PHYS414 - Final Project

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I. Newtonian Solution of a Star

A. Lane-Emden Equation

If we start with hydrostatic equilibrium of stars in Newtonian gravity, we can obtain following ODE's:

$$\frac{m(r)}{dr} = 4\pi r^2 \rho(r) \tag{1}$$

$$\frac{dp(r)}{dr} = \frac{-Gm(r)\rho(r)}{r^2} \tag{2}$$

$p(r)$ and $m(r)$ notates the mass of star within the radius r and the density respectively. Using (2), we can solve for m(r) and take its derivative respect to r:

$$-\frac{dm(r)}{dr} = \frac{d}{dr}(\frac{r^2}{G\rho(r)}\frac{dp(r)}{dr}) \tag{3}$$

We can use the equation for $\frac{dm(r)}{dr}$ in (1) and get:

$$\frac{1}{4\pi r^2}\frac{d}{dr}(\frac{r^2}{G\rho(r)}\frac{dp(r)}{dr}) = -\rho(r) \tag{4}$$

Assuming polytropic equation of state, we can P and ρ by the relation:

$$p = K\rho^{\gamma} = K\rho^{1+\frac{1}{n}} \tag{5}$$

where in is defined as polytropic index. If we take the r derivative of $p(r)$:

$$\frac{dp(r)}{dr} = \frac{(n+1)}{n}K\rho^{\frac{1}{n}}\frac{d\rho(r)}{dr} \tag{6}$$

We can see that it is helpful to define a parameter $p = p_c\theta^n$ for simplicity. Now, plugging the expression for $p(r)$ into (6) into (4), we get:

$$(\frac{K}{G})(n+1)(p_c^{\frac{1-n}{n}})(\frac{1}{4\pi r^2})\frac{d}{dr}(r^2\frac{d\theta(r)}{dr}) = -\theta^n(r) \tag{7}$$

Now, lets define a constant A and scaled radius ξ:

$$A^2 = \frac{(n+1)K\rho_c^{\frac{1}{n}-1}}{4\pi G} \tag{8}$$

$$\xi = r/A \tag{9}$$

So that hydro-static EOS can be neatly written in the desired form:

$$\frac{1}{\xi^2}\frac{d}{d\xi^2}(\xi^2\frac{d\theta(\xi)}{d\xi}) + \theta^n = 0 \tag{10}$$

If we employ Mathematica for exact solution, we get:

$$\theta(\xi) = -\frac{ie^{-i\xi}\left(-1+e^{2i\xi}\right)}{2\xi} \tag{11}$$

which has regular solutions at the center:

$$\theta(\xi) = 1 - \frac{1}{6}\xi^2 + \frac{1}{120}\xi^4 + O(\xi^6) + ..., \tag{12}$$

Furthermore, we can obtain the total mass by integrating (1) but now with scaled parameters:

$$M_{total} = 4\pi A^3 p_c \int_0^{R_{max}/A} \theta^n(\xi)\xi^2 d\xi \tag{13}$$

Using (10), We can replace $\theta^n(\xi)\xi^2$ with $\frac{d\theta(r)}{d\xi}\xi^2$ and get:

$$M_{total} = -4\pi A^3 p_c \frac{1}{\xi}\frac{d\theta(\xi)}{d\xi} \tag{14}$$

Using the definition of A, we can substitute the expression for p_c and obtain a relation of M and R.

$$-4\pi R^3(\frac{4\pi G A^2}{K(n+1)})^{n/(1-n)}\frac{d\theta(\xi)}{d\xi} \tag{15}$$

Employing the definition $R = A\xi_n$:

$$M_{total} = -R^{\frac{3-n}{1-n}}(-4\pi)(\frac{4\pi G}{K(n+1)})^{n/(1-n)}\frac{d\theta(\xi_n)}{d\xi_n}\xi_n^{\frac{n+1}{n-1}} \tag{16}$$

Where M_{total} can we written as a product of a constant β and $R^{\frac{3-n}{1-n}}$:

$$M_{total} = \beta R^{\frac{3-n}{1-n}} \tag{17}$$

thus, we end up with an relation for group of stars that share same polytropic EOS:

$$M_{total} \propto R^{\frac{3-n}{1-n}} \tag{18}$$

B. White Dwarf Data Visualisation

Using the white dwarf data and plot M (in solar mass unit) and R (in earth radius units), we get:

