WFIRST, EUCLID and Their Performance: Will They Be Able to Distinguish the STSQ and the DGP Models from a Cosmological Constant?

Enrico Specogna

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We are entering the so-called era of 'precision cosmology'. In the next 15 to 20 years, several probes are going to dramatically increase not only the quantity but also the quality of our astronomical measurements, which will bring us closer to understand the accelerating nature of the universe's expansion, a truly compelling theme in modern physics. In this work we ask whether two of those probes, WFIRST and EUCLID, will be able to distinguish a dark energy model, *Sharp Transition Scaling Quintessence* and a modified gravity one, the *DGP model*, from the currently accepted cosmological theory: the ΛCDM model. We tried to answer this question looking at the redshift evolution of three observables. They are the luminosity distance of type Ia Supernovae (d_L) and two quantities that parametrise the growth of matter density perturbations: $f\sigma_m$ and the γ factor. Using the predicted accuracies of both probes, we found this: while the measurement of d_L allows to recognise the STSQ model within the redshift range $0.2 < z < 1.7$, $f\sigma_m$ and the γ factor do not. The DGP model, known to be excluded by observations, was used as an example of how modified gravity can be clearly distinguished from the ΛCDM model using γ: it was found that its fractional deviation from the currently known value for γ, 0.55 [1], is 24%. We also attempted to constrain the STSQ model using the Early Dark Energy limits imposed by the Cosmic Microwave Background (CMB) on the STSQ model; it was found that, to avoid a dangerous shift in the first acoustic peak of the CMB, high values for the redshift at the onset of dark energy domination are favoured. We hint at the prospect of using gravitational waves to complement our results, giving a brief look at the promising future of 'multi-messenger astronomy'.

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I. INTRODUCTION

The universe is expanding and it does so in an accelerated fashion. To explain this behaviour, cosmologists postulated the existence of an exotic energy component of the universe, beyond gravitating matter and radiation: dark energy. We currently assume that this component, still not directly detected, is represented by a perfect fluid with a constant density in the context of a cosmological model that we call 'ΛCDM model'; here, Λ refers to an energy density arising from the vacuum (also referred to as 'cosmological constant'). The predictions that this model offers are often spectacularly precise: just think of the almost perfect fit of the Cosmic Microwave Background temperature spectrum into a blackbody curve [2]. There are, however, some major theoretical fallacies in this standard prescription, which can be summarised into two questions that represent the so called 'cosmological constant problem':

1. Why is the density of dark energy 120 times smaller than the predicted value of vacuum energy from quantum field theory?

2. Why does the accelerated expansion start only recently?

Because of these unexplained questions, we are now looking for alternative theories that can suit the accelerated expansion of the universe: one way is to look for a type of dark energy with a density that changes in time. Here, we look at a type of evolving dark energy that arises from an hypothetical scalar field filling the universe: the Sharp Transition Scaling Quintessence - STSQ. This is a single, real, homogeneous scalar field that is minimally coupled to any other kind of component of the universe. This model can give an answer to the first question that we made thanks to its attractor nature, similarly to other quintessence models. STSQ cannot, however, explain why the accelerated expansion only starts recently, so why dark energy becomes the dominating energy component of the universe only at recent cosmological times. As we will see in the following sections, there is a model, Growing Neutrino Quintessence, where this problem can be addressed by assuming that neutrinos can couple with quintessential fields like STSQ.
The focus of this work will be to establish whether the future space telescopes WFIRST and EUCLID will be able to distinguish STSQ from the ΛCDM model by using the following observables: the luminosity distance of type Ia Supernovae and the growth of matter perturbations in the history of the universe, along with a parameter called 'γ' used to described the latter [3].
Another objective of the analysis on the STSQ model presented here is constraining it by looking at the limits imposed by the CMB on the fraction of dark energy that scaling models can have at early times.
Furthermore, another alternative to explain the accelerated expansion of the universe is modified gravity: while dark energy assumes general relativity to be valid at all cosmological scales, the former considers that, instead, local observations only respect Einstein's theory because it is the weak field approximation of some other theory of gravity that describes the universe at the largest scales. The modified gravity model analysed here is the DGP model. As it is excluded by observations, we only present it with an illustrative purpose: the redshift evolution of the γ factor can be shown to be a decisive indicator of a potential modified gravity model in the observations of the growth history of the universe. This is because, unlike a dark energy model like STSQ, DGP's γ evolves in a remarkably different way from the same parameter as expected in a ΛCDM universe, making it possible to distinguish the two and, possibly, discard the one that does not fit the observations.
In general, this work aims at obtaining a sensible prediction on what we can expect from future telescopes and understanding the extent to which different observations can distinguish models like STSQ or DGP gravity from the problematic ΛCDM model.

In Section II we give an outline of the FLRW cosmology. Also, we introduce the concept of scalar fields in modern physics and their use in cosmology as a dark energy perfect fluid, which requires a discussion of Noether's first theorem and the use of the mechanics of variation. In Section III we discuss the theory behind the ΛCDM and the STSQ models, explaining how they can be used to describe the dynamics of the universe, while also giving a physical example that motivates a model like STSQ, which is purely phenomenological: the Growing Neutrino Quintessence. In Section IV we describe the observables used in our analysis: d_L, the luminosity distance, $f\sigma_m$, a variable describing matter density perturbations' time variation and γ. An outline of the constraints that the Cosmic Microwave Background imposes on scaling dark energy models is also given. Section V shows the results of our analysis on the STSQ model, while Section VII shows the results for the DGP model, but not before an introduction to the theory behind it in VI. Finally, before the conclusive comments in Section IX, Section VIII suggests a future extension on this analysis to a new kind of cosmological probe: Gravitational Waves.

II. DARK ENERGY AND SCALAR FIELDS

As it was seen in section I, a cosmological model where dark energy's density is constant in time bears questions that cannot be answered. Then, an evolving dark energy is what we need: possible models where its density does vary with time can be readily found in scalar field models; the STSQ model discussed here is one of them. For this reason, in this section there will be an introduction to the theory of scalar fields. A description of how they change in time (i.e. their equation of motion) and how we can extrapolate physical quantities from them (i.e. pressure and density) is given for quintessence models. Before that, however, the basic elements of FLRW cosmology are laid out, as it is the framework which we assume dark energy to be working in. The expressions shown here are the most important to our discussion and they consist of two independent equations describing the dynamics of space-time and energy in a FLRW universe, namely the *Friedman equation* and the *density equation*.

A. The Basics of FLRW Cosmology

There are two fundamental ideas on which the standard model of cosmology is built: the *cosmological principle* and the spatial *flatness* of the universe. According to the first, the universe is spatially both *homogeneous* and *isotropic*; the second one is a statement about its overall curvature. While this principle was originally taken just as an assumption that simplifies the process of finding solutions to the field equations of general relativity (Appendix A 1), it is now backed by various, strong pieces of evidence. The most notable of those is perhaps the *Cosmic Microwave Background - CMB* radiation and its temperature [4]; this radiation is made up of photons that fill the universe isotropically, at a temperature $T \approx 2.73K$ that is extremely uniform, only having some very small fluctuations to a characteristic scale $\delta T/T \sim 10^{-5}$. In turn, isotropy at every point in space implies homogeneity and both then give strong evidence for the applicability of the cosmological principle. By also assuming the spatial flatness in general relativity, we can reduce Einstein's field equations to find the dynamics of space-time and of the energy content *in* space-time for a spatially homogeneous, isotropic and flat universe. The metric for such a universe is (in Cartesian coordinates) the *Robertson-Walker metric* g_{RW}:

$$g_{RW} = \mathrm{diag}\left(1,\, -a^2(t),\, -a^2(t),\, -a^2(t)\right), \tag{1}$$

where $a(t)$ is the *scale factor*, a parameter that describes the expansion of the universe or, more precisely, of its spatial dimensions. $a(t)$ is dimensionless; it can be scaled to any value and in cosmology $a(t) = 1$ is often chosen as the scale factor's value at present, a_0. To find the dynamics of space-time means to find an equation of motion for $a(t)$; that is the *Friedman equation* and it is found by applying the cosmological principle and the spatial flatness to Einstein's field equations:

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\sum_i \rho_i, \tag{2}$$

where H is the *Hubble parameter*, G is the universal constant of gravitation and ρ_i is the energy density of each component of the universe; the dot notation (as in $\dot{a}$) always indicates a time derivative. There is another useful quantity related to a and very useful in cosmology: the redshift, z. It is connected to a as follows: $a \propto 1/(z+1)$.
In FLRW cosmology, the universe's components i are treated as perfect fluids with pressure p_i and density ρ_i such that, for each one of them, we can also define an *equation of state* of the following form:

$$w_i = \frac{p_i}{\rho_i}. \tag{3}$$

w_i is the *barotropic parameter* of the fluid i and it is an important quantity, as it is what allows us to differentiate between different types of fluid (non-relativistic matter, radiation, ...). Also, by rearranging eq.(2) we can work out the total density ($\sum_i \rho_i$) needed for a flat universe, also known as *critical density*: $\rho_c = (3H^2/8\pi G)$. It is possible to use ρ_c to define a quantity widely used in cosmology known as *density parameter*, which is used to compare the density of some component i to the total or, in a flat universe, critical density: $\Omega_i = \rho_i/\rho_c$. Here, only gravitating matter ρ_m and an *accelerating component* ρ_{acc} will be considered, the latter being either dark energy or an *effective* accelerating component arising from modified gravity. Radiation ρ_r is only included in the discussion on Early Dark Energy in Section IV C, as its density quickly decays after radiation-matter equivalence (at $z_{eq} \sim 3402$ [5]), becoming negligible at recent times ($\Omega_{r,0} = 5.38 \times 10^{-5}$, which is the current density parameter of the CMB radiation as it dominates over all of the other radiation sources [6]). Now, by applying again the two fundamental principles mentioned to the continuity equation in general relativity (which expresses the conservation for energy flowing in a finite region of space-time, Appendix A 2), we obtain a second important relation for our discussion, the *density equation*:

$$\dot{\rho}_i + 3H(\rho_i + p_i) = 0. \tag{4}$$

By integrating it, one can obtain the scaling of ρ_i with a: for gravitating matter we have $\rho_m \propto a^{-3}$, while for radiation we have $\rho_m \propto a^{-4}$. The scaling with a for quintessence dark energy will depend on the evolution of the scalar field ϕ describing it, as we will see in the following paragraphs.

B. What are Scalar Fields ?

The concept of field in physics originates from Faraday's idea to explain the action at a distance of electric and magnetic fields on electric charges. In modern physics, this has been extended to the description of both general relativity, quantum mechanics and many of the theories that are related to them, like cosmology and quantum field theory for example. Two important types of field are vector and scalar fields. The choice of a scalar field often allows to describe a physical system in a simple way, as it can be considered both as a fundamental field (like the Higgs field) or as something arising from the pairing/interaction of multiple, fundamental fields (like the Cooper pair field, a pairing of two fermionic fields) [7]. Scalar fields are particularly relevant to our discussion as they are often used to describe dynamical dark energy. To understand how a scalar field is made, we start from extending the spatial coordinates $x_i(t)$ used in mechanics (i is an index spanning the three spatial dimensions); we just need to rewrite them as ϕ_a and to add the spatial dependence $\mathbf{x}$:

$$x_i(t) \rightarrow \phi_a(\mathbf{x}, t). \tag{5}$$

ϕ_a can be interpreted as a wave and the subscript a can range through finite or infinite sets of values; this means that the number of its degrees of freedom can be finite or infinite, depending on the system that we want to describe. Since the quintessence models of dark energy are represented as a single scalar field, we only need one component for it, so that we can just write our field down as ϕ (therefore dropping the subscript a). In a more formal way, scalar fields are defined by the way they change under certain transformations; in particular, a scalar field differs from a vector field as it does not transform under a Lorentz transformation $\Lambda^{\mu}{}_{\nu}$ (the transformation between inertial frames in special relativity):

$$\phi(x^{'\mu}) = \phi(x^{\nu}), \text{ where } x^{'\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu}. \tag{6}$$

Here, x^{ν} is the rest frame while the 4-vector $x^{'\mu}$ represents the frame that is boosted with respect to x^{ν}; μ, ν are both indices that can take $(0, 1, 2, 3)$ as values, indicating space and time coordinates. The definition of ϕ given so far still does not clarify how it can be used to model dark energy: what we need is to find a connection between scalar fields and the physical quantities useful in cosmology, precisely those involved in eqs.(2) and (4). This is what it is done starting from the next paragraph.

C. Noether's First Theorem

The first important step to take, in order to connect the cosmological variables like the energy density ρ and the pressure p to scalar fields, is the application of *Noether's first theorem* to the scalar field used here for dark energy which, again, we simply call ϕ. This theorem is a statement that links continuous transformations to conserved quantities in physics. Firstly, to show how the theorem works, we need to remind ourselves that ϕ is a dynamic variable, which means that we can describe it with a function called Lagrangian: $\mathcal{L}_{\phi}$. The Lagrangian considered here will only be a function of ϕ and $\partial_{\mu}\phi$ because this ensures the Lorentz invariance for $\mathcal{L}_{\phi}$, a requirement that is necessary to satisfy the principle of relativity [8]. Then, $\mathcal{L}_{\phi} \equiv \mathcal{L}_{\phi}(\phi, \partial_{\mu}\phi)$; the index μ indicates the coordinates $\mathbf{x}, t$ and the notation ∂_{μ} indicates a partial derivative with respect to μ, which is a convention that will be often used here. The equation of motion for a field described by such a Lagrangian is [8][9]:

$$\partial_{\mu}\left(\frac{\partial \mathcal{L}_{\phi}}{\partial(\partial_{\mu}\phi)}\right) - \frac{\partial \mathcal{L}_{\phi}}{\partial \phi} = 0. \tag{7}$$

Secondly, we take a transformation of ϕ such that $\phi \to \phi + \delta\phi$ (which can stem from, for example, a translation in time or space): then, according to Noether's theorem, as long as this transformation is a *continuous symmetry* of the system, a conserved quantity exists, like energy or momentum.
Mathematically, a transformation is a continuous symmetry of $\mathcal{L}_{\phi}$ if its variation $\delta\mathcal{L}_{\phi}$ is equal to the total derivative of a set of functions $\mathcal{J}^{\mu}$: $\partial_{\mu}\mathcal{J}^{\mu}$.
So, if $\delta\mathcal{L}_{\phi} = \partial_{\mu}\mathcal{J}^{\mu}$ is satisfied, we have a *conserved current* J^{μ} or, in other words:

$$\delta\mathcal{L}_{\phi} = \partial_{\mu}\mathcal{J}^{\mu} \implies \partial_{\mu}J^{\mu} = 0. \tag{8}$$

The shape of $\mathcal{J}^{\mu}$ will depend on the exact form of $\mathcal{L}_{\phi}$ and the transformation considered.
It can be shown that J^{μ} can be rewritten in terms of $\mathcal{J}^{\mu}$ and the Lagrangian:

$$J^{\mu} = \frac{\partial \mathcal{L}_{\phi}}{\partial(\partial_{\mu}\phi)}\, \delta\phi - \mathcal{J}^{\mu}, \tag{9}$$

which we can use, depending on the type of transformation $\delta\phi$ considered, to find J^{μ} and its properties.

