Theory and applications of reversible jump MCMC sampling

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Trial lecture
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09.11.2018

Outline

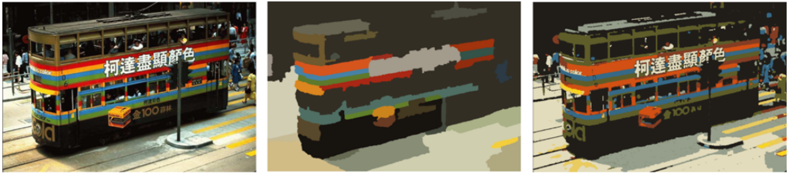
Standard Bayesian inference

Transdimensional Bayesian inference

Simple examples of RJMCMC

Mixtures of Gaussians case study

Discussion



Standard Bayesian inference

Problem

- Considers a parameter $\theta \in \Theta$ of interest;
- Provided likelihood $p(Y|X,\theta) = p(D|\theta)$ and prior $p(\theta)$;
- Aims at computing the posterior $p(\theta|D)$ as:

$$p(\theta|D) = \frac{p(D|\theta)p(\theta)}{\int_{\theta' \in \Theta} p(D|\theta')p(\theta')d\theta'}.$$

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Methods

- Tractable $p(D) = \int_{\theta' \in \Theta} p(D|\theta')p(\theta')d\theta'$, for example:
  - $Y|X,\theta \sim N(\theta, \sigma^2)$, $\theta \sim N(\mu_0, \sigma_0^2)$;
  - $Y|X,\theta \sim Bernoulli(\theta)$, $\theta \sim Beta(\alpha, \beta)$;
- $p(\theta|D)$ as an ergodic distribution of a Markov Chain:
  - Gibbs sampler;
  - Metropolis-Hastings sampler.

Bayesian inference. Metropolis-Hastings

- A Markov Chain with an ergodic distribution $\pi(\theta) = p(\theta|D)$;
- Transition kernel $p(\theta''|\theta') = q(\theta''|\theta')\alpha(\theta'', \theta')$:
  - $q(\theta''|\theta')$ - proposal probability;
  - $\alpha(\theta'', \theta')$ - acceptance probability:

$$\alpha(\theta'', \theta') = \min\left(1, \frac{\pi(\theta'')q(\theta'|\theta'')}{\pi(\theta')q(\theta''|\theta')}\right);$$

- The detailed balance $\forall A, B \subseteq \Theta$ is satisfied.

$$\blacktriangleright \int_A \int_B \pi(\theta')p(\theta''|\theta')d\theta' d\theta''$$
$$= \int_A \int_B \pi(\theta')q(\theta''|\theta') \min\left(1, \frac{\pi(\theta'')q(\theta'|\theta'')}{\pi(\theta')q(\theta''|\theta')}\right) d\theta' d\theta''$$
$$= \int_B \int_A \pi(\theta'')q(\theta'|\theta'') \min\left(\frac{\pi(\theta')q(\theta''|\theta')}{\pi(\theta'')q(\theta'|\theta'')}, 1\right) d\theta'' d\theta'$$
$$= \int_B \int_A \pi(\theta'')p(\theta'|\theta'')d\theta'' d\theta' \blacktriangleleft$$

Transdimensional Bayesian inference. Problem statement

Problem

- Considers a countable set of parameters $\theta_k \in \Theta_k, k \in \mathcal{K}$;
- Provided $p(D|\theta_k, k)$, $p(\theta_k|k)$ and $p(k)$, $\forall k \in \mathcal{K}$;
- Aims at computing the joint posterior $p(\theta_k, k|D)$ as:

$$p(\theta_k, k|D) = \frac{p(D|\theta_k, k)p(\theta_k|k)p(k)}{\sum_{k' \in \mathcal{K}} p(k') \int_{\theta' \in \Theta} p(D|\theta'_k, k')p(\theta'_k|k')d\theta'}.$$

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Examples

- Variable dimension models:
  - Model and parameter exploration within some model space $\mathcal{K}$;
  - Inferring on a mixture of Gaussians with feasible mixtures $\mathcal{K}$;
  - *"The number of things you don't know is one of the things you don't know"* [Green and Hastie, 2009]. E.g. tracking unknown number of targets.

