

s-Arc-transitive solvable Cayley graphs

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October 11, 2023

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Let Γ be a finite simple undirected graph. Let $G \leq \text{Aut}\Gamma$.

Definition

- **s-arc:** $(s+1)$ -tuple of vertices $\alpha_0, \alpha_1, \dots, \alpha_s$ where α_i is adjacent to α_{i+1} and $\alpha_{j-1} \neq \alpha_{j+1}$ for $0 \leq i \leq s-1$ and $1 \leq j \leq s-1$.
- **(G,s) -arc-transitive:** G is transitive on the set of s -arcs of Γ .
- **(G,s) -transitive:** (G,s) -arc-transitive but not $(G,s+1)$ -arc-transitive.

For short, **s-transitive** means $(\text{Aut}\Gamma, s)$ -transitive for graphs.

Lemma

- s -arc-transitive ($1 \leq s$) $\implies k$ -arc-transitive ($1 \leq k \leq s$).
- *the s -arc-transitivity of a graph is inherited by the normal quotients.*

Definition

- **Cayley graph:** $\text{Cay}(G, S)$ with vertex set G and edges $yx^{-1} \in S$.
- **solvable Cayley graph:** G is solvable.

Lemma

Γ is Cayley of $G \iff \exists G \lesssim \text{Aut}\Gamma$ which is vertex-regular.

Definition

A transitive permutation group G is said to be **quasiprimitive** if its nontrivial normal subgroups are transitive, and **bi-quasiprimitive** if its nontrivial normal subgroups have at most 2 orbits and there exists one which has exactly 2 orbits.

The O’Nan-Scott-Praeger theorem¹ divides the quasiprimitive permutation groups into 8 types: holomorph affine (HA), holomorph simple (HS), holomorph compound (HC), almost simple (AS), simple diagonal (SD), compound diagonal (CD), product action (PA) and twisted wreath product (TW).

¹Cheryl E Praeger. “Finite Quasiprimitive Graphs”. In: *Surveys in Combinatorics*, 1997. London Mathematical Society Lecture Note Series. Cambridge University Press, 1997, 6586.

Background

- 1947, Tutte: No 6-arc-transitive trivalent graphs.
- 1981, Weiss: No s -arc-transitive graphs with $\text{val} \geq 3$ for $s = 6$ and $s \geq 8$.
- 2019, Li C.H. & Xia B.Z.:
Connected **non-bipartite** 3-arc-transitive solvable Cayley graph with $\text{val} \geq 3$ is a normal cover of the Hoffman-Singleton graph or the Petersen graph.
- 2021, Zhou J.X.: No such normal covers, so $s \leq 2$ sharply.

Problem

*Studying connected **bipartite** s -arc-transitive solvable Cayley graphs, and determining the upper bound on s .*

For convenience, one may assume $s \geq 3$ and $\text{val} \geq 3$.

Main Result

Theorem

Every connected s -arc-transitive solvable Cayley graph with $s \geq 3$ and val ≥ 3 is a normal cover of one of the following graphs:

- ① the complete bipartite graph $K_{n,n}$ with $n \geq 3$;
- ② the geometry incidence graph $\mathcal{GI}(5, 2, 2)$;
- ③ the standard double cover of the Hoffman-Singleton graph;
- ④ a graph Σ with valency $p^f + 1$ such that $\mathrm{PSL}_3(p^f).2 \leq \mathrm{Aut}\Sigma \leq \mathrm{Aut}(\mathrm{PSL}_3(p^f))$ and $\mathbb{Z}_p^{2f} : \mathrm{SL}_2(p^f) \triangleleft (\mathrm{Aut}\Sigma)_\alpha$, where $p^f \geq 3$ is a prime power and α is a vertex.

In particular, the sharp upper bound on s is 4.

These normal covers are being investigated in a sequel. (usually challenge)

Examples

(1) $K_{n,n}$ with $n \geq 3$

- 3-transitive
- $K_{n,n} \cong \text{Cay}(G, G \setminus H)$ where G is solvable and $H < G$ of index 2

(2) $\mathcal{GI}(5, 2, 2)$

- the incidence graph of $(\mathcal{P}, \mathcal{L})$ where \mathcal{P} (resp. \mathcal{L}) is the set of 2-subspace (resp. 3-subspace) of \mathbb{F}_2^5
- $\text{Aut}(\mathcal{GI}(5, 2, 2)) = \text{GL}_5(2). \langle \sigma \rangle$ is vertex-transitive
- $G_\alpha = 2^6 : (\text{GL}_2(2) \times \text{GL}_3(2))$, $G_{\alpha\beta} = 2^8 : (S_3 \times S_3)$, $\text{val} = \frac{|G_\alpha|}{|G_{\alpha\beta}|} = 7$
- 3-transitive², Theorem 3.4
- $G = RG_\alpha$ with $R \cong 31 : 5 : 2$ vertex-regular³, Theorem 1.1

²Cai Heng Li, Zai Ping Lu, and Gaixia Wang. "Arc-transitive graphs of square-free order and small valency". In: *Discrete Mathematics* 339.12 (2016), pp. 2907–2918.

³Cai Heng Li and Binzhou Xia. *Factorizations of Almost Simple Groups with a Solvable Factor, and Cayley Graphs of Solvable Groups*. Vol. 279. 1375. Mem. AMS, 2022.

Examples

(3) $\text{HS}_{50}^{(2)}$

- HS_{50} is 3-transitive

$\text{Aut}(\text{HS}_{50}) = \text{P}\Sigma\text{U}(3, 5)$ has a solvable vertex-transitive subgroup

It is NOT a Cayley graph

- **standard double cover:** $\Gamma^{(2)} = (\tilde{V}, \tilde{E})$ of $\Gamma = (V, E)$, where
 $\tilde{V} = V \times \{1, 2\}$, $\tilde{E} = \{\{(v, 1), (w, 2)\} \mid \{v, w\} \in E\}$
 $\text{Aut}(\Gamma^{(2)}) \geq \text{Aut}\Gamma \times \langle \sigma \rangle$, Γ s-arc-transitive $\implies \Gamma^{(2)}$ s-arc-transitive
- By MAGMA, \exists_1 connected G -arc-transitive 7-valent graph Γ of order 50 with $(G, G_\alpha) = (\text{PSU}_3(5), A_7)$ or $(\text{PSU}_3(5).{\mathbb Z}_2, S_7)$. $\Gamma \cong \text{HS}_{50}$
- By MAGMA, \exists_1 connected G -arc-transitive 7-valent graph Γ of order 100 with $(G, G_\alpha) = (\text{PSU}_3(5).{\mathbb Z}_2, A_7)$. $\Gamma \cong \text{HS}_{50}^{(2)}$

(Q: Could a normal cover of $\text{HS}_{50}^{(2)}$ be solvable Cayley? see Zhou's paper)

Examples

(4) $\mathcal{PH}(3, q)$

- the incidence graph of $\text{PG}(2, q) = (\mathcal{P}, \mathcal{L})$, where \mathcal{P} (resp. \mathcal{L}) is the set of 1-subspace (resp. 2-subspace) of \mathbb{F}_q^3
- $\text{Aut}(\mathcal{PH}(3, q)) = \text{PGL}_3(q). \langle \sigma \rangle = \text{Aut}(\text{PSL}_3(q))$
- $(\alpha_0, \dots, \alpha_4)$ with $\alpha_0 \in \mathcal{P}$ corresponds to a basis v_1, v_2, v_3 such that

$$\alpha_0 = \langle v_1 \rangle, \alpha_1 = \langle v_1, v_2 \rangle, \alpha_2 = \langle v_2 \rangle, \alpha_3 = \langle v_2, v_3 \rangle, \alpha_4 = \langle v_3 \rangle$$

all ordered bases are equiv. under linear transformation

\implies 4-arc-transitive

- a Cayley graph⁴ of $D_{2(q^2+q+1)}$, for example Heawood graph.

⁴Dragan Marušič. "On 2-arc-transitivity of Cayley graphs". In: *Journal of Combinatorial Theory, Series B* 87.1 (2003), pp. 162–196.

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Overview of the proof

Let $\Gamma = \text{Cay}(H, S)$ be connected, (G, s) -transitive, $\text{val} \geq 3$, with H solvable, $H \leq G \leq \text{Aut}\Gamma$, $s \geq 3$.

Let $M \triangleleft G$ (may trivial) be maximal with ≥ 3 orbits on $V\Gamma$.

