CS2x1:Data Structures and Algorithms

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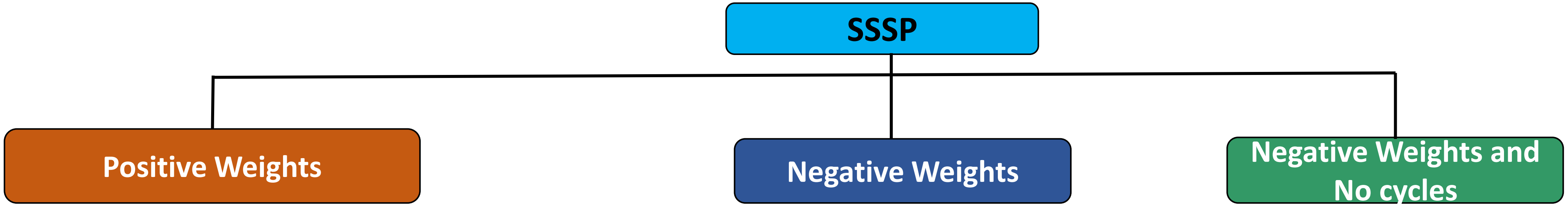
Recap: SSSP



Dijkstra | ***Bellman-Ford*** | ***Directed Acyclic Graph (DAG)***

```
DIJKSTRA (G, w, s) {
INITIALIZE-SINGLE-SOURCE (G, s)
S = ∅
Q = G. V
while Q ≠ ∅ ;
   u = EXTRACT-MIN(Q)
   S = S ∪ {u}
   for each vertex v ϵ Q. Adj[u]
     RELEAX (u, v, w)
}
```

```
INITIALIZE-SINGLE-SOURCE (G, s) {
for each v ϵ G.V
   v. d = ∞
   v. π = NIL
s. d = 0
```

```
RELAX (u, v, w) {
   if v.d > u.d + w(u, v)
      v. d =  u.d + w(u, v)
      v. π = u
```

```
BELLMAN-FORD (G, w, s) {
INITIALIZE-SINGLE-SOURCE (G, s)
for i = 1 to |G.V| - 1
      for each edge (u, v) ∈ G.E
            RELAX (u, v, w)
for each edge (u, v) ∈ G.E
            if v. d > u. d + w(u, v)
              return False
return True
```

Total time complexity: <u>O(V) + O (V log V) + O(V) + O(E logV)</u>
: O(ElogV)

Total time complexity: <u>O(V) + O (VE) + O(E)</u>
: O(VE)

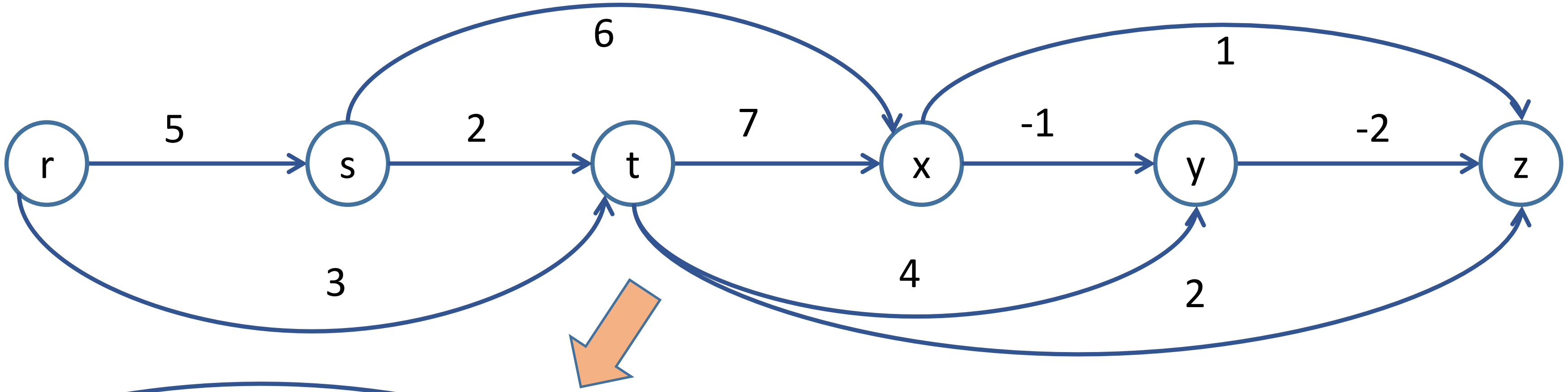
Recap: SSSP-DAG

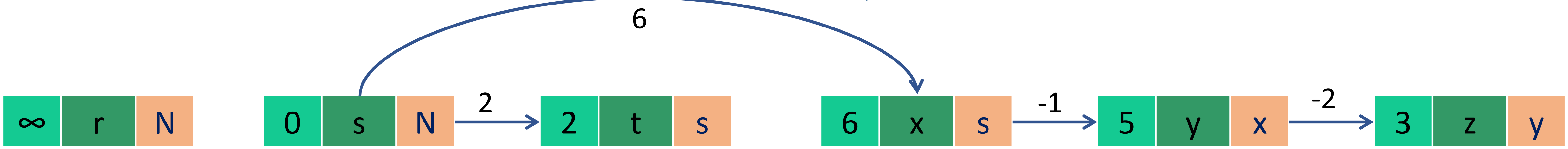
- Directed Graph; No cycles
- Topological sort can be applied on any DAG → O (V+E)
- Works for negative edges

```
DAG-SHORTEST-PATH (G, w, s) {
Topological sort the vertices of G
INITIALIZE-SINGLE-SOURCE (G, s)
for each vertex u, taken in topologically sorted order
    for each edge v ∈ G. Adj [u]
        RELAX (u, v, w)
```

```
INITIALIZE-SINGLE-SOURCE (G, s) {
for each v ϵ G.V
    v. d = ∞
    v. π = NIL
s. d = 0
```

```
RELAX (u, v, w) {
if v. d > u. d + w(u, v)
    v. d =  u. d + w(u, v)
    v. π = u
```





SSSP: DAG → Time Complexity Analysis

DAG-SHORETEST-PATH (G, w, s) {

Topological sort the vertices of G — **O (V + E)**

INITIALIZE-SINGLE-SOURCE (G, s) — **O (V)**

for each vertex *u*, taken in topologically sorted order

    for each edge $v \in$ G. *Adj* [*u*]

        RELAX (*u, v, w*)

O (V+E);

Total time complexity: <u>O (V+E)</u>

INITIALIZE-SINGLE-SOURCE (G, s) {

for each $v \in G.V$

    $v.\ d = \infty$

    $v.\ \pi$ = NIL

$s.\ d = 0$

RELAX (*u, v, w*) {

    if v. d > u. d + w(u, v)

        v. *d* = u. d + w(u, v)

