EC9604: Advanced Macroeconomics[1]

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Contents

[1]These notes are intended to summarize the main concepts, definitions and results covered in the core macroeconomics course of the PhD program at the University of Western Ontario, EC9604. These notes only include selected sections of books or articles relevant to the course used here only in part for reference and teaching purposes. Please let me know of any errors that persist in the document.

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Part I

Random Variables, Probability, and Markov Processes

Consider an experiment that can have several (but finite) outcomes. For example trowing a dice can turn out in getting any number from 1 to 6, or asking someone out can generate an affirmative response, a negative one or perhaps a maybe, or no response at all. A probability function is a function that assigns a value to each possible outcome while satisfying certain rules.

Its clear that when the outcomes are finite, outcomes form a set $S = \{s_1, \ldots, s_n\}$, a probability is a list $(\pi_1, \ldots, \pi_n)$ such that $\Pr(s_i) = \pi_i$:

(a) $\pi_i \geq 0$ for all i.

(b) $\sum \pi_i = 1$.

It is natural to define other outcomes that are formed by unions of the former ones, like getting an even number when trowing the dice (the union of getting a two a four and a six) or getting a positive answer or a maybe when asking someone out. It is clear that the probability of these new outcomes is defined by the sum of probabilities of the original outcomes used to define them.

Formally we could say that for any set $A \subseteq S$ we define $I_A = \{i | s_i \in A\}$ and then a function $\mu : 2^S \to [0, 1]$ as:

$$\mu(A) = \Pr(A) = \sum_{i \in I_A} \pi_i$$

Furthermore we can define the expected value of a real valued function $f : S \to \mathbb{R}$ as $E[f] = \sum \mu(\{s_i\}) f(s_i)$.

This same discussion can be carried out if the possible outcomes are countably infinite, but it is difficult to generalize it otherwise. The objective now is to study which properties does this kind of function satisfy and how it is generalized to deal with cases where outcomes are arbitrary. The key for this is to realize that a probability is a function that maps sets into the interval $[0, 1]$, hence the study of functions that map sets into non-negative numbers will provide the necessary theory, these functions are called measures, for obvious reasons.

The following sections draw on the short exposition of measure theory contained in Chapter 7 of Stokey, Lucas, and Prescott (1989) and complements it with portions of

Kolmogorov and Fomin (2012) (chapters 7 to 10). Both these references are introductory although they present all the relevant results. All the material is also covered in a more advanced manner in Kolmogorov and Fomin (1999).

The aim of the course is not to dwell in the mathematical details of the theory but rather present the most useful results for applications in economic theory, because of this many of the proofs will be omitted only including those that are either instructive of the way the theory is developed. Kolmogorov and Fomin (2012) is a good source for detailed (and easy to understand) proofs.

We end with a discussion of Markov processes, defined following Stokey, Lucas, and Prescott (1989, Ch. 8). These processes are key for the stochastic dynamic problems that we will study in the first part of the course

1. Measure

1.1. Measurable spaces (σ-algebras)

Before we define a measure recall that a measure has for domain a collection of sets. For a measure to have some desirable properties this collection of sets cannot be left unrestricted. It turns out that the appropriate family of sets to be consider is that of σ-algebra.

Definition 1.1. (σ-algebra) Let S be a set and $\mathcal{A} \subseteq 2^S$ a family of its subsets. $\mathcal{A}$ is a σ-algebra if and only if:

(a) $\emptyset, S \in \mathcal{A}$.

(b) $A \in \mathcal{A}$ implies $A^c = S \backslash A \in \mathcal{A}$. We say that $\mathcal{A}$ is closed under complement.

(c) $A_n \in \mathcal{A}$ for $n = 1, \ldots$ implies $\cup A_n \in \mathcal{A}$. We say that $\mathcal{A}$ is closed under countable union.

(i) $\mathcal{A}$ is closed under countable intersection because $\cap A_n = \left(\cup A_n^c\right)^c$.

If $\mathcal{A}$ is only closed under finite union (or intersection) then $\mathcal{A}$ is an algebra.

A σ-algebra imposes certain consistency to the family of sets under consideration. The way to interpret it is that only subsets of the σ-algebra can be known, hence measured. Because of property (i) it is possible to know when none or all of the outcomes occurred. Also if there is an outcome that occurred it must be possible to determine if it didn't. Finally if it is possible to determine that some outcomes occurred individually it can also be determined if at least one or all of them were realized.

It is instructive to consider two simple examples of σ-algebras that arise from throwing a 4 sided dice, then $S = \{1, 2, 3, 4\}$. One (trivial) σ-algebra is:

$$\mathcal{A} = \left\{\emptyset, S\right\}$$

Another one is the σ-algebra generated by the collection $\{\{1\}, \{2\}, \{3\}, \{4\}\}$, then:

$$\mathcal{A} = \left\{ \begin{array}{c} \{1\}, \{2\}, \{3\}, \{4\}, \{2, 3, 4\}, \{1, 3, 4\}, \{1, 2, 4\}, \{1, 2, 3\}, \\ \{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 3\}, \{1, 4\}, \{2, 4\}, \emptyset, S \end{array} \right\}$$

In this case $\mathcal{A} = 2^S$, but this is not necessarily true, imagine that one can only determine if an even number was thrown, then the outcomes are $\{\{1, 3\}, \{2, 4\}\}$, the σ-algebra is:

$$\mathcal{A} = \left\{\{1, 3\}, \{2, 4\}, \emptyset, S\right\}$$

When S has uncountably many elements this process cannot be exemplified as easily but one can always define the σ-algebra generated by a subset $\mathcal{A} \subseteq 2^S$ as the intersection of all σ-algebras that contain $\mathcal{A}$. Clearly the arbitrary intersection of σ-algebras is again a σ-algebra.

Now that we have defined a σ-algebra its possible to say what a measurable set and a measurable space are:

Definition 1.2. (Measurable Space) A pair $(S, \mathcal{A})$ where S is any set and $\mathcal{A}$ is a σ-algebra is called a measurable space. A set $A \in \mathcal{A}$ is called $\mathcal{A}$-measurable.

We say that $A \subseteq S$ is measurable with respect to a σ-algebra $\mathcal{A}$ if its elements are identifiable, that is, if the outcomes represented in A can be told apart from other outcomes given the information in $\mathcal{A}$. For example the set $A = \{4\}$ is not measurable in the last example above, because its impossible to know if a 4 was the outcome of the throw.

A σ-algebra of special importance is the Borel σ-algebra.

Definition 1.3. (Borel σ-algebra) Let $S = \mathbb{R}$ and $\mathcal{A}$ be the set of open and half open intervals. The Borel algebra, noted by $\mathcal{B}$, is the σ-algebra generated by $\mathcal{A}$. A set $B \in \mathcal{B}$ is called a Borel set.

The Borel algebra could have been defined equivalently with the closed and half closed intervals (use complement). In general one can define the Borel algebra for any metric space (S, ρ) as the smallest σ-algebra containing all the open balls. In the case of the Euclidean spaces it can also be generated with open rectangles.

What follows is to define the measure of a measurable set.

1.2. Measures

1.2.1. Measures in σ-algebras

Given a measurable space $(S, \mathcal{A})$ a measure is nothing but a function $\mu : \mathcal{A} \to \overline{\mathbb{R}}$ with certain restrictions that guarantee its consistency:

Definition 1.4. (Measure) Let $(S, \mathcal{A})$ be a measurable space. A measure is an extended real-valued function $\mu : \mathcal{A} \to \overline{\mathbb{R}}$ such that:

(a) $\mu(\emptyset) = 0$

(b) $\mu(A) \geq 0$ for all $A \in \mathcal{A}$.

(c) μ is countably additive. If $\{A_n\}_{n=1}^{\infty}$ is a countable, disjoint sequence in $\mathcal{A}$, then:

$$\mu(\cup A_n) = \sum \mu(A_n)$$

If furthermore $\mu(S) < \infty$ then μ is said to be a finite measure, and if $\mu(S) = 1$ then μ is said to be a probability measure.

Definition 1.5. (Measure Space) A triple $(S, \mathcal{A}, \mu)$ where S is a set, $\mathcal{A}$ is a σ-algebra of its subsets and μ is a measure on $\mathcal{A}$ is called a measure space. The triple is called a probability space if μ is a probability measure.

An important concept is that of almost everywhere and almost surely. These are qualifiers for a given proposition that can be evaluated in sets of $\mathcal{A}$.

Definition 1.6. (Almost Everywhere and Almost Surely) Let $(S, \mathcal{A}, \mu)$ be a measure space. A proposition is said to hold almost everywhere (a.e.) or almost surely (a.s.) if there exists a set $A \in \mathcal{A}$ such that $\mu(A) = 0$ and the proposition holds in A^c.

An example of the use of a.e. or a.s. is when treating functions that are similar to each other. One can say that two functions are equivalent a.e. or that a function is continuous a.e. Then the functions f and g satisfy $f(x) = g(x)$ and $A = \{x | f(x) \neq f(y)\}$ satisfies $\mu(A) = 0$. In measure theory the behavior of functions a.e. is all that matters, then we can treat functions that have anomalies as long as those anomalies occur only in sets of measure zero.

There are some properties of a measure that are useful to keep in mind, a crucial one is used for Bayes law and the definition of conditional probability.

Proposition 1.1. *Let* $(S, \mathcal{A}, \mu)$ *be a measure space and* $B \in \mathcal{A}$ *a set. Define* $\lambda : \mathcal{A} \to \overline{\mathbb{R}}$ *as* $\lambda(A) = \mu(A \cap B)$*. Then* λ *is a measure on* $(S, \mathcal{A})$*. If in addition* $\mu(B) < \infty$ *then* $\tilde{\lambda}$ *defined as* $\tilde{\lambda}(A) = \mu(A \cap B)/\mu(B)$ *is a probability measure on* $(S, \mathcal{A})$*.*

Proof. First, if $A, B \in \mathcal{A}$ then $A \cap B \in \mathcal{A}$, this follows from a σ-algebra being closed under countable intersection, by letting $A_1 = A$ and $A_n = B$ for $n \geq 2$ the result obtains. It is left to check the three properties of a measure:

(a) $\lambda(\emptyset) = \mu(\emptyset \cap B) = \mu(\emptyset) = 0$.

(b) $\lambda(A) = \mu(A \cap B) \geq 0$.

(c) Let $\{A_n\}_{n=1}^{\infty}$ be a countable, disjoint sequence in $\mathcal{A}$, then the sequence $\{A_n \cap B\}_{n=1}^{\infty}$ is also disjoint and that:
$$\lambda(\cup A_n) = \mu((\cup A_n) \cap B) = \mu(\cup (A_n \cap B)) = \sum \mu(A_n \cap B) = \sum \lambda(A_n)$$

(d) If $\mu(B) < \infty$ then all the previous results hold for $\tilde{\lambda}$ by dividing everything by $\mu(B)$. Furthermore $\tilde{\lambda}(S) = \frac{\mu(S \cap B)}{\mu(B)} = \frac{\mu(B)}{\mu(B)} = 1$.

□

Another useful property is given by the following proposition, it reflects the intuitive property of measures being 'increasing':

Proposition 1.2. *Let* $(S, \mathcal{A}, \mu)$ *be a measure space and* $A, B \in \mathcal{A}$ *sets. If* $A \subseteq B$ *then* $\mu(A) \leq \mu(B)$*, if in addition* μ *is finite then* $\mu(B \backslash A) = \mu(B) - \mu(A)$*.*

Proof. Because $A \subseteq B$, there exits $C = B \backslash A = B \cap A^c$ such that $A \cup C = B$ and $A \cap C = \emptyset$. Then
$$\mu(A) + \mu(C) = \mu(B).$$
Moreover, $\mu(A) \leq \mu(B)$ because $\mu(C) \geq 0$. If μ is finite then all elements above are well defined and: $\mu(B \backslash A) = \mu(B) - \mu(A)$.

□

The following property is widely used to establish properties of limits of functions, and of the Lebesgue integral:

Proposition 1.3. *Let* $(S, \mathcal{A}, \mu)$ *be a measure space:*

(a) *If* $\{A_n\}$ *is an increasing sequence in* $\mathcal{A}$*, that is, if* $A_n \subseteq A_{n+1}$ *for all n, then:*
$$\mu(\cup A_n) = \lim \mu(A_n)$$

(b) *If $\{B_n\}$ is an decreasing sequence in $\mathcal{A}$, that is, if $B_n \supseteq B_{n+1}$ for all n, then:*

$$\mu(\cap B_n) = \lim \mu(B_n)$$

Proof. Stokey, Lucas, and Prescott (1989, Sec. 7.2). Satisfying these two properties makes a measure continuous.

□

1.2.2. Measures in algebras and extensions [Optional]

So far we have defined a measure on an σ-algebra, but a σ-algebra is usually a large collection of sets and defining a function on such a set while preserving the consistency required for a measure is not an easy task. An alternative is given by defining measures on algebras, which are smaller and less complicated collections of sets. It can be shown that these measures preserve all the desirable properties of the more complicated spaces, and also allow for an extension to σ-algebras, once the measure is properly constructed.

We start by defining a measure on an algebra.

Definition 1.7. (Measure) Let $(S, \mathcal{A})$ be a measurable space. A measure is an extended real-valued function $\mu : \mathcal{A} \to \overline{\mathbb{R}}$ such that:

(a) $\mu(\emptyset) = 0$

(b) $\mu(A) \geq 0$ for all $A \in \mathcal{A}$.

(c) If $\{A_n\}_{n=1}^{\infty}$ is a countable, disjoint sequence in $\mathcal{A}$, and $\cup A_n \in \mathcal{A}$, then:

$$\mu(\cup A_n) = \sum \mu(A_n)$$

If furthermore $\mu(S) < \infty$ then μ is said to be a finite measure, and if $\mu(S) = 1$ then μ is said to be a probability measure.

Condition (iii) also includes finite union of disjoint sets as a special case.

Definition 1.8. (σ-finite measure) Let S be a set, $\mathcal{A}$ an algebra of its subsets and μ a measure defined on $\mathcal{A}$. If there is a countable sequence of sets in $\mathcal{A}$, $\{A_n\}$, such that $\mu(A_n) \leq \infty$ and $S = \cup A_n$ then μ is σ-finite

It is now possible to extend the notion of this measure to a σ-algebra.

Theorem 1.1. (Caratheodory extension theorem) Let S be a set, $\mathcal{A}$ an algebra of its subsets and μ a measure defined on $\mathcal{A}$. Let $\mathcal{A}^\star$ be the smallest σ-algebra containing $\mathcal{A}$. There exists a measure $\mu^\star$ on $\mathcal{A}^\star$ such that $\mu^\star(A) = \mu(A)$ for all $A \in \mathcal{A}$.

The problem of uniqueness is also solved.

Theorem 1.2. (Hahn extension theorem) Let S be a set, $\mathcal{A}$ an algebra of its subsets, μ a measure defined on $\mathcal{A}$ and $\mathcal{A}^\star$ the minimal σ-algebra of $\mathcal{A}$. If μ is σ-finite then the extension $\mu^\star$ is unique.

To see how these theorems and the extension of a measure are used consider defining a measure on the Borel σ-algebra. It seems logical to define the measure of an interval $A = (a, b)$ as $\mu(A) = b - a$ if $b \geq a$ and $\mu(A) = 0$ otherwise (because the interval would be empty). Yet the Borel σ-algebra contains sets beyond simple intervals, and the countable union of intervals can give rise to weird sets. An answer to this problem is given by defining a measure on the Borel algebra, formed by all types of intervals and their finite unions. Defining a measure on this set seems straightforward:

(a) $\mu(\emptyset) = 0$

(b) $\mu((a, b)) = \mu\left(\left[a, b\right]\right) = \mu\left((a, b]\right) = \mu\left(\left[a, b)\right) = b - a$

(c) $\mu((-\infty, \infty)) = \mu\left((-\infty, b]\right) = \mu([a, \infty)) = \infty$

(d) $\mu(\cup(a_n, b_n)) = \sum(b_n - a_n)$ if the intervals are disjoint.

The function μ can be verified to be a measure on the Borel algebra, and hence an extension to the Borel σ-algebra exists. If we restrict our attention to $S = \left[a, b\right]$ and the intervals contained in it we can define a σ-finite measure, obtaining uniqueness of the extension. This is how we can deal with complicated environments.

Once the measure is extended to the σ-algebra all the results obtained above apply.

1.2.3. Completion of a measure [Optional]

One small detail is left to be checked. Sometimes there is a set $B \subseteq S$ such that $B \subseteq A \in \mathcal{A}$ and $\mu(A) = 0$, but if $B \notin \mathcal{A}$ then its measure is undefined, while it should be clearly zero. The completion of a σ-algebra to include these type of 'harmless' sets is what follows. As before, including sets or behaviors of measure zero is of no consequence.

Definition 1.9. (Completion of a σ-algebra) Let $(S, \mathcal{A}, \mu)$ be a measure space. Define a collection $\mathcal{C}$ as:

$$\mathcal{C} = \left\{C \subset S | \exists_{A \in \mathcal{A}} \mu(A) = 0 \quad \wedge \quad C \subset A\right\}$$

The completion of σ-algebra $\mathcal{A}$ is:

$$\mathcal{A}' = \left\{ B' \subseteq S \mid B' = (A \cup C_1) \setminus C_2 \qquad A \in \mathcal{A} \quad \wedge \quad C_1, C_2 \in \mathcal{C} \right\}$$

By letting $C_1 = C_2 = \emptyset$ we get $\mathcal{A} \subseteq \mathcal{A}'$, $\mathcal{A}'$ includes all sets in 2^S that differ from a set in $\mathcal{A}$ by a set of measure 0.

Definition 1.10. (Completion of a measure) Let $(S, \mathcal{A}, \mu)$ be a measure space and $\mathcal{A}'$ the completion of $\mathcal{A}$. $\mu\left(B'\right) = \mu(B)$ for any $B' \in \mathcal{A}'$ that differs from $B \in \mathcal{A}$ by a set of measure 0.

The Caratheodory and Hahn extension theorems also apply for completions.

2. Measurable functions

A measurable function is a type of function for which it is possible to know (to measure) the conditions (the set) that originates certain outcomes. One can think of a function as mapping certain events in a given measure space to outcomes in another measure space. A function is measurable if the sets that induce a given outcome are measurable. Formally:

Definition 2.1. (Measurable function) Let $(S, \mathcal{A}, \mu)$ and $\left(S', \mathcal{A}', \mu'\right)$ be measure spaces and $f : S \to S'$ a function. f is measurable if and only if $f^{-1}\left(A'\right) \in \mathcal{A}$ for all $A' \in \mathcal{A}'$.

A special case of notable importance is that of $\left(S', \mathcal{A}', \mu'\right) = (\mathbb{R}, \mathcal{B}, \lambda)$, where λ is the Lebesgue measure on the plane. This are real valued functions. In this case the $\mathcal{B}$-measurable sets in $\mathbb{R}$ can be characterized in the following way:

Theorem 2.1. *Let $(S, \mathcal{A}, \mu)$ be a measure space and $f : S \to \mathbb{R}$. f is μ-measurable if and only if $f^{-1}((-\infty, c)) = \{x \in S | f(x) < c\} \in \mathcal{A}$ for all $c \in \mathbb{R}$.*

Proof. This theorem is stated as the definition of a real valued function f being μ-measurable in Stokey, Lucas, and Prescott (1989), but a formal proof is presented in Kolmogorov and Fomin (2012, Sec. 28, Thm. 1). It can also be stated with any of the inequalities $\geq, \leq, >, <$.

□

Also when the measure space in question is a probability space one can characterize formally what a random variable is.

Definition 2.2. (Random variable) Let $(S, \mathcal{A}, P)$ be a probability space and $f : S \to \mathbb{R}$ a real valued function. f is a random variable if and only if f is measurable, that is, if and only if $f^{-1}(B) \in \mathcal{A}$ for all $B \in \mathcal{B}$, where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}$. We further establish the same notation:

(a) An outcome is an element $s \in S$.

(b) An event is a measurable subset of S: $A \in \mathcal{A}$.

(c) The real number $f(s)$ is a realization of the random variable.

(d) The probability measure for f is then: $\mu(B) = P\left(f^{-1}(B)\right) = P\left(\{s \in S | f(s) \in B\}\right)$, for $B \in \mathcal{B}$.

(e) The distribution function for f is: $G(b) = \mu\left((-\infty, b]\right)$, for $b \in \mathbb{R}$.

Generally it is very hard to find a function that is not measurable. The details of the example will depend on the spaces considered. For example if $f : \mathbb{R} \to \mathbb{R}$ and $\mathcal{A}$ is the set of all open (or closed) sets in $\mathbb{R}$ the definition of measurability is equivalent to that of continuity (the pre-image of an open set has to be open) and then all functions that are not continuous are not measurable. It is clear that more complete σ-algebras make more difficult to generate counterexamples. The following three results show how difficult it is to generate them:

Proposition 2.1. *Let $f : \mathbb{R} \to \mathbb{R}$.*

(a) *If f is continuous then f is measurable with respect to the Borel sets.*

(b) *If f is monotone then f is measurable with respect to the Borel sets.*

Proof. Each case is proven:

(a) Let f be continuous. Consider the set $f^{-1}((-\infty, c))$ for any $c \in \mathbb{R}$. The set $(-\infty, c)$ is open, because f is continuous then its pre-image is open, then it is a Borel set. Then its measurable.

(b) Let f be monotone increasing. Consider the set $f^{-1}((-\infty, c))$ for arbitrary $c \in \mathbb{R}$. Then, $f^{-1}((-\infty, c)) = (-\infty, a)$ or $f^{-1}((-\infty, c)) = (-\infty, a]$ or $f^{-1}((-\infty, c)) = (-\infty, \infty)$ or $f^{-1}((-\infty, c)) = \emptyset$ for some $a \in \mathbb{R}$. Monotonicity ensures that if $a \in f^{-1}((-\infty, c))$ and $b \leq a$ then $b \in f^{-1}((-\infty, c))$. Suppose its not, then there exists numbers $b \leq a$ such that $f(b) > c \geq f(a)$, contradicting monotonicity.

All these sets are in $\mathcal{B}$, then f is $\mathcal{B}$-measurable.

□

Corollary 2.1. *The composition of measurable functions is measurable. In particular the composition of a continuous function with a measurable function is measurable.*

Proposition 2.2. *Let $S = \{s_1, s_2, \ldots\}$ be a countable set (potentially infinite) and $\mathcal{A} = 2^S$ a σ-algebra on S. Then all functions $f : S \to \mathbb{R}$ are measurable.*

Proof. The proof is immediate because the pre-image of a Borel set is a subset of S, then it belongs to $\mathcal{A} = 2^S$.

□

In a more general way one can establish the measurability of a function by relating to a class of well behave 'simple' functions. The base for this class is the indicator function.

Definition 2.3. (Indicator Function) Let $(S, \mathcal{A})$ be a measurable space. An indicator function $\chi_A : S \to \mathbb{R}$ is:

$$\chi_A(s) = \begin{cases} 1 & \text{if } s \in A \\ 0 & \text{if } s \notin A \end{cases}$$

Clearly χ_A is measurable if and only if $A \in \mathcal{A}$.

Definition 2.4. (Simple Function) Let $(S, \mathcal{A})$ be a measurable space. A simple function is a function that takes at most countably many values. When the function takes finitely many values it can be expressed as:

$$\phi(s) = \sum_{i=1}^{n} \alpha_i \chi_{A_i}(s)$$

where $\{A_i\}$ is a sequence of subsets of S and $\alpha_i \in \mathbb{R}$.

Characterizing the measurability of simple functions is slightly more complicated.

Proposition 2.3. *A simple function taking values $\{y_1, y_2, \ldots\}$ is measurable if and only if the sets $A_i = \{s \in S | \phi(s) = y_n\}$ are measurable.*

Proof. Both directions are proven.

(a) Let ϕ be measurable, and $\{y_n\} \in \mathcal{B}$, then its pre-image is measurable wrt $\mathcal{A}$.

(b) Let the sets be measurable, that is$A_i \in \mathcal{A}$, and consider $B \in \mathcal{B}$ a Borel set. Then

$$\phi^{-1}(B) = \{s \in S | \phi(s) = y_i \in B\} = \bigcup_{y_i \in B} A_i.$$

Because each $A_i \in \mathcal{A}_i$ and the union is taken over no more than countably many sets we have $\bigcup_{y_i \in B} A_i \in \mathcal{A}$ by definition of a σ-algebra. This proves measurability of $\phi^{-1}(B)$.

□

In what follows all simple functions will be considered measurable. The importance of simple functions is given by the applications of the following proposition.

Proposition 2.4. *Let $(S, \mathcal{A})$ be a measurable space and let $\{f_n\}$ be a sequence of measurable functions converging pointwise to f, that is* $\lim f_n(s) = f(s)$ *for all s. Then f is also measurable.*

Proof. The proof can be found in Stokey, Lucas, and Prescott (1989, Sec. 7.3) or in Kolmogorov and Fomin (2012, Sec. 28.1).

□

Corollary 2.2. *If f is non-negative one can choose the sequence $\{f_n\}$ to be strictly increasing.*

Corollary 2.3. *If f is bounded one can choose the sequence $\{f_n\}$ to converge uniformly.*

The main application is the following result that gives a characterization of measurable functions in terms of simple functions:

Proposition 2.5. *A function f is measurable if and only if it an be represented as the limit of a uniformly converging sequence of measurable simple functions.*

Proof. The first direction is immediate from the previous proposition. If f is the limit of measurable functions then f is also measurable.

Let f be measurable. It is left to construct a converging sequence of simple functions that converges to f. wlog let $f(s) \geq 0$ for all s, then by the Archimedean principle there exists a non-negative integer m such that

$$\frac{m}{n} \leq f(s) < \frac{m+1}{n}$$

Let $f_n(s) = m/n$, because n is fixed and $m \in \mathbb{N} \cup \{0\}$ it follows that f_n can take at most countably many values, hence it is simple. f_n is also measurable because

$$f_n^{-1}((-\infty, c)) = \{s \in S | f_n(s) \leq c\} = \left\{s \in S | f_n(s) \leq \frac{m^\star}{n}\right\} = \left\{s \in S | f_n(s) < \frac{m^\star + 1}{n}\right\}$$

For $m^\star$ chosen by the Archimedean principle. The last set is $f^{-1}\left(\left(-\infty, \frac{m^\star+1}{n}\right)\right)$ which is measurable by assumption. Then f_n is measurable for all n.

Finally, $f_n \to f$ uniformly because

$$|f_n(s) - f(s)| \leq \left|\frac{m}{n} - \frac{m+1}{n}\right| = \frac{1}{n}.$$

□

Other results will follow and are left stated without proof:

Proposition 2.6. *Let f, g be measurable functions and $\alpha \in \mathbb{R}$ then:*

(a) *$f + g$ is measurable.*

(b) *αf is measurable.*

(c) *fg is measurable.*

(d) *$1/f$ is measurable provided that $f(s) \neq 0$.*

Finally continuity of functions is used to strengthen the intuition around measurability.

Proposition 2.7. *Let f, g be equivalent function defined on an interval E, that is they are equal a.e. If f and g are continuous then they coincide.*

Proof. Suppose not, then there exists $x \in E$ such that $f(x) \neq g(x)$. Let $\epsilon = |f(x) - g(x)|$, because f and g are continuous there exists δ such that for $x' \in B_\delta(x)$ it holds that $\left|f(x) - f\left(x'\right)\right| < \frac{\epsilon}{2}$ and $\left|g(x) - g\left(x'\right)\right| < \frac{\epsilon}{2}$. Then for all $x' \in B_\delta(x)$ it holds that $f\left(x'\right) \neq g\left(x'\right)$, but $B_\delta(x)$ has strictly positive measure, contradicting f and g being equivalent.

□

Proposition 2.8. *A function f equivalent to a measurable function g is measurable.*

Proof. Because the functions are equivalent the sets $\{x|f(x) \leq c\}$ and $\{x|g(x) \leq c\}$ can differ in at most by a set of measure zero. Then if the second set is measurable so is the first one (taking into account the completion of the σ-algebra). This proves measurability.

□

Corollary 2.4. *A function f equivalent to a continuous function is measurable.*

Proof. Immediate from continuous functions being measurable.

□

This implies that if a function is continuous a.e. then it is measurable, again the behavior of functions in sets of measure zero carries no consequence. It turns out that this corollary can be strengthened. The result is powerful and is stated without a proof:

Theorem 2.2. ***(Luzin)*** *Let $f : [a, b] \to \mathbb{R}$ be a function. f is measurable if and only if for all $\epsilon > 0$ there exists a continuous function g such that $\mu\{x \in [a, b] \mid f(x) \neq g(x)\} < \epsilon$.*

This theorem shows that for the case of functions of real variable and real value measurability is equivalent to continuity, except on a set of arbitrarily small size. In other words a measurable function can be made continuous by altering its values on a set of arbitrarily small measure.

