

Fall 2019

## Mathematical Optimization – Problem set 2

<https://moodle-app2.let.ethz.ch/course/view.php?id=4844>

### Problem 1: Finding a Chebychev center of a polyhedron

Let  $P = \{x \in \mathbb{R}^n : Ax \leq b\}$  be a polyhedron, where  $A \in \mathbb{R}^{m \times n}$  and  $b \in \mathbb{R}^m$  for some  $m, n \in \mathbb{Z}_{>0}$ . A ball  $B$  with center  $y \in \mathbb{R}^n$  and radius  $r \in \mathbb{R}_{\geq 0}$  is defined as the set of all points within Euclidean distance at most  $r$  from  $y$ , i.e.,  $B = \{x \in \mathbb{R}^n : \|x - y\|_2 \leq r\}$ . We are interested in finding a ball with largest possible radius that is contained in  $P$ . The center of such a ball is called a *Chebychev center* of  $P$ .

- (a) Assume that  $P$  has a Chebychev center. Write a linear program for the problem of finding such a Chebyshev center and the radius of the corresponding ball, and prove that your formulation is correct.

*Hint: Given a ball  $B$  with center  $y \in \mathbb{R}^n$  and radius  $r \in \mathbb{R}_{\geq 0}$  that is contained in a single halfspace  $\{x \in \mathbb{R}^n : a_i^\top x \leq b_i\}$ , what is the point in  $B$  closest to the hyperplane  $\{x \in \mathbb{R}^n : a_i^\top x = b_i\}$ ?*

- (b) Can the linear program that you found in (a) help deciding whether a Chebychev center exists at all?

### Problem 2: Existence of vertices in full-rank polyhedra

Let  $A \in \mathbb{R}^{m \times n}$  have full column rank, let  $b \in \mathbb{R}^m$ , and consider the polyhedron  $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ .

- (a) For  $v, w \in \mathbb{R}^n$  with  $w \neq 0$ , the set  $L(v, w) := \{v + \lambda w : \lambda \in \mathbb{R}\}$  is called a *line*. Prove that  $P$  does not contain a line, i.e., there are no  $v, w \in \mathbb{R}^n$  with  $w \neq 0$  such that  $L(v, w) \subseteq P$ .
- (b) Prove that precisely one of the following two statements is true.
- (i)  $P$  is empty.
  - (ii)  $P$  has a vertex.

### Problem 3: Finite linear programming optima are attained

Let  $P = \{x \in \mathbb{R}^n : Ax \leq b\}$  be a non-empty polyhedron, where  $A \in \mathbb{R}^{m \times n}$  and  $b \in \mathbb{R}^m$  for some  $m, n \in \mathbb{Z}_{>0}$ . Moreover, let  $c \in \mathbb{R}^n$  such that

$$\gamma := \sup\{c^\top x : x \in P\}$$

is finite. In this problem, we show that there exists a point  $x^* \in P$  such that  $c^\top x^* = \gamma$ . To do so, we proceed by induction on  $\dim(P)$ .

- (a) Observe that if  $\dim(P) = 0$ , the statement is true.
- (b) Prove that if  $\ker(A) \neq \{0\}$ , it is enough to show the statement for  $(\dim(P) - 1)$ -dimensional polyhedra.
- (c) In the case where  $\ker(A) = \{0\}$ , we propose a direct proof using the following claim.

*Claim.* For every  $y \in P$ , there exists a non-singular subsystem  $A'y \leq b'$  of  $Ax \leq b$  such that  $y' := (A')^{-1}b' \in P$  and  $c^\top y' \geq c^\top y$ .

Show that the claim implies that the supremum  $\gamma$  is attained.

- (d) Prove the above claim. To this end, start with a point  $y \in P$  and consider the  $y$ -tight constraints in the system  $Ax \leq b$ . If this subsystem is not full-rank, try to change  $y$  so that the number of tight constraints increases, while the objective does not decrease.

- (e) Combine the above points (a) to (d) to get a full proof of existence of  $x^* \in P$  with  $c^\top x^* = \gamma$ .

*Remark: The above problem shows that the optimal value of a feasible and finite linear program is always attained, which justifies writing linear programs using maxima and minima instead of suprema and infima.*

#### Problem 4: Polytopes and vertices

- (a) Prove that every non-empty polytope has a vertex.  
 (b) Let  $P \subseteq \mathbb{R}^n$  be a polytope and let  $c \in \mathbb{R}^n$ . Prove that the optimum value of the linear program  $\max \{c^\top x : x \in P\}$  is attained by a vertex of  $P$ .  
 (c) Let  $P$  be a polytope. Prove that  $\text{conv}(\text{vertices}(P)) = P$ .

*Hint: You can use the following separation statement without proving it. If  $Y \subseteq \mathbb{R}^n$  is a closed convex set and  $z \in \mathbb{R}^n \setminus Y$ , then there exist  $a \in \mathbb{R}^n$  and  $b \in \mathbb{R}$  such that  $a^\top y < b$  for all  $y \in Y$ , and  $a^\top z > b$ .*

#### Problem 5: Finite convex hull

Let  $X = \{x_1, \dots, x_k\} \subseteq \mathbb{R}^n$  be a finite set. In this problem, we show that  $\text{conv}(X)$  is a polytope. To this end, we first assume that  $\text{conv}(X)$  is full-dimensional, i.e.,  $X$  contains  $n + 1$  affinely independent points. In this case, we define a polyhedron  $P$  as follows: For any  $n$  affinely independent points in  $X$ , consider the unique hyperplane  $a^\top x = b$  containing all of them. If one of the halfspaces defined by  $a^\top x \leq b$  and  $a^\top x \geq b$  contains all the points in  $X$ , we add the corresponding inequality to the description of  $P$ .

- (a) Prove that  $\text{conv}(X) \subseteq P$ .  
 (b) To prove  $P \subseteq \text{conv}(X)$ , we assume for the sake of arriving at a contradiction that there exists a point  $y \in P \setminus \text{conv}(X)$ . Follow the steps below to reach a contradiction.  
 (i) Consider the polyhedron  $D$  defined by

$$D = \{d \in \mathbb{R}^n : (y - x_i)^\top d \geq 1 \ \forall i \in [k]\} .$$

Show that  $D$  has a vertex.

*Hint: Exploit that  $y$  and  $\text{conv}(X)$  can be separated by a hyperplane, and use the characterization obtained in Problem 3(b).*

- (ii) Consider a vertex  $d$  of the polyhedron  $D$ . Prove that the inequality  $d^\top x \leq d^\top y - 1$  appears in the description of  $P$ , and observe that this implies  $y \notin P$ , giving the desired contradiction.  
 (c) Conclude that if  $X$  contains  $n + 1$  affinely independent points, then  $\text{conv}(X)$  is a polytope.  
 (d) Reduce the case where  $\text{conv}(X)$  is not full-dimensional to the one where it is.

*Hint: If  $k := \dim(\text{conv}(X)) < n$ , the points in  $X$  all lie in an affine subspace of dimension  $k$ . Use an affine bijection to transform  $X$  to a set in  $\mathbb{R}^k \times \{0\}^{n-k}$ , and argue that the convex hull of the transformed points is a polytope. To complete the proof, you may use that the affine image of a polytope is a polytope.*

#### Programming exercise

Complete the notebook `02_shortestPath.ipynb` on finding the length of shortest  $s$ - $t$  paths using a linear program.