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Mathematical Optimization – Problem set 3

https://moodle-app2.let.ethz.ch/course/view.php?id=4844

Problem 1: Carathéodory's theorem

Let $P \subseteq \mathbb{R}^n$ be a non-empty polytope and let $x \in P$. We prove Carathéodory's theorem, namely that x is a convex combination of at most $n+1$ many vertices of P.

(a) Let $V = \text{vertices}(P)$ and define

$$Q \coloneqq \left\{ \lambda \in \mathbb{R}^V_{\geq 0} \colon \sum_{v \in V} \lambda_v v = x \quad \text{and} \quad \sum_{v \in V} \lambda_v = 1 \right\} \ .$$

Show that Q is a non-empty polytope.

(b) Prove that Q has a vertex λ^*. Show that λ^* has at most $n+1$ non-zero components, and conclude Carathéodory's theorem.

Problem 2: Representation of polyhedral cones

Given a finite set $\{x_1, \dots, x_k\} \subseteq \mathbb{R}^n$, let

$$C(x_1, \dots, x_k) \coloneqq \left\{ \sum_{i=1}^k \lambda_i x_i \colon \lambda_1, \dots, \lambda_k \geq 0 \right\} \ .$$

(a) Let C be a polyhedral cone. Prove that there exists a finite set $\{x_1, \dots, x_k\} \subseteq \mathbb{R}^n$ such that $C = C(x_1, \dots, x_k)$.

(b) Prove that for any finite set $\{x_1, \dots, x_k\} \subseteq \mathbb{R}^n$, the set $C(x_1, \dots, x_k)$ is a polyhedral cone.

Hint: To prove that $C(x_1, \dots, x_k)$ is polyhedral, start with an inequality description of the polytope $\text{conv}(\{0, x_1, \dots, x_k\})$. *Does this inequality description help to obtain one for $C(x_1, \dots, x_k)$?*

Problem 3: Slicing a polyhedron into polytopes

Let $P \subseteq \mathbb{R}^n$ be a polyhedron that does not contain a line, i.e., there do not exist $v, w \in \mathbb{R}^n$ with $w \neq 0$ such that $L(v, w) \coloneqq \{v + \lambda w \colon \lambda \in \mathbb{R}\}$ is contained in P. Prove that there exists $c \in \mathbb{R}^n \setminus \{0\}$ such that the polyhedron

$$P_\beta \coloneqq \{x \in P \colon c^\top x = \beta\}$$

is a polytope for all $\beta \in \mathbb{R}$.

Problem 4: Decomposing a polyhedron into a polytope and a cone

In this problem, we prove that for any polyhedron $P \subseteq \mathbb{R}^n$, there exists a polytope $Q \subseteq \mathbb{R}^n$ and a polyhedral cone $C \subseteq \mathbb{R}^n$ such that

$$P = Q + C \ .$$

Remark: This is the forward direction of Proposition 1.38 in the script.

(a) Assume that P contains a line $L(v, w) \coloneqq \{v + \lambda w \colon \lambda \in \mathbb{R}\}$ for some $v, w \in \mathbb{R}^n$ with $w \neq 0$. Prove that in this case, the above statement can be reduced to the same statement for polyhedra of strictly smaller dimension, i.e., dimension at most $\dim(P) - 1$.

(b) Assume that P does not contain a line, and let $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ be an inequality description of P. Furthermore, define

$$Q \coloneqq \operatorname{conv}(\operatorname{vertices}(P)) \quad \text{and} \quad C \coloneqq \{x \in \mathbb{R}^n : Ax \leq 0\} \ .$$

Prove that $P = Q + C$.

Hint: In order to prove $P \subseteq Q + C$, let $c \in \mathbb{R}^n$ be a vector guaranteed by Problem 3 and let $p \in P$. Prove that the vertices of $\{x \in P : c^\top x = c^\top p\}$ are contained in $Q + C$, and use this to obtain $p \in Q + C$.

(c) Use (a) and (b) to give a full proof of the initial statement.

