XVII.

THE THEORY OF PERMANENT SCREWS; BEING THE NINTH MEMOIR ON THE THEORY OF SCREWS.

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Introduction.

It will be a convenient way of commencing this Memoir to recite a well-known dynamical proposition, and then to enlarge its enunciation by certain abandonment of restrictions.

Suppose a rigid body free to rotate around a fixed point. There are in general three axes about any one of which the body will continue to rotate when once set in rotation so long as the application of force is withheld. These are known as the permanent axes. The freedom of the body in this case is of a particular nature, included in the more general type known as Freedom of the Third Order. This order is itself merely one subdivision of the class which, including the six orders of freedom, embraces every conceivable form of constraint that can be applied to a rigid system. We propose to investigate the theory of permanent screws for a body constrained in the most general manner.

The movement of the body must be a twist velocity about some one screw θ belonging to a system of screws prescribed by the character of the constraints. In the absence of forces external to those arising from the reactions of the constraints, the movement will not, in general, persist as a twist about θ. The instantaneous screw will usually shift its position so as to occupy consecutive positions in the system. It will, however, be always possible to compel the body to remain twisting about θ. For this purpose a wrench of suitable intensity on an appropriate screw η may have to be applied. Without sacrifice of generality we can arrange that η is one of the system of screws which expresses the freedom of the body. It may sometimes appear that the intensity of the necessary wrench on η vanishes. The body in such a case requires no external coercion to preserve θ as the screw about which it twists, and we speak of θ as a *permanent* screw.

In the earlier parts of the Paper we may discard the restrictions involved in the assumption that the material system is only a single rigid body. The Theory of Screw-chains extends a portion of the inquiry to any material system whatever. Any number of parts connected in any manner must still conform to the general law, that the instantaneous movement can always be represented by a twist about a screw-chain. Generally speaking, the position of the instantaneous screw-chain cannot be maintained without the imposition of special coercion. This is applied by a restraining wrench-chain, the relation of which to the instantaneous screw-chain opens up an instructive branch of research. Sometimes we discover that a restraining wrench-chain is provided by the reaction of the constraints. The instantaneous screw-chain is then permanent, and we are thus led to the extension of the theory to any mechanical system whatever.

Another preliminary matter should be also noticed, because it exhibits the relation of the subject discussed in this Memoir to some other parts of the Theory of Screws. In the ordinary theory of the principal axes of a rigid body there are, as is well known, two distinct properties of such an axis which possess dynamical significance. We may think of a principal axis as the axis of a couple which, when applied impulsively to the body, will set it spinning about this line. We may also think of the principal axis as a direction about which, if a body be once set in rotation, it will continue to rotate. When the first of these properties is suitably generalized it opens up the theory of principal screw-chains of inertia, which I have already explained in previous Memoirs. It is from the other property of the principal axis that the present theory takes its rise. The two general theories are wholly distinct in their mathematical treatment, and in the character of the conceptions to which they give rise. It is therefore to be observed that two different departments in the Theory of Screws happen to coalesce in the very special case of a rigid body rotating around a point.

I.—A PROPERTY OF THE KINETIC ENERGY OF A SYSTEM.

It is obvious that a mere alteration of the azimuth about a fixed axis from which a rigid body is set into rotation will not affect its kinetic energy, provided the position of the axis and the angular velocity both remain unaltered. Enunciated in quite a general form the same principle is as follows:—

Any mechanical system in movement is necessarily twisting about a screw-chain. If we arrest the movement of the system, displace it to an adjacent position on the same screw-chain, and then start the system to twist still on the same screw-chain, with its original twist velocity, the kinetic energy of the system is the same as it was originally.

This principle requires that whatever be the symbols employed, T must satisfy a certain identical equation. I propose to investigate this equation, and its character will be best understood by discussing the question at first with co-ordinates of a perfectly general type for a system with n degrees of freedom.

Let the co-ordinates be $x_1, \ldots x_n$ as representing the position of the system, and of course its instantaneous velocity is indicated by $\dot{x}_1, \ldots \dot{x}_n$. Let O be the initial position of the system, then in the time δt it has reached the position O', whereof the co-ordinates are

$$x_1 + \dot{x}\delta t, \ . \ . \ . \ x_n + \dot{x}_n\delta t.$$

The movement from O to O' must, like every possible movement of a system, consist of a twist about a screw-chain. This is a kinematical fact, wholly apart from the particular system of co-ordinates which may have been adopted. We call this screw-chain θ, and $\dot{\theta}$ denotes the twist velocity with which the system moves around it.

Choose now any n independent screw-chains, about each one of which the system is capable of twisting. Then $\dot{\theta}$ can be decomposed into twist velocity components $\dot{\theta}_1, \ldots \dot{\theta}_n$ about the several screw-chains of reference.

Since everything pertaining to the position or the movement of the system must necessarily admit of being expressed in terms of the co-ordinates of the system, and since the quantities $\dot{\theta}_1$, &c., $\dot{\theta}_n$ are definitely determined by the position and movements of the system, it follows that there must be a group of formulæ

$$\dot{\theta}_1 = f_1(x_1, \ldots x_n, \dot{x}_1, \ldots \dot{x}_n),$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\dot{\theta}_n = f_n(x_1, \ldots x_n, \dot{x}_1, \ldots \dot{x}_n).$$

Let the system now receive the small displacement

$$\delta x_1 = \dot{x}_1 \delta\epsilon, \ldots \delta x_n = \dot{x}_n \delta\epsilon$$

where $\delta\epsilon$ is a small quantity, and let it be set in motion so that it has the same twist velocity as before, about the same screw-chain as before. The test of this is that each of the quantities $\dot{\theta}_1$, $\dot{\theta}_n$ is to remain unaltered. This can only be arranged by a suitable accommodation of $\delta\dot{x}_1, \ldots \delta\dot{x}_n$ in accordance with the equations

$$\frac{df_1}{dx_1}\delta x_1 \ldots + \frac{df_1}{dx_n}\delta x_n + \frac{df_1}{d\dot{x}_1}\delta\dot{x}_1 + \ldots \frac{df_1}{d\dot{x}_n}\delta\dot{x}_n = 0,$$

$$\frac{df_n}{dx_1}\delta x_1 \ldots + \frac{df_n}{dx_n}\delta x_n + \frac{df_n}{d\dot{x}_1}\delta\dot{x}_1 + \ldots \frac{df_n}{d\dot{x}_n}\delta\dot{x}_n = 0.$$

We have also the equation arising from the circumstance that T has not altered, so that

$$\frac{dT}{dx_1}\delta x_1 + \ldots \frac{dT}{dx_n}\delta x_n + \frac{dT}{d\dot{x}_1}\delta\dot{x}_1 + \ldots + \frac{dT}{d\dot{x}_n}\delta\dot{x}_n = 0.$$

Let us assume, for brevity, the symbol Δ, such that

$$\Delta = \dot{x}_1 \frac{d}{dx_1} + \ldots + \dot{x}_n \frac{d}{dx_n}.$$

Then we obtain, by elimination of $\delta\dot{x}_1$, $\delta\dot{x}_2$, and $\delta\epsilon$,

$$\begin{vmatrix} \Delta T, & \frac{dT}{d\dot{x}_1}, & \ldots & \frac{dT}{d\dot{x}_n} \\ \Delta f_1, & \frac{df_1}{d\dot{x}_1}, & \ldots & \frac{df_1}{d\dot{x}_n} \\ \cdot & \cdot & \cdot & \cdot \\ \Delta f_n, & \frac{df_n}{d\dot{x}_1}, & \ldots & \frac{df_n}{d\dot{x}_n} \end{vmatrix} = 0.$$

Such is a general condition to be satisfied by T; and it is of some interest to notice that the expression for the kinetic energy of any system is constrained to obey the law that is implied. It is doubtless true that the equation is a complicated one in ordinary systems of co-ordinates. It is, however, a distinctive feature of the Theory of Screw-chains to exhibit this equation in a form of much simplicity.

For, suppose that

$$\dot{\theta}_1 = \dot{x}_1,$$
$$\cdot \quad \cdot \quad \cdot$$
$$\dot{\theta}_n = \dot{x}_n,$$

then the equation above written reduces to

$$\Delta T = 0.$$

We thus have the following theorem:—

If the co-ordinates of a system $\theta'_1, \ldots \theta'_n$ be the n twists about n screw-chains, belonging to the system of screw-chains which express the liberty of the system, and if $\dot{\theta}_1, \ldots \dot{\theta}_n$ be the twist velocities of the system about these screw-chains, then the kinetic energy T satisfies the equation

$$\dot{\theta}_1 \frac{dT}{d\theta'_1} \ldots + \dot{\theta}_n \frac{dT}{d\theta'_n} = 0.$$

I have thought it more instructive to exhibit the origin of this equation as a special case of the general type of co-ordinates. For a brief demonstration the following simple argument suffices :—

If the system be displaced through $\delta\theta'_1, \ldots \delta\theta'_n$ while the velocities are unaltered, the change of kinetic energy is

$$\frac{dT}{d\theta'_1}\delta\theta'_1 + \ldots + \frac{dT}{d\theta'_n}\delta\theta'_n.$$

If the change of the position be that of a small twist around $\dot\theta_1 \ldots \dot\theta_n$, then

$$\dot\theta_1\delta\epsilon = \delta\theta'_1,$$

$$\dot\theta_n\delta\epsilon = \delta\theta'_n,$$

but from the physical property of the kinetic energy already cited, it appears that this arrangement cannot change the kinetic energy, whence

$$\dot\theta_1\frac{dT}{d\theta'_1}\ldots + \dot\theta_n\frac{dT}{d\theta'_n} = 0.$$

The converse of this theorem will now be proved; but before doing so it will be necessary to call attention to a point in the philosophy of co-ordinate systems of any type whatever. Let us take the general case where the co-ordinates are $x_1, \ldots x_n$, and $\dot x_1, \ldots \dot x_n$. Suppose that $\dot x_2, \ldots \dot x_n$ are all zero, then $\dot x_1$ is the velocity of the system. We shall also take $x_2, \ldots x_n$ to be zero, so that we only consider the position of the system defined by x_1. Think now of the two positions for which $x_1 = 0$ and $x_1 = x'_1$, respectively. Whatever be the character of the constraints it must be possible for the system to pass from the position $x_1 = 0$ to the position $x_1 = x'_1$ by a twist about a screw-chain. The magnitude x'_1 is thus correlated to the position of the system on a screw-chain about which it twists.