3. The Lebesgue integral [Optional]

The Lebesgue integral is in at least two important ways a generalization of the Riemann integral and it serves a crucial purpose of defining what it means to take the expected value of a function with respect to a probability distribution. The first sense in which the Riemann integral is generalized is that the Lebesgue integral is defined over measurable functions, a space that is much richer than that of Riemann integrable functions, the second sense is much more crucial: the Lebesgue integral is defined for functions with domain in arbitrary sets, thus allowing to handle a more abstract and general class of functions.

Intuitively the Lebesgue integral is constructed in a similar way than the Riemann integral. To construct the latter one takes successively finer grids of the domain and evaluate the function at certain points, constructing step functions, one above the function and one below, then two sums are constructed and the value of the integral is defined as the (common) value of the limit of those sums as the length of the grid's spaces goes to zero.

The Lebesgue integral of a function $f : S \to \mathbb{R}_+$ is constructed by taking grids over the range of the function $\{y_i\}_{i=1}^n$ such that $0 = y_1 \leq \ldots \leq y_n$. Then one can define the sets $A_i = \{s \in S | y_i \leq f(s) < y_{i+1}\}$ and using the measure over S define $\lambda(A_i)$ and the sum $\sum y_i \lambda(A_i)$. The Lebesgue integral is then the limit of this sum as the values y_i are closer together.

The introduction before of simple functions makes sense when defining the Lebesgue integral. Its definition seems intuitive for this class of functions and Proposition 2.5 creates a bridge between them and the more general class of measurable functions, thus allowing to extend the Lebesgue integral to this broader family.

In what follows we restrict attention to non-negative, real valued functions.

Definition 3.1. (Lebesgue integral for simple functions) Let $(S, \mathcal{A}, \mu)$ be a measure space and $f : S \to \mathbb{R}_+$ a simple, μ-measurable function that takes no more than countably many values $\{y_1, y_2, \ldots\}$. The Lebesgue integral over the set $A \subseteq S$ is defined as:

$$\int_A f(s)\, d\mu = \sum_n y_n \mu(A_n) \tag{3.1}$$

where the sets A_n are defined as:

$$A_n = \{s \in A | f(s) = y_n\}$$

These sets can be empty if there is no element of s in A for which f takes a given value. The Lebesgue integral is defined as long as the series in (3.1) is absolutely convergent. If f takes finitely many values and μ is finite (or a probability measure) this condition is satisfied.

An example is given by the constant function, $f(s) = 1$ for all $s \in S$, then:

$$\int_A f(s)\, d\mu = \int_A d\mu = \mu(A)$$

It can be shown that the lebesgue integral satisfies some natural properties:

Proposition 3.1. *Let f and g be non-negative, measurable, simple and integrable functions on $(S, \mathcal{A}, \mu)$, a measure space, and $c \geq 0$ a constant. Then:*

(a) $\int_A (f+g)(s)\, d\mu = \int_A f(s)\, d\mu + \int_A g(s)\, d\mu$

(b) $\int_A (cf)(s)\, d\mu = c \int_A f(s)\, d\mu$

(c) *If f is bounded $|f(s)| \leq M$ a.e. then f is integrable and $\left|\int_A f(s)\, d\mu\right| \leq M\mu(A)$.*

Proof. Kolmogorov and Fomin (2012, Sec. 29.1).

□

Definition 3.2. (Lebesgue integral - Nonnegative functions) Let $(S, \mathcal{A}, \mu)$ be a measure space. A measurable function $f : S \to \mathbb{R}$ is said to be integrable on a set A if there exists a sequence $\{f_n\}$ of integrable simple functions converging uniformly to f on A. The Lebesgue integral is defined as:

$$\int_A f(s)\, d\mu = \lim \int_A f_n(s)\, d\mu \tag{3.2}$$

This definition precludes the integral from being infinite, as shown in Kolmogorov and Fomin (2012, Sec. 29.1), the limit above exists provided that the functions f_n are integrable (recall that it was asked of the sum in (3.1) to be finite), moreover it is independent of the choice of sequence approximating f, this sequence can be furthermore be chosen to be strictly increasing (Stokey, Lucas, and Prescott 1989). Yet, the concept of the Lebesgue integral can be easily generalized to allow for infinite values, the definition in Stokey, Lucas, and Prescott (1989) allows for this.

What follows is a list of properties of the Lebesgue integral which should be familiar if there is any knowledge of the behavior of Riemann integrals. They are not of particular interest in this course.

Proposition 3.2. *Properties of the Lebesgue integral for non-negative measurable functions:*

(a) $\int_A (f+g)(s)\,d\mu = \int_A f(s)\,d\mu + \int_A g(s)\,d\mu$

(b) $\int_A (cf)(s)\,d\mu = c\int_A f(s)\,d\mu$

(c) *If g is measurable and integrable and f is bounded by g: $|f(s)| \leq g(s)$ a.e., then f is integrable and $\left|\int_A f(s)\,d\mu\right| \leq \int_A g(s)\,d\mu$.*

  (i) *If f is bounded and measurable then it is integrable.*

(d) *If $f \leq g$ a.e. then $\int f(s)\,d\mu \leq \int g(s)\,d\mu$.*

(e) *If $A \subseteq B$ with $A, B \in \mathcal{A}$ then $\int_A f(s)\,d\mu \leq \int_B f(s)\,d\mu$*

(f) *Let $A = \cup A_n$ where $\{A_n\}$ is a finite or countable sequence of disjoint sets. If f is integrable on A then f is integrable on A_n for all n and:*

$$\int_A f(s)\,d\mu = \sum_n \int_{A_n} f(s)\,d\mu$$

when the series on the right is absolutely convergent.

Finally it is noted that a non-negative integrable function induces a measure on a space, the following proposition makes this clear.

Proposition 3.3. *Let f be a non-negative, integrable function, then $\lambda : \mathcal{A} \to \mathbb{R}$ defined as:*

$$\lambda(A) = \int_A f(s)\,d\mu$$

is a measure on $(S, \mathcal{A})$.

Definition 3.3. (Lebesgue integral) Let $(S, \mathcal{A}, \mu)$ be a measure space. A measurable function $f : S \to \mathbb{R}$ is said to be integrable if the following two integrals are finite:

$$\int f^+(s)\,d\mu \qquad \int f^-(s)\,d\mu$$

where:

$$f^+(s) = \begin{cases} f(s) & \text{if } f(s) \geq 0 \\ 0 & \text{if } f(s) < 0 \end{cases} \qquad f^+(s) = \begin{cases} 0 & \text{if } f(s) \geq 0 \\ -f(s) & \text{if } f(s) < 0 \end{cases}$$

The integral of f is defined as:

$$\int f(s)\, d\mu = \int f^{+}(s)\, d\mu - \int f^{-}(s)\, d\mu \tag{3.3}$$

Recall that when $(S, \mathcal{A}, \mu)$ is a probability space the function f is called a random variable, the definitions above are then the definitions of the expected value of a random variable, this expected value exists when f is integrable, we have seen that a sufficient condition for this is to be bounded a.e. and the measure to be finite, this last condition is satisfied immediately by probability measures.

4. The Stieltjes integral [Optional]

The Lebesgue-Stieltjes integral is a type of integral specially useful in probability theory, because of the resemblance between the Stieltjes measures and probability measures. To introduce the concept consider a real valued random variable that takes values on a closed interval $[a, b]$, this is for example the result of coin toss when catalogued as 0 or 1, the underlying probability space is formed by $S = \{H, T\}$, $\mathcal{A} = \left\{\emptyset, S, \{H\}, \{T\}\right\}$ and a probability measure on $\mathcal{A}$, a function $\mu : \mathcal{A} \to [0, 1]$ such that $\mu(\{H\}), \mu(\{T\}) \geq 0$, $\mu(S) = \mu(\{H\}) + \mu(\{T\}) = 1$ and $\mu(\emptyset) = 0$. The random variable is then a function $f : S \to \mathbb{R}$ such that $f(H) = 0$ and $f(T) = 1$. It seems natural to ask what is the probability that $f(s) = 1$, it is of course given by $\mu(T)$, in the same way can ask for the probability that $f(s) \leq c$ for any value c, the function that answers that question is called the cumulative distribution function. In this example we have:

$$F(c) = \Pr(f(s) \leq c) = \begin{cases} 0 & \text{if } c < 0 \\ \mu(H) & \text{if } 0 \leq c < 1 \\ 1 & \text{if } 1 \leq c \end{cases}$$

Because the measure μ is non-negative it is clear that F has to be a non-decreasing function, it is also continuous from the left, moreover it is possible to recover μ from knowledge of F:

$$\mu(H) = F(0) \qquad \mu(T) = 1 - F(0)$$

The Stieltjes measure is a general way of looking at this last step. It treats the problem of inducing a measure from a non-decreasing left continuous function. The application to probability theory is apparent because we deal with the CDF of a random variable, and not directly with its probability measure, as we saw before it is this latter object the one that defines the expected value.

Now we turn to define formally the Stieltjes integral. Let $F : [a, b] \to \mathbb{R}$ be a non-decreasing and left-continuous function. Let $\mathcal{A}$ be an algebra of all subintervals of $[\alpha, \beta)$ (including open, closed and half-open intervals). Define a measure on $\mathcal{A}$ by:

$$\begin{aligned} m(\alpha, \beta) &= F(\beta) - F(\alpha + 0) \\ m[\alpha, \beta] &= F(\beta + 0) - F(\alpha) \\ m(\alpha, \beta] &= F(\beta + 0) - F(\alpha + 0) \\ m[\alpha, \beta) &= F(\beta) - F(\alpha) \end{aligned}$$

Now consider the Lebesgue extension of m, call it μ_F and the σ-algebra of all μ_F-measurable, call it $\mathcal{A}_F$. This set contains all subintervals of $[\alpha, \beta)$ and hence all the Borel sets of $[\alpha, \beta)$.

Definition 4.1. (Stieltjes measure) The measure μ_F described above is called the (Lebesgue-)Stieltjes measure and F its generating function.

This concept is easily extended to the whole real line. Some examples show the generality of this type of measure:

Example 4.1. Let $F(x) = x$, then the Stieltjes measure is nothing but the Lebesgue measure on the real line, that is, the extension of the concept of length of an interval.

Example 4.2. Let F be a jump function with discontinuity points $\{x_1, x_2, \ldots\}$ and corresponding jumps $\{h_1, h_2, \ldots\}$. The measure is of course:

$$m(\{x_n\}) = h_n \qquad m\left(\{x_1, x_2, \ldots\}^c\right) = 0$$

Then every subset of $[\alpha, \beta)$ is μ_F-measurable because their measure depends only on countable points. Any set A has measure given by:

$$\mu_F(A) = \sum_{x_n \in A} h_n$$

This number exists by assumption. A Stieltjes measure generated by a jump function is called a discrete measure. All discrete random variables have CDF that are jump functions.

Example 4.3. Let F be an absolutely continuous non-decreasing function on $[\alpha, \beta)$. Absolutely continuous functions have a finite derivative a.e. let this derivative be $f = F'$. Then the Stieltjes measure μ_F is defined for all Lebesgue measurable sets and:

$$\mu_F(A) = \int_A f(x)\, dx$$

clearly in this case $\mu_F(\{x\}) = 0$ because $\{x\}$ has Lebesgue measure 0.

The result follows from Lebesgue theorem:

Theorem 4.1. ***(Lebesgue)*** *If F is absolutely continuous on $[a, b]$ then the derivative F' is integrable on $[a, b]$ and:*

$$F(\beta) - F(\alpha) = \int_\alpha^\beta F'(x)\, dx$$

Proof. Kolmogorov and Fomin (2012, Sec. 33, Thm. 6).

□

Applying this theorem here we get:

$$m(\alpha, \beta) = m[\alpha, \beta] = m(\alpha, \beta] = m[\alpha, \beta) = \int_{\alpha}^{\beta} f(x)\, dx$$

Because f is non-negative and integrable wrt all Lebesgue-measurable subsets of $[a, b]$ $\left(\mathcal{B}_{[a,b]}\right)$ we know by proposition (3.3) that

$$\mu_F(A) = \int_A f(x)\, dx$$

is a measure on $\left([a, b], \mathcal{B}_{[a,b]}\right)$ that coincides with m, because the extension is unique we get that μ_F is the Stieltjes measure we are looking for.

This type of measure is called absolutely continuous and is related to continuous random variables.

Now we can define the integral with respect to a Stieltjes measure:

Definition 4.2. (Lebesgue-Stieltjes integral) Let μ_F be Stieltjes measure with generating function F, and let g be a μ_F-measurable function, then the integral is defined as:

$$\int_a^b g(x)\, dF(x) = \int_{[a,b]} g(x)\, d\mu_F$$

If μ_F is discrete with $F(x) = \sum_{x_n \leq x} h(x_n)$, then we have:

$$\int_a^b g(x)\, dF(x) = \sum_n g(x_n)\, h_n$$

If μ_F is absolutely continuous then:

$$\int_a^b g(x)\, dF(x) = \int_a^b g(x)\, f(x)\, dx$$

As hinted above in probability Stieltjes measures arise naturally. Let ξ be a random variable and define $F(x) = \Pr(\xi < x)$, then as noted above F is non-decreasing and continuous from the left, moreover $F(-\infty) = 0$ and $F(\infty) = 1$. The Lebesgue-Stieltjes

measure allows us to define the expected value and variance of the random variable as:

$$E[\xi] = \int_{-\infty}^{\infty} x dF(x) \qquad V[\xi] = \int_{-\infty}^{\infty} (x - E[\xi])^2 \, dF(x) \, .$$

These definitions are valid for discrete and continuous random variables.

5. Markov Processes

As seen in Section 6.2 a great deal of problems can be expressed in a recursive setting, and the use of recursive methods can provide solution to problems that would otherwise be impossible to handle. When dealing with random variables the same topic arises, in particular one can think of a sequence made by the realizations of a random variable, because the sequence is ordered one can also think of each element of the sequence being realized sequentially, in this way its natural to consider the case in which one element of the sequence depends on the value of the previous element. More formally, when the distribution of one element of the sequence depends on the realization of the previous element. Markov processes are processes that behave in this way.

The objective is to introduce shocks to a dynamic program, so we start by considering the deterministic dynamic program of Section 6.2, characterized by the Bellman equation:

$$v(x) = \sup_{y \in \Gamma(x)} \left\{ F(x, y) + \beta v(y) \right\} \tag{5.1}$$

The idea is to add a random variable whose realization z will affect the problem, z is a state of the problem and its drawn each period from a distribution characterized by the measure λ. Formally consider $(Z, \mathcal{Z}, \lambda)$ a probability space, then we can define the problem to be::

$$v(x, z) = \sup_{y \in \Gamma(x)} \left\{ F(x, y, z) + \beta \int v\left(y, z'\right) \lambda\left(dz'\right) \right\} \tag{5.2}$$

Recall that $\lambda : \mathcal{Z} \to \mathbb{R}_+$ maps sets of the σ-algebra $\mathcal{Z}$ to real numbers. The problem above can be solved using the results of Sections (1) to (4), but it is not general enough for our purposes because the distribution of z is fixed, and each draw is taken (each period) from the same distribution.

In general we want the distribution of z' to be influenced by the previous draw z, for this we need a special type of function, $Q : Z \times \mathcal{Z} \to \mathbb{R}$, such that for all $z \in Z$ it holds that $Q(z, \cdot)$ is a probability distribution for z'. This is called a transition function and it allows to express the problem as:

$$v(x, z) = \sup_{y \in \Gamma(x)} \left\{ F(x, y, z) + \beta \int v\left(y, z'\right) Q\left(z, dz'\right) \right\} \tag{5.3}$$

The objective is now to characterize transition functions and the properties of the process that they generate.

5.1. Transition functions

Definition 5.1. (Transition Function) Let $(Z, \mathcal{Z})$ be a measurable space. A transition function is a function $Q : Z \times \mathcal{Z} \to [0, 1]$ such that:

(a) For each $z \in Z$ the function $Q(z, \cdot)$ is a probability measure on $(Z, \mathcal{Z})$.

(b) For each $A \in \mathcal{Z}$ the function $Q(\cdot, A)$ is a $\mathcal{Z}$-measurable function.

The interpretation is that for all current value of the random variable the transition function induces a probability measure for next period's value of the variable. Then $Q(a, A)$ is the probability that $z' \in A$ if the current value of the variable is a.

$$Q(a, A) = \Pr\left(z' \in A | z = a\right)$$

Any transition function defines two operators that will be of great importance later.

Definition 5.2. Let Q be a transition function on a measurable space $(Z, \mathcal{Z})$. Define $\mathcal{F}$ as the set of $\mathcal{Z}$-measurable functions and Λ the set of probability measures on $(Z, \mathcal{Z})$.

(a) The Markov operator of Q is an operator T defined on the set of $\mathcal{Z}$-measurable functions:

$$Tf(z) = \int f\left(z'\right) Q\left(z, dz'\right)$$

for all $z \in Z$. T is the expected value of f in the next period if today's realization is z.

(b) The Adjoint operator of Q is $T^\star$ is an operator defined on probability measures on $(Z, \mathcal{Z})$:

$$T^\star \lambda(A) = \int Q(z, A) \lambda(dz)$$

for all $A \in \mathcal{Z}$. T gives the probability that $z' \in A$ if the current value of z is drawn from probability distribution λ.

These operators are important because they will allow to characterize the distribution of a sequence of random variables starting at some initial distribution. This is the objective when solving a stochastic dynamic programming problem. In order for T and $T^\star$ to be useful it is first necessary to check that they are sufficiently well behaved. The following propositions will establish that the operators can be used recursively and their proof will be instructive of how proofs go in measure theory.

Proposition 5.1. *Let $(Z, \mathcal{Z})$ be a measurable space and Q a transition function on that space with Markov operator T. Then $T : M^+(Z, \mathcal{Z}) \to M^+(Z, \mathcal{Z})$ where $M^+(Z, \mathcal{Z})$ is the space of nonnegative, extended variable $\mathcal{Z}$-measurable functions.*

Proof. The proof is done iteratively, first for indicator functions, then it is generalized to simple functions and then to arbitrary nonnegative measurable functions.

First, for any $f \in M^+$ we have that Tf is a nonnegative function of extended real value, this follows immediately, then it is left to check that Tf is also measurable.

Case 1. Let $A \in \mathcal{Z}$ and $f = \chi_A$, where χ_A is by construction measurable and nonnegative. Then:

$$Tf(z) = \int f\left(z'\right) Q\left(z, dz'\right) = \int \chi_A\left(z'\right) Q\left(z, dz'\right) = \int_A Q\left(z, dz'\right) = Q(z, A)$$

Because $Q(\cdot, A)$ is measurable (as a function of z for fixed A) by definition we establish measurability of Tf.

Case 2. Let f be a simple function then there exists (finitely many) indicator functions such that: $f(z) = \sum_{i=1}^{n} a_i \chi_{A_i}(z)$. Then the Markov operator gives:

$$\begin{aligned} Tf(z) &= \int f\left(z'\right) Q\left(z, dz'\right) \\ &= \int \sum_{i=1}^{n} a_i \chi_{A_i}\left(z'\right) Q\left(z, dz'\right) \\ &= \sum_{i=1}^{n} a_i \int_A \chi_{A_i}\left(z'\right) Q\left(z, dz'\right) \\ &= \sum_{i=1}^{n} a_i T\chi_{A_i} \end{aligned}$$

By the previous case each $T\chi_{A_i}$ is measurable. Then, Tf is also measurable because the sum and scalar product of measurable functions is also measurable .

Case 3. Let f be an arbitrary nonnegative, extended real value, measurable function. By proposition 2.5 we know that f can be expressed as the limit of point-wise convergent sequence of simple functions because it is measurable and nonnegative. So for all z we have:

$$\begin{aligned} Tf(z) &= \int f\left(z'\right) Q\left(z, dz'\right) \\ &= \int \lim \phi_n\left(z'\right) Q\left(z, dz'\right) \\ &= \lim \int \phi_n\left(z'\right) Q\left(z, dz'\right) \\ &= \lim T\phi_n(z) \end{aligned}$$

where the third step of interchanging the limit and the integral follows from Lebesgue's Monotone Convergence theorem.[2] Finally the pointwise limit of measurable functions is a measurable function (Proposition 2.4). The result follows because $T\phi_n$ is measurable by the previous case.

□

Corollary 5.1. *Let $(Z, \mathcal{Z})$ be a measurable space and Q a transition function on that space with Markov operator T. Then $T : B(Z, \mathcal{Z}) \to B(Z, \mathcal{Z})$ where $B(Z, \mathcal{Z})$ is the space of bounded $\mathcal{Z}$-measurable functions.*

Proof. Let f be a bounded measurable function, then if $0 \leq f \leq m$ it holds that $0 \leq Tf \leq m$, because $Q(z, \cdot)$ is a probability measure. Then Tf is bounded. Measurability follows from the proposition above by applying it to $f = f^+ - f^-$.

□

This allows us to apply iteratively the operator to a function because if $f \in B(Z, \mathcal{Z})$ then $Tf \in B(Z, \mathcal{Z})$, which allows to evaluate $T(Tf)$, and so on. It will also be important to apply the adjoint operator iteratively to a probability measure. The following proposition will enable us to do so.

Proposition 5.2. *Let $(Z, \mathcal{Z})$ be a measurable space and Q a transition function on that space with Adjoint operator $T^\star$. Then $T^\star : \Lambda(Z, \mathcal{Z}) \to \Lambda(Z, \mathcal{Z})$ where $\Lambda(Z, \mathcal{Z})$ is the space of probability measures on $(Z, \mathcal{Z})$.*

Proof. Let $\lambda \in \Lambda(Z, \mathcal{Z})$ and consider $T^\star\lambda(A) = \int Q(z, A)\, d\lambda(dz)$.

(a) $T^\star\lambda \geq 0$ for all (z, A) because $Q(z, A) \geq 0$.

(b) $T^\star\lambda(\emptyset) = \int Q(z, \emptyset)\lambda(dz) = \int 0\lambda(dz) = 0$, because $Q(z, \cdot)$ is a probability measure.

(c) $T^\star\lambda(Z) = \int Q(z, Z)\lambda(dz) = \int 1\lambda(dz) = 1$, because $Q(z, \cdot)$ is a probability measure.

(d) It is left to show that $T^\star\lambda$ is countably additive. Let $\{A_i\} \subseteq \mathcal{Z}$ be a sequence of disjoint sets and $A = \cup A_i$, then:

$$\sum_{i=1}^{\infty} T^\star\lambda(A_i) = \sum_{i=1}^{\infty} \int Q(z, A_i)\lambda(dz_i) = \int \left(\sum_{i=1}^{\infty} Q(z, A_i) \right) \lambda(dz_i) = \int Q(z, A)\lambda(dz_i) = T^\star\lambda(A)$$

where $\sum Q(z, A_i) = Q(z, A)$ follows from Q being a σ-additive measure and interchange of the sum and the integral can be done because of the Lebesgue's monotone convergence theorem.

[2] The theorem states that if $\{f_n\}$ is a monotone increasing sequence of nonnegative measurable functions then that converges pointwise to f then $\int f d\mu = \lim \int f_n d\mu$. Recall from proposition 2.5 that the sequence $\{\phi_n\}$ of simple functions can be chosen to be monotone increasing.

□

The following result establishes a duality between the Markov operator and its adjoint, in words it says that the expected value of a function tomorrow can be computed with either operator.

Proposition 5.3. *Let $(Z, \mathcal{Z})$ be a measurable space and Q a transition function on that space. Then for any function $f \in B(Z, \mathcal{Z})$ (or more generally $f \in M^{+}(Z, \mathcal{Z})$) it holds that:*

$$\int (Tf(z))\,\lambda(dz) = \int f\left(z'\right) T^{\star}\lambda\left(dz'\right) = \int\int f\left(z'\right) Q\left(z, dz'\right)\lambda(dz)$$

Then to obtain the expected value of function f tomorrow given a distribution λ of z today the order of integration does not matter.

Proof. Stokey, Lucas, and Prescott (1989, Sec. 8.1).

□

We can now define a sequence of probability measures over the sequence of random variables by iterating over Q with the Markov operator.

$$\begin{aligned} Q^{1}(z, A) &= Q(z, A) \\ &\vdots \\ Q^{n+1}(z, A) &= \left(TQ^{n}(\cdot, A)\right)(z) = \int Q^{n}\left(z', A\right) Q\left(z, dz'\right) \end{aligned}$$

Then if a shock is drawn sequentially from Q the function $Q^{n}(z, A)$ will give the probability of going from initial point z to a value in set A in exactly n periods. Its easy to show that each Q^{n} is a transition function.

Finally its clear that starting from an initial probability (λ) the Adjoint operator can be used to define a sequence of probability measures $\left\{\lambda^{n}\right\}$ as $\lambda^{n} = T^{\star}\lambda^{n-1}$, we interpret λ as the distribution of the state z in the initial period and λ_{n} the (unconditional) distribution of z in the n^{th} period.

As a side note a transition function can have stronger properties that are of great use in stochastic dynamic programming:

Definition 5.3. (Feller Property) A transition function Q has the feller property if its Markov operator maps the set of continuous bounded function into itself. $T : C(Z) \to C(Z)$.

Definition 5.4. (Monotone transition functions) A transition function Q is said to be monotone if its Markov operator maps nondecreasing functions to nondecreasing functions.

5.2. Probability measures on spaces of sequences

The idea now is to study sequences of random variables and their probability distributions, this can be done using the transition function defined above.

The first task at hand is to define a probability distribution on a finite sequence of variables. For this let $(Z, \mathcal{Z})$ be a measurable space and for $t < \infty$ let $\left(Z^t, \mathcal{Z}^t\right) = (Z \times \ldots \times Z, \mathcal{Z} \times \ldots \times \mathcal{Z})$ be a product space. Now let Q be a transition function on $(Z, \mathcal{Z})$. A probability measure on the sequence given z_0, the initial value of the variable is:

Definition 5.5. (Probability measure on finite sequence) $\mu^t : Z \times \mathcal{Z}^t \to [0, 1]$ is the probability distribution for the finite sequence and its defined as:

$$\mu^t(z_0, B) = \int_{A_1} \ldots \int_{A_t} Q(z_{t-1}, dz_t) \cdot Q(z_{t-2}, dz_{t-1}) \cdots Q(z_0, dz_1)$$

where $B = A_1 \times \ldots \times A_t \in \mathcal{Z}^t$ is a rectangle in $\mathcal{Z}^t$. It can be shown that it is sufficient to define μ^t only for this type of set, because it can then be extended uniquely to measurable sets on $\mathcal{Z}^t$ by the Caratheodory and Hahn extension theorems.

The next task is to handle infinite sequences of realizations of z. To do this we need to be able to induce a σ-algebra $\mathcal{Z}^\infty$ on the set of infinite sequences and then a probability measure on that σ-algebra.

To do this define the set of finite-measurable rectangles. These sets establish outcomes for the variables for the first T periods, leaving unspecified what happens to the sequence afterwards.

Definition 5.6. (Finite-Measurable Rectangles) B is a finite measurable rectangle if its of the form:

$$B = A_1 \times \ldots \times A_T \times Z \times Z \times \ldots$$

for some finite T. Let $\mathcal{C}$ be the set of all finite measurable rectangles. Let $\mathcal{A}^\infty$ be the set of all finite unions of set in $\mathcal{C}$.

It can be shown that $\mathcal{A}^\infty$ is an algebra, then one can define $\mathcal{Z}^\infty$ to be the σ-algebra induced by $\mathcal{A}^\infty$. Then one can define a measure on finite-measurable rectangles $\mathcal{C}$ just as before, extend it to the algebra $\mathcal{A}^\infty$, and the extend the extension to $\mathcal{Z}^\infty$. This proves the existence of a measure for infinite sequences that coincides with our notion of measure for finite-measurable rectangles.

Now we can define what a stochastic process is:

Definition 5.7. (Stochastic Process) Let $(\Omega, \mathcal{F}, P)$ be a probability space. A stochastic process on $(\Omega, \mathcal{F}, P)$ is an increasing sequence of σ-algebras $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \ldots \subseteq \mathcal{F}$, a measurable space $(Z, \mathcal{Z})$and a sequence of functions $\sigma_t : \Omega \to Z$ such that each σ_t is $\mathcal{F}_t$-measurable.

Definition 5.8. (Sample Path) Let $\omega \in \Omega$, then $(\sigma_1(\omega), \sigma_2(\omega), \ldots)$ is called the sample path of the stochastic process given ω.