D. The Dynamics and the Physical Properties of Scalar Fields Applied to Cosmology

An important element in Einstein's field equations is the *energy-momentum tensor* - $T^{\mu}{}_{\nu}$, which represents the matter content of a gravitational system. In our case, the system is the universe and

the content that $T^{\mu}{}_{\nu}$ represents is either gravitating matter, radiation or dark energy.
The fact that this tensor is of rank 2 is no mistake: the generic current shown in eq.(9) has been derived using a single transformation. Indeed, this energy-momentum tensor represents a conservation that can be obtained from Noether's theorem and the transformation which it arises from is a *space-time translation*. In this case, however, the translations can be four different ones (ν can take $0, 1, 2, 3$ as values), giving rise to 4 separate currents that can be represented by the tensor $T^{\mu}{}_{\nu}$, the columns of which are the currents themselves.
We will need this tensor because it allows us to work out the density ρ_ϕ and pressure p_ϕ of the dark energy scalar field used here. To find it, we express the translations as shifts δx^ν along each coordinate x^ν:

$$x^\nu \to x^\nu + \delta x^\nu; \tag{10}$$

we can also rewrite δx^ν as ϵ^ν, where ϵ^ν is the translation 4-vector.
Now, a change in x^ν translates into a change in $\phi(x^\nu)$; the field changes as is $\phi(x^\nu) \to \phi(x^\nu + \epsilon^\nu)$, and it does so by a quantity $\delta\phi$, which we can work out by expanding $\phi(x^\nu + \epsilon^\nu)$ to the first order:

$$\phi(x^\nu + \epsilon^\nu) = \phi(x^\nu) + \epsilon^\nu \frac{\partial \phi}{\partial x^\nu} + \mathcal{O}[\,(\epsilon^\nu)^2\,].$$

By neglecting the terms of order equal or greater than 2 (we assume ϵ^ν to be a very small, non-0 value) and subtracting ϕ from it we get the variation $\delta\phi$:

$$\delta\phi \equiv \phi(x^\nu + \epsilon^\nu) - \phi(x^\nu) = \epsilon^\nu \partial_\nu \phi. \tag{11}$$

These translations also change the Lagrangian, and we can find the respective variation $\delta\mathcal{L}_\phi$; we start by expanding $\mathcal{L}_\phi(x^\nu + \epsilon^\nu)$, just as we did with $\phi(x^\nu + \epsilon^\nu)$:

$$\mathcal{L}_\phi(x^\nu + \epsilon^\nu) = \mathcal{L}_\phi(x^\nu) + \frac{\partial \mathcal{L}_\phi}{\partial x^\nu}\epsilon^\nu + \mathcal{O}[\,(\epsilon^\nu)^2\,].$$

Then, by neglecting the higher order terms $\mathcal{O}[\,(\epsilon^\nu)^2\,]$, we can obtain the variation of $\mathcal{L}_\phi$:

$$\delta\mathcal{L}_\phi \equiv \mathcal{L}_\phi(x^\nu + \epsilon^\nu) - \mathcal{L}_\phi(x^\nu) = \epsilon^\nu \partial_\nu \mathcal{L}_\phi. \tag{12}$$

We now have, from eqs.(11,12), two necessary elements to find the energy-momentum for a dark energy scalar field. Before working out the general form of $T^{\mu}{}_{\nu}$ for a scalar field Lagrangian as the one satisfying eq.(7), we can make a consideration about its shape by simply looking at the cosmological principle. What happens is this: because of spatial isotropy and homogeneity, $T^{\mu}{}_{\nu}$ has to take the form of the energy-momentum tensor for a *perfect fluid*, which is diagonal [10][11]. Then, the energy-momentum tensor will look like the following:

$$T = \mathrm{diag}(\rho, -p, -p, -p), \tag{13}$$

where ρ and p are the energy density and the pressure of the fluid that T describes. Because of this, for gravitating matter, radiation and dark energy we can define an equation of state of the same form as the one in eq.(3). This is a fundamental fact in this analysis: the cosmological principle basically tells us that it is sufficient to look at the diagonal elements of the energy-momentum tensor to work out both ρ and p, which is a very useful simplification that shows one of the advantages of considering a FLRW universe.
Now, to derive $T^{\mu}{}_{\nu}$, it is perhaps better to prove Noether's theorem directly for this transformation, the translation modifying ϕ and $\mathcal{L}_\phi$ as shown in eqs.(11,12). Take a Lagrangian $\mathcal{L}_\phi$ like the one defined in paragraph II C and consider its variation $\delta\mathcal{L}_\phi$ by taking its total derivative:

$$\delta\mathcal{L}_\phi = \frac{\partial \mathcal{L}_\phi}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}_\phi}{\partial(\partial_\mu \phi)}\delta(\partial_\mu \phi).$$

The property $\delta(\partial_\mu \phi) = \partial_\mu(\delta\phi)$ is valid; then, the term with $\delta(\partial_\mu \phi)$ can be rewritten in this way:

$$\begin{aligned}\delta\mathcal{L}_\phi &= \frac{\partial \mathcal{L}_\phi}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}_\phi}{\partial(\partial_\mu \phi)}\partial_\mu(\delta\phi) \\ &= \frac{\partial \mathcal{L}_\phi}{\partial \phi}\delta\phi + \partial_\mu\left(\frac{\partial \mathcal{L}_\phi}{\partial(\partial_\mu \phi)}\delta\phi\right) - \partial_\mu\left(\frac{\partial \mathcal{L}_\phi}{\partial(\partial_\mu \phi)}\right)\delta\phi \\ &= -\left[\partial_\mu\left(\frac{\partial \mathcal{L}_\phi}{\partial(\partial_\mu \phi)}\right) - \frac{\partial \mathcal{L}_\phi}{\partial \phi}\right]\delta\phi + \partial_\mu\left(\frac{\partial \mathcal{L}_\phi}{\partial(\partial_\mu \phi)}\delta\phi\right).\end{aligned}$$

The expression inside the square brackets is nothing but the equation of motion for the Lagrangian that we are considering, eq.(7). Then, if the field ϕ satisfied it, the term in the square brackets vanishes and $\delta\mathcal{L}_\phi$ is:

$$\delta\mathcal{L}_\phi = \partial_\mu\left(\frac{\partial \mathcal{L}_\phi}{\partial(\partial_\mu \phi)}\delta\phi\right).$$

To substitute eq.(12) in place of $\delta\mathcal{L}_\phi$ is to state that the variation of the Lagrangian is indeed a total derivative: $\partial_\nu(\epsilon^\nu \mathcal{L}_\phi)$, where ϵ^ν can be taken into the partial derivative because it does not depend on x^ν. Then, Noether's theorem is valid for a space-time translation and by completing this calculation we should be able to find the conserved current(s) which we are looking for: $T^\mu{}_\nu$. By also substituting eq.(11) in place of $\delta\phi$ we then get:

$$\partial_\nu(\epsilon^\nu \mathcal{L}_\phi) = \partial_\mu\left(\epsilon^\nu \frac{\partial \mathcal{L}_\phi}{\partial(\partial_\mu \phi)}\partial_\nu\phi\right);$$

ϵ^ν can now be taken out of the two derivatives $(\partial_\mu, \partial_\nu)$ on both sides, so that it can be cancelled out. Then, the only way in which both sides can be equal is when the indices of the partial derivatives are equal, when: $\mu = \nu$. This can be ensured by using the *Kronecker delta symbol* $\delta^\mu{}_\nu$, for which the property $\partial_\nu \equiv \partial_\mu \delta^\mu{}_\nu$ is valid. Then, by bringing every element on the left-hand side, we obtain:

$$\partial_\mu\left(\frac{\partial \mathcal{L}_\phi}{\partial(\partial_\mu \phi)}\partial_\nu\phi - \delta^\mu{}_\nu \mathcal{L}_\phi\right) = 0.$$

This is clearly the equation stating the conservation of a current $T^\mu{}_\nu$, which is what is contained inside the bracket:

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}_\phi}{\partial(\partial_\mu \phi)}\,\partial_\nu\phi - \delta^\mu{}_\nu \mathcal{L}_\phi; \tag{14}$$

$\delta^\mu{}_\nu$ here is the equivalent of $\mathcal{J}^\mu$ for this transformation and for this type of Lagrangian.
To summarise, eq.(14) is the energy-momentum tensor arising from a space-time translation of the coordinates x^ν, for a field ϕ that is described by a Lagrangian such that $\mathcal{L}_\phi \equiv \mathcal{L}_\phi(\phi, \partial_\mu\phi)$.
It is now possible to find ρ_ϕ and p_ϕ and all we need is the explicit form for the Lagrangian. The one needed describes a single, real and minimally coupled (non-interacting) scalar field also called quintessence and it looks like what follows:

$$\mathcal{L}_\phi = \frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - V(\phi), \tag{15}$$

where g is the Robertson-Walker metric defined in eq.(1); α and β are indices that, like μ or ν, range through $0, 1, 2, 3$.

By using eq.(15) in eq.(14) we can work out all the components of the tensor; as stated in eq.(13), the non-0 ones are all on the diagonal and since 3 of them are equal to each other (p_ϕ) we will only need to pick $T^0{}_0$ and, for instance, $T^1{}_1$.
Beginning this calculation with $T^0{}_0$, we can easily see how all terms that do not contain $\partial_0\phi$ vanish under the $\partial(\partial_0\phi)$ derivative:

$$
\begin{aligned}
T^0{}_0 = & \frac{\partial[(1/2)(\partial_0\phi)^2 - (1/2)a^2(\partial_1\phi)^2 - (1/2)a^2(\partial_2\phi)^2 - (1/2)a^2(\partial_3\phi)^2 - V(\phi)]}{\partial(\partial_0\phi)}\partial_0\phi \\
& - (1/2)(\partial_0\phi)^2 + (1/2)a^2(\partial_1\phi)^2 + (1/2)a^2(\partial_2\phi)^2 + (1/2)a^2(\partial_3\phi)^2 + V(\phi) \\
= & (\partial_0\phi)(\partial_0\phi) - (1/2)(\partial_0\phi)^2 + (1/2)a^2(\partial_1\phi)^2 + (1/2)a^2(\partial_2\phi)^2 + (1/2)a^2(\partial_3\phi)^2 + V(\phi).
\end{aligned}
$$

Then, since we want to consider a field that is also spatially *homogeneous*, the spatial derivatives (with indices $1, 2, 3$) also vanish, so that:

$$T^0{}_0 = (1/2)(\partial_0\phi)^2 + V(\phi);$$

by writing $\dot{\phi}$ in place of $\partial_0(\phi)$, we get the equation for the field's energy density, ρ_ϕ, the tensor component $T^0{}_0$:

$$\rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi). \tag{16}$$

To work out the pressure of the field p_ϕ, we just need to take one of the remaining diagonal components, as eq.(13) states that $T^1{}_1 = T^2{}_2 = T^3{}_3 = -p_\phi$. We can take $T^1{}_1$; similarly to what has been done for ρ_ϕ, all terms not containing $\partial_1\phi$ vanish under the $\partial(\partial_1\phi)$ derivative, which only leaves this:

$$T^1{}_1 = -a^2(\partial_1\phi)^2 - (1/2)(\partial_0\phi)^2 + (1/2)a^2(\partial_1\phi)^2 + (1/2)a^2(\partial_2\phi)^2 + (1/2)a^2(\partial_3\phi)^2 + V(\phi).$$

Again, all of the spatial derivatives vanish due to the homogeneity of the field; using the same notation for a time derivative as in eq.(16) we then recognise the pressure to be:

$$p_\phi = \frac{\dot{\phi}^2}{2} - V(\phi). \tag{17}$$

The expressions for the pressure and the density just found make it possible to find the equation of state for the perfect fluid represented by ϕ as defined in eq.(3); the barotropic parameter w_ϕ for this dark energy field is then:

$$w_\phi = \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)} \tag{18}$$

The properties just found for ϕ are all functions of the field itself and $\dot{\phi}$. However, p_ϕ, ρ_ϕ and w_ϕ are only useful if they can be related to the physical observables (like the *luminosity distance* introduced in Paragraph IV A) that we need to describe the effect of a quintessence type of dark energy on the universe. This is done by finding the evolution in time of ϕ, which in turn also tells us the dynamics of pressure and energy density. This is accomplished by finding an *equation of motion* for ϕ, a differential equation that has a function $\phi(t)$ as a solution.
Since quintessence is a fluid which we know the pressure and density of, it is natural to use eq.(4)

as a starting point to derive the equation of motion for ϕ, because it describes the time evolution of a perfect fluid ρ_i. By substituting p_ϕ and ρ_ϕ into eq.(4) we obtain:

$$\frac{d\left((\dot{\phi}^2/2)+V(\phi)\right)}{dt}+3H\left((\dot{\phi}^2/2)+V(\phi)+(\dot{\phi}^2/2)-V(\phi)\right)=0,$$

$$\implies \quad \ddot{\phi}\dot{\phi}+\frac{dV(\phi)}{dt}+3H\dot{\phi}^2=0.$$

Now, ϕ depends on time so that $dV(\phi)/dt$ can be expanded using the chain rule:

$$\ddot{\phi}\dot{\phi}+\frac{dV(\phi)}{d\phi}\frac{d\phi}{dt}+3H\dot{\phi}^2=0$$

$$\implies \quad \ddot{\phi}\dot{\phi}+\frac{dV(\phi)}{d\phi}\dot{\phi}+3H\dot{\phi}^2=0.$$

Dividing both sides by $\dot{\phi}$ we then obtain the equation of motion:

$$\ddot{\phi}+3H\dot{\phi}+\frac{dV(\phi)}{d\phi}=0. \tag{19}$$

This is a wave equation describing how a single scalar, homogeneous and minimally coupled field changes in time in a FLRW universe. The term containing H is a damping term also called *Hubble drag* and it reflects the fact that the redshifting caused by the expansion of the universe damps the kinetic energy of the field oscillations [10]. Also, depending on the form of $V(\phi)$, we can have either a slowly or rapidly varying field, or even an oscillatory one: the STSQ model turns out to be, in a part of its evolution, a field that is characterised by a damped oscillation. $V(\phi)$ is what effectively distinguishes one type of quintessence from another and the interplay between the term with $V(\phi)$ and the one with H is fundamental to determine whether dark energy ever starts to dominate the dynamics of the universe or if it remains subdued to other components. $V(\phi)$'s exact form for the STSQ dark energy is shown in Paragraph III B, which is part of the following section.