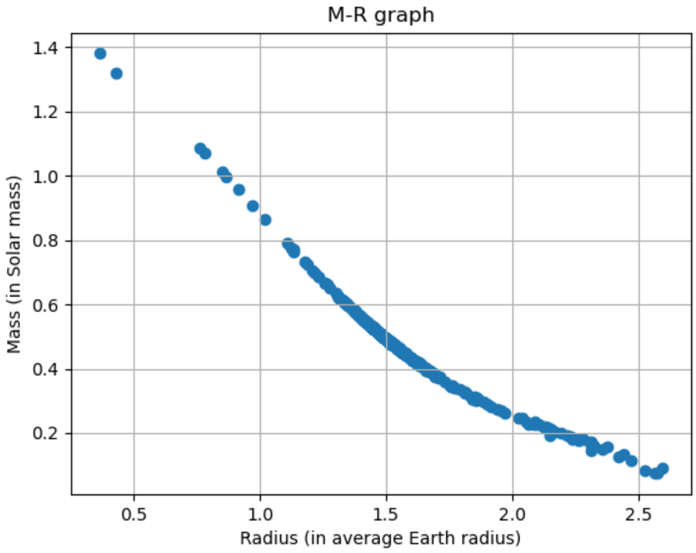

Figure 1: White Dwarf Data Visualization

C. Low-Mass White Dwarves

For a white dwarf, we can express the EOS which is dominated by the electron degeneracy:

$$P = C[x(2x^2 - 3)(x^2 + 1)^{1/2} + 3sinh^{-1}x], \tag{19}$$

$$X = (\frac{\rho}{D})^{\frac{1}{q}} \tag{20}$$

Using mathematica, we can analytically calculate the taylor series of equation (20) for $|x| << 0$:

$$\frac{8Cx^5}{5} - \frac{4Cx^7}{7} + \frac{Cx^9}{3} - \frac{5Cx^{11}}{22} + O\left(x^{12}\right) \tag{21}$$

Omitting higher order terms, we get $P = \frac{8Cx^5}{5}$. We can identify n_* term in

$$P = K_* \rho^{1+\frac{1}{n_*}} \tag{22}$$

as

$$n_* = q/(5 - q). \tag{23}$$

We can obtain a linear relation between $ln(M)$ and $ln(B)$ and apply a linear fit to calculate the slope for filtered data $(ln(M) > 0.25)$:

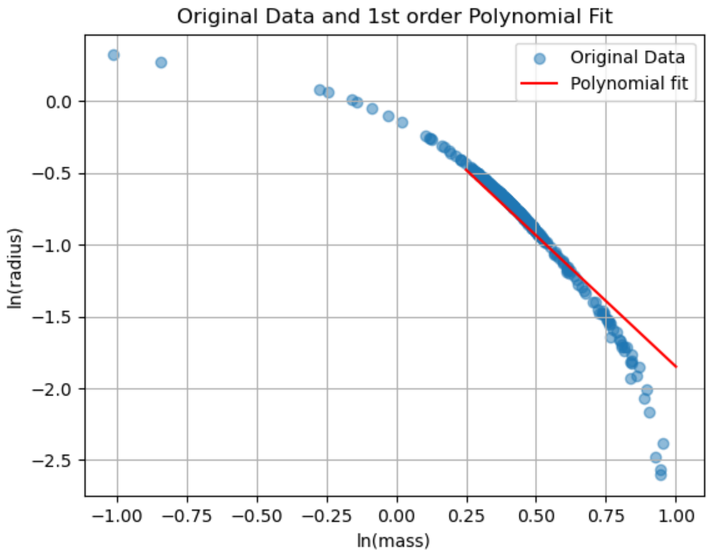

Figure 2: 1st order polynomial fit applied to the R and M ($ln(M) > 0.25$).

