A Task-Based Theory of Occupations with Multidimensional Heterogeneity *

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November 8, 2018

Most Current Version

Abstract

I develop an assignment model of occupations with multidimensional heterogeneity in production tasks and worker skills. Tasks are distributed continuously in the skill space, whereas workers have a discrete distribution with a finite number of types. Occupations arise as bundles of tasks optimally assigned to a type of worker. The model allows us to study how occupations evolve—e.g., changes in their boundaries, wages, and employment—in response to changes in the economic environment, making it useful for analyzing the implications of automation, skill-biased technical change, offshoring, and skill upgrading by workers, among others. I characterize how the wages and marginal product of workers, the substitutability between worker types, and the labor share depend on the assignment. In particular, I show that these properties depend on the productivity of workers in tasks along the boundaries of their occupations. As an application, I study the rise in automation observed in recent decades. Automation is modeled as a choice of the optimal size and location of a mass of identical robots in the task space. The firm trades off the cost of the robots, which varies across the space, against the benefit of reducing the mismatch between tasks' skill requirements and workers' skills. The model rationalizes observed trends in automation and delivers implications for changes in wage inequality, unemployment, and the labor share.

JEL: J23, J24, J31, C78, E24

Key Words: Automation, occupations, assignment, skill mismatch

*I am most grateful to my advisors Fatih Guvenen, Jeremy Lise, Loukas Karabarbounis and David Rahman for their guidance and continuous encouragement. I also want to thank Gueorgui Kambourov, Ellen McGrattan, Jonathan Eaton, Burhan Kuruscu, David Wiczer, Anmol Bhandari, and participants at the Dissertation Interns Workshop at the St. Louis FED for their comments. I also benefitted from long discussions with my classmates, especially Dominic Smith, Keler Marku, Emily Moschini, Juan Herreño, Sergio Salgado, Serdar Birinci, and Kurt See. All errors are my own.

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1 Introduction

Occupational labels are one of the main ways in which we describe jobs. They play a useful role by summarizing the set of tasks performed by a worker. In this way, we know that a dentist will repair cavities, clean teeth and lecture patients about flossing, while a secretary manages schedules and mail among other tasks. Despite their usefulness, relying on occupations as descriptors of the tasks involved in a job can hide changes in the very nature of the job. Unlike the labels, the tasks actually involved in the job undergo continuous change. Dentists and secretaries perform different tasks today than they did just decades ago. Changes in the tasks performed in an occupation carry with them changes in the skills required from the worker, as well as changes in the worker's productivity and compensation.

I develop an assignment model of occupations with multidimensional heterogeneity in production tasks and worker skills (e.g. manual, cognitive, social, etc.). Occupations arise as bundles of productive tasks assigned to workers, in the spirit of Rosen (1978) and Acemoglu & Autor (2011). Tasks are characterized by a vector of skill requirements and are continuously distributed in the skill space, while workers have a discrete distribution with finitely many types. The boundaries of occupations (which define the bundle of tasks) are determined by the distribution of tasks and workers across the skill space, and the production technology. Tasks are assigned to workers so as to minimize the mismatch between the tasks' skill requirements and the workers' skills.

The model allows occupations to react endogenously to changes in the economic environment. Changes in the boundaries of occupations affect in turn the employment and wages of workers by changing their role in production. This makes the model useful for studying the implications of automation, skill-biased technical change, offshoring, and skill upgrading by workers, among others. The adoption of worker replacing technologies (like automation or offshoring) displaces workers from tasks, forcing a reassignment of tasks that changes the role of workers. This change is clear from current trends in manufacturing and clerical occupations.[1] Other forms of technical change, like the increased use of information technologies (IT) and computers in the workplace, affect occupations by changing the role of skills in production, making some skills more relevant for performing tasks and changing the type of worker that is best suited to perform them.[2]

[1] In manufacturing, Acemoglu & Restrepo (2017) estimate that industrial robots have displaced 756,000 workers between 1993 and 2007. Simultaneously, advances in software and AI have made it possible to automate tasks of clerical occupations and of more specialized workers like accountants.

[2] Changes in the skill content of occupations go beyond the adoption of IT by firms. See Atalay et al. (2018) and Deming (2017) for evidence of these changes.

The model makes precise how the marginal productivity of workers, wages and the elasticity of substitution across workers depend on the assignment. Intuitively, these properties depend on the productivity of workers in tasks along the boundaries of their occupations. In this way, shifts of the boundaries of an occupation have a direct effect over workers and output. To see this, consider the productivity of a worker across tasks in her occupation. Tasks along the boundary are those at which the worker is the least productive.[3] Because of that, these tasks are marginal, in the sense that they are the first tasks to be dropped off the occupation if the demand for the worker decreases, and the first ones to be added if the demand for the worker increases. The marginal product of the worker (and hence her wage) is thus determined by how productive the worker is at her boundary tasks; i.e. how much production increases if tasks along the boundary were reassigned to the worker. In a similar way, the elasticity of substitution is also determined by the occupation's boundaries, with a worker only being directly substitutable with her neighbors.

As an application, I use the model to study the rise in automation observed in recent decades. Automation, as other worker replacing technologies like offshoring, takes place at the task level. Because of this, automation takes away some, but not all, of the tasks of an occupation. In a recent study, McKinsey Global Institute (2017) reports that while 50% of tasks are automatable using currently available technology, less than 5% of occupations are fully automatable. Moreover, not all automation technologies are worth adopting, because adoption depends on the gains in productivity relative to the costs of the automation technology. Consequently, automation is more likely to transform rather than to eliminate occupations. Occupations are transformed directly by losing tasks to robots or software, and indirectly through the reassignment of tasks across workers.[4]

I model automation as a choice of the optimal size and location of a mass of identical robots in the task space. Robots replace workers at performing tasks. Automation in the model can be directed. The location of the robots in the task space determines which tasks are automated. The optimal choice of location and mass weights the cost of automation, which varies depending on the location, against the gains in output induced by replacing workers. Automation is thus directed towards regions that exhibit high mismatch between workers' skills and tasks' skills requirements, replacing workers at tasks for which they are

[3] At the same time, tasks along the boundary are the tasks at which the worker is the most productive among all tasks currently unassigned to her.

[4] Other worker replacing technologies, like offshoring, operate through the same effects. See Blinder (2009) and Blinder & Krueger (2013) for offshorability measures based on occupational characteristics.

ill-suited.

As mentioned above, automation induces a reassignment across tasks. Because of this, the workers previously performing the automated tasks are not the only ones affected. It is optimal to reassign tasks so that only the workers with the lowest productivity are displaced by automation, preserving the employment of more productive workers. As a consequence of the reassignment, the mismatch between workers and tasks increases, potentially reducing workers' productivity and wages. As Acemoglu & Restrepo (2018a) point out, wether or not wages decrease depends on how productive robots are at the tasks they overtake. A major increase in productivity due to automation can increase workers' marginal product in their own tasks, increasing wages, while moderate increases in output in the automated tasks can be dominated by the higher mismatch experienced by workers, ultimately reducing their wages.

I estimate the model using occupational data for the U.S. labor market obtained from O*NET, the U.S. Department of Labor Occupational Characteristics Database. For the estimation I divide occupations into five educational requirement categories, namely occupations requiring less than high school, high school, vocational training, college and post-graduate education. The model matches the wage structure across these five educational requirement groups. I complement the model with an estimate of the cost of automation using data on the automatability of occupations from Frey & Osborne (2017), and the cost of industrial robots from the International Federation of Robotics.

The model rationalizes observed trends in automation. I find that it is optimal to automate tasks with a high manual skill requirement, most of them related to manufacturing and construction occupations such as metal workers and construction carpenters. These are occupations that require vocational training or a high school diploma. Yet, the displacing effects of automation fall mostly on workers who performed occupations requiring no education (less than high school). In total, 5.6% of workers are displaced by automation in the model, 3.9% worked in occupations requiring no education, the remaining 1.1% in occupations requiring high school only. Automating tasks with a high manual skill requirement turns out to be optimal despite there being alternative tasks automatable at lower costs. The reason lies in the comparatively high mismatch between workers' skills and the skills demanded by the automated tasks.

The model also implies a downward compression of the wage distribution. This follows from the fall in employment across workers in occupations requiring only high school, and from the reassignment of task among higher educated workers. The increase in output from the automation of tasks is not enough to offset the negative effects of the reassignment.

However, the model overestimates the decline in wages, in large measure due to the coarseness of the categories used in the analysis.

Besides the application to automation, I show how the model can be used to address other changes in the economy. The problem of optimal worker training bears many similarities with the optimal automation problem. Training is captured as changing the worker's skill vector. Thus, the choice in both problems is a location for a worker/robot in the skill space, with the objective of making the worker/robot more adept at certain tasks. I show how to modify the automation problem to deal with worker training. When the task-output production function takes a linear-quadratic form (Galichon, 2016, Ch. 6), it is optimal to train workers to be in the center of their occupations so as to minimize the mismatch between their skills and the tasks' skills requirements.

I also consider the optimal direction of skill-biased technical change. I find that is is optimal to increase the weight of the skills at which the workforce is already more adept, as measured by the mismatch in different skills between workers and tasks' requirements. In other words, it is optimal to specialize in skills, adapting technology to complement the skills for which the workforce is better suited, thus raising productivity. This contrast with automation, where productivity increases by replacing workers at tasks they are not well suited for.

Finally, I extend the model to generate endogenous unemployment of workers by allowing tasks to be left unassigned. Tasks are only performed if workers are productive enough relative to their cost (wages). I show how the value of the minimum wage affects employment in the model and how skill accumulation by workers changes, and potentially expands, the set of tasks performed in the economy. One important consequence of allowing tasks to be left unassigned is that automation ceases to be a pure worker-replacing technology. Automation can now complement workers by taking over tasks that are either not worthwhile for workers to perform, or that are too specialized given the workers' current skills.

Related literature I adopt a task approach to production as in Rosen (1978), Autor et al. (2003) and Acemoglu & Autor (2011). I complement this literature by incorporating multidimensional heterogeneity in tasks and workers, following Lindenlaub (2017). The main methodological difference with Lindenlaub's model resides in my assumption over the discreteness of the distribution of skills across workers. This assumption is motivated by the study of occupations, which arise from the assignment as bundles of tasks assigned to a type of worker. The same assumption has been used before to address different questions. Feenstra & Levinsohn (1995) use a similar setup to mine in the context of a continuum of buyers choosing among a discrete set of products. Stokey (2017) develops a model with one-dimensional heterogeneity, where a continuum of workers are assigned to finitely many tasks, to study the effects of task biased technical change on the wage structure.

This paper is also related to the literature on multidimensional skill mismatch and occupational choice, e.g. Guvenen et al. (2015) and Lise & Postel-Vinay (2015). I complement this literature by providing a framework to study changes in the skill content of occupations. By explicitly modeling the tasks that comprise an occupation my model endogenizes changes in occupations in response to technical change and the skill distribution of the workforce. This comes at a cost, my model abstracts from search or information frictions in the labor market, as well as the dynamic problem of workers.

Finally, the paper adds to the literature on the effects of automation: Acemoglu & Restrepo (2017, 2018b), Aghion et al. (2017), Hemous & Olsen (2018), among others. In particular, I explicitly model the multidimensional nature of skill heterogeneity. This is relevant to determine the automatability of tasks as shown recently by Frey & Osborne (2017).[5] In turn, allowing for varying costs of automation across the task space lets me ask about the direction of automation. In this way, the paper provides a framework where to evaluate which occupations are more likely to be affected by automation, as well as what the consequences of automation can be.

2 Task Assignment Model

I present a model where occupations arise as bundles of tasks assigned to workers, and the boundaries of occupations react endogenously to changes in technology (e.g. automation, skill-biased technical change) and demographics (e.g. the distribution and skills of workers). I use the model to explore how these factors change occupations, and what the effects are on worker productivity, worker compensation, and the incentives to further adopt new technologies.

The model builds on one-dimensional task-based models of production in the spirit of Rosen (1978) and Acemoglu & Autor (2011), where tasks are the basic unit of production, and tasks and workers are defined by a single dimensional measure of their 'complexity' or 'skill'. I extend the basic one-dimensional framework by incorporating multidimensional heterogeneity across workers and tasks, following Lindenlaub (2017)'s multidimensional assignment model. In the model workers are defined by a vector of skills representing their

[5] Autor et al. (2003) also shows how multiple dimensions are relevant for explaining changes in occupations. They focus on the decline of occupations intensive in routine-manual tasks.

cognitive, manual, social ability, etc; tasks are defined by a vector of the skills involved in performing them. Taking into account multiple skills has been shown to be relevant when explaining educational choices (Willis & Rosen, 1979), the role of social skills relative to cognitive skills across occupations (Deming, 2017), and the decline of occupations intensive in routine-manual tasks (Autor et al., 2003).[6]

In the model, production involves the completion of a continuum of tasks by finitely many types of workers.[7] A single type of worker can then perform various tasks; I refer to the set of tasks performed by a worker as the worker's occupation. Which tasks are assigned to each type of worker depends on the distribution of skills among workers and tasks, and on how productive workers are at different tasks. The productivity of a worker at a given task is determined in turn by how well the worker's skills match the skills used in performing the task. In what follows I describe in detail the role of workers, tasks and the production technology. Then I discuss the optimal assignment and the determinants of worker compensation.

Setup

Workers Consider an economy populated by a continuum of workers. A worker is characterized by the skills she possesses, as captured by a vector $x \in \mathcal{S} \subset \mathbb{R}^d$, where $\mathcal{S}$ is the space of skills and $d \geq 1$ is the number of skills. Vector x encodes the level of different skills the worker has, like cognitive, manual, social, etc.

There are N types of workers in the economy: $\{x_1, \ldots, x_N\} \equiv \mathcal{X}$. x_n is the skill vector of workers of type n. There is a mass p_n of workers of type x_n, so that the total mass of workers in the economy is $P = \sum_{n=1}^{N} p_n$. Each worker is endowed with one unit of time. This implies that workers of type n have a total of p_n units of time available to work. Workers can either work or be unemployed. If unemployed, a worker receives a payment $\underline{w} \geq 0$. Workers supply their time inelastically at any wage $w \geq \underline{w}$.

Tasks There is a single final good produced in the economy that aggregates the output of all workers across productive tasks. In particular, production of the final good involves

[6] In the literature on skill formation and education, Cunha et al. (2010) show that taking into account only cognitive skills can lead to wrong policy recommendations regarding investment on education.

[7] Feenstra & Levinsohn (1995) use a similar setup in the context of a continuum of buyers choosing among a discrete set of products. Stokey (2017) develops a one dimensional model with a continuum of workers and finitely many tasks to study the effects of task biased technical change on the wage structure. In her model a bundle of workers is assigned to a given task.

completing a continuum of differentiated tasks. Let $\mathcal{Y} \subseteq \mathcal{S}$ denote the set of tasks used in production. Tasks $y \in \mathcal{Y}$ differ in the skills involved in performing them, and how many times they must be performed. One unit of time is required to perform a task once. To make this precise, I represent a task y by a vector of skills, so that $y \in \mathcal{Y} \subseteq \mathcal{S}$, and denote the density of tasks used in production by $g : \mathcal{Y} \to \mathbb{R}_+$. I assume throughout that:

i $g : \mathcal{Y} \to \mathbb{R}_+$ is an absolutely continuous (a.c.) function with an associated a.c. measure G on $\mathcal{Y}$;

ii there are enough workers to complete all tasks, i.e. $G(\mathcal{Y}) = \int_{\mathcal{Y}} g(y)\, dy \leq P$;

iii the set of tasks $\mathcal{Y}$ is compact.

Task Output Workers vary in their productivity across tasks depending on the match between the skills they possess (x) and the skills involved in performing the task (y). q describes how productive a worker with skills x is when performing task y. These differences play a crucial role in determining the assignment of tasks to workers, and through it the overall productivity of each worker type and the substitutability across workers. As will be discussed later in this Section, the optimal assignment will balance the desire to minimize the mismatch between workers and the tasks they perform, with the capacity constraints imposed by the limited availability of workers.

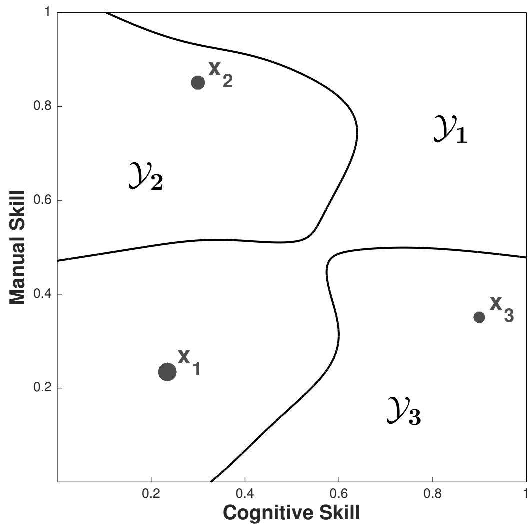
I will denote by $q(x, y)$ the worker/task-specific output generated by a worker of type x_n performing task y. If a task is not assigned to any worker, then no output is generated for that task (abusing notation: $q(\emptyset, y) = 0$ for all $y \in \mathcal{Y}$). Further properties of the task-output function $q : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}_+$ will be specified later.

Assignment As mentioned above, production of the final good combines output from all workers. The output of a worker depends in turn on which task she performs, according to the productivity of the worker implied by q. Because of this, the production of the final good will depend on how tasks are assigned to workers. The assignment of tasks to workers is described by a function $T : \mathcal{Y} \to \mathcal{X}$, so that task y is performed by worker $T(y) \in \mathcal{X}$.[8]

Many tasks can be assigned to the same worker type. I collectively refer to the set of tasks performed by a type of worker as the occupation of the worker. The assignment T

[8]The definition of the assignment function implicitly assumes that all tasks are assigned to a worker in $\mathcal{X}$. This is without loss given the way in which output from all worker/task pairs is aggregated into the final good (see equation 3). I will expand on this in the next subsection where I also discuss how to explicitly include the possibility of not performing some of the tasks.

Figure 1: Assignment Example



Note: *The figure shows an example for an assignment in a two-dimensional skill space (cognitive and manual skills). Three types of workers are considered* $\{x_1, x_2, x_3\}$ *and tasks are uniformly distributed over the unit square, i.e* $\mathcal{Y} = [0,1]^2$ *and* $g(y) = 1$ *for all* y*. The assignment partitions the space into three cells* $\{\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3\}$*. The assignment in this figure is not necessarily optimal or feasible.*

determines which tasks are bundled into the occupation of each worker. The occupation of workers of type x_n is:

$$\mathcal{Y}_n = T^{-1}(x_n) = \{y \in \mathcal{Y} \,|\, x_n = T(y)\} \tag{1}$$

Occupations form a partition of the space of tasks into at most N cells.[9] Figure 1 shows an example of an assignment that partitions the space of tasks into three occupations, corresponding to three worker types.

An assignment is deemed *feasible* if workers have enough time to supply all the time demanded by their occupation. This time is given by the number of tasks in the worker's occupation. The demand for worker n's time is:

$$D_n = \int_{\mathcal{Y}_n} dG \tag{2}$$

An assignment is feasible if $D_n \leq p_n$ for all $n \in \{1, \ldots, N\}$.

[9]It is possible that $\mathcal{Y}_n = \emptyset$, so that no task is assigned to worker x_n.

Final good production The production of a final good aggregates the output from all worker/task pairs through a Cobb-Douglas technology.[10] Given an assignment T, total output is:[11]

$$F(T) = \exp\left(\int_{\mathcal{Y}} \ln q\left(T(y), y\right) dG\right) \tag{3}$$

Optimal assignment

The problem is to find a feasible assignment that maximizes output:

$$\max_{T} F(T) \qquad \text{s.t. } \forall_n\, D_n \leq p_n \tag{4}$$

The assignment determines how tasks are divided into occupations. The exact form of the assignment depends on three factors. First, the distribution of skills in the workforce, which is described by the skill vectors of N types of workers (x_n), and the mass of workers of each type (p_n). Second, the distribution of tasks involved in production, captured by the function g. Finally, the production technology embedded in q, which determines how workers and tasks' characteristics interact in production. Changes in any of these factors translate into changes to the optimal assignment of tasks to workers, thus affecting the boundaries of occupations, and the production of the final good.