Then

- Γ is a normal cover of Γ_M
- Γ_M is $(G/M, s)$ -arc-transitive
- G/M is either quasiprimitive or bi-quasiprimitive on $V\Gamma_M$, $|V\Gamma_M| \geq 3$

Theorem (Praeger, 1993)

Let Γ be a finite connected graph, $G \leq \text{Aut}\Gamma$ is s -arc transitive on Γ for some $s \geq 2$. Suppose $N \triangleleft G$ has more than two orbits on vertices. Then

- ① Γ_N is finite and connected and the group of automorphisms of Γ_N induced by G is s -arc transitive on Γ_N .
- ② N is semiregular on vertices (that is $N_\alpha = 1$ for each vertex α of Γ) and Γ is a cover of Γ_N .

Overview of the proof

Case quasiprimitive: Γ_M non-bipartite. By Li-Xia-Zhou, no such graphs.

Case bi-quasiprimitive: Γ_M bipartite.

Then either $\Gamma_M = K_{n,n}$, or G/M is almost simple. (Reduce to AS type)

Moreover, $\text{soc}(G/M)$ is a classical simple group of Lie type.

(Use factorization of AS groups with a solvable factor)

Let s_0 be the sharp upper bound on s .

- By Example (4), $\mathcal{PH}(3, q)$ is 4-transitive, so $s_0 \geq 4$.
- If there is a connected 5-arc-transitive solvable Cayley graph Γ , then Γ is a normal cover of a 5-arc-transitive graph of type (1)-(4), while analysis above states no such graph.

Hence $s_0 = 4$.

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Reduce to almost simple groups

In the remains of this talk, we assume the followings.

- Let Γ be a connected bipartite (G, s) -transitive graphs of $\text{val} \geq 3$, with bipartitions Δ_1 and Δ_2 and $s \geq 3$.
- Suppose G is bi-quasiprimitive on $V\Gamma$ and contains a vertex-transitive solvable subgroup H .
- Set $G^+ = G_{\Delta_1}$, the setwise stabilizer of G on Δ_1 , and $H^+ = H \cap G^+$.
- Let α, β be adjacent vertices.

Some immediate facts:

- $G^+ = G_{\Delta_1} = G_{\Delta_2}$ has index 2 (hence normal) in G and is transitive on both Δ_1 and Δ_2 , as well as H^+ in H .
- $G = \langle G^+, g \rangle$ where $g \in G \setminus G^+$, g interchanges Δ_1 and Δ_2 , $g^2 \in G^+$

Case: unfaithful action

Now we consider the action of G^+ on Δ_1 and Δ_2 .

Lemma (4.2)

If G^+ acts unfaithfully on Δ_1 or Δ_2 , then $\Gamma = \mathbf{K}_{n,n}$, where $n = |V\Gamma|/2$.

Proof. Let $K = \ker(G^+ \curvearrowright \Delta_1)$.

Then $K^g = \ker(G^+ \curvearrowright \Delta_2)$ and $1 \neq KK^g = K \times K^g \triangleleft G$.

$\because G$ is bi-quasiprimitive, $K \times K^g$ has at most 2 orbits.

Since K^g fixes Δ_2 pointwise, K must be transitive on Δ_2 .

It follows that $\Gamma = \mathbf{K}_{n,n}$. □

We thus additionally assume G^+ is faithful on both Δ_1 and Δ_2 .

$$G^+ \cong (G^+)^{\Delta_1} \cong (G^+)^{\Delta_2}$$

We will prove the above actions are quasiprimitive of almost simple type.

Quasiprimitive \implies same type AS or PA

Lemma (4.3)

Suppose G^+ is quasiprimitive on Δ_1 or Δ_2 . Then

- ① G^+ is quasiprimitive on both Δ_1 and Δ_2
- ② both $(G^+)^{\Delta_1}$ and $(G^+)^{\Delta_2}$ have the same type AS or PA

Proof. WLOG, assume G^+ is quasiprimitive on Δ_1 .

G transitive on $V\Gamma$ and $G^+ \triangleleft G \implies G^+ \curvearrowright \Delta_1, \Delta_2$ are perm. isom.

Hence G^+ is quasiprimitive on Δ_2 of the same type on Δ_1 .

By ref.^{5, Thm 1.2}, $(G^+)^{\Delta_1}$ is of type HA, AS, TW or PA.

For cases HA or TW, $\text{soc}(G^+) \triangleleft G$ is regular on Δ_1 and Δ_2 ,
then Γ is a **G-normal bi-Cayley graph** on $\text{soc}(G^+)$.

By ref.^{6, Lem 3.2}, Γ is at most $(G, 2)$ -arc-transitive, contradicting $s \geq 3$. □

⁵ Michael Giudici, Cai Li, and Cheryl Praeger. "Analysing finite locally s -arc transitive graphs". In: *Trans. AMS* 356 (Aug. 2003), pp. 291–317.

⁶ Marston Conder et al. "Edge-transitive bi-Cayley graphs". In: *Journal of Combinatorial Theory, Series B* 145 (2020), pp. 264–306.

Study $\text{soc}(G^+)$

Lemma (4.4)

$\text{soc}(G^+)$ is nonsolvable and is the unique minimal normal subgroup of G , and $\text{soc}(G^+)_{\alpha} \neq 1$.

Proof.

If $G^+ \curvearrowright \Delta_1$ is quasiprimitive.

By lemma 4.3, G^+ is of type AS or PA.

So $\text{soc}(G^+) \triangleleft_{\min} G^+$ and is nonsolvable and not semiregular.

Let $C = C_G(\text{soc}(G^+))$. Then $C \triangleleft G$ and $\text{soc}(G^+) \cap C = Z(\text{soc}(G^+)) = 1$.

If $C \neq 1$, as $C \cap G^+ = 1$ and $G = G^+.2$, we derive that $C \cong \mathbb{Z}_2$.

Since $|V\Gamma| > 4$, C has more than 2 orbits, contradicting G bi-quasiprimitive.

Hence $C = 1$ and $\text{soc}(G^+)$ is also unique in G .

Now assume G^+ is not quasiprimitive on both Δ_1 and Δ_2 .

Then $\exists M \triangleleft_{\min} G^+$ acting intransitively on Δ_1 hence has > 2 orbits on $V\Gamma$.

Thus $M \not\triangleleftharpoonup G$ and $M^g \neq M$. (Recall $G = \langle G^+, g \rangle$)

Since $g^2 \in G^+$, one has $MM^g = M \times M^g \leq G^+$ is normal in G .

By bi-quasiprimitivity of G , $M \times M^g$ is transitive on Δ_1 and Δ_2 .

By ref.^{7, Thm 1.5}, either

- (a) $M \times M^g$ is regular on Δ_1 and Δ_2 ; or
- (b) $\text{soc}(G^+) = M \times M^g$ is nonsolvable and not semiregular, and is the unique minimal normal subgroup of G .

For case (b), done.

For case (a), Γ is a G -normal bi-Cayley graph on $M \times M^g$, again by ref.^{8, Lem 3.2}, is at most $(G, 2)$ -arc-transitive. A contradiction. □

⁷Cai Heng Li et al. "Finite locally-quasiprimitive graphs". In: *Discrete Mathematics* 246.1 (2002), pp. 197–218.

⁸Conder et al., "Edge-transitive bi-Cayley graphs".

Structure of $\text{soc}(G^+)$

According to Lemma 4.4, we may set that

$$N := \text{soc}(G^+) = T_1 \times T_2 \times \cdots \times T_d \cong T^d \triangleleft_{\min}^{\text{unique}} G.$$

Then N is transitive on Δ_1 and Δ_2 since G is bi-quasiprimitive.

Consider $G^+ \curvearrowright \{T_1, \dots, T_d\}$ by conjugation with kernel K .

Then $N \triangleleft K \leq \text{Aut}(T_1) \times \cdots \times \text{Aut}(T_d)$. And $T_j \triangleleft K$, $1 \leq j \leq d$.

Let $M_j = C_K(T_j)$. Then

$$\cdots \times T_{j-1} \times 1 \times T_{j+1} \times \cdots \leq M_j \leq \cdots \times \text{Aut}(T_{j-1}) \times 1 \times \text{Aut}(T_{j+1}) \times \cdots$$

$M_j \triangleleft K$ and $K/M_j \lesssim \text{Aut}(T_j)$ is almost simple with socle T_j .

Recall H is a vertex-transitive solvable subgroup of G .

Let $H^+ = H \cap G^+$ and H_j be the projection of $H^+ \cap N$ on T_j , $1 \leq j \leq d$.

Then $|H^+ \cap N|$ divides $|H_1||H_2| \cdots |H_d|$.

