s-Arc-transitive solvable Cayley graphs

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   - Overview of the proof
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Preliminaries

Let Γ be a finite simple undirected graph. Let $G \leq \mathrm{Aut}\Gamma$.

Definition

- **s-arc:** (s+1)-tuple of vertices $\alpha_0, \alpha_1, \cdots, \alpha_s$ where α_i is adjacent to α_{i+1} and $\alpha_{j-1} \neq \alpha_{j+1}$ for $0 \leq i \leq s-1$ and $1 \leq j \leq s-1$.
- **(G,s)-arc-transitive:** G is transitive on the set of s-arcs of Γ.
- **(G,s)-transitive:** (G,s)-arc-transitive but not (G,s+1)-arc-transitive.

For short, **s-transitive** means $(\mathrm{Aut}\Gamma, s)$-transitive for graphs.

Lemma

- *s-arc-transitive* $(1 \leq s) \Longrightarrow$ *k-arc-transitive* $(1 \leq k \leq s)$.
- *the s-arc-transitivity of a graph is inherited by the normal quotients.*

Preliminaries

Definition

- **Cayley graph:** $\mathrm{Cay}(G,S)$ with vertex set G and edges $yx^{-1} \in S$.
- **solvable Cayley graph:** G is solvable.

Lemma

Γ is Cayley of G $\iff$ $\exists G \lesssim \mathrm{Aut}\Gamma$ which is vertex-regular.

Definition

A transitive permutation group G is said to be **quasiprimitive** if its nontrivial normal subgroups are transitive, and **bi-quasiprimitive** if its nontrivial normal subgroups have at most 2 orbits and there exists one which has exactly 2 orbits.

The O'Nan-Scott-Praeger theorem[1] divides the quasiprimitive permutation groups into 8 types: holomorph affine (HA), holomorph simple (HS), holomorph compound (HC), almost simple (AS), simple diagonal (SD), compound diagonal (CD), product action (PA) and twisted wreath product (TW).

[1] Cheryl E Praeger. "Finite Quasiprimitive Graphs". In: *Surveys in Combinatorics, 1997*. London Mathematical Society Lecture Note Series. Cambridge University Press, 1997, 6586.

Background

- 1947, Tutte: No 6-arc-transitive trivalent graphs.
- 1981, Weiss: No s-arc-transitive graphs with $\mathrm{val} \geq 3$ for $s = 6$ and $s \geq 8$.
- 2019, Li C.H. & Xia B.Z.:
  Connected **non-bipartite** 3-arc-transitive solvable Cayley graph with $\mathrm{val} \geq 3$ is a normal cover of the Hoffman-Singleton graph or the Petersen graph.
- 2021, Zhou J.X.: No such normal covers, so $s \leq 2$ sharply.

Problem

Studying connected **bipartite** *s-arc-transitive solvable Cayley graphs, and determining the upper bound on s.*

For convenience, one may assume $s \geq 3$ and $\mathrm{val} \geq 3$.

Main Result

Theorem

Every connected s-arc-transitive solvable Cayley graph with $s \geq 3$ and val ≥ 3 is a normal cover of one of the following graphs:

1. *the complete bipartite graph $K_{n,n}$ with $n \geq 3$;*
2. *the geometry incidence graph $\mathcal{GI}(5,2,2)$;*
3. *the standard double cover of the Hoffman-Singleton graph;*
4. *a graph Σ with valency $p^f + 1$ such that $\mathrm{PSL}_3(p^f).2 \leq \mathrm{Aut}\Sigma \leq \mathrm{Aut}(\mathrm{PSL}_3(p^f))$ and $\mathbb{Z}_p^{2f} : \mathrm{SL}_2(p^f) \lhd (\mathrm{Aut}\Sigma)_\alpha$, where $p^f \geq 3$ is a prime power and α is a vertex.*

In particular, the sharp upper bound on s is 4.

These normal covers are being investigated in a sequel. (usually challenge)

Examples

(1) $\mathbf{K}_{n,n}$ with $n \geq 3$

- 3-transitive
- $\mathbf{K}_{n,n} \cong \mathrm{Cay}(G, G \backslash H)$ where G is solvable and $H < G$ of index 2

(2) $\mathcal{GI}(5,2,2)$

- the incidence graph of $(\mathcal{P}, \mathcal{L})$ where $\mathcal{P}$(resp. $\mathcal{L}$) is the set of 2-subspace(resp. 3-subspace) of $\mathbb{F}_2^5$
- $\mathrm{Aut}(\mathcal{GI}(5,2,2)) = \mathrm{GL}_5(2).\langle\sigma\rangle$ is vertex-transitive
- $G_\alpha = 2^6 : (\mathrm{GL}_2(2) \times \mathrm{GL}_3(2))$, $G_{\alpha\beta} = 2^8 : (S_3 \times S_3)$, val$=\frac{|G_\alpha|}{|G_{\alpha\beta}|}$=7
- 3-transitive[2, Theorem 3.4]
- $G = RG_\alpha$ with $R \cong 31 : 5 : 2$ vertex-regular[3, Theorem 1.1]

[2] Cai Heng Li, Zai Ping Lu, and Gaixia Wang. "Arc-transitive graphs of square-free order and small valency". In: *Discrete Mathematics* 339.12 (2016), pp. 2907–2918.

[3] Cai Heng Li and Binzhou Xia. *Factorizations of Almost Simple Groups with a Solvable Factor, and Cayley Graphs of Solvable Groups*. Vol. 279. 1375. Mem. AMS, 2022.

Examples

(3) $\mathrm{HS}_{50}^{(2)}$

- HS_{50} is 3-transitive
  $\mathrm{Aut}(\mathrm{HS}_{50}) = \mathrm{P\Sigma U}(3,5)$ has a solvable vertex-transitive subgroup
  It is NOT a Cayley graph
- **standard double cover:** $\Gamma^{(2)} = (\tilde{V}, \tilde{E})$ of $\Gamma = (V, E)$, where
  $\tilde{V} = V \times \{1,2\}$, $\tilde{E} = \{\{(v,1),(w,2)\} \mid \{v,w\} \in E\}$
  $\mathrm{Aut}(\Gamma^{(2)}) \geq \mathrm{Aut}\Gamma \times \langle\sigma\rangle$, Γ s-arc-transitive $\implies$ $\Gamma^{(2)}$ s-arc-transitive
- By MAGMA, $\exists_1$ connected G-arc-transitive 7-valent graph Γ of order 50 with $(G, G_\alpha) = (\mathrm{PSU}_3(5), A_7)$ or $(\mathrm{PSU}_3(5).\mathbb{Z}_2, S_7)$. $\Gamma \cong \mathrm{HS}_{50}$
- By MAGMA, $\exists_1$ connected G-arc-transitive 7-valent graph Γ of order 100 with $(G, G_\alpha) = (\mathrm{PSU}_3(5).\mathbb{Z}_2, A_7)$. $\Gamma \cong \mathrm{HS}_{50}^{(2)}$

(Q: Could a normal cover of $\mathrm{HS}_{50}^{(2)}$ be solvable Cayley? see Zhou's paper)

(4) $\mathcal{PH}(3, q)$

- the incidence graph of $\mathrm{PG}(2, q) = (\mathcal{P}, \mathcal{L})$, where $\mathcal{P}$(resp. $\mathcal{L}$) is the set of 1-subspace(resp. 2-subspace) of $\mathbb{F}_q^3$
- $\mathrm{Aut}(\mathcal{PH}(3, q)) = \mathrm{P\Gamma L}_3(q).\langle\sigma\rangle = \mathrm{Aut}(\mathrm{PSL}_3(q))$
- $(\alpha_0, \cdots, \alpha_4)$ with $\alpha_0 \in \mathcal{P}$ corresponds to a basis v_1, v_2, v_3 such that

$$\alpha_0 = \langle v_1\rangle, \alpha_1 = \langle v_1, v_2\rangle, \alpha_2 = \langle v_2\rangle, \alpha_3 = \langle v_2, v_3\rangle, \alpha_4 = \langle v_3\rangle$$

  all ordered bases are equiv. under linear transformation $\implies$ 4-arc-transitive
- a Cayley graph[4] of $D_{2(q^2+q+1)}$, for example Heawood graph.

[4] Dragan Marui. "On 2-arc-transitivity of Cayley graphs". In: *Journal of Combinatorial Theory, Series B* 87.1 (2003), pp. 162–196.