Even though the optimal assignment cannot be fully characterized without completely specifying the environment, it is possible to guarantee the existence and uniqueness of a solution by imposing conditions only on the production technology q. The following proposition makes this precise:

Proposition 1. *Consider the problem of finding an optimal assignment in (4).*

If q is such that:

[10] The aggregator does not need to be of the Cobb-Douglas type. Results hold for aggregators of the CES family: $F(T) = \left(\int \left(q\left(T(y), y\right)\right)^{\frac{\sigma-1}{\sigma}} dG(y)\right)^{\frac{\sigma}{\sigma-1}}$, with $\sigma \geq 1$. See Appendix B.

[11] Under this technology, production of the final good only takes place if all tasks are assigned and performed. Recall that if a task is left unassigned $q(\emptyset, y) = 0$. In this sense technology resembles a continuous version of Kremer (1993)'s O-Ring production function. In order to make the comparison precise it is necessary to change the interpretation of q. Consider a continuous production line indexed by $y \in \mathcal{Y}$, at each point in the production line a fatal error can occur that terminates the production process in failure. The arrival rate of an error is given by $\ln q(x, y) \geq 0$ and depends on the point in the production process (y) and the worker assigned to that point (x). The probability that no error arrives at the end of the whole process is given by (3). Thus, $F(T)$ can be interpreted as expected output given an assignment T. See Sobel (1992) for another application of this idea.

1. *All workers can produce in all tasks:* $q(x,y) > 0$ *for all pairs* $(x,y) \in \mathcal{X} \times \mathcal{Y}$.

2. $q(x,\cdot)$ *is upper-semicontinuous in* y *given* $x \in \mathcal{X}$.

3. q *discriminates across workers in almost all tasks: if* $q(x_n,y) = q(x_\ell,y)$ *then* $x_n = x_\ell$ *G-a.e.*

Then there exists a $(G-)$*unique solution* $T^\star$ *to the problem in (4). Moreover, there exist* $\lambda^\star \in \mathbb{R}^N$ *such that* $T^\star$ *is characterized as:*

$$T^\star(y) = \underset{x\in\mathcal{X}}{\text{argmax}} \left\{ \ln q(x,y) - \lambda^\star_{n(x)} \right\} \tag{5}$$

where $n(x)$ *gives the index of a type of worker* $x \in \mathcal{X}$*, and* $\min \lambda^\star_n = 0$.

Proof. The result is established by re-expressing the problem in (4) as an optimal transport problem. The proof is divided into three Lemmas that follow from applying Theorems 5.10 and 5.30 in Villani (2009), summarized in Theorem 1 of Appendix A. All Lemmas are stated and proven in Appendix B.

As is common in assignment problems, I first relax the problem to allow for non-deterministic assignments, see Kantorovich (2006) and Koopmans & Beckmann (1957). An assignment is then a joint measure over workers/task pairs: $\pi : \mathcal{X} \times \mathcal{B}(\mathcal{Y}) \to \mathbb{R}_+$, where $\mathcal{B}(\mathcal{Y})$ denotes the Borel sets of $\mathcal{Y}$. An assignment π is deemed feasible if it is a coupling of measures P and G, see definition 3 in Appendix A. In terms of the assignment problem π must guarantee that workers have enough time to perform all the time demanded by their occupations, and each task is completed at most once. Letting $\Pi(P,G)$ be the set of feasible assignments:

$$\pi \in \Pi(P,G) \longleftrightarrow \forall_n \int_{\mathcal{Y}} d\pi(x_n,y) \leq p_n \qquad \forall_{Y\in\mathcal{B}(\mathcal{Y})} \sum_{n=1}^{N} \int_{y\in Y} d\pi(x_n,y) \leq G(Y) \tag{6}$$

Note that the second condition can be simplified to: $\sum\limits_{n=1}^{N} \pi(x_n,\{y\}) \leq g(y)$.

The problem is now to choose a coupling $\pi \in \Pi(P,G)$ to maximize output. I further simplify the problem by applying natural logarithm to the objective function. Doing so reveals the linearity of the problem in the choice variable π. The relaxed optimization problem is:

$$\max_{\pi\in\Pi(P,G)} \sum_{n=1}^{N} \int_{\mathcal{Y}} \ln q(x_n,y)\; d\pi(x_n,y) \tag{7}$$

Lemma 1 applies Theorem 5.10 of Villani (2009) to establish duality for the problem:

$$\max_{\pi\in\Pi}\sum_{n=1}^{N}\int_{\mathcal{Y}}\ln q\left(x_n,y\right)\,d\pi\left(x_n,y\right)=\inf_{\substack{(\lambda,v)\in\mathbb{R}^N\times L^1(G)\\ w_n+v(y)\geq\ln q(x_n,y)}}\sum_{n=1}^{N}\lambda_n p_n+\int_{\mathcal{Y}}\nu\left(y\right)dG \tag{8}$$

$$=\inf_{\lambda\in\mathbb{R}^N}\sum_{n=1}^{N}\lambda_n p_n+\int_{\mathcal{Y}}\max_{n}\left\{\ln q\left(x_n,y\right)-\lambda_n\right\}dG$$

λ and ν are the multipliers (or potentials) of the problem. Lemma 2 establishes that a solution to the dual problem $(\lambda^{\star},v^{\star})$ exists. The levels of $\lambda^{\star}$ and $\nu^{\star}$ are only determined up to an additive constant. Both the assignment and the value of the dual problem do not change if λ is increased by a constant κ for all workers and ν decreased by the same amount for all tasks. I normalize the value of the minimum $\lambda^{\star}$ to zero. This is convenient when relating the value of $\lambda^{\star}$ to the marginal product of workers and the wages in the decentralization of the optimal assignment.

The first two conditions on the production function q ensure that the value of the primal problem (7) and the dual problem (8) are finite, this is the key step in verifying the conditions for Theorem 5.10 of Villani (2009). In particular, the first condition avoids indeterminacies when evaluating the natural logarithm of q for any worker/task pair.

The solution to the dual problem provides a way to construct the optimal assignment $T^{\star}$. Lemma 3 applies Theorem 5.30 of Villani (2009) to construct $T^{\star}$ as the sub-differential of $v^{\star}$. The third condition on the production function q is crucial to establish single-valuedness of the sub-differential of $v^{\star}$. This gives the formula for the optimal assignment in (5). Galichon (2016, Ch. 5.3) presents an algorithm to solve the dual problem in (8).

□

The first two conditions on q in Proposition 1 are technical and ensure that the theory of duality applies to the problem. The value of $\lambda^{\star}$ is obtained from the solution to the dual problem to 7. The third condition plays a crucial in role in establishing the existence and uniqueness of an optimal assignment function $T^{\star}$. The condition allows to distinguish between workers in each task by demanding injectivity of q in x given y. It plays the same role as the 'twist condition' of Carlier (2003), the condition for positive assortative matching in Lindenlaub (2017), and the single-dimensional Spence-Mirrlees single-crossing property. However, the injectivity condition I assume is less restrictive than the 'twist condition' since it does not involve differentiability of q, moreover, it is simpler to verify in practice since there are finitely many types of workers.

The characterization of the optimal assignment in 5 allows me to solve the problem in a task-by-task basis, and give a more explicit characterization of the occupations in terms of the production technology q:

$$\mathcal{Y}_n=\left\{y\in\mathcal{Y}\,|\,\forall_{\ell}\ \ln q\left(x_n,y\right)-\lambda_n^{\star}\geq\ln q\left(x_{\ell},y\right)-\lambda_{\ell}^{\star}\right\} \tag{9}$$

Tasks are optimally assigned to workers that are more productive at performing them. That is, workers with lower skill mismatch. The role of the multiplier $\lambda^{\star}$ is to penalize the output of a worker in a given task to balance the demand for that type of worker with the limited supply of hours (p_n). The boundaries of an occupation are formed by task $y \in \partial \mathcal{Y}_n$ for which the inequality in (9) is met with equality for some k.

Even though it is possible not to perform a task, by leaving it unassigned, this does not happen under the optimal assignment. It is optimal to assign all tasks because there is no production of the final good if one task is left unassigned (recall that $q(\emptyset, y) = 0$ for all y). In Section 4 I consider an alternative interpretation of the production technology under which tasks can be left unassigned.[12] Doing so gives rise to endogenous unemployment in the model. Unemployment depends on how many (and which) tasks are not performed in the optimal assignment. The results in Proposition 1 do not change by considering the possibility of leaving tasks unassigned, but worker's compensation does change. I expand on this in the next subsection and in Section 4.

Indirect production function

The production technology described above depends not only on how many workers of each type are used, but also on which tasks are performed by each of them; unlike the 'canonical' production function where the roles of each input (in this case each type of worker) are predetermined and unchanging. In the model described above the amount of an input (a type of worker) used in production and what that input is used for are not the same (Autor, 2013). As a consequence, the relation between inputs and output depends on how the tasks are assigned to workers, and how the assignment itself changes as the inputs vary.

The aggregate role of workers in production is captured by the value of the assignment problem (4). The value of the problem defines an indirect production function that depends on the availability of workers in the economy:

$$V(p_1, \ldots, p_N) = \max_{T} F(T) \qquad \text{s.t. } \forall_n \, D_n \leq p_n \tag{10}$$

Function V describes how production changes when the composition of the workforce changes, allowing for workers to be re-assigned optimally across tasks.

The properties of workers in production, such as their marginal product and the

[12] When a task is left unassigned it is taken out of the mix of tasks instead of having output be zero. This amounts to changing the set over which the integral in 3 is taken.

substitutability across different types of workers, are determined by how the assignment reacts to changes in the supply (distribution) of workers. In particular, the properties of workers in production depend on their productivity along the boundaries of occupations, and on how those boundaries react to changes in the environment.

Marginal products and worker compensation

The marginal product of workers of type n is obtained from V as the change in output if the supply of worker's of type n (p_n) were to increase.[13] The marginal product is given by the percentage increase in output obtained from adding more workers of the given type. This is made clear by relating the marginal product to the solution of the dual problem (8):

$$\mathrm{MP}_n = \frac{\partial V\left(p_1, \ldots, p_N\right)}{\partial p_n} = F\left(T^{\star}\right) \lambda_n^{\star} \tag{11}$$

The result follows from the envelope theorem (Milgrom & Segal, 2002) and is proven in Lemma 4 in Appendix B.

To see how the value of $\lambda^{\star}$ relates to the productivity of each type of worker, we must first determine how the assignment responds to an increase in the supply of workers. When the supply of workers of type n increases, the additional workers are only used if tasks are re-assigned to them from other workers. The first tasks to be reassigned are those in the boundaries of occupations. Consider the occupations of two types of workers, n and ℓ, all tasks in the boundary of the occupations, i.e. $y \in \mathcal{Y}_n \cap \mathcal{Y}_\ell$, satisfy:

$$\lambda_n^{\star} - \lambda_\ell^{\star} = \ln q\left(x_n, y\right) - \ln q\left(x_\ell, y\right) \tag{12}$$

Then the difference in the multipliers $\lambda_n^{\star}$ and $\lambda_\ell^{\star}$ is given by the log difference in task output along the boundary between workers n and ℓ. That is, the percentage increase (or decrease) in output if the tasks along the boundary are re-assigned from ℓ to n.[14] It is only optimal to make use of the additional supply of workers if output increases along the boundary of worker's n occupation $(\partial \mathcal{Y}_n)$.

[13] This definition of marginal product takes into account how the assignment changes optimally in response to the increase in the supply of workers of type n. It is also possible to define an arbitrary measure for the marginal product of a type n worker at a given task y, given some arbitrary assignment T. I discuss it in Appendix C.

[14] It is useful to consider an example with finitely many tasks, say $\{y_1, y_2\}$. Then total output is given by $F(T) = q_1\left(x_n\right) q_2\left(x_\ell\right)$. If the assignment changes by having worker x_n perform both tasks the new output is $F\left(T^{'}\right) = \frac{q_2(x_n)}{q_2(x_\ell)} F(T)$. Then $\ln \frac{F\left(T^{'}\right)}{F(T)} = \ln \frac{q_2(x_n)}{q_2(x_\ell)} = \lambda_n - \lambda_\ell$, so that output increases by $100\left(\lambda_n - \lambda_\ell\right)\%$.

If tasks are reassigned to the additional type n workers, workers along the boundaries of $\mathcal{Y}_n$ are displaced. This process generates an excess supply of workers of other types, giving rise to a new round of re-assignment along the boundaries of these workers. Following the process reveals an ordering of workers by productivity, with the least productive worker being displaced by increases in the supply of more productive workers. As a consequence, the least productive worker has zero marginal product.[15] Increases in the supply of that type of workers do not increase output because the additional workers are left unassigned (unemployed).

The total gain in output from the initial increase in supply of workers of type n takes into account the increase in output from all the re-assignments. Using the relation in (12), and recalling that $\min \lambda_k = 0$, we get a total increase in output of λ_n as in (11).

The value of the marginal product affects how workers are compensated. To see this consider how the optimal assignment of tasks to workers can be implemented by a price-taking firm seeking to maximize profits. The firm's problem is:

$$\max_{T} F(T) - \sum_{n=1}^{N} w_n D_n(T)$$

where w_n is the wage paid to a worker of type n, and D_n is the demand for workers of type n, given by (2). This problem is equivalent to the optimal assignment problem in (4) if the wages correspond to the multipliers of the feasibility constraint of each worker type. This is the case if wages are of the form:

$$w_n = F(T^{\star}) \lambda_n^{\star} + \kappa \qquad \text{where } \kappa \geq \underline{w} \tag{13}$$

The wages that decentralize the optimal assignment are given by the marginal product of each worker under the optimal assignment, plus a constant that guarantees that all workers receive at least their outside option. The level of the wages is not pinned down in the problem because only the difference in wages affects the assignment (see equation 9). Recall that in order to produce the firm has to employ a total of $G(\mathcal{Y})$ hours, independently of which workers are hired. So, if all wages increase by κ the total wage bill increases by $\kappa G(\mathcal{Y})$ regardless of the assignment. From the point of view of the firm the constant κ acts as a

[15] The property that the least productive worker has zero marginal product is induced by the capacity constraint on the set of tasks and the times each task can be performed. It is also what motivates the normalization of the multiplier λ in proposition 1.

fixed cost, and thus, it has no effect on the assignment.[16]

An alternative to determine the level of wages in the economy is to allow for tasks to remain unassigned as presented in Section 4, or because of the introduction of automation or offshoring as presented in Section 3.1. The threat of leaving a task unassigned lowers the wage of the least productive worker to its outside option, effectively pinning down $\kappa = \underline{w}$ as the worker with the lowest productivity has zero marginal product. Which tasks are performed under the optimal assignment is then a function of $\underline{w}$. A higher outside option for workers makes unprofitable to perform more tasks, and can induce unemployment among workers of different types.

Substitutability across workers

The substitutability of different types of workers in production also depends on the characteristics of the assignment. How substitutable are 'low' and 'high' skilled workers, or workers specialized in cognitive vs manual skills, depends on which tasks they perform. Intuitively, workers performing similar tasks are more substitutable, as are workers with similar skills. In order to make these results precise I compute the elasticity of substitution under the optimal assignment.

Since there are in general more than two types of workers the appropriate measure of substitutability is given by the Morishima elasticity of substitution (Blackorby & Russell, 1981, 1989).[17] The elasticity of substitution between workers of type n and ℓ is:

$$M_{\ell n} = \mathcal{E}_{\ell n} - \mathcal{E}_{nn} \tag{14}$$

where $\mathcal{E}_{\ell n} = \frac{\mathrm{MP}_n}{D_\ell}\frac{\partial D_\ell}{\partial \mathrm{MP}_n}$ is the cross elasticity of demand for worker k with respect to a change in the marginal product of worker n.[18] Changes in the marginal product of worker x_n are

[16] The indeterminacy of the level of worker compensation is a common feature of assignment models (Sattinger, 1993). Only total surplus and differences across workers are pinned down by the optimality conditions. This result is not a feature of the discreteness in the distribution of workers, see Lindenlaub (2017). The level of worker compensation depends on additional assumptions. For example, having excess workers ($P > G(\mathcal{Y})$) implies that (at least) some type of worker will be partially unassigned (unemployed), driving down the wage for that type of worker to $\underline{w}$. This will be the case when I introduce automation in Section 3.1. Once the wage of one type of worker is known the other wages are implied by differences in their marginal product (see equation 12).

[17] See Baqaee & Farhi (2018) for a recent application of the Morishima elasticity in an input-output network setting.

[18] The Morishima elasticity of substitution measures the effect on the ratio of optimal demands for two inputs (in this case two types of workers, ${}^{D_\ell}/_{D_n}$) given by a (proportional) change of the ratio of marginal products (${}^{\mathrm{MP}_n}/_{\mathrm{MP}_\ell}$). Recall that marginal products and wages move together. When there are more than two workers the direction of the change in the ratio of marginal products matters since the demands for

captured by changes in $\lambda_n^\star$ (see equation 11). Knowing this, it becomes clear from the characterization of occupations in (9) that $\mathcal{E}_{nn} < 0$ and $\mathcal{E}_{n\ell} \geq 0$. That is, increasing $\lambda_n^\star$ decreases the demand for worker x_n and (weakly) increases the demand for other workers. From the point of view of worker compensation, increasing $\lambda_n^\star$ raises the cost of worker n, causing the firm to substitute it for other workers. Because of this the elasticity of substitution is always positive in the model.[19]

Yet, the relevant measure for direct substitutability between workers is the cross-elasticity $\mathcal{E}_{n\ell}$. In a setting with more than two inputs, the ratio ${}^{D_\ell}/_{D_n}$ can change in response to changes in the marginal product of x_n without the demand for worker ℓ being affected. Because of this, the elasticity of substitution between workers n and ℓ is at least equal to $\mathcal{E}_{nn}$, being only greater if there is direct substitution between the two workers, that is, if the demand for worker ℓ changes when the marginal product of n changes. As shown in proposition 2 this happens only if workers n and ℓ share a boundary.

To obtain the magnitude of the elasticity of substitution between two workers it is necessary to determine how much their demands change with the value of $\lambda^\star$. Looking again at the characterization of occupation in (9) the change in the demand will depend in how sensitive the boundaries of the occupation are to changes in $\lambda_n^\star$. The sensitivity of the boundaries depends in turn on the slope of the production function q evaluated at the boundary tasks, see (12). Specifying a functional form on q becomes necessary to completely characterize $\mathcal{E}_{nn}$ and $\mathcal{E}_{kn}$. I follow Feenstra & Levinsohn (1995) and Lindenlaub (2017) in assuming a linear-quadratic form for the production function:

$$q(x, y) = \exp\left(a_x^{'} x + a_y^{'} y - (x - y)^{'} A (x - y)\right) \tag{15}$$

Under (15) the productivity of a worker at a given task depends on the skill mismatch between the worker's skills (x) and the skills involved in performing the task (y), measured by the weighted distance between worker and task's skills.[20] Matrix A controls the weights

inputs changes differently if MP_n or MP_ℓ vary, see Blackorby & Russell (1989, pg 885). Because of this the elasticity is in general asymmetric. I consider a change in the ratio of marginal products in the direction of MP_n:

$$M_{\ell n} = \frac{\partial \ln {}^{D_n}/_{D_\ell}}{\partial \ln \mathrm{MP}_n} = \frac{\mathrm{MP}_n}{D_\ell} \frac{\partial D_\ell}{\partial \mathrm{MP}_n} - \frac{\mathrm{MP}_n}{D_n} \frac{\partial D_n}{\partial \mathrm{MP}_n} = \mathcal{E}_{\ell n} - \mathcal{E}_{nn}$$

[19]Workers satisfy the Kelso & Crawford (1982) gross substitutes condition.

[20]The dependance of production on the mismatch between worker and task skills is similar in spirit to Lazear (2009)'s skill weights approach, where he studies job specific skills, and to the skill mismatch studies of Guvenen et al. (2015) and Lise & Postel-Vinay (2015), who study earnings differential across occupations and the accumulation of skills by workers.

of each skill in the mismatch, it is assumed to be symmetric and positive definite. The higher the weight of a skill the more important it is for production; mismatch in that skill hurts production more. The linear terms ($a_x^{'}x$ and $a_y^{'}y$) capture more skilled workers having an absolute advantage in production, and the value of output generated by tasks involving higher skill levels being higher.

