A Task-Based Theory of Occupations with Multidimensional Heterogeneity *

Sergio Ocampo†

Western University, Department of Economics

October 30, 2021

Abstract

I develop an assignment model of occupations with multidimensional heterogeneity in production tasks and worker skills. Tasks are distributed continuously in the skill space, whereas workers have a discrete distribution with a finite number of types. Occupations arise endogenously as bundles of tasks optimally assigned to a type of worker. The model allows us to study how occupations evolve—e.g., changes in their boundaries, wages, and employment—in response to changes in the economic environment, making it useful for analyzing the implications of automation, skill-biased technical change, offshoring, and skill upgrading by workers, among others. I characterize how the wages, the marginal product of workers, the substitutability between worker types, and the labor share depend on the assignment. In particular, I show that these properties depend on the productivity of workers in tasks along the boundaries of their occupations. I introduce automation as a choice of the optimal size and location of a mass of identical robots in the task space. Automation displaces workers by replacing them in the performance of tasks. This generates a cascading effect on other workers as the boundaries of occupations are redrawn.

JEL: J23, J24, J31, C78, E24

Key Words: Automation, occupations, assignment, skill mismatch

*I am most grateful to Fatih Guvenen, Jeremy Lise, Loukas Karabarbounis and David Rahman for their guidance and continuous encouragement. I also want to thank Anmol Bhandari, Serdar Birinci, Jonathan Eaton, Pawel Gola, Juan Herreño, Gueorgui Kambourov, Burhan Kuruscu, Keler Marku, Ellen McGrattan, Emily Moschini, Sergio Salgado, Kurt See, Dominic Smith, Aaron Sojourner, Nancy Stokey, David Wiczer, and participants at seminars at various universities. All errors are my own.

†Email: socampod@uwo.ca; Web: https://sites.google.com/site/sergiocampod/

Occupations undergo constant change as evidence by long run changes in their skill-content (e.g., Spitz-Oener, 2006; Atalay, Phongthiengtham, Sotelo, and Tannenbaum, 2020). These changes go beyond the composition of employment captured by the decline in routine-based occupations (Autor, Levy, and Murnane, 2003; Autor, Katz, and Kearney, 2006). The introduction of information and communication technologies, and increasing automation and offshoring affect workers by changing the tasks they actually perform in their occupations. Changes in the task composition of occupations carry with them changes in the skills required from the worker, as well as in the worker's productivity and compensation. These changes are felt by workers across the earnings distribution, from plant operators facing the introduction of industrial robots to engineers facing the introduction of new software.

I develop an assignment model of occupations that explicitly incorporates changes in the set of tasks that compose an occupation. In the model, production requires performing a collection of tasks, each generating a task-specific output. The problem is to assign workers to tasks to maximize production. Both workers and tasks are heterogeneous along multiple dimensions as in Lindenlaub (2017). Workers differ in the skills they possess (e.g., manual, cognitive, social, etc.) and tasks differ in the skills that are involved in performing them. The relevance of multiple types of skills in determining labor market outcomes of individuals has been long recognized (Heckman and Sedlacek, 1985; Black and Spitz-Oener, 2010; Deming, 2017). In particular, workers' productivity depends on the mismatch between their skills and the skills involved in the tasks they perform.

To fix ideas, consider the two-dimensional setup depicted in Figure 1, where workers and tasks differ in cognitive and manual skills. Each point in the plane characterizes a task with a different combination of cognitive and manual skills. While some tasks are complex in terms of their cognitive skills and involve no manual skills, others use both types of skills, and so on. Workers are represented by points scattered in the skill space, defining a given combination of skills. Crucially, I assume that there are finitely many types of workers (e.g.,

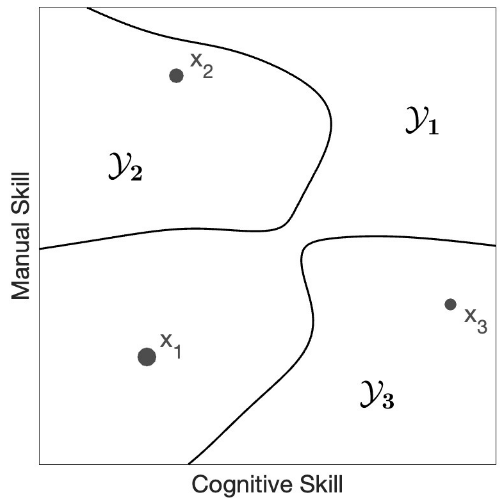


Figure 1: Assignment Example

Note: *The figure shows an example for an assignment in a two-dimensional skill space (cognitive and manual skills). There are three types of workers* $\{x_1, x_2, x_3\}$ *and tasks are continuously distributed over the unit square. The assignment partitions the space into three regions* $\{\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3\}$ *each of which is an occupation.* x_n *performs the tasks in occupation* $\mathcal{Y}_n$*. The assignment in this figure is not necessarily optimal.*

x_1, x_2, x_3), while tasks are continuously distributed in the skill space.[1] Consequently, the assignment of tasks to workers for production divides the space into regions ($\mathcal{Y}_1$, $\mathcal{Y}_2$, and $\mathcal{Y}_3$ in the figure), resulting in bundles of tasks assigned to the same (type of) worker. The tasks in each bundle form the occupation of that worker.

Figure 1 shows the boundaries of occupations implied by an arbitrary assignment. The shape of the boundaries under the optimal assignment depends on the technology for production of task-specific output, which determines how the mismatch between workers' skills and the skills involved in a task affects the workers' productivity, as well as the supply of workers and the demand for tasks. In particular, technology determines in which directions mismatch is more harmful. For example, cognitive mismatch can affect productivity more than manual mismatch, or being over-qualified can be less harmful to productivity than being under-qualified for a task. The optimal assignment seeks to maximize production by minimizing the skill mismatch, subject to the limited supply of workers of each type.

[1]This setup is used in other labor market models like Rosen (1978) and Acemoglu and Autor (2011).

The model makes precise how the marginal productivity of workers, wages and the elasticity of substitution across workers depend on the assignment. These properties depend on the productivity of workers in tasks along the boundaries of their occupations. Boundary tasks are marginal, in the sense that they are the last tasks to be assigned to a worker. Marginal products and wages are thus determined by productivity at the workers' boundary tasks—i.e., how much production increases if additional tasks were reassigned to a worker. The elasticity of substitution between workers is also determined by the boundaries, with a worker only being directly substitutable with their neighbors.

Occupations react endogenously to changes in the economic environment by shifting their boundaries. This makes the model useful for studying the implications of automation, skill-biased technical change, offshoring, and skill upgrading by workers, among others. These phenomena manifest as changes in the production technology, the distribution of workers, or the distribution of tasks. For instance, the use of information technologies (IT) and computers in the workplace makes cognitive skills more relevant in production, which then affects the skill content of occupations.[2]

The reassignment of tasks across workers that follows a change in the environment also affects the distribution of income across workers, potentially changing their rank. For example, the adoption of IT benefits workers with high cognitive skills over those with more manual skills. In the economy described in Figure 1, this increases the wage of workers of type x_3 relative to the wages of workers of type x_1 and x_2. The use of industrial machinery makes differences across workers in terms of their strength (a type of manual skill) less relevant for production. This makes workers x_1 and x_2 more substitutable with each other, reducing the edge that worker x_2 had from her higher manual skills. As a result wage inequality between x_1 and x_2 decreases. These changes can affect the ranking of workers reflecting decreases on the mismatch of some workers over others.

I show how to use the model to study worker replacing technologies like automation

[2]Changes in skill content of occupations go beyond the adoption of IT and the importance of cognitive skills. See Deming (2017), Rendall (2018), and Atalay et al. (2020).

and offshoring.[3] These technologies are directed towards replacing workers at specific tasks (e.g., industrial robots taking spots in the assembly line). Because of this, automation takes away some, but not all, of the tasks of an occupation. In a recent study, McKinsey Global Institute (2017) reports that while 50% of tasks are automatable using currently available technology, less than 5% of occupations are fully automatable. Consequently, automation is more likely to transform rather than to eliminate occupations. In the model, occupations are transformed directly by losing tasks to robots or software, and indirectly through the reassignment of tasks across workers.

I model automation as a choice over the location of a mass of identical 'robots' in the task space. Robots replace workers at performing tasks. Automation can be directed through the location of the robots in the task space. This works in the same way as choosing which tasks to offshore. The optimal choice weighs the cost of automation, which varies depending on the complexity of the tasks being automated, against the gains in output from replacing workers. Automation is thus directed towards regions that exhibit high skill-mismatch between workers and tasks. These regions are located around the boundaries of occupations.

Automation induces a reassignment of tasks across occupations. Because of this, the workers previously performing the automated tasks are not the only ones affected. It is optimal to reassign tasks so that only the workers with the lowest productivity are displaced by automation, preserving the employment of more productive workers. This gives rise to a cascading effect though the distribution of workers. Whether or not wages decrease depends on how productive robots are at the tasks they overtake (Acemoglu and Restrepo, 2018a). A major increase in productivity due to automation can increase workers' marginal product, increasing wages, while moderate increases in output in the automated tasks can

[3]In manufacturing, Acemoglu and Restrepo (2020) estimate that industrial robots have displaced 756,000 workers between 1993 and 2007. Simultaneously, advances in software and AI have made it possible to automate tasks of clerical occupations and of more specialized workers like accountants. Other worker replacing technologies, like offshoring, operate through the same effects. See Blinder (2009) and Blinder and Krueger (2013) for offshorability measures based on occupational characteristics.

be dominated by the higher mismatch experienced by workers, ultimately reducing their wages. The model predicts that workers with higher task displacement have larger reduction in wages, as documented by Acemoglu and Restrepo (2021).

I show how the model can be used to address other changes in the economy. I model the problem of optimal worker training as one of paying a cost to modify a worker's skill vector. Unlike automation, training does not displace workers, although occupations change in response to the new skill distribution. The effect on the distribution of wages is also different. Training a worker reduces the mismatch in the tasks they performs, raising their marginal product. This increases wage differentials with workers who previously earned less, and decreases the differences with workers who previously earned more.

I also consider the direction of skill-enhancing technical change. Technological change is directed towards skills at which the workforce is most adept, as measured by the skill-mismatch between workers and tasks. It is optimal to specialize in skills, increasing productivity by adapting technology to complement the skills for which the workforce is better suited. This contrasts with automation, where productivity increases by replacing workers at tasks they are not well suited for.

Finally, I extend the model by allowing tasks to be left unassigned. Tasks are only performed if workers are productive enough relative to their wages. I show how skill accumulation by workers changes, and potentially expands, the set of tasks performed in the economy. One important consequence of allowing tasks to be left unassigned is that automation ceases to be a pure worker-replacing technology. Automation can now complement workers by taking over tasks that are either not worthwhile for workers to perform, or that are too specialized given the workers' current skills.

Related literature I adopt a task approach to production as in Rosen (1978), Autor, Levy, and Murnane (2003) and Acemoglu and Autor (2011). I complement this literature by incorporating multidimensional heterogeneity in tasks and workers as in Lindenlaub

(2017) and explicitly modeling the assignment of tasks to workers. Introducing multidimensional heterogeneity in my model endogenizes the ranking of workers, letting ranks reflect mismatch in the assignment. The main assumption I place on the model is the discreteness of the distribution of skills across workers. This assumption is motivated by the study of occupations, which arise from the assignment as bundles of tasks to a type of worker, similar to occupations in plant level data as in Combemale, Whitefoot, Ales, and Fuchs (2021). The same assumption has been used in different contexts. For instance, Stokey (2018) develops a model with one-dimensional heterogeneity, where a continuum of workers are assigned to finitely many tasks, to study the effects of task biased technical change on the wage structure.

Methodologically, the closest paper to mine is Feenstra and Levinsohn (1995), who use a similar setup in the context of a continuum of buyers choosing from a discrete set of products. I generalize their model in a labor market context and provide a general proof of the existence and uniqueness of the solution using results from optimal transport theory (Villani, 2009; Galichon, 2016). This allows me to extend their differentiability results. Unlike Feenstra and Levinsohn (1995), I do not consider unobserved heterogeneity across workers and tasks. Finally, the applications to technical change, unemployment, and automation are all new.[4]

This paper is also related to the literature documenting the relevance of multiple skills in explaining educational choices (Willis and Rosen, 1979), differences in wages within demographic categories (Heckman and Sedlacek, 1985), the role of social (non-cognitive) skills relative to cognitive skills in various labor market outcomes (Heckman, Stixrud, and Urzua, 2006; Spitz-Oener, 2006; Black and Spitz-Oener, 2010; Deming, 2017), and the decline of occupations intensive in routine-manual tasks (Autor, Levy, and Murnane, 2003). In particular, this paper is related to papers on multidimensional skill mismatch and occupational choice, and the specificity of human capital to occupations and skills, i.a., Poletaev and Robinson (2008), Kambourov and Manovskii (2009), Gathmann and

[4] In Feenstra and Levinsohn (1995)'s setup, the techniques I develop can be applied to the problem of designing a new product (defined by a vector of characteristics) given a distribution of consumers.

Schönberg (2010), Yamaguchi (2012), Stinebrickner, Sullivan, and Stinebrickner (2019), Guvenen, Kuruscu, Tanaka, and Wiczer (2020), and Lise and Postel-Vinay (2020). This literature treats the assignment of tasks to occupations as exogenous and invariant, and focuses on informational frictions. I endogenize the bundling of tasks into occupations, which depends on technology, and the demand and supply of skills.

Finally, I contribute to the literature on the effects of automation and other worker replacing technologies: Acemoglu and Restrepo (2018b, 2020, 2021), Aghion, Jones, and Jones (2019), Hemous and Olsen (2021), among others. In particular, I explicitly model the multidimensional nature of skill heterogeneity. This is relevant to determine the automatability of tasks as shown recently by Frey and Osborne (2017), Webb (2020), and Martinez and Moen-Vorum (2021). In this way, I provide a framework to analyze the direction and consequences of automation.

1 Task assignment model

I present a model where occupations arise as bundles of tasks assigned to workers, and the boundaries of occupations react endogenously to changes in technology (e.g., automation, skill-biased technical change) and demographics (e.g., the distribution and skills of workers). The model builds on one-dimensional task-based models of production in the spirit of Rosen (1978) and Acemoglu and Autor (2011). I extend the one-dimensional framework by incorporating multidimensional heterogeneity across workers and tasks following Lindenlaub (2017). In the model, workers are defined by a vector of skills (cognitive, manual, social ability, etc) and tasks are defined by a vector of the skills involved in performing them.

Production requires the completion of a continuum of tasks by workers. There are finitely many types of workers who are assigned to tasks by the firm. A single type of worker can then perform various tasks; I refer to the set of tasks performed by a worker as the worker's occupation. The productivity of a worker at a given task is determined by how well the worker's skills match the skills used in performing the task. In what follows I describe in detail the role of workers, tasks and the production technology.

Workers There is a continuum of workers characterized by the skills they possess, captured by a vector $x \in \mathcal{S} \subset \mathbb{R}^d$, where $\mathcal{S}$ is the space of skills and $d \geq 1$ is the number of skills. Vector x encodes the level of the worker's skills, like cognitive, manual, or social skills.

There are N types of workers in the economy: $\{x_1, \ldots, x_N\} \equiv \mathcal{X}$. x_n is the skill vector of workers of type n. There is a mass p_n of workers of type x_n. Each worker is endowed with one unit of time. This implies that workers of type n have a total of p_n units of time available to work. Workers can either be assigned to tasks or not. If unassigned, a worker receives a payment $\underline{w} \geq 0$, understood as the outside option of of the worker. Workers supply their time inelastically at any wage $w \geq \underline{w}$.[5]

Tasks Production requires completing a continuum of differentiated tasks. Let $\mathcal{Y} \subseteq \mathcal{S}$ denote the set of tasks used in production. Tasks $y \in \mathcal{Y}$ differ in the skills involved in performing them, and how many times they must be performed. One unit of time is required to perform a task once. I denote the density of tasks used in production by $g : \mathcal{Y} \to \mathbb{R}_+$. I maintain the following assumption throughout:

Assumption 1. *The set of tasks and the distribution of tasks satisfy the following properties:*

i. $g : \mathcal{Y} \to \mathbb{R}_+$ *is an absolutely continuous probability density function with an associated absolutely continuous measure* G *on* $\mathcal{Y}$*;*

ii. there are enough workers to complete all tasks, i.e., $G(\mathcal{Y}) = \int_{\mathcal{Y}} g(y)\, dy \leq \sum_{n=1}^{N} p_n$*;*

iii. the set of tasks $\mathcal{Y}$ *is compact.*

[5]An alternative interpretation is that of a firm with N workers with different skills $\{x_1, \ldots, x_N\}$ and effective productivities $\{p_1, \ldots, p_N\}$.

Task-output Workers vary in their productivity across tasks depending on the match between the skills they possess (x) and the skills involved in performing the task (y). The function $q : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ gives the worker-task-specific output generated by a worker of type x performing task y. If a task is not assigned to any worker, then no output is generated for that task, abusing notation: $q(\emptyset, y) = 0$ for all $y \in \mathcal{Y}$.

In general, the productivity of a worker at a given task will depend on the mismatch between the worker's and task's skills as captured by some notion of the distance between the worker and the task in the skill space.[6] The optimal assignment balances the desire to minimize the mismatch between workers and the tasks they perform, with the capacity constraints imposed by the limited availability of workers. The exact shape of the assignment depends on the specific distance measure capturing mismatch, but the general properties of the assignment do not. I return to this in Section 2 when I discuss the optimal assignment.

Finally, it will often prove useful to impose a functional form on q to fix ideas and derive further results in the analysis to come:

Assumption 2. *The task output production function satisfies:*

$$q(x, y) = \exp\left(a_x^{'} x + a_y^{'} y - (x - y)^{'} A (x - y)\right), \tag{1}$$

where A is symmetric and positive definite.

Assumption 2 corresponds to the payoff function of Tinbergen (1956), the utility function of Feenstra and Levinsohn (1995), and the production technology of Lindenlaub (2017). Under (1), the mismatch between a worker and a task is measured by the weighted quadratic distance between worker and task's skills. Matrix A controls the weights of each skill in the mismatch. The linear terms, $a_x^{'} x$ and $a_y^{'} y$, capture more skilled workers having an absolute advantage in production, and tasks involving higher skill levels generating more output.

[6] The dependance of production on the mismatch between worker and task skills is similar in spirit to Lazear (2009)'s skill weights approach, where he studies job-specific skill.

Assignment The assignment of tasks to workers is described by a function $T : \mathcal{Y} \to \mathcal{X}$, so that task y is performed by worker $T(y) \in \mathcal{X}$. The set of tasks performed by a type of worker form that worker's occupation. The occupation of workers of type x_n is:

$$\mathcal{Y}_n \equiv T^{-1}(x_n) = \{y \in \mathcal{Y} \,|\, x_n = T(y)\}. \tag{2}$$

Occupations form a partition of the space of tasks into at most N cells.[7] Figure 1 in the introduction showed an example of an assignment that partitions the space of tasks into three occupations, corresponding to three worker types.

An assignment is deemed *feasible* if workers can supply all the time demanded by their occupation. This time is given by the number of tasks in the worker's occupation.

Definition 1. An assignment is *feasible* if the demand for worker $n's$ time, $D_n \equiv \int_{\mathcal{Y}_n} dG$, satisfies $D_n \leq p_n$ for all $n \in \{1, \ldots, N\}$.

Production Production aggregates the output from all worker/task pairs through a Cobb-Douglas technology given an assignment T:[8]

$$F(T) = \exp\left(\int_{\mathcal{Y}} \ln q(T(y), y)\; dG\right). \tag{3}$$

Under this technology, production only takes place if all tasks are assigned and performed. Recall that task output is zero if a task is left unassigned, $q(\emptyset, y) = 0$.[9]

[7] It is possible that $\mathcal{Y}_n = \emptyset$, so that no task is assigned to worker x_n.

[8] The aggregator does not need to be of the Cobb-Douglas type. Results hold for aggregators of the CES family: $F(T) = \left(\int (q(T(y), y))^{\frac{\sigma-1}{\sigma}} dG(y)\right)^{\frac{\sigma}{\sigma-1}}$, with $\sigma > 1$. See Appendix B.

[9] The production technology resembles a continuous version of Kremer (1993)'s O-Ring production function. In order to make the comparison precise, it is necessary to change the interpretation of q. Consider a continuous production line indexed by $y \in \mathcal{Y}$, at each point in the production line a fatal error can occur that terminates the production process in failure. The arrival rate of an error is given by $\ln q(x, y) \geq 0$ and depends on the point in the production process (y) and the worker assigned to that point (x). The probability that no error arrives at the end of the whole process is given by (3). Thus, $F(T)$ can also be interpreted as expected output given an assignment T. See Sobel (1992) for another application of this idea.

2 The optimal assignment of tasks to workers

The problem is to find a feasible assignment that maximizes output:

$$\max_{T} F(T) \qquad \text{s.t. } \forall_n \, D_n \leq p_n. \tag{4}$$

This can be seen as problem of a manager organizing production taking as given the type and quantity of workers at their disposal. It can also be interpreted as the problem of a planner allocating workers to tasks. The allocation coincides with that of firms hiring workers in a decentralized market and assigning them to tasks once hired, with wages determined by the workers' marginal products, see Section 3.2.

It is possible to guarantee the existence and uniqueness of a solution by imposing conditions only on the production technology q. Proposition 1 makes this precise:

Proposition 1. *Consider the optimal assignment problem in (4). If q is such that:*

i. Every worker/task pair is productive: $q(x,y) > 0$ for all pairs $(x,y) \in \mathcal{X} \times \mathcal{Y}$,

ii. $q(x,\cdot)$ is upper-semicontinuous in y given $x \in \mathcal{X}$, and

iii. q discriminates across workers: for all $x_n \neq x_\ell$, $q(x_n,y) \neq q(x_\ell,y)$ G-a.e.

Then, there exists a $(G-)$unique solution $T^\star$ to the problem in (4). Moreover, there exist a unique $\lambda^\star \in \mathbb{R}^N$ with $\min \lambda_n^\star = 0$ such that $T^\star$ is characterized as:

$$T^\star(y) = \underset{x \in \mathcal{X}}{\mathit{argmax}} \left\{ \ln q(x,y) - \lambda^\star_{n(x)} \right\}, \tag{5}$$

where $n(x)$ gives the index of a type of worker $x \in \mathcal{X}$.

Proof. The result is established by expressing the problem in (4) as an optimal transport problem. The proof is divided into three Lemmas that relax the problem in (4) by allowing for non-deterministic assignments and then constructing a solution by means of the dual of the relaxed problem. A non-deterministic assignment is a joint measure over worker/task pairs: $\pi : \mathcal{X} \times \mathcal{B}(\mathcal{Y}) \to \mathbb{R}_+$, where $\mathcal{B}(\mathcal{Y})$ denotes the Borel sets of $\mathcal{Y}$. In contrast to the assignment T described above, a non-deterministic assignment π allows for mixing in the matching of workers and tasks, with the same task being (potentially) performed by more than one type of worker.

The relaxed problem, in terms of π, is linear in the choice of the assignment after a suitable transformation off the objective function:

$$\max_{\pi\in\Pi(P,G)} \sum_{n=1}^{N} \int_{\mathcal{Y}} \ln q\left(x_n, y\right)\, d\pi\left(x_n, y\right) \tag{6}$$

The dual to (6) expresses the problem in terms of the multipliers (or potentials) λ and ν:

$$\begin{aligned}\max_{\pi\in\Pi} \sum_{n=1}^{N} \int_{\mathcal{Y}} \ln q\left(x_n, y\right)\, d\pi\left(x_n, y\right) &= \inf_{\substack{(\lambda,\nu)\in\mathbb{R}^N\times L^1(G)\\ \lambda_n+\nu(y)\geq \ln q(x_n,y)}} \sum_{n=1}^{N} \lambda_n p_n + \int_{\mathcal{Y}} \nu\left(y\right) dG \\ &= \inf_{\nu^x\in\mathbb{R}^N} \sum_{n=1}^{N} \nu_n^x p_n + \int_{\mathcal{Y}} \max_n \left\{\ln q\left(x_n, y\right) - \lambda_n\right\} dG\end{aligned} \tag{7}$$

Under the conditions above, this problem admits a unique solution which characterizes a deterministic assignment $T^\star$ according to equation (5).

Lemmas B.1, B.2, and B.3 formalize these arguments and establish the properties of the solution to the primal and dual problems. The results follow from Theorems 5.10 and 5.30 in Villani (2009) summarized in Theorem A.1 of Appendix A. All Lemmas are stated and proven in Appendix B.

□

Before discussing the characterization of the optimal assignment, I briefly discuss the role of the three conditions in Proposition 1. The first condition ensures that $F\left(x, y\right)$ is defined for any assignment. The second condition guarantees that duality applies to the problem so that the equality in (7) holds. The third condition plays a crucial in role in establishing the existence and uniqueness of an optimal assignment function $T^\star$. The injectivity of q in x given y makes it possible to distinguish between workers in each task. The discreteness o the problem on the worker side considerably relaxes the standard differentiability conditions necessary to obtain a deterministic assignment. The injectivity plays the same role as the 'twist condition' of Carlier (2003) and the condition for positive assortative matching in Lindenlaub (2017).

The characterization of the optimal assignment in (5) allows me to give a more explicit characterization of the occupations in terms of the production technology q:

$$\mathcal{Y}_n = \left\{y \in \mathcal{Y} \,|\, \forall_\ell \ \ln q\left(x_n, y\right) - \lambda_n^\star \geq \ln q\left(x_\ell, y\right) - \lambda_\ell^\star\right\}. \tag{8}$$

Tasks are optimally assigned to workers that are more productive at performing them, workers with lower skill mismatch, subject to the penalty captured by $\lambda^{\star}$ that balances the demand for that type of worker with the limited supply of hours (p_n).

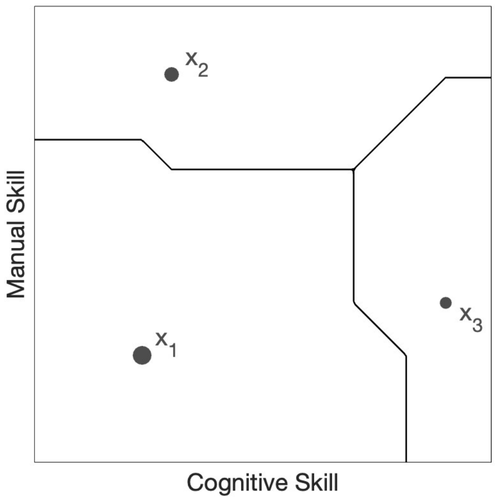
The boundaries of occupations play a central role in determining the properties of the assignment like the the workers' marginal product, compensation, and substitutability. Intuitively, the boundaries are formed by the marginal tasks of a worker, a concept that I will make precise in Section 3.1. Formally, the boundaries of an occupation are formed by task $y \in \partial\mathcal{Y}_n$ for which the inequality in (8) is met with equality for some ℓ:

$$\begin{aligned}\partial\mathcal{Y}_n = \{y \in \mathcal{Y} | \exists_{\ell \neq n} \ln q\left(x_n, y\right) - \lambda_n^{\star} = \ln q\left(x_\ell, y\right) - \lambda_\ell^{\star} \\ \wedge \forall_{m \neq \ell, n} \ln q\left(x_n, y\right) - \lambda_n^{\star} \geq \ln q\left(x_m, y\right) - \lambda_m^{\star}\}\end{aligned} \tag{9}$$

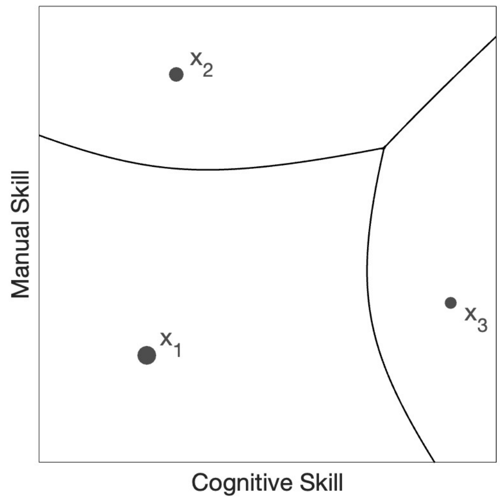
The key feature of the assignment is that the mismatch between workers (pairs) is constant across the boundaries of occupations, as in equation (9). The difference in mismatch is reflected by the difference in the productivity between workers $(\ln q\left(x_n, y\right) - \ln q\left(x_\ell, y\right))$ and captured by the value of the multipliers $(\lambda_n^{\star} - \lambda_\ell^{\star})$.

The shape of the boundaries in the space of skills depends on how the mismatch measure implicit in q weighs the distance between workers and tasks. However, the assignment follows the common principle of minimizing mismatch regardless of the functional form chosen for q. Figure 2 exemplifies this by plotting the optimal assignment under four different measures of distance for the plane. I consider distances belonging to the Minkowski (or L-p) family, which take the form: $d\left(x, y\right) = \left(\sum_{i=1}^{d} |x_i - y_i|^p\right)^{1/p}$. A larger value for p makes the distance measure more sensitive to the dimension of highest mismatch, which affects the curvature of the boundaries.[10]

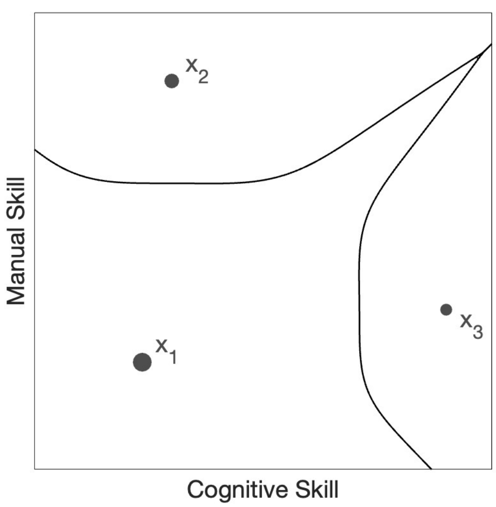
[10] Alternative measures of mismatch can reduce the dimensionality of the problem by encoding the differences between workers and tasks into a single dimension. This is the case when measuring mismatch with the cosine similarity between the skill vectors of workers and tasks. The assignment problem is reduced to finding cutoff values for angles between 0 and $\pi/2$ that characterize rays partitioning the space. The cosine similarity has been used to measure the distance between occupations by Gathmann and Schönberg (2010). I provide details for the cosine similarity mismatch in Appendix C.



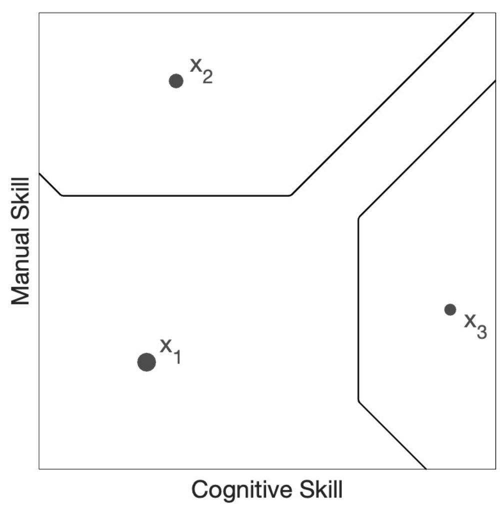
(a) Manhattan distance - L1



(b) Euclidean distance - L2



(c) Minkowski distance of Order 4 - L4



(d) Chebyshev distance - L∞

Figure 2: Assignment Example - Mismatch Loss

Note: *The figures shows the optimal assignments in a two-dimensional skill space (cognitive and manual skills) with three types of workers* $\{x_1, x_2, x_3\}$ *with mass* $P = \{0.4, 0.3, 0.3\}$*. Tasks are uniformly distributed over the unit square, i.e.,* $\mathcal{Y} = [0,1]^2$ *and* $g(y) = 1$*. The production function* q *is given by* $\ln q(x,y) = a_x^{'} x + a_y^{'} y - d(x,y)$*, where* $d(x,y)$ *is a distance measure capturing mismatch. Each sub-figure considers a different distance of the Minkowski distance family:* $d(x,y) = \left(\sum_{i=1}^{d} |x_i - y_i|^p\right)^{1/p}$*. A higher* p *puts more weight on the dimension with the highest mismatch. The Chebyshev distance is obtained as* $\lim_{p\to\infty} \left(\sum_{i=1}^{d} |x_i - y_i|^p\right)^{1/p} = \max_i |x_i - y_i|$*.*

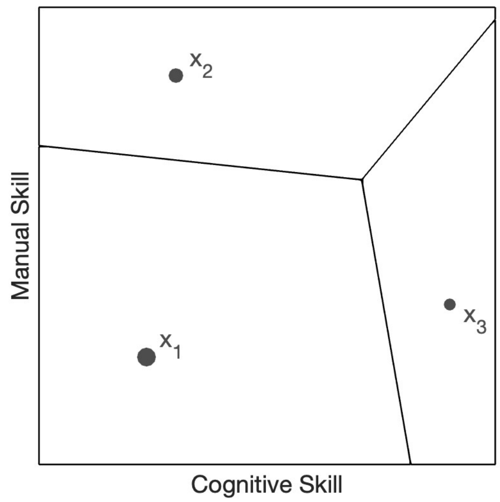


Figure 3: Assignment Example - Quadratic Mismatch Loss

Note: *The figure shows the optimal assignment in a two-dimensional skill space (cognitive and manual skills) with three types of workers* $\{x_1, x_2, x_3\}$ *with mass* $P = \{0.4, 0.3, 0.3\}$. *Tasks are uniformly distributed over the unit square, i.e.,* $\mathcal{Y} = [0,1]^2$ *and* $g(y) = 1$. *The production function* q *is as in (1) with* $A = I_2$.

Imposing Assumption 2 simplifies the characterization of occupations. When q is as in equation (1), the boundaries of occupations take the form of hyperplanes whose normal vectors depend on matrix A and the difference in skills between neighboring workers. The boundary between the occupations of workers x_n and x_ℓ is:

$$y \in \mathcal{Y}_n \cap \mathcal{Y}_\ell \longleftrightarrow 0 = y' \underbrace{A(x_\ell - x_n)}_{\text{Normal Vector}} - \frac{1}{2} \underbrace{\left(x_\ell' A x_\ell - x_n' A x_n + a_x' (x_\ell - x_n) + \lambda_\ell^\star - \lambda_n^\star \right)}_{\text{Intercept}}. \quad (10)$$

Equation (10) reveals an equivalence between the optimal assignment and the partition induced by a power diagram, which enables the use of tools from computational geometry to analyze the properties of the assignment.[11] The optimal assignment partitions the space into convex polyhedra defined by hyperplanes as in Aurenhammer, Hoffmann, and Aronov (1998) and Galichon (2016, ch. 5). Figure 3 exemplifies this by plotting the optimal assignment under the task output function of Assumption 2.

[11] A power diagram partitions a space into cells that minimize the power between a node (x) associated with the cell and the points y in the cell. The power function between two points is $\text{pow}(x,y) = d(x,y)^2 - \mu$, where $d(x,y)$ is a distance and $\mu \in \mathbb{R}$.

Indirect production function The solution to the assignment problem implies an indirect production function that transforms inputs (workers) into goods. Crucially, The amount of an input (a type of worker) used in production and what that input is used for are not the same (Autor, 2013). As a consequence, the relationship between inputs and output depends on how tasks are assigned to workers, and how the assignment itself changes as the amount of inputs varies. The role of workers in production is captured by the value of the assignment problem (4):

$$V\left(p_1, \ldots, p_N\right) = \max_{T} F\left(T\right) \qquad \text{s.t. } \forall_n \, D_n \leq p_n. \tag{11}$$

Function $V : \mathbb{R}_+^N \rightarrow \mathbb{R}_+$ describes how production changes when the composition of the workforce changes, allowing for workers to be re-assigned optimally across tasks.

3 Properties of the assignment

I now examine the properties of the assignment. The multipliers $\lambda^\star$ that characterize the assignment in (5) play a central role by determining the boundaries of occupations and therefore the marginal product and compensation of workers. I show that $\lambda^\star$ also implies an ordering or workers based on their relative productivity at boundary tasks. This ranking plays an important role in shaping the effects of automation and other worker replacing technologies studied in Section 4.

