

# General Theory of Relativity

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## 1 Special Relativity

For the space-time of physics we need four coordinates, the time  $t$  and three space coordinates  $x, y, z$ . We put

$$t = x^0, x = x^1, y = x^2, z = x^3,$$

so that the four coordinates may be written  $x^\mu$ , where the suffix  $\mu$  takes on the values 0, 1, 2, 3. The suffix is written in the upper position so that we may maintain a “balancing” of the suffixes in all the general equations of the theory. The precise meaning of “balancing” will become clear a little later.

Let us take a point close to the point that we originally considered and let its coordinates be  $x^\mu + dx^\mu$ . The four quantities  $dx^\mu$  which form the displacement may be considered as the components of a vector. The laws of special relativity allow us to make linear nonhomogeneous transformations of the coordinates, resulting in linear homogeneous transformations of the  $dx^\mu$ . These are such that, if we choose units of distance and of time such that the velocity of light is unity,

$$(dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \quad (1.1)$$

is invariant.

Any set of quantities  $A^\mu$  that transform under a change of coordinates in the same way as the  $dx^\mu$  form what is called a *contravariant vector*. The invariant quantity

$$(A^0)^2 - (A^1)^2 - (A^2)^2 - (A^3)^2 = (A, A) \quad (1.2)$$

may be called the squared length of the vector. With a second contravariant vector we have the scalar product invariant

$$A^0 B^0 - A^1 B^1 - A^2 B^2 - A^3 B^3 = (A, B) \quad (1.3)$$

In order to get a convenient way of writing such invariants we introduce the device of lowering suffixes. Define

$$A_0 = A^0, A_1 = -A^1, A_2 = -A^2, A_3 = -A^3. \quad (1.4)$$

Then the expression on the left side of (1.2) may be written  $A_\mu A^\mu$ , in which it is understood that a summation is to be taken over the four values of  $\mu$ . With the same notation we can write (1.3) as  $A_\mu B^\mu$  or else  $A^\mu B_\mu$ .

The four quantities  $A_\mu$  introduced by equation (1.4) may also be considered as the components of a vector. Their transformation laws under a change of coordinates are somewhat different from those of  $A^\mu$ , because of the differences in sign, and the vector is called a *covariant vector*.<sup>(1)</sup>

From two contravariant vectors  $A^\mu$  and  $B^\mu$  we may form the sixteen quantities  $A^\mu B^\nu$ . The suffix  $\nu$ , like all the Greek suffixes appearing in this work, also takes on the four values 0, 1, 2, 3. Those sixteen quantities form the components of a tensor of the second rank. It is sometimes called the outer product of the vectors  $A^\mu$  and  $B^\nu$ , as distinct of the scalar product (1.3) which is called the inner product.

The tensor  $A^\mu B^\nu$  is rather a special tensor because there are special relations between its components. But we can add together several tensors constructed in this way to get a general tensor of the second rank; say

$$T^{\mu\nu} = A^\mu B^\nu + A'^\mu B'^\nu + A''^\mu B''^\nu + \dots \quad (1.5)$$

The important thing about the general tensor is that under a transformation of coordinates its components transform in the same way as the components of  $A^\mu B^\nu$ .

We may lower one of the suffixes in  $T^{\mu\nu}$  by applying the lowering process to each of the terms on the right hand side of (1.5). Thus we may form  $T_\mu{}^\nu$  or  $T^\mu{}_\nu$ .

In  $T_\mu{}^\nu$  we may set  $\nu = \mu$  and get  $T_\mu{}^\mu$ . This is to be summed over the four values of  $\mu$ . A summation is always implied over a suffix that occurs twice in a term. Thus  $T_\mu{}^\mu$  is scalar. It is equal to  $T^\mu{}_\mu$ .

We may continue this process and multiply more than two vectors together, taking care that their suffixes are all different. In this way we can construct tensors of higher rank. If the vectors are all contravariant, we get a vector with all its suffixes upstairs. We may then lower any of the suffixes and so get a general tensor with any number of suffixes upstairs and any number downstairs.

We may set a downstairs suffix equal to an upstairs one. We then have to sum over all values of this suffix. The suffix becomes a dummy. We are left with a tensor having two fewer effective suffixes than the original one. This process is called *contraction*. Thus, if we start with the fourth rank tensor  $T^\mu{}_{\nu\rho}{}^\sigma$ , one way of contracting it is to put  $\sigma = \rho$ , which gives the second rank tensor  $T^\mu{}_{\nu\rho}{}^\rho$ ,

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<sup>1</sup>In fact, if we set  $A_\mu = g_{\mu\nu} A^\nu$  where

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

we can see that the coefficients of the transformation  $A'_\mu = \Lambda_\mu{}^\nu A_\nu$  are linearly related to the coefficients of the transformation  $A'^\mu = \Lambda^\mu{}_\nu A^\nu$ .

having only sixteen components, arising from the four values of  $\mu$  and  $\nu$ . We could contract again to get the scalar  $T^\mu_{\mu\rho}{}^\rho$ , with just one component.

At this stage one can appreciate the balancing of suffixes. Any effective suffix occurring in an equation appears once and only once in each term of the equation, and always upstairs or only downstairs. A suffix appearing twice in a term is a dummy, and it must occur once upstairs and once downstairs. It may be replaced by any other Greek letter not already mentioned in the term. Thus  $T^\mu_{\nu\rho}{}^\rho = T^\mu_{\nu\alpha}{}^\alpha$ . A suffix must never occur more than twice in a term.

## 2 Oblique axes

Before passing to the formalism of general relativity it is convenient to consider an intermediate formalism—special relativity referred to oblique rectilinear axes.

If we make a transformation to oblique axes, each of the  $dx^\mu$  mentioned in (1.1) becomes a linear function the new  $dx^\mu$  and the quadratic form (1.1) becomes a general quadratic form in the new  $dx^\mu$ . We may write it

$$g_{\mu\nu}dx^\mu dx^\nu \quad (2.1)$$

with summation understood over both  $\mu$  and  $\nu$ . The coefficients  $g_{\mu\nu}$  appearing here depend on the system of oblique axes. Of course we take  $g_{\mu\nu} = g_{\nu\mu}$  because any difference of  $g_{\mu\nu}$  and  $g_{\nu\mu}$  would not show up in the quadratic form (2.1). There are thus ten independent coefficients  $g_{\mu\nu}$ .<sup>2</sup>

A general contravariant vector has four components  $A^\mu$  which transform like the  $dx^\mu$  under any transformation of the oblique axes. Thus

$$g_{\mu\nu}A^\mu A^\nu$$

is invariant. It is the squared length of the vector  $A^\mu$ .

Let  $B^\mu$  be a second contravariant vector; then  $A^\mu + \lambda B^\mu$  is still another, for any value of the number  $\lambda$ . Its squared length is

$$g_{\mu\nu}(A^\mu + \lambda B^\mu)(A^\nu + \lambda B^\nu) = g_{\mu\nu}A^\mu A^\nu + \lambda(g_{\mu\nu}A^\mu B^\nu + g_{\mu\nu}A^\nu B^\mu) + \lambda^2 g_{\mu\nu}B^\mu B^\nu.$$

This must be an invariant for all values of  $\lambda$ . It follows that the term independent of  $\lambda$  and the coefficients of  $\lambda$  and  $\lambda^2$  must separately be invariants. The coefficient of  $\lambda$  is

$$g_{\mu\nu}A^\mu B^\nu + g_{\mu\nu}A^\nu B^\mu = 2g_{\mu\nu}A^\mu B^\nu$$

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<sup>2</sup>In general the number of independent components in a symmetric matrix of  $n \times n$  dimensions is equal to the number of unordered pairs of  $n$  elements. From the total number of ordered pairs we subtract  $n$  pairs with identical elements, and we are left with  $n^2 - n$ . We divide the last number by two to obtain the number of unordered pairs with distinct elements. The total number of unordered pairs is then

$$T_n = \frac{1}{2}n \cdot (n + 1)$$

which is the  $n$ th triangular number. Then we notice that  $T_{n-1}$  is the number of independent components of an antisymmetric matrix of  $n \times n$  dimensions. In the case of four dimensions this number is  $T_3 = 6$ .

since in the second term in the left we can interchange  $\mu$  and  $\nu$  and then set  $g_{\mu\nu} = g_{\nu\mu}$ . Thus we find that  $g_{\mu\nu}A^\mu B^\nu$  is an invariant. It is the scalar product of  $A^\mu$  and  $B^\mu$ .

