CS2x1:Data Structures and Algorithms

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Recap: All Pair Shortest Path (APSP)

Shortest Path

Single-pair shortest path (SSSP) | **All-pair shortest path (APSP)**

Positive Weights | **Negative Weights**

Dijkstra

- Single source vertex:
  - ✓ Time complexity → O (E log V)
  - ✓ *G: Dense graph* E ≈ V^2
  - ✓ Time complexity → O (V^2 log V)

Bellman-Ford

- Single source vertex:
  - ✓ Time complexity → O (VE)
  - ✓ *G: Dense graph* E ≈ V^2
  - ✓ Time complexity → O (V^3)

- All pair source vertex:

Dijkstra

✓ all-pair → v-vertices → dense graph → O (V^3 log V)

Bellman-Ford

✓ all-pair → v-vertices → dense graph → O (V^4)

Bellman-Ford

✓ all-pair → v-vertices → sparse graph → O (V^2E)

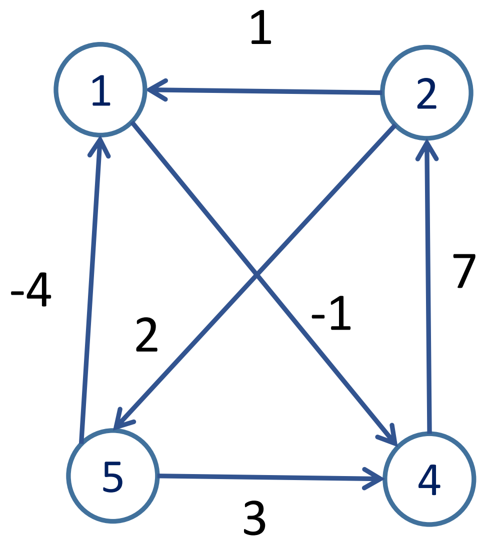
SLOW-ALL-PAIRS-SHORTEST-PATHS (W)

Total time complexity: O(V^4)

EXTEND-SHORTEST-PATHS (L, W)

Total time complexity: O(V^3 log V)

APSP: Floyd-Warshall → dynamic programming

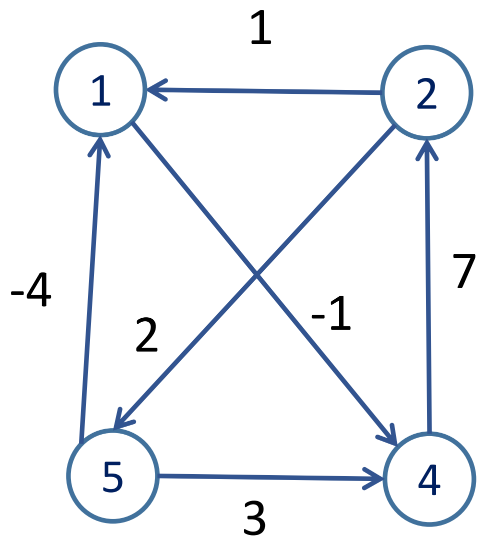


$D^{(k)} = (d_{ij}^{(k)})$

APSP: Floyd-Warshall → dynamic programming (1)

$D^{(k)} = (d_{ij}^{(k)})$

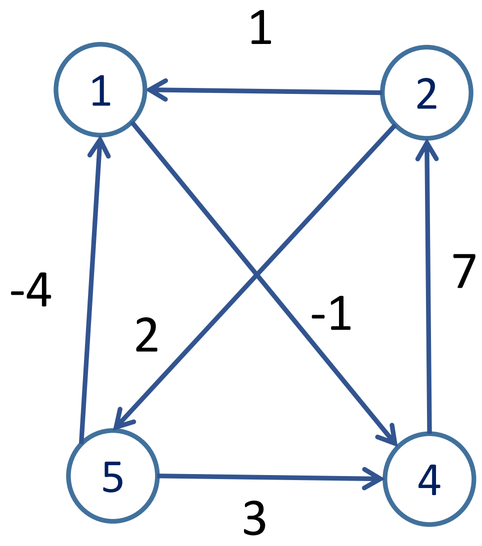
$D^{(1)} =$



APSP: Floyd-Warshall → dynamic programming (2)

$D^{(k)} = (d_{ij}^{(k)})$

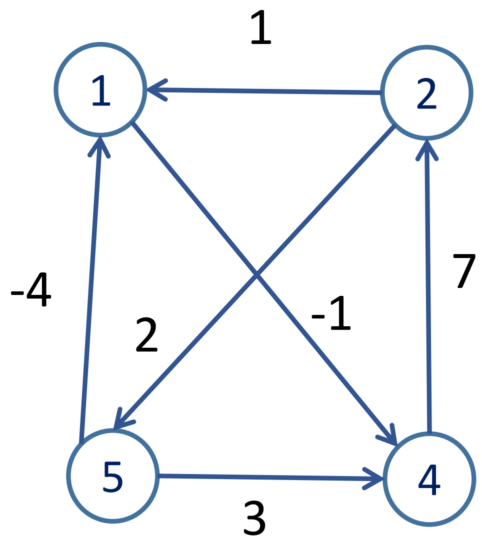
$D^{(2)} =$



APSP: Floyd-Warshall → dynamic programming (3)

$D^{(k)} = (d_{ij}^{(k)})$

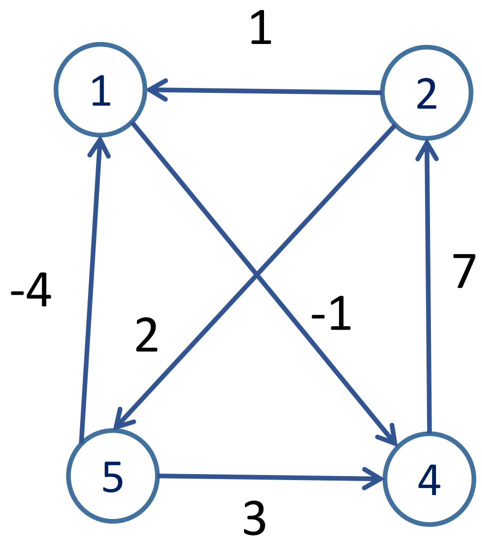
$D^{(3)} =$



APSP: Floyd-Warshall → dynamic programming (4)

