes two values: z_H and z_L, with $z_H > z_L$. When an entrepreneur has idea quality z_i at time t, the entrepreneur has the same productivity at time $t+1$ with probability p. With probability $1-p$, the entrepreneur has productivity $z_{\neg i}$ at time $t+1$.

i. Entrepreneurial productivity follows a Markov process. Construct a transition function for this Markov process and find its stationary distribution.

ii. Solving the dynamic problem:

   (a) Formulate the dynamic entrepreneur's problem.

   (b) Characterize the solution to the entrepreneur's problem in closed-form. Hint: Use Guess-and-Verify. Guess that the idea quality only affects the level of the value function, $V(a,z) = m(z) + n \log a$. You are looking for $m(z)$ and n. You do not have to solve for $m(z)$ in closed form, just specify what characterizes their solution.

   (c) Interpret the value function in terms of wealth endowment, returns, and risk.

   (d) What is the change in the value function with respect to a small percentage increase in the wealth level of the household?

   (e) What is the effect of relaxing the borrowing constraint $\bar{\lambda}$ on the value of the entrepreneur? Explain.

iii. Optimal savings by entrepreneurs

   (a) What is the entrepreneur's Euler equation? Interpret in terms of the entrepreneur's marginal cost and benefit of savings.

   (b) What is the entrepreneur's optimal saving function?

iv. Construct a Markov transition function for the states of the entrepreneurs.

Aggregation

There is a continuum of mass 1 of entrepreneurs. Assume that their productivities are distributed according to the stationary distribution you found above. The aggregate assets of high- and low- productivity entrepreneurs are denoted A_H and A_L. Aggregate capital is $K = A_H + A_L$.

Aggregate productivity is endogenous and depends on who is using the capital. So, $Z = s_H^K z_H + \left(1 - s_H^K\right) z_L$ where $s_H^K = \frac{K_H}{K}$ is the share of capital used by the high-productivity entrepreneurs.

i. **Wages:** There is an inelastic supply of L units of labor every period. Use the demand for labor obtained in part 1 to state the market clearing condition for labor and find the market clearing wage, w, in terms of aggregate capital, K, and aggregate productivity, Z.

ii. **Returns:** Assume that, in equilibrium, the low-productivity entrepreneurs are indifferent between producing or not producing, while high-productivity want to produce and demand capital. That is, low-productivity entrepreneurs make zero profits while high-productivity entrepreneurs are constrained.

    (a) Use this condition to obtain an expression for r in terms of the aggregate capital and the aggregate productivity.

    (b) Use that condition to further solve for $R(z_H)$ and $R(z_L)$ in terms of aggregate variables and individual productivities.

iii. Use the policy functions of the entrepreneurs and the evolution of entrepreneurial productivity to obtain expressions for the evolution of the aggregate assets of high- and low- productivity entrepreneurs, A_H and A_L. That is, evolution equations for $A_{H,t+1}$ and $A_{L,t+1}$ as a function of $A_{H,t}$ and $A_{L,t}$.

iv. Use the evolution equations to obtain an evolution equation for aggregate capital $(K = A_H + A_L)$.

v. Find the steady state value of capital in this economy. How does it differ from the steady state value in the Neo-Classical Growth Model.

Aggregate Productivity

i. Use the evolution equations for A_H and A_L to derive two evolutions equations for the wealth share s_H. These equations jointly determine the steady state of aggregate productivity Z. Show this graphically.

ii. What happens to aggregate productivity when p increases? Prove it and explain the intuition behind your result.

13 Portfolio choice over the life cycle

Consider a continuum of households that live for at most a finite number of periods, T. There is also mortality risk each period so that the conditional probability of surviving for one period to the next is $\delta \in (0,1)$. Household's preferences are given by $\sum_{t=1}^{T} (\beta\delta)^{t-1} \frac{c_t^{1-\sigma}}{1-\sigma}$. The household maximizes expected utility.

The household allocates beginning-of-period wealth, W_t , towards current consumption, c_t, purchases of a risky asset x_t, and a risk-free asset, b_t , according to: $c_t + x_t + b_t \leq W_t$.

Wealth evolves according to: $W_{t+1} = Rx_t + R_f b_t$, where R denotes the (gross) return to the risky asset and R_f denotes the gross return to the risk-free asset. Assume that R takes on two values, R_h and R_ℓ, with probabilities p and $1-p$.

There is an overlapping-generations structure, so that the households who die are replaced by newborn households. The wealth of the dying households is equally shared by newborns, so that all households are born with the same wealth. Note that, because households die randomly, this is equal to the average wealth in the economy.

Portfolio choices

i. Set up the household's problem as a dynamic program.

ii. Financial planners recommend that households' portfolios should become less risky as they age. Evaluate this recommendation. **Hint:** Guess that the value function satisfies $V_t(W) = a_t \frac{W^{1-\sigma}}{1-\sigma}$. Derive the household's optimal asset allocation given this guess. For this it is convenient to re-express the problem in terms of the savings rate s_t and the share of savings into risky assets ϕ_t, so that $c_t = (1-s_t) W_t$ and $x_t = \phi_t s_t W_t$. The question is about the share of savings going into risky assets, call it ϕ_t.

   (a) What if R is i.i.d over time and is drawn from a distribution $F(R)$?

iii. Define the Markov process for household wealth.

