MusaicBox

Table des matières

Musaicbox application is intended to be a POC for main concepts defined and illustrated on 88musaics.org site, plus others (modes, scales, chords…). The implementation chosen for this application are detailed, and some concepts from Musical Set Theory are discussed.

Introduction

The idea behind this application is to offer the user a UI/UX to concretely manipulate abstract concepts, highlight relationship (morphism) between logic spatial and sound structures (*Pitch Class Set*) .

We hope that user will make use of this tool in his composition and instrumental practice activities.

For which audience ?

Musician eager to add to his source of inspiration a muse of a scientific nature, based on idea of reducing any combination of sounds to a set of pitch classes, called *pitch class set* (PCS).[1]

Use Cases

4 entries (four main pages)

- PCS page (with undo/redo.[2])

Useful from one PCS, explore modes, possible chords and other PCS in geometrical transformation relationship (with animation), Forte number, invariant class, and more….

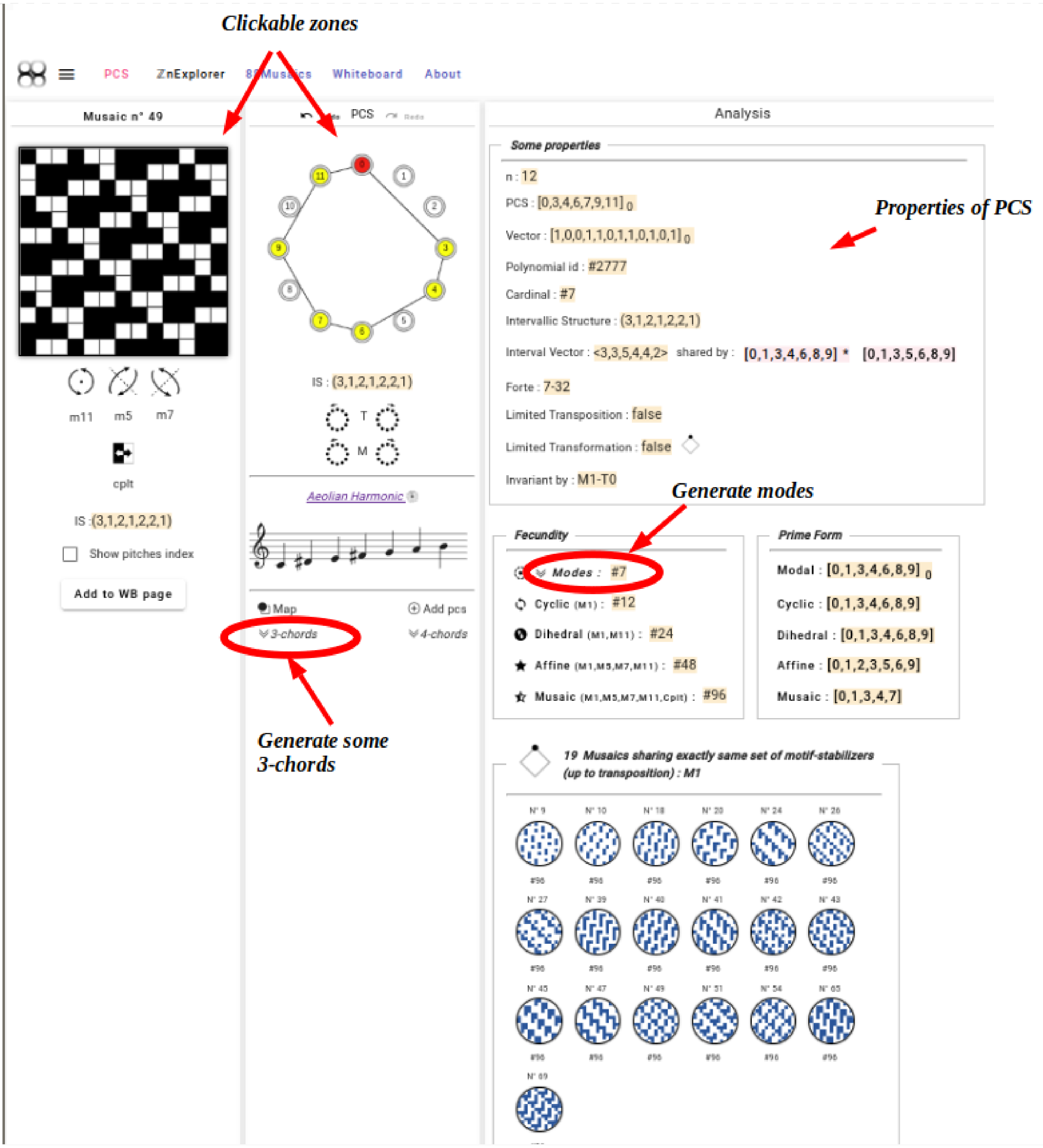


Figure 1. 17 PCS LT in Cyclic Prime Form

- ZnExplorer page

Explore orbits resulting of various group action.

A good illustration of mathematics concepts.

Example : Get Limited Transposition PCS in Cyclic Prime Form : Select only `M1` operation and click on button with label : *Show orbits (352) grouped by Stabilizers signature*. On 352 PCS representatives, 335 are 12 uniques transposed, 17 are less than 12 (called PCS in Limited Transposition)

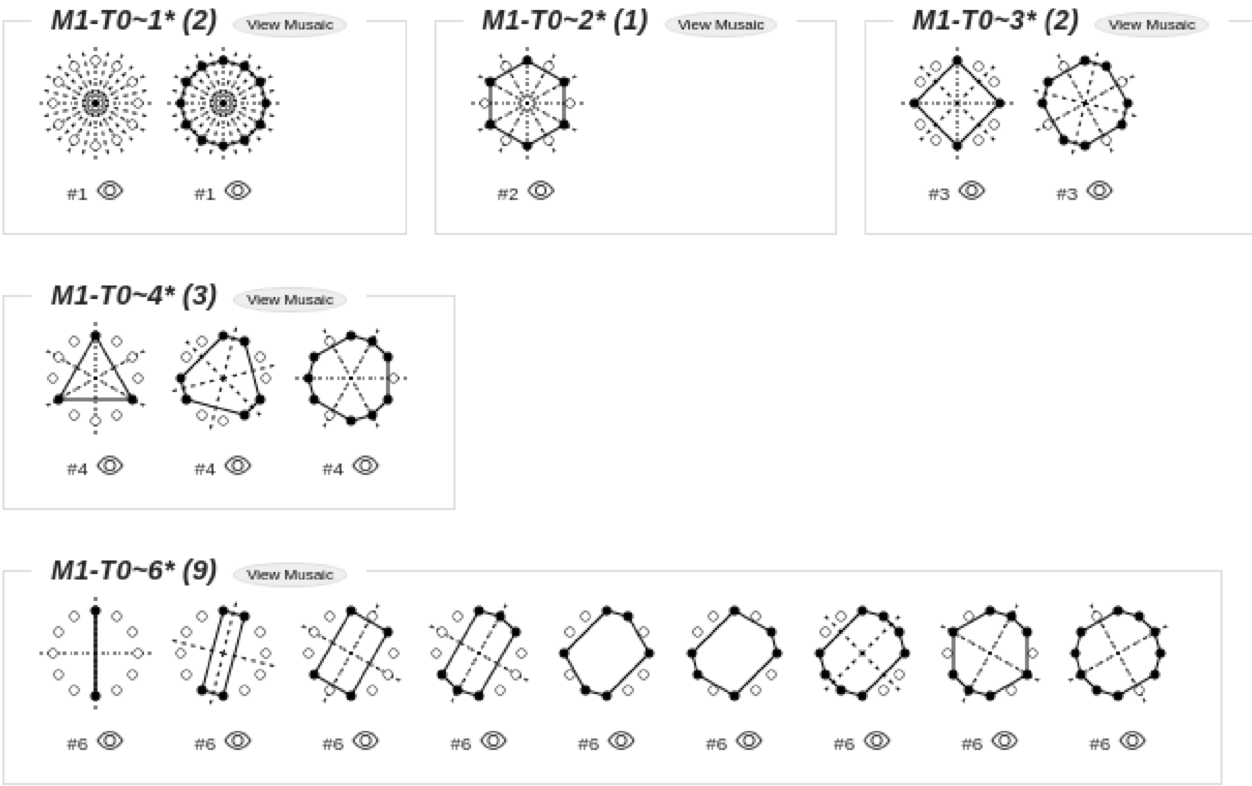


Figure 2. 17 PCS LT in Cyclic Prime Form

- 88 musaics page

Explore musaics shearing same is-stabilizers.[3], having same **octotropes** (partition) and search musaic that include a PCS given (search form on top right menu)

