SOAC Exercise 3 − Numerical Simulation of Wave Propagation

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Introduction

In this exercise we simulate the propagation of waves following an initial perturbation in height according to the linear one-layer shallow water equations across a periodic spatial domain.

After the initial perturbation is introduced to the system, it splits into two waves which travel in both directions with a constant phase velocity and no dispersion. The waves meet back at the same point a certain time later.

What we want to see is how accurately we can simulate this process, and how the simulation performs for different numbers of grid points and time step. Additionally we investigate what effect the use of an unstaggered grid has on the overall scheme.

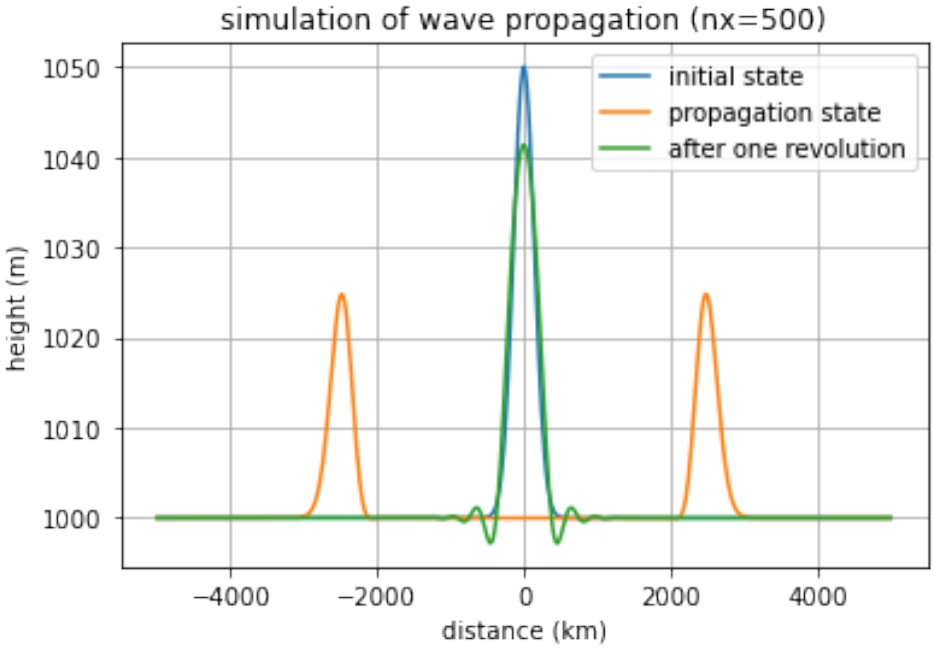

Figure 1: Simulation of wave propagation ($nx = 500$), nx is number of grid points. There is a perturbation in the middle of domain in the initial state, then two waves propagate in opposing direction. With the periodic boundary condition, the waves meet up again after one revolution.

a) Phase speed of the gravity waves

Theoretically, the phase speed of the shallow water gravity waves is given by:

$$c = c_g = \sqrt{gH} \tag{1}$$

In the initial state, phase speed is calculated as $\sqrt{gH} = 100m/s$, if we ignore the perturbation height. Numerical phase velocity depends on the wave number due to numerical dispersion:

$$c^* = \frac{\omega}{k} = c\frac{\sin k\Delta x}{k\Delta x} \tag{2}$$

In our case, the wave number k is very small, so

$$c^* \approx c \tag{3}$$

we can say that the surface waves propagate in both directions with constant phase velocity without loss of form.

b) Time of full revolution

The waves will be at the same position as they were initially after a period T equal to the ratio of the length of the domain to the phase speed of the gravity wave or

$$T = \frac{L}{c} \tag{4}$$

In this case we find that $T = 1.16$ days.