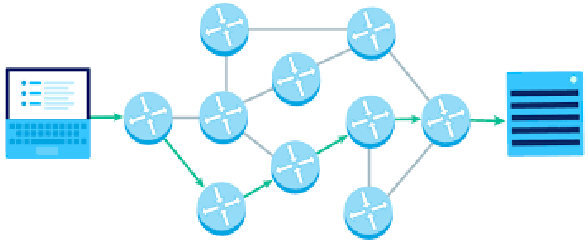
        $v.\ \pi$ = u

Graphs: Shortest Path (SP)

- Digital Mapping Services in Google Maps
- Social Networking Applications
- Telephone Network
- IP routing to find Open shortest Path First
- Designate file server
- Robotic Path

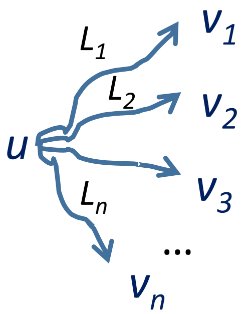
Google Maps

UBER



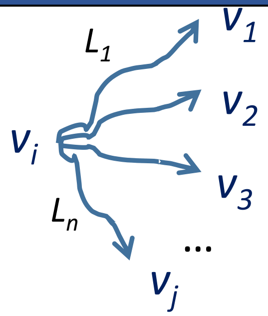
Shortest Path

Single-pair shortest path (SSSP)



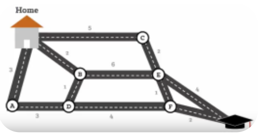
Shortest path
From a single source → to every other vertex

All-pair shortest path (APSP)



Shortest path
From every vertex → to every other vertex

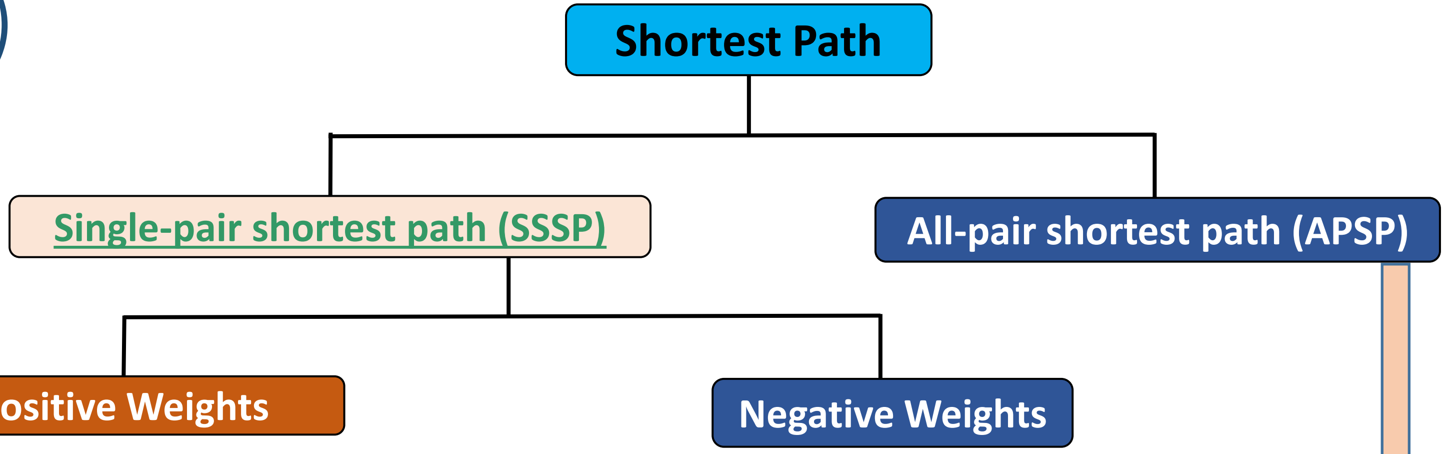
(i) Edge weights are *positive*



(ii) Edge weights are *negative*

TransferWise Splitwise

All Pair Shortest Path (APSP)



Dijkstra

❖ Single source vertex:
- ✓ Time complexity → O (E log V)
- ✓ *G: Dense graph* E ≈ V^2
- ✓ Time complexity → O (V^2 log V)

Bellman-Ford

❖ Single source vertex:
- ✓ Time complexity → O (VE)
- ✓ *G: Dense graph* E ≈ V^2
- ✓ Time complexity → O (V^3)

SLOW-ALL-PAIRS-SHORTEST-PATHS (W)

EXTEND-SHORTEST-PATHS (L, W)

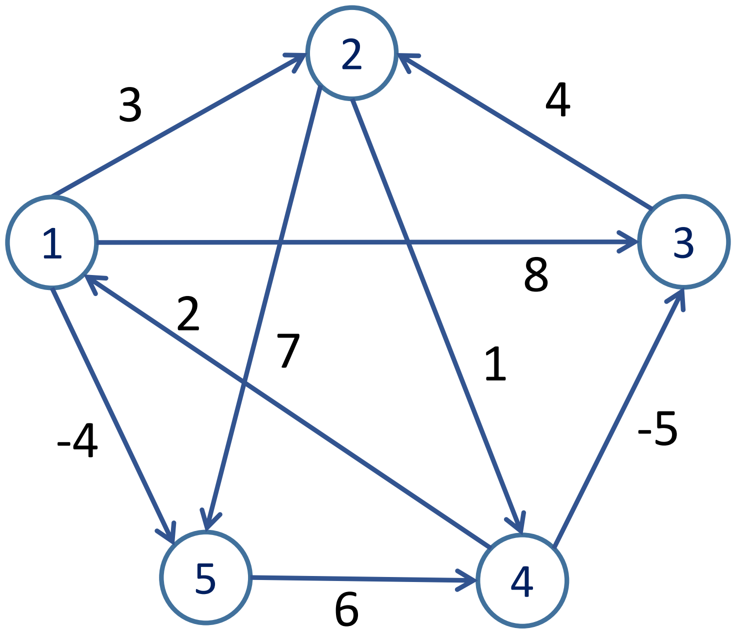
❖ All pair source vertex:

- ✓ all-pair → v-vertices → dense graph → O (V^3 log V) (*Dijkstra*)
- ✓ all-pair → v-vertices → dense graph → O (V^4) (*Bellman-Ford*)
- ✓ all-pair → v-vertices → sparse graph → O (V^2E) (*Bellman-Ford*)
- ✓ *Can this be improved?*

