s-Arc-transitive solvable Cayley graphs

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October 1, 2023

Outline

Preliminaries

Let Γ be a finite simple undirected graph. Let $G \le \mathrm{Aut}\Gamma$.

Definition

- **s-arc:** (s+1)-tuple of vertices $\alpha_0, \alpha_1, \cdots, \alpha_s$ where α_i is adjacent to α_{i+1} and $\alpha_{j-1} \neq \alpha_{j+1}$ for $0 \le i \le s-1$ and $1 \le j \le s-1$.
- **(G,s)-arc-transitive:** G is transitive on the set of s-arcs of Γ.
- **(G,s)-transitive:** (G,s)-arc-transitive but not (G,s+1)-arc-transitive.

For short, **s-transitive** means $(\mathrm{Aut}\Gamma, s)$-transitive for graphs.

Lemma

- *s-arc-transitive* $(1 \le s) \Longrightarrow$ *k-arc-transitive* $(1 \le k \le s)$.
- *the s-arc-transitivity of a graph is inherited by the normal quotients.*

Preliminaries

Definition

- **Cayley graph:** $\mathrm{Cay}(G, S)$ with vertex set G and edges $yx^{-1} \in S$.
- **solvable Cayley graph:** G is solvable.

Lemma

Γ is Cayley of G $\iff$ $\exists G \lesssim \mathrm{Aut}\Gamma$ which is vertex-regular.

Definition

A transitive permutation group G is said to be
quasiprimitive if its nontrivial normal subgroups are transitive, and
bi-quasiprimitive if its nontrivial normal subgroups have at most 2 orbits and there exists one which has exactly 2 orbits.

The O'Nan-Scott-Praeger theorem[1] divides the quasiprimitive permutation groups into 8 types: holomorph affine (HA), holomorph simple (HS), holomorph compound (HC), almost simple (AS), simple diagonal (SD), compound diagonal (CD), product action (PA) and twisted wreath product (TW).

[1] Cheryl E Praeger. "Finite Quasiprimitive Graphs". In: *Surveys in Combinatorics, 1997*. London Mathematical Society Lecture Note Series. Cambridge University Press, 1997, 6586.

Background

- 1947, Tutte: No 6-arc-transitive trivalent graphs.
- 1981, Weiss: No s-arc-transitive graphs with $\mathrm{val} \geq 3$ for $s = 6$ and $s \geq 8$.
- 2019, Li C.H. & Xia B.Z.:
  Connected **non-bipartite** 3-arc-transitive solvable Cayley graph with $\mathrm{val} \geq 3$ is a normal cover of the Hoffman-Singleton graph or the Petersen graph.
- 2021, Zhou J.X.: No such normal covers, so $s \leq 2$ sharply.

Problem

Studying connected **bipartite** *s-arc-transitive solvable Cayley graphs, and determining the upper bound on* s.

For convenience, one may assume $s \geq 3$ and $\mathrm{val} \geq 3$.

Main Result

Theorem

Every connected s-arc-transitive solvable Cayley graph with $s \geq 3$ and val ≥ 3 is a normal cover of one of the following graphs:

1. *the complete bipartite graph $K_{n,n}$ with $n \geq 3$;*
2. *the geometry incidence graph $\mathcal{GI}(5, 2, 2)$;*
3. *the standard double cover of the Hoffman-Singleton graph;*
4. *a graph Σ with valency $p^f + 1$ such that $\mathrm{PSL}_3(p^f).2 \leq \mathrm{Aut}\Sigma \leq \mathrm{Aut}(\mathrm{PSL}_3(p^f))$ and $\mathbb{Z}_p^{2f} : \mathrm{SL}_2(p^f) \lhd (\mathrm{Aut}\Sigma)_\alpha$, where $p^f \geq 3$ is a prime power and α is a vertex.*

In particular, the sharp upper bound on s is 4.