Each σ_t is just a random variable that takes a value on Z given some event on Ω. In almost all cases we will have $(Z, \mathcal{Z}) = (\mathbb{R}, \mathcal{B})$. The selection of the probability space $(\Omega, \mathcal{F}, P)$ is also standard. Because we are interested in the behavior of infinite sequences of the realizations of the random variable we can set $(\Omega, \mathcal{F}, P) = (Z^\infty, \mathcal{Z}^\infty, \mu(z_0, \cdot))$. The restriction that the σ-algebras are increasing follows from the draws being taken sequentially, this σ-algebras will be interpreted as possible histories, and any future history must include all of the possible previous histories from which it could have followed.

Given a stochastic process we can use probability measure P to induce measures on finite sets of sample paths.

Definition 5.9. (Probabilities on Paths) Let $C \in \mathcal{Z}^n$ we can define:

$$P_{t+1,\ldots,t+n}(C) = P\left(\left\{\omega \in \Omega \mid (\sigma_{t+1}(\omega), \ldots \sigma_{t+n}(\omega)) \in C\right\}\right)$$

This is the probability that an event occurs and the sample path lies in C between periods $t+1$ and $t+n$.

Definition 5.10. (Stationary Stochastic Process) A stochastic process is said to be stationary if $P_{t+1,\ldots,t+n}(C)$ is independent of t for all n and C. That is, if it does not matter the point in time where we start the sequence.

Definition 5.11. (Conditional probability) Let $P_{t+1,\ldots,t+n}(C|a_{t-s}, \ldots, a_{t-1}, a_t)$ be the conditional probability of the event $\left\{\omega \in \Omega \mid (\sigma_{t+1}(\omega), \ldots \sigma_{t+n}(\omega)) \in C\right\}$ given that the event $\left\{\omega \in \Omega \mid \sigma_\tau(\omega) = a_\tau\right\}$ happened.

Now we can define what a Markov process is:

Definition 5.12. (Markov Process) A stochastic process is a Markov process if:

$$P_{t+1,\ldots,t+n}(C|a_{t-s}, \ldots, a_t) = P_{t+1,\ldots t+n}(C|a_t)$$

for $t = 1, 2, \ldots$, $n = 1, 2, \ldots$, $s = 1, 2, \ldots, t-1$ and $C \in \mathcal{Z}^n$.

The distribution of the path of a Markov process only depends on the last realization.

A general setting is easy to construct using a transition function Q. Let $(\Omega, \mathcal{F}, P) = (Z^\infty, \mathcal{Z}^\infty, \mu(z_0, \cdot))$ and for each T define $\mathcal{A}^T$ as the collection of all finite-measurable sets:

$$B = A_1 \times \ldots \times A_T \times Z \times Z \times \ldots$$

As before this forms an algebra, let $\mathcal{F}^T$ be the σ-algebra generated by $\mathcal{A}^T$. Clearly $\mathcal{F}^t \subseteq \mathcal{F}^{t+1}$. Then we can define the sequence of functions $\tilde{z}_t : \Omega \to Z$ as:

$$\tilde{z}_t(\omega) = \tilde{z}_t(a_1, a_2, \ldots) = a_t$$

so that it selects the t^{th} realization of the sequence ω. These functions are clearly $\mathcal{F}^t$ measurable, because they don't contain information about future realizations of the variable.

The definition of P through Q can be used to verify that this process is a Markov process. Moreover it holds that:

$$P_{t+1}(C|a_{t-s}, \ldots, a_t) = P_{t+1}(C|a_t) = Q(a_t, C)$$

for $C \in \mathcal{Z}$.

5.3. Markov chains

We now zoom into a special type of Markov process that is particularly useful in applications of dynamic programming. A Markov chain (or finite state Markov chain):

Definition 5.13. (Markov Chain) A Markov chain is a Markov process defined on a space $Z = \{z_1, \ldots, z_l\}$ with finite dimension (finitely many elements).

The relevance of Markov chains resides in two observations. First, they allow for a simple characterization of their transition function, as shown below. Second, most computational methods (and thus applications) of dynamic programming discretize the state space, effectively imposing that the space Z is finite.

Before characterizing the transition function of a Markov chain it is useful to recall that the natural σ-algebra over Z is $\mathcal{Z} = 2^Z$ (the power set), and that the space of probabilities distributions over Z is formed by vectors $p \in \mathbb{R}^l$ such that $p_i \geq 0$ and $\sum_{i=1}^{l} p_i = 1$[3]. The

[3]Formally $p \in \Delta^l$, where $\Delta^l = \left\{ p \in \mathbb{R}^l_+ | \sum_{i=1}^{l} p_i = 1 \right\}$ is the $l-1$ dimensional simplex. This same set is particularly useful in characterizing price systems in finite dimensional exchange economies.

transition function of the Markov process can be then characterized by a matrix:

Definition 5.14. (Markov Matrix / Stochastic Matrix) A square matrix $\Pi = \left[\pi_{ij}\right]$ of dimensions $l \times l$ is considered a Markov (or stochastic) matrix if $\pi_{ij} \geq 0$ for all i and j, and $\sum_{j=1}^{l} \pi_{ij} = 1$ for all i. Equivalently, if its rows are probability distributions on Z: $\pi_i \in \Delta^l$.

The transition function of a Markov chain is a function $Q : Z \times \mathcal{Z} \to \mathbb{R}_+$ that gives the probability of a given set $A \in \mathcal{Z}$ given a current state z_i. We can then construct a Markov matrix by setting:

$$\pi_{ij} = Q\left(z_i, \left\{z_j\right\}\right) = \Pr\left(z_{t+1} = z_j \,|z_t = z_i\right)$$

So π_{ij} is interpreted as the probability that $z_{t+1} = z_j$ conditional on $z_t = z_i$. The row $\pi_i = \left(\pi_{i1}, \ldots, \pi_{il}\right)$ is the conditional probability of z_{t+1}, given that $z_t = z_i$.

We can also go the other way, constructing a transition function Q from a Markov matrix Π. Let $A \in \mathcal{Z}$, because Z is finite we can define $A = \left\{z_{a_1}, \ldots, z_{a_n}\right\}$ where $a_1, \ldots, a_n$ are $n \leq l$ indices. This gives:

$$Q\left(z_i, A\right) = \sum_{j=1}^{n} \pi_{ia_j}$$

We can now define the Markov operator and the adjoint Markov operator of a Markov chain making use of the Markov matrix (Markov!).

(a) Recall that The Markov operator of Q is an operator T defined on the set of $\mathcal{Z}$-measurable functions:

$$Tf(z) = \int f\left(z'\right) Q\left(z, dz'\right)$$

for all $z \in Z$. T is the expected value of f in the next period if today's realization is z. For Markov chains the function f can be reduced to a row-vector $\vec{f} = \left(f(z_1), \ldots, f\left(z_l\right)\right) \in \mathbb{R}^l$, which reduces the integral to:

$$Tf\left(z_i\right) = \vec{f}\pi_i'$$

more generally we have:

$$Tf = \vec{f}\Pi'$$

the i^{th} element of Tf (which is an l-dimensional vector) corresponds to: $E\left[f\left(z_{t+1}\right) | z_t = z_i\right]$.

(b) Recall that the adjoint operator of Q is $T^\star$ is an operator defined on probability measures

on $(Z, \mathcal{Z})$:

$$T^{\star}\lambda(A) = \int Q(z, A)\,\lambda(dz)$$

for all $A \in \mathcal{Z}$. T gives the probability that $z^{'} \in A$ if the current value of z is drawn from probability distribution λ. Because the space is finite we can represent probabilities distributions as vectors in Δ^{l}. Let $p \in \Delta^{l}$ be an initial distribution on Z, we want to know the distribution on Z for the next period ($\hat{p}$):

$$T^{\star}\,p = \hat{p} = p\Pi \qquad \text{where: } \hat{p}_j = \sum_{i=1}^{l} p_i \pi_{ij}$$

The j^{th} element of $T^{\star}\,p$ (which is an l-dimensional row-vector) corresponds to the unconditional probability that $z_{t+1} = z_j$: $\Pr\left(z_{t+1} = z_j\right)$.

It shouldn't be a surprise that the Markov operator is characterized by $\Pi^{'}$ and the adjoint operator by its transpose Π.[4]

As with general Markov processes there is a special interest in the limit behavior of the adjoint operator $\left(\lim_{n\to\infty} \Pi^{n} p\right)$, in particular the existence and properties of an invariant distribution, that is $p^{\star}$ such that $p^{\star} = p^{\star}\Pi$ (generally $\lambda^{\star} = T\lambda^{\star}$). The problem of finding an invariant distribution is frequently cast as an eigenvector problem. $p^{\star}$ is the eigenvector associated with any unit-eigenvalue of Π.

Another property that will be of interest is the presence of Ergodic sets. These are subsets of the space $E \subseteq Z$ that the process never leaves once it takes a value in them. Formally:

Definition 5.15. (Ergodic Set) A set $E \subseteq Z$ is ergodic if and only if $Q\left(z_i, E\right) = 1$ for all $z_i \in E$ and there does not exist a proper subset $E^{'} \subset E$ that is ergodic.

The ergodic sets are important because they tell us sections of the state space that are of interest. Only ergodic sets have positive mass in the invariant distribution.

Following SLP we now show 5 examples of the possible limit behavior of Markov chains. After the examples we state the main results on the existence and uniqueness of ergodic sets, invariant distributions, and the convergence of the sequences $\left\{\frac{1}{n}\sum_{k=0}^{n}\Pi^{k}\right\}$ and $\left\{\Pi^{n}\right\}$. (clearly if the second sequence converges so does the first one).

[4] If vectors are assumed to be columns instead of rows then $Tf = \Pi f$ and $T^{\star}\,p = \Pi^{'}\,p$. The adjoint is characterized as the transpose of the Markov operator in any case.

Example 5.1. (Uniqueness of ergodic set, convergence of $\{\Pi^n\}$) Let $l = 2$ and consider the Markov matrix:

$$\Pi = \begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix}$$

Clearly the only ergodic set is Z because one has positive probability of going to z_1 or z_2 starting in any state. Moreover:

$$\lim_{n\to\infty} \Pi^n = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

The invariant distribution is then $p^\star = (1/2, 1/2)$. Moreover $\lim \left(p_0 \Pi^n \right) = p^\star$ for all $p_0 \in \Delta^2$.

Note: Convergence is easily defined in this setup because the limit is taken element wise.

Example 5.2. (Uniqueness of a ergodic set, convergence of $\{\Pi^n\}$) Let $l = 3$ and $\gamma \in (0, 1)$. Consider the Markov matrix:

$$\Pi = \begin{bmatrix} 1-\gamma & \gamma/2 & \gamma/2 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

There is a unique ergodic set $E = \{z_2, z_3\} \neq Z$. The state z_1 is never reached again once you leave it. One can also show:

$$\Pi^n = \begin{bmatrix} (1-\gamma)^n & (1-(1-\gamma)^n)/2 & (1-(1-\gamma)^n)/2 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

clearly $\{\Pi^n\}$ converges:

$$\lim_{n\to\infty} \Pi^n = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

and the invariant distribution is $p^\star = (0, 1/2, 1/2)$.

Example 5.3. (Cyclical sets , convergence of $\left\{ \frac{1}{n} \sum_{k=0}^{n} \Pi^k \right\}$) Consider an l-dimensional Markov chain and order its states into two subsets, the first one with k elements and the

second one with $l - k$ elements. Suppose the Markov matrix has the form:

$$\Pi = \begin{bmatrix} 0 & \Pi_1 \\ \Pi_2 & 0 \end{bmatrix}$$

where the first matrix Π_1 is of dimension $k \times (l - k)$ and matrix Π_2 of dimension $(l - k) \times k$. Clearly if at one period one is in the first subset the next period one will be in the second subset, and vice-versa. So there are no proper subsets that form an ergodic subset, instead the process cycles from the first subset to the second period by period.

$$\Pi^{2n} = \begin{bmatrix} (\Pi_1\Pi_2)^n & 0 \\ 0 & (\Pi_2\Pi_1)^n \end{bmatrix} \qquad \Pi^{2n+1} = \begin{bmatrix} 0 & (\Pi_1\Pi_2)^n \Pi_1 \\ (\Pi_2\Pi_1)^n \Pi_2 & 0 \end{bmatrix}$$

In this example the sequence $\left\{\Pi^n\right\}$ does not converge but its odd and even elements do, then the sequence $\left\{\frac{1}{n}\sum_{k=0}^{n}\Pi^k\right\}$ does converge.

For example if $l = 4$, $k = 2$ and $\Pi_1 = \Pi_2$:

$$\lim_{n\to\infty} \Pi^{2n} = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix} \qquad \lim_{n\to\infty} \Pi^{2n+1} = \begin{bmatrix} 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{bmatrix}$$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n}\Pi^k = \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}$$

An invariant distribution is found as one of the rows of the last limit: $p^\star = (1/4, 1/4, 1/4, 1/4)$.

Example 5.4. (Two ergodic sets, Infinitely many invariant distributions) Consider an l-dimensional Markov chain and order its states into two subsets, the first one with k elements and the second one with $l - k$ elements. Suppose the Markov matrix has the form:

$$\Pi = \begin{bmatrix} \Pi_1 & 0 \\ 0 & \Pi_2 \end{bmatrix}$$

where the first matrix Π_1 is of dimension $k \times (l - k)$ and matrix Π_2 of dimension $(l - k) \times k$.

Clearly once the process enters the first subset it never leaves it. The same goes for the second subset. Then they are both ergodic. Also $\Pi^n = \begin{bmatrix} \Pi_1^n & 0 \\ 0 & \Pi_2^n \end{bmatrix}$, this sequence converges if and only if $\{\Pi_1^n\}$ and $\{\Pi_2^n\}$ converge. Let $l = 4$, $k = 2$ and $\Pi_1 = \Pi_2 = \begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix}$, then:

$$\lim_{n\to\infty} \Pi^n = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}$$

There are two invariant distributions: $p_1^\star = (1/2, 1/2, 0, 0)$ and $p_2^\star = (0, 0, 1/2, 1/2)$. But any convex combination of them is also an invariant distribution.

Example 5.5. (Two ergodic sets, Infinitely many invariant distributions) Let $l = 3$, $\gamma \in (0, 2)$ and $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1$. Consider the Markov matrix:

$$\Pi = \begin{bmatrix} 1-\gamma & \gamma\alpha & \gamma\beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

As in the second example s_1 is a transient state (once you leave it you never come back), but there are now two ergodic sets $\{s_2\}$ and $\{s_3\}$. We also have:

$$\lim_{n\to\infty} \Pi^n = \lim_{n\to\infty} \begin{bmatrix} (1-\gamma)^n & \left(1-(1-\gamma)^n\right)\alpha & \left(1-(1-\gamma)^n\right)\beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \alpha & \beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The sequence $\{\Pi^n\}$ converges and there are two invariant distributions $p_1^\star = (0, 1, 0)$ and $p_2^\star = (0, 0, 1)$ given by the second and third rows of the limiting matrix. The first row is a convex combination of the limiting distribution.

Now we turn to the general results. The following theorem encompasses all the possible outcomes of a Markov chain. In particular, an ergodic set and a limit distribution always exist, but they need not be unique, and although the sequence $\{\Pi^n\}$ need not converge, the sequence $\left\{\frac{1}{n}\sum_{k=0}^{n}\Pi^k\right\}$ always converges, and its limit gives away the invariant distributions.

Theorem 5.1. *Let* $Z = (z_1, \ldots, z_l)$ *and denote the stochastic matrix by its elements:* $\Pi = \left[\pi_{ij}\right]$. *The powers of* Π *are also denoted by its elements* $\Pi^n = \left[\pi_{ij}^{(n)}\right]$.

(a) *Z can be partitioned into* $M \geq 1$ *ergodic sets and a transient set (an ergodic set always exists).*

(b) *The sequence* $\left\{\frac{1}{n}\sum_{k=0}^{n}\Pi^k\right\}$ *always converges to a Markov matrix* $\overline{\Pi}$.

(i) *For any* $p_0 \in \Delta^l$ *and* $p_k = p_0\Pi^k$ *it holds that:* $\frac{1}{n}\sum_{k=0}^{n} p_k \to p_0\overline{\Pi}$.

(c) *Each row of* $\overline{\Pi}$ *is an invariant distribution, and every invariant distribution is a convex combination of the rows of* $\overline{\Pi}$ *(so* $p_0\overline{\Pi}$ *is an invariant distribution for all* $p_0 \in \Delta^l$*).*

We can strengthen these results by imposing extra structure on Π. We can get uniqueness of the ergodic set and the invariant distribution under a "reachability" condition (at least one state should be reachable in finite time starting from anywhere).

Theorem 5.2. *Let Z and* Π *as in Theorem 5.1.* Π *has a unique ergodic set if and only if there exists a state* z_j *such that for all* $i \in (1, \ldots, l)$ *there exist* $n \geq 1$ *such that* $\pi_{ij}^{(n)} > 0$.

Moreover, if this is the case Π *has a unique invariant distribution* $p^\star$ *and all rows of* $\overline{\Pi}$ *are equal to* $p^\star$ *(so for any* $p_0 \in \Delta^l$ *we have* $p_0\overline{\Pi} = p^\star$*).*

The previous result still does not rule out cyclicality in the ergodic set. We can get this under a "mixing" condition.

Theorem 5.3. *Let Z and* Π *as in Theorem 5.1. For* $n = 1, 2, \ldots$ *and* $j = 1, \ldots, l$ *define* $\epsilon_j^{(n)} = \min_i \pi_{ij}^{(n)}$ *and* $\epsilon^{(n)} = \sum_{j=1}^{l} \epsilon_j^{(n)}$. *Z has a unique ergodic set without cyclically moving subsets if and only if for some* $N \geq 1$ *it holds that* $\epsilon^{(N)} > 0$.

Moreover, if this is the case Π *has a unique invariant distribution* $p^\star$ *and the sequence* $\{\Pi^n\}$ *converges (so for any* $p_0 \in \Delta^l$ *we have* $\lim p_0\Pi^n = p^\star$*).*

In this notation, $\pi_{ij}^{(n)}$ is the probability that state j is reached from state i in n steps. $\epsilon_j^{(n)}$ measures the lowest such probability, because we don't know from which state we start from we need to know that the condition is satisfied for all states. This mixing property is stronger than the first one because we need there to be at least one column of non-zero elements, guaranteeing mixing towards one state (j) starting from any state (i), uniformly in time (the same time (N) for all the initial states).

5.4. Weak convergence of monotone Markov processes

In the previous section we obtained results characterizing the limiting behavior of Markov processes when the state space is finite. Unfortunately these results do not extend immediately to general Markov processes. The problem at hand is to establish when the sequence of distribution functions $\{\lambda_n\}$, constructed as $\lambda_n = T^\star \lambda_{n-1}$ with λ_0 given, converges. We must first define what it means for a sequence of distributions to converge. The simplest definition comes in the form of set-wise convergence (the equivalent of point-wise convergence for functions):

Definition 5.16. (Set-wise Convergence) Let $(Z, \mathcal{Z})$ be a measurable space and $\Lambda(Z, \mathcal{Z})$ the set of probability distributions. Consider a sequence $\{\lambda_n\} \subseteq \Lambda(Z, \mathcal{Z})$, we say that $\lambda_n \overset{s.w}{\to} \lambda \in \Lambda(Z, \mathcal{Z})$ if $\lambda_n(A) \to \lambda(A)$ for all $A \in \mathcal{Z}$.

This notion of convergence is intuitive but it turns out to be too strong for most applications. The following proposition shows why:

Definition 5.17. Let $(Z, \mathcal{Z})$ be a measurable space and $\Lambda(Z, \mathcal{Z})$ the set of probability distributions. Consider a sequence $\{\lambda_n\} \subseteq \Lambda(Z, \mathcal{Z})$. $\{\lambda_n\}$ converges set-wise to λ if and only if $\lim \int f(z)\, d\lambda_n = \int f(z)\, d\lambda$ for all bounded and measurable functions $f \in B(Z, \mathcal{Z})$.

Thus asking for set-wise convergence requires the expected value of a large class of functions to converge. A way to weaken this is to limit the space of functions for which convergence is required.

Definition 5.18. (Weak Convergence) Let (Z, ρ) be a metric space and $\mathcal{Z}$ the Borel set of Z. Define $\Lambda(Z, \mathcal{Z})$ as the set of probability distributions. Consider a sequence $\{\lambda_n\} \subseteq \Lambda(Z, \mathcal{Z})$, we say that $\{\lambda_n\}$ converges weakly to $\lambda \in \Lambda(Z, \mathcal{Z})$ if $\lim \int f(z)\, d\lambda_n = \int f(z)\, d\lambda$ for all bounded and continuous functions $f \in C(Z)$.

The main results we will obtain establish the existence of an invariant distribution under a continuity assumption on the Markov operator (the Feller property). We can then ensure uniqueness if the Markov operator is monotone and a mixing condition is satisfied, along with uniqueness we will obtain the weak convergence of $\left\{T^{\star n}\lambda_0\right\}$.

In what follows we consider $Z \subseteq \mathbb{R}^l$ for $l < \infty$, with $\mathcal{Z}$ the Borel σ-algebra of Z. The Markov process is characterized by its transition function Q, its Markov operator $T : B(Z, \mathcal{Z}) \to B(Z, \mathcal{Z})$ and its adjoint operator $T^\star : \Lambda(Z, \mathcal{Z}) \to \Lambda(Z, \mathcal{Z})$. We also define the inner product $\langle f, \lambda \rangle = \int f(z)\, d\lambda$.

We first expand on the Feller property through the following proposition:

Proposition 5.4. *The following three statements are equivalent:*

(a) *(Feller property) If* $f \in C(Z)$ *then* $Tf \in C(Z)$.

(b) *If* $z_n \to z$ *then* $Q(z_n, \cdot) \to Q(z, \cdot)$ *(that is for all* $A \in \mathcal{Z}$*).*

(c) *If* $\lambda_n \to \lambda$ *then* $T^\star \lambda_n \to \lambda$

Then preserving continuity in conditional expected values (Tf is a conditional expected value) has equivalent statements in terms of the conditional distributions ($Q(z_n, \cdot)$) and unconditional distributions ($T^\star \lambda_n$). It turns out that continuity is enough to guarantee that an invariant distribution exists.

Theorem 5.4. *If* $Z \subseteq \mathbb{R}^l$ *is compact and* Q *satisfies the Feller property then an invariant distribution exists. That is, there is* $\lambda^\star \in \Lambda(Z, \mathcal{Z})$ *such that* $\lambda^\star = T^\star \lambda^\star$.

Yet, continuity is not enough to rule out the existence of many invariant distributions or of cycling sets. Monotonicity is needed for this. As before it is first useful to take a detour on what monotonicity (as in Definition 5.4) implies for distribution functions. We then have to impose an ordering of distribution functions to be able to talk about monotonicity.

Definition 5.19. (First Order Stochastic Dominance) A distribution μ (first order stochastically) dominates λ ($\mu \geq \lambda$) if $\int f(z)\, d\mu \geq \int f(z)\, d\lambda$ for all increasing, bounded and measurable function f.

In what follows we call a sequence $\{\lambda_n\}$ monotone if $\lambda_{n+1} \geq \lambda_n$ for all n, or if $\lambda_{n+1} \leq \lambda_n$ for all n. We can now establish the following result:

Proposition 5.5. *The following three statements are equivalent:*

(a) *(Monotone property) If* $f \in B(Z, \mathcal{Z})$ *is weakly increasing then* Tf *is also weakly increasing.*

(b) *Let* $\lambda, \mu \in \Lambda(Z, \mathcal{Z})$*. If* $\mu \geq \lambda$ *then* $T^\star \mu \geq T^\star \lambda$.

(c) *If* $z \geq z'$ *then* $Q(z, \cdot) \geq Q\left(z', \cdot\right)$ *(in the stochastic dominance sense)*

The last statement is particularly useful because it translates monotonicity of the Markov operator directly into monotonicity of the transition function ("better" states lead to "better" distributions).

Now we introduce the final condition needed for the main result of this section. It is a mixing condition akin to that in Theorems 5.2 and 5.3, along with a restriction on the form of the set Z. To see why it is necessary to go SLP exercises 12.12 and 12.13.

Assumption. *The set* $Z = \left[a, b\right]$ *is a closed and bounded rectangle in* $\mathbb{R}^l$ *characterized by* a *and* b,[5] *and there exists* $z \in Z$, $\epsilon > 0$ *and* $N \geq 1$ *such that:*

$$Q^N\left(a, \left[c, b\right]\right) \geq \epsilon \qquad Q^N\left(b, \left[a, c\right]\right) \geq \epsilon$$

Under this assumption it is possible to reach the "upper" region of the rectangle, $\left[c, b\right]$, in finite time starting from the "lower" corner (a), and it is possible to reach the "lower" region of the rectangle, $[a, c]$, in finite time starting from the "upper" corner (b). It is possible to show that if one can move through the set from the corners it is possible to do it from anywhere (under a monotonicity assumption).

Proposition 5.6. *Let* Q *satisfy monotonicity and the previous assumption for some tuple* (c, ϵ, N)*, then:*

$$Q^N\left(z, \left[c, b\right]\right) \geq \epsilon \qquad Q^N\left(z, \left[a, c\right]\right) \geq \epsilon \qquad \text{for all } z \in Z$$

Finally we establish the convergence result.

Theorem 5.5. *Let* $S = \left[a, b\right] \in \mathbb{R}^l$ *be a rectangle and satisfy the assumption above. If* Q *is monotone and satisfies the Feller property, then* Q *has a unique invariant distribution* $\lambda^\star$ *and* $T^{\star n}\lambda_0 \to \lambda^\star$ *for all* $\lambda_0 \in \Lambda\left(Z, \mathcal{Z}\right)$.

[5] A set $Z \subseteq \mathbb{R}^l$ is a closed and bounded rectangle if there are two vectors $a, b \in \mathbb{R}^l$ such that$a \leq b$ and $Z = \left[a_1, b_1\right] \times \ldots \times \left[a_l, b_l\right]$.

Part II

Dynamic Programming

This part of the course develops the mathematical tools and notation used in dynamic programming problems of the type encountered in applications across macroeconomics. necessary to study how random variables affect optimization problems.

We start with an overview of non-stochastic (or deterministic) dynamic programming problems and the main mathematical results that allow us to solve them. The exposition follows almost verbatim Stokey, Lucas, and Prescott (1989, Ch. 3). The main objective here is just having a reference for the main results in dynamic programming.

Having established the basics we extend the notation and the basic results to be able to deal with stochastic problems. This follows Stokey, Lucas, and Prescott (1989, Ch. 9). The objective of this section is to mention the main results without getting into details as they are all extensions of the non-stochastic case.

Finally, we will use these tools to tackle centralized and decentralized economies in the context of the Neoclassical Growth Model. We will first layout the decentralized version of the model and define equilibrium. Having done that we can discuss how to solve the model using the centralized problem. Finally we discuss the nature of the solution in terms of Markov chains and their properties.

6. Non-Stochastic Dynamic Programming [Optional]

The topic of this section is how to state and solve (deterministic) dynamic programming problems. That is how to solve a Bellman equation of the form:

$$v(x) = \sup_{y \in \Gamma(x)} \left\{ F(x, y) + \beta v(y) \right\}$$

where the solution is given by a function v satisfying the equation. The final objective is to establish conditions for a solution to exist and characterize the properties of such a solution. To do this some mathematical background has to be set up, this is done in Section 6.1 where the contraction mapping theorem is stated and proven, and sufficient conditions for an operator to be a contraction are established.

Once the basic tools are in place the problem at hand is to express usual sequential problems (stated in terms of infinite sums) in a recursive way, the equivalence between the two representations of the problem is established by optimality principle which is presented in Section 6.2, along with it the conditions for existence of a solution and the properties it can inherit from the objective function F and the correspondence Γ are listed.

All the exposition of the theoretical aspects follows (very) closely Section 3.2 and all of chapter 4 of Stokey, Lucas, and Prescott (1989).[6] Most proofs are relegated to the book because their treatment would require more time than the one the course has.

6.1. Contraction Mapping Theorem

Three results are covered in this section that will be essential for studying dynamic programming (DP) problems. These results are the contraction mapping theorem, its corollary and the Blackwell sufficiency conditions. Before stating them recall the definition of a complete metric space and of a contraction mapping (or simply contraction) in a metric space:

Definition 6.1. A metric space is a pair (S, ρ) of a set and a metric (or distance) $\rho : S \times S \to \mathbb{R}$ such that for all $x, y, z \in S$:

(a) $\rho(x, y) \geq 0$ and $\rho(x, y) = 0 \iff x = y$.

(b) $\rho(x, y) = \rho(y, x)$.

(c) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$.

[6] There is no reason to deviate.

A metric space is furthermore complete if all Cauchy sequences in S converge to an element in S.

