Quantum Computing Assignment 3

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Problem 3.2.1. Show that

$$|\psi(t_2)\rangle = e^{-iH(t_2-t_1)/\hbar}|\psi(t_1)\rangle$$

is a solution of the time-dependent Schrödinger Equation.

Solution.
The time-dependent Schrödinger Equation in some Hilbert space $\mathcal{H}$ for some system described by the wave-function $|\Psi\rangle \in \mathcal{H}$ is given by:

$$i\hbar\frac{\delta}{\delta t}|\Psi\rangle = H|\Psi\rangle$$

where H is the Hamiltonian Operator of the Hilbert space in question. If we choose t_1 as a constant while allowing t_2 to vary, and measure time as $t = t_2 - t_1$ (and also ignore relativistic effects that could cause weird, idiosyncratic notions of time-like basis vectors), we can find the time derivative of $|\psi(t_2)\rangle$ easily enough by inspection.

$$\frac{\delta}{\delta t}|\psi(t_2)\rangle = \frac{-i}{\hbar}He^{-iH(t_2-t_1)/\hbar}|\psi(t_1)\rangle$$

since if t_1 is constant, $\delta|\psi(t_1)\rangle/\delta t = 0$. Now simply note that $i\hbar = (-i/\hbar)^{-1}$, and so therefore by trivial algebraic substitution:

$$i\hbar\frac{\delta}{\delta t}|\psi(t_2)\rangle = H|\psi(t_2)\rangle$$

which makes $|\psi(t_2)\rangle$ a solution of the time-dependent Schrödinger Equation. $\square$

Problem 3.4.1(a). Prove that if the operators P_i satisfy $P_i^\dagger = P_i$ and $P_i^2 = P_i$, then $P_iP_j = 0$ for all $i \neq j$.

Solution.
Any Hermitian projection operator P in Hilbert space $\mathcal{H}$ can be written in the form

$$P = |\psi_n\rangle\langle\psi_n|$$

for some $n \in \dim\mathcal{H}$ where $|\psi_n\rangle$ are orthonormal, and in particular part of an orthonormal basis for $\mathcal{H}$. Then for the operators P_iP_j, it follows that

$$P_iP_j = |\psi_i\rangle\langle\psi_i|\psi_j\rangle\langle\psi_j| = |\psi_i\rangle\delta_{ij}\langle\psi_j|$$

because of the orthonormality restraint on each $|\psi_n\rangle$. Therefore, to obtain non-zero results from operation of the P_iP_j operator, it is necessary that $i = j$. $\square$

Problem 3.4.3. Verify that a measurement of the Pauli observable X is equivalent to a complete measurement with respect to the basis $\left\{\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right\}$

Solution.
The eigenvalues of X are 1 and -1, corresponding to the normalized eigenvectors $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle$, respectively. Thus a measurement of it is equivalent to a complete measurement in the basis spanned by $\frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$ ☐

Problem 3.5.1. Find the density matrices of the following states

(a) $\left\{(|0\rangle, \frac{1}{2}), (|1\rangle, \frac{1}{2})\right\}$

(b) $\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

(c) $\left\{(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle, \frac{1}{2}), (\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle, \frac{1}{2})\right\}$

Solution.

(a)
$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

(b)
$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

(c)
$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

☐

Problem C1. Given is the state vector $|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$, $\alpha_0, \alpha_1 \in \mathbb{C}$. Find the (θ, ψ) coordinates of this state on the Bloch sphere.

