Notes on GTM251

Yuandong Li

September 11, 2023

Contents

1 Introduction

1.1 History

1.2 CFSG

Every finite simple group is isomorphic to one of the followings:

(i) a cyclic group C_p of prime order p;

(ii) an alternating group A_n for $n \geq 5$;

(iii) a classical group:

- linear: $\mathrm{PSL}_n(q)$, $n \geq 2$, except $\mathrm{PSL}_2(2)$ and $\mathrm{PSL}_2(3)$;
- unitary: $\mathrm{PSU}_n(q)$, $n \geq 3$, except $\mathrm{PSU}_3(2)$;
- symplectic: $\mathrm{PSp}_{2n}(q)$, $n \geq 2$, except $\mathrm{PSp}_4(2)$;
- orthogonal: $\mathrm{P}\Omega_{2n+1}(q)$, $n \geq 3$, q odd; $\mathrm{P}\Omega^+_{2n}(q)$, $\mathrm{P}\Omega^-_{2n}(q)$, $n \geq 4$;

where q is a power p^a of a prime p;

(iv) an exceptional group of Lie type:

$$G_2(q), q \geq 3;\ F_4(q);\ E_6(q);\ {}^2E_6(q);\ {}^3D_4(q);\ E_7(q);\ E_8(q)$$

with q a prime power, or

$${}^2B_2(2^{2n+1}),\ {}^2G_2(3^{2n+1},\ {}^2F_4(2^{2n+1}),\ n \geq 1;$$

or the Tits group ${}^2F_4(2)'$;

(v) one of 26 sporadic simple groups:

- the five Mathieu groups M_{11}, M_{12}, M_{22}, M_{23}, M_{24};
- the seven Leech lattice groups Co_1, Co_2, Co_3, McL, HS, Suz, J_2;
- the three Fischer groups Fi_{22} , Fi_{23}, Fi'_{24} ;
- the five Monstrous groups $\mathbb{M}$, $\mathbb{B}$, Th, HN, He;
- the six pariahs J_1, J_3, J_4, O'N, Ly, Ru.

Conversely, every group in this list is simple, and the only repetitions in this list are:

$$\begin{aligned} \mathrm{PSL}_2(4) \cong &\mathrm{PSL}_2(5) \cong A_5; \\ &\mathrm{PSL}_2(7) \cong \mathrm{PSL}_3(2); \\ &\mathrm{PSL}_2(9) \cong A_6; \\ &\mathrm{PSL}_4(2) \cong A_8; \\ &\mathrm{PSU}_4(2) \cong \mathrm{PSp}_4(3). \end{aligned}$$

introduce, construction, orders, simplicity, action(reveal subgroup structure)

1.3 App of CFSG

The symmetric difference set of almost simple subgroups and maximal subgroups of A_n is listed out, while listing their intersection is impossible.

2 The Alternating Groups

2.1 Introduction

$\mathrm{Aut}(A_n) \cong S_n$ for $n \geq 7$ but for $n = 6$ there is an exceptional outer automorphism of S_6.

subgroup structure (O'Nan-Scott Thm)