Let us suppose that the co-ordinates are of such a kind that the identical equation which T must necessarily satisfy has the form

$$\dot x_1\frac{dT}{dx'_1}\ldots + \dot x_n\frac{dT}{dx'_n} = 0.$$

Then, for the particular displacement corresponding to the first co-ordinate, $\dot{x}_2, \ldots \dot{x}_n$ are all zero, and

$$\frac{dT}{dx'_1} = 0;$$

and as T must involve $\dot{x}_1$ in the second power, we have

$$T = H\dot{x}_1^2,$$

where H is independent of x'_1.

Let $\dot{\theta}_1$ be the twist velocity about the screw-chain corresponding to the first co-ordinate, then, of course, A being a constant,

$$T = A\dot{\theta}_1^2,$$

whence

$$A\dot{\theta}_1^2 = H\dot{x}_1^2,$$

$$\sqrt{A}\,\dot{\theta}_1 = \sqrt{H}\,\dot{x}_1,$$

whence by integration and adjustment of units and origins

$$\theta_1 = x_1.$$

We thus see that while the displacement corresponding to the first co-ordinate must always be a twist about a screw-chain, whatever be the actual nature of the metric element chosen for the co-ordinate, yet that when the identical equation admits of the form

$$\dot{\theta}_1 \frac{dT}{d\theta'_1} \ldots + \dot{\theta}_n \frac{dT}{d\theta'_n} = 0,$$

the metric element chosen can be nothing else than the *amplitude of the twist* about the screw-chain.

We have thus proved the following general theorem:—

If the identical equation, satisfied by T, admits of expression in the form

$$\dot{\theta}_1 \frac{dT}{d\theta'_1} \ldots + \dot{\theta}_n \frac{dT}{d\theta'_n} = 0,$$

then the co-ordinates must be twists about n screw-chains of reference.

It is worth noticing that the identical equation in this form is independent of linear transformation.

Suppose, for instance, we write

$$\theta'_1 = (11)\,\phi'_1 \ldots + (1n)\,\phi'_n,$$
$$\theta'_n = (n_1)\,\phi'_1 \ldots + (un)\,\phi'_n.$$

Then, by differentiation

$$\dot{\theta}_1 = (11)\,\dot{\phi}_1 \ldots + (1n)\,\dot{\phi}_n.$$

Thus, the two sets of variables are cogredients, and by the theory of linear transformations

$$\dot{\theta}_1 \frac{dT}{d\theta'_1} \ldots + \dot{\theta}_n \frac{dT}{d\theta'_n} = \dot{\phi}_1 \frac{dT}{d\phi'_1} \ldots + \dot{\phi}_n \frac{dT}{d\phi'_n}.$$

The expression admitting of this property is known in algebra as an *emanant.*

We could, however, have foreseen the permanence of the form of the identical equation from the fact, that if *any* set of n independent screw-chains belonging to the system be chosen, the equation must assume this form.

The existence of such a simple mode of expression for this property of T, suggests the natural character of screw-chain co-ordinates.

II.—The General Equations of Motion with Screw-chain Co-ordinates.

Let the components of the wrench-chain, when resolved on the screw-chains of reference, have for intensities η''_1, η''_n. Let $p_1, \ldots p_n$ be the pitches of the chains of reference, by which is meant that $2p_1$ is the work done on that screw-chain by a twist of unit amplitude against a wrench of unit intensity on the same screw-chain. Then, choosing a set of co-reciprocal screw-chains of reference, we have, from Lagrange's equations,

$$\frac{d}{dt}\left(\frac{dT}{d\dot{\theta}_1}\right) - \frac{dT}{d\theta'_1} = 2p_1\eta''_1,$$
$$\cdot \quad \cdot \quad \cdot$$
$$\frac{d}{dt}\left(\frac{dT}{d\dot{\theta}_n}\right) - \frac{dT}{d\theta'_n} = 2p_n\eta''_n.$$

These equations admit of a transformation by the aid of the vanishing emanant

$$\dot\theta_1 \frac{dT}{d\theta'_1} \dots + \dot\theta_n \frac{dT}{d\theta'_n} = 0. \qquad \text{(I.)}$$

Differentiating this equation by $\dot\theta_1$, we find

$$\frac{dT}{d\theta'_1} + \dot\theta_1 \frac{d^2T}{d\dot\theta_1 d\theta'_1} + \dots + \dot\theta_n \frac{d^2T}{d\dot\theta_n d\theta'_n} = 0, \qquad \text{(II.)}$$

but

$$\frac{d}{dt}\left(\frac{dT}{d\dot\theta_1}\right) = \ddot\theta_1 \frac{d^2T}{d\dot\theta_1^2} + \ddot\theta_2 \dots + \ddot\theta_n \frac{d^2T}{d\dot\theta_1 d\dot\theta_n} + \dot\theta_1 \frac{d^2T}{d\dot\theta_1 d\theta'_n} + \dots + \dot\theta_n \frac{d^2T}{d\ddot\theta_n d\theta'_n},$$

whence, substituting from (II.),

$$\frac{d}{dt}\left(\frac{dT}{d\dot\theta_1}\right) = \ddot\theta_1 \frac{d^2T}{d\dot\theta_1^2} \dots + \ddot\theta_n \frac{d^2T}{d\dot\theta_1 d\dot\theta_n} - \frac{dT}{d\theta'_1}.$$

Thus, Lagrange's equations may be written

$$\ddot\theta_1 \frac{d^2T}{d\dot\theta_1^2} + \ddot\theta^2 \frac{d^2T}{d\dot\theta_1 d\dot\theta_2} \dots + \ddot\theta_n \frac{d^2T}{d\dot\theta_1 d\dot\theta_n} = 2p_1\left(\eta''_1 + \frac{1}{p_1}\frac{dT}{d\theta'_1}\right),$$

$$\cdot \quad \cdot \quad \cdot$$

$$\ddot\theta_1 \frac{d^2T}{d\dot\theta_1 d\dot\theta_n} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad + \ddot\theta_n \frac{d^2T}{d\dot\theta_n^2} = 2p_n\left(\eta''_n + \frac{1}{p_n}\frac{dT}{d\theta'_n}\right).$$

These equations can be simplified still further by a judicious choice of the screw-chains of reference without interfering with the generality of the investigation. We have already assumed the screw-chains of reference to be co-reciprocal. If, however, we select that particular group which forms the principal screw-chains of inertia, then every pair are conjugate screw-chains of inertia besides being reciprocal.

In this case T takes the form

$$T = M(u_1^2\dot\theta_1^2 + \dots + u_n^2\dot\theta_n^2) + \theta'_1 \frac{dT}{d\dot\theta'_1} + \dots + \theta'_n \frac{dT}{d\theta'_n};$$

neglecting the small quantities $\theta'_1 \dots$ &c., we have

$$\frac{d'^2T}{d\dot\theta_1^2} = 2Mu_1^2 \dots \frac{d^2T}{d\dot\theta_n^2} = 2Mu_n^2,$$

$$\frac{d^2T}{d\dot\theta_1 d\dot\theta_2} = 0, \ \text{\&c.}$$

Introducing these values we have

$$Mu_1^2\ddot{\theta}_1 = p_1\left(\eta''_1 + \frac{1}{p_1}\frac{dT}{d\theta'_1}\right),$$

$$\cdot \quad \cdot \quad \cdot$$

$$Mu_n^2\ddot{\theta}_n = p_n\left(\eta''_n + \frac{1}{p_n}\frac{dT}{d\theta'_n}\right).$$

These may be regarded as a generalization for any mechanical system whatever of the well-known Eulerian equations for the rotation of a rigid body around a fixed point.

If there be no external forces then the equations of movement assume the extremely simple form (since η''_1, &c., η''_n are all zero),

$$Mu_1^2\ddot{\theta}_1 = \frac{dT}{d\theta'_1};$$

$$\cdot \quad \cdot \quad \cdot$$

$$Mu_n^2\ddot{\theta}_n = \frac{dT}{d\theta'_n}.$$

III.—The Restraining Wrench-chain.

If a system be twisting about a screw-chain θ, where no external forces are in action, the system will, in general, forsake θ and gradually adopt one screw-chain after another. It is, however, possible, by the application of a suitable wrench-chain to compel the system to keep to θ. We call this the *restraining wrench-chain*, and we proceed to study its character.

The co-ordinates of the wrench-chain required are obtained by imposing the conditions

$$\ddot{\theta}_1 = 0; \quad \ddot{\theta}_2 = 0, \ldots \ddot{\theta}_n = 0.$$

We therefore infer from the general equations just given, that if η''_1, &c., η''_n are the co-ordinates of the restraining wrench-chain

$$\eta_1'' + \frac{1}{p_1}\frac{dT}{d\theta'_1} = 0,$$

$$\cdot \quad \cdot \quad \cdot$$

$$\eta_n'' + \frac{1}{p_n}\frac{dT}{d\theta'_n} = 0,$$

whence we deduce the following theorem:—

If the position of a system be referred to co-reciprocal screw-chains of reference, then

$$-\frac{1}{p_1}\frac{dT}{d\theta'_1}, \ldots -\frac{1}{p_n}\frac{dT}{d\theta'_n}$$

are the co-ordinates of the restraining wrench-chain which would coerce the system into continuing to twist about the same screw-chain θ.*

We obtain a confirmation of this theorem by the following method of viewing the subject. It must be possible to coerce the system to twist about θ by the imposition of special constraints. The reactions of these constraints will constitute, in fact, the restraining wrench-chain. It is, however, a characteristic feature that, as the system is, *ex hypothesi*, still at liberty to twist about θ, the reaction of any constraints which are consistent with this freedom must lie on a screw-chain reciprocal to θ.

The condition that two screw-chains, θ and η, shall be reciprocal is

$$+p_1\theta_1\eta_1, \ldots +p_n\theta_n\eta_n = 0\,;$$

but this is clearly satisfied if for η_1, &c., we substitute

$$-\frac{1}{p_1}\frac{dT}{d\theta'_1}, \ldots -\frac{1}{p_n}\frac{dT}{d\theta'_n};$$

for the equation then becomes

$$\theta_1\frac{dT}{d\theta'_1}, \ldots +\theta_n\frac{dT}{d\theta'_n} = 0,$$

which, when multiplied by $\dot{\theta}$, reduces to the known identity

$$\dot{\theta}_1\frac{dT}{d\theta'_1}, \ldots +\dot{\theta}_n\frac{dT}{d\theta'_n} = 0.$$

This gives a physical meaning to the emanant identity, which, as we have seen, is characteristic of screw-chain co-ordinates. From this point of view it becomes merely an expression of the fact that the restraining wrench-chain must be reciprocal to the instantaneous screw-chain.

The kinetic energy will remain unaltered by a derangement of the

* A particular case of this, or what is equivalent thereto, is given in Williamson and Tarleton's *Dynamics*, 2nd ed., p. 432.

system which consists of a twist on any screw-chain reciprocal to the restraining screw-chain, the twist velocities remaining unaltered.

For, if $\theta'_1, \ldots \theta'_n$ be the co-ordinates of the displacement, the change in T is

$$\theta'_1 \frac{dT}{d\theta'_1}, \ldots + \theta_n' \frac{dT}{d\theta'_n},$$

which may be written

$$p_1\theta'_1 \frac{1}{p_1} \frac{dT}{d\theta'_1}, \ldots + p_n\theta'_n \frac{1}{p_n} \frac{dT}{d\theta'_n}$$

but this will be zero if, and only if, the screw-chain $\theta'_1, \ldots \theta'_n$ be reciprocal to the screw-chain

$$\frac{1}{p_1} \frac{dT}{d\theta'_1}, \ldots \frac{1}{p_n} \frac{dT}{d\theta'_n}.$$

IV.—THE ACCELERATING SCREW-CHAIN.

When the system has forsaken the instantaneous screw-chain θ, and advanced to another screw-chain ϕ, there must be a twist velocity about some screw-chain ρ, which, when compounded with the twist velocity about θ, gives the twist velocity about ϕ. When ϕ and θ are indefinitely close, then ρ is the accelerating screw-chain.

To study the subject we take the n principal screw-chains of inertia as the screws of reference. If external forces are absent, we have

$$Mu_1^2 \ddot{\theta}_1 = \frac{dT}{d\theta'_1},$$
$$\&c.,$$
$$Mu_n^2 \ddot{\theta}_n = \frac{dT}{d\theta'_n}.$$

It is plain that the co-ordinates of the accelerating screw-chain are $\ddot{\theta}_1, \ldots \ddot{\theta}_n$, whence we have the following theorem:—

If a mechanical system be twisting around a screw-chain θ, and if external forces are absent, the co-ordinates of the corresponding accelerating screw-chain are proportional to

$$\frac{1}{u_1^2}\cdot\frac{dT}{d\theta'_1}, \ldots \frac{1}{u_n^2}\frac{dT}{d\theta'_n}.$$

We can prove this otherwise, as follows:—

From the theory of screw-chains it is known that if a quiescent system receive an impulsive wrench-chain with co-ordinates

$$\frac{u_1^2}{p_1}\rho_1, \ldots \frac{u_n^2}{p_n}\rho_n,$$

the system commences to twist about the screw-chain, of which the co-ordinates are

$$\rho_1, \ldots \rho_n.$$

If, by the imposition of a restraining wrench, the system continues to remain twisting about θ, the restraining wrench has neutralized the acceleration ; it follows that the restraining wrench-chain, regarded as impulsive, must have generated an instantaneous twist velocity on the accelerating screw-chain, equal and opposite to the acceleration that would otherwise have taken place. The co-ordinates of this impulsive wrench are proportional to

$$\frac{1}{p_1}\frac{dT}{d\theta'_1}, \ldots \frac{1}{p_n}\frac{dT}{d\theta'_n}.$$

The corresponding instantaneous screw-chain is obtained by multiplying these expressions severally by

$$\frac{p_1}{u_1^2}, \ldots \frac{p_n}{u_n^2},$$

and thus we find, as before, for the co-ordinates of the accelerating screw-chain

$$\frac{1}{u_1^2}\frac{dT}{d\theta'_1}, \ldots \frac{1}{u_n^2}\frac{dT}{d\theta'_n}.$$

We can now express the relation between the accelerating screw-chain and the instantaneous screw-chain.

We have, from the expressions already given,

$$M(u_1^2\dot\theta_1\ddot\theta_1 + \ldots + u_n^2\dot\theta_n\ddot\theta_n) = \dot\theta_1\frac{dT}{d\theta'_1} + \ldots\ \dot\theta_n\frac{dT}{d\theta'_n}.$$

But the right-hand side is the emanant which we know to be zero, whence

$$u_1^2\dot\theta_1\ddot\theta_1 + \&c., \quad + u_n^2\dot\theta_n\ddot\theta_n = 0.$$

This shows that $\dot\theta_1, \ldots \dot\theta_n$, and $\ddot\theta_1, \ldots \ddot\theta_n$ are on conjugate screw-chains of inertia, and hence we infer that when a mechanical system is moving without external forces, the instantaneous screw-chain and the accelerating screw-chain are conjugate screw-chains of inertia.

The construction of the accelerating screw-chain is thus obtained :—

When a single rigid body is in movement a cylindroid can be drawn through two consecutive instantaneous screws, and this cylindroid osculates the motion. We can thus conceive a chain of cylindroids osculating the movement of any system whatever. If an instantaneous screw chain θ be chosen on these cylindroids, then it is always possible to inscribe on the same cylindroids one, but only one screw-chain, which is conjugate to θ with respect to inertia. This is the accelerating screw-chain.

V.—PERMANENT SCREWS.

Reverting to the general system of equations, we investigate the condition that θ shall be a permanent screw. It is obvious that if $\ddot\theta_1$, $\ddot\theta_2$, &c., $\ddot\theta_n$ are all zero, then

$$\frac{dT}{d\theta'_1}, \ldots \frac{dT}{d\theta'_n}$$

must each be zero. Conversely, we are able to show that if the differential coefficients just written are all zero, then the quantities $\ddot\theta_1, \ldots \ddot\theta_n$ must each vanish.

For this is obviously true unless the determinant

$$\begin{vmatrix} \frac{d^2T}{d\dot{\theta}_1^2}, & \frac{d^2T}{d\dot{\theta}_1 d\dot{\theta}_2}, & \ldots, & \frac{d^2T}{d\dot{\theta}_1 d\dot{\theta}_n}, \\ \frac{d^2T}{d\dot{\theta}_1 d\theta_2}, & & & \\ \frac{d^2T}{d\dot{\theta}_1 d\dot{\theta}_n}, & & , & \frac{d^2T}{d\dot{\theta}_n^2} \end{vmatrix}$$

should be zero. Remembering that T is a homogeneous function in the second degree of the quantities $\dot{\theta}_1, \ldots \dot{\theta}_n$, the evanescence of the determinant just written would indicate that T admitted of expression by means of $n-1$ square terms, such as

$$\pm L_1^2 \pm L_2^2 \ldots \pm L_{n-1}^2.$$

This vanishes if

$$L_1 = 0; \quad L_2 = 0, \ \&c.; \quad L_{n-1} = 0;$$

each of these is a linear equation in $\dot{\theta}_1, \ldots \dot{\theta}_n$, and consequently a real system of values for $\dot{\theta}_1$, $\dot{\theta}_n$ must satisfy these equations, and render T zero. It would thus appear that a real motion of the system would correspond to a state of zero kinetic energy. This is, of course, impossible; it therefore follows that the determinant must not vanish, and consequently we have the following theorem:—

If the screw-chains of reference be co-reciprocal, then the necessary and the sufficient conditions for θ to be a permanent screw are indicated by the equations

$$\frac{dT}{d\theta'_1} = 0; \ \ldots \ \frac{dT}{d\theta'_n} = 0.$$

There are n of these equations, but they are not independent. The emanant identity shows that if $n-1$ of them be satisfied, the co-ordinates so found must, generally speaking, satisfy the last equation also.

As the quantities $\theta'_1, \ldots \theta'_n$ are small, we may generally expand T in powers, as follows:—

$$\begin{aligned} T = T_0 + \theta'_1 T_1 + \ldots \theta'_n T_n \\ + \theta'^2_1 T_{11} + \ldots + 2\theta'_1\theta'_2 T_{12} + \ldots \end{aligned}$$

The equation

$$\frac{dT}{d\theta'_1} = 0$$

therefore becomes

$$T_1 + 2\theta'_1 T_{11} + 2\theta'_2 T_{12} + 1 \ldots = 0,$$

and as θ'_1, θ'_2, &c., are indefinitely small, this reduces to

$$T_1 = 0,$$

where T_1 is a homogeneous function of $\dot\theta_1, \dot\theta_2, \ldots \dot\theta_n$ in the second degree.

For the study of the permanent screws we have, therefore, n equations of the second degree in the co-ordinates of the instantaneous screw-chain, and any screw-chain will be permanent if its co-ordinates render the several differential coefficients zero. We may write the necessary co-ordinates that have to be fulfilled, as follows:—

Let us denote, by Roman capitals, the several differential coefficients of T with respect to the variables. Then the emanant identity is

$$\dot\theta_1 \mathrm{I} + \dot\theta_2 \mathrm{II} + \dot\theta_3 \mathrm{III} + \ldots = 0,$$

and we may develop any single expression, such as III, in the following form:—

$$\mathrm{III} = \mathrm{III}_{11}\dot\theta_1^2 + \mathrm{III}_{22}\dot\theta_2^2 + \mathrm{III}_{33}\dot\theta_3^2 + 2\mathrm{III}_{12}\dot\theta_1\dot\theta_2 + \ldots 2\mathrm{III}_{14}\dot\theta_1\dot\theta_4.$$

As the emanant is to vanish identically, we must have the coefficients of the several terms, such as $\dot\theta_{13}$, $\dot\theta_1^2\dot\theta_2$, $\dot\theta_1\dot\theta_2\dot\theta_3$, &c., all zero, the result being three types of equation—

$$\begin{array}{lll} \mathrm{I}_{11} = 0, & \mathrm{I}_{22} + \mathrm{II}_{12} = 0, & \mathrm{I}_{23} + \mathrm{II}_{13} + \mathrm{III}_{12} = 0, \\ \mathrm{II}_{22} = 0, & \mathrm{II}_{11} + \mathrm{I}_{12} = 0, & \&c., \\ \mathrm{III}_{33} = 0, & \mathrm{II}_{33} + \mathrm{III}_{23} = 0, & \&c., \\ \mathrm{IV}_{44} = 0, & \&c., & \&c. \end{array}$$

Of the first of these classes of equations, $\mathrm{I}_{11} = 0$, there are n, of the second there are $n(n-1)$, and of the third, $\dfrac{n(n-1)(n-2)}{1.2.3}$, in all, $\dfrac{n(n+1)(n+2)}{1.2.3}$.

VI.—A SINGLE FREE RIGID BODY.

In the case of a single free rigid body there is another identical equation besides that implied in the vanishing of the emanant. The origin of this equation is thus explained :—

Let T be the kinetic energy of a free rigid body twisting for the moment around a screw θ. It is obvious that T will be a function of the six co-ordinates, $\theta'_1, \ldots \theta'_6$, which express the position of the body, and also of $\dot\theta_1, \ldots \dot\theta_6$, the co-ordinates of the twist velocity,

$$T = f(\theta'_1, \ldots \theta'_6, \quad \dot\theta_1, \ldots \dot\theta_6).$$

It is plain, from symmetry, that the kinetic energy will be unaltered if the motion of the body be arrested, and if, after having received a displacement by a twist of *any pitch* on the same axes as the instantaneous screw, it be again set in motion with the primitive twist velocity. Let ϵ be the amplitude of the twist, then the obvious property which we have just stated involves the following analytical property :—

That T must remain unchanged if we substitute as follows, for θ'_1,

$$\theta'_1 + \left(\dot\theta_1 + \frac{p_x}{4p_1} \cdot \frac{dR}{d\dot\theta_1}\right)\epsilon,$$

and similarly for θ'_2, &c., θ'_6.

