University of Minnesota
Math Refresher
Summer 2015

Problem Set 1

1. Find the inf and sup (if they exist) of the set S in the following cases:
   - $S = \left\{\frac{1}{n} : n \in N\right\}$.
   - $S = \left\{\frac{1}{n} - \frac{1}{m} : n, m \in N\right\}$.
   - $S = \left\{1 - \frac{(-1)^n}{n} : n, m \in N\right\}$.

2. Let A and B be nonempty bounded subsets of $\Re$. Define $A+B$ as $A+B = \{a + b : a \in A, b \in B\}$. Show that sup(A+B)=supA+supB.

3. Let S be a nonempty bounded subset of $\Re$. Show that there is an increasing sequence $\{x_m\}$ in S such that $x_m \to supS$, and a decreasing sequence $\{x_m\}$ in S such that $y_m \to infS$.

4. Show that every real Cauchy sequence is bounded.

5. Show that every real Cauchy sequence converges.

6. Let x,y $\in$R^n. Show that for any two sequences $\{x_m\}$, $\{y_m\}$ such that $x_m \to x$ and $y_m \to y$, we have $d(x_m, y_m) \to d(x, y)$.

7. Given two sequences $\{x_m\}$, $\{y_m\}$ in R. Show that:
   - $limsup\,(x_m + y_m) \leq limsup\,(x_m) + limsup\,(y_m)$
   - $liminf\,(x_m + y_m) \geq liminf\,(x_m) + liminf\,(y_m)$

8. Give an example of a function $f : \Re \to \Re$ that is continuous at exactly two points, or show that no such function can exist (Exercise 48, Sundaram 1.7)

9. Show that it is possible for two functions $f : \Re \to \Re$ and $g : \Re \to \Re$ to be discontinuous but for their product $g \cdot f$ to be continuous. What about their composition $f \circ g$? (Exercise 49, Sundaram 1.7).

10. Define $f : \Re_+ \to \Re$ as: $f(0) = 0$ and $f(x) = x \cdot \sin\left(\frac{1}{x}\right)$ otherwise.

    Where is this function continuous? Prove it (Exercise 51, Sundaram 1.7).

11. Define $f : \Re_+ \to \Re$ as:

    $f(x) = \begin{cases} x & \text{when } x \text{ is irrational.} \\ 1 - x & \text{when } x \text{ is rational.} \end{cases}$ (Exercise 53, Sundaram 1.7)

12. Define $f : \Re \to \Re$ as:

$$f(x) = \begin{cases} 0 & \text{when } x \text{ is irrational or } x = 0. \\ \frac{1}{q} & \text{when } x = \frac{p}{q} \text{ (reduced fraction) and } x \neq 0. \end{cases}$$

13. Let S be a metric space and $q \in S$. Show that the distance function $d(p, q)$ is a uniformly continuous function of p.

14. Determine whether the following functions are uniformly continuous on $\Re_+$:

    $a)$ $f(x) = \sin x$.

    $b)$ $f(x) = 1/x$.