III. TWO TYPES OF DARK ENERGY: THE ΛCDM AND THE STSQ MODELS

We now introduce the dark energy model studied here: the STSQ model. As we aim to assess whether WFIRST and EUCLID will be able to distinguish a STSQ from a ΛCDM universe, we also present the main features of the latter. The STSQ model is, in a way, just a toy model of dark energy: it draws inspiration from other scaling quintessence models, among which there is *Growing Neutrino Quintessence*, which is also presented.

A. The ΛCDM Model

The ΛCDM model of dark energy can describe the accelerated expansion of the universe in a simple and effective way. However, because of the difficulties with this model outlined in Section I, we are now looking for other ways to describe dark energy. The idea of the ΛCDM model is essentially that there is a fixed, constant value of energy arising from the vacuum, $\rho_{de,0} = 2.51 \times 10^{-47}$ GeV^4 [6], and it can be used to explain why the attractive effect of gravity is not only lightened but literally overcome at the largest scales, making the universe expand in an accelerated fashion. Because of this constancy, the equation of state for this model, w_{de}, can be found from eq.(4) to be:

$w_{de} = -1$. As for any other dark energy model, these features lead to the universe expanding in a particular way, which we can observe by looking at the redshift evolution of luminosity distances and the growth of matter density perturbations, described in Section IV. One of our aims is to assess whether future space probes can distinguish between this model and the STSQ model described in the following paragraph.

B. The STSQ Model

With the theory of scalar fields applied to cosmology outlined in section II, it is now possible to describe the dark energy scalar field studied here: *Sharp Transition Scaling Quintessence - STSQ*. This is a quintessence model, which means that the dynamics of this field is fully described by the Lagrangian in eq.(15). Different models of quintessence will have different expressions for their potential energy $V(\phi)$ and plenty of different examples for $V(\phi)$ exist.
However, depending on the behaviour of the solutions for ϕ given by the equation of motion eq.(19), we can roughly classify them in two categories: thawing and freezing [12][13]. While in the first class ϕ only starts evolving at late cosmological times, being almost 'frozen' at early epochs, the second class does quite the opposite. A freezing field evolves slower as the universe grows older, which is what happens with the STSQ model.
In particular, STSQ looks similar to a type of freezing model called *scaling quintessence* which has a potential energy that is generally described by two exponentials summed together. Depending on the value of ϕ, one exponential allows for dark energy to scale with respect to the dominant universe component at early cosmological epochs (radiation or matter) while the other becomes dominant only at late times; this allows for the quintessence field to break free from the scaling condition and to freeze to a final set value at late cosmological times, *gradually* overtaking the other components of the universe (which instead decay, as shown in Paragraph II A) and causing the accelerated expansion that we now observe.

However, for the STSQ model we have a *sharp transition* in the behaviour of ϕ; $V(\phi)$ switches its functional form abruptly, as soon as ϕ overtakes ϕ_c, a value of the field that we can call *critical* (see fig.(2)):

$$V(\phi) = \begin{cases} M^4 \exp\left(-\frac{\lambda\phi}{M}\right) & \phi \leq \phi_c \\ M^4 \exp\left(-\frac{\lambda\phi_c}{M}\right) \exp\left(\frac{\eta\lambda(\phi-\phi_c)}{M}\right) & \phi > \phi_c \end{cases}. \tag{20}$$

The consequence of such a transition is that at a *critical redshift*, z_c, ϕ changes from a scaling behaviour (when $z \geq z_c$) to a damped oscillation around ϕ_c (when $z < z_c$), to which ϕ asymptotically tends as one can see in fig.(1). Here, η and λ are parameters of the STSQ model while M is the reduced Planck mass: $M = 2.43 \times 10^{18}$ GeV. The oscillation is another feature of STSQ that differentiates it from other freezing quintessence models. The damped oscillation that takes place after z_c indicates that the field tends to become a cosmological constant, where the equation of state is $w = -1$ and the density of the field stays constant: $\rho_\phi \approx V(\phi_c) = \rho_{de,0}$, as shown in fig.(2). However, before z_c, as shown in fig.(3), the equation of state of the field can be seen to tend to the same value as the equation of state of matter: $w_\phi = 0$. This appears evident by looking at eq.(4): as we demand that ρ_ϕ scales with the density of the dominating universe's component ρ_m (matter) at scaling (i.e. they need to have the same functional dependence on a), so that $\rho_\phi \propto a^{-3}$, eq.(4) will give the same solution for w_i for both components, because their time derivatives will also be equal to each other. The scaling solution is such that the density parameter of dark energy at scaling $\Omega_{\phi,scaling}$ is fixed to the following value [14]:

$$\Omega_{\phi,scaling} = \frac{n}{\lambda^2} \approx \frac{2\Omega_{de,0}}{2\Omega_{de,0} + (z_c+1)^3\Omega_{m,0}}, \tag{21}$$

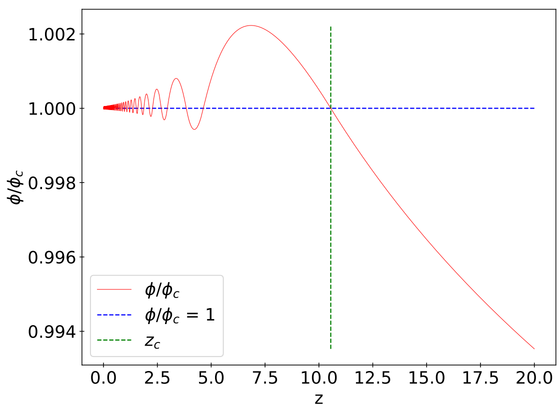

FIG. 1. Redshift evolution of ϕ/ϕ_c that shows how the field starts to oscillate around ϕ_c (here ϕ/ϕ_c=1) after the critical redshift $z_c = 10.55$.

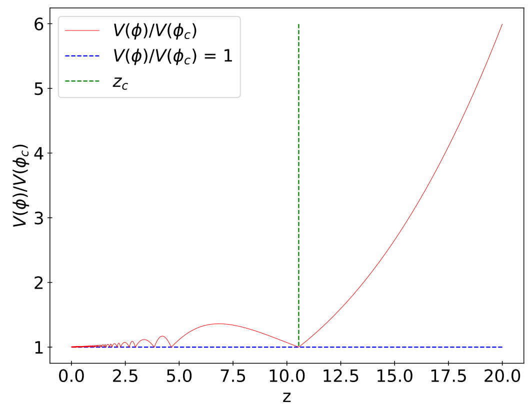

FIG. 2. Redshift evolution of $V(\phi)/V(\phi_c)$ showing how ϕ's potential energy starts tending asymptotically to $V(\phi_c)$ after $z_c = 10.55$.

which can be worked out by using the scaling condition for ρ_ϕ [14]: again, $\rho_\phi \propto a^{-3}$; $\Omega_{de,0}$ and $\Omega_{m,0}$ are the current values for dark energy's and matter's density parameters respectively. From eq.(21) it is clear that the parameter λ is set by choosing z_c; ϕ_c can also be shown to be fixed by λ and so by z_c. Knowing the connection between λ and z_c it is now possible to work out ϕ_c too (see Appendix B 1): $\phi_c = \frac{M}{\lambda} \ln\left(\frac{M^4}{\rho_{de,0}}\right)$, which, for $z_c = 10.55$, is 2.04×10^{19} GeV.

What makes STSQ worth comparing to ΛCDM dark energy is how it naturally explains the the smallness of the cosmological constant: it allows for an attractor solution for $\lambda > \sqrt{n}$, which means that z_c can be in principle as high as one desires and still explain the smallness of $\rho_{de,0}$ today. Then, the amount of dark energy during the scaling epoch given by $\Omega_{\phi,scaling}$ could be arbitrarily small and thanks to the attractor behaviour of ϕ its density will always break free from the scaling with respect to matter, allowing ϕ to freeze and to eventually overcome the density of matter.

It turns out that, because of the Early Dark Energy constraints set by the Cosmic Microwave Background, high values of z_c are indeed favoured, as it can be seen in Paragraph V B.

Nonetheless, STSQ still cannot explain why the density fraction of dark energy becomes relevant only at recent times: this means that the choice of z_c is made simply knowing that, from what we observe, the transition of a dynamical dark energy towards a cosmological constant only happens in the recent history of the universe. Again, STSQ is a purely phenomenological model.

However, *Growing Neutrino Quintessence* seems to have the capability of describing a scaling behaviour at early times similarly to STSQ, while offering at the same time a physical explanation behind the transition that, in the STSQ case, is defined as *sharp*.

C. Growing Neutrino Quintessence

As mentioned in the previous section, scaling quintessence models are represented by a potential energy $V(\phi)$ that is the sum of two exponentials and each of those is dominant at different cosmological epochs. While one describes the scaling period, the other allows the field to evolve out of the scaling condition and to eventually settle to a constant value. The potential energy of STSQ does this as well, even though the change is sharp; what is more, STSQ is a purely phenomenological model as it does not include any physical mechanism that would justify the transition from scaling to a cosmological constant. However, a dark energy model called *Growing Neutrino Quintessence*

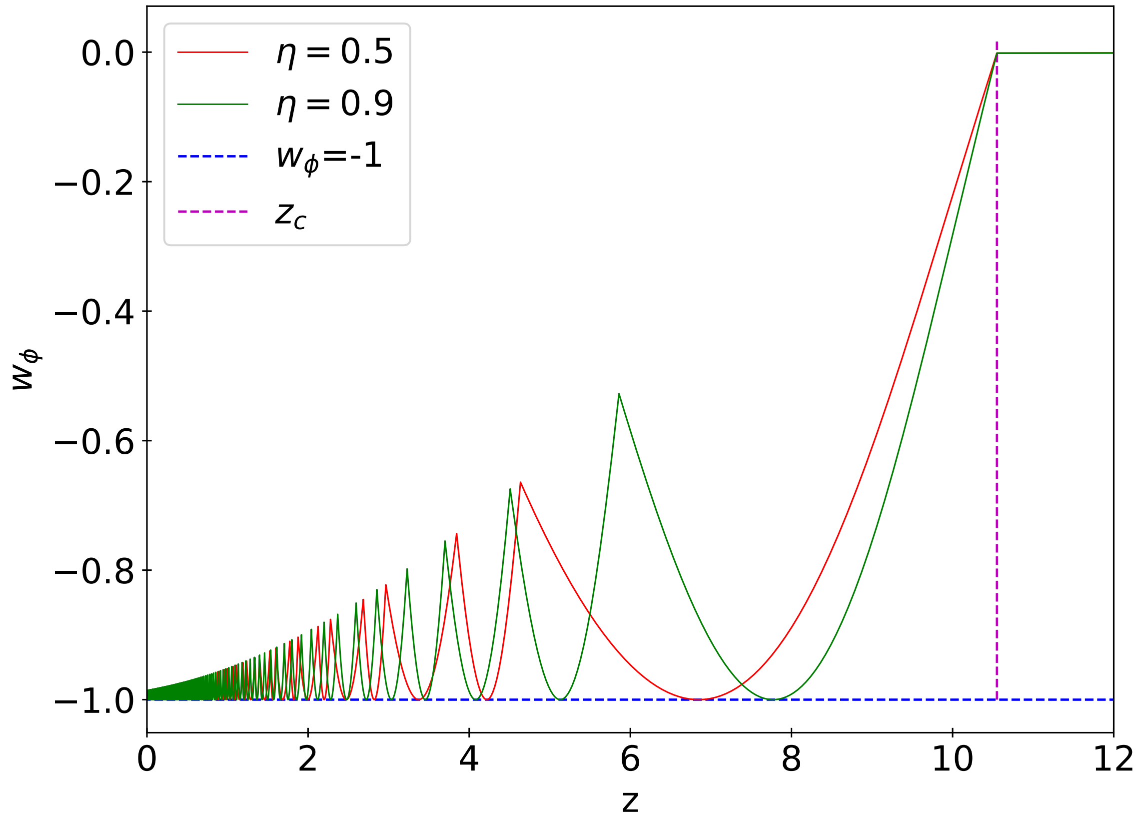

FIG. 3. Evolution of w_ϕ for two different values of the η parameter of the STSQ model: as z_c is reached the field enters a phase of damped oscillation, leading its equation of state to change from a constant (0) to an oscillating value that tends to -1 (a cosmological constant) asymptotically.