These normal covers are being investigated in a sequel. (usually challenge)

Examples

(1) $\mathbf{K}_{n,n}$ with $n \geq 3$

- 3-transitive
- $\mathbf{K}_{n,n} \cong \mathrm{Cay}(G, G \setminus H)$ where G is solvable and $H < G$ of index 2

(2) $\mathcal{GI}(5,2,2)$

- the incidence graph of $(\mathcal{P}, \mathcal{L})$ where $\mathcal{P}$(resp. $\mathcal{L}$) is the set of 2-subspace(resp. 3-subspace) of $\mathbb{F}_2^5$
- $\mathrm{Aut}(\mathcal{GI}(5,2,2)) = \mathrm{GL}_5(2).\langle\sigma\rangle$ is vertex-transitive
- $G_\alpha = 2^6 : (\mathrm{GL}_2(2) \times \mathrm{GL}_3(2))$, $G_{\alpha\beta} = 2^8 : (S_3 \times S_3)$, val$=\frac{|G_\alpha|}{|G_{\alpha\beta}|}=7$
- 3-transitive[2, Theorem 3.4]
- $G = RG_\alpha$ with $R \cong 31 : 5 : 2$ vertex-regular[3, Theorem 1.1]

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[2] Cai Heng Li, Zai Ping Lu, and Gaixia Wang. "Arc-transitive graphs of square-free order and small valency". In: *Discrete Mathematics* 339.12 (2016), pp. 2907–2918.

[3] Cai Heng Li and Binzhou Xia. *Factorizations of Almost Simple Groups with a Solvable Factor, and Cayley Graphs of Solvable Groups*. Vol. 279. 1375. Mem. AMS, 2022.

Examples

(3) $\mathrm{HS}_{50}^{(2)}$

- HS_{50} is 3-transitive
  $\mathrm{Aut}(\mathrm{HS}_{50}) = \mathrm{P\Sigma U}(3,5)$ has a solvable vertex-transitive subgroup
  It is NOT a Cayley graph
- **standard double cover:** $\Gamma^{(2)} = (\tilde{V}, \tilde{E})$ of $\Gamma = (V, E)$, where
  $\tilde{V} = V \times \{1,2\}$, $\tilde{E} = \{\{(v,1),(w,2)\} \mid \{v,w\} \in E\}$
  $\mathrm{Aut}(\Gamma^{(2)}) \geq \mathrm{Aut}\Gamma \times \langle\sigma\rangle$, Γ s-arc-transitive $\implies$ $\Gamma^{(2)}$ s-arc-transitive
- By MAGMA, $\exists_1$ connected G-arc-transitive 7-valent graph Γ of order 50 with $(G, G_\alpha) = (\mathrm{PSU}_3(5), A_7)$ or $(\mathrm{PSU}_3(5).\mathbb{Z}_2, S_7)$. $\Gamma \cong \mathrm{HS}_{50}$
- By MAGMA, $\exists_1$ connected G-arc-transitive 7-valent graph Γ of order 100 with $(G, G_\alpha) = (\mathrm{PSU}_3(5).\mathbb{Z}_2, A_7)$. $\Gamma \cong \mathrm{HS}_{50}^{(2)}$

(Q: Could a normal cover of $\mathrm{HS}_{50}^{(2)}$ be solvable Cayley? see Zhou's paper)

Examples

(4) $\mathcal{PH}(3, q)$

- the incidence graph of $\mathrm{PG}(2, q) = (\mathcal{P}, \mathcal{L})$, where $\mathcal{P}$(resp. $\mathcal{L}$) is the set of 1-subspace(resp. 2-subspace) of $\mathbb{F}_q^3$
- $\mathrm{Aut}(\mathcal{PH}(3, q)) = \mathrm{P\Gamma L}_3(q).\langle\sigma\rangle = \mathrm{Aut}(\mathrm{PSL}_3(q))$
- $(\alpha_0, \cdots, \alpha_4)$ with $\alpha_0 \in \mathcal{P}$ corresponds to a basis v_1, v_2, v_3 such that

$$\alpha_0 = \langle v_1\rangle, \alpha_1 = \langle v_1, v_2\rangle, \alpha_2 = \langle v_2\rangle, \alpha_3 = \langle v_2, v_3\rangle, \alpha_4 = \langle v_3\rangle$$

  all ordered bases are equiv. under linear transformation
  $\Longrightarrow$ 4-arc-transitive
- a Cayley graph[4] of $D_{2(q^2+q+1)}$, for example Heawood graph.

[4] Dragan Marui. "On 2-arc-transitivity of Cayley graphs". In: *Journal of Combinatorial Theory, Series B* 87.1 (2003), pp. 162–196.