Problem 5: Minkowski sum of a polytope and a cone
In this problem, we prove that for any polytope $Q \subseteq \mathbb{R}^n$ and any polyhedral cone $C \subseteq \mathbb{R}^n$, the Minkowski sum

$$P \coloneqq Q + C$$

is a polyhedron. We propose a proof by induction on the dimension n of the ambient space.

Remark: This is the backward direction of Proposition 1.38 in the script.

(a) Show that if $\dim(P) < n$, it is enough to prove the statement in an ambient space of dimension strictly less than n.

(b) Prove that if $\dim(P) = n$ and $Q \neq \emptyset$, it is enough to consider the case where $\dim(Q) = n$, too.

(c) Assume that C has a line, and show that in this case, the statement can be reduced to the same statement in in an ambient space of dimension strictly less than n.

(d) Assume that C is non-trivial but does not contain a line, and that $\dim(P) = \dim(Q) = n$. We prove the statement directly in this case.

(i) Prove that there exist $v \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$ such that $P \cap \{x \in \mathbb{R}^n : v^\top x = \beta\}$ is a non-empty polytope and $Q \subseteq \{x \in \mathbb{R}^n : v^\top x \leq \beta\}$.

Hint: Exploit the result of Problem 3.

Let $x_1, \dots, x_k \in \mathbb{R}^n$ such that $C = C(x_1, \dots, x_k) \coloneqq \left\{\sum_{i=1}^k \lambda_i x_i : \lambda \in \mathbb{R}^k_{\geqslant 0}\right\}$. For all $i \in [k]$ and every $q \in \operatorname{vertices}(Q)$, using v and β obtained in step (i), define $w_{q,i} \in \mathbb{R}^n$ such that

$$\{w_{q,i}\} = \{q + \lambda x_i : \lambda \in \mathbb{R}_{\geq 0}\} \cap \{x \in \mathbb{R}^n : v^\top x = \beta\} \ . \tag{1}$$

(ii) Prove that the points $w_{q,i} \in \mathbb{R}^n$ are well-defined for all $q \in \operatorname{vertices}(Q)$ and all $i \in [k]$, i.e., show that the intersection in (1) contains precisely one point for all choices of q and i.

Let $W_\beta \coloneqq \{w_{q,i} : q \in \operatorname{vertices}(Q),\ i \in [k]\}$ and define $P_\beta \coloneqq \operatorname{conv}(\operatorname{vertices}(Q) \cup W_\beta)$. Note that P_β is full-dimensional because it contains Q, and consider an inequality description of P_β that consists of facet-defining constraints only.

(iii) Show that $v^\top x \leqslant \beta$ (or a scaled version of this constraint) appears in this description of P_β, and thus we can write P_β the form

$$P_\beta = \{x \in \mathbb{R}^n : Ax \leq b,\ v^\top x \leq \beta\}$$

for some $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, where every constraint is facet-defining.

Using the A and b obtained in step (iii), we define $\overline{P} \coloneqq \{x \in \mathbb{R}^n : Ax \leq b\}$. We will show that in fact, $P = \overline{P}$, which implies that P is a polyhedron, as desired.

(iv) Prove that $P \subseteq \overline{P}$.

Hint: It is clear that $Q \subseteq \overline{P}$. Show that $Ax_i \leq 0$ for all $i \in [k]$, and use this to get $P \subseteq \overline{P}$.

(v) To prove that $\overline{P} \subseteq P$, assume for the sake of contradiction that there exists a point $y \in \overline{P} \setminus P$. Show that this implies existence of a point $z \in \overline{P} \setminus P$ with $v^\top z \leq \beta$, and use z to obtain a contradiction.

Hint: To obtain z, apply a separation theorem to obtain a halfspace $\{x \in \mathbb{R}^n : d^\top x \leq \delta\}$ that contains P but not y. Prove that $\max_{x \in P} d^\top x$ is attained by a vertex q of Q and show that z can be chosen as a point on the line segment connecting q and y.

(e) Use points (a) to (d) to give a full proof of the initial statement.

Programming exercise

Work through the notebook `03_basisExchange.ipynb` that introduces working with matrices in python and makes you implement a basis exchange procedure for systems of linear equations in tableau form.