SOAC Exercise 1

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Abstract

Here we examine the numerical accuracy of various numerical schemes for the inertial oscillation equation. Numerical stability and truncation error are compared, and the Heun scheme is found to be better than other schemes.

1 Introduction

In geophysical fluid dynamics, the inertial oscillation can be described by the following simplified equations:

$$\frac{\partial u(t)}{\partial t} = fv(t) \tag{1}$$

$$\frac{\partial v(t)}{\partial t} = -fu(t) \tag{2}$$

Combining the two equations, and given the initial values of u and v, we can solve the equations analytically, where the velocities are given by a trigonometric function depending on time t:

$$u(t) = C_1 \sin(ft) + C_2 \cos(ft) \tag{3}$$

$$v(t) = C_1 \cos(ft) - C_2 \sin(ft) \tag{4}$$

If $u(0) = 10\,\mathrm{ms}^{-1}$ and $v(0) = 0\,\mathrm{ms}^{-1}$, then $C_1 = 0$ and $C_2 = 10$, and

$$u(t) = 10\cos(ft) \tag{5}$$

$$v(t) = -10\sin(ft) \tag{6}$$

Numerically, using Taylor series approximation, we can choose different time schemes and compare their numerical stability and truncation error. As explained in the class, the Euler forward scheme has a "first order" error in Δt. Now we will analyse and compare the leap-frog, Matsuno, and Heun scheme.

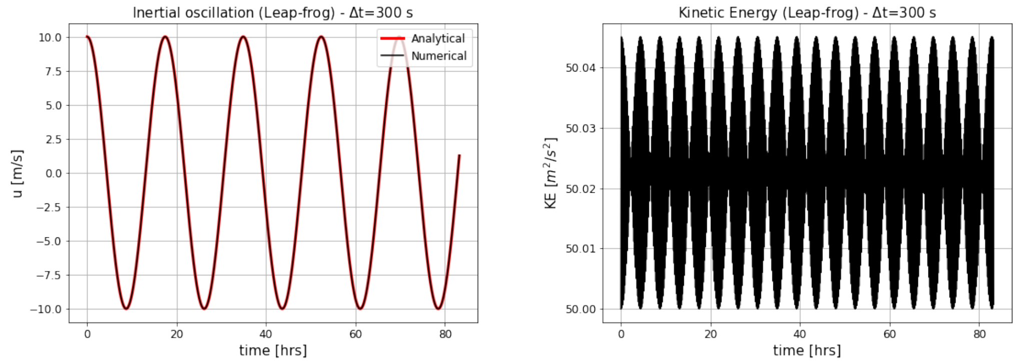
2 Leap-frog

The leap-frog scheme is a centered-time difference scheme. The equations for updating velocity are as follows:

$$\frac{u(t+\Delta t) - u(t-\Delta t)}{2\Delta t} = \frac{\partial u}{\partial t}(t) + \frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial t^3}(t) + O\left(\Delta t^3\right) \tag{7}$$

$$u(t+\Delta t) = u(t-\Delta t) + 2\Delta t f v(t)) \tag{8}$$

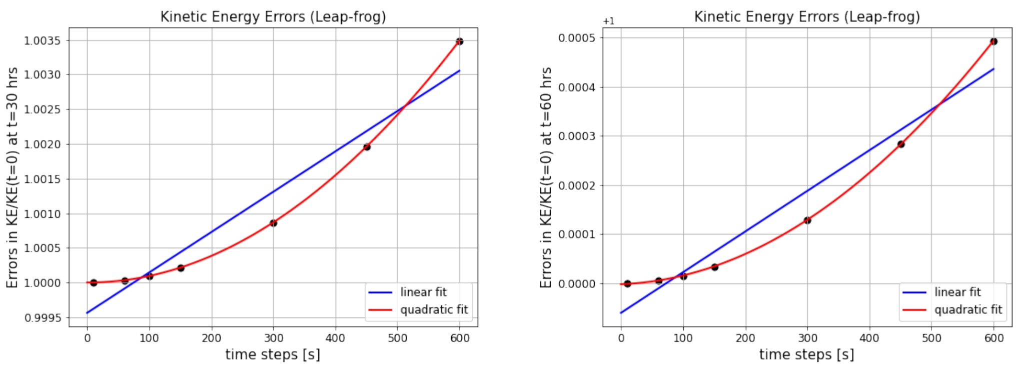
The truncation error is second order in Δt. To update velocity, we can initialise this scheme with one Euler-forward step.



(a) Leap-frog scheme: inertial oscillation

(b) Kinetic energy

Figure 1: Leap-frog scheme: velocity u (left) and Kinetic energy (right)



(a) Ke errors at $t = 30$hrs

(b) Ke errors at $t = 60$hrs

Figure 2: KE errors for different time steps at 30hrs and 60hrs

In Figure 1, we see the numerical solution of velocity u is almost overlapping with analytical one. Kinetic energy is not conserved, but always close to and fluctuating periodically around the analytical value 50.

As shown in Figure 2, the larger time step, the larger kinetic energy error. We also compare the kinetic energy at different times: the error at 30 hrs is much larger than that at 60 hrs.