In these expressions p_x denotes an arbitrary pitch, while R is the function

$$\dot\theta_1^2 + \dot\theta_2^2 \ldots + \dot\theta_6^2 + 2(12)\,\dot\theta_1\dot\theta_2 + 2\,(23)\,\dot\theta_2\dot\theta_3, \text{ \&c.},$$

where (12) is the cosine of the angle between the first and second screws of reference, and similarly for (23), &c.

Substituting in T, we obtain

$$\left(\dot\theta_1 + \frac{p_x}{4p_1} \cdot \frac{dR}{d\theta'_1}\right)\frac{dT}{d\theta'_1}, + \ldots + \left(\dot\theta_6 + \frac{p_x}{4p_6} \cdot \frac{dR}{d\dot\theta_6}\right)\frac{dT}{d\theta'_6} = 0.$$

As this must be true for every value of p_x, we must have, besides the vanishing emanant, the condition

$$\frac{1}{p_1}\frac{dR}{d\dot{\theta}_1}\cdot\frac{dT}{d\theta'_1}+\ldots+\frac{1}{p_6}\frac{dR}{d\dot{\theta}_6}\cdot\frac{dT}{d\theta'_6}=0.$$

It is clear that the form of T must be such as to satisfy this equation.

The physical meaning of this equation is easily obtained, for it may be written thus,

$$p_1\frac{1}{p_1}\frac{dR}{d\dot{\theta}_1}\times\frac{1}{p_1}\frac{dT}{d\theta'_1}+\ldots+p_6\frac{1}{p_6}\frac{dR}{d\dot{\theta}_6}\times\frac{1}{p_6}\frac{dT}{d\dot{\theta}'_6}=0.$$

This means that the screw of which the co-ordinates are

$$\frac{1}{p_1}\frac{dR}{d\dot{\theta}_1},\ldots\frac{1}{p_6}\frac{dR}{d\dot{\theta}_6},$$

and the screw of which the co-ordinates are

$$\frac{1}{p_1}\frac{dT}{d\theta'_1},\ldots\frac{1}{p_6}\frac{dT}{d\theta'_6}$$

must be reciprocal.

The former denotes a screw of infinite pitch parallel to θ, the latter denotes the restraining screw. The result may be thus stated:—

The restraining screw corresponding to any given instantaneous screw is the same for any other instantaneous screw on the same axis of any pitch whatever.

We now compute the co-ordinates of the restraining wrench for a free rigid body.

Suppose the body to have a standard position from which we displace it by small twists $\theta'_1, \ldots \theta'_6$ around the six principal screws of inertia. While the body is in this position it receives a twist velocity of which the co-ordinates relatively to the six principal screws of inertia are $\dot{\theta}_1, \ldots \dot{\theta}_6$.

To compute the kinetic energy we proceed as follows:—Let a point lie initially at x, y, z, then, after the displacement of the system prior to starting, it is moved to X, Y, Z, where

$$X = a(\theta'_1-\theta'_2)+y(\theta'_5+\theta'_6)-z(\theta'_3+\theta'_4)+x,$$
$$Y = b(\theta'_3-\theta'_4)+z(\theta'_1+\theta'_2)-x(\theta'_5+\theta'_6)+y,$$
$$Z = c(\theta'_5-\theta'_6)+x(\theta'_3+\theta'_4)-y(\theta'_1+\theta'_2)+z,$$

in which a, b, c are the radii of gyration on the principal axes. The six principal screws of inertia lie, of course, two by two on each of the three principal axes, with pitches $+a$, $-a$ on the first, $+b$, $-b$ on the second, and $+c$, $-c$ on the third.

In consequence of the twist velocity with the components $\dot{\theta}_1, \ldots \dot{\theta}_6$, each point X, Y, Z receives a velocity of which the components are

$$a(\dot{\theta}_1 - \dot{\theta}_2) + Y(\dot{\theta}_5 + \dot{\theta}_6) - Z(\dot{\theta}_3 + \dot{\theta}_4),$$

$$b(\dot{\theta}_3 - \dot{\theta}_4) + Z(\dot{\theta}_1 + \dot{\theta}_2) - X(\dot{\theta}_5 + \dot{\theta}_6),$$

$$c(\dot{\theta}_5 - \dot{\theta}_6) + X(\dot{\theta}_3 + \dot{\theta}_4) - Y(\dot{\theta}_1 + \dot{\theta}_2).$$

Before substitution for X, Y, Z it will be convenient to use certain abbreviations,

$$\theta'_1 - \theta'_2 = \epsilon_1; \quad \theta'_1 + \theta'_2 = \lambda_1,$$

$$\theta'_3 - \theta'_4 = \epsilon_2; \quad \theta'_3 + \theta'_4 = \lambda_2,$$

$$\theta'_5 - \theta'_6 = \epsilon_3; \quad \theta'_5 + \theta'_6 = \lambda_3.$$

$$\dot{\theta}_1 - \dot{\theta}_2 = \rho_1; \quad \dot{\theta}_1 + \dot{\theta}_2 = \omega_1,$$

$$\dot{\theta}_3 - \dot{\theta}_4 = \rho_2; \quad \dot{\theta}_3 + \dot{\theta}_4 = \omega_2,$$

$$\dot{\theta}_5 - \dot{\theta}_6 = \rho_3; \quad \dot{\theta}_5 + \dot{\theta}_6 = \omega_3.$$

After a few transformations we obtain

$$\tfrac{1}{2}\int v^2 dm \div M = a^2\dot{\theta}_1^2 + a^2\dot{\theta}_2^2 + b^2\dot{\theta}_3^2 + b^2\dot{\theta}_4^2 + c^2\dot{\theta}_5^2 + c^2\dot{\theta}_6^2,$$

$$+ ab\epsilon_2\rho_1\omega_3 - ac\epsilon_3\rho_1\omega_2 - \lambda_1\omega_2\omega_3(b^2 - c^2),$$

$$+ bc\epsilon_3\rho_2\omega_1 - ba\epsilon_1\rho_2\omega_3 - \lambda_2\omega_3\omega_1(c^2 - a^2),$$

$$+ ca\epsilon_1\rho_3\omega_2 - cb\epsilon_2\rho_3\omega_1 - \lambda_3\omega_1\omega_2(a^2 - b^2),$$

whence we easily find

$$\frac{dT}{d\theta'_1} = + ac\rho_3\omega_2 - ab\rho_2\omega_3 - (b^2 - c^2)\omega_2\omega_3.$$

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If $\eta''_1, \ldots \eta_6$ be the co-ordinates of the restraining wrench, then, as we have seen,

$$\eta''_1 = -\frac{1}{p_1} \cdot \frac{dT}{d\theta'_1},$$

whence we deduce the following fundamental expressions for the co-ordinates of the restraining wrench:—

$$p_1\eta''_1 = -acp_3\omega_2 + abp_2\omega_3 + (b^2 - c^2)\,\omega_2\omega_3,$$

$$p_2\eta''_2 = +acp_3\omega_2 - abp_2\omega_3 + (b^2 - c^2)\,\omega_2\omega_3,$$

$$p_3\eta''_3 = -abp_1\omega_3 + cbp_3\omega_1 + (c^2 - a^2)\,\omega_3\omega_1,$$

$$p_4\eta''_4 = +abp_1\omega_3 - cbp_3\omega_1 + (c^2 - a^2)\,\omega_3\omega_1,$$

$$p_5\eta''_5 = -bcp_2\omega_1 + acp_1\omega_2 + (a^2 - b^2)\,\omega_1\omega_2,$$

$$p_6\eta''_6 = +bcp_2\omega_1 - acp_1\omega_2 + (a^2 - b^2)\,\omega_1\omega_2.$$

As usual, I here write

$$p_1 = +a\,; \quad p_2 = -a\,; \quad p_3 = +b\,; \quad p_4 = -b\,; \quad p_5 = +c\,; \quad p_6 = -c.$$

We verify at once that

$$p_1\eta''_1\dot{\theta}_1 + \ldots p_6\eta''_6\dot{\theta}_6 = 0,$$

but this is shown otherwise to be true, because the restraining screw must be reciprocal to the instantaneous screw.

These equations enable us to study the correspondence between each instantaneous screw θ and the corresponding restraining screw η. It is to be noted that this correspondence is not of the homographic, or one-to-one type, such as we meet with in the study of the Principal Screws of Inertia, and in other parts of the Theory of Screws. The correspondence now to be considered has a different character.

If any θ be given, then, no doubt, one η is given definitely, but the converse is not true. If η be selected arbitrarily there will not in general be any possible θ. If, however, there be any one θ, then every screw on the same axis as θ will also correspond to the same η.

We proceed to find the condition which η must satisfy.

From the equations in the last article we can eliminate the six quantities, $\theta_1, \ldots \theta_6$; we can also write $\eta''_1 = \eta''\eta_1; \ldots \eta''_n = \eta''\eta_n$ where η'' is the intensity of the restraining wrench and $\eta_1, \ldots \eta_6$, the co-ordinates of the screw on which it acts.

We have

$$a(\eta''_1 + \eta''_2) = 2ab\rho_2\omega_3 - 2ac\rho_3\omega_2,$$

$$a(\eta''_1 - \eta''_2) = (b^2 - c^2)\,\omega_2\omega_3,$$

whence

$$\frac{b^2 - c^2}{a}\,\frac{\eta_1 + \eta_2}{\eta_1 - \eta_2} = b\,\frac{\rho_2}{\omega_2} - c\,\frac{\rho_3}{\omega_3},$$

and from the two similar equations we obtain, by addition,

$$\frac{b^2 - c^2}{a}\,\frac{\eta_1 + \eta_2}{\eta_1 - \eta_2} + \frac{c^2 - a^2}{b}\,\frac{\eta_3 + \eta_4}{\eta_3 - \eta_4} + \frac{a^2 - b^2}{c}\,\frac{\eta_5 + \eta_6}{\eta_5 - \eta_6} = 0.$$

It might have been thought that any screw could be the residence of a restraining wrench, provided the corresponding instantaneous screw were fitly appointed. On the other hand it would be noted that to each restraining screw must correspond a singly infinite number of possible instantaneous screws. As the choice of an instantaneous screw had five degrees of infinity, it was to be presumed that the restraining screws could only have four degrees of infinity, *i. e.* that the co-ordinates of a restraining screw must satisfy some equation, or, in other words, that they must belong to a screw system of the fifth order, as we have now shown them to do.