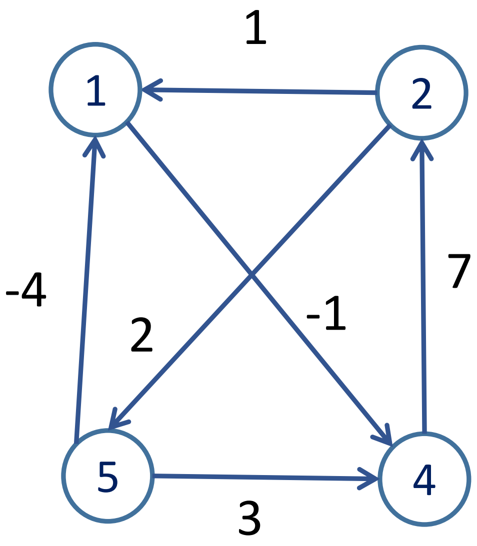
$D^{(k)} = (d_{ij}^{(k)})$

$D^{(4)} =$

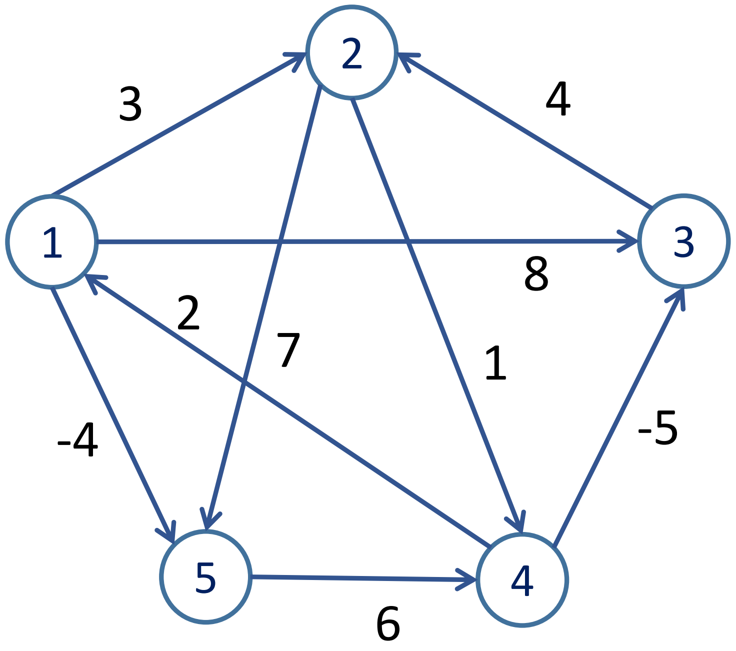


APSP: Floyd-Warshall → dynamic programming (5)

$D^{(k)} = (d_{ij}^{(k)})$



APSP: Floyd-Warshall



Floyd-WARSHALL (W) {

1 n = W. rows

2 $D^{(0)}$ = W

3 for k = 1 to n

4 let $D^{(k)} = (d_{ij}^{(k)})$ be a new $n \times n$ matrix

5 for i = 1 to n

6 for j = 1 to n

7 $d_{ij}^{(k)} = \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})$

8 return $D^{(n)}$

- W(i,j) = 0; *if i=j*
  = ∞; *if there is no edge between i and j*
  = "weight of edge"

- $d_{ij}^{(k)} = w_{ij}$; *if k = 0*
  $= \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})$; if $k \geq 1$

- $\pi_{ij}^{(0)} = NIL$; *if* $i = j$ *or* $w_{ij} = \infty$,
  $= i$; *if* $i \neq j$ *and* $w_{ij} < \infty$

Step 1: n = W. rows = 5

Step 2: $D^{(0)}$ = W

$$\begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

$$\pi^{(0)} = \begin{pmatrix} NIL & 1 & 1 & NIL & 1 \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & NIL & NIL \\ 4 & NIL & 4 & NIL & NIL \\ NIL & NIL & NIL & 5 & NIL \end{pmatrix}$$

APSP: Floyd-Warshall (2)

$$D^{(0)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

Floyd-WARSHALL (W) {

1 n = W. rows
2 $D^{(0)}$ = W
3 for k = 1 to n
4 let $D^{(k)} = (d_{ij}^{(k)})$ be a new $n \times n$ matrix
5 for i = 1 to n
6 for j = 1 to n
7 $d_{ij}^{(k)} = \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})$
8 return $D^{(n)}$

Step 3: k = 1
Step 4: $D^{(1)} = d_{ij}^{(1)}$
Step 5: i = 1
Step 6: j = 1 to 5
Step 7:

$d_{11}^{(1)} = \min(d_{11}^{(0)}, d_{11}^{(0)} + d_{11}^{(0)}) \rightarrow \min(0, 0+0) \rightarrow 0$
$d_{12}^{(1)} = \min(d_{12}^{(0)}, d_{11}^{(0)} + d_{12}^{(0)}) \rightarrow \min(3, 0+3) \rightarrow 3$
$d_{13}^{(1)} = \min(d_{13}^{(0)}, d_{11}^{(0)} + d_{13}^{(0)}) \rightarrow \min(8, 0+8) \rightarrow 8$
$d_{14}^{(1)} = \min(d_{14}^{(0)}, d_{11}^{(0)} + d_{14}^{(0)}) \rightarrow \min(\infty, 0+\infty) \rightarrow \infty$
$d_{15}^{(1)} = \min(d_{15}^{(0)}, d_{11}^{(0)} + d_{15}^{(0)}) \rightarrow \min(-4, 0+-4) \rightarrow -4$

Step 3: k = 1
Step 4: $D^{(1)} = d_{ij}^{(1)}$
Step 5: i = 4
Step 6: j = 1 to 5
Step 7:

$d_{41}^{(1)} = \min(d_{41}^{(0)}, d_{41}^{(0)} + d_{11}^{(0)}) \rightarrow \min(2, 2+0) \rightarrow 2$
$d_{42}^{(1)} = \min(d_{42}^{(0)}, d_{41}^{(0)} + d_{12}^{(0)}) \rightarrow \min(\infty, 2+3) \rightarrow 5$
$d_{43}^{(1)} = \min(d_{43}^{(0)}, d_{41}^{(0)} + d_{13}^{(0)}) \rightarrow \min(-5, 2+8) \rightarrow -5$
$d_{44}^{(1)} = \min(d_{44}^{(0)}, d_{41}^{(0)} + d_{14}^{(0)}) \rightarrow \min(0, 2+\infty) \rightarrow 0$
$d_{45}^{(1)} = \min(d_{45}^{(0)}, d_{41}^{(0)} + d_{15}^{(0)}) \rightarrow \min(\infty, 2+-4) \rightarrow -2$

- $\pi_{ij}^{(k)} = \pi_{ij}^{(k-1)}$; *if* $d_{ij}^{(k-1)} \le d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$,
  $= \pi_{kj}^{(k-1)}$ *i; if* $d_{ij}^{(k-1)} > d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$

$$\pi^{(1)} = \begin{pmatrix} \text{NIL} & 1 & 1 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & \text{NIL} & \text{NIL} \\ 4 & \text{NIL} & 4 & \text{NIL} & \text{NIL} \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{pmatrix}$$

APSP: Floyd-Warshall (3)

$$D^{(1)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

Floyd-WARSHALL (W) {

1 n = W. rows

2 $D^{(0)}$ = W

3 for k = 1 to n

4 let $D^{(k)} = (d_{ij}^{(k)})$ be a new $n \times n$ matrix

5 for i = 1 to n

6 for j = 1 to n

7 $d_{ij}^{(k)} = \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})$

8 return $D^{(n)}$

Step 3: k = 2

Step 4: $D^{(2)} = d_{ij}^{(2)}$

Step 5: i = 1

Step 6: j = 1 to 5

Step 7:

