• PROGRAM OF "PHYSICS"

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PHYSICS I
(General Mechanics)
02 credits (30 periods)

PHYSICS I

Chapter 5
Rotation of a Rigid Object About a Fixed Axis

Rotational Kinematics

Torque and Angular Acceleration

Moments of Inertia

Rotational Kinetic Energy

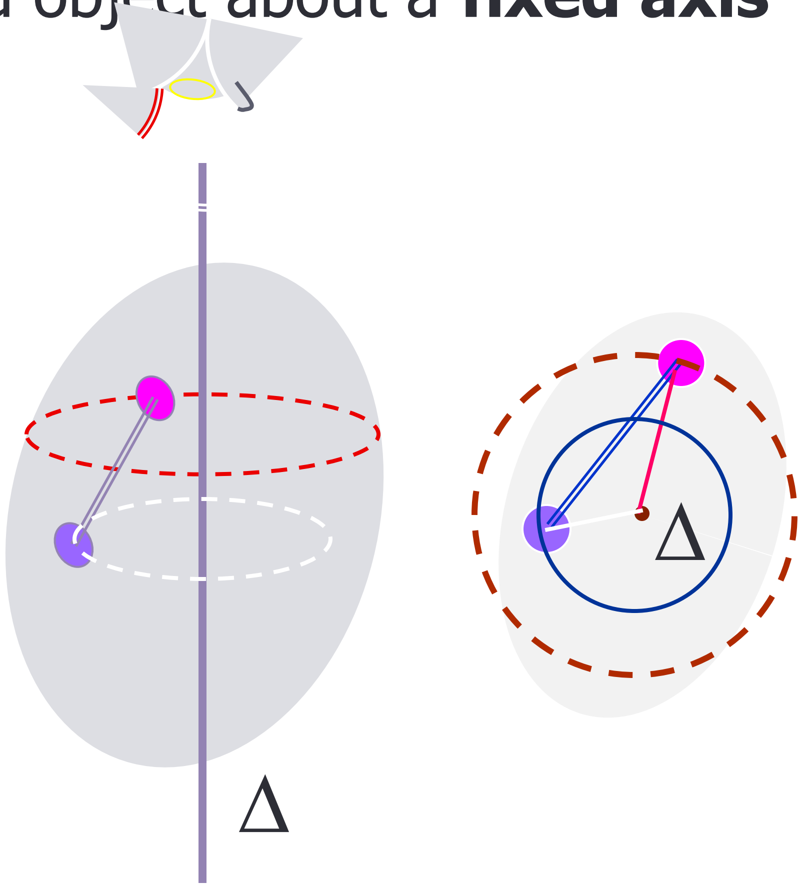
Rolling Motion of a Rigid Object

Angular Momentum of a Rotating Rigid Object

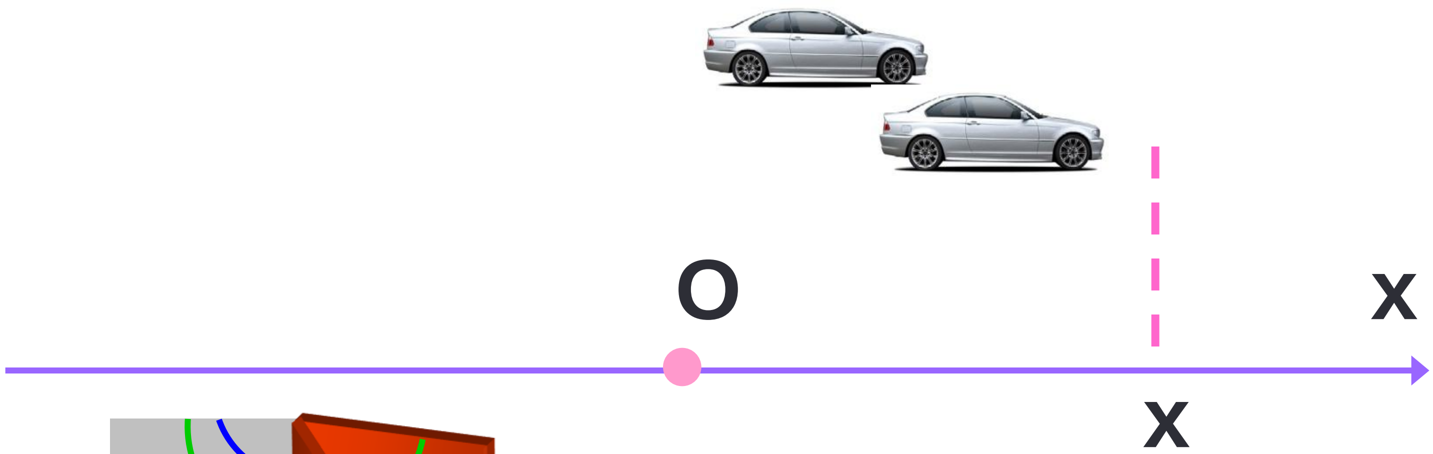
Conservation of Angular Momentum

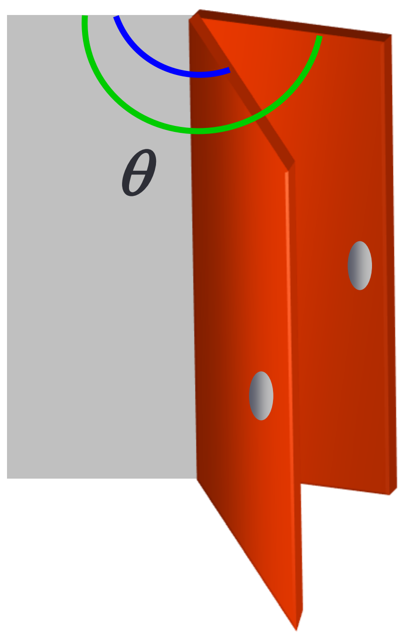
