

# The O’Nan-Scott Theorem

## 1 Some Lemmas

## 2 The proof of the O’Nan-Scott Theorem

Last week we introduced some lemmas and proved part of the O’Nan-Scott Theorem. This week we will finish the proof of the O’Nan-Scott Theorem.

**Notation:** Let  $H$  be a subgroup of  $S_n$  not containing  $A_n$ ,  $N$  be a minimal normal subgroup of  $H$ , and  $K$  be the stabilizer in  $H$  of a point.

$H$  intransitive  $\implies$  case (i).

$H$  transitive imprimitive  $\implies$  case (ii).

Now we assume  $H$  primitive. And hence the discussion zooms into  $\text{soc}(H)$ .

$\exists N$  abelian  $\implies$  case (iv) affine.

Additionally we assume  $\forall N$  nonabelian.

If  $H$  has more than one minimal normal subgroups  $N_1 \neq N_2$ .

It can be shown that  $\exists x \in S_n$  conjugates  $N_1$  to  $N_2$ . **specify x**

By corollary 2.11,  $x$  also conjugates  $N_2 = C_H(N_1)$  to  $N_1 = C_H(N_2)$ . **(Why?)**

Hence  $H < \langle H, x \rangle$ , which has a unique minimal normal subgroup  $N_1 \times N_2$ .

Additionally we assume  $H$  has a unique minimal normal subgroup  $N$ , which is nonabelian.

$N$  simple  $\implies C_H(N) = 1 \implies H \overset{\text{conj.}}{\curvearrowright} N$  faithfully  $\implies$  case (vi) AS.

$N = T^m = T_1 \times \cdots \times T_m$  with  $m > 1 \implies H \overset{\text{conj.}}{\curvearrowright} \{T_1, \dots, T_m\}$  transitively, and  $K$  as well.

Let  $K_i := p_i(K \cap N) \leq T_i$  the projection of  $K$  onto  $T_i$ . Then  $K \cap N \leq K_1 \times \cdots \times K_m$ .

We divide the discussion into 2 cases. Before that, we claim the following fact.

Claim:  $K$  normalizes  $K_1 \times \cdots \times K_m$ .

*Proof.* Since  $K \cap N \triangleleft K$ ,  $\forall k \in K$ ,  $\forall x \in K \cap N$ ,

we have  $x = p_1(x) \cdots p_m(x)$ , and  $p_1(x)^k \cdots p_m(x)^k = x^k = p_1(x^k) \cdots p_m(x^k) \in K \cap N$ .

Then  $p_i(x)^k = p_j(x^k)$  whenever  $T_i^k = T_j$ . (In direct product, equal iff. all coordinates equal.)

$\forall y \in K_1 \times \cdots \times K_m$ ,  $\exists x_1, \dots, x_m \in K \cap N$  s.t.  $y = p_1(x_1) \cdots p_m(x_m)$ .

Then  $y^k = p_1(x_1)^k \cdots p_m(x_m)^k = p_1(x_{l_1}^k) \cdots p_m(x_{l_m}^k) \in K_1 \times \cdots \times K_m$ , where  $T_i = T_{l_i}^k$ .  $\square$

Case  $K_i \neq T_i$  for some  $i$ :

Now  $K \cap N \leq K_1 \times \cdots \times K_m < N$ .

By corollary 2.15,  $K_1 \times \cdots \times K_m = K \cap N$  and  $K$  permutes  $K_i$ ’s transitively. Let  $k := |T_i : K_i|$ .

Then  $H = (T_1 \times \cdots \times T_m) \rtimes K \leq S_k \wr S_m \curvearrowright [T_1 : K_1] \times \cdots \times [T_m : K_m] \implies$  case (iii) PA.

Case  $K_i = T_i$  for all  $i$ :

Support of  $(t_1, \dots, t_m) \in N$  is defined as  $\text{supp}(t) := \{i \mid t_i \neq 1\}$ .

$\Omega_1 :=$  a non-empty min.(with set inclusion) supp. of an elt in  $K \cap N$ .

Claim:  $\Omega_1$  a block of  $K, H \curvearrowright [m]$ .

*Proof.* 1 and all elts in  $K \cap N$  with support  $\Omega_1$  (i.e.  $t_i \neq 1$  and  $t_j = 1 \ \forall i \in \Omega_1, \forall j \notin \Omega_1$ )

forms a normal subgp of  $K \cap N$ , which maps onto a normal subgp of hence  $T_i$  itself  $\forall i \in \Omega_1$ .

$\Omega_1 \cap \Omega_2 \neq \emptyset \implies \exists x, y$  s.t.  $[x, y] \neq 1$  has support contained in  $\Omega_1 \cap \Omega_2$ , that is  $\Omega_1$  by min.  $\square$

$|\Omega_1| = 1 \implies N \leq K$ , a contradiction.

$|\Omega_1| = m \implies K \cap N = \{(t, \dots, t) \mid t \in T\}$  WLOG.  $N \curvearrowright [N : K \cap N] \implies$  case (v)diagonal.

$\forall i, \forall x, y \in K \cap N, p_i(x) = p_i(y) \implies p_i(xy^{-1}) = 1 \implies xy^{-1} = 1$  i.e.  $p_i|_{K \cap N}$  inj.

$\forall x, y \in K \cap N$  with nonempty support (hence  $[m]$ ), if  $p_i(x) = p_i(y)$  for some  $i$ , then for all  $i \in [m]$  since otherwise  $1 \neq xy^{-1} \in K \cap N$  has smaller support. Note that  $K$  acts transitively on  $\{K_1, \dots, K_m\}$ , so  $p_i(x)^k = p_j(x^k)$  for some  $k \in K$ . Thus  $p_i(x) = p_1(x^{k_i})^{k_i^{-1}}$  where  $T_i^{k_i} = T_1$

$|\Omega_1| = k \neq 1, m$ :

Suppose  $\Omega_1$  is in a block system  $\{\Omega_1, \dots, \Omega_l\}$  of  $K$  on  $[m]$ . Let  $N_j = \times_{i \in \Omega_j} T_i$  for  $j = 1, \dots, l$ .

Then  $N = N_1 \times \dots \times N_l \cong T^{kl}$ . For each  $N_j$ ,  $N_j \cap K$  is a diagonal subgroup of  $N_j$ .

$$\implies N = \left( \times_{i \in \Omega_1} T_i \right)^l \cong T^{kl}, N \cap K = \left( \text{diag} \left( \times_{i \in \Omega_1} T_i \right) \right)^l \cong T^l.$$

The action of each  $\times_{i \in \Omega_1} T_i$  is diagonal of degree  $r = |T|^{k-1}$ .  $H \leq S_r \wr S_l \curvearrowright [r]^l \implies$  case (iii)PA.