Progress of Theoretical Physics, Vol. 49, No. 1, January 1973

# Theory of Interacting Spin-3/2 Particle

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(Received June 2, 1972)

A consistent theory of the interacting spin 3/2 particle is presented. Instead of the constraint conditions, which are the origin of troubles in a theory of interacting higher spin particles, we suppress redundant particles by making their masses infinite. Our theory is general enough to extend to particles with $s>3/2$.

## § 1. Introduction, results and discussion

As is well known, we have serious difficulties in the theory of interacting higher spin ($s\geq 3/2$) fields both in classical and quantum theories:

A) The consistency of covariant quantization in the presence of external electromagnetic field is incompatible with the positive-definiteness requirement of the equal-time commutation relations.[1] This incompatibility appears even in the more general cases.

B) The interacting classical wave equation possesses acausal modes of propagation.[2]

C) The time ordered product is non-covariant. Because of this non-covariance there is a difference between vacuum expectation value of the $T$ product and the covariant propagator. This difference must be explained by the complicated normal dependent interaction Hamiltonian. When we consider interaction Hamiltonian as power series of the coupling constant, for the particle with $s\geq 3/2$, this series may be infinite. Therefore, we do not have field theoretical basis for Feynman rules for the system including higher spin particles. Moreover we have no clear outlook for the Lehmann-Symanzik-Zimmermann reduction formula.

These three difficulties are very serious. Some people even suggest that higher spin particle cannot be elementary. These come from the constraint conditions introduced to suppress redundant fields.

In this article we propose non-constraint theory of higher spin field and show that our theory is free from the difficulties A), B), C). We suppress redundant fields not by constraints but by making the masses of the corresponding particles infinite. This is the extension of the $\xi$-limiting theory proposed by Lee and Yang[3] for the interacting spin one particle.

In § 2 we give the Lagrangian in our theory and discuss the canonical quantization procedure. The equal-time commutators of the field operators are quite different from those of usual theory, so that the difficulty A) disappears.

In § 3 we study the quantization based on the idea of Peierls[4] and Takahashi-Umezawa.[5] This method of quantization is useful to develop the theory manifestly covariant and to separate covariantly the contributions of the redundant fields. We find that the special suppression procedure is needed to get a causal theory. This fact seems to relate to the difficulty B) in usual theory. The time ordered product is covariant. Then, our theory is free from difficulty C). In § 3 we also obtain the various Green functions of the respective fields. Through the Green functions we see that the redundant particles play a role of regulator. When the masses of the redundant particles are finite our theory is renormalizable. By taking the limit the divergence appears for some Feynman diagrams. This divergence is nothing but the ultra-violet divergence in higher spin field.

In § 4 we give explicit forms of the wave functions of the respective fields, their orthonormality relations and completeness conditions.

In § 5 we give Fourier expansion of field operator and see that the redundant spin 1/2 particles are both ghosts. The difficulty A) may be due to the incomplete separation of the redundant fields. We also write down the Hamiltonian operator in Fock representation and LSZ reduction formulas. We shall discuss only the spin 3/2 field in detail. However our method is also applicable to the general cases without significant change.

## § 2. Canonical formulation

We consider the general spinor-vector field. The most general equation of motion consisting of at most first-order derivative is the following:

$$\Lambda_{\mu\nu}(\partial)\psi_\nu(x)=0\,, \tag{2·1}$$

where

$$\begin{aligned}\Lambda_{\mu\nu}(\partial)=&-(\gamma_\lambda\partial_\lambda+m)\delta_{\mu\nu}-A(\gamma_\mu\partial_\nu+\gamma_\nu\partial_\mu)\\&-B\gamma_\mu\gamma_\lambda\partial_\lambda\gamma_\nu-Cm\gamma_\mu\gamma_\nu\,.\end{aligned} \tag{2·2}$$

Equation (2·2) can be rewritten as

$$\Lambda_{\mu\nu}(\partial)=-(\Gamma_\lambda)_{\mu\nu}\partial_\lambda-M_{\mu\nu} \tag{2·3}$$

with

$$(\Gamma_\lambda)_{\mu\nu}=\gamma_\lambda\delta_{\mu\nu}+A(\gamma_\mu\delta_{\nu\lambda}+\gamma_\nu\delta_{\mu\lambda})+B\gamma_\mu\gamma_\lambda\gamma_\nu \tag{2·4}$$

and

$$M_{\mu\nu}=m\delta_{\mu\nu}+Cm\gamma_\mu\gamma_\nu\,. \tag{2·5}$$

In (2·2) we put the coefficients of $\gamma_\mu\partial_\nu$ and $\gamma_\nu\partial_\mu$ equal to each other so as to make the Lagrangian

$$L=\bar{\psi}_\mu \Lambda_{\mu\nu}\psi_\nu=-\bar{\psi}_\mu\{(\Gamma_\lambda)_{\mu\nu}\partial_\lambda+M_{\mu\nu}\}\psi_\nu \tag{2·6}$$

with

$$\bar{\psi}_\mu=\psi_\mu{}^*\gamma_4 \tag{2·7}$$

real. In Eq. (2·1) if,

$$A\neq -1/2\,, \tag{2·8a}$$

$$B=(3/2)A^2+A+1/2\,, \tag{2·8b}$$

$$C=-(3A^2+3A+1), \tag{2·8c}$$

we immediately get

$$(\gamma_\nu\partial_\nu+m)\psi_\mu=0\,, \tag{2·9a}$$

$$\gamma_\mu\psi_\mu=0\,, \tag{2·9b}$$

$$\partial_\mu\psi_\mu=0\,. \tag{2·9c}$$

This is the Rarita-Schwinger equation describing the irreducible spin 3/2 field. The parameter $A$ comes from the ambiguity of the wave function in the point transformation,