Outline

Overview of the proof

Let $\Gamma = \mathrm{Cay}(H, S)$ be connected, (G, s)-transitive, $\mathrm{val} \geq 3$, with H solvable, $H \leq G \leq \mathrm{Aut}\Gamma$, $s \geq 3$.
Let $M \lhd G$ (may trivial) be maximal with ≥ 3 orbits on $V\Gamma$.
Then

- Γ is a normal cover of Γ_M
- Γ_M is $(G/M, s)$-arc-transitive
- G/M is either quasiprimitive or bi-quasiprimitive on $V\Gamma_M$, $|V\Gamma_M| \geq 3$

Theorem (Praeger,1993)

Let Γ be a finite connected graph, $G \leq \mathrm{Aut}\Gamma$ is s-arc transitive on Γ for some $s \geq 2$. Suppose $N \lhd G$ has more than two orbits on vertices. Then

1. *Γ_N is finite and connected and the group of automorphisms of Γ_N induced by G is s-arc transitive on Γ_N.*
2. *N is semiregular on vertices (that is $N_\alpha = 1$ for each vertex α of Γ) and Γ is a cover of Γ_N.*

Overview of the proof

Case quasiprimitive: Γ_M non-bipartite. By Li-Xia-Zhou, no such graphs.
Case bi-quasiprimitive: Γ_M bipartite.
Then either $\Gamma_M = \mathbf{K}_{n,n}$, or G/M is almost simple. (Reduce to AS type)
Moreover, $\mathrm{soc}(G/M)$ is a classical simple group of Lie type.
(Use factorization of AS groups with a solvable factor)

Let s_0 be the sharp upper bound on s.

- By Example (4), $\mathcal{PH}(3, q)$ is 4-transitive, so $s_0 \geq 4$.
- If there is a connected 5-arc-transitive solvable Cayley graph Γ, then Γ is a normal cover of a 5-arc-transitive graph of type (1)-(4), while analysis above states no such graph.

Hence $s_0 = 4$.

Outline

Reduce to almost simple groups

In the remains of this talk, we assume the followings.

- Let Γ be a connected bipartite (G, s)-transitive graphs of val≥ 3, with bipartitions Δ_1 and Δ_2 and $s \geq 3$.
- Suppose G is bi-quasiprimitive on $V\Gamma$ and contains a vertex-transitive solvable subgroup H.
- Set $G^+ = G_{\Delta_1}$, the setwise stabilizer of G on Δ_1, and $H^+ = H \cap G^+$.
- Let α, β be adjacent vertices.

Some immediate facts:

- $G^+ = G_{\Delta_1} = G_{\Delta_2}$ has index 2 (hence normal) in G and is transitive on both Δ_1 and Δ_2, as well as H^+ in H.
- $G = \langle G^+, g\rangle$ where $g \in G \backslash G^+$, g interchanges Δ_1 and Δ_2, $g^2 \in G^+$

Case: unfaithful action

Now we consider the action of G^+ on Δ_1 and Δ_2.

Lemma (4.2)

If G^+ acts unfaithfully on Δ_1 or Δ_2, then $\Gamma = \mathbf{K}_{n,n}$, where $n = |V\Gamma|/2$.

Proof. Let $K = \ker(G^+ \curvearrowright \Delta_1)$.
Then $K^g = \ker(G^+ \curvearrowright \Delta_2)$ and $1 \neq KK^g = K \times K^g \lhd G$.
$\because$ G is bi-quasiprimitive, $K \times K^g$ has at most 2 orbits.
Since K^g fixes Δ_2 pointwise, K must be transitive on Δ_2.
It follows that $\Gamma = \mathbf{K}_{n,n}$. $\square$

We thus additionally assume G^+ is faithful on both Δ_1 and Δ_2.

$$G^+ \cong (G^+)^{\Delta_1} \cong (G^+)^{\Delta_2}$$

We will prove the above actions are quasiprimitive of almost simple type.

Quasiprimitive $\implies$ same type AS or PA

Lemma (4.3)

Suppose G^+ is quasiprimitive on Δ_1 or Δ_2. Then

1. *G^+ is quasiprimitive on both Δ_1 and Δ_2*
2. *both $(G^+)^{\Delta_1}$ and $(G^+)^{\Delta_2}$ have the same type AS or PA*