might explain it. This is a scaling quintessence model as well and it is built upon the speculation that neutrinos with a mass that varies in time ('growing') couple with the quintessence field. In other words, the mass of neutrinos $m_g(\phi)$ depends directly on ϕ: $m_g(\phi) = \overline{m}_g \exp\left(-\frac{\lambda\phi}{M}\right)$, being λ a generic constant similar to the λ used for STSQ and $\overline{m}_g$ is a constant. This directly leads to a modification of the field's Lagrangian in eq.(15), which will need to include a term for the coupling of ϕ with the neutrinos. A different Lagrangian also gives a different equation of motion for this type of quintessence [15]:

$$\ddot{\phi} + 3H\dot{\phi} + \left(\frac{dV(\phi)}{d\phi} + \frac{\beta\rho_g}{M}\right) = 0, \tag{22}$$

where β is a positive constant quantifying the strength of the neutrinos-quintessence coupling. ρ_g is the density of neutrinos and we assume that it dilutes more slowly than ordinary matter, so that it is ρ_g that dominates the dynamics of the universe straight after matter domination, and not the quintessence field as in the STSQ model. One can integrate the potential energy term in the brackets of eq.(22) to work out the effective potential V_{eff} of the Growing Neutrino Quintessence model:

$$V_{\text{eff}} = M^4 \exp\left(-\frac{\lambda\phi}{M}\right) + n_g \overline{m}_g \exp\left(\frac{\beta\phi}{M}\right), \tag{23}$$

where n_g is the number density of neutrinos. The first exponential term in eq.(23) is the same as in the STSQ and other scaling models: it is the one allowing for scaling. The second one allows to break free from the scaling, which then leads to the accelerated expansion and just like STSQ it makes ϕ oscillate around a fixed value. The parameter β depends on the properties of this growing matter. β influences the scaling of ρ_g with a and so it is directly related to the redshift at matter-neutrino equivalence [15], which means that this dark energy model can potentially explain why the acceleration of the universe only starts at recent cosmological times. This is what makes this model interesting: while scaling models already explain why the size of the dark energy density is so small, Growing Neutrino Quintessence explains the 'coincidence problem' (why does the acceleration only start recently?) too, therefore solving the main theoretical issues of a cosmological constant and representing a valid alternative to it.

IV. THE OBSERVABLES: THE LUMINOSITY DISTANCE, THE GROWTH FACTORS AND THE CONSTRAINTS OF EARLY DARK ENERGY

The study of the STSQ model carried out here will make use of some important observables: *luminosity distance* d_L, the *growth factors* $f\sigma_m$ and γ and the redshift evolution that one would expect in a universe where dark energy is a STSQ field. We also review the constraints on the STSQ parameter z_c imposed by Early Dark Energy (EDE) in the last paragraph of this section, which can determine the validity of the STSQ model as an alternative to the cosmological constant. Once these are defined, we will use them to compare the STSQ model to the ΛCDM model described in the previous section and we will be able to determine whether the future mission WFIRST is able to distinguish them within different redshift ranges. This last part is what is done in Section V B.

A. The Luminosity Distance

The WFIRST mission will measure the luminosity distance from a well-known type of standard candles: type Ia Supernovae. These are bright events that most likely happen when a white dwarf in a binary system accretes matter from its binary companion; if the accreted mass overtakes a limit called *Chandrasekhar mass*, the result is a shock wave that bounces on the degenerate core and clashes with the outer layer of the star that became a white dwarf. This leads to a powerful nuclear reaction, which translates into the release of an enormous amount of energy by a Supernova, such that its luminosity outshines the galaxy where it is located. The fact that this Supernova occurs always when the same limit is overtaken (the Chandrasekhar mass) means that its intrinsic luminosity is always the same, with very little dispersion, and that we can use this event as a *standard candle* [16]. Standard candles are events where the absolute luminosity does not depend on the relative, measured one and in the case of type Ia Supernovae it is measurable because of its relation to the time-scale in which the peak luminosity decreases in its intensity; inferring the latter gives us the former. Then, knowing what the absolute luminosity is and measuring the relative luminosity of a supernova, it is possible to obtain the d_L from this source.

The luminosity distance is not the actual, physical distance from this luminous source; the latter quantity gets in fact distorted by the effect of both the universe's expansion and a decrease in the number of photons per unit area as the (physical) distance increases [17]. These all contribute to decrease the energy of the electromagnetic radiation reaching our telescopes, making the observed object look more redshifted and thus further away than what it actually is. However, since what gets actually measured by a probe like WFIRST is indeed the luminosity distance, it is the quantity

that we will use in our analysis.
For a standard candle at a redshift z, d_L is:

$$d_L(z) = a_0(1+z)\int_0^z \frac{dz'}{H(z')}; \tag{24}$$

as mentioned in Paragraph II A, a_0 is the current value for scale factor, which we can choose to be 1. The key in this equation is that depending on the model of dark energy assumed, H will be a specific solution of eq.(2) describing that model, making d_L evolve in a specific way that may or may not be distinguishable from other models: this will depend on WFIRST's precision in measuring d_L. When we refer to the precision of the distance measurement, what we are talking about is the fractional deviation Δd_L of the luminosity distance in the two models:

$$\Delta d_L = \left| \frac{d_{L,\mathrm{STSQ}} - d_{L,\Lambda\mathrm{CDM}}}{d_{L,\Lambda\mathrm{CDM}}} \right|, \tag{25}$$

where $d_{L,\Lambda\mathrm{CDM}}$ and $d_{L,\mathrm{STSQ}}$ are the luminosity distances simulated using the ΛCDM model and the STSQ models respectively. WFIRST's predicted 1σ accuracy (precision) for this measurement is 0.18% in the $0.2 < z < 1.7$ range [18].

B. The Growth of Matter Density Perturbations

Two other important observables considered here are the *growth factors* $f\sigma_m$ and γ. The expansion of the universe does not only affect distances but also the perturbations of non-relativistic matter (with density ρ_m) at scales smaller than the cosmological ones. These perturbations $\delta_m(\mathbf{x}, t)$ are represented in linear perturbation theory as [19]:

$$\delta_m(\mathbf{x}, t) \equiv \frac{\rho_m(\mathbf{x}, t)}{\rho_m(t)} - 1, \tag{26}$$

where $\rho_m(\mathbf{x}, t)$ is the matter density at some point $\mathbf{x}$ and time t, while $\rho_m(t)$ is the matter density of the region surrounding the point $\mathbf{x}$ averaged at t. The evolution of δ_m is given by the following equation [3]:

$$\ddot{\delta}_m + 2H\dot{\delta}_m - \frac{\rho_m}{2M^2}\delta_m = 0. \tag{27}$$

As the universe expands, δ_m can be said to change in proportion to a *linear growth function* called $G(z)$ (or $\delta_m \propto G(z)$ [19]), which can be found by solving the following equation [1] (indeed, it is identical to eq.(27)):

$$\ddot{G}(z) + 2H\dot{G}(z) - \frac{3}{2}\Omega_{m,0}H_0^2(1+z)^3G(z) = 0, \tag{28}$$

where H_0 and $\Omega_{m,0}$ are the current values of the Hubble constant and matter's density parameter. While a solution for $G(z)$ describes the redshift (or time) evolution of matter perturbations, it is actually not what WFIRST observes, which instead measures the quantity $f(z)\sigma_m$. σ_m is the root mean square of a matter perturbation over a set radius of space of length $8h^{-1}$Mpc [20] (h is the dimensionless Hubble parameter: $h = 0.674$ [6]) and it is proportional to $G(z)$, so that we can rewrite the previous parameters combination $f(z)\sigma_m$ as $f(z)G(z)$; we can do this because what we will use to assess the distinguishability of different acceleration models by WFIRST is a fractional error, so that any proportionality constant relating $G(z)$ and σ_m cancels out. $f(z)$ is a function of $G(z)$ and H, which can be well approximated by a parametrisation of $\Omega_m(z)$ [1]:

$$f(z) = \frac{\dot{G}(z)}{HG(z)} \approx \Omega_m^\gamma(z). \tag{29}$$

Then, by finding G and its time derivative it is possible to work out $f(z)$ and, from eq.(29), also γ. The latter is a parameter that depends on z and it can be very important when discussing modified gravity: while it seems to be almost insensitive to different models of dark energy, its z evolution varies notably among different models beyond general relativity [3], as we will see for the DGP model in Section VII.
WFIRST will be conducting an imaging of more than 500 million galaxies over a sky area of 2000 deg^2 and this will provide it with the possibility of not only looking at numerous type Ia Supernovae but also making weak lensing measurements; this allows to probe the clustering of matter between the observer and the lensed source galaxies, which is directly connected with the growth of matter perturbations and can measure it in the form of the $f(z)\sigma_m$ combination [18].
Similarly, EUCLID will image over 15 billion galaxies over a 15,000 deg^2 area and by looking at weak lensing along with other probes like galactic clustering it will be able to measure the γ parameter [21]. Just as with d_L, the redshift evolution γ and $f(z)\sigma_m$ given by different acceleration models will be used to distinguish between them. Similarly to d_L, the precision of WFIRST's measurement of $f(z)\sigma_m$ can also be expressed with the its fractional deviation:

$$\Delta f(z)\sigma_m = \left| \frac{(f(z)\sigma_m)_{\mathrm{STSQ}} - (f(z)\sigma_m)_{\Lambda\mathrm{CDM}}}{(f(z)\sigma_m)_{\Lambda\mathrm{CDM}}} \right|, \tag{30}$$

where $(f(z)\sigma_m)_{\mathrm{STSQ}}$ and $(f(z)\sigma_m)_{\Lambda\mathrm{CDM}}$ are the $f(z)\sigma_m$ combinations simulated assuming either a STSQ or ΛCDM model respectively. WFIRST's predicted 1σ accuracy for this combination is 1.2% in the $1 < z < 2$ range. As for the γ parameter, the EUCLID mission is predicted to measure it with an error $\delta\gamma = 0.007$. In this case the error is not fractional, which means that $\delta\gamma$ can just be expressed as:

$$\delta\gamma = \gamma_{\mathrm{STSQ}} - \gamma_{\Lambda\mathrm{CDM}}; \tag{31}$$

of course, γ_{STSQ} and $\gamma_{\Lambda\mathrm{CDM}}$ are γs of the STSQ and ΛCDM models respectively. The total span of the error is then twice $\delta\gamma$ and it can be written as $\Delta\gamma = 2\delta\gamma = 0.014$.

C. Early Dark Energy's Constraints on Scaling Quintessence

Luminosity distances and the growth of matter density perturbations are both useful instruments to assess whether future probes will be able to tell the difference between a scaling quintessence like STSQ and the ΛCDM model. However, by considering the constraints on *Early Dark Energy - EDE*, we can also test the validity of a model like STSQ. In the context of scaling models, EDE is the fraction Ω_e of dark energy that stays fixed during the scaling epoch (in Paragraph III B Ω_e was written as $\Omega_{\phi,scaling}$). Its amount and the duration of the scaling from a redshift z_e to z_c (the redshift at Early Dark Energy's onset and at the end of the scaling respectively) can both influence various observable phenomena of the universe, from the perturbation growth described in the last paragraph to the cosmic microwave background (CMB) [22]. The latter turns out to be one of the most precise probes to study EDE [22]. EDE can affect the position and amplitude of CMB's peaks, which is what we can detect in principle. The change in their position can be caused by two different effects that can be described by looking at CMB's first acoustic peak (the term *acoustic* refers to the fact that waves propagate in the photons-baryons plasma before recombination in a somewhat similarly way to how acoustic waves travel in the air) [23]. The first peak's angular position θ_* is given by [6]:

$$\theta_* = \frac{s_*}{d_*}, \tag{32}$$

where s_* is the physical distance covered by the acoustic waves before recombination, also called sound horizon, and it depends on H, z_* (the redshift at recombination) and the speed of the

acoustic waves in the pre-recombination plasma, c_s. d_* is the physical distance of the surface of last scattering from us, or the distance to the part of the universe that we still observe to be in the recombination phase [5].
Assuming that the ΛCDM model rules the universe's expansion, θ_* can be measured to be $100\,\theta_* = 1.04092 \pm 0.00031$ [24]: variations in s_* and d_* can potentially change this value. Assuming dark energy to only be relevant after recombination ($z_* \sim 1090$ [6]), we just need to look at the change in d_* as EDE only modifies s_* in a redshift $z > z_*$.
d_* is given by [25]:

$$d_* = \int_0^{z_*} \frac{dz}{H(z)}. \tag{33}$$

Here is why it can change: taking Early Dark Energy into account has a direct effect on H at different redshifts and as one can see from eq.(33), a change in H will mean that d_* also changes, which in turn might generate a θ_* that is not consistent with the value given in [24]. This is the point where using the CMB can constrain the validity of the STSQ model: by fixing z_e, one can vary z_c to check whether the θ_* simulated by this model is inconsistent with [24], as the fractional accuracy of d_* is directly related to the fractional accuracy in the measurement of θ_*, $\Delta\theta_*$ (see Appendix C 1 for a proof):

$$\Delta\theta_* \approx \Delta d_* = \left| \frac{d_{*,\mathrm{STSQ}} - d_{*,\Lambda\mathrm{CDM}}}{d_{*,\Lambda\mathrm{CDM}}} \right|. \tag{34}$$

$d_{*,\mathrm{STSQ}}$ and $d_{*,\Lambda\mathrm{CDM}}$ are the physical distances to recombination predicted by the STSQ and ΛCDM models respectively.
In other words, the maximum shift of θ_* within the STSQ model cannot exceed $\Delta\theta_{*,\max} = 0.0006$, to a 2σ confidence limit (this is obtained directly from θ_*'s measurement in [24]).
The Early Dark Energy constraint on Ω_e, $\Omega_e < 0.0036$ [26], sets the minimum z_c value for a scaling dark energy; for the STSQ model, this is given by eq.(21) and, by using the constraint in [26], we get a minimum redshift of around $z_c = 10.55$: z_c can then only be varied to bigger values than this one for our analysis.
d_* can be simulated by splitting the redshift range $z_* - z_0$ into two luminosity distance integrations and then adding them together:

$$d_* = \int_{z_*}^{z_e} \frac{dz}{H(z)} + \int_{z_e}^{z_0} \frac{dz}{H(z)}. \tag{35}$$

In the case of the STSQ model, while distance simulation in the $z_e - z_0$ range uses the field's evolution given the potential energy in eq.(20), the one in the $z_* - z_e$ range instead assumes that the density of dark energy is a constant equal to its value at z_e. So, for $z > z_e$:

$$\Omega_e(z) = \Omega_e(z_e)\frac{(z_e+1)^3}{(z+1)^3}. \tag{36}$$

At redshifts between recombination z_* and z_e, radiation can have some influence on the expansion and, even if minimal, we are taking account of that for our EDE analysis on STSQ.