Exclude PA type

Lemma (4.5)

Suppose $d \geq 2$. Then $\exists j \in \{1, 2, \dots, d\}$ s.t. M_j is transitive on Δ_1 or Δ_2 ; in particular, if G^+ is quasiprimitive on Δ_1 or Δ_2 , then it is not of type PA.

Proof.

Assume G^+ is quasiprimitive of type PA on Δ_1 or Δ_2 .

Then $\exists \mathcal{B} = \Omega^d$ a maximal block system s.t. $G^+ \curvearrowright \mathcal{B}$ primitive by PA.

For $(w_1, w_2, \dots, w_d) \in \Omega^d$, $(x_1, x_2, \dots, x_d) \in M_j$, we have

$$(w_1, w_2, \dots, w_d)^{(x_1, x_2, \dots, x_d)} = (w_1^{x_1}, w_2^{x_2}, \dots, w_d^{x_d}).$$

From previous analysis, $x_j = 1$. Then transitivity of M_j implies $|\Omega| = 1$.

Need to prove the transitivity of some M_j .

Before that we prepare some lemmas.

Theorem (2.2, Thompson-Wielandt)

Let Γ be a connected (G, s) -transitive graph with $s \geq 2$, and let α and β be adjacent vertices of Γ . Then one of the following holds, where p is a prime:

- ① $s \leq 3$, $G_{\alpha\beta}^{[1]} = 1$, and $G_\alpha^{[1]} \cong (G_\alpha^{[1]})^{\Gamma(\beta)} \triangleleft G_{\alpha\beta}^{\Gamma(\beta)} \cong G_{\alpha\beta}^{\Gamma(\alpha)}$;
- ② $s \leq 3$, $G_{\alpha\beta}^{[1]}$ is a nontrivial p -group, $\mathrm{PSL}_d(p^f) \triangleleft G_\alpha^{\Gamma(\alpha)}$ for some $d \geq 2$, and $\mathrm{val}(\Gamma) = (p^{df} - 1)/(p^f - 1)$;
- ③ $s \geq 4$, $\mathrm{PGL}_2(p^f) \triangleleft G_\alpha^{\Gamma(\alpha)}$, $\mathrm{val}(\Gamma) = p^f + 1$, and one of the following occurs:
 - i $s=4$, and $G_\alpha = p^{2f} : \frac{p^f-1}{(3,p^f-1)} \cdot \mathrm{PGL}_2(p^f) \cdot \mathcal{O}$ with $|\mathcal{O}| \mid (3, p^f - 1)f$;
 - ii $s=5$, and $G_\alpha = p^{3f} : \mathrm{GL}_2(2^f) \cdot \mathbb{Z}_e$ with $e|f$;
 - iii $s=7$, and $G_\alpha = (p^{2f} \times (p^f)^{1+2}) : \mathrm{GL}_2(3^f) \cdot \mathbb{Z}_e$ with $e|f$.

Lemma (2.3)

For a connected $(G, 3)$ -arc-transitive graph Γ and adjacent vertices α, β , at least one of the following holds:^a, Thm 4.2

- ① $G_\alpha^{[1]}$ is transitive on $\Gamma(b) \setminus \{\alpha\}$.
- ② $G_\alpha = A_7$ or S_7 .
- ③ $\mathbb{Z}_p^f \triangleleft G_\alpha^{\Gamma(\alpha)} \leq \text{AGL}_1(p^f)$ with p a prime.

^aCai Heng Li, kos Seress, and Shu Jiao Song. "s-Arc-transitive graphs and normal subgroups". In: *Journal of Algebra* 421 (2015), pp. 331–348.

Lemma (2.5)

Let Γ be a connected (G, s) -transitive graph with $s \geq 3$, and suppose $N \triangleleft G$ and not semiregular. Then one of the following holds:

- ① both $G_\alpha^{\Gamma(\alpha)}$ and $N_\alpha^{\Gamma(\alpha)}$ are almost simple 2-transitive;
- ② $G_\alpha^{\Gamma(\alpha)}$ is affine and $N_\alpha^{\Gamma(\alpha)}$ is primitive.

Proof of Lemma 2.5:

- $(G, 3)$ -arc-transitive $\implies G_\alpha^{\Gamma(\alpha)}$ is 2-transitive (AS or affine)
- $N \triangleleft G$, $N_\alpha \neq 1$ and connectivity $\implies 1 \neq N_\alpha^{\Gamma(\alpha)} \triangleleft G_\alpha^{\Gamma(\alpha)}$ transitive

Claim:

$G_{\alpha\beta}^{[1]} = 1$ and $G_\alpha^{[1]} \curvearrowright \Gamma(\beta) \setminus \{\alpha\}$ transitive $\implies N_\alpha \curvearrowright \Gamma(\alpha)$ 2-transitive

- check $G_\beta^{[1]} / G_{\alpha\beta}^{[1]} \cong (G_\beta^{[1]})^{\Gamma(\alpha)} \triangleleft N_{\alpha\beta}^{\Gamma(\alpha)}$
- $G_\beta^{[1]}$ (hence $N_{\alpha\beta}$) $\curvearrowright \Gamma(\alpha) \setminus \{\beta\}$ transitive

Better change the conditions to $G_{\alpha\beta}^{[1]} = 1$ and 3-arc-transitive?

$G_\alpha^{\Gamma(\alpha)}$ AS:

By ref.⁹, Thm5.3(S)Notes(2), either $N_\alpha^{\Gamma(\alpha)}$ is 2-transitive, or $N_\alpha^{\Gamma(\alpha)} = \mathrm{PSL}_2(8)$ primitive with $\mathrm{val}(\Gamma) = 28$. The latter one is impossible.

$G_\alpha^{\Gamma(\alpha)}$ affine:

① $s \geq 4$:

① $\mathrm{val}(\Gamma) = 3$, $N_\alpha^{\Gamma(\alpha)} \geq \mathbb{Z}_3$ primitive

② $\mathrm{val}(\Gamma) = 4$ and $G_\alpha^{\Gamma(\alpha)} = \mathrm{PGL}_2(3) \cong S_4$,

① $N_\alpha^{\Gamma(\alpha)} = A_4$ or S_4 is 2-transitive

② $N_\alpha^{\Gamma(\alpha)} \cong \mathbb{Z}_2^2$ is regular, $N_{\alpha\beta} \curvearrowright \Gamma(\alpha) \cup \Gamma(\beta)$ trivial, connect \implies

$N_{\alpha\beta} = 1$, hence $N_\alpha = \mathbb{Z}_2^2$, but by ref.¹⁰, lem^{2.5}, no such normal subgroup

② $s = 3$: $\stackrel{\text{Thm2.2}}{\implies} G_{\alpha\beta}^{[1]} = 1$. Consider $G_\alpha^{[1]} \curvearrowright \Gamma(\beta) \setminus \{\alpha\}$

trans $\stackrel{\text{Claim}}{\implies} N_\alpha^{\Gamma(\alpha)}$ 2-transitive

intrans. $\stackrel{\text{lem2.3}}{\implies} \mathbb{Z}_{p^f} \triangleleft G_\alpha^{\Gamma(\alpha)} \leq \mathrm{AGL}_1(p^f)$, so $G_\beta^{[1]} \cong (G_\beta^{[1]})^{\Gamma(\alpha)} \triangleleft G_{\alpha\beta}^{\Gamma(\alpha)} \leq \mathbb{Z}_{p^f-1} \cdot \mathbb{Z}_f$
but this leads to a contradiction. □

⁹Peter Cameron. "Finite Permutation Groups and Finite Simple Groups". In: *Bulletin of The London Mathematical Society - BULL LOND MATH SOC* 13 (Jan. 1981), pp. 1–22.

¹⁰C.H. Li, Z.P. Lu, and H. Zhang. "Tetralivalent edge-transitive Cayley graphs with odd number of vertices". In: *Journal of Combinatorial Theory, Series B* 96.1 (2006).

Lemma (2.5)

Let Γ be a connected (G, s) -transitive graph with $s \geq 3$, and suppose $N \triangleleft G$ and not semiregular. Then one of the following holds:

- ① both $G_\alpha^{\Gamma(\alpha)}$ and $N_\alpha^{\Gamma(\alpha)}$ are almost simple 2-transitive;
- ② $G_\alpha^{\Gamma(\alpha)}$ is affine and $N_\alpha^{\Gamma(\alpha)}$ is primitive.

By lemma 4.4, $N \triangleleft G$ and $N_\alpha \neq 1$, $\forall \alpha \in V\Gamma$.