APSP: Slow and extend (1)

```
SLOW-ALL-PAIRS-SHORTEST-PATHS (W) {
n = W. rows
L^(1) = W
for m = 2 to n-1
    let for L^(m) be a new n x n matrix
    L^(m) = EXTEND-SHORTEST-PATHS (L^(m-1), W)
return L^(n-1)
```

```
EXTEND-SHORTEST-PATHS (L, W) {
n = L. rows
let L' = (l'_ij ) be a new n x n matrix
for i = 1 to n
    for j = 1 to n
        l'_ij = ∞
        for k = 1 to n
            i'_ij = min (l'_ij , l_ik + w_kj )
    return L'
```



- $W(i,j) = 0$; *if i=j*
  $= \infty$; *if there is no edge between i and j*
  = "weight of edge"

Step 1: n = W. rows = 5
Step 2: $L^{(1)}$ = W

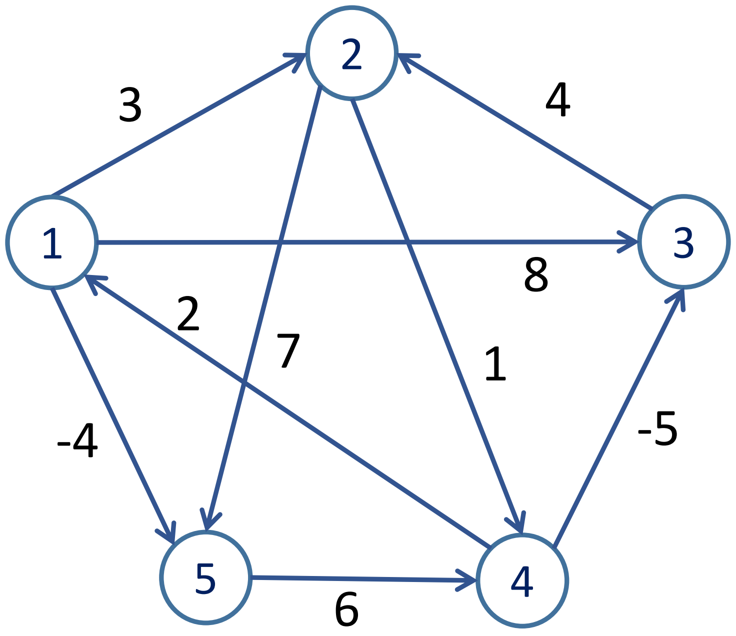
$$\begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

APSP: Slow and extend (2)

```
SLOW-ALL-PAIRS-SHORTEST-PATHS (W) {
n = W. rows
L^(1) = W
for m = 2 to n-1
    let for L^(m) be a new n x n matrix
    L^(m) = EXTEND-SHORTEST-PATHS (L^(m-1), W)
return L^(n-1)
```

```
EXTEND-SHORTEST-PATHS (L, W) {
n = L. rows
let L' = (l'_ij ) be a new n x n matrix
for i = 1 to n
    for j = 1 to n
        l'_ij = ∞
        for k = 1 to n
            i'_ij = min (l'_ij , l_ik + w_kj )
    return L'
```

$$L^{(1)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$



Step 1: n = W. rows = 5

Step 2: $L^{(1)}$ = W

Step 3: m = 2 to 5 - 1

Step 4: $L^{(2)}$ = EXTEND-SHORTEST-PATHS ($L^{(1)}$, W) $\rightarrow$

$$\begin{pmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \end{pmatrix}$$

APSP: Slow and extend (3)

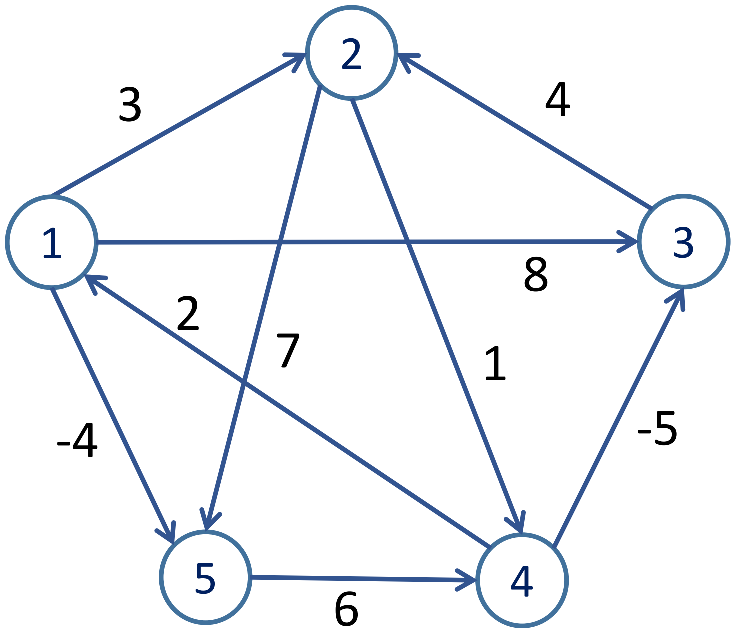
EXTEND-SHORTEST-PATHS (L, W) {

```
n = L. rows
let L' = (l'_ij ) be a new n x n matrix
for i = 1 to n
  for j = 1 to n
      l'_ij = ∞
      for k = 1 to n
          i'_ij = min (l'_ij , l_ik + w_kj )
return L'
```

$$L^{(1)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

Step 4: $L^{(2)}$ = EXTEND-SHORTEST-PATHS ($L^{(1)}$, W)

$$\begin{pmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \end{pmatrix}$$



$i'_{11} = min\ (l'_{11} , l_{1k} + w_{k1})$; k → 1 to 5

$, l_{11} + w_{11}$	$0 + 0 = 0$
$, l_{12} + w_{21}$	$3 + \infty = \infty$
$, l_{13} + w_{31}$	$8 + \infty = \infty$
$, l_{14} + w_{41}$	$\infty + 2 = \infty$
$, l_{15} + w_{51}$	$-4 + \infty = \infty$

$i'_{11} = 0;$

$i'_{12} = min\ (l'_{12} , l_{1k} + w_{k2})$; k → 1 to 5

$, l_{11} + w_{12}$	$0 + 3 = 3$
$, l_{12} + w_{22}$	$3 + 0 = 3$
$, l_{13} + w_{32}$	$8 + 4 = 12$
$, l_{14} + w_{42}$	$\infty + \infty = \infty$
$, l_{15} + w_{52}$	$-4 + \infty = \infty$

$i'_{12} = 3;$

$i'_{13} = min\ (l'_{13} , l_{1k} + w_{k3})$; k → 1 to 5

$, l_{11} + w_{13}$	$0 + 8 = 8$
$, l_{12} + w_{23}$	$3 + \infty = \infty$
$, l_{13} + w_{33}$	$8 + 0 = 8$
$, l_{14} + w_{43}$	$\infty + -5 = \infty$
$, l_{15} + w_{53}$	$-4 + \infty = \infty$