$d_{11}^{(2)} = \min(d_{11}^{(1)}, d_{12}^{(1)} + d_{21}^{(1)}) \rightarrow \min(0, 3+\infty) \rightarrow 0$

$d_{12}^{(2)} = \min(d_{12}^{(1)}, d_{12}^{(1)} + d_{22}^{(1)}) \rightarrow \min(3, 3+0) \rightarrow 3$

$d_{13}^{(2)} = \min(d_{13}^{(1)}, d_{12}^{(1)} + d_{23}^{(1)}) \rightarrow \min(8, 3+\infty) \rightarrow 8$

$d_{14}^{(2)} = \min(d_{14}^{(1)}, d_{12}^{(1)} + d_{24}^{(1)}) \rightarrow \min(\infty, 3+1) \rightarrow 4$

$d_{15}^{(2)} = \min(d_{15}^{(1)}, d_{12}^{(1)} + d_{25}^{(1)}) \rightarrow \min(-4, 0+-4) \rightarrow -4$

Step 3: k = 2

Step 4: $D^{(2)} = d_{ij}^{(2)}$

Step 5: i = 3

Step 6: j = 1 to 5

Step 7:

$d_{31}^{(2)} = \min(d_{31}^{(1)}, d_{32}^{(1)} + d_{21}^{(1)}) \rightarrow \min(\infty, 4+\infty) \rightarrow \infty$

$d_{32}^{(2)} = \min(d_{32}^{(1)}, d_{32}^{(1)} + d_{22}^{(1)}) \rightarrow \min(4, 4+0) \rightarrow 4$

$d_{33}^{(2)} = \min(d_{33}^{(1)}, d_{32}^{(1)} + d_{23}^{(1)}) \rightarrow \min(0, 4+\infty) \rightarrow 0$

$d_{34}^{(2)} = \min(d_{34}^{(1)}, d_{32}^{(1)} + d_{24}^{(1)}) \rightarrow \min(\infty, 4+1) \rightarrow 5$

$d_{35}^{(2)} = \min(d_{35}^{(1)}, d_{32}^{(1)} + d_{25}^{(1)}) \rightarrow \min(\infty, 4+7) \rightarrow 11$

- $\pi_{ij}^{(k)} = \pi_{ij}^{(k-1)}$; *if* $d_{ij}^{(k-1)} \le d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$,
  $= \pi_{kj}^{(k-1)}$ *; if* $d_{ij}^{(k-1)} > d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$

$$\pi^{(2)} = \begin{pmatrix} \text{NIL} & 1 & 1 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & \text{NIL} & \text{NIL} \\ 4 & 1 & 4 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{pmatrix}$$

APSP: Floyd-Warshall (4)

Floyd-WARSHALL (W) {
1 n = W. rows
2 $D^{(0)}$ = W
3 for k = 1 to n
4 let $D^{(k)} = (d_{ij}^{(k)})$ be a new $n \times n$ matrix
5 for i = 1 to n
6 for j = 1 to n
7 $d_{ij}^{(k)} = \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})$
8 return $D^{(n)}$

$$D^{(2)} \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

- $\pi_{ij}^{(k)} = \pi_{ij}^{(k-1)}$; *if* $d_{ij}^{(k-1)} \le d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$,
  $= \pi_{kj}^{(k-1)}$ *i; if* $d_{ij}^{(k-1)} > d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$

Step 3: k = 3
Step 4: $D^{(1)} = d_{ij}^{(1)}$
Step 5: i = 4
Step 6: j = 1 to 5
Step 7:
$d_{41}^{(3)} = \min(d_{41}^{(2)}, d_{43}^{(2)} + d_{31}^{(2)}) \rightarrow \min(2, -5+\infty) \rightarrow 2$
$d_{42}^{(3)} = \min(d_{42}^{(2)}, d_{43}^{(2)} + d_{32}^{(2)}) \rightarrow \min(5, -5+4) \rightarrow -1$
$d_{43}^{(3)} = \min(d_{43}^{(2)}, d_{43}^{(2)} + d_{33}^{(2)}) \rightarrow \min(-5, -5+0) \rightarrow -5$
$d_{44}^{(3)} = \min(d_{44}^{(2)}, d_{43}^{(2)} + d_{34}^{(2)}) \rightarrow \min(0, -5+5) \rightarrow 0$
$d_{45}^{(3)} = \min(d_{45}^{(2)}, d_{43}^{(2)} + d_{35}^{(2)}) \rightarrow \min(-2, -5+11) \rightarrow -2$

$$\pi^{(3)} = \begin{pmatrix} \text{NIL} & 1 & 1 & 2 & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & 2 & 2 \\ 4 & 1 & 4 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{pmatrix}$$

APSP: Floyd-Warshall (5)

$$D^{(3)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

Floyd-WARSHALL (W) {

1 n = W. rows

2 $D^{(0)}$ = W

3 for k = 1 to n

4 let $D^{(k)} = (d_{ij}^{(k)})$ be a new $n \times n$ matrix

5 for i = 1 to n

6 for j = 1 to n

7 $d_{ij}^{(k)} = \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})$

8 return $D^{(n)}$

Step 3: k = 4

Step 4: $D^{(4)} = d_{ij}^{(4)}$

Step 5-7:

$d_{13}^{(4)} = \min(d_{13}^{(3)}, d_{14}^{(3)} + d_{43}^{(3)}) \rightarrow \min(8, 4+ -5) \rightarrow -1$

$d_{21}^{(4)} = \min(d_{21}^{(3)}, d_{24}^{(3)} + d_{41}^{(3)}) \rightarrow \min(\infty, 1+2) \rightarrow 3$

$d_{23}^{(4)} = \min(d_{23}^{(3)}, d_{24}^{(3)} + d_{43}^{(3)}) \rightarrow \min(0, 1+ -5) \rightarrow -4$

$d_{25}^{(4)} = \min(d_{25}^{(3)}, d_{24}^{(3)} + d_{45}^{(3)}) \rightarrow \min(7, 1+-2) \rightarrow -1$

$d_{31}^{(4)} = \min(d_{31}^{(3)}, d_{34}^{(3)} + d_{41}^{(3)}) \rightarrow \min(\infty, 5+2) \rightarrow 7$

$d_{35}^{(4)} = \min(d_{35}^{(3)}, d_{34}^{(3)} + d_{45}^{(3)}) \rightarrow \min(11, 5+-2) \rightarrow 3$

$d_{51}^{(4)} = \min(d_{51}^{(3)}, d_{54}^{(3)} + d_{41}^{(3)}) \rightarrow \min(\infty, 6+ 2) \rightarrow 8$

$d_{52}^{(4)} = \min(d_{52}^{(3)}, d_{54}^{(3)} + d_{42}^{(3)}) \rightarrow \min(\infty, 6+-1) \rightarrow 5$

$d_{53}^{(4)} = \min(d_{53}^{(3)}, d_{54}^{(3)} + d_{43}^{(3)}) \rightarrow \min(\infty, 6+ -5) \rightarrow 1$

$$\pi^{(4)} = \begin{pmatrix} NIL & 1 & 1 & 2 & 1 \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & 2 & 2 \\ 4 & 3 & 4 & NIL & 1 \\ NIL & NIL & NIL & 5 & NIL \end{pmatrix}$$

