Robust Contracts in Common Agency

Online Appendix

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1 Individual Limited Liability

The model presented in the main text assumes that contracts are subject to common limited liability, so that the contracts satisfy $w_1(y) + w_2(y) \geq 0$ for all $y \in Y$. We can instead think of a stronger requirement and ask that the contract offered by each principal have to satisfy $w_i(y) \geq 0$ for all $y \in Y$ and $i \in \{1, 2\}$. In this case a principal cannot charge the agent, regardless of what the other principal is paying . Unlike previously, there are no equilibrium transfers between principals (through the agent).

Importantly, changing limited liability does not change our analysis on the principal's best response. We show thatLRS contracts are still best responses:

$$w_i(y) = \alpha_i y_i - (1 - \alpha_i) w_j(y) - \alpha_i k_i \qquad \forall y \in Y \tag{1}$$

Theorem 1 of the main text then goes through with just one modification, namely the limited liability requirement. The formal statement and proof are provided at the end of this section. We now state and prove a series of lemmas to establish the result. They follow closely the arguments in the proof of Theorem 1 in the text, with the appropriate modifications.

The following proposition relates the expected payments to the agent under any technology with its value under $\mathcal{A}_0$.

Proposition 1. *Let $(F, c) \in A^\star(w|\mathcal{A})$. For $\mathcal{A} \supseteq \mathcal{A}_0$, it holds that:*

$$E_F[w_1(y) + w_2(y)] \geq V_A(w|\mathcal{A}_0)$$

Moreover, if $(F, c) \in A^\star(w|\mathcal{A})$ then $F \in \mathcal{F}$ where:

$$\mathcal{F} = \{F \in \Delta(Y) \,|E_F[w_1(y) + w_2(y)] \geq V_A(w|\mathcal{A}_0)\}$$

Proof. To see the first inequality let $(F, c) \in A^{\star}(w|\mathcal{A})$ for $\mathcal{A} \supseteq \mathcal{A}_0$:

$$E_F[w_1(y) + w_2(y)] \geq E_F[w_1(y) + w_2(y)] - c \geq V_A(w|\mathcal{A}) \geq V_A(w|\mathcal{A}_0)$$

Then $F \in \mathcal{F}$. $\square$

Lemma 1 characterizes the principal's payoff for a given contract scheme using the set $\mathcal{F}$ (defined in the proposition above). Note that $\mathcal{F}$ only depends on the contract and the known set of actions $\mathcal{A}_0$. The following results are valid for any scheme w that provides positive guarantees for principal i

We formally define them as follows:

Eligibility: A contract w is *eligible* for principal i if: $V_i(w) > 0$.

Lemma 1. *Let w be an eligible contract for principal i, then $V_i(w) = \min_{F \in \mathcal{F}} E_F[y_i - w_i(y)]$. Moreover if $F \in \underset{F \in \mathcal{F}}{argmin} E_F[y_i - w_i(y)]$ then $E_F[w_1(y) + w_2(y)] = V_A(w|\mathcal{A}_0)$.*

Proof. We first establish the first claim: Let w be an eligible contract scheme then: $V_i(w) = \min_{F \in \mathcal{F}} E_F[y_i - w_i(y)]$.

It must be that: $V_i(w) \geq \min_{F \in \mathcal{F}} E_F[y_i - w_i(y)]$. Using the definition of $V_i(w)$:

$$V_i(w) = \inf_{\mathcal{A} \supseteq \mathcal{A}_0} \min_{(F,c) \in A^{\star}(w|\mathcal{A})} E_F[y_i - w_i(y)] \geq \min_{F \in \mathcal{F}} E_F[y_i - w_i(y)]$$

Where the inequality follows because if $(F, c) \in A^{\star}(w|\mathcal{A})$ then $F \in \mathcal{F}$.

To prove equality suppose that $V_i(w) > \min_{F \in \mathcal{F}} E_F[y_i - w_i(y)]$, and let $F' \in \underset{F \in \mathcal{F}}{\operatorname{argmin}} E_F[y_i - w_i(y)]$. We have that $E_{F'}[w_1(y) + w_2(y)] \geq V_A(w|\mathcal{A}_0)$. There are two options:

1. F' does not place full support on the values of y that maximize $w_1 + w_2$.

   Let $\hat{y} \in \operatorname{argmax}\{w_1(y) + w_2(y)\}$, and $\hat{F} = \delta_{\hat{y}}$ be a distribution with full mass on $\hat{y}$. Let $\epsilon \in [0, 1]$ and $F_\epsilon = (1 - \epsilon) F' + \epsilon \hat{F}$.

   Note then that for all ϵ there exists a $\xi_\epsilon > 0$ such that: $E_{F_\epsilon}[w_1(y) + w_2(y)] - \xi_\epsilon > V_A(w|\mathcal{A}_0)$.

Define and $\mathcal{A}_\epsilon = \mathcal{A}_0 \cup \{(F_\epsilon, \xi_\epsilon)\}$. It follows that the unique optimal action of the agent in $\mathcal{A}_\xi$ is $(F_\epsilon, \xi_\epsilon)$. Then:

$$V_i(w) \leq V_i(w|\mathcal{A}_\epsilon) = E_{F_\epsilon}[y_i - w_i(y)] = (1-\epsilon) E_{F'}[y_i - w_i] + \epsilon E_{\hat{F}}[y_i - w_i]$$

This condition holds for all $\epsilon > 0$. Letting $\epsilon \to 0$ we arrive at a contradiction:

$$V_i(w) \leq E_{F'}[y_i - w_i] = \underset{F \in \mathcal{F}}{\operatorname{argmin}} E_F[y_i - w_i]$$

2. F' places full support on the values of y that maximize $w_1 + w_2$. There are still two possible cases:

    (a) $E_{F'}[w_1 + w_2] > V_A(w|\mathcal{A}_0)$. Then there exists $\xi > 0$ and a technology $\mathcal{A}' = \mathcal{A}_0 \cup \left\{\left(F', \xi\right)\right\}$ such that $\left(F', \xi\right)$ is the unique optimal action for the agent in $\mathcal{A}'$. Then we arrive at a contradiction:

    $$V_i(w) \leq V_i\left(w|\mathcal{A}'\right) = E_{F'}[y_i - w_i] = \underset{F \in \mathcal{F}}{\operatorname{argmin}} E_F[y_i - w_i]$$

    (b) $E_{F'}[w_1 + w_2] = V_A(w|\mathcal{A}_0)$. This implies $V_A(w|\mathcal{A}_0) = \max_{y \in Y}\{w_1 + w_2\}$ which can only be satisfied if F' is available in $\mathcal{A}_0$ at zero cost. By the positive cost assumption this implies that $F = \delta_{(0,0)}$ and that $w_1(0,0) + w_2(0,0) = \max_{y \in Y}\{w_1(y) + w_2(y)\}$. In this case the unique optimal action for the agent under any technology is $(\delta_0, 0)$, so the value of the principal is $V_i(w) = -w_i(0,0) \leq 0$, where the inequality follows from limited liability. This contradicts eligibility.

Now we establish the second claim: Let w be an eligible contract scheme for principal i. If $F \in \underset{F \in \mathcal{F}}{\operatorname{argmin}} E_F[y_i - w_i]$ then $E_F[w_1 + w_2] = V_A(w|\mathcal{A}_0)$.

To prove this, let $F' \in \underset{F \in \mathcal{F}}{\operatorname{argmin}} E_F[y_i - w_i(y)]$ and suppose for a contradiction that $E_F[w_1(y) + w_2(y)] > V_A(w|\mathcal{A}_0)$.

Let $\epsilon \in [0,1]$ and define $F_\epsilon = (1-\epsilon) F' + \epsilon\delta_0$. For low enough ϵ it holds that: $E_{F_\epsilon}[w_1(y) + w_2(y)] > V_A(w|\mathcal{A}_0)$. Then there exists $\xi_\epsilon > 0$ such that $\{(F_\epsilon, \xi_\epsilon)\} = A^\star(w|\mathcal{A}_\epsilon)$ where $\mathcal{A}_\epsilon = \mathcal{A}_0 \cup \{(F_\epsilon, \xi_\epsilon)\}$. The payoff to the principal is then:

$$V_i(w|\mathcal{A}_\epsilon) = (1-\epsilon) E_F[y_i - w_i(y)] + \epsilon(-w_i(0,0)) \leq (1-\epsilon) E_F[y_i - w_i(y)] = (1-\epsilon) V_i(w) < V_i(w)$$

This gives a contradiction. □

Given the known action set $\mathcal{A}_0$, the next lemma links the principal's guaranteed payoff to the agent's payoff in an affine way. This link allows the principal to increase her own guaranteed payoff by controlling the payoff given to the agent. Lemma 2 also offers a relation between any contract w_i, the outcome y_i and the contract w_j offered by the other principal.

The affine link between the agent's payoff and the principal's payoff is a crucial element in providing incentives for the agent. Given the partial knowledge over the agent's set of actions the principals' optimal strategy is to tie their payoff to that of the agent, thus aligning the agent's objectives with their own. This is the same mechanism at the heart of the optimal contracts in Hurwicz & Shapiro (1978) and Carroll (2015), and will be crucial in establishing the optimality of affine (LRS) contracts in the setting we develop.

Lemma 2. *Let w be an eligible contract. There exits k, λ with $\lambda > 0$ such that for all $y \in Y$:*

$$w_i(y) \leq \frac{1}{1+\lambda} y_i - \frac{\lambda}{1+\lambda} w_j(y) - \frac{1}{1+\lambda} k \tag{2}$$

$$V_i(w) = k + \lambda V_A(w|\mathcal{A}_0) \tag{3}$$

Proof. This lemma is proven with the following two propositions. In both propositions define:

Let $S \subseteq \mathbb{R}^2$ be the convex hull of all points $(w_1(y) + w_2(y), y_i - w_i(y))$ for $y \in Y$.

Let $T \subseteq \mathbb{R}^2$ be the set of all pairs (u, v) such that $u > V_A(w|\mathcal{A}_0)$ and $v < V_i(w)$.

Proposition. $S \cap T = \emptyset$.

Proof. Let $(u, v) \in T$ then let $F \in \underset{F \in \mathcal{F}}{\text{argmin}} E_F [y_i - w_i (y)]$, by definition of T and Lemma (1):

$$\begin{aligned} u &> V_A (w|\mathcal{A}_0) = E_F [w_1 (y) + w_2 (y)] \\ v &< V_i (w) = E_F [y_i - w_i (y)] \end{aligned}$$

now, suppose for a contradiction that $(u, v) \in S$, then there exists $F' \in \Delta (Y)$ such that:

$$u = E_{F'} [w_1 (y) + w_2 (y)] \qquad \text{and} \qquad v = E_{F'} [y_i - w_i (y)]$$

Note that F' guarantees a payoff to the agent larger than $V_A (w|\mathcal{A}_0)$ so $F' \in \mathcal{F}$ but:

$$E_F [y_i - w_i (y)] > E_{F'} [y_i - w_i (y)]$$

which contradicts minimality of F. Then $S \cap T = \emptyset$ $\square$

Since $S \cap T = \emptyset$ we can apply the separating hyperplane theorem which implies that there exist constants (k, λ, μ) such that $(\lambda, \mu) \neq (0, 0)$ and:

$$k + \lambda u - \mu v \leq 0 \qquad (u, v) \in S \tag{4}$$

$$k + \lambda u - \mu v \geq 0 \qquad (u, v) \in T \tag{5}$$

Let $F^{\star} \in \underset{F \in \mathcal{F}}{\text{argmin}} E_F [y_i - w_i (y)]$. Note that the pair $(E_{F^{\star}} [w_1 (y) + w_2 (y)], E_{F^{\star}} [y_i - w_i (y)])$ lies in the closures of both S and T. Then:

$$k + \lambda E_{F^{\star}} [w_1 (y) + w_2 (y)] - \mu E_{F^{\star}} [y_i - w_i (y)] = 0 \tag{6}$$

It is left to show that $\lambda, \mu > 0$.

Note that $(u, v) \in T$ admits u arbitrarily high and v arbitrarily low. So for (5) to hold it must be that $\lambda \geq 0$ and $\mu \geq 0$. There are then two cases to rule out:

1. Suppose $\mu = 0$, then it must be that $\lambda > 0$ (since $(\lambda, \mu) \neq (0, 0)$). From (4) and (5)

$$u \leq -\frac{k}{\lambda} \quad (u, v) \in S \qquad \text{and} \qquad u \geq -\frac{k}{\lambda} \quad (u, v) \in T$$

   So $\max_{y \in Y} [w_1(y) + w_2(y)] = \max_{u \in S} u \leq -\frac{k}{\lambda} \leq \inf_{u \in T} u = V_A(w|\mathcal{A}_0)$. Which implies:

$$\max_{y \in Y} [w_1(y) + w_2(y)] = V_A(w|\mathcal{A}_0)$$

   This can only happen if the agent has an action $(F, 0) \in \mathcal{A}_0$ such that $E_F[w_1(y) + w_2(y)] = \max[w_1(y) + w_2(y)]$, the only action in $\mathcal{A}_0$ with zero cost is $(\delta_0, 0)$, so $\max(w_1(y) + w_2(y)) = w_1(0, 0) + w_2(0, 0)$. This is also the unique action in $A^\star(w|\mathcal{A}_0)$ so:

$$V_i(w) \leq V_i(w|\mathcal{A}_0) = -w_i(0, 0) \leq 0$$

   This violates eligibility ($V_i(w) > 0$).

2. Suppose $\lambda = 0$, then it must be that $\mu > 0$ (since $(\lambda, \mu) \neq (0, 0)$). From (4) and (5)

$$v \geq \frac{k}{\mu} \quad (u, v) \in S \qquad \text{and} \qquad v \leq \frac{k}{\mu} \quad (u, v) \in T$$

   So $\min_{y \in Y} [y_i - w_i(y)] = \min_{v \in S} v \geq \frac{k}{\mu} \geq \sup_{v \in T} v = V_i(w)$. But we know that $\min_{y \in Y} [y_i - w_i(y)] \leq 0 - w(0, 0) \leq 0$ this implies $V_i(w) \leq 0$ which contradicts eligibility. So $\lambda > 0$.

Note that since λ and μ are greater than zero μ can be normalized to 1, giving from (4):

$$k + \lambda(w_i(y) + w_j(y)) - (y_i - w_i(y)) \leq 0$$

And from (6):

$$V_i(w) = k + \lambda V_A(w|\mathcal{A}_0)$$

□

The following two lemmas (3 and 4) use the relation between the principals' contracts derived in Lemma 2 to construct an alternative contract that dominates the original one, in the sense that it weakly increases principal i's guaranteed payoff. Since the relation obtained in (2) is affine in output and the other principal's contract, the alternative contract constructed below will inherit that form. These contracts form the LRS contracts defined in (31).

Lemma 3. *Let $w = (w_i, w_j)$ with w_i satisfying (2) and (3). Then the contract*

$$w_i^{'}(y) = \frac{1}{1+\lambda} y_i - \frac{\lambda}{1+\lambda} w_j(y) - \frac{1}{1+\lambda} k \tag{7}$$

satisfies $V_i\left(w_i^{'}, w_j\right) \geq V_i(w)$.

Proof. Clearly $w_i^{'}$ satisfies (2) as an equality, rearrange it as:

$$\left(y_i - w_i^{'}(y)\right) = k + \lambda\left(w_i^{'}(y) + w_j(y)\right)$$

then let $(F, c) \in A^{\star}(w|\mathcal{A})$ for any $\mathcal{A} \supseteq \mathcal{A}_0$ and taking expectations one gets:

$$E_F\left[y_i - w_i^{'}(y)\right] \geq k + \lambda V_A\left(\left(w_i^{'}, w_j\right)|\mathcal{A}_0\right) \tag{8}$$

This applies to any (F, c) under any technology, so this guarantees a payoff for principal i.

Note that $w_i^{'}(y) \geq w_i(y)$ for all $y \in Y$ so the agent is always at least as well off under $w_i^{'}$. Then from equations (3) and (8):

$$E_F\left[y_i - w_i^{'}(y)\right] \geq k + \lambda V_A(w|\mathcal{A}_0) = V_i(w)$$

Since this holds for all $(F,c) \in A^{\star}(w|\mathcal{A})$, by Lemma 1:

$$V_i\left(\left(w_i^{'}, w_j\right)|\mathcal{A}\right) = \min_{F \in A^{\star}(w|\mathcal{A})} E_F\left[y_i - w_i^{'}(y)\right] \geq V_i(w)$$

Then $V_i(w)$ is a lower bound for $V_i\left(\left(w_i^{'}, w_j\right)|\mathcal{A}\right)$ under arbitrary $\mathcal{A} \supseteq \mathcal{A}_0$. Thus $V_i\left(w_i^{'}, w_j\right) \geq V_i(w)$ by definition. □

Lemma 4. *Let $\left(w_i^{'}, w_j\right)$ with $w_i^{'}$ an affine contract on y_i and w_j, there is an affine contract $w_i^{''}$ that does at least as well as $w_i^{'}$ for principal i: $V_i\left(w_i^{''}, w_j\right) \geq V_i\left(w_i^{'}, w_j\right)$, with strict inequality unless $\min_y w_i^{'}(y) = 0$.*

Proof. Note that by limited liability $\min_y w_i^{'}(y) \geq 0$ let $\beta = \min_y w_i^{'}(y)$ and $w_i^{''}(y) = w_i^{'}(y) - \beta$ which is a valid contract $\left(w_i^{''}(y) \geq 0\right)$ and is affine on y_i and w_j. Note that $A^{\star}\left(\left(w_i^{''}, w_j\right)|\mathcal{A}\right) = A^{\star}\left(\left(w_i^{'}, w_j\right)|\mathcal{A}\right)$ for all $\mathcal{A} \supseteq \mathcal{A}_0$. This implies $V_i\left(w_i^{''}, w_j\right) \geq V_i\left(w_i^{'}, w_j\right)$, with strict inequality if $\beta > 0$. □

The last two lemmas (5 and 6) establish the form of the principal's payoffs under LRS contracts and the existence of an optimal contract in that class.