$i'_{13} = 8;$

APSP: Slow and extend (4)

EXTEND-SHORTEST-PATHS (L, W) {

$n = L$. rows

let $L' = (l'_{ij})$ be a new $n \times n$ matrix

for $i = 1$ to n

for $j = 1$ to n

$l'_{ij} = \infty$

for $k = 1$ to n

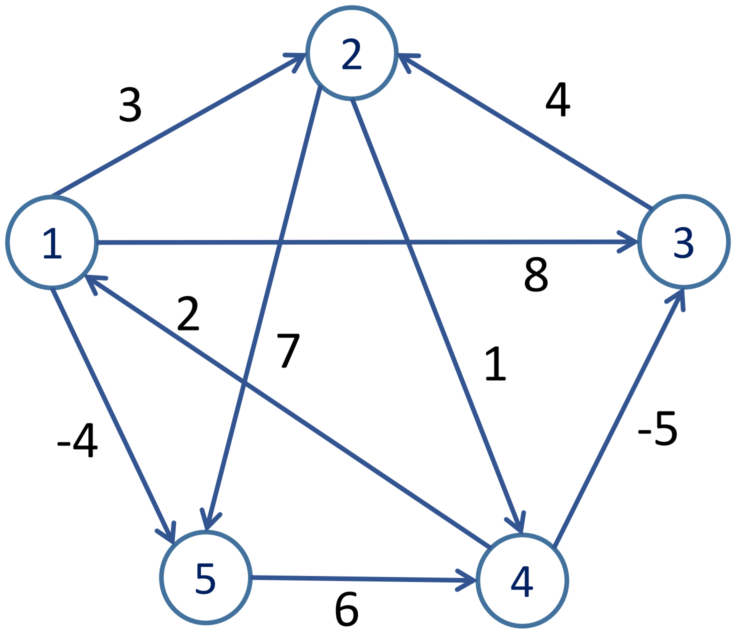
$l'_{ij} = \min(l'_{ij}, l_{ik} + w_{kj})$

return L'

$$L^{(1)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

Step 4: $L^{(2)}$ = EXTEND-SHORTEST-PATHS ($L^{(1)}$, W)

$$\begin{pmatrix} 0 & 3 & 8 & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \end{pmatrix}$$



$i'_{14} = \min(l'_{11}, l_{1k} + w_{k4})$; $k \rightarrow 1$ to 5

$, l_{11} + w_{14}$ $0 + \infty = \infty$

$, l_{12} + w_{24}$ $3 + 1 = 4$

$, l_{13} + w_{34}$ $8 + \infty = \infty$

$, l_{14} + w_{44}$ $\infty + 0 = \infty$

$, l_{15} + w_{54}$ $-4 + 6 = 2$

$i'_{14} = 2$;

$i'_{42} = \min(l'_{12}, l_{4k} + w_{k2})$; $k \rightarrow 1$ to 5

$, l_{41} + w_{12}$ $2 + 3 = 5$

$, l_{42} + w_{22}$ $\infty + 0 = \infty$

$, l_{43} + w_{32}$ $-5 + 4 = -1$

$, l_{44} + w_{42}$ $0 + \infty = \infty$

$, l_{45} + w_{52}$ $\infty + \infty = \infty$

$i'_{42} = -1$;

$i'_{53} = \min(l'_{13}, l_{5k} + w_{k3})$; $k \rightarrow 1$ to 5

$, l_{51} + w_{13}$ $\infty + 8 = 8$

$, l_{52} + w_{23}$ $\infty + \infty = \infty$

$, l_{53} + w_{33}$ $\infty + 0 = 8$

$, l_{54} + w_{43}$ $6 + -5 = 1$

$, l_{55} + w_{53}$ $0 + \infty = \infty$

$i'_{53} = 1$;

APSP: Slow and extend (5)

```
SLOW-ALL-PAIRS-SHORTEST-PATHS (W) {
n = W. rows
L^(1) = W
for m = 2 to n-1
    let for L^(m) be a new n x n matrix
    L^(m) = EXTEND-SHORTEST-PATHS (L^(m-1), W)
return L^(n-1)
```

```
EXTEND-SHORTEST-PATHS (L, W) {
n = L. rows
let L' = (l'_ij ) be a new n x n matrix
for i = 1 to n
  for j = 1 to n
        l'_ij = ∞
        for k = 1 to n
             i'_ij = min (l'_ij , l_ik  + w_kj )
  return L'
```

$$L^{(1)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

$m = 2$

$$L^{(2)} = \begin{pmatrix} 0 & 3 & 8 & 2 & -4 \\ 3 & 0 & -4 & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & \infty & 1 & 6 & 0 \end{pmatrix}$$

$m = 3$

$$L^{(3)} = \begin{pmatrix} 0 & 3 & 8 & 2 & -4 \\ 3 & 0 & -4 & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & \infty & 1 & 6 & 0 \end{pmatrix}$$

$m = 4$

$$L^{(4)} = \begin{pmatrix} 0 & 3 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix}$$

APSP: Slow and extend → Time complexity analysis

SLOW-ALL-PAIRS-SHORTEST-PATHS (W) {

n = W. rows
$L^{(1)}$ = W
for m = 2 to $n-1$ ← **O (V)**
    let for $L^{(m)}$ be a new $n \times n$ matrix
    $L^{(m)}$ = EXTEND-SHORTEST-PATHS ($L^{(m-1)}$, W) ← **Single vertex → O (V^3)** **V-vertices → O (V^4)**
return $L^{(n-1)}$

EXTEND-SHORTEST-PATHS (L, W) {

n = L. rows
let L' = (l'_{ij}) be a new $n \times n$ matrix
for i = 1 to n
    for j = 1 to n
        $l'_{ij} = \infty$
        for k = 1 to n
            $i'_{ij} = min\,(l'_{ij}\,, l_{ik} + w_{kj})$
    return L'

Total time complexity: <u>O(V^4)</u>

FASTER-ASAP

FASTER-ALL-PAIRS-SHORTEST-PATHS (W) {

n = W. rows

$L^{(1)}$ = W

m = 1

while $m < n\text{-}1$

  let for $L^{(2m)}$ be a new n x n matrix

  $L^{(2m)}$ = EXTEND-SHORTEST-PATHS ($L^{(m)}$, $L^{(m)}$)

  $m = 2m$

return $L^{(m)}$

$$L^{(2)} = \textit{EXTEND-SHORTEST-PATHS}\ (L^{(1)}, L^{(1)})$$

$$L^{(4)} = \textit{EXTEND-SHORTEST-PATHS}\ (L^{(2)}, L^{(2)})$$

$$L^{(8)} = \textit{EXTEND-SHORTEST-PATHS}\ (L^{(4)}, L^{(4)})$$

$$L^{(16)} = \textit{EXTEND-SHORTEST-PATHS}\ (L^{(8)}, L^{(8)})$$

.