1 Rotational Kinematics

- A **rigid object** is one that is **nondeformable**—that is, it is an object in which the separations between all pairs of particles **remain constant**
- We treat the rotation of a rigid object about a **fixed axis**
- Every point on the object undergoes **circular motion** about the point O
- Every point of the object undergoes the **same angle** in any given time interval

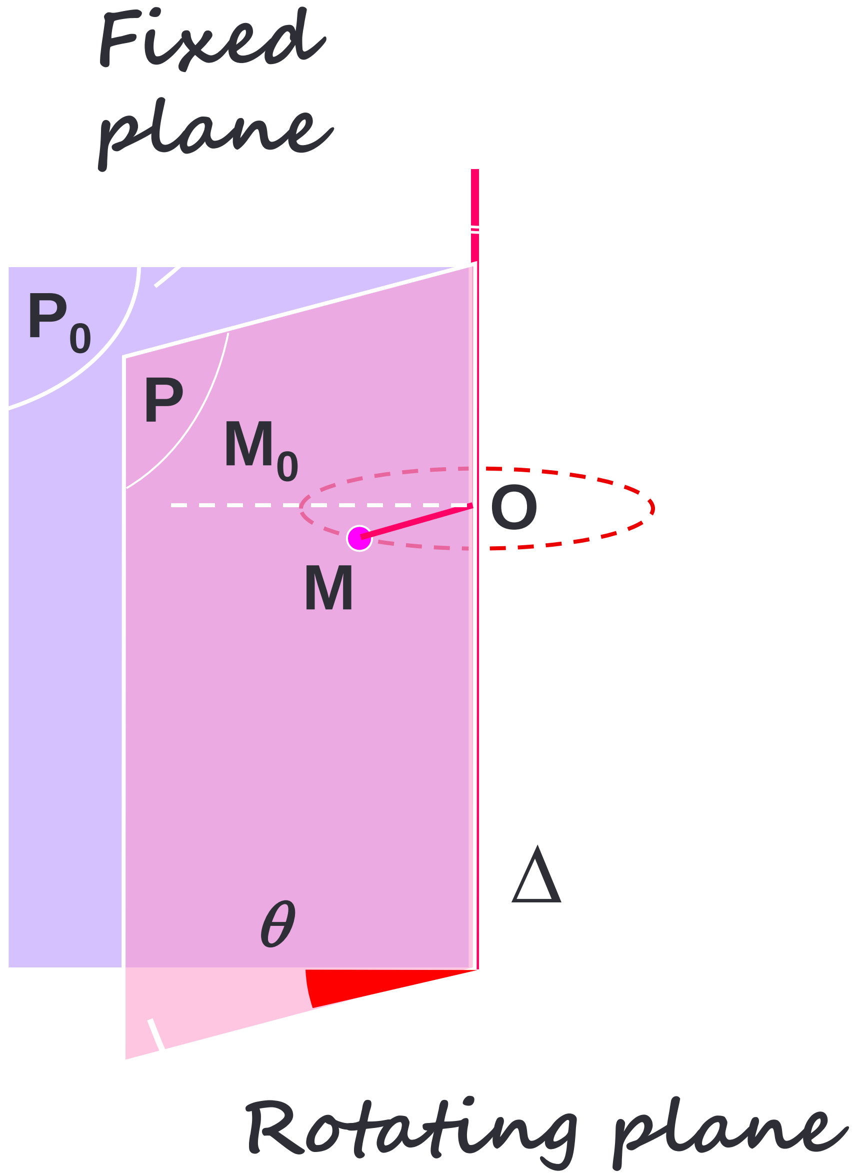


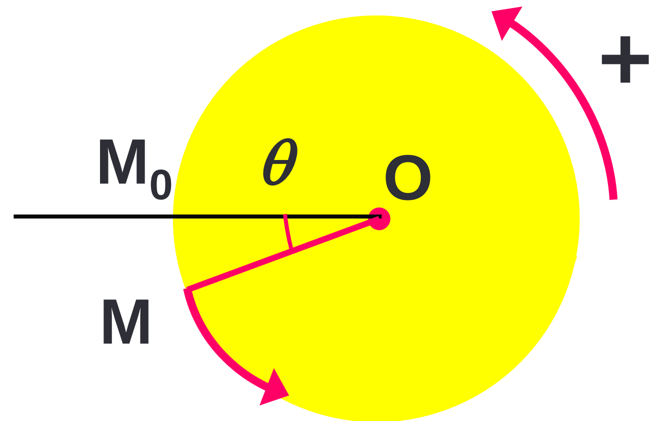
How to determine the position of a rotating object ?





