Quantum Computing Assignment 5

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Problem E1. Pick one of the 4 Bell states $|\beta_{ij}\rangle, i, j = 0, 1$ (given on p. 75 of the textbook), and show that it is entangled.

Solution.
Most generally, to show that a 2-Qubit state $|\psi\rangle$ is entangled it must be shown that there are no values $(\alpha_0, \alpha_1, \beta_0, \beta_1) \in \mathbb{R}^4$ such that $|\psi\rangle = (\alpha_0|0\rangle + \alpha_1|1\rangle)(\beta_0|0\rangle + \beta_1|1\rangle)$. Using the Bell state $|\beta_{00}\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$, we can see that this places the constraints that

$$\frac{1}{\sqrt{2}}|00\rangle = \alpha_0\beta_0|00\rangle \to \alpha_0 = \frac{1}{\beta_0\sqrt{2}}$$

and

$$\frac{1}{\sqrt{2}}|11\rangle = \alpha_1\beta_1|11\rangle \to \alpha_1 = \frac{1}{\beta_1\sqrt{2}}$$

However, since the Bell state has no $|01\rangle$ or $|10\rangle$ components, we also have the constraints:

$$0|10\rangle = \alpha_1\beta_0|10\rangle \to 0 = \alpha_1\beta_0$$

and

$$0|01\rangle = \alpha_0\beta_1|01\rangle \to 0 = \alpha_0\beta_1$$

This means that at least two of the values $(\alpha_0, \alpha_1, \beta_0, \beta_1) \in \mathbb{R}^4$ must be 0. However, according to the earlier constraints, setting any one of these values to 0 requires that some other value approach ∞, which - strictly speaking - causes the group to lie outside $\mathbb{R}^4$. Therefore, $|\beta_{00}\rangle$ cannot be written as the tensor product of two constituent states, which means that it is an entangled state. □

Problem E2. Orthogonality on the Bloch sphere. Consider two points on the Bloch sphere: $|\psi\rangle$ with coordinates (θ, ϕ) and $|\chi\rangle$ with coordinates $(\pi - \theta, \phi + \pi)$. Show that $|\chi\rangle$ and $|\psi\rangle$ are orthogonal, i.e., $\langle\chi|\psi\rangle = 0$.

Solution.
States on the Bloch sphere with some coordinates (in spherical coordinates with $r = 1$) (α, β) are represented in the form

$$\cos\left(\frac{\alpha}{2}\right)|0\rangle + e^{i\beta}\sin\left(\frac{\alpha}{2}\right)|1\rangle.$$

To prove orthogonality, we can convert each state to this form and directly take the inner product ($\langle\gamma|\sigma\rangle = (|\gamma|^*)|\sigma|$)

$$\langle\chi|\psi\rangle = \chi^*\psi = \left(\cos\left[\frac{\theta}{2}\right]|0\rangle + e^{i\phi}\sin\left[\frac{\theta}{2}\right]|1\rangle\right)^* \left(\cos\left[\frac{\pi-\theta}{2}\right]|0\rangle + e^{i(\phi+\pi)}\sin\left[\frac{\pi-\theta}{2}\right]|1\rangle\right)$$

$$= \left(\cos\left[\frac{\theta}{2}\right]\langle 0| + e^{-i\phi}\sin\left[\frac{\theta}{2}\right]\langle 1|\right) \left(\cos\left[\frac{\pi-\theta}{2}\right]|0\rangle + e^{i(\phi+\pi)}\sin\left[\frac{\pi-\theta}{2}\right]|1\rangle\right)$$

Because the states $|0\rangle$ and $|1\rangle$ are defined to be orthonormal, we know that $\langle i|j\rangle = \delta_{ij}$ and so this product can be simplified:

$$\langle\chi|\psi\rangle = \cos\left[\frac{\theta}{2}\right]\cos\left[\frac{\pi-\theta}{2}\right] + e^{i(\phi+\pi-\phi)}\sin\left[\frac{\theta}{2}\right]\sin\left[\frac{\pi-\theta}{2}\right]$$

Now employ some trigonometric identities, including the double-angle formula:

$$\cos\left[\frac{\pi-x}{2}\right] = \sin\left[\frac{x}{2}\right] \rightarrow \langle\chi|\psi\rangle = \cos\left[\frac{\theta}{2}\right]\sin\left[\frac{\theta}{2}\right] + e^{i\pi}\sin\left[\frac{\theta}{2}\right]\sin\left[\frac{\pi-\theta}{2}\right]$$

$$\frac{\sin[2x]}{2} = \sin[x]\cos[x] \rightarrow \langle\chi|\psi\rangle = \frac{\sin[\theta]}{2} + e^{i\pi}\sin\left[\frac{\theta}{2}\right]\sin\left[\frac{\pi-\theta}{2}\right]$$

$$e^{i\pi} = -1 \rightarrow \langle\chi|\psi\rangle = \frac{\sin[\theta]}{2} - \sin\left[\frac{\theta}{2}\right]\sin\left[\frac{\pi-\theta}{2}\right]$$

$$\sin\left[\frac{\pi-x}{2}\right] = \cos\left[\frac{x}{2}\right] \rightarrow \langle\chi|\psi\rangle = \frac{\sin[\theta]}{2} - \sin\left[\frac{\theta}{2}\right]\cos\left[\frac{\theta}{2}\right]$$

$$\frac{\sin[2x]}{2} = \sin[x]\cos[x] \rightarrow \langle\chi|\psi\rangle = \frac{\sin[\theta]}{2} - \frac{\sin[\theta]}{2} = 0$$

and therefore $|\chi\rangle$ and $|\psi\rangle$ are orthogonal. $\square$