.

.

$\lceil \log(n-1) \rceil$ **Total time complexity: <u>O(V³logV)</u>**

EXTEND-SHORTEST-PATHS (L, W) {

$n = L$. rows

let $L' = (l'_{ij})$ be a new n x n matrix

for i = 1 to n

  for j = 1 to n

    $l'_{ij} = \infty$

    for k = 1 to n

      $i'_{ij} = min\ (l'_{ij}, l_{ik} + w_{kj})$

return L'

SLOW-ALL-PAIRS-SHORTEST-PATHS (W) {

n = W. rows

$L^{(1)}$ = W

for m = 2 to $n\text{-}1$

  let for $L^{(m)}$ be a new n x n matrix

  $L^{(m)}$ = EXTEND-SHORTEST-PATHS ($L^{(m-1)}$, W)

return $L^{(n-1)}$

$$L^{(2)} = \textit{EXTEND-SHORTEST-PATHS}\ (L^{(1)}, W)$$

$$L^{(3)} = \textit{EXTEND-SHORTEST-PATHS}\ (L^{(2)}, W)$$

$$L^{(4)} = \textit{EXTEND-SHORTEST-PATHS}\ (L^{(3)}, W)$$

$$L^{(5)} = \textit{EXTEND-SHORTEST-PATHS}\ (L^{(4)}, W)$$

.

.

.

$$L^{(n-1)} = \textit{EXTEND-SHORTEST-PATHS}\ (L^{(n-2)}, W)$$

Total time complexity: <u>O(V⁴)</u>

Recap: All Pair Shortest Path (APSP)

Shortest Path

- **Single-pair shortest path (SSSP)**
  - **Positive Weights**
  - **Negative Weights**
- **All-pair shortest path (APSP)**

Dijkstra

❖ Single source vertex:

✓ Time complexity → O (E log V)

✓ *G: Dense graph* E ≈ V^2

✓ Time complexity → O (V^2 log V)

Bellman-Ford

❖ Single source vertex:

✓ Time complexity → O (VE)

✓ *G: Dense graph* E ≈ V^2

✓ Time complexity → O (V^3)

❖ All pair source vertex:

✓ all-pair → *Dijkstra* v-vertices → dense graph → O (V^3 log V)

✓ all-pair → *Bellman-Ford* v-vertices → dense graph → O (V^4)

✓ all-pair → *Bellman-Ford* v-vertices → sparse graph → O (V^2E)

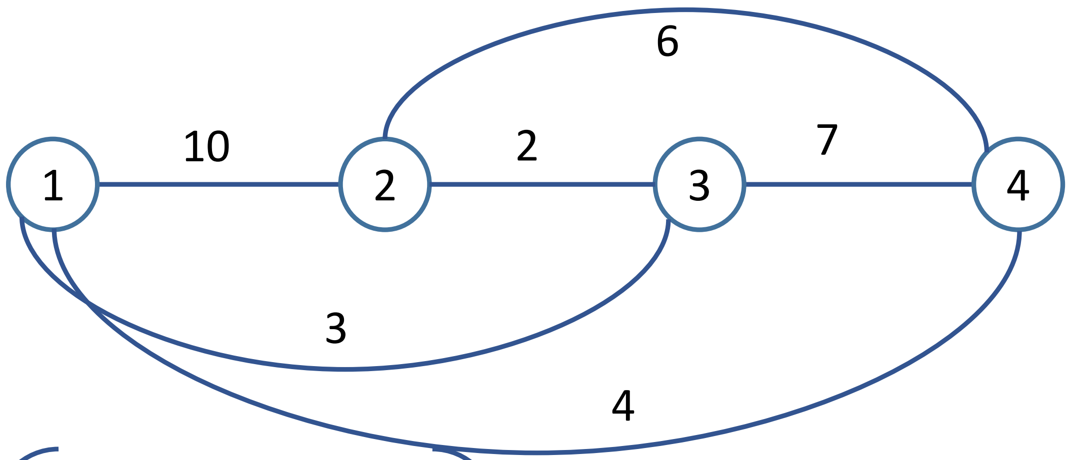
SLOW-ALL-PAIRS-SHORTEST-PATHS (W)

Total time complexity: O(V^4)

EXTEND-SHORTEST-PATHS (L, W)

Total time complexity: O(V^3 log V)

APSP: Floyd-Warshall → dynamic programming (1)



$D^{(k)} = (d_{ij}^{(k)})$

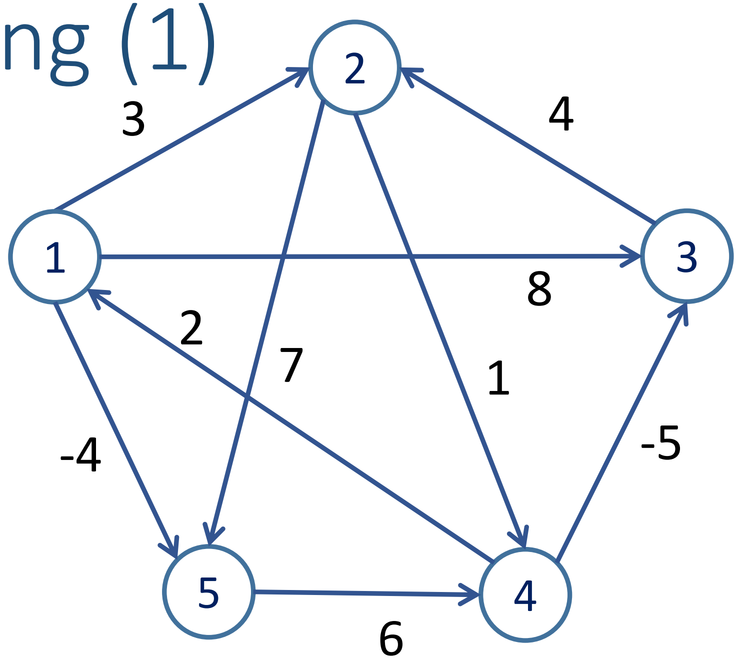
$$D^{(0)} = (d_{ij}^{(0)}) \begin{pmatrix} 0 & 10 & 3 & 4 \\ 10 & 0 & 2 & 6 \\ 3 & 2 & 0 & 7 \\ 4 & 6 & 7 & 0 \end{pmatrix}$$

$$D^{(2)} = (d_{ij}^{(2)}) \begin{pmatrix} 0 & 10 & 3 & 4 \\ 10 & 0 & 2 & 6 \\ 3 & 2 & 0 & 7 \\ 4 & 6 & 7 & 0 \end{pmatrix}$$

$$D^{(1)} = (d_{ij}^{(1)}) \begin{pmatrix} 0 & 10 & 3 & 4 \\ 10 & 0 & 2 & 6 \\ 3 & 2 & 0 & 7 \\ 4 & 6 & 7 & 0 \end{pmatrix}$$

$$D^{(3)} = (d_{ij}^{(3)}) \begin{pmatrix} 0 & 5 & 3 & 4 \\ 5 & 0 & 2 & 6 \\ 3 & 2 & 0 & 7 \\ 4 & 6 & 7 & 0 \end{pmatrix}$$