Let  $g$  be the determinant of  $g_{\mu\nu}$ . It must not vanish; otherwise the four axes would not provide independent directions in space-time and would no be suitable as axes. For the orthogonal axes of the preceding section the diagonal elements of  $g_{\mu\nu}$  are 1,  $-1$ ,  $-1$ ,  $-1$  and the nondiagonal elements are zero. Thus  $g = -1$ . With oblique axes  $g$  must still be negative, because the oblique axes can be obtained from the orthogonal ones by a continuous process, resulting in  $g$  varying continuously, and  $g$  cannot pass through the value zero.

Define the covariant vector  $A_\mu$ , with a downstairs suffix, by

$$A_\mu = g_{\mu\nu}A^\nu. \quad (2.2)$$

Since the determinant  $g$  does not vanish, these equations can be solved for  $A^\nu$  in terms of the  $A_\mu$ . Let the result be

$$A^\nu = g^{\mu\nu}A_\mu. \quad (2.3)$$

Each  $g^{\mu\nu}$  equals the cofactor of the corresponding  $g_{\mu\nu}$ , divided by the determinant itself. It follows that  $g^{\mu\nu} = g^{\nu\mu}$ .

Let us substitute for the  $A^\nu$  in (2.2) their values given by (2.3). We must replace the dummy  $\mu$  in (2.3) by some other Greek letter, say  $\rho$ , in order not to have three  $\mu$ 's in the same term. We get

$$A_\mu = g_{\mu\nu}g^{\nu\rho}A_\rho.$$

Since this equation must hold for any four quantities  $A_\mu$ , we can infer

$$g_{\mu\nu}g^{\nu\rho} = g_\mu^\rho, \quad (2.4)$$

where

$$\begin{aligned} g_\mu^\rho &= 1 && \text{for } \mu = \rho, \\ &= 0 && \text{for } \mu \neq \rho. \end{aligned} \quad (2.5)$$

The formula (2.2) may be used to lower any upper suffix occurring in a tensor. Similarly, (2.3) can be used to raise any downstairs suffix. If a suffix is lowered and raised again, the result is the same as the original tensor, on account of (2.4) and 2.5. Note that  $g_\mu^\nu$  just produces a substitution of  $\rho$  for  $\mu$  or of  $\mu$  for  $\rho$ ,

$$g_\mu^\rho A^\mu = A^\rho,$$

and of  $\mu$  for  $\rho$ ,

$$g_\mu^\rho A_\rho = A_\mu.$$

if we apply the rule to raising a suffix to the  $\mu$  in  $g_{\mu\rho}$  we get

$$g^\alpha{}_\nu = g^{\alpha\mu}g_{\mu\nu}.$$

This agrees with (2.4) if we take into account that in  $g^\alpha{}_\nu$  we may write the suffixes one above the other because of the symmetry of  $g_{\mu\nu}$ . Further we may raise the suffix  $\nu$  by the same rule and get

$$g^{\alpha\beta} = g^{\nu\beta} g^\alpha{}_\nu,$$

a result which follows immediately from (2.5). The rules for raising and lowering suffixes apply to all the suffixes in  $g_{\mu\nu}$ ,  $g^\mu{}_\nu$ ,  $g^{\mu\nu}$ .

### 3 Curvilinear coordinates

We now pass on to a system of curvilinear coordinates. We shall deal with quantities which are located at a point in space. Such a quantity may have various components, which are then referred to the axes at that point. There may be a quantity of the same nature at all points in space. It then becomes a field quantity.

If we take such quantity  $Q$  (or one of its components if it has several), we can differentiate it with respect to any of the four coordinates. We write the result

$$\frac{\partial Q}{\partial x^\mu} = Q_{,\mu}$$

A downstairs suffix preceded by a comma will always denote a derivative in this way. We put the suffix  $\mu$  downstairs in order to balance the upstairs  $\mu$  in the denominator on the left. We can see that the suffixes balance by noting that the change in  $Q$ , when we pass from a point  $x^\mu$  to the neighboring point  $x^\mu + \delta x^\mu$  is

$$\delta Q = Q_{,\mu} \delta x^\mu. \quad (3.1)$$

We shall have vectors and tensors located at a point, with various components referring to the axes at that point. When we change our system of coordinates, the components will change according to the same laws as in the preceding section, depending on the change of axes at the point concerned. We shall have a  $g_{\mu\nu}$  and a  $g^{\mu\nu}$  to lower and raise suffixes, as before. But *they are no longer constants*. They vary from point to point. They are field quantities.

Let us see the effect of a particular change in the coordinate system. Take new curvilinear coordinates  $x'^\mu$ , each a function of the four  $x^\mu$ 's. They may be written more conveniently  $x'^\mu$ , with the prime attached to the suffix rather than the main symbol.

Making a small variation in the  $x^\mu$ , we get the four quantities  $\delta x^\mu$  forming a contravariant vector. Referred to the new axes, this vector has the components

$$\delta x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} \delta x^\nu = x'^\mu{}_{,\nu} \delta x^\nu$$

with the notation of (3.1). This gives the law for the transformation of any contravariant vector  $A^\nu$ ; namely,

$$A'^\mu = x'^\mu{}_{,\nu} A^\nu \quad (3.2)$$

Interchanging the two systems of axes and changing the suffixes, we get

$$A^\lambda = x_{,\mu'}^\lambda A^{\mu'} \quad (3.3)$$

We know from the laws of partial differentiation that

$$\frac{\partial x^\lambda}{\partial x^{\mu'}} \frac{\partial x^{\mu'}}{\partial x^\nu} = g_\nu^\lambda. \quad (3.4)$$

To see how a covariant vector  $B_\mu$  transforms, we use the condition that  $A^\mu B_\mu$  is invariant. Thus with the help of (3.3)

$$A^{\mu'} B_{\mu'} = A^\lambda B_\lambda = x_{,\mu'}^\lambda A^{\mu'} B_\lambda$$

This result must hold for all values of the four  $A^{\mu'}$ ; therefore we can equate the coefficients of  $A^{\mu'}$  and get

$$B_{\mu'} = x_{,\mu'}^\lambda B_\lambda \quad (3.5)$$

We can now use the formulas (3.2) and (3.5) to transform any tensor with any upstairs and downstairs suffixes. We just have to use coefficients like  $x_{,\nu'}^{\mu'}$  for each upstairs suffix and  $x_{,\nu'}^\lambda$  for each downstairs suffix and make all the suffixes balance. For example

$$T^{\alpha'\beta'}_{\gamma'} = x_{,\lambda}^{\alpha'} x_{,\mu}^{\beta'} x_{,\gamma'}^\nu T^{\lambda\mu}_\nu. \quad (3.6)$$

Any quantity that transforms according to this law is a tensor. This may be taken as the definition of a tensor.