$$\psi_\mu'=(\delta_{\mu\nu}+\alpha\gamma_\mu\gamma_\nu)\psi_\nu\,. \tag{2·10}$$

Now we discuss the general case where the parameters do not satisfy (2·8). We treat all $\psi_\mu$ as dynamical variables. The canonically conjugate momenta $\pi_\mu$ of $\psi_\mu$ are

$$\pi_\mu\equiv\partial L/\partial(\partial\psi_\mu/\partial x_0)=i\bar{\psi}_\nu(\Gamma_4)_{\nu\mu}\,. \tag{2·11}$$

From the standard commutation relations

$$\{\psi_\mu(x),\pi_\nu(x')\}_{x_0=x_0'}=i\delta^3(\boldsymbol{x}-\boldsymbol{x}')\delta_{\mu\nu}\,, \tag{2·12}$$

we get

$$\{\psi_\mu(x),\bar{\psi}_\nu(x')\}_{x_0=x_0'}=(\Gamma_4)^{-1}_{\mu\nu}\delta^3(\boldsymbol{x}-\boldsymbol{x}')\,. \tag{2·13}$$

Here $\Gamma_4{}^{-1}$ means the inverse of $\Gamma_4$:

$$\begin{aligned}(\Gamma_4)^{-1}_{\mu\nu}=\{&(1+2A+3A^2-2B)\gamma_4\delta_{\mu\nu}-(A^2-A-2B)(\gamma_\mu\delta_{4\nu}+\gamma_\nu\delta_{4\mu})\\&-2(2A+A^2+2B)\gamma_4\delta_{4\mu}\delta_{4\nu}+(A^2-B)\gamma_\mu\gamma_4\gamma_\nu\}/(1+2A+3A^2-2B).\end{aligned} \tag{2·14}$$

Although $\Gamma_4$ is singular in the case of (2·8), as will be seen later in our theory, $\Gamma_4$ is regular. Following the standard method, we obtain the Hamiltonian

$$\begin{aligned}H&=\pi_\mu(\partial\psi_\mu/\partial x_0)-L\\&=\bar{\psi}_\mu\{(\Gamma_i)_{\mu\nu}\partial_i+M_{\mu\nu}\}\psi_\nu\,,\end{aligned} \tag{2·15}$$

and the energy momentum tensor, where Latin indicies denote spatial components.

The symmetrical energy momentum tensor in our theory satisfies Schwinger's commutator condition in its simplest form contrary to usual theory.[6)]

## § 3. Extended method of Peierls' quantization

We discuss Peierls' quantization for the free reducible spinor-vector field $\psi_\mu$. We start with the investigation of $\Lambda_{\mu\nu}^{-1}$ defined by

$$\Lambda_{\mu\nu}(ip)\Lambda_{\nu\lambda}^{-1}(ip)=\Lambda_{\mu\nu}^{-1}(ip)\Lambda_{\nu\lambda}(ip)=\delta_{\mu\lambda}\,. \tag{3·1}$$

$\Lambda_{\mu\nu}^{-1}(ip)$ should have the following form:

$$\begin{aligned}\Lambda_{\mu\nu}^{-1}(ip)=&I_1(p^2)\delta_{\mu\nu}+I_2(p^2)\gamma_\mu\gamma_\nu+I_3(p^2)(i\gamma_\mu p_\nu+i\gamma_\nu p_\mu)\\&-I_4(p^2)p_\mu p_\nu+I_5(p^2)i\gamma_\lambda p_\lambda\delta_{\mu\nu}+I_6(p^2)\gamma_\mu i\gamma_\lambda p_\lambda\gamma_\nu\\&-I_7(p^2)(\gamma_\mu\gamma_\lambda p_\lambda p_\nu+p_\mu\gamma_\lambda p_\lambda\gamma_\nu)-I_8(p^2)p_\mu p_\nu i\gamma_\lambda p_\lambda.\end{aligned} \tag{3·2}$$

Substituting (3·2) into (3·1), we get

$$I_1=-m/(p^2+m^2), \tag{3·3a}$$

$$\begin{aligned}I_2=&m[m^4C(1+4C)+m^2p^2(6CA+4B^2+A^2+7CA^2+2BA-2B)\\&+p^4\{(A^2+A)(1+2A+3A^2-2B)-(1+A)^2(A+2B+C)\}]/D(p^2),\end{aligned} \tag{3·3b}$$

$$\begin{aligned}I_3=&[2m^2p^2(A+2B+C)(A^2-2B-2C)-\{(2C-A)m^2+(A^2-A-2B)p^2\}\\&\times\{(1+4C)m^2+(1+2A+3A^2-2B)p^2\}]/D(p^2),\end{aligned} \tag{3·3c}$$

$$\begin{aligned}I_4=&-4m[(A^2+A-C)(1+4C)m^2+\{(1+2A+3A^2-2B)(A^2+A-C)\\&-(A+2B+C)(A^2+2A+2B)\}p^2]/D(p^2),\end{aligned} \tag{3·3d}$$