- $\pi_{ij}^{(k)} = \pi_{ij}^{(k-1)}$; *if* $d_{ij}^{(k-1)} \leq d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$,
  $= \pi_{kj}^{(k-1)}$ *i; if* $d_{ij}^{(k-1)} > d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$

APSP: Floyd-Warshall (6)

Floyd-WARSHALL (W) {
1 n = W. rows
2 $D^{(0)}$ = W
3 for k = 1 to n
4 let $D^{(k)} = (d_{ij}^{(k)})$ be a new n x n matrix
5 for i = 1 to n
6 for j = 1 to n
7 $d_{ij}^{(k)} = \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})$
8 return $D^{(n)}$

Step 3: k = 5
Step 4: $D^{(5)} = d_{ij}^{(5)}$

$$D^{(4)} \begin{pmatrix} 0 & 3 & -1 & 4 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix}$$

Step 5-7:

$d_{12}^{(5)} = \min(d_{12}^{(4)}, d_{15}^{(4)} + d_{52}^{(4)}) \rightarrow \min(3, -4+5) \rightarrow 1$

$d_{13}^{(5)} = \min(d_{13}^{(4)}, d_{15}^{(4)} + d_{53}^{(4)}) \rightarrow \min(-1, -4+1) \rightarrow -3$

$d_{14}^{(5)} = \min(d_{14}^{(4)}, d_{15}^{(4)} + d_{54}^{(4)}) \rightarrow \min(4, -4+6) \rightarrow 2$

- $\pi_{ij}^{(k)} = \pi_{ij}^{(k-1)}$; *if* $d_{ij}^{(k-1)} \leq d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$,
  $= \pi_{kj}^{(k-1)}$ *i; if* $d_{ij}^{(k-1)} > d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$

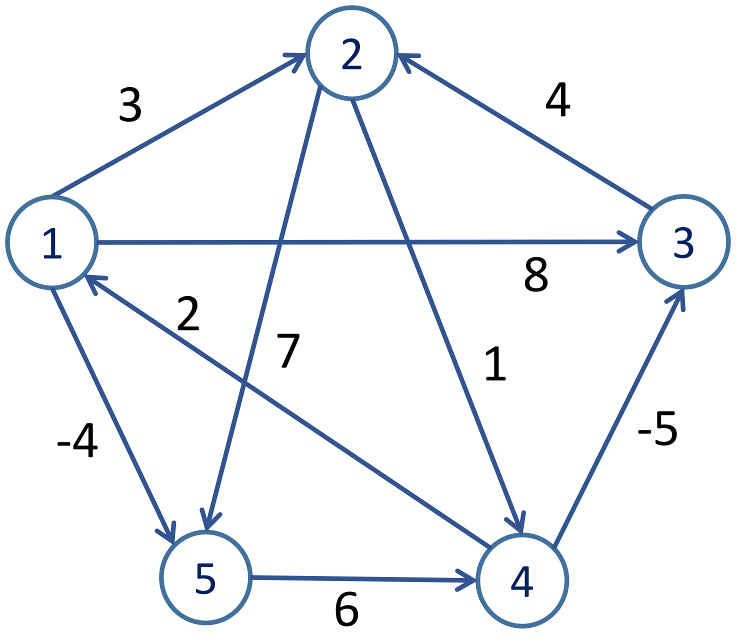
$$D^{(5)} \begin{pmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix} \Rightarrow \pi^{(5)} = \begin{pmatrix} \text{NIL} & 3 & 4 & 5 & 1 \\ 4 & \text{NIL} & 4 & 2 & 1 \\ 4 & 3 & \text{NIL} & 2 & 1 \\ 4 & 3 & 4 & \text{NIL} & 1 \\ 4 & 3 & 4 & 5 & \text{NIL} \end{pmatrix}$$

APSP: Floyd-Warshall (7)

```
Floyd-WARSHALL (W) {
n = W. rows
D(0) = W
for k = 1 to n
    let D(k) = (dij(k)) be a new n x n matrix
    for i = 1 to n
        for j = 1 to n
            dij(k) = min (dij(k-1) , dik(k-1) + dkj(k-1) )
return D(n)
```

$$D^{(5)} = \begin{pmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix}$$

$$\pi^{(5)} = \begin{pmatrix} NIL & 3 & 4 & 5 & 1 \\ 4 & NIL & 4 & 2 & 1 \\ 4 & 3 & NIL & 2 & 1 \\ 4 & 3 & 4 & NIL & 1 \\ 4 & 3 & 4 & 5 & NIL \end{pmatrix}$$

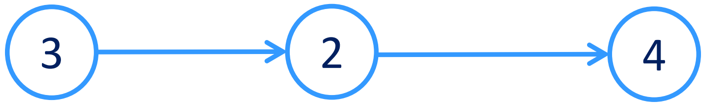


$D_{24}^{(5)} = 1$ $\pi_{24}^{(5)} = 2$

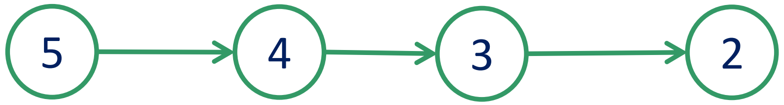
Example#1:

Example#2: $D_{34}^{(5)} = 5$ $\pi_{34}^{(5)} = 2$

$\pi_{32}^{(5)} = 3$



Example#3: $D_{52}^{(5)} = 5$



$\pi_{54}^{(5)} = 5$ $\pi_{53}^{(5)} = 4$ $\pi_{52}^{(5)} = 2$

APSP: Floyd-Warshall → time complexity

Floyd-WARSHALL (W) {

n = W. rows

$D^{(0)}$ = W

for k = 1 to n ← **O (V)** **n= # of vertices = |V|**

    let $D^{(k)} = (d_{ij}^{(k)})$ be a new $n \times n$ matrix

    for i = 1 to n ← **O (V)**

        for j = 1 to n ← **O (V)**

            $d_{ij}^{(k)} = \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})$ ← **O(V^3)**

return $D^{(n)}$

Total time complexity: <u>O(V^3) → O(n^3)</u>

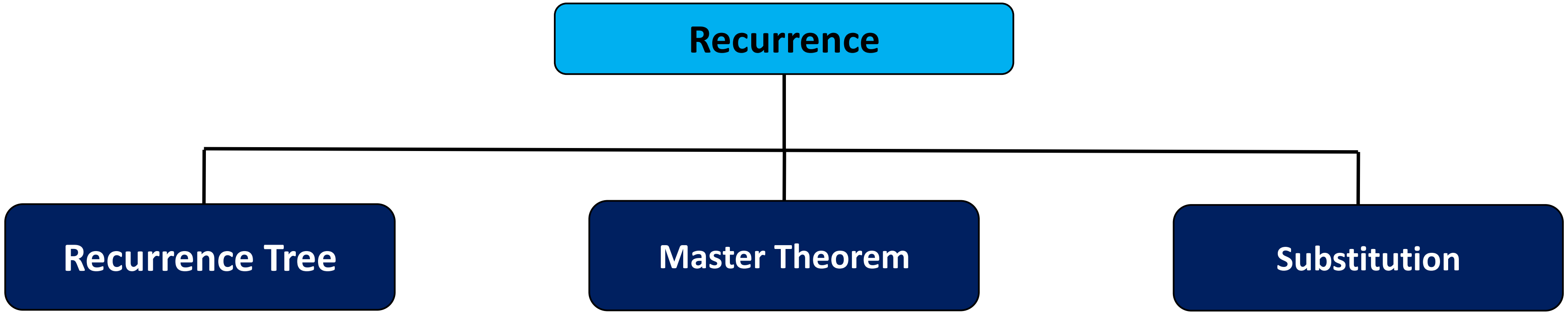
List of Topics [C201]