Lemma 5. *For w an eligible contract scheme such that w_i is an LRS contract given w_j satisfying limited liability with equality. w_i is characterized by $\alpha \in (0,1]$. Then:*

$$V_i(w) = \frac{1-\alpha}{\alpha} V_A(w|\mathcal{A}_0) + k = \max_{(F,c) \in \mathcal{A}_0} \left((1-\alpha) E_F[y_i + w_j(y)] - \frac{1-\alpha}{\alpha} c\right) + \alpha k \tag{9}$$

This also holds for $\alpha = 0$ if we interpret the term $\frac{1-\alpha}{\alpha} c$ as 0 when $c = 0$ and ∞ for $c > 0$.

Proof. Let $F \in \underset{F \in \mathcal{F}}{\operatorname{argmin}} E_F[y_i - w_i(y)]$ by Lemma 1 one has:

$$V_i(w) = E_F[y_i - w_i(y)] = \frac{1-\alpha}{\alpha} E_F[w_1(y) + w_2(y)] + k = \frac{1-\alpha}{\alpha} V_A(w|\mathcal{A}_0) + k$$

The second equality follows by replacing $V_A(w|\mathcal{A}_0)$. □

Lemma 6. *In the class of LRS contracts that satisfy limited liability with equality there exists an optimal one for principal i.*

Proof. From Lemma 5 we can express $V_i(w)$ directly as a function of α as in (9). Recall that $k(\alpha) = \min_y \left[y_i - \frac{1-\alpha}{\alpha} w_j(y)\right]$ is function is continuous in α for a given w_j. Moreover, The function $(1-\alpha) E_F[y_i + w_j(y)] - \frac{1-\alpha}{\alpha} c$ is continuous in α, thus its maximum over $\mathcal{A}_0$ is continuous as well. Since the RHS in equation (9) is continuous in α it achieves a maximum in $[0,1]$. This α gives the optimal guarantee over all contracts of this class. $\square$

Theorem 1. *For any contract w_j there exists LRS contract $\overline{w}_i$ such that $\overline{w}_i \in BR_i(w_j)$, where $\min_y \bar{w}_i(y) = 0$. That is, there is always a LRS contract that is* ***robust*** *for principal i.*

Proof. By Lemma 6 among the class of LRS contracts there is an optimal one, call it $w_i^\star$. Suppose there is an arbitrary contract w_i that does strictly better than $w_i^\star$: $V_i(w_i, w_j) > V_i(w_i^\star, w_j)$. Note that $V_i(w_i^\star, w_j) \geq V_i(y_i, w_j) \geq 0$. Hence it must be the case that w_i is eligible, i.e. $V_i(w_{i,} w_j) > 0$. Then by Lemmas 2, 3 and 4 there exists an LRS contract $w_i^{'}$ such that $V_i\left(w_i^{'}, w_j\right) \geq V_i(w_i, w_j)$. This contradicts $w_i^\star$ being optimal among LRS contracts. $\square$

Corollary 1. *If $\mathcal{A}_0$ has the full support property then any robust contract for principal i is a LRS contract, or she cannot guarantee a positive payoff.*

Proof. Suppose w_i is an optimal contract for principal i and define $w_i^{'}$ as in Lemma 3. Note that $w_i^{'}$ satisfies:

$$E_F\left[y_i - w_i^{'}(y)\right] \geq k + \lambda V_A\left(\left(w_i^{'}, w_j\right) | \mathcal{A}_0\right)$$

Since w_i satisfies Equation (3) from Lemma 2 we can replace for k to obtain:

$$E_F\left[y_i - w_i^{'}(y)\right] \geq V_p(w) + \lambda\left(V_A\left(\left(w_i^{'}, w_j\right) | \mathcal{A}_0\right) - V_A\left((w_i, w_j) | \mathcal{A}_0\right)\right) \tag{10}$$

Because of full support, since $w_i^{'}(y) \geq w_i(y)$ pointwise and any action under $\mathcal{A}_0$ gives a (weakly) higher payoff to the agent under $w_i^{'}$ than under w_i, it follows that $V_A\left(\left(w_i^{'}, w_j\right) | \mathcal{A}_0\right) \geq V_A\left((w_i, w_j) | \mathcal{A}_0\right)$, with strict inequality unless $w_i^{'}$ is identical to w_i.

Since the equation (10) holds for all F we get: $V_i\left(w_i^{'}, w_j\right) \geq V_i\left(w\right)$, with strict inequality when w_i is not identical to $w_i^{'}$. Then $w_i = w_i^{'}$, or else optimality would be contradicted. Moreover, w_i has to be a LRS contract, or else by Lemma 4 there is a LRS contract that strictly improves on w_i. $\square$

1.1 Existence of Equilibrium - Examples

In the examples below we consider different specifications for f. In all cases we consider the problem of principal i when $w_j = (1-\theta_j) y_j + \theta_j (\overline{y}_i - y_i)$. As shown in Lemma 11 of Appendix A.2 in the main text, the corresponding LRS contract for principal i is: $w_i = (1-\theta_i) y_i + \theta_i \left(\overline{y}_j - y_j\right)$, where $\theta_i = (1-\alpha)(1-\theta_j)$.

Example 1. Constant marginal cost (linear cost)

To better understand the determinants of the share θ we consider the case where the agent's production technology exhibits constant marginal cost of production in total output. Note that if two actions have the same expected total output the agent will always pick the one with lower cost. Below we assume that this lowest cost is a constant fraction of total expected surplus. We formalize this notion in the following assumption

Assumption 1. *For any $x \in [0, \bar{y}_1 + \bar{y}_2]$ there exists $(F, c) \in \mathcal{A}_0$ such that $E_F[y_1 + y_2] = x$ and*

$$\gamma x = \min\left\{c \middle| (F, c) \in \mathcal{A}_0 \text{ and } E_F[y_1 + y_2] = x\right\}$$

where $\gamma < 1$ is the marginal cost.

Note that this allows to replace the maximization of the agent over $(F, c) \in \mathcal{A}_0$ with one over the expected value of total output $x \in [0, \bar{y}_1 + \bar{y}_2]$. Under Assumption 1 we can characterize the equilibrium strategies of the principals and the agent.

Proposition 2. *Under Assumption 1 if principal j plays the contract $w_j(y) = (1-\theta_j) y_j + \theta_j (\overline{y}_i - y_i)$ for some $\theta_j \in [0, 1]$, then principal i best responds with a contract of the form $w_i(y) = (1-\theta_i) y_i + \theta_i \left(\overline{y}_j - y_j\right)$ with:*

$$\theta_i = \begin{cases} (1-\theta_j) - \sqrt{(1-\theta_j)\,\gamma \frac{\overline{y}_1 + \overline{y}_2}{\overline{y}_i}} & \text{if } \theta_j < 1 - \gamma \frac{\overline{y}_1 + \overline{y}_2}{\overline{y}_i} \\ 0 & otw \end{cases} \tag{11}$$

Moreover, an equilibrium exists and in equilibrium , if the true technology is $\mathcal{A}_0$, the agent chooses (F, c) such that $E_F[y_1 + y_2] = \overline{y}_1 + \overline{y}_2$ and $c = \gamma(\overline{y}_1 + \overline{y}_2)$.

Proof. Under Assumption 1 the cost function has the form: $f(x) = \gamma x$ for some constant $\gamma > 0$. The value of the agent and his optimal action are:

$$V_A(w|\mathcal{A}_0) = \max_{x \in X} \{ \ ((1 - \theta_1 - \theta_2) - \gamma) x\} + \theta_1 \overline{y}_2 + \theta_2 \overline{y}_1 \qquad x^\star = \begin{cases} \overline{x} & \text{if } 1 - \theta_1 - \theta_2 > \gamma \\ 0 & \text{if } 1 - \theta_1 - \theta_2 < \gamma \\ X & \text{if } 1 - \theta_1 - \theta_2 = \gamma \end{cases}$$

□

Then the best response of principal i is characterized by:

$$\text{BR}_i(w_j) = \underset{\theta_i \in [0, 1-\theta_j]}{\text{argmax}} \left\{ \begin{cases} \theta_i \left(\overline{x} - \overline{y}_j\right) - \frac{\theta_i}{1-\theta_1-\theta_2} \gamma \overline{x} & \text{if } 1 - \theta_1 - \theta_2 > \gamma \\ -\theta_i \overline{y}_j & \text{if } 1 - \theta_1 - \theta_2 \leq \gamma \end{cases} \right\}$$

The function in the first case is strictly concave, its critical value if $\overline{x} > \overline{y}$ is given by:

$$\theta_i^\star = (1 - \theta_j) - \sqrt{\frac{(1 - \theta_j)\gamma \overline{x}}{\overline{x} - \overline{y}_j}}$$

This is an interior solution if:

$$1 - \theta_j - \theta_i^\star > \gamma \qquad \text{and} \qquad 0 \leq \theta_i^\star \leq (1 - \theta_j)$$

these conditions are satisfied if and only if: $\frac{\overline{x} - \overline{y}_j}{\overline{x}} > \frac{\gamma}{1-\theta_j}$. This condition amounts to there being enough expected output to pay for the cost of the agent and the fees. The best response of principal i is:

$$\mathrm{BR}_i\left(\theta_j\right)=\begin{cases}1-\theta_j-\sqrt{\frac{(1-\theta_j)\gamma\overline{x}}{\overline{x}-\overline{y}_j}} & \text{if} \quad (1-\theta_j)\left(\overline{x}-\overline{y}_j\right)>\gamma\overline{x}\\ 0 & \text{otw}\end{cases}$$

The best response of each principal is then single valued. As in Lemma 11 of Appendix A.2 in the main text, this implies the existence of an equilibrium.

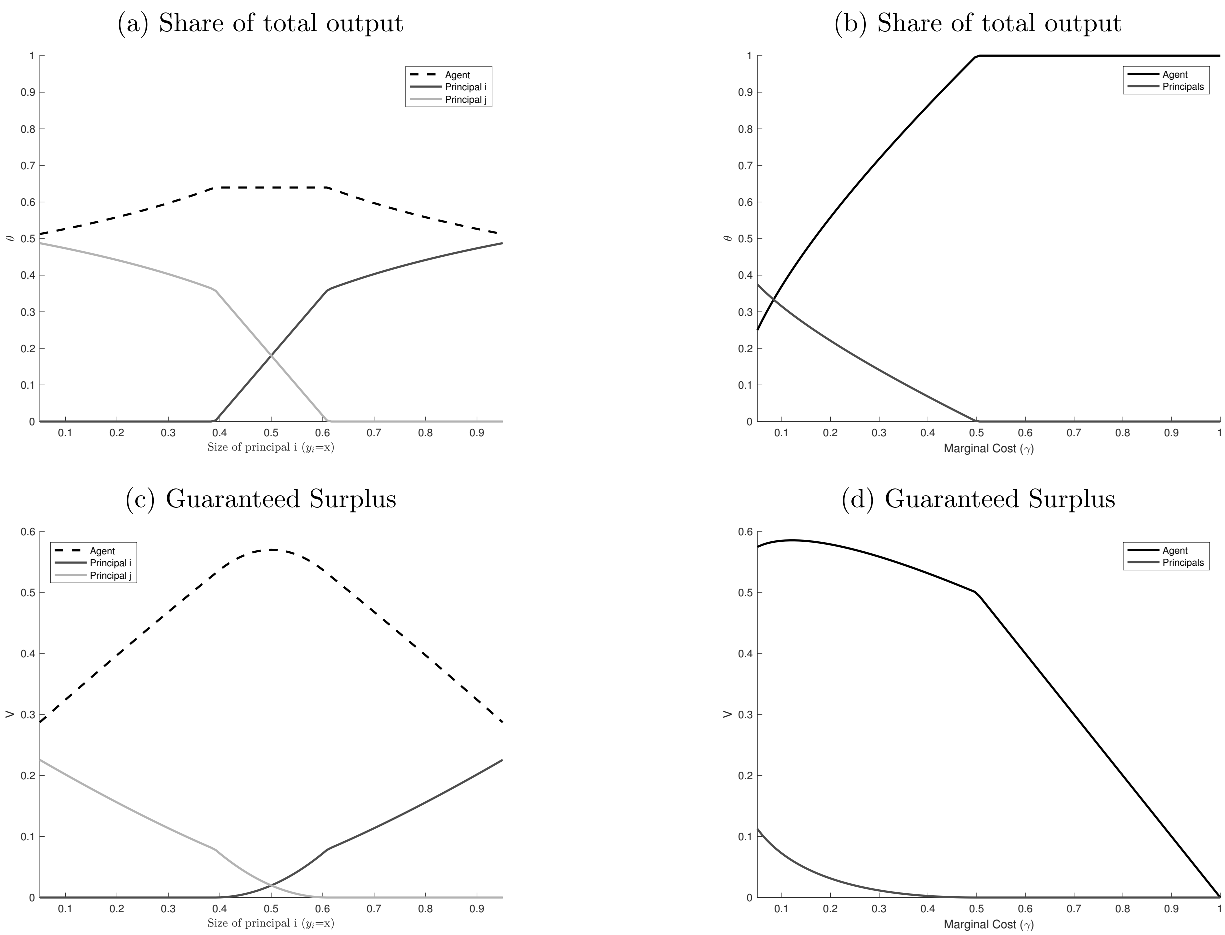
Recall that when $\theta_i=0$ principal i's guaranteed payoff, V_i, is zero as well. If this is the case in equilibrium we say that the principal has been driven out of the game. Effectively the principal renounces her output by setting $w_i\left(y\right)=y_i$. In particular, we see from equation (11) that if $\overline{y}_i<\gamma\left(\overline{y}_1+\overline{y}_2\right)$ the principal cannot guarantee herself a positive payoff, regardless of θ_j. For a principal to be able to profit in the game, she must be able to cover the (total) production cost of the agent. Clearly, when $w_i\left(y\right)=y_i$, the principal can always opt for the zero contract ($w_i\left(y\right)=0$). This is another way to opt out of the game since the principal cannot guarantee herself a positive payoff without incentivizing the agent.

We can now analyze the equilibrium contracts and payoffs for different values $(\overline{y}_1,\overline{y}_2)$ and γ. This allows for determining the effect of changes in competitor's size (Figures 1a, 1c) and productivity (Figures 1b, 1d) on the equilibrium outcomes.

The share of output that a principal can appropriate for herself decreases as her competitor becomes larger. Eventually if a principal is too small relative to her competition she is driven driven out of the game and cannot guarantee any positive payoffs. When a principal is relatively large she opts for increasing her share of total output, this lowers the share of the agent but increases his fees. As the productivity of the agent goes down (γ increases), higher incentives are needed to induce him to produce. This is achieved by reducing the share of output going to the principals. Eventually both principals end up giving up their output in equilibrium. In this case, competition drives their guaranteed payoffs to zero.

Example 2. Constant cost

Figure 1: Constant Marginal Cost of Production



The figures show the equilibrium of the game in LRS contracts under Assumption 1. We let $\bar{y}_1 = \overline{x}$ *and* $\bar{y}_2 = 1 - \overline{x}$*. Figures 1a and 1c vary* $\overline{x}$ *and fix* $\gamma = {}^1\!/_4$*. Figures 1b and 1d vary* γ *and fix* $\overline{x} = {}^1\!/_2$*.*

Assume now that the agent is indifferent between actions, so that $f(x) = \gamma$, with $\gamma > 0$, if $x > 0$, and $f(0) = 0$. This function is not convex. Since the agent's payoff under LRS contracts is increasing in expected total output, hence the agent will choose to induce the maximum expected total output, as long as it covers the cost γ.

$$x^{\star}(\theta_1, \theta_2) = \begin{cases} \overline{x} & \text{if} \quad (1-\theta_1-\theta_2)\,\overline{x} > \gamma \\ 0 & \text{otw} \end{cases}$$

Then the best response of principal i is characterized by:

$$\mathrm{BR}_i(w_j) = \underset{\theta_i \in [0, 1-\theta_j]}{\operatorname{argmax}} \left\{ \begin{cases} \theta_i\left(\overline{x} - \overline{y}_j\right) - \frac{\theta_i}{1-\theta_1-\theta_2}\gamma & \text{if } (1-\theta_1-\theta_2)\,\overline{x} > \gamma \\ -\theta_i \overline{y}_j & \text{if } (1-\theta_1-\theta_2)\,\overline{x} \le \gamma \end{cases} \right\}$$

The function in the first case is strictly concave, its critical value if $\overline{x} > \overline{y}$ is given by:

$$\theta_i^{\star} = (1-\theta_j) - \sqrt{\frac{(1-\theta_j)\,\gamma}{\overline{x} - \overline{y}_j}}$$

This is an interior solution if:

$$(1-\theta_j-\theta_i^{\star})\,\overline{x} > \gamma \qquad \text{and} \qquad 0 \le \theta_i^{\star} \le (1-\theta_j)$$

these conditions are satisfied if and only if: $\overline{x} - \overline{y}_j > \frac{\gamma}{1-\theta_j}$. This condition amounts to there being enough expected output to pay for the cost of the agent and the fees. The best response of principal i is:

$$\mathrm{BR}_i(\theta_j) = \begin{cases} 1-\theta_j - \sqrt{\frac{(1-\theta_j)\gamma}{\overline{x}-\overline{y}_j}} & \text{if} \quad (1-\theta_j)\left(\overline{x} - \overline{y}_j\right) > \gamma \\ 0 & \text{otw} \end{cases}$$

The best response of each principal is then single valued. As in Lemma 11 of Appendix

A.2 in the main text, this implies the existence of an equilibrium.