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2. Main Result & Examples

3. Proof of the Main result
   - Overview of the proof
   - Reduce to almost simple groups
   - Almost simple groups

Overview of the proof

Let $\Gamma = \mathrm{Cay}(H, S)$ be connected, (G, s)-transitive, $\mathrm{val} \geq 3$, with H solvable, $H \leq G \leq \mathrm{Aut}\Gamma$, $s \geq 3$.
Let $M \lhd G$ (may trivial) be maximal with ≥ 3 orbits on $V\Gamma$.
Then

- Γ is a normal cover of Γ_M
- Γ_M is $(G/M, s)$-arc-transitive
- G/M is either quasiprimitive or bi-quasiprimitive on $V\Gamma_M$, $|V\Gamma_M| \geq 3$

Theorem (Praeger,1993)

Let Γ be a finite connected graph, $G \leq \mathrm{Aut}\Gamma$ is s-arc transitive on Γ for some $s \geq 2$. Suppose $N \lhd G$ has more than two orbits on vertices. Then

1. *Γ_N is finite and connected and the group of automorphisms of Γ_N induced by G is s-arc transitive on Γ_N.*
2. *N is semiregular on vertices (that is $N_\alpha = 1$ for each vertex α of Γ) and Γ is a cover of Γ_N.*

Overview of the proof

Case quasiprimitive: Γ_M non-bipartite. By Li-Xia-Zhou, no such graphs.
Case bi-quasiprimitive: Γ_M bipartite.
Then either $\Gamma_M = \mathbf{K}_{n,n}$, or G/M is almost simple. (Reduce to AS type)
Moreover, $\mathrm{soc}(G/M)$ is a classical simple group of Lie type.
(Use factorization of AS groups with a solvable factor)

Let s_0 be the sharp upper bound on s.

- By Example (4), $\mathcal{PH}(3, q)$ is 4-transitive, so $s_0 \geq 4$.
- If there is a connected 5-arc-transitive solvable Cayley graph Γ, then Γ is a normal cover of a 5-arc-transitive graph of type (1)-(4), while analysis above states no such graph.

Hence $s_0 = 4$.

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Reduce to almost simple groups

In the remains of this talk, we assume the followings.

- Let Γ be a connected bipartite (G, s)-transitive graphs of val≥ 3, with bipartitions Δ_1 and Δ_2 and $s \geq 3$.
- Suppose G is bi-quasiprimitive on $V\Gamma$ and contains a vertex-transitive solvable subgroup H.
- Set $G^+ = G_{\Delta_1}$, the setwise stabilizer of G on Δ_1, and $H^+ = H \cap G^+$.
- Let α, β be adjacent vertices.

Some immediate facts:

- $G^+ = G_{\Delta_1} = G_{\Delta_2}$ has index 2 (hence normal) in G and is transitive on both Δ_1 and Δ_2, as well as H^+ in H.
- $G = \langle G^+, g\rangle$ where $g \in G\backslash G^+$, g interchanges Δ_1 and Δ_2, $g^2 \in G^+$

Case: unfaithful action

Now we consider the action of G^+ on Δ_1 and Δ_2.

Lemma (4.2)

If G^+ acts unfaithfully on Δ_1 or Δ_2, then $\Gamma = \mathbf{K}_{n,n}$, where $n = |V\Gamma|/2$.

Proof. Let $K = \ker(G^+ \curvearrowright \Delta_1)$.
Then $K^g = \ker(G^+ \curvearrowright \Delta_2)$ and $1 \neq KK^g = K \times K^g \lhd G$.
$\because$ G is bi-quasiprimitive, $K \times K^g$ has at most 2 orbits.
Since K^g fixes Δ_2 pointwise, K must be transitive on Δ_2.
It follows that $\Gamma = \mathbf{K}_{n,n}$. $\square$

We thus additionally assume G^+ is faithful on both Δ_1 and Δ_2.

$$G^+ \cong (G^+)^{\Delta_1} \cong (G^+)^{\Delta_2}$$

We will prove the above actions are quasiprimitive of almost simple type.

Quasiprimitive $\Longrightarrow$ same type AS or PA

Lemma (4.3)

Suppose G^+ is quasiprimitive on Δ_1 or Δ_2. Then

1. *G^+ is quasiprimitive on both Δ_1 and Δ_2*
2. *both $(G^+)^{\Delta_1}$ and $(G^+)^{\Delta_2}$ have the same type AS or PA*

Proof. WLOG, assume G^+ is quasiprimitive on Δ_1.
G transitive on $V\Gamma$ and $G^+ \lhd G \implies G^+ \curvearrowright \Delta_1, \Delta_2$ are perm. isom.
Hence G^+ is quasiprimitive on Δ_2 of the same type on Δ_1.
By ref.[5, Thm 1.2], $(G^+)^{\Delta_1}$ is of type HA, AS, TW or PA.
For cases HA or TW, $\mathrm{soc}(G^+) \lhd G$ is regular on Δ_1 and Δ_2,
then Γ is a **G-normal bi-Cayley graph** on $\mathrm{soc}(G^+)$.
By ref.[6, Lem 3.2], Γ is at most $(G, 2)$-arc-transitive, contradicting $s \geq 3$. □

[5] Michael Giudici, Cai Li, and Cheryl Praeger. "Analysing finite locally s-arc transitive graphs". In: *Trans. AMS* 356 (Aug. 2003), pp. 291–317.

[6] Marston Conder et al. "Edge-transitive bi-Cayley graphs". In: *Journal of Combinatorial Theory, Series B* 145 (2020), pp. 264–306.

Study $\mathrm{soc}(G^+)$

Lemma (4.4)

$\mathrm{soc}(G^+)$ is nonsolvable and is the the unique minimal normal subgroup of G, and $\mathrm{soc}(G^+)_\alpha \neq 1$.

Proof.

If $G^+ \curvearrowright \Delta_1$ is quasiprimitive.
By lemma 4.3, G^+ is of type AS or PA.
So $\mathrm{soc}(G^+) \underset{min}{\lhd} G^+$ and is nonsolvable and not semiregular.
Let $C = C_G(\mathrm{soc}(G^+))$. Then $C \lhd G$ and $\mathrm{soc}(G^+) \cap C = Z(\mathrm{soc}(G^+)) = 1$.
If $C \neq 1$, as $C \cap G^+ = 1$ and $G = G^+.2$, we derive that $C \cong \mathbb{Z}_2$.
Since $|V\Gamma| > 4$, C has more than 2 orbits, contradicting G bi-quasiprimitive.
Hence $C = 1$ and $\mathrm{soc}(G^+)$ is also unique in G.

Now assume G^+ is not quasiprimitive on both Δ_1 and Δ_2.
Then $\exists M \underset{min}{\lhd} G^+$ acting intransitively on Δ_1 hence has > 2 orbits on $V\Gamma$.
Thus $M \not\lhd G$ and $M^g \neq M$. (Recall $G = \langle G^+, g\rangle$)
Since $g^2 \in G^+$, one has $MM^g = M \times M^g \leq G^+$ is normal in G.
By bi-quasiprimitivity of G, $M \times M^g$ is transitive on Δ_1 and Δ_2.
By ref.[7,Thm 1.5], either

(a) $M \times M^g$ is regular on Δ_1 and Δ_2; or
(b) $\mathrm{soc}(G^+) = M \times M^g$ is nonsolvable and not semiregular, and is the unique minimal normal subgroup of G.

For case (b), done.
For case (a), Γ is a G-normal bi-Cayley graph on $M \times M^g$, again by ref.[8,Lem 3.2], is at most $(G, 2)$-arc-transitive. A contradiction. □

[7] Cai Heng Li et al. "Finite locally-quasiprimitive graphs". In: *Discrete Mathematics* 246.1 (2002), pp. 197–218.
[8] Conder et al., "Edge-transitive bi-Cayley graphs".

Structure of $\mathrm{soc}(G^+)$

According to Lemma 4.4, we may set that

$$N := \mathrm{soc}(G^+) = T_1 \times T_2 \times \cdots \times T_d \cong T^d \underset{min}{\overset{unique}{\lhd}} G.$$

Then N is transitive on Δ_1 and Δ_2 since G is bi-quasiprimitive.

Consider $G^+ \curvearrowright \{T_1, \cdots, T_d\}$ by conjugation with kernel K.
Then $N \lhd K \leq \mathrm{Aut}(T_1) \times \cdots \times \mathrm{Aut}(T_d)$. And $T_j \lhd K$, $1 \leq j \leq d$.
Let $M_j = C_K(T_j)$. Then

$$\cdots \times T_{j-1} \times 1 \times T_{j+1} \times \cdots \leq M_j \leq \cdots \times \mathrm{Aut}(T_{j-1}) \times 1 \times \mathrm{Aut}(T_{j+1}) \times \cdots$$

$M_j \lhd K$ and $K/M_j \lesssim \mathrm{Aut}(T_j)$ is almost simple with socle T_j.

Recall H is a vertex-transitive solvable subgroup of G.
Let $H^+ = H \cap G^+$ and H_j be the projection of $H^+ \cap N$ on T_j, $1 \leq j \leq d$.
Then $|H^+ \cap N|$ divides $|H_1||H_2| \cdots |H_d|$.

Exclude PA type

Lemma (4.5)

Suppose $d \geq 2$. Then $\exists j \in \{1, 2, ..., d\}$ s.t. M_j is transitive on Δ_1 or Δ_2; in particular, if G^+ is quasiprimitive on Δ_1 or Δ_2, then it is not of type PA.

Proof.
Assume G^+ is quasiprimitive of type PA on Δ_1 or Δ_2.
Then $\exists \mathcal{B} = \Omega^d$ a maximal block system s.t. $G^+ \curvearrowright \mathcal{B}$ primitive by PA.
For $(w_1, w_2, ..., w_d) \in \Omega^d$, $(x_1, x_2, ..., x_d) \in M_j$, we have

$$(w_1, w_2, ..., w_d)^{(x_1, x_2, ..., x_d)} = (w_1^{x_1}, w_2^{x_2}, ..., w_d^{x_d}).$$

From previous analysis, $x_j = 1$. Then transitivity of M_j implies $|\Omega| = 1$.
Need to prove the transitivity of some M_j.

Before that we prepare some lemmas.

Theorem (2.2, Thompson-Wielandt)

Let Γ be a connected (G,s)-transitive graph with $s \geq 2$, and let α and β be adjacent vertices of Γ. Then one of the following holds, where p is a prime:

1. $s \leq 3$, $G_{\alpha\beta}^{[1]} = 1$, *and* $G_{\alpha}^{[1]} \cong (G_{\alpha}^{[1]})^{\Gamma(\beta)} \lhd G_{\alpha\beta}^{\Gamma(\beta)} \cong G_{\alpha\beta}^{\Gamma(\alpha)}$;
2. $s \leq 3$, $G_{\alpha\beta}^{[1]}$ *is a nontrivial p-group,* $\mathrm{PSL}_d(p^f) \lhd G_{\alpha}^{\Gamma(\alpha)}$ *for some* $d \geq 2$, *and* $\mathrm{val}(\Gamma) = (p^{df} - 1)/(p^f - 1)$;
3. $s \geq 4$, $\mathrm{PGL}_2(p^f) \lhd G_{\alpha}^{\Gamma(\alpha)}$, $\mathrm{val}(\Gamma) = p^f + 1$, *and one of the following occurs:*
   1. s=4, *and* $G_\alpha = p^{2f} : \frac{p^f-1}{(3,p^f-1)}.\mathrm{PGL}_2(p^f).\mathcal{O}$ *with* $|\mathcal{O}| \mid (3, p^f - 1)f$;
   2. s=5, *and* $G_\alpha = p^{3f} : \mathrm{GL}_2(2^f).\mathbb{Z}_e$ *with* $e|f$;
   3. s=7, *and* $G_\alpha = (p^{2f} \times (p^f)^{1+2}) : \mathrm{GL}_2(3^f).\mathbb{Z}_e$ *with* $e|f$.

Lemma (2.3)

For a connected $(G,3)$-arc-transitive graph Γ and adjacent vertices α, β, at least one of the following holds:[a, Thm 4.2]

1. $G_\alpha^{[1]}$ *is transitive on* $\Gamma(b) \backslash \{\alpha\}$.
2. $G_\alpha = A_7$ *or* S_7.
3. $\mathbb{Z}_p^f \lhd G_\alpha^{\Gamma(\alpha)} \leq \mathrm{A\Gamma L}_1(p^f)$ *with* p *a prime.*

[a] Cai Heng Li, kos Seress, and Shu Jiao Song. "s-Arc-transitive graphs and normal subgroups". In: *Journal of Algebra* 421 (2015), pp. 331–348.

Lemma (2.5)

Let Γ be a connected (G,s)-transitive graph with $s \geq 3$, and suppose $N \lhd G$ and not semiregular. Then one of the following holds:

1. *both* $G_\alpha^{\Gamma(\alpha)}$ *and* $N_\alpha^{\Gamma(\alpha)}$ *are almost simple 2-transitive;*
2. $G_\alpha^{\Gamma(\alpha)}$ *is affine and* $N_\alpha^{\Gamma(\alpha)}$ *is primitive.*

Proof of Lemma 2.5:

- $(G,3)$-arc-transitive $\implies$ $G_\alpha^{\Gamma(\alpha)}$ is 2-transitive (AS or affine)
- $N \lhd G$, $N_\alpha \neq 1$ and connectivity $\implies$ $1 \neq N_\alpha^{\Gamma(\alpha)} \lhd G_\alpha^{\Gamma(\alpha)}$ transitive

Claim:

$$G_{\alpha\beta}^{[1]} = 1 \text{ and } G_\alpha^{[1]} \curvearrowright \Gamma(\beta)\backslash\{\alpha\} \text{ transitive } \implies N_\alpha \curvearrowright \Gamma(\alpha) \text{ 2-transitive}$$

- check $G_\beta^{[1]}/G_{\alpha\beta}^{[1]} \cong (G_\beta^{[1]})^{\Gamma(\alpha)} \lhd N_{\alpha\beta}^{\Gamma(\alpha)}$
- $G_\beta^{[1]}$ (hence $N_{\alpha\beta}$) $\curvearrowright \Gamma(\alpha)\backslash\{\beta\}$ transitive

Better change the conditions to $G_{\alpha\beta}^{[1]} = 1$ and 3-arc-transitive?

$G_\alpha^{\Gamma(\alpha)}$ **AS:**

By ref.[9, Thm5.3(S)Notes(2)], either $N_\alpha^{\Gamma(\alpha)}$ is 2-transitive, or $N_\alpha^{\Gamma(\alpha)} = \mathrm{PSL}_2(8)$ primitive with $\mathrm{val}(\Gamma) = 28$. The latter one is impossible.

$G_\alpha^{\Gamma(\alpha)}$ **affine:**

1. $s \geq 4$:
   1. $\mathrm{val}(\Gamma) = 3$, $N_\alpha^{\Gamma(\alpha)} \geq \mathbb{Z}_3$ primitive
   2. $\mathrm{val}(\Gamma) = 4$ and $G_\alpha^{\Gamma(\alpha)} = \mathrm{PGL}_2(3) \cong S_4$,
      1. $N_\alpha^{\Gamma(\alpha)} = A_4$ or S_4 is 2-transitive
      2. $N_\alpha^{\Gamma(\alpha)} \cong \mathbb{Z}_2^2$ is regular, $N_{\alpha\beta} \curvearrowright \Gamma(\alpha) \cup \Gamma(\beta)$ trivial, connect $\Longrightarrow$ $N_{\alpha\beta} = 1$, hence $N_\alpha = \mathbb{Z}_2^2$, but by ref.[10, lem 2.5], no such normal subgroup
2. $s = 3$: $\overset{Thm2.2}{\Longrightarrow} G_{\alpha\beta}^{[1]} = 1$. Consider $G_\alpha^{[1]} \curvearrowright \Gamma(\beta) \backslash \{\alpha\}$

trans $\overset{Claim}{\Longrightarrow} N_\alpha^{\Gamma(\alpha)}$ 2-transitive

intrans. $\overset{lem2.3}{\Longrightarrow} \mathbb{Z}_p^f \lhd G_\alpha^{\Gamma(\alpha)} \leq \mathrm{A\Gamma L}_1(p^f)$, so $G_\beta^{[1]} \cong (G_\beta^{[1]})^{\Gamma(\alpha)} \lhd G_{\alpha\beta}^{\Gamma(\alpha)} \leq \mathbb{Z}_{p^f-1}.\mathbb{Z}_f$ but this leads to a contradiction. □

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[9] Peter Cameron. "Finite Permutation Groups and Finite Simple Groups". In: *Bulletin of The London Mathematical Society - BULL LOND MATH SOC* 13 (Jan. 1981), pp. 1–22.

[10] C.H. Li, Z.P. Lu, and H. Zhang. "Tetravalent edge-transitive Cayley graphs with odd number of vertices". In: *Journal of Combinatorial Theory, Series B* 96.1 (2006).

Lemma (2.5)

Let Γ be a connected (G, s)-transitive graph with $s \geq 3$, and suppose $N \lhd G$ and not semiregular. Then one of the following holds:

1. *both $G_\alpha^{\Gamma(\alpha)}$ and $N_\alpha^{\Gamma(\alpha)}$ are almost simple 2-transitive;*
2. *$G_\alpha^{\Gamma(\alpha)}$ is affine and $N_\alpha^{\Gamma(\alpha)}$ is primitive.*

By lemma 4.4, $N \lhd G$ and $N_\alpha \neq 1$, $\forall \alpha \in V\Gamma$.
Applying lemma 2.5, we have Γ is N-locally-primitive.
Hence Γ is K-locally-primitive since $N \leq K$.

Lemma (2.4)

Let Γ be a connected bipartite G-locally-primitive graph with G-orbits Δ_1 and Δ_2 on $V\Gamma$ and each $|\Delta_i| > 1$. Suppose $\exists N \lhd G$ s.t. N is intransitive on Δ_1 and Δ_2. Then[a], *Lem 5.1*

1. *Γ is a normal cover of Γ_N.*
2. *N is semiregular on $V\Gamma$, $G^{V\Gamma_N} \cong G/N$ and $G_\alpha \cong (G/N)_v$ for $\alpha \in V\Gamma$ and $v \in V\Gamma_N$.*
3. *Γ_N is G/N-locally-primitive. Futhermore, if Γ is locally (G,s)-arc-transitive with $s \geq 2$, then Γ_N is locally $(G/N, s)$-arc-transitive.*

[a] Giudici, Li, and Praeger, "Analysing finite locally *s*-arc transitive graphs".