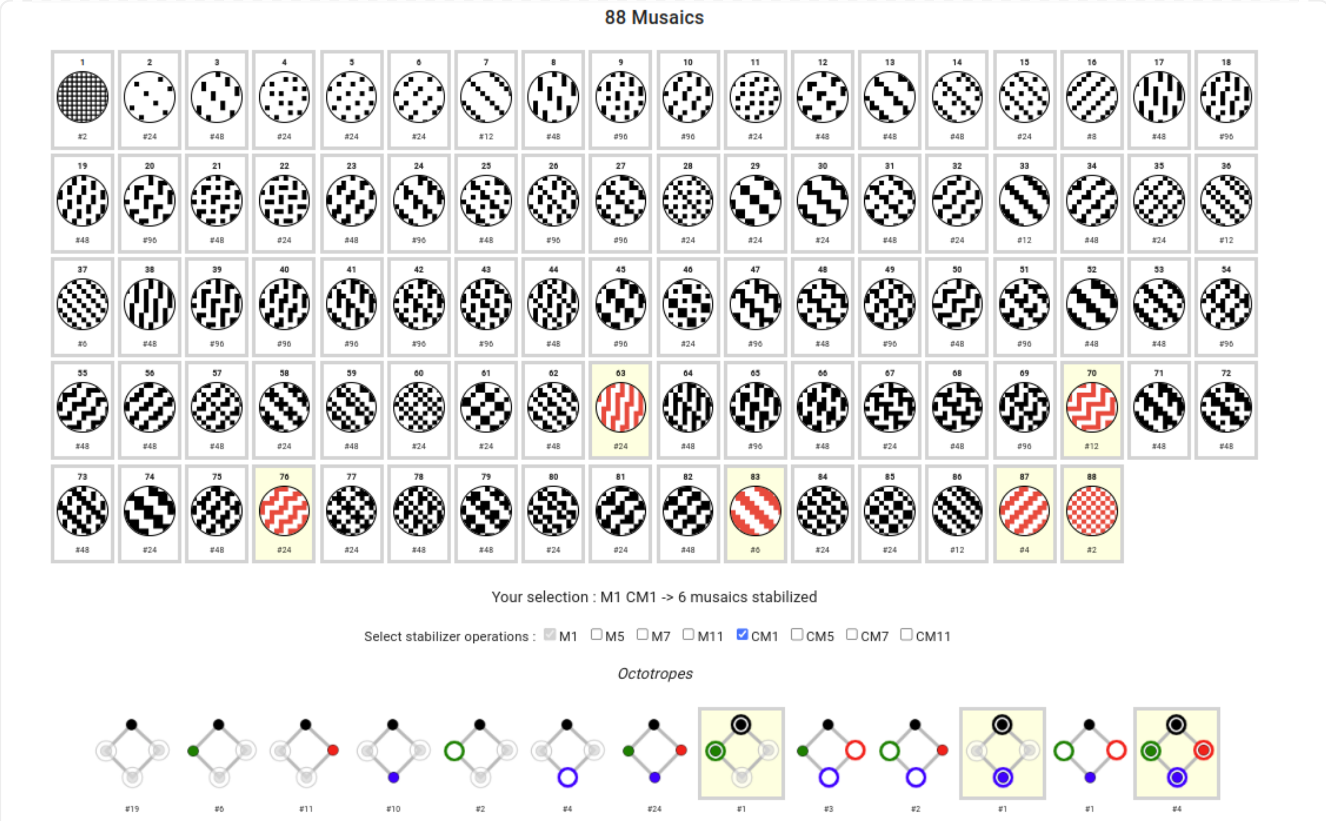


Figure 3. Example page 88 musaics

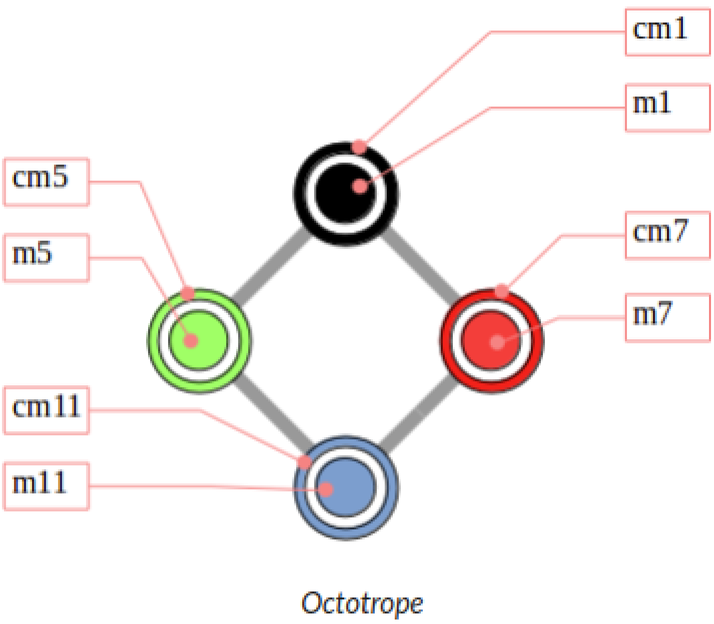
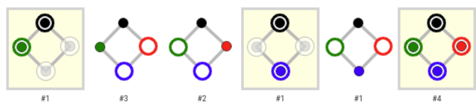


Figure 4. Octotrope, a geometry figure for is-stabilizers classes

Example : There is 6 musaics which are invariant by complement operation (a stabilizer) : Select only `CM1` operation (C for *complement* M1 for *neutral multiplication operation*). Set of these 6 musaics is partitioned by 3 octotropes (is-stabilizers classes) : `M1,M5,CM1,CM5`, `M1,M11,CM1,CM11` and `M1,M5,M7,M11,CM1,CM5,CM7,CM11`



Push selection musaics to "Whiteboard" page (right click), or explore one musaic into "PCS" page.

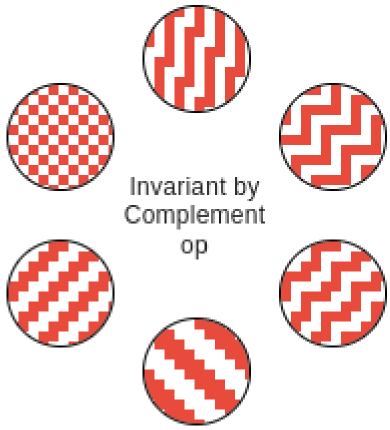


Figure 5. Musaics invariant by CM1 pushed on Whiteboard page

- whiteboard page (with Undo/Redo.[2] and Copy/Cut/Paste)

Organize PCS in various representations on 2D surface.