Proof.WLOG, assume G^+ is quasiprimitive on Δ_1.
G transitive on $V\Gamma$ and $G^+ \lhd G \implies G^+ \curvearrowright \Delta_1, \Delta_2$ are perm. isom.
Hence G^+ is quasiprimitive on Δ_2 of the same type on Δ_1.
By ref.[5,Thm 1.2], $(G^+)^{\Delta_1}$ is of type HA, AS, TW or PA.
For cases HA or TW, $\mathrm{soc}(G^+) \lhd G$ is regular on Δ_1 and Δ_2,
then Γ is a **G-normal bi-Cayley graph** on $\mathrm{soc}(G^+)$.
By ref.[6,Lem 3.2], Γ is at most $(G, 2)$-arc-transitive, contradicting $s \geq 3$. □

[5] Michael Giudici, Cai Li, and Cheryl Praeger. "Analysing finite locally s-arc transitive graphs". In: *Trans. AMS* 356 (Aug. 2003), pp. 291–317.

[6] Marston Conder et al. "Edge-transitive bi-Cayley graphs". In: *Journal of Combinatorial Theory, Series B* 145 (2020), pp. 264–306.

Lemma (4.4)

$\mathrm{soc}(G^+)$ *is nonsolvable and is the the unique minimal normal subgroup of* G*, and* $\mathrm{soc}(G^+)_\alpha \neq 1$.

Proof.

If $G^+ \curvearrowright \Delta_1$ is quasiprimitive.
By lemma 4.3, G^+ is of type AS or PA.
So $\mathrm{soc}(G^+) \underset{min}{\lhd} G^+$ and is nonsolvable and not semiregular.
Let $C = C_G(\mathrm{soc}(G^+))$. Then $C \lhd G$ and $\mathrm{soc}(G^+) \cap C = Z(\mathrm{soc}(G^+)) = 1$.
If $C \neq 1$, as $C \cap G^+ = 1$ and $G = G^+.2$, we derive that $C \cong \mathbb{Z}_2$.
Since $|V\Gamma| > 4$, C has more than 2 orbits, contradicting G bi-quasiprimitive.
Hence $C = 1$ and $\mathrm{soc}(G^+)$ is also unique in G.

Now assume G^+ is not quasiprimitive on both Δ_1 and Δ_2.
Then $\exists M \underset{min}{\lhd} G^+$ acting intransitively on Δ_1 hence has > 2 orbits on $V\Gamma$.
Thus $M \not\lhd G$ and $M^g \neq M$. (Recall $G = \langle G^+, g\rangle$)
Since $g^2 \in G^+$, one has $MM^g = M \times M^g \leq G^+$ is normal in G.
By bi-quasiprimitivity of G, $M \times M^g$ is transitive on Δ_1 and Δ_2.
By ref.[7, Thm 1.5], either

a. $M \times M^g$ is regular on Δ_1 and Δ_2; or
b. $\mathrm{soc}(G^+) = M \times M^g$ is nonsolvable and not semiregular, and is the unique minimal normal subgroup of G.

For case (b), done.
For case (a), Γ is a G-normal bi-Cayley graph on $M \times M^g$, again by ref.[8, Lem 3.2], is at most $(G, 2)$-arc-transitive. A contradiction. □

[7] Cai Heng Li et al. "Finite locally-quasiprimitive graphs". In: *Discrete Mathematics* 246.1 (2002), pp. 197–218.

[8] Conder et al., "Edge-transitive bi-Cayley graphs".

Structure of $\mathrm{soc}(G^+)$

According to Lemma 4.4, we may set that

$$N := \mathrm{soc}(G^+) = T_1 \times T_2 \times \cdots \times T_d \cong T^d \overset{unique}{\underset{min}{\lhd}} G.$$

Then N is transitive on Δ_1 and Δ_2 since G is bi-quasiprimitive.

Consider $G^+ \curvearrowright \{T_1, \cdots, T_d\}$ by conjugation with kernel K.
Then $N \lhd K \leq \mathrm{Aut}(T_1) \times \cdots \times \mathrm{Aut}(T_d)$. And $T_j \lhd K$, $1 \leq j \leq d$.
Let $M_j = C_K(T_j)$. Then

$$\cdots \times T_{j-1} \times 1 \times T_{j+1} \times \cdots \leq M_j \leq \cdots \times \mathrm{Aut}(T_{j-1}) \times 1 \times \mathrm{Aut}(T_{j+1}) \times \cdots$$

$M_j \lhd K$ and $K/M_j \lesssim \mathrm{Aut}(T_j)$ is almost simple with socle T_j.

