Quantum Computing Assignment 4

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February 13, 2017

Problem 3.4.1.

(b) Prove that any pure state $|\psi\rangle$ can be decomposed as $|\psi\rangle = \sum_i \alpha_i |\psi_i\rangle$ where $\alpha_i = \sqrt{p(i)}, p(i) = \langle\psi|P_i|\psi\rangle$, and $|\psi_i\rangle = \frac{P_i|\psi\rangle}{\sqrt{p(i)}}$

(c) Also prove that $\langle\psi_i|\psi_j\rangle = \delta_{i,j}$.

Solution.

(b) In order for some pure state $|\psi\rangle$ to be in the same Hilbert space as the system it's attempting to describe, it must be the case that it can be formed from linear combinations of any valid basis vectors for that space. It must also be true that the probability of finding the system in any one of these basis states at any point in space-time must total to 100% or the system is not fully described and thus the basis is invalid. Therefore, for any decomposition of the Identity into $\sum_i P_i$ projection operators, the sum of expectation values must satisfy

$$\sum_i [p[i]] = \sum_i [\langle\psi|P_i|\psi\rangle] = 1$$

which is trivially satisfied if each constituent $|\psi_i\rangle$ that make up $|\psi\rangle$ takes the form $\frac{P_i|\psi\rangle}{\sqrt{p[i]}}$. In that case, the sum of expectation values can be reduced to:

$$|\psi\rangle = \sum_i [\alpha_i|\psi_i\rangle] = \sum_i \left[\sqrt{p[i]}|\psi_i\rangle\right]$$

$$= \sum_i \left[\sqrt{p[i]}\frac{P_i|\psi\rangle}{\sqrt{p[i]}}\right] = \sum_i [P_i|\psi\rangle] = I|\psi\rangle = |\psi\rangle$$

(c) If any given $|\psi_i\rangle$ is represented by projection of $|\psi\rangle$ where projection operators are given as the result of decomposition of the identity, it must be the case that each has been projected into orthogonal subspaces of the Hilbert space encompassing $|\psi\rangle$. Therefore, two constituents of $|\psi\rangle$, $|\psi_i\rangle$ and $|\psi_j\rangle$ are not orthogonal if and only if they have been projected to the same subspace, which is true if and only if they are the same state. $\square$

Problem 3.5.3. Consider any linear transformation T on a Hilbert space $\mathcal{H}$ of dimension N. This linear transformation T induces a transformation $\rho \mapsto T\rho T^\dagger$ on the set of linear operators on the Hilbert space $\mathcal{H}$. Prove that the above transformation is also linear.

Solution.
First, if T is linear, so is $T^\dagger$. Then, for any linear operation $(\rho|\psi\rangle) \in \mathcal{H}$, this induced transformation manifests as $(T\rho T^\dagger|\psi\rangle) \in \mathcal{H}$. Let $|\psi_a = T^\dagger|\psi\rangle$. Obviously, $|\psi_a\rangle$ has been obtained by linearly transforming $|\psi\rangle$ via $T^\dagger$. We have been given ρ is a linear transformation, and that T is a linear transformation. Thus we know that $T\rho|\psi_a\rangle$ linearly transforms $|\psi_a\rangle$, and it follows that the outcome of this is related via linear transformations (because that's all that has taken place) to $|\psi\rangle$. So it holds that $\rho \mapsto T\rho T^\dagger$ is a linear transformation. $\square$

Problem D1. Consider the operator on the Hilbert space H_4

$$\rho = \frac{1}{4}(1-c)I_4 + c(|0\rangle \otimes |0\rangle)(\langle 0| \otimes \langle 0|)$$

where $0 \leq c \leq 1$ and $|0\rangle = (1,0)^T$. Does ρ define a density matrix? (Why or why not?)

Solution.
It's simplest (though not fastest) to see ρ in matrix form, and then decide if it is a density matrix.

$$\rho = \frac{1-c}{4}I_4 + c\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right)\left(\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}\right)$$

$$= \frac{1-c}{4}I_4 + \begin{bmatrix}c & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{bmatrix} = \begin{bmatrix}\frac{1+3c}{4} & 0 & 0 & 0\\0 & \frac{1-c}{4} & 0 & 0\\0 & 0 & \frac{1-c}{4} & 0\\0 & 0 & 0 & \frac{1-c}{4}\end{bmatrix}$$

Because ρ is diagonal, its determinant is the product of its diagonal elements. That is,

$$|\rho| = \frac{-3c^4}{256} + \frac{c^3}{32} - \frac{3c^2}{128} + \frac{1}{256}$$

which for c within the given bounds is non-zero. Furthermore, the trace of ρ is $\frac{1+3c}{4} + 3\frac{1-c}{4} = 1$ so ρ can be said to be a valid density matrix. $\square$

Problem D2. Suppose that we expand a density matrix for N qubits in terms of tensor products of Pauli spin matrices as

$$\rho = \frac{1}{2^N}\sum_{j_0=0}^{3}\sum_{j_1=0}^{3}\cdots\sum_{j_{N-1}=0}^{3} c_{j_0 j_1 \cdots j_{N-1}}\sigma_{j_0} \otimes \sigma_{j_1} \cdots \otimes \sigma_{j_{N-1}},$$

where $\sigma_0 = I_2$.

(i) What is the condition on the expansion coefficients if we impose $\rho = \rho^\dagger$?

(ii) What is the condition on these coefficients if we impose tr $\rho = 1$?

(iii) Find (calculate) tr $(\rho\sigma_{k_0} \otimes \sigma_{k_1} \cdots \sigma_{k_{N-1}})$.

Solution.

(i) If it must be the case that $\rho = \rho^\dagger$, it must be the case that any given expansion coefficient $c = c^* \iff c \in \mathbb{R}$.

(ii) Since $\mathrm{tr}(S + T) = \mathrm{tr}(S) + \mathrm{tr}(T)$, it must be the case that if $\mathrm{tr}(\rho) = 1$ the sum of all coefficients must be 1. That is

$$\sum_i [c_i] = 1$$

(iii) Actually no idea.

$\square$