- Introduction:
  - *Data structures*
  - *Abstract data types*
  - *Analysis of algorithms.*

- Creation and manipulation of data structures:
  - *Arrays; Stacks; Queues; Linked lists; Trees; Heaps; Hash tables; Balanced trees [AVL]*; *Graphs*.

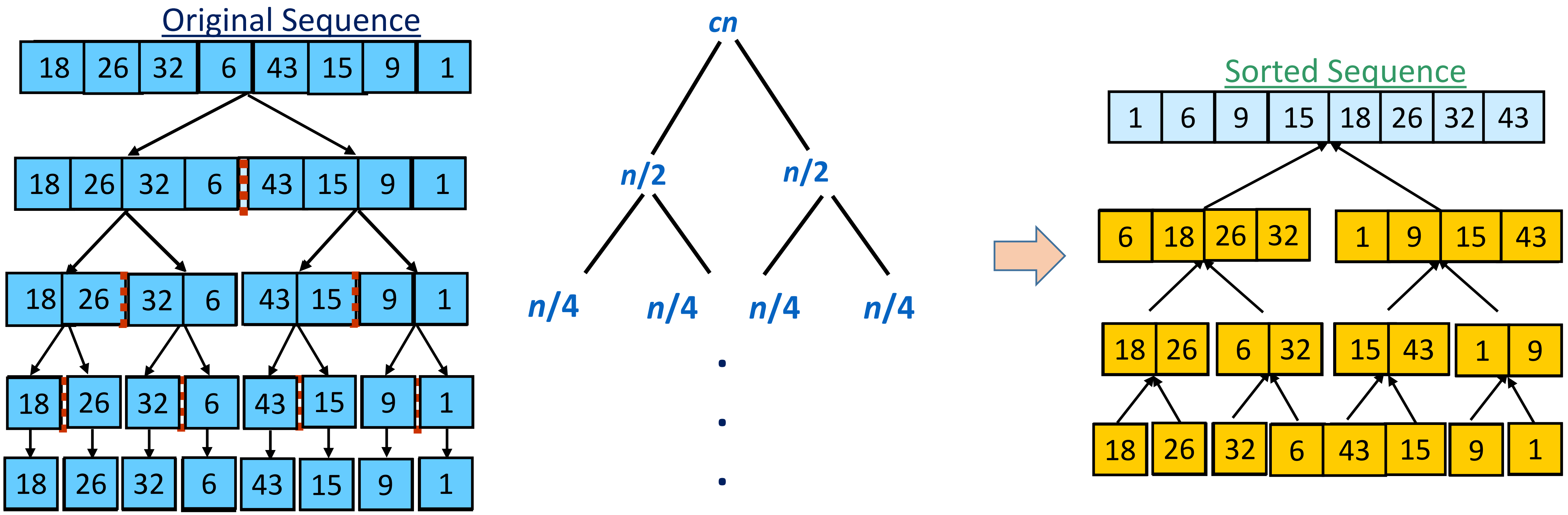
- Algorithms for sorting and searching, *depth-first and breadth-first search, shortest paths and minimum spanning tree*.

Algorithms analysis



Recurrence

- Recurrence relation: A recurrence is an equation or inequality that describes a function in terms of its values on smaller inputs.
  - For example, merge sort → T(n) = 2 T(n/2) + cn; divide-and-conquer

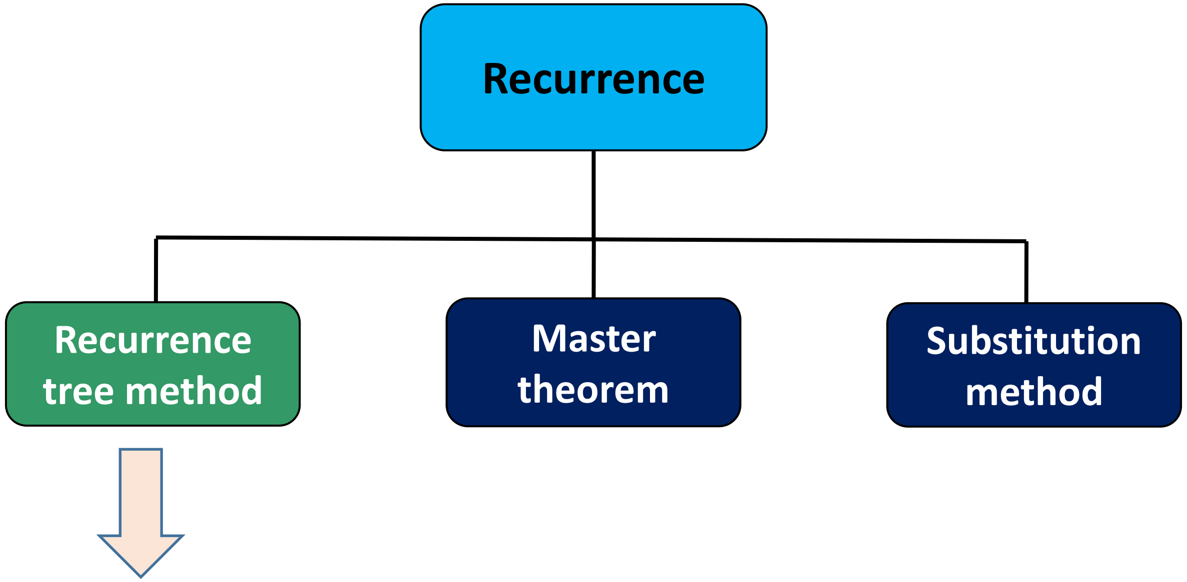


Basic math

- Logarithms
  - ✓ $log\ x^y = y\ logx$
  - ✓ $log\ xy = logx + logy$
  - ✓ $a^{logx_b} = x^{loga_b}$
- Arithmetic series
  - ✓ $\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \ n(n+1)/2$
- Geometric series
  - ✓ $\sum_{k=1}^{n} x^k = 1 + x^1 + x^2 + x^3 + \cdots + x^n = \frac{x^{n+1}-1}{x-1} \qquad (x \neq 1)$
  - ✓ $a + ar + ar^2 + \cdots = \sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}\ \ (for\ |r| < 1)$
- Harmonic series
  - ✓ $\sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \cdots + \frac{1}{n}\ \sim logn$

Recurrence Tree

- *Recursion Tree Method* is a pictorial representation, which is in the form of a tree where at each level nodes are expanded.
- It is useful when the *divide-and-conquer* algorithm is used.
- It is sometimes difficult to come up with a good guess. In Recursion tree, each root and child represents the cost of a single sub problem.



