In-Class Exercise

1. Use Floyd's Algorithm to compute D and P from the following array W:

	1	2	3	4
1	0	1	5	10
2	5	0	2	6
3	4	9	0	5
4	7	2	1	0

Answer

$D^{(0)} =$

	1	2	3	4
1	0	1	5	10
2	5	0	2	6
3	4	9	0	5
4	7	2	1	0

$D^{(1)} =$

	1	2	3	4
1	0	1	5	10
2	5	0	2	6
3	4	**5**	0	5
4	7	2	1	0

When $k = 1$, nothing in row/column 1 changes!

$D^{(1)}[2][3] = \min(D^{(0)}[2][3], D^{(0)}[2][1] + D^{(0)}[1][3])$
$= \min(2, 5 + 5) = 2$

$D^{(1)}[2][4] = \min(D^{(0)}[2][4], D^{(0)}[2][1] + D^{(0)}[1][4])$
$= \min(6, 5 + 10) = 6$

$D^{(1)}[3][2] = \min(D^{(0)}[3][2], D^{(0)}[3][1] + D^{(0)}[1][2])$
$= \min(9, 4 + 1) = 5$

$D^{(1)}[3][4] = \min(D^{(0)}[3][4], D^{(0)}[3][1] + D^{(0)}[1][4])$
$= \min(5, 4 + 10) = 5$

$D^{(1)}[4][3] = \min(D^{(0)}[4][3], D^{(0)}[4][1] + D^{(0)}[1][3])$
$= \min(1, 7 + 10) = 1$

$D^{(1)}[4][2] = \min(D^{(0)}[4][2], D^{(0)}[4][1] + D^{(0)}[1][2])$
$= \min(2, 7 + 1) = 2$

Answer

$D^{(1)} =$

	1	2	3	4
1	0	1	5	10
2	5	0	2	6
3	4	5	0	5
4	7	2	1	0

$D^{(2)} =$

	1	2	3	4
1	0	1	**3**	**7**
2	5	0	2	6
3	4	5	0	5
4	7	2	1	0

When $k = 2$, nothing in row/column 2 changes!

$D^{(2)}[1][3] = \min(D^{(1)}[1][3], D^{(1)}[1][2] + D^{(1)}[2][3])$
$= \min(5, 1 + 2) = 3$

$D^{(2)}[1][4] = \min(D^{(1)}[1][4], D^{(1)}[1][2] + D^{(1)}[2][4])$
$= \min(10, 1 + 6) = 7$

$D^{(2)}[3][1] = \min(D^{(1)}[3][1], D^{(1)}[3][2] + D^{(1)}[2][1])$
$= \min(4, 5 + 5) = 4$

$D^{(2)}[3][4] = \min(D^{(1)}[3][4], D^{(1)}[3][2] + D^{(1)}[2][4])$
$= \min(5, 5 + 6) = 5$

$D^{(2)}[4][1] = \min(D^{(1)}[4][1], D^{(1)}[4][2] + D^{(1)}[2][1])$
$= \min(7, 2 + 5) = 7$

$D^{(2)}[4][3] = \min(D^{(1)}[4][3], D^{(1)}[4][2] + D^{(1)}[2][3])$
$= \min(1, 2 + 2) = 2$

Answer

$D^{(2)} =$

	1	2	3	4
1	0	1	3	7
2	5	0	2	6
3	4	5	0	5
4	7	2	1	0

$D^{(3)} =$

	1	2	3	4
1	0	1	3	7
2	5	0	2	6
3	4	5	0	5
4	**5**	2	1	0

When $k = 3$, nothing in row/column 3 changes!

$D^{(3)}[1][2] = \min(D^{(2)}[1][2], D^{(2)}[1][3] + D^{(2)}[3][2])$
$= \min(1, 3 + 5) = 1$

$D^{(3)}[1][4] = \min(D^{(2)}[1][4], D^{(2)}[1][3] + D^{(2)}[3][4])$
$= \min(7, 3 + 5) = 7$

$D^{(3)}[2][1] = \min(D^{(2)}[2][1], D^{(2)}[2][3] + D^{(2)}[3][1])$
$= \min(5, 2 + 4) = 5$

$D^{(3)}[2][4] = \min(D^{(2)}[2][4], D^{(2)}[2][3] + D^{(2)}[3][4])$
$= \min(6, 2 + 5) = 6$

$D^{(3)}[4][1] = \min(D^{(2)}[4][1], D^{(2)}[4][3] + D^{(2)}[3][1])$
$= \min(7, 1 + 4) = 5$

$D^{(3)}[4][2] = \min(D^{(2)}[4][2], D^{(2)}[4][3] + D^{(2)}[3][2])$
$= \min(2, 1 + 5) = 2$

Answer

$D^{(3)} =$

	1	2	3	4
1	0	1	3	7
2	5	0	2	6
3	4	5	0	5
4	5	2	1	0

$D^{(4)} =$

	1	2	3	4
1	0	1	3	7
2	5	0	2	6
3	4	5	0	5
4	5	2	1	0

When $k = 4$, nothing in row/column 4 changes!