Entrepreneurial firms problem

The risky asset, x_t, is the capital invested in a firm that produces with a constant-returns-to-scale technology, combining capital and labor

$$y_i = z_i f(x_i, \ell_i),$$

where z_i is distributed iid across firms and across time. Labor is hired in competitive markets at wage w.

Each investor invests in a different firm. Crucially, investors do not know the productivity of the firm at the time of investing. Labor is instead chosen after z_i is observed.

i. Pose the profit maximization problem of the firm. Recall that this problem takes as given the level of capital (x) and so the only choice is labor demand.

ii. Solve the problem to obtain the optimal labor demand and the optimal profit level of an entrepreneur. **Hint:** You can solve for things generally using Euler's theorem for homogenous functions, or you can assume that $f(x, \ell) = x^{\alpha}\ell^{1-\alpha}$ and solve explicitly.

iii. Use the optimal profits of the firm to obtain the distribution of returns to risky capital investment R.

Rest of the economy and equilibrium

Workers The is also a continuum of infinitely lived workers.They derive utility exclusively from consumption, supply one unit of labor each, have no savings and consume all of their wages each period. There is a mass L of workers.

Corporate Sector The corporate firm operates the same technology as the entrepreneurial firms but with productivity z_c. So that

$$Y_c = z_c f(K_c, L_c).$$

The firm obtains capital in the bond market at rate R_f, and hires workers at wage w.

i. State all market clearing conditions.

ii. Define a stationary recursive competitive equilibrium for this economy. **Hint:** Prices and aggregates are constant in a stationary recursive competitive equilibrium. You can assume that z takes on two values, z_h and z_ℓ, with probabilities p and $1-p$.

iii. Solve for the equilibrium wage as a function of the aggregate capital stock.

iv. Show that the production side of the economy aggregates, with total output a function of aggregate productivity, capital, and labor. **Hint:** Use the labor demand of individual firms to aggregate labor demand and market clearing to obtain the wage. Then use it to aggregate output.

14 Savings over the life cycle

Consider a benchmark life-cycle model where individuals differ in the permanent components of their income and returns. Each individual is characterized by a tuple (a_0, I_Y, I_R) that describes a wealth endowment, a_0, and two permanent characteristics, (I_Y, I_R), that affect income $Y_t(I_Y, I_R)$ and (gross) returns $R_t(I_Y, I_R)$.

The problem of an individual is,

$$\max_{\{c_t, a_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t u(c_t) + \beta^{T+1} v(R_{T+1} a_{T+1}) \quad \text{s.t. } c_t + a_{t+1} = R_t a_t + Y_t$$

where $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ and $v(x) = \Upsilon \frac{x^{1-\gamma}}{1-\gamma}$.

i. Define future wealth, H_t, as the discounted value of future labor income $\{Y_{t+1}, \ldots, Y_T\}$. Use the age-dependent interest rate to discount. Derive a recursive expression for H_t.

ii. Define un-borrowed funds as $b_t = a_t + H_t$ and show that the relevant state of the problem is total (or virtual) wealth: $W_t = R_t a_t + Y_t + H_t$.

iii. Let $1 - s_t = \frac{c_t}{W_t}$ be the average consumption rate of an individual at age t. Use this to derive an updating function for W_{t+1} in terms of W_t.

iv. Pose a dynamic programming problem for the consumer ($V_t(W)$ as a function of $V_{t+1}\left(W'\right)$). Do not forget to specify the terminal condition.

v. Solve for the optimal savings rate. **Hint:** Guess that $V_t(W) = \alpha_t \frac{W_t^{1-\gamma}}{1-\gamma}$ and solve for α_t.

vi. Show that the savings rate of the consumer satisfies a recursive formulation

$$(1 - s_t)^{-1} = 1 + \left(\beta R_{t+1}^{1-\gamma}\right)^{\frac{1}{\gamma}} (1 - s_{t+1})^{-1}$$

vii. Show that the growth rate of consumption is independent of income $\{Y_t\}$ and show how the effect of R_t on the growth rate of consumption depends on γ.

viii. Show that a permanent increase in returns (returns go up at every age by the same proportion) increases the average consumption rate if $\gamma > 1$.

15 Capital-income risk (Angeletos, 2007)

This question is based on Angeletos (2007) at the Review of Economic Dynamics.

Consider an economy with a continuum of infinitely-lived entrepreneurs. Each entrepreneur supplies (inelastically) one unit of labor in the market and also has a firm. These entrepreneurial firms produce using a common constant-returns-to-scale technology that employs capital and labor. Entrepreneurs are subject to idiosyncratic productivity shocks, z_i, that affect their production. The technology of production for firm i is $y_i = z_i f(k_i, \ell_i)$. The entrepreneurial productivity z_i is drawn every period from a distribution ψ (independently across entrepreneurs). The average productivity across agents is always 1 ($E[z_i] = \int z\psi(z)\,dz = 1$).

Crucially, the entrepreneurs must choose capital before they know the realized value of their productivity z_i.

Entrepreneurs can hire labor in the market at a wage rate w, but cannot rent capital, so entrepreneurs can only invest in the firm they own [the reason is that the productivity of each entrepreneur is private information, and entrepreneurs would not know what type of firm they are investing in if they were to lend capital to another firm]. There is, however a non-state-contingent bond that the entrepreneurs can use to save and borrow, call it b_i. This is a pure financial asset in zero net supply in the economy. The (gross) rate of return on the bond is R.

