

# Bessel's Inequality

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## Theorem

Let  $(V, \langle \cdot, \cdot \rangle)$  be an inner product space.

Let  $\|\cdot\|$  be the inner product norm for  $(V, \langle \cdot, \cdot \rangle)$ .

Let  $E = \{e_n : n \in \mathbb{N}\}$  be a countably infinite orthonormal subset of  $V$ .

Then, for all  $h \in V$ :

$$\sum_{n=1}^{\infty} |\langle h, e_n \rangle|^2 \leq \|h\|^2$$

### Corollary 1

Let  $E$  be a orthonormal subset of  $V$ .

Then, for each  $h \in V$ , the set:

$$\{e \in E : \langle h, e \rangle \neq 0\}$$

is countable.

### Corollary 2

Let  $E$  be a orthonormal subset of  $V$ .

Then, for all  $h \in V$ :

$$\sum_{e \in E} |\langle h, e \rangle|^2 \leq \|h\|^2$$

## Proof

Note that for any natural number  $n$  we have, applying sesquilinearity of the inner product:

$$\begin{aligned} \left\| h - \sum_{k=1}^n \langle h, e_k \rangle e_k \right\|^2 &= \left\langle h - \sum_{k=1}^n \langle h, e_k \rangle e_k, h - \sum_{j=1}^n \langle h, e_j \rangle e_j \right\rangle \\ &= \left\langle h, h - \sum_{j=1}^n \langle h, e_j \rangle e_j \right\rangle - \left\langle \sum_{k=1}^n \langle h, e_k \rangle e_k, h - \sum_{j=1}^n \langle h, e_j \rangle e_j \right\rangle \\ &= \langle h, h \rangle - \left\langle h, \sum_{j=1}^n \langle h, e_j \rangle e_j \right\rangle - \left\langle \sum_{k=1}^n \langle h, e_k \rangle e_k, h \right\rangle + \left\langle \sum_{k=1}^n \langle h, e_k \rangle e_k, \sum_{j=1}^n \langle h, e_j \rangle e_j \right\rangle \end{aligned}$$

$$\begin{aligned}
&= \|h\|^2 - \left\langle h, \sum_{j=1}^n \langle h, e_j \rangle e_j \right\rangle - \overline{\left\langle h, \sum_{j=1}^n \langle h, e_j \rangle e_j \right\rangle} + \left\| \sum_{k=1}^n \langle h, e_k \rangle e_k \right\|^2 \\
&= \|h\|^2 - \left\langle h, \sum_{j=1}^n \langle h, e_j \rangle e_j \right\rangle - \overline{\left\langle h, \sum_{j=1}^n \langle h, e_j \rangle e_j \right\rangle} + \sum_{k=1}^n \|\langle h, e_k \rangle e_k\|^2 \\
&= \|h\|^2 - \left\langle h, \sum_{j=1}^n \langle h, e_j \rangle e_j \right\rangle - \overline{\left\langle h, \sum_{j=1}^n \langle h, e_j \rangle e_j \right\rangle} + \sum_{k=1}^n |\langle h, e_k \rangle|^2
\end{aligned}$$

We have:

$$\begin{aligned}
\left\langle h, \sum_{j=1}^n \langle h, e_j \rangle e_j \right\rangle &= \sum_{j=1}^n \langle h, \langle h, e_j \rangle e_j \rangle && \text{sesquilinearity of inner product} \\
&= \sum_{j=1}^n \overline{\langle \langle h, e_j \rangle e_j, h \rangle} && \text{conjugate symmetry of inner product} \\
&= \sum_{j=1}^n \overline{\langle e_j, h \rangle \langle h, e_j \rangle} \\
&= \sum_{j=1}^n \langle h, e_j \rangle \overline{\langle h, e_j \rangle} && \text{conjugate symmetry of inner product} \\
&= \sum_{j=1}^n |\langle h, e_j \rangle|^2 && \text{Product of Complex Number with Conjugate}
\end{aligned}$$

Therefore:

$$\begin{aligned}
\left\| h - \sum_{k=1}^n \langle h, e_k \rangle e_k \right\|^2 &= \|h\|^2 - \sum_{j=1}^n |\langle h, e_j \rangle|^2 - \overline{\sum_{j=1}^n |\langle h, e_j \rangle|^2} + \sum_{k=1}^n |\langle h, e_k \rangle|^2 \\
&= \|h\|^2 - 2 \sum_{j=1}^n |\langle h, e_j \rangle|^2 + \sum_{k=1}^n |\langle h, e_k \rangle|^2 && \begin{array}{l} \text{since } |\langle h, e_j \rangle|^2 \text{ is real} \\ \text{for each } j, \text{ we have} \\ \sum_{j=1}^n |\langle h, e_j \rangle|^2 \in \mathbb{R} \end{array} \\
&= \|h\|^2 - \sum_{k=1}^n |\langle h, e_k \rangle|^2
\end{aligned}$$

Since:

$$\left\| h - \sum_{k=1}^n \langle h, e_k \rangle e_k \right\|^2 \geq 0$$

we have:

$$\sum_{k=1}^n |\langle h, e_k \rangle|^2 \leq \|h\|^2$$

Since:

$$|\langle h, e_k \rangle|^2 \geq 0 \text{ for each } k$$

we have that:

$$\text{the sequence } \left\langle \sum_{k=1}^n |\langle h, e_k \rangle|^2 \right\rangle_{n \in \mathbb{N}} \text{ is increasing.}$$

So:

the sequence  $\left\langle \sum_{k=1}^n |\langle h, e_k \rangle|^2 \right\rangle_{n \in \mathbb{N}}$  is bounded and increasing.

So from Monotone Convergence Theorem (Real Analysis): Increasing Sequence, we have that:

the sequence  $\left\langle \sum_{k=1}^n |\langle h, e_k \rangle|^2 \right\rangle_{n \in \mathbb{N}}$  converges.

Since:

$$\sum_{k=1}^n |\langle h, e_k \rangle|^2 \leq \|h\|^2 \text{ for each } n$$

we then have from Limits Preserve Inequalities:

$$\|h\|^2 \geq \lim_{n \rightarrow \infty} \sum_{k=1}^n |\langle h, e_k \rangle|^2 = \sum_{k=1}^{\infty} |\langle h, e_k \rangle|^2$$

■

## Source of Name

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This entry was named for Friedrich Wilhelm Bessel.

## Sources

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- 1990: John B. Conway: *A Course in Functional Analysis* (2nd ed.) ... [\(previous\)](#) ... [\(next\)](#): I Hilbert Spaces: §4. Orthonormal Sets of Vectors and Bases: 4.8 Bessel's Inequality

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