$D^{(4)}[1][2] = \min(D^{(3)}[1][2], D^{(3)}[1][4] + D^{(3)}[4][2])$
$= \min(1, 7 + 2) = 1$

$D^{(4)}[1][3] = \min(D^{(3)}[1][3], D^{(3)}[1][4] + D^{(3)}[4][3])$
$= \min(3, 7 + 1) = 3$

$D^{(4)}[2][1] = \min(D^{(3)}[2][1], D^{(3)}[2][4] + D^{(3)}[4][1])$
$= \min(5, 6 + 5) = 5$

$D^{(4)}[2][3] = \min(D^{(3)}[2][3], D^{(3)}[2][4] + D^{(3)}[4][3])$
$= \min(2, 6 + 1) = 2$

$D^{(4)}[3][1] = \min(D^{(3)}[3][1], D^{(3)}[3][4] + D^{(3)}[4][1])$
$= \min(4, 5 + 5) = 4$

$D^{(4)}[3][2] = \min(D^{(3)}[3][2], D^{(3)}[3][4] + D^{(3)}[4][2])$
$= \min(5, 5 + 2) = 5$

$D^{(4)}$ is also the final answer

In-Class Exercise

Find the optimal order, and its cost, for evaluating the product of the following matrices:

A_1	$\times$	A_2	$\times$	A_3	$\times$	A_4	$\times$	A_5
10×4		4×5		5×20		20×2		2×50

Show the final arrays M and P.

$$M[i][j] = \underset{i \leq k \leq j-1}{minimum} (M[i][k] + M[k+1][j] + d_{i-1}d_k d_j)$$

$P[i][j]$ = the value of k when $M[i][j]$ is chosen

Answer Step 1

The first diagonal can have all its values set to 0, since we will not multiply a matrix by itself

i.e. M[1][1] contains the minimum # of multiplications required to multiply arrays 1 through 1, so we place 0 in this index

M:

	1	2	3	4	5
1	0				
2		0			
3			0		
4				0	
5					0

$d_0 = 10$ $d_1 = 4$ $d_2 = 5$ $d_3 = 20$ $d_4 = 2$ $d_5 = 50$

Answer Step 2

Second diagonal

Find M[1][2] $i = 1, j = 2$. k ranges from 1 to 1

M[1][1] + M[2][2] + $d_0 d_1 d_2$
$= 0 + 0 + 10 \times 4 \times 5 = 200$

$k = 1$ when we found value for M[1][2], so P[1][2] = 1

Find M[2][3] $i = 2, j = 3$. k ranges from 2 to 2

M[2][2] + M[3][3] + $d_1 d_2 d_3$
$= 0 + 0 + 4 \times 5 \times 20 = 400$

$k = 2$ when we found value for M[2][3], so P[2][3] = 1

M:

	1	2	3	4	5
1	0	200			
2		0	400		
3			0		
4				0	
5					0

$d_0 = 10$ $d_1 = 4$ $d_2 = 5$ $d_3 = 20$ $d_4 = 2$ $d_5 = 50$

Answer Step 2

Second diagonal

Find M[3][4] $i = 3, j = 4$. k ranges from 3 to 3

M[3][3] + M[4][4] + $d_2d_3d_4$

$= 0 + 0 + 5 \times 20 \times 2 = 200$

$k = 3$ when we found value for M[3][4], so P[3][4] = 1

Find M[4][5] $i = 4, j = 5$. k ranges from 4 to 4

M[4][4] + M[5][5] + $d_3d_4d_5$

$= 0 + 0 + 20 \times 2 \times 50 = 400$

$k = 4$ when we found value for M[4][5], so P[4][5] = 4

M:

	1	2	3	4	5
1	0	200			
2		0	400		
3			0	200	
4				0	2000
5					0

$d_0 = 10 \quad d_1 = 4 \quad d_2 = 5 \quad d_3 = 20 \quad d_4 = 2 \quad d_5 = 50$

Answer Step 3

Third diagonal

Find M[1][3] $i = 1, j = 3$. k ranges from 1 to 2

min (M[1][1] + M[2][3] + $d_0d_1d_3$

M[1][2] + M[3][3] + $d_0d_2d_3$

= min(0 + 400 + 10 × 4 × 20,

200 + 0 + 10 × 5 × 20)

= min (1200, 1200)

Since both equal 1200, we can arbitrarily select the first one. Therefore, $k = 1$ when we found value for M[1][3], so P[1][3] = 1

M:

	1	2	3	4	5
1	0	200	1200		
2		0	400		
3			0	200	
4				0	2000
5					0

$d_0 = 10 \quad d_1 = 4 \quad d_2 = 5 \quad d_3 = 20 \quad d_4 = 2 \quad d_5 = 50$