$$I_5=1/(p^2+m^2), \tag{3·3e}$$

$$\begin{aligned}I_6=&[(1+2A+3A^2-2B)(A^2-B)p^4+m^2p^2(-2A^3+A^2+6AC\\&+8A^2C-BA^2+6AB+2C+6B^2+2C^2)\\&+(m^4/2)\{(A+2B+C)+(1+4C)(3C-A)\}]/D(p^2),\end{aligned} \tag{3·3f}$$

$$\begin{aligned}I_7=&m[m^2\{2(A+2B+C)(2C-A)+(1+4C)(A^2-2B-2C)\}\\&+p^2\{2(A+2B+C)(A^2-A-2B)+(A^2-2B-2C)\\&\times(1+2A+3A^2-2B)\}]/D(p^2),\end{aligned} \tag{3·3g}$$

$$\begin{aligned}I_8=&2[\{(1+4C)(A^2+A-C)+(A+2B+C)(1+2A)^2\}m^2\\&+(1+2A+3A^2-2B)(A^2+2A+2B)p^2]/D(p^2),\end{aligned} \tag{3·3h}$$

where

$$D(p^2)=[\{(1+4C)m^2+(1+2A+3A^2-2B)p^2\}^2+4m^2p^2(A+2B+C)^2](p^2+m^2). \tag{3·3i}$$

That is, $\Lambda^{-1}$ have three poles at

$$p^2 = -m^2 \tag{3·4a}$$

and

$$p^2 = -M^{(\pm)2} = -m^2[\{A+2B+C \pm \{(A+2B+C)^2 + (1+2A+3A^2-2B)(1+4C)\}^{1/2}\} \\ \times (1+2A+3A^2-2B)^{-1}]^2. \tag{3·4b}$$

Our spinor-vector field $\psi_\mu$ describes three kinds of particles whose masses are $m$, $M^{(+)}$, $M^{(-)}$. The particle with mass $m$ is the spin 3/2 particle which we want to have, whereas the particles with masses $M^{(\pm)}$ are the redundant spin 1/2 particles which we must suppress. For simplicity we impose the following condition among the parameters:

$$A+2B+C=0\,. \tag{3·5}$$

Then

$$M^{(+)2} = M^{(-)2} = \frac{1+4C}{1+3A+3A^2+C} m^2 \equiv M^2 \tag{3·6}$$

and

$$\Lambda_{\mu\nu}^{-1} = -\frac{d_{\mu\nu}(ip)}{(p^2+m^2)} - \frac{d_{\mu\nu}^{(+)}(ip)}{(p^2+M^2)} - \frac{d_{\mu\nu}^{(-)}(ip)}{(p^2+M^2)}\,, \tag{3·7}$$

where

$$\begin{aligned} d_{\mu\nu}(ip) &= m\delta_{\mu\nu} - (m/3)\gamma_\mu\gamma_\nu + (1/3)(i\gamma_\mu p_\nu + i\gamma_\nu p_\mu) + (4/3m)p_\mu p_\nu \\ &\quad - i\gamma_\lambda p_\lambda \delta_{\mu\nu} - (1/3)\gamma_\mu i\gamma_\lambda p_\lambda \gamma_\nu - (1/3m)(\gamma_\mu\gamma_\lambda p_\lambda p_\nu + p_\mu \gamma_\lambda p_\lambda \gamma_\nu) \\ &\quad - (2/3m^2)p_\mu p_\nu i\gamma_\lambda p_\lambda\,, \end{aligned} \tag{3·8a}$$

$$\begin{aligned} d_{\mu\nu}^{(\pm)}(ip) &= \frac{m}{12(1+3A+3A^2+C)}\Big[2(1+C) \\ &\quad \pm \left(\frac{1+4C}{1+3A+3A^2+C}\right)^{1/2}(2+3A-C)\Big]\gamma_\mu\gamma_\nu \\ &\quad - \frac{1}{6(1+3A+3A^2+C)}\left[(1+3A-2C) \pm \left(\frac{1+4C}{1+3A+3A^2+C}\right)^{1/2}\right] \\ &\quad \times (i\gamma_\mu p_\nu + i\gamma_\nu p_\mu) + \frac{1}{3m}\left[-2 \pm \left(\frac{1+4C}{1+3A+3A^2+C}\right)^{1/2}\right] p_\mu p_\nu \\ &\quad + \frac{1}{12(1+3A+3A^2+C)}\Big[(2+3A-C) \\ &\quad \pm 2\left(\frac{1+4C}{1+3A+3A^2+C}\right)^{1/2}(1+3A+3A^2+C)\Big] i\gamma_\mu\gamma_\lambda p_\lambda \gamma_\nu \end{aligned}$$

$$-\frac{1}{6m(1+4C)}\left[-(1+4C)\pm\left(\frac{1+4C}{1+3A+3A^2+C}\right)^{1/2}(2C-3A-1)\right]$$
$$\times(\gamma_\mu\gamma_\lambda p_\lambda p_\nu+p_\mu\gamma_\lambda p_\lambda\gamma_\nu)-\frac{1}{3m^2(1+4C)}\left[-(1+4C)\right.$$
$$\left.\pm 2\left(\frac{1+4C}{1+3A+3A^2+C}\right)^{1/2}(1+3A+3A^2+C)\right]p_\mu p_\nu i\gamma_\lambda p_\lambda\,. \qquad (3\cdot 8\text{b})$$

In Eq. (3·7) the separation to $d^{(+)}$ and $d^{(-)}$ in the numerator of $1/(p^2+m^2)$ has been made before putting (3·6).