If $M_j \lhd K$ is intransitive on both Δ_1 and Δ_2 for each $j = 1, ..., d$, then M_j is semiregular hence $M_j \cap K_\alpha = 1$, and $K_\alpha \cong (K/M_j)_v \lesssim \mathrm{Aut}(T)$. Since $\frac{K_\alpha}{M_j \cap K_\alpha} \cong \frac{K_\alpha \ker \varphi_j}{\ker \varphi_j}$, K_α is a diagonal subgroup, i.e.

$$K_\alpha = \{(a, a, ..., a) | a \in P\}, \;\; P \leq \mathrm{Aut}(T).$$

$$N_\alpha = N \cap K_\alpha = (T_1 \times \cdots \times T_d) \cap K_\alpha = \{(a, a, ..., a) | a \in P \cap T\} \lesssim T,$$

thus $|T|^{d-1}$ divides $|N : N_\alpha| = |\Delta_1|$,

$H^+ \curvearrowright \Delta_1$ transitively $\implies$ $|T|^{d-1}$ divides $|H^+| = |H^+ \cap K||H^+K/K|$.
$N \curvearrowright \Delta_1$ transitively $\implies$

$$K/N = K_\alpha N/N \cong K_\alpha/N_\alpha \cong P/(P \cap T) \cong PT/T \lesssim \mathrm{Out}(T).$$

$\therefore$ $|H^+ \cap K|/|H^+ \cap N|$ divides $|\mathrm{Out}(T)|$.
$\because$ $|H^+ \cap N|$ divides $|H_1 \times \cdots \times H_d|$,
Therefore,

$$|T|^{d-1} \text{ divides } |H_1 \times \cdots \times H_d||\mathrm{Out}(T)||H^+K/K|.$$

Case: $H^+ \not\leq K$: $\{T_1, ..., T_k\}$ an H^+-conj.-orbit. Then $H_1 \cong \cdots \cong H_k$.

$$M := M_1 \cap \cdots \cap M_k \qquad (\Delta_i)_M := \{M\text{-orbits on } \Delta_i\}$$

Consider $\overline{K} := K/M \curvearrowright (\Delta_i)_M$. Then

- M_j intrans. $\Longrightarrow$ $|(\Delta_i)_M| > 1$
- $\overline{K_v}$ with $v \in (\Delta_i)_M$ is diagonal
- $\overline{H^+}$ and $\overline{N} \cong T^k$ are transitive on $(\Delta_i)_M$
- $\overline{H^+}\overline{K}/\overline{K} \lesssim S_k$
- projection of $\overline{H^+} \cap \overline{N}$ on T_j is isomorphic to H_j for $1 \leq j \leq k$

Similar as previous page,

$$|T|^{k-1} \text{ divides } |H_1 \times \cdots \times H_k||\mathrm{Out}(T)||S_k| = |H_1|^k|\mathrm{Out}(T)|k!$$

Contradict to ref.[11, page 76] why does lemma 2.6 hold by ref.?

[11] Li and Xia, *Factorizations of Almost Simple Groups with a Solvable Factor, and Cayley Graphs of Solvable Groups.*

Case: $H^+ \leq K$**:** H^+ normalizes $T_1, ..., T_d$, $\therefore T_1 \lhd H^+T_1 =: Y = H^+Y_\alpha$.

$$Y_\alpha = Y_\alpha/(Y\alpha \cap T_1) \cong Y_\alpha T_1/T_1 \leq Y/T_1 = H^+T_1/T_1 \cong H^+/(H^+ \cap T_1)$$

$\therefore Y$ is a product of two solvable subgroups. By ref.[12, Thm 1.1],

- either $T_1 = \mathrm{PSL}_2(p^f)$ with $p^f \geq 5$
- or $T_1 = \mathrm{PSU}_3(8),\ \mathrm{PSU}_4(2),\ \mathrm{PSL}_4(2),\ M_{11},\ \mathrm{PSL}_3(p^f)$ with $p^f = 3, 4, 5, 7, 8$

Meanwhile,

$H^+ \curvearrowright \Delta_i$ transitive $\Longrightarrow K = H^+K_\alpha \Longrightarrow K/M_j = \frac{H^+M_j}{M_j}(K/M_j)_{v_j}$

(factorization of AS group K/M_j with socle above and a solvable factor)

$\therefore \exists$ prime $r \mid |T|$ but $\nmid |H_j| = |(H^+M_j/M_j) \cap \mathrm{soc}(K/M_j)|$ by ref.[12,Thm 1.1].

Recall previous equation and note that $|H^+K/K| = 1$ here,

we have $|T|_r^{d-1}$ divides $|\mathrm{Out}(T)|_r$, which is impossible by ref.[13, Thm 8.17].

[12] L.C. Kazarin. "On groups which are the product of two solvable subgroups". In: *Communications in Algebra* 14.6 (1986), pp. 1001–1066.

[13] Li and Xia, *Factorizations of Almost Simple Groups with a Solvable Factor, and Cayley Graphs of Solvable Groups.*

Lemma (4.6)

G^+ is quasiprimitive on Δ_1 and Δ_2.

Proof. Assume $G^+ \curvearrowright \Delta_1$ is NOT quasiprimitive.
By ref.[14],

- $d = 2l$ and $\exists M \underset{min}{\lhd} G^+$ semiregular and $N = M \times M^g$,
- $N_\alpha \lesssim T$ is a diagonal subgroup of N.

What could T be?

Note that $Y = MH^+$ has factorization $Y = H^+Y_\alpha$ into solvable factors. Then ref.[15] implies that either $T = \mathrm{PSL}_2(p^f)$ with $p^f \geq 5$, or $T = \mathrm{PSU}_3(8)$, $\mathrm{PSU}_4(2)$, $\mathrm{PSL}_4(2)$, M_{11} or $\mathrm{PSL}_3(p^f)$ with $p^f \in \{3, 4, 5, 7, 8\}$.

[14] Li et al., "Finite locally-quasiprimitive graphs".

[15] Kazarin, "On groups which are the product of two solvable subgroups".

Case (a): $N_\alpha \cong T$.
Recall $G_\alpha^{\Gamma(\alpha)}$ is 2-transitive hence almost simple here.
Then lemma 2.5 implies $N_\alpha^{\Gamma(\alpha)} \cong T$ is 2-transitive.
Hence Γ is locally $(N, 2)$-arc-transitive.

WLOG, assume $M = T_1 \times \cdots \times T_l$ and

$$P = T_1 \times \cdots \times T_{l-1} \times T_1^g \times \cdots \times T_{l-1}^g \lhd N$$

Then $P_\alpha = P \cap N_\alpha = 1$ and hence length of P-orbits is $|P| = |T|^{2l-1} < |T|^{2l-1} = |\Delta_i|$, P is intransitive on both Δ_1 and Δ_2.
Then lemma 2.4 implies Γ_P is locally $(N/P, 2)$-arc-transitive.

However, $N/P \cong T_l \times T_l^g \cong T^2$ and $(N/P)_v \cong N_\alpha \cong T$ implies that N/P is quasiprimitive of type HS on the bipartitions of Γ_P. Why?
This contradicts ref.[16, Thm 1.2].

[16] Giudici, Li, and Praeger, "Analysing finite locally *s*-arc transitive graphs".

Case (b): $1 \neq N_\alpha < T$.
Let L be the maximal subgroup of T containing N_α.

- By lemma 4.5, $\exists M_j$ is transitive on Δ_1 or Δ_2. $|N : N_\alpha| = |\Delta_1| \mid |M_j|$.

$$|T|/|N_\alpha| \text{ and } |T|/|L| \text{ divide } |\mathrm{Out}(T)|^{2l-1}. \quad (\therefore T \neq M_{11})$$

- Consider $m(T)$ the smallest index of the solvable subgroups in T.

$$m(T)^{2l} \leq |G_\alpha|$$

- Since M, M^g are the only 2 minimal normal subgroups of G^+, G^+/K is transitive on $\{T_1, ..., T_l\}$ and has 2 orbits on $\{T_1, ..., T_{2l}\}$.
  Hence $l \mid |G : K|$ and $G^+/K \lesssim S_l \wr \mathbb{Z}_2$.
  Thereby $l \mid |G : N|$ and $G^+ \lesssim N.(\mathrm{Out}(T)^l : S_l)^2.\mathbb{Z}_2$.

Now we discuss what L could be.

Definition (p.p.d.)

For positive integers a, $m \geq 2$, a prime r is called a **primitive prime divisor** of $a^m - 1$ if r divides $a^m - 1$ but not $a^i - 1$ for each $i = 1, 2, ..., m - 1$.

