The O'Nan-Scott Theorem

1 Some Lemmas

2 The proof of the O'Nan-Scott Theorem

Last week we introduced some lemmas and proved part of the O'Nan-Scott Theorem. This week we will finish the proof of the O'Nan-Scott Theorem.

Notation: Let H be a subgroup of S_n not containing A_n, N be a minimal normal subgroup of H, and K be the stabilizer in H of a point.

H intransitive $\implies$ case (i).

H transitive imprimitive $\implies$ case (ii).

Now we assume H primitive. And hence the discussion zooms into $\mathrm{soc}(H)$.

$\exists N$ abelian $\implies$ case (iv)affine.

Additionally we assume $\forall N$ nonabelian.

If H has more than one minimal normal subgroups $N_1 \neq N_2$.

It can be shown that $\exists x \in S_n$ conjugates N_1 to N_2. specify x

By corollary 2.11, x also conjugates $N_2 = C_H(N_1)$ to $N_1 = C_H(N_2)$. (Why?)

Hence $H < \langle H, x \rangle$, which has a unique minimal normal subgroup $N_1 \times N_2$.

Additionally we assume H has a unique minimal normal subgroup N, which is nonabelian.

N simple $\implies C_H(N) = 1 \implies H \overset{\text{conj.}}{\curvearrowright} N$ faithfully $\implies$ case (vi)AS.

$N = T^m = T_1 \times \cdots \times T_m$ with $m > 1 \implies H \overset{\text{conj.}}{\curvearrowright} \{T_1, \cdots, T_m\}$ transitively, and K as well.

Let $K_i := p_i(K \cap N) \leq T_i$ the projection of K onto T_i. Then $K \cap N \leq K_1 \times \cdots \times K_m$.

We divide the discussion into 2 cases. Before that, we claim the following fact.

<u>Claim:</u> K normalizes $K_1 \times \cdots \times K_m$.

Proof. Since $K \cap N \lhd K$, $\forall k \in K$, $\forall x \in K \cap N$,

we have $x = p_1(x) \cdots p_m(x)$, and $p_1(x)^k \cdots p_m(x)^k = x^k = p_1(x^k) \cdots p_m(x^k) \in K \cap N$.

Then $p_i(x)^k = p_j(x^k)$ whenever $T_i^k = T_j$. (In direct product, equal iff. all coordinates equal.)

$\forall y \in K_1 \times \cdots \times K_m$, $\exists x_1, \cdots, x_m \in K \cap N$ s.t. $y = p_1(x_1) \cdots p_m(x_m)$.

Then $y^k = p_1(x_1)^k \cdots p_m(x_m)^k = p_1(x_{l_1}^k) \cdots p_m(x_{l_m}^k) \in K_1 \times \cdots \times K_m$, where $T_i = T_{l_i}^k$. $\square$

Case $K_i \neq T_i$ for some i:

Now $K \cap N \leq K_1 \times \cdots \times K_m < N$.

By corollary 2.15, $K_1 \times \cdots \times K_m = K \cap N$ and K permutes K_i's transitively. Let $k := |T_i : K_i|$.

Then $H = (T_1 \times \cdots \times T_m) \rtimes K \leq S_k \wr S_m \curvearrowright [T_1 : K_1] \times \cdots \times [T_m : K_m] \implies$ case (iii)PA.

Case $K_i = T_i$ for all i:

Support of $(t_1, \cdots, t_m) \in N$ is defined as $\mathrm{supp}(t) := \{i \mid t_i \neq 1\}$.

$\Omega_1 :=$ a non-empty min.(with set inclusion) supp. of an elt in $K \cap N$.

Claim: Ω_1 a block of $K, H \curvearrowright [m]$ which is induced by $K, H \curvearrowright \{T_1, \cdots, T_m\}$.