Entrepreneurs seek to maximize the expected discounted value of their utility. However, they have recursive Epstein-Zin preferences, instead of the usual time-additive preferences with CRRA utility. The recursive preferences they have exhibit constant elasticity of inter-temporal substitution (CEIS) and constant relative risk aversion (CRRA):

$$u_t = U(c_t) + \beta U\left(\text{CE}\left[U^{-1}(u_{t+1})\right]\right),$$

where the certainty equivalent of the future utility u_{t+1} is $\text{CE}[x] = \Upsilon^{-1}(E[\Upsilon(x)])$. The utility functions U and Υ aggregate consumption across dates and states, respectively; they are given by

$$U(c) = \frac{c^{1-\frac{1}{\theta}}}{1-\frac{1}{\theta}} \qquad \Upsilon(x) = \frac{x^{1-\gamma}}{1-\gamma},$$

where θ is the elasticity of inter-temporal substitution and γ is the coefficient of relative risk aversion.

The recursive problem of an entrepreneur is

$$V_t(s) = \max\ U(c) + \beta U\left(\Upsilon^{-1}\left(\int \Upsilon\left(U^{-1}\left(V_{t+1}\left(s'\right)\right)\right)\psi(z)\,dz'\right)\right)$$

where s is the state vector. Note that the value function depends on time because the aggregates of the economy can change (deterministically) over time. Choosing the state is as always crucial.

i. Define a recursive competitive equilibrium for this economy. Assume in your definition that prices evolve over deterministically over time.

ii. Pose the profit maximization problem of the entrepreneurial firm. Recall that this problem takes as given the level of capital. Solve the problem to obtain the optimal labor demand and the optimal profit level of an entrepreneur. This is Lemma 1 of the paper. You can solve for things generally, or you can assume that $f(k, \ell) = k^{\alpha} \ell^{1-\alpha}$ and solve explicitly.

iii. Write down the budget constraint of the entrepreneurs in terms of its portfolio choice over capital and bonds.

iv. Define the financial wealth of the household (equation 7 in the paper) and write the entrepreneurs' dynamic problem in terms of it.

v. Solve the portfolio allocation problem of the entrepreneurs using guess-and-verify. This is Lemma 2 of the paper. You can solve for things generally or assume that $\theta = 1/\gamma$ to simplify the recursive problem of the entrepreneurs.

   (a) Interpret the variables ζ, ϕ, and ρ that characterize the solution.

   (b) Provide simplified expressions for the case in which $\theta = 1/\gamma$. Interpret the optimal choice of the entrepreneur.

vi. Show that the economy admits exact aggregation. That is, find expressions for the aggregate variables and prices in terms of the solution to the entrepreneurs' problem. This is proposition 1 of the paper. You can simplify results if you assume that $\theta = 1$.

vii. Focus now on the steady state of the model, when aggregate variables and prices do not change over time. Prove Proposition 2.

viii. Explain equations (22) and (23) in your own words.

ix. Explain why it must be that $R\beta < 1$ and $f_K(K, 1) \geq R$ in steady state .

x. Explain the result in Proposition 3 and the intuition behind it.

16 Lucas tree economy

Consider an economy with infinitely lived farmers, each endowed with a "tree." These trees are of the Lucas variety, a type of tree indigenous to northern Illinois. These trees give fruit exactly once a year, all of them at exactly the same time, and producing exactly the same number of fruits. Moreover, they never whither. However, the harvest obtained from each tree varies from year to year. The fruit is exquisite but it is also extremely delicate. It cannot be stored or preserved.

The variability of the fruit-output of the Lucas trees has long attracted the attention of researchers because of the surprising patterns in fruit yield across years. Botanists, biologists, and applied mathematicians that have studied these peculiar trees have shown that they can only produce fruit in one of N amounts. More surprisingly is that the number of fruits harvested from a tree one year only depends on the number produced in the previous year, although not in a deterministic way. Rather, the previous harvest determines the probability that a given number of fruit is produced in the following harvest. As of today there is no evidence of any other plant with such an intriguing pattern. Amazing!

i. Let $z_{i,t}$ be the fruit harvested from a Lucas tree in period t. Use the language of Markov processes to describe the findings of the Botanists, biologists, and applied mathematicians.

Lets go back to our farmers. They like to eat fruit and are impatient. Their preferences for fruit consumption in a given period are captured by a function $u(c_t)$. They discount the future at a constant rate $\beta < 1$. They are the only ones in this economy and the trees that each of them have are the only source of fruit.

2. Pose the Planner's problem and characterize its allocation of fruit in this economy.

The farmers interact in a decentralized market. There is a market active in each period in which Farmers can trade shares on their trees. Each tree has one unit of perfectly divisible shares ($s_{i,t} \in [0,1]$). The price of a unit share for a tree is $q(z)$ and depends (in general) on the fruit output of a tree. Each farmer therefore has s_{it} shares of trees and is therefore entitled to their fruits and to sell those shares in the market (or buy new ones).

3. Write down a recursive problem for the farmer.

4. Equilibrium:

   (a) Define an equilibrium for this economy.

   (b) How does the equilibrium allocation compare to the planner's?

