Quantum Computing Assignment 2

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February 6, 2017

Problem B1. Let A be an $m \times m$ matrix and B an $n \times n$ matrix, both with complex entries. Let the eigenvalues and eigenvectors of A be $\lambda_1, \lambda_2, \ldots, \lambda_m$ and $|u_j\rangle, j = 1, 2, \ldots m$ and those of B be $\mu_1, \mu_2, \ldots, \mu_n$ and $|v_k\rangle, k = 1, 2, \ldots n$. Let c_1, c_2, and c_3 be real parameters. Find the eigenvalues and eigenvectors of the matrix

$$C = c_1 A \otimes B + c_2 A \otimes I_n + c_3 I_m \otimes B$$

in terms of those of A and B. Here I_n is the $n \times n$ identity matrix.

Solution.
The eigenvalues $\alpha_1, \alpha_2, \ldots, \alpha_{nm}$ and eigenvectors $|t_i\rangle$, $i = 1, 2, \ldots, nm$ of C can be found by taking the sum of the eigenvectors and eigenvalues of each summand. The eigenvalues of $A \otimes B$ can be found by realizing that they must satisfy $A|u\rangle \otimes B|v\rangle = \lambda|u\rangle \otimes \mu|v\rangle = \lambda\mu|uv\rangle$. As for the terms including identity, anything can be chosen for the eigenvectors of identity, so I'll use $\lambda|u\rangle$ for the first, and $\mu|v\rangle$ for the second, which leaves us with eigenvalues of C as $c_1\lambda\mu + c_2\lambda^2 + c_3\mu^2$ for all λ and μ and eigenvectors $c_1|uv\rangle + c_2|u\rangle^2 + c_3|v\rangle^2$ for all u and v □

Problem B2. Consider the operator

$$\rho = \frac{1}{4}(1 - \epsilon)I_4 + \epsilon(|0\rangle \otimes |0\rangle)(\langle 0| \otimes \langle 0|)$$

where $0 \leq \epsilon \leq 1$ and $|0\rangle = (1, 0)^T$.

(a) Write it in the matrix representation.

(b) What are $\mathrm{tr}(\rho)$ and $\mathrm{tr}(\rho^2)$?

Solution.

(a)

$$\begin{bmatrix} \frac{1+3\epsilon}{4} & 0 & 0 & 0 \\ 0 & \frac{1-\epsilon}{4} & 0 & 0 \\ 0 & 0 & \frac{1-\epsilon}{4} & 0 \\ 0 & 0 & 0 & \frac{1-\epsilon}{4} \end{bmatrix}$$

(b)

$$\mathrm{tr}(\rho) = \frac{1+3\epsilon}{4} + \frac{1-\epsilon}{4} + \frac{1-\epsilon}{4} + \frac{1-\epsilon}{4} = 1$$

$$\mathrm{tr}(\rho^2) = \frac{(1+3\epsilon)^2}{16} + \frac{(1-\epsilon)^2}{16} + \frac{(1-\epsilon)^2}{16} + \frac{(1-\epsilon)^2}{16} = \frac{3\epsilon^2+1}{4}$$

□

Problem B3. Show that the eigenvalues of a projector P are all either 0 or 1.

Solution.
If P is a projection operator, then it holds that $P^2{=}P$. By spectral mapping theorem, we know that for an eigenvalue λ of P, λ^2 is an eigenvalue of P^2. Given these two facts, we know that the eigenvalues of P have the property that $\lambda = \lambda^2$, which only has two solutions: $\lambda = 1$ and $\lambda = 0$.

□

Problem B4. Show that if P is a projector, the operator $I - P$ is also a projector. (Here I denotes the identity operator.)

Solution.
If $I - P$ is a projector, then it must be the case that $(I - P)^2 = I - P$

$$(I-P)^2 = (I-P)(I-P) = I(I-P) - P(I-P) = I - P - P + P^2 = I - P$$

Thus, $I - P$ is a projector. □

Problem B5. Prove that $\mathrm{Tr}(AB)$ is real if A and B are Hermitian.

Solution.
If A and B are Hermitian, then $(BA)^\dagger = A^\dagger B^\dagger = AB$, and since $\mathrm{Tr}(AB){=}\mathrm{Tr}(BA)$, the diagonal elements of AB (and BA) must be equal to their complex conjugates, and therefore they must be real. □

Problem B6. Show that if U is unitary, then $i(I - U)/(I + U)$ is Hermitian.

Solution.
The simplest way to check this is to figure out if $i(I - U)/(I + U)$ matches the definition of a Hermitian matrix: A is Hermitian $\iff A = A^\dagger$.

$$(i(I-U)/(I+U))^\dagger = -i\left((I-U)(I+U)^{-1}\right)^\dagger$$

$$= -i\left((I+U)^{-1}\right)^\dagger (I-U)^\dagger$$

$$I^\dagger = I \rightarrow (i(I-U)/(I+U))^\dagger = -i(I+U^\dagger)^{-1}(I-U^\dagger)$$

Using the definition of a unitary matrix (A is unitary $\iff AA^\dagger = I$), the identity can be written in terms of U as $I = UU^\dagger = U^\dagger U$. Furthermore, the definition of the identity lets us write $U^\dagger = U^\dagger I$ Thus the expression for $(i(I-U)/(I+U))^\dagger$ can be rewritten

$$-i(U^\dagger U + U^\dagger I)^{-1}(U^\dagger U - U^\dagger I)$$

$$= -i(U^\dagger(U+I))^{-1}U^\dagger(U-I)$$
$$= iU(U+I)^{-1}U^\dagger(I-U)$$
$$= i(I-U)/(I+U)$$

So $i(I-U)/(I+U)$ is Hermitian.

$\square$