Using the new parameter

$$\xi=-(1+3A+3A^2+C) \qquad (3\cdot 9)$$

instead of $C$, (3·6) is rewritten as

$$M^2=m^2\{4+3(1+2A)^2/\xi\}\,. \qquad (3\cdot 10)$$

Then, to make the mass $M$ of redundant particles infinite, we must take

$$2A+1\neq 0 \qquad (3\cdot 11\text{a})$$

and

$$\xi\rightarrow +0\,. \qquad (3\cdot 11\text{b})$$

Since we already have (3·5), (3·11a) and (3·11b) mean that the irreducible condition (2·8) is satisfied in the limit. However as is specified in (3·11b), we must take a limit from a positive value of $\xi$. If we take a limit from a negative value, our equation of motion becomes acausal. Then, it is not certain whether the equation of motion (2·1) with (2·8) is causal. In order that (2·1) is obviously causal we must assume the limit (3·11b). This fact may be the reason of the difficulty B) in usual theory. The same situation exists in the spin one particle.[3] The difficulty B) for the spin one particle has been reported by Minkowski and Seiler.[7]

According to Peierls' quantization procedure, we assume

$$\{\psi_\mu(x),\bar{\psi}_\nu(x')\}=id_{\mu\nu}(\partial)\Delta(x-x';m)$$
$$+id_{\mu\nu}^{(+)}(\partial)\Delta(x-x';M)+id_{\mu\nu}^{(-)}(\partial)\Delta(x-x';M), \qquad (3\cdot 12)$$

where $\Delta(x-x';m)$ is the invariant delta function with mass $m$. In (3·12) if we put $x_0=x_0'$ we get exactly the same equation as (2·12).

We also get

$$\langle 0|T\{\psi_\mu(x),\bar{\psi}_\nu(x')\}|0\rangle=id_{\mu\nu}(\partial)\Delta_c(x-x';m)$$
$$+id_{\mu\nu}^{(+)}(\partial)\Delta_c(x-x';M)+id_{\mu\nu}^{(-)}(\partial)\Delta_c(x-x';M)$$
$$+(i/2)[\varepsilon(x_0-x_0'),d_{\mu\nu}(\partial)]\Delta(x-x';m)$$
$$+(i/2)[\varepsilon(x_0-x_0'),d_{\mu\nu}^{(+)}(\partial)]\Delta(x-x';M)$$

$$+(i/2)\left[\varepsilon(x-x_0'),\, d_{\mu\nu}^{(-)}(\partial)\right]\Delta(x-x';M), \tag{3·13}$$

where $\Delta_c=-(i/2)\Delta_F$ is the causal delta function. Inserting the expression given by (3·8) into (3·13) we easily see that the last three normal dependent terms cancel out. Then

$$\langle 0|T\{\psi_\mu(x),\bar{\psi}_\nu(x')\}|0\rangle = id_{\mu\nu}(\partial)\Delta_c(x-x';m)$$
$$+id_{\mu\nu}^{(+)}(\partial)\Delta_c(x-x';M)+id_{\mu\nu}^{(-)}(\partial)\Delta_c(x-x';M). \tag{3·14}$$

The $T$-product is covariant in spite of the fact that each $d(\partial)$ includes third-order derivatives. Thus the correspondence of the canonical formulation and the covariant formulation is established.

So far we have discussed only free field equations. However it is obvious that our theory does not give rise to any difficulty in the presence of the interaction. As to B) we remark that $\det|\Gamma_\mu n_\mu|\neq 0$ for any time like $n_\mu$.[2] The normal dependent Hamiltonian stated in C) is not required in order to get a covariant theory.[8] There exists the unitary transformation which connects operators of the Heisenberg representation with those of the interaction representation in one-to-one correspondent manner. That is, in the Takahashi-Umezawa notation,[9] we have

$$\psi(x)=\psi(x/\sigma)=S^{-1}[\sigma]\psi(x)S[\sigma]. \tag{3·15}$$

Furthermore we see from (3·8) that $p^2$ and $p^3$ terms (leading terms at $p\to\infty$) of the propagator are supressed by means of the presence of the redundant spin 1/2 fields. The redundant spin 1/2 fields play a role of regulator. Therefore, when $\xi$ is finite, our theory is renormalizable. In a calculation of certain Feynman diagrams, we happen to have terms such as $\log\xi$, $\xi^{-1}$, $\xi^{-2}$, etc., which cannot be removed by the renormalization procedure. Then our theory is unrenormalizable in the limit. However this does not mean that we are never able to calculate higher-order corrections. Finite results may be obtained by rearranging the perturbation expansion as was done by Lee.[10]

