Notes on GTM251

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References:

Main: The finite simple groups - Wilson (GTM 251)

Perm.: Permutation Groups - J.D. Dixon, B. Mortimer (GTM 163)

Finite permutation groups - Wielandt

Class.: The Subgroup Structure of the Finite Classical Groups - Kleidman & Liebeck

The Maximal Subgroups of the Low-Dimensional Finite Classical Groups - J.N. Bray, et al.

[Notes] Classical Groups without Orthogonal (2021fall) - C.H. Li, P.C. Hua

More: (notes and papers to be referred)

1.1 History

Galois(1830s): A_n, $\mathrm{PSL}_2(p)$, realized the importance

Jordan-Hölder: $1 = G_0 \lhd G_1 \lhd \cdots \lhd G_{n-1} \lhd G_n = G$, where G_i/G_{i-1} is simple

Camille Jordan (1870): $\mathrm{PSL}_n(q)$

Sylow theorem (1872): the first tools for classifying finite simple groups

Mathieu(1860s): M_{11}, M_{12}, M_{22}, M_{23}, M_{24}

L.E. Dickson(1901): classical groups, inspired by Lie algebras

Chevalley(1955): a uniform construction of $\mathrm{PSL}_{n+1}(q)$, $\mathrm{P}\Omega_{2n+1}(q)$, $\mathrm{PSp}_{2n}(q)$, $\mathrm{P}\Omega^{+}_{2n}(q)$

"twisting": ${}^3D_4(q)$, ${}^2E_6(q)$

Feit-Thompson(1963): odd order is soluble, hence nonab. FSG has an involution

1960s: proof of CSFG began

1970s: 20 sporadic simple groups dicovered

1980s: CSFG was "almost" complete

3 generations of proof of CSFG:

1. 1982 Gorenstein, abandon after vol 1, too long, bugs in quasithin case
2. 1992 Lyons, Solomon, vol 1-6 done, bug fixed, vol 7? also too long
3. Aschbacher, et al., find some geometric characters to simplify the proof, fusion system?

1.2 CFSG

Every finite simple group is isomorphic to one of the followings:

(i) a cyclic group C_p of prime order p;

(ii) an alternating group A_n for $n \geq 5$;

(iii) a classical group:

- linear: $\mathrm{PSL}_n(q)$, $n \geq 2$, except $\mathrm{PSL}_2(2)$ and $\mathrm{PSL}_2(3)$;
- unitary: $\mathrm{PSU}_n(q)$, $n \geq 3$, except $\mathrm{PSU}_3(2)$;
- symplectic: $\mathrm{PSp}_{2n}(q)$, $n \geq 2$, except $\mathrm{PSp}_4(2)$;
- orthogonal: $\mathrm{P}\Omega_{2n+1}(q)$, $n \geq 3$, q odd; $\mathrm{P}\Omega^+_{2n}(q)$, $\mathrm{P}\Omega^-_{2n}(q)$, $n \geq 4$;

where q is a power p^a of a prime p;

(iv) an exceptional group of Lie type:

$$G_2(q), q \geq 3;\ F_4(q);\ E_6(q);\ {}^2E_6(q);\ {}^3D_4(q);\ E_7(q);\ E_8(q)$$

with q a prime power, or

$${}^2B_2(2^{2n+1}),\ {}^2G_2(3^{2n+1},\ {}^2F_4(2^{2n+1}),\ n \geq 1;$$

or the Tits group ${}^2F_4(2)'$;

(v) one of 26 sporadic simple groups:

- the five Mathieu groups M_{11}, M_{12}, M_{22}, M_{23}, M_{24};
- the seven Leech lattice groups Co_1, Co_2, Co_3, McL, HS, Suz, J_2;
- the three Fischer groups Fi_{22} , Fi_{23}, Fi'_{24} ;
- the five Monstrous groups $\mathbb{M}$, $\mathbb{B}$, Th, HN, He;
- the six pariahs J_1, J_3, J_4, O'N, Ly, Ru.

Conversely, every group in this list is simple, and the only repetitions in this list are:

$$\begin{aligned}\mathrm{PSL}_2(4) \cong & \mathrm{PSL}_2(5) \cong A_5;\\ & \mathrm{PSL}_2(7) \cong \mathrm{PSL}_3(2);\\ & \mathrm{PSL}_2(9) \cong A_6;\\ & \mathrm{PSL}_4(2) \cong A_8;\\ & \mathrm{PSU}_4(2) \cong \mathrm{PSp}_4(3).\end{aligned}$$

introduction, construction, orders, simplicity, **action(reveal subgroup structure)**

1.3 After CFSG

1.3.1 Permutation group theory

Classify

- multiply-transitive groups
- 2-transitive groups
- primitive permutation groups (O'Nan-Scott Thm): reduce to AS case

1.3.2 Maximal subgroups of simple groups

A_n : O'Nan-Scott, Liebeck-Praeger-Saxl
(The symmetric difference set of AS subgroups and maximal subgroups of A_n is listed out, while listing their intersection is impossible.)

Classical: began with Aschbacher,1984, see Kleidman-Liebeck and Low-dimension.

Exceptional: Done recently by David Craven, see arXiv

Sporadic: Done. See a survey by Wilson and recent work on arXiv for the Monster.