Lemma (2.1, Zsigmondy)

For positive integers a, $m \geq 2$, $a^m - 1$ has a primitive prime divisor r if $(a, m) \neq (2, 6)$ and $(2^e - 1, 2)$ with $e \geq 2$ an integer.
Moreover, $r \equiv 1 \pmod{m}$, and in particular $r > m$.

Case (b.1): L is nonsolvable. $(\therefore T \neq \mathrm{PSL}_3(3))$

- $T = \mathrm{PSL}_2(p^f)$ with $p^f \geq 5$: By ref.[17], L may be
  1. $L = A_5$ with $p^f = p \equiv \pm 1 (mod\ 5)$ or $p^f = p^2 \equiv -1 (mod\ 5)$
  2. $L = \mathrm{PSL}_2(p^e)$ with f/e odd, or $L = \mathrm{PGL}_2(p^e)$ with f/e even.

  Both contradict $|T : L| \mid |\mathrm{Out}(T)|^{2l-1}$.
- $T = \mathrm{PSL}_4(2)$: By ATLAS, $\mathrm{Out}(T) = 2$, $m(T) = 35$,

$$L = 2^3 : \mathrm{PSL}_3(2),\ S_6,\ (A_5 \times 3) : 2,\ A_7.$$

  Recall $|T : L| \mid |\mathrm{Out}(T)|^{2l-1}$, $N_\alpha = L = A_7$, searching G_α in ref.[18,Thm3.4] gives $l \geq 6$ hence $35^{2l} \leq |G_\alpha| \leq |S_6 \times S_7|$. Contradiction!
- $T = \mathrm{PSL}_3(p^f)$, $\mathrm{PSU}_3(8)$, $\mathrm{PSU}_4(2)$ with $p^f = 4, 5, 7, 8$: by ATLAS, contradicting that $|T : L|$ divides $|\mathrm{Out}(T)|^{2l-1}$. (Table 1)

[17] Leonard Eugene Dickson. "Linear Groups, with an Exposition of the Galois Field Theory". In: 1958.

[18] Li, Lu, and Wang, "Arc-transitive graphs of square-free order and small valency".

Case (b.2): L is solvable.
Now $N_\alpha^{\Gamma(\alpha)}$ is solvable and $G_\alpha^{\Gamma(\alpha)}$ is affine, $\mathrm{val}(\Gamma)$ is a prime power.

One can exclude $T \neq \mathrm{PSL}_2(p^f)$ by ATLAS as previous. (see Table 2)

Suppose $T = \mathrm{PSL}_2(p^f)$, $p^f \geq 5$. Then $|\mathrm{Out}(T)| = (2, p-1)f$. By ref.[19],

$$L \in \{D_{2\frac{p^f-1}{(2,p-1)}}, D_{2\frac{p^f+1}{(2,p-1)}}, \mathbb{Z}_p^f : \mathbb{Z}_{\frac{p^f-1}{(2,p-1)}}\}, \text{ or } L \in \{A_4, S_4\} \text{ and } f = 1.$$

A_4, S_4: Then $f = 1$ and $\mathrm{Out}(T) \leq \mathbb{Z}_2$, but $|T : L|$ is divisible by $p \geq 5$.

$D_{2\frac{p^f-1}{(2,p-1)}}$: $|T : L| = p^f(p^f + 1)/2$

If r a ppd. of $p^{2f} - 1$, then $r > 2f$ divides $|T : L|$ but not $|\mathrm{Out}(T)|$.
Or $r = 2, p$ resp. for $(p, 2f) = (2, 6)$, $(2^m - 1, 2)$ does the same.

$D_{2\frac{p^f+1}{(2,p-1)}}$: Similar as above.

$\mathbb{Z}_p^f : \mathbb{Z}_{\frac{p^f-1}{(2,p-1)}}$: $|T : L| = p^f + 1$, $p^{2f} - 1$ has a ppd $r | p^f + 1$ but not $(2, p-1)f$.

[19] Dickson, "Linear Groups, with an Exposition of the Galois Field Theory".

Summary

Recall hypothesis in the beginning of this section:

- Let Γ be a connected bipartite (G, s)-transitive graphs of val≥ 3, with bipartitions Δ_1 and Δ_2 and $s \geq 3$.
- Suppose G is bi-quasiprimitive on $V\Gamma$ and contains a vertex-transitive solvable subgroup H.
- Set $G^+ = G_{\Delta_1}$, the setwise stabilizer of G on Δ_1, and $H^+ = H \cap G^+$.

Proposition (4.7)

Under such hypothesis, either $\Gamma = \mathbf{K}_{n,n}$, or G is an almost simple group and G^+ is quasiprimitive on both Δ_1 and Δ_2.

Proof. If G^+ acts unfaithfully on Δ_1 or Δ_2, lemma 4.2 gives $\Gamma = \mathbf{K}_{n,n}$.
Faithful case:
Lemma 4.3 shows either G^+ is not quasiprimitive on Δ_1 or Δ_2, or quasiprimitive on both with same type AS or PA.
Lemma 4.6 exclude the former. Lemma 4.5 exclude PA in the latter.
Finally lemma 4.4 implies G is AS.

Outline

1. Preliminaries & Background
2. Main Result & Examples
3. Proof of the Main result
   - Overview of the proof
   - Reduce to almost simple groups
   - Almost simple groups

Factorization of AS groups

Theorem

[a] *Suppose that G is an almost simple group with socle L and $G = HK$ for a solvable subgroup H and core-free subgroup K of G. Then one of the following statements holds.*

- (a) *Both H and K are solvable, and the triple (G, H, K) is described in Proposition 4.1.*
- (b) *$L = A_n$, and the triple (G, H, K) is described in Proposition 4.3.*
- (c) *L is a sporadic simple group, and the triple (G, H, K) is described in Proposition 4.4.*
- (d) *L is a classical group of Lie type, and the triple $(L, H \cap L, K \cap L)$ lies in Table 1.1 or Table 1.2.*

[a] Li and Xia, *Factorizations of Almost Simple Groups with a Solvable Factor, and Cayley Graphs of Solvable Groups*.

Factorizing G and G^+

Assume G is AS with socle T, and G^+ is quasiprimitive on Δ_1 and Δ_2.
Set $G = T.\mathcal{O}$ and $G^+ = T.\mathcal{O}'$, with $\mathcal{O}' < \mathcal{O} \leq \mathrm{Out}(T)$ and $|\mathcal{O} : \mathcal{O}'| = 2$.

$T \curvearrowright \Delta_i$ transitive $\implies G^+ = G_\alpha T \implies G_\alpha / T_\alpha \cong G_\alpha T/T = G^+/T = \mathcal{O}'$

By Frattini argument, $G = HG_\alpha$ and $G^+ = H^+G_\alpha$.
Both factorization have a solvable factor H, H^+.

Lemma (5.1)

$T \neq \mathrm{PSL}_2(q)$ with q a prime power.

Proof. Check the classification in ref.[20], in which the only 3-arc-transitive graph is the Petersen graph, non-bipartite. Why no $\mathrm{Pet}^{(2)}$? □

[20] C. E. Praeger A. Hassani L. R. Nochefranca. "Two-arc transitive graphs admitting a two-dimensional projective linear group". In: 2.4 (1999), pp. 335–353.

Lemma (5.2)

If G_α solvable, then $T = \mathrm{PSL}_3(3)$ and $\Gamma = \mathcal{PH}(3, 3^3)$ is 4-transitive of valency 4.

Proof. By ref.[21, Prop. 4.1], both (G, H, G_α) and (G^+, H^+, G_α) lie in

TABLE 4.1.

