A Practitioners' Note on The Shapley-Owen-Shorrocks Decomposition*

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Abstract

Decomposing empirical or economic phenomena into the contributions of different inputs is a frequent goal of economic analysis. However, in many settings, the quantity of interest depends on many inputs which are aggregated non-linearly. In these settings, decompositions need not sum to one and often depend on the order in which inputs are "zero-ed out." In this note we describe a simple, but convenient alternative. We show that using the Shapley-Owen value, extended to inequality decompositions in Shorrocks (1999, 2013), provides an additive decomposition that sums to one and is easily interpretable in terms of the contribution of different inputs (or groups of them) to some aggregate outcome. We provide several examples to help implement the approach. We believe this is exceptionally well-suited to decompositions in rich-structural models of economic phenomena which are typically non-linear.

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In this short note, we aim to provide a simple overview of the Shapley-Owen-Shorrocks decomposition. The objective of the decomposition is to obtain values measuring the contribution of several inputs (or groups of them) to some aggregate outcome. When the outcome can be expressed as a linear function of inputs the decomposition is immediate: the value of of each input can be obtained by setting all other inputs to their baseline value, say zero. But, when the outcome is the product of a non-linear aggregation, as is the case in structural models of economic decision making, the order in which inputs are "zero-ed out" matters.

The decomposition, developed in Shorrocks (1999, 2013) for the study of inequality, uses the concept of the Shapley value to treat each input (a variable, policy function, or price) as a player and the outcome of interest as the surplus being generated by the inputs collective "action." In this way, the Shapley value of each input provides its *contribution*, adding up to the aggregate outcome. The same concept applies when a group of inputs moves together, as is the case when changing all prices or initial conditions in counterfactuals, following the generalization of Shapley (1953) in Owen (1977) to unions of players. We have found this decomposition useful in our own work, as well as suggesting to colleagues, collaborators, and students.

Despite being common in some areas of economics, or within some research networks, we believe it is an underused tool. We see this approach as providing an intuitive decomposition when there are non-trivial interactions between various factors, as is the case for virtually all economic phenomena. Our goal is to transparently define the decomposition and illustrate its uses through fully worked-out toy examples. We hope that this will contribute to wider use.

Before proceeding we provide a short and in-exhaustive list of the uses of the decomposition in economics research. It has been used in for decomposing the R^2 in regression analysis (Israeli 2007; Huettner and Sunder 2012); and to decompose the contribution of groups of regressors to the variance of earnings as in Allen and Arkolakis (2014); in structural models, it has been used to quantify the roles of changing labor market and demographic conditions for the take up of disability insurance in a structural life cycle model (Michaud and Wiczer 2018); in the literature on earnings dynamics, it has been used to decompose higher order moments of earnings changes, distinguishing the roles of employer and occupational switchers (Carrillo Tudela, Visschers, and Wiczer 2022), and the role of changes in earnings risk for homeownership (Paz Pardo 2024); in Hubmer, Halvorsen, Salgado, and Ozkan (2024) the decomposition is used in accounting for the roles of returns, savings, income, and inheritances in wealth accumulation; in Audoly, M$^{\text{c}}$Gee, Ocampo, and Paz Pardo (2024) we use it to decompose the explanatory power of groups of regressors in a multinomial logit model; in Athreya, Gordon, Jones, and Neelakantan (2025) it is used to decompose differences in wealth accumulation by race; in firm dynamics it has been used to account for drivers of output volatility (Kabir and Tan 2024); Kwon, Lee, and Pouliot (2024) highlight the relationship between the Shapley value and least squares and perform the decomposition on groups of inputs in the context of a Roy model; Millard (2025) decomposes the gap in post-secondary education between individuals with and without an early-onset disability.

1. The Decomposition

Given an arbitrary function $Y = f(X_1, X_2, ..., X_n)$, the Shapley-Owen-Shorrocks decomposition is a method to decompose the value of $f(\cdot)$ into each of its arguments $X_1, X_2, ..., X_n$. Intuitively, the contribution of each argument if it were to be *"removed"* from the function. However, because the function can be nonlinear, the order in which the arguments are removed matters in general for the decomposition. The function f can be the outcome of a regression, like the predicted values or sum of square residuals, or the output of a structural model, such as a counterfactual value for a variable given a list of model parameters or components, or a transformation of the sample, for example the Gini coefficient.