## § 4. Wave functions, their orthonormalities and completeness

In this section, we give explicit forms of the respective wave functions. We denote spin 3/2 wave function and two kinds of spin 1/2 wave functions with helicity $r$ by $U_{r,\mu}(\boldsymbol{p},m)$ and $U_{r,\mu}^{(\pm)}(\boldsymbol{p},M)$, respectively. We construct then, from the helicity diagonalized wave functions of spin 1/2 and spin 1 particles, using the usual composition law:

$$U_{3/2,\mu}(\boldsymbol{p},m)=u^+(\boldsymbol{p},m)e_\mu^+(\boldsymbol{p},m), \tag{4·1a}$$

$$U_{1/2,\mu}=(1/3)^{1/2}(2^{1/2}u^+e_\mu^0+u^-e_\mu^+), \tag{4·1b}$$

$$U_{-1/2,\mu}=(1/3)^{1/2}(u^+e_\mu^-+2^{1/2}u^-e_\mu^0), \tag{4·1c}$$

$$U_{-3/2,\mu}=u^-e_\mu^-, \tag{4·1d}$$

$$U^{(+)}_{1/2,\mu}(\boldsymbol{p},M)=(1/3)^{1/2}[-u^{+}(\boldsymbol{p},M)e_{\mu}{}^{0}(\boldsymbol{p},M)$$
$$+2^{1/2}u^{-}(\boldsymbol{p},M)e_{\mu}{}^{+}(\boldsymbol{p},M)], \tag{4·2a}$$

$$U^{(+)}_{-1/2,\mu}=(1/3)^{1/2}[-2^{1/2}u^{+}e_{\mu}{}^{-}+u^{-}e_{\mu}{}^{0}], \tag{4·2b}$$

$$U^{(-)}_{1/2,\mu}(\boldsymbol{p},M)=u^{+}(\boldsymbol{p},M)e_{\mu}(\boldsymbol{p},M), \tag{4·3a}$$

$$U^{(-)}_{-1/2,\mu}=u^{-}e_{\mu}\,. \tag{4·3b}$$

By using the angular variables defined as

$$\boldsymbol{p}=p(\cos\phi\sin\theta,\,\sin\phi\sin\theta,\,\cos\theta), \tag{4·4}$$

$u^{\pm}$, $e_{\mu}{}^{\pm 0}$ and $e_{\mu}$ we have used are given in the forms

$$u^{+}(\boldsymbol{p},m)=\left[\frac{(\boldsymbol{p}^2+m^2)^{1/2}+m}{2(\boldsymbol{p}^2+m^2)^{1/2}}\right]^{1/2}\begin{pmatrix}\cos(\theta/2)\\ e^{i\phi}\sin(\theta/2)\\ \cos(\theta/2)p/[(\boldsymbol{p}^2+m^2)^{1/2}+m]\\ e^{i\phi}\sin(\theta/2)p/[(\boldsymbol{p}^2+m^2)^{1/2}+m]\end{pmatrix}, \tag{4·5a}$$

$$u^{-}(\boldsymbol{p},m)=\left[\frac{(\boldsymbol{p}^2+m^2)^{1/2}+m}{2(\boldsymbol{p}^2+m^2)^{1/2}}\right]^{1/2}\begin{pmatrix}-e^{-i\phi}\sin(\theta/2)\\ \cos(\theta/2)\\ e^{-i\phi}\sin(\theta/2)p/[(\boldsymbol{p}^2+m^2)^{1/2}+m]\\ -\cos(\theta/2)p/[(\boldsymbol{p}^2+m^2)^{1/2}+m]\end{pmatrix}, \tag{4·5b}$$

$$e_{\mu}{}^{+}(\boldsymbol{p},m)=(e^{i\phi}/2^{1/2})(\cos\theta\cos\phi-i\sin\phi,\,\cos\theta\sin\phi+i\cos\phi,\,-\sin\theta,\,0), \tag{4·6a}$$

$$e_{\mu}{}^{-}(\boldsymbol{p},m)=(e^{-i\phi}/2^{1/2})(-\cos\theta\cos\phi-i\sin\phi,\,-\cos\theta\sin\phi+i\cos\phi,\,\sin\theta,\,0), \tag{4·6b}$$

$$e_{\mu}{}^{0}(\boldsymbol{p},m)=-[(\boldsymbol{p}^2+m^2)^{1/2}/m][\sin\theta\cos\phi,\,\sin\theta\sin\phi,\,\cos\theta,\,ip/(\boldsymbol{p}^2+m^2)^{1/2}], \tag{4·6c}$$

$$e_{\mu}(\boldsymbol{p},m)=[\boldsymbol{p},\,(\boldsymbol{p}^2+m^2)^{1/2}]. \tag{4·6d}$$

Now we immediately see that $U_{r,\mu}$ satisfies the equation of motion (2·1) but $U^{(\pm)}_{r,\mu}$ does not. We must modify $U^{(\pm)}_{r,\mu}$ so as to satisfy (2·1) using the ambiguity of wave functions $U_{r,\mu}$ under the point transformation (2·10). That is, the correct wave functions for spin 1/2 parts should have the form

$$U^{(\pm)\prime}_{r,\mu}(\boldsymbol{p},M)=N^{(\pm)}(\delta_{\mu\nu}+\alpha^{(\pm)}\gamma_{\mu}\gamma_{\nu})\,U^{(\pm)}_{r,\nu}. \tag{4·7}$$

Using the identities

$$3^{1/2}\gamma_5 U^{(+)}_{\pm 1/2,\mu}(\boldsymbol{p},M)+U^{(-)}_{\pm 1/2,\mu}(\boldsymbol{p},M)=i\gamma_{\mu}u^{\pm}(\boldsymbol{p},M), \tag{4·8a}$$

$$i\gamma_{\mu}U^{(-)}_{\pm 1/2,\mu}(\boldsymbol{p},M)=-u^{\pm}(\boldsymbol{p},M), \tag{4·8b}$$

$$i\gamma_{\mu}U^{(+)}_{\pm 1/2,\mu}(\boldsymbol{p},M)=3^{1/2}\gamma_5 u^{\pm}(\boldsymbol{p},M), \tag{4·8c}$$

from the equation of motion (2·1) and the normalization condition

$$|\bar{U}_{r,\mu}^{(\pm)\prime}(\boldsymbol{p}, M)\,(\Gamma_4)_{\mu\nu} U_{r,\nu}^{(\pm)\prime}(\boldsymbol{p}, M)| = 1\,, \tag{4·9}$$