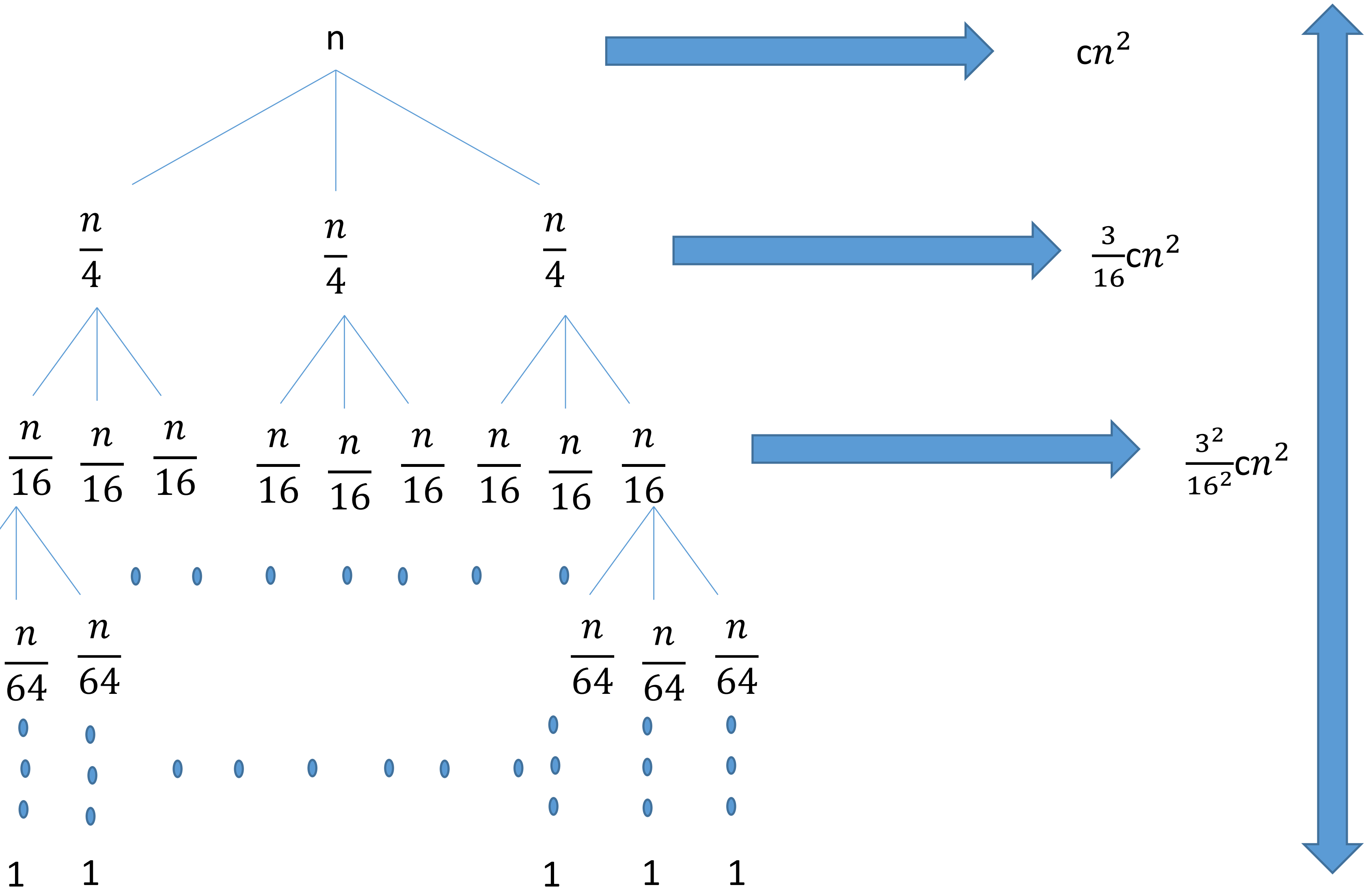
$$T(n) = 3T(\frac{n}{4}) + cn^2$$

Recurrence Tree

$T(n) = 3T(\frac{n}{4}) + cn^2$

$T(n/4) = 3T(\frac{n}{16}) + c(n^2/16)$

$T(n/16) = 3T(\frac{n}{64}) + c(n^2/16^2)$



Recurrence Tree

T(n) = 3T($\frac{n}{4}$)+ cn^2

Geometric series:

$a + ar + ar^2 + \cdots = \sum_{k=0}^{\infty} ar^k = \frac{a}{1-r} \quad (for\ |r| < 1)$

$$T(n) = cn^2 + \frac{3}{16}cn^2 + \frac{3^2}{16^2}cn^2 + \frac{3^3}{16^3}cn^2 + \cdots$$

$$= cn^2\left(1 + \frac{3}{16} + \frac{3^2}{16^2} + \frac{3^3}{16^3} + \cdots\right)$$

$$= cn^2\,(16/13)$$

$$= O(cn^2)$$

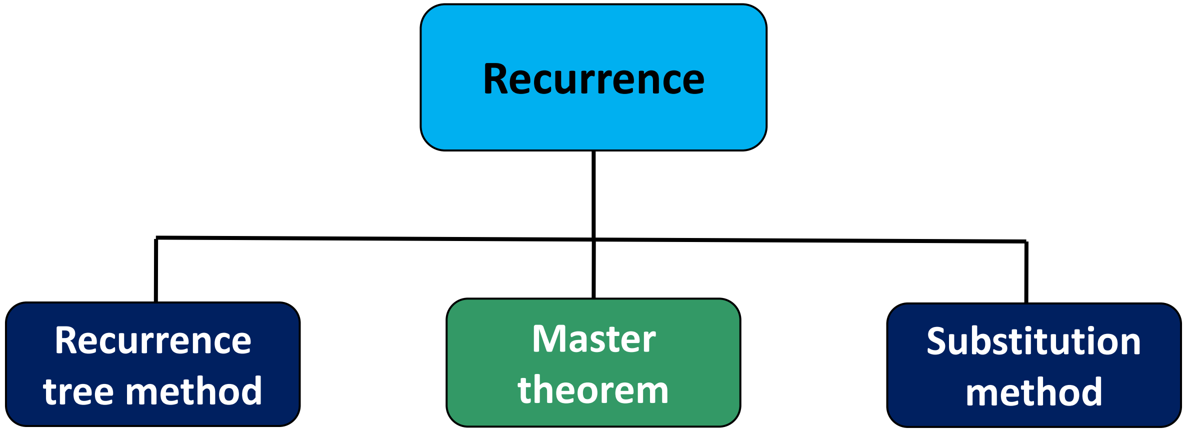
a=1 ; r= 3/16

Master Theorem (1)

- The Master theorem method is used for solving the following types of recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + \Theta(n^k log^p n)$$

Where, a ≥ 1, b > 1, k ≥ 0 and *p is a real number*, *n* is the size of the problem, *a* is the number of sub problems in the recursion, and *n/b* is the size of each sub problem.