It should be noted that it has a meaning for a tensor to be symmetrical or antisymmetrical between two suffixes like  $\lambda$  and  $\mu$ , because this property of symmetry is preserved with the change of coordinates.<sup>(3)</sup>

The formula (3.4) may be written

$$x_{,\alpha'}^\lambda x_{,\nu}^{\beta'} g_{\beta'}^{\alpha'} = g_\nu^\lambda$$

It just shows that  $g_\nu^\lambda$  is a tensor. We have also, for any vectors  $A^\mu$ ,  $B^\nu$ ,

$$g_{\alpha'\beta'} A^{\alpha'} B^{\beta'} = g_{\mu\nu} A^\mu B^\nu = g_{\mu\nu} x_{,\alpha'}^\mu x_{,\beta'}^\nu A^{\alpha'} B^{\beta'}.$$

Since this holds for all values of  $A^{\alpha'}$ ,  $B^{\beta'}$ , we can infer

$$g_{\alpha'\beta'} = x_{,\alpha'}^\mu x_{,\beta'}^\nu g_{\mu\nu}. \quad (3.7)$$

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<sup>3</sup>In fact if  $T^{\mu\nu}$  is tensor and  $(T^{\mu\nu} = \pm T^{\nu\mu})$ . By the transformation law

$$T^{\mu'\nu'} = x_{,\alpha'}^{\mu'} x_{,\beta}^{\nu'} T^{\alpha\beta}.$$

Using the property of symmetry

$$T^{\mu'\nu'} = \pm x_{,\alpha'}^{\mu'} x_{,\beta}^{\nu'} T^{\beta\alpha} = \pm x_{,\beta}^{\nu'} x_{,\alpha'}^{\mu'} T^{\beta\alpha}$$

and reordering the dummy suffixes

$$T^{\mu'\nu'} = \pm x_{,\alpha}^{\nu'} x_{,\beta}^{\mu'} T^{\alpha\beta} = \pm T^{\nu'\mu'}.$$

This shows that  $g_{\mu\nu}$  is a tensor. Similarly,  $g^{\mu\nu}$  is a tensor. They are called the *fundamental tensors*.

If  $S$  is any scalar field quantity, it can be considered as a function of the four  $x^\mu$  or the four  $x^{\mu'}$ . From the laws of partial differentiation

$$S_{,\mu'} = S_{,\lambda} x^\lambda_{,\mu'}.$$

Hence the  $S_{,\lambda}$  transform like the  $B_\lambda$  of equations (3.5) and thus *the derivative of a scalar field is a covariant vector field.*<sup>(4)</sup>

## 4 Nontensors

We can have a quantity  $N^\mu_{\nu\rho\dots}$  with various up and down suffixes, which is not a tensor. If it is a tensor, it must transform under a change of coordinate system according to the law exemplified by (3.6). With any other law it is a nontensor. A tensor has the property that if all the components vanish in a system of coordinates, they vanish in every system of coordinates. This may not hold for a nontensor.

For a nontensor we can raise and lower suffixes by the same rules as for a tensor. Thus, for example,

$$g^{\alpha\nu} N^\mu_{\nu\rho} = N^{\mu\alpha}_{\rho}.$$

The consistency of these rules is quite independent of the transformation laws to a different system of coordinates. Similarly, we can contract a nontensor by putting an upper and lower suffix equal.

We may have tensors and nontensors appearing together in the same equation. The rules for balancing suffixes apply equally to tensors and nontensors.

### 4.1 The Quotient Theorem

Suppose that  $P_{\lambda\mu\nu}$  is such that  $A^\lambda P_{\lambda\mu\nu}$  is a tensor *for any vector*  $A^\lambda$ . Then  $P_{\lambda\mu\nu}$  is a tensor.

To prove it, write  $A^\lambda P_{\lambda\mu\nu} = Q_{\mu\nu}$ . We are given that this is a tensor; therefore

$$Q_{\beta\gamma} = Q_{\mu'\nu'} x^{\mu'}_{,\beta} x^{\nu'}_{,\gamma}.$$

Thus

$$A^\alpha P_{\alpha\beta\gamma} = A^{\lambda'} P_{\lambda'\mu'\nu'} x^{\mu'}_{,\beta} x^{\nu'}_{,\gamma}.$$

Since  $A^\lambda$  is a vector, we have from (3.2),

$$A^{\lambda'} = A^\alpha x^\lambda_{,\alpha'}.$$

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<sup>4</sup>As a matter of fact, a covariant (cogradient) vector can be defined as an array of four quantities  $A_\mu$  that transform like the components of the gradient of a scalar. A contravariant (contragradient) vector can be defined as an array of four quantities that transform like the differentials  $dx^\mu$ .

So

$$A^\alpha P_{\alpha\beta\gamma} = A^\alpha x_{,\alpha}^{\lambda'} P_{\lambda'\mu'\nu'} x_{,\beta}^{\mu'} x_{,\gamma}^{\nu'}.$$

This equation must hold for all values of  $A^\alpha$ , so

$$P_{\alpha\beta\gamma} = P_{\lambda'\mu'\nu'} x_{,\alpha}^{\lambda'} x_{,\beta}^{\mu'} x_{,\gamma}^{\nu'}.$$

showing that  $P_{\alpha\beta\gamma}$  is a tensor.

The theorem also holds if  $P_{\alpha\beta\gamma}$  is replaced by a quantity with any number of suffixes, and if some of the suffixes are upstairs.

## 5 Curved Space

One can easily imagine a curved two-dimensional space as a surface immersed in Euclidean three-dimensional space. In the same way, one can have a curved four-dimensional space immersed in a flat space of a larger number of dimensions. Such a curved space is called a Riemann space. A small region of it is approximately flat.

Einstein assumed that physical space is of this nature and thereby laid the foundation for his theory of gravitation.

For dealing with curved space one cannot introduce a rectilinear system of axes. One has to use curvilinear coordinates, such as those dealt with in Section 3. The whole formalism of that section can be applied to curved space, because all the equations are local ones which are not disturbed by the curvature.

The invariant distance  $ds$  between a point  $x^\mu$  and a neighboring point  $x^\mu + dx^\mu$  is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

like (2.1).  $ds$  is real for a timelike interval and imaginary for a spacelike interval.

With a network of curvilinear coordinates the  $g_{\mu\nu}$ , given as functions of the coordinates, fix all the elements of distance; so they fix the metric. They determine both the coordinate system and the curvature of the space.

## 6 Parallel Displacement

Suppose we have a vector  $A^\mu$  located at a point  $P$ . **If the space is curved, we cannot give a meaning to a parallel vector at a different point  $Q$ ,** as one can easily see if one thinks of the example of a curved two-dimensional space in a three-dimensional Euclidean space. However, if we take a point  $P'$  close to  $P$ , there is a parallel vector at  $P'$ , with an uncertainty of the second order, counting the distance from  $P$  to  $P'$  as the first order. Thus we can give a meaning to displacing the vector  $A^\mu$  from  $P$  to  $P'$  keeping it parallel to itself and keeping the length constant.

We can transfer the vector continuously along a path by this process of parallel displacement. Taking a path from  $P$  to  $Q$ , we end up with a vector at  $Q$  which is parallel to the original vector at  $P$ , **with respect to this path.**



But a different path will give a different result. There is no absolute meaning to a parallel vector at  $Q$ . If we transport the vector at  $P$  by parallel displacement around a closed loop, we shall end up with a vector at  $P$  which is usually in a different direction.

We can get equations for the parallel displacement of a vector by supposing our four-dimensional physical space immersed in a flat space of a higher number of dimensions; say  $N$ . In this  $N$ -dimensional space we introduce rectilinear coordinates  $z^n$  ( $n = 1, \dots, N$ ). Those coordinates do not need to be orthogonal, only rectilinear. Between two neighboring points there is an invariant distance  $ds$  given by

$$ds^2 = h_{mn} dz^m dz^n, \quad (6.1)$$

summed for  $n, m = 1, 2, \dots, N$ . The  $h_{nm}$  are constants, unlike the  $g_{\mu\nu}$ . We may use them to lower suffixes in the  $N$ -dimensional space; thus

$$dz_n = h_{mn} dz^m.$$

Physical space forms a four-dimensional “surface” in the flat  $N$ -dimensional space. Each point  $x^\mu$  in the surface determines a definite point  $y^n$  in the  $N$ -dimensional space. Each coordinate  $y^n$  is a function of the four  $x$ ’s; say  $y^n(x)$ . The equations of the surface would be given by eliminating the four  $x$ ’s from the  $N$   $y^n(x)$ ’s. There are  $N - 4$  such equations.