Nevertheless, one drawback in the leap-frog scheme is that it is unreliable because odd and even time steps are decoupled.

3 Multi-step schemes

To overcome some of the difficulties encountered by the Euler forward and leap-frog schemes we can try multi-step schemes. In these schemes, we make a "first guess" estimate for the new velocity (the predictor step), and then correct that first guess with information from the current value of the other velocity (the corrector step).

For the predictor step, we use the equations

$$u_{n+1}^* - u_n = \Delta t F_n \tag{9}$$

$$v_{n+1}^* - v_n = \Delta t F_n \tag{10}$$

where u_{n+1}^* represents the *first guess* value of u we mentioned above. It's worth highlighting that this formulation looks similar to the Euler Forward scheme.

Then, in the corrector step we factor in the first guess value in a new calculation for the function. So, the *corrected* guess for u, for instance, is a factor of some multiple α of the current function F_n and some multiple β of the newly calculated function F_{n+1}^*.

$$u_{n+1} - u_n = \Delta t \left(\alpha F_n + \beta F_{n+1}^*\right) \tag{11}$$

$$v_{n+1} - v_n = \Delta t \left(\alpha F_n + \beta F_{n+1}^*\right) \tag{12}$$

It is assumed that $\alpha + \beta = 1$. What is left to decide is the exact value of each, which will be done in the following sections.

3.1 Matsuno Scheme

In the Matsuno scheme it is assumed that $\beta = 1$ (and therefore $\alpha = 0$). Using these values, we can simplify the multi-step formulae as given in Equations 13 and 14.

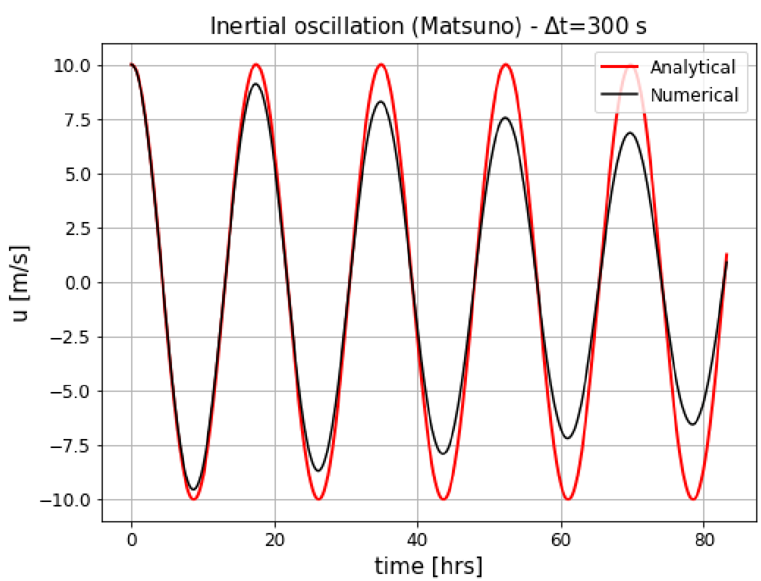


Figure 3: Matsuno scheme: inertial oscillation

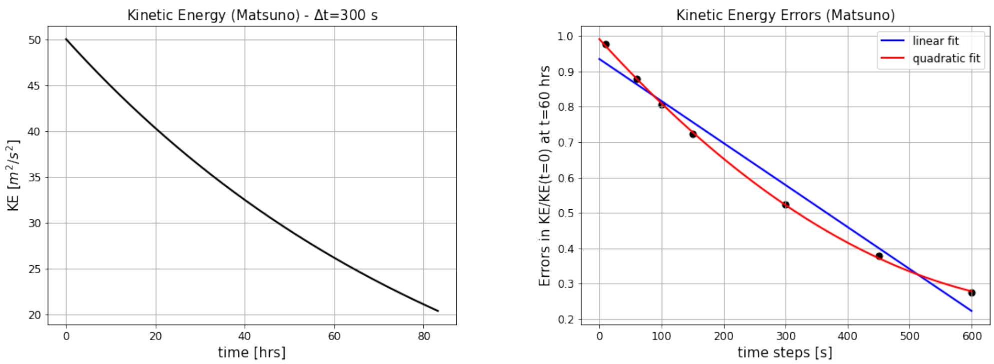
$$u_{n+1}^* = u_n + f v \Delta t \tag{13}$$

$$v_{n+1}^* = v_n - f u \Delta t \tag{14}$$

$$u_{n+1} = u_n + f v_{n+1}^* \Delta t \tag{15}$$

$$v_{n+1} = v_n - f u_{n+1}^* \Delta t \tag{16}$$

It can be seen in Figure 3 that the amplitude of the oscillation decreases instead of remaining constant. This is an example of numerical instability. Additionally, in Figure 4a we can see that the kinetic energy of the numerical oscillation is not conserved; in fact, it decreases with time. After a period of 60 hours, we can further see (Figure 4b) that for a variety of time steps there is a significant divergence from the initial kinetic energy. For instance, with just 100 time steps, the kinetic energy is 80% of its initial value, while for 300 time steps the energy is only a little over 50% of its initial value. Therefore it may not be the best scheme to use.