Example 3. Government contracting (first price auction)

Consider a setup where two competing firms bid for a government contract (such as a contract for the provision of services to the government, the construction of a public good, or the privatization of a government asset). The government announces that the contracting process has a fixed cost $c > 0$, and that the contract will be awarded with the objective of maximizing the government's profits. The cost of the contract can be interpreted as the social benefit of carrying out the project that the contract stipulates, or the valuation of a government asset that is being privatized. Both firms have their own valuation of the contract, we denote them by $\overline{y}_1$ and $\overline{y}_2$. We assume without loss that $\overline{y}_1 > \overline{y}_2 > c$.

If the government is known to be corrupt the firms would have reasons to doubt the announcement. For instance, the government can potentially (and secretly) favor one of the firms, or 'under the table' payments can make the cost vary depending on who is awarded the contract; it is also possible that the government is willing to randomize between the firms and lower the cost, this might be the case if bids are hard to assess and the government can lower costs at the expense of adding error to the contracting process, or if technicalities can arise that create the chance of a lower bid to be awarded the contract.[1]

The possible outcomes of the contracting process are that firm 1 is awarded the contract, firm 2 is awarded the contract, or the process is declared null and neither firm gets it. Note that, in a perfect information setting, this setup is that of a first price auction. The game would then have no solution since firm 1 would try to marginally outbid firm 2, leaving the bids undefined. Instead we show that there is a unique equilibrium in robust contracts for this game.

[1]Randomness in who is assigned the contract can also arise from last minute changes in the rules (not uncommon in developing countries), or from challenges made in courts to the rules or the decision of the government. It is worth pointing out that randomization is not itself necessary for our results. The firms could simply be worried that the government can allocate the good with certainty to the other contractor for a zero cost. This is in fact the worst case scenario they face.

To formalize the model consider a payoff space $Y = \{(0,0), (\overline{y}_1, 0), (0, \overline{y}_2)\}$ and a known set of actions for the agent given by:

$$\mathcal{A}_0 = \left\{(\delta_0, 0), \left(\delta_{\overline{y}_1}, c\right), \left(\delta_{\overline{y}_2}, c\right)\right\}$$

We now solve for the equilibrium when firms are allow to offer bids that depend who the contract is awarded to. LRS contracts will have the following form:

$$w_i = \begin{cases} \theta_i \overline{y}_j & \text{if } y = (0,0) \\ \overline{y}_i - \theta_i\left(\overline{y}_i - \overline{y}_j\right) & \text{if } y = (\overline{y}_i, 0) \\ 0 & \text{if } y = \left(0, \overline{y}_j\right) \end{cases}$$

Note that since $\theta_i \geq 0$ and $\overline{y}_j > 0$ it holds that $w_i\left(0, \overline{y}_j\right) \leq w_i(0,0) \leq w_i(\overline{y}_i, 0)$. That is, the principals always pay more if they win the auction, followed by no one winning and lastly if the auction is won by the other principal.

It is left to characterize the optimal actions of the government and the two firms, given the form of the contracts. The government's problem is:

$$V_A(w|\mathcal{A}_0) = \max\left\{\ \theta_i\overline{y}_j + \theta_j\overline{y}_i,\ \ (1-\theta_i)\,\overline{y}_i + \theta_i\overline{y}_j - c,\ \ (1-\theta_j)\,\overline{y}_j + \theta_j\overline{y}_i - c\right\}$$

Note that for any strategy of the firms (θ_1, θ_2) the government will either award the contract to the firm with the highest valuation (firm 1) or not award it at all.

This implies that the best response of firm 2 is to set $\theta_2 = 0$ or to offer the zero contract. In turn, this gives rise to two equilibria of the game. One in which firm 2 sets $\theta_2 = 0$, thus bidding $w_2(y) = y_2$, where firm 1 optimally sets

$$\theta_1 = \begin{cases} 1 - \sqrt{\frac{c}{\overline{y}_1 - \overline{y}_2}} & \text{if } c\left(\frac{\overline{y}_1 - \overline{y}_2}{\overline{y}_1}\right) < \overline{y}_1 \wedge \overline{y}_2 + c < \overline{y}_1 \\ 0 & \text{otw} \end{cases}$$

The condition above is just guaranteeing that the government will prefer awarding the contract to firm 1 over declaring null the process, and that the valuation of firm 1 is enough to pay the cost to the government and compensate for not awarding the contract to firm 2 (this ensures that $\theta_1 \geq 0$).

And another equilibrium in which firm 2 walks away from the bid, setting $w_2(y) = 0$, where firm 1 optimally sets $w_1(y) = (1-\theta_1)\, y_1$ with $\theta_1 = 1 - \sqrt{\frac{c}{\overline{y}_1}}$. For this to be an equilibrium it must be a best response by principal 2 to offer the zero contract when principal 1 offers this contract. That is the case when:

$$\overline{y}_2 < \sqrt{c\overline{y}_1}$$

If $\overline{y}_1 = \overline{y}_2$ then there are no eligible contracts for the firms, since the agent will be indifferent between them and neither firm can guarantee to be awarded the contract. Because of this the only equilibrium is for both of them to set $w_i(y) = y_i$. In all cases the government ends up awarding the contract to the firm with the highest valuation.

Example 4. Provision of public goods (Public vs private contracting domain)

Consider an agent that produces one unit of a public good with variable quality $q \in [0,1]$. The cost function of the agent depends on q and is given by $f(q) = \gamma q$. Each principals values the public good with $y_i = \nu_i q$. The output space is then:

$$Y = \left\{ (y_1, y_2) \in \mathbb{R}_+^2 | \exists_{q \in [0,1]} \quad y_1 = \upsilon_1 q \quad \wedge \quad y_2 = \upsilon_2 q \right\}$$

This abandons the assumption on the set Y being a cross product. Output is now assumed to be perfectly correlated across principals. This will only change the intercept of the LRS contract. The efficient outcome is of course to provide the good at highest quality if $\nu_1 + \nu_2 \geq \gamma$.

In this case there is no competition factor since output is perfectly correlated across principals. Each principal "partially" free rides on the other by lowering compensation by a fraction of the other principal's payoff. Moreover the agent optimally chooses to set $q = 1$. An interesting feature of this equilibrium is that no matter how different the valuations are, *all principals get the same share of expected output and the same guaranteed payoff. Moreover the agent picks the efficient action.*

Proposition 3. ***(Public good provision - Public common agency)*** *In the public common agency equilibrium both principals offer contracts of the form:*

$$w_i(y) = (1-\theta)\,y_i - \theta y_j \qquad \textit{where } \theta \textit{ is such that} \qquad \frac{1-\theta}{(1-2\theta)^2} = \frac{\nu_i+\nu_j}{\gamma}$$

if $\frac{(\nu_j-\nu_i)^2}{\max\{\nu_i,\nu_j\}} \leq \gamma \leq \nu_1 + \nu_2$.

Proof. We first note that the LRS contracts in equilibrium change because of our assumption on the output space. If principal j offers a contract $w_j = (1-\theta_j)\,y_j - \theta_j y_i$, then the LRS contract of principal i (given by (31)) is increasing in both y_i and y_j as long as:

$$(\alpha + (1-\alpha)\,\theta_j)\,\nu_i - (1-\alpha)\,(1-\theta_j)\,\nu_j \geq 0 \tag{12}$$

In this case the minimum is achieved when $y_i = y_j = 0$. This implies $k = 0$, and thus: $w_i = (1-\theta_i)\,y_i - \theta_i y_j$, with $\theta_i = (1-\alpha)\,(1-\theta_j)$ and no fees payed to the agent. Condition (12) is verified later.

The value of the agent is given by:

$$V_A(w|\mathcal{A}_0) = \max\{0 \quad , \quad (1-\theta_1-\theta_2)\,(\nu_1+\nu_2) - \gamma\}$$

The agent will choose either to induce the highest quality of not to produce at all.

The best response of principal i is then:

$$\mathrm{BR}_i(w_j) = \underset{\theta_i \in [0,1-\theta_j]}{\operatorname{argmax}} \left\{ \begin{cases} \theta_i(\nu_1+\nu_2) - \frac{\theta_i}{1-\theta_1-\theta_2}\gamma & \text{if } (1-\theta_1-\theta_2)(\nu_1+\nu_2) > \gamma \\ -\theta_i \overline{y}_j & \text{if } (1-\theta_1-\theta_2)(\nu_1+\nu_2) \leq \gamma \end{cases} \right\}$$

The interior solution assuming that the agent produces is given by:

$$\theta_i^\star = (1-\theta_j) - \sqrt{\frac{(1-\theta_j)\gamma}{\nu_1+\nu_2}}$$

Moreover, in equilibrium it must be that:

$$\frac{1-\theta_j}{(1-\theta_i-\theta_j)^2} = \frac{\nu_i+\nu_j}{\gamma} \qquad \wedge \qquad \frac{1-\theta_i}{(1-\theta_i-\theta_j)^2} = \frac{\nu_i+\nu_j}{\gamma}$$

which implies that $\theta_i = \theta_j = \theta$, where θ is such that: $\frac{1-\theta}{(1-2\theta)^2} = \frac{\nu_i+\nu_j}{\gamma}$. This characterizes the equilibrium contract. It is left to verify the assumptions, namely condition (12) which is satisfied if $\frac{(\nu_j-\nu_i)^2}{\max\{\nu_i,\nu_j\}} \leq \gamma$, and profitability of the agent ($(1-\theta_1-\theta_2)(\nu_1+\nu_2) > \gamma$), feasibility of the share θ $\left(0 \leq \theta \leq \frac{1}{2}\right)$ and profitability of the principals, which are always satisfied. $\square$

2 Multiple Principals

The model considered in the main text can be extended to multiple principals. Our main result is preserved in this case. Letting N be the number of principals we have:

$$\mathrm{BR}_i(w_{-i}) = \underset{w_i}{\operatorname{argmax}} V_i(w_i, w_{-i}) \tag{13}$$

where $w_{-i}(y) = (w_1(y), \ldots, w_{i-1}(y), w_{i+1}(y), \ldots, w_N(y))$.

Theorem 2. *For any set of contracts w_{-i}, there exists an LRS contract $\overline{w}_i$ such that $\overline{w}_i \in BR_i(w_j)$, where $\min_{y\in Y}\left\{\bar{w}_i(y) + \sum_{j\neq i} w_j(y)\right\} = 0$ or $\min_{y\in Y}\{\bar{w}_i(y)\} = 0$ according to limited liability. That is, there is always a LRS contract that is **robust** for principal i.*

If $\mathcal{A}_0$ satisfies the full support property, then any robust contract for principal i is a LRS contract or principal i cannot guarantee a payoff higher than $\sum_{j\neq i} w_j(0,0)$ or 0, according to limited liability.

Proof. The proof is virtually identical to that of Theorem 1 in the main text. Lemmas 22 to 27 follow by defining the aggregate competing contract $w^c(y) = \sum_{j\neq i} w_j(y)$. □

We can characterize them as in Propositions 1 and 3 of the main text, depending on the limited liability restrictions:

Proposition 4. *Let w be a LRS contract scheme satisfying limited liability with equality. There exist $(\theta_1, ..., \theta_N)$ and $(k_1, \ldots, k_N)$ such that the for all $i \in \{1, 2, \ldots, N\}$ contracts are:*

$$w_i(y) = (1-\theta_i)\, y_i - \theta_i \sum_{j\neq i} y_j - k_i \qquad \textit{Liminted Liability} \qquad (14)$$

$$w_i(y) = (1-\theta_i)\, y_i + \theta_i \sum_{j\neq i} \left(\overline{y}_j - y_j\right) \qquad \textit{Individual Limited Liability} \qquad (15)$$

where $\sum_{i=1}^{N} k_i = 0$ when limited liability is placed over the aggregate payment to the agent. θ_i is the share of total output and total guaranteed surplus going to principal i in equilibrium. Guaranteed surplus is computed relative to the payoffs under inaction.

Proof. By Theorem 2 there is a LRS contract in the best response of each principal. Then consider contracts of the following form for all i:

$$w_i(y) = y_i - \frac{1-\alpha_i}{\alpha_i} \sum_{j=1}^{n} w_j(y) - k_i$$

Letting $\beta_i = \frac{1-\alpha_i}{\alpha_i}$ we obtain the following expression for the sum of contracts:

$$\sum_{i=1}^{n} w_i(y) = \frac{\sum (y_i - k_i)}{1 + \sum \beta_i}$$

When limited liability is placed over the aggregate payment to the agent this implies $\sum\limits_{i=1}^{N} k_i = 0$. Replacing into the contract we get:

$$w_i(y) = y_i - \frac{\beta_i}{1+\sum \beta_i} \sum (y_i - k_i) - k_i$$

When limited liability applies to each individual contract it must be that $\min w_i(y) = 0$, the minimum is achieved when $y_i = 0$ and $y_j = \overline{y}_j$ for $j \neq i$, then one can solve for k_i:

$$k_i \quad = \quad -\frac{\beta_i}{1+\sum \beta_i} \sum_{j \neq i} \overline{y}_j + \frac{\beta_i}{1+\sum \beta_i} \left(\sum k_i \right)$$

Replacing one last time we get the equilibrium wage and defining $\theta_i = \frac{\beta_i}{1+\sum \beta_i}$:

$$w_i(y) = (1-\theta_i) y_i + \theta_i \sum_{j \neq i} \left(\overline{y}_j - y_j \right) \tag{16}$$

From Lemma 26 we can establish that the share of total guaranteed surplus going to principal i in equilibrium is equal to θ_i. To see this note from equilibrium contract, equation (16), that principal i's payoff given inaction is $-\frac{\beta_i}{1+\sum \beta_i} \sum_{j \neq i} \overline{y}_j$ and that total surplus given inaction is by construction zero. Then we have:

$$\theta_i = \frac{V_i(w) + \frac{\beta_i}{1+\sum \beta_i} \sum\limits_{j \neq i} \overline{y}_j}{\sum\limits_i V_i(w) + V_A(w|\mathcal{A}_0)} = \frac{\beta_1 V_A(w|\mathcal{A}_0) + k_1 + \frac{\beta_i}{1+\sum \beta_i} \sum\limits_{j \neq i} \overline{y}_j}{(1+\sum \beta_i) V_A(w|\mathcal{A}_0) + \sum k_i} = \frac{\beta_i}{1+\sum \beta_i}$$

□

In order to further characterize the equilibrium we first present the best response in LRS contracts of principal i, given LRS contracts played by the other principals, characterized by θ_{-i}:

$$\mathrm{BR}_i(\theta_{-i}) = \underset{\theta_i \in [0,1]}{\operatorname{argmax}} \left[\max_{(F,c) \in \mathcal{A}_0} \left\{ E_F \left[\theta_i \left(\sum y_i - \sum_{j \neq i} \overline{y}_j \right) \right] - \frac{c}{1-\sum \theta_i} \right\} \right] \tag{17}$$

From the FOC of the principal's problem we can obtain an expression for θ_i given θ_{-i} and a pair (F, c):

$$(1+\theta_i)\,\Gamma_i = \frac{1}{1-\sum \theta_i}$$

where $\Gamma_i = \frac{E_F[\sum y_i]-\sum_{j\neq i}\overline{y}_j}{c}$ and (F, c) are maximizers of $V_A(w)$. As before $\Gamma_i > 0$ it is necessary for the principal to have an interior solution. This implies that $\mathcal{A}_0$ must be such that there exists a pair (F, c) that satisfies:

$$E_F[y_i] > \sum_{j\neq i} E_F\left[\overline{y}_j - y_j\right]$$

This condition is stronger than non-triviality and increasingly difficult to satisfy as the number of principals increases.

To compute an interior equilibrium where a subset I of n_I principals have $\theta_i \in (0,1)$, and for all principal $k \notin I$ $\theta_i = 0$, note that the equilibrium condition above induces a linear system of $n_I - 1$ equations, holding $i \in I$ fixed these equations are of the form:

$$(1-\theta_j) = \frac{\Gamma_i}{\Gamma_j}(1-\theta_i)$$

for $j \in I$. Then we get:θ. For this to be an interior equilibrium $\theta_i \in (0,1)$ it is needed that:

$$\frac{1}{\Gamma_i^2} + \frac{1}{\Gamma_i}(n_I - 1) \leq \left(\sum_{i\in I} \frac{1}{\Gamma_i}\right)$$

To get a sense of this expression it is instructive to consider the case of a symmetric solution, then $\Gamma_i = \Gamma_j$ and the expression is reduced to $\Gamma_i \geq 1$.

3 Double Limited Liability

As mentioned before our equilibrium contracts require principals to pay a fee to the agent. This fee depends on the maximum potential payment that other principals can make, thus, in equilibrium, principals offer potentially large payments to the agent. These payments are motivated by competition among the principals. This form of competition can lead to solutions where both principals force each other to up their payments and reduce their final payoff. This has two practical implications: first, principals can have negative ex post payoffs; second, one principal can try to drive out the other by increasing her own payments to the agent. Both these implications can be dealt with by introducing limited liability on the principals. We show now that the core of our results does not rely on the principals to offer unbounded rewards to the agent.

Imposing limited liability on the principals amounts to restricting contracts so that $y_i - w_i(y) \geq 0$ for all $y \in Y$. Under this extra assumption only the definition of LRS contracts changes, adding a cap to the amount that the principal can pay to the agent.

Linear Revenue Sharing contracts (Principal's limited liability): A contract w_i is a LRS contract for principal i if, given a competing contract w_j, it ties the principal's ex-post payoff linearly to the total revenue of the agent. That is, for some $\alpha \in (0,1]$ and $k \in \mathbb{R}$:

$$y_i - w_i(y) = \min\left((1-\alpha)(y_i + w_j(y)) - \alpha k, 0\right) \tag{18}$$

The relation of the value of the principal and the agent (equation (7) in the paper) and Theorem 1 in the paper remain true as shown in detail below.