Transdimensional Bayesian inference. Model and parameter exploration

1. Consider a class of models $\Omega : m_1(Y|X, \theta_1), ..., m_k(Y|X, \theta_k)$;
2. Put priors for all models $p(m_1), ..., p(m_k)$ and their parameters $p(\theta_1|m_1), ..., p(\theta_k|m_k)$;
3. Obtain the joint posterior distribution of models and parameters $p(m_1, \theta_1|D), ..., p(m_k, \theta_k|D)$;
4. Rao-Blackwellized inference on Δ in the joint space of models and parameters:
   $p(\Delta|D) = \int_\Omega p(m|D) \int_\Theta p(\Delta|m, \theta, D)p(\theta|m, D)d\theta dm$;
5. Easy to extend by means of considering several classes of models $\Omega_1, ..., \Omega_r$ with priors $p(\Omega_1), ..., p(\Omega_r)$.

Transdimensional Bayesian inference. Auxiliary states

- A Markov Chain with an ergodic distribution $p(\theta_k, k|D)$;
- States $\{\theta_k^{(i)} \in \Theta_k, k\}, k \in \mathcal{K}$ might have different domains;
- Use auxiliary states $x \sim g$ and $y \sim f$;
- Construct deterministically a new state $(\theta'', y) = h(\theta', x)$;
- Obtaining a diffeomorphism $(\theta', x) \to (\theta'', y) : \Theta'_x \to \Theta''_y$:
  - Bijection;
  - Differentiable;
  - Dimension matching $\dim \Theta'_x = \dim \Theta''_y$;
- $\left|\frac{\partial(\theta'', y)}{\partial(\theta', x)}\right|$ gives the ratio of the area in Θ''_y space to that in Θ'_x;
- Correct for that in $\alpha((\theta'', k''), (\theta', k'))$ to ensure the detailed balance:

$$\alpha((\theta'', k''), (\theta', k')) = \min\left(1, \frac{\pi(\theta'', k'')f(y)}{\pi(\theta', k')g(x)}\left|\frac{\partial(\theta'', y)}{\partial(\theta', x)}\right|\right).$$

Transdimensional Bayesian inference. Auxiliary states

The detailed balance $\forall A \subseteq \Theta'', B \subseteq \Theta'$ is satisfied:

$$\blacktriangleright \int_{\tilde{A}} \int_{B} \pi(\theta', k') g(x) \min \left(1, \frac{\pi(\theta'', k'') f(y)}{\pi(\theta', k') g(x)} \left| \frac{\partial(\theta'', y)}{\partial(\theta', x)} \right| \right) d\theta' dx$$

$$= \int_{\tilde{A}} \int_{B} \pi(\theta'', k'') f(y) \min \left(\frac{\pi(\theta', k') g(x)}{\pi(\theta'', k'') f(y)} \left| \frac{\partial(\theta', x)}{\partial(\theta'', y)} \right|, 1 \right) \times$$

$$\times \left| \frac{\partial(\theta'', y)}{\partial(\theta', x)} \right| d\theta' dx$$

$$= \int_{\tilde{B}} \int_{A} \pi(\theta'', k'') f(y) \min \left(\frac{\pi(\theta', k') g(x)}{\pi(\theta'', k'') f(y)} \left| \frac{\partial(\theta', x)}{\partial(\theta'', y)} \right|, 1 \right) d\theta'' dy \blacktriangleleft$$

Here by the change of multiple variables in integration:

$$d\theta'' dy = \left| \frac{\partial(\theta'', y)}{\partial(\theta', x)} \right| d\theta' dx$$

Transdimensional Bayesian inference. Mixtures of proposals

- A Markov Chain with an ergodic distribution $p(\theta_k, k|D)$;
- Assume that mixtures of proposals g_i, f_i, with $i \sim Q(\cdot|\theta', k')$;
- Use auxiliary states $x \sim g_i$ and $y \sim f_i$;
- Correct for that in $\alpha((\theta'', k''), (\theta', k'))$ to ensure the detailed balance:

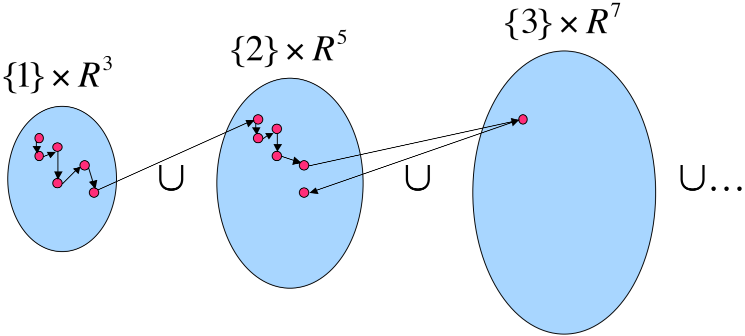
$$\alpha((\theta'', k''), (\theta', k')) = \min\left(1, \frac{\pi(\theta'', k'')Q(i|\theta'', k'')f_i(y)}{\pi(\theta', k')Q(i|\theta', k')g_i(x)}\left|\frac{\partial(\theta'', y)}{\partial(\theta', x)}\right|\right)$$

- Then the detailed balance is satisfied and it can be shown:

$$\int_{\tilde{A}}\int_B \pi(\theta', k')Q(i|\theta'', k'')g_i(x)\alpha_i((\theta'', k''), (\theta', k'))d\theta' dx$$
$$= \int_{\tilde{A}}\int_B \pi(\theta'', k'')Q(i|\theta', k')f_i(y)\alpha_i((\theta', k'), (\theta'', k''))d\theta'' dy.$$

Transdimensional Bayesian inference. Illustration

- A countable set of parameters $\theta_k \in \Theta_k, k \in \mathcal{K}$;
- Each $\Theta_k = \mathcal{R}^{n_k}$;
- The search space $\cup_{k \in \mathcal{K}}(\{k\} \times \mathcal{R}^{n_k})$;
- Illustration from [Dellaert, 2005].



Transdimensional Bayesian inference. Convergence [Brooks and Giudici, 2000]

- Run I chains with different starting points for $2 \cdot T$ iterations;
- Retain θ_i^t for chain $i = \{1, ..., I\}$ at time $t = \{T+1, ..., 2 \cdot T\}$;
- Estimate the total variation V of θ under the target by $\widehat{V}$;
- Estimate the between model variation B_m of θ by $\widehat{B}_m$;
- Estimate the within model variation W_m of θ by $\widehat{W}_m$;
- Estimate the within chain variation W_c of θ by $\widehat{W}_c$;
- $B_m W_c$ gives the within chain variation split between models;
- $W_m W_c$ gives the variance within both chains and models.

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- Estimate the within chain variation W_c of θ by $\widehat{W}_c$;
- $B_m W_c$ gives the within chain variation split between models;
- $W_m W_c$ gives the variance within both chains and models.

1. $\widehat{V}$ and $\widehat{W}_c$ should approximate V well and can be compared;
2. $\widehat{W}_m$ and $\widehat{W}_m \widehat{W}_c$ for within model variance can be compared;
3. $\widehat{B}_m$ and $\widehat{B}_m \widehat{W}_c$ for between model variance can be compared;
4. Detect if running multiple chains and multiple models improve the mixing.

Transdimensional Bayesian inference. Example 1 [Zabaras, 2017]

One dimensional random walk MH

- Consider $\theta_1, \theta_2 \in \mathcal{R}$, $\mathcal{K} = \{1, 2\}$ and $x, y \sim g \in \mathcal{R}$ such that:
  - $\theta_2 = \theta_1 + x, y = -x$;
  - $\theta_1 = \theta_2 + y, x = -y$;
- Then the acceptance ratio becomes:

$$\alpha((\{2\}, \theta_2), (\{1\}, \theta_1)) = \min\left(1, \frac{\pi(\{2\}, \theta_2) g(y)}{\pi(\{1\}, \theta_1) g(x)} \left| \frac{\partial(\theta_2, y)}{\partial(\theta_1, x)} \right| \right)$$
$$= \min\left(1, \frac{\pi(\{2\}, \theta_2) g(\theta_1 - \theta_2)}{\pi(\{1\}, \theta_1) g(\theta_2 - \theta_1)}\right);$$

- Determinant of the Jacobian is equal to 1;
- Considers the same dimensions of θ_1 and θ_2.