The functional form in (15) greatly simplifies the characterization of occupations in the optimal assignment.[21] In particular, boundaries take the form of hyperplanes whose normal vectors depend on matrix A and the difference in skills between neighboring workers. This is made clear by replacing (15) in condition (12). The boundary between the occupations of workers x_n and x_ℓ is:

$$y \in \mathcal{Y}_n \cap \mathcal{Y}_\ell \longleftrightarrow 0 = y^{'} \underbrace{A\left(x_\ell - x_n\right)}_{\text{Normal Vector}} - \frac{1}{2} \underbrace{\left(x_\ell^{'} A x_\ell - x_n^{'} A x_n + a_x^{'}\left(x_\ell - x_n\right) + \lambda_\ell^\star - \lambda_n^\star\right)}_{\text{Intercept}} \quad (16)$$

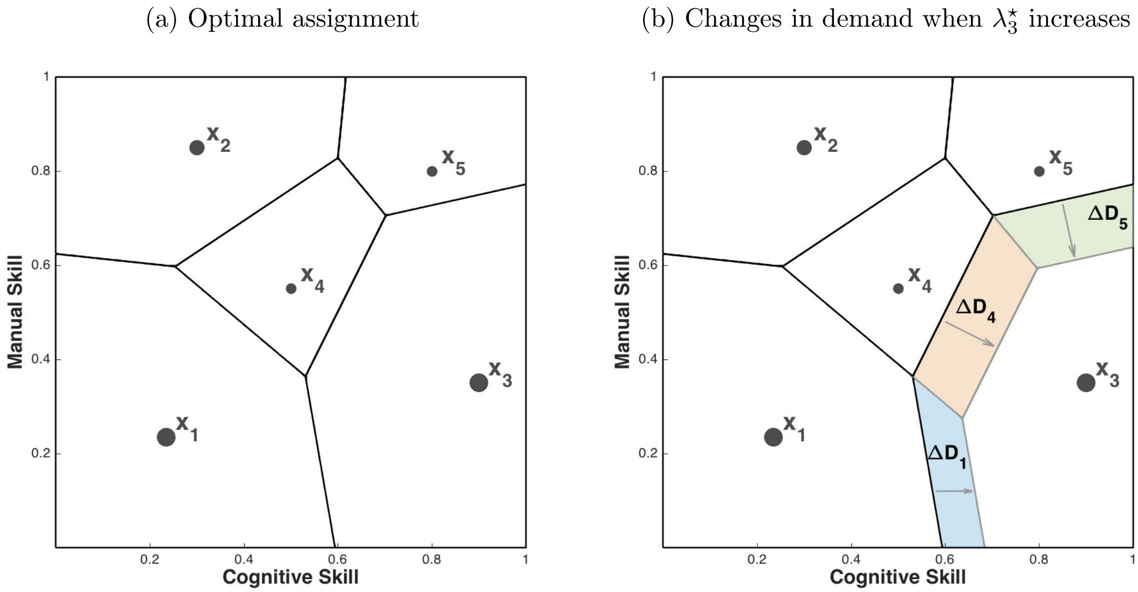
Figure 2a shows the form of the optimal assignment when q is given by (15). A worker will perform the tasks closest to her skills, for which she has the least mismatch, conditioned on the limited supply of workers (feasibility constraint in 4). The location of the boundaries depends on the value of the multipliers $\lambda^\star$, but the slope depends on the relation between workers' skills and the production technology embodied by A. Figure 2b illustrates this by increasing the value of $\lambda_3^\star$. When $\lambda_n^\star$ increases the boundaries of the occupation of worker n will shift 'inward' in a parallel fashion, reducing the demand for x_n and increasing the demand for all its neighbors. If an occupation $\mathcal{Y}_m$ does not share a border with $\mathcal{Y}_n$, it is not directly affected by changes in $\lambda_n^\star$ (see the boundaries of $\mathcal{Y}_2$ in 2b).

The geometric structure induced by adopting the functional form in (15) makes it possible to characterize the change in demand following a change in $\lambda^\star$ in a general way (Feenstra & Levinsohn, 1995). The change in demand is always given by the area of a (hyper)trapezoid, formed as the plane that defines the boundary between occupations moves (see Figure 2b).

I exploit the geometric structure of the problem to compute closed form expressions for the derivates of demand (Proposition 2). The cross derivative of demand depends on how exposed two workers are to one-another, measured by the length of their boundary, and how

[21]Under (15) there is an equivalence between the optimal assignment and the partition induced by a power diagram. A power diagram partitions a space into cells that minimize the power between a node (x) associated to the cell and the points y in the cell. The outcome is a partition of the space into convex polyhedra defined by hyperplanes. The power function between two points is $\text{pow}\left(x,y\right) = d\left(x,y\right)^2 - \mu$, where $d\left(x,y\right)$ is a distance and $\mu \in \mathbb{R}$. This relation is noted by Galichon (2016, ch. 5) and is treated formally by Aurenhammer et al. (1998).

Figure 2: Assignment Example - Quadratic Mismatch Loss



Note: *The figures shows the assignment in a two-dimensional skill space (cognitive and manual skills). Five types of workers are considered* $\{x_1, \ldots, x_5\}$ *with mass* $P = \{0.3, 0.2, 0.3, 0.1, 0.1\}$. *Tasks are uniformly distributed over the unit square, i.e* $\mathcal{Y} = [0,1]^2$ *and* $g(y) = 1$ *for all* y. *The production function* q *is given by (15) with* $A = I_2$, *the value of* a_x *and* a_y *does not change the optimal assignment.*

similar their skills are, measured by the weighted distance between their skill vectors (x_n and x_ℓ). When the demand for a worker changes, it is optimal to make the adjustment along the boundaries. Because of this, workers with longer boundaries are more substitutable, moreover, workers are only directly substitutable with their occupational neighbors. How much the boundary reacts to a change in demand depends on how similar workers are at performing tasks. The closer the skills of the workers are, the more substitutable along their boundary. Finally, the second part of the proposition follows from noting that the set of tasks is fixed, so the total effect of the change in demand as λ_n changes must be zero.

Proposition 2. *Let* $\lambda \in \mathbb{R}^N$ *be a vector of multipliers. If* q *is continuous then* D_n *is continuously differentiable with respect to* λ *and:*

i $\forall_{\ell \neq n} \quad \frac{\partial D_n}{\partial \lambda_\ell} = \frac{area(\mathcal{Y}_n \cap \mathcal{Y}_\ell)}{2\sqrt{(x_n - x_\ell)' A' A (x_n - x_\ell)}} = \frac{\int_{\mathcal{Y}_n \cap \mathcal{Y}_\ell} dG}{2\sqrt{(x_n - x_\ell)' A' A (x_n - x_\ell)}} \geq 0$

ii $\frac{\partial D_n}{\partial \lambda_n} = -\sum_{\ell \neq n} \frac{\partial D_\ell}{\partial \lambda_n} < 0$

The proof of Proposition 2 is presented in Appendix B, it extends the results of Feenstra & Levinsohn (1995) by applying Reynolds' transport theorem (see Theorem 2 in Appendix A) to compute the change in demand for arbitrary configurations of workers (x).

Using part two of Proposition 2 the expression for the Morishima elasticity becomes:

$$M_{\ell n} = \left(1 + \frac{D_\ell}{D_n}\right) \mathcal{E}_{\ell n} + \sum_{m \neq n, \ell} \frac{D_m}{D_n} \mathcal{E}_{mn} \tag{17}$$

The elasticity of substitution between workers x_n and x_ℓ is a weighted average of the cross elasticities of demand of all workers, with the weights given by the demand of each type of worker relative to worker n's demand. The elasticity includes the direct substitution effect between n and ℓ, and the secondary effects induced by the substitution of worker n for other workers $(m \neq n, \ell)$. When two workers do not share a boundary $(\mathcal{Y}_n \cap \mathcal{Y}_\ell = \emptyset)$ the direct effect disappears since the cross demand elasticity is zero, but the elasticity of substitution is not zero because it takes into account the changes in the assignment through the boundaries of $\mathcal{Y}_n$. When there are only two types of workers the second term vanishes in (17), and, noting that $\frac{\partial D_n}{\partial \lambda_n} = -\frac{\partial D_\ell}{\partial \lambda_n} = \frac{\partial D_\ell}{\partial \lambda_\ell} = -\frac{\partial D_n}{\partial \lambda_n}$, we get symmetry.

3 Directed technical change

Changes in technology are a major factor shaping the way in which tasks are assigned to workers. For instance, the increase of information technology (IT) in the workplace has shifted focus from manual to cognitive skills, and changed the distribution of tasks across occupations (e.g. clerical and secretarial jobs). More directly, automation technologies and offshoring have replaced workers in performing certain tasks across manufacturing jobs, customer services, and accounting among others.

I consider two forms of technical change and study how they affect the division of tasks into occupations. Innovation in worker replacing technology (such as robots, software, AI, offshoring) lead to the automation of tasks and a reassignment of (remaining) tasks to workers. Innovation in skill-enhancing technology, such as IT in the modern workplace, or the power loom in the 18th and 19th centuries, changes the productivity of workers across tasks, inducing a reassignment of tasks to reduce mismatch across occupations. In both cases, technical change is followed by changes in the role of workers in production, affecting their productivity and substitution patterns. The assignment also determines how substitutable workers are with alternative forms of production (e.g. robots, software).

Both types of technical change can be directed (towards specific tasks or skills) with the aim of increasing production. In both cases, production is increased the most by reducing the mismatch in between tasks and workers, whether by directing automation towards the tasks with the highest mismatch or by increasing the weight on skills for which the workforce is better suited.

The answer to which tasks are optimally automated, and which skills become more important for production, depends on the joint distribution of skills requirements across tasks and skill endowments of workers, and its interaction within the production technology. Moreover, the answer depends on how changes in technology influence the assignment of tasks to workers. I consider the directed technical change problem in the rest of this section.

3.1 Directed automation

I introduce automation technology in the form of a robot that can replace workers in performing tasks.[22] The robot is modeled as a flexible technology that can be adapted to perform different types of tasks. This captures a key property of current technologies like industrial robots or advanced AI programs, which can be reprogrammed or adapted to carry out a variety of tasks (Acemoglu & Restrepo, 2017; Frey & Osborne, 2017). It also relates to other technologies, like offshoring, which, as automation, replace workers at the task they perform in their 'local' labor market (Blinder, 2009; Blinder & Krueger, 2013). The automation problem consists of designing a robot and optimally assigning tasks among the workers and the robot to maximize production. Tasks assigned to the robot are automated.

I treat the robot as a new type of worker. The key difference is that it is possible to choose the robot's skills and supply. I denote by $r \in \mathbb{R}^d$ the skills of the robot and by $p_r \geq 0$ its supply. The automation technology is embodied by a cost function $\Omega : \mathbb{R}^d_+ \times \mathbb{R}_+ \to \mathbb{R}$, so that the cost of producing a mass p_r of a robot with skills r is given by $\Omega(r, p_r)$. Many changes in the patterns of automation can be seen as changes in the cost of automation (Ω). For instance, recent advances in artificial intelligence are reducing the cost of automating tasks intensive in cognitive skills (McKinsey Global Institute, 2017; Frey & Osborne, 2017), while previous innovations like the conveyer belt allowed for the automation of tasks involving manual skills.

Once the robot is designed the set of available workers is expanded to include it: $\mathcal{X}_R = \{x_1, \ldots, x_N, r\}$. Accordingly, the assignment is now described by a function $T_R : \mathcal{Y} \to \mathcal{X}_R$. The assignment of tasks to the robot will of course depend on how productive the robot is

[22] I focus on the short term effects of automation keeping the distribution of tasks fixed, abstracting from the potential gains from adding new tasks, or from performing more the existing tasks with the displaced workers. Acemoglu & Restrepo (2018b) study the effects of automation in an environment with changes in the set of tasks.

relative to the available workers. It is better to design robots so that they replace workers at tasks where skill mismatch is high, and worker productivity is low. These tasks are located along the boundaries of occupations. Automation is thus less likely to occur at 'core' tasks of an occupation, for which the worker is best suited. I denote by $q_R : \mathbb{R}^d \times \mathcal{Y} \to \mathbb{R}$ the production technology of the robot, so that a robot r performing task y produce $q_R(r, y)$.

When tasks are automated the total demand for labor decreases,[23] inducing unemployment among workers. Which workers become unemployed depends on the way in which the assignment reacts to the introduction of the robot. As tasks are assigned to the robot, the workers who would have performed those tasks are directly displaced. Yet, these workers do not necessarily become unemployed since they can take over the tasks of other workers. The end result of this process depends on the substitutability and relative productivities of the workers in the economy. It is the workers with the lowest marginal product who will become displaced (unemployed) as a response to the introduction of the robot, even if the tasks in their occupation are not directly affected by automation. This follows from the order in which workers are substituted from one another described in the previous section when discussing the marginal product of workers.

The automation problem is to choose jointly the skills and mass of the robot (r, p_r), and the new assignment (T_R) to maximize output, net of the automation cost (Ω):

$$\max_{\{r, p_r, T_R\}} \quad F_R(T_R, r) - \Omega(r, p_r) \qquad \text{s.t. } \forall_n\, D_n \leq p_n \qquad D_R \leq p_r \tag{18}$$

where:

$$F_R(T_R, r) = \exp\left(\int\limits_{\mathcal{Y} \setminus \mathcal{Y}_R} \ln q\left(T_R(y), y\right)\, dG + \int\limits_{\mathcal{Y}_R} \ln q_R(r, y)\, dG \right) \tag{19}$$

and

$$\mathcal{Y}_R = T_R^{-1}(r) \qquad D_R = \int_{\mathcal{Y}_R} dG$$

It is convenient to think of the problem in two steps, first solving for an optimal assignment given a set of workers and a robot, and then choosing the optimal skills and mass of the robot taking into account the effect on the optimal assignment. In this way the problem of finding an optimal assignment can be simplified making use of the results in Proposition 1. Taking as given the robot skills and mass (r, p_r), the optimal assignment is

[23] This is a consequence of the assumption that the set of tasks to be performed $(\mathcal{Y})$ is fixed, as is the distribution of tasks (G).

necessarily characterized by a vector $\mu^{\star} \in \mathbb{R}^{N+1}$:[24]

$$T_R(y) = x_n \longleftrightarrow \forall_{\ell} \ \ln q(x_n, y) - \mu_n^{\star} \geq \ln q(x_{\ell}, y) - \mu_{\ell}^{\star} \\ \wedge \quad \ln q(x_n, y) - \mu_n^{\star} \geq \ln q_R(r, y) - \mu_R^{\star} \tag{20}$$

This assignment satisfies the capacity constraints of all robots and the workers.

Abusing notation the problem is then:

$$\max_{\{r, p_r\}} \quad F_R(\mu^{\star}(r, p_r), r) - \Omega(r, p_r) \tag{21}$$

where $\mu^{\star}$ depends on the value of r and p_r, and takes into account how the optimal assignment reacts to changes in the robot skills and mass. The first order conditions of the problem are now derived using the envelope theorem of Milgrom & Segal (2002) and Reynolds' transport theorem (Theorem 2 in the Appendix):[25]

$$\nabla\left(F_R(\mu^{\star}(r, p_r), r) - \Omega(r, p_r)\right) = \nabla F_R(\mu^{\star}, r) - \nabla\Omega(r, p_r) = 0_{d+1\times 1}$$

I first focus on the the derivative of output with respect to the robot's skills:

$$\frac{\partial\left(F_R(\mu^{\star}(r, p_r), r) - \Omega(r, p_r)\right)}{\partial r} = F_R(\mu^{\star}(r, p_r), r) \int\limits_{\mathcal{Y}_R} \frac{\partial \ln q_R(r, y)}{\partial r} dG - \frac{\partial\Omega(r, p_r)}{\partial r} = 0_{d\times 1} \tag{22}$$

The marginal cost of changing the robot's skills is balanced with the gain in output the change in skills induces.

The first term in (22) accounts for the change in output across all tasks assigned to the robot. Changing the robot's skills changes the productivity of the robot across tasks, in general increasing it for some tasks and decreasing it for others, as changing r can increase mismatch for some of the tasks in $\mathcal{Y}_R$. Thus, the first term gives the net gain in output from a change in the robot's skills. Unlike previous results, all of the tasks assigned to the robot matter, and not only those in the boundary of the automated region.

[24]This strategy has been exploited extensively by the optimal sensor placement literature under quadratic loss functions. Under that loss function the optimal assignment is necessarily a power diagram, see Aurenhammer et al. (1998, Thm . 1) and Xin et al. (2016, Thm. 1).

[25]See Xin et al. (2016) for further applications in the theory of optimal power diagrams with capacity constraints. Proposition 3 in Appendix B provides an alternative derivation for the result based on de Goes et al. (2012). The alternative proof is more tedious, but being more explicit it makes it clear how changing the robot's skills affects output.

It is convenient to use an explicit functional form for q_R to fix ideas. Take for instance the production functioned presented in (15). Assuming that $q_R(r,y) = q(r,y)$, the first order condition can be expressed as:

$$\frac{\partial \left(F_R\left(\mu^{\star}\left(r,p_r\right),r\right) - \Omega\left(r,p_r\right)\right)}{\partial r} = 2F_R D_R\left(\frac{a_x}{2} - A\left(r - b_R\right)\right) - \frac{\partial\Omega\left(r,p_r\right)}{\partial r} = 0_{d\times 1}$$

where $b_R = \frac{\int_{\mathcal{Y}_R} ydG}{D_R}$ is the centroid (or barycenter) of the automated area. Absent other considerations it is optimal to set the robot's skills to the centroid of the automated region, this minimizes the (quadratic) loss from skill mismatch, thus maximizing the robot's output.[26] The robot's skills deviate from the centroid to account for gains from having higher skills (a_x), and for the cost of automation ($\partial\Omega(r,p_r)/\partial r$).

The first order condition with respect to p_r takes the usual form of equating marginal product to marginal cost. As in (11), the marginal product is $\text{MP}_R = F_R\mu_R^{\star}$:

$$\frac{\partial F}{\partial p_r} = F_R\left(\mu^{\star}\left(r,p_r\right),r\right)\mu_R^{\star} - \frac{\partial\Omega\left(r,p_r\right)}{\partial p_r} = 0 \tag{23}$$

Note, however, that the automation problem in (18) is not concave in r and thus condition (22) is only necessary and not sufficient (Urschel, 2017). The first order condition is descriptive of the properties that the robot's skills must satisfy relative to the automated region, but it does not pin down the set of tasks to be automated. Regardless, the problem can be solved numerically using a version of Lloyd's algorithm (Lloyd, 1982). The algorithm consists on finding the optimal assignment for a given value of r and p_r, then adjusting r and p_r to satisfy their respective first order conditions. The process is repeated until convergence. This algorithm has been proven to converge monotonically to a local minimum of the objective function (see Du et al. (2010) and references therein). Urschel (2017) gives sufficient conditions that can be checked for convergence to a global minimum.[27]

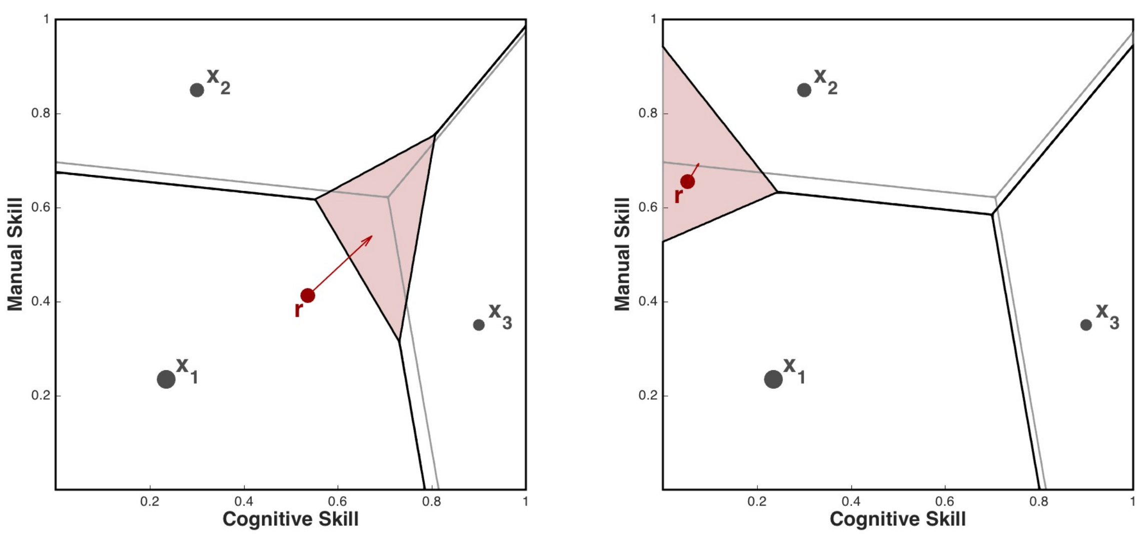
Figure 3 presents the solution to the automation problem assuming that q and q_R are given by (15), and that the automation cost is quadratic in the robot's skills: $\Omega(r) = r' A_R r$.