5. Find an expression for the price of shares (q) using the farmers' first order condition. Interpret your solution.

17 Monopolistic Competition and Price Adjustment Costs

Consider an economy with a continuum of producers. All producers are identical except for the fact that they produce differentiated varieties of goods that are then aggregated into final consumption:

$$Y = \left(\int_0^1 y_i^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{\frac{\varepsilon}{\varepsilon-1}}.$$

The demand for each producer's good is

$$\frac{y_i}{Y} = \left(\frac{P_i}{P}\right)^{-\varepsilon}.$$

i. Derive an expression for the aggregate price of the economy as a function of the prices of individual producers.

   The producers' problem is to maximize their profits choosing prices. Their output comes from using their linear-labor-technology: $y_i = zn_i$. Their problem is, in terms of output,

$$\Pi_i = \max_{P_i} \; P_i y_i - \frac{W}{z} y_i$$

ii. Derive the optimal pricing choice of firms.

iii. What is the aggregate price and aggregate profits in the economy?

iv. Real wages $w = W/P$:

   (a) What is the real wage in this economy? (how many units of final good can be bought with one unit of labor)

   (b) Are workers being paid the marginal product of their labor? Explain.

v. Is the aggregate (nominal) price level P defined in the model? What does this imply for Money Neutrality? Explain.

   Consider now that this economy operates for a second period. However, firms now face an adjustment cost that makes it costly to change their prices. They have to take this into consideration when setting their prices. Their (discounted) profits are

$$\Pi = P_{i,1}y_{i,1} - \underbrace{\frac{w_1}{z_1}}_{\text{Real Mrg. Cost}} y_{i,1}P_1 - \underbrace{\frac{\lambda}{2}\left(\frac{P_{i,1}}{P_{i,0}} - 1\right)^2}_{\text{Real Adjustment Costs}} P_1 + \beta\frac{P_1}{P_2}\left[P_{i,2}y_{i,2} - \underbrace{\frac{w_2}{z_2}}_{\text{Real Mrg. Cost}} y_{i,2}P_2 - \underbrace{\frac{\lambda}{2}\left(\frac{P_{i,2}}{P_{i,1}} - 1\right)^2}_{\text{Real Adjustment Costs}} P_2\right]$$

vi. Optimal prices:

(a) Characterize the firm's optimal price P_1. Take as given $P_{i,0}$ and express the firms' first order condition in terms of relative prices $\left({}^{P_i}/_{P}\right)$ and inflation $\left(1+\pi_t = {}^{P_t}/_{P_{t-1}}\right)$. Verify that if $\lambda = 0$ then you get the same result as above when there were no adjustment costs.

(b) Interpret the three terms of the first order condition. Hint: The first term is the same as in the normal monopolist problem. The other two terms are determined by the effects of price adjustment and depend on inflation.

vii. Equilibrium. All the firms in this economy are identical. Further assume that $P_{i,0} = P_0$ for all firms. Then the equilibrium is symmetric, with all firms making the same choices and we have that $P_{i,t} = P_t$ for all firms.

(a) Use this result to simplify the optimal pricing equation. You should obtain the following result:

$$0 = -\left(\varepsilon - 1\right)\left(1 - \frac{\varepsilon}{\varepsilon - 1}\text{mc}_1\right)Y_1 - \lambda\left[\pi_1\left(1+\pi_1\right) - \beta\pi_2\left(1+\pi_2\right)\right]$$

This is the New-Keynesian Phillips Curve. It establishes a relationship between inflation (and future inflation expectations) and the marginal cost of production (given here bye $\text{mc}_t = {}^{w_t}/_{z_t}$). Without adjustment costs for prices this equations simply determines the level of the real marginal costs (real wage). The New-Keynesian Phillips Curve implies a relationship between the marginal costs, output (demand), and inflation.

(b) What happens to inflation if the marginal cost goes up $(\text{mc}_1 \uparrow)$? How does the magnitude of the effect change with λ? Explain what is the mechanism for this in the model and its intuition.

(c) What happens to inflation if output demand goes up $(Y_1 \uparrow)$? Explain what is the mechanism for this in the model and its intuition.

In answering these questions you can set $\beta = 0$ so that you don't have to worry about the behavior of future inflation.

18 Equivalence of Productivity and Preferences

Consider an economy with a continuum of potential producers and a representative consumer.

- All producers are identical except for the fact that they produce differentiated varieties of goods. In order to produce they must pay an entry cost $\psi > 0$. Potential producers simultaneously decide whether to enter and produce.
- The preferences for these goods are captured by the following function:

$$Y = \left(\int_0^\infty \gamma_i^{\frac{1}{\varepsilon}} y_i^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}},$$

  where γ_i is a preference shifter for variety i of the good.

  - The goods are ordered according to γ, so that lower i varieties have a larger γ, i.e., $\gamma_i > \gamma_j$ for $i < j$. The highest γ is therefore γ_0.
  - The consumer only consumes the goods of producers who actually enter but the preferences are defined over all possible goods.
- The consumer supplies 1 unit of labor inelastically in a perfectly competitive labor market.

i. Demand for varieties:

  (a) Derive an expression for the demand for variety i of goods (y_i) in terms of its price (p_i), the level of aggregate utility (Y), and the shadow price of utility (P).