Give the musician the ability to add text, organize PCS (multiple selection, position, representation, zoom, …), save page content to a file and restore content from a local file.

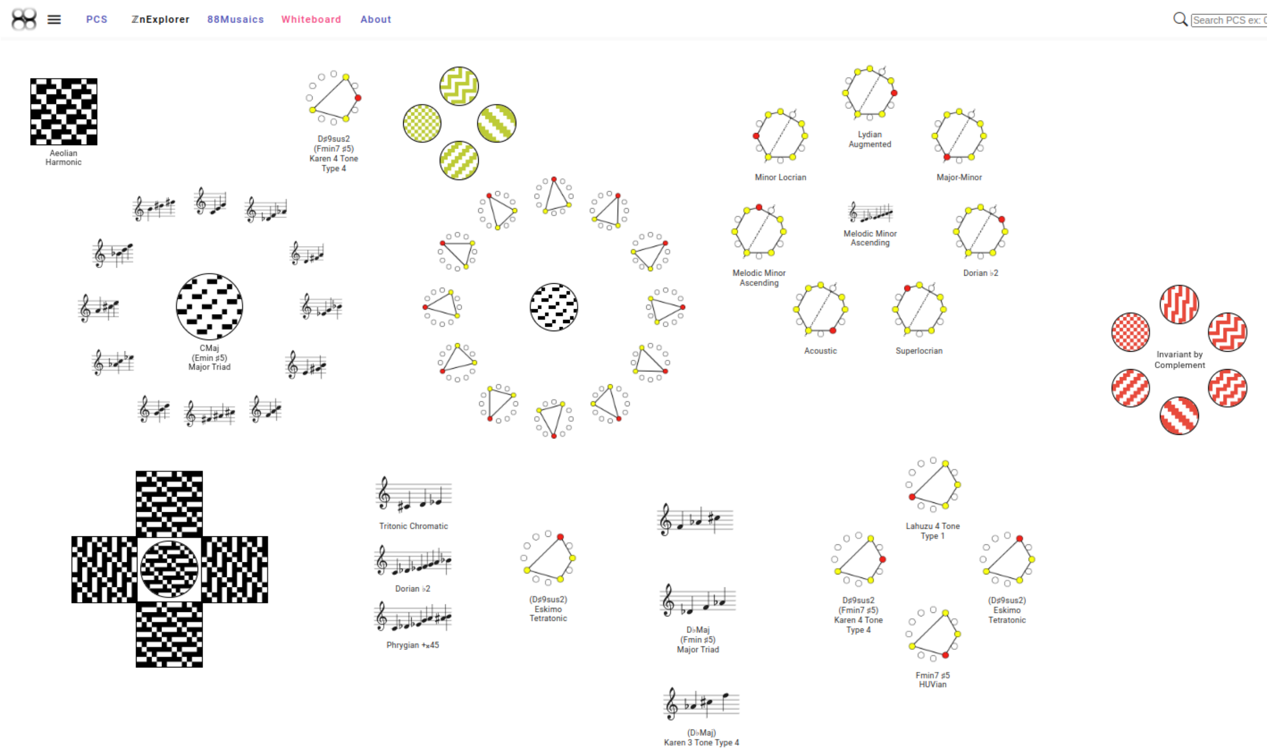


Figure 6. Example Whiteboard page 1

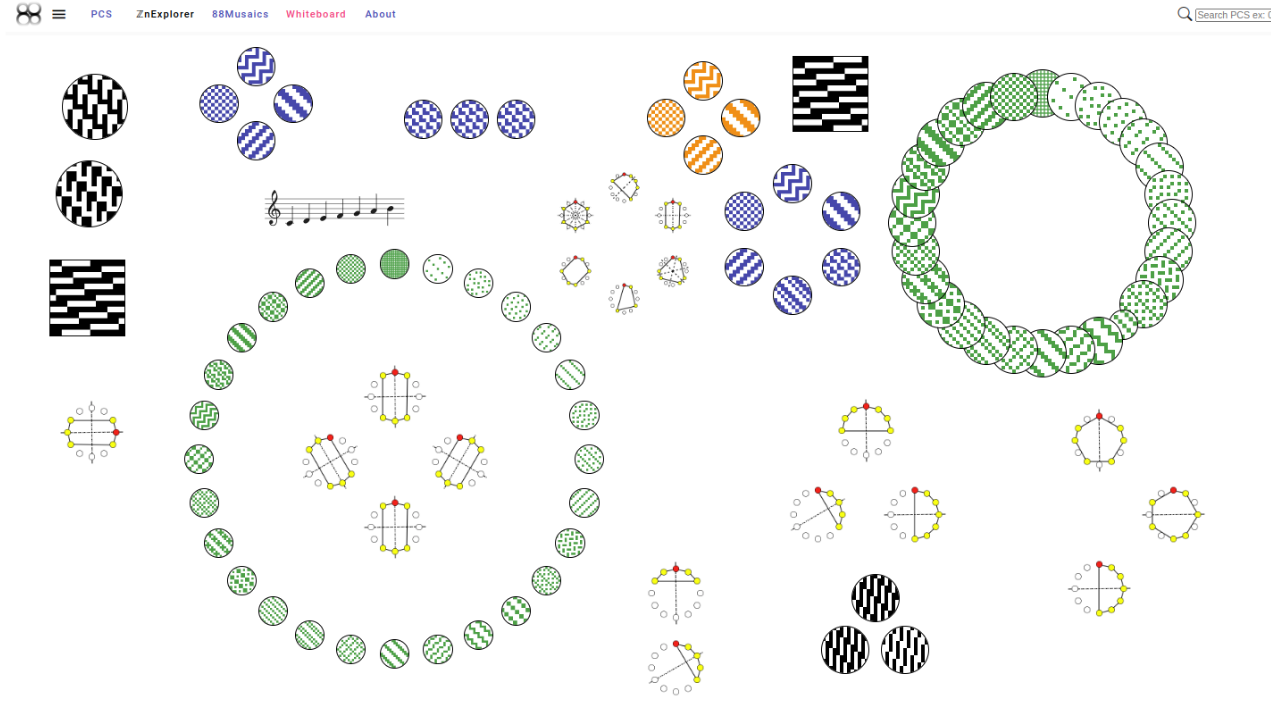


Figure 7. Example Whiteboard page 2

PCS

Abstraction

PCS is abstraction, a *pitches class set* where a pitch class refer to all pitches related to each other by octave and/or enharmonic equivalence.[4]

Enumeration

With set of 12 elements `E` = `{0,1,2,3,4,5,6,7,8,9,10,11}`

Is there 2^{12} = **4096 pcs**, from empty set `{}` to ful set `{0,1,2,3,4,5,6,7,8,9,10,11}` passing by all possible ordered pcs configurations as `{}`, `{0}`, `{1}`, …, `{0,4,7}`, `{1,5,7}`,… , etc

4096 pcs can be ordered by their number of pitches (cardinality).

- Empty set : 1
- Monad set : 12 (12 pitch classes)
- Dyad set : 66
- Triad : 220 (among them, the 12 major triads)
- …
- Full set : 1 (chromatic set)

Their distribution by cardinality is given by line 12 of Pascal’s triangle.

