

# Notes on GTM251

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## 1 Introduction

### Pace:

Lesson 1: Chapter 1 (Overview)

Lesson 2: §2.1-§2.4 (Group action,  $A_n$ )

Lesson 3: §2.5-§2.7 (O’Nan-Scott, maximal subgroups of  $S_n$  and  $A_n$ , cover)

Lesson 4: §3.1-§3.3 ( $\mathrm{PSL}_n(q)$ )

Lesson 5: §3.4 (forms: bilinear, sesquilinear, quadratic)

Lesson 6: §3.5 ( $\mathrm{PSp}_{2m}(q)$ )

Lesson 7: §3.6 ( $\mathrm{PSU}_n(q)$ )

Lesson 8: §3.7 ( $\mathrm{P}\Omega_m(q)$ , odd  $q$ )

Lesson 9: §3.8 ( $\mathrm{P}\Omega_{2n}(q)$ , even  $q$ )

Lesson 10: §3.10 (maximal subgroups of classical groups)

### References:

Main: The finite simple groups - Wilson (GTM 251)

Perm.: Permutation Groups - J.D. Dixon, B. Mortimer (GTM 163)

Finite permutation groups - Wielandt

Class.: The Subgroup Structure of the Finite Classical Groups - Kleidman & Liebeck

The Maximal Subgroups of the Low-Dimensional Finite Classical Groups - J.N. Bray, et al.

[Notes] Classical Groups without Orthogonal (2021fall) - C.H. Li, P.C. Hua

More: (notes and papers to be referred)

## 1.1 History

Galois(1830s):  $A_n$ ,  $\text{PSL}_2(p)$ , realized the importance

Jordan-Hölder:  $1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_{n-1} \triangleleft G_n = G$ , where  $G_i/G_{i-1}$  is simple

Camille Jordan (1870):  $\text{PSL}_n(q)$

Sylow theorem (1872): the first tools for classifying finite simple groups

Mathieu(1860s):  $M_{11}$ ,  $M_{12}$ ,  $M_{22}$ ,  $M_{23}$ ,  $M_{24}$

L.E. Dickson(1901): classical groups, inspired by Lie algebras

Chevalley(1955): a uniform construction of  $\text{PSL}_{n+1}(q)$ ,  $\text{P}\Omega_{2n+1}(q)$ ,  $\text{PSp}_{2n}(q)$ ,  $\text{P}\Omega_{2n}^+(q)$

"twisting":  ${}^3D_4(q)$ ,  ${}^2E_6(q)$

Feit-Thompson(1963): odd order is soluble, hence nonab. FSG has an involution

1960s: proof of CSFG began

1970s: 20 sporadic simple groups dicovered

1980s: CSFG was "almost" complete

3 generations of proof of CSFG:

1. 1982 Gorenstein, abandon after vol 1, too long, bugs in quasithin case
2. 1992 Lyons, Solomon, vol 1-6 done, bug fixed, vol 7? also too long
3. Aschbacher, et al., find some geometric characters to simplify the proof, fusion system?

## 1.2 CFSG

Every finite simple group is isomorphic to one of the followings:

- (i) a cyclic group  $C_p$  of prime order  $p$ ;
- (ii) an alternating group  $A_n$  for  $n \geq 5$ ;
- (iii) a classical group:
  - linear:  $\text{PSL}_n(q)$ ,  $n \geq 2$ , except  $\text{PSL}_2(2)$  and  $\text{PSL}_2(3)$ ;
  - unitary:  $\text{PSU}_n(q)$ ,  $n \geq 3$ , except  $\text{PSU}_3(2)$ ;
  - symplectic:  $\text{PSp}_{2n}(q)$ ,  $n \geq 2$ , except  $\text{PSp}_4(2)$ ;
  - orthogonal:  $\text{P}\Omega_{2n+1}(q)$ ,  $n \geq 3$ ,  $q$  odd;  $\text{P}\Omega_{2n}^+(q)$ ,  $\text{P}\Omega_{2n}^-(q)$ ,  $n \geq 4$ ;

where  $q$  is a power  $p^a$  of a prime  $p$ ;

- (iv) an exceptional group of Lie type:

$$G_2(q), q \geq 3; F_4(q); E_6(q); {}^2E_6(q); {}^3D_4(q); E_7(q); E_8(q)$$

with  $q$  a prime power, or

$${}^2B_2(2^{2n+1}), {}^2G_2(3^{2n+1}), {}^2F_4(2^{2n+1}), n \geq 1;$$

or the Tits group  ${}^2F_4(2)'$ ;

- (v) one of 26 sporadic simple groups:

- the five Mathieu groups  $M_{11}, M_{12}, M_{22}, M_{23}, M_{24}$ ;
- the seven Leech lattice groups  $\text{Co}_1, \text{Co}_2, \text{Co}_3, \text{McL}, \text{HS}, \text{Suz}, \text{J}_2$ ;
- the three Fischer groups  $\text{Fi}_{22}, \text{Fi}_{23}, \text{Fi}'_{24}$ ;
- the five Monstrous groups  $\mathbb{M}, \mathbb{B}, \text{Th}, \text{HN}, \text{He}$ ;
- the six pariahs  $\text{J}_1, \text{J}_3, \text{J}_4, \text{O}'\text{N}, \text{Ly}, \text{Ru}$ .

Conversely, every group in this list is simple, and the only repetitions in this list are:

$$\begin{aligned} \text{PSL}_2(4) &\cong \text{PSL}_2(5) \cong A_5; \\ \text{PSL}_2(7) &\cong \text{PSL}_3(2); \\ \text{PSL}_2(9) &\cong A_6; \\ \text{PSL}_4(2) &\cong A_8; \\ \text{PSU}_4(2) &\cong \text{PSp}_4(3). \end{aligned}$$

introduction, construction, orders, simplicity, **action(reveal subgroup structure)**

### 1.3 After CFSG

#### 1.3.1 Permutation group theory

Classify

- multiply-transitive groups
- 2-transitive groups
- primitive permutation groups (O’Nan-Scott Thm): reduce to AS case

#### 1.3.2 Maximal subgroups of simple groups

$A_n$  : O’Nan-Scott, Liebeck-Praeger-Saxl

(The symmetric difference set of AS subgroups and maximal subgroups of  $A_n$  is listed out, while listing their intersection is impossible.)

Classical: began with Aschbacher, 1984, see Kleidman-Liebeck and Low-dimension.

Exceptional: Done recently by David Craven, see arXiv

Sporadic: Done. See a survey by Wilson and recent work on arXiv for the Monster.

## 2 The Alternating Groups

### 2.1 Introduction

$\text{Aut}(A_n) \cong S_n$  for  $n \geq 7$  but for  $n = 6$  there is an exceptional outer automorphism of  $S_6$ .  
subgroup structure (O’Nan-Scott Thm)