We can obtain a confirmation of the expression for the co-ordinates of η in a somewhat interesting manner. It is obvious, from the physical aspect of the question, that so long as θ retains the same direction and situation, its pitch is immaterial so far as η is concerned. For the rigid body twisting about a screw has no tendency to depart from the screw in so far as its velocity of *translation* is concerned. It is the *rotation* which necessitates the restraining wrench if the motion is to be preserved about the same instantaneous screw. We ought, therefore, to find that the expressions for the co-ordinates of η remained unaltered if we substituted for $\dot{\theta}_1, \ldots \dot{\theta}_6$; the co-ordinates of any other screw on the same straight line as θ. These are

$$\dot{\theta}_1 + \frac{H}{a}(\dot{\theta}_1 + \dot{\theta}_2), \qquad \dot{\theta}_2 - \frac{H}{a}(\dot{\theta}_1 + \dot{\theta}_2),$$

$$\dot{\theta}_3 + \frac{H}{b}(\dot{\theta}_3 + \dot{\theta}_4), \qquad \dot{\theta}_4 - \frac{H}{b}(\dot{\theta}_3 + \dot{\theta}_4),$$

$$\dot{\theta}_5 + \frac{H}{c}(\dot{\theta}_5 + \dot{\theta}_6), \qquad \dot{\theta}_6 - \frac{H}{c}(\dot{\theta}_5 + \dot{\theta}_6),$$

where H is arbitrary.

Whence we obtain the following substitutions:—

$$\text{for } \rho_1 \text{ we put } \rho_1 + \frac{2H}{a}\omega_1,$$

$$,, \quad \rho_2 \quad ,, \quad \rho_2 + \frac{2H}{b}\omega_2,$$

$$,, \quad \rho_3 \quad ,, \quad \rho_3 + \frac{2H}{c}\omega_3,$$

while ω_1, ω_2, ω_3 are unchanged.

Introducing these into the values for η''_1, it becomes

$$-ac\omega_2\left(\rho_3 + \frac{2H}{c}\omega_3\right) + ab\omega_3\left(\rho_2 + \frac{2H}{b}\omega_2\right) + (b^2 - c^2)\,\omega_2\omega_3,$$

from which H disappears, and the required result is proved.

It is important to observe that in the case of a free rigid body the restraining screw intersects the instantaneous screw at right angles, for the restraining screw is reciprocal to the instantaneous screw, and, consequently, if ϵ be the angle between the two screws, and d their distance apart,

$$(p_\eta + p_\theta)\cos\epsilon - d\sin\epsilon = 0.$$

We have seen that this must be true for every value of p_θ, whence

$$\cos\epsilon = 0\,; \quad d = 0.$$

We can prove this theorem otherwise,

$$\eta''_1 + \eta''_2 = 2b\rho_2\omega_3 - 2c\rho_3\omega_2,$$

$$\eta''_3 + \eta''_4 = 2c\rho_3\omega_1 - 2a\rho_1\omega_3,$$

$$\eta''_5 + \eta''_6 = 2a\rho_1\omega_2 - 2b\rho_2\omega_1,$$

multiplying up by ω_1, ω_2, ω_3, and adding, we get

$$(\eta_1+\eta_2)(\theta_1+\theta_2)+(\eta_3+\eta_4)(\theta_3+\theta_4)+(\eta_5+\eta_6)(\theta_5+\theta_6)=0.$$

This proves that η and θ are rectangular, but we already know that they are reciprocal, and therefore they intersect at right angles.

The expressions for the restraining wrenches can be illustrated by taking as a particular case an instantaneous screw which passes through the centre of inertia.

The equations to the axis of the screw are

$$\frac{a\rho_1+y\omega_3-z\omega_2}{\omega_1}=\frac{b\rho_2+z\omega_1-x\omega_3}{\omega_2}=\frac{c\rho_3+x\omega_2-y\omega_1}{\omega_3}.$$

If x, y, z are all simultaneously zero, then

$$\frac{a\rho_1}{\omega_1}=\frac{b\rho_2}{\omega_2}=\frac{c\rho_3}{\omega_3},$$

and these are, accordingly, the conditions that the instantaneous screw passes through the centre of inertia.

With these substitutions the co-ordinates become

$$p_1\eta''_1=(b^2-c^2)\,\omega_2\omega_3;\quad p_3\eta''_3=(c^2-a^2)\,\omega_3\omega_1;\quad p_5\eta''_5=(a^2-b^2)\,\omega_1\omega_2,$$

$$p_2\eta''_2=(b^2-c^2)\,\omega_2\omega_3;\quad p_4\eta''_4=(c^2-a^2)\,\omega_3\omega_1;\quad p_6\eta''_6=(a^2-b^2)\,\omega_1\omega_2;$$

remembering that $p_1=+a$; p_2+-a, &c, we have

$$\eta''_1+\eta''_2=0;\quad \eta''_3+\eta''_4=0;\quad \eta''_5+\eta''_6=0;$$

but these are the conditions that the pitch of η shall be infinite; in other words the restraining wrench is a couple.

From the equations already given, we can find the co-ordinates of the instantaneous screw in terms of those of the restraining screw.

We have

$$H=\sqrt{\frac{abc\,(\eta''_1-\eta''_2)(\eta''_3-\eta''_4)(\eta''_5-\eta''_6)}{2\,(b^2-c^2)(c^2-a^2)(a^2-b^2)}},$$

and

$$\omega_1 = H \frac{b^2 - c^2}{a(\eta''_1 - \eta''_2)};$$

$$\omega_2 = H \frac{c^2 - a^2}{b(\eta''_3 - \eta''_4)};$$

$$\omega_3 = H \frac{a^2 - b^2}{c(\eta''_5 - \eta''_6)}.$$

If we make

$$L\dot{\theta}^2 = a\omega_1^2 + b\omega_2^2 + c\omega_3^2.$$

$$h^3\dot{\theta}^2 = a^3\omega_1^2 + b^3\omega_2^2 + c^3\omega_3^2.$$

Then we have

$$\theta_1 = \omega_1 \left(\frac{p_1 + p_\theta}{2p_1} + \frac{La^2 - h^3}{2abc}\right),$$

$$\theta_2 = \omega_1 \left(\frac{p_2 + p_\theta}{2p_2} - \frac{La^2 - h^3}{2abc}\right),$$

$$\theta_3 = \omega_2 \left(\frac{p_3 + p_\theta}{2p_3} + \frac{Lb^2 - h^3}{2abc}\right),$$

$$\theta_4 = \omega_2 \left(\frac{p_4 + p_\theta}{2p_4} - \frac{Lb^2 - h^3}{2abc}\right),$$

$$\theta_5 = \omega_3 \left(\frac{p_5 + p_\theta}{2p_5} + \frac{Lc^2 - h^3}{2abc}\right),$$

$$\theta_6 = \omega_3 \left(\frac{p_6 + p_\theta}{2p_6} - \frac{Lc^2 - h^3}{2abc}\right).$$

In this, p_θ is the pitch of θ, and is, of course, an indeterminate quantity.

If the freedom of a body be restricted, then any screw will be permanent, provided its restraining screw belong to the reciprocal system. For the body will not depart from the original instantaneous screw except by an acceleration. This must be on a screw which stands to the restraining screw in the relation of instantaneous to impulsive, but in the case supposed these two screws are reciprocal, therefore they cannot be so related, and therefore there is no acceleration.

VII.—TWO DEGREES OF FREEDOM.

We now commence to investigate the circumstances with regard to a constrained system. There is nothing to be said as to the restraining wrench when the freedom is of the first order. Of course, in this case, as every movement of the body can only be a twist about the screw which prescribes its freedom, the restraining wrench is provided by the reactions of the constraints. It is only where the body has liberty to abandon its original instantaneous screw that the theory of the restraining wrench becomes significant.

If a rigid body has *two degrees of freedom*, then it is free to twist about every screw on a certain cylindroid. If the body be set initially in motion by a twist velocity about some one screw on the surface, then, in general, it will not remain twisting about this screw. A movement will take place by which the instantaneous axis gradually comes into coincidence with other screws on the cylindroid. If we impose a restraining wrench η, of course θ can be maintained as the instantaneous screw; η is reciprocal to θ. It may be compounded with any reactions of the constraints of the system. Thus, given θ, there is an entire screw system of the fifth order, any one screw of which may be taken as the restrainer. Of this system there is one, but only one, which lies on the cylindroid itself. There are many advantages in taking it as the restraining wrench, and it entails no sacrifice of generality; we therefore have the following statement:—To each screw on the cylindroid, regarded as an instantaneous screw, will correspond one screw, also on the cylindroid, as a restraining screw.

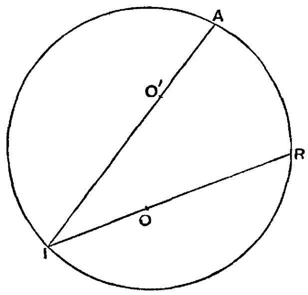

Fig. 1.

The position of this restraining screw is found at once, by the property that it must be reciprocal to the instantaneous screw. If we employ the circular representation for the screws on the cylindroid (fig. 1), and if O be the pole of the axis of pitch, then it is known that the extremities of any chord, such as IR drawn through O, will correspond to two reciprocal screws. If therefore I be the instantaneous screw, then R must be the restraining screw. If a body be set in motion by a twist velocity about I, it will be possible, by a suitable wrench applied on the screw corresponding to R to compel the body to continue to twist about I.

Let O' be the pole of the axis of inertia, then, if IA be a chord drawn through O', the points I and A correspond to a pair of conjugate screws of inertia. It further appears that A is the instantaneous screw corresponding to an impulsive wrench on R. Therefore the effect of the wrench on R when applied to control the body twisting about I is to compound its movement with a nascent twist velocity about A. Therefore A must be the accelerating screw corresponding to I. We thus see that—

Of two conjugate screws of inertia, for a rigid system with two degrees of freedom, either is the accelerator for a body animated by a twist velocity about the other.