[26] This result is shared by the literature on the optimality of centroidal Voronoi diagrams, and is exploited extensively in optimal sensor placement problems. It is also linked to K-means and other vector quantization methods.

[27] In practice there are only finitely many candidates for a global minimum, making the selection of the solution simple. It is optimal to automate tasks around one of the vertices of the partition induced by the initial assignment (without automation). Aurenhammer (1987) shows there at most 2n-5 of these vertices in a diagram when the production function is quadratic in x and y, $d = 2$ and $n \geq 3$.

Figure 3: Directed Automation Example - Quadratic Automation Cost

(a) Symmetric weights (b) Asymmetric weights



Note: *The figures show the result of the automation problem taking the robot's mass as given in a two-dimensional skill space (cognitive and manual skills). Three types of workers are considered* $\{x_1, x_2, x_3\}$ *with mass* $P = \{0.5, 0.3, 0.2\}$*. Tasks are uniformly distributed over the unit square, i.e* $\mathcal{Y} = [0,1]^2$ *and* $g(y) = 1$ *for all* y*. The production function* q *is given by (15) with* $A = I_2$*,* $a_x = [0.2, 0.1]^{'}$ *and* $a_y = [0,0]^{'}$*. The automation cost function is:* $\Omega(r) = r^{'} A_R r$*, with* A_R *diagonal. The mass of the robot is fixed at* $p_r = 0.05$*. The assignment without the robot is presented in grey.*

The two panels differ only on the weights of cognitive and manual skills in the automation cost function. To keep the example simple I fix the mass of the robot exogenously.

Panel 3a assumes symmetric weights. It is then optimal to automate the tasks around the center vertex of the original assignment (without the robot). These are the tasks with the highest mismatch. Yet, because of the cost of endowing the robot with high cognitive and manual skills, it is not optimal to have place the robot's skills in the automated area. The introduction of the robot displaces all three workers from the tasks being automated, but not all types of workers become unemployed. The assignment reacts endogenously to the automation, favoring the more productive workers (x_2 and x_3) over the least productive worker (x_1). The boundaries of the occupations adjust, re-assigning tasks along the boundaries of $\mathcal{Y}_1$ to workers of type x_2 and x_3. Only x_1 is displaced after tasks are reassigned.

Panel 3b assumes asymmetric weights, with a higher weight on automating cognitive tasks. It is no longer optimal to automate the tasks in the center vertex due to the high cost of automating cognitive skills. Nevertheless, the automated tasks are still located along the boundary of occupations. In this case around the vertex formed by $\mathcal{Y}_1$, $\mathcal{Y}_2$ and the boundary of the task space. Since these tasks involve less cognitive skills it is possible to locate the robot's skills closer to the centroid of the automated region. As in panel 3a automation takes away tasks from workers, in this case only from x_1 and x_2. The assignment reacts to automation by reassigning tasks along the boundary of $\mathcal{Y}_1$ towards more productive workers. Again, x_1 is the only type of worker displaced by automation.

The two examples in Figure 3 capture a general feature of the automation problem: it is optimal to automate tasks around the vertices of the original assignment (without the robot) since those are the tasks with the highest mismatch. Which tasks are optimally automated is jointly determined by the original assignment and the properties of the cost function.

Worker training

The problem of worker training bears many similarities to the automation problem described above. In particular, the main question behind worker training, which skills should a worker have, is the same question behind the automation problem. The answer in both cases comes from the desire to reduce mismatch between tasks and workers. The same way that the robot's skills are chosen to minimize the mismatch in the automated area, the worker's skills are chosen to minimize mismatch across the tasks in her occupation. Crucially, as the skills of the worker change the assignment will change, altering the tasks in the worker's occupation.

Formally, the problem of optimal worker training is the same as that of choosing the robot's skills in (18), after appropriately modifying the cost function. Consider the problem of training worker n by choosing new skills $\tilde{x} \in \mathcal{S}$:

$$\max_{\{\tilde{x},T\}} \quad F(T,\tilde{x}) - \Gamma(\tilde{x}|x_n,p_n) \qquad \text{s.t. } \forall_\ell\, D_\ell \leq p_\ell \tag{24}$$

where the cost of changing skills (Γ) depends on the workers' current skills and mass. Following the same steps as in the automation problem, the first order condition of the problem is:

$$F(\lambda^\star(\tilde{x}),\tilde{x}) \int_{\mathcal{Y}_n} \frac{\partial \ln q(\tilde{x},y)}{\partial \tilde{x}} dG(y) - \frac{\partial \Gamma(\tilde{x}|x_n,p_n)}{\partial \tilde{x}} = 0_{d\times 1} \tag{25}$$

The interpretation is the same as before, with the first term capturing the net gains in output from changing the workers' skills. The objective is to minimize skill mismatch across the tasks in the worker's occupation given the cost of changing the workers' skills. If q is given by (15) this is achieved by setting $\tilde{x}$ to the centroid of the occupation, and adjusting for the weight of skills in production (a_x) and the marginal cost of changing the worker's skills. Even if acquiring skills was costless, it is not always optimal to increase the worker's skills, doing so can generate its own costs as mismatch increases with respect to the boundary tasks of the worker's occupation. The problem is further complicated by the ambiguous effects on total output, since training one worker can induce higher mismatch for other workers, as the assignment changes. Because of this, condition (25) is only necessary, and not sufficient, for characterizing the optimal worker training.

The worker training problem is particularly useful when thinking about the introduction of new tasks. New tasks are likely to involve skills for which no worker is particularly well suited, inducing higher mismatch at early stages of adoption. It is then optimal to train workers to acquire skills that better match the changes in their occupations brought up by the new tasks.[28] The introduction of new technologies, like computers and IT, changes occupations by directly modifying the tasks carried out by workers. This is potentially a major disruption since the workforce is likely not to have the right combination of skills to perform the new tasks. This hurts the population groups who experience the highest mismatch, while benefiting those whose skills align more with the new technology. In order

[28]This resembles the skill process of workers in Lise & Postel-Vinay (2015), where workers converge to the skill requirements of their occupations as they spend more time performing it.

to reduce the mismatch workers must train into new skills, more aligned with the new tasks. This training process will in turn modify occupations, changing the bundling of tasks and the roles of each type of worker in production.

3.2 Skill enhancing technology

Technical change can also complement the current skills of workers. This is the case with the introduction of software that complements cognitive over manual skills in the completion of tasks, or heavy machinery, such as cranes, that complements dexterity over brawn. Unlike automation, this type of technical change affects the productivity of workers across tasks without displacing them. But, as with automation, technical change is followed by a reassignment of tasks geared towards reducing the mismatch between workers and the tasks they perform. Workers who are more adept at the skills favored by the new technologies increase their productivity, while other workers lose their comparative edge. Changes in the boundaries of occupations are thus directed towards reducing mismatch in the skills complemented by new technologies.

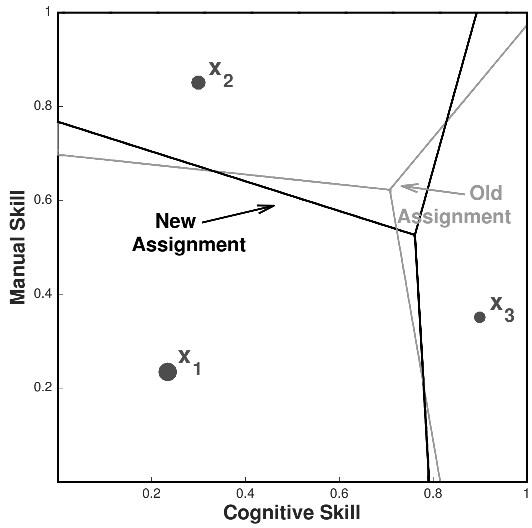
Just as with automation this type of technical change can also be directed. Which skills to favor depends on the joint distribution of workers and task, and the mismatch across different skills. In order to maximize production it is optimal to weight more those skills for which mismatch is lowest, concentrating technology on enhancing the skills at which the workforce already excels. This contrasts with the way in which automation is directed. Instead of replacing workers at the tasks they are ill-suited for, technology enhances the worker's productivity by weighting the skills with the better match, while reducing the importance of the skills that the workforce lacks.

To make the discussion precise I impose additional structure on how skill mismatch affects production. Consider two skills, cognitive and manual, and a production technology q as in (15). The relative importance of skills is then governed by matrix A, which I will assume to be diagonal taking the form:

$$A = \begin{bmatrix} \alpha & 0 \\ 0 & 1-\alpha \end{bmatrix}$$

where $\alpha \in [0, 1]$. Higher α makes cognitive match more important for production, while simultaneously reducing the importance of manual skill match. The problem is to choose the value of α optimally to maximize output, taking into account the cost of changing

Figure 4: Example - Increase in the Weight of Cognitive Skills



Note: *The figure shows the result of an increase in the weight of cognitive skills in a two-dimensional skill space (cognitive and manual skills). Three types of workers are considered* $\{x_1, x_2, x_3\}$ *with mass* $P = \{0.5, 0.3, 0.2\}$*. Tasks are uniformly distributed over the unit square, i.e* $\mathcal{Y} = [0,1]^2$ *and* $g(y) = 1$ *for all* y*. The production function* q *is given by (15) with* $A = diag(\alpha, 1-\alpha)$*, the value of* a_x *and* a_y *does not change the optimal assignment.*

technology and the changes in the assignment of tasks to workers:

$$\max_{\{T,\alpha\}} F(T,\alpha) - \Upsilon(\alpha) \qquad \text{s.t. } \forall_n \, D_n \leq p_n \tag{26}$$

The optimality condition for α can be obtained using the same techniques as before. The optimal α satisfies:

$$F(T,\alpha)(M_m - M_c) - \frac{\partial \Upsilon(\alpha)}{\partial \alpha} \geq 0 \tag{27}$$

Where M_s is total mismatch in skill s: $M_s = \sum_{n=1}^{N} \int_{\mathcal{Y}_n} (x_{n,s} - y_s)^2 \, dy$. The first term captures how much production would increase if α increases. The net gain in production is determined by the difference in total mismatch by skill, which depends on the assignment and the distribution of tasks and workers. If, for a given assignment, there is more mismatch in the manual dimension ($M_m > M_c$), the workforce is biased towards cognitive skills. It is then optimal to direct technical change towards cognitive skills by increasing α. In this way technology reinforces the workforce's bias by giving more weight to skills for which there is a better match. The gain in output is balanced by the marginal cost of changing α. Absent that cost it is optimal to shift all the weight towards one of the skills. Specializing production to depend only on the skill with the lowest mismatch in the workforce.

Figure 4 shows how the assignment of tasks to workers changes when the weight of cognitive skills increases. The boundaries of occupations shift and become less sensitive to differences in manual skills, discriminating across workers based on differences in their cognitive skills (as $\alpha \to 1$ the boundaries become vertical). As this happens workers' marginal product and substitutability change. Worker x_3 becomes less substitutable with others, as her cognitive skills differ from those of workers x_1 and x_2; recall from Proposition 2 that the elasticity of substitution decreases with the weighted distance between workers' skills. On the other hand, workers x_1 and x_2 become more substitutable, since they differ mostly in their manual skills, which are now less important in production.

These changes relate to observed patterns following the adoption of IT in production. There is a higher premium of for workers with high cognitive skill (like college graduates), and a lower premium for manual intensive workers relative to low skill workers (Katz & Murphy, 1992). As shown in Section 2, the difference in marginal products (and compensation) across workers is a function of the differences in output at the boundaries. From equation 12:

$$\underbrace{\ln q\left(x_n, y\right) - \ln q\left(x_\ell, y\right)}_{\text{Diff. in Output}} = \underbrace{a_x^{'}\left(x_n - x_\ell\right)}_{\text{Diff. in Skills}} - \underbrace{\left(\left(x_n - y\right)^{'} A\left(x_n - y\right) - \left(x_\ell - y\right)^{'} A\left(x_\ell - y\right)\right)}_{\text{Diff. in Mismatch}}$$

When technology weights more cognitive skills differences in cognitive skills are amplified through the mismatch term, while differences in manual skills are down-weighted. As a consequence differences in marginal products and compensation become more influenced by differences in cognitive skills.

4 Tasks and Unemployment

In Section 2 I assume that if a task is not assigned to any worker there is no output ($q\left(\emptyset, y\right) = 0$). This, together with the way output is aggregated into the final good (equation 3) implies that there is no production unless all tasks are assigned. An alternative approach is to consider the aggregation only across tasks which are performed, the ones assigned to a worker. In this way it is possible to leave tasks unassigned without shutting down the production of the final good. A consequence of leaving tasks unassigned is that some workers are left unemployed. Which workers are unemployed, as well as the level of unemployment, depends on the assignment.

To formalize this idea consider an alternative to the final good technology described in equation 3:

$$F(T) = \exp\left(\int\limits_{\mathcal{Y}\setminus\mathcal{Y}_\emptyset} \ln q(T(y), y)\, dG\right) - 1 \tag{28}$$

where the assignment T is extended so that tasks can be unassigned, i.e. $T : \mathcal{Y} \to \mathcal{X} \cup \{\emptyset\}$, and $\mathcal{Y}_\emptyset$ denotes the set of tasks left unassigned, i.e. $\mathcal{Y}_\emptyset = T^{-1}(\{\emptyset\})$. In this way only the tasks that are assigned are considered in the aggregation. The level of the production needs to be adjusted since leaving tasks unassigned opens the possibility for a free lunch. If only a measure-zero set of tasks is assigned the integral in (28) is equal to zero, regardless of the assignment, and output is therefore 1. The subtraction takes care of this.

Its immediate that the result form the aggregation in (28) is equivalent to having $q(\emptyset, y) = 1$ in the original formula 3, extending T to take values over $\mathcal{X}$ and the unassigned option. That way tasks that are left unassigned (assigned to the empty set) don't add to the integral, obtaining the integral in (28) as a result. Adopting this convention turns out to be useful because it allows me to apply Proposition 1 in the same way as in Section 2. Leaving a task unassigned is equivalent to assigning it to a worker '$\emptyset$', which is in infinite supply, has an outside option of zero, and produces $q(\emptyset, y) = 1$ in all tasks.

The main difference with the results of Section 2 is that the level of the worker's outside option ($\underline{w}$) affects the assignment. To simplify calculations I will assume in this section that the outside option is given by a fraction of total output: $\underline{w}(T) = \underline{\lambda} F(T)$.[29] Under this assumption there exists a vector $\lambda^\star \in \mathbb{R}^N_+$ such that $\min \lambda^\star_n = 0$ and occupations are given by:

$$\mathcal{Y}_n = \{y \in \mathcal{Y} \,|\, \forall_\ell \; \ln q(x_n, y) - \lambda^\star_n \geq \ln q(x_\ell, y) - \lambda^\star_\ell \quad \wedge \quad \ln q(x_n, y) - \lambda^\star_n \geq \underline{\lambda}\} \tag{29}$$

This is the equivalent to condition (9), it differs in the introduction of the second inequality, which compares the output of worker n in the task with the minimum payment the worker must receive. The second inequality comes from ensuring that it is profitable to assign the task; the outcome if the task is unassigned is $\ln q(\emptyset, y) = 0$, and the compensation of the worker depends on $\lambda^\star_n + \underline{\lambda}$. The unassigned tasks are:

$$\mathcal{Y}_\emptyset = \{y \in \mathcal{Y} \,|\, \forall_n \; \ln q(x_n, y) - \lambda^\star_n < \underline{\lambda}\} \tag{30}$$

[29] Without this assumption it is not possible to determine the value of λ independently of the assignment T. The term $\underline{\lambda}$ in (29) has to be replaced by $\underline{w}/F(T)$.

A necessary condition for a task to be assigned s that $q(x_n, y) \geq 1$, and the higher $\underline{\lambda}$ is, the fewer tasks are assigned for production.

As in Section 2, the marginal product of a worker is given as in equation (11) and its compensation is given by $w_n = \lambda_n^\star F(T^\star) + \underline{w}$.

To fix ideas consider q as in (15), depending on the quadratic mismatch between worker and task's skills. Under that technology:

$$\ln q(x, y) = a_x^{'} x + a_y^{'} y - (x - y)^{'} A (x - y)$$

This provides a clear geometrical interpretation for which tasks are left unassigned. Workers will be assigned to a task only if it the mismatch is no greater than $a_x^{'} x + a_y^{'} y - \underline{\lambda}$.[30] This condition guarantees that enough output is produced by the worker for it to be profitable to perform the task and cover the worker's outside option. However, the condition does not imply that the task will be assigned to the worker, this depends on the comparison between workers' productivity as in Section 2 (see the first inequality in equation 29).

Which tasks to perform will depend critically on which tasks are more productive given current technology. This idea is captured by a_y, which determines which tasks generate more output, regardless of which worker performs them.[31] A higher cognitive weight in a_y makes cognitive intensive tasks more likely to be performed. For example, one of the effects of the increased use of information technology is to make cognitive intensive tasks more productive; as a consequence, it becomes optimal to perform more cognitive intensive tasks. Opposite changes can occur on the relevance of manual intensive tasks in production, shifting workers from manual to cognitive intensive tasks.

Figure 5 shows the optimal assignment in the model allowing for task to be unassigned, and workers to be unemployed. The two panels differ on the weight of task skills in determining the output of a task, as measured by a_y. I assume that $a_y = \overline{a}_y [\alpha_y, 1 - \alpha_y]^{'}$ and I vary the relative importance of skills by choosing the weight $\alpha_y \in [0, 1]$. A higher value of α_y makes cognitive intensive tasks more productive (tasks along the 45º line do not change their productivity with α_y).

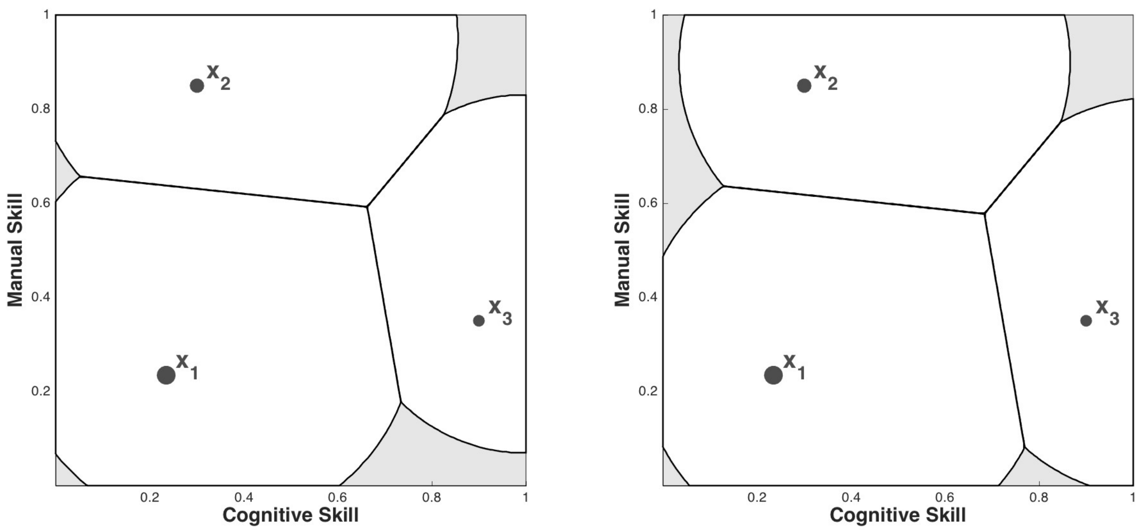
Panel 5a presents the assignment under equal skill weights in a_y. The grey areas represent unassigned tasks. It is optimal not to perform tasks for which agents have high mismatch, as in Section 3.1 these tasks are located along the boundaries of the task space, and the

[30]With $a_y = 0$, a tasks will be assigned only if it lies in a 'circle' of radius $\sqrt{a_x^{'} x}$ around the skills of the worker. The shape of the 'circle' depends on the weights in matrix A.