Definition 6.2. Let (S, ρ) be a metric space and $T : S \to S$ a function mapping S into itself. T is a contraction (with modulus β) if and only if there exists $\beta \in (0, 1)$ such that for all $x, y \in S$:

$$\rho(Tx, Ty) \leq \beta \rho(x, y)$$

The iterates of T are the mappings $\{T^n\}$ defined by $T^0 x = x$ and $T^n x = T\left(T^{n-1} x\right)$ fro $n = 1, 2, \ldots$.

The contraction mapping theorem establishes the existence and uniqueness of a fixed point in S for any contraction mapping, moreover it provides a simple algorithm to approximate the fixed point from any arbitrary point in the space. A fixed point is a point $x \in S$ such that $x = Tx$.

Theorem 6.1. ***(Contraction Mapping Theorem)*** *Let (S, ρ) be a complete metric space and $T : S \to S$ a contraction with modulus β, then:*

(a) *T has exactly one fixed point $v \in S$.*

(b) *For any $v_0 \in S$ and $n = 0, 1, \ldots$ it holds that:*

$$\rho\left(T^n v_0, v\right) \leq \beta^n \rho(v_0, v)$$

Proof. The outline of the proof is to establish that the sequence $\{v_n\} \subset S$ with $v_n = T^n v_0$ is Cauchy and the use completeness of the space to argue that its limit is the fixed point of the mapping.

Let $v_0 \in S$ and define $v_{n+1} = Tv_n$ so that $v_n = T^n v_0$. Because T is a contraction mapping:

$$\rho(v_2, v_1) = \rho(Tv_1, Tv_0) \leq \beta \rho(v_1, v_0)$$

By induction we get:

$$\rho(v_{n+1}, v_n) \leq \beta^n \rho(v_1, v_0)$$

Then for $m > n$ we get:

$$\begin{aligned}
\rho(v_m, v_n) &\leq \rho(v_m, v_{m-1}) + \rho(v_{m-1}, v_{m-2}) + \ldots + \rho(v_{n+1}, v_n) \\
&\leq \left(\beta^{m-1} + \beta^{m-2} + \ldots + \beta^n\right) \rho(v_1, v_0) \\
&= \beta^n \left(\beta^{m-n-1} + \beta^{m-n-2} + \ldots + 1\right) \rho(v_1, v_0) \\
&\leq \frac{\beta^n}{1 - \beta} \rho(v_1, v_0)
\end{aligned}$$

Because $\frac{\rho(v_1,v_0)}{1-\beta}$ is fixed, and finite, and $\beta^n \to 0$ its clear that for any $\epsilon > 0$ there exists N large enough for $\rho(v_m, v_n) \leq \epsilon$ for all $m, n \geq N$. Then $\{v_n\}$ is Cauchy and there exists $v \in S$ such that $v_n \to v$ because S is complete.

Now we show that v is a fixed point of T. For all n and v_0:

$$\begin{aligned}\rho(Tv, v) &\leq \rho\left(Tv, T^n v\right) + \rho\left(T^n v, v\right) \\ &\leq \beta\rho\left(v, T^{n-1}v\right) + \rho\left(T^n v, v\right) \\ &= \beta\rho(v, v_{n-1}) + \rho(v_n, v)\end{aligned}$$

It follows that $\rho(v, v_{n-1}) \to 0$ and $\rho(v_n, v) \to 0$ because $v_n \to v$. Because this is done for arbitrary n we get $\rho(Tv, v) \leq \epsilon$ for all $\epsilon > 0$ which implies $\rho(Tv, v) = 0$. By definition this is $Tv = v$, a fixed point.

To show uniqueness suppose for a contradiction that there is $v' \neq v$ such that $Tv' = v'$, then:

$$0 \neq \rho\left(v', v\right) = \rho\left(Tv', Tv\right) \leq \beta\rho\left(v', v\right)$$

But this contradicts $\beta < 1$. Then v is the unique fixed point.

The second part of the theorem follows by induction. For $n = 0$,

$$\rho(Tv_0, v) = \rho(Tv_0, Tv) \leq \beta\rho(v_0, v)$$

and for any $n \geq 1$,

$$\rho\left(T^n v_0, v\right) = \rho\left(T^n v_0, Tv\right) \leq \beta\rho\left(T^{n-1}v_0, v\right)$$

The result follows.

□

The contraction mapping theorem is a very powerful and simple theorem, yet its results can be strengthened by further characterizing the fixed point. So far it has been established its existence in S and its uniqueness, the following corollary to the theorem allows to locate the fixed point in a given subset of S.

Corollary 6.1. *Let (S, ρ) be a complete metric space and $T : S \to S$ a contraction mapping with fixed point $v \in S$.*

(a) *If $S' \subseteq S$ is closed and $T\left(S'\right) \subseteq S'$, then $v \in S'$.*

(b) *If in addition there exists $S'' \subseteq S'$ such that $T\left(S'\right) \subseteq S''$, then $v \in S''$.*

Proof. Let $v_0 \in S'$ and $\left\{T^n v_0\right\}$ a sequence in S' so that $T^n v_0 \to v$, because S' is closed it follows that $v \in S'$. If in addition $T\left(S'\right) \subseteq S''$ then it follows that $v = Tv \in S''$.

□

Finally a set of sufficiency conditions are established for a mapping on the space of bounded functions to be a contraction. In most economic applications these conditions are trivial to check.

Theorem 6.2. ***(Blackwell conditions)*** *Let $X \subseteq \mathbb{R}^l$ and $B(X)$ be the the space of bounded functions on X ($f : X \to \mathbb{R}$) with the sup-norm. Let $T : B(X) \to B(X)$, T is a contraction (with modulus β) if it satisfies the following two conditions:*

(a) *(monotonicity) Let $f, g \in B(X)$ and $f(x) \leq g(x)$ for all $x \in X$. Then $Tf(x) \leq Tg(x)$ for all $x \in X$.*

(b) *(discounting) There exists $\beta \in (0,1)$ such that $T(f+a)(x) \leq Tf(x) + \beta a$ for all $f \in B(X)$, $x \in X$ and $a \geq 0$.*

Proof. If $f(x) \leq g(x)$ for all x we say that $f \leq g$.

Let $f, g \in B(X)$, by definition of the sup-norm $f(x) - g(x) \leq \|f - g\|$ for all $x \in X$, then $f(x) \leq g(x) + \|f - g\|$, using the notation defined at the beginning of the proof this is $f \leq g + \|f - g\|$, where $\|f - g\| > 0$ is a scalar. Then by hypothesis we have:

$$Tf \leq T(g + \|f - g\|) \leq Tg + \beta \|f - g\| \longrightarrow Tf - Tg \leq \beta \|f - g\|$$

But it also holds that $g(x) - f(x) \leq \|f - g\|$ which implies $Tg - Tf \leq \beta \|f - g\|$. Joining we have, for all $x \in X$:

$$|Tf(x) - Tg(x)| \leq \beta \|f - g\|$$

Taking sup we get:

$$\|Tf - Tg\| \leq \beta \|f - g\|$$

which establishes that T is a contraction.

□

Extended Blackwell conditions. I also present a modified version of Blackwell's sufficiency conditions for vector valued functions. I first define the relevant set of functions.

Proposition 6.1. *Let $X \subset \mathbb{R}^n$ and $B(X) = \left\{ f | f : X \to \mathbb{R} \wedge \exists_{M_f} \forall_{x \in X} |f(x)| \leq M_f \right\}$ the set of bounded functions defined on the set X. The space $S = B(X) \times B(X)$ equipped with the norm $\|f\| = \max\left\{ \|f_1\|_\infty, \|f_2\|_\infty \right\} = \max\left\{ \sup_{x \in X} |f_1(x)|, \sup_{x \in X} |f_2(x)| \right\}$ is a normed vector space. It is also a metric space with the metric $\rho(f, g) = \|f - g\|$.*

Proof. The proof proceeds by showing that $\|\cdot\|$ is a norm.

(a) Clearly $\|f\| \geq 0$ and if $f(x) = 0$ for all $x \in X$ then $\|f\| = 0$. Finally:

$$\begin{aligned} \|f\| &= 0 \\ \max\left\{\sup_{x\in X}|f_1(x)|, \sup_{x\in X}|f_2(x)|\right\} &= 0 \end{aligned}$$

which happens if and only if $\sup_{x\in X}|f_1(x)| = 0$ and $\sup_{x\in X}|f_2(x)| = 0$. Again, this happens if and only if $f_1(x) = f_2(x) = 0$ for all $x \in X$. That is, if $f(x) = 0$ for all $x \in X$.

(b) $\|\alpha f\| = \max\left\{\sup_{x\in X}|\alpha f_1(x)|, \sup_{x\in X}|\alpha f_2(x)|\right\} = |\alpha| \max\left\{\sup_{x\in X}|f_1(x)|, \sup_{x\in X}|f_2(x)|\right\} = |\alpha|\,\|f\|$

(c) Triangle Inequality:

$$\begin{aligned} \|f+g\| &= \max\left\{\sup_{x\in X}|f_1(x)-g_1(x)|, \sup_{x\in X}|f_2(x)-g_2(x)|\right\} \\ &\leq \max\left\{\left(\sup_{x\in X}|f_1(x)| + \sup_{x\in X}|g_1(x)|\right), \left(\sup_{x\in X}|f_2(x)| + \sup_{x\in X}|g_2(x)|\right)\right\} \\ &\leq \max\left\{\sup_{x\in X}|f_1(x)|, \sup_{x\in X}|f_2(x)|\right\} + \max\left\{\sup_{x\in X}|g_1(x)|, \sup_{x\in X}|g_2(x)|\right\} \\ &= \|f\| + \|g\| \end{aligned}$$

The first inequality follows from properties of the absolute value and the second one from the inequality:

$$\sup_{x\in X}|f_i(x)| + \sup_{x\in X}|g_i(x)| \leq \max\left\{\sup_{x\in X}|f_1(x)| + \sup_{x\in X}|g_1(x)|, \sup_{x\in X}|f_2(x)| + \sup_{x\in X}|g_2(x)|\right\}$$

(d) Under the above three conditions $\|f\|$ is a norm.

(e) Clearly the sum and scalar product of bounded functions is bounded.

□

Proposition 6.2. *Consider* (S, ρ) *with* $S = B(X) \times B(X)$ *and* $\rho(f,g) = \|f-g\|$. (S, ρ) *is a complete space.*

Proof. The proof starts by showing that a Cauchy sequence in S is formed by Cauchy sequences in $B(X)$. Then the completeness of $B(X)$ is used to establish the result.

(a) Let $\{f_n\} \subset S$ be a Cauchy sequence and $\epsilon > 0$. There exists N such that $\forall_{n,m>N} \|f_n - f_m\| < \epsilon$ which is:

$$\max\left\{\sup_{x\in X}|f_{1n}(x) - f_{1m}(x)|, \sup_{x\in X}|f_{2n}(x) - f_{2m}(x)|\right\} < \epsilon$$

$$\sup_{x\in X}\left|f_{1n}(x)-f_{1m}(x)\right|<\epsilon \qquad \wedge \qquad \sup_{x\in X}\left|f_{2n}(x)-f_{2m}(x)\right|<\epsilon$$

This implies that the sequences $\{f_{1n}\}\subset B(X)$ and $\{f_{2n}\}\subset B(X)$ are Cauchy with respect to the sup-norm $\|f\|_\infty=\sup_{x\in X}|f(x)|$.

(b) Because the space $(B(X),\|f\|_\infty)$ is complete, the above implies that the sequences $\{f_{1n}\}$ and $\{f_{2n}\}$ are convergent in the sup norm. $\exists_{f_1,f_2\in B(X)} f_{1n}\to f_1\wedge f_{2n}\to f_2$. Denote $f:S\to\mathbb{R}^2$ as $f(x)=\left[f_1(x)\ \ f_2(x)\right]'$.

(c) Let $\epsilon>0$. By convergence of $\{f_{1n}\}$ and $\{f_{2n}\}$ there exist numbers N_1 and N_2 such that:

$$\forall_{n\geq N_1}\left\|f_{1n}-f_1\right\|_\infty<\epsilon \qquad \wedge \qquad \forall_{n\geq N_2}\left\|f_{2n}-f_2\right\|_\infty<\epsilon$$

Then for $N=\max\{N_1,N_2\}$ it holds that:

$$\forall_{n\geq N}\left\|f_{1n}-f_1\right\|_\infty<\epsilon \qquad \wedge \qquad \left\|f_{2n}-f_2\right\|_\infty<\epsilon$$

which is:

$$\forall_{n\geq N}\sup_{x\in X}\left|f_{1n}(x)-f_1(x)\right|<\epsilon \qquad \wedge \qquad \sup_{x\in X}\left|f_{2n}(x)-f_2(x)\right|<\epsilon$$

implying then:

$$\forall_{n\geq N}\max\left\{\sup_{x\in X}\left|f_{1n}(x)-f_1(x)\right|,\sup_{x\in X}\left|f_{2n}(x)-f_2(x)\right|\right\}<\epsilon$$

which is:

$$\forall_{n\geq N}\left\|f_n-f\right\|<\epsilon$$

(d) The above proves that a Cauchy sequence converges on S over the given norm.

□

Theorem 6.3. ***(Extended Blackwell)*** *Consider (S,ρ) with $S=B(X)\times B(X)$ and $\rho(f,g)=\|f-g\|$. Let $T:S\to S$ be an operator satisfying*

(a) *(Monotonicity) $f,g\in S$ and $f(x)\leq g(x)$, for all $x\in X$, implies $Tf(x)\leq Tg(x)$, for all $x\in X$, (where $f(x)\leq g(x)$ is taken in the vector sense, i.e. $f_1(x)\leq g_1(x)$ and $f_2(x)\leq g_2(x)$).*

(b) *(Discounting) there exists some $\beta\in(0,1)$ such that $T(f+A)\leq Tf(x)+\beta A$ for $f\in S$, $A=[a\ a]'\in\mathbb{R}^2_+$ and $x\in X$.*

Then T is a contraction in S with modulus β.

Proof. The proof follows closely that of Blackwell's conditions

(a) Let $f, g \in S$, and define $A = \begin{bmatrix} \|f-g\| & \|f-g\| \end{bmatrix}'$, it holds that:

$$\begin{aligned} f_1(x) - g_1(x) &\leq |f_1(x) - g_1(x)| \leq \sup_{x \in X} |f_1(x) - g_1(x)| \\ &\leq \max\left\{ \sup_{x \in X} |f_1(x) - g_1(x)|, \sup_{x \in X} |f_2(x) - g_2(x)| \right\} = \|f-g\|_\infty \end{aligned}$$

By a similar argument $f_2(x) - g_2(x) \leq \|f-g\|_\infty$ then it holds that: $f(x) \leq g(x) + A$ for all $x \in X$

(b) By monotonicity and discounting:

$$Tf(x) \leq T(g+a)(x) \leq Tg(x) + \beta A$$

which holds for all $x \in X$.

(c) The same argument applies to show that $g(x) \leq f(x) + A$ and $Tg(x) \leq Tf(x) + \beta A$ for all $x \in X$.

(d) Joining:

$$Tf_i(x) - Tg_i(x) \leq \beta \|f-g\| \qquad \wedge \qquad Tg_i(x) - Tf_i(x) \leq \beta \|f-g\|$$

which implies:

$$|Tf_1(x) - Tg_1(x)| \leq \beta \|f-g\| \qquad \wedge \qquad |Tf_2(x) - Tg_2(x)| \leq \beta \|f-g\|$$

and then:

$$\sup_{x \in X} |Tf_1(x) - Tg_1(x)| \leq \beta \|f-g\| \qquad \wedge \qquad \sup_{x \in X} |Tf_2(x) - Tg_2(x)| \leq \beta \|f-g\|$$

(e) Finally:

$$\begin{aligned} \|Tf - Tg\|_\infty &= \max\left\{ \sup_{x \in X} |Tf_1(x) - Tg_1(x)|, \sup_{x \in X} |Tf_2(x) - Tg_2(x)| \right\} \\ &\leq \max\left\{ \beta \|f-g\|, \beta \|f-g\| \right\} = \beta \|f-g\| \end{aligned}$$

This is the definition of T being a contraction with modulus β.

□

6.2. The Bellman Equation

We start with the infinite horizon consumption savings model. This is the workhorse model of modern macroeconomics and is known as the neoclassical growth model. There are two (related) ways of setting up the problem. One resembles the finite horizon problem already discussed, it is called sequence problem, the other form is to cast the problem as the solution to a functional equation, this dynamic programming approach has several advantages that will be presented in the next section.

As before, consider a discrete time, consumption-savings problem where the agent can either consume or save (invest) in capital that will be productive in the following period. The agent derives utility from consumption according to utility function u and discounts the future at a constant rate $\beta < 1$. Production only uses capital and the technology is described by a function f.

The problem of an agent endowed with k_0 units of capital is:

$$v(k_0) = \max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \qquad \text{s.t. } c_t + k_{t+1} \leq f(k_t) \quad c_t, k_t \geq 0 \quad k_0 \text{ given}$$

Provided that u is strictly increasing, a sustained assumption, we can eliminate consumption as before to get:

$$v(k_0) = \max_{\{k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t u(f(k_t) - k_{t+1}) \qquad \text{s.t. } 0 \leq k_{t+1} \leq f(k_t) \quad k_0 \text{ given}$$

In the sequence problem, much like in the finite horizon problem before, the objective is to look for an infinite sequence that solves the problem and attains the maximum. This can prove to be too difficult in practice.

The dynamic programming problem takes a different approach. Instead of trying to solve the problem for all periods simultaneously the objective is to solve the problem one period at a time. That is, given the capital stock at the beginning of the period take an optimal investment decision for the next period. The problem is that, in order to make the decision, its necessary to know the extra value for the agent of the capital to be saved, we need a function that represents preferences over next period's capital.

The DP starts by assuming that we already know such a function. It is called a value function and is defined as v above. The value function is the maximum value given to the agent if she starts in a given period with initial capital k. Knowing v it is possible to cast the following problem:

$$\max_{0 \leq k_1 \leq f(k_0)} \left\{ u(f(k_0) - k_1) + \beta v(k_1) \right\}$$

If we knew v the problem above could be solved. The solution to the problem is a policy function $g : \mathbb{R}_+ \to \mathbb{R}_+$ that gives the optimal capital next period given a capital level today. That is $k_1 = g(k_0)$.

It should be clear now that if $v(k_1)$ gives the maximum value starting in period 1 and the problem above maximizes that value and the value in period 0 (given by $u(f(k_0) - k_1)$) then the value of the whole problem is given by the maximum above. But that is the definition

of v, then:

$$v(k_0) = \max_{0 \leq k_1 \leq f(k_0)} \left\{ u(f(k_0) - k_1) + \beta v(k_1) \right\}$$

This is a functional equation, where f and u are known functions, k_1 is a variable of choice and k_0 is given. Then this is an equation in the function v, the solution to this equation is the value function needed to solve the problem (to find the policy function).

In general solving functional equations is not easy, but this type of functional equation can be reinterpreted to both establish the existence of a solution and to obtain a method to find it.

Let u and f be bounded and continuous functions and define an operator $T : C(X) \to C(X)$ as:

$$Tv(k) = \max_{0 \leq k' \leq f(k)} \left\{ u\left(f(k) - k'\right) + \beta v\left(k'\right) \right\}$$

boundedness of Tv is immediate for the sum of bounded is also bounded. Continuity is a consequence of the ToM, the objective function is continuous and the correspondence $\Gamma(k) = \left\{ k' | 0 \leq k' \leq f(k) \right\}$ is continuous and compact valued because f is continuous and bounded.

The solution to the functional equation, v, is then a fixed point of the mapping T. It is left to verify that T is a contraction to establish the existence and uniqueness of the solution to the neoclassical growth model. It turns out that Blackwell's sufficient conditions are immediate:

(a) (monotonicity) Let $v, w \in C(X)$ and $v(k) \leq w(k)$ for all k. Then:

$$Tv(k) = \max_{0 \leq k' \leq f(k)} \left\{ u\left(f(k) - k'\right) + \beta v\left(k'\right) \right\} \leq \max_{0 \leq k' \leq f(k)} \left\{ u\left(f(k) - k'\right) + \beta w\left(k'\right) \right\} = Tw(k)$$

(b) (discounting) Let $v \in C(X)$ and $a > 0$. Then:

$$T(v + a)(k) = \max_{0 \leq k' \leq f(k)} \left\{ u\left(f(k) - k'\right) + \beta \left(v\left(k'\right) + a \right) \right\} = \max_{0 \leq k' \leq f(k)} \left\{ u\left(f(k) - k'\right) + \beta v\left(k'\right) \right\} + \beta a$$

In particular:

$$T(v + a)(k) \leq Tv(k) + \beta a$$

It is possible to further characterize v and the policy function g, for that extra results are needed.

6.3. A general framework and the principle of optimality

The problem to be studied in terms of infinite sequences is of the form:

$$v^{\star}(x_0) = \sup_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \qquad \text{s.t. } x_{t+1} \in \Gamma(x_t) \tag{6.1}$$

Corresponding to this problem is the following functional equation:

$$v(x) = \sup_{y \in \Gamma(x)} \left\{ F(x, y) + \beta v(y) \right\} \tag{6.2}$$

Above, X is the set of possible values for x, and it is not necessarily an euclidean space, $\Gamma : X \rightrightarrows X$ is a correspondence that assigns feasible values of the choice variable and $F : \text{Gr}(\Gamma) \to \mathbb{R}$ is a return or payoff function. $\beta > 0$ is a discount factor.

Some conditions have to be met for both problems to give the same solution, in the sense that $v(x) = v^{\star}(x)$ and that the optimal choice of one problem is the the same as the choice for the other. This equivalence between both problems is called the *principle of optimality*. After the validity of the principle has been established the properties of the solution to FE can be studied.

The conditions for the principle of optimality are stated below and the two propositions that constitute the principle are shown without proof.

It will be convenient to define the set of all possible feasible sequences for x, given an starting point x_0.

Definition 6.3. The set of all possible feasible sequences starting at $x_0 \in X$ is:

$$\Pi(x_0) = \left\{ \{x_t\}_{t=0}^{\infty} \,|\, x_{t+1} \in \Gamma(x_t) \quad \wedge \quad x_0 \text{ given} \right\}$$

and $\underline{x} = (x_0, x_1, x_2, \ldots)$ is an element.

Assumption A.1: . Γ is a nonempty valued correspondence.

Assumption A.2: . For all $x_0 \in X$ and $\underline{x} \in \Pi(x_0)$ the following limit exists (although it might be infinite):

$$\lim_{n \to \infty} \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1})$$

Remark. Assumption A.2 holds if F is bounded and $\beta \in (0, 1)$.

Under assumptions A.1 and A.2 $\Pi(x_0)$ is nonempty valued and problem (6.1) is well posed, moreover they are enough to guarantee that the function $v^\star$ satisfies equation (6.2).

Proposition 6.3. *Let X, Γ, F and β satisfy assumption A.1 and A.2, then $v^\star$ is a solution to the FE (6.2):*

$$v^\star(x) = \sup_{y \in \Gamma(x)} \left\{ F(x, y) + \beta v^\star(y) \right\}$$

For $v^\star$ to be the only solution to the FE an extra condition is needed.

Proposition 6.4. *Let X, Γ, F and β satisfy assumption A.1 and A.2, if v is a solution to the FE (6.2) and for all $x_0 \in X$ and $\underline{x} \in \Pi(x_0)$ it holds that:*

$$\lim_{n \to \infty} \beta^n v(x_n) = 0$$

then $v = v^\star$.

The previous two propositions establish equivalence between the value of the two problems. It can also be shown that the optimizer of the SP problem also solves the FE in the following sense:

Proposition 6.5. *Let X, Γ, F and β satisfy assumption A.1 and A.2. Let $\underline{x}^\star \in \Pi(x_0)$ be a feasible plan that attains the supremum in (6.1), then:*

$$v^\star\left(x_t^\star\right) = F\left(x_t^\star, x_{t+1}^\star\right) + \beta v^\star\left(x_{t+1}^\star\right) \tag{6.3}$$

Again, under an extra boundedness condition a plan that solves the problem in (6.2) also solves the problem in the SP.

Proposition 6.6. *Let X, Γ, F and β satisfy assumption A.1 and A.2. Let $\underline{x}^\star \in \Pi(x_0)$ be a feasible plan that satisfies equation (6.3) and for which $\limsup \beta^t v^\star\left(x_t^\star\right) \leq 0$, then $\underline{x}^\star$ attains the supremum in (6.1) for initial state x_0.*

Now we can define the optimal policy correspondence as:

$$G^\star(x) = \left\{ y \in \Gamma(x) \,|\, v^\star(x) = F(x, y) + \beta v^\star(y) \right\}$$

We say that a plan $\underline{x}$ is generated by G if it satisfies $x_{t+1} \in G(x_t)$. The previous two propositions imply that any optimal plan of the sequence problem is generated by $G^\star$ and that if a plan is generated by $G^\star$ and satisfies the additional boundedness condition then it is also optimal.

Now we can concentrate in studying the properties of the DP in (6.2).

6.4. Bounded problems

Now we concentrate in establishing properties of the solution to the following problem:

$$v(x) = \max_{y \in \Gamma(x)} \left\{ F(x, y) + \beta v(y) \right\} \tag{6.4}$$

$$G(x) = \left\{ y \in \Gamma(x) \,|\, v(x) = F(x, y) + \beta v(y) \right\}$$

where v is the value function and G the policy correspondence.

Assumptions A.1 and A.2 have to be met for the implications of this sections to be valid on the original sequence problem. Additional assumptions are also imposed that ensure that the previous ones are met.

Assumption A.3: . X is a a convex subset of $\mathbb{R}^l$ and Γ is a nonempty, compact valued and continuous correspondence.

Assumption A.4: . The function $F : \mathrm{Gr}(\Gamma) \to \mathbb{R}$ is bounded and continuous and $\beta \in (0, 1)$.

Because F is bounded and continuous it is natural to think that the solution to equation (6.4) lies in the set $C(X)$. What follows it to establish the existence of a solution by means of the contraction mapping theorem.

Define a mapping $T : C(X) \to C(X)$ as:

$$Tf(x) = \max_{y \in \Gamma(x)} \left\{ F(x, y) + \beta f(y) \right\} \tag{6.5}$$

The solution to (6.4) is then a $v \in C(X)$ such that $v = Tv$. The following proposition establishes that T is a contraction from $C(X)$ into itself and also some properties of the policy correspondence G.

Proposition 6.7. *Let X, Γ, F and β satisfy assumption A.3 and A.4, and consider $C(X)$ the space of continuous bounded function on X along with the sup norm. Then:*

(a) *T defined in (6.5) maps $C(X)$ into itself.*

(b) *T defined in (6.5) has a unique fixed point $v \in C(X)$, and for all $v_0 \in C(X)$*

$$\left\| T^n v_0 - v \right\| \leq \beta^n \left\| v_0 - v \right\|$$

(c) *Given v the optimal policy correspondence $G(x) = \left\{ y \in \Gamma(x) \,|\, v(x) = F(x, y) + \beta v(y) \right\}$ is nonempty, compact valued and u.h.c.*

Proof. Each part is established separately.

(a) Under A.3 and A.4 and given f continuous and bounded the function $F(x, y) + \beta f(y)$ is continuous in (x, y) and Γ satisfies all assumptions of the ToM, thus establishing that Tf is continuous.

Because F and f are bounded then Tf is bounded as well. There exists $M \geq 0$ such that $-M \leq F(x, y) + \beta f(y) \leq M$ for all (x, y), then for all x we have: $-M \leq \max_{y \in \Gamma(x)} \{F(x, y) + \beta f(y)\} \leq M$ which establishes boundedness of Tf.

Then $Tf \in C(X)$ for any $f \in C(X)$.

(b) Blackwell conditions are met:

(i) (monotonicity) Let $f, g \in C(X)$ and $f(x) \leq g(x)$ for all x. Then:

$$Tf(x) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta f(y)\} \leq \max_{y \in \Gamma(x)} \{F(x, y) + \beta g(y)\} = Tg(x)$$

(ii) (discounting) Let $f \in C(X)$ and $a > 0$. Then:

$$T(f + a)(x) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta (f(y) + a)\} = \max_{y \in \Gamma(x)} \{F(x, y) + \beta f(y)\} + \beta a$$

In particular:

$$T(f + a)(x) \leq Tf(x) + \beta a$$

Then T is a contraction. By the contraction mapping theorem the result follows.

(c) The properties of G follow from the ToM which applies as shown before.

□

Additional assumption will help to characterize v and G better. The corollary of the contraction mapping theorem is the tool to be used now. First monotonicity can be inherited by the solution.

Assumption A.5:. For all y $F(\cdot, y)$ is strictly increasing in its first l arguments.

Assumption A.6:. Γ is monotone in the sense that if $x \leq x'$ the $\Gamma(x) \subseteq \Gamma\left(x'\right)$.