By differentiating the  $y^n(x)$  with respect to the parameters  $x^\mu$ , we get

$$\frac{\partial y^n(x)}{\partial x^\mu} = y_{,\mu}^n.$$

For two neighboring points in the surface differing by  $\delta x^\mu$ , we have

$$\delta y^n = y_{,\mu}^n \delta x^\mu \quad (6.2)$$

The squared distance between them is, from (6.1)

$$\delta s^2 = h_{mn} y_{,\mu}^m y_{,\nu}^n \delta x^\mu \delta x^\nu.$$

We may write it

$$\delta s^2 = y_{,\mu}^n y_{n,\nu} \delta x^\mu \delta x^\nu.$$

Hence

$$g_{\mu\nu} = y_{,\mu}^n y_{n,\nu}. \quad (6.3)$$

Take a contravariant vector  $A^\mu$  in physical space, located at the point  $x$ . Its components  $A^\mu$  are like the  $\delta x^\mu$  of (6.2). Thus

$$A^n = y_{,\mu}^n A^\mu. \quad (6.4)$$

Now, shift the vector  $A^n$ , keeping it parallel to itself (which means, of course, keeping its components constant), to a neighboring point  $x + dx$  in the surface. It will no longer lie in the surface at the new point, on account of the curvature

of the surface. But we can project it on to the surface, to get a definite vector lying on the surface.

The projection process consists in splitting the vector into two parts, a tangential part and a normal part, and discarding the normal part. Thus

$$A^n = A_{\text{tan}}^n + A_{\text{nor}}^n. \quad (6.5)$$

Now, if  $K^\mu$  denotes the components of  $A_{\text{tan}}^n$  referred to the  $x$  coordinate system in the surface, we have, corresponding to (6.4),

$$A_{\text{tan}}^n = K^\mu y_{,\mu}^n(x + dx), \quad (6.6)$$

with the coefficients  $y_{,\mu}^n$  taken at the new point  $x + dx$ .

$A_{\text{nor}}^n$  is defined to be orthogonal to every tangential vector at the point  $x + dx$ , and thus to every vector like the right-hand side of (6.6), no matter what the  $K^\mu$  are. Thus<sup>(5)</sup>

$$A_{\text{nor}}^n y_{n,\mu}(x + dx) = 0.$$

If we now multiply (6.5) by  $y_{n,\nu}(x + dx)$  the  $A_{\text{nor}}^n$  term drops out and we are left with

$$\begin{aligned} A^n y_{n,\nu}(x + dx) &= K^\mu y_{,\mu}^n(x + dx) y_{n,\nu}(x + dx) \\ &= K^\mu g_{\mu\nu}(x + dx) \end{aligned}$$

from (6.3). Thus, to the first order in  $dx$

$$\begin{aligned} K_\nu(x + dx) &= A^n [y_{n,\nu}(x) + y_{n,\nu,\sigma} dx^\sigma] \\ &= A^\mu y_{,\mu}^n [y_{n,\nu} + y_{n,\nu,\sigma} dx^\sigma] \\ &= A_\nu + A^\mu y_{,\mu}^n y_{n,\nu,\sigma} dx^\sigma \end{aligned}$$

This  $K_\nu$  is the result of parallel displacement of  $A_\nu$  to the point  $x + dx$ . We may put

$$K_\nu - A_\nu = dA_\nu,$$

so  $dA_\nu$  denotes the change in  $A_\nu$  under parallel displacement. Then we have

$$dA_\nu = A^\mu y_{,\mu}^n y_{n,\nu,\sigma} dx^\sigma \quad (6.7)$$

## 7 Christoffel Symbols

By differentiating (6.3) we get (**omitting the second comma with two differentiations**)

$$\begin{aligned} g_{\mu\nu,\sigma} &= y_{,\mu\sigma}^n y_{n,\nu} + y_{,\mu}^n y_{n,\nu\sigma} \\ &= y_{n,\mu\sigma} y_{,\nu}^n + y_{n,\nu\sigma} y_{,\mu}^n \end{aligned} \quad (7.1)$$

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<sup>5</sup>Here, Dirac assumes that the metric is positive (or negative) which is not the case for the space-time of relativity. Therefore, this mathematical argument supports, at most, an analogy between Einstein's space-time and a Riemann space.

since we can move the suffix  $n$  freely up and down, on account of the constancy of the  $h_{mn}$ . Interchanging  $\mu$  and  $\sigma$  in (7.1) we get

$$g_{\sigma\nu,\mu} = y_{n,\mu\sigma}y_{,\nu}^n + y_{n,\nu\mu}y_{,\sigma}^n \quad (7.2)$$

Interchanging  $\nu$  and  $\sigma$  in (7.1)

$$g_{\mu\sigma,\nu} = y_{n,\mu\nu}y_{,\sigma}^n + y_{n,\sigma\nu}y_{,\mu}^n \quad (7.3)$$

Now take (7.1)+(7.3)-(7.2) and divide by 2. The result is

$$\frac{1}{2}(g_{\mu\nu,\sigma} + g_{\mu\sigma,\nu} - g_{\nu\sigma,\mu}) = y_{n,\nu\sigma}y_{,\mu}^n. \quad (7.4)$$

Put

$$\Gamma_{\mu\nu\sigma} = \frac{1}{2}(g_{\mu\nu,\sigma} + g_{\mu\sigma,\nu} - g_{\nu\sigma,\mu}) \quad (7.5)$$

It is called a Christoffel symbol of the first kind. It is symmetrical with respect to the last two suffixes. It is a nontensor. A simple consequence of (7.5) is

$$\Gamma_{\mu\nu\sigma} + \Gamma_{\nu\mu\sigma} = g_{\mu\nu,\sigma}. \quad (7.6)$$

We see now that (6.7) can be written as

$$dA_\nu = A^\mu \Gamma_{\mu\nu\sigma} dx^\sigma. \quad (7.7)$$

All reference to the  $N$ -dimensional space has now disappeared, as the Christoffel symbol involves only the metric  $g_{\mu\nu}$  of physical space.

We can infer that the length of a vector is unchanged by parallel displacement. We have

$$\begin{aligned} d(g^{\mu\nu} A_\mu A_\nu) &= g^{\mu\nu} A_\nu dA_\mu + g^{\mu\nu} A_\mu dA_\nu + A_\mu A_\nu g^{\mu\nu}{}_{,\sigma} dx^\sigma \\ &= A^\mu dA_\mu + A^\nu dA_\nu + A_\mu A_\nu g^{\mu\nu}{}_{,\sigma} dx^\sigma \\ &= A^\mu A^\nu \Gamma_{\nu\mu\sigma} dx^\sigma + A^\nu A^\mu \Gamma_{\mu\nu\sigma} dx^\sigma + A_\mu A_\nu g^{\mu\nu}{}_{,\sigma} dx^\sigma \\ &= A^\mu A^\nu g_{\mu\nu,\sigma} dx^\sigma + A_\mu A_\nu g^{\mu\nu}{}_{,\sigma} dx^\sigma \end{aligned} \quad (7.8)$$

Now  $g^{\alpha\mu}{}_{,\sigma} g_{\mu\nu} + g^{\alpha\mu} g_{\mu\nu,\sigma} = (g^{\alpha\mu} g_{\mu\nu})_{,\sigma} = g_{\nu,\sigma}^\alpha = 0$ . Multiplying by  $g^{\beta\nu}$ , we get

$$g^{\alpha\beta}{}_{,\sigma} = -g^{\alpha\mu} g^{\beta\nu} g_{\mu\nu,\sigma}. \quad (7.9)$$

This is a useful formula giving the derivative of  $g^{\alpha\beta}$  in terms of the derivative of  $g_{\mu\nu}$ . It allows us to infer

$$A_\alpha A_\beta g^{\alpha\beta}{}_{,\sigma} = -A^\mu A^\nu g_{\mu\nu,\sigma}$$

and so the expression (7.8) vanishes. Thus the length of a vector is constant. In particular, a null vector (i.e. a vector of zero length) remains a null vector under parallel displacement.

The constancy of the length of the vector follows also from geometrical arguments. When we split up the vector  $A^\mu$  into tangential and normal parts

according to (6.5), the normal part is infinitesimal and is orthogonal to the tangential part. It follows that, to the first order, the length of the whole vector equals that of its tangential part.