V. THE RESULTS OF THE ANALYSIS ON THE STSQ MODEL

As the theory behind the STSQ model along with the observables that we can use to study its measurability and viability have been outlined, we can now show the results of our investigation on whether the STSQ and ΛCDM model can be distinguished by WFIRST and EUCLID and on the EDE constraint on it. Firstly however, an overview of the Python program used for this analysis is given.

A. The Python Program Used

To assess the distinguishability of not only different types of dark energy but also of one type of modified gravity (the DGP model), a Python program has been developed. Its essential purpose is to find the solution of four differential equations. One is the equation of motion of quintessence, eq.(19), with the potential for the STSQ model, eq.(20). The other three are the equations for the evolution of the matter density perturbation: while eq.(28) can be used to describe both STSQ and the ΛCDM model, the DGP model is described by a modified equation. See Appendix D 2 for their expressions in terms of Linder's formalism [3], where we rewrite the growth factor as g by factoring a out: $g \sim G/a$.
The code is built around one main file called **main_class.py**: it contains a class called **model**, which is the class called by all the other files. These files calculate the fractional deviations of d_L and $f\sigma_m$ between the ΛCDM model and STSQ. The γ factor is also calculated and it serves, for instance, as a comparison of the dark energy models with the modified gravity one studied here: the DGP model. The EDE constraints are also checked for the STSQ model. All the files are available and described more in detail at the following GitHub repository link: STSQ Dark Energy Model Analysis.

Here, we outline the workings of the most important file of the program, the **model** class. This is the core of the whole program as it contains all the elements that are important to solve the differential equations mentioned above. We start from the initial conditions z_i, g_i, $\dot{g}_i$, ϕ_i and $\dot{\phi}_i$.

When calling the class, one also has to specify its parameters which are z_i, η, z_c and N, the number of reiterations for the Runge-Kutta algorithm. The latter is an algorithm that solves the differential equations by taking the initial conditions mentioned above and updating them for the next value of z, which differs from z_i by a small value δz that is set by N (a greater N increases the smallness and precision of each step in the algorithm). A Runge-Kutta algorithm is written for every acceleration model considered (there are in fact three functions containing the algorithm, which are called *DGP*, *stsq* and *CDM*) and, while updating the initial conditions, it uses them to calculate other cosmological variables such as H, $\Omega_{\rm acc}$ (a generic density parameter to define the accelerating component of the Friedman equation(2)), Ω_m, $f\sigma_m$, d_L, and γ (all of which are defined in the other functions written in **main_class.py**). Both the initial conditions and the cosmological parameters are then appended to an array of lists defined at the end of each Runge-Kutta function (like the *DGP* function); once this is done and the initial conditions are updated, the latter are sent back to the beginning of the Runge-Kutta algorithm, which again appends them to the lists and updates them in a continuous cycle that ends when $z = 0$ is reached. In fig.(4) there is a description of the first loop: as soon it is over (the lower blue/violet box), the updated initial conditions get substituted to the original ones (the higher blue/violet box) and serve as new initial values for a new loop (and new cosmological parameters) in the Runge-Kutta algorithm. In other words, as more loops are performed, more of these values are appended to the lists and their redshift evolution is simulated. Whenever a file other than **main_class.py** calls this class it initiates this process and, depending on the list in the array of lists that gets called by the file in question (i.e. depending on the list of the cosmological parameter called), we can plot and study a certain variable, like H against z.
A final remark: this code can be flexibly changed to simulate other types of dark energy quintessence: as long as the shape $V(\phi)$ is known, it can just be substituted to the function(s) containing $V(\phi)$ in **main_class.py**, namely V and V_2.

B. Results on the Distinguishability of the STSQ Model and on the EDE Constraints

To summarise the results, we start by looking at the fractional deviations. For both the luminosity distance and the growth factor analysis, the simulation has run using a z_i (the initial value of the redshift) fixed at $z_i = 20$. The solutions of eq.(19) could be found starting from the condition imposed on the initial value of the field ϕ_i, which depends on z_i and λ (and so on z_c too), as one can

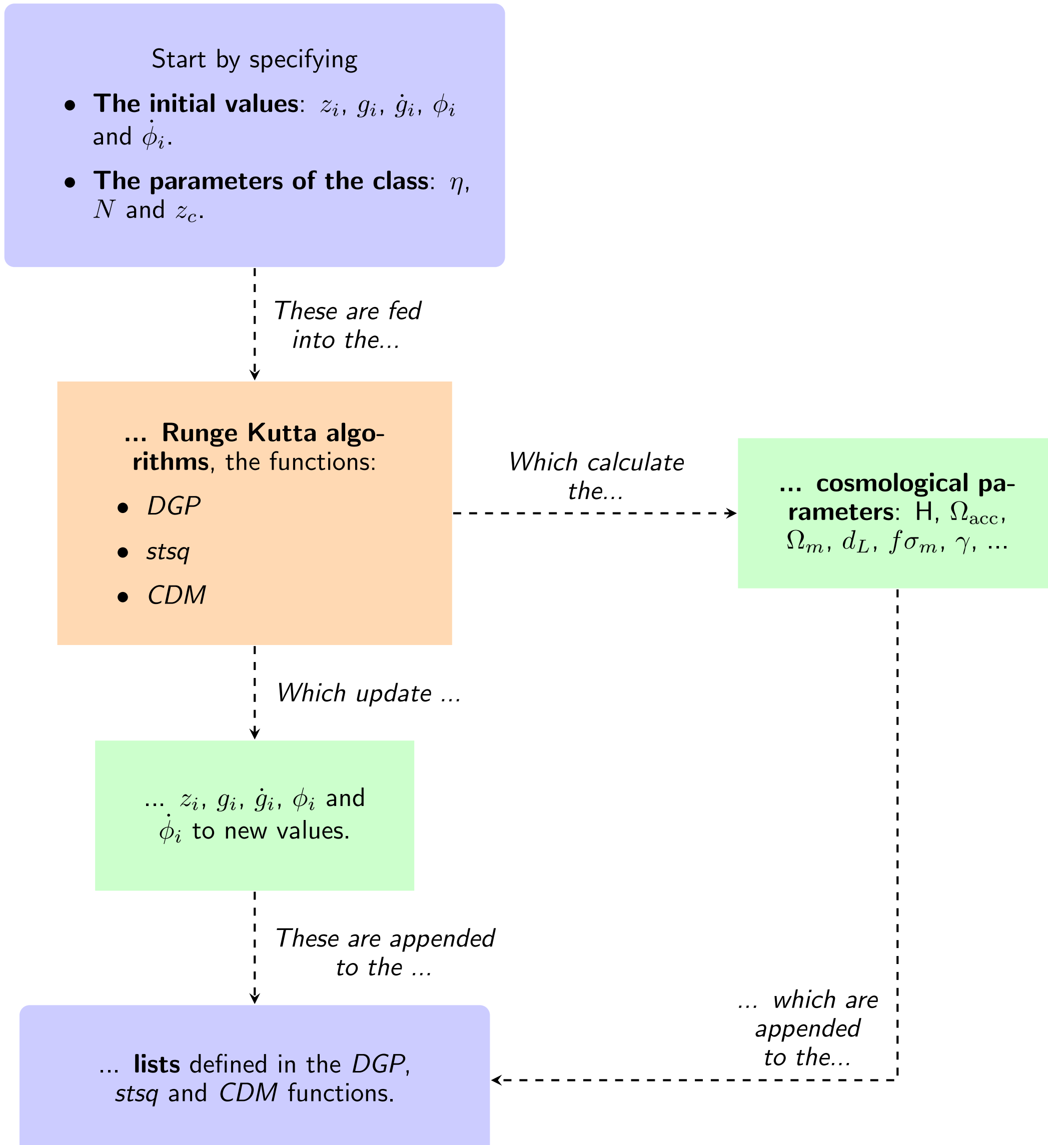

FIG. 4. A schematic representation of the first loop followed by the **model** class in the **main_class.py** file.

see in Appendix D 1. The redshift evolution of the growth factor G could be found by turning $G(z)$ in eq.(28) into the variable $g(z)$ used by Linder in [3]: $g(z) \sim G(z)/a$. The advantage of this choice of notation is that $g(z)$ is almost fixed at matter domination [3], so that the initial value of $\dot{g}$, $\dot{g}_i$, can be set to 0. g_i, the initial value of g, can be arbitrarily set to 1 as what we are going to calculate is its fractional deviation anyway. The equation of motion for g, solved using the initial conditions just mentioned, can be found in Appendix D 2 for the STSQ, the ΛCDM and the DGP models. Indeed, the fixed choice of z_i changes for the EDE discussion, as in that case we investigate the effect of early dark energy given different boundaries of the scaling era of the STSQ model: z_i and z_c.

Let us start from the analysis of d_L. Again, what we intend to find is the redshift evolution of the quantity Δd_L defined in paragraph IV A. By plotting the fractional deviation for different values of η, $\eta = 0.1$, $\eta = 0.5$ and $\eta = 0.9$ we obtained fig.(5), where $z_c = 10.55$. In all of the three cases we see that, in the redshift interval $0.2 < z < 1.7$, WFIRST's fractional deviation ($\Delta d_L = 0.0018$) is always overtaken, respectively at $z = 0.57$, $z = 0.36$ and $z = 0.33$. This means that

WFIRST will be able to measure the difference between the STSQ and the ΛCDM models at a 1σ confidence level (at least for $z > 0.57$). The higher is η, the easier it is to measure the difference between the two models.
The same operation was done for a fixed value of η, $\eta = 0.9$. By varying z_c through the values of 12, 15 and 17 it was found that, in the redshift range $0.2 < z < 1.7$, it is again possible to distinguish the two dark energy models. For the three critical redshifts mentioned, WFIRST's Δd_L is overcome at $z = 0.56$, $z = 1.04$ and $z = 1.10$ respectively; this means that STSQ can be identified in the redshift range considered, at least for $z > 1.10$ as one can see in fig.(6). For a fixed η, it will be easier to find evidence for STSQ if z_c is low (but not less than 10.55 because of the EDE constraint), as the model becomes measurable at a low redshift: 0.56. Let us now

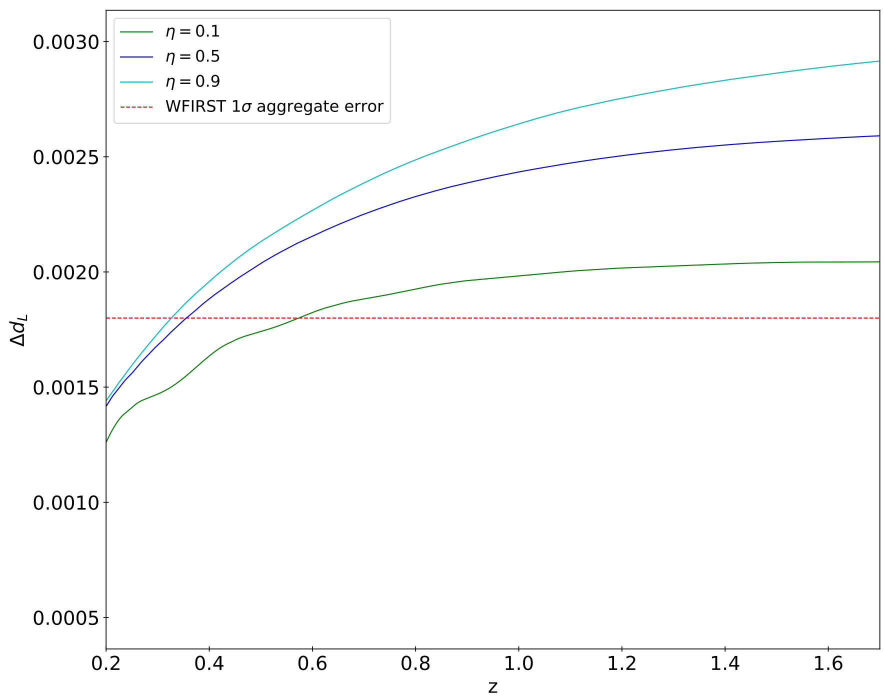

FIG. 5. Redshift evolution of the fractional deviation of d_L between STSQ and the ΛCDM model, at $z_c = 10.55$ (the minimum critical redshift set by the EDE constraints). WFIRST's aggregate error (or fractional accuracy) is $\Delta d_L = 0.0018$ for the range $0.2 < z < 1.7$.

