General linear groups

Lemma 0.1 (Iwasawa)**.** *If finite group G satisfies the following conditions, then G is simple.*

i. $G' = G$;

ii. G *is primitive on some set* Ω;

iii. $\exists A \trianglelefteq G_\alpha$ *such that* A *is solvable;*

iv. $G = A^G$.

i.e. A perfect primitive group G, being the normal closure of an abelian normal subgroup A of its point stabilizer, is simple.

Proof. Suppose that $1 \neq N \trianglelefteq G$. Then, by primitivity, N is transitive on Ω and hence $G = G_\alpha N$. For any $g \in G$, $g = hn$ for some $h \in G_\alpha$ and $n \in N$.

Then $a^g = a^{hn} = a^n$, $\forall a \in A$, since $A \trianglelefteq G_\alpha$. Moreover, $a^n = a(n^{-1})^a n \in AN$ since $N \trianglelefteq G$. Thus $G = A^G = AN$.

Now, $G/N = AN/N = A/(A \cap N)$ is solvable. Meanwhile, $(G/N)' = G'N/N = GN/N = G/N$. Thus $G/N = 1$ and $G = N$, G is simple. $\square$