ICMU Mini-course on Gaussian Processes
Homework 1: The Multivariate Normal Distribution

Please upload your completed homework using this link by the end of the day on Saturday, February 7. The document name should include your last name. Scans or clear photographs of hand-written answers are acceptable if they are uploaded as a single document. If you upload multiple documents with the same name (e.g., if you wish to update a previously submitted document), we will grade the latest version.

1. This question is about linear transformations of multivariate normal random vectors and singularity. Let X be a p-dimensional random vector such that $X \sim N_p(\mu, \Sigma)$. Consider the linear transformation $Y = AX + b$, where A is an $m \times p$ constant matrix and b is an $m \times 1$ constant vector.

   (a) Derive the distribution of Y and clearly state the resulting mean vector and covariance matrix.

   (b) Suppose $p = 3$ and the covariance matrix Σ has eigenvalues $\{4, 2, 0\}$.

      i. Describe the geometric shape of this distribution in $\mathbb{R}^3$.

      ii. Explain why the joint probability density function (pdf) for X cannot be written in its standard form. Refer specifically to the properties of $\det(\Sigma)$ and the existence of Σ^{-1}.

2. This question concerns the conditional distributions of multivariate normal random vectors and the Schur complement. Suppose the $(p+q) \times 1$ vector X is distributed as $N_{p+q}(\mu, \Sigma)$. We partition X, μ, and Σ as follows:

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \quad \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

   where X_1 is $p \times 1$ and X_2 is $q \times 1$. Assume Σ_{22} is non-singular.

   (a) Define the transformation $Z = X_1 - \Sigma_{12}\Sigma_{22}^{-1}X_2$. Show that Z and X_2 are independent.

   (b) Use the result from part (a) to prove that the conditional distribution of X_1 given $X_2 = X_2$ is multivariate normal.

   (c) State the conditional mean $\mathrm{E}[X_1 | X_2 = X_2]$ and the conditional covariance $\mathrm{Var}(X_1 | X_2 = X_2)$. How does the conditional variance change as the value of X_2 changes?

3. A common pitfall is assuming that zero covariance implies independence. This question explores the specific conditions under which that implication holds.

   (a) Let $X = (X_1, X_2, \ldots, X_p)^\top$ follow a multivariate normal distribution $N_p(\mu, \Sigma)$. Prove that if $\mathrm{Cov}(X_i, X_j) = 0$ for all $i \neq j$, then the components $\{X_i\}$ are mutually independent. *Hint: Compare the joint pdf or moment generating function (mgf) to the product of the marginals.*

   (b) Provide a counter-example to show that if two random variables X and Y are individually normally distributed and $\mathrm{Cov}(X, Y) = 0$, they are not necessarily independent.

   (c) Explain why your counter-example in part (b) does not contradict the result in part (a).