[31]When all tasks must be performed the value of a_y does not affect the assignment. This is immediate from replacing (15) into 9.

Figure 5: Assignment Example - Unemployment

(a) Symmetric Weights - $\alpha_y = 1/2$ (b) Asymmetric Weights - $\alpha_y = 3/4$



Note: *The figures show the assignment in a two-dimensional skill space (cognitive and manual skills). Three types of workers are considered* $\{x_1, x_2, x_3\}$ *with mass* $P = \{0.5, 0.3, 0.2\}$. *Tasks are uniformly distributed over the unit square, i.e* $\mathcal{Y} = [0,1]^2$ *and* $g(y) = 1$ *for all* y. *The production function* q *is given by (15) with* $A = I_2$, $a_x = [0.2, 0.1]'$ *and* $a_y = \overline{a}_y [\alpha_y, 1-\alpha_y]'$, *with* $\overline{a}_y \in \mathbb{R}_+$ *and* $\alpha_y \in [0,1]$. *The worker's outside option is 0.*

vertices of the assignment. Even though the wights on a_y are symmetric and the weights on a_x favor cognitive skills, most of the unassigned tasks involve high cognitive skills. This is because of the distributions of skills in the population. In the example there are relatively few x_3 workers, and so, performing the high-cognitive tasks comes at the cost of greater mismatch for workers x_1 and x_2, as the boundaries between them and x_3 would have to shift rightwards. Its worth noting that the assignment is such that only worker x_1 is unemployed. x_1 is the least productive worker type.

In Panel 5b the weights on skills change, making cognitive intensive tasks more productive, and manual intensive tasks less productive. As a response to this change workers x_1 and x_3 take over tasks in the bottom-right corner of the space, at the expense of tasks along the vertical axis. The higher productivity makes it worthwhile to reassign workers towards cognitive intensive tasks; doing so shifts the boundaries of $\mathcal{Y}_2$ towards x_1 and x_3, and away from the vertical axis. Unemployment is still concentrated in workers of type x_1.

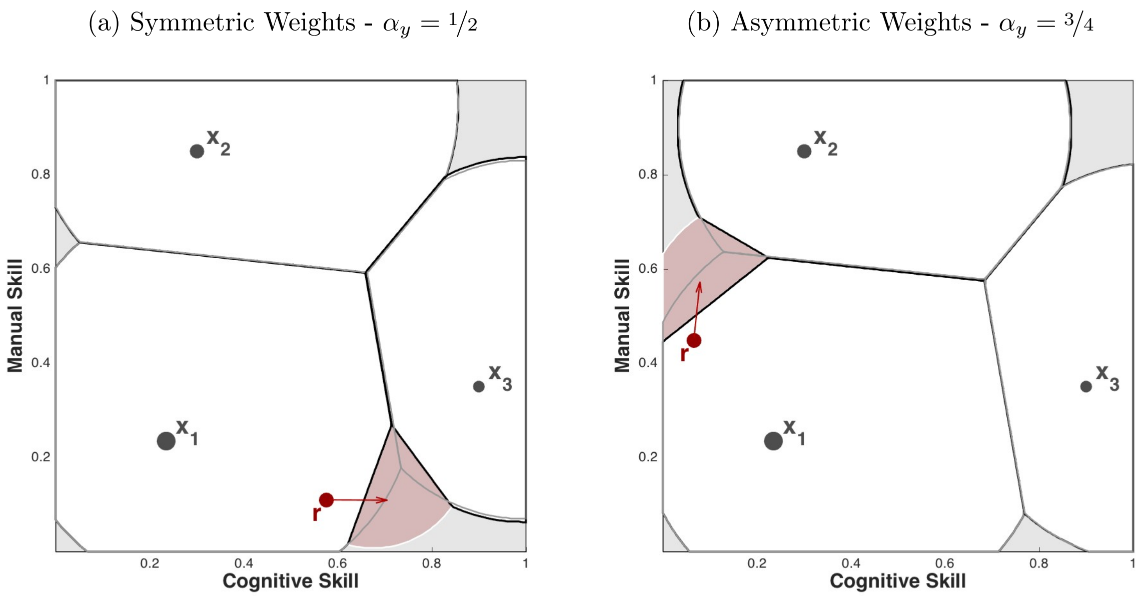
Automation and unassigned tasks

The characterization of automation as a worker replacing technology given in Section 3.1 changes once tasks can be left unassigned. It is now possible to direct automation towards the tasks which were previously unassigned, that is, tasks which are not worthwhile for workers to perform (because of low productivity), or tasks for which workers don't have the appropriate skills (high mismatch). If this happens automation does not displace workers. Moreover, performing additional tasks necessarily increases output, potentially raising worker's marginal products and wages.

Whether or not it is optimal to automate unassigned tasks or to displace workers depends comparing the cost of automating a tasks with the mismatch in the worker-task assignment. Even though the mismatch is highest for unassigned tasks, it might be too costly to engineer the technology necessary to automate those tasks. In general it will turn out to be optimal to automate tasks along the boundaries of occupations and unassigned tasks. As a result automation ends up partially displacing workers. To illustrate this I expand the example in Figure 5 by solving the optimal automation problem. The results are presented in Figure 6. Production technology is the same for workers and the robot, and is given by (15). The cost of automation is quadratic in skills as in the example in Figure 3.

Panels 6a and 6b present similar results, with the robot being placed so as to automate part of the cognitive/manual intensive tasks that were unassigned. The robot is only partially

Figure 6: Assignment Example - Unemployment and Automation



Note: *The figures show the assignment in a two-dimensional skill space (cognitive and manual skills). Three types of workers are considered* $\{x_1, x_2, x_3\}$ *with mass* $P = \{0.5, 0.3, 0.2\}$. *Tasks are uniformly distributed over the unit square, i.e* $\mathcal{Y} = [0,1]^2$ *and* $g(y) = 1$ *for all* y. *The production function* q *is given by (15) with* $A = I_2$, $a_x = [0.2, 0.1]^{'}$ *and* $a_y = \overline{a}_y [\alpha_y, 1-\alpha_y]^{'}$, *with* $\overline{a}_y \in \mathbb{R}_+$ *and* $\alpha_y = 1/2$. *The worker's outside option is 0. The automation cost function is:* $\Omega(r) = r^{'} A_R r$, *with* A_R *diagonal. The mass of the robot is fixed at* $p_r = 0.03$. *The assignment without the robot is presented in grey.*

displacing workers since it is taking over unassigned tasks. Thus, the mass of unemployed workers increases, but less than the mass of tasks being automated (0.01 vs 0.03). Output increases due to the production of new tasks and the reduction in mismatch in some of the old tasks.

The two panels in Figure 6 also show how the incentives for automation change as technology favoring the production of certain types of tasks change. If technological change favors cognitive intensive tasks over manual intensive tasks, workers are reassigned away from the latter and into the former (see Figure 5). Consequently, production can be increased by directing automation towards manual intensive tasks, in a way that disrupts the optimal assignment of tasks to workers the least as possible. In this scenario technological change makes new tasks available for workers, while leaving other tasks unassigned, automation follows by taking over tasks that are no longer worthwhile for workers to perform. [32]

5 Empirical Application

I now use the model developed in Section 2 to examine U.S. occupational data. First I estimate the model using data on occupation characteristics and wages. I then use data on the automatability of occupations to infer the cost of automation. Finally, I make use of the results in Section 3.1 to solve for the optimal direction of automation.

Data sources The main source of data for the estimation of the model is the 2010 version of O*NET.[33] The O*NET is the U.S. Department of Labor Occupational Characteristics Database, it contains information on attributes of 974 occupations. Attributes characterize the knowledge, skills, and abilities that are used to perform the tasks that make up an occupation. The data reports the importance of 277 such attributes, as rated by analysts with expertise in each occupation.

I complement the O*NET data with tabulations from the 2010 Occupational Employment

[32] This idea is similar in spirit to Acemoglu & Restrepo (2018b)'s race between man and machine. As in their paper, changes in technology lead to a reassignment of workers towards more complex (and newer) tasks, while relatively simpler (and older) tasks are automated, displacing workers in the process. My model differs in a key aspect from theirs: the set of tasks to be performed is held fixed throughout. Yet, the environment I present can be reinterpreted, by considering the space of existing tasks to be larger than the set of tasks currently performed. Technological change, as well as changes in the skills of the workforce, continuously change the set of tasks that workers perform, moving towards more complex tasks (previously unassigned), and away from simpler tasks, that can then be automated.

[33] Data is available at: https://www.onetcenter.org/db_releases.html

Statistics (OES), provided by the Bureau of Labor Statistics.[34] The OES include data on employment and average annual wages by occupation for non-military occupations. It covers a total of 796 occupations with total employment of 127.1 million workers.

I merge the two datasets matching the SOC and title of each occupation. The resulting data contains 800 occupations with total employment of 119 million workers. The discrepancy between the number of occupations in the original OES data and the final sample I use is explained by the higher detail of occupations in the O*NET data. In particular, the OES data lumps smaller or specialized occupations into an 'all other' category. I am able to match these categories to individual occupations contained in the O*NET sample. The loss of employment is also explained by the 'all other' category. Not all of these occupations have a counterpart in the O*NET sample, unmatched occupations are dropped from my sample since I don't observe any of their attributes.

Occupations' skill requirements To obtain a measure of the skill requirements of each occupation I proceed in two steps, similar to the ones used by Guvenen et al. (2015) and Lindenlaub (2017). First, I categorize attributes into skill groups. Second, I reduce the dimension of each group by taking the first principal component of the group of attributes as my measure of skill.[35] For the exercise below I will only consider two skill groups, namely cognitive and manual skills. In total I use 69 cognitive attributes and 47 manual attributes, the complete list of attributes is reported in Appendix D. Figure 7a shows the skill measure of the 800 occupations in the sample in the cognitive-manual skill space.[36]

Distribution of tasks I take the distribution of skill requirements across occupations as informative of the underlying (continuous) distribution of skill requirements across tasks. I construct a non-parametric estimate of the distribution of skill requirements (g) by smoothing the (weighted) distribution of skill requirements of occupations with a Gaussian kernel. The level curves of the resulting function are presented in Figure 7b. Darker regions have a higher density.

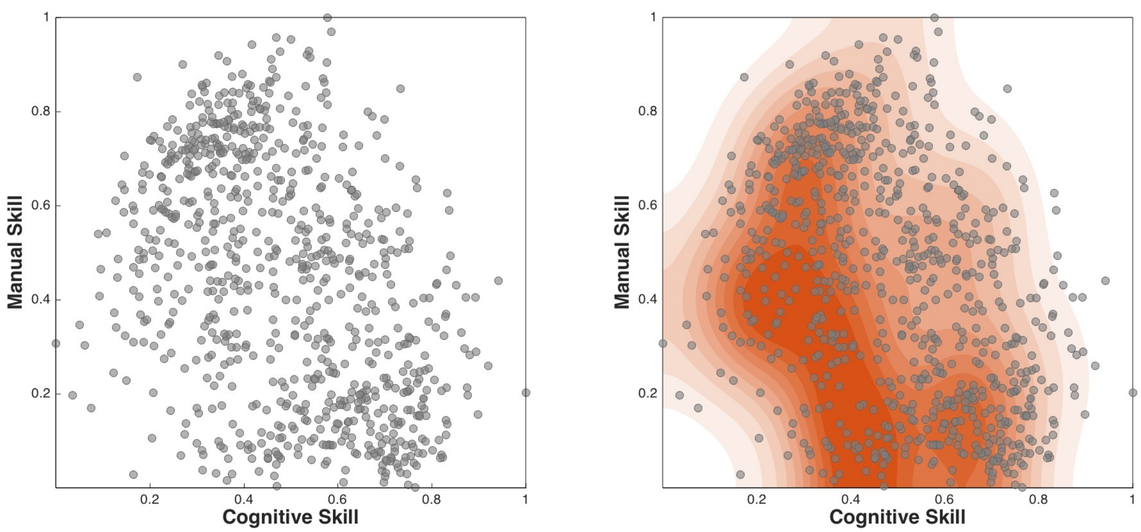
[34] Data is available at: https://www.bls.gov/oes/tables.htm

[35] I weight occupations by employment when performing principal component analysis on them. The results are not noticeably affected if I do not weight occupations. I have also repeated this exercise by focusing on 6 hand-picked attributes for each skill group, and using their average value as the measure of skill. There is no change in the general distribution of skills.

[36] The correlation between the constructed measure of skills is -0.23.

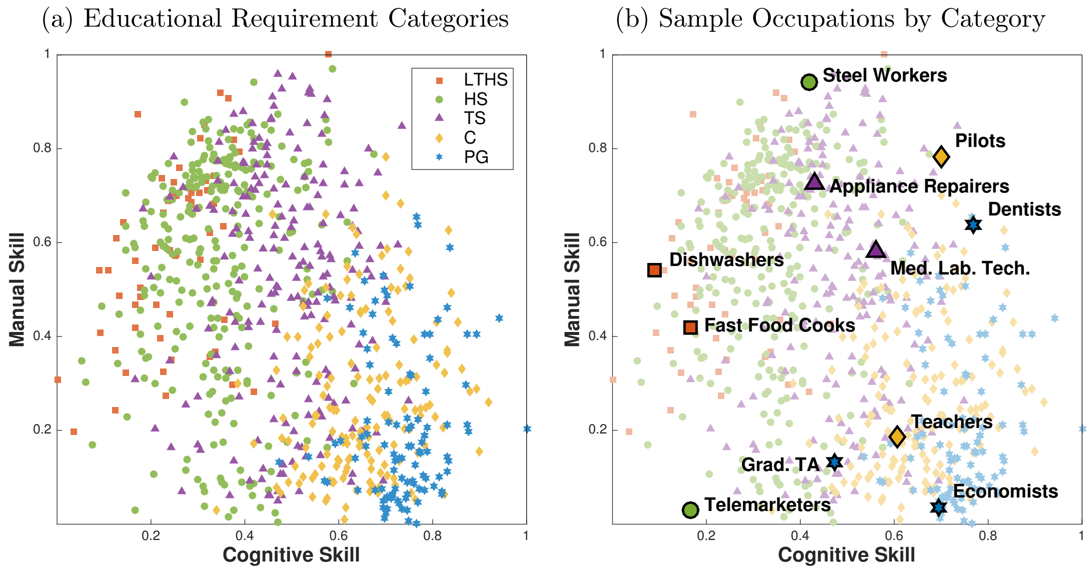
Figure 7: Occupation Skill Measures and Task Distribution

(a) Occupation Skill Measure (b) Tasks' Skill Requirement Distribution



Note: *The left panel shows the cognitive and manual skill components of the 800 occupations in my sample. Each point in the graphs corresponds to one occupation. Occupational skill requirements are computed respectively as the first principal component of cognitive and manual occupational attributes.*
The right panel shows the level curves of the distribution of tasks' skill requirements, inferred from the distribution of skill components of occupations, weighted by employment.

Figure 8: Occupations by Educational Requirement Categories



Note: *The left panel shows the cognitive and manual skill components of the 800 occupations in my sample by educational requirement categories. Each point in the graphs corresponds to one occupation. Occupational skill requirements are computed respectively as the first principal component of cognitive and manual occupational attributes. Educational requirement categories are: less than high-school (LTHS), high-school (HS), trade-school (TS), college (C), and post-graduate (PG). The right panel highlights sample occupations in each group.*

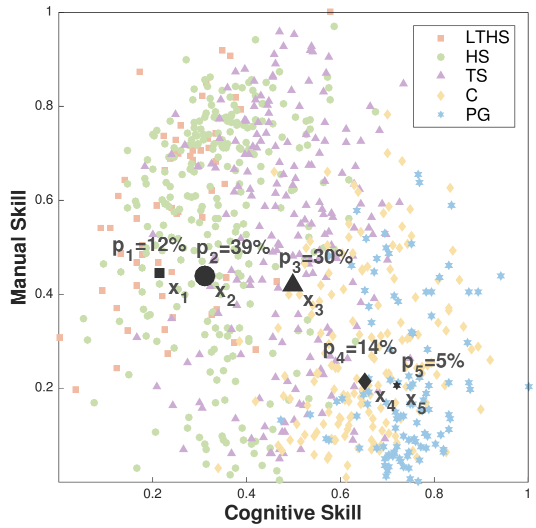
Distribution of workers I use occupational data to infer the distribution of workers in the economy. To do so I group the occupations in my sample into five educational categories, which correspond to the educational requirement reported by O*NET for each occupation.[37] Categories are: less than high-school, high-school, vocational training (trade-school), college and post-graduate education.

Figure 8a presents the occupations in the sample grouped by educational requirement. The ordering of groups in the cognitive-manual skill space becomes apparent form the figure, with occupations requiring less than high-school education concentrated in the north-west boundary of the scatter, and successive categories of higher education moving towards higher cognitive skill requirement. Occupations requiring college and post-graduate education have the highest cognitive skill requirement and the lowest manual skill requirement. Figure 8b highlights occupations sample occupations of each group.

I interpret each category of O*NET occupations as the (meta-)occupation of a type of

[37] The code for the educational requirement of the occupation is 2D1. For occupations with missing educational requirement I use O*NET's job zone classification to impute the value. Job zones are a mixture of educational and experience requirements for an occupation. The five job zone categories overlap to a great extent with the educational requirement variable.

Figure 9: Distribution of Workers by Educational Requirement Categories



Note: *The figure shows the estimate for worker's skill (x_n) and mass (p_n) and the underlying occupations in the five educational requirement categories. Educational requirement categories are: less than high-school (LTHS, x_1), high-school (HS, x_2), trade-school (TS, x_3), college (C, x_4), and post-graduate (PG, x_5). Worker's skill (x_n) is computed as the employment-weighted average of the cognitive and manual skill requirements of the occupations in the worker's category. Worker's mass (p_n) is computed as the employment share of the occupations in the worker's category.*

worker. I infer the mass (p_n) and skills (x_n) of each type of worker from the occupations in its category. Thus, the mass of the each type of worker is the share of employment of the occupations in the category; the skills of each type of worker are obtained as the employment-weighted average of the skill requirements of the occupations in the category.[38] Figure 9 presents the resulting distribution of workers in the skill space. Each group should be interpreted as the average worker of occupations that require a given level of education, they do not correspond to the average worker with that level of education.

Production technology and assignment Task output is parametrized as in (15) with $\alpha_y = [0\,,\,0]'$ and A a diagonal matrix:

$$q\left(x,y\right) = \exp\left(a_x' x - \left(x-y\right)' A \left(x-y\right)\right) \qquad \text{where: } A = \begin{bmatrix} A_{cc} & 0 \\ 0 & A_{mm} \end{bmatrix} \tag{31}$$

[38]Other studies, like Lindenlaub (2017), also use O*NET's occupational skill requirements as a measure of worker's skills. In contrast, studies like Lise & Postel-Vinay (2015) and Guvenen et al. (2015) use worker-side data from the NLSY to compute worker's skills.

There are then four parameters to estimate characterizing the role of mismatch in production (A_{cc} and A_{mm}) and the effect of skills in wages (a_x). I estimate the parameters in a two step procedure. First choosing the ratio ${}^{A_{cc}}/_{A_{mm}}$ to minimize the classification error between the model's assignment and the data over the category of occupations. Then, the scale of the mismatch (A_{mm}) and the value of a_x are estimated to match wages by occupational category.

Recall from equation (16) that under this technology the boundaries of occupations are given by hyperplanes, whose normal vectors are defined by A. Thus the ratio ${}^{A_{cc}}/_{A_{mm}}$ fully determines the assignment given the estimated distribution of tasks' skill requirements (g), and the distribution of workers (x_m, p_n). For a given value of ${}^{A_{cc}}/_{A_{mm}}$ it is possible to classify each occupation in the sample according to the type of worker it is assigned to. I choose the value of ${}^{A_{cc}}/_{A_{mm}}$ to minimize the classification error between the model's assignment and the observed educational requirement of the occupation.