Solution.
Most generally, any state vector $|\Psi\rangle$ on the Bloch sphere can be written in the form

$$|\Psi\rangle = \cos\left[\frac{\theta}{2}\right]|0\rangle + \sin\left[\frac{\theta}{2}\right]e^{i\psi}|1\rangle$$

For the given state vector, θ is easily found as $\theta = 2\arccos[\alpha_0]$. The phase factor on the 1 state is a little uglier. We know the form, and we have an expression for θ, so we can directly see that

$$\alpha_1 = \sin\left[\frac{\theta}{2}\right]e^{i\psi} = \sin\left[\frac{2\arccos[\alpha_0]}{2}\right]e^{i\psi}$$

$$= \sqrt{1-\alpha^2}e^{i\psi} \rightarrow \frac{\alpha_1}{\sqrt{1-\alpha_0^2}} = e^{i\psi} \rightarrow \psi = -i\ln\left[\frac{\alpha_1}{\sqrt{1-\alpha_0^2}}\right]$$

This most likely indicates that the lack of an explicit phase term means that $\psi = 0$. So we have the coordinates

$$(2\arccos[\alpha_0], 0)$$

□

Problem C2. Let $X = \sigma_x$, $Y = \sigma_y$, and $Z = \sigma_z$ denote the usual Pauli spin matrices. Show that $[Y, Z] = 2iX$ and $[Z, X] = 2iY$. Recall that the commutator of two operators A and B is given by $[A, B] = AB - BA$.

Solution.

- $[Y, Z] = YZ - ZY$

$$\begin{aligned} &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 2i \\ 2i & 0 \end{bmatrix} = 2i\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 2iX \end{aligned}$$

- $[Z, X] = ZX - XZ$

$$\begin{aligned} &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} = 2i\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = 2iY \end{aligned}$$

□

Problem C3. Show that any 2×2 matrix A can be represented as a linear combination of the Pauli spin matrices X, Y, and Z, and the (2×2) identity matrix I.

Solution.
The most straightforward proof (though perhaps not simplest) is to analyze entry by entry. Let the matrix A be represented by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Now we must ask if each entry can be formed by linear combinations of the entries of the Pauli matrices and the 2×2 Identity.

- a
  For any complex number a, is it true that it can be decomposed such that $a = \alpha + \beta$. Since the complex plane's dimension is matched by the free variables here, this is trivially true.

- b
  For any complex number b, is it true that it can be decomposed such that $b = \alpha - i\beta$. Once again, it is trivial that two free variables will span $\mathbb{C}$

- c
  For any complex number c, is it true that it can be decomposed such that $c = \alpha + i\beta$. Trivially equivalent to b.

- d
  For any complex number d, is it true that it can be decomposed such that $d = \alpha - \beta$. Still two free variables.

So trivially any 2×2 matrix can be composed of linear combinations of the Pauli matrices and the 2×2 Identity. □

Problem C4. Consider a composite system consisting of two qubits. Find the Schmidt decomposition of the states

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \text{ and } \frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle).$$

Solution.

- $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$
  Let the Schimdt bases be
  $$|\Psi_0^A\rangle = |0\rangle \quad |\Psi_1^A\rangle = |1\rangle$$
  $$|\Psi_0^B\rangle = |0\rangle \quad |\Psi_1^B\rangle = |1\rangle$$
  Then this state can be written as
  $$\frac{1}{\sqrt{2}}|\Psi_0^A\rangle \otimes |\Psi_0^B\rangle + \frac{1}{\sqrt{2}}|\Psi_1^A\rangle \otimes |\Psi_1^B\rangle$$
  with $p_0 = p_1 = 1/2$.

- $\frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle)$
  This one is much less trivial. First construct a naiëve matrix form M of the state such that M_{ij} is the sum of the coefficients of state $|ij\rangle$:
  $$M = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Now we can decompose this using the matrix's eigenvectors as:

$$\frac{1}{2}(1+\sqrt{5})|\Psi_0^A\rangle \otimes |\Psi_0^B\rangle + \frac{1}{2}(1-\sqrt{5})|\Psi_1^A\rangle \otimes |\Psi_1^B\rangle$$

where the Schmidt bases are

$$|\Psi_0^A\rangle = \frac{1}{2}(1+\sqrt{5})|0\rangle \quad |\Psi_1^A\rangle = \frac{1}{2}(1-\sqrt{5})|0\rangle$$

$$|\Psi_0^B\rangle = |0\rangle \quad |\Psi_1^B\rangle = |1\rangle$$

and $p_0 = p_1 = \frac{1}{2}$

$\square$