Lecture 4: Mathematical Induction

Appendix A

Sets

- A **set** is a collection of objects.
- We denote sets by capital letters such as S:

$$S = \{ 1, 2, 3, 4 \}$$

- In this example, S is the set containing the first four positive integers.

- Is the following set valid?

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- Is the following set valid?

$$S = \{ 1, 2, 3, 3, 4 \}$$

- An element is either in a set or not. It is *redundant* to list an element twice.
  - $S = \{ 1, 2, 3, 3, 4 \}$ is actually $S = \{ 1, 2, 3, 4 \}$

Sets

- What does the following set contain?

$S = \{ 3, 6, 9, \ldots, 3i, \ldots \}$

Sets

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$S = \{ 3, 6, 9, \ldots, 3i, \ldots \}$

- When we write out a few items and then a general item, we are describing an **infinite set**.
- In this case, the set contains every positive multiple of 3.

We can also represent the same set with a formal description of the objects it contains:

$S = \{ n \text{ such that } n = 3i \text{ for some positive integer } i \}$

Sets

- The objects in a set are called **elements** or **members** of the set.
- If x is a member of set S, we write $x \in S$.
- If x is not a member of S, we write $x \notin S$.

For example:

$S = \{ 1, 2, 3, 4 \}$ $\quad 2 \in S$ and $\quad 5 \notin S$

- Two sets are equal if they have the same elements: $S = T$
- If two sets are not equal, we say $S \neq T$

Sets

Suppose we have the two following sets. What can be said of their relationship?

$S = \{ 1, 2, 3 \}$ $\quad$ $T = \{ 1, 2, 3, 5, 6 \}$

Sets

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$S = \{ 1, 2, 3 \}$ $T = \{ 1, 2, 3, 5, 6 \}$

- S is a **subset** of T because every element in S is also in T.

  $S \subseteq T$

Is S a subset of S?

Sets

Suppose we have the two following sets. What can be said of their relationship?

$S = \{ 1, 2, 3 \}$ $T = \{ 1, 2, 3, 5, 6 \}$

- S is a **subset** of T because every element in S is also in T.

  $S \subseteq T$

Is S a subset of S?

- Yes! Every set is a subset of itself.
- However, if a set S is a subset of another set T, but is *not* equal to T, we say S is a **proper subset** of T:
- $S \subset T$

Sets

$S = \{ 1, 4, 5, 6 \} \qquad T = \{ 1, 3, 5 \}$

- What is $S \cap T$?

Sets

$S = \{ 1, 4, 5, 6 \}$ $T = \{ 1, 3, 5 \}$

- What is $S \cap T$?
- The **intersection** of S and T is the set of all elements that are in <u>both</u> S and T.

$\therefore S \cap T = \{ 1, 5 \}$

Sets

$S = \{ 1, 4, 5, 6 \}$ $T = \{ 1, 3, 5 \}$

- What is $S \cup T$?

Sets

$S = \{ 1, 4, 5, 6 \}$ $T = \{ 1, 3, 5 \}$

- What is $S \cup T$?
- The **union** of S and T is the set of all elements that are in <u>either</u> S or T.

$\therefore S \cup T = \{ 1, 3, 4, 5, 6 \}$

Sets

$S = \{ 1, 4, 5, 6 \}$ $T = \{ 1, 3, 5 \}$

- What is $S - T$? What is $T - S$?

Sets

$S = \{ 1, 4, 5, 6 \}$ $T = \{ 1, 3, 5 \}$

- What is $S - T$? What is $T - S$?
- The **difference** between S and T is the set of all elements that are in S but are <u>not</u> in T.

$S - T = \{ 4, 6 \}$

$T - S = \{ 3 \}$

Mathematical Induction

Suppose we want to prove an equality with an **infinite domain** to be true.