Transdimensional Bayesian inference. Example 2 [Zabaras, 2017]

Equilikely birth/death moves within MH

- Consider $\theta_1 \in \mathcal{R}$, $\theta_2 \in \mathcal{R}^2$ and $\mathcal{K} = \{1, 2\}$;
- Consider $x \sim g \in \mathcal{R}$ and $\theta_2 = h(\theta_1, x) = (\theta_1, x)$;
- Then the inverse is $(\theta_1, x) = h^{-1}(\theta_2) = \theta_2$ with probability 1;
- Acceptance of the **birth** move $\{1\} \times \mathcal{R} \to \{2\} \times \mathcal{R}^2$:

$$\alpha((\{2\}, \theta_2), (\{1\}, \theta_1)) = \min\left(1, \frac{\pi(\{2\}, \theta_2) f(y)}{\pi(\{1\}, \theta_1) g(x)} \left|\frac{\partial(\theta_2, y)}{\partial(\theta_1, x)}\right|\right)$$
$$= \min\left(1, \frac{\pi(\{2\}, \theta_2) \times 1}{\pi(\{1\}, \theta_1) g(x)} \left|\frac{\partial\theta_2}{\partial(\theta_1, x)}\right|\right) = \min\left(1, \frac{\pi(\{2\}, \theta_2)}{\pi(\{1\}, \theta_1) g(x)}\right);$$

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- Then the inverse is $(\theta_1, x) = h^{-1}(\theta_2) = \theta_2$ with probability 1;
- Acceptance of the **death** move $\{2\} \times \mathcal{R}^2 \to \{1\} \times \mathcal{R}^1$:

$$\alpha((\{1\}, \theta_1), (\{2\}, \theta_2)) = \min\left(1, \frac{\pi(\{1\}, \theta_1) g(x)}{\pi(\{2\}, \theta_2) f(y)} \left| \frac{\partial(\theta_1, x)}{\partial(\theta_2, y)} \right| \right)$$
$$= \min\left(1, \frac{\pi(\{1\}, \theta_1) g(x)}{\pi(\{2\}, \theta_2)} \left| \frac{\partial(\theta_1, x)}{\partial \theta_2} \right| \right) = \min\left(1, \frac{\pi(\{1\}, \theta_1) g(x)}{\pi(\{2\}, \theta_2)} \right);$$

- Here $x = \theta_{2,2}$;
- Determinant of the Jacobian is equal to 1.

Transdimensional Bayesian inference. Example 3 [Zabaras, 2017]

Equilikely split/merge moves within MH

- Consider $\theta_1 \in \mathcal{R}$, $\theta_2 \in \mathcal{R}^2$ and $\mathcal{K} = \{1, 2\}$;
- Consider $x \sim g \in \mathcal{R}$ and $\theta_2 = h(\theta_1, x) = (\theta_1 - x, \theta_1 + x)$;
- Then the inverse is $(\theta_1, x) = h^{-1}(\theta_2) = \left(\frac{\theta_{2,1} - \theta_{2,2}}{2}, \frac{\theta_{2,1} + \theta_{2,2}}{2}\right)$;
- Acceptance of the **split** move $\{1\} \times \mathcal{R} \to \{2\} \times \mathcal{R}^2$:

$$\alpha((\{2\}, \theta_2), (\{1\}, \theta_1)) = \min\left(1, \frac{\pi(\{2\}, \theta_2)}{\pi(\{1\}, \theta_1)g(x)} \times 2\right);$$

Transdimensional Bayesian inference. Example 3 [Zabaras, 2017]