The constancy of the length of any vector requires the constancy of the scalar product  $g^{\mu\nu}A_\mu B_\nu$  of any two vectors  $A$  and  $B$ . This can be inferred from the constancy of the length of  $A + \lambda B$  for any value of the parameter  $\lambda$ .

It is frequently useful to raise the first suffix of the Christoffel symbol so as to form

$$\Gamma_{\nu\sigma}^\mu = g^{\mu\lambda}\Gamma_{\lambda\mu\sigma}.$$

It is then called a Christoffel symbol of the second kind. It is symmetrical between its two lower suffixes. As explained in Section 4, this raising is quite permissible, even for a nontensor.

The formula (7.7) may be written

$$dA_\nu = \Gamma_{\nu\sigma}^\mu A_\mu dx^\sigma \quad (7.10)$$

It is the standard formula referring to covariant components. For a second vector  $B^\nu$  we have

$$\begin{aligned} d(A_\nu B^\nu) &= 0 \\ A_\nu dB^\nu &= -B^\nu dA_\nu = -B^\nu \Gamma_{\nu\sigma}^\mu A_\mu dx^\sigma \\ &= -B^\mu \Gamma_{\mu\sigma}^\nu A_\nu dx^\sigma \end{aligned}$$

This must hold for any  $A_\nu$ , so we get

$$dB^\nu = -\Gamma_{\mu\sigma}^\nu B^\mu dx^\sigma. \quad (7.11)$$

This is the standard formula for parallel displacement referring to contravariant components.

## 8 Geodesics

Take a point with coordinates  $z^\mu$  and suppose it moves along a track; we then have a function of some parameter  $\tau$ . Put  $dz^\mu/d\tau = u^\mu$ .

There is a vector  $u^\mu$  at each point of the track. Suppose that as we go along the track the vector  $u^\mu$  gets shifted by parallel displacement. Then the whole track is determined if we are given the initial point and the initial value of the vector  $u^\mu$ . We just have to shift the initial point from  $z^\mu$  to  $z^\mu + u^\mu d\tau$ , then shift the vector  $u^\mu$  to this new point by parallel displacement, then shift the point again in the direction fixed by the new  $u^\mu$ , and so on. Not only is the track determined, but also the parameter  $\tau$  along it. A track produced this way is called a geodesic.

If the vector  $u^\mu$  is a null vector, it always remains a null vector and the track is called a null geodesic. If the vector  $u^\mu$  is initially timelike (i.e.,  $u^\mu u_\mu > 0$ ), it is always timelike and we have a timelike geodesic. Similarly if  $u^\mu$  is initially spacelike ( $u^\mu u_\mu < 0$ ), it is always spacelike and we have a spacelike geodesic.

We get the equations of a geodesic by applying (7.11) with  $B^\nu = u^\nu$  and  $dx^\sigma = dz^\sigma$ . Thus

$$\frac{du^\nu}{d\tau} + \Gamma_{\mu\sigma}^\nu u^\mu \frac{dz^\sigma}{d\tau} = 0 \quad (8.1)$$

or

$$\frac{d^2 z}{d\tau^2} + \Gamma_{\mu\sigma}^\nu \frac{dz^\mu}{d\tau} \frac{dz^\sigma}{d\tau} = 0 \quad (8.2)$$

For a timelike geodesic we may multiply the initial  $u^\nu$  by a factor so as to make its length unity. This merely requires a change in the scale of  $\tau$ . The vector  $u^\mu$  now always has the length unity. It is just the velocity vector  $v^\mu = dz^\mu/ds$ , and the parameter  $\tau$  has become the proper time  $s$ .

Equation (8.1) then becomes

$$\frac{dv^\mu}{ds} + \Gamma_{\nu\sigma}^\mu u^\nu u^\sigma = 0. \quad (8.3)$$

Equation (8.2) becomes

$$\frac{d^2 z^\mu}{ds^2} + \Gamma_{\nu\sigma}^\mu \frac{dz^\nu}{ds} \frac{dz^\sigma}{ds} = 0. \quad (8.4)$$

We make the physical assumption that the world line of a particle not acted on by any forces, except gravitational, is a timelike geodesic. This replaces Newton's first law of motion. Equation (8.4) fixes the acceleration and provides the equations of motion.

We also make the assumption that the path of a ray of light is a null geodesic. It is fixed by equation (8.2) referring to some parameter along the path. The proper time  $s$  cannot now be used because  $ds$  vanishes.

## 9 The Stationary Property of Geodesics

A geodesic that is not a null geodesic has the property that  $\int ds$ , taken along a section of the track with the end points  $P$  and  $Q$ , is stationary if one makes a small variation of the track keeping the end points fixed.

Let us suppose that each point of the track, with the coordinates  $z^\mu$  is shifted so that its coordinates become  $z^\mu + \delta z^\mu$ . If  $dx^\mu$  denotes an element along the track,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

Thus

$$\begin{aligned} 2ds\delta ds &= dx^\mu dx^\nu \delta g_{\mu\nu} + g_{\mu\nu} dx^\mu \delta dx^\nu + g_{\mu\nu} dx^\nu \delta dx^\mu \\ &= dx^\mu dx^\nu \delta g_{\mu\nu} + 2g_{\mu\lambda} dx^\mu \delta dx^\lambda. \end{aligned}$$

Now

$$\delta dx^\lambda = d(\delta x^\lambda).$$

Thus, with the help of  $dx^\mu = v^\mu ds$ ,

$$\delta(ds) = \left( \frac{1}{2} g_{\mu\nu,\lambda} v^\mu v^\nu \delta x^\lambda + g_{\mu\lambda} v^\mu \frac{d\delta x^\lambda}{ds} \right) ds$$

Hence

$$\delta \int ds = \int \delta(ds) = \int \left[ \frac{1}{2} g_{\mu\nu, \lambda} v^\mu v^\nu \delta x^\lambda + g_{\mu\lambda} v^\mu \frac{d\delta x^\lambda}{ds} \right] ds$$

By partial integration (of the second term), using the condition that  $\delta x^\lambda = 0$  at the end points  $P$  and  $Q$ , we get

$$\delta \int ds = \int \left[ \frac{1}{2} g_{\mu\nu, \lambda} v^\mu v^\nu - \frac{d}{ds} (g_{\mu\lambda} v^\mu) \right] \delta x^\lambda ds \quad (9.1)$$

The condition for this to vanish with arbitrary  $\delta x^\lambda$  is

$$\frac{d}{ds} (g_{\mu\lambda} v^\mu) - \frac{1}{2} g_{\mu\nu, \lambda} = 0 \quad (9.2)$$

Now

$$\begin{aligned} \frac{d}{ds} (g_{\mu\lambda} v^\mu) &= g_{\mu\lambda} \frac{dv^\mu}{ds} + g_{\mu\lambda, \nu} v^\mu v^\nu \\ &= g_{\mu\lambda} \frac{dv^\mu}{ds} + \frac{1}{2} (g_{\lambda\mu, \nu} + g_{\lambda\nu, \mu}) v^\mu v^\nu. \end{aligned}$$

Thus the condition (9.2) becomes

$$g_{\mu\lambda} \frac{dv^\mu}{ds} + \Gamma_{\lambda\mu\nu} v^\mu v^\nu = 0.$$

Multiplying this by  $g^{\lambda\sigma}$ , it becomes

$$\frac{dv^\sigma}{ds} + \Gamma_{\mu\nu}^\sigma v^\mu v^\nu = 0,$$

which is just the condition (8.3) for the geodesic.

This work shows that for a geodesic, (9.1) vanishes and  $\int ds$  is stationary. Conversely, if we assume that  $\int ds$  is stationary, we can infer that the track is a geodesic. Thus we may use the stationary condition as the definition of a geodesic, except in the case of a null geodesic.

## 10 Covariant Differentiation

Let  $S$  be a scalar field. Its derivative  $S_{,y}$  is a covariant vector, as we saw in Section 3. Now let  $A_\mu$  be a vector field. Is its derivative  $A_{\mu, \nu}$  a tensor?