Consider a model with two principals $i \in \{1,2\}$ and one agent A, all risk neutral. The payoff space for the principals is $Y_1 \times Y_2 \subset \mathbb{R} \times \mathbb{R}$, it is assumed that Y_i is compact and that $\min\{Y_i\} = 0$. The agent has access to a technology $\mathcal{A} \subset \Delta(Y) \times \mathbb{R}_+$.An action is therefore a pair (F, c), where F is a probability distribution over payoffs $y = (y_1, y_2)$ and $c \geq 0$ is the cost of the action. $\Delta(Y)$ is endowed with the weak-$*$ topology and $\Delta(Y) \times \mathbb{R}$ with the natural product topology.

The game has two stages. First both principals offer a contract to the agent; this is done simultaneously and in a non-cooperative fashion. Second, the agent chooses an action in its technology set $\mathcal{A}$. Finally payments realize. The principals do not know $\mathcal{A}$, but they both know a subset $\mathcal{A}_0$ of $\mathcal{A}$. For now we assume that both principals know the same $\mathcal{A}_0$, but this assumption is not necessary for any of the results below. Only three other assumptions are placed on the set $\mathcal{A}_0$:

Non-triviality: $\exists_{(F,c)\in\mathcal{A}_0} E_F[y_1+y_2]-c>0$. This guarantees that the principals can benefit from hiring the agent.

Positive Cost: If $(F,c)\in\mathcal{A}\supseteq\mathcal{A}_0$ and $c=0$ then $F=\delta_0$, where δ_0 is the degenerate distribution on $y=(0,0)$. This implies that generating output requires some cost for any action in $\mathcal{A}_0$.

Full support: A technology $\mathcal{A}$ has the full support property if for all $(F,c)\in\mathcal{A}$ such that $(F,c)\neq(\delta_0,0)$, $\operatorname{supp}(F)=Y_1\times Y_2$.

Contracts: A contract by principal i is a continuous function $w_i : Y_1\times Y_2\to[0,\infty)$. In what follows let $w_i=w_i(y_1,y_2)$. **Contracts must also satisfy limited liability on the principals' side, i.e.** $y_i-w_i\geq 0$**.** A contract scheme is a vector of functions $w=(w_1,w_2)$.

Given a contract scheme and a technology $\mathcal{A}$, the agent will choose from the set of actions that maximize its expected payoff. The set of optimal actions and the value they give are:

$$A^{\star}(w|\mathcal{A})=\operatorname*{argmax}_{(F,c)\in\mathcal{A}} E_F[w_1+w_2]-c \qquad V_A(w|\mathcal{A})=\max_{(F,c)\in\mathcal{A}} E_F[w_1+w_2]-c \tag{19}$$

We define the value of a principal given a contract scheme w is given by the minimum payoff guarantee offered by the contract:

$$V_i(w)=\inf_{\mathcal{A}\supseteq\mathcal{A}_0} V_i(w|\mathcal{A}) \tag{20}$$

where $V_i(w|\mathcal{A})$ is the value for a given technology $\mathcal{A}$ that is is given by:

$$V_i(w|\mathcal{A}) = \min_{(F,c)\in A^{\star}(w|\mathcal{A})} E_F[y_i - w_i] \tag{21}$$

We restrict our attention to contracts that are eligible to a principal in the sense that they guarantee more that the trivial payoff 0. Formally:

Eligibility: A contract w is eligible for principal i if: $V_i(w) > 0$.

Finally we can define the best response of principal i to a contract w_j offered by principal j as:

$$\mathrm{BR}_i(w_j) = \underset{w_i}{\mathrm{argmax}} \quad V_i(w_i, w_j) \tag{22}$$

The following proposition relates the expected payments to the agent under any technology with its value under $\mathcal{A}_0$.

Proposition. *Let $(F,c) \in A^{\star}(w|\mathcal{A})$.For $\mathcal{A} \supseteq \mathcal{A}_0$, it holds that:*

$$E_F[w_1(y) + w_2(y)] \geq V_A(w|\mathcal{A}_0)$$

Moreover, if $(F,c) \in A^{\star}(w|\mathcal{A})$ then $F \in \mathcal{F}$ where:

$$\mathcal{F} = \{F \in \Delta(Y) \,|E_F[w_1 + w_2] \geq V_A(w|\mathcal{A}_0)\}$$

Proof. To see the first inequality let $(F,c) \in A^{\star}(w|\mathcal{A})$ for $\mathcal{A} \supseteq \mathcal{A}_0$:

$$E_F[w_1(y) + w_2(y)] \geq E_F[w_1(y) + w_2(y)] - c \geq V_A(w|\mathcal{A}) \geq V_A(w|\mathcal{A}_0)$$

Then $F \in \mathcal{F}$. $\square$

Lemma 7 characterizes the principal's payoff for a given contract scheme using the set $\mathcal{F}$ (defined in the proposition above). Note that $\mathcal{F}$ only depends on the contract and the known set of actions $\mathcal{A}_0$. In this way we replace the complexity of the definition of $V_i(w)$ in (20) with an object that depends only on known elements. The following results are valid for any scheme w that is eligible for principal i.

Lemma 7. *Let w be an eligible contract for principal i, then $V_i(w) = \min_{F \in \mathcal{F}} \quad E_F[y_i - w_i]$. Moreover if $F \in \underset{F\in\mathcal{F}}{argmin} E_F[y_i - w_i]$ then $E_F[w_1 + w_2] = V_A(w|\mathcal{A}_0)$.*

Proof. We first establish the first claim: Let w be an eligible contract scheme then: $V_i(w) = \min_{F\in\mathcal{F}} E_F[y_i - w_i(y)]$.

It must be that: $V_i(w) \geq \min_{F\in\mathcal{F}} E_F[y_i - w_i(y)]$. Using the definition of $V_i(w)$:

$$V_i(w) = \inf_{\mathcal{A}\supseteq\mathcal{A}_0} \min_{(F,c)\in A^\star(w|\mathcal{A})} E_F[y_i - w_i(y)] \geq \min_{F\in\mathcal{F}} E_F[y_i - w_i(y)]$$

Where the inequality follows because if $(F, c) \in A^\star(w|\mathcal{A})$ then $F \in \mathcal{F}$.

To prove equality suppose that $V_i(w) > \min_{F\in\mathcal{F}} E_F[y_i - w_i(y)]$, and let $F' \in \underset{F\in\mathcal{F}}{\operatorname{argmin}} E_F[y_i - w_i(y)]$. We have that $E_{F'}[w_1(y) + w_2(y)] \geq V_A(w|\mathcal{A}_0)$. There are two options:

1. F' does not place full support on the values of y that maximize $w_1 + w_2$.

   Let $\hat{y} \in \operatorname{argmax}\{w_1(y) + w_2(y)\}$, and $\hat{F} = \delta_{\hat{y}}$ be a distribution with full mass on $\hat{y}$. Let $\epsilon \in [0, 1]$ and $F_\epsilon = (1-\epsilon) F' + \epsilon\hat{F}$.

   Note then that for all ϵ there exists a $\xi_\epsilon > 0$ such that: $E_{F_\epsilon}[w_1(y) + w_2(y)] - \xi_\epsilon > V_A(w|\mathcal{A}_0)$.

   Define and $\mathcal{A}_\epsilon = \mathcal{A}_0 \cup \{(F_\epsilon, \xi_\epsilon)\}$. It follows that the unique optimal action of the agent in $\mathcal{A}_\xi$ is $(F_\epsilon, \xi_\epsilon)$. Then:

$$V_i(w) \leq V_i(w|\mathcal{A}_\epsilon) = E_{F_\epsilon}[y_i - w_i(y)] = (1-\epsilon) E_{F'}[y_i - w_i] + \epsilon E_{\hat{F}}[y_i - w_i]$$

This condition holds for all $\epsilon > 0$. Letting $\epsilon \to 0$ we arrive at a contradiction:

$$V_i(w) \leq E_{F'}[y_i - w_i] = \underset{F \in \mathcal{F}}{\text{argmin}} E_F[y_i - w_i]$$

2. $F^{'}$ places full support on the values of y that maximize $w_1 + w_2$. There are still two possible cases:

   (a) $E_{F'}[w_1 + w_2] > V_A(w|\mathcal{A}_0)$. Then there exists $\xi > 0$ and a technology $\mathcal{A}^{'} = \mathcal{A}_0 \cup \{(F^{'}, \xi)\}$ such that $(F^{'}, \xi)$ is the unique optimal action for the agent in $\mathcal{A}^{'}$. Then we arrive at a contradiction:

   $$V_i(w) \leq V_i\left(w|\mathcal{A}^{'}\right) = E_{F'}[y_i - w_i] = \underset{F \in \mathcal{F}}{\text{argmin}} E_F[y_i - w_i]$$

   (b) $E_{F'}[w_1 + w_2] = V_A(w|\mathcal{A}_0)$. This implies $V_A(w|\mathcal{A}_0) = \max_{y \in Y}\{w_1 + w_2\}$ which can only be satisfied if $F^{'}$ is available in $\mathcal{A}_0$ at zero cost. By the positive cost assumption this implies that $F = \delta_{(0,0)}$ and that $w_1(0,0) + w_2(0,0) = \max_{y \in Y}\{w_1(y) + w_2(y)\}$. In this case the unique optimal action for the agent under any technology is $(\delta_0, 0)$, so the value of the principal is $V_i(w) = -w_i(0,0) \leq 0$, where the inequality follows from limited liability. This contradicts eligibility.

Now we establish the second claim: Let w be an eligible contract scheme for principal i. If $F \in \underset{F \in \mathcal{F}}{\text{argmin}} E_F[y_i - w_i]$ then $E_F[w_1 + w_2] = V_A(w|\mathcal{A}_0)$.

To prove this, let $F^{'} \in \underset{F \in \mathcal{F}}{\text{argmin}} E_F[y_i - w_i(y)]$ and suppose for a contradiction that $E_F[w_1(y) + w_2(y)] > V_A(w|\mathcal{A}_0)$.

Let $\epsilon \in [0,1]$ and define $F_\epsilon = (1-\epsilon) F^{'} + \epsilon\delta_0$. For low enough ϵ it holds that: $E_{F_\epsilon}[w_1(y) + w_2(y)] > V_A(w|\mathcal{A}_0)$. Then there exists $\xi_\epsilon > 0$ such that $\{(F_\epsilon, \xi_\epsilon)\} = A^\star(w|\mathcal{A}_\epsilon)$ where $\mathcal{A}_\epsilon = \mathcal{A}_0 \cup \{(F_\epsilon, \xi_\epsilon)\}$. The payoff to the principal is then:

$$V_i(w|\mathcal{A}_\epsilon) = (1-\epsilon) E_F[y_i - w_i(y)] + \epsilon(-w_i(0,0)) \leq (1-\epsilon) E_F[y_i - w_i(y)] = (1-\epsilon) V_i(w) < V_i(w)$$

This gives a contradiction. $\square$

Given the known action set $\mathcal{A}_0$, the next lemma links the principal's guaranteed payoff to the agent's payoff in an affine way. This link allows the principal to increase her own guaranteed payoff by controlling the payoff given to the agent. Lemma 8 also offers a relation between any contract w_i, the outcome y_i and the contract w_j offered by the other principal.

Lemma 8. *Let w be an eligible contract. There exits k, λ with $\lambda > 0$ such that for all $y \in Y$:*

$$\begin{aligned} w_i(y) &\leq \frac{1}{1+\lambda} y_i - \frac{\lambda}{1+\lambda} w_j(y) - \frac{1}{1+\lambda} k && (23) \\ V_i(w) &= k + \lambda V_A(w|\mathcal{A}_0) && (24) \end{aligned}$$

Proof. This lemma is proven with the following two propositions. In both propositions define:

Let $S \subseteq \mathbb{R}^2$ be the convex hull of all points $(w_1(y) + w_2(y), y_i - w_i(y))$ for $y \in Y$.

Let $T \subseteq \mathbb{R}^2$ be the set of all pairs (u, v) such that $u > V_A(w|\mathcal{A}_0)$ and $v < V_i(w)$. Note T is convex.

$\square$

Proposition. $S \cap T = \emptyset$.

Proof. Let $(u, v) \in T$ then let $F \in \underset{F \in \mathcal{F}}{\operatorname{argmin}} E_F[y_i - w_i]$, by definition of T and Lemma (7):

$$\begin{aligned} u &> V_A(w|\mathcal{A}_0) = E_F[w_i + w_j] \\ v &< V_i(w) = E_F[y_i - w_i] \end{aligned}$$

now, suppose for a contradiction that $(u, v) \in S$, then there exists $F' \in \Delta(Y)$ such that:

$$\begin{aligned} u &= E_{F'}[w_i + w_j] \\ v &= E_{F'}[y_i - w_i] \end{aligned}$$

Note that $F^{'}$ guarantees a payoff to the agent larger than $V_A\left(w|\mathcal{A}_0\right)$ so $F^{'}\in\mathcal{F}$ but:

$$E_F\left[y_i-w_i\right]>E_{F'}\left[y_i-w_i\right]$$

which contradicts minimality of F. Then $S\cap T=\emptyset$

Since $S\cap T=\emptyset$ we can apply the separating hyperplane theorem which implies that there exist constants (k,λ,μ) such that $(\lambda,\mu)\neq(0,0)$ and:

$$k+\lambda u-\mu v\leq 0 \qquad (u,v)\in S \tag{25}$$

$$k+\lambda u-\mu v\geq 0 \qquad (u,v)\in T \tag{26}$$

Let $F^{\star}\in\underset{F\in\mathcal{F}}{\operatorname{argmin}}E_F\left[y_i-w_i\right]$. Note that the pair $\left(E_{F^{\star}}\left[w_1+w_2\right],E_{F^{\star}}\left[y_i-w_i\right]\right)$ lies in the closures of both S and T. Then:

$$k+\lambda E_{F^{\star}}\left[w_1+w_2\right]-\mu E_{F^{\star}}\left[y_i-w_i\right]=0 \tag{27}$$

It is left to show that $\lambda,\mu>0$.

Note that $(u,v)\in T$ admits u arbitrarily high and v arbitrarily low. So for (26) to hold it must be that $\lambda\geq 0$ and $\mu\geq 0$. There are then two cases to rule out:

□

1. Suppose $\mu=0$, then it must be that $\lambda>0$ (since $(\lambda,\mu)\neq(0,0)$). Also from (25) and (26)

$$u\leq-\frac{k}{\lambda} \qquad (u,v)\in S$$
$$u\geq-\frac{k}{\lambda} \qquad (u,v)\in T$$

So $\max\limits_{y\in Y}\left[w_1\left(y\right)+w_2\left(y\right)\right]=\max\limits_{u\in S_1}u\leq-\frac{k}{\lambda}\leq\inf\limits_{u\in T_1}u=V_A\left(w|\mathcal{A}_0\right)$. Which implies:

$$\max_{y\in Y}\left[w_1\left(y\right)+w_2\left(y\right)\right]=V_A\left(w|\mathcal{A}_0\right)$$

This can only happen if the agent has an action $(F,0)\in\mathcal{A}_0$ such that $E_F\left[w_1\left(y\right)+w_2\left(y\right)\right]=\max\left[w_1\left(y\right)+w_2\left(y\right)\right]$, the only action in $\mathcal{A}_0$ with zero cost is $(\delta_0,0)$, so $\max\left(w_1\left(y\right)+w_2\left(y\right)\right)=w_1\left(0,0\right)+w_2\left(0,0\right)$. This is also the unique action in $A^\star\left(w|\mathcal{A}_0\right)$ so:

$$V_i\left(w\right)\leq V_i\left(w|\mathcal{A}_0\right)=-w_i\left(0,0\right)\leq 0$$

This violates eligibility $\left(V_i\left(w\right)>0\right)$.

(a) Suppose $\lambda=0$, then it must be that $\mu>0$ (since $(\lambda,\mu)\neq(0,0)$). Also from (25) and (26)

$$\begin{aligned} v&\geq\frac{k}{\mu} & &(u,v)\in S\\ v&\leq\frac{k}{\mu} & &(u,v)\in T\end{aligned}$$

So $\min\limits_{y\in Y}\left[y_i-w_i\left(y\right)\right]=\min\limits_{v\in S_1}v\geq\frac{k}{\mu}\geq\sup\limits_{v\in T_1}v=V_i\left(w\right)$. But we know that $\min\limits_{y\in Y}\left[y_i-w_i\left(y\right)\right]\leq 0-w\left(0,0\right)\leq 0$ this implies $V_i\left(w\right)\leq 0$ which contradicts eligibility. So $\lambda>0$.

Proof. Note that since λ and μ are greater than zero μ can be normalized to 1, giving from (25):

$$k+\lambda\left(w_i\left(y\right)+w_j\left(y\right)\right)-\left(y_i-w_i\left(y\right)\right)\leq 0$$

And from (27):

$$V_i\left(w\right)=k+\lambda V_A\left(w|\mathcal{A}_0\right)$$

□

Corollary. *Let w be an eligible contract of where $w_i\left(y\right)=\min\left(\frac{1}{1+\lambda}y_i-\frac{\lambda}{1+\lambda}w_j\left(y\right)-\frac{1}{1+\lambda}k,y_i\right)$*

with $\lambda > 0$. *Then*

$$V_i(w) \quad = \quad k + \lambda V_A(w|\mathcal{A}_0) \tag{28}$$

Proof. Let $F^* \in \underset{F \in \mathcal{F}}{\operatorname{argmin}} E_F[y_i - w_i]$. By Lemma 7we have that

$$\begin{aligned} k + \lambda V_A(w|\mathcal{A}_0) - V_i(w) &= k + \lambda E_{F^\star}[w_1 + w_2] - E_{F^\star}[y_i - w_i] \\ &= k + (1+\lambda) E_{F^*}(w_i) + \lambda E_{F^*}(w_j) - E_{F^*}(y_i) \\ &= k + (1+\lambda) E_{F^*}\left(\min\left(\frac{1}{1+\lambda} y_i - \frac{\lambda}{1+\lambda} w_j(y) - \frac{1}{1+\lambda} k, y_i\right)\right) + \lambda E_{F^*}(w_j) - E \end{aligned}$$

Suppose for a contradiction that F^* places some positive probability $\delta > 0$ on a set $\bar{Y} \subset Y$ such that such that $\frac{1}{1+\lambda} y_i - \frac{\lambda}{1+\lambda} w_j(y) - \frac{1}{1+\lambda} k > y_i$ for $y \in \overline{Y}$.