The Shapley-Owen-Shorrocks decomposition is the unique decomposition satisfying four important properties. ***(i)*** The decomposition is exact decomposition under addition, letting C_j denote the contribution of argument X_j to the value of the function $f(\cdot)$,

$$\sum_{j=1}^{n} C_j = f(X_1, X_2, ..., X_n), \tag{1}$$

so that $C_j/f(\cdot)$ can be interpreted as the proportion of $f(.)$ that can be attributed to X_j.[1] ***(ii)*** The decomposition is symmetric with respect to the order of the arguments. That is, the order in which the variable X_j is removed from $f(\cdot)$ does not alter the value of C_j. ***(iii)*** The decomposition assigns zero contribution to factors that have a null-effect (an irrelevance normalization). If a factor X_j never changes the outcome of the function, in

[1]The interpretation holds as long as f is non-negative. If f can take negative values, then the interpretation of C_j under the exact additive rule can be misleading as some arguments can have $C_j < 0$.

an abuse of notation $\partial_j f(\cdot) = 0$ everywhere, then $C_j = 0$. ***(iv)*** The decomposition says that the attribution operator is linear in the index you are decomposing. This is a useful closure requirement that implies contributions rescale linearly with a rescaling of the outcome function and are linear for combinations of outcomes.

The contribution of input X_j is then

$$C_j = \sum_{k=0}^{n-1} \frac{(n-k-1)!k!}{n!} \left(\sum_{s \subseteq S_k \setminus \{X_j\} : |s|=k} \left[f(s \cup X_j) - f(s) \right] \right), \tag{2}$$

where n is the total number of arguments in the original function f, $S_k \setminus \{X_j\}$ is the set of all *"sub-models"* that contain k arguments and exclude argument X_j.[2] For example,

$$\begin{aligned} S_{n-1} \setminus X_n &= f(X_1, X_2, \ldots, X_{n-1}) \\ S_1 \setminus X_n &= \{f(X_1), f(X_2), \ldots, f(X_{n-1})\}. \end{aligned}$$

The decomposition in (2) accounts for all possible permutations of the decomposition order. Thus, $\frac{(n-k-1)!k!}{n!}$ can be interpreted as the probability that one of the particular sub-model with k variables is randomly selected when all model sizes are all equally likely. For example, if $n = 3$, there are sub-models of size $\{0, 1, 2\}$. In particular, there are 2^2 permutation of models that exclude each variable: $\{\underbrace{(0,0)}_{k=0}, \underbrace{(1,0), (0,1)}_{k=1}, \underbrace{(1,1)}_{k=2}\}$.

$$\begin{aligned} k = 0 : \frac{(n-k-1)!k!}{n!} &= \frac{(3-0-1)!0!}{3!} = \frac{1}{3} \\ k = 1 : \frac{(n-k-1)!k!}{n!} &= \frac{(3-1-1)!1!}{3!} = \frac{1}{6} \\ k = 2 : \frac{(n-k-1)!k!}{n!} &= \frac{(3-2-1)!2!}{3!} = \frac{1}{3} \end{aligned}$$

We provide a simple implementation of the decomposition in Matlab at the end of the document and outline the algorithm below.

[2] We abuse notation here. A "*sub-model*" is an evaluation of function f with only some of its arguments. This language is motivated by the function corresponding in practice to the outcome of a regression or structural model. Formally, when we write $f(X_1)$, we mean $f(X_1, \emptyset_2, \ldots, \emptyset_n)$, where we assume the j-th argument of the function can always take on a null value denoted $\emptyset_j$. In our regression example below, this null value corresponds to a zero valued regressor or parameter. In the case of the structural model, this null value can correspond to setting some parameters to a predetermined value or excluding certain model components, like the adjustment of prices or a specific shock agents face.

Shapley-Owen-Shorrocks Algorithm

Inputs: Variable inputs and values across "sub-models."

(a) X: a binary $2^n \times n$ matrix. Gives inputs in "sub-models."

(b) F: a $n \times 1$ vector of model values.

The rows of X and F corresponding to "sub-models". A row of zeroes represents the reference value or null-model. A row of ones represent the full model with all inputs.

Output: C, a $n \times 1$ vector of input contributions.

Computing contribution for j^{th} input:

- Initialize contribution to 0, $C(j) = 0$.
- Loop over sub-model sizes, $k = 0, \ldots, n-1$.

  i Define weight for this class of sub-models $\omega_k = \frac{(n-k-1)!k!}{n!}$.

  ii Find sub-models $S_k \setminus \{X_j\}$:Rows of X with k inputs and without j.

  iii Loop over $s \in S_k \setminus \{X_j\}$ updating the contribution of input j:

$$C(j) = C(j) + \omega_k \times (F(s \cup \{j\}) - F(s))$$

2. Linear Example

We begin with the benchmark case of a linearly-aggregated outcome. In this case the natural decomposition is just the individual value of each input as they are already additive. We take this case to illustrate how the weighting scheme in (2) works.

