Notes on GTM251

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1 The Alternating Groups

1.1 Introduction

$\mathrm{Aut}(A_n) \cong S_n$ for $n \geq 7$ but for $n = 6$ there is an exceptional outer automorphism of S_6.hahahssww subgroup structure (O'Nan-Scott Thm) ddsss hahah

1.2 Permutations

This is about PG hhhhh sddd

1.3 The O'Nan-Scott Theorem

1.3.1 Some Lemmas

1.3.2 The proof of the O'Nan-Scott Theorem

Last week we introduced some lemmas and proved part of the O'Nan-Scott Theorem. This week we will finish the proof of the O'Nan-Scott Theorem.

Notation: Let H be a subgroup of S_n not containing A_n, N be a minimal normal subgroup of H, and K be the stabilizer in H of a point.

H intransitive $\implies$ case (i).

H transitive imprimitive $\implies$ case (ii).

Now we assume H primitive. And hence the discussion zoom into $\mathrm{soc}(H)$.

$\exists N$ abelian $\implies$ case (iv)affine.

Additionally we assume $\forall N$ nonabelian.

If H has more than one minimal normal subgroups $N_1 \neq N_2$.

It can be shown that $\exists x \in S_n$ conjugates N_1 to N_2. specify x

By corollary 2.11, x also conjugates $N_2 = C_H(N_1)$ to $N_1 = C_H(N_2)$. (Why?)

Hence $H < \langle H, x \rangle$, which has a unique minimal normal subgroup $N_1 \times N_2$.

Additionally we assume H has a unique minimal normal subgroup N, which is nonabelian.

N simple $\implies C_H(N) = 1 \implies H \overset{\text{conj.}}{\curvearrowright} N$ faithfully $\implies$ case (vi)AS.

$N = T^m = T_1 \times \cdots \times T_m$ with $m > 1 \implies H \overset{\text{conj.}}{\curvearrowright} \{T_1, \cdots, T_m\}$ transitively, and K as well.

Let $K_i := p_i(K \cap N) \leq T_i$ the projection of K onto T_i. Then $K \cap N \leq K_1 \times \cdots \times K_m$.

Case $K_i \neq T_i$ for some i:

Now $K_1 \times \cdots \times K_m < N$ hence $N_H(K_1 \times \cdots \times K_m) < H$.

<u>Claim:</u> K normalizes $K_1 \times \cdots \times K_m$.

Since $K \cap N \lhd K$, $\forall k \in K$, $\forall x \in K \cap N$,

we have $x = p_1(x) \cdots p_m(x)$, and $p_1(x)^k \cdots p_m(x)^k = x^k = p_1(x^k) \cdots p_m(x^k) \in K \cap N$.

Then $p_i(x)^k = p_j(x^k)$ whenever $T_i^k = T_j$. (In direct product, equal iff. all coordinates equal.)

$\forall y \in K_1 \times \cdots \times K_m$, $\exists x_1, \cdots, x_m \in K \cap N$ s.t. $y = p_1(x_1) \cdots p_m(x_m)$.

Then $y^k = p_1(x_1)^k \cdots p_m(x_m)^k = p_1(x_{l_1}^k) \cdots p_m(x_{l_m}^k) \in K_1 \times \cdots \times K_m$, where $T_i = T_{l_i}^k$.

By corollary 2.15, $K_1 \times \cdots \times K_m = K \cap N$ and K permutes K_i's transitively. Let $k := |T_i : K_i|$.

Then $H = (T_1 \times \cdots \times T_m) \rtimes K \leq S_k \wr S_m \curvearrowright [T_1 : K_1] \times \cdots \times [T_m : K_m] \implies$ case (iii)PA.

Case $K_i = T_i$ for all i: