

ICMU Mini-course on Gaussian Processes

Homework 1: The Multivariate Normal Distribution

Please upload your completed homework using [this link](#) by the end of the day on Saturday, February 7. The document name should include your last name. Scans or clear photographs of hand-written answers are acceptable if they are uploaded as a single document. If you upload multiple documents with the same name (e.g., if you wish to update a previously submitted document), we will grade the latest version.

1. This question is about linear transformations of multivariate normal random vectors and singularity. Let \mathbf{X} be a p -dimensional random vector such that $\mathbf{X} \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Consider the linear transformation $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}$, where \mathbf{A} is an $m \times p$ constant matrix and \mathbf{b} is an $m \times 1$ constant vector.
 - (a) Derive the distribution of \mathbf{Y} and clearly state the resulting mean vector and covariance matrix.
 - (b) Suppose $p = 3$ and the covariance matrix $\boldsymbol{\Sigma}$ has eigenvalues $\{4, 2, 0\}$.
 - i. Describe the geometric shape of this distribution in \mathbb{R}^3 .
 - ii. Explain why the joint probability density function (pdf) for \mathbf{X} cannot be written in its standard form. Refer specifically to the properties of $\det(\boldsymbol{\Sigma})$ and the existence of $\boldsymbol{\Sigma}^{-1}$.
2. This question concerns the conditional distributions of multivariate normal random vectors and the Schur complement. Suppose the $(p+q) \times 1$ vector \mathbf{X} is distributed as $N_{p+q}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. We partition \mathbf{X} , $\boldsymbol{\mu}$, and $\boldsymbol{\Sigma}$ as follows:

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix}, \quad \boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{bmatrix}, \quad \boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{bmatrix}$$

where \mathbf{X}_1 is $p \times 1$ and \mathbf{X}_2 is $q \times 1$. Assume $\boldsymbol{\Sigma}_{22}$ is non-singular.

- (a) Define the transformation $\mathbf{Z} = \mathbf{X}_1 - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\mathbf{X}_2$. Show that \mathbf{Z} and \mathbf{X}_2 are independent.
 - (b) Use the result from part (a) to prove that the conditional distribution of \mathbf{X}_1 given $\mathbf{X}_2 = \mathbf{x}_2$ is multivariate normal.
 - (c) State the conditional mean $E[\mathbf{X}_1|\mathbf{X}_2 = \mathbf{x}_2]$ and the conditional covariance $\text{Var}(\mathbf{X}_1|\mathbf{X}_2 = \mathbf{x}_2)$. How does the conditional variance change as the value of \mathbf{x}_2 changes?
3. A common pitfall is assuming that zero covariance implies independence. This question explores the specific conditions under which that implication holds.
 - (a) Let $\mathbf{X} = (X_1, X_2, \dots, X_p)^\top$ follow a multivariate normal distribution $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Prove that if $\text{Cov}(X_i, X_j) = 0$ for all $i \neq j$, then the components $\{X_i\}$ are mutually independent. *Hint: Compare the joint pdf or MGF to the product of the marginals.*
 - (b) Provide a counter-example to show that if two random variables X and Y are individually normally distributed and $\text{Cov}(X, Y) = 0$, they are not necessarily independent.
 - (c) Explain why your counter-example in part (b) does not contradict the result in part (a).