look at the constraints on $f(z)\sigma_m(z)$, which was defined in paragraph IV B. Just as a reminder: G can be approximated with σ_m, which is why we will refer to $f(z)\sigma_m(z)$ rather than $f(z)G(z)$ while describing the results of the analysis on $f(z)\sigma_m(z)$. It was found that WFIRST's fractional accuracy for the $f(z)\sigma_m(z)$ combination, $\Delta f(z)\sigma_m(z) = 0.012$, can never be overcome in the range $1 < z < 2$. This is the case if we fix $z_c = 10.55$ and vary η through 0.1, 0.5, 0.9 or even if we fix $\eta = 0.9$ and vary z_c through 12, 15 and 17. By looking at figures (7) and (11, in Appendix D 3), it is then clear how WFIRST will not be able to recognise STSQ, at least not to a 1σ confidence limit in the redshift range $1 < z < 2$. This is significant, as it also sets a clear distinction between type Ia Supernovae and weak lensing as probes: the first ones seem to be decisive in the search for a model like STSQ, as the measurement of d_L by WFIRST is precise enough to distinguish it from a cosmological constant in a good part of the redshift range $0.2 < z < 1.7$. However, WFIRST is just not precise enough in the use weak lensing as a probe to identify the STSQ model. This result

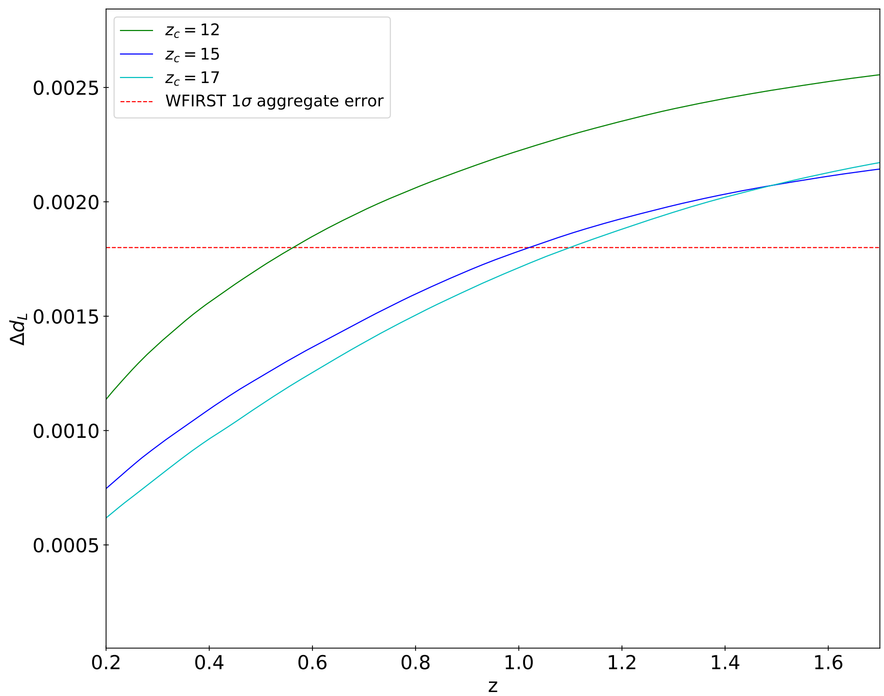

FIG. 6. Redshift evolution of the fractional deviation of d_L between STSQ and the ΛCDM model, for $\eta = 0.9$. WFIRST's aggregate error (or fractional accuracy) is $\Delta d_L = 0.0018$ for the range $0.2 < z < 1.7$.

on the use of weak lensing is confirmed by looking at the γ parameter measured by another future probe, EUCLID. γ still describes matter perturbation's growth, as shown in paragraph IV B. This parameter is taken to be a constant at later redshifts [1][3]: $\gamma = 0.55$. As we see from fig.(8) (and fig.(12) in Appendix D 3) this is in fact the behaviour of γ that we can expect from the STSQ and the ΛCDM models, as their curves tend to become linear after $z \sim 5$. In particular, after $z = 6.02$ all the different curves for the different η values of the STSQ model (0.1, 0.5, 0.9) enter the aggregate error span of EUCLID, $\Delta\gamma = 0.014$, meaning that they cannot be distinguished from ΛCDM dark energy. As the redshift range where the curves describing the STSQ model are within EUCLID's span in the measurement of γ does include WFIRST's $1 < z < 2$ range, we can conclude that the two future measurements might be consistent, as none would be able to distinguish the two models using weak lensing, at least for $1 < z < 2$.

Finally, we look at the constraints imposed by Early Dark Energy on the STSQ model. As seen in paragraph IV C, the constant dark energy fraction Ω_e that exists during the scaling epoch of scaling models of dark energy can influence the angular position θ_* of the first acoustic peak of the CMB. θ_* can shift with respect to the the value that Planck has measured fitting the CMB data with the ΛCDM model: if this shift is greater that the 2σ fractional deviation in the measurement, $\Delta\theta_* = 0.0006$, then the STSQ model cannot be valid. As we know, the shift can happen both when dark energy is a relevant component of the universe before recombination and after recombination. Here, we only consider the latter case. In Paragraph IV C, it was shown that the fractional accuracy of θ_*, $\Delta\theta_*$, can approximated to Δd_*, the fractional deviation of the physical distance to the surface of last scattering, as long as the shift is small. We simulated d_* using the STSQ model and we did so for different values of z_e (the redshift when the scaling starts), which is

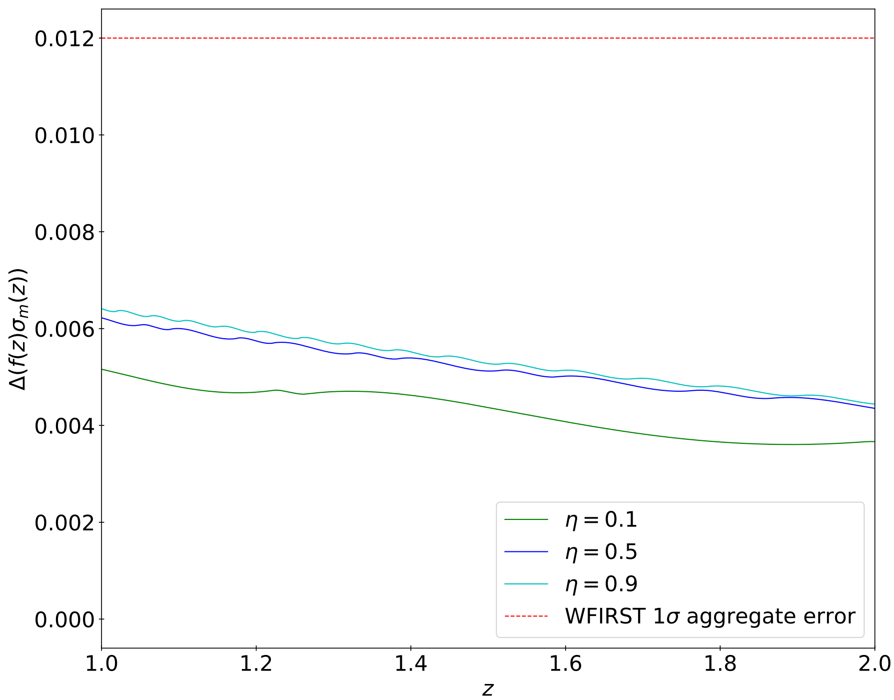

FIG. 7. Redshift evolution of the fractional deviation of $f(z)\sigma_m(z)$ between STSQ and the ΛCDM model; $z_c = 10.55$ and different values of η are considered. WFIRST's aggregate error (or fractional accuracy) is $\Delta f(z)\sigma_m(z) = 0.012$ for the range $1 < z < 2$.

always smaller than z_* as we only consider the case where dark energy matters after z_*. For each of z_* we calculated values of Δd_* for different values of η and z_c, which are summarised in tables (I,II). There seems to be a common trend: fixing η, if z_c is increased then Δd_* becomes smaller, and viceversa. Similarly, if z_c is fixed, Δd_* grows as η grows and viceversa. These are true for all of the tables, which means that a high value for η and a low one for z_c can potentially create a deviation that is beyond 0.0006. An explanation for the fact that high values of z_c are favoured,

	$\eta = 0.1$	$\eta = 0.5$	$\eta = 0.9$
$z_c = 15$	7.56×10^{-4}	1.21×10^{-3}	1.59×10^{-3}
$z_c = 20$	5.08×10^{-4}	1.05×10^{-3}	1.58×10^{-3}
$z_c = 40$	1.36×10^{-4}	3.11×10^{-4}	5.78×10^{-4}

	$\eta = 0.1$	$\eta = 0.5$	$\eta = 0.9$
$z_c = 20$	5.30×10^{-4}	1.00×10^{-3}	1.70×10^{-3}
$z_c = 50$	7.54×10^{-5}	2.10×10^{-4}	2.8×10^{-4}
$z_c = 80$	1.87×10^{-5}	5.96×10^{-5}	8.39×10^{-5}

TABLE I. The two tables summarize the values of $\Delta\theta_*$ obtained by simulating d_* from $z_{dec} = 1089$ to $z_0 = 0$. The left table has a fixed $z_e = 50$, the right one has $z_e = 100$.

as they contribute to smaller shift, can be found by considering that a higher z_c corresponds to a shorter range of redshifts where scaling takes place, so that the STSQ field reaches the state of a cosmological constant earlier in the history of the universe, which in turn leads to a greater expansion, closer to ΛCDM model (in general, the physical distances in the latter are greater than those predicted by the STSQ model at equal redshift). For example, look at the left table of the table pair (I), where $z_e = 100$: in the $\eta = 0.1$ column, $\Delta\theta_*$ reaches the minimum at $z_c = 80$, which is also what gives the minimum redshift range of scaling (between z_e and z_c), $\delta z = z_e - z_c = 20$. Another consideration can be made; take the right table of the table pair (I) along with the left

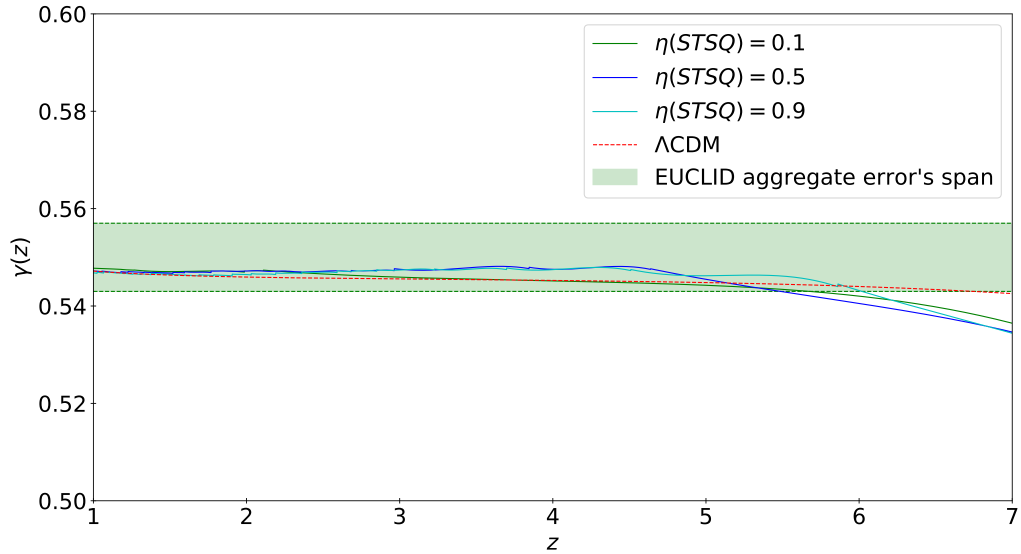

FIG. 8. Redshift evolution of the γ factor for STSQ and the ΛCDM model, at $z_c = 10.55$. EUCLID's error in the measurement of γ is expressed as a range (span) $\Delta\gamma = 0.014$.

	$\eta = 0.1$	$\eta = 0.5$	$\eta = 0.9$
$z_c = 50$	7.69×10^{-5}	2.10×10^{-4}	3.40×10^{-4}
$z_c = 100$	9.90×10^{-6}	2.88×10^{-5}	4.94×10^{-5}
$z_c = 250$	6.63×10^{-7}	2.08×10^{-6}	3.13×10^{-6}

	$\eta = 0.1$	$\eta = 0.5$	$\eta = 0.9$
$z_c = 100$	1.04×10^{-5}	3.01×10^{-5}	5.06×10^{-5}
$z_c = 300$	4.17×10^{-7}	1.39×10^{-6}	1.95×10^{-6}
$z_c = 750$	2.74×10^{-8}	6.72×10^{-8}	1.32×10^{-7}

TABLE II. The two tables summarize the values of $\Delta\theta_*$ obtained by simulating d_* from $z_{dec} = 1089$ to $z_0 = 0$. The left table has a fixed $z_e = 300$, the right one has $z_e = 800$.

and the right tables the table pair (II). Take the values of Δd_* for $\eta = 0.1$ for which the scaling redshift range (δz) is 50. These represent the same redshift duration of the scaling period but at different cosmological epochs, respectively starting from $z_e = 100$, 300 and 800. It is evident that, the earlier we get, the smaller is Δd_* going to be: they are 1.87×10^{-5}, 6.63×10^{-7} and 2.74×10^{-8} respectively. This then indicates the cosmological epoch where the scaling phase is placed reduces Δd_* just as the scaling's redshift duration does.

To summarize, there are two important things to consider when constraining the STSQ model: one is the redshift duration of the scaling, which we call δz here, and the other is the cosmological epoch in which the scaling takes place. It was found that if scaling happens at an early redshift and for a short δz range, Δd_* is minimized, therefore ensuring that the shift $\Delta\theta_*$ does not invalidate the STSQ model.

It is not possible however to give a conclusive answer for the ranges of η, z_c and z_e in which the STSQ model is valid: this required a careful analysis of the parameter space where η, z_c and z_e are coordinates, which is not done here. A future continuation of this analysis might entail a simulation that calculates Δd_* for different values of these three paramaters, allowing to find the volumes in that parameter space where Δd_* and therefore $\Delta\theta_*$ are lower than 0.0006, constraining the validity of the STSQ model.