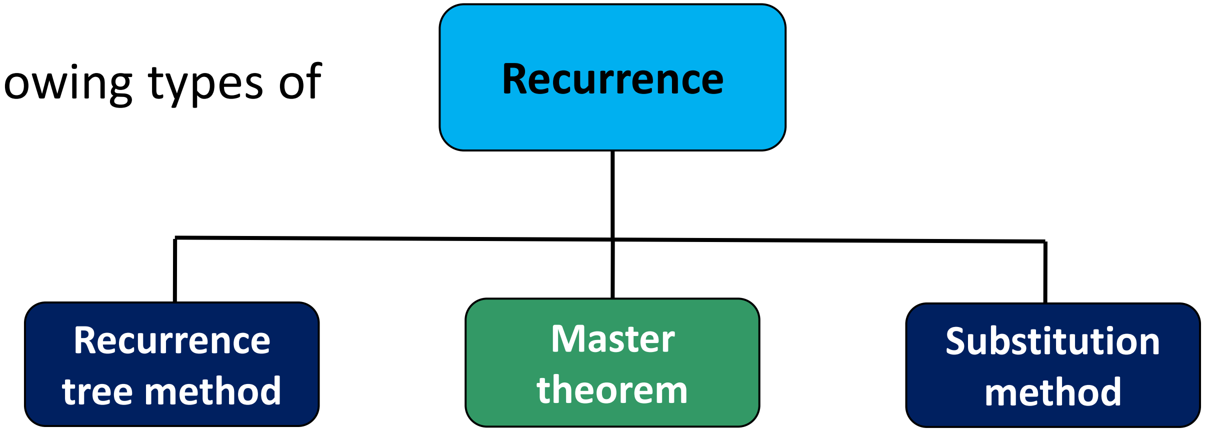


Master Theorem (2)

- The Master theorem method is used for solving the following types of recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + \Theta(n^k log^p n)$$

Using the three following cases, it solves $T(n)$



Case1: If $a > b^k$, then $T(n) = \Theta(n^{log_b^a})$

Case2: If $a = b^k$

a. If p > -1, then $T(n) = \Theta(n^{log_b^a} log^{p+1} n)$
b. If p = -1, then $T(n) = \Theta(n^{log_b^a} \log log\, n)$
c. If p < -1, then $T(n) = \Theta(n^{log_b^a})$

Case3: If $a < b^k$

a. If p ≥ 0, then $T(n) = \Theta(n^k log^p n)$
b. If p < 0, then $T(n) = \Theta(n^k)$

Exercise: Master Theorem (1)

$T(n) = 3T\left(\frac{n}{2}\right) + n^2$

Solution:
a=3; b=2; k=2; p=0

$a < b^k$ => 3<4, p=0, then apply *case 3. a*

$\Theta(n^2)$

$T(n) = 6T\left(\frac{n}{3}\right) + n^2 logn$

Solution:
a=6; b=3; k=2; p=1

$a < b^k$ => 6<9,
p=1 then apply *case 3. a*

$\Theta(n^2\, logn)$

- The Master theorem method is used for solving the following types of recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + \Theta(n^k log^p n)$$

Using the three following cases, it solves $T(n)$

Case1: If $a > b^k$, then $T(n) = \Theta(n^{log_b^a})$

Case2: If $a = b^k$

a. If p > -1, then $T(n) = \Theta(n^{log_b^a}\, log^{p+1} n)$
b. If p = -1, then $T(n) = \Theta(n^{log_b^a}\, \log log\, n)$
c. If p < -1, then $T(n) = \Theta(n^{log_b^a})$

Case3: If $a < b^k$

a. If p ≥ 0, then $T(n) = \Theta(n^k log^p n)$
b. If p < 0, then $T(n) = \Theta(n^k)$

Exercise: Master Theorem (2)

$T(n) = 2T\left(\frac{n}{2}\right) + nlogn$

Solution:
a=2; b=2; k=1; p=1

$a = b^k$ => 2 = 2, p > -1 then apply *case 2. a*

$$\Theta(\mathrm{n}log^2 n)$$

$T(n) = 2T\left(\frac{n}{2}\right) + n/logn$

Solution:
a=2; b=2; k=1; p= -1

$a = b^k$ => 2 = 2, p = -1 then apply *case 2. b*

$$\Theta(\mathrm{n}loglogn)$$

- The Master theorem method is used for solving the following types of recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + \Theta(n^k log^p n)$$

Using the three following cases, it solves $T(n)$

Case1: If $a > b^k$, then $T(n)$ = $\Theta(n^{log_b^a})$

Case2: If $a = b^k$

a. If p > -1, then $T(n) = \Theta(n^{log_b^a} log^{p+1} n)$
b. If p = -1, then $T(n) = \Theta(n^{log_b^a} \log log\, n)$
c. If p < -1, then $T(n) = \Theta(n^{log_b^a})$

Case3: If $a < b^k$

a. If p ≥ 0, then $T(n) = \Theta(n^k log^p n)$
b. If p < 0, then $T(n) = \Theta(n^k)$

Exercise: Master Theorem (3)

$T(n) = 4T\left(\frac{n}{2}\right) + logn$

Solution:
a=4; b=2; k=0; p=1

$a > b^k$, p > -1 then apply *case 1*

$\Theta(n^2)$

$T(n) = 3T\left(\frac{n}{3}\right) + \sqrt{n}$

Solution:
a=3; b=3; k=1/2; p= 0

$a > b^k$
$\Theta(\mathrm{n})$

- The Master theorem method is used for solving the following types of recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + \Theta(n^k log^p n)$$

Using the three following cases, it solves $T(n)$

Case1: If $a > b^k$, then $T(n)$ = $\Theta(n^{log_b^a})$

Case2: If $a = b^k$

a. If p > -1, then $T(n) = \Theta(n^{log_b^a} log^{p+1} n)$
b. If p = -1, then $T(n) = \Theta(n^{log_b^a} \log log\, n)$
c. If p < -1, then $T(n) = \Theta(n^{log_b^a})$

Case3: If $a < b^k$

a. If p ≥ 0, then $T(n) = \Theta(n^k log^p n)$
b. If p < 0, then $T(n) = \Theta(n^k)$