From the fit, we can calculate the slope and use the relation for n_* as follows:

$$slope = \frac{3 - n_*}{1 - n_*} = -3.3387 \tag{24}$$

From theory, we assume q is an integer and find that smallest positive integer satisfying (23) is $q = 3$, which gives $n = 1.5$. Now, we can calculate the Lane-Emden equation and obtain unknown parameters:

$$\xi = 3.6578223569228965 \tag{25}$$
$$\theta = 1.0176017680895436e^{-16} \tag{26}$$
$$\theta'(\xi) = -0.202860682607365 \tag{27}$$
$$\tag{28}$$

We can now calculate K_* since we already have an explicit formula (eq. (16)) for K_* in terms of ξ and $\theta'(\xi)$.

$$K_* = 0.030964561408303514 \tag{29}$$

Moreover, we can calculate p_c since we know ξ and $\theta'(\xi)$. Using equation (14) results:

$$\rho_c = -\frac{M\xi}{4\pi R^3(\theta'(\xi))} \tag{30}$$

Using (30), we can plot ρ_c vs. M graph:

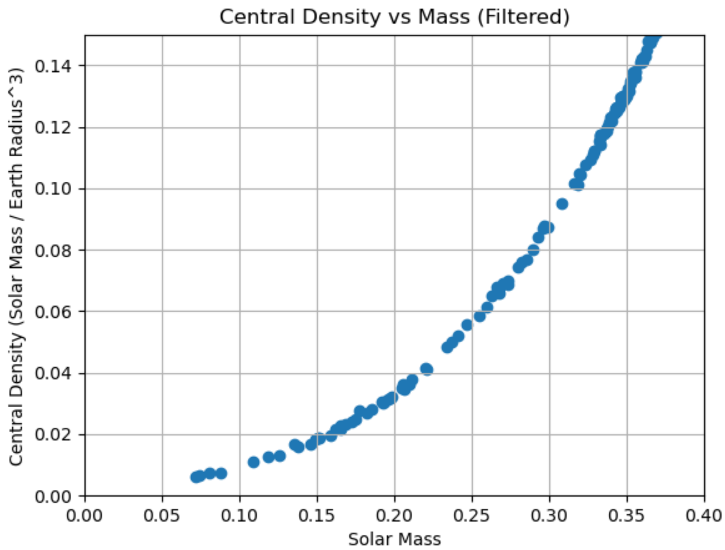

Figure 3: Low-Mass White Dwarf: Central Density (ρ_c) vs Mass in logarithmic base.

II. Einstein

A. Mass vs. Radius

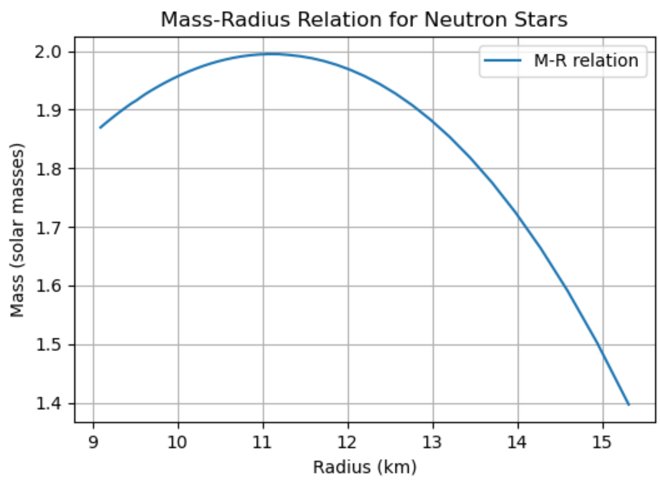

Figure 4: Mass vs. Radius with differing ρ_c

50 ρ_c values are sampled between 10^{-3} and 9×10^{-3}. Resulting Relation can be seen above.

B. Fractional Binding Energy

We have the following relation for baryonic mass:

$$m'_P = 4\pi \left(1 - \frac{2m}{r}\right)^{-\frac{1}{2}} r^2 \rho \tag{31}$$

where we can define fractional binding energy (Δ) as:

$$\Delta \equiv \frac{M_P - M}{M}. \tag{32}$$

and obtain the relation in Figure 5.