Consider a linear model with 3 variables:

$$Y = f(X_1, X_2, X_3) = \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 \tag{3}$$

Consider the the partial effect of X_3 on Y. There are 4 possible models that exclude X_3, one with no variable, 2 with one variable and one with 2 variables

$$k = 0 : 0$$
$$k = 1 : \{\beta_0 + \beta_1 X_1, \beta_0 + \beta_2 X_2\}$$
$$k = 2 : \beta_0 + \beta_1 X_1 + \beta_2 X_2$$

In all 4 models, the partial effect of including X_3 is always $f(s \cup X_3) - f(s) = \beta_3 X_3 \quad \forall s$. Hence, linearity is such that the order in which variables are included to construct C_3 does not matter:

$$C_3 = \sum_{k=0}^{2} \frac{(n-k-1)!k!}{n!} \sum_{s \subseteq S_k \setminus \{X_3\}:|s|=k} \left[f(s \cup X_j) - f(s) \right]$$
$$= \sum_{k=0}^{2} \frac{(3-k-1)!k!}{3!} \sum_{s \subseteq S_k \setminus \{X_3\}:|s|=k} \beta_3 X_3 = \beta_3 X_3 \tag{4}$$

Notice that this would be the same if the original function took an arbitrary number of variables: $Y = f(X_1, X_2, \ldots, X_j) = \sum_{j=1}^{n} \beta_j X_j$. The only difference is that the number of sub-models grows exponentially: 2^{n-1}, but the partial effect of including X_j for some $j \in \{0, 1, \ldots, n\}$ is always $\beta_j X_j$.

Thus, in the linear case, the decomposition is mathematically identical to the usual regression decomposition. We average over the same object in each permutation because the effect does not depend on the order in this special case.

3. Nonlinear example I

We illustrate the value of this decomposition with a simple nonlinear model including $n = 3$ variables:

$$Y = f(X_1, X_2, X_3) = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 X_2. \tag{5}$$

The objective is to decompose the value of Y into the contribution (or partial effect) of each variable.

Removing X_1. There are four possible models that exclude X_1—one with no variable, two with one variable, and one with two variables:

$$k = 0 : \beta_0$$
$$k = 1 : \{\beta_0 + \beta_2 X_2\ ,\ \beta_0\}$$
$$k = 2 : \beta_0 + \beta_2 X_2 + \beta_3 X_3 X_2$$

In all four models, the partial effect of including X_1 is always $f(s \cup X_1) - f(s) = \beta_1 X_1$.

This reflects the fact that the order in which variables are included does not matter to construct C_1:

$$C_1 = \sum_{k=0}^{2} \frac{(3-k-1)!k!}{3!} \left(\sum_{s \subseteq S_k \setminus \{X_3\}:|s|=k} \left[f(s \cup X_j) - f(s) \right] \right) = \beta_1 X_1 \tag{6}$$

This would be the same for any argument X_j entering linearly into f an arbitrary number of variables: $Y = f(X_1, X_2, X_3, X_4, ..., X_n) = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 X_2 + \sum_{j=4}^{n} \beta_j X_j$. The only difference is that the number of sub-models grows exponentially, 2^{n-1}, but the partial effect of including X_j for some $j \in \{4, ..., n\}$ is always $C_j = \beta_j X_j$.

Removing X_2**.** In this case, the partial effect can be decomposed into all the possible ways X_2 can be added into the model, $f(s \cup X_2) - f(s)$, these are

$$\begin{aligned}
k = 0\ (\emptyset_1, \emptyset_3) &: \beta_0 + \beta_2 X_2 - \beta_0 = \beta_2 X_2 \\
k = 1\ (X_1, \emptyset_3) &: \beta_0 + \beta_1 X_1 + \beta_2 X_2 - (\beta_0 + \beta_1 X_1) = \beta_2 X_2 \\
k = 1\ (\emptyset_1, X_3) &: \beta_0 + \beta_2 X_2 + \beta_3 X_2 X_3 - \beta_0 = \beta_2 X_2 + \beta_3 X_2 X_3 \\
k = 2\ (X_1, X3) &: \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_2 X_3 - (\beta_0 + \beta_1 X_1) = \beta_2 X_2 + \beta_3 X_2 X_3
\end{aligned}$$

Here, the partial effects of adding X_2 are not the same across sub-models because X_2 enters nonlinearly into the original model. The symmetric property of the decomposition takes care of this.