In the case of freedom of the second order we are enabled to obtain the form of T, from the fact that the emanant vanishes, that is,

$$\dot{\theta}_1 \frac{dT}{d\theta'_1} + \dot{\theta}_2 \frac{dT}{d\theta'_2} = 0.$$

If we assume that T is a homogeneous function of the second degree in $\dot{\theta}_1$ and $\dot{\theta}_2$, the solution of this equation gives

$$T = L\dot{\theta}_1^2 + 2S\dot{\theta}_1\dot{\theta}_2 + M\dot{\theta}_2^2 + H(\theta'_1\dot{\theta}_2 - \theta'_2\dot{\theta}_1)^2 + (\theta'_1\dot{\theta}_2 - \theta'_2\dot{\theta}_1)(A\dot{\theta}_1 + B\dot{\theta}_2).$$

If we further suppose that θ'_1 and θ'_2 are so small that their squares may be neglected, then the term multiplied by H may be discarded, and we have

$$T = L\dot{\theta}_1^2 + 2S\dot{\theta}_1\dot{\theta}_2 + M\dot{\theta}_2^2 + (\theta'_1\dot{\theta}_2 - \theta'_2\dot{\theta}_1)(A\dot{\theta}_1 + B\dot{\theta}_2),$$

whence

$$\frac{dT}{d\theta'_1} = + \dot{\theta}_2(A\dot{\theta}_1 + B\dot{\theta}_2)\,; \quad \frac{dT}{d\theta'_2} = - \dot{\theta}_1(A\dot{\theta}_1 + B\dot{\theta}_2).$$

Thus we get, for the co-ordinates of the restraining screw, supposing the screws of reference to be reciprocal,

$$\eta''_1 = \frac{1}{p_1}\frac{dT}{d\theta'_1} = + \frac{\dot{\theta}_2}{p_1}(A\dot{\theta}_1 + B\dot{\theta}_2); \quad \eta''_2 = \frac{1}{p_2}\frac{dT}{d\theta'_2} = - \frac{\dot{\theta}_1}{p_2}(A\dot{\theta}_1 + B\dot{\theta}_2).$$

We easily verify that

$$p_1\eta''_1\dot{\theta}_1 + p_2\eta''_2\dot{\theta}_2 = 0,$$

which is, of course, merely another way of expressing that η and θ are reciprocal.

It may be useful to show how the form of T, just obtained, can be derived from direct calculation. I merely set down here the steps of the work and the final result.

Let us take any two screws on the cylindroid α and β, and let their co-ordinates, when referred to the absolute screws of inertia, be

$$\alpha_1, \ldots \alpha_6,$$

and

$$\beta_1, \ldots \beta_6.$$

Then any other screw on the cylindroid, about which the body has been displaced by a twist, will have, for co-ordinates,

$$\begin{gathered} \alpha_1\theta'_1 + \beta_1\theta'_2, \\ \alpha_2\theta'_1 + \beta_2\theta'_2, \\ \cdots \\ \alpha_6\theta'_1 + \beta_6\theta'_2, \end{gathered}$$

and the screw about which the body is twisting, with a twist velocity $\dot{\theta}$, will have, for co-ordinates,

$$\begin{gathered} \alpha_1\dot{\theta}_1 + \beta_1\dot{\theta}_2, \\ \cdots \\ \alpha_6\dot{\theta}_1 + \beta_6\dot{\theta}_2. \end{gathered}$$

We find, for the kinetic energy, so far as the terms involving θ'_1 and θ'_2 are concerned, the expression

$$(\theta'_1\dot{\theta}_2 - \theta'_2\dot{\theta}_1)(A\dot{\theta}_1 + B\dot{\theta}_2)$$

where

$$\begin{aligned} A = &+ bc(\alpha_1 + \alpha_2)[(\alpha_5 - \alpha_6)(\beta_3 - \beta_4) - (\alpha_3 - \alpha_4)(\beta_5 - \beta_6)] \\ &\qquad + (b^2 - c^2)(\alpha_3 + \alpha_4)(\alpha_5 + \alpha_6)(\beta_1 + \beta_2), \\ &+ ca(\alpha_3 + \alpha_4)[(\alpha_1 - \alpha_2)(\beta_5 - \beta_6) - (\alpha_5 - \alpha_6)(\beta_1 - \beta_2)] \\ &\qquad + (c^2 - a^2)(\alpha_5 + \alpha_6)(\alpha_1 + \alpha_2)(\beta_3 + \beta_4), \\ &+ ab(\alpha_5 + \alpha_6)[(\alpha_3 - \alpha_4)(\beta_1 - \beta_2) - (\alpha_1 - \alpha_2)(\beta_3 - \beta_4)] \\ &\qquad + (a^2 - b^2)(\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4)(\beta_5 + \beta_6). \end{aligned}$$

$$B = + bc(\beta_1 + \beta_2)[(\alpha_5 - \alpha_6)(\beta_3 - \beta_4) - (\alpha_3 - \alpha_4)(\beta_5 - \beta_6)]$$
$$- (b^2 - c^2)(\alpha_1 + \alpha_2)(\beta_3 + \beta_4)(\beta_5 + \beta_6)$$
$$+ ca(\beta_3 + \beta_4)[(\alpha_1 - \alpha_2)(\beta_5 - \beta_6) - (\alpha_5 - \alpha_6)(\beta_1 - \beta_2)]$$
$$- (c^2 - a^2)(\alpha_3 + \alpha_4)(\beta_5 + \beta_6)(\beta_1 + \beta_2)$$
$$+ ab(\beta_5 + \beta_6)[(\alpha_3 - \alpha_4)(\beta_1 - \beta_2) - (\alpha_1 - \alpha_2)(\beta_3 - \beta_4)]$$
$$- (a^2 - b^2)(\alpha_5 + \alpha_6)(\beta_1 + \beta_2)(\beta_3 + \beta_4).$$

We now write the equations of motion for a body with two degrees of freedom, and unacted upon by any force, the reference being made to the principal screws of inertia.

We have, from the general equations (p. 622),

$$Mu_1^2 \ddot{\theta}_1 = \frac{dT}{d\theta'_1},$$
$$Mu_2^2 \ddot{\theta}_2 = \frac{dT}{d\theta'_2}.$$

Introducing the value just obtained for T,

$$Mu_1^2 \ddot{\theta}_1 = + \dot{\theta}_2 (A\theta_1 + B\dot{\theta}_2),$$
$$Mu_2^2 \ddot{\theta}_2 = - \dot{\theta}_1 (A\dot{\theta}_1 + B\dot{\theta}_2).$$

There must be one screw on the cylindroid, for which

$$A\dot{\theta}_1 + B\dot{\theta}_2 = 0.$$

This screw will have the accelerations $\ddot{\theta}_1$ and $\ddot{\theta}_2$, both zero, and thus we have the following theorem:—

If a rigid system has two degrees of freedom, then, among the screws about which it is at liberty to twist, there is one, and in general only one, which has the property of a permanent screw.

The existence of a single permanent screw in the case of freedom of the second order seems a noteworthy point in abstract Dynamics. The analogy here ceases entirely between the permanent screws and the principal screws of inertia. Of the latter there are two on the cylindroid.

Let N (fig. 2) be the critical point on the circle which corresponds to the permanent screw. Let P be a screw θ, the twist velocity about which is $\dot{\theta}$. Let u_θ be a linear parameter appropriate to the screw θ, such that $Mu_\theta^2\dot{\theta}^2$ is the kinetic energy.

If the body be set twisting about N it will stay there. If set twisting about any other screw θ the instantaneous screw will gradually change. We shall first study the rate at which the instantaneous screw shifts its position.

Let Q be an adjacent position, then, by Ptolemy's theorem,

$$\dot{\theta}^2 PQ + \dot{\theta}_1\dot{\phi}_2 = \dot{\theta}_2\dot{\phi}_1,$$

but

$$\dot{\phi}_1 = \dot{\theta}_1 + \ddot{\theta}_1\delta t,$$

$$\dot{\phi}_2 = \dot{\theta}_2 + \ddot{\theta}_2\delta t,$$

whence

$$\dot{\theta}^2 PQ = (\dot{\theta}_2\ddot{\theta}_1 - \dot{\theta}_1\ddot{\theta}_2)\,\delta t,$$

$$\frac{PQ}{\delta t} = \frac{\dot{\theta}_2\ddot{\theta}_1 - \dot{\theta}_1\ddot{\theta}_2}{\dot{\theta}^2},$$

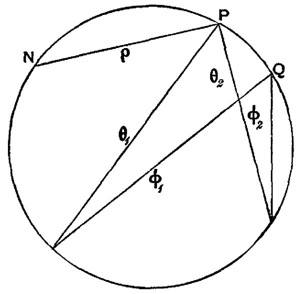

Fig. 2.

which is, accordingly, the rate at which P will change its position. If we substitute for $\ddot{\theta}_1$ and $\ddot{\theta}_2$ their values already found in the last article, we obtain for the velocity of P the expression

$$\frac{1}{M}\frac{u_1^2\dot{\theta}_1^2 + u_2^2\dot{\theta}_2^2}{u_1^2 u_2^2\dot{\theta}^2}(A\dot{\theta}_1 + B\dot{\theta}_2).$$

If N be the position of the permanent screw, and if ρ be the length of the chord PN, then the expression just written assumes the form

$$k\rho\dot{\theta}u_\theta^2$$

where k is a constant.

This expression illustrates the character of the screw corresponding to N. If ρ be zero, then the expression for this velocity vanishes. This means that P has no tendency to abandon N; in other words, that the screw corresponding to N is permanent.

We may present the velocity of P in a purely geometrical aspect, as follows:—Let XT (fig. 3) be the axis of inertia, then u_θ^2 is proportional to the perpendicular from P on T. It follows that the velocity of P for a given twist velocity is proportional to the product of PN and PT.

If O' be the pole of XT with regard to the circle, then the acceleration

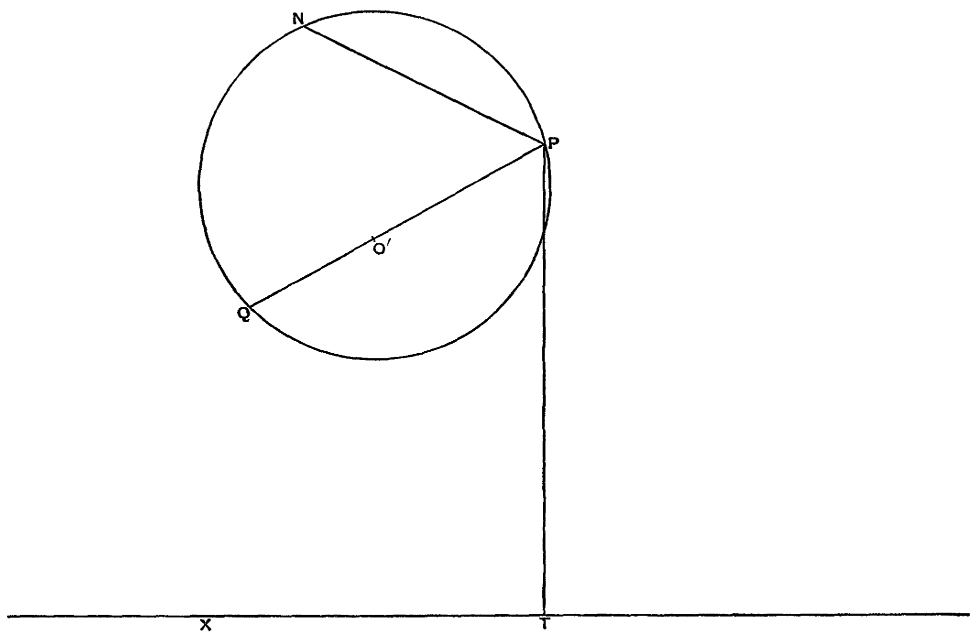

Fig. 3.

is a twist about the screw corresponding to Q, and the magnitude of the acceleration varies as $\dot{\theta}^2 \times PO' \times PN$.