Example: Prove that $1 + 2 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 1$ for any $n \geq 1$

Proving a few steps is simple:

- $n = 1$: $1 + 2^1 = 2^{1+1} - 1$ or $1 + 2 = 4 - 1$ or $3 = 3$
- $n = 2$: $1 + 2^1 + 2^2 = 2^{2+1} - 1$ or $1 + 2 + 4 = 8 - 1$ or $7 = 7$
- etc.

However, to prove this true for *all* values of n would take infinite steps!

- Instead, **mathematical induction** helps us prove these types of equalities in a fairly simple way.

Mathematical induction

Suppose we are climbing a ladder, and we know two things to be true:

1. We can climb the first rung.
2. When we are on any rung of the ladder, we can climb to the next rung.

What can we conclude from these two statements?

Mathematical induction

Suppose we are climbing a ladder, and we know two things to be true:

1. We can climb the first rung.
2. When we are on any rung of the ladder, we can climb to the next rung.

What can we conclude from these two statements?

We can climb a ladder of infinite size!

- We can get to the first rung by statement 1.
- We can get to the second rung from the first rung by statement 2
- We can get to the third rung from the second rung by statement 2
- …
- We can get to rung n from rung n - 1 by statement 2

Mathematical Induction

Mathematical induction works the same way.

If we have a statement P(n) which we want to prove is true for any positive integer n, we perform the following steps:

1. Prove that P(1) is true. **(Induction Base)**
2. Write out P(k) and assume it's true. **(Induction Hypothesis)**
3. Write out P(k + 1) and show that if P(k) is true, P(k + 1) must also be true **(Induction Step)**

Induction Example

Example: Prove that $1 + 2 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 1$ for any $n \geq 1$

- **Step 1**: Prove base case, P(1)
  - $1 + 2^1 = 2^{1+1} - 1$ or $3 = 2^2 - 1$, which is true.
- **Step 2**: Assume the inductive hypothesis to be true, P(k)
  - $1 + 2 + 2^2 + \ldots + 2^k = 2^{k+1} - 1$
- **Step 3**: Prove that **if** P(k) is true, P($k + 1$) must be true. First, write out P($k + 1$):
  - $1 + 2 + 2^2 + \ldots + 2^{k+1} \stackrel{?}{=} 2^{k+1+1} - 1$

Induction Example (cont)

P(*k*): $1 + 2^1 + 2^2 + \ldots + 2^k = 2^{k+1} - 1$

P(*k*+1): $1 + 2^1 + 2^2 + \ldots + \mathbf{2^{\boldsymbol{k}}} + 2^{k+1} \stackrel{?}{=} 2^{k+1+1} - 1$

We can rewrite the left side of P(k+1) to include the next-to-last term, $\mathbf{2^{\boldsymbol{k}}}$

Induction Example (cont)

P(k): $1 + 2^1 + 2^2 + \ldots + 2^k = 2^{k+1} - 1$

P(k+1): $1 + 2^1 + 2^2 + \ldots + 2^k + 2^{k+1} \stackrel{?}{=} 2^{k+1+1} - 1$

We can then find the left side of P(k) within P(k+1). Replace it with the right side of P(k), since we have assumed they're equal:

P(k+1): $2^{k+1} - 1 + 2^{k+1}$

$= 2(2^{k+1}) - 1$ (adding like terms)

$= 2^{k+1+1} - 1$

We end with $2^{k+1+1} - 1$ which is the right side of P(k+1).
We have shown that if P(k) is true then P(k+1) must also be true, thus completing the proof.

Induction Example (cont)

Given this equality:

$1 + 2 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 1$ for any $n \geq 1$

We have proven:

1. It is true when $n = 1$
2. If it is true when $n = k$, it is also true when $n = k + 1$

<u>Therefore</u>

The equality is true when $n = 1$ by statement 1

The equality is true when $n = 2$ by statement 2

The equality is true when $n = 3$ by statement 2

etc.

In-Class Exercise

Use mathematical induction to prove:

- $1 + 2 + \ldots + n = [n(n + 1)] / 2$