$\alpha^{(\pm)}$ and $N^{(\pm)}$ are determined. The results are

$$\alpha^{(+)} = \frac{m-(1+3A)M}{6M(1+2A)}\,, \tag{4·10a}$$

$$\alpha^{(-)} = \frac{m-(1+A)M}{2M(1+2A)}\,, \tag{4·10b}$$

$$N^{(+)} = \left[\frac{M(M-2m)}{m^2}\right]^{1/2} \tag{4·11a}$$

and

$$N^{(-)} = \left[\frac{M(M+2m)}{3m^2}\right]^{1/2}. \tag{4·11b}$$

Instead of $U_{r,\mu}$ and $U_{r,\mu}^{(\pm)\prime}$, we define $u^{(\alpha)}(\boldsymbol{p})$ by

$$u_\mu{}^{(1)}(\boldsymbol{p}) = U_{3/2,\mu}(\boldsymbol{p}, m)\,, \tag{4·12a}$$

$$u_\mu{}^{(2)}(\boldsymbol{p}) = U_{1/2,\mu}(\boldsymbol{p}, m)\,, \tag{4·12b}$$

$$u_\mu{}^{(3)}(\boldsymbol{p}) = U_{-1/2,\mu}(\boldsymbol{p}, m)\,, \tag{4·12c}$$

$$u_\mu{}^{(4)}(\boldsymbol{p}) = U_{-3/2,\mu}(\boldsymbol{p}, m)\,, \tag{4·12d}$$

$$u_\mu{}^{(5)}(\boldsymbol{p}) = U_{1/2,\mu}^{(+)\prime}(\boldsymbol{p}, M)\,, \tag{4·12e}$$

$$u_\mu{}^{(6)}(\boldsymbol{p}) = U_{-1/2,\mu}^{(+)\prime}(\boldsymbol{p}, M)\,, \tag{4·12f}$$

$$u_\mu{}^{(7)}(\boldsymbol{p}) = U_{1/2,\mu}^{(-)\prime}(\boldsymbol{p}, M) \tag{4·12g}$$

and

$$u_\mu{}^{(8)}(\boldsymbol{p}) = U_{-1/2,\mu}^{(-)\prime}(\boldsymbol{p}, M)\,. \tag{4·12h}$$

The wave functions for antiparticles are written as

$$V_{r,\mu}(\boldsymbol{p}) = \gamma_5 U_{r,\mu}(\boldsymbol{p})\,, \tag{4·13a}$$

$$V_{r,\mu}^{(\pm)\prime}(\boldsymbol{p}) = \gamma_5 U_{r,\mu}^{(\pm)\prime}(\boldsymbol{p})\,, \tag{4·13b}$$

$$v_\mu{}^{(\pm)}(\boldsymbol{p}) = \gamma_5 u^{(\alpha)}(\boldsymbol{p})\,. \tag{4·13c}$$

These functions satisfy the orthogonality relations

$$\bar{u}_\mu{}^{(\alpha)}(\boldsymbol{p})\,(\Gamma_4)_{\mu\nu} u_\nu{}^{(\beta)}(\boldsymbol{p}) = \varepsilon_\alpha \delta_{\alpha\beta}\,, \tag{4·14a}$$

$$\bar{v}_\mu{}^{(\alpha)}(-\boldsymbol{p})\,(\Gamma_4)_{\mu\nu} u_\nu{}^{(\beta)}(\boldsymbol{p}) = 0\,, \tag{4·14b}$$

$$\bar{u}_\mu{}^{(\alpha)}(\boldsymbol{p})\,(\Gamma_4)_{\mu\nu} v_\nu{}^{(\beta)}(-\boldsymbol{p}) = 0\,, \tag{4·14c}$$

$$\bar{v}_\mu{}^{(\alpha)}(-\boldsymbol{p})\,(\Gamma_4)_{\mu\nu} v_\nu{}^{(\beta)}(-\boldsymbol{p}) = \varepsilon_\alpha \delta_{\alpha\beta}\,, \tag{4·14d}$$

with

$$\varepsilon_\alpha = \begin{cases} +1 & \text{for } \alpha=1,2,3,4\,, \\ -1 & \text{for } \alpha=5,6,7,8\,, \end{cases} \tag{4·15}$$

and the completeness relations

$$\sum_r U_{r,\mu}(\boldsymbol{p})\bar{U}_{r,\nu}(\boldsymbol{p}) = \int d_{\mu\nu}(ip)\Delta^+(p,m)dp_0\,, \tag{4·16a}$$

$$\sum_r V_{r,\mu}(-\boldsymbol{p})\bar{V}_{r,\nu}(-\boldsymbol{p}) = \int d_{\mu\nu}(ip)\Delta^-(p,m)dp_0\,, \tag{4·16b}$$

$$\sum_r U_{r,\mu}^{(\pm)\prime}(\boldsymbol{p})\bar{U}_{r,\nu}^{(\pm)\prime}(\boldsymbol{p}) = -\int d_{\mu\nu}^{(\pm)}(ip)\Delta^+(p,M)dp_0\,, \tag{4·16c}$$