row	G	H	K
1	$\mathrm{PSL}_2(7).\mathcal{O}$	$7{:}\mathcal{O}$, $7{:}(3\times\mathcal{O})$	S_4
2	$\mathrm{PSL}_2(11).\mathcal{O}$	$11{:}(5\times\mathcal{O}_1)$	$\mathrm{A}_4.\mathcal{O}_2$
3	$\mathrm{PSL}_2(23).\mathcal{O}$	$23{:}(11\times\mathcal{O})$	S_4
4	$\mathrm{PSL}_3(3).\mathcal{O}$	$13{:}\mathcal{O}$, $13{:}(3\times\mathcal{O})$	$3^2{:}2.\mathrm{S}_4$
5	$\mathrm{PSL}_3(3).\mathcal{O}$	$13{:}(3\times\mathcal{O})$	$\mathrm{A\Gamma L}_1(9)$
6	$\mathrm{PSL}_3(4).(\mathrm{S}_3\times\mathcal{O})$	$7{:}(3\times\mathcal{O}).\mathrm{S}_3$	$2^4{:}(3\times\mathrm{D}_{10}).2$
7	$\mathrm{PSL}_3(8).(3\times\mathcal{O})$	$73{:}(9\times\mathcal{O}_1)$	$2^{3+6}{:}7^2{:}(3\times\mathcal{O}_2)$
8	$\mathrm{PSU}_3(8).3^2.\mathcal{O}$	$57{:}9.\mathcal{O}_1$	$2^{3+6}{:}(63{:}3).\mathcal{O}_2$
9	$\mathrm{PSU}_4(2).\mathcal{O}$	$2^4{:}5$	$3_+^{1+2}{:}2.(\mathrm{A}_4.\mathcal{O})$
10	$\mathrm{PSU}_4(2).\mathcal{O}$	$2^4{:}\mathrm{D}_{10}.\mathcal{O}_1$	$3_+^{1+2}{:}2.(\mathrm{A}_4.\mathcal{O}_2)$
11	$\mathrm{PSU}_4(2).2$	$2^4{:}5{:}4$	$3_+^{1+2}{:}\mathrm{S}_3$, $3^3{:}(\mathrm{S}_3\times\mathcal{O})$, $3^3{:}(\mathrm{A}_4\times 2)$, $3^3{:}(\mathrm{S}_4\times\mathcal{O})$
12	M_{11}	$11{:}5$	$\mathrm{M}_9.2$

where $\mathcal{O}\leqslant \mathrm{C}_2$, and $\mathcal{O}_1,\mathcal{O}_2$ are subgroups of $\mathcal{O}$ such that $\mathcal{O}=\mathcal{O}_1\mathcal{O}_2$.

[21] Li and Xia, *Factorizations of Almost Simple Groups with a Solvable Factor, and Cayley Graphs of Solvable Groups.*

1-3: Excluded by lemma 5.1.

4-5: $G = \mathrm{PSL}_3(3).2$, $G_\alpha = 3^2 : 2.S_4$ or $\mathrm{A\Gamma L}_1(9) \cong 3^2 : 8.2$
$G_\alpha^{\Gamma(\alpha)}$ 2-transitive $\Longrightarrow$ $\mathrm{val}(\Gamma) = \cancel{3}, 4, \cancel{9}$. By ref.[22], $G_\alpha = 3^2 : 2.S_4$
Hence $|V\Gamma| = 26$, and $\Gamma = \mathcal{PH}(3, 3^3)$ by ref.[23, Table 1].

6: $G = \mathrm{PSL}_3(4).(S_3 \times \mathbb{Z}_2)$ and $G_\alpha = 2^4 : (3 \times D_{10}).2$
3-arc-transitive $\Longrightarrow$ $\mathrm{val}(\Gamma) = 5$, but $G_\alpha = \mathbb{Z}_4 \times (\mathbb{Z}_5 : \mathbb{Z}_4)$ by ref.[24]

<u>Claim:</u> If $\mathrm{val}(\Gamma) \geq 5$, then $|G_\alpha|$ divides $|G_\alpha^{\Gamma(\alpha)}|^2/\mathrm{val}(\Gamma)$.

7: $G = \mathrm{PSU}_3(8).(3 \times \mathbb{Z}_2)$, $G_\alpha^{\Gamma(\alpha)} = 2^3 : 7 : 3$, $\mathrm{val}(\Gamma) = 8$, contradiction.

8: $G = \mathrm{PSU}_3(8).3^2.2$, $G_\alpha^{\Gamma(\alpha)} = 2^3 : 7 : 3$ with $\mathrm{val}(\Gamma) = 2^3$ or
$G_\alpha^{\Gamma(\alpha)} = 2^6 : 63 : 3$ with $\mathrm{val}(\Gamma) = 2^6$, both contradiction

9-11: $G = \mathrm{PSU}_4(2)$, $G_\alpha = 3_+^{1+2} : 2.A_4$, $\mathrm{val}(\Gamma) = \cancel{4}$(ref.[15]), $\cancel{9}$(3-arc-trans.)

12: $G^+ = M_{11} \implies G = M_{11}.2$ which is not in the table.

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[22] Li, Lu, and Zhang, "Tetravalent edge-transitive Cayley graphs with odd number of vertices".

[23] Ying Cheng and James Oxley. "On weakly symmetric graphs of order twice a prime". In: *Journal of Combinatorial Theory, Series B* 42.2 (1987), pp. 196–211.

[24] Richard Weiss. "An application of p-factorization methods to symmetric graphs". In: *Mathematical Proceedings of the Cambridge Philosophical Society* 85.1 (1979), 4348.

Lemma (5.3)

T is not an alternating simple group.

Proof. Both (G, H, G_α) and (G^+, H^+, G_α) lie in the following. All possibilities can be excluded by lemma 2.3.

Proposition 4.3. *Suppose $n \geqslant 5$ and $\mathrm{soc}(G) = \mathrm{A}_n$ acting naturally on $\Omega_n = \{1, \dots, n\}$. If $G = HK$ for a solvable subgroup H and core-free non-solvable subgroup K of G, then one of the following holds.*

(a) $\boxed{\mathrm{A}_n \trianglelefteq G \leqslant \mathrm{S}_n \text{ with } n \geqslant 6,\ H \text{ is transitive on } \Omega_n, \text{ and } \mathrm{A}_{n-1} \trianglelefteq K} \leqslant \mathrm{S}_{n-1}$.

(b) $\boxed{\mathrm{A}_n \trianglelefteq G \leqslant \mathrm{S}_n \text{ with } n = p^f \text{ for some prime } p,\ H \text{ is 2-homogeneous on } \Omega_n, \text{ and } \mathrm{A}_{n-2} \trianglelefteq K \leqslant \mathrm{S}_{n-2} \times \mathrm{S}_2; \text{ moreover, either } H \leqslant \mathrm{A\Gamma L}_1(p^f) \text{ or } (H, n) \text{ lies}}$ *in the table:*

H	n
$5^2{:}\mathrm{SL}_2(3)$, $5^2{:}\mathrm{Q}_8.6$, $5^2{:}\mathrm{SL}_2(3).4$	5^2
$7^2{:}\mathrm{Q}_8.\mathrm{S}_3$, $7^2{:}\mathrm{SL}_2(3).6$	7^2
$11^2{:}\mathrm{SL}_2(3).5$, $11^2{:}\mathrm{SL}_2(3).10$	11^2
$23^2{:}\mathrm{SL}_2(3).22$	23^2
$3^4{:}2^{1+4}.5$, $3^4{:}2^{1+4}.\mathrm{D}_{10}$, $3^4{:}2^{1+4}.\mathrm{AGL}_1(5)$	3^4

(c) $\boxed{\mathrm{A}_n \trianglelefteq G \leqslant \mathrm{S}_n \text{ with } n = 8 \text{ or } 32,\ \mathrm{A}_{n-3} \trianglelefteq K \leqslant \mathrm{S}_{n-3} \times \mathrm{S}_3}$ *and* $(H, n) = (\mathrm{AGL}_1(8), 8)$, $(\mathrm{A\Gamma L}_1(8), 8)$ *or* $(\mathrm{A\Gamma L}_1(32), 32)$.

(d) $\mathrm{A}_6 \trianglelefteq G \leqslant \mathrm{S}_6$, $H \leqslant \mathrm{S}_4 \times \mathrm{S}_2$, *and* $K = \mathrm{PSL}_2(5)$ *or* $\mathrm{PGL}_2(5)$.

intransitive

(e) $\mathrm{A}_6 \trianglelefteq G \leqslant \mathrm{S}_6$, $H \leqslant \mathrm{S}_3 \wr \mathrm{S}_2$, *and* $K = \mathrm{PSL}_2(5)$ *or* $\mathrm{PGL}_2(5)$.

(f) $n = 6$ *or* 8, *and* (G, H, K) *lies in* Table $\boxed{4.2}$.

Table 4.2.

row	G	H	K
1	M_{10}	$3^2{:}\mathrm{Q}_8$	$\mathrm{PSL}_2(5)$
2	$\mathrm{PGL}_2(9)$	$\mathrm{AGL}_1(9)$	$\mathrm{PSL}_2(5)$
3	$\mathrm{P\Gamma L}_2(9)$	$\mathrm{A\Gamma L}_1(9)$	$\mathrm{PSL}_2(5)$
4	$\mathrm{P\Gamma L}_2(9)$	$\mathrm{AGL}_1(9)$, $3^2{:}\mathrm{Q}_8$, $\mathrm{A\Gamma L}_1(9)$	$\mathrm{PGL}_2(5)$
5	A_8	15, $3 \times \mathrm{D}_{10}$, $\mathrm{\Gamma L}_1(16)$	$\mathrm{AGL}_3(2)$
6	S_8	D_{30}, $\mathrm{S}_3 \times 5$, $\mathrm{S}_3 \times \mathrm{D}_{10}$, $3 \times \mathrm{AGL}_1(5)$, $\mathrm{S}_3 \times \mathrm{AGL}_1(5)$	$\mathrm{AGL}_3(2)$

Lemma (5.4)

T is not a sporadic simple group.

Proof. Both (G, H, G_α) and (G^+, H^+, G_α) lie in the following. The only two possibilities can be excluded by lemma 2.3.