Proposition 6.8. *Let X, Γ, F and β satisfy assumption A.3 to A.6, and let v be the unique solution to (6.4), then v is strictly increasing.*

Proof. Let $C'(X) \subseteq C(X)$ be the set of bounded, continuous and non-decreasing functions and $C''(X) \subseteq C'(X)$ the set of strictly increasing functions. Clearly $C'(X)$ is closed. By the corollary of the contraction mapping theorem it suffices to show that $T\left(C'(X)\right) \subseteq C''(X)$.

Let $f \in C^{'}(X)$ and consider $x < x^{'}$. We want to show that Tf is strictly increasing. This follows with A.5 and A.6:

$$Tf(x) = \max_{y \in \Gamma(x)} F(x, y) + \beta f(y) \leq \max_{y \in \Gamma\left(x^{'}\right)} F(x, y) + \beta f(y) < \max_{y \in \Gamma\left(x^{'}\right)} F\left(x^{'}, y\right) + \beta f(y) = Tf\left(x^{'}\right)$$

where the first inequality follows from $\Gamma(x) \subseteq \Gamma\left(x^{'}\right)$, a larger choice set implies a higher than or equal maximum, the second inequality follows from F being strictly increasing.

□

It is also possible to induce convexity as follows:

Assumption A.7: . F is strictly concave in both arguments.

Assumption A.8:. Γ has a convex graph.

Proposition 6.9. *Let X, Γ, F and β satisfy assumption A.3, A.4, A.7 and A.8, and let v be the unique solution to (6.4), then v is strictly concave and G is a continuous single valued function.*

Proof. Let $C^{'}(X)$ be the set of concave, bounded and continuous functions and $C^{''}(X)$ the set of strictly concave, bounded and continuous functions. The set $C^{'}(X) \subseteq C(X)$ is closed and that $C^{''}(X) \subseteq C^{'}(X)$. To show that v is strictly concave we use the corollary of the contraction mapping theorem.

We want to show that for all $f \in C^{'}(X)$ it follows that $Tf \in C^{''}(X)$ where

$$Tf(x) = Tf(x) = \max_{y \in \Gamma(x)} F(x, y) + \beta f(y)$$

So let f be weakly concave on x, bounded and continuous. let $x_1, x_2 \in X$ and $\lambda \in (0, 1)$ and define $x_\lambda = \lambda x_1 + (1 - \lambda) x_2$. Let $y_i \in G\left(x_i\right) \subseteq \Gamma\left(x_i\right)$ and, by A.8, $y_\lambda = \lambda y_1 + (1 - \lambda)\, y_2 \in \Gamma(x_\lambda)$:

$$\begin{aligned}
Tf(x_\lambda) &= \max_{y \in \Gamma(x_\lambda)} F(x_\lambda, y) + \beta f(y) \\
&\geq F\left(x_\lambda, y_\lambda\right) + \beta f\left(y_\lambda\right) \\
&\geq F\left(x_\lambda, y_\lambda\right) + \lambda \beta f\left(y_1\right) + (1 - \lambda)\, \beta f\left(y_2\right) \\
&> \lambda F\left(x_1, y_1\right) + (1 - \lambda)\, F\left(x_2, y_2\right) + \lambda \beta f\left(y_1\right) + (1 - \lambda)\, \beta f\left(y_2\right) \\
&= \lambda T f(x_1) + (1 - \lambda)\, T f(x_2)
\end{aligned}$$

where the first inequality follows from $y_\lambda$being feasible at x_λ, the second one from f being concave and the third one from A.7. The final equality is obtained rearranging and recalling the optimality of y_1 and y_2 under x_1 and x_2 respectively. Joining results we get

$$Tf(x_\lambda) > \lambda T f(x_1) + (1 - \lambda)\, T f(x_2)$$

Then the "image" of any concave function is a strictly concave function. This proves that $Tf \in C''(X)$. Then by the corollary of the contraction mapping theorem the unique fix point of T belongs to C''. That is, v is strictly concave.

Finally, consider the policy function $G(x) = \underset{y \in \Gamma(x)}{\operatorname{argmax}} F(x, y) + v(y)$. Because F and v are strictly concave the single valuedness and continuity of G follow as an immediate consequence of the ToM under convexity part (ii).

□

Finally there are conditions for v to be differentiable, allowing the use of first order conditions.

Assumption A.9: . F is continuously differentiable on the interior of its domain, $\mathrm{Gr}(A)$.

Proposition 6.10. *Let X, Γ, F and β satisfy assumption A.3, A.4, and A.7 to A.9, and let v be the unique solution to (6.4). If $x_0 \in IntX$ and $g(x_0) \in Int\Gamma(x_0)$ then v is continuously differentiable at x_0 with derivatives given by:*

$$v_i(x_0) = F_i(x_0, g(x_0))$$

Proof. Stokey, Lucas, and Prescott (1989, sec. 4.2, pp. 85).

□

7. Stochastic Dynamic Programming

We now want to establish the basics of dynamic programming problems where (some of) the states are subject to stochastic variation. We want to allow the variation in the stochastic variables to be serially correlated and for that we use the concepts developed for Markov processes in Section 5. The results below are a summary of those in Chapter 9 of Stokey, Lucas, and Prescott (1989).

The basic problem takes one of two forms:

$$v(k,z) = \max_{k' \in [0, zf(k)]} \left\{ U\left(zf(k) - k'\right) + \beta \int_z v\left(k', z'\right) Q\left(z, dz'\right) \right\} \tag{7.1}$$

$$v(k,z) = \max_{y \in [0, f(k)]} \left\{ U\left(f(k) - y\right) + \beta \int_z v\left(y - z', z'\right) Q\left(z, dz'\right) \right\} \tag{7.2}$$

The key mathematical feature in the two models is the expectation over future values of the state z taking as given the state's current value. The conditional distribution of z' is given by the transition function Q.

The key economic difference between the two problems is the degree of control over the future value of the states. In the first problem the future value of state k is perfectly controlled and there is only uncertainty over the future value of z. In the second problem the value of the first state is only known until the following period. Only the choice y is known by the decision maker, but the value of $k' = y - z'$ depends on z'.

The second problem is of course more general and it also allows for more flexible numerical methods for the solution of the functional equation. See Phelan and Eslami (2022).

The general form of the stochastic dynamic programming problem is cumbersome because of the required notation to take into account the stochastic process induced by the choices of the decision maker. However, the end result is basically the same as in the non-stochastic case of Section 6. Most importantly, the principle of optimality and the contraction mapping theorem still apply.

In what follows we cover the basic definitions that will be useful to establish what constitutes a solution to the stochastic dynamic programming problem. We will later use these to construct a competitive equilibrium in decentralized economies. We will do this for the first type of problem shown above. Even though it is less general, the basic concepts are clearer in it and it maps better to the neoclassical growth model, leading example to come.

We start by defining measurable spaces for the endogenous state $(X, \mathcal{X})$ and the

exogenous (stochastic) state $(Z, \mathcal{Z})$, as well as their product that forms the state space of the problem $(S, \mathcal{S}) = (X \times Z, \mathcal{X} \times \mathcal{Z})$. We also impose that Q is a stationary transition function, particularly that it satisfies the Feller property (definition 5.3) preserving continuity and boundedness.

The decision problem consists in choosing the future value of the endogenous state, a choice y from a feasible set characterized by the correspondence $\Gamma : X \times Z \to Z$. The graph of Γ is $A = \{(x, y, z) \,|\, y \in \Gamma(x, z)\}$. The payoff function is $F : A \to \mathbb{R}$ and the decision maker discounts future payoffs at a rate $\beta \geq 0$.

7.1. The Sequential Problem

The sequential problem is where most of the notational complications arise. The decision maker takes as given some initial state (x_0, z_0) and then makes *contingency plans* that specify actions in future periods depending on the realization of shocks up until then (the information available). Importantly, the plans depend only on the history of shocks (the exogenous state) as the endogenous state, x, is chosen every period, except for the first. The objective is to maximize the present discounted expected payoff. The expectation is over the possible sample paths for the realization of the shocks. This requires dusting off the notation on stochastic processes developed after definition 5.7.

Definition 7.1. (Contingency Feasible Plan) Let $\left(Z^t, \mathcal{Z}^t\right)$ be a measurable space over the partial history of shocks in periods 1 through t. A feasible plan is a value $\pi_0 \in \Gamma(x_0, z_0)$ and a sequence of measurable functions $\pi_t : Z^t \to \Gamma\left(\pi_{t-1}\left(z^{t-1}\right), z_t\right)$. Denote by $\Pi(s_0)$ the set of plans feasible given $s_0 = (x_0, z_0)$.

The measurability conditions over the (choice) functions π_t implies that we can compute probabilities over the outcomes (decisions) given the probabilities on paths implied by Q. Recall that Q is a function over $(Z, \mathcal{Z})$ so that, given s_0, we can define the probability measure $\mu^t(z_0, \cdot) : \mathcal{Z}^t \to [0, 1]$ as in definition 5.5. This function establishes probabilities over sequences of shocks z^t and allows to compute expectations over payoffs $F(\cdot)$ provided that F is measurable with respect to the appropriate σ-algebra on A (defined as $\mathcal{A} = \{C \in \mathcal{X} \times \mathcal{X} \times \mathcal{Z} | C \in A\}$) and that the function F is integrable (either by being always positive or negative, or by being integrable with respect to μ^t given any feasible plan $\pi \in \Pi(s_0)$). These conditions allow us to define the (period-t) discounted payoffs

$$u_0(\pi, s_0) = F(x_0, \pi_0, z_0);$$
$$u_n(\pi, s_0) = F(x_0, \pi_0, z_0) + \sum_{t=1}^{n} \beta^t \int_{Z^t} F\left(\pi_{t-1}\left(z^{-1}\right), \pi_t\left(z^t\right), z_t\right) \mu^t\left(z_0, dz^t\right);$$

and its limit

$$u(\pi, s_0) = \lim_{n\to\infty} u_n(\pi, s_0).$$

This finally allow us to define the sequential problem of the decision maker:

$$v^\star(s) = \max_{\pi\in\Pi(s)} u(\pi, s) \tag{7.3}$$

Just posing this problem is a mouthful. Solving it is even worse. First, just evaluating the probabilities along the sample path becomes increasingly taxing. For instance, even if the exogenous state could take only two values, there are over a thousand possible histories after 10 periods (and the number gets prohibitively large for 100 periods). Each of these histories has its own contingent plan, that is evaluated depending on its probability.

When the decision maker does not have perfect control over the endogenous state variable as in the second problem in (7.2) we must also define a law of motion for the endogenous state variable

$$\phi : X \times X \times Z \to X$$

so that $x' = \phi\left(x, y, z'\right)$ is the next period's endogenous state. This law of motion interacts with the contingent choices of the control, y, to generate a contingent plan, that is now

Definition 7.2. (Contingency Feasible Plan') Let $\left(Z^t, \mathcal{Z}^t\right)$ be a measurable space over the partial history of shocks in periods 1 through t. A feasible plan is a value $\pi_0 \in \Gamma(x_0, z_0)$ and a sequence of measurable functions $\pi_t : Z^t \to \Gamma\left(x_t^\pi\left(z^t\right), z_t\right)$, where $x_t^\pi\left(z^t\right) \equiv \phi\left(x_{t-1}^\pi\left(z^{t-1}\right), \pi_{t-1}\left(z^{t-1}\right), z_t\right)$ is defined as the (contingent) path of the endogenous variable given the plan π. Denote by $\Pi(s_0)$ the set of plans feasible given $s_0 = (x_0, z_0)$.

The implication of the changes to the problem is that we now have two stochastic processes. One is the exogenous state variable z, that we treat just as before. In the first type of problem this process is enough to describe the problem as it completely describes the process of choices, y, and hence of the endogenous state $x' = y$. In this more general case we have to interface the process for z through the law of motion ϕ to obtain the process for x given a plan π.

7.2. The Recursive Problem

The alternative to the sequential problem is to exploit the Markov nature of the exogenous (stochastic) state. Because z_t follows a Markov process we do not actually have to carry around the whole history of shocks and instead we can subsume it in its current realization

z_t. However, this is not enough to determine the choice of the decision maker. In the sequential problem, the sequence of the exogenous state z^t, along with the initial value of the endogenous state x_0, contained all the information needed to know the current state given a contingent plan. If we drop the history of the exogenous state in favor of just its current realization, we have to introduce the current value of x that contains the (relevant) information about the previous shocks.

The problem is now

$$v(x,z) = \max_{y \in \Gamma(x,z)} \left\{ F(x,y,z) + \beta \int v\left(y, z'\right) Q\left(z, dz'\right) \right\}, \tag{7.4}$$

with its associated policy function (or in general a correspondence)

$$G(x,z) = \left\{ y \in \Gamma(x,z) \,|\, (x,z) = F(x,y,z) + \beta \int v\left(y, z'\right) Q\left(z, dz'\right) \right\}. \tag{7.5}$$

The policy function G is the object of interest for most problems as it describes the actions of the decision maker and, as we will discuss below, it induces a distribution over the outcomes of the model at the core of representative and heterogeneous agent models.

In the second type of problem in (7.2) the recursive formulation introduces the law of motion ϕ:

$$v(x,z) = \max_{y \in \Gamma(x,z)} \left\{ F(x,y,z) + \beta \int v\left(\phi\left(x,y,z'\right), z'\right) Q\left(z, dz'\right) \right\}. \tag{7.6}$$

7.3. The Principle of Optimality

The first half of the principle of optimality comes easily. We can construct the optimal contingency plans π_t out of G. The general construction is simple but required worrying about measurability when G is a correspondence. When G is a function we can simply construct the contingent plans recursively as

$$\pi_0 = G(s_0), \ldots, \pi_t\left(z^t\right) = G\left(\pi_{t-1}\left(z^t\right), z_t\right), \ldots$$

This construction of π is immediately feasible and measurable. It is left to show that a contingent plan constructed this way satisfies the optimal value of the sequential problem, $v^\star$. The following theorem gives conditions for this, the proof is in Theorem 9.2 of Stokey, Lucas, and Prescott (1989).

Theorem 7.1. *Let $(X, \mathcal{X})$, $(Z, \mathcal{Z})$, Q, F, Γ, and β be given. Γ is non-empty valued and allows for a*

measurable selection. F is $\mathcal{A}$-measurable and integrable (see Assumption 9.2 in Stokey, Lucas, and Prescott 1989). Let $v^\star$ be as in (7.3) and v as in (7.4) such that

$$\lim_{t\to\infty}\int_{Z^t}\beta^t v\left(\pi_{t-1}\left(z^{t-1}\right),z_t\right)\mu^t\left(z_0,dz^t\right)=0$$

for all feasible plans $\pi\in\Pi(s_0)$ and all initial states $s_0\in X\times Z$. Finally, let G be defined as in (7.5) and suppose that it is non-empty and allows for a measurable selection.

Then, $v=v^\star$ and any plan $\pi^\star$ generated by G achieves the maximum in (7.3).

The result can be strengthened to show that a contingent plan is optimal if and only if it is generated by G. But the details do not add to the intuition of these problems. It can also be shown to apply to the more general problem in (7.2) under slight modifications to the assumptions of the theorem.

Section 9.2 of Stokey, Lucas, and Prescott (1989) focuses on bounded problems of the type most commonly encountered in economic applications and establishes how to apply a version of the contraction mapping theorem that leads to the construction of solutions to the dynamic programing problem in (7.4) and shows that the Envelope theorem applies to these problems (under additional continuity assumptions) so as to obtain the derivatives of the value function v in therms of the payoff function F.

7.4. Markov Processes over States

Consider again the dynamic programming problem in (7.4) and suppose that its solution is a policy function G (single-valued). The objective is to describe the behavior and properties of the sequence of states $\{s_t\}=\{(x_t,z_t)\}$ as a function of the properties of the transition function of z, Q, and the policy function G. More precisely, the objective is to show that $\{s_t\}$ follows itself a Markov process with some transition function P.

The main result here is

Theorem 7.2. *Let $(X,\mathcal{X})$ and $(Z,\mathcal{Z})$ be measurable spaces with their product space $(S,\mathcal{S})$ also measurable. Let $Q:Z\times\mathcal{Z}\to[0,1]$ be a transition function for the exogenous state z and $G:S\to X$ a policy function for the endogenous state x.*

If G is measurable with respect to $\mathcal{Z}$ then the function $P:S\times\mathcal{S}\to[0,1]$ defined as

$$P\left((x,z),A\times B\right)=\begin{cases}Q(z,B) & \text{if } G(x,z)\in A\\ 0 & \text{if } G(x,z)\notin A\end{cases}$$

is a transition function on the state space $(S,\mathcal{S})$ defining a Markov process.

So, out of the solution of the dynamic problem comes a stochastic process for states. In this case the process is quite straightforward: the exogenous state evolves according to $Q(z, \cdot)$ and the endogenous state evolves deterministically because the decision maker has full control over it $\left(\phi\left(x, y, z^{'}\right) = y\right)$. Because the evolution of the endogenous state is deterministic it either is in a set $A \in \mathcal{X}$, or it is not. If it is, then the transition function P just needs to care about the conditional probability of $z^{'} \in B \in \mathcal{Z}$ given the current value of the exogenous state, z.

This construction of the Markov process will be crucial for the study of heterogenous agent models in part III.

We now turn to the properties of the Markov process characterized by P. Unsurprisingly, P inherits its properties from Q.

Theorem 7.3. *Consider the following conditions:*

(a) *X is a convex Borel set in $\mathbb{R}^{\ell}$ with Borel subsets $\mathcal{X}$;*

(b) *either Z is countable with $\mathcal{Z} = 2^{Z}$ its power set (set of all subsets) or Z is a compact (Borel) set in $\mathbb{R}^{k}$with Borel subsets $\mathcal{Z}$ and Q satisfies the Feller property;*

(c) *G is continuous.*

If all conditions are satisfied, then P satisfies the Feller property.

This result is important because, as seen in Section 5, the Feller property is key for most results in Markov processes that we care about. All the conditions in the theorem are used so that continuity makes sense. When Z is countable continuity is immediate on z and so Q already satisfies the Feller property. The insistence on Borel sets plays a similar role when the sets are uncountable.

The main offshoot of this Theorem is establishing the existing of an invariant distribution for the states under the solution of the dynamic programming problem. This is what we refer to as the *stationary distribution* of the states. See Theorem 5.4. When P also satisfies the Monotone property the invariant distribution is unique and the distribution of states converges to it, as in Theorem 5.5 when X and Z are rectangles in Euclidean spaces.

When the endogenous state evolves according to the law of motion $\phi\left(x, y, z^{'}\right)$ we must adjust the construction of the transition function for the states. The key difference is that $x^{'}$ is now stochastic and the is therefore a distribution over its possible values given the previous state (x, z). This distribution is constructed from G and Q similar to the case above.

To better understand the construction of the Markov process take a step back. The transition function $P((x, z), A \times B)$ is answering the question of how likely is it that

$\left(x', z'\right) \in A \times B$ given the values of (x, z). But the value of $x' = \phi\left(x, y, z'\right)$ depends on z'. In fact, z' is the only unknown given that we have (x, z). So, we can ask what is the set of values of z' that would make $x' \in A$ hold and then see if those values of z' are also in B. Call that set H and define it (in general as a correspondence) as

$$H\left((x, z), A\right) = \left\{z' \in Z \,|\, x' = \phi\left(x, G\left(x, z\right), z'\right) \in A\right\}.$$

The probability that $\left(x', z'\right) \in A \times B$ given (x, z) is then the probability that $z' \in H \cap B$ given z, so we define the transition function as

$$P\left((x, z), A \times B\right) = Q\left(z, H\left((x, z), A\right) \cap B\right).$$

Under appropriate assumptions P inherits the Feller property from Q as above.

8. Stochastic Recursive Competitive Equilibrium

We now turn to study dynamic economies subject to shocks using the tools developed above. The type of economies we are interested in are those in which individual agents interact with each other through markets. The main feature of these economies is therefore the price-taking behavior of agents. This is as opposed to models of imperfect competition or models of search friction with bilateral trading covered later in the course.

We start by describing the Neoclassical Growth Model, which constitutes the backbone of most models used in macroeconomics. The economy is populated by a representative firm and a representative household.[7] For now, there is no government. We first outline the problem of the firm and the household and then discuss how to cast them recursively, how to understand the stationary equilibrium, and how to compute the solution.

The firm produces using a constant-returns-to-scale technology that combines capital and labor. The firm chooses capital and labor to maximize its profits every period taking as given its current productivity (z_t) the period's prices: the rental rate of capital (r_t) and the wage rate (w_t). In this formulation, the problem of the firm is static:

$$\pi_t = \max_{\left\{k_t, \ell_t^d\right\}} f\left(z_t, k_t, \ell_t^d\right) - (r_t + \delta) k_t - w_t \ell_t^d.$$

The household chooses contingent plans for consumption and labor (or equivalently leisure) taking as given the return on their assets (r_t) and the wage rate (w_t). The household owns the firms and hence receives the profits the firm generates (π_t), which are also taken as given by the household. In the deterministic case, the household problem is

$$\max_{\{c_t, \ell_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c, \ell) \qquad \text{s.t. } c_t + a_{t+1} = (1 + r_t) a_t + w_t \ell_t + \pi_t,$$

for some initial value of assets a_0. The solution of the household problem is complex because the consumption and labor plans have to be contingent on any sequence of $\{(r_t, w_t, \pi_t)\}$ that can arise. We will return to this problem later.

[7] A more formal and complete formulation of the model would introduce a continuum of agents (or households) and firms who populate the economy. The households would be price takers and their preferences would have to be homothetic. The price-taking assumption makes their constraints linear, or put another way homogeneous of degree one, the homotheticity ensures that their choices are scale free, so that the choice of an agent with half the income of another agent is to consume half as much of every good. The firms would also be price takers and operate a technology that has constant-returns-to-scale (homogeneous of degree one). This ensures that the production choices of the firms scale one-to-one. These assumptions are enough for aggregation into a representative household and a representative firm.

Equilibrium requires that markets clear along any sequence of prices, so that

$$\ell_t = \ell_t^d; \qquad a_t = k_t; \qquad c_t + a_{t+1} = f(z_t, k_t, \ell_t) + (1+\delta) a_t.$$

However, the variable z_t is exogenous (it is not determined by any decision maker in the economy) and random. In particular $\{z_t\}$ follows a Markov Process with transition function Q and initial value z_0. This requires some extra notation can help describe the economy. To make things simpler we assume (for now) that z takes on finitely many values. Let $z^t = (z_0, z_1, \ldots, z_t)$ be the history of shocks in the economy, taking z_0 as given, and S^t the space of all histories, and $\mu_t\left(z^t\right)$ give the probability of history z^t $\left(\sum_{z^t \in S^t} \mu\left(z^t\right) = 1\right)$. Then, the problem of the household can be formally written as

$$v(a_0) = \max_{\{a_t(\cdot), c_t(\cdot), \ell_t(\cdot)\}} \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t u\left(c_t\left(z^t\right), \ell^t\left(z^t\right)\right) \mu_t\left(z^t\right)$$

$$\text{s.t.} \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} p_t\left(z^t\right) \left[\left(1 + r_t\left(z^t\right)\right) a_t\left(z^{t-1}\right) + w_t\left(z^t\right) \ell_t\left(z^t\right) + \pi_t\left(z^t\right) - c_t\left(z^t\right) - a_{t+1}\left(z^t\right)\right],$$

where we abuse notation by writing $a_0\left(z^{-1}\right) = k_0$ and $p_t\left(z^t\right)$ is the price of future resources (the stochastic discount factor).

An Arrow-Debreu equilibrium of this economy is therefore defined as sequences of functions that depend on histories of shocks. That is, an equilibrium is a set sequences for quantities $\left\{a_t(\cdot), c_t(\cdot), \ell_t(\cdot), k_t(\cdot), \ell_t^d(\cdot), \pi_t(\cdot)\right\}$ and prices $\left\{p_t(\cdot), r_t(\cdot), w_t(\cdot)\right\}$ such that

(a) Households maximize the present discounted value of utility with $\{a_t(\cdot), c_t(\cdot), \ell_t(\cdot)\}$, taking as given prices $\left\{p_t(\cdot), r_t(\cdot), w_t(\cdot)\right\}$ and transfers $\left\{\pi_t\left(s^t\right)\right\}$.

(b) Firms maximize their period profits with $\left\{k_t(\cdot), \ell_t^d(\cdot)\right\}$ taking as given prices $\{r_t(\cdot), w_t(\cdot)\}$

$$\pi_t\left(z^t\right) = f\left(z_t, k_t\left(z^t\right), \ell_t^d\left(z^t\right)\right) - \left(r_t\left(z^t\right) + \delta\right) k_t\left(z^t\right) - w_t\left(z^t\right) \ell_t^d\left(z^t\right).$$

(c) Markets clear for every history

$$\ell_t\left(z^t\right) = \ell_t^d\left(z^t\right); \qquad a_t\left(z^{t-1}\right) = k_t\left(z^t\right);$$

$$c_t\left(z^t\right) + a_{t+1}\left(z^t\right) = f\left(z_t, k_t\left(z^t\right), \ell_t\left(z^t\right)\right) + (1+\delta) a_t\left(z^{t-1}\right).$$

(d) The initial conditions are satisfied, so that $a_0\left(z^{-1}\right) = k_0$.

In the Arrow-Debreu equilibrium all trading happens at time 0, taking as given (k_0, z_0). We can alternatively define a sequential markets equilibrium for this economy. This definition avoids introducing the stochastic discount factor. WE instead have the equilibrium be a set sequences for quantities $\left\{a_t(\cdot), c_t(\cdot), \ell_t(\cdot), k_t(\cdot), \ell_t^d(\cdot), \pi_t(\cdot)\right\}$ and prices $\{r_t(\cdot), w_t(\cdot)\}$ such that

(a) Households maximize the present discounted value of utility with $\{a_t(\cdot), c_t(\cdot), \ell_t(\cdot)\}$, taking as given prices $\{r_t(\cdot), w_t(\cdot)\}$ and transfers $\left\{\pi_t\left(s^t\right)\right\}$.

$$v(a_0) = \max_{\{a_t(\cdot), c_t(\cdot), \ell_t(\cdot)\}} \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t u\left(c_t\left(z^t\right), \ell^t\left(z^t\right)\right) \mu_t\left(z^t\right)$$
$$\text{s.t. } \left(1 + r_t\left(z^t\right)\right) a_t\left(z^{t-1}\right) + w_t\left(z^t\right) \ell_t\left(z^t\right) + \pi_t\left(z^t\right) \geq c_t\left(z^t\right) a_{t+1}\left(z^t\right) \qquad \forall_t \forall_{z^t \in Z^t},$$

(b) Firms maximize their period profits with $\left\{k_t(\cdot), \ell_t^d(\cdot)\right\}$ taking as given prices $\{r_t(\cdot), w_t(\cdot)\}$

$$\pi_t\left(z^t\right) = f\left(z_t, k_t\left(z^t\right), \ell_t^d\left(z^t\right)\right) - \left(r_t\left(z^t\right) + \delta\right) k_t\left(z^t\right) - w_t\left(z^t\right) \ell_t^d\left(z^t\right).$$

(c) Markets clear for every history

$$\ell_t\left(z^t\right) = \ell_t^d\left(z^t\right); \qquad a_t\left(z^{t-1}\right) = k_t\left(z^t\right);$$

$$c_t\left(z^t\right) + a_{t+1}\left(z^t\right) = f\left(z_t, k_t\left(z^t\right), \ell_t\left(z^t\right)\right) + (1 + \delta) a_t\left(z^{t-1}\right).$$

(d) The initial conditions are satisfied, so that $a_0\left(z^{-1}\right) = k_0$.

The sequential problem of the agents in this economy must therefore keep track of an impossible large state vector because the optimal choices depend on its complete history. Tackling this problem proves to be impractical if not impossible. Because of that we seek to re-formulate the household's problem in its recursive form. Doing so will also provide us with a definition of recursive competitive equilibrium (RCE).

8.1. Recursive Competitive Equilibrium

The objective now is to cast the problem recursively. This allows us to provide a clearer definition of equilibrium and (later) to solve the problem using the tools of dynamic programming developed above and the computational tools introduced below.

The main question of the recursive problem is what constitute the relevant state of the economy. Crucially, the household is a price taker: they have no clue about the aggregate effect of their choices. Equivalently you can think of the household as atomistic, so that they know that their (individual) actions have no effect on the aggregates of the economy. However, the states must provide enough information to solve the problem and to forecast how the states themselves evolve.