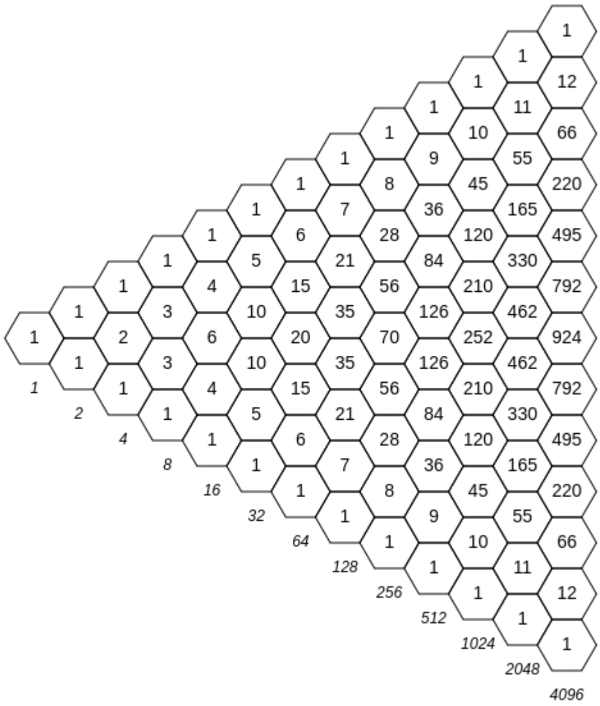


Figure 8. Pascal's triangle

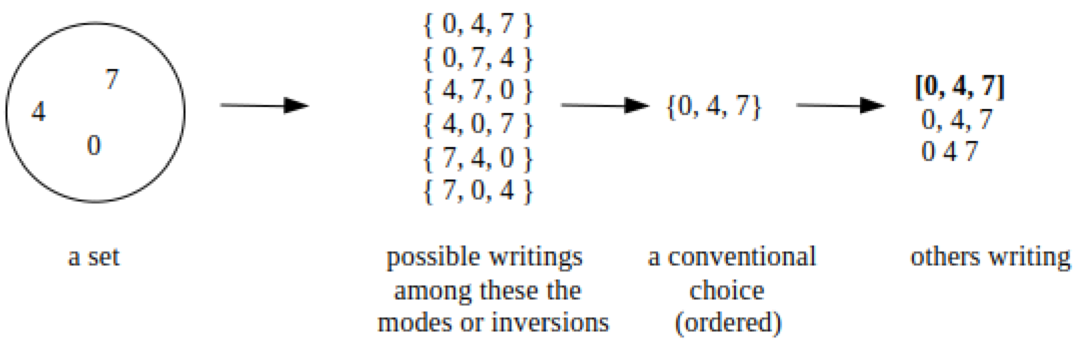
PCS Representations

A *Pitch Set Class* (PCS, or pcs) may have multiple representations.

Textual

A PCS is, by definition, an unordered set of PC, even if, in practice, by convention, we always present them textually ordered, in ascending order.

Let us take the PCS composed of the PCs {0, 4, 7} (C, E and G). Formally the textual representations below are all equal.



Example for C,E,G : `{0,4,7}` or `[0,4,7]` (from musical set theory)

It is customary to represent pitches class into a PCS in an ordered manner, called *Normal Form*, from smallest to largest. For example: [0, 4, 7] not [4, 0, 7]

Vector

Example for [0, 4, 7] : `[1,0,0,0,1,0,0,1,0,0,0,0]` (ordered list of binary values)

Note : can be also vector of boolean values : `[true,false,false,false,true,false,false,true,false,false,false,false]`

Integer

From binary vector representation, we sum of power of 2, where value is 1 (or true) into vector (algorithm called *polynomial function*)

Example for [0, 4, 7] : `1 + 16 + 128` = `145` (decimal value)

Table 1. Example Polynomial Identifier ([0,4,7])

Power of 2	1	2	4	8	16	32	64	128	252	512	1024	2048	*pid*
pcs	**1**	0	0	0	**1**	0	0	**1**	0	0	0	0	
pid	**1**	0	0	0	**16**	0	0	**128**	0	0	0	0	***145***

Examples :

```
pid('[]') = 0 (empty set)
pid('[0,1,2,3,4,5,6,7,8,9,10,11]') = 4095 (chromatic set)
pid('[1,3,5,7,9,11]') = 2730 (whole tone scale)
```

As each pcs into the 4096 is unique, each of these pcs has a unique value by polynomial function. We call this value *pid* for polynomial identifier.

Geometry

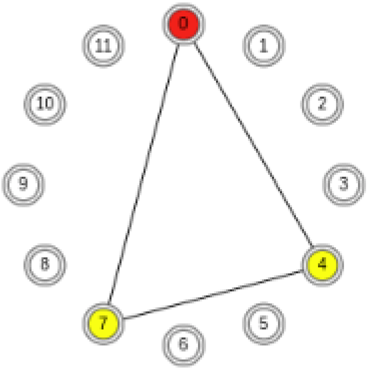
Example with [0,4,7]

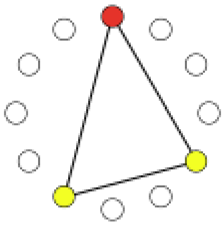
Linear

■□□□■□□■□□□□

Circle / Clock

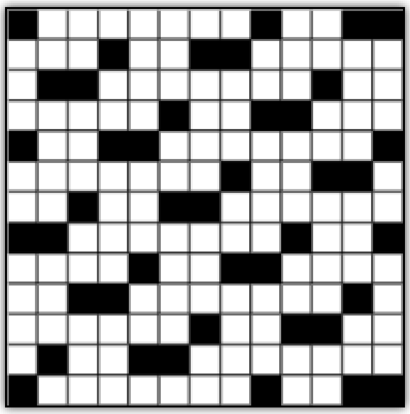
Optional with polygon inscribed.





Musaic

0	5	10	3	8	1	6	11	4	9	2	7	0
1	6	11	4	9	2	7	0	5	10	3	8	1
2	7	0	5	10	3	8	1	6	11	4	9	2
3	8	1	6	11	4	9	2	7	0	5	10	3
4	9	2	7	0	5	10	3	8	1	6	11	4
5	10	3	8	1	6	11	4	9	2	7	0	5
6	11	4	9	2	7	0	5	10	3	8	1	6
7	0	5	10	3	8	1	6	11	4	9	2	7
8	1	6	11	4	9	2	7	0	5	10	3	8
9	2	7	0	5	10	3	8	1	6	11	4	9
10	3	8	1	6	11	4	9	2	7	0	5	10
11	4	9	2	7	0	5	10	3	8	1	6	11
0	5	10	3	8	1	6	11	4	9	2	7	0



DIS : Musaic, Tonnetz and more

More generally, a matrix regular representation is instance of a *Dual Interval Space* ([DIS]), a two-dimensional array of pitches where “rows” are separated by the same interval and the “columns” by one other but also same (non-zero) interval.