We must examine how  $A_{\nu, \nu}$  transforms under a change of coordinate system. With the notation in Section 3,  $A_\mu$  transforms to

$$A_{\mu'} = A_\rho x_{, \mu'}^\rho$$

like equation (3.5), and hence

$$\begin{aligned} A_{\mu', \nu'} &= \left( A_\rho x_{, \mu'}^\rho \right)_{, \nu'} \\ &= A_{\rho, \sigma} x_{, \nu'}^\sigma x_{, \mu'}^\rho + A_\rho x_{, \mu' \nu'}^\rho. \end{aligned}$$

The last term should not be here if we were to have the correct transformation law for a tensor. Thus  $A_{\mu,\nu}$  is not a tensor.

We can, however, modify the process of differentiation so as to get a tensor. Let us take the vector  $A_\mu$  at the point  $x$  and shift it to  $x + dx$  by parallel displacement. It is still a vector. We may subtract it from the vector  $A_\mu$  at  $x + dx$  and the difference will be a vector. It is, to the first order

$$A_\mu(x + dx) - [A_\mu(x) + \Gamma_{\mu\nu}^\alpha A_\alpha dx^\nu] = (A_{\mu,\nu} - \Gamma_{\mu\nu}^\alpha A_\alpha) dx^\nu.$$

This quantity is a vector, for any vector  $dx^\nu$ ; hence, by the quotient theorem of Section 4, the coefficient

$$A_{\mu\nu} - \Gamma_{\mu\nu}^\alpha A_\alpha$$

is a tensor. One can easily verify directly that it transforms correctly under a change of coordinate system.

It is called the covariant derivative of  $A_\mu$  and written

$$A_{\mu;\nu} = A_{\mu,\nu} - \Gamma_{\mu\nu}^\alpha A_\alpha. \quad (10.1)$$

The sign : before a lower suffix will always denote a covariant derivative, just as the comma denotes an ordinary derivative.

Let  $B_\nu$  be a second vector. We define the outer product  $A_\mu B_\nu$  to have the covariant derivative<sup>(6)</sup>

$$(A_\mu B_\nu)_\sigma = A_{\mu;\sigma} B_\nu + A_\mu B_{\nu;\sigma} \quad (10.2)$$

Evidently it is a tensor with three suffixes. It has the value

$$(A_\mu B_\nu)_\sigma = (A_{\mu,\sigma} - \Gamma_{\mu\sigma}^\alpha A_\alpha) B_\nu + A_\mu (B_{\nu,\sigma} - \Gamma_{\nu\sigma}^\alpha B_\alpha)$$

Let  $T_{\mu\nu}$  be a tensor with two suffixes. It is expressible as a sum of terms like  $A_\mu B_\nu$ , so, its covariant derivative is

$$T_{\mu\nu;\sigma} = T_{\mu\nu,\sigma} - \Gamma_{\mu\sigma}^\alpha T_{\alpha\nu} - \Gamma_{\nu\sigma}^\alpha T_{\mu\alpha}. \quad (10.3)$$

The rule can be extended to the covariant derivative of a tensor  $Y_{\mu\nu\dots}$  with any number of suffixes downstairs

$$Y_{\mu\nu\dots;\sigma} = Y_{\mu\nu\dots,\sigma} - \text{a } \Gamma \text{ term for each suffix.} \quad (10.4)$$

In each of these  $\Gamma$  terms we must make the suffixes balance, which is sufficient to fix how the suffixes go.

The case of a scalar is included in the general formula (10.4) with the number of suffixes in  $Y$  zero.

$$Y_{;\sigma} = Y_{,\sigma}. \quad (10.5)$$

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<sup>6</sup>Notice that the definition of the covariant derivative of a second rank tensor is not motivated because the notion of *parallel displacement of tensor* doesn't make sense.

Let's apply (10.3) to the fundamental tensor  $g_{\mu\nu}$ . It gives

$$\begin{aligned} g_{\mu\nu;\sigma} &= g_{\mu\nu,\sigma} - \Gamma_{\mu\sigma}^{\alpha} g_{\alpha,\nu} - \Gamma_{\nu\sigma}^{\alpha} g_{\mu\alpha} \\ &= g_{\mu\nu,\sigma} - \Gamma_{\nu\mu\sigma} - \Gamma_{\nu\mu\sigma} = 0 \end{aligned}$$

from (7.6). Thus the  $g_{\mu\nu}$ , count as constants under covariant differentiation.

Formula (10.2) is the usual rule that one uses for differentiating a product. We assume this usual rule holds also for the covariant derivative of the scalar product of two vectors. Thus

$$(A^{\mu}B_{\mu})_{;\sigma} = A^{\mu}_{;\sigma}B_{\mu} + A^{\mu}B_{\mu;\sigma}.$$

We get, according to (10.5) and (10.1),

$$(A^{\mu}B_{\mu})_{;\sigma} = A^{\mu}_{;\sigma}B_{\mu} + A^{\mu}(B_{\mu,\sigma} - \Gamma_{\mu\sigma}^{\alpha}B_{\alpha}).$$

and hence

$$A^{\mu}_{;\sigma}B_{\mu} = A^{\mu}_{;\sigma}B_{\mu} - A^{\alpha}\Gamma_{\alpha\sigma}^{\mu}B_{\mu}.$$

Since this holds for any  $B_{\mu}$ , we get

$$A^{\mu}_{;\sigma}B_{\mu} = A^{\mu}_{;\sigma}B_{\mu} + A^{\alpha}\Gamma_{\alpha\sigma}^{\mu}B_{\mu}, \quad (10.6)$$

which is the basic formula for the covariant derivative of a contravariant vector. The same Christoffel symbol occurs as in the basic formula (10.1) for a covariant vector, but now there is a + sign. The arrangement of suffixes is completely determined by the balancing requirement.

We can extend the formalism so as to include the covariant derivative of any tensor with any number of upstairs or downstairs suffixes. A  $\Gamma$  terms appears for each suffix, with a + sign if the suffix is upstairs and a - sign if it is downstairs. If we contract two suffixes in the tensor, the corresponding  $\Gamma$  terms cancel.

The formula for the covariant derivative of a product,

$$(XY)_{;\sigma} = X_{;\sigma}Y + XY_{;\sigma}, \quad (10.7)$$

holds quite generally, with  $X$  and  $Y$  any kind of tensor quantities. On account of the  $g_{\mu\nu}$  counting as constants, we can shift suffixes up or down before covariant differentiation and the result is the same as if we shifted them afterwards.

The covariant derivative of a nontensor has no meaning.

The laws of physics must be valid in all systems of coordinates. Thus must be expressible as tensor equations. Whenever they involve the derivative of a field quantity, it must be a covariant derivative. The field equations of physics must all be rewritten with the ordinary derivatives replaced by covariant derivatives. For example, the d'Alembert equation  $\square V = 0$  for a scalar  $V$  becomes, in covariant form

$$g^{\mu\nu}V_{;\mu;\nu} = 0.$$

This gives, from (10.1) and (10.5),

$$g^{\mu\nu}(V_{;\mu;\nu} - \Gamma_{\mu\nu}^{\alpha}V_{\alpha}) = 0. \quad (10.8)$$



Even if one is working with flat space (which means neglecting the gravitational field) and one is using curvilinear coordinates, one must write one's equations in terms of covariant derivatives if one wants them to hold in all systems of coordinates.

## 11 The Curvature Tensor

With the product law (10.7) we see that covariant differentiation is very similar to ordinary differentiation. But there is an important property of ordinary differentiation, that if we perform two differentiations in succession their order does not matter, which does not, in general, hold for covariant differentiation.