Proposition 4.4. *Let $L = \mathrm{soc}(G)$ be a sporadic simple group. If $G = HK$ for a solvable subgroup H and core-free non-solvable subgroup K of G, then one of the following holds.*

(a) $\boxed{\mathrm{M}_{12} \leqslant G \leqslant \mathrm{M}_{12}.2,\ H \text{ is transitive on } [G{:}\mathrm{M}_{11}], \text{ and } K = \mathrm{M}_{11}}$

(b) *$G = \mathrm{M}_{24}$, H is transitive on $[\mathrm{M}_{24}{:}\mathrm{M}_{23}]$, and $K = \mathrm{M}_{23}$.*

(c) *(G, H, K) lies in* Table 4.3.

Conversely, each triple (G, H, K) in parts (a)–(c) *gives a factorization $G = HK$.*

Table 4.3.

row	G	H	K
1	M_{11}	11, $11{:}5$	M_{10}
2	M_{11}	M_9, $\mathrm{M}_9.2$, $\mathrm{AGL}_1(9)$	$\mathrm{PSL}_2(11)$
3	M_{12}	$\mathrm{M}_9.2$, $\mathrm{M}_9.\mathrm{S}_3$	$\mathrm{PSL}_2(11)$
4	$\mathrm{M}_{12}.2$	$\mathrm{M}_9.2$, $\mathrm{M}_9.\mathrm{S}_3$	$\mathrm{PGL}_2(11)$
5	$\mathrm{M}_{22}.2$	$11{:}2$, $11{:}10$	$\mathrm{PSL}_3(4){:}2$
6	M_{23}	23	M_{22}
7	M_{23}	$23{:}11$	M_{22}, $\mathrm{PSL}_3(4){:}2$, $2^4{:}\mathrm{A}_7$
8	$\mathrm{J}_2.2$	$5^2{:}4$, $5^2{:}(4\times 2)$, $5^2{:}12$, $5^2{:}\mathrm{Q}_{12}$, $5^2{:}(4\times\mathrm{S}_3)$	$\mathrm{G}_2(2)$
9	HS	$5_+^{1+2}{:}8{:}2$	M_{22}
10	HS.2	$5_+^{1+2}{:}(4\wr\mathrm{S}_2)$	M_{22}, $\mathrm{M}_{22}.2$
11	HS.2	$5^2{:}4$, $5^2{:}(4\times 2)$, $5^2{:}4^2$, $5_+^{1+2}{:}4$, $5_+^{1+2}{:}(4\times 2)$, $5_+^{1+2}{:}4^2$, $5_+^{1+2}{:}8{:}2$	$\mathrm{M}_{22}.2$
12	He.2	$7_+^{1+2}{:}6$, $7_+^{1+2}{:}(6\times 2)$, $7_+^{1+2}{:}(6\times 3)$, $7_+^{1+2}{:}(\mathrm{S}_3\times 3)$, $7_+^{1+2}{:}(\mathrm{S}_3\times 6)$	$\mathrm{Sp}_4(4).4$
13	Suz.2	$3^5{:}12$, $3^5{:}((11{:}5)\times 2)$	$\mathrm{G}_2(4).2$

Lemma (5.5)

If T is a classical simple group of Lie type, then $s \leq 4$, and one of the following is true.

1. $T = \mathrm{PSL}_5(2)$, *and* $\Gamma \cong \mathcal{GI}(5,2,2)$.
2. $T = \mathrm{PSU}_3(5)$, *and* $\Gamma \cong \mathrm{HS}_{50}^{(2)}$.
3. Γ *is of valency* $p^f + 1$, $\mathrm{PSL}_3(p^f).2 \leq G \leq \mathrm{Aut}(\mathrm{PSL}_3(p^f))$ *and* $\mathbb{Z}_p^{2f} : \mathrm{SL}_2(p^f) \lhd G_\alpha \leq P_1$ *or* P_2 *where* P_1 *and* P_2 *are the stabilizers of* T *on a 1-dimension subspace and a 2-dimension subspace, respectively.*

Proof. If G_α solvable, then $T = \mathrm{PSL}_3(3)$, $\Gamma = \mathcal{GI}(3,1,3^3)$, satisfies (3). If G_α is nonsolvable, note that G_α has a 2-transitive representation, hence has a minimal normal subgroup in the following list.[25]

[25] Cameron, "Finite Permutation Groups and Finite Simple Groups".

We list here the simple groups T which can occur as minimal normal subgroups of 2-transitive groups of degree n, according to Theorem 5.3. The number k is the maximum degree of transitivity of a group G with socle T.

T	n	k	Remarks
$A_n, n \geqslant 5$	n	n	Two representations if $n = 6$
$\mathrm{PSL}(d, q), d \geqslant 2$	$(q^d-1)/(q-1)$	3 if $d = 2$	$(d, q) \neq (2, 2), (2, 3)$
		2 if $d > 2$	Two representations if $d > 2$
$\mathrm{PSU}(3, q)$	q^3+1	2	$q > 2$
${}^2B_2(q)$ (Suzuki)	q^2+1	2	$q = 2^{2a+1} > 2$
${}^2G_2(q)$ (Ree)	q^3+1	2	$q = 3^{2a+1} > 3$
$\mathrm{PSp}(2d, 2)$	$2^{2d-1}+2^{d-1}$	2	$d > 2$
$\mathrm{PSp}(2d, 2)$	$2^{2d-1}-2^{d-1}$	2	$d > 2$
$\mathrm{PSL}(2, 11)$	11	2	Two representations
$\mathrm{PSL}(2, 8)$	28	2	
A_7	15	2	Two representations
M_{11} (Mathieu)	11	4	
M_{11} (Mathieu)	12	3	
M_{12} (Mathieu)	12	5	Two representations
M_{22} (Mathieu)	22	3	
M_{23} (Mathieu)	23	4	
M_{24} (Mathieu)	24	5	
HS (Higman–Sims)	176	2	Two representations
Co_3 (Conway)	276	2	

Notice that $1 \neq T_\alpha \lhd G_\alpha$ and $T_\alpha \not\leq G_\alpha^{[1]}$.
Thus T_α has a composition factor listed above.

Check $(T, H^+ \cap T, T_\alpha)$ in following list.

TABLE 1.1.

row	L	$H \cap L \leqslant$	$K \cap L \trianglerighteq$	remark
1	$\mathrm{PSL}_n(q)$	$\hat{}\mathrm{GL}_1(q^n){:}n = \frac{q^n-1}{(q-1)d}{:}n$	$q^{n-1}{:}\mathrm{SL}_{n-1}(q)$	$d=(n,q-1)$
2	$\mathrm{PSL}_4(q)$	$q^3{:}\frac{q^3-1}{d}.3 < \mathrm{P}_k$	$\mathrm{PSp}_4(q)$	$d=(4,q-1)$, $k\in\{1,3\}$
3	$\mathrm{PSp}_{2m}(q)$	$q^{m(m+1)/2}{:}(q^m-1).m < \mathrm{P}_m$	$\Omega^-_{2m}(q)$	$m\geqslant 2$, q even
4	$\mathrm{PSp}_4(q)$	$q^3{:}(q^2-1).2 < \mathrm{P}_1$	$\mathrm{Sp}_2(q^2)$	q even
5	$\mathrm{PSp}_4(q)$	$q^{1+2}{:}\frac{q^2-1}{2}.2 < \mathrm{P}_1$	$\mathrm{PSp}_2(q^2)$	q odd
6	$\mathrm{PSU}_{2m}(q)$	$q^{m^2}{:}\frac{q^{2m}-1}{(q+1)d}.m < \mathrm{P}_m$	$\mathrm{SU}_{2m-1}(q)$	$m\geqslant 2$, $d=(2m,q+1)$
7	$\Omega_{2m+1}(q)$	$(q^{m(m-1)/2}.q^m){:}\frac{q^m-1}{2}.m < \mathrm{P}_m$	$\Omega^-_{2m}(q)$	$m\geqslant 3$, q odd
8	$\mathrm{P}\Omega^+_{2m}(q)$	$q^{m(m-1)/2}{:}\frac{q^m-1}{d}.m < \mathrm{P}_k$	$\Omega_{2m-1}(q)$	$m\geqslant 5$, $d=(4,q^m-1)$, $k\in\{m,m-1\}$
9	$\mathrm{P}\Omega^+_8(q)$	$q^6{:}\frac{q^4-1}{d}.4 < \mathrm{P}_k$	$\Omega_7(q)$	$d=(4,q^4-1)$, $k\in\{1,3,4\}$

TABLE 1.2.

