University of Minnesota
Math Refresher
SUMMER 2015

Problem Set 2

1. Show that $\frac{1}{n} \cdot \|x\|_1 \leq \|x\|_2 \leq \sqrt{n} \cdot \|x\|_\infty$, for $x \in \Re^n$.

2. Let $b, c, p \in \Re$ such that $0 < b < 1$, $p > 0$ and $c > 0$. Show that:

   - $b^n \to 0$ (Bartle, exercise 14.H).
   - $nb^n \to 0$.
   - $c^{\frac{1}{n}} \to 1$.
   - $\frac{1}{n^p} \to 0$.
   - $\sqrt[n]{p} \to 1$.

   Hint: see section 14.8 in Bartle's and 3.20 in Rudin's. Use the same strategy.

3. Suppose $\{f_n\}$ is a equicontinuous sequence of functions on a compact set K, and $\{f_n\}$ converges pointwise on K. Prove that $\{f_n\}$ converges uniformly on K. (Exercise 16, section 7, Rudin's).

4. Define f_n on $\Re$ by: $f_n(x) = \frac{nx}{1+(nx)^2}$

   Show that f_n converges pointwise. Is the convergence uniform in $\Re$? (Exercise 17.D, Bartle's).

5. Show Proposition 5.2 in the Handouts.

6. Show Remark 5.1 in the Handouts.

7. Show the following statements:

   - A set $A \in \Re^n$ is open iff it is the countable union of a collection of open balls in $\Re^n$.
   - Show that if a set $A \in \Re^n$ is closed then it is the countable intersection of a collection of open sets in $\Re^n$.

8. Show that the set $I = [0, 1]$ is connected.

9. Classify the following sets (open, closed, connected, compact (without using Heine-Borel) or none). Show your answer.

   - $A = \{(x, y) \in \Re^2 : x^2 + y^2 = 1\}$

- $B = \{(x, y, z) \in \Re^3 : x^2 + y^2 = 1\}$
- $C = \{(x, y, z) \in \Re^3 : x^2 + y^2 + z^2 > 1\}$

10. Let f be a bounded and continuous function from $\Re^n$ to $\Re$ such that $f(x_0) > 0$. Show that f is strictly positive in a neighborhood of x_0.

11. If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty, then $\bigcap K_\alpha$ is nonempty.

12. Use the previous result to show that if $\{K_n\}$ is a sequence of nonempty compact sets such that $K_n \supset K_{n+1}$, for $n = 1, 2, 3...$, then $\bigcap_1^\infty K_n$ is nonempty.

13. Show the Nearest Point Theorem: Let F be a nonempty subset of $\Re^n$ and let $x \in \Re$ be a point outside F. Then there exists at least one point $y \in F$ such that $\|z - x\| \geq \|y - x\|$ for all $z \in F$. *Hint: define the distance from x to F as $d = inf\{\|x - z\| : z \in F\}$ and consider the sets $F_k = \left\{z \in F : \|x - z\| \leq d + \frac{1}{k}\right\}$, then use the previous exercise.*

14. Replace the word "compact" in Theorem 5.2 in the Handouts by "closed" and then by "bounded" and show (with a counterexample) that in those cases the theorem doesn't hold.

15. Show Theorem 1.76 (Inverse Function Theorem) from Sundaram's.

16. Prove that $D \arcsin y = \frac{1}{\sqrt{1-y^2}}$ for $y \in (-1, 1)$.