Having estimated ${}^{A_{cc}}/_{A_{mm}}$ it is possible to use wage data to estimate the remaining parameters. To do this we first relate the value of A_{mm} and a_x to the multipliers λ associated with the optimal assignment. From (16) it is possible to obtain an equation for λ_n as a function of skills and mismatch with respect to the lowest paid worker:

$$\lambda_n = a_{x'}\left(x_n - \underline{x}\right) - \underbrace{\left(x_n - y_n\right)' A \left(x_n - y_n\right)}_{x_n \text{ mismatch}} + \underbrace{\left(\underline{x} - \underline{y}\right)' A \left(\underline{x} - \underline{y}\right)}_{\underline{x} \text{ mismatch}} \tag{32}$$

where $\underline{x}$ are the skills of the lowest paid worker, and y_n and $\underline{y}$ are a boundary tasks of workers x_n and $\underline{x}$ respectively.

The second and third terms in (32) give the mismatch of workers x_n and $\underline{x}$ at the boundaries of their occupations. Although mismatch is not directly observable, it can be backed out using only the estimate of ${}^{A_{cc}}/_{A_{mm}}$ since it determines the boundaries of the assignment. Its easy to show from (16) that the multipliers of the assignment with $a_x = \vec{0}$ and $A^0 = \operatorname{diag}\left({}^{A_{cc}}/_{A_{mm}}, 1\right)$ provide an exact measure of the mismatch terms in (32):

$$\begin{aligned} A_{mm}\left(\lambda_n^0 - \underline{\lambda}^0\right) &= A_{mm}\left(\left(\underline{x} - \underline{y}\right)' A^0 \left(\underline{x} - \underline{y}\right) - \left(x_n - y_n\right)' A^0 \left(x_n - y_n\right)\right) \\ &= \left(\underline{x} - \underline{y}\right)' A \left(\underline{x} - \underline{y}\right) - \left(x_n - y_n\right)' A \left(x_n - y_n\right) \end{aligned}$$

where λ^0 is the multiplier vector of the auxiliary assignment and $\underline{\lambda}^0$ is the multiplier of the lowest paid worker.[39]

[39]No that even though $\underline{\lambda} = 0$ it is not necessarily the case that $\underline{\lambda}^0 = 0$. λ^0 measures only the mismatch along the boundaries relative to the worker with the highest mismatch. $\lambda^0 = 0$ only for the worker(s) with

Table 1: Estimates of Production Technology

A_{cc}	A_{mm}	a_x^c	a_x^m
7.25	1.98	1.61	0.28

Note: *Estimated values for the parameters of the production technology in (31). A_{cc} and A_{mm} give the weight of cognitive and manual mismatch respectively. a_x^c and a_x^m give the weight of worker's cognitive and manual skills on marginal products and wages.*

Finally, it is possible to construct an empirical measure of the multipliers λ from (13):

$$\hat{\lambda}_n = \frac{w_n - \min_\ell (w_\ell)}{F(T)}$$

where the value of κ is pinned down by the lowest observed wage since $\min_\ell(\lambda) = 0$, and $F(T)$ is a measure of total output.[40] Wages are taken as the employment-weighted average of the average annual wage across occupations in each educational requirement category. The estimates of A_{mm} and a_x are then obtained from the fitting the following linear relation:

$$\hat{\lambda}_n = a_{x'}(x_n - \underline{x}) + A_{mm}\left(\lambda_n^0 - \underline{\lambda}^0\right)$$

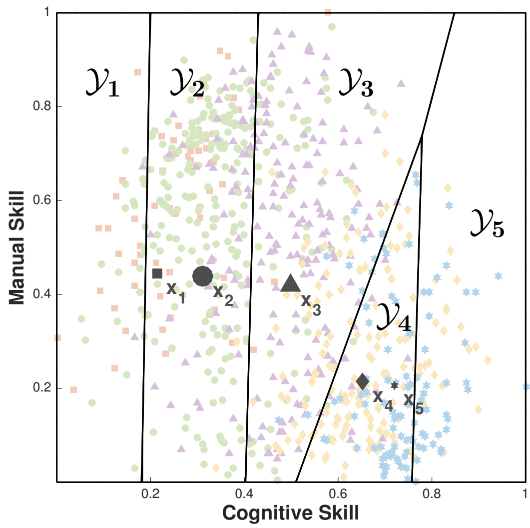
Table 1 shows the estimates for the parameters of the production technology for task output, and Figure 10 shows the assignment given those estimates. The value weight on cognitive mismatch (A_{cc}) is 3.65 times the weight on manual mismatch (A_{mm}), reflecting the division across educational requirement categories in terms of incrementally higher cognitive skill requirements (see Figure 8a), and the higher dispersion of manual skill requirements in each category. The value of a_x reflects the pattern of higher wages for more cognitive demanding occupational categories, thus having $a_x^c > a_x^m$. An increase in cognitive skill of 0.01 implies an increase in wages of 1.61% more of total output, compared to a 0.28% increase for an equal increase in manual skills.

Table 2 presents the wage of each educational requirement category in the data and the ones implies by the assignment under the estimated task output technology. The wage of

the highest mismatch. In contrast λ also includes the importance of worker's skills in production, measured by a_x. The worker with the highest mismatch is not necessarily the lowest paid worker if her skills are high enough.

[40]Following Karabarbounis & Neiman (2014) I take output as corporate sector value added, measured as 60% of GDP for 2010. This implies a labor share of 62% for the full BLS sample, and 57% for my matched O*NET sample. The difference is explained by the loss of employment and the lower average wage of my sample, $42000 compared with $44000 2010 dollars.

Figure 10: Assignment



Note: *The figure shows the assignment of tasks to workers in the cognitive-manual skill space given the output task technology in (31) and the parameter estimates in Table* 1.

Table 2: Wages in the estimated model

Wages	Less Than High School	High School	Trade School	College	Post-Graduate
Data	22.8k	29.9k	48.9k	79.7k	89.6k
Model	22.8k	26.7k	50.7k	73.7k	87.0k

Note: *The table presents the average annual wages in 2010 dollars across occupations in each educational requirement category, and the implied wages for each worker type under the assignment shown in Figure* 10.

the less than high school category is matched by construction by setting κ so as to match it. The other wages are obtained as in (13). In general, wages in the model underestimate the level of observed wages, with the highest discrepancy in high school wages (10.8% lower in the model than in the data).[41]

Cost of automation Occupations vary in how automatable they are depending on the combination of skills that are required to perform the tasks that compose them. I use estimates of the automatability of occupations provided by Frey & Osborne (2017) to estimate the cost of automation.[42] The main assumption is that the cost of automation is inversely related to the degree of automatability of an occupation. Figure 11a shows the occupations in my sample with darker points reflecting lower indices of automatability. I infer the shape of the cost function non-parametrically by fitting a Gaussian kernel to Frey & Osborne (2017)'s automatability index on the cognitive-manual skill space. Figure 11b shows the level curves of the estimated cost function.

I scale the automation cost function to match the average cost of an industrial robot per replaced worker to the cost of automation in manufacturing occupations (SOC code 51). I obtain the average cost of an industrial robot from the International Federation of Robotics (IFR) annual report.[43] The cost is \$147,883 for 2010. I take the worker replacement ratio from Acemoglu & Restrepo (2017) who estimate that an industrial robot replaces between 4 and 6.2 workers. I take the lower bound of their estimate. Finally, I assume that the cost function is linear in the mass of the robot.

Automation problem Figure 12 shows the assignment under the optimal robot placement. It is optimal to automate manual intensive task along the upper edge of the skill space, placing the robot at $r = [0.43\,,\,0.98]$. The automated region accounts for 5.6% of the tasks ($p_r = 0.056$), displacing the same share of workers.[44] The cost of the robot is \$157,500, or \$39,400 per unit of replaced-workers. Output increases 3.17% as a result of the decrease in mismatch. Output net of automation costs increases 0.25%.

Both less than highs school ($\Delta D_1 = -0.039$) and high school workers ($\Delta D_1 = -0.017$)

[41] The understatement of high-school, college and post-graduate wages in the model reduces the labor share to 56.7%, compared to 57.3% in the sample.

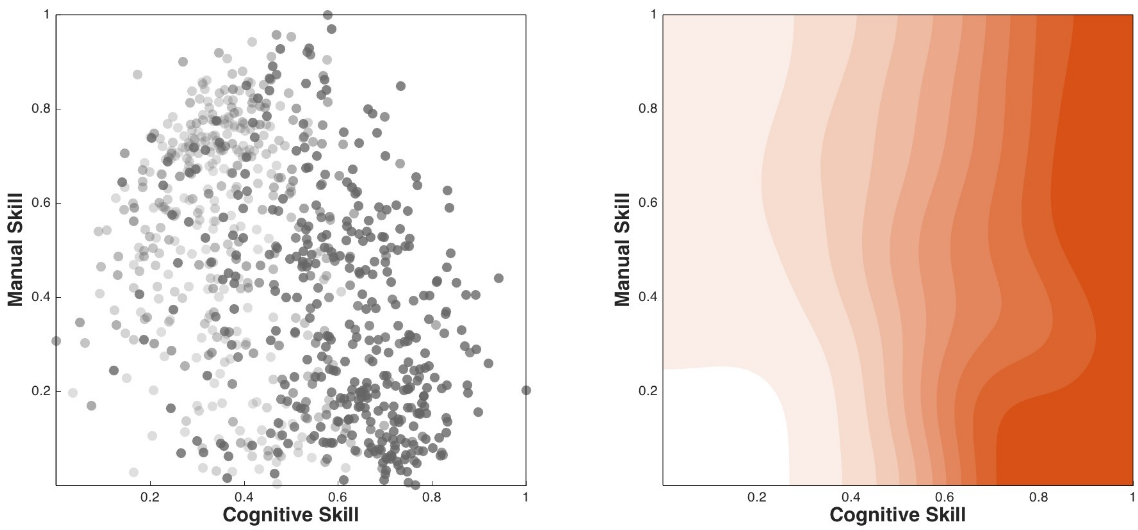
[42] Frey & Osborne (2017) provide an index of automatability for the occupations in the 2010 O*NET based attributes related to 'computerization bottlenecks'. The index goes from 0 to 1 and gives the likelihood that an occupation is fully automatable given its attributes.

[43] See the executive summary in https://ifr.org/free-downloads/

[44] Occupations that fall in the automated region include steel and metal workers, machine operators in the plastic industry, construction carpenters and continuous mining machine operators.

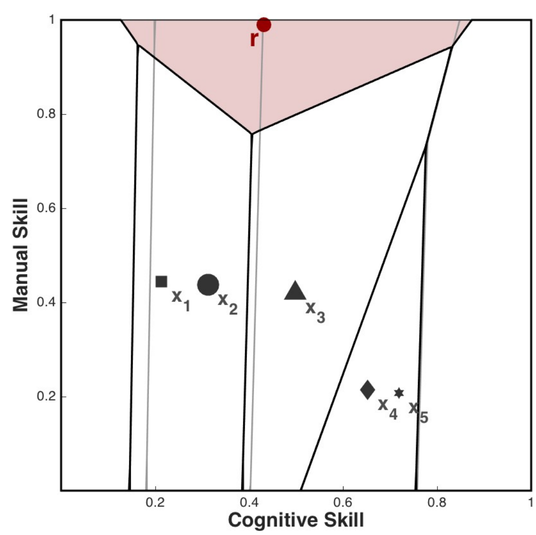
Figure 11: Cost of Automation

(a) Automatability Index by Occupation (b) Cost of Automation - Level Curves



Note: *The left panel shows the occupations in Frey & Osborne (2017)'s sample discriminated by their automatability index. The index measures the likelihood that an occupation is automatable given current technology. Darker occupations correspond to lower indices of automatability, or occupations which are less likely to be automatable.*
The right panel shows the level curves of the cost function inferred from the automatability index of occupations. Darker regions correspond to higher automation costs.

Figure 12: Assignment with Optimal Robot Placement



Note: *The figure shows the assignment of tasks to workers and the robot in the cognitive-manual skill space given the estimated automation cost. The shaded region corresponds to automated tasks, it accounts for 5.6% of tasks. Both x_1 and x_2 workers are displaced by automation.*

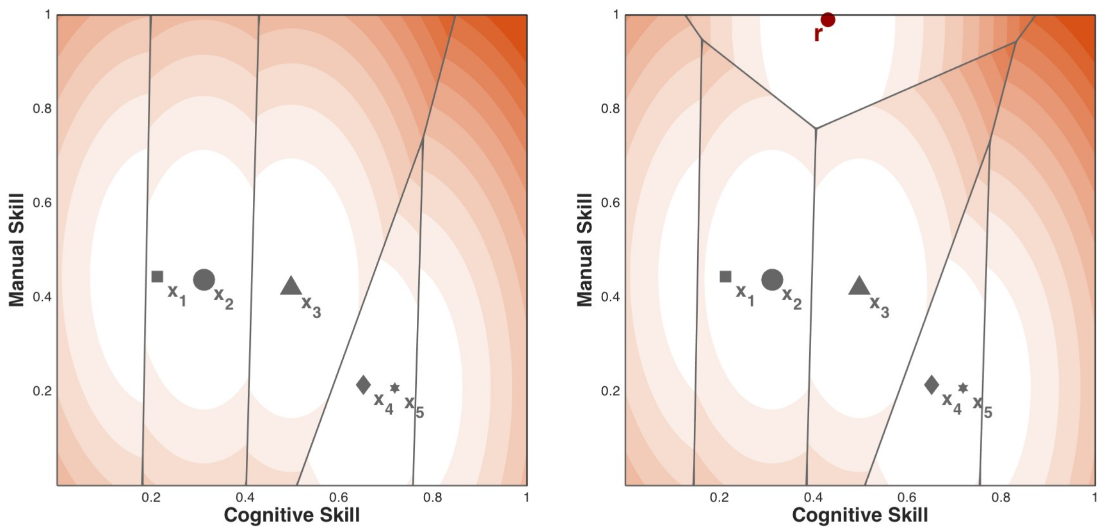
Table 3: Wages after Automation

	High School	Trade School	College	Post-Graduate
Δ%Wages	-14.4%	-13.3%	-8.3%	-6.8%

Note: *The table presents the percentage change of wages for each worker type after automation. Less than high school workers (x_1) do not have changes in their wage.*

Figure 13: Mismatch and the Assignment of Tasks to Workers

(a) Mismatch without Automation (b) Mismatch with Automation



Note: *The figures show the level curves of mismatch across tasks under the assignment without automation (left panel) and with automation (right panel). Darker regions correspond to higher mismatch.*

are displaced under the new assignment. As a consequence wages are compressed at the bottom, reflecting the drop in marginal productivity of high school workers. Under the new assignment both x_1 and x_2 workers have zero marginal product. The wages of the other workers decreases as well, as seen in Table 3. Recall from (12) that marginal products are given by differences in productivity across workers. As the assignment changes and workers with lower productivity are displaced two effects come into play. First, the mismatch at the boundaries decreases for the displaced workers. Second, the new assignment implies higher mismatch for workers affected by automation, but reassigned to new tasks (as x_3). Both effects decreases the difference between workers' productivity at the boundaries, compressing wages. Although in general these effects can be counteracted by the increase in output is not large enough as to increase wages.

Finally, it is worth noting that the tasks being automated are not those for which the cost of automation is lowest. As was mentioned in Section 3.1, the automation problem balances the cost of automation with the benefits that stem from the decrease in mismatch in the assignment. It is by decreasing mismatch between tasks' skill requirements and workers' skills that output increases. Figure 13 shows the level curves of the mismatch across tasks for the assignment with and without automation. The reduction of mismatch compensates for cost of the robot along the upper end of the skill space.

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A Mathematical preliminaries

I include definitions and theorems that are relevant for the proofs in the text and in Appendix B.

Definition 1. [Probability Space] A probability space is a triplet $(A, \mathcal{A}, \mu)$ of a set A, a σ-algebra $\mathcal{A}$ on that set and a probability measure $\mu : \mathcal{A} \to [0, 1]$. When the σ-algebra is understood (generally as the Borel σ-algebra) it is omitted.

Definition 2. [Polish Space] A set A is a polish space if it is separable (allows for a dense countable subset) and metrizable topological space (there exists at least one metric that induces the topology).

Definition 3. [Coupling] Let $(\mathcal{Y}, G)$ and $(\mathcal{X}, P)$ be two probability spaces. A coupling π of G and P is a joint distribution on $(\mathcal{X} \times \mathcal{Y})$ such that $\int_{\mathcal{X}\times Y} d\pi\,(x, y) = G\,(Y)$ for all $Y \in \mathcal{B}\,(\mathcal{Y})$ and $\int_{X\times\mathcal{Y}} d\pi\,(x, y) = P\,(X)$ for all $X \in \mathcal{B}\,(\mathcal{X})$, where $\mathcal{B}\,(A)$ denotes the Borel sets of A. So π gives G and P as marginals. Let $\Pi\,(P, G)$ be the set of all couplings of P and G. When the assignment is given by an assignment function the coupling is deterministic.

Definition 4. [h-transform] Let $h : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ be a function. The h-transform of a function $f : \mathcal{X} \to \mathbb{R}$ is given by:

$$f^h\,(y) = \sup_{x\in\mathcal{X}} \{h\,(x, y) - f\,(x)\}$$

Definition 5. [h-convex] A function $f : \mathcal{X} \to \mathbb{R}$ is said to be h-convex if there exists a function $g : \mathcal{Y} \to \mathbb{R}$ such that:

$$f\,(x) = \sup_{y\in\mathcal{Y}} \{h\,(x, y) - g\,(y)\}$$

Definition 6. [h-subdifferential] The h-subdifferential of a function $v : \mathcal{Y} \to \mathbb{R}$ is defined as the set $\partial^h v\,(y) = \left\{x \in \mathcal{X} \mid v\,(y) + v^h\,(x) = h\,(x, y)\right\}$.

The following theorem joins results from optimal transport on the existence of a solution to the Monge-Kantorovich problem and the applicability of Kantorovich's duality to the mass transportation problem:

Theorem 1. *Villani (2009, Thm. 5.10 and Thm 5.30) Let $(\mathcal{Y}, G)$ and $(\mathcal{X}, P)$ be two Polish probability spaces and let $h : \mathcal{X} \times \mathcal{Y} \to \mathbb{R} \cup \{-\infty\}$ be an upper semicontinuous function.*

Consider the optimal transport problem:

$$\sup_{\pi \in \Pi(G,P)} \int_{\mathcal{X}\times\mathcal{Y}} h(x,y)\, d\pi(x,y)$$

where function $h(x,y)$ describes the gain (or surplus) of transporting a unit of mass from y to x, and $\Pi(G,P)$ denotes the set of couplings of G and P.

If there exist real valued lower semicontinuous functions $a \in L^1(P)$ and $b \in L^1(G)$:

$$\forall_{(x,y)\in\mathcal{X}\times\mathcal{Y}} \quad h(x,y) \le a(x) + b(y)$$

then:

1. *There is duality:*

$$\begin{aligned}
\sup_{\pi \in \Pi(G,P)} \int_{\mathcal{X}\times\mathcal{Y}} h(x,y)\, d\pi(x,y) &= \inf_{\substack{(\lambda,v)\in L^1(P)\times L^1(G)\\ \lambda(x)+v(y)\ge h(x,y)}} \int_{\mathcal{X}} \lambda(x)\, dP(x) + \int_{\mathcal{Y}} v(y)\, dG(y) \\
&= \inf_{w\in L^1(P)} \int_{\mathcal{X}} \lambda(x)\, dP(x) + \int_{\mathcal{Y}} \lambda^h(y)\, dG(y) \\
&= \inf_{v\in L^1(G)} \int_{\mathcal{X}} v^h(x)\, dP(x) + \int_{\mathcal{Y}} v(y)\, dG(y)
\end{aligned}$$

   where $\Pi(G,P)$ is the set of couplings of G and P and f^h denotes the h-transform of function f:

$$f^h(y) = \sup_{x\in\mathcal{X}} h(x,y) - f(x)$$

   The functions w and v are h-convex since they are the h-transform of one another.

2. *If, furthermore, h is real valued ($h : \mathcal{X}\times\mathcal{Y} \to \mathbb{R}$) and the solution to the Monge-Kantorovich problem is finite $\left(\max_{\pi\in\Pi(G,P)} \int_{\mathcal{X}\times\mathcal{Y}} h(x,y)\, d\pi(x,y) < \infty\right)$ then there is a measurable h-monotone set $\Gamma \subset \mathcal{X}\times\mathcal{Y}$[45] such that for any $\pi \in \Pi(G,P)$ the following statements are equivalent:*

   (a) *π is optimal.*

   (b) *π is h-cyclically monotone.*

   (c) *There is a h-convex function λ such that $\lambda(x) + \lambda^h(y) = h(x,y)$ π-almost surely.*

[45] If a, b and h are continuous then Γ is closed.