Applying lemma 2.5, we have Γ is N -locally-primitive.

Hence Γ is K -locally-primitive since $N \leq K$.

Lemma (2.4)

Let Γ be a connected bipartite G -locally-primitive graph with G -orbits Δ_1 and Δ_2 on $V\Gamma$ and each $|\Delta_i| > 1$. Suppose $\exists N \triangleleft G$ s.t. N is intransitive on Δ_1 and Δ_2 . Then^{a,Lem 5.1}

- ① Γ is a normal cover of Γ_N .
- ② N is semiregular on $V\Gamma$, $G^{V\Gamma_N} \cong G/N$ and $G_\alpha \cong (G/N)_v$ for $\alpha \in V\Gamma$ and $v \in V\Gamma_N$.
- ③ Γ_N is G/N -locally-primitive. Furthermore, if Γ is locally (G, s) -arc-transitive with $s \geq 2$, then Γ_N is locally $(G/N, s)$ -arc-transitive.

^aGiudici, Li, and Praeger, "Analysing finite locally s -arc transitive graphs".

If $M_j \triangleleft K$ is intransitive on both Δ_1 and Δ_2 for each $j = 1, \dots, d$, then M_j is semiregular hence $M_j \cap K_\alpha = 1$, and $K_\alpha \cong (K/M_j)_v \lesssim \text{Aut}(T)$. Since $\frac{K_\alpha}{M_j \cap K_\alpha} \cong \frac{K_\alpha \ker \varphi_j}{\ker \varphi_j}$, K_α is a diagonal subgroup, i.e.

$$K_\alpha = \{(a, a, \dots, a) | a \in P\}, \quad P \leq \text{Aut}(T).$$

$$N_\alpha = N \cap K_\alpha = (T_1 \times \cdots \times T_d) \cap K_\alpha = \{(a, a, \dots, a) | a \in P \cap T\} \lesssim T,$$

thus $|T|^{d-1}$ divides $|N : N_\alpha| = |\Delta_1|$,

$H^+ \curvearrowright \Delta_1$ transitively $\implies |T|^{d-1}$ divides $|H^+| = |\textcolor{blue}{H^+ \cap K}| |H^+ K / K|$.
 $N \curvearrowright \Delta_1$ transitively \implies

$$K/N = K_\alpha N / N \cong K_\alpha / N_\alpha \cong P / (P \cap T) \cong PT / T \lesssim \text{Out}(T).$$

$\therefore |H^+ \cap K| / |\textcolor{blue}{H^+ \cap N}|$ divides $|\text{Out}(T)|$.

$\because |H^+ \cap N|$ divides $|H_1 \times \cdots \times H_d|$,

Therefore,

$|T|^{d-1}$ divides $|H_1 \times \cdots \times H_d| |\text{Out}(T)| |H^+ K / K|$.

Case: $H^+ \not\leq K$: $\{T_1, \dots, T_k\}$ an H^+ -conj.-orbit. Then $H_1 \cong \cdots \cong H_k$.

$$M := M_1 \cap \cdots \cap M_k \quad (\Delta_i)_M := \{M\text{-orbits on } \Delta_i\}$$

Consider $\overline{K} := K/M \curvearrowright (\Delta_i)_M$. Then

- M_j intrans. $\implies |(\Delta_i)_M| > 1$
- $\overline{K_v}$ with $v \in (\Delta_i)_M$ is diagonal
- $\overline{H^+}$ and $\overline{N} \cong T^k$ are transitive on $(\Delta_i)_M$
- $\overline{H^+K}/\overline{K} \lesssim S_k$
- projection of $\overline{H^+} \cap \overline{N}$ on T_j is isomorphic to H_j for $1 \leq j \leq k$

Similar as previous page,

$$|T|^{k-1} \text{ divides } |H_1 \times \cdots \times H_k| |\text{Out}(T)| |S_k| = |H_1|^k |\text{Out}(T)| k!$$

Contradict to ref.¹¹, page 76

why does lemma 2.6 hold by ref.?

¹¹Li and Xia, Factorizations of Almost Simple Groups with a Solvable Factor, and Cayley Graphs of Solvable Groups

Case: $H^+ \leq K$: H^+ normalizes T_1, \dots, T_d , $\therefore T_1 \triangleleft H^+$. $T_1 =: Y = H^+ Y_\alpha$.

$$Y_\alpha = Y_\alpha / (Y_\alpha \cap T_1) \cong Y_\alpha T_1 / T_1 \leq Y / T_1 = H^+ T_1 / T_1 \cong H^+ / (H^+ \cap T_1)$$

$\therefore Y$ is a product of two solvable subgroups. By ref.¹², Thm 1.1,

- either $T_1 = \mathrm{PSL}_2(p^f)$ with $p^f \geq 5$
- or $T_1 = \mathrm{PSU}_3(8)$, $\mathrm{PSU}_4(2)$, $\mathrm{PSL}_4(2)$, M_{11} , $\mathrm{PSL}_3(p^f)$ with $p^f = 3, 4, 5, 7, 8$

Meanwhile,

$$H^+ \curvearrowright \Delta; \text{ transitive} \implies K = H^+ K_\alpha \implies K / M_j = \frac{H^+ M_j}{M_j} (K / M_j)_{v_j}$$

(factorization of AS group K / M_j with socle above and a solvable factor)

$\therefore \exists$ prime $r \mid |T|$ but $\nmid |H_j| = |(H^+ M_j / M_j) \cap \mathrm{soc}(K / M_j)|$ by ref.^{12, Thm 1.1}.

Recall previous equation and note that $|H^+ K / K| = 1$ here,

we have $|T|_r^{d-1}$ divides $|\mathrm{Out}(T)|_r$, which is impossible by ref.¹³, Thm 8.17.

¹²L.C. Kazarin. "On groups which are the product of two solvable subgroups". In: *Communications in Algebra* 14.6 (1986), pp. 1001–1066.

¹³Li and Xia, *Factorizations of Almost Simple Groups with a Solvable Factor, and Cayley Graphs of Solvable Groups* 

Quasiprimitive $G^+ \curvearrowright \Delta_1$ and Δ_2

Lemma (4.6)

G^+ is quasiprimitive on Δ_1 and Δ_2 .

Proof. Assume $G^+ \curvearrowright \Delta_1$ is NOT quasiprimitive.

By ref.¹⁴,

- $d = 2l$ and $\exists M \triangleleft_{min} G^+$ semiregular and $N = M \times M^g$,
- $N_\alpha \lesssim T$ is a diagonal subgroup of N .

What could T be?

Note that $Y = MH^+$ has factorization $Y = H^+ Y_\alpha$ into solvable factors.

Then ref.¹⁵ implies that either $T = \mathrm{PSL}_2(p^f)$ with $p^f \geq 5$, or $T = \mathrm{PSU}_3(8)$, $\mathrm{PSU}_4(2)$, $\mathrm{PSL}_4(2)$, M_{11} or $\mathrm{PSL}_3(p^f)$ with $p^f \in \{3, 4, 5, 7, 8\}$.

¹⁴Li et al., "Finite locally-quasiprimitive graphs".

¹⁵Kazarin, "On groups which are the product of two solvable subgroups".

Case (a): $N_\alpha \cong T$.

Recall $G_\alpha^{\Gamma(\alpha)}$ is 2-transitive hence almost simple here.

Then lemma 2.5 implies $N_\alpha^{\Gamma(\alpha)} \cong T$ is 2-transitive.

Hence Γ is locally $(N, 2)$ -arc-transitive.

WLOG, assume $M = T_1 \times \cdots \times T_l$ and

$$P = T_1 \times \cdots \times T_{l-1} \times T_1^g \times \cdots \times T_{l-1}^g \triangleleft N$$

Then $P_\alpha = P \cap N_\alpha = 1$ and hence length of P -orbits is

$|P| = |T|^{2l-1} < |T|^{2l-1} = |\Delta_i|$, P is intransitive on both Δ_1 and Δ_2 .

Then lemma 2.4 implies Γ_P is locally $(N/P, 2)$ -arc-transitive.

However, $N/P \cong T_l \times T_l^g \cong T^2$ and $(N/P)_v \cong N_\alpha \cong T$ implies that N/P is quasiprimitive of type HS on the bipartitions of Γ_P . **Why?**
This contradicts ref.¹⁶, Thm 1.2.

¹⁶Giudici, Li, and Praeger, "Analysing finite locally s -arc transitive graphs".

Case (b): $1 \neq N_\alpha < T$.

Let L be the maximal subgroup of T containing N_α .

