Math for Simscape Robot Fish

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The Lighthill Equation
This "robot fish" is modeled as an N-joint manipulator tasked with tracking a time-dependent plane curve known as the Lighthill curve [1]:

$$Y(x,t) = (c_1x + c_2x^2)sin(kx - \omega t) + c_1xsin(\omega t)$$

The Lighthill curve is supposed to model a fish's swimming waveform (or gait) in the XY-plane. Each fish's gait has its own unique amplitude profile, wavenumber, and angular frequency. These parameters are also a function of swimming speed, acceleration, etc. The most important parameter to the gait shape is the wavenumber, related to k by 2π, for which different values describe 4 different gait types: thunniform (0, 1/4), carangiform (1/4, 3/4), sub-carangiform (3/4, 5/4), and anguilliform (5/4, ∞).

Making it a Dynamics Problem
The simscape model is composed of 8 body links + 1 tail fin link. The 8 body links of constant length $L = 6.25$ cm are connected by the inner N = 7 revolute joints J_{1-7}. There are also revolute joints J_0 which connects the frontmost link to the world frame and J_t connects the 8th link to the tailfin. J_{1-t} each have torsional spring and dampers with constant coefficients $k = 200$ $[g \cdot cm^2/s^2]$ and $b = 5$ $[g \cdot cm/s^2]$. J_{1-t} are subject to proportional-derivative (PD) torque input τ_i $[g \cdot cm^2/s^2]$ which stabilizes the error $e_i = q_i - q_{d,i}$ where q_i are the measured joint angles [rad] and $q_{d,i}$ are the desired (or reference) joint angles [rad]. The reference angles are the angles subtended by the chain of manipulator links of length L whose joints intersect the Lighthill curve. First the reference joint positions at time t are found using the bisection method to solve the equation:

$$(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2 = L^2$$

for x_i at each joint with $y_i = Y(x_i, t)$ and $x_{1-2} = 0$ so that the Lighthill curve starts from J_2. Then the reference angles are derived from trigonometry of the reference positions:

$$q_{d,i} = tan^{-1}(\frac{y_{i+1} - y_i}{x_{i+1} - x_i}) - \sum_{n=1}^{i-1} q_{d,n}$$

with $q_{d,1-2} = 0$. This gives a vector of reference angles q_d which can be tracked over time.

Time evolution of the manipulator joint angles is determined by the matrix differential equation:

$$M(q)\ddot{q} + B\dot{q} + Kq = \tau$$

M(q) is a diagonal matrix which holds the instantaneous moments of inertia of the joints along its main diagonal. B and K are constant spring and damper matrices. M(q), B, and K are all 7x7. Note M(q) depends on q (the vector of measured angles) because the angles affect the distance between non-adjacent joints. The exact expression for elements M_{ii} is impracticly complicated (exactly like deriving the dynamics of an 7-tuple pendulum). Luckily the small angle approximation can be used so that M(q) no longer depends on q and becomes a constant with:

$$M_{ij} = \delta_{ij} \sum_{k=1}^{8} m_k L^2 (i-k)^2$$

where point masses $m_k = [290, 590, 380, 210, 150, 95, 54, 64][g]$ sit at J_{1-t}.

The inertia matrix M can then be used directly to calculate a suitable torque vector:

$$\tau = -M(K_d \dot{e} + K_p e)$$

with error signal $e = q - q_d$ which is a PD controller with constant gain matrices K_p and K_d. To reduce controller noise the the angular velocity $\dot{q}$ is pre-filtered with a 2nd order low pass filter:

$$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

where $\omega_n = 50$ rad/s and $\zeta = 0.9$.

Reference

1. Optimization of the Kinematic Model for Biomimetic Robotic Fish with Rigid Headshaking Mitigation. MDPI Robotics 2017, 6, 30, 2-4.