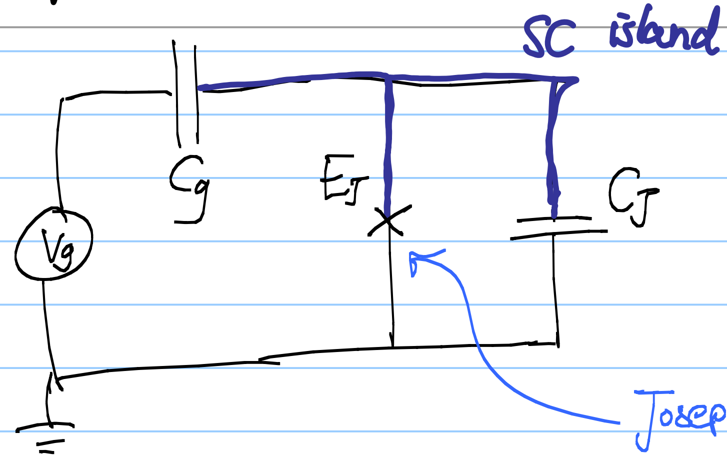
The Cooper pair box

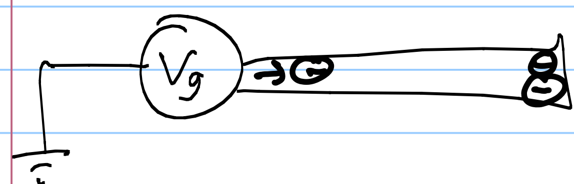
Note Title 11/11/2008

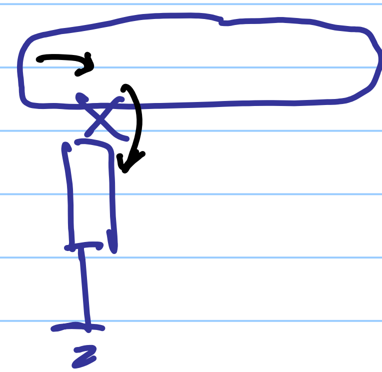


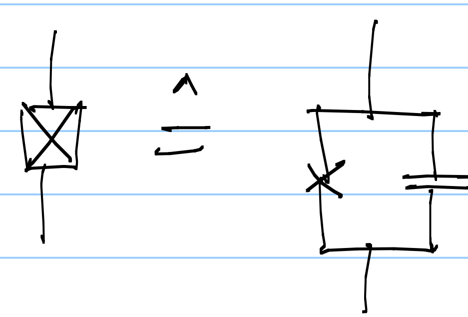
E_J : Josephson junction
C_g : gate capacitance
C_J : junction capacitance

charging energy :

$$E_c = \frac{e^2}{2C_\Sigma}, \qquad C_\Sigma = C_g + C_J$$







offset charge $n_g = C_g V_g / 2e$

$$H = 4E_c (n - n_g)^2 - E_J \cos\varphi$$

"operator" as well

operator!
number of additional Cooper pairs on the SC island

⚠ Note: n and φ are conjugate, i.e. "$[n, \varphi] = i$"

Harmonic oscillator:

$$H = \left(i\frac{d}{d\xi}\right)^2 + \xi^2 \qquad = p^2 + x^2$$

position ("x")

momentum ("p")

▷ position and momentum are sort of the same here!

Momentum in position basis is $p = \frac{\hbar}{i}\frac{d}{dx}$

$$\rightarrow \quad [p, x] = \frac{\hbar}{i}\left[\frac{d}{dx}, x\right] = \frac{\hbar}{i}\left(\frac{d}{dx}x - x\frac{d}{dx}\right) = \frac{\hbar}{i} = -i\hbar$$

$\left(x = -\frac{\hbar}{i}\frac{d}{dp} \right.$ this is what we get when representing x in the momentum basis)

(1) Phase basis: Here φ will be our position, just like x

$$n \to i\frac{d}{d\varphi}$$

$$[n, \varphi] = i\left[\frac{d}{d\varphi}, \varphi\right] = i$$

$$H = \underbrace{4E_c\left(i\frac{d}{d\varphi} + n_g\right)^2}_{\text{kinetic energy term}} - \underbrace{E_J \cos\varphi}_{\text{potential } V(\varphi)}$$

(2) Charge basis:

⚠ charge is quantized : $\hat{n}$ has a discrete spectrum

$$\varphi \to \cancel{-i\frac{d}{dn}}$$ fails because n is discrete

$$e^{i\varphi} \longrightarrow \sum_{n=-\infty}^{+\infty} |n\rangle\langle n+1|$$

$$\hat{n}|n\rangle = n|n\rangle \qquad n = -\infty \cdots +\infty$$

charge operator

$\{|n\rangle\}$ is an ONB

e.g. some particular charge state $|m\rangle$

$$e^{i\varphi}|m\rangle = \sum_{n=-\infty}^{+\infty} |n\rangle \underbrace{\langle n+1|m\rangle}_{\delta_{n+1,m}} = |m-1\rangle$$

$n+1 = m \Rightarrow n = m-1$

cmp. h.o. case: $a|m\rangle = \sqrt{m}\,|m-1\rangle$

$\Rightarrow$ $e^{i\varphi}$ is a little bit like the harm. osc. lowering operator, EXCEPT FOR