- By lemma 4.5, $\exists M_j$ is transitive on Δ_1 or Δ_2 . $|N : N_\alpha| = |\Delta_1| \mid |M_j|$.

$|T|/|N_\alpha|$ and $|T|/|L|$ divide $|\text{Out}(T)|^{2l-1}$. ($\therefore T \neq M_{11}$)

- Consider $m(T)$ the smallest index of the solvable subgroups in T .

$$m(T)^{2l} \leq |G_\alpha|$$

- Since M, M^g are the only 2 minimal normal subgroups of G^+ ,
 G^+/K is transitive on $\{T_1, \dots, T_l\}$ and has 2 orbits on $\{T_1, \dots, T_{2l}\}$.
Hence $l \mid |G : K|$ and $G^+/K \lesssim S_l \wr \mathbb{Z}_2$.
Thereby $l \mid |G : N|$ and $G^+ \lesssim N \cdot (\text{Out}(T)^l : S_l)^2 \cdot \mathbb{Z}_2$.

Now we discuss what L could be.

Definition (p.p.d.)

For positive integers a , $m \geq 2$, a prime r is called a **primitive prime divisor** of $a^m - 1$ if r divides $a^m - 1$ but not $a^i - 1$ for each $i = 1, 2, \dots, m - 1$.

Lemma (2.1, Zsigmondy)

For positive integers a , $m \geq 2$, $a^m - 1$ has a primitive prime divisor r if $(a, m) \neq (2, 6)$ and $(2^e - 1, 2)$ with $e \geq 2$ an integer.

Moreover, $r \equiv 1 \pmod{m}$, and in particular $r > m$.

Case (b.1): L is nonsolvable. ($\therefore T \neq \mathrm{PSL}_3(3)$)

- $T = \mathrm{PSL}_2(p^f)$ with $p^f \geq 5$: By ref.¹⁷, L may be
 - ① $L = A_5$ with $p^f = p \equiv \pm 1 \pmod{5}$ or $p^f = p^2 \equiv -1 \pmod{5}$
 - ② $L = \mathrm{PSL}_2(p^e)$ with f/e odd, or $L = \mathrm{PGL}_2(p^e)$ with f/e even.Both contradict $|T : L| \mid |\mathrm{Out}(T)|^{2l-1}$.
- $T = \mathrm{PSL}_4(2)$: By ATLAS, $\mathrm{Out}(T) = 2$, $m(T) = 35$,

$$L = 2^3 : \mathrm{PSL}_3(2), S_6, (A_5 \times 3) : 2, A_7.$$

Recall $|T : L| \mid |\mathrm{Out}(T)|^{2l-1}$, $N_\alpha = L = A_7$, searching G_α in ref.^{18, Thm3.4} gives $l \geq 6$ hence $35^{2l} \leq |G_\alpha| \leq |S_6 \times S_7|$. Contradiction!

- $T = \mathrm{PSL}_3(p^f)$, $\mathrm{PSU}_3(8)$, $\mathrm{PSU}_4(2)$ with $p^f = 4, 5, 7, 8$: by ATLAS, contradicting that $|T : L|$ divides $|\mathrm{Out}(T)|^{2l-1}$. (Table 1)

¹⁷Leonard Eugene Dickson. "Linear Groups, with an Exposition of the Galois Field Theory". In: 1958.

¹⁸Li, Lu, and Wang, "Arc-transitive graphs of square-free order and small valency"

Case (b.2): L is solvable.

Now $N_\alpha^{\Gamma(\alpha)}$ is solvable and $G_\alpha^{\Gamma(\alpha)}$ is affine, $\text{val}(\Gamma)$ is a prime power.

One can exclude $T \neq \text{PSL}_2(p^f)$ by ATLAS as previous. (see Table 2)

Suppose $T = \text{PSL}_2(p^f)$, $p^f \geq 5$. Then $|\text{Out}(T)| = (2, p-1)f$. By ref.¹⁹,

$$L \in \{D_{2\frac{p^f-1}{(2,p-1)}}, D_{2\frac{p^f+1}{(2,p-1)}}, \mathbb{Z}_p^f : \mathbb{Z}_{\frac{p^f-1}{(2,p-1)}}\}, \text{ or } L \in \{A_4, S_4\} \text{ and } f = 1.$$

A_4, S_4 : Then $f = 1$ and $\text{Out}(T) \leq \mathbb{Z}_2$, but $|T : L|$ is divisible by $p \geq 5$.

$$D_{2\frac{p^f-1}{(2,p-1)}} : |T : L| = p^f(p^f + 1)/2$$

If r a ppd. of $p^{2f} - 1$, then $r > 2f$ divides $|T : L|$ but not $|\text{Out}(T)|$.

Or $r = 2, p$ resp. for $(p, 2f) = (2, 6), (2^m - 1, 2)$ does the same.

$$D_{2\frac{p^f+1}{(2,p-1)}} : \text{Similar as above.}$$

$$\mathbb{Z}_p^f : \mathbb{Z}_{\frac{p^f-1}{(2,p-1)}} : |T : L| = p^f + 1, p^{2f} - 1 \text{ has a ppd } r | p^f + 1 \text{ but not } (2, p-1)f.$$

¹⁹Dickson, "Linear Groups, with an Exposition of the Galois Field Theory".

Summary

Recall hypothesis in the beginning of this section:

- Let Γ be a connected bipartite (G, s) -transitive graphs of $\text{val} \geq 3$, with bipartitions Δ_1 and Δ_2 and $s \geq 3$.
- Suppose G is bi-quasiprimitive on $V\Gamma$ and contains a vertex-transitive solvable subgroup H .
- Set $G^+ = G_{\Delta_1}$, the setwise stabilizer of G on Δ_1 , and $H^+ = H \cap G^+$.

Proposition (4.7)

Under such hypothesis, either $\Gamma = K_{n,n}$, or G is an almost simple group and G^+ is quasiprimitive on both Δ_1 and Δ_2 .

Proof. If G^+ acts unfaithfully on Δ_1 or Δ_2 , lemma 4.2 gives $\Gamma = K_{n,n}$.

Faithful case:

Lemma 4.3 shows either G^+ is not quasiprimitive on Δ_1 or Δ_2 , or quasiprimitive on both with same type AS or PA.

Lemma 4.6 exclude the former. Lemma 4.5 exclude PA in the latter.

Finally lemma 4.4 implies G is AS.

Outline

1 Preliminaries & Background

2 Main Result & Examples

3 Proof of the Main result

- Overview of the proof
- Reduce to almost simple groups
- Almost simple groups

Factorization of AS groups

Theorem

^a Suppose that G is an almost simple group with socle L and $G = HK$ for a solvable subgroup H and core-free subgroup K of G . Then one of the following statements holds.

- (a) Both H and K are solvable, and the triple (G, H, K) is described in Proposition 4.1.
- (b) $L = An$, and the triple (G, H, K) is described in Proposition 4.3.
- (c) L is a sporadic simple group, and the triple (G, H, K) is described in Proposition 4.4.
- (d) L is a classical group of Lie type, and the triple $(L, H \ L, K \ L)$ lies in Table 1.1 or Table 1.2.

^aLi and Xia, Factorizations of Almost Simple Groups with a Solvable Factor, and Cayley Graphs of Solvable Groups.

Factorizing G and G^+

Assume G is AS with socle T , and G^+ is quasiprimitive on Δ_1 and Δ_2 .
Set $G = T.\mathcal{O}$ and $G^+ = T.\mathcal{O}'$, with $\mathcal{O}' < \mathcal{O} \leq \text{Out}(T)$ and $|\mathcal{O} : \mathcal{O}'| = 2$.

$$T \curvearrowright \Delta_i \text{ transitive} \implies G^+ = G_\alpha T \implies G_\alpha / T_\alpha \cong G_\alpha T / T = G^+ / T = \mathcal{O}'$$

By Frattini argument, $G = HG_\alpha$ and $G^+ = H^+G_\alpha$.
Both factorization have a solvable factor H, H^+ .

Lemma (5.1)

$T \neq \text{PSL}_2(q)$ with q a prime power.

Proof. Check the classification in ref.²⁰, in which the only 3-arc-transitive graph is the Petersen graph, non-bipartite. Why no Pet⁽²⁾? □

²⁰C. E. Praeger A. Hassani L. R. Nochefranca. "Two-arc transitive graphs admitting a two-dimensional projective linear group". In: 2.4 (1999), pp. 335–353.