Let us first consider a scalar field  $S$ . We have from the formula (10.1),

$$\begin{aligned} S_{;\mu;\nu} &= S_{;\mu,\nu} - \Gamma_{\mu\nu}^{\alpha} S_{;\alpha} \\ &= S_{,\mu\nu} - \Gamma_{\mu\nu}^{\alpha} S_{,\alpha}. \end{aligned} \quad (11.1)$$

**This is symmetrical between  $\mu$  and  $\nu$ , so in this case the order of the covariant differentiation does not matter.**

Now let us take a vector  $A_{\nu}$  and apply two covariant differentiations to it. From the formula (10.3) with  $A_{\nu;\rho}$  for  $T_{\nu\rho}$  we get

$$\begin{aligned} A_{\nu;\rho;\sigma} &= A_{\nu;\rho,\sigma} - \Gamma_{\nu\sigma}^{\alpha} A_{\alpha;\rho} - \Gamma_{\rho\sigma}^{\alpha} A_{\nu;\alpha} \\ &= (A_{\nu,\rho} - \Gamma_{\nu\rho}^{\alpha} A_{\alpha})_{;\sigma} - \Gamma_{\nu\sigma}^{\alpha} (A_{\alpha,\rho} - \Gamma_{\alpha\rho}^{\beta} A_{\beta}) - \Gamma_{\rho\sigma}^{\alpha} (A_{\nu,\alpha} - \Gamma_{\nu\alpha}^{\beta} A_{\beta}) \\ &= A_{\nu,\rho\sigma} - \Gamma_{\nu\rho}^{\alpha} A_{\alpha,\sigma} - \Gamma_{\nu\sigma}^{\alpha} A_{\alpha,\rho} - \Gamma_{\rho\sigma}^{\alpha} A_{\nu,\alpha} \\ &\quad - A_{\beta} (\Gamma_{\nu\rho,\sigma}^{\beta} - \Gamma_{\nu\sigma}^{\alpha} \Gamma_{\alpha\rho}^{\beta} - \Gamma_{\rho\sigma}^{\alpha} \Gamma_{\nu\alpha}^{\beta}) \end{aligned}$$

Interchange  $\rho$  and  $\sigma$  here and subtract from the previous expression. The result is

$$A_{\nu;\rho;\sigma} - A_{\nu;\sigma;\rho} = A_{\beta} R_{\nu\rho\sigma}^{\beta} \quad (11.2)$$

where

$$R_{\nu\rho\sigma}^{\beta} = \Gamma_{\nu\sigma,\rho}^{\beta} - \Gamma_{\nu\rho,\sigma}^{\beta} + \Gamma_{\nu\sigma}^{\alpha} \Gamma_{\alpha\rho}^{\beta} - \Gamma_{\nu\rho}^{\alpha} \Gamma_{\alpha\sigma}^{\beta} \quad (11.3)$$

The left hand of (11.2) is a tensor. It follows that the right-hand side of (11.2) is a tensor. This holds for any vector  $A_{\beta}$ ; therefore, by the quotient theorem in Section 4,  $R_{\nu\rho\sigma}^{\beta}$  is a tensor. It is called the Riemann-Christoffel tensor or the curvature tensor.

It has the obvious property

$$R_{\nu\rho\sigma}^{\beta} = -R_{\nu\sigma\rho}^{\beta} \quad (11.4)$$

Also we easily see from (11.3) that

$$R_{\nu\rho\sigma}^{\beta} + R_{\rho\sigma\nu}^{\beta} + R_{\sigma\nu\rho}^{\beta} = 0 \quad (11.5)$$

Let us lower the suffix  $\beta$  and put it as the first suffix. We get

$$R_{\mu\nu\sigma\rho} = g_{\mu\beta} R_{\nu\sigma\rho}^{\beta} = g_{\mu\beta} \Gamma_{\nu\sigma,\rho}^{\beta} + \Gamma_{\nu\alpha}^{\alpha} \Gamma_{\mu\alpha\rho} - \langle \rho\sigma \rangle,$$

where the symbol  $\langle \rho\sigma \rangle$  is used to denote the preceding terms with  $\rho$  and  $\sigma$  interchanged. Thus

$$\begin{aligned} R_{\mu\nu\sigma\rho} &= R_{\mu\nu\sigma,\rho} - g_{\mu\beta,\rho}\Gamma_{\nu\sigma}^{\beta} + \Gamma_{\mu\beta\rho}\Gamma_{\nu\sigma}^{\beta} - \langle \rho\sigma \rangle \\ &= \Gamma_{\mu\nu\sigma,\rho} - \Gamma_{\beta\mu\rho}\Gamma_{\nu\sigma}^{\beta} - \langle \rho\sigma \rangle \end{aligned}$$

from (7.6). So from (7.5)

$$R_{\mu\nu\sigma\rho} = \frac{1}{2} (g_{\mu\sigma,\nu\rho} - g_{\nu\sigma,\mu\rho} - g_{\mu\rho,\nu\sigma} - g_{\nu\rho,\mu\sigma}) + \Gamma_{\beta\mu\sigma}\Gamma_{\nu\rho}^{\beta} - \Gamma_{\beta\mu\rho}\Gamma_{\nu\sigma}^{\beta} \quad (11.6)$$

Some further symmetries now show up; namely,

$$R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} \quad (11.7)$$

and

$$R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu} = R_{\sigma\rho\nu\mu} \quad (11.8)$$

The result of all these symmetries is that, of the 256 components of  $R_{\mu\nu\rho\sigma}$ , only 20 are independent.

## 12 The Condition for Flat Space

If space is flat, we may choose a system of coordinates that is rectilinear and then the  $g_{\mu\nu}$  are constant. The tensor  $R_{\mu\nu\rho\sigma}$  then vanishes.

Conversely, if  $R_{\mu\nu\rho\sigma}$  vanishes, one can prove that the space is flat. Let us take a vector  $A_{\mu}$  situated at the point  $x$  and shift it by parallel displacement to the point  $x+dx$ . Then shift it by parallel displacement to the point  $x+dx+\delta x$ . If  $R_{\mu\nu\rho\sigma}$  vanishes, the result must be the same as if we had shifted it first from  $x$  to  $x+\delta x$ , then to  $x+\delta x+dx$ . Thus we can shift the vector to a distant point and the result we get is independent of the path to the distant point. Therefore, if we shift the original vector  $A_{\mu}$  at  $x$  to all points by parallel displacement, we get a vector field that satisfies  $A_{\mu;\nu} = 0$ , or

$$A_{\mu,\nu} = \Gamma_{\mu\nu}^{\alpha} A_{\alpha}. \quad (12.1)$$

Can such a vector field be the gradient of a scalar? Let us put  $A_{\mu} = S_{\mu}$  in (12.1). We get

$$S_{\mu,\nu} = \Gamma_{\mu\nu}^{\alpha} S_{\alpha} \quad (12.2)$$

By virtue of the symmetry of  $\Gamma_{\mu\nu}^{\alpha}$  in the lower suffixes, we have the same value for  $S_{\mu,\nu}$  as  $S_{\nu,\mu}$  and the equations (12.2) are integrable.

Let us take four independent scalars satisfying (12.2) and let them to be the coordinates  $x^{\alpha'}$  of a new system of coordinates. Then

$$x_{,\mu\nu}^{\alpha'} = \Gamma_{\mu\nu}^{\sigma} x_{,\sigma}^{\alpha'}.$$

According the transformation law (3.7),

$$g_{\mu\lambda} = g_{\alpha'\beta'} x_{,\mu}^{\alpha'} x_{,\lambda}^{\beta'}.$$

Differentiating this equation with respect to  $x^\nu$ , we get

$$\begin{aligned}
g_{\mu\lambda,\nu} - g_{\alpha'\beta',\nu} x_{,\mu}^{\alpha'} x_{,\lambda}^{\beta'} &= g_{\alpha'\beta'} \left( x_{,\mu\nu}^{\alpha'} x_{,\lambda}^{\beta'} + x_{,\mu}^{\alpha'} x_{,\lambda\nu}^{\beta'} \right) \\
&= g_{\alpha'\beta'} \left( \Gamma_{\mu\nu}^{\sigma} x_{,\sigma}^{\alpha'} x_{,\lambda}^{\beta'} + x_{,\mu}^{\alpha'} \Gamma_{\lambda\nu}^{\sigma} x_{,\sigma}^{\beta'} \right) \\
&= g_{\sigma\lambda} \Gamma_{\mu\nu}^{\sigma} + g_{\mu\sigma} \Gamma_{\lambda\nu}^{\sigma} \\
&= \Gamma_{\lambda\mu\nu} + \Gamma_{\mu\lambda\nu} = g_{\mu\lambda,\nu}
\end{aligned}$$

from (7.6). Thus

$$g_{\alpha'\beta',\nu} x_{,\mu}^{\alpha'} x_{,\lambda}^{\beta'} = 0.$$

It follows that  $g_{\alpha'\beta',\nu} = 0$ . Referred to the new system of coordinates, the fundamental tensor is constant. Thus we have flat space referred to rectilinear coordinates.