APSP: Floyd-Warshall → dynamic programming (1)



Floyd-WARSHALL (W) {

1 n = W. rows

2 $D^{(0)}$ = W

3 for k = 1 to n

4 let $D^{(k)} = (d_{ij}^{(k)})$ be a new $n \times n$ matrix

5 for i = 1 to n

6 for j = 1 to n

7 $d_{ij}^{(k)} = \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$

8 return $D^{(n)}$

- W(i,j) = 0; *if i=j*
  = ∞; *if there is no edge between i and j*
  = "weight of edge"

- $d_{ij}^{(k)} = w_{ij}$ *; if k = 0*
  $= \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$; if $k \geq 1$

- $\pi_{ij}^{(0)} = NIL$ *; if* $i = j$ *or* $w_{ij} = \infty$,
  $=$ *i; if* $i \neq j$ *and* $w_{ij} < \infty$

Step 1: n = W. rows = 5

Step 2: $D^{(0)}$ = W →

$$\begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

$$\pi^{(0)} = \begin{pmatrix} NIL & 1 & 1 & NIL & 1 \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & NIL & NIL \\ 4 & NIL & 4 & NIL & NIL \\ NIL & NIL & NIL & 5 & NIL \end{pmatrix}$$

APSP: Floyd-Warshall (2)

$$D^{(0)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

Floyd-WARSHALL (W) {

1 n = W. rows

2 $D^{(0)}$ = W

3 for k = 1 to n

4 let $D^{(k)} = (d_{ij}^{(k)})$ be a new $n \times n$ matrix

5 for i = 1 to n

6 for j = 1 to n

7 $d_{ij}^{(k)} = \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})$

8 return $D^{(n)}$

Step 3: k = 1

Step 4: $D^{(1)} = d_{ij}^{(1)}$

Step 5: i = 1

Step 6: j = 1 to 5

Step 7:

$d_{11}^{(1)} = \min(d_{11}^{(0)}, d_{11}^{(0)} + d_{11}^{(0)}) \rightarrow \min(0, 0+0) \rightarrow 0$

$d_{12}^{(1)} = \min(d_{12}^{(0)}, d_{11}^{(0)} + d_{12}^{(0)}) \rightarrow \min(3, 0+3) \rightarrow 3$

$d_{13}^{(1)} = \min(d_{13}^{(0)}, d_{11}^{(0)} + d_{13}^{(0)}) \rightarrow \min(8, 0+8) \rightarrow 8$

$d_{14}^{(1)} = \min(d_{14}^{(0)}, d_{11}^{(0)} + d_{14}^{(0)}) \rightarrow \min(\infty, 0+\infty) \rightarrow \infty$

$d_{15}^{(1)} = \min(d_{15}^{(0)}, d_{11}^{(0)} + d_{15}^{(0)}) \rightarrow \min(-4, 0+-4) \rightarrow -4$

Step 3: k = 1

Step 4: $D^{(1)} = d_{ij}^{(1)}$

Step 5: i = 4

Step 6: j = 1 to 5

Step 7:

$d_{41}^{(1)} = \min(d_{41}^{(0)}, d_{41}^{(0)} + d_{11}^{(0)}) \rightarrow \min(2, 2+0) \rightarrow 2$

$d_{42}^{(1)} = \min(d_{42}^{(0)}, d_{41}^{(0)} + d_{12}^{(0)}) \rightarrow \min(\infty, 2+3) \rightarrow 5$

$d_{43}^{(1)} = \min(d_{43}^{(0)}, d_{41}^{(0)} + d_{13}^{(0)}) \rightarrow \min(-5, 2+8) \rightarrow -5$

$d_{44}^{(1)} = \min(d_{44}^{(0)}, d_{41}^{(0)} + d_{14}^{(0)}) \rightarrow \min(0, 2+\infty) \rightarrow 0$

$d_{45}^{(1)} = \min(d_{45}^{(0)}, d_{41}^{(0)} + d_{15}^{(0)}) \rightarrow \min(\infty, 2+-4) \rightarrow -2$

- $\pi_{ij}^{(k)} = \pi_{ij}^{(k-1)}$; *if* $d_{ij}^{(k-1)} \le d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$,
  $= \pi_{kj}^{(k-1)}$ *i; if* $d_{ij}^{(k-1)} > d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$

$$\pi^{(1)} = \begin{pmatrix} \text{NIL} & 1 & 1 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & \text{NIL} & \text{NIL} \\ 4 & \text{NIL} & 4 & \text{NIL} & \text{NIL} \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{pmatrix}$$

APSP: Floyd-Warshall (3)

$$D^{(1)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

Floyd-WARSHALL (W) {

1 n = W. rows

2 $D^{(0)}$ = W

3 for k = 1 to n

4 let $D^{(k)} = (d_{ij}^{(k)})$ be a new $n \times n$ matrix

5 for i = 1 to n

6 for j = 1 to n

7 $d_{ij}^{(k)} = \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})$

8 return $D^{(n)}$

Step 3: k = 2
Step 4: $D^{(2)} = d_{ij}^{(2)}$
Step 5: i = 1
Step 6: j = 1 to 5
Step 7:

$d_{11}^{(2)} = \min(d_{11}^{(1)}, d_{12}^{(1)} + d_{21}^{(1)}) \rightarrow \min(0, 3+\infty) \rightarrow 0$

$d_{12}^{(2)} = \min(d_{12}^{(1)}, d_{12}^{(1)} + d_{22}^{(1)}) \rightarrow \min(3, 3+0) \rightarrow 3$

$d_{13}^{(2)} = \min(d_{13}^{(1)}, d_{12}^{(1)} + d_{23}^{(1)}) \rightarrow \min(8, 3+\infty) \rightarrow 8$

$d_{14}^{(2)} = \min(d_{14}^{(1)}, d_{12}^{(1)} + d_{24}^{(1)}) \rightarrow \min(\infty, 3+1) \rightarrow 4$