VI. MODIFIED GRAVITY: THE DGP MODEL

Dark energy is not the only type of physical theory that can be used to explain the accelerating nature of the universe's expansion. There is another class of theories which, differently from dark energy, do not rely on general relativity: this class is called *Modified Gravity.* These theories are modifications of Einstein's theory of gravity and thus represent an alternative to it. The modified gravity that we will look at here is the *Dvali-Gabadadze-Porrati (DGP) Model.* Coherently with the general idea of modified gravity, the DGP model also modifies gravity at large scales, also explaining why the effects of gravitation appear to be so hard to notice at very low redshifts [27]. The DGP model is based on the idea that gravity emerges from a Minkowski space that is a 5-dimensional one rather than the 4D one which we are used to think of in general relativity; at small scales, the 4D behaviour is recovered [27]. The consequences of this model can be translated into cosmology in a way that makes it easily comparable to dark energy: this is because it changes the density equation, the Friedman equation and the matter density perturbation equation in such a way that they appear to be just slight modifications of the equations that we have already defined for dark energy, eqs.(2,27). If we call the new Friedman parameter H_{DGP} and the new matter density perturbation δ_m we can rewrite the equations that dictate their dynamics as [28] (assuming both the cosmological principle and the spatial flatness of the universe):

$$H_{DGP}^2 - \frac{H_{DGP}}{r_c} = \frac{\rho_m}{3M^2}, \tag{37}$$

and,

$$\ddot{\delta}_m + 2H_{DGP}\,\dot{\delta}_m - \frac{\rho_m}{2M^2}\left(1 + \frac{1}{3\beta}\right)\delta_m = 0. \tag{38}$$

The Friedman equation for the DGP model is basically a quadratic and, by solving it for H_{DGP}, we find:

$$H_{DGP} = \frac{1}{2}\left[H_0(1-\Omega_{m,0}) + \sqrt{H_0^2(1-\Omega_{m,0})^2 + \frac{4\rho_m}{3M^2}}\right], \tag{39}$$

where H_0 and $\Omega_{m,0}$ are the current values for the Hubble parameter and the density parameter of matter. Here, β is just a contraction of the following expression:

$$\beta = 1 - 2r_c\,H_{DGP}\left(1 + \frac{\dot{H}_{DGP}}{3H_{DGP}^2}\right). \tag{40}$$

r_c is an important parameter for the DGP model: it is called *crossover scale* and, interestingly, we can recover the FLRW equations for H and δ_m, eqs.(2,27), if $r_c \to \infty$. The crossover scale describes the scale at which the correction to gravity imposed by the DGP model starts becoming relevant [29], so that setting it to infinity means indeed saying that the DGP scale is never reached, so that the universe is wholly described the equations of FLRW cosmology. r_c is, in general, finite. We can find its expression by considering the H_{DGP}/r_c term in eq.(37) as an effective accelerating density $\rho_{\rm acc}$ described by $\rho_{\rm acc} = 3M^2H_{DGP}/r_c$ (similarly to how curvature can be treated as an 'extra component' in non-flat universes). From this, it can be shown that r_c has this shape [29]:

$$r_c = \frac{1}{H_0(1-\Omega_{m,0})}, \tag{41}$$

These are all the equations that are going to be needed to analyse the DGP model. However, we are only going to simulate the redshift evolution of the γ growth parameter, as the DGP model is excluded by observations. In [29] it was found the single parameter describing this model, namely r_c, cannot be fitted uniquely when we use different observational probes, including the CMB and high-redshift Supernovae. Nonetheless, we can use the DGP model to show how the γ parameter can be vital to distinguish modified gravity models from dark energy.

VII. THE GROWTH OF MATTER DENSITY PERTURBATIONS IN THE DGP MODEL

The Python program used to simulate γ for the DGP model is the same one described in paragraph V A. The function containing the Runge-Kutta for it is called *DGP* and it is built to solve the equation for matter density perturbation's time evolution. To do so, eq.(38) was transformed so that the variable used was g rather δ_m, which again factors a out; here, we also assume that the initial value of $\dot{g}_i$ is 0 and g_i is again taken to be 1, an arbitrary value (refer to Appendix D 2 for the density perturbation equation derived in terms on g for the DGP model). As mentioned in the previous paragraph, here we only simulate γ because the DGP model is observationally excluded: simulating d_L or $f\sigma_m$ and compare the fractional deviations of these observables with their values generated by the ΛCDM model would not be meaningful because we already know from [29] that telescopes like WFIRST, EUCLID or Planck would not observe a model like DGP gravity. However, purely as a demonstration, looking at the evolution of γ gives a good idea of how this parameter can be of primary importance to distinguish dark energy from modified gravity. In fig.(9) we see that the curve for γ in the case of the DGP model is remarkably

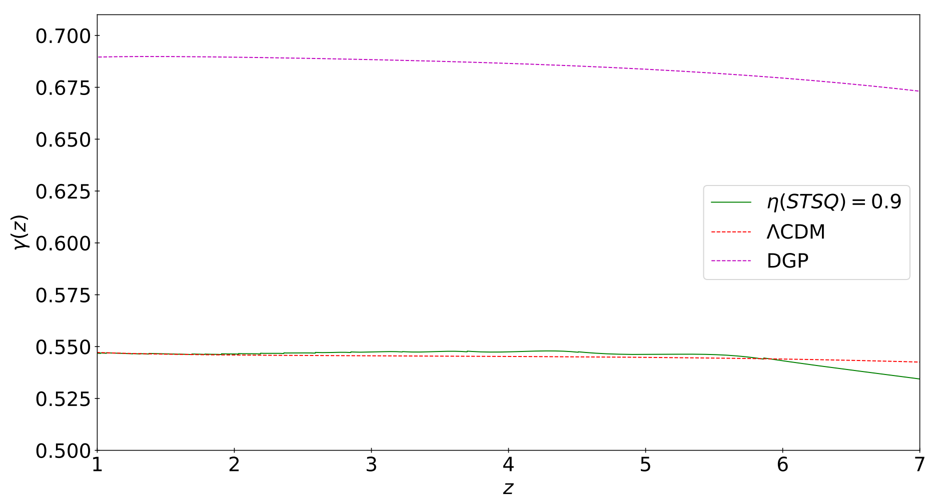

FIG. 9. Redshift evolution of the γ factor for STSQ, DGP gravity and the ΛCDM model, at $z_c = 17$ and $\eta = 0.9$.

different from its evolution in the STSQ and ΛCDM models. We see the redshift range in which γ can be approximated as constant and its deviation from both the dark energy models is close to $\gamma_{DGP} - \gamma_{\Lambda CDM} \approx 0.13$, which means that, knowing that the current accepted value for γ is 0.55 [1], the fractional deviation of a modified gravity model like DGP gravity from ΛCDM is $\sim 24\%$. This strong deviation tends to be a characteristic feature of modified gravity models [3][30]. Therefore, the use of γ as a cosmological parameter to study the accelerated expansion of the universe through the lens of the growth of matter's density perturbations is shown here to be extremely promising. A future possible extension on this analysis could include an analysis on the fractional deviations on d_L and $f\sigma_m$ as predicted by another, observationally viable modified gravity model. For example, modified gravity models like $f(R)$ gravity, which is based on modifying the term containing the Ricci scalar R in the action of gravity (the same one that can be used to work out Einstein's field equations), is a model which, theoretically, should have a visible impact on the linear perturbations growth similarly the DGP model itself.

VIII. A FUTURE PROSPECT TO STUDY THE UNIVERSE'S EXPANSION HISTORY: GRAVITATIONAL WAVES AND THE LISA MISSION

A future extension to the comparison of the redshift evolution of the luminosity distance between the STSQ and the ΛCDM could include another type of probe: gravitational waves. Gravitational waves (GWs) can be thought of as an equivalent of electromagnetic waves in the gravitational context: somewhat similarly to how accelerated charges produce electromagnetic waves, moving massive bodies and systems that interact gravitationally, like stellar binaries, can produce 'ripples' in space-time called gravitational waves. These waves can be used as 'standard sirens', which is again a term indicating an analogy with electromagnetic standard candles: similarly to type Ia Supernovae, the objects generating gravitational waves such as 'compact mergers' (black hole-neutron star or black hole-black hole binaries) can be used as distance indicators. The luminosity distance

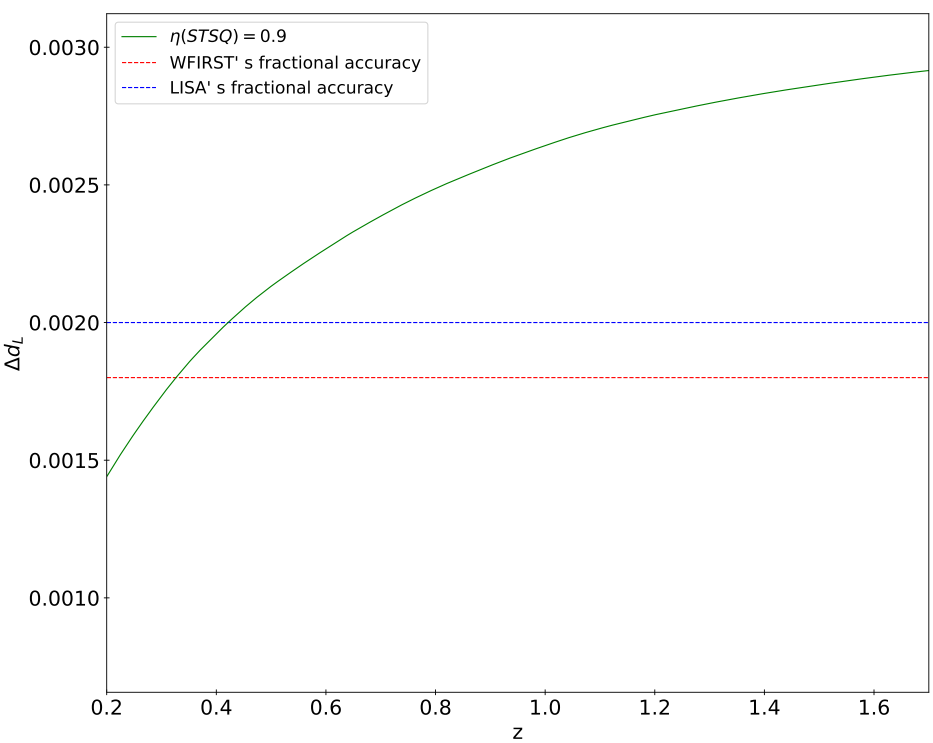

FIG. 10. Redshift evolution of STSQ's fractional deviation for the ΛCDM model for $z_c = 10.55$ and $\eta = 0.9$; the fractional uncertainty of LISA is added at $\Delta d_L = 0.002$.

is enclosed into the gravitational wave signal [31]: this is the time evolution $h(t)$ of the gravitational wave's 'strain - h', which is defined as $h \approx \delta L/L$, being L a length in space that is being stretched or compressed by a quantity δL by the gravitational wave. The future mission LISA, scheduled to be launched in space in 2034, will measure these waves for a multitude of purposes ranging from studying the number, formation and evolution of compact mergers in the Milky Way to measuring the masses of massive black holes at the centre of galaxies [32]. The latter in particular shows an important reason to appreciate the potential of gravitational waves: as they interact weakly with matter, they can probe domains that cannot be probed using electromagnetic radiation, like galactic centres. However, what matters here is the cosmological potential of GWs: for the luminosity distance measurements, the three probes sent into space for the LISA mission will act as a giant interferometer (each pair of probes is separated by 2.5×10^6 km) that will measure the tiny gravitational waves' strain ($\delta L/L \sim 10^{-23}$) of these gravitational signals. This will enable us to determine the distance of black hole binaries (BHBs) up to an accuracy of 0.2% per single event, at a redshift $z \sim 1$ [33]. This accuracy is comparable to the 1σ accuracy for the measurement of d_L of WFIRST (0.18%) at a redshift range $0.2 < z < 1.7$ (which includes LISA's $z \sim 1$) and therefore this represents a valid way to, for instance, try to identify a model like STSQ, as done in the study presented here. As shown in fig.(10), LISA might actually be able to identify the STSQ model as at $z \sim 1$ LISA's fractional difference has already been clearly overtaken. A future extension for this project would then be to consider gravitational waves for different redshift ranges, to verify whether the STSQ or other scaling models can be distinguished from a cosmological constant. Even though it is true that the fractional deviation mentioned above for GWs sources at $z \sim 1$ can worsen due to the effects of the lensing of GWs [33], it has to be stressed how GWs have no need for calibration that uses a distance ladder while electromagnetic sources do: this is a fundamental advantage that can drastically reduce systematic errors in the measurement of d_L.

Gravitational waves are indeed another probe worth analysing beyond the Supernovae and the galactic clustering observed by WFIRST and EUCLID or the CMB by Planck. There is yet another compelling reason to look at them: the distance ladder-independent measurement of d_L combined with the redshift of the same source is in fact what we need to work out H_0, the current Hubble constant value. Measuring gravitational waves in the local universe has the potential of dissipating the uncertainity created around the true value of H_0 by the so-called 'Hubble tension', which refers to the discrepancy in its measurement by different probes. While Planck measured $H_0 = 67.4$ km/(Mpc s) using the CMB [24], standard candles measured something quite different: from Cepheid Variables it was found in [34] that $H_0 = 73.2$ km/(Mpc s). This difference is considered as one of the main hints at a model for the universe that is different from the standard ΛCDM cosmology. An example of a gravitational wave event used to measure H_0 is given in [35]. In this work a value $H_0 = 75$ km/(Mpc s) was found, which, looking at the error in its measurement, is actually consistent with both the CMB and the standard candles measurement. One of the main objectives of the LISA mission is the measurement of this constant. With the gravitational events that it will detect in the future, along with other interferometers like VIRGO and LIGO, the measurement of H_0 will improve and this will probably put a final word on the Hubble tension: if both of the results obtained through electromagnetic means become inconsistent with the measurements of GWs, then this will be a clear hint that the ΛCDM cannot explain the observations and that, in the end, we need to look for a new cosmology.

