The Classical Groups

1 Introduction

'Classical' simple groups: linear groups, unitary groups, symplectic groups, orthogonal groups.

Mainly obtained by taking $G'/Z(G')$ from suitible matrix groups G.

Definition	Simplicity	Subgroups	Automorphisms & Covering groups	Isomorphisms
$PSL_n(q)$	Iwasawa's lemma	geometry	(briefly mentioned)	projective spaces

Symplectic groups: easy to understand, orders, simplicity, subgroups, covering groups, automorphisms, generic isomorphism $\mathrm{Sp}_2(q) \cong \mathrm{SL}_2(q)$, exceptional isomorphism $\mathrm{Sp}_4(2) \cong S_6$.

Unitary groups: similar to symplectic groups.

Orthogonal groups:

- fundamental differences between the cases of $\mathrm{char}F = 2$ or odd
- subquotient is not usually simple
- to get usually simple groups, using spinor norm for odd char (see Clifford algebras and spin groups), and quasideterminant for char 2
- generic isomorphisms $\mathrm{P}\Omega_6^+(q) \cong \mathrm{PSL}_4(q)$, $\mathrm{P}\Omega_6^-(q) \cong \mathrm{PSU}_4(q)$, $\mathrm{P}\Omega_5(q) \cong \mathrm{PSp}_4(q)$ all derive from the Klein correspondence

A simple version of Aschbacher-Dynkin theorem is proved, relying heavily on representation theory.

More explicit versions for individual classes of groups see Kleidman and Liebeck's book.

Some exceptional behavior of small classical groups is related to exceptional Weyl groups.

2 Finite fields

TODO: ref to Hua and Peter's notes

Definition 2.1. *field* $+, \cdot$

Lemma 2.2. *all non-zero elts have the same + order.*

Proof. exercise □

Definition 2.3. *characteristic, prime subfield*

Lemma 2.4. $|F| = p^d$

Proof. F is a vector space over F_0. □

Lemma 2.5. $F^\times$ *is cyclic.*

Proposition 2.6. *$\forall$ prime p and $d \in \mathbb{N}_+$, $\exists_1 F$ of order p^d, denoted as $\mathbb{F}_{p^d}$.*

Lemma 2.7. $\mathrm{Aut}(\mathbb{F}_{p^d}) \cong \mathbb{Z}_d$.

3 General linear groups