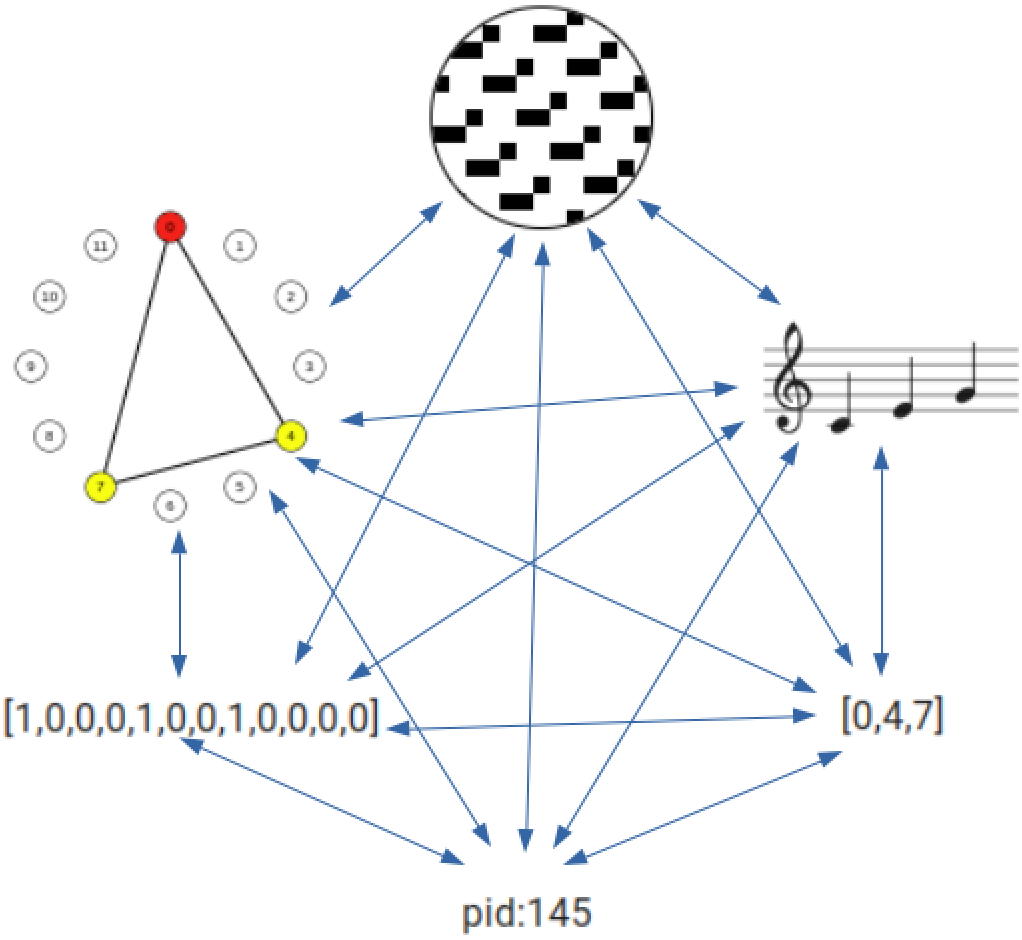
Examples : DIS(y,x) where y is row interval and x is column interval. Violin is DIS(1,7), guitar in P4 Tuning is DIS(1,5), Tonnetz is DIS(4, 7), etc.

Instrument in DIS(x,y) are in *regular interface* family.

Together

All PCS representations are interchangeable by bijective connection.

Example with Set, Integer, Vector, musaic, clock and score notation.



PCS Identity

Any ordered PCS (in normal form) is unique, but it is not a sufficient quality to sort them.

Order relation

It would be useful to be able to sort the pcs among themselves. To do this, we need to define a total order relation that verifies:

`∀ x, y ∈ P(E), (id(x) ≤ id(y) and id(y) ≤ id(x)) => x = y`

In others all, if two PCS have same identity value, then we are dealing with the same PCS.

Question is : How to represent the identity relationship (id function)?

State

By definition, a PCS is a collection of PC. Type is not atomic, and may have some algorithmic efficiency problem, so we prefer a scalar identity.

To implement order relation, we use integer representation, to go through the order of natural integer.

Integer

Polynomial function

Polynomial function is a good candidate for sorting the PCS among themselves.

However, there remains a bias.

Examples :

Table 2. Example problem when compare identity with Polynomial Identifier

pcs1	pcs2	pid(pcs1)	pid(pcs2)	pcs1 < pcs2
[ ]	[0]	*0*	*1*	***true***
[0,4,7]	[1,5,8]	*145*	*290*	***true***
[0,3,7]	[2,6,11]	*137*	*2116*	***true***
[0,11]	[0,3,7]	*2049*	*137*	***false*** (???) *waiting true*

In the first line, we admitted that a piece with a smaller cardinal than another piece will be considered smaller than the latter. But this is contradicted by the last line.

Augmented Polynomial function

In order to solve the inconsistency of the polynomial function for sort pcs lists, we increase this function by another value that takes into account the cardinality of the set. The first value[5], outside *pid* domain, is 2^n

So, augmented polynomial function, which takes into account the cardinal, is : `pid + 2`^{12}` * cardinal`

Table 3. Example Compare with Augmented Polynomial Identifier

pcs1	pcs2	augPid(pcs1)	augPid(pcs2)	pcs1 < pcs1
[ ]	[0]	*0*	*4097*	***true***
[0,4,7]	[1,5,8]	*12433*	*12578*	***true***
[0,3,7]	[2,6,11]	*12425*	*14404*	***true***

pcs1	pcs2	augPid(pcs1)	augPid(pcs2)	pcs1 < pcs1
[0,11]	[0,3,7]	*10241*	*12425*	***true*** (ok)

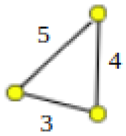
In MusaicBox code, Augmented Polynomial Identifier is called *id*, and *pid* is kept because is commonly used.

Equivalence relation

Example of *proposition* : "*To be a major triad*"

There are only 12 pcs, among the 4096 pcs, where this proposition is true.

Table 4. Different representations of major triad

structure name	intervals	clock	musaic
Major triad	major third then perfect fifth then perfect forth	specific inscribed  polygon	specific motif 

Such proposition can take form of an *equivalence relation* : *R* = "*to share same structure*"

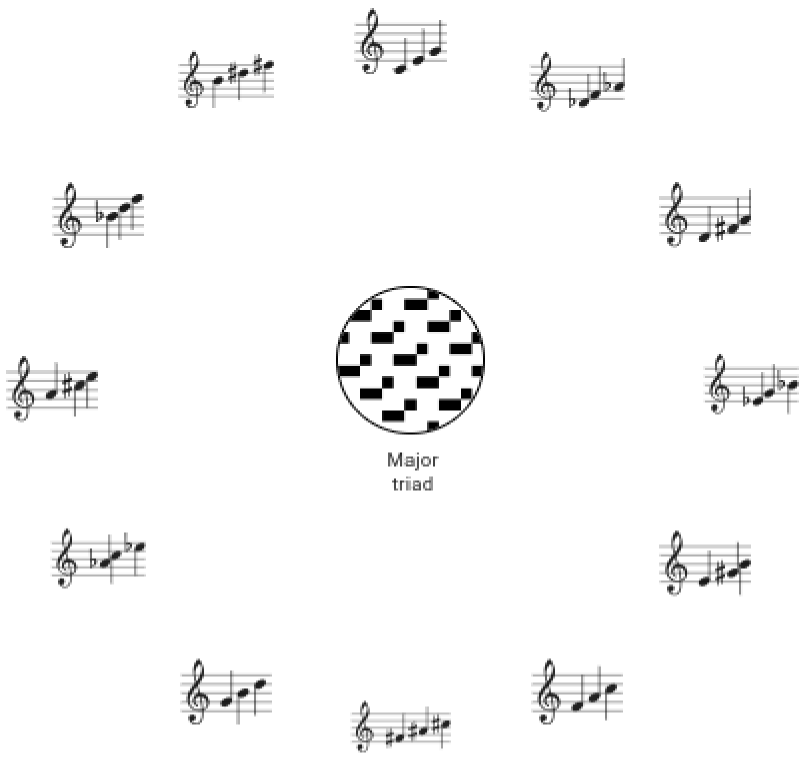
Some characteristics of *R* :

- symmetric : `∀ x, y ∈ P(E), x\ R\ y => y\ R\ x`
- reflexive : `∀ x ∈ P(E), x\ R\ x`
- transitive : `∀ x, y, z ∈ P(E), (x\ R\ y and y\ R\ z) => x\ R\ z`

