Research Ideas on Symmetrical Graph Theory

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1 construction of 2-arc-transitive non-Cayley normal cover of $K_{2^n,2^n}$

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2 characterize locally 2-arc-transitive graphs of order p^n

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3 $\mathrm{Cay}(G,S)$ with $\mathrm{Aut}(G,S)$ 2-transitive on S

Theorem 3.1 (Godsil,1981). $N_{\mathrm{Aut}\Gamma}(\hat{G}) = \hat{G} \rtimes \mathrm{Aut}(G,S)$

Conjecture 3.1 (Xu, 1998). *Almost all Cayley graphs are normal.*

Lemma 3.1.1. *If* $\mathrm{Aut}(G,S)$ *is* s*-transitive on* S*, then* Γ *is* $(N_{\mathrm{Aut}\Gamma}(\hat{G}), s)$*-arc-transitive.*

Theorem 3.2 (Praeger,1999). *cubic normalizer-arc-transitive* $\mathrm{Cay}(G,S)$ *on nonab. simple* $G = T$ *satisfies* $\mathrm{Aut}\Gamma = \hat{G} \rtimes \mathrm{Aut}(G,S)$

Theorem 3.3 (Li,1998). *normalizer-2-arc-transitive* $\mathrm{Cay}(G,S)$ *on nonab. char. simple* $G = T^l$ *satisfies* $\mathrm{Aut}\Gamma = \hat{G} \rtimes \mathrm{Aut}(G,S)$

Conjecture 3.2. $\Gamma = \mathrm{Cay}(G,S)$ *is connected. If* $\mathrm{Aut}(G,S)$ *is 2-transitive on* S*, then either* $\mathrm{Aut}\Gamma = \hat{G} \rtimes \mathrm{Aut}(G,S)$*, or* Γ *is a normal cover of* $\mathbf{K}_{p^e,p^e}$

For solvable G, Zhou find exceptional normal cover of $\mathbf{K}_{2^f}$ with $f \geq 3$.

Problem 3.1. *nonsolvable* G

Problem 3.2. *2-transitive* $\rightarrow$ *(quasi)primitive*

4 bi-quasiprimitive 2-arc-transitive graph

see Praeger, Ivanov 1993: affine (bi-)primitive 2-arc-transitive graphs

Problem 4.1. *Generalize to (bi-)quasiprimitive 2-arc-trans. graphs*

5 locally primitive (s-arc-transitive) Cayley graphs

[cyclic, dihedral, solvable, alternating, simple]
more complex G? bi-Cayley?

5.1 arc-transitive (bi-)Cayley graphs

arc-transitive Cayley graphs on T, odd prime val with solvable stabilizers
Extended to bi-Cayley with same condition, while focus on bipartite
Extended to bi-Cayley of bi-quasiprimitive instead of solvable stabilizer

Problem 5.1 (*). *Study arc-transitive bi-quasiprimitive bi-Cayley graphs on T of val=pq or p^e*

release Lemma 2.4 (G_v solvable $\rightarrow G_{vu}^{[1]} = 1$)
such that Lemma 3.3 can be extended using Li-Wang-Xia

5.2 s-arc-transitive solvable Cayley graphs

$s \leq 2$ sharply for non-bipartite $s \leq 4$ sharply for bipartite

Problem 5.2 (*). *Study locally (quasi)primitive solvable Cayley graphs.*

locally primitive graphs of order p^e are studied by Li,Pan,Ma 2009
locally primitive metacyclic normal Cayley graphs are studied by Pan 09
locally primitive abelian Cayley graphs are studied by Li,Lou,Pan 2011
locally primitive dihedral Cayley graphs are studied by Pan, 2014

5.3 s-arc-transitive Cayley graphs on non-solvable groups

Locally primitive (s-arc-transitive) Cayley graphs of finite simple groups is classified by Li,Wang,Xia

method:
$\bar{X} := X/M$ quasiprimitive, reduct to AS
$\bar{X} = \bar{G}\bar{X}_\alpha$, use factorization of AS groups
If $\bar{G}$ or $\bar{X}_\alpha$ is solvable, then use Li-Xia.
Else both non-solvable, use Li-Wang-Xia.