Answer Step 3

Third diagonal

Find M[2][4] $i = 2, j = 4$. k ranges from 2 to 3

min (M[2][2] + M[3][4] + $d_1d_2d_4$

M[2][3] + M[4][4] + $d_1d_3d_4$

= min(0 + 200 + 4 × 5 × 2,

400 + 0 + 4 × 20 × 2)

= min (240, 560) = 240

k = 2 when we found value for M[2][4], so P[2][4] = 2

M:

	1	2	3	4	5
1	0	200	1200		
2		0	400	240	
3			0	200	
4				0	2000
5					0

$d_0 = 10$ $d_1 = 4$ $d_2 = 5$ $d_3 = 20$ $d_4 = 2$ $d_5 = 50$

Answer Step 3

Third diagonal

Find M[3][5] $i = 3, j = 5$. k ranges from 3 to 4

min (M[3][3] + M[4][5] + $d_2d_3d_5$

M[3][4] + M[5][5] + $d_2d_4d_5$

= min (0 + 2000 + 5 × 20 × 50,

200 + 0 + 5 × 2 × 50)

= min (7000, 700) = 700

k = 4 when we found value for M[3][5], so P[3][5] = 4

M:

	1	2	3	4	5
1	0	200	1200		
2		0	400	240	
3			0	200	700
4				0	2000
5					0

$d_0 = 10$ $d_1 = 4$ $d_2 = 5$ $d_3 = 20$ $d_4 = 2$ $d_5 = 50$

Answer Step 4

Fourth diagonal

Find M[1][4] $i = 1, j = 4$. k ranges from 1 to 3

min (M[1][1] + M[2][4] + $d_0d_1d_4$

M[1][2] + M[3][4] + $d_0d_2d_4$

M[1][3] + M[4][4] + $d_0d_3d_4$

= min (0 + 240 + 10 × 4 × 2,

200 + 200 + 10 × 5 × 2,

1200 + 0 + 10 × 20 × 2)

= min (320, 500, 1600) = 320

$k = 1$ when we found value for M[1][4], so P[1][4] = 1

M:

	1	2	3	4	5
1	0	200	1200	320	
2		0	400	240	
3			0	200	700
4				0	2000
5					0

$d_0 = 10$ $d_1 = 4$ $d_2 = 5$ $d_3 = 20$ $d_4 = 2$ $d_5 = 50$

Answer Step 4

Fourth diagonal

Find M[2][5] $i = 2, j = 5$. k ranges from 2 to 4

min (M[2][2] + M[3][5] + $d_1d_2d_5$

M[2][3] + M[4][5] + $d_1d_3d_5$

M[2][4] + M[5][5] + $d_1d_4d_5$

= min (0 + 700 + 4 × 5 × 50,

400 + 2000 + 4 × 20 × 50,

240 + 0 + 4 × 2 × 50)

= min (1700, 6400, 640) = 640

k = 4 when we found value for M[2][5], so P[2][5] = 4

M:

	1	2	3	4	5
1	0	200	1200	320	
2		0	400	240	640
3			0	200	700
4				0	2000
5					0

$d_0 = 10$ $d_1 = 4$ $d_2 = 5$ $d_3 = 20$ $d_4 = 2$ $d_5 = 50$

Answer Step 5

Fifth diagonal

Find M[1][5] $\quad i = 1, j = 5$. k ranges from 1 to 4

min (M[1][1] + M[2][5] + $d_0d_1d_5$

M[1][2] + M[3][5] + $d_0d_2d_5$

M[1][3] + M[4][5] + $d_0d_3d_5$

M[1][4] + M[5][5] + $d_0d_4d_5$

= min (0 + 640 + 10 × 4 × 50,

200 + 700 + 10 × 5 × 50,

1200 + 2000 + 10 × 20 × 50,

320 + 0 + 10 × 2 × 50)

= min (2640, 3400, 13200, 1320) = 1320

$k = 4$ when we found value for M[1][5], so P[1][5] = 4

M:

	1	2	3	4	5
1	0	200	1200	320	1320
2		0	400	240	640
3			0	200	700
4				0	2000
5					0

We are done!

$d_0 = 10 \quad d_1 = 4 \quad d_2 = 5 \quad d_3 = 20 \quad d_4 = 2 \quad d_5 = 50$

Final P Array

We are multiplying A1A2A3A4A5

To see where the first split occurs when multiplying A1-A5, we check P[1][5] and find 4. The first split occurs *after* 4:

(A1A2A3A4)A5

We now check P[1][4] to see where the first split occurs when multiplying A1 - A4 and find 1. The second split occurs after 1:

(A1(A2A3A4))A5

P:

	1	2	3	4	5
1		1	1	1	4
2			2	2	4
3				3	4
4					4
5					

Final P Array

We are multiplying A1A2A3A4A5

We now check P[2][3] to see where the first split occurs when multiplying A2 - A3 and find 2. The third split occurs after 2:

(A1(A2(A3A4)))A5

We have now found the optimal multiplication order!

P:

	1	2	3	4	5
1		1	1	1	4
2			2	2	4
3				3	4
4					4
5					