row	L	$H \cap L \leqslant$	$K \cap L$
1	$\mathrm{PSL}_2(11)$	$11{:}5$	A_5
2	$\mathrm{PSL}_2(16)$	D_{34}	A_5
3	$\mathrm{PSL}_2(19)$	$19{:}9$	A_5
4	$\mathrm{PSL}_2(29)$	$29{:}14$	A_5
5	$\mathrm{PSL}_2(59)$	$59{:}29$	A_5
6	$\mathrm{PSL}_4(3)$	$2^4{:}5{:}4$	$\mathrm{PSL}_3(3)$, $3^3{:}\mathrm{PSL}_3(3)$
7	$\mathrm{PSL}_4(3)$	$3^3{:}13{:}3$	$(4\times\mathrm{PSL}_2(9)){:}2$
8	$\mathrm{PSL}_4(4)$	$2^6{:}63{:}3$	$(5\times\mathrm{PSL}_2(16)){:}2$
9	$\mathrm{PSL}_5(2)$	$31{:}5$	$2^6{:}(\mathrm{S}_3\times\mathrm{PSL}_3(2))$
10	$\mathrm{PSp}_4(3)$	$3^3{:}\mathrm{S}_4$	$2^4{:}\mathrm{A}_5$
11	$\mathrm{PSp}_4(3)$	$3^{1+2}_+{:}2.\mathrm{A}_4$	A_5, $2^4{:}\mathrm{A}_5$, S_5, A_6, S_6
12	$\mathrm{PSp}_4(5)$	$5^{1+2}_+{:}4.\mathrm{A}_4$	$\mathrm{PSL}_2(5^2)$, $\mathrm{PSL}_2(5^2){:}2$
13	$\mathrm{PSp}_4(7)$	$7^{1+2}_+{:}6.\mathrm{S}_4$	$\mathrm{PSL}_2(7^2)$, $\mathrm{PSL}_2(7^2){:}2$
14	$\mathrm{PSp}_4(11)$	$11^{1+2}_+{:}10.\mathrm{A}_4$	$\mathrm{PSL}_2(11^2)$, $\mathrm{PSL}_2(11^2){:}2$
15	$\mathrm{PSp}_4(23)$	$23^{1+2}_+{:}22.\mathrm{S}_4$	$\mathrm{PSL}_2(23^2)$, $\mathrm{PSL}_2(23^2){:}2$
16	$\mathrm{Sp}_6(2)$	$3^{1+2}_+{:}2.\mathrm{S}_4$	A_8, S_8
17	$\mathrm{PSp}_6(3)$	$3^{1+4}_+{:}2^{1+4}.\mathrm{D}_{10}$	$\mathrm{PSL}_2(27){:}3$
18	$\mathrm{PSU}_3(3)$	$3^{1+2}_+{:}8$	$\mathrm{PSL}_2(7)$
19	$\mathrm{PSU}_3(5)$	$5^{1+2}_+{:}8$	A_7
20	$\mathrm{PSU}_4(3)$	$3^4{:}\mathrm{D}_{10}$, $3^4{:}\mathrm{S}_4$, $3^4{:}3^2{:}4$, $3^{1+4}_+.2.\mathrm{S}_4$	$\mathrm{PSL}_3(4)$
21	$\mathrm{PSU}_4(8)$	$513{:}3$	$2^{12}{:}\mathrm{SL}_2(64).7$
22	$\Omega_7(3)$	$3^5{:}2^4.\mathrm{AGL}_1(5)$	$\mathrm{G}_2(3)$
23	$\Omega_7(3)$	$3^{3+3}{:}13{:}3$	$\mathrm{Sp}_6(2)$
24	$\Omega_9(3)$	$3^{6+4}{:}2^{1+4}.\mathrm{AGL}_1(5)$	$\Omega^-_8(3)$, $\Omega^-_8(3).2$
25	$\Omega^+_8(2)$	$2^2{:}15.4 < \mathrm{A}_9$	$\mathrm{Sp}_6(2)$
26	$\Omega^+_8(2)$	$2^6{:}15.4$	A_9
27	$\mathrm{P}\Omega^+_8(3)$	$3^6{:}2^4.\mathrm{AGL}_1(5)$	$\Omega_7(3)$
28	$\mathrm{P}\Omega^+_8(3)$	$3^6{:}(3^3{:}13{:}3)$, $3^{3+6}{:}13.3$	$\Omega^+_8(2)$

Note the isomorphism $\mathrm{PSp}_4(3)\cong\mathrm{PSU}_4(2)$ for rows 10 and 11.

One of the following occurs:

1. $T = \mathrm{PSL}_n(p^f)$ and $\mathbb{Z}_p^{f(n-1)} : \mathrm{SL}_{n-1}(p^f) \lhd T_\alpha \leq P_1$ or P_{n-1}, where $n \geq 3$, $(n, p^f) \neq (3,2)$ $(3, 3)$ or $(4, 2)$
2. $T = \mathrm{PSp}_4(p^f)$ and $\mathrm{PSL}_2(p^{2f}) \lhd T_\alpha \lhd \mathrm{PSL}_2(p^{2f}).\mathbb{Z}_2$
3. $T = \mathrm{PSU}_4(p^f)$, and $\mathrm{PSU}_3(p^f) \lhd T_\alpha$
4. $(T, T \cap H^+, T_\alpha)$ lies in row 6-21, 23, 25-26 in Table 1.2

Case (a):

If $G_\alpha^{\Gamma(\alpha)}$ is affine, then ref.[26, Table7.3] implies contradiction.
Thus $G_\alpha^{\Gamma(\alpha)}$ is AS with socle $\mathrm{PSL}_{n-1}(p^f)$, and $\mathrm{val}(\Gamma) = \frac{p^{f(n-1)}-1}{p^f-1}$.

$$\mathrm{val}(\Gamma) - 1 \text{ divides } |G_\alpha^{[1]}| \text{ divides } \frac{|P_1||\mathrm{Out}(T)|}{2|\mathrm{PSL}_{n-1}(p^f)|} = fp^{f(n-1)}(p^f - 1).$$

Thus $p^{f(n-2)} - 1$ divides $fp^{f(n-2)}(p^f - 1)^2$.

Now $n \geq 4$ implies $p^{f(n-2)} - 1$ has a ppd r, while r divides the left but not right side of above formula. A contradicition.

Thus $n = 3$, $\mathrm{val}(\Gamma) = p^f + 1$ and $\mathbb{Z}_p^{2f} : \mathrm{SL}_2(p^f) \lhd G_\alpha$.
So $s \leq 4$ by Theorem 2.2.

[26] Peter J. Cameron. *Permutation Groups*. London Mathematical Society Student Texts. Cambridge University Press, 1999.

Case (b): $G_\alpha^{\Gamma(\alpha)}$ is almost simple with socle $\mathrm{PSL}_2(p^{2f})$, $\mathrm{val}(\Gamma) = p^{2f} + 1$,

$|G_\alpha^{[1]}|$ divides $\dfrac{|G_\alpha|}{G_\alpha^{\Gamma(\alpha)}} = \dfrac{|T_\alpha||\mathcal{O}'|}{G_\alpha^{\Gamma(\alpha)}}$ divides $2|\mathcal{O}'|$ divides $|\mathrm{Out}(\mathrm{PSp}_4(p^f))| = 2f$

However, by lemma 2.3, $p^{2f} = \mathrm{val}(\Gamma) - 1$ divides $|G_\alpha^{[1]}|$. A contradiction.

Case (c): Similar as above,

$$p^{3f} = \mathrm{val}(\Gamma) - 1 \text{ divides } |\mathrm{Out}(T)|/2 = f(4, p^f + 1).$$

Case (d):

row 6-8: MAGMA shows no such graph

row 9: $T = \mathrm{PSL}_5(2)$, $T_\alpha = 2^6 : (S_3 \times \mathrm{PSL}_3(2))$. Example(2): $\mathcal{GI}(5,2,2)$.

row 19: $T = \mathrm{PSU}_3(5)$, $T_\alpha = A_7$. Example(3): $\mathrm{HS}_{50}^{(2)}$.

row 20: $T = \mathrm{PSU}_4(3)$, $T_\alpha = \mathrm{PSL}_3(4)$, $\mathrm{val}(\Gamma) = 21$.
But $|\mathrm{Out}(T)| = 8$ not divisible by $\mathrm{val}(\Gamma) - 1$.

row 21: $T = \mathrm{PSU}_4(8)$, $T_\alpha = \mathbb{Z}_2^{12} : \mathrm{SL}_2(64).\mathbb{Z}_7$, $\mathrm{Out}(T) = \mathbb{Z}_6$.
By ref.[27, Table7.3], $G_\alpha^{\Gamma(\alpha)}$ is AS with socle $\mathrm{PSL}_2(64)$ and $\mathrm{val}(\Gamma) = 65$.
MAGMA shows no such graph.

else: T_α is AS with $\mathrm{soc}(T_\alpha) \neq A_7$, $|\mathrm{Out}(T)| < 8$. $|G_\alpha^{[1]}|$ divides $\frac{|\mathrm{Out}(T)|}{2}$.
But nonsolvable G_α implies $\mathrm{val}(\Gamma) \geq 5$. A contradiction.

[27] Cameron, *Permutation Groups*.

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