3.1 Marginal products and worker ranks

The marginal product of workers of type n is obtained from the indirect product function V as the change in output if the supply of type n workers (p_n) were to increase. In this way, the marginal product takes into account how the assignment changes optimally in response to the increase in the supply of workers of type n.[12]

Proposition 2. *The marginal product of type n workers is:*

$$MP_n \equiv \frac{\partial V(p_1, \ldots, p_N)}{\partial p_n} = F(T^\star)\,\lambda_n^\star. \tag{12}$$

Proof. The result follows from the relationship between the multiplier λ of the relaxed problem (6) and its dual (7) and the multipliers of the original problem (4). The relationship is obtained by applying the envelope theorem (Milgrom and Segal, 2002) and the definition of the indirect production function in (11). See Lemma B.4 in Appendix B for a detailed derivation of the result. □

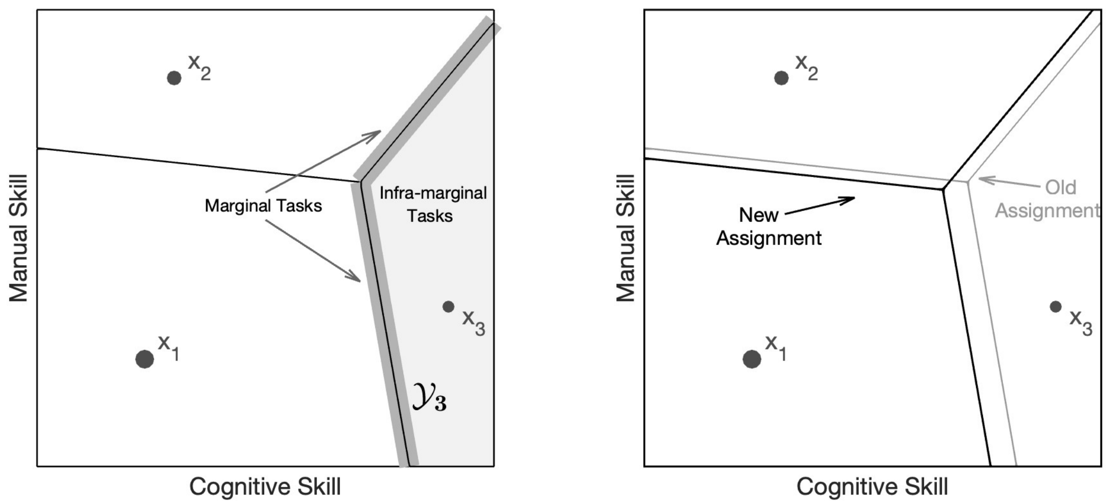
The relationship between the value of $\lambda^\star$ and the marginal product of workers comes from how the assignment responds to an increase in the supply of workers. When p_n increases, the additional workers increase output only if tasks are re-assigned to them from other workers. The first tasks to be reassigned are those in the boundaries of occupations. Consider the occupations of two types of workers, n and ℓ. Tasks in the boundary of the occupations, $y \in \mathcal{Y}_n \cap \mathcal{Y}_\ell$, satisfy:

$$\lambda_n^\star - \lambda_\ell^\star = \ln q(x_n, y) - \ln q(x_\ell, y). \tag{13}$$

Then, the difference in the multipliers $\lambda_n^\star$ and $\lambda_\ell^\star$ is given by the log difference in task output along the boundary between workers n and ℓ. That is, the percentage increase (or decrease) in output if the tasks along the boundary are re-assigned from ℓ to n.[13] Thus, it is only optimal to make use of the additional supply of workers if output increases along the boundary of the worker's occupation. In this sense, these are the marginal tasks of a worker as in Figure 4a.

[12] It is also possible to define a measure of the marginal product of a type n worker at a given task y, given some arbitrary assignment T. See Appendix B.4.

[13] It is useful to consider an example with finitely many tasks, say $\{y_1, y_2\}$, one assigned to worker x_n and the other to worker x_ℓ. Then, total output is $F(T) = q_1(x_n)\, q_2(x_\ell)$. If the assignment changes by having worker x_n perform both tasks the new output is $F\left(T'\right) = \frac{q_2(x_n)}{q_2(x_\ell)} F(T)$. Then, $\ln \frac{F\left(T'\right)}{F(T)} = \ln \frac{q_2(x_n)}{q_2(x_\ell)} = \lambda_n - \lambda_\ell$, so that output increases by approximately $100\,(\lambda_n - \lambda_\ell)\,\%$.



(a) Optimal assignment

(b) Changes in demand when $\lambda_3^\star$ increases

Figure 4: Marginal Tasks and Task Reassignment

Note: *The figures shows the assignment in a two-dimensional skill space (cognitive and manual skills). Five types of workers are considered* $\{x_1, \ldots, x_5\}$ *with mass* $P = \{0.3, 0.2, 0.3, 0.1, 0.1\}$. *Tasks are uniformly distributed over the unit square, i.e.,* $\mathcal{Y} = [0,1]^2$ *and* $g(y) = 1$. *The production function* q *is as in (1) with* $A = I_2$. *The left panel highlights the marginal tasks of worker* x_3. *The right panel shows the reassignment of tasks following an increase in the supply of* x_3 *workers. The boundaries of* $\mathcal{Y}_3$ *are pushed outwards, but also the shared boundary of* $\mathcal{Y}_2$ *and* $\mathcal{Y}_1$ *following the cascading effect of the reassignment.*

The reassignment process that originates in the increase in the supply of a worker gives rise to a cascading effect affecting the assignment of other workers, as in Figure 4b. As tasks are reassigned towards type n workers, workers along the boundaries of $\mathcal{Y}_n$ are displaced. This process generates an excess supply of workers of other types, giving rise to a new round of re-assignment along the boundaries of these workers' occupations. Following the cascading process reveals an ordering of workers by productivity, with the least productive worker being displaced by increases in the supply of more productive workers. As a consequence, the least productive worker has zero marginal product. Increases in the supply of that type of workers do not increase output because the additional workers are left unassigned.[14]

The total gain in output from the initial increase in the supply of workers of type n takes into account the increase in output from all the re-assignments. Using the relation in (13),

[14]The property that the least productive worker has zero marginal product is induced by the capacity constraint on the set of tasks and the times each task can be performed. This property motivates the normalization of the multiplier λ in Proposition 1.

and recalling that $\min \lambda_n^\star = 0$, we get a total increase in output of $\lambda_n^\star$ as in Lemma 2.

The value of $\lambda^\star$ depends on the differences in skills and skill mismatch of workers relative to the least productive worker. Under Assumption 2 it is possible to make this precise. Manipulating equation (10) gives:

$$\lambda_n^\star = \underbrace{a_{x'}\left(x_n - \underline{x}\right)}_{\text{Difference in Skills}} - \underbrace{\underbrace{\left(x_n - y_n\right)' A \left(x_n - y_n\right)}_{x_n \text{ mismatch at boundary}} - \underbrace{\left(\underline{x} - \underline{y}\right)' A \left(\underline{x} - \underline{y}\right)}_{\underline{x} \text{ mismatch at boundary}}}_{\text{Difference in Mismatch}}. \tag{14}$$

where $\underline{x}$ are the skills of the lowest paid worker (the worker with $\lambda_m^\star = 0$), and y_n and $\underline{y}$ are boundary tasks of workers x_n and $\underline{x}$ respectively.

Finally, it is worth stressing the role of the multipliers $\lambda^\star$ in determining a ranking of workers. An a-priori ranking of workers does not exist because they are heterogeneous along multiple dimensions. However, the optimal assignment implies a ranking based on the relative productivity of workers at their boundary tasks. The ranking is completely captured by $\lambda^\star$ and has direct implications for how a reassignment happens following a change in the environment. I will return to this in Section 4.1 when I introduce worker replacing technologies like automation.

3.2 Worker compensation

The value of the marginal product determines how workers are compensated. Consider the problem of a price-taking firm seeking to maximize profits. The firm's problem is to choose both the demand for each type of worker and the assignment of tasks to workers:

$$\max_T F(T) - \sum_{n=1}^{N} w_n D_n(T), \tag{15}$$

where w_n is the wage paid to a worker of type n, and D_n is the demand for workers of type n, obtained as in Definition 1. This problem results in the same assignment as the optimal assignment problem in (4) if the workers' wages correspond to their marginal products plus a constant that guarantees that all workers receive at least their outside option:

$$w_n = F(T^\star)\,\lambda_n^\star + \kappa, \quad \text{where } \kappa \geq \underline{w}. \tag{16}$$

The level of wages is not pinned down because only the difference in wages affects the assignment (see equation 8).[15] Nevertheless, differences in wages are informative about the role of skill mismatch in determining the marginal product of workers. The wage of a worker relative to the lowest paid worker gives their marginal product which depends on the worker's mismatch and skill level as in (13) and (14).

Finally, the marginal product of workers also pins down the labor share. The characterization of the marginal products makes it clear that workers do not appropriate all the output they produce, but only the output they generate in marginal (boundary) tasks plus the compensation for their outside option. The labor share is:

$$LS \equiv \frac{\sum\limits_{n=1}^{N} w_n D_n}{F(T)} = \sum_{n=1}^{N} \lambda_n^\star p_n + \frac{\kappa G(\mathcal{Y})}{F(T)}. \tag{17}$$

3.3 Substitutability across workers

The substitutability of workers in production, as measured by their elasticity of substitution, plays an important role in policy analysis, and in determining the effects of changes to technology and the distribution of skill supply and demand. Intuitively, workers performing similar tasks are more substitutable, as are workers with similar skills. I make these results precise by computing the elasticity of substitution under the optimal assignment.

The appropriate measure of substitutability in a setup with more than two types of

[15] The indeterminacy of the level of wages is a common feature of assignment models (Sattinger, 1993). This result is not a feature of the discreteness in the distribution of workers, see Lindenlaub (2017). The level of worker compensation depends on additional assumptions. For example, having excess workers ($P > G(\mathcal{Y})$) implies that (at least) some type of worker will be partially unassigned (unemployed), driving down the wage for that type of worker to $\underline{w}$. This will be the case when I introduce automation in Section 4.1.

workers is the Morishima elasticity of substitution (Blackorby and Russell, 1981, 1989).[16] The Morishima elasticity measures the change on the ratio of demands for two inputs (in this case two workers, $^{D_\ell}/_{D_n}$) after a change of the marginal products (MP_n, MP_ℓ), which in the model match changes in wages.

Definition 2. The elasticity of substitution between workers of type n and ℓ is:

$$M_{\ell n} \equiv \frac{\partial \ln {}^{D_n}/_{D_\ell}}{\partial \ln \mathrm{MP}_n} = \mathcal{E}_{\ell n} - \mathcal{E}_{nn}, \tag{18}$$

where $\mathcal{E}_{\ell n} = \frac{\mathrm{MP}_n}{D_\ell}\frac{\partial D_\ell}{\partial \mathrm{MP}_n}$ is the cross elasticity of demand for worker k with respect to a change in the marginal product of worker n.[17]

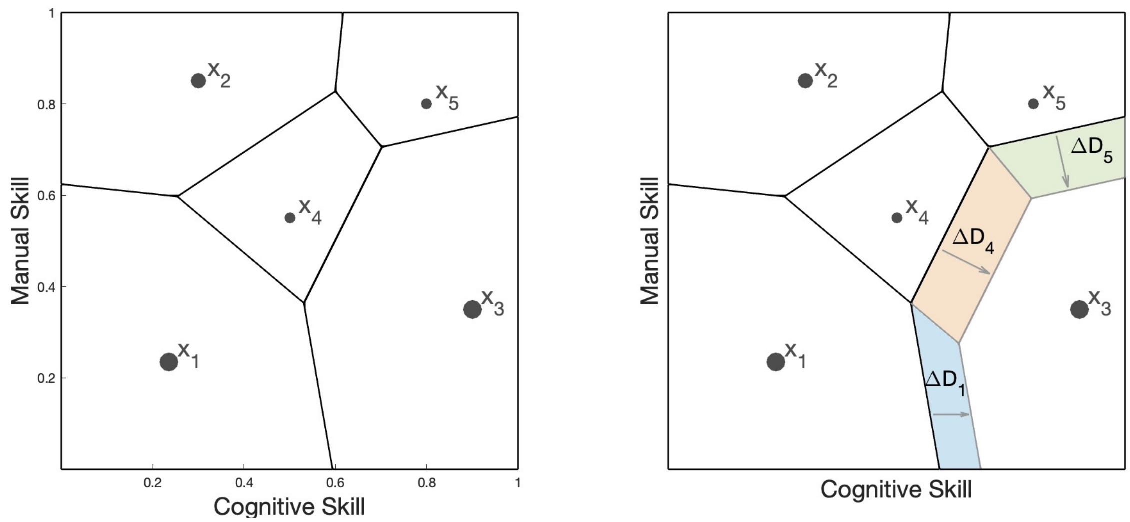
A direct implication of the assignment is that workers are gross substitutes in the sense of Kelso and Crawford (1982). An increase in the marginal product of worker x_n is captured by an increase in $\lambda_n^\star$, which results in a decrease of their demand, D_n, and (weakly) increases the demand for other workers, see (8). It follows that $\mathcal{E}_{nn} < 0$ and $\mathcal{E}_{n\ell} \geq 0$. From the point of view of worker compensation, increasing $\lambda_n^\star$ raises the cost of worker n, causing the firm to substitute it for other workers. The elasticity of substitution is always positive.

Moreover, only neighbors are directly substitutable for one another and the cross-elasticity is zero between any two workers who do not share a boundary. A change in $\lambda_n^\star$ requires a change in the difference of task outputs along the boundary. Because of this, only the neighbors of worker x_n are directly affected when $\lambda_n^\star$ changes. However, to obtain the magnitude of the cross-elasticities, $\mathcal{E}_{n\ell}$, it is necessary to determine how much workers' demands change with the value of $\lambda^\star$. The sensitivity of the boundaries depends on the slope of the production function q evaluated at the boundary tasks, see (13).

The geometric structure induced by imposing Assumption 2 makes it possible to further characterize the change in demand following a change in $\lambda^\star$. In this case, the change in

[16] See Baqaee and Farhi (2019) for a recent application of the Morishima elasticity.

[17] With more than two inputs the direction of the change in the ratio of marginal products matters because the demands for inputs changes differently depending on whether MP_n or MP_ℓ vary (Blackorby and Russell, 1989, pg 885). Because of this, the elasticity is in general asymmetric so that $M_{\ell n} \neq M_{n\ell}$.



(a) Optimal assignment (b) Changes in demand when $\lambda_3^\star$ increases

Figure 5: Assignment Example - Quadratic Mismatch Loss

Note: *The figures shows the assignment in a two-dimensional skill space (cognitive and manual skills). Five types of workers are considered* $\{x_1, \ldots, x_5\}$ *with mass* $P = \{0.3, 0.2, 0.3, 0.1, 0.1\}$*. Tasks are uniformly distributed over the unit square, i.e.,* $\mathcal{Y} = [0,1]^2$ *and* $g(y) = 1$*. The production function* q *is as in (1) with* $A = I_2$*.*

demand is given by the area of a (hyper)trapezoid, formed as the plane that defines the boundary between occupations moves. Figure 5b illustrates this by increasing the value of $\lambda_3^\star$. The boundaries of $\mathcal{Y}_3$ shift 'inward' in a parallel fashion, reducing the demand for x_3 and increasing the demand for all its neighbors. In contrast, the boundary of $\mathcal{Y}_2$, which does not share a boundary with $\mathcal{Y}_3$, does not change.

The following Proposition formalizes the above results:

Proposition 3. *Let* $\lambda \in \mathbb{R}^N$ *be a vector of multipliers. If* q *is continuous then* D_n *is continuously differentiable with respect to* λ *and:*

i. If $\mathcal{Y}_n \cap \mathcal{Y}_\ell = \emptyset$ *then* $\frac{\partial D_n}{\partial \lambda_\ell} = \frac{\partial D_\ell}{\partial \lambda_n} = 0$ *and* $\mathcal{E}_{n\ell} = \mathcal{E}_{\ell n} = 0$*;*

ii. $\frac{\partial D_n}{\partial \lambda_n} = -\sum\limits_{\ell \neq n} \frac{\partial D_\ell}{\partial \lambda_n} < 0.$

If Assumption 2 holds then:

iii. $\forall_{\ell \neq n} \quad \frac{\partial D_n}{\partial \lambda_\ell} = \frac{area(\mathcal{Y}_n \cap \mathcal{Y}_\ell)}{2\sqrt{(x_n - x_\ell)' A' A (x_n - x_\ell)}} = \frac{\int_{\mathcal{Y}_n \cap \mathcal{Y}_\ell} dG}{2\sqrt{(x_n - x_\ell)' A' A (x_n - x_\ell)}} \geq 0.$

Proof. The first result follows immediately from the characterization of occupations in (8). The second result from the feasibility of the assignment, which ensures that $\sum_{n=1}^{N} D_n = \int_{\mathcal{Y}} dG$ so that the sum of demands is constant. Then:

$$\frac{\partial D_n}{\partial w_n} + \sum_{m \neq n} \frac{\partial D_m}{\partial w_n} = 0$$

The third result extends the findings of Feenstra and Levinsohn (1995) for arbitrary configurations of workers (x) by applying Reynolds' transport theorem (see Theorem A.2 in Appendix A). I present the complete proof in Lemma B.5 of Appendix B.2.

□

The first results in Proposition 3 formalizes the idea that only neighbors are directly substitutable. The second result exploits the feasibility of the assignment to find a relationship between the change in demand across workers. This relationship gives rise to a sharper formula for the elasticity of substitution:

Corollary 1. *The Morishima elasticity of substitution between workers x_n and x_ℓ is a weighted average of the cross-elasticities of demand of all workers, with the weights given by the demand of each type of worker relative to worker n's demand.*

$$M_{\ell n} = \left(1 + \frac{D_\ell}{D_n}\right) \mathcal{E}_{\ell n} + \sum_{m \neq n, \ell} \frac{D_m}{D_n} \mathcal{E}_{mn}. \tag{19}$$

The last result of Proposition 3 exploits the geometric structure of the problem under Assumption 2 to compute closed-form expressions for the derivates of demand. The cross derivative of demand depends on how exposed two workers are to one-another, measured by the length of their boundary, and how similar their skills are, measured by the weighted distance between their skill vectors (x_n and x_ℓ). This captures the idea that workers with more common tasks, i.e., with longer boundaries, are more substitutable. How much the boundary reacts to a change in demand depends on how similar workers are at performing tasks as captured by the denominator. The closer the skills of the workers are, the more substitutable they are along their boundary.

4 Directed technical change

Changes in technology are a major factor in shaping the way in which tasks are assigned to workers. For instance, the increase of information technology (IT) in the workplace has shifted focus from manual to cognitive skills, and changed the distribution of tasks across occupations (e.g., clerical and secretarial jobs). More directly, automation technologies and offshoring have replaced workers in performing certain tasks across manufacturing jobs, customer services, accounting, among others.

I consider two forms of technical change and study how they affect the division of tasks into occupations. Innovation in worker replacing technology (such as robots, software, AI, offshoring) lead to the automation of tasks and the reassignment of (remaining) tasks to workers. Innovation in skill-enhancing technology, such as IT in the modern workplace, or the power loom in the 18th and 19th centuries, changes the productivity of workers across tasks, inducing a reassignment of tasks to reduce mismatch across occupations.

Technical change can be directed (towards specific tasks or skills) with the aim of increasing production. Production increases the most by reducing the mismatch between tasks and workers, whether by directing automation towards the tasks with the highest mismatch or by increasing the weight on skills for which the workforce is better suited.

4.1 Worker replacing technologies

I introduce worker replacing technologies in the form of a robot that can replace workers in performing tasks. The robot is modeled as a flexible technology that can be adapted to perform different types of tasks. This captures a key property of current technologies like industrial robots or AI programs, which can be reprogrammed or adapted to carry out a variety of tasks (Frey and Osborne, 2017; Acemoglu and Restrepo, 2020). It also relates to technologies like offshoring, which, as automation, replace workers at the task they perform (Blinder, 2009; Blinder and Krueger, 2013).