Recall H is a vertex-transitive solvable subgroup of G.
Let $H^+ = H \cap G^+$ and H_j be the projection of $H^+ \cap N$ on T_j, $1 \leq j \leq d$.
Then $|H^+ \cap N|$ divides $|H_1||H_2| \cdots |H_d|$.

Exclude PA type

Lemma (4.5)

Suppose $d \geq 2$. Then $\exists j \in \{1, 2, ..., d\}$ s.t. M_j is transitive on Δ_1 or Δ_2; in particular, if G^+ is quasiprimitive on Δ_1 or Δ_2, then it is not of type PA.

Proof.

Assume G^+ is quasiprimitive of type PA on Δ_1 or Δ_2.
Then $\exists \mathcal{B} = \Omega^d$ a maximal block system s.t. $G^+ \curvearrowright \mathcal{B}$ primitive by PA.
For $(w_1, w_2, ..., w_d) \in \Omega^d$, $(x_1, x_2, ..., x_d) \in M_j$, we have

$$(w_1, w_2, ..., w_d)^{(x_1, x_2, \ldots, x_d)} = (w_1^{x_1}, w_2^{x_2}, ..., w_d^{x_d}).$$

From previous analysis, $x_j = 1$. Then transitivity of M_j implies $|\Omega| = 1$.
Need to prove the transitivity of some M_j.

By lemma 4.4, $N \lhd G$ and $N_\alpha \neq 1$, $\forall \alpha \in V\Gamma$. By the following lemma,

Lemma (2.5)

Let Γ be a connected (G, s)-transitive graph with $s \geq 3$, and suppose $N \lhd G$ and not semiregular. Then one of the following holds:

1. *both $G_\alpha^{\Gamma(\alpha)}$ and $N_\alpha^{\Gamma(\alpha)}$ are almost simple 2-transitive;*
2. *$G_\alpha^{\Gamma(\alpha)}$ is affine and $N_\alpha^{\Gamma(\alpha)}$ is primitive.*

we have Γ is N-locally-primitive, hence K-locally-primitive since $N \leq K$.

Lemma (2.4)

Let Γ be a connected bipartite G-locally-primitive graph with G-orbits Δ_1 and Δ_2 on $V\Gamma$ and each $|\Delta_i| > 1$. Suppose $\exists N \lhd G$ s.t. N is intransitive on Δ_1 and Δ_2. Then[a, Lem 5.1]

1. *Γ is a normal cover of Γ_N.*
2. *N is semiregular on $V\Gamma$, $G^{V\Gamma_N} \cong G/N$ and $G_\alpha \cong (G/N)_v$ for $\alpha \in V\Gamma$ and $v \in V\Gamma_N$.*
3. *Γ_N is G/N-locally-primitive. Futhermore, if Γ is locally (G, s)-arc-transitive with $s \geq 2$, then Γ_N is locally $(G/N, s)$-arc-transitive.*

[a] Giudici, Li, and Praeger, "Analysing finite locally s-arc transitive graphs".

If $M_j \lhd K$ is intransitive on both Δ_1 and Δ_2 for each $j = 1, ..., d$, then M_j is semiregular hence $M_j \cap K_\alpha = 1$, and $K_\alpha \cong (K/M_j)_v \lesssim \mathrm{Aut}(T)$. Furthermore, K_α is a diagonal subgroup, i.e. Why?

$$K_\alpha = \{(a, a, ..., a) | a \in P\}, \ P \leq \mathrm{Aut}(T).$$

$$N_\alpha = N \cap K_\alpha = (T_1 \times \cdots \times T_d) \cap K_\alpha = \{(a, a, ..., a) | a \in P \cap T\} \lesssim T,$$

thus $|T|^{d-1}$ divides $|N : N_\alpha| = |\Delta_1|$,

$H^+ \curvearrowright \Delta_1$ transitively $\implies$ $|T|^{d-1}$ divides $|H^+| = |H^+ \cap K||H^+K/K|$.
$N \curvearrowright \Delta_1$ transitively $\implies$

$$K/N = K_\alpha N/N \cong K_\alpha/N_\alpha \cong P/(P \cap T) \cong PT/T \lesssim \mathrm{Out}(T).$$

$\therefore$ $|H^+ \cap K|/|H^+ \cap N|$ divides $|\mathrm{Out}(T)|$.
$\because$ $|H^+ \cap N|$ divides $|H_1 \times \cdots \times H_d|$,
Therefore,

$$|T|^{d-1} \text{ divides } |H_1 \times \cdots \times H_d||\mathrm{Out}(T)||H^+K/K|.$$

Case: $H^+ \not\leq K$**:**

Case: $H^+ \leq K$**:**

Quasiprimitive $G^+ \curvearrowright \Delta_1$ and Δ_2

Lemma (4.6)

G^+ is quasiprimitive on Δ_1 and Δ_2.