$$\begin{aligned}
C_2 &= \underbrace{\frac{1}{3}\beta_2 X_2}_{k=0} + \underbrace{\frac{1}{6}(\beta_2 X_2) + \frac{1}{6}(\beta_2 X_2 + \beta_3 X_2 X_3)}_{k=1} + \underbrace{\frac{1}{3}(\beta_2 X_2 + \beta_3 X_2 X_3)}_{k=2} \\
&= \beta_2 X_2 + \frac{1}{2}\beta_3 X_2 X_3
\end{aligned} \tag{7}$$

The result is quite intuitive. $\beta_2 X_2$ appears in all sub-models; hence, its probability of appearing in the decomposition is 1. $\beta_3 X_2 X_3$ appears in two of the four sub-models; hence, its probability of appearing is 1/2. Weighting each term by its probability of appearing in the decomposition ensures symmetry.

Removing X_3**.** We proceed in the same way for X_3 as we did for X_2. There are four sub-models. In two of them, the effect of adding X_3 is null, because X_2 is not in the model. In the two remaining sub-models, the effect is $\beta_3 X_2 X_3$. Hence,

$$C_3 = \frac{1}{2}\beta_3 X_2 X_3. \tag{8}$$

Finally, we verify the decomposition:

$$\begin{aligned}
C_1 + C_2 + C_3 &= \beta_1 X_1 + \left(\beta_2 X_2 + \frac{1}{2}\beta_3 X_2 X_3\right) + \left(\frac{1}{2}\beta_3 X_2 X_3\right) \\
&= \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_2 X_3 \\
&= f(X_1, X_2, X_3) - \beta_0 \\
&= f(X_1, X_2, X_3) - f(\emptyset_1, \emptyset_2, \emptyset_3).
\end{aligned}$$

Reference value for the decomposition. The decomposition is additive with respect to the "null" model where none of the variables is included. This is made apparent in the previous result, where the decomposition does not include the value of β_0.

4. Nonlinear example II: $\mathbf{R^2}$

Finally, we consider a decomposition of the coefficient of determination in the linear model. Our own use of the decomposition applies this for a nonlinear model, combining the insights from this and the preceding example (see Audoly et al. 2024). We note that this application of the decomposition was first proposed by Israeli (2007) and Huettner and Sunder (2012). Recent applications include Nikolova and Cnossen (2020); Engstrom, Hersh, and Newhouse (2021); Biasi and Ma (2022); and Biasi, Lafortune, and Schönholzer (2025); among others.

Consider a linear regression model with n regressors and $i = 1, \ldots, M$ observations,

$$y_i = \mathbf{x}_i'\beta + u_i = \beta_0 + \sum_{j=1}^{n} \beta_j x_{ij} + u_i, \tag{9}$$

and define the average value of y as $\overline{y} \equiv \sum_{i=1}^{M} y_i/M$ and the predicted value

$$\hat{y}_i = \mathbf{x}_i'\hat{\beta} = \hat{\beta}_0 + \sum_{j=1}^{n} \hat{\beta}_j x_{ij}, \tag{10}$$

where we assume that all regressors have zero mean so that $\hat{\beta}_0 = \overline{y}$.

The function of interest is $f(X_1, \ldots, X_K) = R^2$, defined as the explained sum of squares SSE over the total sum of squares SST

$$R^2(X_1, X_2, .., X_n) = \frac{SSE}{SST} = \frac{\sum_{i=1}^{M}(\hat{y}_i - \overline{y})^2}{\sum_{i=1}^{M}(y_i - \overline{y})^2}. \tag{11}$$

This makes it clear that the function being decomposed is nonlinear even though the model that generates it is itself linear.

Reference value for the decomposition. The reference value for the R^2 in the Shapley-Owen-Shorrocks decomposition is given by the model without regressors, satisfying

$$R^2(\emptyset) = \frac{\sum_{i}^{M}(\hat{\beta}_0 - \overline{y})^2}{\sum_{i}^{M}(y_i - \overline{y})^2} = 0, \tag{12}$$

so that, in this case, the decomposition recovers the level of the R^2 of the full model (with all variables), unlike the previous example.