This implies that $0 > \frac{\lambda y_i + \lambda w_j(y) + k}{1+\lambda} \implies \frac{-k}{\lambda} > y_i + w_j(y)$, where the RHS is the agents payment if output is y.

Now consider $\hat{y} \in Y$ for which $\frac{1}{1+\lambda}\hat{y}_i - \frac{\lambda}{1+\lambda} w_j(\hat{y}) - \frac{1}{1+\lambda} k = \hat{y}_i$. Then the agents payment at $\hat{y}$ is (computed in two ways)

$$\hat{y}_i + w_j(\hat{y}) = \frac{1}{1+\lambda}\hat{y}_i - \frac{\lambda}{1+\lambda} w_j(\hat{y}) - \frac{1}{1+\lambda} k + w_j(\hat{y})$$

Rearranging we get that

$$\begin{aligned} \frac{\lambda}{1+\lambda}\left(\hat{y}_i + w_j(\hat{y})\right) &= -\frac{1}{1+\lambda} k \\ \hat{y}_i + w_j(\hat{y}) &= -\frac{k}{\lambda} > y_i + w_j(y) \end{aligned}$$

Also it must be the case that F^* puts positive probability on a $\tilde{y} \in Y$ for which the payoff to principal i is positive (by eligibility).

Now consider F' that is the same as F^* but shifts all the weight δ in $\overline{Y}$ to $\hat{y}$. Then $E_{F'}(w_i + w_j) > V_A(w|\mathcal{A}_0)$.

Now consider F'' that is the same as F' but shifts a small but positive weight from $\tilde{y}$ to $\hat{y}$ such that we still have $E_{F''}(w_i + w_j) \geq V_A(w|\mathcal{A}_0)$. Note that $F'' \in \mathcal{F}$. But also the payoff to principal i under F'' is worse than that under F' and F^* which violates the minimality of F^*.

Hence F^* places full support on $y \in Y$ for which $\frac{1}{1+\lambda}y_i - \frac{\lambda}{1+\lambda}w_j(y) - \frac{1}{1+\lambda}k \leq y_i$. Then we have

$$
\begin{aligned}
k + \lambda V_A(w|\mathcal{A}_0) - V_i(w) &= k + \lambda E_{F^\star}[w_1 + w_2] - E_{F^\star}[y_i - w_i] \\
&= k + (1+\lambda) E_{F^\star}[w_i] + \lambda E_{F^\star}[w_2] - E_{F^\star}[y_i] \\
&= k + (1+\lambda) E_{F^\star}\left[\frac{1}{1+\lambda}y_i - \frac{\lambda}{1+\lambda}w_j(y) - \frac{1}{1+\lambda}k\right] + \lambda E_{F^\star}[w_2] - E_{F^\star}[y_i] \\
&= 0
\end{aligned}
$$

Rearranging

$$V_i(w) = k + \lambda V_A(w|\mathcal{A}_0)$$

□

The following two lemmas (9 and 10) use the relation between the principals' contracts derived in Lemma 8 to construct an alternative contract that dominates the original one, in the sense that it weakly increases principal i's guaranteed payoff. Since the relation obtained in (23) is affine in output and the other principal's contract, the alternative contract constructed below will inherit that form.

Lemma 9. *Let $w = (w_i, w_j)$ with w_i satisfying (23) and (24). Then the contract*

$$w_i^{'}(y) = \frac{1}{1+\lambda} y_i - \frac{\lambda}{1+\lambda} w_j(y) - \frac{1}{1+\lambda} k$$

satisfies $V_i\left(w_i^{'}, w_j\right) \geq V_i(w)$.

Proof. Clearly $w_i^{'} \leq \frac{1}{1+\lambda} y_i - \frac{\lambda}{1+\lambda} w_j(y) - \frac{1}{1+\lambda} k$, rearrange it as:

$$\left(y_i - w_i^{'}(y)\right) = k + \lambda\left(w_i^{'}(y) + w_j(y)\right)$$

then let $(F, c) \in A^{\star}(w|\mathcal{A})$ for any $\mathcal{A} \supseteq \mathcal{A}_0$ and taking expectations one gets:

$$E_F\left[y_i - w_i^{'}(y)\right] = k + \lambda E_F\left[w_i^{'}(y) + w_j(y)\right] \geq k + \lambda V_A\left(\left(w_i^{'}, w_j\right)|\mathcal{A}_0\right)$$

$$E_F\left[y_i - w_i^{'}(y)\right] \geq k + \lambda V_A\left(\left(w_i^{'}, w_j\right)|\mathcal{A}_0\right) \tag{29}$$

This applies to any optimal (F, c) under any technology, so this guarantees a payoff for principal i.

Note that $w_i^{'}(y) \geq w_i(y)$ for all $y \in Y$ so the agent is always at least as well off under $w_i^{'}$ and it doesn't violate the agent's limited liability . Then:

$$V_A\left(\left(w_i^{'}, w_j\right)|\mathcal{A}_0\right) \geq V_A(w|\mathcal{A}_0)$$

Joining with (29):

$$E_F\left[y_i - w_i^{'}(y)\right] \geq k + \lambda V_A(w|\mathcal{A}_0) = V_i(w)$$

Since this holds for all $(F, c) \in A^{\star}(w|\mathcal{A})$:

$$V_i\left(\left(w_i^{'}, w_j\right)|\mathcal{A}\right) = \min_{F \in A^{\star}(w|\mathcal{A})} E_F\left[y_i - w_i^{'}(y)\right] \geq V_i(w)$$

Finally:

$$V_i\left(w_i^{'}, w_j\right) = \inf_{\mathcal{A}\supseteq\mathcal{A}_0} V_i\left(\left(w_i^{'}, w_j\right)|\mathcal{A}\right) \geq V_i\left(w\right)$$

$\square$

Lemma 10. *Let* $\left(w_i^{'}, w_j\right)$ *with* $w_i^{'}$ *be the affine contract on* y_i *and* w_j *satisfying 9. There is an affine contract* $w_i^{''}$ *that does at least as well as* $w_i^{'}$ *for principal i:* $V_i\left(w_i^{''}, w_j\right) \geq V_i\left(w_i^{'}, w_j\right)$*, with strict inequality unless* $\min_y w_i^{'}\left(y\right) = 0$*.*

Proof. Note that by limited liability $\min_y w_i^{'}\left(y\right) \geq 0$ let $\beta = \min_y w_i^{'}$ and $w_i^{''}\left(y\right) = w_i^{'}\left(y\right) - \beta$ which is a valid contract $\left(w_i^{''}\left(y\right) \geq 0\right)$ and is affine on y_i and w_j. Note that $A^{\star}\left(\left(w_i^{''}, w_j\right)|\mathcal{A}\right) = A^{\star}\left(\left(w_i^{'}, w_j\right)|\mathcal{A}\right)$ for all $\mathcal{A} \supseteq \mathcal{A}_0$ since subtracting a constant doesn't change the agent's incentives. This implies $V_i\left(w_i^{''}, w_j\right) \geq V_i\left(w_i^{'}, w_j\right)$, with strict inequality if $\beta > 0$. $\square$

Lemma 11. *Let* $w^{''} = \left(w_i^{''}, w_j\right)$ *be the contract in 10and* w_i *satisfying (23) and (24). Then the contract*

$$\begin{aligned} w_i^{PLL}\left(y\right) &= \min\left(w_i^{''}, y_i\right) \\ &= \min\left(\frac{1}{1+\lambda}y_i - \frac{\lambda}{1+\lambda}w_j\left(y\right) - \frac{1}{1+\lambda}k^{''}, y_i\right) \end{aligned}$$

where $k^{''}$ *is such that* $\min_y w_i^{''}\left(y\right) = 0$ *satisfies* $V_i\left(w_i^{PLL}, w_j\right) \geq V_i\left(w\right)$*.*

Proof. First note that $k^{''} \geq k$. Also note that

$$w_i^{PLL}\left(y\right) \leq \frac{1}{1+\lambda}y_i - \frac{\lambda}{1+\lambda}w_j\left(y\right) - \frac{1}{1+\lambda}k^{''}$$

rearrange it as:

$$\left(y_i - w_i^{PLL}\left(y\right)\right) \geq k^{''} + \lambda\left(w_i^{PLL}\left(y\right) + w_j\left(y\right)\right)$$

then let $(F,c) \in A^{\star}(w|\mathcal{A})$ for any $\mathcal{A} \supseteq \mathcal{A}_0$ and taking expectations one gets:

$$E_F\left[y_i - w_i^{PLL}(y)\right] \geq k'' + \lambda E_F\left[w_i^{PLL}(y) + w_j(y)\right] \geq k'' + \lambda V_A\left(\left(w_i^{PLL}, w_j\right)|\mathcal{A}_0\right)$$

$$\begin{aligned} E_F\left[y_i - w_i^{PLL}(y)\right] &\geq k + \lambda V_A\left(\left(w_i^{PLL}, w_j\right)|\mathcal{A}_0\right) + \left(k'' - k\right) \\ &= k + \lambda V_A\left(\left(w_i^{PLL} + \frac{\left(k'' - k\right)}{\lambda}, w_j\right)|\mathcal{A}_0\right) \end{aligned} \tag{30}$$

This applies to any optimal (F,c) under any technology, so this guarantees a payoff for principal i.

Note that

$$\begin{aligned} w_i^{PLL} + \frac{\left(k'' - k\right)}{\lambda} &\geq \min\left(w_i'', y_i\right) + \frac{\left(k'' - k\right)}{\lambda} \\ &= \min\left(\frac{1}{1+\lambda}y_i - \frac{\lambda}{1+\lambda}w_j(y) - \frac{1}{1+\lambda}k'', y_i\right) + \frac{\left(k'' - k\right)}{\lambda} \\ &> \min\left(\frac{1}{1+\lambda}y_i - \frac{\lambda}{1+\lambda}w_j(y) - \frac{1}{1+\lambda}k'', y_i\right) + \frac{\left(k'' - k\right)}{1+\lambda} \\ &= \min\left(\frac{1}{1+\lambda}y_i - \frac{\lambda}{1+\lambda}w_j(y) - \frac{1}{1+\lambda}k'' + \frac{\left(k'' - k\right)}{1+\lambda}, y_i + \frac{\left(k'' - k\right)}{1+\lambda}\right) \\ &= \min\left(\frac{1}{1+\lambda}y_i - \frac{\lambda}{1+\lambda}w_j(y) - \frac{1}{1+\lambda}k, y_i + \frac{\left(k'' - k\right)}{1+\lambda}\right) \\ &= min\left(w_i', y_i + \frac{\left(k'' - k\right)}{1+\lambda}\right) \\ &\geq w_i \end{aligned}$$

for all $y \in Y$ because $w_i' \geq w_i$ and by since w_i satisfies principals limited liability then $w_i \leq y_i \leq y_i + \frac{(k''-k)}{1+\lambda}$.

So the agent is always at least as well off under $w_i^{PLL} + \frac{(k''-k)}{\lambda}$ as he was under w_i. Then:

$$V_A\left(\left(\left(w_i^{PLL} + \frac{\left(k'' - k\right)}{\lambda}, w_j\right), w_j\right)|\mathcal{A}_0\right) \geq V_A\left(w|\mathcal{A}_0\right)$$

Joining with (30):

$$E_F\left[y_i - w_i^{PLL}(y)\right] \geq k + \lambda V_A(w|\mathcal{A}_0) = V_i(w)$$

Since this holds for all $(F, c) \in A^\star(w|\mathcal{A})$:

$$V_i\left(\left(w_i^{PLL}, w_j\right)|\mathcal{A}\right) = \min_{F \in A^\star(w|\mathcal{A})} E_F\left[y_i - w_i^{PLL}(y)\right] \geq V_i(w)$$

Finally:

$$V_i\left(w_i^{PLL}, w_j\right) = \inf_{\mathcal{A} \supseteq \mathcal{A}_0} V_i\left(\left(w_i^{PLL}, w_j\right)|\mathcal{A}\right) \geq V_i(w)$$

□

Definition. Given a contract w_j, a contract w_i is an LRS contract if there exists $\alpha \in [0,1]$ and k such that:

$$w_i(y) = \min\left(\alpha y_i - (1-\alpha) w_j(y) - \alpha k, y_i\right) \tag{31}$$

and $\min_y \left(\alpha y_i - (1-\alpha) w_j(y) - \alpha k\right) = 0$.

For a given w_j let $\mathcal{W}_i(w_j)$ be the set of LRS contracts for principal i.

The last two lemmas (12 and 13) establish the form of the principal's payoffs under LRS contracts and the existence of an optimal contract in that class.

Lemma 12. *Let w be an eligible LRS contract scheme characterized by $\alpha \in (0,1]$, then:*

$$V_i(w) = \frac{1-\alpha}{\alpha} V_A(w|\mathcal{A}_0) + k$$

This also holds for $\alpha = 0$ if we interpret the term $\frac{1-\alpha}{\alpha} c$ as 0 when $c = 0$ and ∞ for $c > 0$.

Proof. This follows immediately by 3 by setting $\alpha = \frac{1}{1+\lambda}$. □

Lemma 13. *In the class of LRS contracts there exists an optimal one for principal i.*

Proof. First note that

$$
\begin{aligned}
V_A\left(w|\mathcal{A}_0\right) &= \max_{(F,c)\in\mathcal{A}_0} E_F\left[w_i\left(y\right)+w_j\left(y\right)-c\right] \\
&= \max_{(F,c)\in\mathcal{A}_0} E_F\left[\min\left(\alpha y_i-\left(1-\alpha\right)w_j\left(y\right)-\alpha k,y_i\right)+w_j\left(y\right)-c\right]
\end{aligned}
$$

is continuous in α.

The function $\min\left(\alpha y_i-\left(1-\alpha\right)w_j\left(y\right)-\alpha k,y_i\right)+w_j\left(y\right)-c$ is continuous in α and so by some functional analysis result it should be that $E_F\left[\min\left(\alpha y_i-\left(1-\alpha\right)w_j\left(y\right)-\alpha k,y_i\right)+w_j\left(y\right)-c\right]$ is also continuous in α , thus its maximum over $\mathcal{A}_0$ is continuous as well.

Recall that:

$$
k\left(\alpha\right)=\min_y\left[y_i-\frac{1-\alpha}{\alpha}w_j\left(y\right)\right]
$$

This function is continuous in α for a given continuous w_j.

This implies that

$$
\frac{1-\alpha}{\alpha}V_A\left(w|\mathcal{A}_0\right)+k
$$

is continuous in α hence it achieves a maximum in $[0,1]$. This α gives the optimal guarantee over all contracts of this class.

Now let $\alpha^*\in\arg\max_{\alpha\in[0,1]}\frac{1-\alpha}{\alpha}V_A\left(w|\mathcal{A}_0\right)+k$. If the LRS contract characterized by α^* is eligible then this contract is optimal in the class of LRS contracts. If not, then all LRS contracts provide a non-positive (and non-negative by PLL) guarantee for principal i. Hence any LRS contract provides zero guarantee and thus is optimal . $\square$

Theorem 3. *For any contract w_j there exists $\alpha\in[0,1]$ such that:*

$$
w_i\left(y\right)=\min\left(\alpha y_i-\left(1-\alpha\right)w_j\left(y\right)-\alpha k\left(\alpha\right),y_i\right)\qquad w_i\left(w_j\right)\in BR_i\left(w_j\right)
$$

where $k\left(\alpha\right)$ is such that $\min_y\left(\alpha y_i-\left(1-\alpha\right)w_j\left(y\right)-\alpha k\left(\alpha\right)\right)=0$. That is, there is a LRS contract in the best response of principal i.

Proof. By Lemma 13 among the class of LRS contracts there is an optimal one, call it $w_i^\star$. Suppose there is an arbitrary contract w_i that does strictly better than $w_i^\star$: $V_i(w_i, w_j) > V_i(w_i^\star, w_j)$. Note that $V_i(w_i^\star, w_j) \geq V_i(y_i, w_j) \geq 0$. Hence it must be the case that w_i is eligible, i.e. $V_i(w_{i,} w_j) > 0$. Then by Lemmas 8, 9 , 10, and (11) there exists a LRS contract $w_i^{'}$ such that $V_i\left(w_i^{'}, w_j\right) \geq V_i(w_i, w_j)$. This contradicts $w_i^\star$ being optimal among the LRS contracts. □

Corollary 2. *Suppose$\mathcal{A}_0$ has the full support property. For any given w_j for which there exists an eligible contract for principal i then, $BR_i(w_j) \subseteq \mathcal{W}_i(w_j)$, that is, any optimal contract for principal i is LRS.*

Proof. Suppose w_i is an optimal contract for principal i.