Lemma (5.2)

If G_α solvable, then $T = \mathrm{PSL}_3(3)$ and $\Gamma = \mathcal{PH}(3, 3^3)$ is 4-transitive of valency 4.

Proof. By ref.²¹, Prop. 4.1, both (G, H, G_α) and (G^+, H^+, G_α) lie in

TABLE 4.1.

row	G	H	K
1	$\mathrm{PSL}_2(7).\mathcal{O}$	$7:\mathcal{O}, 7:(3 \times \mathcal{O})$	S_4
2	$\mathrm{PSL}_2(11).\mathcal{O}$	$11:(5 \times \mathcal{O}_1)$	$A_4.\mathcal{O}_2$
3	$\mathrm{PSL}_2(23).\mathcal{O}$	$23:(11 \times \mathcal{O})$	S_4
4	$\mathrm{PSL}_3(3).\mathcal{O}$	$13:\mathcal{O}, 13:(3 \times \mathcal{O})$	$3^2:2.S_4$
5	$\mathrm{PSL}_3(3).\mathcal{O}$	$13:(3 \times \mathcal{O})$	$\mathrm{AGL}_1(9)$
6	$\mathrm{PSL}_3(4).(S_3 \times \mathcal{O})$	$7:(3 \times \mathcal{O}).S_3$	$2^4:(3 \times D_{10}).2$
7	$\mathrm{PSL}_3(8).(3 \times \mathcal{O})$	$73:(9 \times \mathcal{O}_1)$	$2^{3+6}:7^2:(3 \times \mathcal{O}_2)$
8	$\mathrm{PSU}_3(8).3^2.\mathcal{O}$	$57:9.\mathcal{O}_1$	$2^{3+6}:(63:3).\mathcal{O}_2$
9	$\mathrm{PSU}_4(2).\mathcal{O}$	$2^4:5$	$3_+^{1+2}:2.(A_4.\mathcal{O})$
10	$\mathrm{PSU}_4(2).\mathcal{O}$	$2^4:D_{10}.\mathcal{O}_1$	$3_+^{1+2}:2.(A_4.\mathcal{O}_2)$
11	$\mathrm{PSU}_4(2).2$	$2^4:5:4$	$3_+^{1+2}:S_3, 3^3:(S_3 \times \mathcal{O}),$ $3^3:(A_4 \times 2), 3^3:(S_4 \times \mathcal{O})$
12	M_{11}	$11:5$	$M_9.2$

where $\mathcal{O} \leqslant C_2$, and $\mathcal{O}_1, \mathcal{O}_2$ are subgroups of \mathcal{O} such that $\mathcal{O} = \mathcal{O}_1\mathcal{O}_2$.

²¹Li and Xia, Factorizations of Almost Simple Groups with a Solvable Factor, and Cayley Graphs of Solvable Groups

- 1-3: Excluded by lemma 5.1.
- 4-5: $G = \mathrm{PSL}_3(3).2$, $G_\alpha = 3^2 : 2.S_4$ or $\mathrm{AGL}_1(9) \cong 3^2 : 8.2$
 $G_\alpha^{\Gamma(\alpha)}$ 2-transitive $\implies \mathrm{val}(\Gamma) = 3, 4, \emptyset$. By ref.²², $G_\alpha = 3^2 : 2.S_4$
Hence $|V\Gamma| = 26$, and $\Gamma = \mathcal{PH}(3, 3^3)$ by ref.²³, Table 1.
- 6: $G = \mathrm{PSL}_3(4).(S_3 \times \mathbb{Z}_2)$ and $G_\alpha = 2^4 : (3 \times D_{10}).2$
3-arc-transitive $\implies \mathrm{val}(\Gamma) = 5$, but $G_\alpha = \mathbb{Z}_4 \times (\mathbb{Z}_5 : \mathbb{Z}_4)$ by ref.²⁴

Claim: If $\mathrm{val}(\Gamma) \geq 5$, then $|G_\alpha|$ divides $|G_\alpha^{\Gamma(\alpha)}|^2 / \mathrm{val}(\Gamma)$.

- 7: $G = \mathrm{PSU}_3(8).(3 \times \mathbb{Z}_2)$, $G_\alpha^{\Gamma(\alpha)} = 2^3 : 7 : 3$, $\mathrm{val}(\Gamma) = 8$, contradiction.
- 8: $G = \mathrm{PSU}_3(8).3^2.2$, $G_\alpha^{\Gamma(\alpha)} = 2^3 : 7 : 3$ with $\mathrm{val}(\Gamma) = 2^3$ or
 $G_\alpha^{\Gamma(\alpha)} = 2^6 : 63 : 3$ with $\mathrm{val}(\Gamma) = 2^6$, both contradiction
- 9-11: $G = \mathrm{PSU}_4(2)$, $G_\alpha = 3_+^{1+2} : 2.A_4$, $\mathrm{val}(\Gamma) = 4$ (ref.¹⁵), \emptyset (3-arc-trans.)
- 12: $G^+ = M_{11} \implies G = M_{11}.2$ which is not in the table.

²²Li, Lu, and Zhang, "Tetravalent edge-transitive Cayley graphs with odd number of vertices".

²³Ying Cheng and James Oxley. "On weakly symmetric graphs of order twice a prime". In: *Journal of Combinatorial Theory, Series B* 42.2 (1987), pp. 196–211.

²⁴Richard Weiss. "An application of p-factorization methods to symmetric graphs". In: *Mathematical Proceedings of the Cambridge Philosophical Society* 85.1 (1979), 4348.

Lemma (5.3)

T is not an alternating simple group.

Proof. Both (G, H, G_α) and (G^+, H^+, G_α) lie in the following. All possibilities can be excluded by lemma 2.3.

PROPOSITION 4.3. Suppose $n \geq 5$ and $\text{soc}(G) = A_n$ acting naturally on $\Omega_n = \{1, \dots, n\}$. If $G = HK$ for a solvable subgroup H and core-free non-solvable subgroup K of G , then one of the following holds.

- (a) $A_n \trianglelefteq G \leq S_n$ with $n \geq 6$, H is transitive on Ω_n , and $A_{n-1} \trianglelefteq K \leq S_{n-1}$.
- (b) $A_n \trianglelefteq G \leq S_n$ with $n = p^f$ for some prime p , H is 2-homogeneous on Ω_n , and $A_{n-2} \trianglelefteq K \leq S_{n-2} \times S_2$; moreover, either $H \leq \text{AGL}_1(p^f)$ or (H, n) lies in the table:

H	n
$5^2:\text{SL}_2(3), 5^2:\text{Q}_8.6, 5^2:\text{SL}_2(3).4$	5^2
$7^2:\text{Q}_8.S_3, 7^2:\text{SL}_2(3).6$	7^2
$11^2:\text{SL}_2(3).5, 11^2:\text{SL}_2(3).10$	11^2
$23^2:\text{SL}_2(3).22$	23^2
$3^4 \cdot 2^{1+4}.5, 3^4 \cdot 2^{1+4}.D_{10}, 3^4 \cdot 2^{1+4}.\text{AGL}_1(5)$	3^4

- (c) $A_n \trianglelefteq G \leq S_n$ with $n = 8$ or 32 , $A_{n-3} \trianglelefteq K \leq S_{n-3} \times S_3$ and $(H, n) = (\text{AGL}_1(8), 8), (\text{AGL}_1(8), 8)$ or $(\text{AGL}_1(32), 32)$.
- (d) $A_6 \trianglelefteq G \leq S_6$, $H \leq S_4 \times S_2$, and $K = \text{PSL}_2(5)$ or $\text{PGL}_2(5)$.
- (e) $A_6 \trianglelefteq G \leq S_6$, $H \leq S_3(S_2)$, and $K = \text{PSL}_2(5)$ or $\text{PGL}_2(5)$.
- (f) $n = 6$ or 8 , and (G, H, K) lies in Table 4.2.

TABLE 4.2.

row	G	H	K
1	M_{10}	$3^2:\text{Q}_8$	$\text{PSL}_2(5)$
2	$\text{PGL}_2(9)$	$\text{AGL}_1(9)$	$\text{PSL}_2(5)$
3	$\text{PFL}_2(9)$	$\text{AGL}_1(9)$	$\text{PSL}_2(5)$
4	$\text{PFL}_2(9)$	$\text{AGL}_1(9), 3^2:\text{Q}_8, \text{AGL}_1(9)$	$\text{PGL}_2(5)$
5	A_8	$15, 3 \times D_{10}, \Gamma L_1(16)$	$\text{AGL}_3(2)$
6	S_8	$D_{30}, S_3 \times 5, S_3 \times D_{10}, 3 \times \text{AGL}_1(5), S_3 \times \text{AGL}_1(5)$	$\text{AGL}_3(2)$

Lemma (5.4)

T is not a sporadic simple group.