The importance of the theory relating to the existence of a single permanent screw on the cylindroid makes it worth while to investigate the question from another point of view.

Let us think of any cylindroid S placed quite arbitrarily with respect to the position of the rigid body. A certain restraining screw η will correspond to each screw θ on S. As θ moves over the cylindroid, so must the corresponding screw η describe some other ruled surface S'. The two surfaces, S and S', will thus have two corresponding systems of screws, whereof every two correspondents are reciprocal. One screw can be discovered on S', which is reciprocal, not alone to its corresponding θ, but to all the screws on the cylindroid. A wrench on this η can be provided by the reactions of the constraints, and, consequently, the constraints will in this case, arrest the tendency of the body to depart from θ as the instantaneous screw. It follows that this particular θ is the permanent screw.

We can obtain some interesting results by performing the actual calculation of the relations between η and the cylindroid.

A set of forces applied to a rigid system has components X, Y, Z at a point, and three corresponding moments F, G, H in the rectangular planes of reference.

Let p be the pitch of the screw on which the wrench thus represented lies, and let x, y, z be the co-ordinates of any point on this screw. Then, in the plane of Z the moments of the forces are $xY - yX$, and if to this be added pX, the whole must equal H.

Thus we have the three equations, so well known in statics,

$$F = pX + yZ - zY,$$

$$G = pY + zX - xZ,$$

$$H = pZ + xY - yX.$$

The centrifugal acceleration on a point P is, of course, $\omega^2 PH$, where ω is the angular velocity, and PH the perpendicular let fall on the axis. The three components of this force are X', Y', Z', where

$$X' = \omega^2 \sin\theta\,(x \sin\theta - y \cos\theta),$$

$$Y' = \omega^2 \cos\theta\,(y \cos\theta - x \sin\theta),$$

$$Z' = \omega^2\,(z - m \sin 2\theta),$$

and the three moments are F', G', H' where

$$F' = \omega^2 \sin\theta\,(yz \sin\theta + xz \cos\theta - 2my \cos\theta),$$

$$G' = \omega^2 \cos\theta\,(-\,yz \sin\theta - xz \cos\theta + 2mx \sin\theta),$$

$$H' = \omega^2\,((y^2 - x^2) \sin\theta \cos\theta + xy \cos 2\theta).$$

We are now to integrate these expressions over the entire mass, and we employ the following abbreviations :—

$$\int xdm = M\alpha\,; \quad \int ydm = M\beta\,; \quad \int zdm = M\gamma\,;$$

$$\int x^2dm = Ma^2\,; \quad \int y^2dm = Mb^2\,; \quad \int z^2dm = M\gamma^2\,;$$

$$\int xydm = Mr^2\,; \quad \int xzdm = Mq^2\,; \quad \int yzdm = Mp^2\,;$$

$$X = \int X'dm\,; \quad Y = \int Y'dm\,; \quad Z = \int Z'dm\,;$$

$$F = \int F'dm\,; \quad G = \int G'dm\,; \quad H = \int H'dm\,;$$

then, omitting the factor $M\omega^2$, we have

$$X = +\,(\alpha \sin\theta - \beta\cos\theta)\sin\theta,$$

$$Y = -\,(\alpha \sin\theta - \beta\cos\theta)\cos\theta,$$

$$Z = \gamma - m\sin 2\theta.$$

$$F = +\sin\theta\,(p^2\sin\theta + q^2\cos\theta) - 2m\beta\sin\theta\cos\theta,$$

$$G = -\cos\theta\,(p^2\sin\theta + q^2\cos\theta) + 2m\alpha\sin\theta\cos\theta,$$

$$H = (a^2 - b^2)\sin\theta\cos\theta + r^2\cos 2\theta.$$

We can easily verify that

$$FY - GX = 2mXY.$$

We now examine the points on the cylindroid intersected by the axis of the screw

$$F = pX + yZ - zY,$$

$$G = pY + zX - xZ,$$

$$H = pZ + xY - yX.$$

We write the equations of the cylindroid in the form

$$x = R\cos\phi\,; \quad y = R\sin\phi\,; \quad Z = m\sin^2\phi,$$

then, eliminating p and R, and making

$$U = X^2 + Y^2 + Z^2,$$

$$V = FX + GY + HZ,$$

we find, after a few reductions,

$$\tan^3\phi\,(YV - GU) + \tan^2\phi\,(XV - FU + 2mXU)$$
$$+ \tan\phi\,(YV - GU - 2m\,YU) + XV - FU = 0.$$

This cubic corresponds, of course, to the three generators of the cylindroid which the ray intersects.

If we put

$$FY - GX = 2mXY.$$

then the cubic becomes, by eliminating m,

$$(Y\tan\phi + X)(X(YV - GU)\tan^2\phi + (XV - FU)\,Y) = 0.$$

The factor $Y\tan\phi + X$ simply means that the restraining screw cuts the instantaneous screw at right angles.

The two other screws in which η intersects the cylindroid are given by the equation

$$(XYV - XGU)\tan^2\phi + (XYV - FUY) = 0.$$

These two screws are of equal pitch, and the value of the pitch is

$$\frac{p_1(XYV - XGU) + p_2(FUY - XYV)}{U(FY - GX)};$$

where p_1 and p_2 are the pitches of the two principal screws on the cylindroid, the expression becomes, after a few reductions,

$$-\frac{V}{U} + \frac{(p^2 - \alpha p_1)\sin\theta + (q^2 + \beta p_2)\cos\theta}{\alpha\sin\theta - \beta\cos\theta}.$$

This is the pitch of the two equal pitch screws on the cylindroid which η intersects. If η is to be reciprocal to the cylindroid, then, of course, the pitch of η itself should be equal and opposite in value to this expression. Hence the permanent screw on the cylindroid is given by

$$(p^2 - \alpha p_1)\sin\theta + (q^2 + \beta p_2)\cos\theta = 0.$$

We notice here the somewhat remarkable circumstance, that if

$$p^2 - \alpha p_1 = 0, \quad \text{and} \quad q^2 + \beta p_2 = 0,$$

then *all* the screws on the cylindroid are permanent screws.

It hence appears that if two screws on a cylindroid are permanent, then every screw on the cylindroid is permanent.

The investigation of these points opens up an interesting theory which we do not here further refer to.

VIII.—THREE DEGREES OF FREEDOM.

Let us now specially consider the case of a rigid body which has freedom of the third order. On account of the evanescence of the emanant we have

$$\dot{\theta}_1 \frac{dT}{d\theta'_1} + \dot{\theta}_2 \frac{dT}{d\theta'_2} + \dot{\theta}_3 \frac{dT}{d\theta'_3} = 0.$$

It is well known that if U, V, W be three conics whose equations submit to the condition

$$xU + yV + zW = 0,$$

those conics must have three common intersections.

It therefore follows that the three equations

$$\frac{dT}{d\theta'_1} = 0, \quad \frac{dT}{d\theta'_2} = 0, \quad \frac{dT}{d\theta'_3} = 0$$

must have three common screws. These are, of course, the permanent screws, and, accordingly, we have the theorem :—

A rigid system which has freedom of the third order has, in general, three permanent screws.

There will be a special convenience for the study of the subject in taking these three screws as the screws of reference. We shall use the plane representation of the three-system, and the equations of the conics will be

$$A_1\dot{\theta}_2\dot{\theta}_3 + B_1\dot{\theta}_3\dot{\theta}_1 + C_1\dot{\theta}_1\dot{\theta}_2 = 0, \text{ or } U = 0,$$

$$A_2\dot{\theta}_2\dot{\theta}_3 + B_2\dot{\theta}_3\dot{\theta}_1 + C_2\dot{\theta}_1\dot{\theta}_2 = 0, \text{ ,, } V = 0,$$

$$A_3\dot{\theta}_2\dot{\theta}_3 + B_3\dot{\theta}_3\dot{\theta}_1 + C_3\dot{\theta}_1\dot{\theta}_2 = 0, \text{ ,, } W = 0;$$

but, as

$$\dot{\theta}_1 U + \dot{\theta}_2 V + \dot{\theta}_3 W = 0,$$

identically we must have

$$B_1 = 0; \quad A_2 = 0; \quad A_3 = 0;$$

$$C_1 = 0; \quad C_2 = 0; \quad B_3 = 0;$$

and also

$$A_1 + B_2 + C_3 = 0.$$

For symmetry we may write

$$A_1 = \mu - \nu; \quad B_2 = \nu - \lambda; \quad C_3 = \lambda - \mu.$$

We thus find that when T is referred to the three principal screws of the system, its expression must be

$$T = a\dot{\theta}_1^2 + b\dot{\theta}_2^2 + c\dot{\theta}_3^2 + 2f\dot{\theta}_2\dot{\theta}_3 + 2g\dot{\theta}_1\dot{\theta}_3 + 2h\dot{\theta}_1\dot{\theta}_2,$$

$$+ (\mu - \nu)\, \theta'_1\dot{\theta}_2\dot{\theta}_3 + (\nu - \lambda)\, \theta'_2\dot{\theta}_3\dot{\theta}_1 + (\lambda - \mu)\, \theta'_3\dot{\theta}_1\dot{\theta}_2.$$

We now express the general dynamical equations of a three-system.

Let η'' be the intensity of any wrench acting on a screw η belonging to the system, and let $2\varpi_{1\eta}$ represent the virtual coefficient between η and the first of the three screws of reference.