(d) *There exist* $\lambda : \mathcal{X} \to \mathbb{R}$ *and* $v : \mathcal{Y} \to \mathbb{R}$ *such that* $\lambda(x) + v(y) \geq h(x, y)$ *with equality* π*-almost surely.*

(e) π *is concentrated in* Γ*.*

3. *If,* h *is real valued* ($h : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$) *and there are functions* $c \in L^1(P)$ *and* $d \in L^1(G)$ *such that:*

$$\forall_{(x,y)\in\mathcal{X}\times\mathcal{Y}} \quad c(x) + d(y) \leq h(x, y)$$

then the dual problem has a solution. There is a function w *that attains the infimum.*

4. *(this part from Villani (2009, Thm. 5.30))If:*

(a) h *is real valued* ($h : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$)

(b) *the solution to the Monge-Kantorovich problem is finite:*

$$\max_{\pi\in\Pi(G,P)} \int_{\mathcal{X}\times\mathcal{Y}} h(x, y)\, d\pi(x, y) < \infty$$

(c) *for any* h*-convex function* $v : \mathcal{Y} \to \mathbb{R} \cup \{-\infty\}$ *the subdifferential* $\partial^h v(y)$ *is single valued* G*-almost everywhere*

Then

(a) *there is a unique (in law) optimal coupling* π *of* (G, P)*.*

(b) *the optimal coupling is deterministic:* $T : \mathcal{Y} \to \mathcal{X}$*.*

(c) *the optimal coupling is characterized by the existence of a function* h*- convex function* v *such that* $T(y) = \partial^h v(y)$*.*

Finally, Reynold's transport theorem is used extensively in the text:

Theorem 2. ***[Reynolds' Transport Theorem]*** *The rate of change of the integral of a scalar function* f *within a volume* V *is equal to the volume integral of the change of* f*, plus the boundary integral of the rate at which* f *flows though the boundary* ∂V *of outward unit normal* n*:*

$$\nabla \int_V f(x)\, dV = \int_V \nabla f(x)\, dV + \int_{\partial V} f(x)\,(\nabla x \cdot n)\, dA$$

B Proofs from main text

Consider the set up of Section 2. There are N types of workers $\{x_1, \ldots x_N\} \equiv \mathcal{X}$, there is a mass p_n of workers of type x_n. The mass of workers is described by a (discrete) measure P so that $P(x_n) = p_n$. There is a continuum of tasks $y \in \mathcal{Y}$ distributed continuously according to an absolutely continuous measure $G : \mathcal{B}(\mathcal{Y}) \rightarrow \mathbb{R}_+$. $\mathcal{Y}$ is assumed compact.

Output is produced by completing tasks. A worker of type x_n performing task y produced $q(x_n, y)$. $q : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R}$ is a real valued function. Output for all worker/task pairs is aggregated into a final good:

$$F(\pi) = \begin{cases} \left(\sum_{n=1}^{N} \int (q(x_n, y))^{\frac{\sigma-1}{\sigma}} d\pi(x_n, y)\right)^{\frac{\sigma}{\sigma-1}} & \text{if } \sigma > 1 \\ \exp\left(\sum_{n=1}^{N} \int\limits_{\mathcal{Y}} \ln q(x_n, y)\ d\pi(x_n, y)\right) & \text{if } \sigma = 1 \end{cases}$$

where $\pi \in \Pi(P, G)$ is a coupling of P and G (see definition 3). The coupling π describes the assignment: a mass $\pi(x, y)$ of workers of type x is assigned to task y.

The problem is to maximize output of the final good by choosing an assignment of tasks to workers π. I first transform the objective function so that the problem takes the form of a Monge-Kantorovich problem:

$$\max_{\pi \in \Pi(P,G)} \int h(x, y|\sigma)\, d\pi(x, y) \tag{33}$$

where $h(x, y|\sigma) = \begin{cases} (q(x, y))^{\frac{\sigma-1}{\sigma}} & \text{if } \sigma > 1 \\ \ln q(x, y) & \text{if } \sigma = 1 \end{cases}$.

The following proposition establishes duality for this problem:

Lemma 1. *If q satisfies the following properties:*

1. *$\sigma > 1$ or all workers can produce in some task: $\forall_x \exists y \quad q(x, y) > 0$*

2. *$q(x, \cdot)$ is upper-semicontinuous in y given $x \in \mathcal{X}$.*

Then, the following equalities hold:

$$\max_{\pi\in\Pi(P,G)}\int\left(h\left(x,y|\sigma\right)\right)^{\frac{\sigma-1}{\sigma}}d\pi\left(x,y\right)=\inf_{\substack{(\lambda,v)\in\mathbb{R}^N\times L^1(G)\\ \lambda_n+v(y)\geq q(x_n,y|\sigma)}}\sum_{n=1}^{N}\lambda_n p_n+\int_{\mathcal{Y}}v\left(y\right)dG\left(y\right)$$

$$=\inf_{\lambda\in\mathbb{R}^N}\sum_{n=1}^{N}\lambda_n p_n+\int_{\mathcal{Y}}\max_{n}\left\{q\left(x_n,y|\sigma\right)-\lambda_n\right\}dG\left(y\right)$$

Proof. This follows from applying theorem 1 (Villani, 2009, Thm. 5.10). Note that $\mathcal{Y}\subset\mathbb{R}^n$ and $\mathcal{X}$ is finite they are both Polish spaces. $h\left(x,y|\sigma\right)$ is upper semicontinuous because $f\left(x\right)=x^{\frac{\sigma-1}{\sigma}}$ and $f\left(x\right)=\ln x$ are continuous and monotone increasing, and q is upper semicontinuous.

It is left to verify that there exist real valued lower semicontinuous functions $a\in L^1\left(P\right)$ and $b\in L^1\left(G\right)$:

$$\forall_{(x,y)\in\mathcal{X}\times\mathcal{Y}}\quad h\left(x,y|\sigma\right)\leq a\left(x\right)+b\left(y\right)$$

For this let $a\left(x\right)=\max_{y\in\mathcal{Y}}\left\{h\left(x,y|\sigma\right)\right\}$ and $b\left(y\right)=0$. The max in the definition of a is well defined because h is upper semicontinuous and $\mathcal{Y}$ is compact, furthermore a is finite (either $\sigma>1$ or, if $\sigma=1$, h is finite for at least some y guaranteeing a a final value). Function a is immediately continuous with respect to the discrete topology. The desired equalities follow from part 1 of Theorem 1.

□

The dual problem is then to find a value associated with each type of worker $\{\lambda_1,\ldots,\lambda_N\}$. The problem is:

$$\inf_{\lambda\in\mathbb{R}^N}\sum_{n=1}^{N}\lambda_n p_n+\int_{\mathcal{Y}}v\left(y\right)dG\left(y\right)\qquad\text{where: }v\left(y\right)=\max_{n\in\{1,\ldots,N\}}\left\{h\left(x,y|\sigma\right)-\lambda_n\right\}\tag{34}$$

I show that the dual problem has a solution and I use that solution to construct a solution to the Monge-Kantorovich problem in (33). Furthermore the solution will take the form of a deterministic transport, the implied assignment function is the solution to problem (4) in the main text. Part 3 of Theorem 1 establishes that solution to the dual problem (34) exists.

Lemma 2. *If q satisfies the following properties:*

1. *$\sigma>1$ or all workers can produce in all tasks: $q\left(x,y\right)>0$ for all pairs $\left(x,y\right)\in\mathcal{X}\times\mathcal{Y}$.*

2. *$q\left(x,\cdot\right)$ is upper-semicontinuous in y given $x\in\mathcal{X}$.*

Then there exists $\lambda^\star\in\mathbb{R}^N$ such that:

$$\lambda^\star\in\underset{\lambda\in\mathbb{R}^N}{argmin}\sum_{n=1}^{N}\lambda_n p_n+\int_{\mathcal{Y}}\left(\max_{n\in\{1,\ldots,N\}}\left\{h\left(x,y|\sigma\right)-\lambda_n\right\}\right)dG\left(y\right)$$

Proof. This follows from applying part 3 of theorem 1 (Villani, 2009, Thm. 5.10). The function $h(x, y|\sigma)$ is required to be real valued. When $\sigma > 1$ this is verified since q is real valued. When $\sigma = 1$ it is verified under the additional condition that $q(x, y) > 0$ for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$.

It is left to find functions $c \in L^1(P)$ and $d \in L^1(G)$ such that:

$$\forall_{(x,y)\in\mathcal{X}\times\mathcal{Y}} \quad c(x) + d(y) \leq h(x, y|\sigma)$$

For this let $c(x) = 0$ and $d(y) = \min_n \{h(x, y|\sigma)\}$. The minimum is well defined since $\mathcal{X}$ is finite. □

The final part of Proposition 1 is obtained from applying Theorem 5.30 of Villani (2009), reproduced as part 4 of Theorem 1. The result is established under the conditions that both $(F(x, y))^{\frac{\sigma-1}{\sigma}}$ and the Monge-Kantorovich problem (33) have finite value and the F-subdifferential of w is single valued G-almost everywhere.

Lemma 3. *If q is such that:*

1. *$\sigma > 1$ or all workers can produce in all tasks: $q(x, y) > 0$ for all pairs $(x, y) \in \mathcal{X} \times \mathcal{Y}$.*
2. *$q(x, \cdot)$ is upper-semicontinuous in y given $x \in \mathcal{X}$.*
3. *q discriminates across workers in almost all tasks: if $q(x_n, y) = q(x_m, y)$ then $x_n = x_m$ G-a.e.*

Then there exists $\lambda^\star \in \mathbb{R}^N$ that solves the dual problem (34). Moreover, let T be defined as:

$$T(y) = \underset{x\in\mathcal{X}}{argmax}\left\{h(x, y|\sigma) - \lambda^\star_{n(x)}\right\}$$

T is single-valued G-almost everywhere and it induces a deterministic coupling $\pi^\star : \mathcal{X} \times \mathcal{B}(\mathcal{Y}) \rightarrow \mathbb{R}_+$ that is the unique (in law) solution to the Monge-Kantorovich problem (33). $\pi^\star$ is:

$$\pi^\star(x_n, Y) = \int_{Y\cap T^{-1}(x_n)} dG$$

Function T is an assignment function and it is the solution to the Monge transportation problem (4).

Proof. The proof follows from applying part 4 of Theorem 1 (from Villani (2009, Thm. 5.30)). Finiteness of $h(x, y|\sigma)$ is guaranteed if $\sigma > 1$, or if $\sigma = 1$ and $q(x, y) > 0$ for all pairs $(x, y) \in \mathcal{X} \times \mathcal{Y}$. Finiteness of the value of the Monge-Kantorovich problem is guaranteed since $\mathcal{Y}$ and $\mathcal{X}$ are both compact, and q is upper semicontinuous on y.

It is left to verify that for any h-convex function $v : \mathcal{Y} \to \mathbb{R} \cup \{-\infty\}$ the h-subdifferential $\partial^h v(y)$ is single valued G-almost everywhere. The h- subdifferential for a given y is given by:

$$\partial^h v(y) = \left\{ x \in \mathcal{X} \mid v^h(x) + v(y) = h(x, y|\sigma) \right\} \qquad \text{where} \qquad v^h(x) = \sup_y \{h(x, y|\sigma) - v(y)\}$$

Since v is h-convex we can instead use its conjugate function $v^h(x_n) = \lambda_n$. Then the h-subdifferential is then equivalently given by:

$$\partial^h v(y) = \operatorname*{argmax}_{x \in \mathcal{X}} \left\{ h(x, y|\sigma) - \lambda_{n(x)} \right\}$$

Since $q(\cdot, y)$ is injective in x given y G-a.e., and $\mathcal{X}$ is finite, we get that $\partial^h v(y)$ is generically a singleton.

□

The following lemma establishes the relation between the multipliers of the transformed problem (33) and multipliers of the original problem (7).

Lemma 4. *Consider two constrained maximization problems:*

$$V(m) = \max_x \quad F(x) \qquad s.t. \;\; h(x) = m \tag{35}$$

$$W(m) = \max_x \quad g(F(x)) \qquad s.t. \;\; h(x) = m \tag{36}$$

where $F : \mathcal{X} \to \mathbb{R}$, $h : \mathcal{X} \to \mathbb{R}^n$, $m \in \mathbb{R}^n$ and $g : \mathbb{R} \to \mathbb{R}$ is strictly monotone. Let $\lambda \in \mathbb{R}^n$ be the multiplier associated with the constraints in (35), and $\mu \in \mathbb{R}^n$ the multiplier associated with the constraints in (36). Then: $\mu = g'(F(x^\star))\lambda$, where $x^\star$ is a solution for (35) and (36).

Proof. Because g is strictly monotone both problems have the same argmax, call it $x^\star(m)$. The value of each problem is:

$$V(m) = F(x^\star(m)) \qquad W(m) = g(F(x^\star(m)))$$

By the envelope theorem (Milgrom & Segal, 2002) we know that:

$$\lambda = \frac{\partial V(m)}{\partial m} = \frac{\partial F(x^\star)}{\partial x} \frac{\partial x^\star(m)}{\partial m} \qquad \mu = \frac{\partial W(m)}{\partial m} = \frac{\partial g(F(x^\star))}{\partial F} \frac{\partial F(x^\star)}{\partial x} \frac{\partial x^\star(m)}{\partial m}$$

Joining gives the result: $\mu = g'(F(x^\star))\lambda$.

□

I now present the proof for the differentiability of demand:

Proposition 2. *Let $\lambda \in \mathbb{R}^N$ be a vector of multipliers. If q is continuous then D_n is continuously differentiable with respect to w and:*

i $\frac{\partial D_n}{\partial w_m} = \frac{area(\mathcal{Y}_n(w) \cap \mathcal{Y}_m(w))}{2\sqrt{(x_n - x_m)' A' A (x_n - x_m)}} \geq 0$

ii $\frac{\partial D_n}{\partial w_n} = -\sum_{m \neq n} \frac{\partial D_m}{\partial w_n} < 0$

Proof. First note that since the space of tasks $\mathcal{Y}$ is fixed it holds that $\sum_{n=1}^{N} D_n = \int_{\mathcal{Y}} dy$ so the sum of demands is constant. Then:

$$\frac{\partial D_n}{\partial w_n} + \sum_{m \neq n} \frac{\partial D_m}{\partial w_n} = 0$$

which gives part (ii) of the proposition, the relation between the demand's own price derivative and the cross derivatives of other demands.

The rest of the proof follows from an application of Reynolds' Transport Theorem (Theorem 2). In order to apply Reynolds' theorem recall that $D_m = \int_{\mathcal{Y}_m} \rho(y)\, dy$, where ρ is the density of tasks in the space. In our case $\rho(y) = 1$. So the volume is $\mathcal{Y}_m$ and the function is the density of tasks.

The second term in the theorem measures the rate at which the density flows in and out of the volume. The density flows out and into other workers as tasks are reassigned. Consider the flow into of $\mathcal{Y}_m$ and out of $\mathcal{Y}_k$. The flow is in the direction $\frac{A(x_k - x_m)}{\sqrt{(x_k - x_m)' A' A (x_k - x_m)}}$ and through the shared boundary of $\mathcal{Y}_m$ and $\mathcal{Y}_k$, given by $\mathcal{Y}_m \cap \mathcal{Y}_k$. Note that when prices change the hyperplanes that define the boundaries of the demand sets move in parallel.

Applying the theorem:

$$\frac{\partial D_m}{\partial w_n} = \int_{\mathcal{Y}_m} \frac{\partial \rho(y)}{\partial w_n} dy + \sum_{k \neq m} \int_{\mathcal{Y}_m \cap \mathcal{Y}_k} \rho(y) \left(\frac{\partial y \cdot \frac{A(x_k - x_m)}{\sqrt{(x_k - x_m)' A' A (x_k - x_m)}}}{\partial w_n} \right) dy$$

Note that for all $y \in \mathcal{Y}_m \cap \mathcal{Y}_k$ lie in a plane perpendicular to $A(x_k - x_m)$. Then they can be always expressed as $y = y_\lambda + a\vec{v}$ where $a \in \mathbb{R}$, $\vec{v}$ is a vector perpendicular to $A(x_k - x_m)$ and $y_\lambda = (1 - \lambda) x_k + \lambda x_m$ is such that $y_\lambda \in \mathcal{Y}_m \cap \mathcal{Y}_k$. Then the change $y \in \mathcal{Y}_m \cap \mathcal{Y}_k$ is equal to the change in y_λ.

$$\frac{\partial D_m}{\partial w_n} = \sum_{k \neq m} \int_{\mathcal{Y}_m \cap \mathcal{Y}_k} \rho(y) \left(\frac{\partial y_\lambda \cdot \frac{A(x_k - x_m)}{\sqrt{(x_k - x_m)' A' A (x_k - x_m)}}}{\partial w_n} \right) dy$$

The value of λ is obtained from the equation for the hyperplane that defines $\mathcal{Y}_m \cap \mathcal{Y}_k$:

$$\lambda = \frac{(x_m - x_k)' A (x_m - x_k) + w_m - w_k}{2 (x_m - x_k)' A (x_m - x_k)}$$

so:

$$\frac{\partial y_\lambda \cdot \frac{x_k - x_m}{\sqrt{(x_k-x_m)'(x_k-x_m)}}}{\partial w_n} = \begin{cases} \frac{1}{2\sqrt{(x_n-x_m)' A' A (x_n-x_m)}} & \text{if } k=n \\ 0 & \text{otw} \end{cases}$$

Replacing:

$$\frac{\partial D_m}{\partial w_n} = \frac{\int_{\mathcal{Y}_m \cap \mathcal{Y}_n} \rho(y)\, dy}{2\sqrt{(x_n - x_m)'(x_n - x_m)}} = \frac{\text{area}\left(\mathcal{Y}_n(w) \cap \mathcal{Y}_m(w)\right)}{2\sqrt{(x_n - x_m)'(x_n - x_m)}}$$

which completes the proof.

□

Proposition 3. *Consider the automation problem in 18 and let $\mu \in \mathbb{R}^{N+1}$ characterize an assignment according to 20. If q is differentiable then the first order conditions of the problem are:*

$$F_R(\mu, r) \int_{\mathcal{Y}_R} \frac{\partial \ln q(r,y)}{\partial r} dy - \frac{\partial \Omega(r, p_r)}{\partial r} = 0 \qquad [r]$$

$$F_R(\mu, r)\, \mu_R - \frac{\partial \Omega(r, p_r)}{\partial p_r} = 0 \qquad [p_r]$$

Proof. After replacing T_R for μ in the problem, and abusing notation, the corresponding Lagrangian is:

$$\max_{\{r, p_r, \mu, \Lambda\}} \mathcal{L} = F_R(\mu, r) - \Omega(r, p_r) + \sum_{n=1}^{N} \Lambda_n (p_n - D_n) + \Lambda_R (p_r - D_R) \tag{37}$$

The multipliers of the workers/robot capacity constraints are given by the vector $\Lambda \in \mathbb{R}^{N+1}$.

The first order condition of interest is with respect to the skills of the robot:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{\partial F_R(\mu, r)}{\partial r} - \frac{\partial \Omega(r, p_r)}{\partial r} - \sum_{n=1}^{N} \Lambda_n \frac{\partial D_n}{\partial r} - \Lambda_R \frac{\partial D_R}{\partial r} \tag{38}$$

Following de Goes et al. (2012) and using the result in Lemma 4 the first order condition becomes:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{\partial F_R(\mu, r)}{\partial r} - \frac{\partial \Omega(r, p_r)}{\partial r} - F_R(\mu, r) \left(\sum_{n=1}^{N} \mu_n \frac{\partial D_n}{\partial r} - \mu_R \frac{\partial D_R}{\partial r} \right) \tag{39}$$

I proceed by computing separately the first term of the first order condition:

$$\frac{\partial F_R(\mu, r)}{\partial r} = F_R(\mu, r) \left(\sum_{n=1}^{N} \frac{\partial \int_{\mathcal{Y}_n} \ln q(x_n, y)\, dy}{\partial r} + \frac{\partial \int_{\mathcal{Y}_R} \ln q(r, y)\, dy}{\partial r} \right)$$

Each of the derivatives follows from Reynold's theorem.

$$\begin{aligned}\frac{\partial\int_{\mathcal{Y}_n}\ln q\left(x_n,y\right)dy}{\partial r}&=\int_{\mathcal{Y}_n}\frac{\partial\ln q\left(x_n,y\right)}{\partial r}dy+\int_{\mathcal{Y}_n\cap\mathcal{Y}_R}\ln q\left(x_n,y\right)\frac{\partial y\cdot c_{nr}}{\partial r}dy\\&=\int_{\mathcal{Y}_n\cap\mathcal{Y}_R}\ln q\left(x_n,y\right)\frac{\partial y\cdot c_{nr}}{\partial r}dy\end{aligned}$$

where $c_{nr}=\frac{2A(x_n-r)}{\sqrt{(x_n-r)'A(x_n-r)'}}$ is the normal vector to the direction in which the boundary is moving.