Orbit

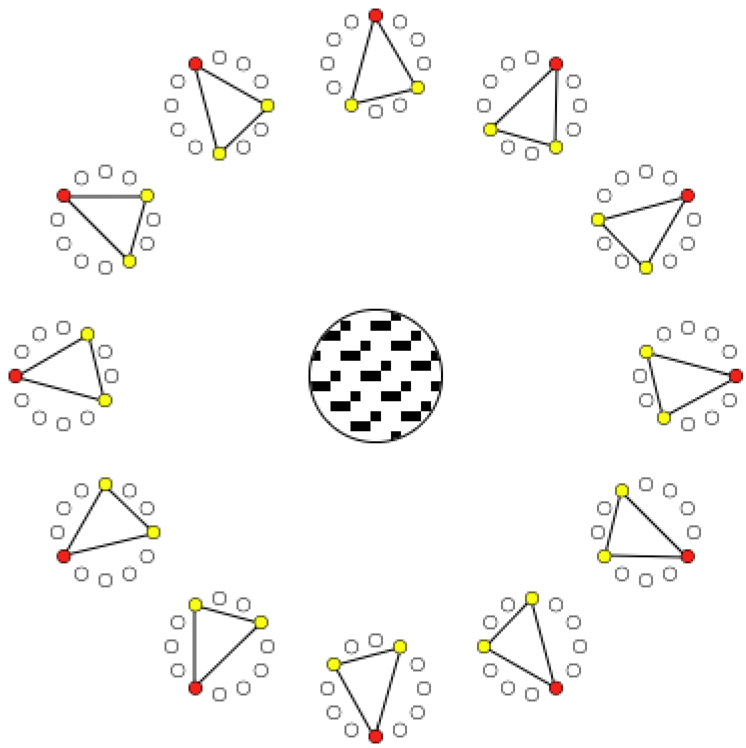
Orbit is a set where all of its elements (pcs) are connected by the same equivalence relation.[6]

Example : Orbit cyclic of major triad as `{ C,E,G }` is a set with equivalence relations "*having same structure of major triad*". This set is composed of 12 elements (`{ C, E, G }, { Cb, F, Ab }, …, { B, D#, F# }`)



Below, same, with clock representation.

Figure 9. Orbit Cyclic in score view and Major Triad Motif



If we organize the 4096 pcs set into subsets with equivalence relation "*having same structure (of inscribed polygon)*", we obtain 352 types of polygons[7] therefore 352 subsets (orbits) forming a partition of 4096 pcs set.

Figure 10. Orbit Cyclic in clock view and their inscribed polygon

Table 5. Enumeration via line 12 of Pascal triangle

PCS cardinal	**Orbit cardinal**	*Coefficient of Transposition*						**Pascal triangle line 12**
		1	2	3	4	6	12	*line 12*
0	1	1						1

PCS cardinal	**Orbit cardinal**	*Coefficient of Transposition*						**Pascal triangle line 12**
1	1						1	12
2	6					1	5	66
3	19				1		18	220
4	43			1		2	40	495
5	66						66	792
6	80		1		1	3	75	924
7	66						66	792
8	43			1		2	40	495
9	19				1		18	220
10	6					1	5	66
11	1						1	12
12	1	1						1
total	**352**	2	1	2	3	9	335	**4096**
	352 orbits of cyclic group (17 + 335)	*17 cyclic orbits PCS in Limited Transposition* (cardinal < 12)					335 orbits of cardinal 12	4096 orbits of cardinal 1 (trivial group)

Set of all these sets is known as `P(E)` (**power set**), and `cardinal(P(E))` $= 2^n =$ **`4096`**, cardinality ordered by line 12 of Pascal triangle.

Pour partitionner l'ensemble des 4096 PCS,

En musique, il est généralement admis que nous pouvons changer la hauteur d'une oeuvre sans en changer fondamentalement sa nature. Dans la musique tonale, cette action est appelée "changement de tonalité".

Exemple : petite mélodie en 2 tonalités

On peut avoir 12 versions de cette mélodie (dans les 12 tonalités). L'ensemble de ces 12 versions est

Equivalence relation : "Having same prime form"

En identifiant n'importe quel son à une meme classe de hauteur (octave), modulo *n*, nous pouvons réduire tout extrait musical à un ensemble classes de hauteur.

It is about gathering all the elements sharing the same characteristic, in the same set called **equivalence orbit**, or **X orbit**, or **orbit** if equivalence context is clear.

An orbit is a set which can be empty (contains empty pcs) or contain all elements.

> For a given equivalence relation, an element belongs to only one orbit. Orbits, as a result of a group action, form a partition of the set on which it acts (4096 PCS).

Octave/Enharmonic equivalence

Reduce to 12 pitches class and its 4096 PCS combinaisons (2^{12}).

Trivial group has **4096** orbits, each orbit has max only one pcs (cardinal = 1)

Intervallic structure equivalence up to shift

This is form a cyclic group (group action on Z12).

All pcs of a given orbit share **same intervallic structure up to circular shift**, **obtained by transposition**.

In other words, PCS in clock representation having the **same inscribed polygon**.

Cyclic group has **352** orbits.[8]

Dihedral equivalence

In this group, all PCs of a given orbit share the same interval structure of itself or its **inverse**.

Dihedral group has **224** orbits.

Affine equivalence

In this group, any pcs of a given orbit share with others pcs into this orbit, same intervallic structure of itself or this inverse or this transformed by **multiplication by 5 or 7 and their inverse**.

Affine group has **156** orbits.

Musaic equivalence

In this group, any pcs of a given orbit are in affine equivalence with itself or **affine complement**.

Musaic group has **88** orbits.

Prime Form

A quality that allows, without ambiguity, to designate a representative among the elements of an orbit.

To put it simply, it is the **smallest element of an orbit**.

Can be represented by a function PrimeForm : EquivalenceRelation x pcs → pcs (from an equivalence relation and a pcs given we obtain one and oly one pcs representative of equivalence relation orbit.
Given R, an equivalence relation, and pcs1, pcs2 (two pcs), if PrimeForm(R, pcs1) == PrimeForm(R, pcs2), then pcs1 and pcs2 belong to the same R equivalence orbit.

What Prime Form is

Given an equivalence relation orbit (of pcs), there will always be a unique pcs *smaller* than others into same orbit (thanks to the order relation).

Originally [Forte], prime denotes a pcs in normal form and "most packed on the left (0)"

[Rahn] John Rahn proposes a more rational approach, based on vector representation of a pcs (and its image function in an integer result of polynomial function)

What Prime Form is not

Prime form is a "technical" characteristic of one element into an orbit, **without musical resonance**.

In absolute terms, any pcs into an orbit can be a representative of their orbit. By convention, we select the *minimal element*

Modal prime form

It is a pcs of cyclic orbit that, if possible, highlights its symmetry (else is cyclic prime form).