$$\sum_r V_{r,\mu}^{(\pm)\prime}(-\boldsymbol{p})\bar{V}_{r,\nu}^{(\pm)\prime}(-\boldsymbol{p}) = -\int d_{\mu\nu}^{(\pm)}(ip)\Delta^-(p,M)dp_0\,, \tag{4·16d}$$

where $d_{\mu\nu}$ and $d_{\mu\nu}^{(\pm)}$ are those given by (3·8) and $\Delta^+(p,m)$ and $\Delta^-(p,m)$ are positive and negative frequency parts of $\Delta(p,m)$. From (4·16) we can say that $d_{\mu\nu}$ and $d_{\mu\nu}^{(\pm)}$ derived from $\Lambda_{\mu\nu}^{-1}(ip)$ are projection operators of the spin 3/2 field and the redundant spin 1/2 fields. Summing the four relations of (4·16), we have

$$\sum_\alpha \varepsilon_\alpha[u_\mu^{(\alpha)}(\boldsymbol{p})\bar{u}_\nu^{(\alpha)}(\boldsymbol{p}) + v_\mu^{(\alpha)}(-\boldsymbol{p})\bar{v}_\nu^{(\alpha)}(-\boldsymbol{p})] = \int[d_{\mu\nu}(ip)\Delta(p,m) + \{d_{\mu\nu}^{(+)}(ip)+d_{\mu\nu}^{(-)}(ip)\}\Delta(p,M)]dp_0 = (\Gamma_4)_{\mu\nu}^{-1}. \tag{4·17}$$

Finally, it should be remarked that, because of the identities (4·8), we have the identity

$$(\delta_{\mu\nu}+\alpha\gamma_\mu\gamma_\nu)U_{r,\nu}^{(+)} = -(1/3)^{1/2}[\delta_{\mu\nu}-(1+3\alpha)\gamma_\mu\gamma_\nu]\gamma_5 U_{r,\nu}^{(-)}. \tag{4·18}$$

Then, we cannot say that which is which, even if we consider that the states described by $U_{r,\mu}^{(\pm)\prime}$ are the ones composed of the spin 1/2 state of the spinor and the spin 1, 0 states of the vector. We can only say that there are two kinds of spin 1/2 states.

## § 5. Fourier expansion and LSZ formulas

Using the orthonormality relations (4·14) and the completeness relation (4·17), we can expand $\psi_\mu(x)$ in terms of the annihilation operators $a^{(\alpha)}(\boldsymbol{p})$ $b^{(\alpha)}(\boldsymbol{p})$ and the creation operators $a^{(\alpha)\dagger}(\boldsymbol{p})$ $b^{(\alpha)\dagger}(\boldsymbol{p})$:

$$\psi_\mu(x) = V^{-1/2}\sum_{\boldsymbol{p}} e^{i\boldsymbol{p}\cdot\boldsymbol{x}}\sum_\alpha[a^{(\alpha)}(\boldsymbol{p})u_\mu^{(\alpha)}(\boldsymbol{p}) + b^{(\alpha)\dagger}(-\boldsymbol{p})v_\mu^{(\alpha)}(-\boldsymbol{p})]. \tag{5·1}$$

From (2·13), (4·14) and (5·1), we obtain

$$\{a^{(\alpha)\dagger}(\boldsymbol{p}), a^{(\beta)}(\boldsymbol{p}')\} = \{b^{(\alpha)\dagger}(-\boldsymbol{p}), b^{(\beta)}(-\boldsymbol{p}')\} = \varepsilon_\alpha\delta_{\alpha\beta}\delta_{\boldsymbol{p},\boldsymbol{p}'}, \tag{5·2a}$$

$$\{a^{(\alpha)\dagger}(\boldsymbol{p}), b^{(\beta)}(-\boldsymbol{p}')\} = \{b^{(\alpha)\dagger}(-\boldsymbol{p}), a^{(\beta)}(\boldsymbol{p}')\} = 0\,. \tag{5·2b}$$

Then, the redundant spin 1/2 particles are both ghosts. Substituting (5·1) into

(2·15) and integrating with respect to $\boldsymbol{x}$, we get total Hamiltonian in Fock representation

$$\int Hd^3x=\sum_{\boldsymbol{p}}\sum_{\alpha}(\boldsymbol{p}^2+m_\alpha{}^2)^{1/2}\varepsilon_\alpha[a^{(\alpha)\dagger}(\boldsymbol{p})a^{(\alpha)}(\boldsymbol{p})-b^{(\alpha)}(-\boldsymbol{p})b^{(\alpha)\dagger}(-\boldsymbol{p})], \tag{5·3}$$

where

$$m_\alpha=\begin{cases}m & \text{for } \alpha=1,2,3,4\,,\\ M & \text{for } \alpha=5,6,7,8\,.\end{cases} \tag{5·4}$$