In a similar way:

$$\frac{\partial\int_{\mathcal{Y}_R}\ln q\left(r,y\right)dy}{\partial r}=\int_{\mathcal{Y}_R}\frac{\partial\ln q\left(r,y\right)}{\partial r}dy+\sum_{n=1}^{N}\int_{\mathcal{Y}_n\cap\mathcal{Y}_R}\ln q\left(x_n,y\right)\frac{\partial y\cdot c_{rn}}{\partial r}dy$$

where $c_{rn}=-c_{nr}$. Joining and reorganizing we get:

$$\frac{1}{F_R\left(\mu,r\right)}\frac{\partial F_R\left(\mu,r\right)}{\partial r}=\int_{\mathcal{Y}_R}\frac{\partial\ln q\left(r,y\right)}{\partial r}dy+\sum_{n=1}^{N}\int_{\mathcal{Y}_n\cap\mathcal{Y}_R}\left(\ln q\left(x_n,y\right)-\ln q\left(r,y\right)\right)\frac{\partial y\cdot c_{nr}}{\partial r}dy$$

Note now that by the definition of the boundary $\ln q\left(x_n,y\right)-\ln q\left(r,y\right)=\mu_n-\mu_r$ for all $y\in\mathcal{Y}_n\cap\mathcal{Y}_R$. Then:

$$\frac{1}{F_R\left(\mu,r\right)}\frac{\partial F_R\left(\mu,r\right)}{\partial r}=\int_{\mathcal{Y}_R}\frac{\partial\ln q\left(r,y\right)}{\partial r}dy+\sum_{n=1}^{N}\left(\mu_n-\mu_r\right)\int_{\mathcal{Y}_n\cap\mathcal{Y}_R}\frac{\partial y\cdot c_{nr}}{\partial r}dy$$

Finally note that $\frac{\partial D_n}{\partial r}=\int_{\mathcal{Y}_n\cap\mathcal{Y}_R}\frac{\partial y\cdot c_{nr}}{\partial r}dy$, which follows from applying Reynold's Theorem (again) to D_n.

$$\frac{1}{F_R\left(\mu,r\right)}\frac{\partial F_R\left(\mu,r\right)}{\partial r}=\int_{\mathcal{Y}_R}\frac{\partial\ln q\left(r,y\right)}{\partial r}dy+\sum_{n=1}^{N}\left(\mu_n-\mu_r\right)\frac{\partial D_n}{\partial r}$$

When the location of the robot (r) is changed there is a change in output due to the change in mismatch inside the region previously assigned to the robot ($\mathcal{Y}_R$), that is given by the first term. There is also a change in the demand for workers, only workers who are neighbors of the robot are affected. When their demand is affected the demand of the robot changes in the opposite direction. The demand for worker n changes by $\frac{\partial D_n}{\partial r}$, that is valued by the planner at $\lambda_n-\lambda_r$. Recall that λ_n is the shadow price of the supply of a worker.

It is left to spell out the first term:

$$\begin{aligned}\int_{\mathcal{Y}_R}\frac{\partial\ln q\left(r,y\right)}{\partial r}dy&=\int_{\mathcal{Y}_R}\frac{\partial a_x'r-\left(r-y\right)'A\left(r-y\right)}{\partial r}dy\\&=\int_{\mathcal{Y}_R}\left(a_x-2Ar+2Ay\right)dy\\&=2D_R\left(\frac{a_x}{2}-A\left(r-b_R\right)\right)\end{aligned}$$

where $b_R=\frac{\int_{\mathcal{Y}_R}ydy}{D_R}$ is the barycenter (centroid, average or center of mass) of the tasks assigned to r.

It is now possible to obtain the first order condition of the problem with respect to the location of the robot. Note that since the total demand is constant it holds that:

$$\frac{\partial D_R}{\partial r} = -\sum_{n=1}^{N} \frac{\partial D_n}{\partial r}$$

then:

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{\partial F_R(\mu, r)}{\partial r} - \frac{\partial \Omega(r, p_r)}{\partial r} - F_R(\mu, r)\left(\sum_{n=1}^{N} (\mu_n - \mu_R)\frac{\partial D_n}{\partial r}\right) \tag{40}$$

Replacing for $\frac{\partial F_R(\mu,r)}{\partial r}$ we get:

$$\frac{\partial \mathcal{L}}{\partial r} = 2F_R(\mu, r)\, D_R\left(\frac{a_x}{2} - A(r - b_R)\right) - \frac{\partial \Omega(r, p_r)}{\partial r} \tag{41}$$

The first order condition does not include the effect of r on the demand for workers since the gains cancel with the reductions/increases of slack in the feasibility constraints.

This is a necessary condition for an optimum. It does not fully characterize the solution. In fact there can be, in general, multiple solutions to the problem. The first order condition is also silent about the location of the region assigned to r. Instead it prescribes the relation between the region's centroid and the location of r. It is convenient to see what happens when $a_x = 0$ and $\frac{\partial \Omega(r,p_r)}{\partial r} = 0$. Then the necessary condition reduces to make r equal to the barycenter of its region.

The first order condition with respect to p_r is:

$$\frac{\partial F}{\partial p_r} = F_R(\mu, r)\, \mu_R - \frac{\partial \Omega(r, p_r)}{\partial p_r}$$

The first order condition with respect to μ requires more work, but it follows from applying again Reynolds' Transport Theorem.

$$\frac{\partial F}{\partial \mu_n} = p_n - D_n$$

□

C Marginal product

The marginal product of a worker gives the change in output if more workers of that type are used in production. The change in output depends on the tasks that are assigned to the additional workers.[46] Because of this it is possible to define the marginal product at a given task, and under some initial assignment. In the main text I consider the notion of equilibrium marginal products, where the assignment is not taken as given, but it is allowed to react optimally to changes in the supply of workers.

Consider the marginal product of a worker of type x_k at task $\overline{y}$, given an assignment T. Since task $\overline{y}$ has no mass, output does not change if the task is re-assigned to a worker of type x_k. The marginal product is measured by adding a mass of workers of type x_k and assigning them to a region around task $\overline{y}$, replacing the workers previously assigned to those tasks. The marginal product at $\overline{y}$ is obtained as the change in output when the mass of added workers tends to zero.

Proposition 4. ***[Marginal Product]*** *Let T be a deterministic assignment and fix a task $\overline{y} \in \mathcal{Y}_n^{\mathrm{o}}$. The marginal product of a worker of type x_k at task $\overline{y}$ is:*

$$MP\left(x_k, \overline{y}|T\right) = F\left(T\right)\left(\ln q\left(x_k, \overline{y}\right) - \ln q\left(x_n, \overline{y}\right)\right)$$

where $F\left(T\right) = \exp\left(\int \ln q\left(T\left(y\right), y\right) dG\right)$ and $T\left(\overline{y}\right) = x_n$.

When task $\overline{y}$ is re-assigned from x_n to x_k output changes by $\ln q\left(x_k, \overline{y}\right) - \ln q\left(x_n, \overline{y}\right)$. The marginal product takes into account the opportunity cost of assigning task $\overline{y}$ to x_k, which comes from the capacity constraint of tasks. The derivative of output takes into account the scale of production at the current assignment. Task $\overline{y}$ is required to be in the interior of $\mathcal{Y}_n$ for technical reasons. If $\overline{y} \in \mathcal{Y}_n \cap \mathcal{Y}_m$ it becomes necessary to specify the region around $\overline{y}$ to which x_k will be assigned.

The proof of the result is complicated because the task $\overline{y}$ has dimension zero in the space of tasks, which has dimension $d \geq 1$. Before showing the general proof for the result I consider the one-dimensional case, where the argument is simpler. I further assume that $y \sim U\left([0,1]\right)$. When $d = 1$ the production function can be written as:

$$F\left(T\right) = \exp\left(\int_0^1 \ln q\left(T\left(y\right), y\right) dy\right)$$

[46] Unlike traditional production functions, the amount of an input used by the firm in production and what that input is used for are not the same.

Fix a task $\overline{y} \in (0,1)$ and consider adding a mass ϵ of workers of type x_k. Workers are assigned to the set $C_{\overline{y},\epsilon} = \left\{y \mid |y-\overline{y}| < \frac{\epsilon}{2}\right\} = \left[\overline{y}-\frac{\epsilon}{2}, \overline{y}+\frac{\epsilon}{2}\right]$. The new assignment is:

$$T_\epsilon(y) = \begin{cases} T(y) & \text{if } y \notin C_{\overline{y},\epsilon} \\ 0 & \text{if } y \in C_{\overline{y},\epsilon} \wedge x \neq x_k \\ 1 & \text{if } y \in C_{\overline{y},\epsilon} \wedge x = x_k \end{cases}$$

The change in output is:

$$F(T_\epsilon) - F(T) = F(T)\left(\exp\left(\int_{\overline{y}-\frac{\epsilon}{2}}^{\overline{y}+\frac{\epsilon}{2}} \left(\ln q(x_k, y) - \ln q(T(y), y)\right) dy\right) - 1\right)$$

The marginal product is:

$$\mathrm{MP}(x_k, \overline{y}|T) = \left.\frac{\partial F(R_\epsilon)}{\partial \epsilon}\right|_{\epsilon=0} = \lim_{\epsilon \to 0} \frac{F(R_\epsilon) - F(T)}{\epsilon}$$

replacing and applying L'Hôpital's rule:

$$\mathrm{MP}(x_k, \overline{y}|T) = F(R) \left.\frac{\partial \exp\left(\int_{\overline{y}-\frac{\epsilon}{2}}^{\overline{y}+\frac{\epsilon}{2}} \left(\ln q(x_k, y) - \ln q(T(y), y)\right) dy\right)}{\partial \epsilon}\right|_{\epsilon=0}$$

The derivative follows from Leibniz's rule. Generically $\overline{y} \in \mathcal{Y}_n^{\mathrm{o}}$ and:

$$\begin{aligned} \mathrm{MP}(x_k, \overline{y}|T) = F(T) \Bigg[& \frac{1}{2}\left(\ln q\left(x_k, \overline{y}+\frac{\epsilon}{2}\right) - \ln q\left(T(y), \overline{y}+\frac{\epsilon}{2}\right)\right) \\ & + \frac{1}{2}\left(\ln q\left(x_k, \overline{y}-\frac{\epsilon}{2}\right) - \ln q\left(T(y), \overline{y}-\frac{\epsilon}{2}\right)\right)\Bigg]_{\epsilon=0} \\ = F(T) & \left(\ln q(x_k, \overline{y}) - \ln q(x_n, \overline{y})\right) \end{aligned}$$

If $\overline{y} \in \mathcal{Y}_n \cap \mathcal{Y}_m$ the marginal product takes into account that x_k replaces different types of workers around $\overline{y}$:

$$\mathrm{MP}(x_k, \overline{y}|T) = F(T)\left(\ln q(x_k, \overline{y}) - \frac{\ln q(x_n, \overline{y}) + \ln q(x_m, \overline{y})}{2}\right)$$

In multiple dimensions the treatment of the boundary cases becomes intractable, except in very specific cases for which similar expressions are obtained.

I now provide the general proof of the result.

Proof. Recall that the space of skills is of dimension d. Changing the assignment of tasks to workers in any region of dimension less than d will have no impact on output. To compute the effect on output of the added workers it is necessary to proceed one dimension at a time. Consider a region formed as a hypercube around $\overline{y}$, with sides of length ϵ_i, denote this region by $C_{\overline{y},\epsilon} = \left\{ y \,|\, \forall_i \,|y_i - \overline{y}_i| \leq \frac{\epsilon_i}{2} \right\}$. Note that as all $\epsilon_i \to 0$ the region $C_{\overline{y},\epsilon} \to \{\overline{y}\}$. The assignment is modified as in the one-dimensional example:

$$T_\epsilon(y) = \begin{cases} T(y) & \text{if } y \notin C_{\overline{y},\epsilon} \\ 0 & \text{if } y \in C_{\overline{y},\epsilon} \wedge x \neq x_k \\ 1 & \text{if } y \in C_{\overline{y},\epsilon} \wedge x = x_k \end{cases}$$

The difference in production between the two assignments is:

$$F(T_\epsilon) - F(T) = F(T) \left(\exp \left(\int_{\overline{y}_1 - \frac{\epsilon_1}{2}}^{\overline{y}_1 + \frac{\epsilon_1}{2}} \cdots \int_{\overline{y}_d - \frac{\epsilon_d}{2}}^{\overline{y}_d + \frac{\epsilon_d}{2}} \left(\ln q(x_k, y) - \ln q(T(y), y) \right) dy \right) - 1 \right)$$

I proceed by computing the change in output when the region $C_{\overline{y},\epsilon}$ changes. The change has to be computed one dimension at a time. If all dimensions are changed simultaneously the change in F goes to zero (this can be verified directly using Reynold's transport theorem- Theorem 2). The change in output when $C_{\overline{y},\epsilon}$ changes in the d^{th} dimension is:

$$\begin{aligned} \frac{\partial F(T_\epsilon)}{\partial \epsilon_d} = F(T) \Bigg(\int_{\overline{y}_1 - \frac{\epsilon_1}{2}}^{\overline{y}_1 + \frac{\epsilon_1}{2}} \cdots \int_{\overline{y}_{d-1} - \frac{\epsilon_d}{2}}^{\overline{y}_{d-1} + \frac{\epsilon_d}{2}} & \frac{1}{2} \left(\ln q \left(x_k, \begin{pmatrix} y_1 \\ \vdots \\ \overline{y}_d + \frac{\epsilon}{2} \end{pmatrix} \right) - \ln q \left(T \begin{pmatrix} y_1 \\ \vdots \\ \overline{y}_d + \frac{\epsilon}{2} \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ \overline{y}_d + \frac{\epsilon}{2} \end{pmatrix} \right) \right) \\ & + \frac{1}{2} \left(\ln q \left(x_k, \begin{pmatrix} y_1 \\ \vdots \\ \overline{y}_d - \frac{\epsilon}{2} \end{pmatrix} \right) - \ln q \left(T \begin{pmatrix} y_1 \\ \vdots \\ \overline{y}_d - \frac{\epsilon}{2} \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ \overline{y}_d - \frac{\epsilon}{2} \end{pmatrix} \right) \right) dy_1 \ldots dy_{d-1} \Bigg) \end{aligned}$$

Applying the same procedure iteratively we obtain the change in output as x_k is assigned to tasks around $\overline{y}$ in all directions:

$$\mathrm{MP}(x_k, \overline{y}|T) = \left. \frac{\partial^d F(T_\epsilon)}{\partial \epsilon_1 \cdots \partial \epsilon_d} \right|_{\epsilon=0} = F(T) \left(\ln q(x_k, \overline{y}) - \ln q(x_n, \overline{y}) \right)$$

$\square$

D List of Cognitive and Manual Attributes

Cognitive		Manual	
Code	Attribute	Code	Attribute
Worker Characteristics - Abilities			
1A1a2	Written Comprehension	1A1e1	Speed of Closure
1A1a4	Written Expression	1A1e2	Flexibility of Closure
1A1b1	Fluency of Ideas	1A1e3	Perceptual Speed
1A1b2	Originality	1A1f1	Spatial Orientation
1A1b3	Problem Sensitivity	1A1f2	Visualization
1A1b4	Deductive Reasoning	1A1g1	Selective Attention
1A1b5	Inductive Reasoning	1A1g2	Time Sharing
1A1b6	Information Ordering	1A1g1	Selective Attention
1A1b7	Category Flexibility	1A1g2	Time Sharing
1A1c1	Mathematical Reasoning	1A2a1	Arm-Hand Steadiness
1A1c2	Number Facility	1A2a2	Manual Dexterity
1A1d1	Memorization	1A2a3	Finger Dexterity
		1A2b1	Control Precision
		1A2b2	Multi-limb Coordination
		1A2b3	Response Orientation
		1A2b4	Rate Control
		1A2c1	Reaction Time
		1A2c2	Wrist-Finger Speed
		1A2c3	Speed of Limb Movement
		1A3a	Physical Strength Abilities
		1A3a1	Static Strength
		1A3a2	Explosive Strength
		1A3a3	Dynamic Strength
		1A3a4	Trunk Strength
		1A3b1	Stamina
		1A3c1	Extent Flexibility
		1A3c2	Dynamic Flexibility
		1A3c3	Gross Body Coordination
		1A3c4	Gross Body Equilibrium

Worker Characteristics - Interests			
1B1b	Investigative		
1B1c	Artistic		
1B1e	Enterprising		
Worker Requirements - Basic Abilities			
2A1a	Reading Comprehension		
2A1b	Active Listening		
2A1c	Writing		
2A1d	Speaking		
2A1e	Mathematics		
2A1f	Science		
2A2a	Critical Thinking		
2A2b	Active Learning		
2A2c	Learning Strategies		
2A2d	Monitoring		
Worker Requirements - Cross-Functional Skills			
2B2i	Complex Problem Solving	2B3d	Installation
2B3a	Operations Analysis	2B3h	Operation and Control
2B3b	Technology Design	2B3j	Equipment Maintenance
2B3c	Equipment Selection	2B3l	Repairing
2B3e	Programming		
2B3g	Operation Monitoring		
2B3k	Troubleshooting		
2B4e	Judgment and Decision Making		
2B4g	Systems Analysis		
2B4h	Systems Evaluation		
Resource Management Skills			
2B5a	Time Management		
2B5b	Management of Financial Resources		
2B5c	Management of Material Resources		
2B5d	Management of Personnel Resources		
Knowledge			
2C1a	Administration and Management	2C1b	Clerical
2C1c	Economics and Accounting	2C2a	Production and Processing

2C3a	Computers and Electronics	2C3c	Design
2C3b	Engineering and Technology	2C3d	Building and Construction
2C4a	Mathematics	2C3e	Mechanical
2C4b	Physics	2C9a	Telecommunications
2C4c	Chemistry	2C10	Transportation
2C4d	Biology		
2C4f	Sociology and Anthropology		
2C4g	Geography		
2C5a	Medicine and Dentistry		
2C6	Education and Training		
2C7a	English Language		
2C7b	Foreign Language		
2C7c	Fine Arts		
2C7d	History and Archeology		
2C7e	Philosophy and Theology		
2C8a	Public Safety and Security		
2C8b	Law and Government		
Generalized Work Activities			
4A1b2	Inspecting Equipment- Structures- or Material	4A1a2	Monitor Processes- Materials- or Surroundings
4A1b3	Estimating the Quantifiable Characteristics of Products- Events- or Information	4A1b1	Identifying Objects- Actions- and Events
4A2a1	Judging the Qualities of Things- Services- or People	4A3a1	Performing General Physical Activities
4A2a2	Processing Information	4A3a2	Handling and Moving Objects
4A2a3	Evaluating Information to Determine Compliance with Standards	4A3a3	Controlling Machines and Processes
4A2a4	Analyzing Data or Information	4A3a4	Operating Vehicles- Mechanized Devices- or Equipment
4A2b1	Making Decisions and Solving Problems	4A3b4	Repairing and Maintaining Mechanical Equipment
4A2b2	Thinking Creatively	4A3b5	Repairing and Maintaining Electronic Equipment

4A2b3	Updating and Using Relevant Knowledge	4A3b6	Documenting/Recording Information
4A2b4	Developing Objectives and Strategies	4A4c1	Performing Administrative Activities
4A2b5	Scheduling Work and Activities		
4A2b6	Organizing- Planning- and Prioritizing Work		
4A3b1	Interacting With Computers		
4A4c3	Monitoring and Controlling Resources		