

# Covering Groups

## 1 Schur Multiplier

$A_n$  as quotient group of  $2.A_n$ .

Let  $+\pi$  and  $-\pi$  be the two preimages of  $\pi \in A_n$  under the natural quotient map. **But there is no canonical choice of which element gets which sign.**

Let  $+1$  be the identity in  $2.A_n$ . For each  $\pi \in A_n$ , we define  $+\pi$  to be the element which multiplied together with  $+1$  gives itself, and  $-\pi$  for the other one.

**Definition 1.1.**  $\tilde{G}$  is a covering group of  $G$  if  $Z(\tilde{G}) \leq \tilde{G}'$  and  $\tilde{G}/Z(\tilde{G}) \cong G$ .

If  $|Z(\tilde{G})| = 2, 3$ , then the covering group is called double, triple cover.

**Theorem 1.2** (Schur). Every finite perfect group  $G$  has a unique maximal covering group  $\tilde{G}$ , with the property that every other covering group is a quotient of  $\tilde{G}$ . We call  $\tilde{G}$  the **universal covering group** of  $G$  and  $Z(\tilde{G})$  the **Schur multiplier** of  $G$ , denoted as  $M(G)$ .

**Example 1.3** (non-perfect).  $\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle \alpha \rangle \times \langle \beta \rangle$  has four maximal covering groups: one  $Q_8$  and three  $D_8$ .

$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}, Z(Q_8) = \{\pm 1\}, Q_8/Z(Q_8) \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

$$D_8 = \langle a \rangle : \langle b \rangle, Z(D_8) = \langle a^2 \rangle, D_8/Z(D_8) = \langle Z(D_8)a \rangle \times \langle Z(D_8)ab \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

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**Example 1.4.** The alternating groups  $A_n$  ( $n > 4$ ) only have double covers, except for  $A_6$  and  $A_7$  both with the same Schur multiplier  $\mathbb{Z}_6$ .

## 2 Double Covers of $A_n$ and $S_n$

Now we define  $2.S_n$ .

Firstly, let  $G$  be a set of order  $2n!$ , with a map  $\varphi$  onto  $S_n$  such that each  $\pi \in S_n$  has exactly two preimages denoted as  $+\pi$  and  $-\pi$ .

Intuitively, we should define the multiplication of  $G$  as  $+\pi + \sigma = +(\pi\sigma)$  and  $+\pi - \sigma = -(\pi\sigma)$ .

WLOG, we denote  $+(1\ 2)$  as  $[1\ 2]$  and  $-(1\ 2)$  as  $-[1\ 2]$ . Then for each transposition  $\pi \in S_n$ , taking  $(+\pi)^{-1}$  to be a preimage of  $\pi$ , define the products (of 3 elements in  $G$ )  $[i\ j]^{+\pi}$  and  $[i\ j]^{-\pi}$  to be a same preimage of  $(i^\pi\ j^\pi)$ , we denote it as  $-[i^\pi\ j^\pi]$ . That is  $[i\ j]^{\pm\pi} = -[i^\pi\ j^\pi]$ .

Then define the preimage  $+(a_i, a_{i+1}, \dots, a_j)$  by

$$[a_i\ a_{i+1}\ \dots\ a_j] = [a_i\ a_{i+1}][a_i\ a_{i+2}] \cdots [a_i\ a_j].$$

Finally by multiplying together disjoint cycles in the usual way, we obtain all elements.

The multiplication defined above is well-defined. That is, if we compute the same product in two different ways, we get the same result. A proof using construction of double cover of orthogonal group is given in Section 3.9.

Note that, by the rule  $[i\ j]^{\pm\pi} = -[i^\pi\ j^\pi]$  to define  $[i\ j]$ 's, all the elements  $\pm[i\ j]$  are conjugate. Thus they are all square to 1 or -1 simultaneously. Whatever, the elements like  $\pm[1\ 2][3\ 4]$  always square to -1.

Therefore, we obtain two distinct double cover of  $S_n$ . We denote  $2.S_n^+$  (actually  $\mathbb{Z}_2 \times S_n$ ) as the one with  $||[i\ j]|| = 2$  (where  $(i\ j)$  is identified with a coset  $Z(2.S_n)a$  with  $a$  of order 2), and  $2.S_n^-$  as the other one with  $||[i\ j]|| = 4$  ( $(i\ j)$  a coset of an element of order 4). However, both of them has the same subgroup  $2.A_n \cong \mathbb{Z}_2 \times A_n$  of index 2, which is the unique double cover of  $A_n$ .

**Example 2.1.**

- $2.S_4^+ \cong \mathbb{Z}_2 \times S_4$  is the group of symmetries of the cube, where  $S_4$  permutes the four diagonals of the cube, and  $\mathbb{Z}_2$  permutes the two opposite vertices in a diagonal simultaneously.
- $2.S_4^- \cong \text{GL}_2(3)$  where  $S_4$  permutes the four lines through the origin in  $\mathbb{F}_3^2$ , and  $\mathbb{Z}_2$  (i.e. scalar matrices) permutes the two points on each line simultaneously.

Note that  $\text{SL}(2, 3) = 2.A_4$  is the unique subgroup of order 24 in  $\text{GL}_2(3)$ .

### 3 Triple Covers of $A_6$ and $A_7$