By virtue of the covariance of the $T$ product and orthonormalities of wave functions discussed in the previous section, we have LSZ formulas:

$$a_{\rm out}^{(\alpha)}(\boldsymbol{p})T[0(x_1)\cdots]-(-)^nT[0(x_1)\cdots]a_{\rm in}^{(\alpha)}(\boldsymbol{p})$$
$$=-i\varepsilon_\alpha V^{-1/2}\int d^4xe^{-ipx}\overline{u}_\mu{}^{(\alpha)}(\boldsymbol{p})\Lambda_{\mu\nu}(\partial)T[\psi_\mu(x),0(x_1),\cdots], \tag{5·5a}$$

$$(-)^nT[0(x_1)\cdots]a_{\rm in}^{(\alpha)\dagger}(\boldsymbol{p})-a_{\rm out}^{(\alpha)\dagger}(\boldsymbol{p})T[0(x_1)\cdots]$$
$$=-i\varepsilon_\alpha V^{-1/2}\int d^4xT[\overline{\psi}_\mu(x),0(x_1),\cdots]\Lambda_{\mu\nu}(-\overleftarrow{\partial})u_\nu{}^{(\alpha)}(\boldsymbol{p})e^{ipx}, \tag{5·5b}$$

$$b_{\rm out}^{(\alpha)\dagger}(-\boldsymbol{p})T[0(x_1)\cdots]-(-)^nT[0(x_1)\cdots]b_{\rm in}^{(\alpha)\dagger}(-\boldsymbol{p})$$
$$=-i\varepsilon_\alpha V^{-1/2}\int d^4xe^{ipx}\overline{v}_\mu{}^{(\alpha)}(\boldsymbol{p})\Lambda_{\mu\nu}(\partial)T[\psi_\nu(x),0(x_1)\cdots], \tag{5·5c}$$

$$(-)^nT[0(x_1)\cdots]b_{\rm in}^{(\alpha)}(-\boldsymbol{p})-b_{\rm out}^{(\alpha)}(-\boldsymbol{p})T[0(x_1)\cdots]$$
$$=-i\varepsilon_\alpha V^{-1/2}\int d^4xT[\overline{\psi}_\mu(x),0(x_1)\cdots]\Lambda_{\mu\nu}(-\overleftarrow{\partial})v_\nu{}^{(\alpha)}(\boldsymbol{p})e^{-ipx}, \tag{5·5d}$$

where $n$ is the number of $\psi_\mu$ and $\overline{\psi}_\mu$ in $0(x_1)\cdots$ and $p_\mu$ is the energy momentum vector with $p_0=(\boldsymbol{p}^2+m^2)^{1/2}$, $0(x_1)\cdots$ are local operators and $d^4x=dx_1dx_2dx_3dx_0$. Of course, we are interested only in (5·5) with $\alpha=1\sim4$. We have similar formulas for Retarded products.

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**Note added in proof:**

After completing this work Professor Y. Takahashi kindly pointed out us the existence of a closely related work by H. Munczek, Phys. Rev. **164** (1967), 1794.

Ours is different from his work about the treatments of two spin 1/2 particles whose masses are denoted by $M_1$ and $M_2$. In his work one of the spin 1/2 particles is quantized with positive metric and the other with negative metric, whereas, in ours both are quantized with negative metric. His propagator has double pole in the case of $M_1=M_2$. This is not the case in ours.

Let us discuss the reason why these differences occur. Introducing $a$ and $b$ by

$$a=(A+2B+C)/(1+2A+3A^2-2B),$$

$$b=[(A+2B+C)^2+(1+4C)(1+2A+3A^2-2B)]^{1/2}/(1+2A+3A^2-2B),$$

Eq. (3·4b) becomes $M^2=(a\pm b)^2$. Then we have following four possibilities of defining $M_1$ and $M_2$:

| | | |
|---|---|---|
| Case A | $M_1=-a+b\,,$ | $M_2=-a-b\,,$ |
| Case B | $M_1=-a+b\,,$ | $M_2=a+b\,,$ |
| Case C | $M_1=a-b\,,$ | $M_2=a+b\,,$ |
| Case D | $M_1=a-b\,,$ | $M_2=-a-b\,.$ |

The spin 1/2 part of $\Lambda_{\mu\nu}^{-1}$, that is, $\Lambda_{\mu\nu}^{-1}+d_{\mu\nu}/(p^2+m^2)$ is obtained by the uses of (3·3a)∼(3·3h). When $A=-1$, it has the form:

Case A

$$\frac{m\gamma_\mu+2ip_\mu}{6(M_1-M_2)\,m^2}\left[(-M_2+2m)\,\frac{i\gamma\cdot p-M_1}{p^2+M_1^2}+(M_1-2m)\,\frac{i\gamma\cdot p-M_2}{p^2+M_2^2}\right](m\gamma_\nu+2ip_\nu),$$

Case B

$$\frac{m\gamma_\mu+2ip_\mu}{6(M_1+M_2)\,m^2}\left[(M_2+2m)\,\frac{i\gamma\cdot p-M_1}{p^2+M_1^2}+(M_1-2m)\,\frac{i\gamma\cdot p+M_2}{p^2+M_2^2}\right](m\gamma_\nu+2ip_\nu).$$

The expressions for the cases C and D are given by replacing $M_i$ with $-M_i$ in those of the cases A and B respectively. Thus we find that there are two different cases. The case A is corresponding to Munczek's one and the case B is to ours. If $M_1=M_2$ in our case, the propagator becomes

$$(m\gamma_\mu+2ip_\mu)(i\gamma\cdot p-2m)(m\gamma_\nu+2ip_\nu)/[6m^2(p^2+M_1^2)].$$

$M_1$ appears only in the denominator $P^2+M_1^2$. This is exactly the same as what happens for vector meson.[3] The difference between two cases appears when we consider the diagram including closed loops.