In general we have $s_t = (a_t, z_t, \Gamma_t)$ be the state of the individual representative household, it includes the household's capital (a), the aggregate productivity (z), and the distribution of states in the economy (Γ). Keeping track of the distribution of states is in principle necessary in order to compute prices. The sequential problem circumvents this problem by making all variables (prices included) depend on the full history of exogenous shocks. That, of course, contains all the information necessary. However, the recursive problem cannot depend the history of states, $s^t = (s_0, s_1, \ldots, s_t)$.

To solve this issue we use the structure of the economy. The key is that the economy we are studying can be aggregated. This means that the underlying distribution of households does not matter and that only the aggregate (average) capital is relevant: $K_t = \int a_t d\Gamma_t$. Because of this, an individual household only needs to keep track of aggregate capital and not of the whole distribution. Crucially, knowing the aggregate capital is enough to compute the relevant prices in the economy. This makes the state $s_t = (a, z, K)$.[8]

The household's recursive problem is then

$$
\begin{aligned}
V(\underbrace{a}_{\text{Ind. State}}, \underbrace{z, K}_{\text{Agg. States}}) = \max_{\{c,\ell,a'\}} \; & u(c,\ell) + \beta \int_{z'} V\left(a', z', K'\right) Q\left(z, dz'\right) \\
\text{s.t.} \quad & c + a' = (1 + R(z,K))\, a + W(z,K)\, \ell + \pi(z,K)\,; \\
& K' = G_k(z,K)\,.
\end{aligned}
$$

This problem looks a lot like the dynamic programming problems we h ave discussed before, with the exception that it has to include functions that map the aggregate states into prices, $r = R(z,K)$ and $w = W(z,K)$, and that incorporate the evolution of the aggregate state, $G_k(z,K)$. These functions are taken as given by the household and are found as part of the equilibrium defined below.

An Recursive Competitive Equilibrium (RCE) is a set of a value function V, policy functions g_k and g_ℓ, updating functions G_k and price functions R and W such that:

[8] The state is often written as (k, z, K) emphasizing the difference between the "little k" faced by an individual household and the "big K" faced by the economy as a whole.

(a) The value function V and policy functions g_a and g_ℓ solve the household's dynamic programming problem, taking as given the updating and price functions G_k, R, and W.

(b) The firms maximize profits taking prices as given. This implies that the pricing functions R and W satisfy the firm's first order conditions

$$R(z, K) = f_k(z, K, G_\ell(z, K)) - \delta \qquad W(z, K) = f_\ell(z, K, G_\ell(z, K)),$$

where $L = G_\ell(z, K)$ gives the aggregate equilibrium level of labor as a function of the aggregate states.

(c) Updating functions G_k and G_ℓ are consistent with individual optimization

$$G_k(z, K) = g_a(K, z, K);$$
$$G_\ell(z, K) = g_\ell(K, z, K).$$

Crucially, consistency only has to apply in equilibrium. This allows us device an algorithm to solve for the RCE. The key is that consistency does not have to hold as you converge to the equilibrium because the household dynamic problem can be solved given any update functions.

Algorithm

input : Guess for updating functions (G_k, G_ℓ)
output : $V, g_k, g_\ell, G_k, G_\ell$

1. Solve the DP problem of the agent given G_k, G_ℓ:
   $(V, g_k, g_\ell) = T(V; G_k, G_\ell)$ (a fixed point problem) ;
2. Update updating functions:
   $G_k(z, K) = g_k(K, z, K) \qquad G_\ell(z, K) = g_\ell(K, z, K)$;
3. Check convergence in updating functions ;
4. Repeat (1)-(3) until convergence ;

Algorithm 1: RCE Algorithm

Solving the problem requires solving the fixed point characterizing the solution to the household's Bellman equation. Unfortunately, this implies that the curse of dimensionality applies because you have to solve the agent's problem off-equilibrium. That is, you need to know $g_k(a, z, K)$ for any combination of (a, K), even though in equilibrium $a = K$. For efficient economies, where the first welfare theorem applies, we can avoid this problem by focusing on the planner's problem and then constructing the equilibrium prices. However,

most applications involve economies with market failures, or distortions (like taxes!) that prevent us from doing this.

8.2. Stationary Equilibrium: What does it mean?

We now discuss a key property of the equilibrium. When the process for the exogenous shocks is stationary and the problem of the firm and the household satisfy certain regularity conditions, the equilibrium converges to a stationary equilibrium. The objective of this subsection is to discuss what that means.

First we discuss informally what the regularity conditions are. What we want is to establish conditions that produce "well-behaved" policy functions for the endogenous states (capital in this case), that is, continuous and monotone. The problem must also satisfy standard Inada and transversality conditions that guarantee that it is effectively bounded.

Continuity is inherited from the continuity of payoff functions (in this case u and f) and the properties of the feasible correspondence. See the Theorem of the Maximum for more on this. It also requires that the Markov Process for z satisfies the Feller property (because of the expectation in the value function).

Monotonicity requires having a sense of what is "better" in the context of the problem. We interpret z as productivity and hence it makes sense to interpret higher values of z as being better. For the resulting solution to be monotone in the states (k, z, K) we need the payoffs to be monotone (as we usually assume), the transition function Q to be monotone (see Definition 5.4 and Proposition 5.5).

Crucially, the conditions imposed over the Markov process for z already guarantee that it is a stationary process with a unique invariant distribution to which it converges, regardless of its initial condition, z_0. The question is whether the stochastic process followed by the equilibrium variables (capital) are also stationary and converge to an invariant distribution. The stationary equilibrium is then just a recursive competitive equilibrium for which the stochastic process of the endogenous variables is stationary.

The concept of the stationary equilibrium is immediate in the non-stochastic case. Then, policy functions map (deterministically) a value of the endogenous state (capital) into a new value for itself. A sequential markets equilibrium and a recursive competitive equilibrium are defined just as in the previous subsection, except that they do not depend on the history of shocks (as there are none). The equivalent of being stationary and having an invariant distribution is then to have a single steady-state value that satisfies the equilibrium conditions. Intuitively, the deterministic case is like the stochastic one with degenerate distributions, so the invariant degenerate distribution puts full probability on a single value

of the variable. That is the steady state.

In the stochastic case, the equilibrium is composed by functions that map the realization of the stochastic process for productivity (z) into values for quantities (K) and prices (r, w). When these functions are measurable with respect to the underlying productivity process they form themselves a stochastic processes (the sequence of the random variables for quantities and prices). The construction of these stochastic processes is carried out as in Section 7.4. Then, the stationary equilibrium is an endogenously determined probability distribution over the state variables, with the properties of a Markov chain induced by the policy functions and the exogenous process for shocks.

8.3. Computing the Equilibrium: Value function iteration (the discrete case)

The objective is now to illustrate how to compute the equilibrium. We take advantage of the fact that the the economy we described is efficient and that we can therefore use the planner's problem to construct all the equilibrium functions. We further simplify the problem by getting rid of the labor choice. This simplifies the exposition. The planner's problem is to choose aggregate quantities subject to feasibility:

$$\begin{aligned} V(K,z) = \max_{\{C,K\}} \; & u(C) + \beta \int_{z'} V\left(K', z'\right) Q\left(z, dz'\right) \\ \text{s.t.} \quad & C + K' = f(z,K) + (1-\delta)K. \end{aligned}$$

The solution to the planner's problem immediately gives us the equilibrium for individual quantities $\{c, a, k\}$ and prices $\{r\}$ by setting

$$c = C \qquad k = a = K \qquad r = f_k(z,K,L) - \delta.$$

We can solve the problem using value function iteration. The key is that, unlike the household problem above, we do not need to condition on the functions for prices or aggregates.

Algorithm

Result: Fixed Point of Bellman Operator T

$n = 0$; $V^0 \in S$; $dist_V = 1$;
while $n \leq N$ & $dist_V > tol_V$ **do**
    $V^{n+1} = TV^n$;
    $dist_V = d\left(V^{n+1}, V^n\right)$;
end
if $dist_V \leq tol_V$ **then**
    Obtain g from TV^n;
else
    You are in trouble... something went wrong;
end

Algorithm 2: Value Function Iteration

While actually solving the problem as posed can be challenging (because of the difficulties in making continuous choices and taking expectations) it is possible to approximate it with a related (and much simpler problem) in which the whole problem is discretized. This is the simplest implementation of value function iteration. The key advantage is that there are no continuous choices (or integral), instead, the problem consists in choosing the best value of capital from a pre-specified and fixed grid (hence its common name of grid search).

The approximation of the (continuous) dynamic programming problem with a discrete one does not require the use of derivatives and is robust to complications such as kinks in the choice set, or asymmetries in the functions being used. It is also very easy to implement. However, it is not (in general) a very precise approximation, and it has a low rate of convergence, making it slow. This problem is compounded by the curse of dimensionality, which bites particularly hard for discrete problems because they require large state spaces in order to improve the accuracy of the approximation (more on how to gauge accuracy at the end of this section).

The discrete problem is

$$V\left(k_i, z_j\right) = \max_{k' \in \{k_1, \ldots, k_I\}} U\left(z_j k_i^{\alpha} + (1-\delta)\, k_i - k'\right) + \beta \sum_{j'=1}^{J} \Pi_{jj'} V\left(k', z_{j'}\right)$$

where we have replaced the constraint, leaving consumption implicitly defined by the choice of capital which is now discrete. Conveniently, everything in the problem is now a

vector or a matrix:

$$V = \left[V_{ij}\right]; \qquad \vec{k} = \left[k_1, \ldots, k_I\right]^T; \qquad \vec{z} = \left[z_1, \ldots, z_J\right]^T; \qquad \Pi = \left[\Pi_{jj'} = Q\left(z_j, \left\{z_{j'}\right\}\right)\right];$$

$$U = \left[U_{ijh} = u\left(z_j k_i^{\alpha} \qquad \vec{k} = \left[k_1, \ldots, k_I\right]^T - k_h'\right)\right];$$

This allows us to solve the problem of choosing $k' \in \{k_1, \ldots, k_I\}$ for every pair of $\left(k_i, z_j\right)$ in two different ways. Either looping through all the pairs of states, or collapsing the matrix of payoffs along its third dimension.

Algorithm

Function T(*V_old,k_grid,z_grid,*α, β)**:**

n_k = length(k_grid)
n_z = length(z_grid)
V = zeros(n_k, n_z); G_{kp} = zeros(n_k, n_z); G_c = zeros(n_k, n_z)

for *i = 1:*n_k **do**
  for *j = 1:*n_z **do**
    V_aux = zeros(n_k)
    for *h = 1:*n_k **do**
      V_aux[h] = u(k_grid[i],z_grid[j],k_grid[h];α, β) + βsum($\Pi_{j,j'}$V_old[h,j'])
    end
    V[i,j], G_{kp}[i,j] = findmax(V_aux)
    G_c[i,j] = f(k_grid[i],z_grid[j]) + (1-δ)k_grid[i] - k_grid[G_{kp}[i,j]]
  end
end
return V, G_{kp}, G_c

Algorithm 3: Bellman Operator: Discrete grid with loops

This algorithm can be sped up in many programming languages by operating directly on matrices, instead of relying on loops. This also leads to a more concise program.

In order to evaluate the accuracy of the solution we make use of **Euler Residuals**. These are the residuals in the first order conditions of the actual problem, which should be zero

for the correct solution.

$$\textbf{Res}\left(k, z_j | g\right) = \underbrace{\frac{\beta \sum_{j'=1}^{J} \Pi_{jj'} U'\left(f\left(z_{j'}, g\left(k, z_j\right)\right) + (1-\delta)k - g\left(g\left(k, z_j\right), z_{j'}\right)\right) \frac{\partial f\left(z_{j'}, g\left(k, z_j\right)\right)}{dk}}{U'\left(f\left(z_j, k\right) + (1-\delta)k - g\left(k, z_j\right)\right)} - 1}_{\text{\% Error in Euler Equation}}$$

We can evaluate these residuals for values of capital in the grid used to solve the problem. The Euler residuals can help diagnose if there are parts of the state space that need to be denser (say having more grid points near low-levels of capital where the curvature of the problem is higher) or whether the approximation to the solution is satisfactory in general.

Having approximated the solution to the dynamic programming problem we can construct a Markov process for the states in the economy and obtain their stationary distribution. In the special case of discrete grid search, this is facilitated by the fact that the choice of future capital is always in the grid. We can then construct a Markov transition matrix for the state vector of the economy. In this case, the state is $s = (k, z)$, and the state space can be express as a long vector

$$\vec{s} = \left((k_1, z_1), (k_1, z_2), \ldots, (k_1, z_J), (k_2, z_1), \ldots, (k_2, z_J), \ldots, (k_I, z_1), \ldots, (k_I, z_J)\right).$$

The Markov transition matrix is therefore a square matrix with $I \times J$ rows and columns. We can build the matrix following the steps in Section 7.4. The key is that z evolves independently following the transition matrix Π, while k evolves deterministically. The matrix is then

$$\Psi_{\left(ij, i'j'\right)} = \Pi_{jj'} \chi_{\left\{i' = G_{kp}(i,j)\right\}}.$$

The properties of the stochastic process for capital and productivity then follow from this matrix. For instance, the stationary distribution is obtained from the eigenvector associated with the matrix unit eigenvalue.

8.4. Recursive Competitive Equilibrium Example: Sovereign Default

Sovereign default models form a large literature on international economics and are also a great example of stochastic dynamic programming. The choice to default required dynamics in order to have an opportunity and a reason to borrow and to introduce a relevant tradeoff around default. The model must also be stochastic in order to induce the situations in which a decision maker borrows and then finds themselves in a situation

where they opt to default. These models are also inherently inefficient, preventing the use of the planner's problem. The reason is that markets must be incomplete in order for the decision maker not to be able to fully insure against risk. Further, the borrowing and default decisions depend on prices, which are taken as given by the decision maker, but that respond endogenously (in equilibrium) to the decision maker's choices.

The basic sovereign default model follows Arellano (2008). It is a stochastic endowment economy. Output (or income) follows an exogenous (discrete) Markov process described by an underlying state $s \in \{s_1, \ldots, s_N\} = S$. The decision maker, say the government, chooses borrowing/saving and whether to default on debt. The decision is made taking as given a price schedule for debt (q) that depends on the state of the economy and the debt of the government.

The dynamic programming problem is then split in two. First there is the (discrete) choice of whether to default. The state of the government is the pair (s, b), where s is the exogenous state of the economy and b is the level of outstanding bonds to be paid to the government (so $b > 0$ means savings and $b < 0$ means debt). If the government pays they get to access the lending markets and gets a value of $V(s, b)$ but if it defaults it is thrown into financial autarky and gets a value $V^A(s)$ (that no longer depends on b because there is no debt and no access no markets). The value of the government, $V^\star$, reflects the upper envelope of this choice,

$$V^\star(s, b) = \max_{d \in \{0,1\}} \left\{ (1 - d(s, b)) V(s, b) + d(s, b) V^A(s) \right\}.$$

The value of having access to the financial markets is

$$\begin{aligned} V(s, b) = \max_{\{c, b'\}} & \left\{ \frac{c(s, b)^{1-\sigma} - 1}{1 - \sigma} + \beta \sum_{s' \in S} \pi\left(s, s'\right) V^\star\left(s', b'\right) \right\} \\ \text{s.t.} \quad & c(s, b) \leq y(s) + b - q(s, b) b'(s, b) \\ & -B \leq b'(s, b) \qquad \text{[B: borrowing limit]} \\ & 0 \leq c(s, b). \end{aligned}$$

The value of going into autarky is

$$V^A(s) = \frac{(h(y(s)))^{1-\sigma} - 1}{1 - \sigma} + \beta \sum_{s' \in S} \pi(s, s') \left(\theta V^\star(s', 0) + (1 - \theta) V^A(s') \right),$$

where $h(y) < y$ is a function that penalizes output, representing the costs of autarky.

There is no choice for the government as there are no markets that allow it to smooth consumption. The government returns to the markets with probability $\theta \geq 0$.

There is also a sector of financial intermediaries that operate in perfect competition. They are risk neutral and lend in an actuarially fair manner, meaning that their prices reflect the expected costs of default and so they break even in expectation. The profits of one of these financial intermediaries are

$$\text{Pr} = qb' - \frac{1-\delta}{1+r}b',$$

where δ is the (endogenously determined) probability of default taken as given by the intermediary. This probability comes, in equilibrium, from the optimal default choice of the government, $g^D(s, b)$, and satisfies

$$\delta = E_{s'}\left[g^D\left(s', b'\right)|s\right] \qquad \text{where } g^D\left(s', b'\right) = \begin{cases} 1 & \text{if default} \\ 0 & \text{if no default} \end{cases}.$$

Free entry gives the zero profit (break even) condition that Pr = 0 and so the debt price is (in equilibrium):

$$q\left(s, b'\right) = \begin{cases} \frac{1 - \sum_{s' \in S} \pi(s') g^D(s', b')}{R} & \text{if } b' < 0 \\ \frac{1}{R} & \text{if } b' \geq 0 \end{cases}$$

A **Recursive Competitive Equilibrium** is then a set of value functions $\left\{V^\star, V^A, V\right\}$, policy functions $\left\{g^c, g^b, g^D\right\}$, and a price functional $\left\{q\left(s, b'\right)\right\}$ such that

(a) The value functions solve the Bellman equations of the government and the policy functions achieve the maximum in those equations taking the price q as given.

(b) The price q satisfies the zero profit or break even condition of the financial intermediaries

Part III

Heterogeneous Agent Models

9. An Endowment Economy

Huggett (1993)

10. A Production Economy

Aiyagari (1994)

11. A Production Economy with Aggregate Risk

Krusell and Smith (1998)

12. Implications for Wealth Distributions

Benhabib and Bisin (2018)Stachurski and Akira Toda (2019, 2020)

13. An Economy with Production-Risk

Angeletos (2007)

14. An Economy with Heterogeneous Returns

Guvenen et al. (2023)

Part IV

Stochastic Calculus

This part of the course develops the mathematical tools necessary to study how random variables affect optimization problems. We start with an overview of the most relevant stochastic processes that we will encounter in the applications to come. The most important result is Ito's Lemma, which defines the way in which we can take derivatives of functions that depend on diffusions. Then we can apply Ito's Lemma to problems of dynamic optimization, with special attention to stopping time problems. Finally we apply it to the characterization of the distribution of a random variable. This is done by means of the Kolmogorov forward equation.

All these sections follow closely Dixit and Pindyck (1994), with some portions adapted from Stokey (2009).

15. Stochastic Processes

15.1. Definitions

The idea now is to study sequences of random variables. A stochastic process is similar to a random variable, with the difference that it also depends on time. Adding the time dimension adds notation, but it does not change any of the main ideas. For convenience we will first go over the definition of a random variable.

Definition 15.1. (Random variable) Let $(\Omega, \mathcal{A}, P)$ be a probability space and $x : \Omega \to \mathbb{R}$ a real valued function. x is a random variable if and only if x is measurable, that is, if and only if $x^{-1}(B) \in \mathcal{A}$ for all $B \in \mathcal{B}$, where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}$. We further establish the same notation:

(a) An outcome is an element $\omega \in \Omega$.

(b) An event is a measurable subset of Ω: $A \in \mathcal{A}$.

(c) The real number $x(\omega)$ is a realization of the random variable.

(d) The probability measure for x is then: $\mu(B) = P\left(x^{-1}(B)\right) = P\left(\left\{\omega \in \Omega | x(\omega) \in B\right\}\right)$, for $B \in \mathcal{B}$.

(e) The distribution function for f is: $G(b) = \mu\left((-\infty, b]\right)$, for $b \in \mathbb{R}$.

Now we can work on adding the time dimension to the definition of a random variable. In general time can be discrete or continuous, but in what follows we will assume that time is continuous starting at 0 and going on forever, so $t \in [0, \infty)$. Intuitively a stochastic process is formed by function $x : [0, \infty) \times \Omega \to \mathbb{R}$ that gives a realization for every outcome and time. At every point in time the random variable takes a variable, the sequence of those values forms the realization (path) of the stochastic process.

The question is on how to measure the possible outcomes of the random variable through time. We need a way of determining where the random variable is at a certain point in time, and where it has been, but that does not provide information about the value of future realizations. This is achieved using a filtration.

Definition 15.2. (Filtration) Let $\mathcal{A}$ be a σ-algebra. The set $\mathbb{A} = \left\{\mathcal{A}_t | t \geq 0\right\}$ is a filtration if $\mathcal{A}_t \subseteq \mathcal{A}$ and $\mathcal{A}_s \subseteq \mathcal{A}_t$ for all $t \geq 0$ and $s \leq t$. $\mathcal{A}_t$ is the set of events known at time t.

Now we can define a stochastic process as a function that is measurable in a filtered space.

Definition 15.3. (Stochastic Process) Let $(\Omega, \mathbb{A}, P)$ be a filtered probability space with a time index $t \in \mathbb{R}_+$, and let $\mathcal{B}_+$ be the Borel sets of $\mathbb{R}_+$. A stochastic process is a function $x : [0, \infty) \times \Omega \to \mathbb{R}$ that is measurable with respect to $\mathcal{B}_+ \times \mathcal{A}$ (that is, x is jointly measurable in (t, ω)). Moreover:

(a) For all $t \in \mathbb{R}_+$ and $\omega \in \Omega$, $x(t, \omega)$ is measurable with respect to $\mathcal{A}_t$, where $\mathcal{A}_t$ is in the filtration $\mathbb{A}$.

(b) For all $t \in \mathbb{R}_+$, $x(t, \cdot) : \Omega \to \mathbb{R}$ is an ordinary random variable on the probability space $(\Omega, \mathcal{A}_t, P_t)$.

(c) For all $\omega \in \Omega$, $x(\cdot, \omega) : \mathbb{R}_+ \to \mathbb{R}$ is a Borel measurable function. This is called the **sample path** of x.

15.2. Discrete time examples

It is not hard to come up with examples of discrete time stochastic processes. They are often used to model the behavior of many stationary economic variables by means of ARMA(p,q) representations, as well non-stationary variables usually related to random walks.

To fix ideas we start with the simple example of a (fair) coin toss. There are two possible outcomes, so $\Omega = \{H, T\}$, when tossing the coin is always possible to know which outcome occurred, and whether or not the coin was tossed, this gives: $\mathcal{A} = \left\{\{H\}, \{T\}, \emptyset, \Omega\right\}$. Finally the probability distribution P assigns values to sets in the σ-algebra $\mathcal{A}$:

$$P(\{H\}) = P(\{T\}) = \frac{1}{2} \qquad P(\emptyset) = 0 \qquad P(\Omega) = 1$$

Now we can define a random variable $\epsilon : \Omega \to \mathbb{R}$ as: $\epsilon(H) = 1$ and $\epsilon(T) = -1$. ϵ is a random variable with respect to the probability space $(\Omega, \mathcal{A}, P)$. As will be the case almost always we can dispense of the outcome space Ω for most applications and just refer to the random variable and the probability distribution induced over its values. In this way we have: $\epsilon \in \{-1, 1\}$ with $\Pr(\epsilon = 1) = \Pr(\epsilon = -1) = 1/2$.

Furthermore we can extend this example to define the stochastic process that comes up from the repeated coin toss. In this case time is discrete and finite $t \in \{1, 2, 3\}$ and at each time a coin is tossed, then the random variable variable ϵ_t is defined as the value of ϵ given the outcome of the t^{th} coin toss. The sequence $\{\epsilon_t\}_{t=1}^{3}$ is a stochastic process with respect to the filtered probability space $(\Omega, \mathbb{A}, P)$, where:

$$\Omega = \{(H, H, H), (H, H, T), (H, T, H), (H, T, T), (T, H, H), (T, T, H), (T, H, T), (T, T, T)\}$$

$$\mathcal{A} = 2^{\Omega} \qquad P(\omega) = \frac{1}{8} \quad \forall_{\omega \in \Omega}$$

The filtration is established taking into account that at each point in time only the outcome of current and past tosses is known:

$$\mathcal{A}_1 = \left\{\emptyset, \Omega, \{(H,H,H),(H,H,T),(H,T,H),(H,T,T)\}, \{(T,H,H),(T,T,H),(T,H,T),(T,T,T)\}\right\}$$

$$\mathcal{A}_2 = \left\{\emptyset, \Omega, \{(H,H,H),(H,H,T)\}, \{(T,H,H),(T,H,T)\}, \{(H,T,H),(H,T,T)\}, \{(T,T,H),(T,T,T)\}\right\}$$

$$\mathcal{A}_3 = \mathcal{A}$$

So, in the first σ-algebra all outcomes for which the first toss comes up heads are indistinguishable from each other, in the second σ-algebra one can distinguish between outcomes that have the sequence $\{H, T\}$ and $\{H, H\}$, but no information is given about the outcome of third toss. This same ideas apply if time goes on forever, so we can define our stochastic process over $t \in \mathbb{N}$.

In the previous example the stochastic process obtained satisfies the property of being iid (identically and independently distributed). The values of the stochastic process at each point in time are independent from its previous values, and they all have the same probabilities of occurring.

We now use our stochastic process $\{\epsilon_t\}$ to define a random walk. Random walks are particularly useful to understand the behavior of continuous time stochastic processes. As we will see the building block of most of them is the continuous time approximation of a random walk.

Example 15.1. (Random Walk Process) Consider a stochastic process x. Denote by x_t the value of x at time t, and fix the initial value x_0. x_t is assumed to evolve according to:

$$x_t = x_{t-1} + \epsilon_t \qquad \text{for } t \geq 1$$

ϵ_t is a random variable that can take two values $\{-1, 1\}$, and its probability distribution is independent of time, so that: $\Pr(\epsilon_t = 1) = \Pr(\epsilon_t = -1) = \frac{1}{2}$.

Given the starting value x_0 the variable x_t can only take on discrete values. For instance, for $x_0 = 0$ and t odd they are $\{-t, \ldots, -1, 0, 1, \ldots, t\}$, and for t even they are $\{-t, \ldots, -2, 0, 2, \ldots, t\}$. These values tell you which paths of the process cam be known at time t.

Finally, this process has no drift. Given an initial value x_0 the expected value of x_t

for any t is x_0 ($E[x_t] = x_0$), this follows from the expected value of each change being $E[x_t - x_{t-1}] = E[\epsilon_t] = 0$.

This process can be generalized in many ways. The most useful one for our purposes is to allow for drift, which can be done by changing the probabilities of the random variable ϵ_t, letting $\Pr(\epsilon_t = 1) = p$ and $\Pr(\epsilon_t = -1) = 1 - p$ achieves the desired result. If $p > 1/2$ the process will have positive drift.

15.3. Brownian motion (Wiener processes)

A Brownian motion, or Weiner process, is a continuous time stochastic process ($W(t)$) that satisfies three properties:

(a) $W(t)$ has continuous sample paths.

(b) $W(t)$ has stationary independent increments.

(c) Increments of $W(t)$ over a finite interval of time are normally distributed with variance that increases linearly in time.

The first property implies that a Brownian motion has no jumps, so as the time interval goes to zero the change in the process must also go to zero. The second and third properties imply that the change in $W(t)$ over some interval of length Δt must satisfy:

$$\Delta W = \epsilon_t \sqrt{\Delta t} \qquad \epsilon_t \sim N(0, 1)$$

which we write as $dW = \epsilon_t \sqrt{dt}$ as $\Delta t \to 0$. This implies that:

$$E[dW] = E[\epsilon_t]\sqrt{dt} = 0 \qquad V[dW] = E\left[\epsilon_t^2\right] dt = dt$$

Moreover we assume that ϵ_t is serially uncorrelated, i.e., $E[\epsilon_t \epsilon_s] = 0$ for $t \neq s$, so the values of dW for any two different time intervals are independent.

Its easy to note the relation between the Brownian motion and the random walk processes. In discrete time we had $\Delta x_t = x_t - x_{t-1} = \epsilon_t \Delta t$, where $\Delta t = 1$. We will use this fact when approximating Brownian motions using random walks as $\Delta t \to 0$.

To see that this representation implies the third property consider a time interval that starts at t and ends at T, and divide into n intervals of length $\Delta t = T/n$. Then we have:

$$W(t + T) - W(t) = \sum_{i=1}^{n} \epsilon_i \sqrt{\Delta t}$$

What we want to show is that $W(t+T) - W(t) \sim N(0,t)$. To prove this we can use the Central Limit Theorem:

Theorem 15.1. ***(Central Limit Theorem)*** *If* $\{\epsilon_1, \epsilon_2, \epsilon_3, \ldots\}$ *are iid (but not necessarily normal) with* $E\left[\epsilon_i\right] = \mu < \infty$ *and* $V\left[\epsilon_i\right] = \sigma^2 < \infty$*, then* $Z_n = \sqrt{n}\frac{\sum_{i=1}^{n}\epsilon_i - n\mu}{\sigma} \to N(0,1)$ *as* $n \to \infty$.