Proof. Both (G, H, G_α) and (G^+, H^+, G_α) lie in the following. The only two possibilities can be excluded by lemma 2.3.

PROPOSITION 4.4. Let $L = \text{soc}(G)$ be a sporadic simple group. If $G = HK$ for a solvable subgroup H and core-free non-solvable subgroup K of G , then one of the following holds.

- (a) $M_{12} \leqslant G \leqslant M_{12}.2$, H is transitive on $[G:M_{11}]$, and $K = M_{11}$
- (b) $G = M_{24}$, H is transitive on $[M_{24}:M_{23}]$, and $K = M_{23}$.
- (c) (G, H, K) lies in Table 4.3.

Conversely, each triple (G, H, K) in parts (a)–(c) gives a factorization $G = HK$.

TABLE 4.3.

row	G	H	K
1	M_{11}	11, 11:5	M_{10}
2	M_{11}	$M_9, M_9.2, \text{AGL}_1(9)$	$\text{PSL}_2(11)$
3	M_{12}	$M_9.2, M_9.S_3$	$\text{PSL}_2(11)$
4	$M_{12}.2$	$M_9.2, M_9.S_3$	$\text{PGL}_2(11)$
5	$M_{22}.2$	11:2, 11:10	$\text{PSL}_3(4):2$
6	M_{23}	23	M_{22}
7	M_{23}	23:11	$M_{22}, \text{PSL}_3(4):2, 2^4.A_7$
8	$J_2.2$	$5^2:4, 5^2:(4 \times 2), 5^2:12, 5^2:\text{Q}_{12}, 5^2:(4 \times S_3)$	$G_2(2)$
9	HS	$5_+^{1+2}.8:2$	M_{22}
10	HS.2	$5_+^{1+2}.(4:S_2)$	$M_{22}, M_{22}.2$
11	HS.2	$5^2:4, 5^2:(4 \times 2), 5^2:4^2, 5_+^{1+2}:4, 5_+^{1+2}.(4 \times 2), 5_+^{1+2}:4^2, 5_+^{1+2}:8:2$	$M_{22}.2$
12	He.2	$7_+^{1+2}.6, 7_+^{1+2}.(6 \times 2), 7_+^{1+2}.(6 \times 3), 7_+^{1+2}.(S_3 \times 3), 7_+^{1+2}.(S_3 \times 6)$	$\text{Sp}_4(4).4$
13	Suz.2	$3^5:12, 3^5:(11:5) \times 2$	$G_2(4).2$

Lemma (5.5)

If T is a classical simple group of Lie type, then $s \leq 4$, and one of the following is true.

- ① $T = \mathrm{PSL}_5(2)$, and $\Gamma \cong \mathcal{GI}(5, 2, 2)$.
- ② $T = \mathrm{PSU}_3(5)$, and $\Gamma \cong \mathrm{HS}_{50}^{(2)}$.
- ③ Γ is of valency $p^f + 1$, $\mathrm{PSL}_3(p^f).2 \leq G \leq \mathrm{Aut}(\mathrm{PSL}_3(p^f))$ and $\mathbb{Z}_p^{2f} : \mathrm{SL}_2(p^f) \triangleleft G_\alpha \leq P_1$ or P_2 where P_1 and P_2 are the stabilizers of T on a 1-dimension subspace and a 2-dimension subspace, respectively.

Proof. If G_α solvable, then $T = \mathrm{PSL}_3(3)$, $\Gamma = \mathcal{GI}(3, 1, 3^3)$, satisfies (3). If G_α is nonsolvable, note that G_α has a 2-transitive representation, hence has a minimal normal subgroup in the following list.²⁵

²⁵Cameron, "Finite Permutation Groups and Finite Simple Groups".

We list here the simple groups T which can occur as minimal normal subgroups of 2-transitive groups of degree n , according to Theorem 5.3. The number k is the maximum degree of transitivity of a group G with socle T .

T	n	k	Remarks
$A_n, n \geq 5$	n	n	Two representations if $n = 6$
$\mathrm{PSL}(d, q), d \geq 2$	$(q^d - 1)/(q - 1)$	3 if $d = 2$ 2 if $d > 2$	$(d, q) \neq (2, 2), (2, 3)$ Two representations if $d > 2$
$\mathrm{PSU}(3, q)$	$q^3 + 1$	2	$q > 2$
${}^2B_2(q)$ (Suzuki)	$q^2 + 1$	2	$q = 2^{2a+1} > 2$
${}^2G_2(q)$ (Ree)	$q^3 + 1$	2	$q = 3^{2a+1} > 3$
$\mathrm{PSp}(2d, 2)$	$2^{2d-1} + 2^{d-1}$	2	$d > 2$
$\mathrm{PSp}(2d, 2)$	$2^{2d-1} - 2^{d-1}$	2	$d > 2$
$\mathrm{PSL}(2, 11)$	11	2	Two representations
$\mathrm{PSL}(2, 8)$	28	2	
A_7	15	2	Two representations
M_{11} (Mathieu)	11	4	
M_{11} (Mathieu)	12	3	
M_{12} (Mathieu)	12	5	Two representations
M_{22} (Mathieu)	22	3	
M_{23} (Mathieu)	23	4	
M_{24} (Mathieu)	24	5	
HS (Higman-Sims)	176	2	Two representations
Co_3 (Conway)	276	2	

Notice that $1 \neq T_\alpha \triangleleft G_\alpha$ and $T_\alpha \not\leq G_\alpha^{[1]}$.

Thus T_α has a composition factor listed above.

Check $(T, H^+ \cap T, T_\alpha)$ in following list.

TABLE 1.2.

TABLE 1.1.

row	L	$H \cap L \leqslant$	$K \cap L \triangleright$	remark
1	$\mathrm{PSL}_n(q)$	$\mathrm{^tGL}_1(q^n):n = \frac{q^n-1}{(q-1)d} \cdot n$	$q^{n-1} \cdot \mathrm{SL}_{n-1}(q)$	$d = (n, q-1)$
2	$\mathrm{PSL}_4(q)$	$q^3 \cdot \frac{q^3-1}{d} \cdot 3 < P_k$	$\mathrm{PSp}_4(q)$	$d = (4, q-1), k \in \{1, 3\}$
3	$\mathrm{PSp}_{2m}(q)$	$q^{m(m+1)/2} \cdot (q^m - 1) \cdot m < P_m$	$\Omega_{2m}^-(q)$	$m \geq 2, q \text{ even}$
4	$\mathrm{PSp}_4(q)$	$q^3 \cdot (q^2 - 1) \cdot 2 < P_1$	$\mathrm{Sp}_2(q^2)$	$q \text{ even}$
5	$\mathrm{PSp}_4(q)$	$q^{1+2} \cdot \frac{q^2-1}{2} \cdot 2 < P_1$	$\mathrm{PSp}_2(q^2)$	$q \text{ odd}$
6	$\mathrm{PSU}_{2m}(q)$	$q^{m^2} \cdot \frac{q^{2m}-1}{(q+1)d} \cdot m < P_m$	$\mathrm{SU}_{2m-1}(q)$	$m \geq 2, d = (2m, q+1)$
7	$\Omega_{2m+1}(q)$	$(q^{m(m-1)/2} \cdot q^m) \cdot \frac{q^m-1}{2} \cdot m < P_m$	$\Omega_{2m}^-(q)$	$m \geq 3, q \text{ odd}$
8	$\mathrm{P}\Omega_{2m}^+(q)$	$q^{m(m-1)/2} \cdot \frac{q^{m-1}}{d} \cdot m < P_k$	$\Omega_{2m-1}(q)$	$m \geq 5, d = (4, q^m - 1), k \in \{m, m-1\}$
9	$\mathrm{P}\Omega_8^+(q)$	$q^6 \cdot \frac{q^4-1}{d} \cdot 4 < P_k$	$\Omega_7(q)$	$d = (4, q^4 - 1), k \in \{1, 3, 4\}$