(1) no prefactor

(2) goes all the way down to $-\infty$

$$\cos\varphi = \frac{1}{2}\left(e^{i\varphi} + e^{-i\varphi}\right) = \frac{1}{2}\sum_{n=-\infty}^{+\infty}\Big[\,|n\rangle\langle n+1| + |n+1\rangle\langle n|\,\Big]$$

$$\left(|a\rangle\langle b|\right)^{\dagger} = |b\rangle\langle a|$$

$$e^{-i\varphi} = \left(e^{i\varphi}\right)^{\dagger}$$

$$H = 4E_c(n-n_g)^2 - \frac{E_J}{2}\sum_{n=-\infty}^{+\infty}\Big[\,|n\rangle\langle n+1| + |n+1\rangle\langle n|\,\Big]$$

We can write this as a matrix and diagonalize it.

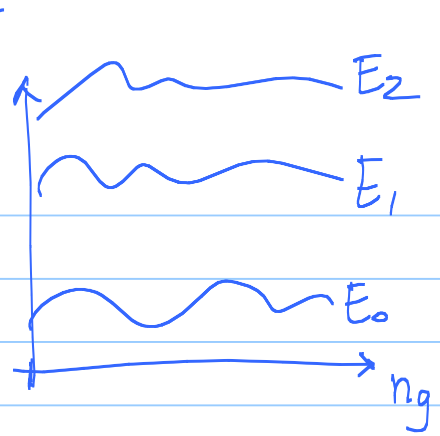
$$\frac{H}{E_c} = 4(n-n_g)^2 - \frac{E_J}{E_c}\,\frac{1}{2}\sum_{n=-\infty}^{+\infty}\Big[\,|n\rangle\langle n+1| + |n+1\rangle\langle n|\,\Big]$$

$\langle n | H | n+4 \rangle = 0 \ldots$

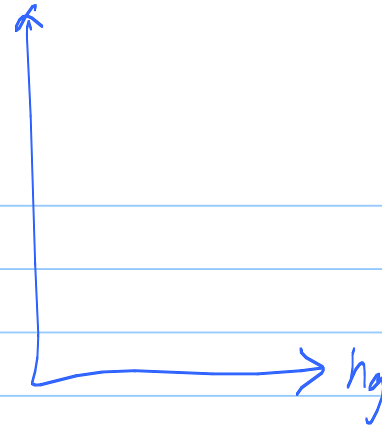
Cutoff: let the overall matrix size $N = 2n+1$
such that

$$
(H)_{mn} = \begin{pmatrix}
\ddots & \ddots & & & & \\
-\frac{E_J}{2E_c} & 4(-2-n_g)^2 & -\frac{E_J}{2E_c} & & & \\
& \ddots & 4(-1-n_g)^2 & \ddots & & \\
& & & 4(0-n_g)^2 & & \\
& & & & 4(+1-n_g)^2 & \\
& & & & & 4(+2-n_g)^2 \\
& & & & & & \ddots
\end{pmatrix}
$$

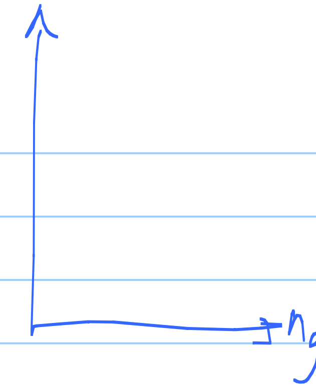
Task: – Construct this matrix!
– Look at the first three eigenenergies
 * for several E_J/E_c (0.5, 10, 30)
 * as a function of n_g



$E_J/E_C = 0.5$



$E_J/E_C = 10$



$E_J/E_C = 30$

Reminder

Consider an operator A. B is the Hermitian conjugate of A, i.e. $B = A^\dagger$

$$:\Leftrightarrow \quad \forall\, |\psi\rangle, |\bar{\psi}\rangle \qquad \langle \bar{\psi} | A \psi \rangle = \langle B \bar{\psi} | \psi \rangle$$

e.g. $$\int_{-\infty}^{+\infty} dx\, \bar{\psi}^*(x)\, A(x)\, \psi(x) = \int dx\, \left(B(x)\, \bar{\psi}(x)\right)^* \psi(x)$$

Now, let's check that indeed $(\underset{A}{|a\rangle\langle b|})^\dagger = \underset{B}{|b\rangle\langle a|}$

Let's choose some $|\psi\rangle$ and $|\bar{\psi}\rangle$

(1) $A|\psi\rangle = |a\rangle \underbrace{\langle b|\psi\rangle}_{\text{number}}$

$\langle\bar{\psi}|A\psi\rangle = \langle\bar{\psi}|a\rangle\langle b|\psi\rangle$

(2) $B|\bar{\psi}\rangle = |b\rangle\langle a|\bar{\psi}\rangle$

$\langle B\bar{\psi}| = (|B\bar{\psi}\rangle)^\dagger = (|b\rangle\langle a|\bar{\psi}\rangle)^\dagger$

$= \langle\bar{\psi}|a\rangle\langle b|$

recall: $\langle a|b\rangle^* = \langle b|a\rangle$

$\Rightarrow \langle B\bar{\psi}|\psi\rangle = \langle\bar{\psi}|a\rangle\langle b|\psi\rangle$

are equal!