c) Different grid resolutions

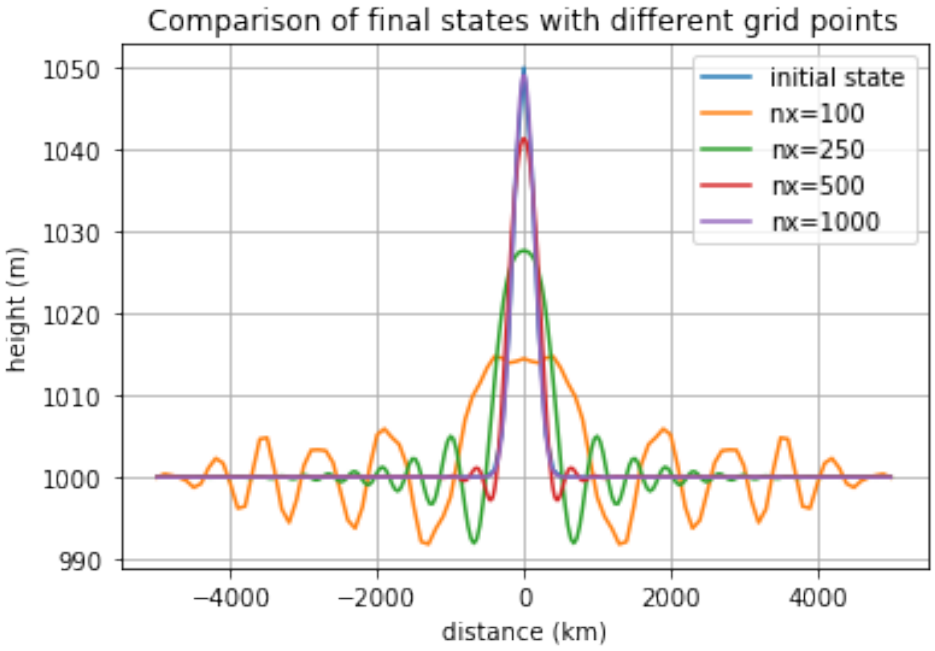

Figure 2: Comparison of final states with different grid points

Given the period T mentioned in the previous section, we now wish to evaluate the response of the numerical scheme for different numbers of grid points between 100 and 1000. For the purposes of this question we use values of 100, 250, 500, and 1000.

The results may be seen in Fig. 2. Here we can clearly see that for fewer grid points the final state is quite different to the initial state, whereas for more grid points (500 or 1000) the final state is more similar to the initial state.

It is clear that a higher resolution results in a more accurate simulation. This is because for a lower number of grid points, we will necessarily have a larger value for dx. As is usual for many simulations, having a particularly large step size can mean that the simulation can't really keep up with the measurements and will begin to deviate significantly from the initial conditions.

We can see a specific example of this in the equation for numerical phase velocity (Eq 2). Under ideal circumstances, the numerical phase velocity would approximate the real phase velocity. However, this numerical phase velocity is dependent on a balance between k and dx. We already know that k must be small for such long waves as we are dealing with; and therefore, dx must be large (O(105) in order to compensate. But it were to become too large, then the numerical phase velocity would begin to deviate significantly from the real phase velocity.

d) Does reducing dt help?

No, reducing dt does not help. This conclusion is shown in Fig. 3 that when we change dt to a factor of 5 there is not an evident change in the shape of the signal in the final state. This can be explained in the dispersion equation.

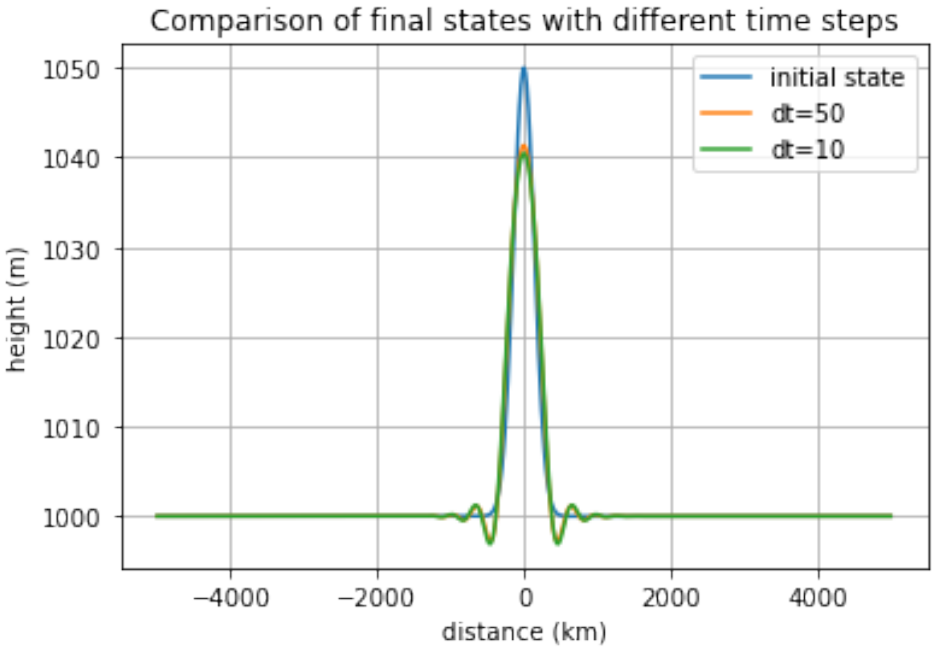

Figure 3: Comparison of final states with different time steps

If we plug in the analytical solution of h and u into the numerical SWE equations, we will arrive at a following equation:

$$\frac{\sin \omega \Delta t}{\Delta t}^2 = gH \frac{\sin k \Delta x}{\Delta x}^2 \tag{5}$$

