Covering Groups

1 Schur Multiplier

A_n as quotient group of $2.A_n$.

Let $+\pi$ and $-\pi$ be the two preimages of $\pi \in A_n$ under the natural quotient map. But there is no canonical choice of which element gets which sign.

Let $+1$ be the identity in $2.A_n$. For each $\pi \in A_n$, we define $+\pi$ to be the element which multipilied together with $+1$ gives itself, and $-\pi$ for the other one.

Definition 1.1. *$\tilde{G}$ is a covering group of G if $Z(\tilde{G}) \leq \tilde{G}'$ and $\tilde{G}/Z(\tilde{G}) \cong G$.*

If $|Z(\tilde{G})| = 2, 3$, then the covering group is called double, triple cover.

Theorem 1.2 (Schur)**.** *Every finite perfect group G has a unique maximal covering group $\tilde{G}$, with the property that every other covering group is a quotient of $\tilde{G}$. We call $\tilde{G}$ the **universal covering group** of G and $Z(\tilde{G})$ the **Schur multiplier** of G, denoted as $M(G)$.*

Example 1.3 (non-perfect)**.** *$\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle \alpha \rangle \times \langle \beta \rangle$ has four maximal covering groups: one Q_8 and three D_8.*

$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$, $Z(Q_8) = \{\pm 1\}$, $Q_8/Z(Q_8) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

$D_8 = \langle a \rangle : \langle b \rangle$, $Z(D_8) = \langle a^2 \rangle$, $D_8/Z(D_8) = \langle Z(D_8)a \rangle \times \langle Z(D_8)ab \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Only one of 3 different subgroups of $D_8/Z(D_8)$ could map to the normal subgroup $\langle a \rangle$ of D_8. Thus there are 3 different covering groups of $\mathbb{Z}_2 \times \mathbb{Z}_2$. However, Q_8 has more symmetric structure coinciding with $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Example 1.4. *The alternating groups A_n $(n > 4)$ only have double covers, except for A_6 and A_7 both with the same Schur multiplier $\mathbb{Z}_6$.*

2 Double Covers of A_n and S_n

Now we define $2.S_n$.

Firstly, let G be a set of order $2n!$, with a map φ onto S_n such that each $\pi \in S_n$ has exactly two preimages denoted as $+\pi$ and $-\pi$.

Intuitively, we should define the multiplication of G as $+\pi + \sigma = +(\pi\sigma)$ and $+\pi - \sigma = -(\pi\sigma)$.

WLOG, we denote $+(1\ 2)$ as $[1\ 2]$ and $-(1\ 2)$ as $-[1\ 2]$. Then for each transposition $\pi \in S_n$, taking $(+\pi)^{-1}$ to be a preimage of π, define the products (of 3 elements in G) $[i\ j]^{+\pi}$ and $[i\ j]^{-\pi}$ to be a same preimage of $(i^\pi\ j^\pi)$, we denote it as $-[i^\pi\ j^\pi]$. That is $[i\ j]^{\pm\pi} = -[i^\pi\ j^\pi]$.

Then define the preimage of $(a_i, a_{i+1}, \cdots, a_j)$ by

$$[a_i\ a_{i+1}\ \cdots\ a_j] = [a_i\ a_{i+1}][a_i\ a_{i+2}] \cdots [a_i\ a_j].$$

Finally by multiplying together disjoint cycles in the usual way, we obtain all elements.

The multiplication defined above is well-defined. That is, if we compute the same product in two different ways, we get the same result. A proof using construction of double cover of orthogonal group is given in Section 3.9.

Note that, by the rule $[i\ j]^{\pm\pi} = -[i^\pi\ j^\pi]$ to define $[i\ j]$'s, all the elements $\pm[i\ j]$ are conjugate. Thus they are all square to 1 or -1 simotaneously. If $(\pm[i\ j])^2 = 1$, then the elements like $\pm[1\ 2][3\ 4]$ square to -1. And vice versa.

Therefore, we obtain two distinct double cover of S_n. We denote $2.S_n^+$ (actually $\mathbb{Z}_2 \times S_n$) as the one with $|[i\ j]| = 2$ (where $(i\ j)$ is identified with a coset $Z(2.S_n)a$ with a of order 2), and $2.S_n^-$ as the other one with $|[i\ j]| = 4$ ($(i\ j)$ a coset of an element of order 4). However, both of them has the same subgroup $2.A_n \cong \mathbb{Z}_2 \times A_n$ of index 2, which is the unique double cover of A_n.

3 Triple Covers of A_6 and A_7