□

Define w_i^{PLL} as in Lemma 11. Note that from equation 30 for any $(F, c) \in A^\star(w|\mathcal{A})$ for any $\mathcal{A} \supseteq \mathcal{A}_0$ it satisfies:

$$E_F\left[y_i - w_i^{PLL}(y)\right] \geq k + \lambda V_A\left(\left(w_i^{PLL} + \frac{(k^{''} - k)}{\lambda}, w_j\right)|\mathcal{A}_0\right)$$

Also note that $k'' - k \geq 0$ as in the proof of Lemma11 Note that w_i satisfies Equation (24) from Lemma 8:

$$V_i(w) = k + \lambda V_A((w_i, w_j)|\mathcal{A}_0)$$

Replacing for k:

$$E_F\left[y_i - w_i^{PLL}(y)\right] \geq V_i(w) + \lambda\left(V_A\left(\left(w_i^{PLL} + \frac{(k^{''} - k)}{\lambda}, w_j\right)|\mathcal{A}_0\right) - V_A((w_i, w_j)|\mathcal{A}_0)\right)$$

Because of full support, since $w_i^{PLL} + \frac{(k^{''}-k)}{\lambda} \geq w_i(y)$ point wise and any action under $\mathcal{A}_0$ gives a (weakly) higher payoff to the agent under $w_i^{PLL} + \frac{(k^{''}-k)}{\lambda}$ than under w_i, it follows that

$V_A\left(\left(w_i^{PLL}+\frac{(k''-k)}{\lambda},w_j\right)|\mathcal{A}_0\right)\geq V_A\left((w_i,w_j)\,|\mathcal{A}_0\right)$, with strict inequality unless $w_i^{PLL}+\frac{(k''-k)}{\lambda}$ is identical to w_i.

Since the equation above holds for all optimal F under any technology it must be true that:

$$V_i\left(w_i^{PLL},w_j\right)\geq V_i\left(w\right)+\lambda\left(V_A\left(\left(w_i^{PLL}+\frac{(k''-k)}{\lambda},w_j\right)|\mathcal{A}_0\right)-V_A\left((w_i,w_j)\,|\mathcal{A}_0\right)\right)>V_i\left(w\right)$$

where the strict inequality follows when w_i is not identical to $w_i^{PLL}+\frac{(k''-k)}{\lambda}$.

Then $w_i=w_i^{PLL}+\frac{(k''-k)}{\lambda}$ (or else optimality would be contradicted).

It must be that w_i is LRS (i.e $\frac{(k''-k)}{\lambda}=0$) otherwise, w_i^{PLL} gives the principal a strictly greater guarantee compared to $w_i^{PLL}+\frac{(k''-k)}{\lambda}$ as the incentives of the agent are not changed and in w_i^{PLL} the limited liability for the agent binds.

Any optimal contract is LRS.

4 Private common agency

The following proposition relates the expected payments to the agent under any technology with its value under $\mathcal{A}_0$.

Proposition. *Let $(F, c) \in A^\star (w|\mathcal{A})$ for $\mathcal{A} \supseteq \mathcal{A}_0$, it holds that:*

$$E_F [w_1 (y_1) + w_2 (y_2)] \geq V_A (w|\mathcal{A}_0)$$

Moreover, if $(F, c) \in A^\star (w|\mathcal{A})$ then $F \in \mathcal{F}$ where:

$$\mathcal{F} = \{F \in \Delta (Y) | E_F [w_1 + w_2] \geq V_A (w|\mathcal{A}_0)\}$$

Proof. To see the first inequality let $(F, c) \in A^\star (w|\mathcal{A})$ for $\mathcal{A} \supseteq \mathcal{A}_0$:

$$E_F [w_1 (y_1) + w_2 (y_2)] \geq E_F [w_1 (y_1) + w_2 (y_2)] - c = V_A (w|\mathcal{A}) \geq V_A (w|\mathcal{A}_0)$$

Then $F \in \mathcal{F}$. $\square$

In this section we characterize the behavior of a principal facing extreme uncertainty on the set of actions of the agent. The principal acts taken as given the other principal's action. We proceed in a similar fashion as Carroll (2015) to establish the following lemmas.

Lemma 14 characterizes the principal's payoff for a given contract scheme using the set $\mathcal{F}$ (defined in the proposition above). Note that $\mathcal{F}$ only depends only on the contract and the known set of actions $\mathcal{A}_0$. In this way we replace the complexity of the definition of $V_i (w)$ in Equation (2) in the paper with an object that depends only on known elements. The following results are valid for any scheme w eligible for principal i.

Lemma 14. *Let w be an eligible contract for principal i, then $V_i (w) = \min_{F \in \mathcal{F}} E_F [y_i - w_i]$. Moreover if $F \in \underset{F \in \mathcal{F}}{argmin} E_F [y_i - w_i]$ then $E_F [w_1 + w_2] = V_A (w|\mathcal{A}_0)$.*

Proof. The proof is identical to that of Lemma (22). $\square$

Lemma 15 links the principal's payoff guarantee to the agent's payoff given the known action set $\mathcal{A}_0$ in an affine way. This link allows the principal to increase its own guaranteed payoff by controlling the payoff given to the agent. The lemma also offers a relation between any contract w_i, the outcome y_i and the contract w_j offered by the other principal.

The affine link between the agent's payoff and the principal's payoff is a crucial element in providing incentives for the agent. Given the lack of knowledge over the agent's set of actions the principals' optimal strategy is to tie their payoff to that of the agent, thus aligning the agent's objectives with their own. This is the same mechanism that lies at the center of Hurwicz & Shapiro (1978) and Carroll (2015) optimal contracts, and will be crucial to establish the optimality of affine contracts in the setting we develop.

Lemma 15. *Let w be an eligible contract. There exits k, λ with $\lambda > 0$ such that for all $y \in Y$:*

$$
\begin{aligned}
w_i\left(y_i\right) &\leq \frac{1}{1+\lambda} y_i-\frac{\lambda}{1+\lambda} \overline{w}_j-\frac{1}{1+\lambda} k && (32)\\
V_i\left(w\right) &= k+\lambda V_A\left(w \mid \mathcal{A}_0\right) && (33)
\end{aligned}
$$

where $\overline{w}_j=\max_{y_j \in Y_j} w_j\left(y_j\right)$.

Proof. This lemma is proven with the following two propositions. In both propositions define:

Let $S \subseteq \mathbb{R}^2$ be the convex hull of all points $\left(w_i\left(y_i\right)+\overline{w}_j, y_i-w_i\left(y_i\right)\right)$ for $y_i \in Y_i$ and $\overline{w}_j=\max_{y_j \in Y_j} w_j\left(y_j\right)$.

Let $T \subseteq \mathbb{R}^2$ be the set of all pairs (u, v) such that $u>V_A\left(w \mid \mathcal{A}_0\right)$ and $v<V_i(w)$. Note T is convex.

Proposition. $S \cap T=\emptyset$.

Proof. Let $(u,v) \in T$ then let $F \in \underset{F \in \mathcal{F}}{\text{argmin}} E_F[y_i - w_i]$, by definition of T and Lemma (14):

$$\begin{aligned} u &> V_A(w|\mathcal{A}_0) = E_F[w_i(y_i) + w_j(y_j)] \\ v &< V_i(w) = E_F[y_i - w_i(y_i)] \end{aligned}$$

now, suppose for a contradiction that $(u,v) \in S$, then there exists $F' \in \Delta(Y)$ such that:

$$\begin{aligned} u &= E_{F'}[w_i(y_i)] + \overline{w}_j \\ v &= E_{F'}[y_i - w_i(y_i)] \end{aligned}$$

Without loss F' is such that $E_{F'}[w_j(y_j)] = \overline{w}_j$.[2] Then:

$$u = E_{F'}[w_i(y_i) + w_j(y_j)] > V_A(w|\mathcal{A}_0)$$

That is, F' guarantees a payoff to the agent larger than $V_A(w|\mathcal{A}_0)$ so $F' \in \mathcal{F}$ but:

$$E_F[y_i - w_i] > E_{F'}[y_i - w_i]$$

which contradicts minimality of F. Then $S \cap T = \emptyset$ $\square$

Since $S \cap T = \emptyset$ we can apply the separating hyperplane theorem which implies that there exist constants (k, λ, μ) such that $(\lambda, \mu) \neq (0,0)$ and:

$$k + \lambda u - \mu v \leq 0 \qquad (u,v) \in S \tag{34}$$

$$k + \lambda u - \mu v \geq 0 \qquad (u,v) \in T \tag{35}$$

Let $F^\star \in \underset{F \in \mathcal{F}}{\text{argmin}} E_F[y_i - w_i(y_i)]$ such that $E_{F^\star}[w_j(y_j)] = \overline{w}_j$. This $F^\star$ always exists since

[2]This uses our assumption on the output space being of the form $Y = Y_1 \times Y_2$.

the objective function $E_F\left[y_i - w_i\left(y_i\right)\right]$ only depends on y_i, moreover, recall that

$$\mathcal{F} = \left\{F \in \Delta\left(Y\right) | E_F\left[w_1\left(y_1\right) + w_2\left(y_2\right)\right] \geq V_A\left(w|\mathcal{A}_0\right)\right\}$$

Then if $F \in \mathcal{F}$ the distribution $F^{\star}$ with the same marginal over y_i and full probability over $\overline{w}_j$ also belongs to $\mathcal{F}$.

Note that the pair $\left(E_{F^{\star}}\left[w_i\left(y_i\right) + w_j\left(y_j\right)\right], E_{F^{\star}}\left[y_i - w_i\left(y_i\right)\right]\right)$ lies in the closures of both S and T. Then:

$$k + \lambda E_{F^{\star}}\left[w_1 + w_2\right] - \mu E_{F^{\star}}\left[y_i - w_i\right] = 0 \tag{36}$$

It is left to show that $\lambda, \mu > 0$.

Note that $(u, v) \in T$ admits u arbitrarily high and v arbitrarily low. So for (35) to hold it must be that $\lambda \geq 0$ and $\mu \geq 0$. There are then two cases to rule out:

1. Suppose $\mu = 0$, then it must be that $\lambda > 0$ (since $(\lambda, \mu) \neq (0, 0)$). Also from (34) and (35)

$$\begin{aligned} u &\leq -\frac{k}{\lambda} \qquad (u, v) \in S \\ u &\geq -\frac{k}{\lambda} \qquad (u, v) \in T \end{aligned}$$

So $\max_{y_i \in Y_i}\left[w_i\left(y_i\right) + \overline{w}_j\right] = \max_{u \in S} u \leq -\frac{k}{\lambda} \leq \inf_{u \in T} u = V_A\left(w|\mathcal{A}_0\right)$. Which implies:

$$\max_{y_i \in Y_i}\left[w_i\left(y_i\right) + \overline{w}_j\right] = V_A\left(w|\mathcal{A}_0\right)$$

This can only happen if the agent has an action $(F, 0) \in \mathcal{A}_0$ such that $E_F\left[w_1\left(y_1\right) + w_2\left(y_2\right)\right] = \overline{w}_1 + \overline{w}_2$, by the positive cost assumption, the only action in $\mathcal{A}_0$ with zero cost is $(\delta_0, 0)$, so $\overline{w}_1 + \overline{w}_2 = w_1\left(0\right) + w_2\left(0\right)$. This is also the unique action in $A^{\star}\left(w|\mathcal{A}_0\right)$ so:

$$V_i\left(w\right) \leq V_i\left(w|\mathcal{A}_0\right) = -w_i\left(0\right) \leq 0$$

This violates eligibility $(V_i(w) > 0)$.

2. Suppose $\lambda = 0$, then it must be that $\mu > 0$ (since $(\lambda, \mu) \neq (0,0)$). Also from (34) and (35)

$$v \geq \frac{k}{\mu} \qquad (u,v) \in S$$
$$v \leq \frac{k}{\mu} \qquad (u,v) \in T$$

So $\min_{y_i \in Y_i} [y_i - w_i(y_i)] = \min_{v \in S} v \geq \frac{k}{\mu} \geq \sup_{v \in T} v = V_i(w)$. But we know that $\min_{y_i \in Y_i} [y_i - w_i(y_i)] \leq 0 - w(0) \leq 0$ this implies $V_i(w) \leq 0$ which contradicts eligibility. So $\lambda > 0$.

Note that since λ and μ are greater than zero μ can be normalized to 1, giving from (34):

$$k + \lambda (w_i(y_i) + \overline{w}_j) - (y_i - w_i(y_i)) \leq 0$$

And from (36) and Lemma (14):

$$V_i(w) = k + \lambda V_A(w|\mathcal{A}_0)$$

□

The following two lemmas (16 and 17) use the relation between the principals' contracts derived in the previous lemma to construct an alternative contract that dominates the original one in the sense that it guarantees a higher or equal payoff to principal i. Since the relation obtained in (32) is affine on output and the other principal's contract the alternative contract constructed below will inherit that form.

Lemma 16. *Let $w = (w_i, w_j)$ with w_i satisfying (32) and (33). Then the contract*

$$w_i^{'}(y_i) = \frac{1}{1+\lambda} y_i - \frac{\lambda}{1+\lambda} \overline{w}_j - \frac{1}{1+\lambda} k$$

satisfies $V_i\left(w_i^{'}, w_j\right) \geq V_i\left(w\right)$.

Proof. Clearly $w_i^{'}$ satisfies (32) as an equality, rearrange it as:

$$\left(y_i - w_i^{'}\left(y_i\right)\right) = k + \lambda\left(w_i^{'}\left(y_i\right) + \overline{w}_j\right)$$

then let $(F,c) \in A^{\star}\left(w|\mathcal{A}\right)$ for any $\mathcal{A} \supseteq \mathcal{A}_0$ and taking expectations one gets:

$$E_F\left[y_i - w_i^{'}\left(y_i\right)\right] = k+\lambda E_F\left[w_i^{'}\left(y_i\right) + \overline{w}_j\right] \geq k+\lambda E_F\left[w_i^{'}\left(y_i\right) + w_j\left(y_i\right)\right] \geq k+\lambda V_A\left(\left(w_i^{'}, w_j\right)|\mathcal{A}_0\right)$$

$$E_F\left[y_i - w_i^{'}\left(y_i\right)\right] \geq k + \lambda V_A\left(\left(w_i^{'}, w_j\right)|\mathcal{A}_0\right) \tag{37}$$

This applies to any (F,c) under any technology, so this guarantees a payoff for principal i.

Note that $w_i^{'}\left(y_i\right) \geq w_i\left(y_i\right)$ for all $y_i \in Y_i$ so the agent is always at least as well off under $w_i^{'}$ and it doesn't violate the agent's limited liability. Then:

$$V_A\left(\left(w_i^{'}, w_j\right)|\mathcal{A}_0\right) \geq V_A\left(w|\mathcal{A}_0\right)$$

Joining with (37):

$$E_F\left[y_i - w_i^{'}\left(y_i\right)\right] \geq k + \lambda V_A\left(w|\mathcal{A}_0\right) = V_i\left(w\right)$$

Since this holds for all $(F,c) \in A^{\star}\left(w|\mathcal{A}\right)$:

$$V_i\left(\left(w_i^{'}, w_j\right)|\mathcal{A}\right) = \min_{F \in A^{\star}(w|\mathcal{A})} E_F\left[y_i - w_i^{'}\left(y_i\right)\right] \geq V_i\left(w\right)$$

Finally:

$$V_i\left(w_i^{'}, w_j\right) = \inf_{\mathcal{A} \supseteq \mathcal{A}_0} V_i\left(\left(w_i^{'}, w_j\right)|\mathcal{A}\right) \geq V_i\left(w\right)$$

□

Lemma 17. *Let* $\left(w_i^{'}, w_j\right)$ *with* $w_i^{'}$ *an affine contract on* y_i, *there is an affine contract* $w_i^{''}$ *that does at lest as well as* $w_i^{'}$ *for principal* i: $V_i\left(w_i^{''}, w_j\right) \geq V_i\left(w_i^{'}, w_j\right)$, *with strict inequality* *unless* $\min_{y_i} w_i^{'}(y_i) = 0$.

Proof. Note that by limited liability $\beta = \min_{y_i} w_i^{'}(y_i) \geq 0$. Let $w_i^{''}(y) = w_i^{'}(y) - \beta$ which is a valid contract $\left(w_i^{''}(y) \geq 0\right)$ and is affine on y_i. Note that $A^{\star}\left(\left(w_i^{''}, w_j\right)|\mathcal{A}\right) = A^{\star}\left(\left(w_i^{'}, w_j\right)|\mathcal{A}\right)$ for all $\mathcal{A} \supseteq \mathcal{A}_0$ since subtracting a constant doesn't change the agent's incentives. This implies $V_i\left(w_i^{''}, w_j\right) \geq V_i\left(w_i^{'}, w_j\right)$, with strict inequality if $\beta > 0$. $\square$

The previous two lemmas show affine contracts weakly dominate any eligible contract. We will show that contracts that are linear in the principal's output improve on them.

Linear contracts: A contract w_i is linear, given a contract w_j, if:

$$w_i(y_i) = \alpha y_i$$

where $\alpha \in [0,1]$. Note that $\min_{y} \; w_i(y) = 0$ and that w_i does not depend on w_j.

Let $\mathcal{W}_i$ be the set of all linear contracts of principal i. Note that any eligible contract (w_i, w_j) can be (weakly) improved for principal i by a contract of the form $\left(w_i^{'}, w_j\right)$ where $w_i^{'} \in \mathcal{W}_i$.

The last two lemmas (18 and 19) establish the form of the principal's payoffs under linear contracts and the existence of an optimal contract in that class.

Lemma 18. *Let $F \in \underset{F\in\mathcal{F}}{argmin} E_F[y_i - w_i]$ for w an eligible contract scheme such that $w_i \in \mathcal{W}_i$ is characterized by $\alpha \in (0,1]$, then:*

$$V_i(w) = \frac{1-\alpha}{\alpha} V_A(w|\mathcal{A}_0) - \frac{1-\alpha}{\alpha}\overline{w}_j = \max_{(F,c)\in\mathcal{A}_0}\left(\frac{1-\alpha}{\alpha}\left(E_F\left[\alpha_i y_i - \left(\overline{w}_j - w_j(y)\right)\right] - c\right)\right)$$

This also holds for $\alpha = 0$ if we interpret the term $\frac{1-\alpha}{\alpha}c$ as 0 when $c = 0$ and ∞ for $c > 0$.