The automation problem consists of designing a robot and optimally assigning tasks among the workers and the robot to maximize production. Tasks assigned to the robot are automated. I denote by $r \in \mathbb{R}^d$ the skills of the robot and by $p_r \geq 0$ its supply. The automation technology is embodied by a cost function $\Omega : \mathbb{R}_+^d \times \mathbb{R}_+ \to \mathbb{R}$, so that the cost of producing a mass p_r of a robot with skills r is given by $\Omega(r, p_r)$. Many changes in the patterns of automation can be seen as changes in the cost of automation, Ω. For instance, recent advances in artificial intelligence are reducing the cost of automating tasks intensive in cognitive skills (McKinsey Global Institute, 2017; Frey and Osborne, 2017; Martinez and Moen-Vorum, 2021), while previous innovations like industrial robot-arms allowed for the automation of tasks involving manual skills.

Once the robot is designed, the set of available workers is expanded to include it: $\mathcal{X}_R \equiv \{x_1, \ldots, x_N, r\}$. Accordingly, the assignment is now described by a function $T_R : \mathcal{Y} \to \mathcal{X}_R$. Robots are designed so that they replace workers at tasks where skill mismatch is high, and worker productivity is low. These tasks are located along the boundaries of occupations. Automation is thus less likely to occur at 'core' tasks of an occupation, for which the worker is best suited. I denote by $q_R : \mathbb{R}^d \times \mathcal{Y} \to \mathbb{R}$ the production technology of the robot so that a robot r performing task y produce $q_R(r, y)$.

When tasks are automated the total demand for labor decreases, inducing unemployment among workers.[18] Which workers become unemployed depends on the way in which the assignment reacts to the introduction of the robot. As tasks are assigned to the robot, the workers who would have performed those tasks are directly displaced. Yet, these workers do not necessarily become unemployed since they can take over the tasks of other workers. The end result of this process depends on the relative productivities of the workers as captured by $\lambda^\star$. As in Section 3.1, it is the workers with the lowest marginal product who will become displaced (unemployed) as a response to the introduction of the robot, even if the tasks in their occupation are not directly affected by automation.

[18] This is a consequence of the assumption that the set of tasks to be performed ($\mathcal{Y}$) is fixed, as is the distribution of tasks (G). I partly relax this assumption in Section 5.

However, a large enough displacement of tasks can change the ranking of workers by increasing the mismatch of displaced worker after reassignment. In this way, low levels of task displacement reduce workers' wages, but larger levels can change the ranking of workers along with larger reductions in wages. These patters are in line with findings for the U.S. from Acemoglu and Restrepo (2021).

The automation problem is to choose jointly the skills and mass of the robot (r, p_r), and the new assignment (T_R) to maximize output, net of the automation cost (Ω):

$$\max_{\{r,p_r,T_R\}} \quad F_R(T_R) - \Omega(r, p_r) \qquad \text{s.t. } \forall_n\, D_n \leq p_n \qquad D_R \leq p_r, \tag{20}$$

where:

$$F_R(T_R) = \exp\left(\int_{\mathcal{Y}\setminus\mathcal{Y}_R} \ln q(T_R(y), y)\; dG + \int_{\mathcal{Y}_R} \ln q_R(r, y)\; dG\right) \tag{21}$$

and

$$\mathcal{Y}_R = T_R^{-1}(r), \qquad D_R = \int_{\mathcal{Y}_R} dG. \tag{22}$$

It is convenient to think of the problem in two steps, first solving for an optimal assignment given a set of workers and a robot, and then choosing the optimal skills and mass of the robot. The problem of finding an optimal assignment can be simplified making use of the results in Proposition 1. Taking as given the robot's skills and mass (r, p_r), the optimal assignment is characterized by a vector $\mu^\star \in \mathbb{R}^{N+1}$:[19]

$$\begin{aligned} T_R^\star(y) = x_n \longleftrightarrow & \forall_\ell \ln q(x_n, y) - \mu_n^\star \geq \ln q(x_\ell, y) - \mu_\ell^\star \\ & \wedge \quad \ln q(x_n, y) - \mu_n^\star \geq \ln q_R(r, y) - \mu_R^\star. \end{aligned} \tag{23}$$

The problem becomes:

$$\max_{\{r,p_r\}} \quad V_R(r, p_r) - \Omega(r, p_r), \tag{24}$$

[19]This strategy has been exploited by the optimal sensor placement literature under quadratic loss functions, see Aurenhammer et al. (1998, Thm . 1) and Xin et al. (2016, Thm. 1).

where $V_R(r, p_r) = F_R(T_R^\star)$ takes into account how the optimal assignment reacts to changes in the robot skills and mass. I obtain the first order conditions of the problem using the envelope theorem of Milgrom and Segal (2002) and Reynolds' transport theorem:[20]

$$\nabla V_R(r, p_r) - \nabla \Omega(r, p_r) = 0_{d+1\times 1}. \tag{25}$$

I first focus on the the derivative of output with respect to the robot's skills:

$$\frac{\partial (V_R(r, p_r) - \Omega(r, p_r))}{\partial r} = V_R(r, p_r) \int\limits_{\mathcal{Y}_R} \frac{\partial \ln q_R(r, y)}{\partial r} dG - \frac{\partial \Omega(r, p_r)}{\partial r} = 0_{d\times 1}. \tag{26}$$

The first term in (26) accounts for the change in output across all tasks assigned to the robot. It gives the net gain in output from a change in the robot's skills as mismatch changes with r across tasks. Unlike previous results, all of the tasks assigned to the robot matter, and not only those in the boundary of the automated region.

It is convenient to use an explicit functional form for q_R to fix ideas. Assuming that $q_R(r, y)$ is as in (1), the first order condition becomes:

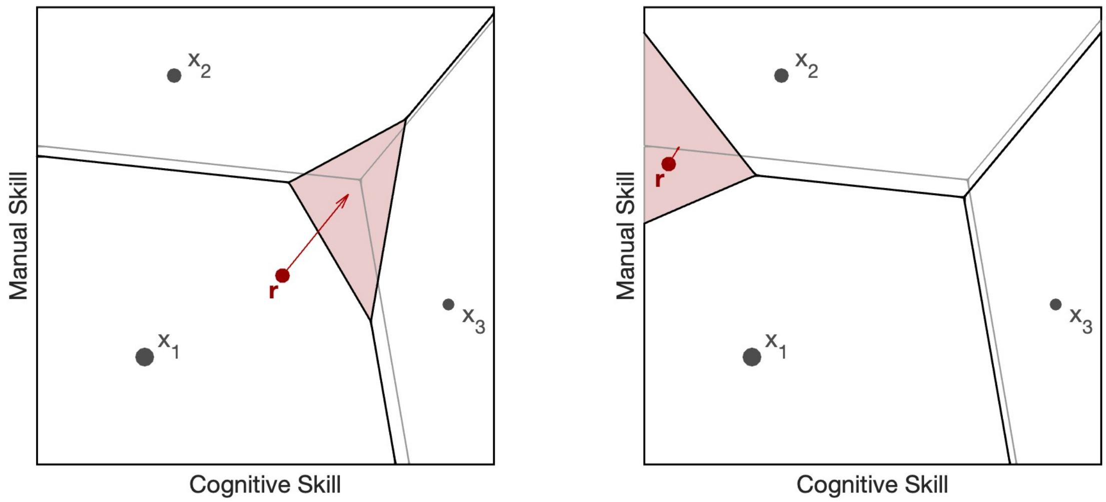
$$\frac{\partial (F_R(\mu^\star(r, p_r), r) - \Omega(r, p_r))}{\partial r} = 2F_R D_R \left(\frac{a_x}{2} - A(r - b_R)\right) - \frac{\partial \Omega(r, p_r)}{\partial r} = 0_{d\times 1}. \tag{27}$$

where $b_R = \frac{\int_{\mathcal{Y}_R} y dG}{D_R}$ is the centroid (or barycenter) of the automated area. Absent other considerations it is optimal to set the robot's skills to the centroid of the automated region, this minimizes the (quadratic) loss from skill mismatch, thus maximizing the robot's output.[21] The robot's skills deviate from the centroid to account for gains from having higher skills (a_x), and for the cost of automation ($^{\partial \Omega(r,p_r)}/_{\partial r}$).

The first order condition with respect to p_r takes the usual form of equating marginal

[20]See Xin et al. (2016) for further applications in the theory of optimal power diagrams with capacity constraints. Proposition 4 in Appendix B provides an alternative derivation for the result based on de Goes et al. (2012). The alternative proof is lengthier but more explicit, making it clear how changing the robot's skills affects output.

[21]This result is shared by the literature on the optimality of centroidal Voronoi diagrams. It is also used in K-means and other vector quantization methods.



(a) Symmetric weights

(b) Asymmetric weights

Figure 6: Directed Automation Example - Quadratic Automation Cost

Note: *The figures show the result of the automation problem taking the robot's mass as given in a two-dimensional skill space (cognitive and manual skills). Three types of workers are considered* $\{x_1, x_2, x_3\}$ *with mass* $P = \{0.5, 0.3, 0.2\}$. *Tasks are uniformly distributed over the unit square, i.e.,* $\mathcal{Y} = [0,1]^2$ *and* $g(y) = 1$. *The production function q is as in (1) with* $A = I_2$, $a_x = [0.2, 0.1]'$ *and* $a_y = [0, 0]'$. *The automation cost function is:* $\Omega(r) = r' A_R r$, *with* A_R *diagonal. The mass of the robot is fixed at* $p_r = 0.05$. *The assignment without the robot is presented in grey.*

product to marginal cost. As in (12), the marginal product is $\mathrm{MP}_R = F_R \mu_R^\star$:

$$F_R\left(T_R^\star\right)\mu_R^\star - \frac{\partial \Omega\left(r, p_r\right)}{\partial p_r} = 0. \tag{28}$$

The first order condition is descriptive of the properties that the robot's skills must satisfy relative to the automated region, but it does not pin down the set of tasks to be automated. The automation problem in (20) is not concave in r and thus condition (26) is only necessary and not sufficient (Urschel, 2017). Regardless, the problem can be solved numerically using a version of Lloyd's algorithm (Lloyd, 1982).This algorithm has been proven to converge monotonically to a local minimum of the objective function (Du et al., 2010). Urschel (2017) gives sufficient conditions for convergence to a global minimum.[22]

[22]In practice there are only finitely many candidates for a global minimum, making the selection of the solution simple. It is optimal to automate tasks around one of the vertices of the partition induced by the initial assignment (without automation). Aurenhammer (1987) shows there at most 2n-5 of these vertices in a diagram when the production function is quadratic in x and y, $d = 2$ and $n \geq 3$.

Figure 6 presents the solution to the automation problem assuming that q and q_R satisfy Assumption 2, and that the automation cost is $\Omega(r) = r' A_R r$. The two panels differ only on the weights of cognitive and manual skills in the automation cost function. The two examples in Figure 6 capture a general feature of the automation problem: it is optimal to automate tasks around the vertices of the original assignment because those are the tasks with the highest mismatch.

Panel 6a assumes symmetric weights. It is then optimal to automate the tasks around the center vertex of the original assignment. These are the tasks with the highest mismatch. Yet, because of the cost of endowing the robot with high cognitive and manual skills, it is not optimal to have placed the robot's skills in the automated area. The introduction of the robot displaces all three workers with respect to the original assignment, but only x_1 is displaced after reassignment.

Panel 6b assumes asymmetric weights, with a higher weight on automating cognitive tasks. It is now optimal to automate the tasks around the vertex formed by $\mathcal{Y}_1$, $\mathcal{Y}_2$ and the boundary of the task space. These tasks involve less cognitive skills so it is possible to locate the robot's skills closer to the centroid of the automated region. As in panel 6a automation induces a reassignment of tasks along the boundary of $\mathcal{Y}_1$ towards more productive workers.

Wages and the labor share The effect of automation on wages is ambiguous. First, automation induces a reassignment across tasks. Because of this, the workers previously performing the automated tasks are not the only ones affected. The reassignment weakly increases the mismatch between workers and tasks. Introducing the robot relaxes the assignment problem, and weakly decreases the value of the multipliers associated with each worker, i.e., $\mu_n^\star < \lambda_n^\star$ for $n = \{1, \ldots, N\}$. The increase in the mismatch reduces wages. Second, automation reduces the skill mismatch for the tasks being automated, increasing overall output. This increases the marginal product and wages of all workers.

Whether or not wages decrease depends on how productive robots are at the tasks they overtake (Acemoglu and Restrepo, 2018a). A major increase in productivity due to automation can increase workers' marginal product, increasing wages, while moderate increases in output in the automated tasks can be dominated by the higher mismatch experienced by workers, ultimately reducing their wages.

Regardless of the change in wages, the labor share decreases. This follows directly from the definition of the labor share in (17). The labor share decreases as the value of the multipliers, $\mu^{\star}$, and the total demand for labor, $G\left(\mathcal{Y}\backslash\mathcal{Y}_R\right)$, go down, both changes reduce the labor costs of production.

Lemma 1. *The labor share* $LS = \sum_{n=1}^{N} w_n D_n / F(T_R)$ *decreases with automation.*

Proof. The decrease in the total demand for labor, $G\left(\mathcal{Y}\backslash\mathcal{Y}_R\right) < G\left(\mathcal{Y}\right)$, and (weak) increase in output, $F\left(T_R^{\star}\right) \geq F\left(T^{\star}\right)$, follow from the automation problem in (20). The effect on the labor share follows from its definition in (17).

□

4.2 Worker training

I introduce the problem of optimal worker training in a similar way to the automation problem described above. The objective in both cases is to reduce the mismatch between tasks and workers, now by modifying workers' skills. Crucially, as the skills of the worker change the assignment also changes, altering the tasks in the workers' occupation.

Formally, the problem of training worker n by choosing new skills $\tilde{x} \in \mathcal{S}$ is:

$$\max_{\left\{\tilde{x},\tilde{T}\right\}} \quad F\left(\tilde{T},\tilde{x}\right) - \Gamma\left(\tilde{x}|x_n,p_n\right) \qquad \text{s.t. } \forall_{\ell}\, D_{\ell} \leq p_{\ell} \tag{29}$$

where the cost of changing skills (Γ) depends on the workers' current skills and mass. The first order condition of the problem is:

$$F\left(\tilde{T}^{\star}\right) \int_{\mathcal{Y}_n} \frac{\partial \ln q\left(\tilde{x},y\right)}{\partial \tilde{x}} dG\left(y\right) - \frac{\partial \Gamma\left(\tilde{x}|x_n,p_n\right)}{\partial \tilde{x}} = 0_{d\times 1}. \tag{30}$$

The interpretation is the same as in the automation problem. The objective is to minimize skill mismatch across the tasks in the worker's occupation given the cost of changing the workers' skills. If q is given by (1) this is achieved by setting $\tilde{x}$ to the centroid of the occupation, and adjusting for the weight of skills in production (a_x) and the marginal cost of changing the worker's skills. Even if acquiring skills was costless, it is not always optimal to increase the worker's skills, doing so can generate its own costs as mismatch increases with respect to the boundary tasks of the worker's occupation. The problem is further complicated by the ambiguous effects on total output, since training one worker can induce higher mismatch for other workers, as the assignment changes. Because of this, condition (30) is only necessary, and not sufficient, for characterizing the optimal worker training. However, Lloyd's algorithm still applies.

The worker training problem is particularly useful when thinking about the introduction of new tasks. New tasks are likely to involve skills for which no worker is particularly well suited, inducing higher mismatch at early stages of adoption. It is then optimal to train workers to acquire skills that better match the changes in their occupations brought up by the new tasks. The introduction of new technologies, like computers and IT, changes occupations by directly modifying the tasks carried out by workers. At the time of the introduction, the workforce is likely not to have the right combination of skills to perform the new tasks. In order to reduce the mismatch workers must train into new skills. This training process will, in turn, modify occupations, changing the bundling of tasks and the roles of each type of worker in production.

4.3 Skill enhancing technology

Technical change can also complement the current skills of workers. This is the case with the introduction of software that complements cognitive over manual skills in the completion of tasks, or heavy machinery, such as cranes, that complements dexterity over brawn. Unlike automation, this type of technical change affects the productivity of workers across tasks without displacing them. But, as with automation, technical change is followed by a reassignment of tasks. Changes in the boundaries of occupations are directed toward reducing the mismatch in the skills complemented by new technologies.

Skill-enhancing technical change can also be directed. It is optimal to direct new technologies towards the the skills with the lowest mismatch, concentrating technology on enhancing the skills at which the workforce already excels. This contrasts with the way in which automation is directed. Instead of replacing workers at the tasks they are ill-suited for, technology enhances the worker's productivity by increasing the weight of the skills with a better match, while reducing the importance of the skills that the workforce lacks.

To make the discussion precise, I impose additional structure on how skill mismatch affects production. Consider two skills, cognitive and manual, and a production technology q as in (1) where the relative importance of skills is governed by a diagonal matrix $A = \text{diag}\left(\alpha, 1-\alpha\right)$, where $\alpha \in [0,1]$. Higher α makes cognitive match more important for production, while simultaneously reducing the importance of manual skill match.

The problem is to choose the value of α taking into account changes in the assignment of tasks to workers. I assume that the cost of changing α is proportional to how much is being produced, this simplifies the analysis by making the magnitude of the cost comparable to the gains from additional production and captures the idea that changing more productive processes is more expensive than changing less productive ones. The problem is:

$$\max_{\{T,\alpha\}} F\left(T,\alpha\right) - \Upsilon\left(\alpha\right) F\left(T,\alpha\right) \qquad \text{s.t. } \forall_n\, D_n \leq p_n. \tag{31}$$

The optimality condition for α can be obtained using the same techniques as before. The optimal α satisfies:

$$\left(M_m - M_c\right) - \frac{\partial \Upsilon\left(\alpha\right)}{\partial \alpha} \geq 0, \tag{32}$$

where M_s is total mismatch in skill s: $M_s \equiv \sum_{n=1}^{N} \int_{\mathcal{Y}_n} \left(x_{n,s} - y_s\right)^2 dy$. The first term captures how much production would increase if α increases. The net gain in production is determined

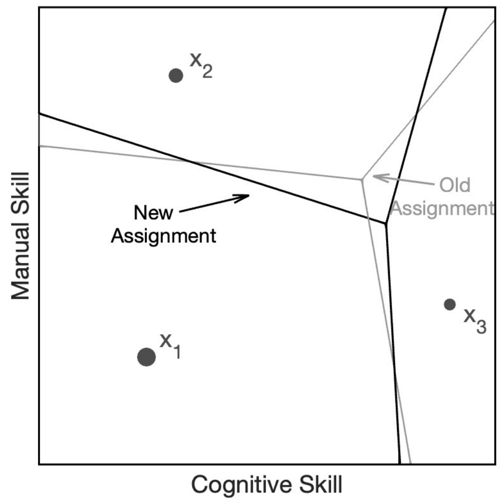


Figure 7: Example - Increase in the Weight of Cognitive Skills

Note: *The figure shows the result of an increase in the weight of cognitive skills in a two-dimensional skill space (cognitive and manual skills). Three types of workers are considered* $\{x_1, x_2, x_3\}$ *with mass* $P = \{0.5, 0.3, 0.2\}$*. Tasks are uniformly distributed over the unit square, i.e.,* $\mathcal{Y} = [0,1]^2$ *and* $g(y) = 1$*. The production function* q *is as in (1) with* $A = diag(\alpha, 1-\alpha)$*.*

by the difference in total mismatch by skill, which depends on the assignment and the distribution of tasks and workers. If, for a given assignment, there is more mismatch in the manual dimension ($M_m > M_c$), the workforce is biased towards cognitive skills. It is then optimal to direct technical change towards cognitive skills by increasing α. In this way, technology reinforces the workforce's bias by giving more weight to skills for which there is a better match. Absent that cost it is optimal to shift all the weight towards one of the skills. Specializing production to depend only on the skill with the lowest mismatch.