Problem 5.3. *(*) Locally primitive (s-arc-transitive) Cayley graphs on T^l*

6 Weiss's Conjecture

Conjecture 6.1. *Suppose* $\Gamma = (V, E)$ *is a finite undirected graph,* $G \leq \mathrm{Aut}\Gamma$ *such that* G *is vertex-transitive and locally primitive. Then* $|G_x|$ *is bounded by a number depending only on* $|\Gamma(x)|$.

Now, in addition, Γ is (G,s)-transitive. Then the number of locally s-arcs divides $|G_x|$. But this bound on s is dependent on $|\Gamma(x)|$.

Problem 6.1 (open?)**.** *bound* $s \leq 3$ *for* $|\Gamma(x)| \geq 3$ *and* $G_{x,y}^{[1]} = 1$.

structure of $G_x/O_p(G_x^{[1]})$ described for arc-trans loc. qp, especially G_x for arc-trans. loc. 2-trans.

Problem 6.2. *Suppose* $\Gamma = (V, E)$ *is a finite undirected graph,* G *is transitive on* V *and s-arcs respectively with* $G_{x,y}^{[1]} = 1$.
Determine the upper bound on s.

s-arc-transitive graphs with $s \geq 2$ are locally-primitive.

7 arc-transitive locally quasiprimitive graph

Graphs mentioned here are locally finite G-graphs, i.e., vertex stabilizers G_α are finite.

7.1 locally quasiprimitive G-graphs

Problem 7.1. *Determine the structure of* G_x.

$\because$ locally quasiprimitivity $\implies$ edge-transitivity,
$\therefore G \curvearrowright V\Delta$ has at most 2 orbits (adjacent vertices in distinct orbits).

Problem 7.2. *Bound* $|G_\alpha|$ *and determine amalgams* $(G_\alpha, G_\beta; G_{\alpha,\beta})$

- Tutte: bounded $|G_x|$ when G is vertex-transitive on finite trivalent graphs
- Goldschmidt: bounded $|G_x|$ for locally finite trivalent G-graphs, determined amalgams when locally primitive
- locally finite and locally s-arc transitive graphs with $s \geq 6$ in which each vertex has val≥ 3 the amalgams are weak BN-pairs, known.
- for $s \geq 4$ a partial classification of the amalgams was recently obtained

general results on the structure of G_x: Thompson-Wielandt Theorem.

G loc. fin. + loc. q-prim. + V-trans. $\overset{T-W}{\implies} G_{x,y}^{[1]}$ is a p-group

7.2 arc-trans. loc. q-prim. G-graphs

Now assume Δ loc. fin. + loc. q-prim. + arc-trans.

In [19] Weiss conjectured that the order of $O_p(G_x^{[1]})$ is bounded by a function of $|\Delta(x)|$ when G_x^{Δ} is also a primitive permutation group on Δ.

Problem 7.3. *Obtain the structure of $O_p(G_x^{[1]})$ or bound its order, assuming something on $G_x^{\Delta(x)}$ and its action.*

Amalgams of arc-trans. loc. 2-trans. graphs with $G_{x,y}^{[1]} \neq 1$ studied.

Problem 7.4. *Determining amalgams of arc-transitive G-graphs of a given valency. (recently)*

raises

Problem 7.5. *Study $G_x^{[1]}$ when $G_{x,y}^{[1]} = 1$, and about $G_x^{[1]}/O_p(G_x^{[1]})$ in general.*

This paper gives

Theorem 7.1. *...*

Theorem 7.2. *describes the structure of $G_x/O_p(G_x^{[1]})$*

applied to obtain

Theorem 7.3. *the structure of G_x of a class of arc-trans. loc. fin. and loc. 2-trans. G-graphs with trivial edge kernel.*

8 Homogenous graphs

works on this area usually published on top journals such as JCTB

8.1 construction of TW type 3-connected homogeneous graph

3-ch

9 Generalized quadrangles