s-Arc-transitive solvable Cayley graphs

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September 12, 2023

Outline

Preliminaries

Let Γ be a finite simple undirected graph. Let $G \leq \mathrm{Aut}\Gamma$.

Definition

- **s-arc:** (s+1)-tuple of vertices $\alpha_0, \alpha_1, \cdots, \alpha_s$ where α_i is adjacent to α_{i+1} and $\alpha_{j-1} \neq \alpha_{j+1}$ for $0 \leq i \leq s-1$ and $1 \leq j \leq s-1$.
- **(G,s)-arc-transitive:** G is transitive on the set of s-arcs of Γ.
- **(G,s)-transitive:** (G,s)-arc-transitive but not (G,s+1)-arc-transitive.

For short, **s-transitive** means $(\mathrm{Aut}\Gamma, s)$-transitive for graphs.

Lemma

- *s-arc-transitive* $(1 \leq s) \Longrightarrow$ *k-arc-transitive* $(1 \leq k \leq s)$.
- *the s-arc-transitivity of a graph is inherited by the normal quotients.*

Preliminaries

Definition

- **Cayley graph:** $\mathrm{Cay}(G,S)$ with vertex set G and edges $yx^{-1} \in S$.
- **solvable Cayley graph:** G is solvable.

Lemma

Γ is Cayley of G $\iff$ $\exists G \lesssim \mathrm{Aut}\Gamma$ which is vertex-regular.

Background

- 1947, Tutte: No 6-arc-transitive trivalent graphs.
- 1981, Weiss: No s-arc-transitive graphs with $\mathrm{val} \geq 3$ for $s = 6$ and $s \geq 8$.
- 2019, Li C.H. & Xia B.Z.:
  Connected **non-bipartite** 3-arc-transitive solvable Cayley graph with $\mathrm{val} \geq 3$ is a normal cover of the Hoffman-Singleton graph or the Peterson graph.
- 2021, Zhou J.X.: No such normal covers, so $s \leq 2$ sharply.

Problem

Studying connected **bipartite** *s-arc-transitive solvable Cayley graphs, and determining the upper bound on s.*

For convenience, one may assume $s \geq 3$ and $\mathrm{val} \geq 3$.

Main Result

Theorem

Every connected s-arc-transitive solvable Cayley graph with $s \geq 3$ and val ≥ 3 is a normal cover of one of the following graphs:

1. *the complete bipartite graph $K_{n,n}$ with $n \geq 3$;*
2. *the geometry incidence graph $\mathcal{GI}(5,2,2)$;*
3. *the standard double cover of the Hoffman-Singleton graph;*
4. *a graph Σ with valency $p^f + 1$ such that $\mathrm{PSL}_3(p^f).2 \leq \mathrm{Aut}\Sigma \leq \mathrm{Aut}(\mathrm{PSL}_3(p^f))$ and $\mathbb{Z}_p^{2f} : \mathrm{SL}_2(p^f) \lhd (\mathrm{Aut}\Sigma)_\alpha$, where $p^f \geq 3$ is a prime power and α is a vertex.*

In particular, the sharp upper bound on s is 4.

These normal covers are being investigated in a sequel. (usually challenge)

Examples

(1) $\mathbf{K}_{n,n}$ with $n \geq 3$

- 3-transitive
- $\mathbf{K}_{n,n} \cong \mathrm{Cay}(G, G \backslash H)$ where G is solvable and $H < G$ of index 2

(2) $\mathcal{GI}(5,2,2)$

- the incidence graph of $(\mathcal{P}, \mathcal{L})$ where $\mathcal{P}$(resp. $\mathcal{L}$) is the set of 2-subspace(resp. 3-subspace) of $\mathbb{F}_2^5$
- $\mathrm{Aut}(\mathcal{GI}(5,2,2)) = \mathrm{GL}_5(2).\langle\sigma\rangle$ is vertex-transitive
- $G_\alpha = 2^6 : (\mathrm{GL}_2(2) \times \mathrm{GL}_3(2))$, $G_{\alpha\beta} = 2^8 : (S_3 \times S_3)$, val$=\frac{|G_\alpha|}{|G_{\alpha\beta}|}$=7
- 3-transitive[1, Theorem 3.4]
- $G = RG_\alpha$ with $R \cong 31 : 5 : 2$ vertex-regular[2, Theorem 1.1]

[1] Cai Heng Li, Zai Ping Lu, and Gaixia Wang. "Arc-transitive graphs of square-free order and small valency". In: *Discrete Mathematics* 339.12 (2016), pp. 2907–2918.

[2] Cai Heng Li and Binzhou Xia. *Factorizations of Almost Simple Groups with a Solvable Factor, and Cayley Graphs of Solvable Groups*. Vol. 279. 1375. Mem. AMS, 2022.

Examples

(3) $\mathrm{HS}_{50}^{(2)}$

- **standard double cover:** $\Gamma^{(2)} = (\tilde{V}, \tilde{E})$ of $\Gamma = (V, E)$, where $\tilde{V} = V \times \{1, 2\}$, $\tilde{E} = \{\{(v, 1), (w, 2)\} \mid \{v, w\} \in E\}$

Examples

(4) $\mathcal{PH}(3, q)$

- the incidence graph of $\mathrm{PG}(2, q) = (\mathcal{P}, \mathcal{L})$, where $\mathcal{P}$(resp. $\mathcal{L}$) is the set of 1-subspace(resp. 2-subspace) of $\mathbb{F}_q^3$
- $\mathrm{Aut}(\mathcal{PH}(3, q)) = \mathrm{P\Gamma L}_3(q).\langle\sigma\rangle = \mathrm{Aut}(\mathrm{PSL}_3(q))$
- $(\alpha_0, \cdots, \alpha_4)$ with $\alpha_0 \in \mathcal{P}$ corresponds to a basis v_1, v_2, v_3 such that

$$\alpha_0 = \langle v_1\rangle, \alpha_1 = \langle v_1, v_2\rangle, \alpha_3 = \langle v_2\rangle, \alpha_4 = \langle v_2, v_3\rangle, \alpha_5 = \langle v_3\rangle$$

  all ordered bases are equiv. under linear transformation
  $\Longrightarrow$ 4-arc-transitive
- a Cayley graph[3] of $D_{2(q^2+q+1)}$

[3] Dragan Marui. "On 2-arc-transitivity of Cayley graphs". In: *Journal of Combinatorial Theory, Series B* 87.1 (2003), pp. 162–196.

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1. Preliminaries & Background
2. Main Result & Examples
3. Proof of the Main result
   - Overview of the proof
   - Reduct to almost simple groups
   - As case

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