Figure 7 shows how the assignment of tasks to workers changes when the weight of cognitive skills increases. The boundaries of occupations shift and become less sensitive to differences in manual skills, discriminating across workers based on differences in their cognitive skills (as $\alpha \to 1$ the boundaries become vertical). As this happens workers' marginal products and substitutability change. Worker x_3 becomes less substitutable with others, as her cognitive skills differ from those of workers x_1 and x_2; recall from Proposition 3 that the elasticity of substitution decreases with the weighted distance between workers' skills. On the other hand, workers x_1 and x_2 become more substitutable, since they differ mostly in their manual skills, which are now downweighted.

5 Unassigned tasks and unemployment

I now extend the model to allow tasks to be left unassigned. Having unassigned tasks also allows automation technology to improve productivity without replacing workers, by performing tasks that are not otherwise assigned. How many tasks are unassigned depends on the productivity of workers relative to their outside option, which determines the pay of the lowest payed worker.

I modify the aggregator in equation (3) so that it only aggregates over output of assigned tasks. In this way, it is possible to leave tasks unassigned without shutting down production. Formally, the aggregation technology is now:

$$F\left(T\right)=\exp\left(\int\limits_{\mathcal{Y}\setminus\mathcal{Y}_{\emptyset}}\ln q\left(T\left(y\right),y\right)\,dG\right)-1, \tag{33}$$

where $T:\mathcal{Y}\rightarrow\mathcal{X}\cup\{\emptyset\}$ so that tasks can be unassigned, and $\mathcal{Y}_{\emptyset}=T^{-1}\left(\{\emptyset\}\right)$ denotes the set of tasks left unassigned. F expresses output relative to no assignment ($T\left(y\right)=\emptyset$ for all y), so that if all tasks are left unassigned output is zero.

Its immediate that the result from the aggregation in (33) is equivalent to having $q\left(\emptyset,y\right)=1$ in the original formula (3), extending T to take values over $\mathcal{X}$ and the unassigned option. That way, tasks that are left unassigned don't add to the integral, obtaining (33) as a result. Adopting this convention turns out to be useful because it allows me to apply Proposition 1. Leaving a task unassigned is equivalent to assigning it to a worker '$\emptyset$', which is in infinite supply, has an outside option of zero, and produces $q\left(\emptyset,y\right)=1$ in all tasks.

The main difference with the results of Section 2 is that the level of the worker's outside option ($\underline{w}$) affects the assignment. To simplify calculations, I will further assume that the outside option is a fraction $\underline{\lambda}$ of total output: $\underline{w}\left(T\right)=\underline{\lambda}F\left(T\right)$.[23] Then, there exists a vector

[23]Without this assumption it is not possible to determine the value of λ independently of the assignment T. The term $\underline{\lambda}$ in (34) has to be replaced by $\underline{w}/F(T)$.

$\lambda^{\star} \in \mathbb{R}_{+}^{N}$ such that $\min \lambda_n^{\star} = 0$ and occupations are given by:

$$\mathcal{Y}_n = \left\{ y \in \mathcal{Y} \,|\, \forall_\ell \ \ln q\left(x_n, y\right) - \lambda_n^{\star} \geq \ln q\left(x_\ell, y\right) - \lambda_\ell^{\star} \quad \wedge \quad \ln q\left(x_n, y\right) - \lambda_n^{\star} \geq \underline{\lambda} \right\}. \tag{34}$$

This condition differs from (8) in the introduction of the second inequality, which compares the output of worker n in the task with the minimum payment the worker must receive. This ensures that it is profitable to assign the task. The unassigned tasks are:

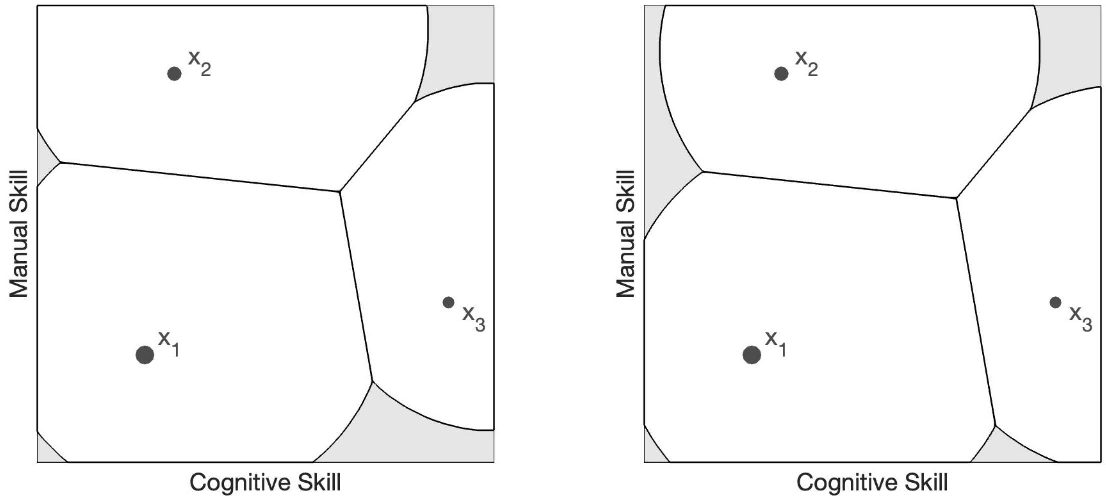
$$\mathcal{Y}_{\emptyset} = \left\{ y \in \mathcal{Y} \,|\, \forall_n \ \ln q\left(x_n, y\right) - \lambda_n^{\star} < \underline{\lambda} \right\}. \tag{35}$$

The higher $\underline{\lambda}$ is, the fewer tasks are assigned for production. The marginal product of a worker is given as in equation (12) of Section 3.1 and their compensation is given by $w_n = \lambda_n^{\star} F\left(T^{\star}\right) + \underline{w} = \left(\lambda_n^{\star} + \underline{\lambda}\right) F\left(T^{\star}\right)$.

To fix ideas, consider q as in (1), depending on the quadratic mismatch between worker and task's skills. This provides a clear geometrical interpretation for which tasks are left unassigned. Workers will be assigned to a task only if it the mismatch is no greater than $a_x^{'}x + a_y^{'}y - \underline{\lambda}$.[24] This guarantees that enough output is produced by the worker for it to cover the workers' outside option. However, the condition does not imply that the task will be assigned to the worker, this depends on the comparison between workers' productivities as in Section 2 (see the first inequality in equation 34).

Which tasks to perform will depend critically on which tasks are more productive given current technology. This idea is captured by a_y, which determines which tasks generate more output, regardless of which worker performs them. A higher cognitive weight in a_y, say because of the use of information technologies, makes cognitive intensive tasks more likely to be performed. Opposite changes can occur on the relevance of manual intensive tasks in production, shifting workers from manual to cognitive intensive tasks.

[24] With $a_y = 0$, a task will be assigned only if it lies in a ellipse of radius $\sqrt{a_x^{'}x}$ around the skills of the worker. The shape of the ellipse depends on the weights in matrix A.



(a) Symmetric Weights - $\alpha_y = 1/2$ (b) Asymmetric Weights - $\alpha_y = 3/4$

Figure 8: Assignment Example - Unassigned Tasks

Note: *The figures show the assignment in a two-dimensional skill space (cognitive and manual skills). Three types of workers are considered $\{x_1, x_2, x_3\}$ with mass $P = \{0.5, 0.3, 0.2\}$. Tasks are uniformly distributed over the unit square, i.e., $\mathcal{Y} = [0,1]^2$ and $g(y) = 1$. The production function q is as in (1) with $A = I_2$, $a_x = [0.2, 0.1]'$ and $a_y = \overline{a}_y [\alpha_y, 1-\alpha_y]'$, with $\overline{a}_y \in \mathbb{R}_+$ and $\alpha_y \in [0,1]$. The worker's outside option is 0.*

Figure 8 shows the optimal assignment when tasks can be left unassigned. The two panels differ on the weight of task skills in determining the output of a task, as measured by a_y. I assume that $a_y = \overline{a}_y [\alpha_y, 1-\alpha_y]'$ and vary the relative importance of skills by choosing $\alpha_y \in [0,1]$. A higher value of α_y makes cognitive intensive tasks more productive.

Panel 8a presents the assignment under equal skill weights in a_y. The grey areas represent unassigned tasks. As in Section 4.1, these tasks are located along the boundaries of the task space, and the vertices of the assignment. Most of the unassigned tasks involve high cognitive skills. This is because of the distributions of skills among workers. In the example, there are relatively few x_3 workers, and so, performing the high-cognitive tasks comes at the cost of a greater mismatch for workers x_1 and x_2. Its worth noting that the assignment is such that only worker x_1 is unemployed. x_1 is the least productive worker type.

In Panel 8b the weights on skills change, making cognitive intensive tasks more productive, and manual intensive tasks less productive. As a response to this change workers x_1 and x_3 take over tasks in the bottom-right corner of the space, at the expense of tasks along the manual axis. Higher productivity makes it worthwhile to reassign workers towards cognitive intensive tasks. Unemployment is still concentrated in workers of type x_1.

The role of the outside option As in Section 3, the value of the workers' outside option $\underline{w}$ equals the minimum wage in the economy. But, unlike the problem in Section 3, the value of $\underline{w}$ affects the assignment. An increase in the value of the outside option, $\underline{w}$, reduces employment, by limiting the set of tasks that are profitable to produce at the current wages.

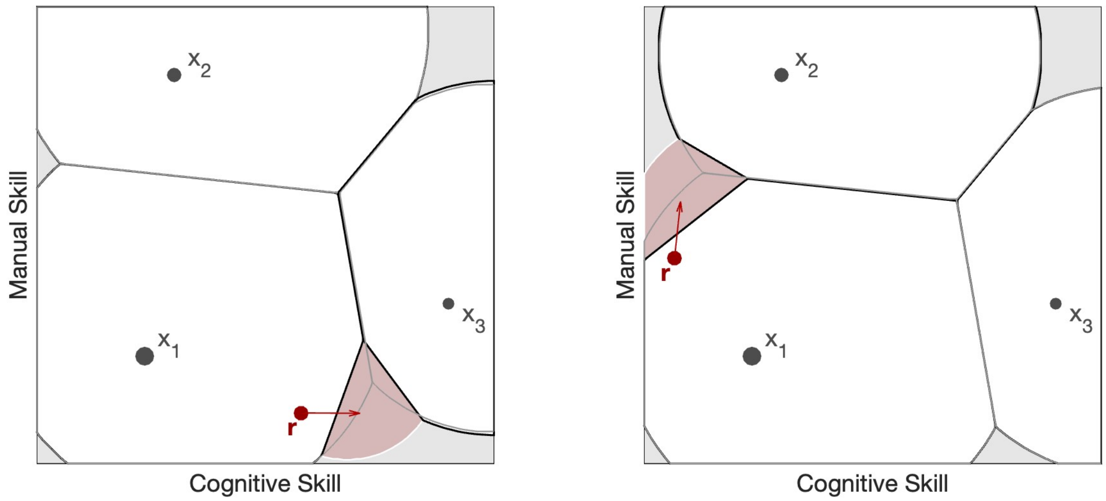
The net effect on wages is nevertheless ambiguous. As the assignment of tasks changes so does the mismatch of workers at their boundary tasks. Mismatch necessarily goes down for the type of worker(s) that are not fully employed, but it might increase for other workers. Moreover, changes in the assignment can lead to a compression of the wage distribution. This is because wages reflect marginal productivities relative to the least productive workers (see equation 14). These effects must be weighted against the increase in the level of wages coming from $\underline{w}$ to determine the net effect of the increase of the outside option.[25]

5.1 Automation and unassigned tasks

The characterization of automation as a worker replacing technology given in Section 4.1 changes once tasks can be left unassigned. It is now possible to direct automation towards unassigned tasks, that is, tasks which are not worthwhile for workers to perform (because of low productivity), or tasks for which workers don't have the appropriate skills (high mismatch). If this happens, automation does not displace workers. Moreover, performing additional tasks increases output, potentially raising workers' marginal products and wages.

In general, it is optimal to automate tasks along the boundaries of occupations and unassigned tasks. As a result, automation ends up partially displacing workers. To illustrate this I expand the example in Figure 8 by solving the optimal automation problem. The resulting assignment is presented in Figure 9. Production technology is the same for workers

[25]The effects on the wage distribution are similar to the ones documented for Brazil by Engbom and Moser (2018) by shifting up the level of wages from below.



(a) Symmetric Weights - $\alpha_y = 1/2$ (b) Asymmetric Weights - $\alpha_y = 3/4$

Figure 9: Assignment Example - Unemployment and Automation

Note: *The figures show the assignment in a two-dimensional skill space (cognitive and manual skills). Three types of workers are considered* $\{x_1, x_2, x_3\}$ *with mass* $P = \{0.5, 0.3, 0.2\}$. *Tasks are uniformly distributed over the unit square, i.e.,* $\mathcal{Y} = [0,1]^2$ *and* $g(y) = 1$. *The production function* q *is as in (1) with* $A = I_2$, $a_x = [0.2, 0.1]'$ *and* $a_y = \bar{a}_y [\alpha_y, 1-\alpha_y]'$, *with* $\bar{a}_y \in \mathbb{R}_+$ *and* $\alpha_y = 1/2$. *The worker's outside option is 0. The automation cost function is:* $\Omega(r) = r' A_R r$, *with* A_R *diagonal. The mass of the robot is fixed at* $p_r = 0.03$. *The assignment without the robot is presented in grey.*

and the robot and is given by (1). The cost of automation is quadratic in skills as in the example in Figure 6.

Panels 9a and 9b present similar results, with the robot being placed so as to automate part of the cognitive/manual intensive tasks that were unassigned. The robot is only partially displacing workers. Thus, the mass of unassigned workers increases, but less than the mass of tasks being automated. Output increases due to the production of new tasks and the reduction in the mismatch in some of the old tasks.

The two panels in Figure 9 also show how the incentives for automation change as the production technology changes. If technology favors cognitive intensive tasks over manual intensive tasks, workers are reassigned away from the latter and into the former (see Figure 8). Consequently, production can be increased by directing automation towards manual intensive tasks, in a way that avoids disrupting the optimal assignment of tasks to workers. In this scenario technological change makes new tasks available for workers, while automation follows by taking over tasks that are no longer worthwhile for workers to perform.[26]

6 Concluding remarks

I develop a framework to study occupations, where production takes place by assigning workers to tasks in a multidimensional setting. Occupations arise from the assignment process, instead of being taken as a preexisting feature of production. Because of this, the framework incorporates endogenous changes in the boundaries of occupations in response to changes in the economic environment. This flexibility is particularly important when addressing the consequences of worker replacing technologies like automation or offshoring. These technologies replace workers in some, but not all, of the tasks they perform, transforming occupations.

The model makes precise the role of tasks in defining the marginal product, compensation, and substitutability of workers. All these properties are shaped by how productive workers are at the tasks along the boundaries of their occupations. These are the tasks for which the workers are the least productive, and at which they are directly substitutable for other workers. The model also makes it possible to ask about the optimal direction of automation, i.e., which type of tasks should be automated.

[26] This process is similar to the one in Acemoglu and Restrepo (2018b). Changes in technology lead to a reassignment of workers towards more complex (and newer) tasks, while relatively simpler (and older) tasks are automated, displacing workers in the process.

References

Acemoglu, D. and D. Autor (2011). Skills, Tasks and Technologies: Implications for Employment and Earnings. In D. Card and O. Ashenfelter (Eds.), *Handbook of Labor Economics*, Volume 4 of *Handbook of Labor Economics*, Chapter 12, pp. 1043 – 1171. Elsevier.

Acemoglu, D. and P. Restrepo (2018a). Modeling Automation. *AEA Papers and Proceedings 108*, 48–53.

Acemoglu, D. and P. Restrepo (2018b, June). The Race between Man and Machine: Implications of Technology for Growth, Factor Shares, and Employment. *American Economic Review 108*(6), 1488–1542.

Acemoglu, D. and P. Restrepo (2020). Robots and jobs: Evidence from us labor markets. *Journal of Political Economy 128*(6), 2188–2244.

Acemoglu, D. and P. Restrepo (2021, June). Tasks, automation, and the rise in us wage inequality. Working Paper 28920, National Bureau of Economic Research.

Aghion, P., B. F. Jones, and C. I. Jones (2019). *Artificial Intelligence and Economic Growth.* University of Chicago Press.

Atalay, E., P. Phongthiengtham, S. Sotelo, and D. Tannenbaum (2020, April). The evolution of work in the united states. *American Economic Journal: Applied Economics 12*(2), 1–34.

Aurenhammer, F. (1987, feb). Power Diagrams: Properties, Algorithms and Applications. *SIAM Journal on Computing 16*(1), 78–96.

Aurenhammer, F., F. Hoffmann, and B. Aronov (1998, Jan). Minkowski-Type Theorems and Least-Squares Clustering. *Algorithmica 20*(1), 61–76.

Autor, D. H. (2013, Sep). The "Task Approach" to Labor Markets: An Overview. *Journal for Labour Market Research 46*(3), 185–199.

Autor, D. H., L. F. Katz, and M. S. Kearney (2006, May). The polarization of the u.s. labor market. *American Economic Review 96*(2), 189–194.

Autor, D. H., F. Levy, and R. J. Murnane (2003). The Skill Content of Recent Technological Change: An Empirical Exploration. *The Quarterly Journal of Economics 118*(4), 1279–1333.

Baqaee, D. R. and E. Farhi (2019). The macroeconomic impact of microeconomic shocks: Beyond hulten's theorem. *Econometrica 87*(4), 1155–1203.

Black, S. E. and A. Spitz-Oener (2010). Explaining Women's Success: Technological Change and the Skill Content of Women's Work. *The Review of Economics and Statistics 92*(1), 187–194.

Blackorby, C. and R. R. Russell (1981). The Morishima Elasticity of Substitution; Symmetry, Constancy, Separability, and its Relationship to the Hicks and Allen Elasticities. *The Review of Economic Studies 48*(1), 147–158.

Blackorby, C. and R. R. Russell (1989). Will the Real Elasticity of Substitution Please Stand Up? (A Comparison of the Allen/Uzawa and Morishima Elasticities). *The American Economic Review 79*(4), 882–888.

Blinder, A. (2009). How Many US Jobs Might be Offshorable? *World Economics 10*(2), 41–78.

Blinder, A. S. and A. B. Krueger (2013). Alternative Measures of Offshorability: A Survey Approach. *Journal of Labor Economics 31*(2), S97–S128.

Carlier, G. (2003). Duality and Existence for a Class of Mass Transportation Problems and Economic Applications. pp. 1–21.

Combemale, C., K. S. Whitefoot, L. Ales, and E. R. H. Fuchs (2021, 08). Not all technological change is equal: how the separability of tasks mediates the effect of technology change on skill demand. *Industrial and Corporate Change*. dtab026.

de Goes, F., K. Breeden, V. Ostromoukhov, and M. Desbrun (2012, November). Blue Noise Through Optimal Transport. *ACM Transactions on Graphics 31*(6), 171:1–171:11.

Deming, D. J. (2017). The Growing Importance of Social Skills in the Labor Market. *The Quarterly Journal of Economics 132*(4), 1593–1640.

Du, Q., M. Gunzburger, and L. Ju (2010). Advances in Studies and Applications of Centroidal Voronoi Tessellations. *Numerical Mathematics: Theory, Methods and Applications 3*(2), 119–142.

Engbom, N. and C. Moser (2018). Earnings Inequality and the Minimum Wage: Evidence from Brazil. Institute Working Paper 7, Opportunity and Inclusive Growth Institute, Federal Reserve Bank of Minneapolis.

Feenstra, R. C. and J. A. Levinsohn (1995). Estimating Markups and Market Conduct with Multidimensional Product Attributes. *The Review of Economic Studies 62*(1), 19–52.

Frey, C. B. and M. A. Osborne (2017). The Future of Employment: How Susceptible Are Jobs to Computerisation? *Technological Forecasting and Social Change 114*, 254 – 280.

Galichon, A. (2016). *Optimal Transport Methods in Economics*. Princeton University Press.

Gathmann, C. and U. Schönberg (2010). How General Is Human Capital? A Task-Based Approach. *Journal of Labor Economics 28*(1), 1–49.

Guvenen, F., B. Kuruscu, S. Tanaka, and D. Wiczer (2020, January). Multidimensional skill mismatch. *American Economic Journal: Macroeconomics 12*(1), 210–44.

Heckman, J., J. Stixrud, and S. Urzua (2006). The Effects of Cognitive and Noncognitive Abilities on Labor Market Outcomes and Social Behavior. *Journal of Labor Economics 24*(3), 411–482.

Heckman, J. J. and G. Sedlacek (1985). Heterogeneity, Aggregation, and Market Wage Functions: An Empirical Model of Self-Selection in the Labor Market. *Journal of Political Economy 93*(6), 1077–1125.

Hemous, D. and M. Olsen (2021). The Rise of the Machines: Automation, Horizontal Innovation and Income Inequality. *American Economic Journal: Macroeconomics, Forthcoming*.

Kambourov, G. and I. Manovskii (2009). Occupational Specificity of Human Capital. *International Economic Review 50*(1), 63–115.

Kantorovich, L. V. (2006, Mar). On the Translocation of Masses. *Journal of Mathematical Sciences 133*(4), 1381–1382.

Kelso, A. S. and V. P. Crawford (1982). Job Matching, Coalition Formation, and Gross Substitutes. *Econometrica 50*(6), 1483–1504.

Koopmans, T. C. and M. Beckmann (1957). Assignment Problems and the Location of Economic Activities. *Econometrica 25*(1), 53–76.

Kremer, M. (1993). The O-Ring Theory of Economic Development. *The Quarterly Journal of Economics 108*(3), 551–575.

Lazear, E. (2009). Firm-Specific Human Capital: A Skill-Weights Approach. *Journal of Political Economy 117*(5), 914–940.

Lindenlaub, I. (2017). Sorting Multidimensional Types: Theory and Application. *The Review of Economic Studies 84*(2), 718–789.

Lise, J. and F. Postel-Vinay (2020, August). Multidimensional skills, sorting, and human capital accumulation. *American Economic Review 110*(8), 2328–76.

Lloyd, S. P. (1982). Least Squares Quantization in PCM. *IEEE Transactions on Information Theory 28*, 129–137.

Martinez, J. and J. Moen-Vorum (2021). Automation potential and diffusion.

McKinsey Global Institute (2017). What the Future of Work Will Mean for Jobs, Skills and Wages.

Milgrom, P. and I. Segal (2002). Envelope Theorems for Arbitrary Choice Sets. *Econometrica 70*(2), 583–601.

Poletaev, M. and C. Robinson (2008). Human Capital Specificity: Evidence from the Dictionary of Occupational Titles and Displaced Worker Surveys, 1984-2000. *Journal of Labor Economics 26*(3), 387–420.

Rendall, M. (2018). Brain versus Brawn: The Realization of Women's Comparative Advantage.

Rosen, S. (1978). Substitution and Division of Labour. *Economica 45*(179), 235–250.

Sattinger, M. (1993). Assignment Models of the Distribution of Earnings. *Journal of Economic Literature 31*(2), 831–880.

Sobel, J. (1992). How to Count to One Thousand. *The Economic Journal 102*(410), 1–8.

Spitz-Oener, A. (2006). Technical Change, Job Tasks, and Rising Educational Demands: Looking outside the Wage Structure. *Journal of Labor Economics 24*(2), 235–270.

Stinebrickner, T. R., P. Sullivan, and R. Stinebrickner (2019). Job Tasks, Time Allocation, and Wages. *Journal of Labor Economics (Forthcoming)*.

Stokey, N. L. (2018). Technology, skill, and the wage structure. *Journal of Human Capital 12*(2), 343–384.

Tinbergen, J. (1956). On the Theory of Income Distribution. *Weltwirtschaftliches Archiv 77*, 155–175.

Urschel, J. C. (2017). On The Characterization and Uniqueness of Centroidal Voronoi Tessellations. *SIAM Journal on Numerical Analysis 55*(3), 1525–1547.

Villani, C. (2009). *Optimal Transport: Old and New*. Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg.

Webb, M. (2020). The impact of artificial intelligence on the labor market.

Willis, R. J. and S. Rosen (1979). Education and Self-Selection. *Journal of Political Economy 87*(5, Part 2), S7–S36.

Xin, S.-Q., B. Lévy, Z. Chen, L. Chu, Y. Yu, C. Tu, and W. Wang (2016, November). Centroidal Power Diagrams with Capacity Constraints: Computation, Applications, and Extension. *ACM Trans. Graph. 35*(6), 244:1–244:12.

Yamaguchi, S. (2012). Tasks and Heterogeneous Human Capital. *Journal of Labor Economics 30*(1), 1–53.

Appendix

Table of Contents

A Mathematical Preliminaries

The following definitions and theorems are relevant for the proofs in the text and in Appendix B.

Definition A.1. [Probability Space] A probability space is a triplet $(A, \mathcal{A}, \mu)$ of a set A, a σ-algebra $\mathcal{A}$ on that set and a probability measure $\mu : \mathcal{A} \to [0,1]$. When the σ-algebra is understood (generally as the Borel σ-algebra) it is omitted.

Definition A.2. [Polish Space] A set A is a polish space if it is separable (allows for a dense countable subset) and metrizable topological space (there exists at least one metric that induces the topology).

Definition A.3. [Coupling] Let $(\mathcal{Y}, G)$ and $(\mathcal{X}, P)$ be two probability spaces. A coupling π of G and P is a joint distribution on $(\mathcal{X} \times \mathcal{Y})$ such that $\int_{\mathcal{X}\times Y} d\pi\left(x,y\right) = G\left(Y\right)$ for all $Y \in \mathcal{B}\left(\mathcal{Y}\right)$ and $\int_{X\times\mathcal{Y}} d\pi\left(x,y\right) = P\left(X\right)$ for all $X \in \mathcal{B}\left(\mathcal{X}\right)$, where $\mathcal{B}\left(A\right)$ denotes the Borel sets of A. So π gives G and P as marginals. Let $\Pi\left(P,G\right)$ be the set of all couplings of P and G. When the assignment is given by an assignment function the coupling is deterministic.

Definition A.4. [h-transform] Let $h : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ be a function. The h-transform of a function $f : \mathcal{X} \to \mathbb{R}$ is given by:

$$f^h\left(y\right) = \sup_{x\in\mathcal{X}} \left\{h\left(x,y\right) - f\left(x\right)\right\}$$

Definition A.5. [h-convex] A function $f : \mathcal{X} \to \mathbb{R}$ is said to be h-convex if there exists a function $g : \mathcal{Y} \to \mathbb{R}$ such that:

$$f\left(x\right) = \sup_{y\in\mathcal{Y}} \left\{h\left(x,y\right) - g\left(y\right)\right\}$$

Definition A.6. [h-subdifferential] The h-subdifferential of a function $v : \mathcal{Y} \to \mathbb{R}$ is defined as the set $\partial^h v\left(y\right) = \left\{x \in \mathcal{X} \mid v\left(y\right) + v^h\left(x\right) = h\left(x,y\right)\right\}$.

The following theorem joins results from optimal transport on the existence of a solution to the Monge-Kantorovich problem and the applicability of Kantorovich's duality to the mass transportation problem:

Theorem A.1. *Villani (2009, Thm. 5.10 and Thm 5.30) Let $(\mathcal{Y}, G)$ and $(\mathcal{X}, P)$ be two Polish probability spaces and let $h : \mathcal{X} \times \mathcal{Y} \to \mathbb{R} \cup \{-\infty\}$ be an upper semicontinuous function.*

Consider the optimal transport problem:

$$\sup_{\pi\in\Pi(G,P)} \int_{\mathcal{X}\times\mathcal{Y}} h\left(x,y\right) d\pi\left(x,y\right)$$

where function $h\left(x,y\right)$ describes the gain (or surplus) of transporting a unit of mass from y to x, and $\Pi\left(G,P\right)$ denotes the set of couplings of G and P.

If there exist real valued lower semicontinuous functions $a \in L^1\left(P\right)$ and $b \in L^1\left(G\right)$:

$$\forall_{(x,y)\in\mathcal{X}\times\mathcal{Y}} \quad h\left(x,y\right) \leq a\left(x\right) + b\left(y\right)$$

then:

1. *There is duality:*

$$\sup_{\pi\in\Pi(G,P)} \int_{\mathcal{X}\times\mathcal{Y}} h(x,y)\, d\pi(x,y) = \inf_{\substack{(\lambda,v)\in L^1(P)\times L^1(G)\\ \lambda(x)+v(y)\geq h(x,y)}} \int_{\mathcal{X}} \lambda(x)\, dP(x) + \int_{\mathcal{Y}} v(y)\, dG(y)$$
$$= \inf_{w\in L^1(P)} \int_{\mathcal{X}} \lambda(x)\, dP(x) + \int_{\mathcal{Y}} \lambda^h(y)\, dG(y)$$
$$= \inf_{v\in L^1(G)} \int_{\mathcal{X}} v^h(x)\, dP(x) + \int_{\mathcal{Y}} v(y)\, dG(y)$$

where $\Pi(G,P)$ is the set of couplings of G and P and f^h denotes the h-transform of function f:

$$f^h(y) = \sup_{x\in\mathcal{X}} h(x,y) - f(x)$$

The functions w and v are h-convex since they are the h-transform of one another.

2. *If, furthermore, h is real valued ($h : \mathcal{X}\times\mathcal{Y}\to\mathbb{R}$) and the solution to the Monge-Kantorovich problem is finite $\left(\max_{\pi\in\Pi(G,P)} \int_{\mathcal{X}\times\mathcal{Y}} h(x,y)\, d\pi(x,y) < \infty\right)$ then there is a measurable h-monotone set $\Gamma\subset\mathcal{X}\times\mathcal{Y}$[27] such that for any $\pi\in\Pi(G,P)$ the following statements are equivalent:*

   (a) *π is optimal.*

   (b) *π is h-cyclically monotone.*

   (c) *There is a h-convex function λ such that $\lambda(x)+\lambda^h(y) = h(x,y)$ π-almost surely.*

   (d) *There exist $\lambda : \mathcal{X}\to\mathbb{R}$ and $v : \mathcal{Y}\to\mathbb{R}$ such that $\lambda(x)+v(y)\geq h(x,y)$ with equality π-almost surely.*

   (e) *π is concentrated in Γ.*

3. *If, h is real valued ($h : \mathcal{X}\times\mathcal{Y}\to\mathbb{R}$) and there are functions $c\in L^1(P)$ and $d\in L^1(G)$ such that:*

$$\forall_{(x,y)\in\mathcal{X}\times\mathcal{Y}} \quad c(x)+d(y)\leq h(x,y)$$

then the dual problem has a solution. There is a function w that attains the infimum.

4. *(this part from Villani (2009, Thm. 5.30))If:*

   (a) *h is real valued ($h : \mathcal{X}\times\mathcal{Y}\to\mathbb{R}$)*

   (b) *the solution to the Monge-Kantorovich problem is finite:*

$$\max_{\pi\in\Pi(G,P)} \int_{\mathcal{X}\times\mathcal{Y}} h(x,y)\, d\pi(x,y) < \infty$$

[27] If a, b and h are continuous then Γ is closed.

(c) *for any h-convex function $v : \mathcal{Y} \to \mathbb{R} \cup \{-\infty\}$ the subdifferential $\partial^h v(y)$ is single valued G-almost everywhere*

Then

(a) *there is a unique (in law) optimal coupling π of (G, P).*

(b) *the optimal coupling is deterministic: $T : \mathcal{Y} \to \mathcal{X}$.*

(c) *the optimal coupling is characterized by the existence of a function h- convex function v such that $T(y) = \partial^h v(y)$.*

Finally, Reynold's transport theorem is used extensively in the text:

Theorem A.2. ***[Reynolds' Transport Theorem]*** *The rate of change of the integral of a scalar function f within a volume V is equal to the volume integral of the change of f, plus the boundary integral of the rate at which f flows though the boundary ∂V of outward unit normal n:*

$$\nabla \int_V f(x)\, dV = \int_V \nabla f(x)\, dV + \int_{\partial V} f(x)\, (\nabla x \cdot n)\, dA$$

B Proofs

B.1 Existence and uniqueness of optimal task assignment

Outline of the poof As is common in assignment problems, I first relax the problem in 4 to allow for non-deterministic assignments, see Kantorovich (2006) and Koopmans and Beckmann (1957). An assignment is then a joint measure over workers/task pairs: $\pi : \mathcal{X} \times \mathcal{B}(\mathcal{Y}) \to \mathbb{R}_+$, where $\mathcal{B}(\mathcal{Y})$ denotes the Borel sets of $\mathcal{Y}$. An assignment π is deemed feasible if it is a coupling of measures P and G, see definition A.3 in Appendix A. In terms of the assignment problem π must guarantee that workers have enough time to perform all the time demanded by their occupations, and each task is completed at most once. Letting $\Pi(P,G)$ be the set of feasible assignments:

$$\pi \in \Pi(P,G) \longleftrightarrow \forall_n \int_{\mathcal{Y}} d\pi(x_n,y) \leq p_n \qquad \forall_{Y \in \mathcal{B}(\mathcal{Y})} \sum_{n=1}^{N} \int_{y\in Y} d\pi(x_n,y) \leq G(Y) \tag{B.1}$$

Note that the second condition can be simplified to: $\sum_{n=1}^{N} \pi(x_n,\{y\}) \leq g(y)$.

The problem is now to choose a coupling $\pi \in \Pi(P,G)$ to maximize output. I further simplify the problem by applying natural logarithm to the objective function. Doing so reveals the linearity of the problem in the choice variable π. The relaxed optimization problem is:

$$\max_{\pi\in\Pi(P,G)} \sum_{n=1}^{N} \int_{\mathcal{Y}} \ln q(x_n,y)\ d\pi(x_n,y) \tag{B.2}$$

Lemma B.1 applies Theorem 5.10 of Villani (2009) to establish duality for the problem:

$$\begin{aligned}\max_{\pi\in\Pi} \sum_{n=1}^{N} \int_{\mathcal{Y}} \ln q(x_n,y)\ d\pi(x_n,y) &= \inf_{\substack{(\lambda,\nu)\in\mathbb{R}^N\times L^1(G)\\ \lambda_n+v(y)\geq \ln q(x_n,y)}} \sum_{n=1}^{N} \lambda_n p_n + \int_{\mathcal{Y}} \nu(y)\, dG \\ &= \inf_{\lambda\in\mathbb{R}^N} \sum_{n=1}^{N} \lambda_n p_n + \int_{\mathcal{Y}} \max_n \{\ln q(x_n,y) - \lambda_n\}\, dG\end{aligned} \tag{B.3}$$

λ and ν are the multipliers (or potentials) of the problem. Lemma B.2 establishes that a solution to the dual problem $(\lambda^\star, v^\star)$ exists. The levels of $\lambda^\star$ and $\nu^\star$ are only determined up to an additive constant. Both the assignment and the value of the dual problem do not change if λ is increased by a constant κ for all workers and ν decreased by the same amount for all tasks. I normalize the value of the minimum $\lambda^\star$ to zero. This is convenient when relating the value of $\lambda^\star$ to the marginal product of workers and the wages in the decentralization of the optimal assignment.

The first two conditions on the production function q ensure that the value of the primal problem (B.2) and the dual problem (B.3) are finite, this is the key step in verifying the conditions for Theorem 5.10 of Villani (2009). In particular, the first condition avoids indeterminacies when evaluating the natural logarithm of q for any worker/task pair.

The solution to the dual problem provides a way to construct the optimal assignment $T^\star$. Lemma B.3 applies Theorem 5.30 of Villani (2009) to construct $T^\star$ as the sub-differential of $v^\star$. The third condition on the production function q is crucial to establish single-valuedness of the sub-differential of $v^\star$. This gives the formula for the optimal assignment in (5). Galichon (2016, Ch. 5.3) presents an algorithm to solve the dual problem in (B.3).

I now turn to the general proof of the problem.

General setting Consider the set up of Section 1. There are N types of workers $\{x_1, \ldots x_N\} \equiv \mathcal{X}$, there is a mass p_n of workers of type x_n. The mass of workers is described by a (discrete) measure P so that $P(x_n) = p_n$. There is a continuum of tasks $y \in \mathcal{Y}$ distributed continuously according to an absolutely continuous measure $G : \mathcal{B}(\mathcal{Y}) \to \mathbb{R}_+$. $\mathcal{Y}$ is assumed compact.