Proof. Assume $G^+ \curvearrowright \Delta_1$ is NOT quasiprimitive.

Case (a): $N_\alpha \cong T$.

Case (b): $1 \neq N_\alpha < T$.

Case (b.1): L is nonsolvable.

Case (b.1): L is solvable.

Summary

Recall hypothesis in the beginning of this section:

- Let Γ be a connected bipartite (G, s)-transitive graphs of val≥ 3, with bipartitions Δ_1 and Δ_2 and $s \geq 3$.
- Suppose G is bi-quasiprimitive on $V\Gamma$ and contains a vertex-transitive solvable subgroup H.
- Set $G^+ = G_{\Delta_1}$, the setwise stabilizer of G on Δ_1, and $H^+ = H \cap G^+$.

Proposition (4.7)

Under such hypothesis, either $\Gamma = \mathbf{K}_{n,n}$, or G is an almost simple group and G^+ is quasiprimitive on both Δ_1 and Δ_2.

Proof. If G^+ acts unfaithfully on Δ_1 or Δ_2, lemma 4.2 gives $\Gamma = \mathbf{K}_{n,n}$.
Faithful case:
Lemma 4.3 shows either G^+ is not quasiprimitive on Δ_1 or Δ_2, or quasiprimitive on both with same type AS or PA.
Lemma 4.6 exclude the former. Lemma 4.5 exclude PA in the latter.
Finally lemma 4.4 implies G is AS.

Outline

$\mathrm{soc}(G) = T \leq G \leq \mathrm{Aut}(T)$

By proposition 4.7, we may assume G is almost simple with socle T, and G^+ is quasiprimitive on Δ_1 and Δ_2.
Set $G = T.\mathcal{O}$ and $G^+ = T.\mathcal{O}'$, with $\mathcal{O}' < \mathcal{O} \leq \mathrm{Out}(T)$ and $|\mathcal{O} : \mathcal{O}'| = 2$.
T is transitive on $\Delta_i \implies G = G_\alpha T \implies G_\alpha / T_\alpha \cong G_\alpha T / T = G/T = \mathcal{O}$
By Frattini argument, $G = HG_\alpha$ and $G^+ = H^+ G_\alpha$.
Both factorization have a solvable factor H, H^+.

Lemma (5.1)

$T \neq \mathrm{PSL}_2(q)$ *with* q *a prime power.*

Proof. Check the classification of connected 2-arc-transitive graphs admitting a two-dimensional linear automorphism group, see ref.[9], in which the only 3-arc-transitive graph is the Petersen graph, a non-bipartite graph. □

[9] C. E. Praeger A. Hassani L. R. Nochefranca. "Two-arc transitive graphs admitting a two-dimensional projective linear group". In: 2.4 (1999), pp. 335–353.

Case: G_α solvable

Lemma (5.2)

If G_α solvable, then $T = \mathrm{PSL}_3(3)$ and $\Gamma = \mathcal{PH}(3, 3^3)$ is 4-transitive of valency 4.

Case: T alternating

Lemma (5.3)

T is not an alternating simple group.

Case: T sporadic

Lemma (5.4)

T is not a sporadic simple group.

Lemma (5.5)

If T is a classical simple group of Lie type, then $s \leq 4$, and one of the following is true.

1. $T = \mathrm{PSL}_5(2)$, *and* $\Gamma \cong \mathcal{GI}(5,2,2)$.
2. $T = \mathrm{PSU}_3(5)$, *and* $\Gamma \cong \mathrm{HS}_{50}^{(2)}$.
3. Γ *is of valency* $p^f + 1$, $\mathrm{PSL}_3(p^f).2 \leq G \leq \mathrm{Aut}(\mathrm{PSL}_3(p^f))$ *and* $\mathbb{Z}_p^{2f} : \mathrm{SL}_2(p^f) \lhd G_\alpha \leq P_1$ *or* P_2 *where* P_1 *and* P_2 *are the stabilizers of* T *on a 1-dimension subspace and a 2-dimension subspace, respectively.*

References

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