$d_{15}^{(2)} = \min(d_{15}^{(1)}, d_{12}^{(1)} + d_{25}^{(1)}) \rightarrow \min(-4, 3+7) \rightarrow -4$

Step 3: k = 2
Step 4: $D^{(2)} = d_{ij}^{(2)}$
Step 5: i = 3
Step 6: j = 1 to 5
Step 7:

$d_{31}^{(2)} = \min(d_{31}^{(1)}, d_{32}^{(1)} + d_{21}^{(1)}) \rightarrow \min(\infty, 4+\infty) \rightarrow \infty$

$d_{32}^{(2)} = \min(d_{32}^{(1)}, d_{32}^{(1)} + d_{22}^{(1)}) \rightarrow \min(4, 4+0) \rightarrow 4$

$d_{33}^{(2)} = \min(d_{33}^{(1)}, d_{32}^{(1)} + d_{23}^{(1)}) \rightarrow \min(0, 4+\infty) \rightarrow 0$

$d_{34}^{(2)} = \min(d_{34}^{(1)}, d_{32}^{(1)} + d_{24}^{(1)}) \rightarrow \min(\infty, 4+1) \rightarrow 5$

$d_{35}^{(2)} = \min(d_{35}^{(1)}, d_{32}^{(1)} + d_{25}^{(1)}) \rightarrow \min(\infty, 4+7) \rightarrow 11$

- $\pi_{ij}^{(k)} = \pi_{ij}^{(k-1)}$ *; if* $d_{ij}^{(k-1)} \leq d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$,
  $= \pi_{kj}^{(k-1)}$ *; if* $d_{ij}^{(k-1)} > d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$

$$\pi^{(2)} = \begin{pmatrix} NIL & 1 & 1 & NIL & 1 \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & NIL & NIL \\ 4 & 1 & 4 & NIL & 1 \\ NIL & NIL & NIL & 5 & NIL \end{pmatrix}$$

APSP: Floyd-Warshall (4)

Floyd-WARSHALL (W) {
1 n = W. rows
2 $D^{(0)}$ = W
3 for k = 1 to n
4 let $D^{(k)} = (d_{ij}^{(k)})$ be a new $n \times n$ matrix
5 for i = 1 to n
6 for j = 1 to n
7 $d_{ij}^{(k)} = \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})$
8 return $D^{(n)}$

$$D^{(2)} \quad \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

- $\pi_{ij}^{(k)} = \pi_{ij}^{(k-1)}$; *if* $d_{ij}^{(k-1)} \le d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$,
  $= \pi_{kj}^{(k-1)}$ *i; if* $d_{ij}^{(k-1)} > d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$

Step 3: k = 3
Step 4: $D^{(1)} = d_{ij}^{(1)}$
Step 5: i = 4
Step 6: j = 1 to 5
Step 7:

$d_{41}^{(3)} = \min(d_{41}^{(2)}, d_{43}^{(2)} + d_{31}^{(2)}) \rightarrow \min(2, -5+\infty) \rightarrow 2$
$d_{42}^{(3)} = \min(d_{42}^{(2)}, d_{43}^{(2)} + d_{32}^{(2)}) \rightarrow \min(5, -5+4) \rightarrow -1$
$d_{43}^{(3)} = \min(d_{43}^{(2)}, d_{43}^{(2)} + d_{33}^{(2)}) \rightarrow \min(-5, -5+0) \rightarrow -5$
$d_{44}^{(3)} = \min(d_{44}^{(2)}, d_{43}^{(2)} + d_{34}^{(2)}) \rightarrow \min(0, -5+5) \rightarrow 0$
$d_{45}^{(3)} = \min(d_{45}^{(2)}, d_{43}^{(2)} + d_{35}^{(2)}) \rightarrow \min(-2, -5+11) \rightarrow -2$

$$\pi^{(3)} = \begin{pmatrix} \text{NIL} & 1 & 1 & 2 & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 2 & 2 \\ \text{NIL} & 3 & \text{NIL} & 2 & 2 \\ 4 & 1 & 4 & \text{NIL} & 1 \\ \text{NIL} & \text{NIL} & \text{NIL} & 5 & \text{NIL} \end{pmatrix}$$

APSP: Floyd-Warshall (5)

$$D^{(3)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

Floyd-WARSHALL (W) {

1 n = W. rows

2 $D^{(0)}$ = W

3 for k = 1 to n

4 let $D^{(k)} = (d_{ij}^{(k)})$ be a new $n \times n$ matrix

5 for i = 1 to n

6 for j = 1 to n

7 $d_{ij}^{(k)} = \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})$

8 return $D^{(n)}$

Step 3: k = 4

Step 4: $D^{(4)} = d_{ij}^{(4)}$

Step 5-7:

$d_{13}^{(4)} = \min(d_{13}^{(3)}, d_{14}^{(3)} + d_{43}^{(3)}) \rightarrow \min(8, 4+ -5) \rightarrow -1$

$d_{21}^{(4)} = \min(d_{21}^{(3)}, d_{24}^{(3)} + d_{41}^{(3)}) \rightarrow \min(\infty, 1+2) \rightarrow 3$

$d_{23}^{(4)} = \min(d_{23}^{(3)}, d_{24}^{(3)} + d_{43}^{(3)}) \rightarrow \min(0, 1+ -5) \rightarrow -4$

$d_{25}^{(4)} = \min(d_{25}^{(3)}, d_{24}^{(3)} + d_{45}^{(3)}) \rightarrow \min(7, 1+-2) \rightarrow -1$

$d_{31}^{(4)} = \min(d_{31}^{(3)}, d_{34}^{(3)} + d_{41}^{(3)}) \rightarrow \min(\infty, 5+2) \rightarrow 7$

$d_{35}^{(4)} = \min(d_{35}^{(3)}, d_{34}^{(3)} + d_{45}^{(3)}) \rightarrow \min(11, 5+-2) \rightarrow 3$

$d_{51}^{(4)} = \min(d_{51}^{(3)}, d_{54}^{(3)} + d_{41}^{(3)}) \rightarrow \min(\infty, 6+ 2) \rightarrow 8$

$d_{52}^{(4)} = \min(d_{52}^{(3)}, d_{54}^{(3)} + d_{42}^{(3)}) \rightarrow \min(\infty, 6+-1) \rightarrow 5$

$d_{53}^{(4)} = \min(d_{53}^{(3)}, d_{54}^{(3)} + d_{43}^{(3)}) \rightarrow \min(\infty, 6+ -5) \rightarrow 1$

$$\pi^{(4)} = \begin{pmatrix} NIL & 1 & 1 & 2 & 1 \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & 2 & 2 \\ 4 & 3 & 4 & NIL & 1 \\ NIL & NIL & NIL & 5 & NIL \end{pmatrix}$$