  (b) What is the elasticity of demand?

  (c) Interpret the role of γ_i in demand.

  The producers' problem is to maximize their profits choosing prices. They are monopolists in their own varieties. Their output comes from using their linear-labor-technology: $y_i = zn_i$. Their problem is, in terms of output,

$$\Pi_i = \max_{p_i} \left\{ p_i y_i - \frac{W}{z} y_i \right\}$$

  subject to the demand curve that you derived above.

ii. Solve the producer's problem:

  (a) Find the optimal price of firm i

  (b) How do profits depend on γ_i? Interpret.

  (c) How are differences in γ_i different from differences in productivity z?

iii. Entry problem: What is the optimal entry rule for producers?

iv. Aggregate price:

(a) Find an expression for P using the optimal prices for each variety and the entry rule.

(b) Interpret in terms of the role of γ and marginal cost of "producing" utility.

(c) Only for this question: Assume that $\gamma_i = 1$ for all producers, but that z is different across them, with lower index i having higher z.

Derive an expression for P.

Comment on the difference between the expression with differences in γ and the expression with differences in z. How are γ and z different?

v. Equilibrium: Define an equilibrium for this economy. Do not forget to state the condition for labor market clearing! **Bonus:** Solve for the equilibrium.

19 Misallocation and TFP Measurement

Consider a capital economy with no labor. There are many types of differentiated capital goods that are demanded by consumers. Consumers preferences for these goods are captured by

$$Q = \left(\int x_i^\mu di \right)^{1/\mu}$$

where x_i is the quantity of variety i of the capital good x. Consumers have a total of 1 unit of income to spend in Q, so that $PQ = 1$, where P is the ideal price index (so that $\int p_i x_i di = PQ$).

Producers of capital goods operate a linear technology and vary in their productivity z_i, so that $x_i(k) = z_i k$. They also differ in their costs of maintaining capital δ_i. So that their total costs are $c_i(k) = (R + \delta_i) k$. There is an exogenous distribution over productivity and costs $\Gamma(z, \delta)$.

Hint: You can solve everything more simply by setting $\delta_i = \delta$.

i. Derive the demand for variety i of the capital good.

ii. Pose and solve the problem of the producer of variety i of the capital good. Include an expression for the optimal price, size (capital), and profits.

iii. The average cost of capital is $R + \overline{\delta}$, where $\overline{\delta} = \int \delta_i d\Gamma$. Define $1 + \tau_i = \frac{R+\delta_i}{R+\overline{\delta}}$ as the wedge between the individual firm's cost of capital and the average cost. Use this wedge to derive an expression for the market clearing average cost of capital $\left(R + \overline{\delta}\right)$ when there is an inelastic supply of K units of capital. **Hint:** Define aggregate productivity as aggregate of the "effective" productivity of an individual firm $\frac{z_i^\mu}{1+\tau_i}$.

iv. Now obtain an expression for aggregate Q in terms of aggregate capital and aggregate productivity, aggregate capital, and a third term capturing the role of τ_i and productivity differences.

v. Is the allocation efficient? For this pose the planner problem (maximizing Q net of depreciation costs given supply of capital, K).

20 A closed-economy with span of control

Consider a static (one period) economy with infinitely many individuals. Individuals differ in their entrepreneurial (or managerial) talent denoted by z_i. Assume that talent is distributed Pareto with minimum value of $\underline{z} = 1$ and a Pareto coefficient of ξ, so that the CDF of the distribution is $\Pr(\tilde{z} \leq z) = 1 - z^{-\xi}$.

Individuals also have an endowment of assets that differs across agents and is distributed independently of their entrepreneurial talent. Assume that the aggregate assets are $\overline{K} = E[a_i] > 0$. There is a market for capital where individuals can lend their assets to firms at a market rate r.

Individuals face an occupational choice. They can use their entrepreneurial talent to start a firm, doing so has a fixed cost $\psi > 0$. If they start a firm they produce final goods with a technology $z_i g(F(k_i, n_i))$, where $g(x) = x^{\alpha}$ $\alpha \in (0,1)$ and $F(k,n) = k^{\gamma} n^{1-\gamma}$. If they do not start a firm they can work. Each individual has one unit of time that they offer inelastically. The market wage is w.

Individuals value consumption. The capital and labor markets are perfectly competitive.

i. Pose the profit maximization of an entrepreneur with talent z. Solve the problem. That is, find functions $\pi^{\star}(z)$, $n^{\star}(z)$, $k^{\star}(z)$ that characterize the optimal profits, and labor and capital demand of a entrepreneur. Take prices r and w as given.

ii. Pose the occupational choice problem of individuals.

iii. Define an equilibrium for this economy.

iv. Solve the equilibrium in this economy. In particular, solve for the prices that clear the markets. You can make any assumptions you deem useful for this (for instance you can do away with capital or the fixed set up costs).

v. How does the distribution of firm productivity (talent) differ from the distribution of talent in the population?

vi. How does talent translate into differences in income? For this compare the distribution of profits and the distribution of talent among entrepreneurs. In particular, if an entrepreneur has 10 times more talent than another, how much more income (profits) do they have?

vii. How does the distribution of firm size by employment differ from the distribution by capital. Interpret the difference.

viii. Analyze the solution. What do the comparative statics of the equilibrium prices and quantities tell you about the economy.

