ICMU Mini-course on Gaussian Processes

Homework 1: The Multivariate Normal Distribution

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1. This question is about linear transformations of multivariate normal random vectors and singularity. Let $\mathbf{X}$ be a p-dimensional random vector such that $\mathbf{X} \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Consider the linear transformation $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}$, where $\mathbf{A}$ is an $m \times p$ constant matrix and $\mathbf{b}$ is an $m \times 1$ constant vector.

   (a) Derive the distribution of $\mathbf{Y}$ and clearly state the resulting mean vector and covariance matrix.

   (b) Suppose $p = 3$ and the covariance matrix $\boldsymbol{\Sigma}$ has eigenvalues $\{4, 2, 0\}$.

      i. Describe the geometric shape of this distribution in $\mathbb{R}^3$.

      ii. Explain why the joint probability density function (pdf) for $\mathbf{X}$ cannot be written in its standard form. Refer specifically to the properties of $\det(\boldsymbol{\Sigma})$ and the existence of $\boldsymbol{\Sigma}^{-1}$.

2. This question concerns the conditional distributions of multivariate normal random vectors and the Schur complement. Suppose the $(p+q) \times 1$ vector $\mathbf{X}$ is distributed as $N_{p+q}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. We partition $\mathbf{X}$, $\boldsymbol{\mu}$, and $\boldsymbol{\Sigma}$ as follows:

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix}, \quad \boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{bmatrix}, \quad \boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{bmatrix}$$

   where $\mathbf{X}_1$ is $p \times 1$ and $\mathbf{X}_2$ is $q \times 1$. Assume $\boldsymbol{\Sigma}_{22}$ is non-singular.

   (a) Define the transformation $\mathbf{Z} = \mathbf{X}_1 - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\mathbf{X}_2$. Show that $\mathbf{Z}$ and $\mathbf{X}_2$ are independent.

   (b) Use the result from Part (a) to prove that the conditional distribution of $\mathbf{X}_1$ given $\mathbf{X}_2 = \mathbf{x}_2$ is multivariate normal.

   (c) State the conditional mean $\mathrm{E}[\mathbf{X}_1 | \mathbf{X}_2 = \mathbf{x}_2]$ and the conditional covariance $\mathrm{Var}(\mathbf{X}_1 | \mathbf{X}_2 = \mathbf{x}_2)$. How does the conditional variance change as the value of $\mathbf{x}_2$ changes?

3. A common pitfall is assuming that zero covariance implies independence. This question explores the specific conditions under which that implication holds.

   (a) Let $\mathbf{X} = (X_1, X_2, \ldots, X_p)^\top$ follow a multivariate normal distribution $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Prove that if $\mathrm{Cov}(X_i, X_j) = 0$ for all $i \neq j$, then the components $\{X_i\}$ are mutually independent. *Hint: Compare the joint pdf or MGF to the product of the marginals.*

   (b) Provide a counter-example to show that if two random variables X and Y are individually normally distributed and $\mathrm{Cov}(X, Y) = 0$, they are not necessarily independent.

   (c) Explain why your counter-example in part (b) does not contradict the result in Part (a).