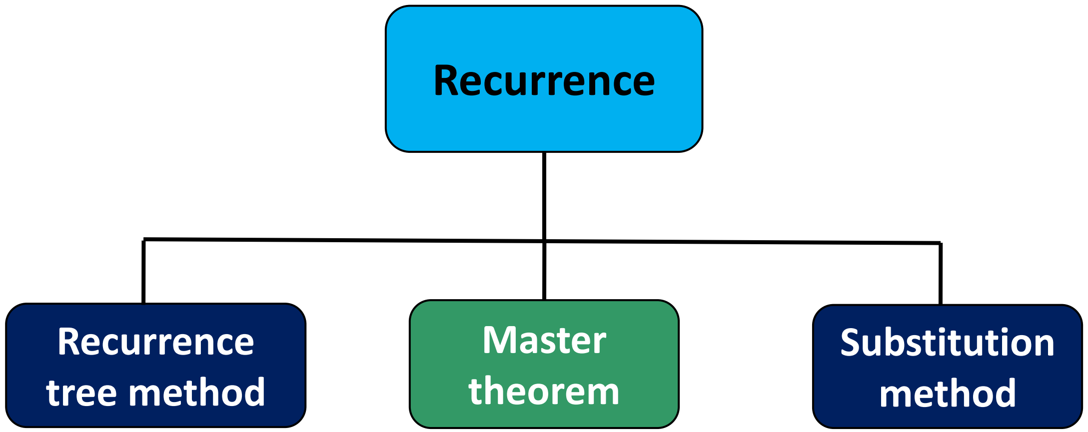
Master Theorem: Subtract and Conquer recurrences

- The Master theorem method is used for solving the following types of recurrence

$$T(n) = \begin{cases} c, & if\ n \leq 1 \\ a\,T(n-b) + f(n), & if\ n > 1 \end{cases}$$

For some constants c, a > 0, b > 0, k ≥ 0, if f(n) is in O (n^k)

$$T(n) = \begin{cases} O(n^k), if\ a < 1 \\ O(n^{k+1}), if\ a = 1 \\ O\left(n^k a^{\frac{n}{b}}\right), if\ a > 1 \end{cases}$$



Exercise: Master Theorem: Subtract and Conquer recurrences (1)

$$T(n) = \begin{cases} 1, & if\ n \leq 0 \\ 3\,T(n-1), & if\ n > 0 \end{cases}$$

Solution: a = 3, b =1, k = 0, f(n) → O(n^0)

T(n) = O(3^n)

T (n) = 3T(n-1)
= 3(3T(n-2))
= 3^2 T(n-2)
= 3^2 (3T(n-3))
= 3^3 T(n-3)
…

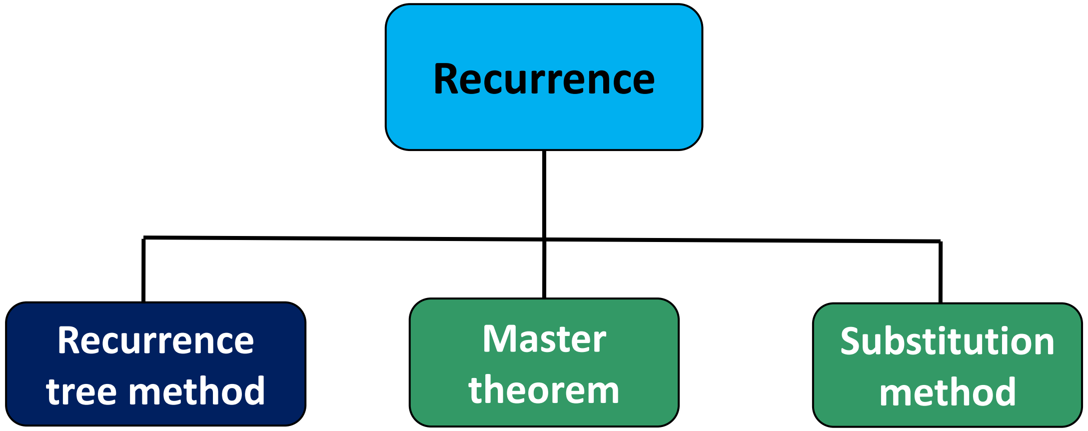
T (n) = 3^nT(n-n) =3^n T(0) = O(3^n)

- The Master theorem method is used for solving the following types of recurrence

$$T(n) = \begin{cases} c, & if\ n \leq 1 \\ a\,T(n-b) + f(n), & if\ n > 1 \end{cases}$$

For some constants c, a > 0, b > 0, k ≥ 0, if f(n) is in O (n^k)

$$T(n) = \begin{cases} O(n^k), if\ a < 1 \\ O(n^{k+1}), if\ a = 1 \\ O(n^k a^{\frac{n}{b}}), if\ a > 1 \end{cases}$$



Exercise: Master Theorem: Subtract and Conquer recurrences (2)

$$T(n) = \begin{cases} 1, & if\ n \le 0 \\ 2\,T(n-1)+1, & if\ n > 0 \end{cases}$$

Solution: a = 2, b =1, k = 0, f(n) → O(n^0)

T(n) = O(2^n)

T (n) = 2T(n-1) +1
= 2(2T(n-2)+1) + 1
= 2^2 T(n-2)+2+1
= 2^2 (2T(n-3)+1)+2+1
= 2^3 T(n-3)+ 2^2 +2+1
…

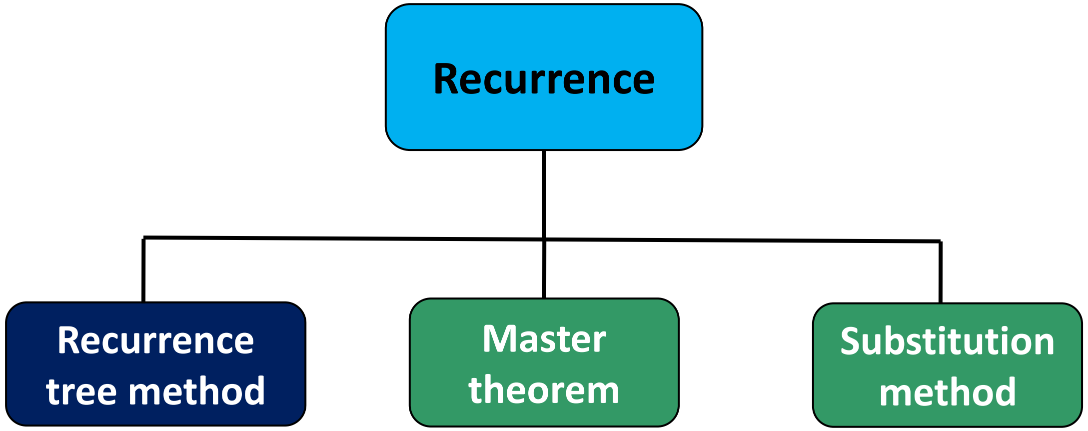
T (k) = 2^kT(n-k) + 2^{k-1} + 2^{k-2} + …. + 2 + 1
= O(2^n)

- The Master theorem method is used for solving the following types of recurrence

$$T(n) = \begin{cases} c, & if\ n \le 1 \\ a\,T(n-b) + f(n), & if\ n > 1 \end{cases}$$

For some constants c, a > 0, b > 0, k ≥ 0, if f(n) is in O (n^k)

$$T(n) = \begin{cases} O(n^k), if\ a < 1 \\ O(n^{k+1}), if\ a = 1 \\ O(n^k a^{\frac{n}{b}}), if\ a > 1 \end{cases}$$



thank you!

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