Output is produced by completing tasks. A worker of type x_n performing task y produced $q(x_n, y)$. $q : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ is a real-valued function. Output for all worker/task pairs is aggregated into a final good:

$$F(\pi) = \begin{cases} \left(\sum_{n=1}^{N} \int (q(x_n, y))^{\frac{\sigma-1}{\sigma}} \, d\pi(x_n, y)\right)^{\frac{\sigma}{\sigma-1}} & \text{if } \sigma > 1 \\ \exp\left(\sum_{n=1}^{N} \int_{\mathcal{Y}} \ln q(x_n, y) \, d\pi(x_n, y)\right) & \text{if } \sigma = 1 \end{cases}$$

where $\pi \in \Pi(P, G)$ is a coupling of P and G (see definition A.3). The coupling π describes the assignment: a mass $\pi(x, y)$ of workers of type x is assigned to task y.

The problem is to maximize output of the final good by choosing an assignment of tasks to workers π. I first transform the objective function so that the problem takes the form of a Monge-Kantorovich problem:

$$\max_{\pi \in \Pi(P,G)} \int h(x, y|\sigma) \, d\pi(x, y) \tag{B.4}$$

where $h(x, y|\sigma) = \begin{cases} (q(x, y))^{\frac{\sigma-1}{\sigma}} & \text{if } \sigma > 1 \\ \ln q(x, y) & \text{if } \sigma = 1 \end{cases}$.

The following proposition establishes duality for this problem:

Lemma B.1. *If q satisfies the following properties:*

1. $\sigma > 1$ *or all workers can produce in some task:* $\forall_x \exists y \quad q(x, y) > 0$
2. $q(x, \cdot)$ *is upper-semicontinuous in y given $x \in \mathcal{X}$.*

Then, the following equalities hold:

$$\max_{\pi\in\Pi(P,G)}\int\left(h\left(x,y|\sigma\right)\right)^{\frac{\sigma-1}{\sigma}}d\pi\left(x,y\right)=\inf_{\substack{(\lambda,v)\in\mathbb{R}^N\times L^1(G)\\ \lambda_n+v(y)\geq q(x_n,y|\sigma)}}\sum_{n=1}^{N}\lambda_n p_n+\int_{\mathcal{Y}}v\left(y\right)dG\left(y\right)$$
$$=\inf_{\lambda\in\mathbb{R}^N}\sum_{n=1}^{N}\lambda_n p_n+\int_{\mathcal{Y}}\max_{n}\left\{q\left(x_n,y|\sigma\right)-\lambda_n\right\}dG\left(y\right)$$

Proof. This follows from applying theorem A.1 (Villani, 2009, Thm. 5.10). Note that $\mathcal{Y}\subset\mathbb{R}^n$ and $\mathcal{X}$ is finite they are both Polish spaces. $h\left(x,y|\sigma\right)$ is upper semicontinuous because $f\left(x\right)=x^{\frac{\sigma-1}{\sigma}}$ and $f\left(x\right)=\ln x$ are continuous and monotone increasing, and q is upper semicontinuous.

It is left to verify that there exist real valued lower semicontinuous functions $a\in L^1\left(P\right)$ and $b\in L^1\left(G\right)$:

$$\forall_{(x,y)\in\mathcal{X}\times\mathcal{Y}}\quad h\left(x,y|\sigma\right)\leq a\left(x\right)+b\left(y\right)$$

For this let $a\left(x\right)=\max_{y\in\mathcal{Y}}\left\{h\left(x,y|\sigma\right)\right\}$ and $b\left(y\right)=0$. The max in the definition of a is well defined because h is upper semicontinuous and $\mathcal{Y}$ is compact, furthermore a is finite (either $\sigma>1$ or, if $\sigma=1$, h is finite for at least some y guaranteeing a a final value). Function a is immediately continuous with respect to the discrete topology. The desired equalities follow from part 1 of Theorem A.1. □

The dual problem is then to find a value associated with each type of worker $\{\lambda_1,\ldots,\lambda_N\}$. The problem is:

$$\inf_{\lambda\in\mathbb{R}^N}\sum_{n=1}^{N}\lambda_n p_n+\int_{\mathcal{Y}}v\left(y\right)dG\left(y\right)\qquad\text{where: }v\left(y\right)=\max_{n\in\{1,\ldots,N\}}\left\{h\left(x,y|\sigma\right)-\lambda_n\right\}\tag{B.5}$$

I show that the dual problem has a solution and I use that solution to construct a solution to the Monge-Kantorovich problem in (B.4). Furthermore, the solution will take the form of a deterministic transport, and the implied assignment function is the solution to the problem (4) in the main text. Part 3 of Theorem A.1 establishes that solution to the dual problem (B.5) exists.

Lemma B.2. *If q satisfies the following properties:*

1. *$\sigma>1$ or all workers can produce in all tasks: $q\left(x,y\right)>0$ for all pairs $\left(x,y\right)\in\mathcal{X}\times\mathcal{Y}$.*
2. *$q\left(x,\cdot\right)$ is upper-semicontinuous in y given $x\in\mathcal{X}$.*

Then there exists $\lambda^{\star}\in\mathbb{R}^N$ such that:

$$\lambda^{\star}\in\underset{\lambda\in\mathbb{R}^N}{\mathit{argmin}}\sum_{n=1}^{N}\lambda_n p_n+\int_{\mathcal{Y}}\left(\max_{n\in\{1,\ldots,N\}}\left\{h\left(x,y|\sigma\right)-\lambda_n\right\}\right)dG\left(y\right)$$

Proof. This follows from applying part 3 of theorem A.1 (Villani, 2009, Thm. 5.10). The function $h(x, y|\sigma)$ is required to be real valued. When $\sigma > 1$ this is verified since q is real valued. When $\sigma = 1$ it is verified under the additional condition that $q(x, y) > 0$ for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$.

It is left to find functions $c \in L^1(P)$ and $d \in L^1(G)$ such that:

$$\forall_{(x,y)\in\mathcal{X}\times\mathcal{Y}} \quad c(x) + d(y) \leq h(x, y|\sigma)$$

For this let $c(x) = 0$ and $d(y) = \min_n \{h(x, y|\sigma)\}$. The minimum is well defined since $\mathcal{X}$ is finite. □

The final part of Proposition 1 is obtained from applying Theorem 5.30 of Villani (2009), reproduced as part 4 of Theorem A.1. The result is established under the conditions that both $(F(x, y))^{\frac{\sigma-1}{\sigma}}$ and the Monge-Kantorovich problem (B.4) have finite value and the F-subdifferential of w is single-valued G-almost everywhere.

Lemma B.3. *If q is such that:*

1. *$\sigma > 1$ or all workers can produce in all tasks: $q(x, y) > 0$ for all pairs $(x, y) \in \mathcal{X} \times \mathcal{Y}$.*
2. *$q(x, \cdot)$ is upper-semicontinuous in y given $x \in \mathcal{X}$.*
3. *q discriminates across workers in almost all tasks: if $q(x_n, y) = q(x_m, y)$ then $x_n = x_m$ G-a.e.*

Then there exists $\lambda^\star \in \mathbb{R}^N$ that solves the dual problem (B.5). Moreover, let T be defined as:

$$T(y) = \underset{x\in\mathcal{X}}{\mathit{argmax}} \left\{h(x, y|\sigma) - \lambda^\star_{n(x)}\right\}$$

T is single-valued G-almost everywhere and it induces a deterministic coupling $\pi^\star : \mathcal{X} \times \mathcal{B}(\mathcal{Y}) \to \mathbb{R}_+$ that is the unique (in law) solution to the Monge-Kantorovich problem (B.4). $\pi^\star$ is:

$$\pi^\star(x_n, Y) = \int_{Y\cap T^{-1}(x_n)} dG$$

Function T is an assignment function and it is the solution to the Monge transportation problem (4).

Proof. The proof follows from applying part 4 of Theorem A.1 (from Villani (2009, Thm. 5.30)). Finiteness of $h(x, y|\sigma)$ is guaranteed if $\sigma > 1$, or if $\sigma = 1$ and $q(x, y) > 0$ for all pairs $(x, y) \in \mathcal{X} \times \mathcal{Y}$. Finiteness of the value of the Monge-Kantorovich problem is guaranteed since $\mathcal{Y}$ and $\mathcal{X}$ are both compact, and q is upper semicontinuous on y.

It is left to verify that for any h-convex function $v : \mathcal{Y} \to \mathbb{R} \cup \{-\infty\}$ the h-subdifferential $\partial^h v(y)$ is single valued G-almost everywhere. The h- subdifferential for a given y is given by:

$$\partial^h v(y) = \left\{x \in \mathcal{X} \mid v^h(x) + v(y) = h(x, y|\sigma)\right\} \qquad \text{where} \qquad v^h(x) = \sup_y \{h(x, y|\sigma) - v(y)\}$$

Since v is h-convex we can instead use its conjugate function $v^h(x_n) = \lambda_n$. Then the h-subdifferential is then equivalently given by:

$$\partial^h v(y) = \underset{x\in\mathcal{X}}{\operatorname{argmax}} \left\{h(x, y|\sigma) - \lambda_{n(x)}\right\}$$

Since $q(\cdot, y)$ is injective in x given y G-a.e., and $\mathcal{X}$ is finite, we get that $\partial^h v(y)$ is generically a singleton.

□

The following lemma establishes the relation between the multipliers of the transformed problem (B.4) and multipliers of the original problem (B.2).

Lemma B.4. *Consider two constrained maximization problems:*

$$V(m) = \max_x \quad F(x) \qquad s.t.\ h(x) = m \tag{B.6}$$

$$W(m) = \max_x \quad g(F(x)) \qquad s.t.\ h(x) = m \tag{B.7}$$

where $F : \mathcal{X} \to \mathbb{R}$, $h : \mathcal{X} \to \mathbb{R}^n$, $m \in \mathbb{R}^n$ and $g : \mathbb{R} \to \mathbb{R}$ is strictly monotone. Let $\lambda \in \mathbb{R}^n$ be the multiplier associated with the constraints in (B.6), and $\mu \in \mathbb{R}^n$ the multiplier associated with the constraints in (B.7). Then: $\mu = g^{'}(F(x^\star))\lambda$, where $x^\star$ is a solution for (B.6) and (B.7).

Proof. Because g is strictly monotone both problems have the same argmax, call it $x^\star(m)$. The value of each problem is:

$$V(m) = F(x^\star(m)) \qquad W(m) = g(F(x^\star(m)))$$

By the envelope theorem (Milgrom and Segal, 2002) we know that:

$$\lambda = \frac{\partial V(m)}{\partial m} = \frac{\partial F(x^\star)}{\partial x}\frac{\partial x^\star(m)}{\partial m} \qquad \mu = \frac{\partial W(m)}{\partial m} = \frac{\partial g(F(x^\star))}{\partial F}\frac{\partial F(x^\star)}{\partial x}\frac{\partial x^\star(m)}{\partial m}$$

Joining gives the result: $\mu = g^{'}(F(x^\star))\lambda$.

□

B.2 Differentiability of demand

Lemma B.5. *Let $\lambda \in \mathbb{R}^N$ be a vector of multipliers. If q satisfies Assumption (2) then D_n is continuously differentiable with respect to λ and:*

$$\frac{\partial D_n}{\partial \lambda_m} = \frac{\text{area}\left(\mathcal{Y}_n(w) \cap \mathcal{Y}_m(w)\right)}{2\sqrt{\left(x_n - x_m\right)' A' A \left(x_n - x_m\right)}} \geq 0$$

Proof. The proof follows from an application of Reynolds' Transport Theorem (Theorem A.2). In order to apply Reynolds' theorem recall that $D_m = \int_{\mathcal{Y}_m} \rho(y)\, dy$, where ρ is the density of tasks in the space. In our case $\rho(y) = 1$. So the volume is $\mathcal{Y}_m$ and the function is the density of tasks.

The second term in the theorem measures the rate at which the density flows in and out of the volume. The density flows out and into other workers as tasks are reassigned. Consider the flow into of $\mathcal{Y}_m$ and out of $\mathcal{Y}_k$. The flow is in the direction $\frac{A(x_k - x_m)}{\sqrt{(x_k - x_m)' A' A (x_k - x_m)}}$ and through the shared boundary of $\mathcal{Y}_m$ and $\mathcal{Y}_k$, given by $\mathcal{Y}_m \cap \mathcal{Y}_k$. Note that when prices change the hyperplanes that define the boundaries of the demand sets move in parallel. Applying the theorem:

$$\frac{\partial D_m}{\partial w_n} = \int_{\mathcal{Y}_m} \frac{\partial \rho(y)}{\partial w_n} dy + \sum_{k \neq m} \int_{\mathcal{Y}_m \cap \mathcal{Y}_k} \rho(y) \left(\frac{\partial y \cdot \frac{A(x_k - x_m)}{\sqrt{(x_k - x_m)' A' A (x_k - x_m)}}}{\partial w_n} \right) dy$$

Note that for all $y \in \mathcal{Y}_m \cap \mathcal{Y}_k$ lie in a plane perpendicular to $A(x_k - x_m)$. Then they can be always expressed as $y = y_\lambda + a\vec{v}$ where $a \in \mathbb{R}$, $\vec{v}$ is a vector perpendicular to $A(x_k - x_m)$ and $y_\lambda = (1 - \lambda) x_k + \lambda x_m$ is such that $y_\lambda \in \mathcal{Y}_m \cap \mathcal{Y}_k$. Then the change $y \in \mathcal{Y}_m \cap \mathcal{Y}_k$ is equal to the change in y_λ.

$$\frac{\partial D_m}{\partial w_n} = \sum_{k \neq m} \int_{\mathcal{Y}_m \cap \mathcal{Y}_k} \rho(y) \left(\frac{\partial y_\lambda \cdot \frac{A(x_k - x_m)}{\sqrt{(x_k - x_m)' A' A (x_k - x_m)}}}{\partial w_n} \right) dy$$

The value of λ is obtained from the equation for the hyperplane that defines $\mathcal{Y}_m \cap \mathcal{Y}_k$:

$$\lambda = \frac{\left(x_m - x_k\right)' A \left(x_m - x_k\right) + w_m - w_k}{2\left(x_m - x_k\right)' A \left(x_m - x_k\right)}$$

so:

$$\frac{\partial y_\lambda \cdot \frac{x_k - x_m}{\sqrt{(x_k - x_m)'(x_k - x_m)}}}{\partial w_n} = \begin{cases} \frac{1}{2\sqrt{(x_n - x_m)' A' A (x_n - x_m)}} & \text{if } k = n \\ 0 & \text{otw} \end{cases}$$

Replacing:

$$\frac{\partial D_m}{\partial w_n} = \frac{\int_{\mathcal{Y}_m \cap \mathcal{Y}_n} \rho(y)\, dy}{2\sqrt{\left(x_n - x_m\right)' \left(x_n - x_m\right)}} = \frac{\text{area}\left(\mathcal{Y}_n(w) \cap \mathcal{Y}_m(w)\right)}{2\sqrt{\left(x_n - x_m\right)' \left(x_n - x_m\right)}}$$

which completes the proof.

$\square$

B.3 Directed automation

Proposition 4. *Consider the automation problem in 20 and let $\mu \in \mathbb{R}^{N+1}$ characterize an assignment according to 23. If q is differentiable then the first order conditions of the problem are:*

$$F_R(\mu, r) \int_{\mathcal{Y}_R} \frac{\partial \ln q(r, y)}{\partial r} dy - \frac{\partial \Omega(r, p_r)}{\partial r} = 0 \qquad [r]$$

$$F_R(\mu, r)\, \mu_R - \frac{\partial \Omega(r, p_r)}{\partial p_r} = 0 \qquad [p_r]$$

Proof. After replacing T_R for μ in the problem, and abusing notation, the corresponding Lagrangian is:

$$\max_{\{r, p_r, \mu, \Lambda\}} \mathcal{L} = F_R(\mu, r) - \Omega(r, p_r) + \sum_{n=1}^{N} \Lambda_n (p_n - D_n) + \Lambda_R (p_r - D_R) \tag{B.8}$$

The multipliers of the workers/robot capacity constraints are given by the vector $\Lambda \in \mathbb{R}^{N+1}$.

The first order condition of interest is with respect to the skills of the robot:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{\partial F_R(\mu, r)}{\partial r} - \frac{\partial \Omega(r, p_r)}{\partial r} - \sum_{n=1}^{N} \Lambda_n \frac{\partial D_n}{\partial r} - \Lambda_R \frac{\partial D_R}{\partial r} \tag{B.9}$$

Following de Goes et al. (2012) and using the result in Lemma B.4 the first order condition becomes:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{\partial F_R(\mu, r)}{\partial r} - \frac{\partial \Omega(r, p_r)}{\partial r} - F_R(\mu, r) \left(\sum_{n=1}^{N} \mu_n \frac{\partial D_n}{\partial r} - \mu_R \frac{\partial D_R}{\partial r} \right) \tag{B.10}$$

I proceed by computing separately the first term of the first order condition:

$$\frac{\partial F_R(\mu, r)}{\partial r} = F_R(\mu, r) \left(\sum_{n=1}^{N} \frac{\partial \int_{\mathcal{Y}_n} \ln q(x_n, y)\, dy}{\partial r} + \frac{\partial \int_{\mathcal{Y}_R} \ln q(r, y)\, dy}{\partial r} \right)$$

Each of the derivatives follows from Reynold's theorem.

$$\begin{aligned} \frac{\partial \int_{\mathcal{Y}_n} \ln q(x_n, y)\, dy}{\partial r} &= \int_{\mathcal{Y}_n} \frac{\partial \ln q(x_n, y)}{\partial r} dy + \int_{\mathcal{Y}_n \cap \mathcal{Y}_R} \ln q(x_n, y) \frac{\partial y \cdot c_{nr}}{\partial r} dy \\ &= \int_{\mathcal{Y}_n \cap \mathcal{Y}_R} \ln q(x_n, y) \frac{\partial y \cdot c_{nr}}{\partial r} dy \end{aligned}$$

where $c_{nr} = \frac{2A(x_n - r)}{\sqrt{(x_n - r)' A (x_n - r)'}}$ is the normal vector to the direction in which the boundary is moving.