Proof. Let $F \in \underset{F\in\mathcal{F}}{\mathrm{argmin}} E_F[y_i - w_i(y_i)]$ such that $E_F[w_j(y_j)] = \overline{w}_j$, where $\overline{w}_j = \max_{y_j\in Y_j} w_j(y_j)$.

By Lemma 14 one has:

$$V_i(w) = E_F[y_i - w_i] \qquad E_F[w_1 + w_2] = V_A(w|\mathcal{A}_0)$$

Then

$$V_i(w) = \frac{1-\alpha}{\alpha} E_F\left[w_i(y_i) + w_j(y_j)\right] - \frac{1-\alpha}{\alpha} E_F\left[w_j(y_j)\right] = \frac{1-\alpha}{\alpha} V_A(w|\mathcal{A}_0) - \frac{1-\alpha}{\alpha}\overline{w}_j$$

Moreover:

$$\begin{aligned}
\frac{1-\alpha}{\alpha} V_A(w|\mathcal{A}_0) - \frac{1-\alpha}{\alpha}\overline{w}_j &= \frac{1-\alpha}{\alpha}\left(\max_{(F,c)\in\mathcal{A}_0} E_F\left[w_i(y_i) + w_j(y_j) - c\right]\right) - \frac{1-\alpha}{\alpha}\overline{w}_j \\
&= \frac{1-\alpha}{\alpha}\left(\max_{(F,c)\in\mathcal{A}_0} E_F\left[\alpha y_i + w_j(y_j) - c\right]\right) - \frac{1-\alpha}{\alpha}\overline{w}_j \\
&= \max_{(F,c)\in\mathcal{A}_0}\left(\frac{1-\alpha}{\alpha}\left(E_F\left[\alpha_i y_i - \left(\overline{w}_j - w_j(y)\right)\right] - c\right)\right)
\end{aligned}$$

□

Lemma 19. *In the class of linear contracts $w_i \in \mathcal{W}_i$ there exists an optimal one for principal i given contract w_j.*

Proof. From lemma (18) one gets that principal i's payoff is given by:

$$V_i(w) = \max_{(F,c)\in\mathcal{A}_0}\left(\frac{1-\alpha}{\alpha}\left(E_F\left[\alpha_i y_i - \left(\overline{w}_j - w_j(y)\right)\right] - c\right)\right)$$

The function $\frac{1-\alpha}{\alpha}\left(E_F\left[\alpha_i y_i - \left(\overline{w}_j - w_j(y)\right)\right] - c\right)$ is continuous in α, moreover it is also continuous in (F, c) (since w_j is a continuous function) and $\mathcal{A}_0$ is a compact set (constant with respect to α). Then V_i is continuous in α as well (by the Theorem of maximum).
Since the RHS is continuous in α it achieves a maximum in $[0, 1]$. This α gives the optimal guarantee over all contracts of this class. □

The lemmas above allow us to characterize the behavior of a principal, in particular they imply that it is always a best response to offer a linear contract, as shown in Theorem 4. The result can be strengthened under the full support property.

Theorem 4. *For any contract w_j there exists $\alpha \in [0,1]$ such that:*

$$w_i(y_i) = \alpha y_i \qquad w_i \in BR_i(w_j)$$

That is, there is a linear contract in the best response of principal i to any contract w_j.

Proof. By Lemma 19 among the class of linear contracts there is an optimal one, call it $w_i^\star$. Suppose there is an arbitrary contract w_i that does strictly better than $w_i^\star$: $V_i(w_i, w_j) > V_i(w_i^\star, w_j)$. By Lemmas 15, 16 and 17 there exists a linear contract $w_i^{'}$ such that $V_i\left(w_i^{'}, w_j\right) \geq V_i(w_i, w_j)$. This contradicts $w_i^\star$ being optimal among the linear contracts. $\square$

Corollary 3. *If $\mathcal{A}_0$ has the full support property then, for any w_j, $BR_i(w_j) \subseteq \mathcal{W}_i$, that is, any optimal contract for principal i is linear.*

Proof. Suppose w_i is an optimal contract for principal i.

- Define $w_i^{'}$ as in Lemma 16. $w_i^{'}$ satisfies:

$$E_F\left[y_i - w_i^{'}(y_i)\right] \geq k + \lambda V_A\left(\left(w_i^{'}, w_j\right)|\mathcal{A}_0\right)$$

Note that w_i satisfies Equation (33) from Lemma 15:

$$V_i(w) = k + \lambda V_A((w_i, w_j)|\mathcal{A}_0)$$

Replacing for k:

$$E_F\left[y_i - w_i^{'}(y_i)\right] \geq V_i(w) + \lambda\left(V_A\left(\left(w_i^{'}, w_j\right)|\mathcal{A}_0\right) - V_A((w_i, w_j)|\mathcal{A}_0)\right)$$

- Because of full support, since $w_i^{'}(y_i) \geq w_i(y_i)$ pointwise and any action under $\mathcal{A}_0$ gives a (weakly) higher payoff to the agent under $w_i^{'}$ than under w_i, it follows that $V_A\left(\left(w_i^{'}, w_j\right)|\mathcal{A}_0\right) \geq V_A((w_i, w_j)|\mathcal{A}_0)$, with strict inequality unless $w_i^{'}$ is identical to w_i.

- Since the equation above holds for all F it must be true that:

$$V_i\left(w_i^{'}, w_j\right) \geq V_i\left(w\right) + \lambda\left(V_A\left(\left(w_i^{'}, w_j\right)|\mathcal{A}_0\right) - V_A\left(\left(w_i, w_j\right)|\mathcal{A}_0\right)\right) > V_i\left(w\right)$$

  where the strict inequality follows when w_i is not identical to $w_i^{'}$.

- Then $w_i = w_i^{'}$ (or else optimality would be contradicted). Then w_i is linear in y_i.

- It must be that w_i is linear, or else by Lemma 17 there is a linear contract that strictly improves w_i.

- Any optimal contract is linear.

$\square$

5 Collusion

To prove this we first obtain versions of Lemmas 22 and 23 for the case of collusion. The results we obtain allow us to apply Carroll (2015)'s Theorem 1 to our collusion environment.

We begin by proving the following proposition relating the expected payments to the agent under any technology with its value under $\mathcal{A}_0$.

Proposition. *Let $(F,c) \in A^\star(w|\mathcal{A})$ for $\mathcal{A} \supseteq \mathcal{A}_0$, it holds that:*

$$E_F[w(y)] \geq V_A(w|\mathcal{A}_0)$$

Moreover, $A^\star(w|\mathcal{A}) \subseteq \mathcal{F}$ where:

$$\mathcal{F} = \{F \in \Delta(Y) \,|E_F[w(y)] \geq V_A(w|\mathcal{A}_0)\}$$

Proof. To see the first inequality let $(F,c) \in A^\star(w|\mathcal{A})$ for $\mathcal{A} \supseteq \mathcal{A}_0$:

$$E_F[w(y)] \geq E_F[w(y)] - c \geq V_A(w|\mathcal{A}) \geq V_A(w|\mathcal{A}_0)$$

Then $A^\star(w|\mathcal{A}) \subseteq \mathcal{F}$ follows from the definition of $\mathcal{F}$. □

Now we prove Lemmas 20 and 21 that will allow us to characterize the optimal contracts under collusion.

Lemma 20. *Let w be an eligible contract then:*

$$V_P(w) = \min_F E_F[y_1 + y_2 - w(y)] \qquad \textit{where } F \textit{ is s.t. } E_F[w(y)] \geq V_A(w|\mathcal{A}_0)$$

moreover if $F \in \underset{F\in\mathcal{F}}{argmin} E_F[y_1 + y_2 - w(y)]$ then $E_F[w(y)] = V_A(w|\mathcal{A}_0)$.

Proof. The proof of this is virtually identical to that of Lemma 22. □

Lemma 21. *Let w be an eligible contract. There exits k, λ with $\lambda > 0$ such that for all $y \in Y$:*

$$
\begin{aligned}
w(y) &\leq \frac{1}{1+\lambda}(y_1 + y_2) - \frac{1}{1+\lambda}k && (38) \\
V_P(w) &= k + \lambda V_A(w|\mathcal{A}_0) && (39)
\end{aligned}
$$

Proof. The proof of this is virtually identical to that of Lemma 23. □

With this we can use the framework developed in Carroll (2015) to obtain:

Theorem 5. *Under collusion there exists a contract that is linear on the sum of payoffs that maximizes V_P.*

$$w(y) = \alpha_c(y_1 + y_2)$$

Proof. This follows from Lemmas (20) and (21) along with Lemmas 2,4,5 and 6 in Carroll (2015), using the same argument as in his main theorem and replacing his y for $y_1 + y_2$. □

Corollary 4. *If $\mathcal{A}_0$ has the full support property then all optimal contracts are of the form:*

$$w(y) = \alpha_c(y_1 + y_2) \qquad \textit{where:} \quad \alpha_c = \sqrt{\frac{c^\star}{E_{F^\star}[y_1 + y_2]}}$$

for $(F^\star, c^\star) \in \underset{(F,c)\in\mathcal{A}_0}{\textit{argmax}} \left[\sqrt{E_F[y_1 + y_2]} - \sqrt{c}\right]^2$. *The payoff of the principals is:*

$$V_P(w) = \left[\sqrt{E_{F^\star}[y_1 + y_2]} - \sqrt{c^\star}\right]^2$$

Proof. Just as in Carroll (2015). □

6 Taxing Multinationals

6.1 Caring only about domestic profits

The following proposition relates the expected payments to the agent under any technology with its value under $\mathcal{A}_0$.

Proposition. *Let* $(F, c) \in A^{\star}(t_1, t_2|\mathcal{A})$*. For* $\mathcal{A} \supseteq \mathcal{A}_0$*, it holds that:*

$$E_F\left[y_1 - t_1(y) + y_2 - t_2(y)\right] \geq V_A(t_1, t_2|\mathcal{A}_0)$$

Moreover, if $(F, c) \in A^{\star}(t_1, t_2|\mathcal{A})$ *then* $F \in \mathcal{F}$ *where:*

$$\mathcal{F} = \{F \in \Delta(Y) \,|E_F\left[y_1 - t_1(y) + y_2 - t_2(y)\right] \geq V_A(t_1, t_2|\mathcal{A}_0)\}$$

Proof. To see the first inequality let $(F, c) \in A^{\star}(t_1, t_2|\mathcal{A})$ for $\mathcal{A} \supseteq \mathcal{A}_0$:

$$E_F\left[y_1 - t_1(y) + y_2 - t_2(y)\right] \geq E_F\left[y_1 - t_1(y) + y_2 - t_2(y)\right] - c \geq V_A(t|\mathcal{A}) \geq V_A(t|\mathcal{A}_0)$$

Then $F \in \mathcal{F}$. $\square$

Lemma 22 characterizes the principal's payoff for a given contract scheme using the set $\mathcal{F}$ (defined in the proposition above). Note that $\mathcal{F}$ only depends on the contract and the known set of actions $\mathcal{A}_0$. In this way we replace the complexity of the definition of $V_i(t)$ with an object that depends only on known elements. The following results are valid for any scheme t that provides positive guarantees for principal i

We formally define them as follows:

Eligibility: A contract t is *eligible* for principal i if: $V_i(t) > 0$.

Lemma 22. *Let* t *be an eligible contract for principal* i*, then* $V_i(t) = \min\limits_{F \in \mathcal{F}} \quad E_F\left[\rho y_i + t_i\right]$*. Moreover if* $F \in \underset{F \in \mathcal{F}}{argmin} E_F\left[\rho y_i + t_i(y)\right]$ *then* $E_F\left[y_1 - t_1(y) + y_2 - t_2(y)\right] = V_A(w|\mathcal{A}_0)$*.*

Proof. The proof is almost identical to that of 1 with the appropriate modification of the payoff function. □

Given the known action set $\mathcal{A}_0$, the next lemma links the principal's guaranteed payoff to the agent's payoff in an affine way. This link allows the principal to increase her own guaranteed payoff by controlling the payoff given to the agent. Lemma 23 also offers a relation between any contract t_i, the outcome (y_i, y_j) and the contract t_j offered by the other principal.

Lemma 23. *Let t be an eligible contract. There exits k, λ with $\lambda > 0$ such that for all $y \in Y$:*

$$t_i(y) \geq \frac{\lambda-\rho}{1+\lambda} y_i + \frac{\lambda}{1+\lambda} y_j - \frac{\lambda}{1+\lambda} t_j(y) + \frac{1}{1+\lambda} k \tag{40}$$

$$V_i(t) = k + \lambda V_A(t|\mathcal{A}_0) \tag{41}$$

Proof. Let $S \subseteq \mathbb{R}^2$ be the convex hull of all points $(y_1 - t_1(y) + y_2 - t_2(y), \rho y_i + t_i)$ for $y \in Y$.

Let $T \subseteq \mathbb{R}^2$ be the set of all pairs (u, v) such that $u > V_A(t|\mathcal{A}_0)$ and $v < V_i(t)$. The proof follows the steps of Lemma 2 □

The following two lemmas (24 and 25) use the relation between the principals' contracts derived in Lemma 23 to construct an alternative contract that dominates the original one, in the sense that it weakly increases principal i's guaranteed payoff. Since the relation obtained in (40) is affine in output and the other principal's contract, the alternative contract constructed below will inherit that form.

Lemma 24. *Let $t = (t_i, t_j)$ with t_i satisfying (40) and (41). Then the contract*

$$t_i'(y) = \frac{\lambda-\rho}{1+\lambda} y_i + \frac{\lambda}{1+\lambda} y_j - \frac{\lambda}{1+\lambda} t_j(y) + \frac{1}{1+\lambda} k$$

satisfies $V_i(t_i', t_j) \geq V_i(t)$.

Proof. The proof is identical to that of Lemma 3. □

Lemma 25. *Let* $\left(t_i^{'}, t_j\right)$ *with* $t_i^{'}$ *an affine contract on* y_i *,* y_j *and* t_j*, there is an affine contract* $t_i^{''}$ *that does at least as well as* $t_i^{'}$ *for principal i:* $V_i\left(t_i^{''}, t_j\right) \geq V_i\left(t_i^{'}, t_j\right)$*, with strict inequality unless* $\max_y t_i^{'}(y) = y_i$.

Proof. The proof is identical to that of Lemma 4. □

The last two lemmas (26 and 27) establish the form of the principal's payoffs under the worldwide taxes and the existence of an optimal contract in that class.

Lemma 26. *For* t *an eligible contract scheme such that* $t_i \in \mathcal{W}_i(t_j)$ *is characterized by* $\alpha \in (0, 1]$*, then:*

$$V_i(t) = \frac{1-\alpha}{\alpha} V_A(t|\mathcal{A}_0) + k = \max_{(F,c)\in\mathcal{A}_0} \left((1-\alpha) E_F[y_i - t_i(y) + y_j - t_j(y)] - \frac{1-\alpha}{\alpha} c \right) + \alpha k$$

This also holds for $\alpha = 0$ *if we interpret the term* $\frac{1-\alpha}{\alpha}c$ *as 0 when* $c = 0$ *and* ∞ *for* $c > 0$.

Proof. The proof is identical to that of Lemma 5. □

Lemma 27. *In the class of WT contracts* $w_i \in \mathcal{W}_i(w_j)$ *there exists an optimal one for principal* i.

Proof. The proof is identical to that of Lemma 6. □

Theorem 6. *For any contract* t_j *there exists* $\alpha \in [0, 1]$ *such that:*

$$t_i^{'}(y) = (1-\alpha-\alpha\rho) y_i + (1-\alpha) y_j - (1-\alpha) t_j(y) + \alpha k \qquad \text{and} \qquad t_i^{'}(t_j) \in BR_i(t_j)$$

where $k(\alpha)$ *is such that* $\min_y \left(y_i - t_i^{'}(y)\right) = 0$.

Proof. The proof is identical to that of Theorem 1 □

6.2 Welfare as a weighted sum of taxes and profits of the multinational

Lemma 28 characterizes the principal's payoff for a given contract scheme using the set $\mathcal{F}$ (defined in the proposition above).

Eligibility: A contract t is *eligible* for principal i if: $V_i(t) > 0$.

Lemma 28. *Let t be an eligible contract for principal i, then $V_i(t) = \min_{F\in\mathcal{F}} E_F[\rho(y_1 - t_1 + y_2 - t_2) + t_i]$. Moreover if $F \in \underset{F\in\mathcal{F}}{argmin} E_F[\rho(y_1 - t_1 + y_2 - t_2) + t_i(y)]$ then $E_F[y_1 - t_1(y) + y_2 - t_2(y)] = V_A(w|\mathcal{A}_0)$.*

Proof. The proof is identical to that of Lemma 1. $\square$

Lemma 29. *Let t be an eligible contract. There exits k, λ with $\lambda > 0$ such that for all $y \in Y$:*

$$\begin{aligned} t_i(y) &\geq \frac{\lambda-\rho}{1+\lambda-\rho}y_i + \frac{\lambda-\rho}{1+\lambda-\rho}y_j - \frac{\lambda-\rho}{1+\lambda-\rho}t_j(y) + \frac{1}{1+\lambda-\rho}k && (42)\\ V_i(t) &= k + \lambda V_A(t|\mathcal{A}_0) && (43)\end{aligned}$$

Proof. Let $S \subseteq \mathbb{R}^2$ be the convex hull of all points $(y_1 - t_1(y) + y_2 - t_2(y), \rho(y_1 - t_1 + y_2 - t_2) + t_i)$ for $y \in Y$.