In theory, analytically, we do not have any dt or dx, which means dt and dx are infinitely small and the dispersion equation is as what we are familiar with. But when $\omega\Delta t$ and $k\Delta x$, these two terms become bigger and finally so big that the dispersion errors appear.

Regarding to the input values in our case, $\sqrt{gH} = 100ms^{-1}$. This means that ω is 100 times larger than k. Furthermore, $dt = 10 \sim 50$ s is way smaller than $dx = 10^3 \sim 10^4$ m, which means a change in dt is relatively small to a change in dx. As a result, the term $\omega\Delta t$ is much smaller than $k\Delta x$, which means that dt is small enough and any reduction of dt does not necessarily lead to a better performance of the model.

$$\omega = \sqrt{gH}k \tag{6}$$

But, why can we not change dt significantly? The reason is that we have to keep the Courant number small enough to resolve the change in the signal, which is h in this case. Due to that, dt has to remain small.

$$\sigma = \frac{\sqrt{gh}\Delta t}{\Delta x} \tag{7}$$

e) Using a staggered grid

In the preceding sections we used a "normal", unstaggered grid. That is, for a grid of nx values divided up into sections of length dx, we evaluate the numerical scheme in between each of these sections. Now we would like to see what happens when for a grid of nx values, we instead evaluate the numerical scheme in the middle of these sections. We can't exactly do this computationally, so we must shift our initial domain L by $\frac{1}{2}dx$. Thus each step taken in x will now be a half-step with respect to the original domain definition.

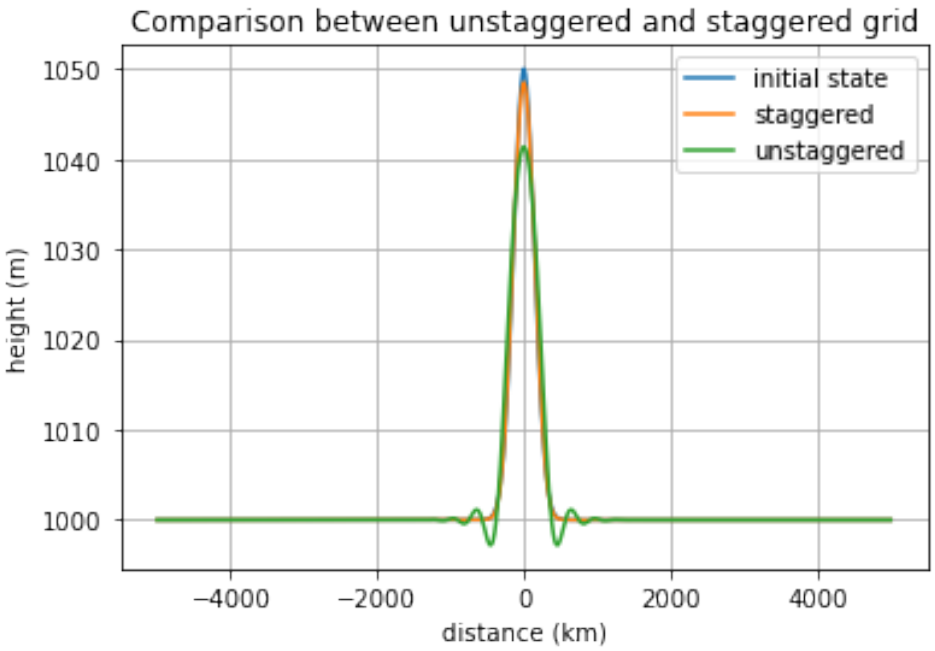

Figure 4: Comparison between unstaggered and staggered grid

We find that this method yields much more accurate results as can be seen in Figure 4.

Discussion

To conclude, we simulate the wave propagation using one dimensional shallow water equations. The higher the spatial grid resolution is, the better the final state is. However, the temporal grid resolution cannot really improve the result. Nevertheless, staggered grid improves the numerical solution.