2 The Alternating Groups

2.1 The O'Nan-Scott Theorem

2.1.1 Some Lemmas

2.1.2 The proof of the O'Nan-Scott Theorem

Last week we introduced some lemmas and proved part of the O'Nan-Scott Theorem. This week we will finish the proof of the O'Nan-Scott Theorem.

Notation: Let H be a subgroup of S_n not containing A_n, N be a minimal normal subgroup of H, and K be the stabilizer in H of a point.

H intransitive $\implies$ case (i).

H transitive imprimitive $\implies$ case (ii).

Now we assume H primitive. And hence the discussion zoom into $\mathrm{soc}(H)$.

$\exists N$ abelian $\implies$ case (iv)affine.

Additionally we assume $\forall N$ nonabelian.

If H has more than one minimal normal subgroups $N_1 \neq N_2$.

It can be shown that $\exists x \in S_n$ conjugates N_1 to N_2. specify x

By corollary 2.11, x also conjugates $N_2 = C_H(N_1)$ to $N_1 = C_H(N_2)$. (Why?)

Hence $H < \langle H, x\rangle$, which has a unique minimal normal subgroup $N_1 \times N_2$.

Additionally we assume H has a unique minimal normal subgroup N, which is nonabelian.

N simple $\implies C_H(N) = 1 \implies H \overset{\text{conj.}}{\curvearrowright} N$ faithfully $\implies$ case (vi)AS.

$N = T^m = T_1 \times \cdots \times T_m$ with $m > 1 \implies H \overset{\text{conj.}}{\curvearrowright} \{T_1, \cdots, T_m\}$ transitively, and K as well.

Let $K_i := p_i(K \cap N) \leq T_i$ the projection of K onto T_i. Then $K \cap N \leq K_1 \times \cdots \times K_m$.

Case $K_i \neq T_i$ for some i:

Now $K \cap N \leq K_1 \times \cdots \times K_m < N$.

<u>Claim:</u> K normalizes $K_1 \times \cdots \times K_m$.

Since $K \cap N \lhd K$, $\forall k \in K$, $\forall x \in K \cap N$,

we have $x = p_1(x) \cdots p_m(x)$, and $p_1(x)^k \cdots p_m(x)^k = x^k = p_1(x^k) \cdots p_m(x^k) \in K \cap N$.

Then $p_i(x)^k = p_j(x^k)$ whenever $T_i^k = T_j$. (In direct product, equal iff. all coordinates equal.)

$\forall y \in K_1 \times \cdots \times K_m$, $\exists x_1, \cdots, x_m \in K \cap N$ s.t. $y = p_1(x_1) \cdots p_m(x_m)$.

Then $y^k = p_1(x_1)^k \cdots p_m(x_m)^k = p_1(x_{l_1}^k) \cdots p_m(x_{l_m}^k) \in K_1 \times \cdots \times K_m$, where $T_i = T_{l_i}^k$.

By corollary 2.15, $K_1 \times \cdots \times K_m = K \cap N$ and K permutes K_i's transitively. Let $k := |T_i : K_i|$.

Then $H = (T_1 \times \cdots \times T_m) \rtimes K \leq S_k \wr S_m \curvearrowright [T_1 : K_1] \times \cdots \times [T_m : K_m] \implies$ case (iii)PA.

Case $K_i = T_i$ for all i:

Support of $(t_1, \cdots, t_m) \in N$ is defined as $\{i \mid t_i \neq 1\}$.

$\Omega_1 :=$ a non-empty min. supp. of an elt in $K \cap N$. $\implies \Omega_1$ a block of $K, H \curvearrowright [m]$.

1 and all elts in $K \cap N$ with support Ω_1 (i.e. $t_i \neq 1$ and $t_j = 1 \; \forall i \in \Omega_1, \forall j \notin \Omega_1$)

forms a normal subgp of $K \cap N$, which maps onto a normal subgp of hence T_i itself $\forall i \in \Omega_1$.

$\Omega_1 \cap \Omega_2 \neq \emptyset \implies \exists x, y$ s.t. $[x, y] \neq 1$ has support contained in $\Omega_1 \cap \Omega_2$, that is Ω_1

$|\Omega_1| = 1 \implies N \leq K$, a contradiction.

$|\Omega_1| = m \implies K \cap N = \{(t, \cdots, t) \mid t \in T\}$ WLOG. $N \curvearrowright [N : K \cap N] \implies$ case (v)diagonal.

$\forall i, \forall x, y \in K \cap N$, $p_i(x) = p_i(y) \implies p_i(xy^{-1}) = 1 \implies xy^{-1} = 1$ i.e. $p_i|_{K \cap N}$ inj.

$|\Omega_1| = k \neq 1, m \implies N = \left(\bigtimes_{i \in \Omega_1} T_i \right)^l \cong T^{kl}$, $N \cap K = \left(\operatorname{diag} \left(\bigtimes_{i \in \Omega_1} T_i \right) \right)^l \cong T^l$.

The action of each $\bigtimes_{i \in \Omega_1} T_i$ is diagonal of degree $r = |T|^{k-1}$. $H \leq S_r \wr S_l \curvearrowright [r]^l \implies$ case (iii)PA.

2.2 Covering Groups

2.2.1 Schur Multiplier

2.2.2 Double Covers of A_n and S_n

2.2.3 Triple Covers of A_6 and A_7

2.3 Coxter Groups