Equilikely split/merge moves within MH

- Consider $\theta_1 \in \mathcal{R}$, $\theta_2 \in \mathcal{R}^2$ and $\mathcal{K} = \{1, 2\}$;
- Consider $x \sim g \in \mathcal{R}$ and $\theta_2 = h(\theta_1, x) = (\theta_1 - x, \theta_1 + x)$;
- Then the inverse is $(\theta_1, x) = h^{-1}(\theta_2) = \left(\frac{\theta_{2,1} - \theta_{2,2}}{2}, \frac{\theta_{2,1} + \theta_{2,2}}{2}\right)$;
- Acceptance of the **split** move $\{1\} \times \mathcal{R} \to \{2\} \times \mathcal{R}^2$:

$$\alpha((\{2\}, \theta_2), (\{1\}, \theta_1)) = \min\left(1, \frac{\pi(\{2\}, \theta_2)}{\pi(\{1\}, \theta_1) g(x)} \times 2\right);$$

- Acceptance of the **merge** move $\{2\} \times \mathcal{R}^2 \to \{1\} \times \mathcal{R}^1$:

$$\alpha((\{1\}, \theta_1), (\{2\}, \theta_2)) = \min\left(1, \frac{\pi(\{1\}, \theta_1) g(x)}{\pi(\{2\}, \theta_2)} \times \frac{1}{2}\right).$$

Image segmentation. Introduction

- We would like to segment different objects on an colored image;
  - E.g. for self-driving vehicles [Romera, 2017];
- Many assume the number of classes for segmentation known;
- This is hardly ever the case in real world;
- Solution: use mixtures of 3 (the number of channels) dimensional Gaussians with unknown number of classes;
- Use RJMCMC for model selection of the number of classes and simultaneous Bayesian inference.

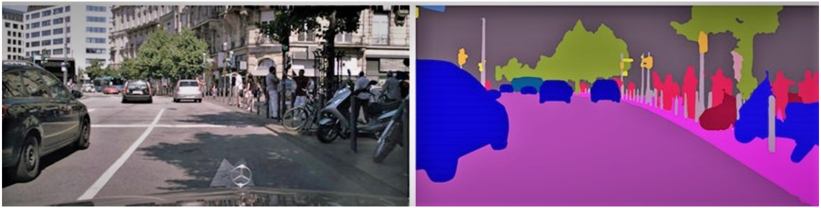

Image segmentation. Mixtures of Gaussians [Kato, 2008]

$$Y_i|\omega_i, \boldsymbol{\mu}_{\omega_i}, \boldsymbol{\Sigma}_{\omega_i}, L, \beta, p_{\omega_i} \overset{\text{ind}}{\sim} N_3(\mu_{\omega_i}, \boldsymbol{\Sigma}_{\omega_i});$$

$$p(\omega_i|\beta, p_{\omega_i}, L) = \frac{1}{Z}\exp\left(-\sum_{j\in\mathcal{P}} V(\omega_i, \omega_j)\right) p_{\omega_i}, p_{\omega_i} = \frac{1}{L};$$

$$V(\omega_i, \omega_j) = \beta \cdot \delta(\omega_i, \omega_j);$$

$$\delta(\omega_i, \omega_j) = (\mathrm{I}(\omega_i \neq \omega_j) - \mathrm{I}(\omega_i = \omega_j));$$

$$L \sim Unif(1, L_{max}), \mu_{\omega_i}|L \sim Unif(\{M\}), \boldsymbol{\Sigma}_{\omega_i}|L \sim Unif(\{S\}).$$

- $\omega_i \in \{1, ..., L\}, i \in \mathcal{P}$ are hidden segmentations for pixels $\mathcal{P}$;
- Here the models are identified by the number of mixtures L;
- Hence we have a model space $\{1, ..., L_{max}\}, L_{max} = 50$;
- $\beta = 2.5$ is the interaction strength parameter;
- $\{M\}$ and $\{S\}$ are the domains for means and covariances;
- p_λ are probability of a pixel be from class λ, $p_1 + ... + p_L = 1$.