In a similar way:

$$\frac{\partial \int_{\mathcal{Y}_R} \ln q(r,y)\,dy}{\partial r} = \int_{\mathcal{Y}_R} \frac{\partial \ln q(r,y)}{\partial r} dy + \sum_{n=1}^{N} \int_{\mathcal{Y}_n \cap \mathcal{Y}_R} \ln q(x_n,y) \frac{\partial y \cdot c_{rn}}{\partial r} dy$$

where $c_{rn} = -c_{nr}$. Joining and reorganizing we get:

$$\frac{1}{F_R(\mu,r)} \frac{\partial F_R(\mu,r)}{\partial r} = \int_{\mathcal{Y}_R} \frac{\partial \ln q(r,y)}{\partial r} dy + \sum_{n=1}^{N} \int_{\mathcal{Y}_n \cap \mathcal{Y}_R} \left(\ln q(x_n,y) - \ln q(r,y)\right) \frac{\partial y \cdot c_{nr}}{\partial r} dy$$

Note now that by the definition of the boundary $\ln q(x_n,y) - \ln q(r,y) = \mu_n - \mu_r$ for all $y \in \mathcal{Y}_n \cap \mathcal{Y}_R$. Then:

$$\frac{1}{F_R(\mu,r)} \frac{\partial F_R(\mu,r)}{\partial r} = \int_{\mathcal{Y}_R} \frac{\partial \ln q(r,y)}{\partial r} dy + \sum_{n=1}^{N} (\mu_n - \mu_r) \int_{\mathcal{Y}_n \cap \mathcal{Y}_R} \frac{\partial y \cdot c_{nr}}{\partial r} dy$$

Finally note that $\frac{\partial D_n}{\partial r} = \int_{\mathcal{Y}_n \cap \mathcal{Y}_R} \frac{\partial y \cdot c_{nr}}{\partial r} dy$, which follows from applying Reynold's Theorem (again) to D_n.

$$\frac{1}{F_R(\mu,r)} \frac{\partial F_R(\mu,r)}{\partial r} = \int_{\mathcal{Y}_R} \frac{\partial \ln q(r,y)}{\partial r} dy + \sum_{n=1}^{N} (\mu_n - \mu_r) \frac{\partial D_n}{\partial r}$$

When the location of the robot (r) is changed, there is a change in output due to the change in mismatch inside the region previously assigned to the robot ($\mathcal{Y}_R$), that is given by the first term. There is also a change in the demand for workers, only workers who are neighbors of the robot are affected. When their demand is affected, the demand of the robot changes in the opposite direction. The demand for worker n changes by $\frac{\partial D_n}{\partial r}$, that is valued by the planner at $\lambda_n - \lambda_r$. Recall that λ_n is the shadow price of the supply of a worker.

It is left to spell out the first term:

$$\begin{aligned}
\int_{\mathcal{Y}_R} \frac{\partial \ln q(r,y)}{\partial r} dy &= \int_{\mathcal{Y}_R} \frac{\partial a_x' r - (r-y)' A (r-y)}{\partial r} dy \\
&= \int_{\mathcal{Y}_R} (a_x - 2Ar + 2Ay)\, dy \\
&= 2D_R \left(\frac{a_x}{2} - A(r - b_R)\right)
\end{aligned}$$

where $b_R = \frac{\int_{\mathcal{Y}_R} y dy}{D_R}$ is the barycenter (centroid, average or center of mass) of the tasks assigned to r.

It is now possible to obtain the first order condition of the problem with respect to the location of the robot. Note that since the total demand is constant it holds that:

$$\frac{\partial D_R}{\partial r} = -\sum_{n=1}^{N} \frac{\partial D_n}{\partial r}$$

then:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{\partial F_R(\mu,r)}{\partial r} - \frac{\partial \Omega(r,p_r)}{\partial r} - F_R(\mu,r)\left(\sum_{n=1}^{N} (\mu_n - \mu_R) \frac{\partial D_n}{\partial r}\right) \tag{B.11}$$

Replacing for $\frac{\partial F_R(\mu,r)}{\partial r}$ we get:

$$\frac{\partial \mathcal{L}}{\partial r} = 2F_R\left(\mu,r\right)D_R\left(\frac{a_x}{2} - A\left(r - b_R\right)\right) - \frac{\partial \Omega\left(r,p_r\right)}{\partial r} \tag{B.12}$$

The first order condition does not include the effect of r on the demand for workers since the gains cancel with the reductions/increases of slack in the feasibility constraints.

This is a necessary condition for an optimum. It does not fully characterize the solution. In fact, there can be, in general, multiple solutions to the problem. The first order condition is also silent about the location of the region assigned to r. Instead, it prescribes the relationship between the region's centroid and the location of r. It is convenient to see what happens when $a_x = 0$ and $\frac{\partial \Omega(r,p_r)}{\partial r} = 0$. Then the necessary condition reduces to make r equal to the barycenter of its region.

The first order condition with respect to p_r is:

$$\frac{\partial F}{\partial p_r} = F_R\left(\mu,r\right)\mu_R - \frac{\partial \Omega\left(r,p_r\right)}{\partial p_r}$$

The first order condition with respect to μ requires more work, but it follows from applying again Reynolds' Transport Theorem.

$$\frac{\partial F}{\partial \mu_n} = p_n - D_n$$

□

B.4 Marginal Product

The marginal product of a worker gives the change in output if more workers of that type are used in production. The change in output depends on the tasks that are assigned to additional workers.[28] Because of this, it is possible to define the marginal product at a given task, and under some initial assignment. In the main text, I consider the notion of equilibrium marginal products, where the assignment is not taken as given, but it is allowed to react optimally to changes in the supply of workers.

Consider the marginal product of a worker of type x_k at task $\overline{y}$, given an assignment T. Since task $\overline{y}$ has no mass, output does not change if the task is re-assigned to a worker of type x_k. The marginal product is measured by adding a mass of workers of type x_k and assigning them to a region around task $\overline{y}$, replacing the workers previously assigned to those tasks. The marginal product at $\overline{y}$ is obtained as the change in output when the mass of added workers tends to zero.

Proposition 5. ***[Marginal Product]*** *Let T be a deterministic assignment and fix a task $\overline{y} \in \mathcal{Y}_n^{o}$. The marginal product of a worker of type x_k at task $\overline{y}$ is:*

$$MP\left(x_k, \overline{y}|T\right) = F\left(T\right)\left(\ln q\left(x_k, \overline{y}\right) - \ln q\left(x_n, \overline{y}\right)\right)$$

where $F\left(T\right) = \exp\left(\int \ln q\left(T\left(y\right), y\right) dG\right)$ and $T\left(\overline{y}\right) = x_n$.

When task $\overline{y}$ is re-assigned from x_n to x_k output changes by $\ln q\left(x_k, \overline{y}\right) - \ln q\left(x_n, \overline{y}\right)$. The marginal product takes into account the opportunity cost of assigning task $\overline{y}$ to x_k, which comes from the capacity constraint of tasks. The derivative of output takes into account the scale of production at the current assignment. Task $\overline{y}$ is required to be in the interior of $\mathcal{Y}_n$ for technical reasons. If $\overline{y} \in \mathcal{Y}_n \cap \mathcal{Y}_m$ it becomes necessary to specify the region around $\overline{y}$ to which x_k will be assigned.

The proof of the result is complicated because the task $\overline{y}$ has dimension zero in the space of tasks, which has dimension $d \geq 1$. Before showing the general proof for the result, I consider the one-dimensional case where the argument is simpler. I further assume that $y \sim U\left([0, 1]\right)$. When $d = 1$ the production function can be written as:

$$F\left(T\right) = \exp\left(\int_0^1 \ln q\left(T\left(y\right), y\right) dy\right)$$

Fix a task $\overline{y} \in (0, 1)$ and consider adding a mass ϵ of workers of type x_k. Workers are

[28] Unlike traditional production functions, the amount of an input used by the firm in production and what that input is used for are not the same.

assigned to the set $C_{\overline{y},\epsilon} = \left\{y \mid |y - \overline{y}| < \frac{\epsilon}{2}\right\} = \left[\overline{y} - \frac{\epsilon}{2}, \overline{y} + \frac{\epsilon}{2}\right]$. The new assignment is:

$$T_\epsilon(y) = \begin{cases} T(y) & \text{if } y \notin C_{\overline{y},\epsilon} \\ 0 & \text{if } y \in C_{\overline{y},\epsilon} \wedge x \neq x_k \\ 1 & \text{if } y \in C_{\overline{y},\epsilon} \wedge x = x_k \end{cases}$$

The change in output is:

$$F(T_\epsilon) - F(T) = F(T)\left(\exp\left(\int_{\overline{y}-\frac{\epsilon}{2}}^{\overline{y}+\frac{\epsilon}{2}} \left(\ln q(x_k, y) - \ln q(T(y), y)\right) dy\right) - 1\right)$$

The marginal product is:

$$\text{MP}(x_k, \overline{y}|T) = \left.\frac{\partial F(R_\epsilon)}{\partial \epsilon}\right|_{\epsilon=0} = \lim_{\epsilon \to 0} \frac{F(R_\epsilon) - F(T)}{\epsilon}$$

replacing and applying L'Hôpital's rule:

$$\text{MP}(x_k, \overline{y}|T) = F(R) \left.\frac{\partial \exp\left(\int_{\overline{y}-\frac{\epsilon}{2}}^{\overline{y}+\frac{\epsilon}{2}} \left(\ln q(x_k, y) - \ln q(T(y), y)\right) dy\right)}{\partial \epsilon}\right|_{\epsilon=0}$$

The derivative follows from Leibniz's rule. Generically $\overline{y} \in \mathcal{Y}_n^{\text{o}}$ and:

$$\begin{aligned} \text{MP}(x_k, \overline{y}|T) = F(T) \Bigg[& \frac{1}{2}\left(\ln q\left(x_k, \overline{y} + \frac{\epsilon}{2}\right) - \ln q\left(T(y), \overline{y} + \frac{\epsilon}{2}\right)\right) \\ & + \frac{1}{2}\left(\ln q\left(x_k, \overline{y} - \frac{\epsilon}{2}\right) - \ln q\left(T(y), \overline{y} - \frac{\epsilon}{2}\right)\right)\Bigg]_{\epsilon=0} \\ = F(T) & \left(\ln q(x_k, \overline{y}) - \ln q(x_n, \overline{y})\right) \end{aligned}$$

If $\overline{y} \in \mathcal{Y}_n \cap \mathcal{Y}_m$ the marginal product takes into account that x_k replaces different types of workers around $\overline{y}$:

$$\text{MP}(x_k, \overline{y}|T) = F(T)\left(\ln q(x_k, \overline{y}) - \frac{\ln q(x_n, \overline{y}) + \ln q(x_m, \overline{y})}{2}\right)$$

In multiple dimensions, the treatment of the boundary cases becomes intractable, except in very specific cases for which similar expressions are obtained.

I now provide the general proof of the result.

Proof. Recall that the space of skills is of dimension d. Changing the assignment of tasks to workers in any region of dimension less than d will have no impact on output. To compute the effect on output of the added workers it is necessary to proceed one dimension at a time. Consider a region formed as a hypercube around $\overline{y}$, with sides of length ϵ_i, denote this region by $C_{\overline{y},\epsilon} = \left\{y \,|\, \forall_i \,|y_i - \overline{y}_i| \leq \frac{\epsilon_i}{2}\right\}$. Note that as all $\epsilon_i \to 0$ the region $C_{\overline{y},\epsilon} \to \{\overline{y}\}$. The assignment is modified as in the one-dimensional example:

$$T_\epsilon(y) = \begin{cases} T(y) & \text{if } y \notin C_{\overline{y},\epsilon} \\ 0 & \text{if } y \in C_{\overline{y},\epsilon} \wedge x \neq x_k \\ 1 & \text{if } y \in C_{\overline{y},\epsilon} \wedge x = x_k \end{cases}$$

The difference in production between the two assignments is:

$$F(T_\epsilon) - F(T) = F(T)\left(\exp\left(\int\limits_{\overline{y}_1-\frac{\epsilon_1}{2}}^{\overline{y}_1+\frac{\epsilon_1}{2}} \cdots \int\limits_{\overline{y}_d-\frac{\epsilon_d}{2}}^{\overline{y}_d+\frac{\epsilon_d}{2}} \left(\ln q(x_k, y) - \ln q(T(y), y)\right) dy\right) - 1\right)$$

I proceed by computing the change in output when the region $C_{\overline{y},\epsilon}$ changes. The change has to be computed one dimension at a time. If all dimensions are changed simultaneously the change in F goes to zero (this can be verified directly using Reynold's transport theorem- Theorem A.2). The change in output when $C_{\overline{y},\epsilon}$ changes in the d^{th} dimension is:

$$\begin{aligned} \frac{\partial F(T_\epsilon)}{\partial \epsilon_d} = F(T) \Bigg(\int\limits_{\overline{y}_1-\frac{\epsilon_1}{2}}^{\overline{y}_1+\frac{\epsilon_1}{2}} \cdots \int\limits_{\overline{y}_{d-1}-\frac{\epsilon_d}{2}}^{\overline{y}_{d-1}+\frac{\epsilon_d}{2}} & \frac{1}{2}\left(\ln q\left(x_k, \begin{pmatrix} y_1 \\ \vdots \\ \overline{y}_d + \frac{\epsilon}{2} \end{pmatrix}\right) - \ln q\left(T\begin{pmatrix} y_1 \\ \vdots \\ \overline{y}_d + \frac{\epsilon}{2} \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ \overline{y}_d + \frac{\epsilon}{2} \end{pmatrix}\right)\right) \\ & + \frac{1}{2}\left(\ln q\left(x_k, \begin{pmatrix} y_1 \\ \vdots \\ \overline{y}_d - \frac{\epsilon}{2} \end{pmatrix}\right) - \ln q\left(T\begin{pmatrix} y_1 \\ \vdots \\ \overline{y}_d - \frac{\epsilon}{2} \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ \overline{y}_d - \frac{\epsilon}{2} \end{pmatrix}\right)\right) dy_1 \ldots dy_{d-1} \Bigg) \end{aligned}$$

Applying the same procedure iteratively we obtain the change in output as x_k is assigned to tasks around $\overline{y}$ in all directions:

$$\mathrm{MP}(x_k, \overline{y}|T) = \left.\frac{\partial^d F(T_\epsilon)}{\partial \epsilon_1 \cdots \partial \epsilon_d}\right|_{\epsilon=0} = F(T)\left(\ln q(x_k, \overline{y}) - \ln q(x_n, \overline{y})\right)$$

□

C Cosine Similarity Assignments

I now consider the optimal assignment of tasks to workers when the mismatch is measured by the cosine similarity between the skill vectors of workers and tasks respectively. The cosine similarity between two vectors is given by the cosine of the angle between the vectors:

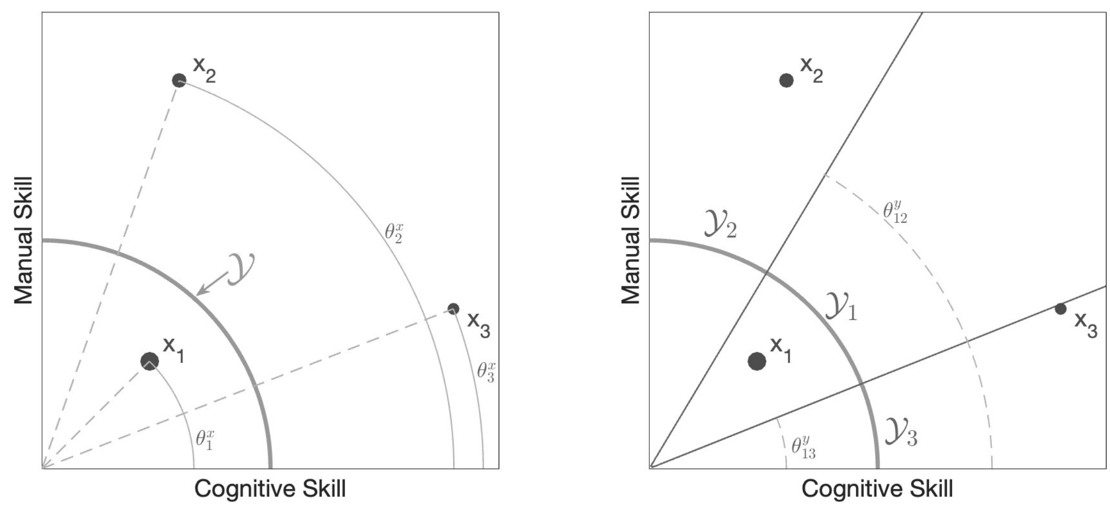
$$\cos(\theta_{xy}) = \frac{x'y}{\|x\|\,\|y\|}, \tag{C.1}$$

where $\|z\| = \sqrt{\sum_{i=1}^{d} z_i^2}$ is the (Euclidean-)norm of a vector. The cosine similarity varies between 0 and 1. The main advantage of the cosine similarity is that if effectively reduces the dimensionality of the problem from d to a single dimension. This is because only the angle between two vectors, and not their magnitudes, matters for this notion of mismatch.

The set up of the model is just as in Section 1 with just two modifications. First, the set of tasks can be restricted (without loss) to tasks on a circle in the positive quadrant (or the surface of a sphere in the positive orthant for higher dimensions). Second, the task output function is:

$$q(x, y) = \exp\left(f^x(x) + \cos(\theta_{xy})\right), \tag{C.2}$$

where $f^x(\cdot)$ captures the role of workers' skills in production. Figure C.1a presents the setup. Each worker x_1, x_2 and x_3 is paired with an angle θ_1^x, θ_2^x, θ_3^x. The cosine similarity between a worker and a task in is the cosine of the difference between their angles.



(a) Model Setup

(b) Optimal Assignment

Figure C.1: Assignment Example - Cosine Similarity Loss

Note: *The left panel depicts the setup of the model when mismatch is measured according to the cosine similarity between workers and tasks as in XX. Because the cosine similarity is independent of magnitudes only the tasks in along gray circumference are considered. There are three types of workers $\{x_1, x_2, x_3\}$ with mass $P = \{0.4, 0.3, 0.3\}$ at angles $\{\theta_1^x, \theta_2^x, \theta_3^x\}$. The right panel shows the optimal assignment. The assignment is characterized by two angles, θ_{12}^y and θ_{13}^y, which determine the boundaries of between $\mathcal{Y}_1$ and $\mathcal{Y}_2$, and $\mathcal{Y}_1$ and $\mathcal{Y}_3$, respectively. These angles partition the space as shown by the two rays that intersect the circumference.*

Figure C.1b shows the optimal assignment. In this setup, the assignment consist on two cutoff angles (θ^y_{12} and θ^y_{12}) which characterize rays partitioning the space of skills. The solution is the same regardless of whether tasks are collapsed onto the circle or they are distributed on the whole plane. The magnitude of the tasks' vectors does not change the cosine distance from the task to the worker and therefore does not affect mismatch. This is undoubtedly a strong assumption, it implies that the magnitude of the skill vectors only affects production by determining an absolute productive of workers which is independent of the assignment. This productivity is captured by the value of $f^x(x)$.

The assignment in Figure C.1b also reveals the logic behind single-dimensional assignments. The productivity of workers as measured by $f^x(x_1)$, $f^x(x_2)$, and $f^x(x_3)$ establishes an ordering of workers prior to the assignment. It is optimal to assign tasks to the most productive worker first to ensure that all of its time is used in production. In the example this is worker x_3. In fact, the value of θ^y_{13} in the assignment is such that a fraction p_3 of the tasks in the circle are assigned to worker x_3. Following this process, it is optimal to assign tasks to worker x_2 next. This gives angle θ^y_{12} such that the tasks between it and $\pi/2$ are assigned to worker x_2. The remaining tasks are assigned to worker x_1, who is the least productive.