Example on pcs : [2, 3, 5, 7, 8] :

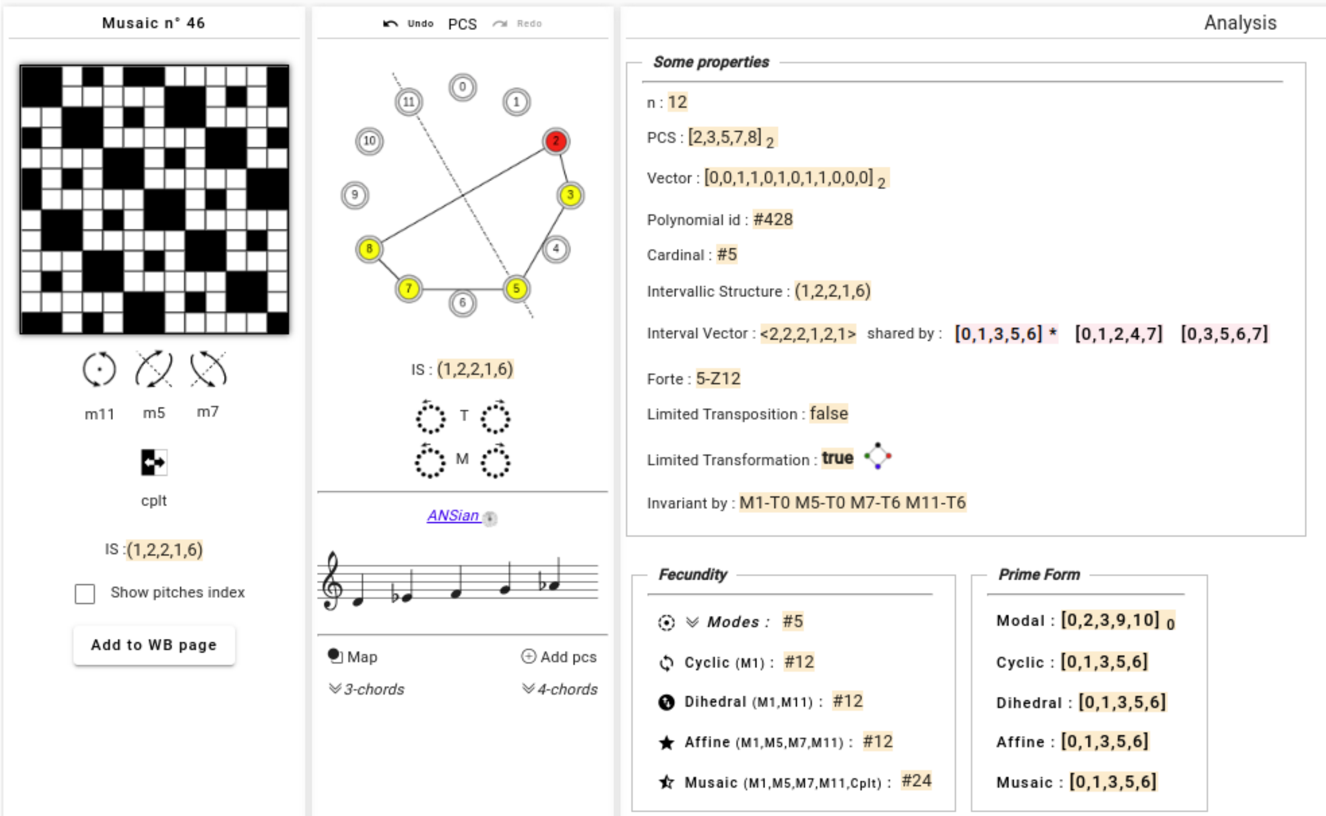


Figure 11. Modal and cyclic prime form

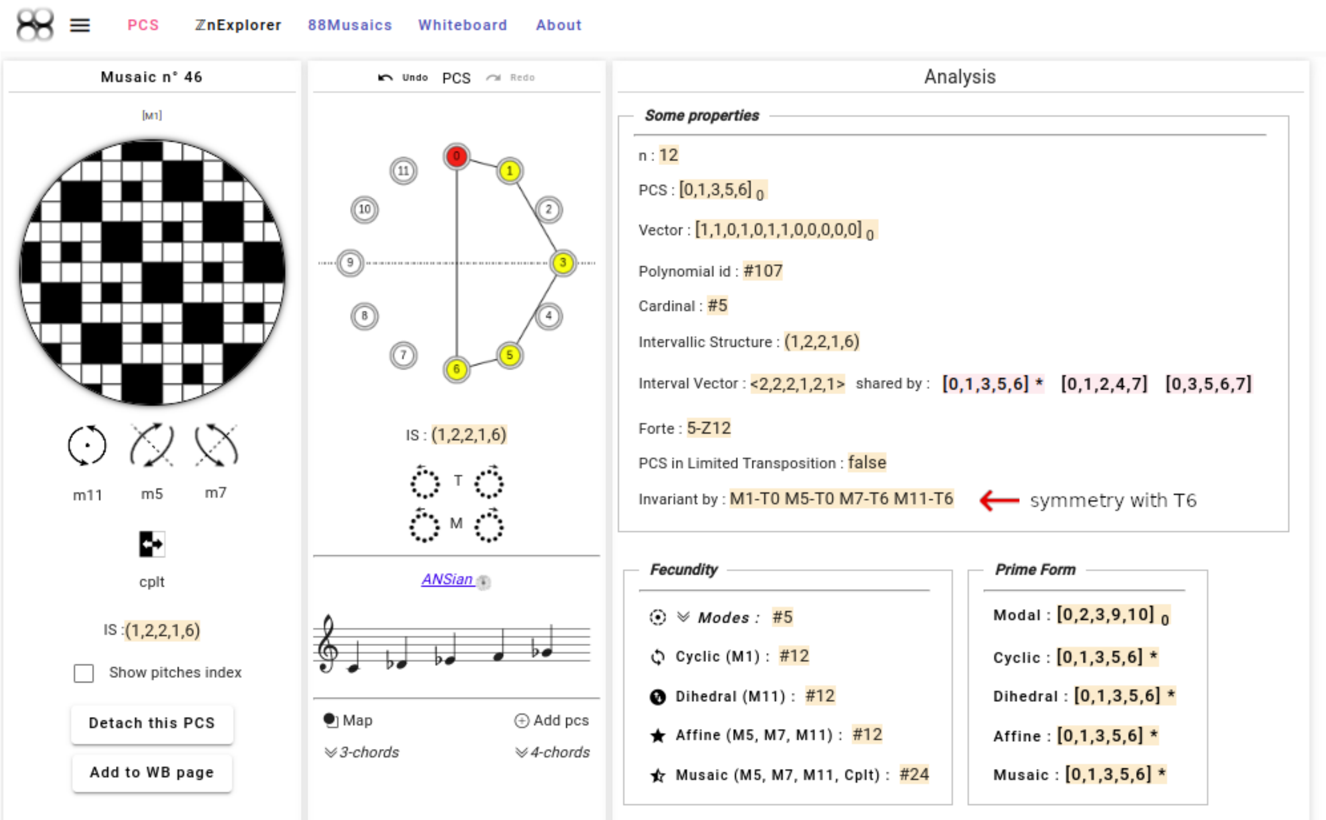


Figure 12. Cyclic prime form

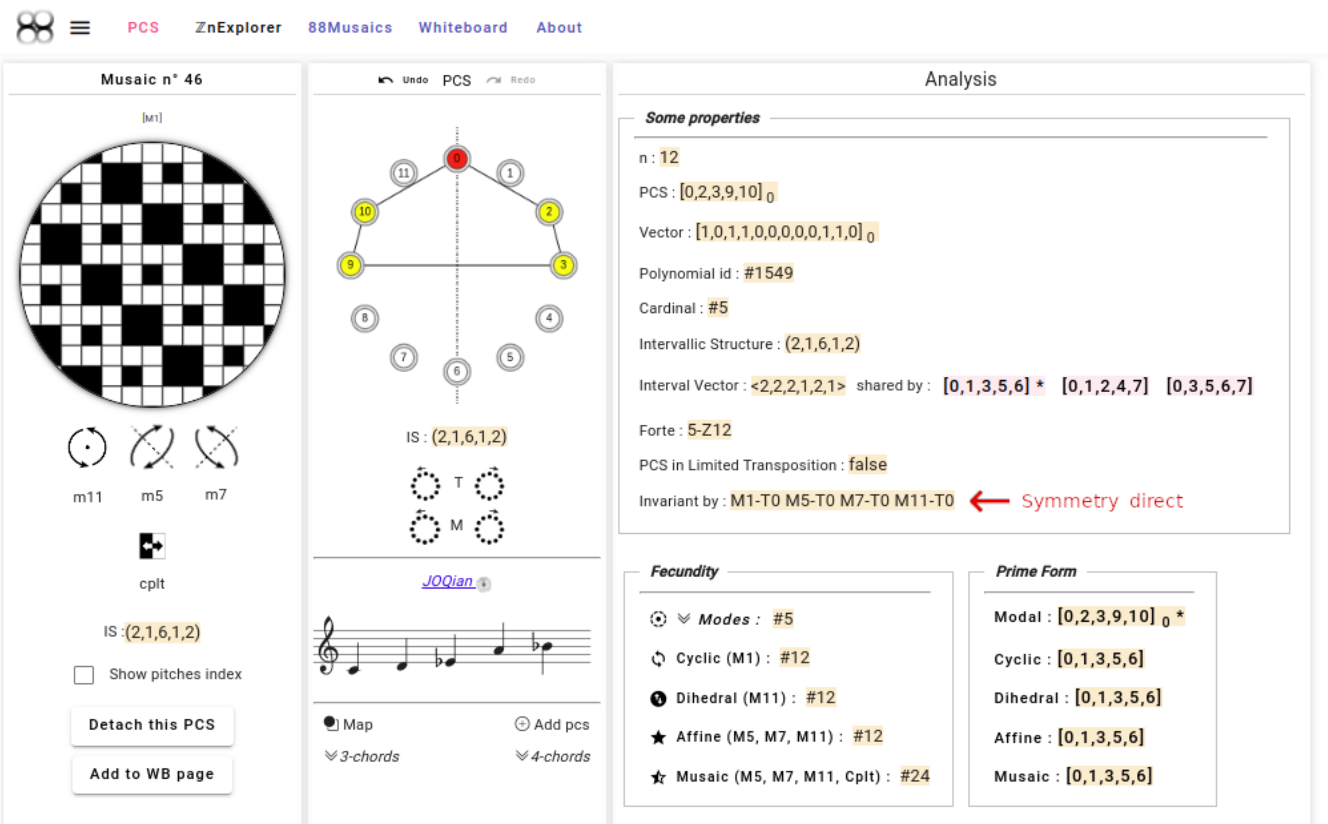


Figure 13. Modal prime form

Same, in other views :

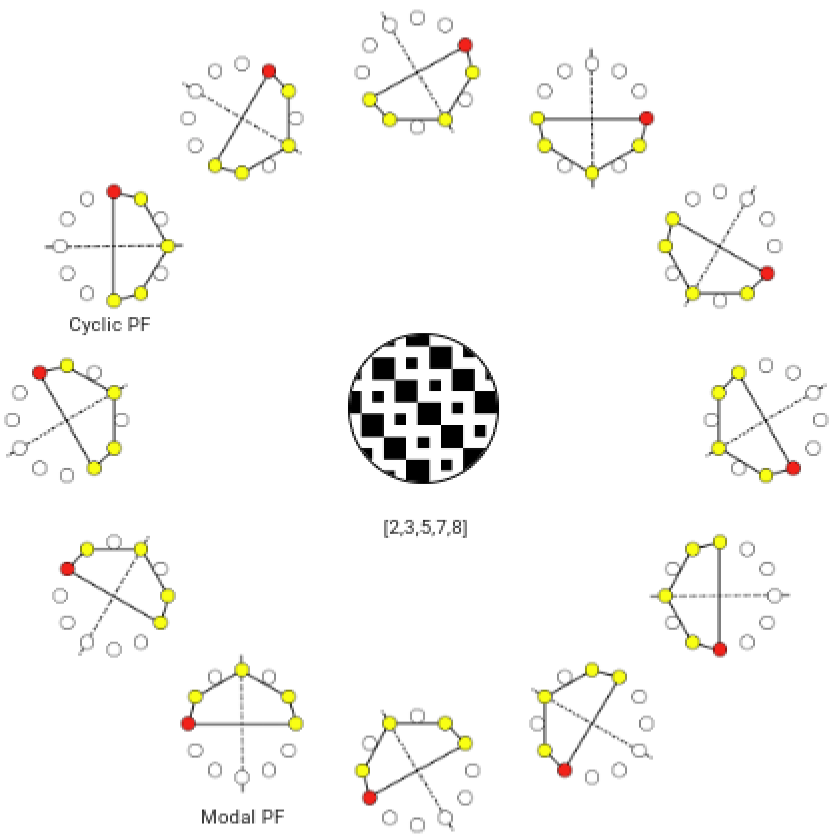


Figure 14. Cyclic orbit and his modal and prime form

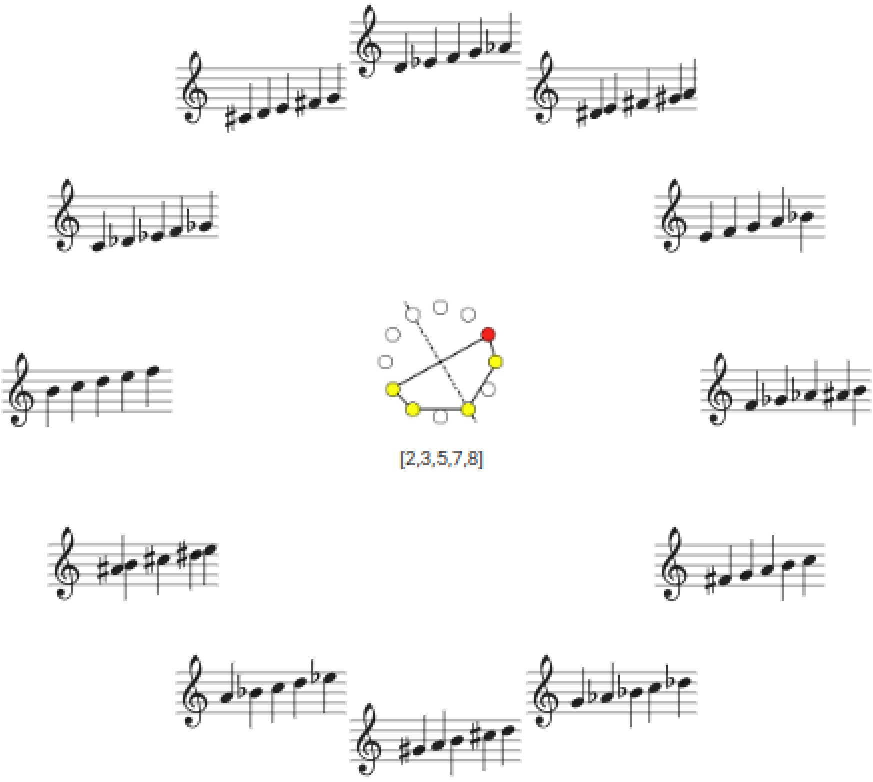


Figure 15. Cyclic orbit and his modal and prime form

References

- [Forte] Forte, Allen. 1973. The Structure of Atonal Music. New Haven: Yale University Press.
- [Rahn] Rahn, John. 1980. Basic Atonal Theory. New York: Longman.
- [DIS] Stephen C.Brown. Dual Interval Space in Twentieth-Century Music, Musaic in armature 1-5 is DIS(1,5).

[1] provided that they accept the postulate of the decomposition of an octave into 12 "equal parts".

[2] redo:Back to the future only possible if the past has not been updated

[3] A stabiliser is a transformation operation which conserve intervallic structure

[4] see Allen Forte, John Rahn…

[5] the last is 2^{12}-1

[6] see *setoid* or *bishop set*

[7] and not 4096/12, because some pcs have less than 12 transposed - limited transposition

[8] 352 > 4096 / 12, because somme pcs are there cardinal cyclic orbit smaller than 12 (pcs in **limited transposition**)