21 Equilibrium in an economy with differentiated goods

Consider a static (one period) closed economy where firms produce differentiated products as in Dixit & Stiglitz. Consumers have preferences for a bundle of goods according to

$$U = \left(\int y_i^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}}$$

i. Let $R = PY$ be the total spending (or total revenue), P be an idea price index and Y the total "aggregate quantity" in the market. P and Y satisfy $PY = \int p_i y_i di = \int r_i di$. Derive the optimal demand curve for y_i in terms of p_i, Y, and P and the revenue (or spending) in good i r_i in terms of total revenue R, p_i, P. Derive an expression for the ideal price index P as a function of the prices of individual varieties.

   The production of goods uses only labor. There are L units of labor in the economy. We use labor as the numeraire so that $w = 1$ and all prices are expressed in terms of units of labor. (so a price p means that someones would have to work p units of labor to pay for the good).

   Firms differ in their productivity z that determines their marginal costs, $1/z$. They all have to pay a a fixed cost φ to setup production for the local market.

ii. Pose the profit maximization problem of a firm with productivity z_i that is choosing the price of their good subject to the demand curve they face. The firm takes as given aggregates. Find expressions for the firm's revenue $r(z)$ and the firms' profits $\pi(z)$ (including fixed costs).

iii. Some times we cannot observe prices and quantities separately in the data. We instead observe revenue and cost-of-goods-sold. This gives rise to a measure of productivity given by the ratio of revenue to costs.

   (a) Calculate this measure and argue whether or not it is a good proxy for the actual productivity of the firm.

   Similarly, if we only observe revenue we do not know if firms with higher revenue produce more or less (in quantities).

   (a) Compare the ratio of revenues and the ratio of quantities produced of two firms with $z_i > z_j$. What can you conclude about quantities and productivities if you see that a firm has higher revenue than another firm?

   In equilibrium (to be characterized later) there is a distribution of firm productivity $z \in (0, \infty)$ with pdf $\mu(z)$ and a mass of $M > 0$.

iv. Express the price aggregator in terms of an integral with respect to productivity (z) and not to firm indices (i). This is possible because your previous results show that all firm with the same productivity charge the same price. The density of firms with productivity z is $M\mu(z)$. Then, use the optimal pricing of firms to express the aggregate price P as a function of M and the average productivity, $\overline{z}$. Show that the relevant measure of the average is a curved-weighted harmonic mean $\overline{x} = \left(\int x^{\varepsilon-1}\mu(x)\,dx\right)^{\frac{1}{\varepsilon-1}}$.

   (a) Show you can do the same for revenue, R, total profits, Π, and total quantity, Y.

   We now want to figure out what determines μ and M. This comes from the entry and exit decisions of firms. The modeling of this follows Hopenhayn (1992). There is a large pool of identical potential entrants deciding whether to become active or not. Firms deciding to become active pay a fixed cost of entry $\varphi_e > 0$ (this is different from the operating cost φ above) and then get a productivity draw z from a CDF Γ. After observing their productivity draws, firms decide whether to remain active or not.

   Assume that Γ is Pareto, so that $\Gamma(z) = 1 - \left(\frac{z}{\underline{z}}\right)^{-\xi}$ for some $\underline{z} > 0$, and a pdf $\gamma(z) = \xi\underline{z}^{\xi}z^{-(\xi+1)}$

v. Characterize the operating decision of firms in terms of cutoff for their productivity, $z^{\star}$. What can you say about it?

   Having the cutoff for operations gives us the distribution of operating firms

$$\mu(z) = \begin{cases} \frac{\gamma(z)}{1-\Gamma(z^{\star})} & \text{if } z \geq z^{\star} \\ 0 & \text{otw} \end{cases}$$

vi. Use this information to solve for the average productivity of operating firms $\overline{z}$.

   There is free entry. This means that, in equilibrium, firms will enter as long as their expected profits cover the fixed entry cost φ_e. Recall that firms do not their productivity before they enter. If they draw a productivity below $z^{\star}$ they get no profits, if they draw a productivity above $z^{\star}$ (with probability $1-\Gamma(z^{\star})$) they get the average profits of surviving firms: $\overline{\pi} = \frac{\Pi}{M}$. You solved for Π above.

vii. State the free entry condition of the firms and explain what it means for the equilibrium level of average profits if you increase the cost of entry.

viii. The solution to the model is the pair of values $z^{\star}$ and $\overline{\pi}$ that simultaneously satisfy the free entry condition and the zero-profit condition for the marginal operating firm. Show that the profits of the marginally operating firm are increasing in productivity and that the average profits are constant with respect to the productivity of the marginal operating firm. Draw a diagram with the free entry and the zero-profit curves as functions of productivity to characterize the solution.

The last step to close the model is to clear the goods and labor market. The free entry condition implies that aggregate profits are zero after paying for entry costs. So, to clear the goods market it must be that $R = L$, that is, total revenue (obtained by firms) must be equal to the total income of consumers (recall that the wage is normalized to 1).

ix. Use the market clearing condition to obtain an expression for the mass of firms M in terms of size of the economy L, the elasticity ε and the profits of firms. Relate the solution to the taste for variety.

