Orthogonality in Abstract Vector Spaces

Learning objectives:

1. Analyze and explain the concept of orthogonal subspaces in higher dimensions and abstract vector spaces:

- Develop a deep understanding of what it means for vectors to be orthogonal in n-dimensional and abstract vector spaces.
- Explain how orthogonal subspaces are defined using inner products and their importance in vector decomposition and signal processing.

2. Apply and implement the Gram-Schmidt process in both real and complex vector spaces:

- Demonstrate the ability to convert any set of linearly independent vectors into an orthogonal or orthonormal set using the Gram-Schmidt process.
- Extend this application to complex vector spaces, highlighting the role of conjugates in orthonormalization.
- Apply the process in real-world applications like quantum mechanics and Fourier analysis.

3. Use orthogonal projections to solve real-world optimization problems, including least squares approximation:

- Calculate the projection of a vector onto a subspace and demonstrate how this minimizes errors in approximating solutions.
- Explain and apply the method to real-world problems, such as data fitting in machine learning models, regression analysis, and solving overdetermined systems using the least squares method.

4. Explore the connection between orthogonality and eigenvalue problems, focusing on matrix diagonalization and applications in PCA:

- Investigate how orthogonal eigenvectors play a critical role in diagonalizing matrices, especially symmetric or Hermitian matrices.
- Apply these principles to eigenvalue problems, explaining the use of orthogonal vectors in practical applications like principal component analysis (PCA) for dimensionality reduction and signal processing.