Then, substituting for T in Lagrange's equations, we have

$$+ a\ddot{\theta}_1 + h\ddot{\theta}_2 + g\ddot{\theta}_3 - (\mu - \nu)\, \dot{\theta}_2\dot{\theta}_3 = \varpi_{1\eta}\eta'',$$

$$+ h\ddot{\theta}_1 + b\ddot{\theta}_2 + f\ddot{\theta}_3 - (\nu - \lambda)\, \dot{\theta}_3\dot{\theta}_1 = \varpi_{2\eta}\eta'',$$

$$+ g\ddot{\theta}_1 + f\ddot{\theta}_2 + c\ddot{\theta}_3 - (\lambda - \mu)\, \dot{\theta}_1\dot{\theta}_2 = \varpi_{3\eta}\eta''.$$

These equations are general. If η be the restraining screw, then an appropriate wrench η'' should be capable of annihilating the acceleration, *i. e.* of rendering

$$\ddot{\theta}_1 = 0; \quad \ddot{\theta}_2 = 0; \quad \ddot{\theta}_3 = 0,$$

whence the position of η, and the intensity η'' are indicated by the equations

$$(\nu - \mu)\, \dot{\theta}_2\dot{\theta}_3 = \varpi_{1\eta}\eta'',$$

$$(\lambda - \nu)\, \dot{\theta}_3\dot{\theta}_1 = \varpi_{2\eta}\eta'',$$

$$(\mu - \lambda)\, \dot{\theta}_1\dot{\theta}_2 = \varpi_{3\eta}\eta''.$$

We can now exhibit the nature of the correspondence between η and θ, for

$$\varpi_{1_\eta} = p_1\eta_1 + \varpi_{12}\eta_2 + \varpi_{13}\eta_3,$$

$$\varpi_{2_\eta} = \varpi_{12}\eta_1 + p_2\eta_2 + \varpi_{32}\eta_3,$$

$$\varpi_{3_\eta} = \varpi_{13}\eta_1 + \varpi_{32}\eta_2 + p_3\eta_3.$$

If we make $H = \dot{\theta}_1\dot{\theta}_2\dot{\theta}_3 \div \eta''$, and omit the dots over θ_1, &c., we have

$$\theta_1 (p_1\eta_1 + \varpi_{12}\eta_2 + \varpi_{13}\eta_3) = H(\gamma - \beta),$$

$$\theta_2 (\varpi_{12}\eta_1 + p_2\eta_2 + \varpi_{32}\eta_3) = H(\alpha - \gamma),$$

$$\theta_3 (\varpi_{13}\eta_1 + \varpi_{32}\eta_2 + p_3\eta_3) = H(\beta - \alpha).$$

A symmetrical method of expressing these three equations will be to reduce them to the homogeneous forms, viz.

$$\theta_1 L + \theta_2 M + \theta_3 N = 0,$$

$$\alpha\theta_1 L + \beta\theta_2 M + \gamma\theta_3 N = 0,$$

where

$$L = \tfrac{1}{2}\frac{dp_\eta}{d\eta_1}; \quad M = \tfrac{1}{2}\frac{dp_\eta}{d\eta_2}; \quad N = \tfrac{1}{2}\frac{dp_\eta}{d\eta_3}.$$

For further development we now employ the plane representative of the system of the screws. We learn that η must lie on the polar of the point $\theta_1, \theta_2, \theta_3$ with respect to the pitch conic, or the locus of all the screws for which

$$p_\eta = 0.$$

We also see that η must lie on the polar of the point $\alpha\theta_1, \beta\theta_2, \gamma\theta_3$ with regard to the pitch conic.

We thus obtain a geometrical construction by which we are enabled to discover the restraining screw when once the instantaneous screw is given.

Two homographic systems are first to be conceived. A point of the first system, of which the co-ordinates are $\theta_1, \theta_2, \theta_3$ has as its correspondent a point in the second system, with co-ordinates $\alpha\theta_1, \beta\theta_2, \gamma\theta_3$. The three double points of the homography correspond, of course, to the permanent screws.

To find the restraining screw η corresponding to a given instantaneous screw θ, we join θ to its homographic correspondent, and the pole of this ray, with respect to the pitch conic, is the position of η.

The pole of the same ray, with regard to the conic of inertia, is the accelerator. It seems hardly possible to have any more vivid geometrical picture of the relation between η and θ than that which these theorems afford.

We have already shown that there is a single steady screw in every mechanical system with two degrees of freedom. We can demonstrate this in a different manner as a deduction from the case of the three-system.

Consider a cylindroid in a three-system, *i. e.* a straight line AB (fig. 4). If the movements of the body be limited to twists about the screws on the cylindroid, there will be reactions about the pole of this ray P with respect to the pitch conic, in addition to the reactions of the three-system.

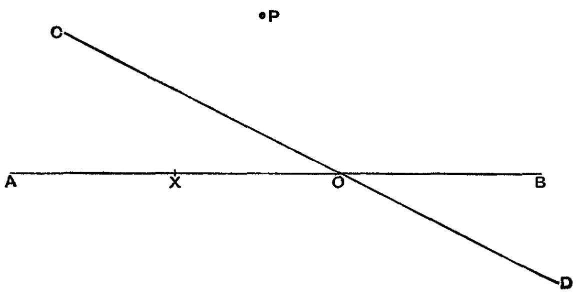

Fig. 4.

The permanent screw on this cylindroid will be one whereon the restraining screw coincides with P. In general, the homographic points corresponding to the points on the ray AB will form a ray CD. The intersection O, regarded as on CD, will be the correspondent of X on AB. The restrainer corresponding to X will therefore lie at P, and will be provided by the constraints. Accordingly, X is a permanent screw on the cylindroid, and it is obvious from the construction that there can be no other screw of the same character.

We can also deduce the expression for T in the two-system, from the expression of the more general type in the three-system; for we have

$$T = T_0 + (\mu - \nu)\,\theta'_1\dot{\theta}_2\dot{\theta}_3 + (\nu - \lambda)\,\theta'_2\dot{\theta}_3\dot{\theta}_1 + (\lambda - \mu)\,\theta'_3\dot{\theta}_1\dot{\theta}_2.$$

Consider any screw on the cylindroid defined by

$$\theta'_1 = P\theta'_2 + Q\theta'_3,$$
$$\dot{\theta}_1 = P\dot{\theta}_2 + Q\dot{\theta}_3;$$

substituting, we obtain,

$$(\theta'_3\dot{\theta}_2 - \theta'_2\dot{\theta}_3)\left[(\lambda - \mu)\, P\dot{\theta}_2 + (\lambda - \nu)\, Q\dot{\theta}_3\right],$$

which we already know to be the form of the function in the case of the two-system.

IX.—REMAINING DEGREES OF FREEDOM.

The permanent screws in freedom of the fourth order are to be investigated in the following manner:—If a screw θ be permanent, the corresponding η must be provided by the reactions of the constraints. All the reactions in freedom of the fourth order lie on the screws of a cylindroid. On a given cylindroid three possible η screws can be found. For, if we substitute $\alpha_1 + \lambda\beta_1$, $\alpha_2 + \lambda\beta_2$, &c., for η_1, η_2, &c., in the equation

$$\frac{b^2 - c^2}{a}\,\frac{\eta_1 + \eta_2}{\eta_1 - \eta_2} + \frac{c^2 - a^2}{b}\,\frac{\eta_3 + \eta_4}{\eta_3 - \eta_4} + \frac{a^2 - b^2}{c}\,\frac{\eta_5 + \eta_6}{\eta_5 - \eta_6} = 0,$$

we obtain a cubic for λ. The three roots of this cubic correspond to three η screws. Take the θ corresponding to one of the η screws, then, of course, θ will not, in general, belong to the four-system. We can, however, assign to θ any pitch we like, and as it intersects η at right angles, it must cut two other screws of equal pitch on the cylindroid. Give to θ a pitch equal and opposite to that of the three latter screws, then θ is reciprocal to the cylindroid, and therefore it belongs to the four-system. We thus have a permanent screw of the system, and accordingly we obtain the following result:—

In the case of a rigid body with freedom of the fourth order there are, in general, three, and only three, permanent screws.

In freedom of the fifth order, the screws about which the system can be twisted are all reciprocal to a single screw ρ. In general, ρ does not lie on the system prescribed by the equation which all possible η co-ordinates satisfy. It is therefore, in general, not possible that the reaction of the constraints can provide an η. There are, however, three permanent screws in the five-system, the existence of which is thus demonstrated:—Draw through the centre of inertia of the body the three principal axes, then, on each of these axes one screw can always be found which is reciprocal to ρ. Each of these will belong to the five-system, and it is obvious from the property of the principal axes, that if the body be set twisting about one of these screws it will have no tendency to depart therefrom, and no call is made on the constraints to provide a restraining wrench.

If the body have freedom of the sixth order it is then perfectly free. Any screw on one of the principal axes through the centre of inertia is a permanent screw, and, consequently, there is in this case a triply infinite number of permanent screws.

The results obtained show that for a rigid body with the several degrees of freedom the permanent screws are as follows:—

		No. of Permanent Screws.
Freedom	I.,	1.
,,	II.,	1.
,,	III.,	3.
,,	IV.,	3.
,,	V.,	3.
,,	VI.,	Triply infinite.

Addendum.

The following are the references to the preceding eight Memoirs on the Theory of Screws:—

1. "The Theory of Screws. A Geometrical Study of the Kinematics, Equilibrium, and small Oscillations of a Rigid Body."—*Transactions*, *Royal Irish Academy*, vol. xxv., Science, pp. 157–217 (1872).
2. "Researches in the Dynamics of a Rigid Body by the aid of the Theory of Screws."—*Philosophical Transactions of the Royal Society*, *London*, 1874, pp. 15–40.
3. "Screw Co-ordinates and their Applications to Problems in the Dynamics of a Rigid Body."—*Transactions*, *R. I. A.*, vol. xxv., pp. 259–327 (1874).

4. "Extension of the Theory of Screws to the Dynamics of any Material System."—*Transactions*, *R. I. A.*, vol. xxviii., pp. 99–136 (1881).
5. "Certain Problems in the Dynamics of a Rigid System moving in Elliptic Space."—*Transactions*, *R. I. A.*, vol. xxviii., pp. 159–184 (1881).
6. "Dynamics and Modern Geometry. A new Chapter in the Theory of Screws."—*Cunningham Memoirs*, *R. I. A.*, No. IV., pp. 1–44 (1886).
7. "On the Plane Sections of the Cylindroid."—*Transactions*, *R. I. A.*, vol. xxix., pp. 1–32 (1887).
8. "How Plane Geometry illustrates General Problems in the Dynamics of a Rigid Body with Three Degrees of Freedom."—*Transactions*, *R. I. A.*, vol. xxix., pp. 247–284 (1888).

I published, in 1876, a work entitled, "Theory of Screws" (8vo, pp. 1–194, Dublin: Hodges, Figgis, & Co.), in which an account of the Theory, in so far as it had then been developed, may be found.

A German translation of this book, and of most of the subsequent Memoirs, has recently been made by Herr Gravelius. This volume contains the most complete account of the Theory that has yet been given in one work. The title is, "Theoretische Mechanik Starrer Systeme auf Grund der Methoden und Arbeiten und mit einem Vorworte von Sir Robert S. Ball. Herausgegeben von Harry Gravelius." 8vo, pp. 1–619. Georg Reimer. Berlin, 1889.