Let $T \subseteq \mathbb{R}^2$ be the set of all pairs (u, v) such that $u > V_A(t|\mathcal{A}_0)$ and $v < V_i(t)$. Note T is convex. The rest follows as in the proof of Lemma 2. $\square$

Lemma 30. *Let $t = (t_i, t_j)$ with t_i satisfying (42) and (43). Then the contract*

$$t_i^{'}(y) = \frac{\lambda-\rho}{1+\lambda-\rho}y_i + \frac{\lambda-\rho}{1+\lambda-\rho}y_j - \frac{\lambda-\rho}{1+\lambda-\rho}t_j(y) + \frac{1}{1+\lambda-\rho}k$$

satisfies $V_i\left(t_i^{'}, t_j\right) \geq V_i(t)$.

Proof. The proof is identical to that of Lemma 3 $\square$

Lemma 31. *Let* $\left(t_i^{'}, t_j\right)$ *with* $t_i^{'}$ *an affine contract on* y_i , y_j *and* t_j, *there is an affine contract* $t_i^{''}$ *that does at least as well as* $t_i^{'}$ *for principal* i: $V_i\left(t_i^{''}, t_j\right) \geq V_i\left(t_i^{'}, t_j\right)$, *with strict inequality unless* $\max_y t_i^{'}(y) = y_i$.

Proof. The proof is identical to that of Lemma 4 $\square$

The last two lemmas (32 and 33) establish the form of the principal's payoffs under the worldwide taxes and the existence of an optimal contract in that class.

Lemma 32. *For* t *an eligible contract scheme such that* $t_i \in \mathcal{W}_i(t_j)$ *is characterized by* $\alpha \in (0, 1]$, *then:*

$$V_i(t) = \frac{1-\alpha}{\alpha} V_A(t|\mathcal{A}_0) + k = \max_{(F,c)\in\mathcal{A}_0} \left((1-\alpha) E_F\left[y_i - t_i(y) + y_j - t_j(y)\right] - \frac{1-\alpha}{\alpha} c \right) + \alpha k$$

This also holds for $\alpha = 0$ *if we interpret the term* $\frac{1-\alpha}{\alpha} c$ *as 0 when* $c = 0$ *and* ∞ *for* $c > 0$.

Proof. The proof is identical to that of Lemma 5. $\square$

Lemma 33. *In the class of WT contracts* $w_i \in \mathcal{W}_i(w_j)$ *there exists an optimal one for principal* i.

Proof. The proof is identical to that of Lemma 6. $\square$

Theorem 7. *For any contract* t_j *there exists* $\alpha \in [0, 1]$ *such that:*

$$t_i^{'}(y) = \frac{1-\alpha-\alpha\rho}{1-\alpha\rho}\left(y_i + y_j - t_j(y)\right) + \frac{\alpha}{1-\alpha\rho} k \qquad \textit{and} \qquad t_i^{'}(t_j) \in BR_i(t_j)$$

where $k(\alpha)$ *is such that* $\min_y \left(y_i - t_i^{'}(y)\right) = 0$.

Proof. The proof is identical to that of Theorem 1. $\square$

7 Lower bound on costs

The model allows for large amounts of output produced for free. The distribution that provides the worst case guarantee is one with zero cost. To rule this out we suppose that the principal knows a lower bound on the cost of producing any given level of expected output. In this section we prove that LRS contracts are a best response to LRS contracts when allowing for a lower bound on costs.

Let $b : \mathbb{R} \to \mathbb{R}_+$ be a convex function satisfying $b(0) = 0$. A technology is a compact set $\mathcal{A} \subset \Delta(Y) \times \mathbb{R}_+$ such that for any $(F, c) \in \mathcal{A}$ we have that $c \geq b(\mathbb{E}_F[y_1 + y_2])$. This holds also for any $(F, c) \in \mathcal{A}_0$ with a strict inequality (i.e. $c > b(\mathbb{E}_F(y))$ if $(F, c) \in \mathcal{A}_0$) This is similar to the positive cost assumption when there was no lower bound on costs. Now suppose that for all $V_i(w)$ is still the infimum of $V_i(w|\mathcal{A})$ over all technologies $\mathcal{A} \supset \mathcal{A}_0$. We furthermore assume that $\mathcal{A}_0$ satisfies the full support property.

The following proposition relates the expected payments to the agent under any technology with its value under $\mathcal{A}_0$.

Proposition 5. *Let $(F, c) \in A^\star(w|\mathcal{A})$. For $\mathcal{A} \supseteq \mathcal{A}_0$, it holds that:*

$$E_F[w_1(y) + w_2(y)] \geq V_A(w|\mathcal{A}_0) + b(E_F[y_1 + y_2])$$

Moreover, if $(F, c) \in A^\star(w|\mathcal{A})$ then $F \in \mathcal{F}$ where:

$$\mathcal{F} = \{F \in \Delta(Y) \,|E_F[w_1(y) + w_2(y)] \geq V_A(w|\mathcal{A}_0) + b(E_F[y_1 + y_2])\}$$

Proof. To see the first inequality let $(F, c) \in A^\star(w|\mathcal{A})$ for $\mathcal{A} \supseteq \mathcal{A}_0$:

$$E_F[w_1(y) + w_2(y)] - b(E_F[y_1 + y_2]) \geq E_F[w_1(y) + w_2(y)] - c \geq V_A(w|\mathcal{A}) \geq V_A(w|\mathcal{A}_0)$$

where the first inequality holds since $c \geq b(E_F[y_1 + y_2])$. Then $F \in \mathcal{F}$. $\square$

The following results are valid for any scheme w that provides positive guarantees for principal i

We formally define them as follows:

Eligibility: A contract w is *eligible* for principal i if: $V_i(w) > 0$.

Lemma 34. *Let w be an eligible contract for principal i, then $V_i(w) = \min\limits_{F\in\mathcal{F}} \quad E_F[y_i - w_i(y)]$. Moreover if $F \in \underset{F\in\mathcal{F}}{argmin} E_F[y_i - w_i(y)]$ then $E_F[w_1(y) + w_2(y)] = V_A(w|\mathcal{A}_0) + b(E_F[y_1 + y_2])$.*

Proof. The proof is broken into the following two propositions.

Proposition. *Let w be an eligible contract then: $V_i(w) = \min\limits_{F\in\mathcal{F}} E_F[y_i - w_i(y)]$*

Proof. First note that it must be that: $V_i(w) \geq \min\limits_{F\in\mathcal{F}} E_F[y_i - w_i(y)]$. Using the definition of $V_i(w)$:

$$V_i(w) = \inf_{\mathcal{A}\supseteq\mathcal{A}_0} \min_{(F,c)\in A^\star(w|\mathcal{A})} E_F[y_i - w_i(y)] \geq \min_{F\in\mathcal{F}} E_F[y_i - w_i(y)]$$

Where the inequality follows because if $(F, c) \in A^\star(w|\mathcal{A})$ then $F \in \mathcal{F}$.

Now we can establish equality. Suppose not, then it must be that: $V_i(w) > \min\limits_{F\in\mathcal{F}} E_F[y_i - w_i(y)]$.

Let $F \in \underset{F\in\mathcal{F}}{\operatorname{argmin}} E_F[y_i - w_i(y)]$.

We have that $E_F[w_1(y) + w_2(y)] \geq V_A(w|\mathcal{A}_0) + b(E_F[y_1 + y_2])$. Then consider the technology $\mathcal{A}' = \mathcal{A}_0 \cup \{(F, b(E_F[y_1 + y_2]))\}$. Then we have that $(F, b(E_F[y_1 + y_2])) \in A^\star\left(w|\mathcal{A}'\right)$, which implies that

$$V_i(w) \leq V_i\left(w|\mathcal{A}'\right) = \min_{(F,c)\in A^\star\left(w|\mathcal{A}'\right)} E_F[y_i - w_i(y)] \leq \min_{F\in\mathcal{F}} E_F[y_i - w_i].$$

□

Proposition. *Let w be an eligible contract for principal i. If $F \in \underset{F\in\mathcal{F}}{argmin} E_F[y_i - w_i(y)]$ then $E_F[w_1(y) + w_2(y)] = V_A(w|\mathcal{A}_0) + b(E_F[y_1 + y_2])$.*

Proof. To prove this, let $F \in \underset{F\in\mathcal{F}}{\operatorname{argmin}} E_F[y_i - w_i]$ and suppose for a contradiction that $E_F[w_1(y) + w_2(y)] > V_A(w|\mathcal{A}_0) + b(E_F[y_1 + y_2])$.

Let $\epsilon \in [0,1]$ and consider $F_\epsilon = (1-\epsilon) F + \epsilon \delta_{(0,0)}$ and $\mathcal{A}_\epsilon = \mathcal{A}_0 \cup \{(F_\epsilon, b(E_{F'}(y_1+y_2)))\}$. It follows that $\{(F_\epsilon, b(E_{F'}(y_1+y_2)))\} = A^\star(w|\mathcal{A}_\epsilon)$ for low enough ϵ since the payoff to the agent is strictly greater choosing F_ϵ at a cost of $b(E_{F'}(y_1+y_2))$, than choosing any $(F,c) \in \mathcal{A}_0$. Note that by convexity $b(E_{F_\epsilon}(y_1+y_2)) \leq (1-\epsilon) b(E_F[y_1+y_2]) + \epsilon b(0)$.

The payoff to the principal is then:

$$V_i\left(w|\mathcal{A}_\epsilon^{'}\right) = (1-\epsilon) E_F[y_i - w_i(y)] - \epsilon w_i(0,0) < E_F[y_i - w_i(y)] = V_i(w) \leq V_i\left(w|\mathcal{A}_\epsilon^{'}\right)$$

which is a contradiction with the definition of $V_i(w)$. The strict inequality follows from $E_F[y_i - w_i(y)] > 0$ by eligibility and $w_i(0,0) \geq 0$ by the agent's limited liability. □

Joining the two propositions the proof of the lemma is completed. □

Now suppose that principal j offers a contract of the form: $w_j(y) = (1-\theta_j) y_j + \theta_j(\bar{y}_i - y_i)$. And consider $w_i : Y \to \mathbb{R}_+$ so that (w_1, w_2) is an eligible contract scheme for principal i. Furthermore suppose that there does not exist $\theta_i \in [0, 1-\theta_j]$ and k such that $w_i(y_1,y_2) = (1-\theta_i) y_i + \theta_i(\bar{y}_j - y_j) + k$. Our objective is to show that in this case there exist an alternative contract $w_i^{'}$ that dominates w_i, where $w_i^{'}(y) = \left(1-\theta_i^{'}\right) y_i + \theta_i^{'}(\bar{y}_j - y_j)$ for some $\theta_i^{'} \in [0, 1-\theta_j]$.

The same separation argument as in the main theorem will follow. However the separation is done in outcome space and not in payoff space.

Define

$$t(x) = \max\{b(x) + V_A(w|\mathcal{A}_0), (1-\theta_j) x + \theta_j \bar{y}_i - V_i(w)\}$$

Clearly $t(x)$ is convex.

Now let $S \in \mathbb{R}^2$ be the convex hull of pairs $(y_1+y_2, w_i(y_1,y_2) + w_j(y_1,y_2))$ for all $(y_1,y_2) \in Y$, and let $T \in \mathbb{R}^2$ be the set of all pairs (x,z) such that x lies in the convex hull of points y_1+y_2, and $z > t(x)$[3]. Both of these sets are convex[4].

[3]Formally $T = \left\{(x,z) \in \mathbb{R}^2 | x \in [\min_Y\{y_1+y_2\}, \max_Y\{y_1+y_2\}] \quad \wedge \quad z > t(x)\right\}$.

[4]The first one is a convex hull, so it is convex, the second one is the intersection of the upper contour set

We claim that S and T are disjoint. If not then there exists $F \in \Delta Y$ such that $E_F \left[w_i \left(y_1, y_2 \right) + w_j \left(y_1, y_2 \right) \right] > t \left(E_F \left[y_1 + y_2 \right] \right)$. In particular we have that

$$E_F \left[w_i \left(y_1, y_2 \right) + w_j \left(y_1, y_2 \right) \right] > b \left(E_F \left[y_1 + y_2 \right] \right) + V_A \left(w | \mathcal{A}_0 \right)$$

Also we have that

$$E_F \left[w_i \left(y_1, y_2 \right) + w_j \left(y_1, y_2 \right) \right] > \left(1 - \theta_j \right) E_F \left[y_1 + y_2 \right] + \theta_j \bar{y}_i - V_i \left(w \right)$$

Replacing by $w_j \left(y \right) = \left(1 - \theta_j \right) y_j + \theta_j \left(\bar{y}_i - y_i \right)$ the second inequality becomes:

$$V_i \left(w \right) > E_F \left[y_i - w_i \left(y_1, y_2 \right) \right]$$

From Lemma 34 we know that $V_i \left(w \right) = \min_{F \in \mathcal{F}} E_F \left[y_i - w_i \left(y_1, y_2 \right) \right]$, but from the first inequality we know that $F \in \mathcal{F}$, this is a contradiction.

Then by the separating hyperplane theorem. There exists λ and μ and k with $\left(\lambda, \mu \right) \neq \left(0, 0 \right)$ such that

$$\begin{aligned} \lambda \left(y_1 + y_2 \right) + \mu z \leq k \qquad & \forall \left(\left(y_1 + y_2 \right), z \right) \in S \\ \lambda \left(y_1 + y_2 \right) + \mu z \geq k \qquad & \forall \left(\left(y_1 + y_2 \right), z \right) \in T \end{aligned} \tag{44}$$

The second inequality implies that $\mu \geq 0$. Now suppose $\mu = 0$ then it must be that $\lambda = 0$, which is a contradiction. This implies that $\mu > 0$.

Now

$$\lambda \left(y_1 + y_2 \right) + \mu z \leq k \qquad \forall \left(\left(y_1 + y_2 \right), z \right) \in S$$

implies that

$$w_i \left(y_1, y_2 \right) + w_j \left(y_1, y_2 \right) \leq \frac{k - \lambda \left(y_1 + y_2 \right)}{\mu}$$

of a convex function (a convex set) with two half spaces (convex sets), so it is convex as well.

Now consider the following wage

$$
\begin{aligned}
w_i^{'}(y_1, y_2) &= \frac{k - \lambda(y_1 + y_2)}{\mu} - w_j(y_1, y_2) \\
&= \theta_i^{'} y_i + \left(1 - \theta_i^{'}\right)(\overline{y}_j - y_j) + k^{'}
\end{aligned}
$$

where $\theta_i^{'} = \theta_j - \frac{\lambda}{\mu}$ and $k^{'} = \frac{k}{\mu} - \theta_j \bar{y}_i - \left(1 - \theta_i^{'}\right)\overline{y}_j$. Note that $w_i^{'} \geq w_i$ pointwise, and recall that $w_i \neq w_i^{'}$ by assumption. Now we need to check that $V_i\left(w_i^{'}\right) \geq V_i(w_i)$.

Consider any technology $\mathcal{A} \supset \mathcal{A}_0$. Then we have that $V_A\left(w^{'}|\mathcal{A}\right) \geq V_A\left(w^{'}|\mathcal{A}_0\right) > V_A(w|\mathcal{A}_0)$. The last inequality follows because $\mathcal{A}_0$ has full support and $w_i^{'}(y) > w_i(y)$ for some $y \in Y$.

Now let $(F, c) \in \mathcal{A}$ such that:

$$
(F, c) = \underset{(F,c) \in A^{\star}\left(w^{'}|\mathcal{A}\right)}{\arg\min} E_F\left[y_i - w_i^{'}(y)\right]
$$

Then $V_i\left(w^{'}|A\right) = E_F\left[y_i - w_i^{'}(y)\right]$. Now we have that from equation 44:

$$
\begin{aligned}
t(E_F[y_1 + y_2]) &\geq E_F\left(\frac{k - \lambda(y_1 + y_2)}{\mu}\right) \\
&= E_F\left[w_1^{'}(y) + w_2(y)\right] \\
&= V_A\left(w^{'}|\mathcal{A}\right) + c \\
&> V_A(w|\mathcal{A}_0) + c \\
&\geq V_A(w|\mathcal{A}_0) + b(E_F[y_1 + y_2])
\end{aligned}
$$

Since the inequality is strict then we have that $t(E_F[y_1 + y_2]) = (1 - \theta_j) E_F[y_1 + y_2] + \theta_j \bar{y}_i - V_i(w)$

Then we have that

$$
\begin{aligned}
V_i\left(w^{'}|A\right) &= E_F\left[y_i - w_i^{'}(y)\right] \\
&= E_F\left[y_i + w_j(y)\right] - E_F\left[w_i^{'}(y) + w_j(y)\right] \\
&= (1-\theta_j) E_F\left[y_1 + y_2\right] + \theta_j \bar{y}_i - E_F\left[w_i^{'}(y) + w_j(y)\right] \\
&= t\left(E_F\left[y_1 + y_2\right]\right) + V_i(w) - E_F\left[w_i^{'}(y) + w_j(y)\right] \\
&\geq V_i(w)
\end{aligned}
$$

Since this holds for all $\mathcal{A} \supset \mathcal{A}_0$. We get that $V_i\left(w^{'}\right) \geq V_i(w)$. So any contract w_i (as described above) can be dominated by a contract of the form:

$$
w_i^{'}(y_1, y_2) = \theta_i^{'} y_i + \left(1 - \theta_i^{'}\right)\left(\overline{y}_j - y_j\right) + k^{'}
$$

This contract can be improved upon by dropping the constant $k^{'}$. Doing so makes it satisfy limited liability with equality (when $y_i = 0$ and $y_j = \overline{y}_j$), it also does not affect the problem of the agent, and it weakly increase the value of the principal (strictly if $k' > 0$).

References

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