Proof. All elts in $K \cap N$ with support Ω_1 (i.e. $t_i \neq 1$ and $t_j = 1$ $\forall i \in \Omega_1, \forall j \notin \Omega_1$) together with 1 forms a normal subgroup of $K \cap N$, which maps onto a normal subgroup of hence T_i itself $\forall i \in \Omega_1$. $\forall g \in K$, if $\Omega_1 \cap \Omega_1^g \neq \emptyset, \Omega_1$, then $\exists x \in K \cap N$ s.t. $\mathrm{supp}(x) = \Omega_1$ and $\mathrm{supp}(x^g) = \Omega_1^g$. Now $[x, x^g] \neq 1$ and $\mathrm{supp}([x, x^g]) \subset \Omega_1 \cap \Omega_1^g$, contradicting to the minimality of Ω_1. □

$|\Omega_1| = 1 \implies N \leq K$, a contradiction.

$|\Omega_1| = m \implies K \cap N = \{(t, \cdots, t) \mid t \in T\}$ WLOG. $N \curvearrowright [N : K \cap N] \implies$ case (v)diagonal.

$\forall i, \forall x, y \in K \cap N$, $p_i(x) = p_i(y) \implies p_i(xy^{-1}) = 1 \implies \mathrm{supp}(xy^{-1}) = \emptyset \implies xy^{-1} = 1$ i.e. $p_i|_{K \cap N}$ is injective. Since $K_i = T_i$, $p_i|_{K \cap N}$ is surjective hence bijective. Thus $K \cap N$ is now a full diagonal subgroup of N. Then the coset action $N \curvearrowright [N : K \cap N]$ is of diagonal type. Now identify each $\alpha^g \in \Omega$ with the corresponding coset $(H_\alpha \cap N)g$. The stabilizer $N_{\alpha^g} = H_{\alpha^g} \cap N = (H_\alpha \cap N)^g$ is also the stabilizer of the coset $(H_\alpha \cap N)g$.

Remark. Actually, $K \cap N$ is a full diagonal subgroup of $N \implies K \cap N = \{(t^{\varphi_1}, \cdots, t^{\varphi_m}) \mid t \in T\}$ where φ_i is an isomorphism from T to T_i. For each $\beta \in \Omega$, we have an $H_\beta \cap N$ and can determine ψ_i's basing on φ_i's, since the $H_\beta \cap N$'s are conjugate to $H_\alpha \cap N$ by the transitive group N. In detail, suppose $\alpha^g = \beta$ for some $g \in N$ and $H_\alpha \cap N = \{(t^{\varphi_1}, \cdots, t^{\varphi_m}) \mid t \in T\}$ and $H_\beta \cap N = \{(t^{\psi_1}, \cdots, t^{\psi_m}) \mid t \in T\}$ where ψ_i, φ_i are isomorphisms from T to T_i. Then $H_\alpha \cap N$ is conjugate to $H_\beta \cap N$ by g. This means we can take $(\varphi_1^{-1}\psi_1, \cdots, \varphi_m^{-1}\psi_m) = \tilde{g} \in \mathrm{Inn}(N)$ on $H_\alpha \cap N$. Thus once $(\varphi_1, \cdots, \varphi_m)$ is given, we could let $\psi : T \to H_\alpha \cap N \to H_\beta \cap N$ be $(\psi_1, \cdots, \psi_m) = (\varphi_1, \cdots, \varphi_m)\tilde{g} = (\varphi_1\tilde{g_1}, \cdots, \varphi_m\tilde{g_m})$.

$|\Omega_1| = k \neq 1, m$:

Suppose Ω_1 is in a block system $\{\Omega_1, \cdots, \Omega_l\}$ of K on $[m]$. Let $N_j = \underset{i \in \Omega_j}{\times} T_i$ for $j = 1, \cdots, l$. Then $N = N_1 \times \cdots \times N_l \cong T^{kl}$. For each N_j, $N_j \cap K$ is a diagonal subgroup of N_j.

$$\implies N = \left(\underset{i \in \Omega_1}{\times} T_i \right)^l \cong T^{kl},\ N \cap K = \left(\mathrm{diag} \left(\underset{i \in \Omega_1}{\times} T_i \right) \right)^l \cong T^l.$$

The action of each $\underset{i \in \Omega_1}{\times} T_i$ is diagonal of degree $r = |T|^{k-1}$. $H \leq S_r \wr S_l \curvearrowright [r]^l \implies$ case (iii)PA.