### 13 The Bianci relations

To deal with the second covariant derivative of a tensor, take first the case in which the tensor is the outer product of two vectors  $A_\mu B_\tau$ . We have

$$\begin{aligned}
(A_\mu B_\tau)_{;\rho;\sigma} &= (A_{\mu;\rho} B_\tau + A_\mu B_{\tau;\rho})_{;\sigma} \\
&= A_{\mu;\rho;\sigma} B_\tau + A_{\mu;\rho} B_{\tau;\sigma} + A_{\mu;\sigma} B_{\tau;\rho} + A_\mu B_{\tau;\rho;\sigma}.
\end{aligned}$$

Now interchange  $\rho$  and  $\sigma$  and subtract. We get from (11.2)

$$(A_\mu B_\tau)_{;\rho;\sigma} - (A_\mu B_\tau)_{;\sigma;\rho} = A_\alpha R_{\mu\rho\alpha}^\alpha B_\tau + A_\mu R_{\tau\rho\sigma}^\alpha B_\alpha.$$

A general tensor  $T_{\mu\tau}$  is expressible as a sum of terms like  $A_\mu B_\tau$ , so it must satisfy

$$T_{\mu\tau;\rho;\sigma} - T_{\mu\tau;\sigma;\rho} = T_{\alpha\tau} R_{\mu\rho\sigma}^\alpha - T_{\mu\alpha} R_{\tau\rho\sigma}^\alpha. \quad (13.1)$$

Now take  $T_{\mu\tau}$  to be the covariant derivative of a vector  $A_{\mu;\tau}$ . We get

$$A_{\mu;\tau;\rho;\sigma} - A_{\mu;\tau;\sigma;\rho} = A_{\alpha;\tau} R_{\mu\rho\sigma}^\alpha + A_{\mu;\alpha} R_{\tau\rho\sigma}^\alpha.$$

In this formula make cyclic permutations of  $\tau, \rho, \sigma$  and add the three equations so obtained. The left hand side gives

$$\begin{aligned}
&A_{\mu;\tau;\rho;\sigma} - A_{\mu;\tau;\sigma;\rho} + \text{cyc perm} \\
&= (A_{\alpha;\tau} R_{\mu\rho\sigma}^\alpha)_{;\tau} + \text{cyc perm} \\
&= A_{\alpha;\tau} R_{\mu\rho\sigma}^\alpha + A_{\mu;\alpha} R_{\tau\rho\sigma}^\alpha + \text{cyc perm}.
\end{aligned} \quad (13.2)$$

The right-hand side gives

$$A_{\alpha;\tau} R_{\mu\rho\sigma}^\alpha + \text{cyc perm}, \quad (13.3)$$

as the remaining terms cancel from (11.5). The first term of (13.2) cancels with (13.3) and we are left with

$$A_{\alpha;\tau} R_{\mu\rho\sigma;\tau}^\alpha + \text{cyc perm} = 0.$$

The factor  $A_\alpha$  occurs throughout this equation and may be canceled out. We are left with

$$R_{\mu\rho\sigma;\tau}^\alpha + R_{\mu\sigma\tau;\rho}^\alpha + R_{\mu\tau\rho;\sigma}^\alpha = 0 \quad (13.4)$$

The curvature tensor satisfies these differential equations as well as other symmetry relations in Section 11. They are known as the Bianci relations.

## 14 The Ricci Tensor

Let us contract two of the suffixes in  $R_{\mu\nu\rho\sigma}$ . If we take two with respect to which it is antisymmetrical, we get zero, of course. If we take any other two we get the same result, apart from the sign because of the symmetries (11.4), (11.7), and (11.8). Let us take the first and last and put

$$R_{\nu\rho\alpha}^\alpha = R_{\nu\rho}.$$

It is called the Ricci tensor.

By multiplying (11.8) by  $g^{\mu\nu}$  we get

$$R_{\nu\rho} = R_{\rho\nu}. \quad (14.1)$$

The Ricci tensor is symmetrical.

We may contract again and form

$$g^{\nu\rho} R_{\nu\rho} = R_\nu^\nu = R,$$

say. This  $R$  is a scalar and is called the scalar curvature or total curvature. It is defined in such way that it is positive for the surface of a sphere in three dimensions, as one can check by a straightforward calculation.

The Bianci relation (13.4) involves five suffixes. Let us contract it twice and get a relation with one nondummy suffix. Put  $\tau = \alpha$  and multiply by  $g^{\mu\rho}$ . The result is

$$g^{\mu\rho} (R_{\mu\rho\sigma;\alpha}^\alpha + R_{\mu\sigma\alpha;\rho}^\alpha + R_{\mu\alpha\rho;\sigma}^\alpha) = 0$$

or

$$(g^{\mu\rho} R_{\mu\rho\sigma}^\alpha)_{;\alpha} + (g^{\mu\rho} R_{\mu\sigma\alpha}^\alpha)_{;\rho} + (g^{\mu\rho} R_{\mu\alpha\rho}^\alpha)_{;\sigma} = 0. \quad (14.2)$$

Now

$$g^{\mu\rho} R_{\mu\rho\sigma}^\alpha = g^{\mu\rho} g^{\alpha\beta} R_{\beta\mu\rho\sigma} = g^{\mu\rho} g^{\alpha\beta} R_{\mu\beta\sigma\rho} = g^{\alpha\beta} R_{\beta\sigma} = R_\sigma^\alpha.$$

One can write  $R_\sigma^\alpha$  with the suffixes one over the other on account of  $R_{\alpha\sigma}$  being symmetrical. Equation (14.2) now becomes

$$R_{\sigma;\alpha}^\alpha + (g^{\mu\rho} R_{\mu\sigma})_{;\rho} - R_{;\sigma} = 0$$

or

$$2R_{\sigma;\alpha}^\alpha - R_{;\sigma} = 0$$

which is the Bianci relation for the Ricci tensor. If we raise the suffix  $\sigma$ , we get

$$\left(R^{\sigma\alpha} - \frac{1}{2}g^{\sigma\alpha}R\right)_{;\alpha} = 0 \quad (14.3)$$

The explicit expression for the Ricci tensor is, from (11.3)

$$R_{\mu\nu} = \Gamma_{\mu\alpha,\nu}^{\alpha} - \Gamma_{\mu\nu,\alpha}^{\alpha} + \Gamma_{\mu\beta}^{\alpha}\Gamma_{\nu\alpha}^{\beta} - \Gamma_{\mu\nu}^{\alpha}\Gamma_{\alpha\beta}^{\beta} \quad (14.4)$$

The first term here does not appear to be symmetrical in  $\mu$  and  $\nu$ , although the other three terms evidently are. To establish that the first term really is symmetrical we need a little calculation.

To differentiate the determinant  $g$  we must differentiate each element  $g_{\lambda\mu}$  and then multiply for the cofactor  $gg^{\lambda\mu}$ . Thus

$$g_{,\nu} = gg^{\lambda\mu}g_{\lambda\mu,\nu}. \quad (14.5)$$

Hence

$$\begin{aligned} \Gamma_{\lambda\mu}^{\mu} &= g^{\lambda\mu}\Gamma_{\lambda\nu\mu} = \frac{1}{2}(g_{\lambda\nu,\mu} + g_{\lambda\mu,\nu} - g_{\mu\nu,\lambda}) \\ &= \frac{1}{2}g^{\lambda\mu}g_{\lambda\mu,\nu} = \frac{1}{2}g^{-1}g_{,\nu} = \frac{1}{2}(\log(g))_{,\nu}. \end{aligned} \quad (14.6)$$

This makes it evident that the first term of (14.4) is symmetrical.