Image segmentation. Splitting classes [Kato, 2008]

1. Choose a class λ' to split with a uniform probability $p_{ss} = \frac{1}{L'}$;
2. Make a split $L'' = L' + 1$ and $\lambda'' = (\lambda''_1, \lambda''_2)$, $x \sim g \in \mathcal{R}^{13}_{[0,1]}$;
3. Increase dimensionality in $\theta'' = \{\vec{p''}, \vec{\boldsymbol{\mu}''}, \vec{\boldsymbol{\Sigma}''}\}$ via $h(\theta', x)$ as:
   - $p''_{\lambda''_1} = p'_{\lambda'} u$, $p''_{\lambda''_2} = p'_{\lambda'}(1-u)$;
   - $\mu''_{\lambda''_1,i} = \mu'_{\lambda',i} + w_i \sqrt{\Sigma_{\lambda',i,i} \frac{1-u}{u}}$, $\mu'_{\lambda'_2,i} = \mu'_{\lambda',i} - w_i \sqrt{\Sigma_{\lambda',i,i} \frac{u}{1-u}}$;
   - $\Sigma''_{\lambda''_1,i,j} = \begin{cases} z_{i,i}(1-w_i^2)\Sigma_{\lambda',i,i}\frac{1}{u}, \text{if } i = j; \\ z_{i,j}\Sigma_{\lambda',i,j}\sqrt{(1-w_i^2)(1-w_j^2)z_{i,i}z_{j,j}}, \text{if } i \neq j; \end{cases}$
   - $\Sigma''_{\lambda''_2,i,j} =$
     $\begin{cases} (1-z_{i,i})(1-w_i^2)\Sigma_{\lambda',i,i}\frac{1}{u}, \text{if } i = j; \\ (1-z_{i,j})\Sigma_{\lambda',i,j}\sqrt{(1-w_i^2)(1-w_j^2)(1-z_{i,i})(1-z_{j,j})}, \text{if } i \neq j; \end{cases}$
   - Here $u, w_i, z_{i,j} \sim Unif(0,1)$ and $x = (u, \boldsymbol{w}, \boldsymbol{z})$;
4. Design of the proposals is based on the moment matching conditions allowing to build reasonably good proposals in RJMCMC
   [Richardson and Green, 1997, Fan and Sisson, 2011].

Image segmentation. Merging classes [Kato, 2008]

1. Choose a pair $(\lambda_1'', \lambda_2'')$ to merge w.r.t. $p_{ms} = \frac{1/d(\lambda_1'',\lambda_2'')}{\sum_{\lambda,\kappa\in\Lambda} 1/d(\lambda,\kappa)}$;
   - Here $d(\lambda, \kappa)$ is the Mahalanobis distance between λ and κ;
2. Make a merge $L' = L'' - 1$ and λ', $y \sim f$;
3. Decrease dimensionality in $\theta' = \{\vec{p'}, \vec{\boldsymbol{\mu}'}, \vec{\boldsymbol{\Sigma}'}\}$ via $h^{-1}(\theta'')$ similarly to the split move based on the solution of the moment matching conditions:
   - $p'_{\lambda'} = p''_{\lambda_1''} + p''_{\lambda_2''}$;
   - $p'_{\lambda'}\boldsymbol{\mu}'_{\lambda'} = p''_{\lambda_1''}\boldsymbol{\mu}''_{\lambda_1''} + p''_{\lambda_2''}\boldsymbol{\mu}''_{\lambda_2''}$;
   - $p'_{\lambda'}(\boldsymbol{\mu}'_{\lambda'}\boldsymbol{\mu'}^{\mathsf{T}}_{\lambda'} + \boldsymbol{\Sigma}'_{\lambda'}) =$ $p''_{\lambda_1''}(\boldsymbol{\mu}''_{\lambda_1''}\boldsymbol{\mu''}^{\mathsf{T}}_{\lambda_1''} + \boldsymbol{\Sigma}''_{\lambda_1''}) + p''_{\lambda_2''}(\boldsymbol{\mu}''_{\lambda_2''}\boldsymbol{\mu''}^{\mathsf{T}}_{\lambda_2''} + \boldsymbol{\Sigma}''_{\lambda_2''})$.