- $\pi_{ij}^{(k)} = \pi_{ij}^{(k-1)}$; *if* $d_{ij}^{(k-1)} \leq d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$,
  $= \pi_{kj}^{(k-1)}$ *i; if* $d_{ij}^{(k-1)} > d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$

APSP: Floyd-Warshall (6)

Floyd-WARSHALL (W) {
1 n = W. rows
2 $D^{(0)}$ = W
3 for k = 1 to n
4 let $D^{(k)} = (d_{ij}^{(k)})$ be a new $n \times n$ matrix
5 for i = 1 to n
6 for j = 1 to n
7 $d_{ij}^{(k)} = \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})$
8 return $D^{(n)}$

Step 3: k = 5
Step 4: $D^{(5)} = d_{ij}^{(5)}$

$$D^{(4)} \begin{pmatrix} 0 & 3 & -1 & 4 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix}$$

Step 5-7:
$d_{12}^{(5)} = \min(d_{12}^{(4)}, d_{15}^{(4)} + d_{52}^{(4)}) \rightarrow \min(3, -4+5) \rightarrow 1$
$d_{13}^{(5)} = \min(d_{13}^{(4)}, d_{15}^{(4)} + d_{53}^{(4)}) \rightarrow \min(-1, -4+1) \rightarrow -3$
$d_{14}^{(5)} = \min(d_{14}^{(4)}, d_{15}^{(4)} + d_{54}^{(4)}) \rightarrow \min(4, -4+6) \rightarrow 2$

- $\pi_{ij}^{(k)} = \pi_{ij}^{(k-1)}$; *if* $d_{ij}^{(k-1)} \le d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$,
  $= \pi_{kj}^{(k-1)}$ *i; if* $d_{ij}^{(k-1)} > d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$

$$D^{(5)} \begin{pmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix}$$

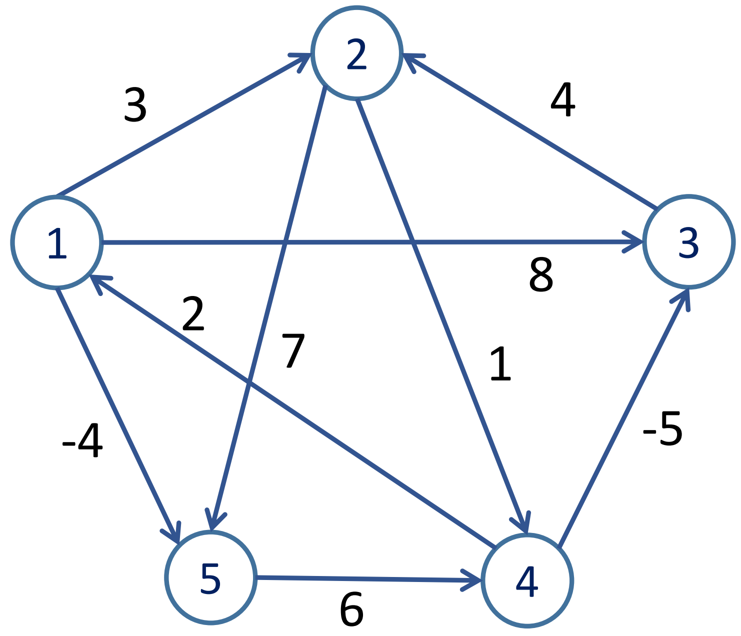
$$\pi^{(5)} = \begin{pmatrix} \text{NIL} & 3 & 4 & 5 & 1 \\ 4 & \text{NIL} & 4 & 2 & 1 \\ 4 & 3 & \text{NIL} & 2 & 1 \\ 4 & 3 & 4 & \text{NIL} & 1 \\ 4 & 3 & 4 & 5 & \text{NIL} \end{pmatrix}$$

APSP: Floyd-Warshall (7)

Floyd-WARSHALL (W) {
n = W. rows
$D^{(0)}$ = W
for k = 1 to n
    let $D^{(k)} = (d_{ij}^{(k)})$ be a new $n \times n$ matrix
    for i = 1 to n
        for j = 1 to n
            $d_{ij}^{(k)} = \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})$
return $D^{(n)}$

$$D^{(5)} = \begin{pmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix}$$

$$\pi^{(5)} = \begin{pmatrix} NIL & 3 & 4 & 5 & 1 \\ 4 & NIL & 4 & 2 & 1 \\ 4 & 3 & NIL & 2 & 1 \\ 4 & 3 & 4 & NIL & 1 \\ 4 & 3 & 4 & 5 & NIL \end{pmatrix}$$

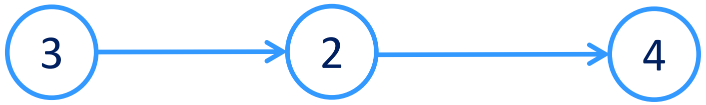


$D_{24}^{(5)} = 1 \quad \pi_{24}^{(5)} = 2$

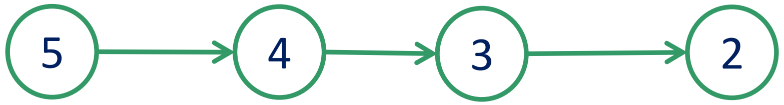
Example#1:

Example#2: $D_{34}^{(5)} = 5 \quad \pi_{34}^{(5)} = 2$

$\pi_{32}^{(5)} = 3$



Example#3: $D_{52}^{(5)} = 5$



$\pi_{54}^{(5)} = 5 \quad \pi_{53}^{(5)} = 4 \quad \pi_{52}^{(5)} = 2$

APSP: Floyd-Warshall → time complexity

```
Floyd-WARSHALL (W) {
n = W. rows
D^(0) = W
for k = 1 to n                                  ← O (V)    n= # of vertices = |V|
    let D^(k) = (d_ij^(k)) be a new n x n matrix
    for i = 1 to n                              ← O (V)
        for j = 1 to n                          ← O (V)
            d_ij^(k) = min (d_ij^(k-1) , d_ik^(k-1) + d_kj^(k-1) )    ← O(V^3)
return D^(n)
```

Total time complexity: $O(V^3) \rightarrow O(n^3)$

APSP: Floyd-Warshall → time complexity

Floyd-WARSHALL (W) {

n = W. rows

$D^{(0)}$ = W

for k = 1 to n **O (V)** **n= # of vertices = |V|**

  let $D^{(k)} = (d_{ij}^{(k)})$ be a new $n \times n$ matrix

  for i = 1 to n **O (V)**

    for j = 1 to n **O (V)**

      $d_{ij}^{(k)} = \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})$ **O(V^3)**

return $D^{(n)}$

Total time complexity: O(V^3) → O(n^3)

thank you!

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