row	L	$H \cap L \leqslant$	$K \cap L$
1	$\mathrm{PSL}_2(11)$	11:5	A_5
2	$\mathrm{PSL}_2(16)$	D_{34}	A_5
3	$\mathrm{PSL}_2(19)$	19:9	A_5
4	$\mathrm{PSL}_2(29)$	29:14	A_5
5	$\mathrm{PSL}_2(59)$	59:29	A_5
6	$\mathrm{PSL}_4(3)$	$2^4 \cdot 5:4$	$\mathrm{PSL}_3(3), 3^3 \cdot \mathrm{PSL}_3(3)$
7	$\mathrm{PSL}_4(3)$	$3^3:13:3$	$(4 \times \mathrm{PSL}_2(9)):2$
8	$\mathrm{PSL}_4(4)$	$2^6:63:3$	$(5 \times \mathrm{PSL}_2(16)):2$
9	$\mathrm{PSL}_5(2)$	31:5	$2^9 \cdot (\mathrm{S}_3 \times \mathrm{PSL}_3(2))$
10	$\mathrm{PSp}_4(3)$	$3^3 \cdot S_4$	$2^4 \cdot A_5$
11	$\mathrm{PSp}_4(3)$	$3^{1+2} \cdot 2.A_4$	$A_5, 2^4 \cdot A_5, S_5, A_6, S_6$
12	$\mathrm{PSp}_4(5)$	$5^{1+2} \cdot 4.A_4$	$\mathrm{PSL}_2(5^2), \mathrm{PSL}_2(5^2):2$
13	$\mathrm{PSp}_4(7)$	$7^{1+2} \cdot 6.S_4$	$\mathrm{PSL}_2(7^2), \mathrm{PSL}_2(7^2):2$
14	$\mathrm{PSp}_4(11)$	$11^{1+2} \cdot 10.A_4$	$\mathrm{PSL}_2(11^2), \mathrm{PSL}_2(11^2):2$
15	$\mathrm{PSp}_4(23)$	$23^{1+2} \cdot 22.S_4$	$\mathrm{PSL}_2(23^2), \mathrm{PSL}_2(23^2):2$
16	$\mathrm{Sp}_6(2)$	$3^{1+2} \cdot 2.S_4$	A_8, S_8
17	$\mathrm{PSp}_6(3)$	$3^{1+4} \cdot 2^{1+4}.D_{10}$	$\mathrm{PSL}_2(27):3$
18	$\mathrm{PSU}_3(3)$	$3^{1+2}:8$	$\mathrm{PSL}_2(7)$
19	$\mathrm{PSU}_3(5)$	$5^{1+2}:8$	A_7
20	$\mathrm{PSU}_4(3)$	$3^4.D_{10}, 3^4.S_4, 3^4 \cdot 3^2:4, 3^{1+4}_+ \cdot 2.S_4$	$\mathrm{PSL}_3(4)$
21	$\mathrm{PSU}_4(8)$	513:3	$2^{12} \cdot \mathrm{SL}_2(64).7$
22	$\Omega_7(3)$	$3^3 \cdot 2^4 \cdot \mathrm{AGL}_1(5)$	$G_2(3)$
23	$\Omega_7(3)$	$3^{3+3}:13:3$	$\mathrm{Sp}_6(2)$
24	$\Omega_9(3)$	$3^{6+4} \cdot 2^{1+4} \cdot \mathrm{AGL}_1(5)$	$\Omega_8^-(3), \Omega_8^-(3).2$
25	$\Omega_8^+(2)$	$2^2:15.4 < A_9$	$\mathrm{Sp}_6(2)$
26	$\Omega_8^+(2)$	$2^6:15.4$	A_9
27	$\mathrm{P}\Omega_8^+(3)$	$3^{6 \cdot 2^4} \cdot \mathrm{AGL}_1(5)$	$\Omega_7(3)$
28	$\mathrm{P}\Omega_8^+(3)$	$3^6 \cdot (3^3:13:3), 3^{3+6}:13.3$	$\Omega_8^+(2)$

Note the isomorphism $\mathrm{PSp}_4(3) \cong \mathrm{PSU}_4(2)$ for rows 10 and 11.

One of the following occurs:

- (a) $T = \mathrm{PSL}_n(p^f)$ and $\mathbb{Z}_p^{f(n-1)} : \mathrm{SL}_{n-1}(p^f) \triangleleft T_\alpha \leq P_1$ or P_{n-1} , where $n \geq 3$, $(n, p^f) \neq (3, 2)$ $(3, 3)$ or $(4, 2)$
- (b) $T = \mathrm{PSp}_4(p^f)$ and $\mathrm{PSL}_2(p^{2f}) \triangleleft T_\alpha \triangleleft \mathrm{PSL}_2(p^{2f}).\mathbb{Z}_2$
- (c) $T = \mathrm{PSU}_4(p^f)$, and $\mathrm{PSU}_3(p^f) \triangleleft T_\alpha$
- (d) $(T, T \cap H^+, T_\alpha)$ lies in row 6-21, 23, 25-26 in Table 1.2

Case (a):

If $G_\alpha^{\Gamma(\alpha)}$ is affine, then ref.^{26, Table 7.3} implies contradiction.

Thus $G_\alpha^{\Gamma(\alpha)}$ is AS with socle $\mathrm{PSL}_{n-1}(p^f)$, and $\mathrm{val}(\Gamma) = \frac{p^{f(n-1)} - 1}{p^f - 1}$.

$$\mathrm{val}(\Gamma) - 1 \text{ divides } |G_\alpha^{[1]}| \text{ divides } \frac{|P_1||\mathrm{Out}(T)|}{2|\mathrm{PSL}_{n-1}(p^f)|} = fp^{f(n-1)}(p^f - 1).$$

Thus $p^{f(n-2)} - 1$ divides $fp^{f(n-2)}(p^f - 1)^2$.

Now $n \geq 4$ implies $p^{f(n-2)} - 1$ has a ppd r , while r divides the left but not right side of above formula. A contradiction.

Thus $n = 3$, $\mathrm{val}(\Gamma) = p^f + 1$ and $\mathbb{Z}_p^{2f} : \mathrm{SL}_2(p^f) \triangleleft G_\alpha$.

So $s \leq 4$ by Theorem 2.2.

²⁶Peter J. Cameron. *Permutation Groups*. London Mathematical Society Student Texts. Cambridge University Press 1999. ↗ ↘

Case (b): $G_\alpha^{\Gamma(\alpha)}$ is almost simple with socle $\mathrm{PSL}_2(p^{2f})$, $\mathrm{val}(\Gamma) = p^{2f} + 1$,

$$|G_\alpha^{[1]}| \text{ divides } \frac{|G_\alpha|}{G_\alpha^{\Gamma(\alpha)}} = \frac{|T_\alpha||\mathcal{O}'|}{G_\alpha^{\Gamma(\alpha)}} \text{ divides } 2|\mathcal{O}'| \text{ divides } |\mathrm{Out}(\mathrm{PSp}_4(p^f))| = 2f$$

However, by lemma 2.3, $p^{2f} = \mathrm{val}(\Gamma) - 1$ divides $|G_\alpha^{[1]}|$. A contradiction.

Case (c): Similar as above,

$$p^{3f} = \mathrm{val}(\Gamma) - 1 \text{ divides } |\mathrm{Out}(T)|/2 = f(4, p^f + 1).$$

Case (d):

row 6-8: MAGMA shows no such graph

row 9: $T = \mathrm{PSL}_5(2)$, $T_\alpha = 2^6 : (S_3 \times \mathrm{PSL}_3(2))$. Example(2): $\mathcal{GI}(5, 2, 2)$.

row 19: $T = \mathrm{PSU}_3(5)$, $T_\alpha = A_7$. Example(3): $\mathrm{HS}_{50}^{(2)}$.

row 20: $T = \mathrm{PSU}_4(3)$, $T_\alpha = \mathrm{PSL}_3(4)$, $\mathrm{val}(\Gamma) = 21$.

But $|\mathrm{Out}(T)| = 8$ not divisible by $\mathrm{val}(\Gamma) - 1$.

row 21: $T = \mathrm{PSU}_4(8)$, $T_\alpha = \mathbb{Z}_2^{12} : \mathrm{SL}_2(64). \mathbb{Z}_7$, $\mathrm{Out}(T) = \mathbb{Z}_6$.

By ref.^{27, Table 7.3}, $G_\alpha^{\Gamma(\alpha)}$ is AS with socle $\mathrm{PSL}_2(64)$ and $\mathrm{val}(\Gamma) = 65$.

MAGMA shows no such graph.

else: T_α is AS with $\mathrm{soc}(T_\alpha) \neq A_7$, $|\mathrm{Out}(T)| < 8$. $|G_\alpha^{[1]}|$ divides $\frac{|\mathrm{Out}(T)|}{2}$.

But nonsolvable G_α implies $\mathrm{val}(\Gamma) \geq 5$. A contradiction.

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