22 International trade: The Melitz model

Consider the same model as in the previous question. We are now going to open the economy. A firm has to pay a fixed cost φ_x in order to export. Moreover, exporting goods is more expensive than selling goods in the local market. This is reflected in a higher marginal cost of τ/z per unit of good exported, $\tau > 1$. These are called iceberg costs in the literature because they are equivalent to having to send τ units of goods for every unit of successful exports (so that $\tau - 1$ goods do not reach the destination, they sink while at sea!). People in other countries have the same preferences for goods as in the local country and have firms with the same technology. The solution across countries is symmetric and so if a firm in the "domestic" economy exports, there is an equivalent firm in the foreign economy that is also exporting.

i. If a firm exports they can charge a different price in the local and the foreign market. What is the optimal price for exported goods?

ii. Firms export if doing so increases their profits. Characterize a cutoff for productivity $z_x^\star$ above which firms export. Assume that $\tau^{\varepsilon-1}\varphi_x > \varphi$. Show that this implies that $z_x^\star > z^\star$ so that only some of the firms who operate export. You can show this graphically with a diagram of profits ($\pi_x(z)$ and $\pi_d(z)$). You can show in the diagram the three regions for productivity (firms do not operate, they operate but don't export, they export). The total profits of firms are $\pi = \pi_d + \pi_x$.

iii. Show that if $\tau^{\varepsilon-1}\varphi_x > \varphi$ then exporting firms have higher measured productivity (revenue over costs) than domestic firms.

iv. Compute the aggregates in the economy (P, R, Π, Y) as functions of the average productivity among all firms $\bar{z} = \left(\frac{1}{M}\left(M_d\bar{z}_d^{\varepsilon-1} + M_x\bar{z}_x^{\varepsilon-1}\right)\right)^{\frac{1}{\varepsilon-1}}$, where $M = M_d + M_x$ is the total number of varieties (domestic+foreign). $\bar{z}_d = \left(\frac{1}{1-\Gamma(z^\star)}\int_{z^\star}^{\infty} z^{\varepsilon-1}\gamma(z)\,dz\right)^{\frac{1}{\varepsilon-1}}$ is the average productivity among domestic producers and $\bar{z}_x = \left(\frac{1}{1-\Gamma(z_x^\star)}\int_{z_x^\star}^{\infty} z^{\varepsilon-1}\gamma(z)\,dz\right)^{\frac{1}{\varepsilon-1}}$ is the average productivity among exporters.

The free entry condition works the same as before, but now the average profits include the likelihood of having a high enough draw to become an exporter. $\bar{\pi} = \bar{\pi}_d + \frac{1-\Gamma(z_x^\star)}{1-\Gamma(z^\star)}\bar{\pi}_x$, where $\frac{1-\Gamma(z_x^\star)}{1-\Gamma(z^\star)}$ is the conditional probability of being an exporter if the firm operates. The free entry condition is $0 \times \Gamma(z^\star) + (1-\Gamma(z^\star))\,\bar{\pi} = \varphi_e$.

v. Use the definition of the cutoffs $(\pi_d(z^\star) = \varphi; \pi_x(z_x^\star) = \varphi_x)$ to express $z_x^\star$ as a function of $z^\star$ and the fixed costs.

vi. Use these results to show that the zero profit curve with trade lies above the zero profit curve in autarky (the closed economy above). Interpret what that implies for the value of the cutoff productivity $z^\star$ and the consequences of opening to trade.

23 Menu Cost Models (Blanco, Boar, Jones, Midrigan, 2025)

Consider an economy with a continuum of ex-ante identical sectors (indexed by s). Firms (indexed by f within each sector) produce using only labor with a technology subject to decreasing returns to scale. There are two types of shocks:

- Sectoral productivity shocks $e_t(s)$ for each sector s.
- Idiosyncratic product quality/productivity shocks $z_t(f,s)$ for firm f in sector s.

The firm production function is

$$y_t(f,s) = e_t(s)\, z_t(f,s)\, (\ell_t(f,s))^{\eta}; \quad \eta < 1$$

Firms choose their prices and demand labor so as to meet demand (the demand for each firm will be specified below).

Crucially, firms face a fixed cost ξ when they change their prices. This cost is random with an iid distribution across firms and over time. With probability λ a firm faces $\xi = 0$, and with probability $1-\lambda$ the firm faces $\xi \sim U\left(0, \overline{\xi}\right)$. The fixed cost is paid in units of labor.

There is a representative household with log-linear preferences

$$E_0 \sum_{t=0}^{\infty} \beta^t \left(\log c_t - h_t\right)$$

where c is final consumption and h is total work hours. The period budget constraint is

$$M_t + q_t B_t = W_t h_t + \Pi_t + M_{t-1} - P_{t-1} c_{t-1} + B_{t-1} + T_t$$

where M is their money holdings, B is a nominal (zero-coupon) bond, W is the nominal wage, P is the prevailing price index for total consumption, Π is the aggregate profit transfer to the households, and T is the government's transfers (or taxes).

The households also face a cash-in-advanced constraint

$$P_t c_t \leq M_t$$

The government prints money so that M grows at a constant rate g_m.

i. Pose the dynamic programming problem of the representative household.

ii. Show that it is optimal for the household to set labor supply so that $W_t = P_t c_t = M_t$

Final goods are produced by perfectly competitive firms that operate a Cobb-Douglas technology

$$y_t = \exp\left(\int \log y_t(s)\, ds\right)$$

3. Obtain the demand for sector s goods.

4. Derive an expression for the aggregate price, P_t, as a function of sectoral prices, $P_t(s)$.

Sectoral output is produced by perfectly competitive sector aggregators that operate a CES technology

$$y_t(s) = \left(\int \left(\frac{y_t(f,s)}{z_t(f,s)}\right)^{\frac{\sigma-1}{\sigma}} df\right)^{\frac{\sigma}{\sigma-1}}$$

Notice that the quality shock z affects both productivity and demand!

5. Obtain the demand for firm f goods in sector s.

6. Derive an expression for the sectoral price, $P_t(s)$, as a function of firm prices, $P_t(f,s)$.

Now consider the firm's optimal pricing problem that consists on choosing whether to adjust prices

$$E_0 \sum_{t=0}^{\infty} \beta^t \frac{1}{P_t c_t}\left[P_t(f,s)\, y_t(f,s) - W_t\left(\frac{y_t(f,s)}{e_t(s)\, z_t(f,t)}\right)^{\frac{1}{\eta}} - \xi_t(f,s)\, \mathbb{I}_t(f,s)\right]$$

where $\mathbb{I}_t(f,s) = 1$ if the firm adjusts prices.

7. Show that in the flexible prices case (setting $\xi = 0$) prices move 1-to-1 with the idiosyncratic shock z_t leaving quality-adjusted prices $(z_t(f,s)\, P_t(f,s))$ and firm revenue unchanged.

8. Show that the firm problem can be written as choosing directly the effective price, $\tilde{P}_t(f,z) \equiv z_t(f,s)\, P_t(f,s)$, without keeping track of z_t at all.

9. We can keep simplifying the problem. To do this we can define an auxiliary variable that will serve as the state of the firm. This variable is the firm **price gap**, defined as the ratio of the price to what the price would have been under flexible prices.

$$x_t(f,s) \equiv \bar{a}^{\eta} \frac{e_t(s)\, \tilde{P}_t(f,s)}{M_t}$$

where $a_t(s) \equiv \frac{W_t}{P_t(s) y_t(s)} \left(\frac{y_t(s)}{e_t(s)}\right)^{\frac{1}{\eta}}$ is sector s' real marginal cost and $\bar{a}$ is its steady state value. With this we can also define the **sectoral price gap**

$$x_t(s) \equiv \left(\int (x_t(f,s))^{1-\sigma}\, df\right)^{\frac{1}{1-\sigma}} = \bar{a}^{\eta} \frac{e_t(s)\, P_t(s)}{M_t}$$

(a) Show that the sectoral price gap can be expressed in terms of the sectoral marginal cost $a_t(s)$. Show that it is decreasing in $a_t(s)$.

(b) Show that the firm problem can be expressed in terms of price gaps alone, so that the relevant state variables are $x_t(f,s)$ and $x_t(s)$.

10. Show that in the flexible prices case (setting $\xi = 0$) the optimal price gap of a firm depends only on the sectoral price gap $x_t(s)$. This is the reset price gap of the firm (the price chosen when re-adjusting prices). Call it $x_t^\star(s)$
The price gaps show strategic complementarity when $\eta < 1$ in the sense that the price of a given firm f depends on the price of its competitors.

(a) Show that the lower η is or the large σ is the stronger the strategic complementarity.

(b) Explain the logic behind strategic complementarities in prices.

We can now formalize the dynamic problem of the firm. Firms have one individual state and two sectoral states. The individual state is the firm's price gap if there is no price adjustment

$$\hat{x}_t(f,s) \equiv \overline{a}^{\eta} \frac{e_t(s)\,\tilde{P}_{t-1}(f,s)}{M_t}$$

and the sectoral states are $x_t(s)$ and $e_t(s)$.

11. Let $V^a(s)$ be the value upon adjustment of a firm in sector s (recall that all firms have the same adjustment level and do not have to keep track of $z_t(f,s)$). And let $V^n(\hat{x},s)$ be the value of no adjustment of a firm with current price gap of $\hat{x}$ in sector s.
Characterize the pricing rule for the new price gap $x_t(f,s)$. That is what values does it take and with which probabilities.

The sectoral price gap evolves according to

$$x_t(s) = \left(\int \left(\Pr\{Adjust\}\, x_t^\star(s) + (1 - \Pr\{Adjust\})\,\hat{x} \right)^{1-\sigma} dF_t(\hat{x};s) \right)^{\frac{1}{1-\sigma}}$$

where F is the distribution of individual price gaps in the sector.

12. For the case of a single sector, define a recursive competitive equilibrium.

13. Sketch the pseudo-algorithm used to find this equilibrium following the logic of Krusell-Smith.

Price dispersion is costly in this economy. We can see this by aggregating across firms in a sector.

14. Define the total amount of labor demanded by a sector as $\ell_t(s) \equiv \int \ell_t(f,s)\,df$. Show that aggregate sectoral output can be expressed as

$$y_t(s) = \phi_t(s)\, e_t(s)\, (\ell_t(s))^{\eta} \qquad \text{where: } \phi_t(s) = \left(\int \left(\frac{x_t(f,s)}{x_t(s)} \right)^{-\frac{\sigma}{\eta}} df \right)^{-\eta}$$

Explain why the dispersion in prices generates misallocation.