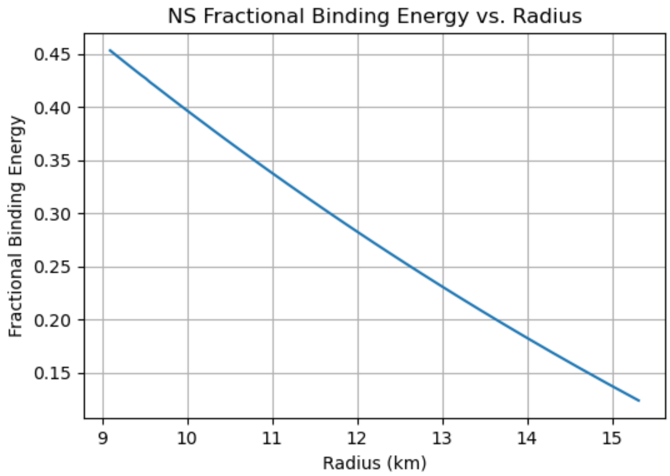

Figure 5: Δ vs radius R

where we can observe that the relation is highly linear.

C. Mass vs Central Density (ρ_c)

Let's begin with simple criterion for NS stability:

$$\frac{dM}{d\rho_c} > 0 \quad \longrightarrow \quad \text{stable} \tag{33}$$

$$\frac{dM}{d\rho_c} < 0 \quad \longrightarrow \quad \text{unstable} \tag{34}$$

Overall, $\frac{dM}{d\rho_c} > 0$, otherwise it forms a black hole. If we visualize M vs ρ_c:

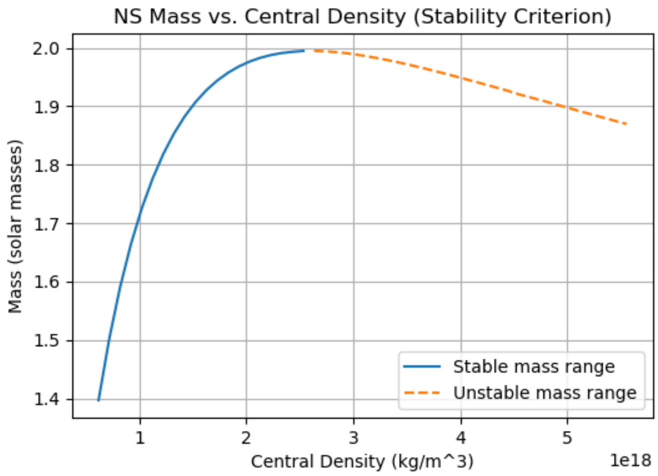

Figure 6: Numerically calculated stable and unstable regions for a Neutron Star

Moreover, we can obtain the maximum allowed mass for NS:

$$M_{max} = 1.9947289179622747 \tag{35}$$

D. Maximum Allowed Mass vs Polytropic Coefficient K

Since we know maximum allowed mass M^* depends on the EOS, we can calculate M^* using fixed polytropic index. We can iterate over ρ_c values for a single K value to solve the TOV. Then, we repeat process for all selected K values to find maximum allowed mass.

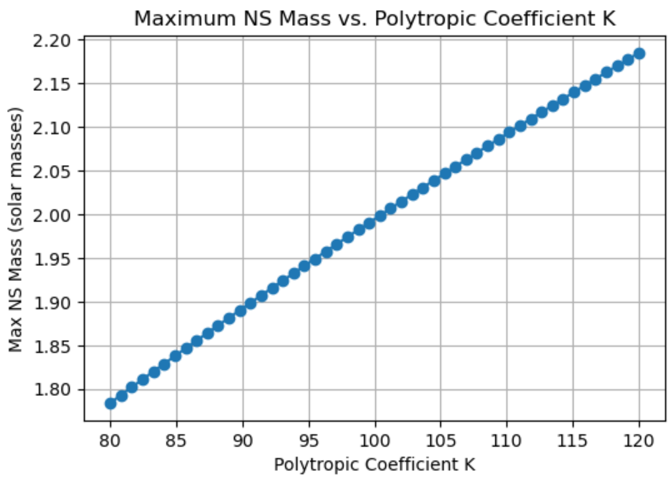

Figure 7: Maximum Allowed Mass M^* vs polytropic index K^*

If we apply cubic interpolation to the data of Figure 7 and leverage root finding using the mass of biggest observed neutron star $M = 2.14$, we get the maximum allowed value of K:

$$K = 115.0947205875922 \tag{36}$$