(a) Kinetic energy

(b) Ke error

Figure 4: Matsuno scheme:Kinetic energy (left) and Ke error (right)

3.2 Heun Scheme

In the Heun scheme it is assumed that $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{2}$. The first two steps are the same as in the Matsuno scheme (see equations 13 and 14). The latter two steps for the predictor equations are given below.

$$u_{n+1} = u_n + \frac{1}{2} f v^*_{n+1} \Delta t + \frac{1}{2} f v \Delta t \tag{17}$$

$$v_{n+1} = v_n - \frac{1}{2} f u^*_{n+1} \Delta t - \frac{1}{2} f u \Delta t \tag{18}$$

It can be seen in figure 5 that the numerical scheme matches the analytical results quite accurately. Kinetic energy changes only very slightly – by just a few thousandths of a joule (see Figure 6a). The error in fact is less than 0.1% for all time steps except for $\Delta t = 600\,\mathrm{s}$, which slightly exceeds that (see Figure 6b). At $\Delta t = 300\,\mathrm{s}$, the error is less than 0.02%. We can therefore say that Heun scheme approximately conserves kinetic energy and is the best scheme found thus far.

4 Conclusion

After comparing all three methods, we arrive at a conclusion that Heun's scheme has the best performance. Matsuno's scheme cannot even simulate the value correctly. The leapfrog case and Heun's cases simulate the velocity quite well with very small difference.

However, the performance of kinetic energy in leapfrog case and Heun case differs greatly. Compared to Heun case, the simulated KE in leapfrog case fluctuates a lot. If we are analysing the KE errors at $t = 60$ hrs, we discover that the KE errors in leapfrog are really small, even better than Heun. But when we look at the $t = 30$ hrs, the KE errors in

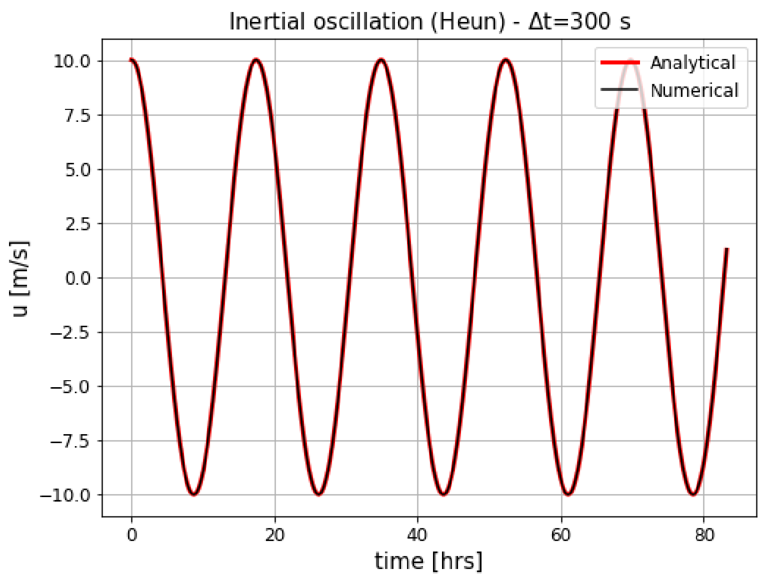
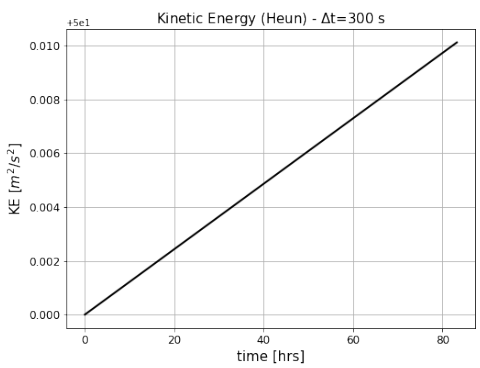
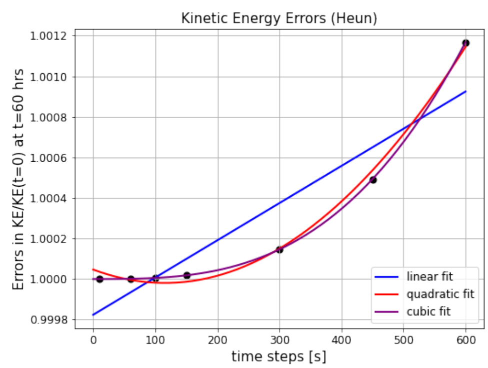


Figure 5: Heun scheme: inertial oscillation



(a) Kinetic energy



(b) Ke error

Figure 6: Heun scheme: Kinetic energy (left) and Ke error (right)

leapfrog becomes way larger than Heun. As a result, leapfrog case does not have a steady performance regarding to kinetic energy errors, which is not good to control the simulation errors. On the contrary, the errors in Heun case is linearly varying, which is easy for us to predict and analyse.

With the arguments above, we conclude that the Heun's scheme is the best choice in this exercise.