IX. CONCLUSION

Through an introduction of Noether's first theorem, we connected scalar fields to cosmology and explained their use as dark energy. When a scalar field ϕ is put in the context of FLRW cosmology, where the cosmological principle and an overall the spatial flatness are both valid assumptions, and gets treated as a perfect fluid, we can obtain the equation of motion for a specific kind of cosmological scalar field: quintessence. Noether's theorem is then what allows us to find the expressions for its density and pressure, which are the essential part of the equation that describes the time evolution of the size of the universe: the Friedman equation, eq.(2).

Depending on the potential energy associated to the quintessence field, one can describe different dark energy models: the one that has been studied here is a scaling model called Sharp Transition Scaling Quintessence - STSQ. As many other dark energy theories, this model does not only aim at explaining the accelerated nature of the expansion of the universe, but it also offers a solution to a problem that arises if we assume that the universe is dominated by a dark energy component with a density and an equation of state that do not vary in time: the smallness of dark energy's density today, which occurs naturally thanks to the attractor nature of the STSQ model. However, STSQ still requires one of its parameters, the redshift at the end of scaling z_c, to be tuned in order to achieve acceleration only at late cosmological times. There exists a physical model that could explain the switch in dark energy's behaviour that brings to acceleration: the Growing Neutrino Quintessence, which was also briefly introduced.

The work presented here attempted to predict whether the STSQ model can be detected by two future space telescopes: WFIRST (also known as the Nancy Grace Roman Space Telescope) and EUCLID. In particular, we described and used three observables that can be used to make this prediction: d_L, which is the luminosity distance of type Ia SNe, the growth factor combination $f\sigma_m$, which describes the growth of matter density perturbations in the history of the universe and the γ growth factor, which also parametrises this growth. By simulating the redshift evolution of these observables for both the ΛCDM and the STSQ models, it was possible to find their fractional deviation and to compare it with the accuracy with which WFIRST and EUCLID are expected to perform. It was found that, while there is the possibility for the STSQ model to be identified within a redshift range $0.2 < z < 1.7$ using the d_L measurements by WFIRST, it is not possible to do so for neither WFIRST's $f\sigma_m$ (at least in the range $1 < z < 2$) nor EUCLID's γ measurements.

The constraints that the Cosmic Microwave Background (CMB) puts on the amount of dark energy that a scaling model can have at early times (just after recombination), as observed by the Planck telescope, was used to constrain the value of z_c for the STSQ model, as it affects the angular position of CMB's first acoustic peak by shifting its value. While it appeared that both z_c and the redshift duration of the scaling impacted on the dimension on this shift, no conclusion could be made on the range that z_c can take, as the shift also depends on η, another parameter of the STSQ model. This can however open the possibility for an analysis that is more focused on the whole parameter space of the STSQ model, which includes η, z_c and the redshift when the scaling starts: z_i, also called z_e.

Dark energy is not the only way to explain the accelerated expansion of the universe. An alternative to dark energy is represented by 'modified gravity' models: these are acceleration theories that try to explain how gravity is weak at small scales while also being the only fundamental force that really matters at the largest scale, without introducing an exotic component like dark energy. Here we introduce one of these modifications of general relativity: the DGP model. Since this model is known to be observationally inconsistent, we present it in a purely demonstrative way: we have simulated the redshift evolution of γ for it to show how, differently from dark energy models, we can use this parameter to potentially identify a modified gravity model like the DGP model, as γ evolves in a remarkably different way in this case.

A final suggestion on a possible continuation of the analysis carried out here is also made, by providing the very basics of the application of Gravitational Waves to cosmology and the future prospects relying on the solution of the current tensions in the ΛCDM model. The attention is focused on the discrepancy in the measurement of H_0, the current value of the Hubble constant.

Gravitational waves are indeed very promising. The future of cosmology fully relies both on them and on the so-called era of 'multimessenger astronomy' that awaits us in the shape of probes such as WFIRST, EUCLID, LIGO, VIRGO, LISA and many others. The combined effort of both gravitational and electromagnetic signals is going to give us data with an unprecedented precision and physical insight: the work presented here is nothing but a brief glimpse into the potential of new discoveries hidden behind those probes.

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Appendix A: ADDITIONS TO SECTION II

1. Einstein's Field Equations

The Einstein's field equations is a system of equations that can be written in the following compact form:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G\, T_{\mu\nu}, \tag{A1}$$

where $G_{\mu\nu}$ is the *Einstein tensor*, Λ is the *cosmological constant*, $g_{\mu\nu}$ is the *metric tensor*, $T_{\mu\nu}$ is the *energy-momentum tensor* and G is the *universal constant of gravitation.*

2. The Continuity Equation in General Relativity

The energy conservation equation in GR is:

$$\nabla_\mu T^\mu{}_\nu = 0, \tag{A2}$$

where ∇_μ is the *covariant derivative.*

Appendix B: ADDITIONS TO SECTION III

1. Derivation of STSQ's Critical Field Value

ϕ_c is found by using this STSQ potential energy:

$$V(\phi_c) = M^4 \exp\left(-\frac{\lambda\phi_c}{M}\right).$$

By taking the natural logarithm of both sides we get:

$$\ln\left(\frac{V(\phi_c)}{M^4}\right) = -\frac{\lambda\phi_c}{M}.$$

Rearranging this expression and demanding that $V(\phi_c) = \rho_{de,0}$, ϕ_c is found to be:

$$\phi_c = \frac{M}{\lambda}\ln\left(\frac{M^4}{\rho_{de,0}}\right) \tag{B1}$$

Appendix C: ADDITIONS TO SECTION IV

1. The Fractional Deviations used for the EDE Constraints

The fractional deviations $\Delta\theta_*$ and Δd_* can be shown to be equivalent. Start from:

$$\theta_{*,\mathrm{STSQ}} = \frac{s_{*,\mathrm{STSQ}}}{d_{*,\mathrm{STSQ}}}.$$

Rewrite $d_{*,\mathrm{STSQ}}$ as $d_{*,\Lambda\mathrm{CDM}} - (d_{*,\Lambda\mathrm{CDM}} - d_{*,\mathrm{STSQ}})$, so that:

$$\theta_{*,\mathrm{STSQ}} = \frac{s_{*,\mathrm{STSQ}}}{d_{*,\Lambda\mathrm{CDM}}}\,\frac{1}{1 - \frac{d_{*,\Lambda\mathrm{CDM}} - d_{*,\mathrm{STSQ}}}{d_{*,\Lambda\mathrm{CDM}}}}.$$

The first term on the right-hand side is approximately $\theta_{*,\Lambda\mathrm{CDM}}$, while the second one can be approximated using the binomial expansion:

$$\theta_{*,\mathrm{STSQ}} \approx \theta_{*,\Lambda\mathrm{CDM}} \left(1 + \frac{d_{*,\Lambda\mathrm{CDM}} - d_{*,\mathrm{STSQ}}}{d_{*,\Lambda\mathrm{CDM}}}\right).$$

By rearranging it we get:

$$\frac{\theta_{*,\mathrm{STSQ}}}{\theta_{*,\Lambda\mathrm{CDM}}} - 1 \equiv \frac{\theta_{*,\mathrm{STSQ}} - \theta_{*,\Lambda\mathrm{CDM}}}{\theta_{*,\Lambda\mathrm{CDM}}} \approx \frac{d_{*,\Lambda\mathrm{CDM}} - d_{*,\mathrm{STSQ}}}{d_{*,\Lambda\mathrm{CDM}}}.$$

By rewriting the left-hand side and the right-hand side of '$\approx$' with the usual notation for the fractional deviations, $\Delta\theta_*$ and Δd_* for θ_* and d_* respectively, we get:

$$\Delta\theta_* \approx \Delta d_*. \tag{C1}$$

Appendix D: ADDITIONS TO SECTION V

1. The Initial Conditions for STSQ

From [14], the initial values of ϕ and $\dot{\phi}$ for STSQ, ϕ_i and $\dot{\phi}_i$, can be shown to be (for a universe with dark energy and pressureless matter as components only):

$$\phi_i = \frac{M}{\lambda} \ln \left[\left(\frac{a_i}{a_0}\right)^3 \frac{2M^4(\lambda^2/3 - 1)}{\rho_{m,0}} \right], \tag{D1}$$

where a_i is the initial (arbitrary) value for a and $\rho_{m,0}$ is the current energy density of matter. Also:

$$\dot{\phi}_i = \sqrt{2M^4 \exp\left(\frac{-\lambda\phi_i}{M}\right)} \tag{D2}$$

2. The Density Perturbation Equations

Following the formalism of Linder for the evolution of density perturbations [3], one can use $g \sim \delta/a$, where g is a new way to express the linear density perturbation δ by factoring a out. The density perturbation equation is then, for the ΛCDM model, found to be:

$$\ddot{g} + 4H_{\Lambda\mathrm{CDM}}\,\dot{g} + \frac{\rho_{\Lambda\mathrm{CDM}}}{M^2}\,g = 0, \tag{D3}$$

where $\rho_{\Lambda\mathrm{CDM}}$ is the density of ΛCDM dark energy and $H_{\Lambda\mathrm{CDM}}$ is the relative Hubble parameter. For the STSQ model, we obtain:

$$\ddot{g} + 4H_\phi\,\dot{g} + \frac{\rho_\phi}{2M^2}\,(1 - w_\phi)\,g = 0, \tag{D4}$$

where ρ_ϕ is STSQ's energy density and H_ϕ is the relative Hubble parameter.
Finally, for the DGP model, we find:

$$\ddot{g} + 4H_{\mathrm{DGP}}\,\dot{g} + \left(\frac{\ddot{a}}{a} + 2H^2_{\mathrm{DGP}} - \frac{\rho_m}{2M^2}\left(1 + \frac{1}{3\beta}\right)\right) g = 0, \tag{D5}$$

where $\ddot{a}/a$ is:

$$\frac{\ddot{a}}{a} = H^2_{\mathrm{DGP}} + aH_{\mathrm{DGP}}H'_{\mathrm{DGP}}.$$

H_{DGP} is the Hubble parameter in the DGP model and the 'prime' index is a derivative with respect to a.

3. Additional Figures for the Dark Energy Results

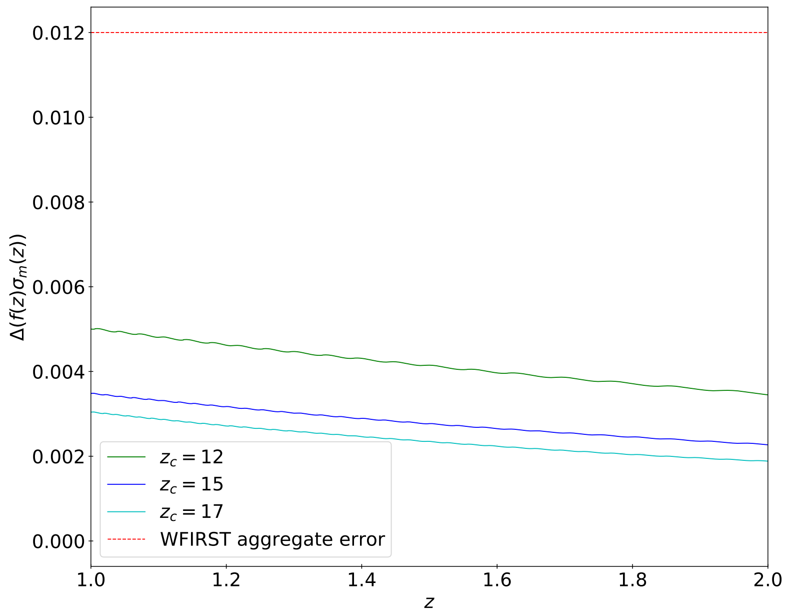

FIG. 11. Redshift evolution of the fractional deviation between STSQ and the ΛCDM model for $f(z)\sigma_m(z)$, for $\eta = 0.9$ and different values of z_c. WFIRST's aggregate error (or fractional accuracy) is $\Delta f(z)\sigma_m(z) = 0.012$ for the range $1 < z < 2$.

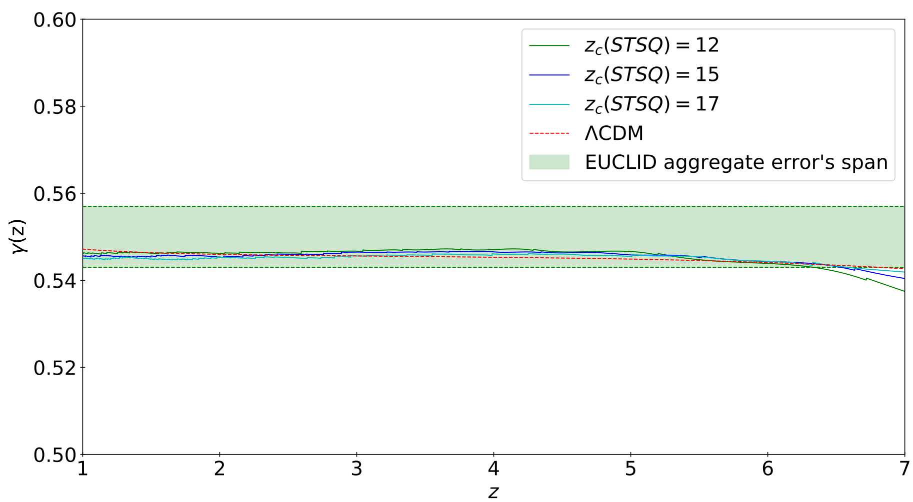

FIG. 12. Redshift evolution of the γ factor for STSQ and the ΛCDM model, for $\eta = 0.9$. EUCLID's error in the measurement of γ is expressed as a range (span) $\Delta\gamma = 0.014$.