Note that ϵ_i already satisfies being iid and $E\left[\epsilon_i\right] = 0$ and $V\left[\epsilon_i\right] = 1$, so $Z_n = \sqrt{n}\sum_{i=1}^{n}\epsilon_i$. Then we can write:

$$W(t+T) - W(t) = \sqrt{T}Z_n$$

By the CLT this converges to a $N(0,T)$ as $n \to \infty$.

A Brownian motion can be generalized to have drift μ and variance σ^2. This is done by adjusting the way the increments of the stochastic process work:

$$dx = \mu dt + \sigma dW$$

In this case the increments are given by a non-stochastic component μdt, which indicates that the process will drift by μ per unit of time deterministically if there are no shocks, and by a stochastic component σdW, where σ is scaling the variance of the increments of the Weiner process W. This process satisfies:

$$E\left[dx\right] = \mu dt \qquad V\left[dx\right] = \sigma^2 dt$$

15.3.1. Random walk approximation of a Brownian motion

As mentioned above we can use the similarities between the increments of a Brownian motion and the increments of a random walk to approximate continuous time processes using discrete time ones. This is important because of two reasons: it helps explain the mechanics of the continuous time model, and it provides an algorithm for simulation in the computer.

Our objective is to approximate the a Brownian motion with drift:

$$dx = \mu dt + \sigma dW$$

We will approximate with a discrete time process y whose increments are h with probability p and $-h$ with probability $1-p$. This gives:

$$E\left[\Delta y\right] = ph - (1-p)h = (2p-1)h$$

$$V\left[\Delta y\right] = E\left[(\Delta y)^2\right] - \left(E\left[\Delta y\right]\right)^2 = \left(1 - (2p-1)^2\right) h^2$$

In order to get the approximation we need to choose values for $h,\ p$ and Δt so that:

$$\mu\Delta t = (2p-1)\, h$$
$$\sigma^2\Delta t = 4p\,(1-p)\, h^2$$

Solving for p we get:

$$p^2 - p + \frac{\sigma^2}{4\left(\sigma^2 + \mu^2\Delta t\right)} = 0$$

The roots of these equation are:

$$\begin{aligned} p &= \frac{1}{2}\left(1 \pm \sqrt{1 - \frac{\sigma^2}{\left(\sigma^2 + \mu^2\Delta t\right)}}\right) \\ &= \frac{1}{2}\left(1 \pm \frac{\mu\sqrt{\Delta t}}{\sqrt{\sigma^2 + \mu^2\Delta t}}\right) \\ &\approx \frac{1}{2}\left(1 \pm \frac{\mu}{\sigma}\sqrt{\Delta t}\right) \end{aligned}$$

where the approximation follows if Δt is small enough relative to σ^2/μ^2, because we are taking Δt close to zero this assumption is satisfied. We further choose only the "+" root because that way $p \geq 1/2$ when $\mu \geq 0$.

Now we can find a value for h:

$$\begin{aligned} \sigma^2\Delta t &= 4p\,(1-p)\, h^2 \\ \sigma^2\Delta t &= 2\left(1 + \frac{\mu}{\sigma}\sqrt{\Delta t}\right)\left(1 - \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{\Delta t}\right)\right) h^2 \\ \sigma^2\Delta t &= \left(1 + \frac{\mu}{\sigma}\sqrt{\Delta t}\right)\left(1 - \frac{\mu}{\sigma}\sqrt{\Delta t}\right) h^2 \\ \sigma^2\Delta t &= \left(1 - \left(\frac{\mu}{\sigma}\right)^2 \Delta t\right) h^2 \\ \sigma^2\Delta t &\approx h^2 \\ \sigma\sqrt{\Delta t} &\approx h \end{aligned}$$

As before we can disregard the term $\left(\frac{\mu}{\sigma}\right)^2 \Delta t$ as long as Δt is small enough relative to σ^2/μ^2.

As an exercise can verify that the first equation also holds:

$$\mu\Delta t = (2p-1)\, h$$

$$\mu\Delta t = (2p-1)\,\sigma\sqrt{\Delta t}$$
$$\frac{\mu}{\sigma}\sqrt{\Delta t} = 2p-1$$
$$\frac{1}{2}\left(1+\frac{\mu}{\sigma}\sqrt{\Delta t}\right) = p$$

In order to simulate a Brownian motion with parameters (μ, σ) we can do as follows:

(a) Set a Δt small relative to $\frac{\sigma^2}{\mu^2}$.

(b) Set $p = \frac{1}{2}\left(1+\frac{\mu}{\sigma}\sqrt{\Delta t}\right)$ and $h = \sigma\sqrt{\Delta t}$.

(c) Simulate the increments of x by drawing realization of a random variable ϵ_t that takes value h with probability p and $-h$ with probability $1-p$.

15.4. Ito processes

Ito processes are the generalization of Brownian motions. Their drift and variance is allowed to depend on the level of the process and the time:

$$dx = \mu(x,t)\,dt + \sigma(x,t)\,dW \tag{15.1}$$

where the functions μ and σ give the value of the mean and standard deviations of the increments of the process x:

$$\mu(x,t) = \lim_{\Delta\to 0^+}\frac{1}{\Delta}E\left[x(t+\Delta)-x(t)\,|x(t)=x\right]$$
$$(\sigma(x,t))^2 = \lim_{\Delta\to 0^+}\frac{1}{\Delta}E\left[(x(t+\Delta)-x(t))^2\,|x(t)=x\right]$$

For future reference note that an Ito process can also be represented as:

$$x(t) = x(0) + \int_0^t \mu(x,s)\,ds + \int_0^t \sigma(x,s)\,dW(s)$$

where the last term is a stochastic integral. Stochastic integrals play an important role in the theory of stochastic processes, for now it suffices to state the following result.

Proposition 15.1. *Let $x(t)$ be an integrable function, then $E\left[\int_0^t x(s)\,dW(s)\right] = 0$.*

This proposition states that the expected value of a stochastic integral is identically zero. The derivation of the result, along with other properties can be found in Stokey (2009, Sec. 3.2).

Two Ito process are of particular importance. They are presented in the examples below.

Example 15.2. (Geometric Brownian notion) A Geometric Brownian motion is an Ito process with $\mu(x, t) = \mu x$ and $\sigma(x, t) = \sigma x$, so:

$$dx = \mu x dt + \sigma x dW$$

A geometric Brownian motion can be thought of as a Brownian motion where the properties apply to percentage increments instead of increments:

$$\frac{dx}{x} = \mu dt + \sigma dW$$

So the percentage increment, dx/x, are normally distributed with mean $\mu\Delta t$ and variance $\sigma^2\Delta t$.

Example 15.3. (Ornstein-Uhlenbeck process) Unlike the previous processes an OU process is mean reverting, similar to an AR(1) process in discrete time. An OU process is an Ito process with $\mu(x, t) = \mu(\overline{x} - x)$ and $\sigma(x, t) = \sigma$. If $x > \overline{x}$ then the process drifts down, and if $x < \overline{x}$ the process drifts up.

$$dx = \mu(\overline{x} - x)\, dt + \sigma dW$$

15.5. Jump processes - Poisson Processes

Jump processes are a type of stochastic process that has discontinuous paths. Jump process change by discrete amounts when a certain outcome occurs. The most important Jump process is the Poisson process, which is just a jump process such that the time of the jumps follows a Poisson distribution. To define it let λ be the mean arrival rate of a jump and u the size of the change of the process (usually $u = 1$, but in general u can be itself a random variable). Then for some process q we have:

$$dq = \begin{cases} 0 & \text{with prob. } 1 - \lambda dt \\ u & \text{with prob. } \lambda dt \end{cases}$$

We can now define a more general process that depends on the Jump process q:

$$dx = f(x, t)\, dt + g(x, t)\, dq \tag{15.2}$$

where absent a jump x evolves deterministically according to the function f, and when there is a jump it moves according to function g,

$$E\left[dx\right] = f(x, t)\, dt + \lambda E_u\left[g(x, t)\, u\right] dt.$$

16. Ito's Lemma

We are often concerned with the behavior of functions of stochastic processes, in particular the differentials of those functions. The number one example at hand is to know how the value of an asset (or an option) evolves over time. Ito's Lemma gives a way to compute those differentials. This relates the functions we are interested in to the stochastic differential equation that governs the underlying stochastic process.

Consider a function $F(x, t)$ that depends on a stochastic process x. x is assumed to be an Ito process following:

$$dx = \mu(x, t)\, dt + \sigma(x, t)\, dW \tag{16.1}$$

Normal calculus rules would give the differential of F as:

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial t} dt$$

Although not always clear, one of the reasons for expressing the differential without resorting to higher order terms is that those terms depend on dt^2, $dt^3 \ldots$. As $dt \to 0$ all higher order terms go to zero faster, and are hence ignored. But stochastic process add a new factor because their components depend of time through $\sqrt{dt}$, so square terms like $(dx)^2$ must also be considered.

A second order Taylor expansion of F gives:

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial t} dt + \frac{1}{2}\left(\frac{\partial^2 F}{\partial x^2}(dx)^2 + \frac{\partial^2 F}{\partial t^2}(dt)^2 + \frac{\partial^2 F}{\partial t^2}(dx)(dt)\right)$$

As shown in Øksendal (2003, Sec. 4.1) $dWdt = dt^2 = 0$, they can be safely ignored because they depend on terms of order higher than dt. That leaves us with:

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial t} dt + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(dx)^2 \tag{16.2}$$

From the definition of our Ito process we get:

$$\begin{aligned}(dx)^2 &= \left(\mu^2(x, t)(dt)^2 + 2\mu(x, t)\,\sigma(x, t)\, dtdW + \sigma^2(x, t)(dW)^2\right) \\ &= \sigma^2(x, t)(dW)^2\end{aligned}$$

We can again drop the terms involving $(dt)^2$ and $(dtdW)$, and also show that $(dW)^2 = dt$ (recall that $E\left[(dW)^2\right] = dt$). The proof is not hard and can be found in Øksendal (2003, Sec.

4.1). Replacing:

$$dF = \left(\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial t}dt\right) + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\left(\sigma^2(x,t)\,dt\right)$$
$$dF = \left(\frac{\partial F}{\partial t} + \mu(x,t)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 F}{\partial x^2}\right)dt + \sigma(x,t)\frac{\partial F}{\partial x}dW \tag{16.3}$$

This derivation (Ito's Formula) means that $y = F(x,t)$ is itself an Ito process with $\mu_y(x,t) = \left(F_t + \mu(x,t)F_x + \frac{1}{2}\sigma^2(x,t)F_{xx}\right)$ and $\sigma_y(x,t) = \sigma(x,t)F_x$ as parameters. Unsurprisingly, the expected value of y is $\mu_y(x,t)$ and its variance is $\sigma_y(x,t)\,dt$.

16.1. Application to geometric brownian motion

We can use Ito's Lemma to obtain the properties of different stochastic processes. For instance the Geometric Brownian motion can be shown to be the exponential of a standard brownian motion, or equivalently it can be shown that the logarithm of a geometric brownian motion is a brownian motion.

Let x be a geometric brownian motion satisfying:

$$dx = \mu x dt + \sigma x dW$$

and $y = \ln x$. By Ito's Lemma:

$$\begin{aligned} dy &= \left(\mu x \cdot \frac{1}{x} + \frac{1}{2}\sigma^2 x^2 \cdot \left(\frac{-1}{x^2}\right)\right)dt + \sigma x \cdot \frac{1}{x}dW \\ &= \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dW \end{aligned}$$

thus y is a brownian motion with parameters $\mu_y = \mu - \frac{1}{2}\sigma^2$ and $\sigma_y = \sigma$. The drift of y is lower than the drift of x, because the logarithm is a concave function Jensen's inequality implies that the expected value of the log is lower.

We can also obtain the expected value of x by noting that:

$$x(t) = x(0) + \int_0^t \mu x(s)\,ds + \int_0^t \sigma x(s)\,dW(s)$$

taking expectations gives:

$$E[x(t)] = x(0) + \int_0^t \mu E[x(s)]\,ds$$

recalling that the third term is a stochastic integral, and hence has expected value equal to zero. From this equation we can derive a first order differential equation for the expected value of x:

$$dE[x] = \mu E[x]\, dx$$

The solution for this equation, given the boundary condition $E[x(0)] = x(0)$ is:

$$E[x] = x(0)\, e^{\mu t}$$

Finding the variance (and other moments) works in the same way. For the variance we want to obtain an expression for x^2, so first consider the function $f(x) = x^2$. By Ito's Lemma we get:

$$df = \left(2\mu x^2 + \sigma^2 x^2\right) dt + 2\sigma x^2 dW$$
$$x^2 = f(x) = x_0^2 + \left(2\mu + \sigma^2\right) \int_0^t x^2(s)\, ds + 2\sigma \int_0^t x^2(s)\, dW(s)$$

We can now take expectations to obtain:

$$E\left[x^2\right] = x_0^2 + \left(2\mu + \sigma^2\right) \int_0^t E\left[x^2(s)\right] ds$$

which leads to a differential equation for $E\left[x^2\right]$:

$$dE\left[x^2\right] = \left(2\mu + \sigma^2\right) E\left[x^2\right]$$
$$E\left[x^2\right] = x^2(0)\, e^{\left(2\mu+\sigma^2\right)t}$$

the variance is then:

$$\begin{aligned} V[x] &= E\left[x^2\right] - E[x]^2 \\ &= x^2(0)\, e^{\left(2\mu+\sigma^2\right)t} - x^2(0)\, e^{2\mu t} \\ &= x^2(0)\, e^{2\mu t}\left(e^{\sigma^2 t} - 1\right) \end{aligned}$$

Some applications are shown below:

Example 16.1. Consider an asset that gives flow payoffs x that evolve according to a geometric brownian motion

$$dx = \mu x dt + \sigma x dW$$

we can compute the expected discounted value of holding that asset easily using the results above:

$$E\left[\int_0^\infty e^{-\rho t}x(t)\,dt\right] = \int_0^\infty e^{-\rho t}E[x(t)]\,dt = \int_0^\infty x(0)\,e^{-(\rho-\mu)t}dt = \frac{x_0}{\rho-\mu}$$

Example 16.2. Now consider an agent that receives flow consumption of x, which evolves again as a geometric brownian motion. The agent's utility is CRRA, so that $u(x) = \frac{x^{1-\theta}}{1-\theta}$. We want to know the expected present value of utility.

$$E\left[\int e^{-\rho t}u(x(t))\,dt\right] = \int e^{-\rho t}E[u(x(t))]\,dt$$

To know it we need to compute $E[u(x(t))]$. From Ito's Lemma we have:

$$\begin{aligned}
du &= \left(\mu x\cdot x^{-\theta} + \frac{1}{2}\sigma^2x^2\cdot\left(-\theta x^{-\theta-1}\right)\right)dt + \sigma x\cdot x^{-\theta}dW\\
du &= (1-\theta)\left(\mu - \frac{\theta}{2}\sigma^2\right)\frac{x^{1-\theta}}{1-\theta}dt + (1-\theta)\,\sigma\frac{x^{1-\theta}}{1-\theta}dW\\
du &= (1-\theta)\left(\mu - \frac{\theta}{2}\sigma^2\right)udt + (1-\theta)\,\sigma u dW
\end{aligned}$$

Thus, u is itself a geometric brownian motion (actually if x is a brownian motion x^k is a geometric brownian motion). Using our previous results we have:

$$E[u] = u(x(0))\,e^{(1-\theta)\left(\mu-\frac{\theta}{2}\sigma^2\right)t}$$

So we have:

$$E\left[\int e^{-\rho t}u(x(t))\,dt\right] = \int e^{-\rho t}E[u(x(t))]\,dt = \frac{(x(0))^{1-\theta}}{(1-\theta)\left(\rho-(1-\theta)\left(\mu-\frac{\theta}{2}\sigma^2\right)\right)}$$

16.2. Poisson Processes

Similar, and simpler, results can be obtained if x follows a Poisson process:

$$dx = f(x,t)\,dt + g(x,t)\,dq$$

and we have a function $H(x,t)$ that depends on x. Unlike the Ito Process the Poisson process does not depend on $\sqrt{dt}$, so higher order terms in the Taylor expansion can be ignored

altogether to get:

$$\begin{aligned} dH &= H_t dt + H_x dx \\ &= \left(H_t + f\left(x,t\right) H_x\right) dt + g\left(x,t\right) H_x dq \end{aligned}$$

The expected value of this change must take into account the probability of a jump in q (given by λdt), so we have:

$$E\left[dH\right] = \left(H_t + f\left(x,t\right) H_x\right) dt + \lambda E_u\left[H\left(x + ug\left(x,t\right), t\right) - H\left(x,t\right)\right] dt \tag{16.4}$$

it follows, by using the identity function that $E\left[dx\right] = f\left(x,t\right) dt + \lambda E_u\left[ug\left(x,t\right)\right] dt$.

We can apply this result to a couple examples taken from Dixit and Pindyck (1994):

Example 16.3. Consider an individual that lives forever and receives a wage $w\left(t\right)$ at each point in time. The wage increases by ϵ at random times, following a Poisson process with arrival rate λ, so:

$$dw = \epsilon dq$$

The individual wants to know the expected discounted value of taking the job we need to compute:

$$V\left(w\right) = E\left[\int_0^\infty e^{-\rho t} w\left(t\right) dt\right]$$

The function V (a value function) has a recursive representation, this is easier to see in the discrete time approximation. Consider a period of length Δt, then:

$$\begin{aligned} V\left(w\left(t\right)\right) &= w\left(t\right)\Delta t + \frac{1}{1+\rho\Delta t} E\left[V\left(w\left(t+\Delta t\right)\right)\right] \\ \left(1+\rho\Delta t\right) V\left(w\left(t\right)\right) &= \left(1+\rho\Delta t\right) w\left(t\right)\Delta t + E\left[V\left(w\left(t+\Delta t\right)\right)\right] \\ \rho\left(\Delta t\right) V\left(w\left(t\right)\right) &= \left(1+\rho\Delta t\right) w\left(t\right)\Delta t + E\left[V\left(w\left(t+\Delta t\right)\right) - V\left(w\left(t\right)\right)\right] \\ \rho V\left(w\left(t\right)\right) &= \left(1+\rho\Delta t\right) w\left(t\right) + \frac{E\left[\Delta V\right]}{\Delta t} \end{aligned}$$

Taking the limit as $\Delta t \to 0$ we get:

$$\rho V = w + \frac{E\left[dV\right]}{dt} \tag{16.5}$$

Staying in the job works just like an asset, with a normal return at rate ρ being equal to the sum of the dividend (in this case given by the wage) and the expected capital gains (from changes in the wage). In the expression above $\frac{E[dV]}{dt} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} E\left[\Delta V\right]$

We can apply the formula from above:

$$E\left[dH\right] = \left(H_t + f\left(x,t\right)H_x\right)dt + \lambda E_u\left[H\left(x + ug\left(x,t\right),t\right) - H\left(x,t\right)\right]dt$$

where $H = V$, $x = w$, $f\left(x,t\right) = 0$ and $g\left(x,t\right) = \epsilon$ and $u = 1$ with certainty:

$$\begin{aligned} E\left[dV\right] &= \lambda\left(V\left(w+\epsilon\right) - V\left(w\right)\right)dt \\ &= \lambda\epsilon\left(\int_0^\infty e^{-\rho t}dt\right)dt \\ &= \frac{\lambda\epsilon}{\rho}dt \end{aligned}$$

This leaves us with an explicit solution for V:

$$V = \frac{w}{\rho} + \frac{\lambda\epsilon}{\rho^2}$$

V is equal to an asset that pays the current wage forever plus the capitalized value of the average raise in wages per unit of time.

Example 16.4. Consider now a firm that produces using capital. As long as capital is operational a flow profit of π is obtained, but capital becomes obsolete when new technologies arrive. These innovations occur at random times following a Poisson process with arrival rate λ. Once the innovation arrives and the capital becomes obsolete the firm goes out of business forever.

The value of the firm follows a process:

$$dV = -Vdq$$

The return can be found as before:

$$\rho V = \pi + \frac{1}{dt}E\left[dV\right]$$

To find $E\left[dV\right]$ we can again use our formula with $H = V$, the identity function:

$$E\left[dV\right] = -\lambda V dt$$

replacing we get

$$\rho V = \pi - \lambda V \quad \text{or} \quad V = \frac{\pi}{\rho + \lambda}.$$

Which this is equivalent to solving:

$$V = \int_0^\infty e^{-(\rho+\lambda)t}\pi dt$$

This should not be a surprise. Consider the case where there are no shocks and the firm can operate forever with certainty. Then V is:

$$V = \int_0^\infty e^{-\rho t}\pi dt = \frac{\pi}{\rho}$$

Now the firm shuts down with a certain probability, given by the arrival of the Poisson shock. Then:

$$V = E\left[\int_0^\infty e^{-\rho t}\pi dt\right] = \int_0^\infty \Pr\left[\text{No shock until time } t\right] e^{-\rho t}\pi dt$$

The probability of there being no shocks is known:

$$\Pr\left[\text{No shock until time } t\right] = e^{-\lambda t}$$

Replacing gives the desired result.

17. Dynamic Programming

In dynamic programming we aim to develop tools for solving problems that involve actions through time, that in turn affect the total value obtained by the agent takin the decisions. The key of dynamic programming is that it focuses on the current decision being taken and its effect on the continuation value for the agent, rather than try to solve for the whole sequence of actions at once.

17.1. Discrete time overview

Dynamic Programming

To build up to the concepts of dynamic programming in continuous time we will first consider a simple discrete time problem of a firm that must invest a fixed amount I to set up the operation of the firm. Once the firm is operational the firm produces one unit of good every period.The current price of the good is known and given by p_0, in the second period the price can go up or down:

$$p_1 = \begin{cases} (1+u)\ p_0 & \text{with prob. } q \\ (1-d)\ p_0 & \text{with prob. } 1-q \end{cases}$$

After that the price is constant. Hence, the firm's decision is whether to invest in the first period, in the second, or not to invest at all. It makes no sense to wait any longer because no new information will arrive after the initial change in price. The firm discounts future payments with an interest rate r.

We can solve the problem by tracing the decisions that the firm can take. In the second period, once the price is known and assuming that the firm is not yet in operation, the firm can either invest or not. If the firm does not invest it gets zero payoff, if it invests it gets:

$$F_1\left(p_1\right) = p_1 + \frac{p_1}{1+r} + \frac{p_1}{(1+r)^2} = p_1 \sum_{i=0}^{\infty} \frac{1}{(1+r)^i} = \frac{1+r}{r}\, p_1$$

The payoff of the firm is then:

$$V_1\left(p_1\right) = \max\left\{F_1\left(p_1\right) - 0, 0\right\}$$

Knowing this is relevant because if the firm does not invest in the first period it can always do so later, so V_1 constitutes the continuation payoff of the firm. The payoff to the firm if it

does not invest in the first period is then:

$$\frac{1}{1+r}E\left[V_1\left(p_1\right)\right] = \frac{1}{1+r}\left(qV_1\left(\left(1+u\right)\ p_0\right) + \left(1-q\right)V_1\left(\left(1-d\right)\ p_0\right)\right)$$

If the firm invests in the first period the payoff is:

$$\begin{aligned} F_0\left(p_0\right) &= p_0 + \frac{1}{1+r}E\left[F_1\left(p_1\right)\right] \\ &= p_0 + \frac{1}{1+r}\left(qF_1\left(\left(1+u\right)\ p_0\right) + \left(1-q\right)F_1\left(\left(1-d\right)\ p_0\right)\right) \\ &= p_0 + \left(\frac{q}{r}\left(1+u\right)\ p_0 + \frac{1-q}{r}\left(1-d\right)\ p_0\right) \\ &= \frac{1}{r}\left(1+r+q\left(u+d\right)-d\right)\ p_0 \end{aligned}$$

So, the value of the firm is:

$$V_0\left(p_0\right) = \max\left\{F_0\left(p_0\right) - I, \frac{1}{1+r}E\left[V_1\left(p_1\right)\right]\right\}$$

In this example we already see the basics of dynamics programming, splitting the problem into the decision at hand (invest or not invest) and the continuation value that they entail. The example also highlights one of the recurring topics of the course: option value. The firm has an option that allows it to invest any of the two dates. Waiting in this problem has value, because investing in the future also means to invest with better information. In fact we can compute the value of this option (to wait) by comparing the value that the firm would have if it was forced to take a decision in the first period:

$$\Omega_0\left(p_0\right) = \max\left\{F_0\left(p_0\right) - I, 0\right\}$$

with the value that includes the possibility of action in the second period:

$$V_0\left(p_0\right) - \Omega_0\left(p_0\right)$$

We can now extend this simple model to allow for action in many periods (more than two). Consider a firm that operates for $T < \infty$ periods. In each period the firm will choose the value of a control variable u that affects (potentially) the per-period payoffs of the firm, namely the profits, and the evolution of a random variable x. x is assumed to follow a Markov process so that the CDF of x_{t+1} is $\Phi_t\left(x_{t+1}|x_t, u_t\right)$. The random variable x is also allowed to affect payoffs, so per-period payoffs are: $\pi\left(u_t, x_t\right)$.

The firm discounts the future at rate $\frac{1}{1+\rho}$ and receives a final payoff pf $\Omega_T(x_T)$ in the last period. The objective is:

$$V_0(x_0) = \max_{\{u_t\}_{t=0}^{T-1}} E\left[\sum_{t=0}^{T-1}\left(\frac{1}{1+\rho}\right)^t \pi(u_t, x_t) + \left(\frac{1}{1+\rho}\right)^T \Omega_T(x_T)\right] \tag{17.1}$$

dynamic programming allows us to write the problem recursively. In the last period we have:

$$V_{T-1}(x_{T-1}) = \max_{u_{T-1}} \pi(u_{T-1}, x_{T-1}) + \left(\frac{1}{1+\rho}\right) E\left[\Omega_T(x_T) | x_{T-1}, u_{T-1}\right] \tag{17.2}$$

For all other periods we can use the notion of continuation payoffs to obtain:

$$V_t(x_t) = \max_{u_t} \pi(u_t, x_t) + \left(\frac{1}{1+\rho}\right) E\left[V_{t+1}(x_{t+1}) | x_t, u_t\right] \tag{17.3}$$

The problem can then be solved by backwards induction, choosing contingent plans for $u_t(x_t)$ one period at a time, instead of tackling the more complicated problem of choosing the whole sequence of $\{u_t\}$.

When time is not finite, there is no terminal date, and we cannot use backwards induction to solve the problem. In this case the value of the firm itself is also independent of time, because each period is just like the next. We then have:

$$V(x) = \max_{u} \pi(u, x) + \left(\frac{1}{1+\rho}\right) E\left[V\left(x'\right) | x, u\right] \tag{17.4}$$

the problem is now to find a function V that satisfies the equation above. The details behind the solution to this problem can be found in Stokey, Lucas, and Prescott (1989).

This setup is very versatile and can be applied to firm problems as the one above, but it is also at the core of modern macroeconomic theory. The following examples make this point in a non-stochastic version of the model.