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   - Here $d(\lambda, \kappa)$ is the Mahalanobis distance between λ and κ;
2. Make a merge $L' = L'' - 1$ and λ', $y \sim f$;
3. Decrease dimensionality in $\theta' = \{\vec{p'}, \vec{\boldsymbol{\mu}'}, \vec{\boldsymbol{\Sigma}'}\}$ via $h^{-1}(\theta'')$ similarly to the split move based on the solution of the moment matching conditions:
   - $p'_{\lambda'} = p''_{\lambda_1''} + p''_{\lambda_2''}$;
   - $p'_{\lambda'}\boldsymbol{\mu}'_{\lambda'} = p''_{\lambda_1''}\boldsymbol{\mu}''_{\lambda_1''} + p''_{\lambda_2''}\boldsymbol{\mu}''_{\lambda_2''}$;
   - $p'_{\lambda'}(\boldsymbol{\mu}'_{\lambda'}\boldsymbol{\mu}'^{\mathsf{T}}_{\lambda'} + \boldsymbol{\Sigma}'_{\lambda'}) =$
     $p''_{\lambda_1''}(\boldsymbol{\mu}''_{\lambda_1''}\boldsymbol{\mu}''^{\mathsf{T}}_{\lambda_1''} + \boldsymbol{\Sigma}''_{\lambda_1''}) + p''_{\lambda_2''}(\boldsymbol{\mu}''_{\lambda_2''}\boldsymbol{\mu}''^{\mathsf{T}}_{\lambda_2''} + \boldsymbol{\Sigma}''_{\lambda_2''})$.

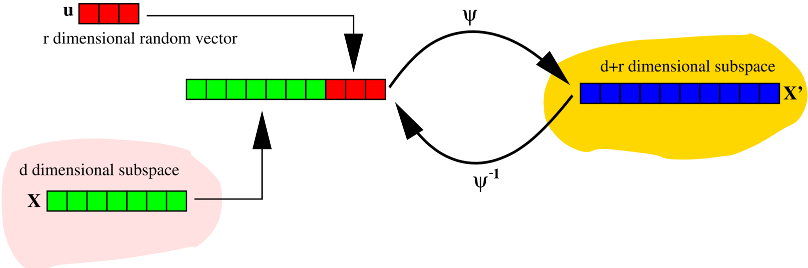


Image segmentation. Acceptance probabilities [Kato, 2008]

- Acceptance probabilities:
  - For split move is $\alpha(L'', \theta'', L', \theta') = \min(1, A)$;
  - For merge move is $\alpha(L', \theta', L'', \theta'') = \min(1, \frac{1}{A})$;
- Here A, based on the standard RJMCMC theory, reduces to:

$$A = \frac{p(L'', \theta'' | \boldsymbol{Y})}{p(L', \theta' | \boldsymbol{Y})} \times \frac{p_m(L'') p_{ms}(\lambda''_1, \lambda''_2)}{p_s(L') p_{ss}(\lambda') p_r} \times \frac{1}{p(u) \prod_{i=1}^{3} p(w_i) \prod_{j=1}^{3} p(z_{i,j})}$$

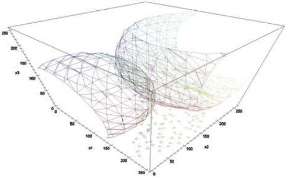
$$\times -p'_{\lambda'} \prod_{i=1}^{3} \left(\frac{\boldsymbol{\Sigma}'_{\lambda',i,i}}{u(1-u)} (1 - w_i^2)(1 - z_{i,i}) z_{i,i} \prod_{j=1}^{3} \frac{\boldsymbol{\Sigma}'_{\lambda',i,j}}{u(1-u)} \right);$$

$$p_r = \prod_{s : \omega''_s \in \{\lambda''_1, \lambda''_2\}} \mathcal{N}_3(Y_s | \mu''_{\omega''_s}, \boldsymbol{\Sigma}''_{\omega''_s}) \times p''_{\omega''_s} \exp(-\beta \sum_{r \in \mathcal{P}} \delta(\omega''_s, \omega''_r)).$$

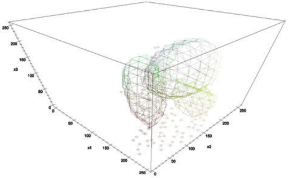
Image segmentation. Illustrative results [Kato, 2008]

Original image

Segmentation result (3 labels)



Initial Gaussians



Final estimation (3 classes)

Image segmentation. What is behind [Zhang et al., 2004]