***Decomposition when* $\mathbf{n} = \mathbf{3}$.** We explicitly calculate the decomposition for $n = 3$ regressors. As before, we abuse notation by only listing the arguments being included in each sub-model. The contribution of each variable is:

$$\begin{aligned} R_1^2 = {} & \frac{1}{3}\left[R^2(X_1) - R^2(\emptyset)\right] + \frac{1}{6}\left(\left[R^2(X_1, X_2) - R^2(X_2)\right] + \left[R^2(X_1, X_3) - R^2(X_3)\right]\right) \\ & + \frac{1}{3}\left[R^2(X_1, X_2, X_3) - R^2(X_2, X_3)\right]; \end{aligned} \tag{13}$$

$$\begin{aligned} R_2^2 = {} & \frac{1}{3}\left[R^2(X_2) - R^2(\emptyset)\right] + \frac{1}{6}\left(\left[R^2(X_1, X_2) - R^2(X_1)\right] + \left[R^2(X_2, X_3) - R^2(X_3)\right]\right) \\ & + \frac{1}{3}\left[R^2(X_1, X_2, X_3) - R^2(X_1, X_3)\right]; \end{aligned} \tag{14}$$

$$\begin{aligned} R_3^2 = {} & \frac{1}{3}\left[R^2(X_3) - R^2(\emptyset)\right] + \frac{1}{6}\left(\left[R^2(X_3, X_2) - R^2(X_2)\right] + \left[R^2(X_1, X_3) - R^2(X_1)\right]\right) \\ & + \frac{1}{3}\left[R^2(X_1, X_2, X_3) - R^2(X_2, X_1)\right]. \end{aligned} \tag{15}$$

Summing across all the contributions we obtain back $R^2(X_1, X_2, X_3)$,

$$R_1^2 + R_2^2 + R_3^2 = R^2 = f(X_1, X_2, X_3). \tag{16}$$

Decomposing the R^2 versus the partial R^2. The value of the contribution differs from the standard definition of partial R^2. This is because the partial R^2 is an all-else-being-equal comparison of excluding regressor X_j. However, it is worth noting that the partial R^2 does not satisfy the exact decomposition requirement or (when applied iteratively) the symmetry requirement.

5. Summary

The Shapley-Owen-Shorrocks decomposition provides an effective alternative when studying non-linear outcomes that come from the interaction of various factors, as is the case in most economic applications. The decomposition is additive, symmetric with respect to factors, and it allows naturally for groups of factors that move together.

While the decomposition is already used by some authors, and in many contexts, there are still examples where it may be useful. We conclude by highlight some of these examples. De Nardi, French, Jones, and McGee (2025) decompose the contribution of various factors to retirement savings in a structural life cycle model, but do not provide an additive decomposition. Nakajima and Telyukova (2020) provide a decomposition under one order of elimination in their main text, but (commendably!) provide robustness to various alternative orders of elimination. Although the cost of implementing the Shapley-Owen-Shorrocks decomposition grows substantially with the number of factors, judiciously grouping them can help minimize costs. Additionally, alternative approaches also impose large computational burdens on researchers.

Finally, the decomposition can also be useful in the context of welfare decompositions from counterfactual exercises, like those in Flodén (2001), Conesa, Kitao, and Krueger (2009), and Moschini and Tran Xuan (2025) that separate the roles of changes in the aggregate level and distributions of consumption and leisure for welfare. The value of these decompositions depends on the order in which consumption and leisure are incorporated, limiting their interpretability. The symmetry of the Shapley-Owen-Shorrocks decomposition provides a viable alternative in this context at little additional cost.

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Shapley-Owen-Shorrocks Implementation in Matlab

```
function C = shapley_owen_shorrocks(X_ind, f_vals)
    % X_ind: binary matrix of size ((2^n) x n)
    % f_vals: 2^n vector of function values
    n       = size(X_ind, 2) ;  % Number of variables
    C       = zeros(n, 1)    ;  % Store contributions
    fact_vec = factorial(0:n); % Precompute factorial terms

    for j = 1:n % Loop over all inputs
      contrib = 0; % Initialize input j contribution

      for k = 0:n-1 % Loop over sub-models without input j
        % Weight for size-k sub-models--adj. by 1 for
           indexing
        w = fact_vec(n-k-1+1)*fact_vec(k+1)/fact_vec(n+1);
        % Find rows of X_ind with k inputs excluding j
        mask_k    = sum(X_ind, 2) == k     ; % k inputs
        mask_j    = X_ind(:, j) == 0       ; % Excluding j
        idx_base = find(mask_k & mask_j) ; % Both

        for idx = idx_base' % Loop over sub-models (k, no j)
          base     = X_ind(idx, :); % Reference sub-model
          base_val = f_vals(idx)   ; % Reference value

          % Add variable j to form S U {j}
          with_j = base;  with_j(j) = 1;
          % Find index of "with_j" sub-model and its value
          idx_with_j = find(ismember(X_ind, with_j, 'rows'));
              if isempty(idx_with_j);
              error('sub-model not found');
              end
          with_j_val = f_vals(idx_with_j);

          % Update contriution
          contrib = contrib + w*(with_j_val - base_val);
        end
    end
        C(j) = contrib; % Update output
    end
end
```