Example 17.1. Consider an economy in which the representative consumer lives forever. There is a good in each period that can be consumed or saved as capital as well as labor. The consumer's utility function is

$$V\left(\bar{k}_0\right) = \sum_{t=0}^{\infty} \beta^t \log c_t$$

Here $0 < \beta < 1$. The consumer is endowed with 1 unit of labor in each period and with $\bar{k}_0$ units of capital in period 0. Capital fully depreciates each period. Feasible allocations

satisfy

$$c_t + k_{t+1} \leq \theta k_t^{\alpha} l_t^{1-\alpha}$$

Here $\theta > 0$ and $0 < \alpha < 1$. We can formulate the problem of maximizing the representative consumer's utility subject to feasibility conditions as a dynamic programming problem. The appropriate Bellman's equation is:

$$\begin{aligned} V(k) &= \max_{c,k',l} \left\{ \log c + \beta V\left(k'\right) \right\} \\ \text{s.t. } & c + k' \leq \theta k^{\alpha} l^{1-\alpha} \\ & c, k' \geq 0 \\ & 0 \leq l \leq 1 \end{aligned}$$

To solve it we guess that the value function has the form $a_0 + a_1 \log k$ and solve for the decisions of the consumer. The constraint will hold with equality because the utility function is strictly increasing in consumption, also production increases with labor and there is no disutility of it, hence there is a corner solution for labor indicating $l = 1$, so with the guess the problem becomes

$$a_0 + a_1 \log k = \max_{k' \in \left[0, \theta k^{\alpha} l^{1-\alpha}\right]} \log\left(\theta k^{\alpha} l^{1-\alpha} - k'\right) + \beta\left(a_0 + a_1 \log k'\right)$$

Then the FOC is

$$\frac{1}{\theta k^{\alpha} l^{1-\alpha} - k'} = \frac{\beta a_1}{k'}$$

solving for k'

$$\begin{aligned} k' &= \beta a_1 \left(\theta k^{\alpha} l^{1-\alpha} - k'\right) \\ &= \frac{\beta a_1 \left(\theta k^{\alpha} l^{1-\alpha}\right)}{1 + \beta a_1} \end{aligned}$$

Then plugging this back into the value function you get

$$a_0 + a_1 \log k = \log\left(\theta k^{\alpha} l^{1-\alpha} - \frac{\beta a_1 \left(\theta k^{\alpha} l^{1-\alpha}\right)}{1 + \beta a_1}\right) + \beta\left(a_0 + a_1 \log\left(\frac{\beta a_1 \left(\theta k^{\alpha} l^{1-\alpha}\right)}{1 + \beta a_1}\right)\right)$$

Collection terms with k you get

$$\begin{aligned} a_1 \log k &= \alpha \log k + \beta a_1 \alpha \log k \\ a_1 (\log k - \beta \alpha \log k) &= \alpha \log k \\ a_1 &= \frac{\alpha}{1 - \beta \alpha} \end{aligned}$$

which means the policy function is

$$\begin{aligned} k' &= \frac{\beta \frac{\alpha}{1-\beta\alpha} \left(\theta k^{\alpha} l^{1-\alpha} \right)}{1 + \beta \frac{\alpha}{1-\beta\alpha}} = \beta \alpha \theta k^{\alpha} l^{1-\alpha} \\ l &= 1 \\ c &= \theta k^{\alpha} l^{1-\alpha} - \beta \alpha \theta k^{\alpha} l^{1-\alpha} \end{aligned}$$

Optimal Stopping Time

There is another type of problem that deserves special treatment. Optimal stopping time problems are at the core of the continuous time applications in the rest of the course. In these problems the agent faces a binary choice (instead of a continuous choice as in the example above), they resemble the example of the firm at the beginning of the Section where the firm has to choose whether or not to invest. This problems are characterized by the inaction of the agent, because the agent usually acts just once, and most of the time the optimal choice is to do nothing. To characterize these problems let $\Omega(x)$ be the termination payoff received once the action is taken (and time is stopped). It depends on the value of state x. The Bellman equation is now:

$$V(x) = \max \left\{ \Omega(x), \max_u \pi(u, x) + \left(\frac{1}{1+\rho} \right) E \left[V\left(x' \right) | x, u \right] \right\} \tag{17.5}$$

We can now define a stopping time as a random variable that signals the decision to stop and take the termination payoff $\Omega(x)$. So:

$$T^{\star} = \left\{ x | \Omega(x) \geq \max_u \pi(u, x) + \left(\frac{1}{1+\rho} \right) E \left[V\left(x' \right) | x, u \right] \right\} \tag{17.6}$$

In general $T^{\star}$ can take many forms, but in most (if not all) of the relevant economic applications it will take the form: $T^{\star} = [\overline{x}, \infty)$, $T^{\star} = (-\infty, \underline{x}]$ or $T^{\star} = (-\infty, \underline{x}] \cup [\overline{x}, \infty)$. As an example we apply these ideas to the problem of search and unemployment, the McCall search model.

Example 17.2. Consider the following infinite horizon model. An agent searches for a job. Each period the agent receives a wage offer from a distribution $F(w)$ with bounded support $W = \left[0, \overline{W}\right]$. If accepted the agent will remain employed at that wage forever. If rejected the worker receives unemployment benefits b. Wage offers are iid over time. The worker preferences are $\sum \beta^t c_t$. Assume no borrowing or lending.

We first set up the workers decision as a dynamic programming problem:

$$V^E(w) = \frac{w}{1-\beta}$$
$$V^U = b + \beta \int \max\left\{V^E(\tilde{w}), V^U\right\} dF(\tilde{w})$$

The decision of a worker when facing a wage offer w is to accept it or reject it, the worker will accept if $V^E(w) > V^u$ and reject otherwise. Then the value of the worker is:

$$V(w) = \max\left[V^E(w), V^U\right]$$
$$V(w) = \max\left[\frac{w}{1-\beta}, b + \beta \int V(\tilde{w})\, dF(\tilde{w})\right]$$

Now we need to show that the decision to take action (accept a job offer) is given by $T^\star = [\overline{w}, \infty)$, where $\overline{w}$ is the reservation wage. To show this, note that V^U is independent of the wage and that V^E is increasing in wages. The reservation wage satisfies:

$$\frac{\overline{w}}{1-\beta} = b + \beta \int V(\tilde{w})\, dF(\tilde{w})$$

This implies that V is constant for $w < \overline{w}$, because the offers are rejected, and it is equal to V^E for $w \geq \overline{w}$:

$$V(w) = \begin{cases} \frac{\overline{w}}{1-\beta} & \text{if } w < \overline{w} \\ \frac{w}{1-\beta} & \text{if } w \geq \overline{w} \end{cases}$$

It is left to find $\overline{w}$. To do this we should first solve for V^U:

$$\begin{aligned} V^U &= b + \beta \int \max\left\{V^E(\tilde{w}), V^U\right\} dF(\tilde{w}) \\ &= b + \beta \int_0^{\overline{w}} \frac{\overline{w}}{1-\beta} dF(\tilde{w}) + \beta \int_{\overline{w}}^{\overline{W}} \frac{\tilde{w}}{1-\beta} dF(\tilde{w}) \\ &= b + \frac{\beta}{1-\beta}\left(\int_0^{\overline{w}} \overline{w} dF(\tilde{w}) + \int_{\overline{w}}^{\overline{W}} \tilde{w} dF(\tilde{w})\right) \end{aligned}$$

$$= b + \frac{\beta}{1-\beta}\left(\overline{w} - \int_{\overline{w}}^{\overline{W}} \overline{w} dF(\tilde{w}) + \int_{\overline{w}}^{\overline{W}} \tilde{w} dF(\tilde{w})\right)$$
$$= b + \frac{\beta}{1-\beta}\left(\overline{w} + \int_{\overline{w}}^{\overline{W}} (\tilde{w} - \overline{w}) dF(\tilde{w})\right)$$

The agent knows she is guaranteed to have $\overline{w}$ forever, finding a job just adds to the value with the wage in excess of $\overline{w}$.

Turning back to determining $\overline{w}$ we can replace V^U to get:

$$\overline{w} = b + \frac{\beta}{1-\beta} \int_{\overline{w}}^{\overline{W}} (\tilde{w} - \overline{w}) dF(\tilde{w})$$

This equation is guaranteed to have a solution for $\overline{w} \in \left[c, \overline{W}\right]$. The LHS is increasing in $\overline{w}$, while the RHS is decreasing in $\overline{w}$.

17.2. Continuous time dynamic programming

We can now turn to develop a general framework to solve dynamic problems in continuous time. To start consider the problem developed in the previous section with periods of length Δt. The agent receives a payoff $\pi(u, x)\Delta t$ every period (where π is the payoff flow), and discounts the future at a rate ρ per unit of time, so the effective discount rate for the period of length Δt is: $\frac{1}{1-\rho\Delta t}$. This leads to the following Bellman-type equation

$$V(x) = \max_u \pi(u, x)\Delta t + \left(\frac{1}{1+\rho\Delta t}\right) E\left[V\left(x'\right)|x, u\right] \tag{17.7}$$

Rearranging we get:

$$\rho V(x) = \max_u (1+\rho\Delta t)\pi(u, x) + \frac{E\left[\left(V\left(x'\right) - V(x)\right)|x, u\right]}{\Delta t} \tag{17.8}$$

Taking the limit as $\Delta t \to 0$ we get our continuous time Bellman equation:

$$\rho V(x) = \max_u \pi(u, x) + \frac{1}{dt} E\left[dV(x)|x, u\right] \tag{17.9}$$

where

$$\frac{E\left[dV\right]}{dt} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} E\left[\Delta V\right]$$

Equation (17.9) works just like a non-arbitrage condition. We can thing of the agent as

holding an asset with value V. The LHS gives the normal rate of return per unit time that the agent requires to hold the asset, given the discount rate ρ. The RHS gives the effective payoff of the asset, composed by the immediate flow payoff π, and the expected capital gains (brought up by changes in the value of the asset).

We can further characterize the problem given knowledge of the stochastic process that x follows. This will allow us to evaluate the expectation in (17.9). If x follows an Ito process, as in equation (15.1), then Ito's Lemma gives the following result:

$$dV = \left(\mu(x,t)V' + \frac{1}{2}\sigma^2(x,t)V''\right)dt + \sigma(x,t)V'dW$$
$$E[dV] = \left(\mu(x,t)V' + \frac{1}{2}\sigma^2(x,t)V''\right)dt$$

Replacing we get the Hamilton-Jacobi-Bellman equation:

$$\rho V(x) = \max_u \pi(u,x) + \mu(x,t)V'(x) + \frac{1}{2}\sigma^2(x,t)V''(x) \tag{17.10}$$

We can take FOC with respect to u and then get a differential equation for V that we can solve.

If x follows a Poisson process, like the one in equation (15.2), we can obtain a similar result. From equation (16.4) we can compute $E[dV]$:

$$E[dV] = \left(f(x,t)V'(x)\right)dt + \lambda E_u\left[V(x+ug(x,t)) - V(x)\right]dt$$

Optimal Stopping Time and the Smooth Pasting Condition

We now go back to the stopping time problem reviewed in Section 17.1. Consider then the problem of an agent that is engaged in some activity (say running a firm). The agent gets a flow payoff of $\pi(x)$ if she continues with the activity, and $\Omega(x,t)$ if she quits the activity (stops). The value of the agent is:

$$V(x,t) = \max\left\{\Omega(x,t), \pi(x)\Delta t + \frac{1}{1+\rho\Delta t}E\left[V(x+dx, t+\Delta t)\right]\right\} \tag{17.11}$$

where x follows a diffusion process and in equation (15.1). We assume that Ω is continuous and weakly increasing in x.

In order to solve the problem we need to find regions of x where it is best for the agent

to continue and those for which it is best to stop. If x is in the continuation region then:

$$V(x) = \pi(x)\Delta t + \frac{1}{1+\rho\Delta t} E\left[V\left(x'\right)\right]$$

From above we know that this implies that for x in the continuation region we have (by applying Ito's lemma):

$$\rho V(x,t) = \pi(x) + V_t(x,t) + \mu(x,t) V_x(x,t) + \frac{1}{2}\sigma^2(x,t) V_{xx}(x,t)$$

For simplicity we assume now that the continuation region has the form $x \geq x^\star(t)$. It is only for $x \geq x^\star(t)$ that the equation above holds. In order to solve it we need to impose certain boundary conditions.

By assumption we know that $V(x) = \Omega(x,t)$ for $x < x^\star(t)$, then, by continuity we can impose that:

$$V\left(x^\star(t), t\right) = \Omega\left(x^\star(t), t\right)$$

this is called "value-matching". Continuity at $x^\star$ is actually necessary for a solution. Suppose for a contradiction that it is optimal to stop for $x < x^\star(t)$, but that $V(x^\star(t), t) < \Omega(x^\star(t), t)$, because V has to be continuous in the domain $x \geq x^\star(t)$ (because it is the solution to a differential equation), and Ω is continuous by definition, then it holds that for x to the right of $x^\star(t)$, but sufficiently close to $x^\star(t)$ it also holds that $V(x,t) < \Omega(x,t)$, which contradicts $x \geq x^\star(t)$ being the continuation region. A similar argument applies for the other inequality.

But this condition is not sufficient to solve the problem, because the value of $x^\star(t)$ is still unknown. The condition that allows us to solve the problem (of jointly finding V and $x^\star$) is to impose further smoothness to our value function, it must not only be continuous, but continuously differentiable. This condition is called "smooth pasting" and it requires the first derivative of the value function to be continuous, that is:

$$V_x\left(x^\star(t), t\right) = \Omega_x\left(x^\star(t), t\right)$$

The reason behind the smooth pasting condition is not at all evident. I will illustrate it below, but I recommend checking Appendix C of Chapter 4 in Dixit and Pindyck (1994), or Stokey (2009, Prop 6.4 pg 124).

To see why the smooth pasting condition arise consider the following case built for a contradiction: the value matching condition holds, but the smooth pasting condition fails, hence V and Ω must meet at a kink. There are two options:

(a) There is an upward kink (forming a concave function). If this is the case then, by continuity, Ω would be higher than F for some value $x > x^\star$. Contradicting that the continuation region starts at $x^\star$.

(b) There is a downward kink (forming a convex function). If this is the case then $x^\star$ cannot be a point of indifference either. There is a better strategy, namely continuing for some time Δt and then choosing what to do. This strategy give higher (expected) payoff.
To see this recall the random walk formulation of the brownian motion, in a time lapse Δt x can either go up by h with probability p or down by $-h$ with probability $1-p$, where:

$$h = \sigma\sqrt{\Delta t} \qquad p = \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{\Delta t}\right)$$

Then the agent can continue if the step is upward and stop if it is downward. The expected payoff of this strategy is:

$$V\left(x^\star(t), t\right) = \pi\left(x^\star(t), t\right)\Delta t + \frac{1}{1+\rho\Delta t}\left[pV\left(x^\star(t) + h, t + \Delta t\right) + (1-p)\,\Omega\left(x^\star(t) - h, t + \Delta t\right)\right]$$

We can take a Taylor expansion around $(x^\star(t), t)$ to approximate the value of $V(x^\star(t) + h, t + \Delta t)$ and $\Omega(x^\star(t) - h, t + \Delta t)$:

$$V\left(x^\star(t) + h, t + \Delta t\right) \approx V\left(x^\star(t), t\right) + V_x\left(x^\star(t), t\right)h + V_t\left(x^\star(t), t\right)\Delta t$$
$$\Omega\left(x^\star(t) - h, t + \Delta t\right) \approx \Omega\left(x^\star(t), t\right) - \Omega_x\left(x^\star(t), t\right)h + \Omega_t\left(x^\star(t), t\right)\Delta t$$

Replacing gives:

$$V\left(x^\star(t), t\right) = \pi\left(x^\star(t), t\right)\Delta t + \frac{1}{1+\rho\Delta t}\left(V\left(x^\star(t), t\right) + \frac{1}{2}\left[\left(V_x\left(x^\star(t), t\right) - \Omega_x\left(x^\star(t), t\right)\right)h + \left(V_t\left(x^\star(t)\right.\right.\right.\right.$$

where we use the value matching condition and the fact that $ph \approx \frac{1}{2}\sigma\sqrt{\Delta t}$ and $p\Delta t \approx \frac{1}{2}\Delta t$.
What matters for evaluating the strategy is the continuation value, and that Δt is of order h^2, so the first two terms in the continuation value $\left(V(x^\star(t), t) + \frac{1}{2}h\left(V_x(x^\star(t), t) - \Omega_x(x^\star(t), t)\right)\right)$ will dictate the behavior of the gain as $\Delta t \to 0$ (or equivalently $h \to 0$). These terms are positive as long as $V_x(x^\star(t), t) > \Omega_x(x^\star(t), t)$, which is the case if there is a downward kink.
Then there cannot be a downward kink, because it would contradict the optimality of the strategy of stopping at $x^\star(t)$.

Example 17.3. Consider a firm that has flow revenues of e^{x_t}, and that can be closed at any

time and sold for a value $\Omega > 0$. The owner of the firm is risk neutral and discounts the future at a rate $\rho > 0$. x_t follows:

$$dx_t = \mu dt + \sigma dW$$

The problem of the firm's owner is then:

$$V(x) = \max\left\{\Omega, e^x \Delta t + \frac{1}{1+\rho\Delta t} E\left[V(x+dx)\right]\right\}$$

where continuation is optimal for $x \geq x^\star$. This problem is independent of time.

As long as x is in the continuation region the value function satisfies the HJB equation:

$$\rho V(x) = e^x + \mu V_x(x) + \frac{1}{2}\sigma^2 V_{xx}(x)$$

This is a second order ordinary differential equation with constant coefficients. Then we know that the solution has the form:

$$V(x) = V^P(x) + A_1 H_1(x) + A_2 H_2(x)$$

where V^P is a particular solution to the differential equation, H_1 and H_2 are homogenous solutions, and $A_1$and A_2 are constants to be determined.

The particular solution is easy to obtain. We can solve for the value of never stopping:

$$V^P(x) = E\left[\int_0^\infty e^{-\rho t} e^x dt\right]$$

We can solve this expectation using the results in example 16.1. We get:

$$V^P(x) = \frac{x}{\rho - \left(\mu + \frac{1}{2}\sigma^2\right)}$$

we assume that $\rho - \left(\mu + \frac{1}{2}\sigma^2\right) > 0$ in order to guarantee the existence of this solution.

The homogenous solutions are obtained from the homogenous equation:

$$\rho H(x) = \mu H_x(x) + \frac{1}{2}\sigma^2 H_{xx}(x)$$

by guessing that $H(x) = e^{\xi x}$ and replacing we get:

$$\rho e^{\xi x} = \mu \xi e^{\xi x} + \frac{1}{2}\sigma^2 \xi^2 e^{\xi x}$$

$$0 = -\rho + \mu\xi + \frac{1}{2}\sigma^2\xi^2$$

our guess is verified for ξ a root of the equation above. There are two roots:

$$\xi_2 = -\frac{\mu + \sqrt{\mu^2 + 2\sigma^2\rho}}{\sigma^2} \qquad \xi_2 = \frac{-\mu + \sqrt{\mu^2 + 2\sigma^2\rho}}{\sigma^2},$$

where $\xi_1 < 0 < 1 < \xi_2$, this follows from $\rho > 0$ and the assumption $\rho - \left(\mu + \frac{1}{2}\sigma^2\right) > 0$. Joining we get the solution for our HJB equation:

$$V(x) = V^P(x) + A_1 e^{\xi_1 x} + A_2 e^{\xi_2 x}$$

Now we must determine the values of A_1 and A_2. To do so we first need to impose certain conditions on our value function.

(a) From optimality in exit it must be that: $V(x) \geq \Omega$.

(b) From feasibility it must be that: $V(x) \leq V^P(x) + \Omega$.

(c) Value matching implies: $V(x^\star) = \Omega$.

We will show that $A_2 = 0$. Suppose for a contradiction that $A_2 > 0$, then as $x \to \infty$ we have $e^{\xi_1 x} \to 0$ (because $\xi_1 < 0$), and $e^{\xi_2 x} \to \infty$ (because $\xi_2 > 0$), because $A_2 > 0$ this implies that V violates its upper bound. Now suppose for a contradiction that $A_2 < 0$, as before $e^{\xi_1 x} \to 0$ and $e^{\xi_2 x} \to \infty$, because $\xi_2 > 1$ the last term will grow faster than the first one, thus violating the lower bound (the value goes to $-\infty$). Then it must be that $A_2 = 0$.

Then we can obtain A_1 from the value matching condition:

$$\begin{aligned} V(x^\star) &= V^P(x^\star) + A_1 e^{\xi_1 x^\star} \\ \left(\Omega - V^P(x^\star)\right) e^{-\xi_1 x^\star} &= A_1 \end{aligned}$$

with this the solution is complete, given a value for $x^\star$. It is left to find such value, for that we make use of the smooth pasting condition:

$$\begin{aligned} V_x(x^\star) &= 0 \\ V_x^P(x^\star) + A_1 \xi_1 e^{\xi_1 x^\star} &= 0 \\ \frac{1}{\rho - \left(\mu + \frac{1}{2}\sigma^2\right)} + \left(\Omega - \frac{x^\star}{\rho - \left(\mu + \frac{1}{2}\sigma^2\right)}\right)\xi_1 &= 0 \end{aligned}$$

$$\frac{1}{\xi_1} + \left(\rho - \left(\mu + \frac{1}{2}\sigma^2\right)\right)\Omega = x^\star$$

18. The Kolmogorov Forward Equation

The last section of this part of the course develops the Kolmogorov Forward Equation, which describes the dynamics of the probability distribution of a random variable (given its initial value). Moreover, it characterizes the stationary distribution of the variable if such distribution exists. This is of particular importance for models with heterogenous agents because the distribution of the agents in the economy is obtained via the KFE.

Given some initial conditions x_0 and t_0 the objective is to characterize the probability distribution function $\varphi(x, t)$:

$$\Pr\left(x_t \in [a, b]\right) = \int_a^b \varphi(u, t)\, du$$

In order to characterize φ we first need to impose a process for x, and then use the random walk approximation. For simplicity:

$$dx = \mu dt + \sigma dW$$

In the random walk approximation the process varies in a period of length Δt by a magnitude of h, it increases with probability p or decreases with probability $1 - p$, where:

$$h = \sigma\sqrt{\Delta t} \qquad p = \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{\Delta t}\right)$$

From time $t - \Delta t$ to time t the process can reach a value x either by growing from $x - h$ or by decreasing from $x + h$. Then the probability (or more intuitively the fraction of the mass) at point x at time t is given by:

$$\varphi(x, t) = p\varphi(x - h, t - \Delta t) + (1 - p)\,\varphi(x + h, t - \Delta t)$$

We can approximate the elements of the right hand side with a second order Taylor expansion:

$$\varphi(x \pm h, t - \Delta t) \approx \varphi(x, t) - \Delta t \frac{\partial \varphi(x, t)}{\partial t} \pm h \frac{\partial \varphi(x, t)}{\partial x} + \frac{1}{2} h^2 \frac{\partial^2 \varphi(x, t)}{\partial x^2}$$

Terms of order higher than Δt are ignored. We can replace to get:

$$0 = -\Delta t \frac{\partial \varphi(x,t)}{\partial t} + (1-2p)\left(h\frac{\partial \varphi(x,t)}{\partial x}\right) + \frac{1}{2}h^2\frac{\partial^2 \varphi(x,t)}{\partial x^2}$$
$$0 = -\Delta t \frac{\partial \varphi(x,t)}{\partial t} - \frac{\mu}{\sigma}\sqrt{\Delta t}\left(\sigma\sqrt{\Delta t}\frac{\partial \varphi(x,t)}{\partial x}\right) + \frac{1}{2}\sigma^2\Delta t\frac{\partial^2 \varphi(x,t)}{\partial x^2}$$
$$0 = -\frac{\partial \varphi(x,t)}{\partial t} - \mu\frac{\partial \varphi(x,t)}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 \varphi(x,t)}{\partial x^2}$$

which gives the KFE:

$$\frac{\partial \varphi(x,t)}{\partial t} = -\mu\frac{\partial \varphi(x,t)}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 \varphi(x,t)}{\partial x^2}$$

If x follows a more general diffusion process we can change the argument above to get:

$$\frac{\partial \varphi(x,t)}{\partial t} = -\frac{\partial\left[\mu(x,t)\varphi(x,t)\right]}{\partial x} + \frac{1}{2}\frac{\partial^2\left[\sigma(x,t)^2\varphi(x,t)\right]}{\partial x^2}$$

The KFE is specially useful for finding the stationary distribution of the process. In this case the distribution does not depend on time so the KFE is:

$$0 = -\frac{\partial\left[\mu(x,t)\varphi(x)\right]}{\partial x} + \frac{1}{2}\frac{\partial^2\left[\sigma(x,t)^2\varphi(x)\right]}{\partial x^2}$$

This equation can be integrated once to get:

$$c_1 = -2\mu(x,t)\varphi(x) + \frac{\partial\left[\sigma(x,t)^2\varphi(x)\right]}{\partial x}$$

where c_1 is a constant of integration (to be determined later). Then we can use the integrating factor:

$$s(x) = e^{-\int^x \frac{2\mu(z,t)}{\sigma^2(z,t)}dz}$$

By multiplying both sides by the integrating factor we get:

$$s(x)\,c_1 = e^{-\int^x \frac{2\mu(z,t)}{\sigma^2(z,t)}dz}\left(-2\mu(x,t)\varphi(x) + \frac{\partial\left[\sigma(x,t)^2\varphi(x)\right]}{\partial x}\right)$$

The RHS can be rewritten noting that:

$$\begin{aligned}
\frac{d}{dx}\left[s(x)\,\sigma^2(x,t)\,\varphi(x)\right] &= \frac{d}{dx}\left[e^{-\int^x \frac{2\mu(z,t)}{\sigma^2(z,t)}dz}\sigma^2(x,t)\,\varphi(x)\right] \\
&= \frac{d}{dx}\left[e^{-\int^x \frac{2\mu(z,t)}{\sigma^2(z,t)}dz}\right]\sigma^2(x,t)\,\varphi(x) + s(x)\,\frac{d}{dx}\left[\sigma^2(x,t)\,\varphi(x)\right] \\
&= -\frac{2\mu(x,t)}{\sigma^2(x,t)}\sigma^2(x,t)\,\varphi(x) + s(x)\,\frac{d}{dx}\left[\sigma^2(x,t)\,\varphi(x)\right] \\
&= -2\mu(x,t)\,\varphi(x) + s(x)\,\frac{d}{dx}\left[\sigma^2(x,t)\,\varphi(x)\right]
\end{aligned}$$

Then we get:

$$s(x)\,c_1 = \frac{d}{dx}\left[s(x)\,\sigma^2(x,t)\,\varphi(x)\right]$$

Integrating again:

$$c_1\int^x s(y)\,dy + c_2 = s(x)\,\sigma^2(x,t)\,\varphi(x)$$

rearranging gives:

$$\varphi(x) = \frac{1}{s(x)\,\sigma^2(x,t)}\left(c_1\int^x s(y)\,dy + c_2\right)$$

where $\int^x f(\xi)\,d\xi = F(x)$, being F the antiderivative of f.

Example 18.1. Dynamics and Barriers Consider a brownian motion with two reflecting barriers $\overline{x}$ and $\underline{x}$. The process behaves as $dx = \mu dt + \sigma dW$ for $x \in (\underline{x}, \overline{x})$, but is kept in those bounds by force. In terms of the random walk representation that means that starting at $\overline{x} - h$ the process stays at $\overline{x} - h$ with probability p, instead of taking a step up, and goes down to $\overline{x} - 2h$ with probability $1 - p$. Similarly for $\overline{x} + h$.

The KFE applies for any point in the interior of the domain, so for $x \in (\underline{x}, \overline{x})$ we have:

$$\frac{\partial\varphi(x,t)}{\partial t} = -\mu\frac{\partial\varphi(x,t)}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2\varphi(x,t)}{\partial x^2}$$

Moreover, because we are interested in the stationary behavior of the process we know that the distribution does not depend on time, which results in:

$$0 = -\mu\frac{\partial\varphi(x)}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2\varphi(x)}{\partial x^2}$$

or better:

$$\varphi'(x) = \frac{1}{2}\frac{\sigma^2}{\mu}\varphi''(x)$$

We can solve this equation:

$$\varphi(x) = Ae^{\gamma x} + B$$

where $\gamma = \frac{2\mu}{\sigma^2}$ and A and B are constants to be determined. To find them we can make use of the boundary conditions implied by the barriers.

From the random walk approximation we can derive the following equation for the upper bound:

$$\begin{aligned}\varphi(\overline{x} - h) &= p\varphi(x - h) + p\varphi(x - 2h)\\ (1 - p)\varphi(\overline{x} - h) &= p\varphi(x - 2h)\end{aligned}$$

Using now a second order Taylor expansion around $\overline{x} - h$:

$$\begin{aligned}(1 - p)\varphi(\overline{x} - h) &= p\left(\varphi(\overline{x} - h) - h\varphi'(\overline{x} - h) + \frac{1}{2}h^2\varphi''(\overline{x} - h)\right)\\ (1 - 2p)\varphi(\overline{x} - h) &= -ph\varphi'(\overline{x} - h) + p\frac{1}{2}h^2\varphi''(\overline{x} - h)\\ -\frac{\mu}{\sigma}\sqrt{\Delta t}\varphi(\overline{x} - h) &= -\frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{\Delta t}\right)\sigma\sqrt{\Delta t}\varphi'(\overline{x} - h) + \frac{1}{4}\left(1 + \frac{\mu}{\sigma}\sqrt{\Delta t}\right)\sigma^2\Delta t\varphi''(\overline{x} - h)\\ -\frac{2\mu}{\sigma^2}\varphi(\overline{x} - h) &= -\left(1 + \frac{\mu}{\sigma}\sqrt{\Delta t}\right)\varphi'(\overline{x} - h) + \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{\Delta t}\right)\sigma\sqrt{\Delta t}\varphi''(\overline{x} - h)\end{aligned}$$

taking $\Delta t \to 0$ we get:

$$\begin{aligned}\frac{2\mu}{\sigma^2}\varphi(\overline{x}) &= \varphi'(\overline{x})\\ \gamma\varphi(\overline{x}) &= \varphi'(\overline{x})\end{aligned}$$

Replacing for the solution of φ we find that $B = 0$. Then A is found to guarantee that φ integrates to one. This results in:

$$\varphi(\overline{x}) = \frac{\gamma e^{\gamma x}}{e^{\gamma\overline{x}} - e^{\gamma\underline{x}}}$$

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