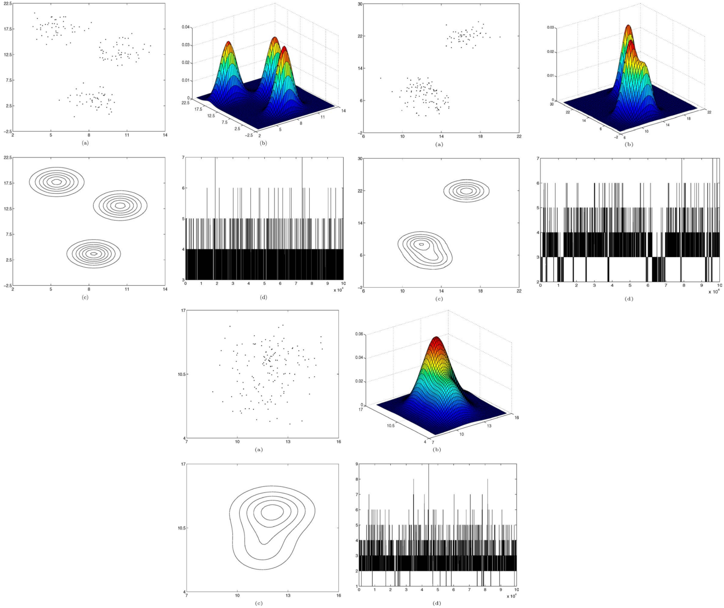
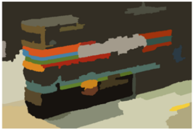


Image segmentation. Further results [Kato, 2008]

Original image



JSEG (37 labels)

RJMCMC (10 labels)

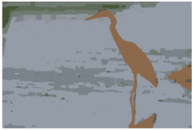
Original image

JSEG (50 labels)

RJMCMC (8 labels)

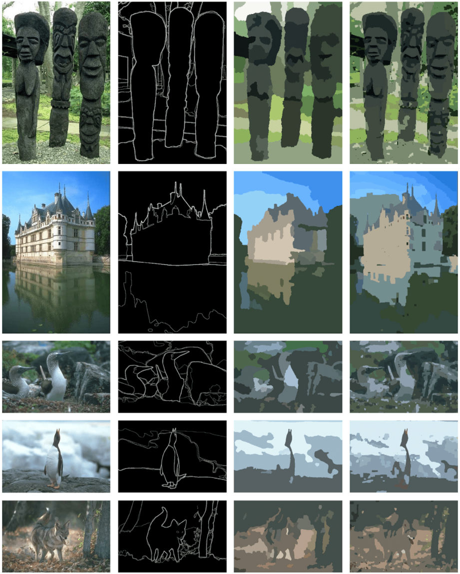
Original image

JSEG (14 labels)



RJMCMC (9 labels)

Image segmentation. Further results [Kato, 2008]



Original image | Human segmentation | JSEG | RJMCMC

RJMCMC. Interesting applications

- Tracking a variable number of interacting targets [Khan et al., 2005];
- Change points detection analysis [Zhao and Chu, 2010];
- Bayesian model determination [Green, 1995];
- Building configurations of Bayesian ANN [Andrieu et al., 2000];
- Inference on hidden Markov models [Robert et al., 2000];
- Synthesizing open worlds with constraints [Yeh et al., 2012].

RJMCMC. Actual problems and open questions

- Designing efficient RJMCMC is challenging;
- Multi-modal space exploration is especially challenging;
- Constructing proposals and calculation of the acceptance probabilities is often tedious in real life applications;
- Efficient subsampling techniques must be developed to be able to work with *"tall data"*;
- Convergence assessment and validity of the output analysis is far from obvious in ultrahigh-dimensional settings.

RJMCMC. Modifications

- Use moments matching proposals [Richardson and Green, 1997, Fan and Sisson, 2011];
- Use a secondary Markov chain to make a proposal [Al-Awadhi et al., 2004];
- Target tempering to enhance mixing properties [Karagiannis and Andrieu, 2013];
- Parallel tempering (population based) chains [Jasra et al., 2007, Fouskakis et al., 2009];
- State space augmentation to the largest model [Brooks et al., 2003];
- General usage of auxiliary states in RJMCMC [Brooks et al., 2003].

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