Angle θ: coordinate for rotation



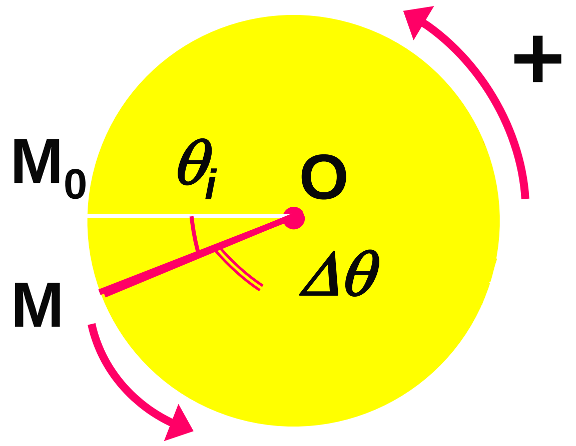


Angular coordinate :

$$\boxed{(OM_0\,,\,OM) = \theta}$$

The *angular displacement* is defined as the angle the object rotates through during some time interval

$$\boxed{\Delta\theta = \theta_f - \theta_i}$$



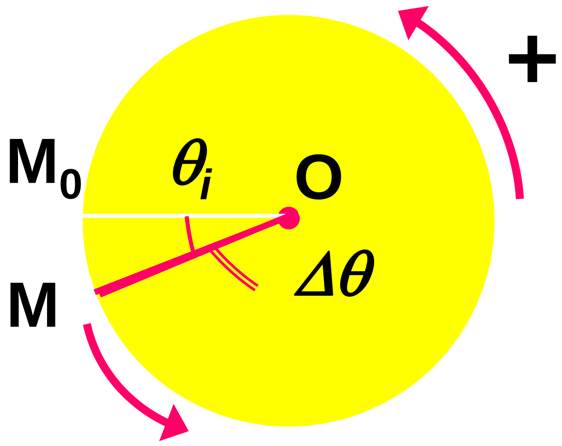
The average angular velocity (speed), ω, of a rotating rigid object is the ratio of the angular displacement to the time interval

$$\bar{\omega} = \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta\theta}{\Delta t}$$

The *instantaneous* angular velocity (speed) is defined as the limit of the average speed as the time interval approaches zero

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

radians/sec (rad/s)



Angular speed will be

<u>positive</u> if θ is increasing (counterclockwise)

<u>negative</u> if θ is decreasing (clockwise)

The average angular acceleration of a rotating object is defined as the ratio of the change in the angular speed $\Delta\omega$ to the time interval Δt :

$$\bar{\alpha} = \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{\Delta\omega}{\Delta t}$$

(rad/s^2)

The instantaneous angular acceleration is defined as the limit of the ratio $\Delta\omega/\Delta t$ as Δt approaches zero :

$$\boxed{\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}}$$

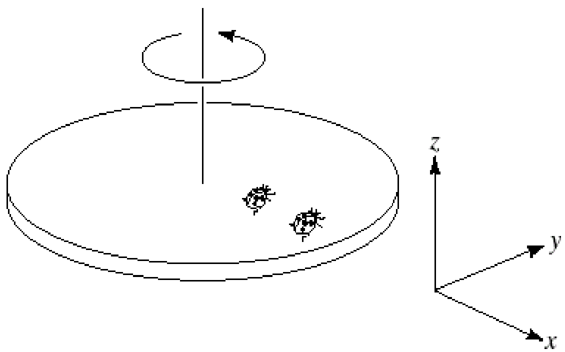
Notes about angular kinematics:

When rotating about a fixed axis, every particle on a rigid object rotates through the same angle and has the same angular speed and the same angular acceleration.

Test

A ladybug sits at the outer edge of a merry-go-round, and a gentleman bug sits halfway between her and the axis of rotation. The merry-go-round makes a complete revolution once each second. The gentleman bug's angular speed is

1. half the ladybug's.
2. the same as the ladybug's.
3. twice the ladybug's.
4. impossible to determine



Note: both insects have an angular speed of 1 rev/s

- **UNIFORM ROTATIONAL MOTION**

$$\omega = const$$

$$\alpha = 0$$

$$\omega = \theta' = const \ ;$$

$$\theta = \omega t + \theta_0$$

- **ROTATIONAL MOTION WITH CONSTANT ANGULAR ACCELERATION**

$$\alpha = const \qquad \omega = \alpha t + \omega_0$$

$$\theta = \frac{\alpha}{2} t^2 + \omega_0 t + \theta_0$$

$$\omega^2 - \omega_0^2 = 2\alpha(\theta - \theta_0)$$

$\alpha.\omega > 0 \rightarrow$ **Increasing speed**

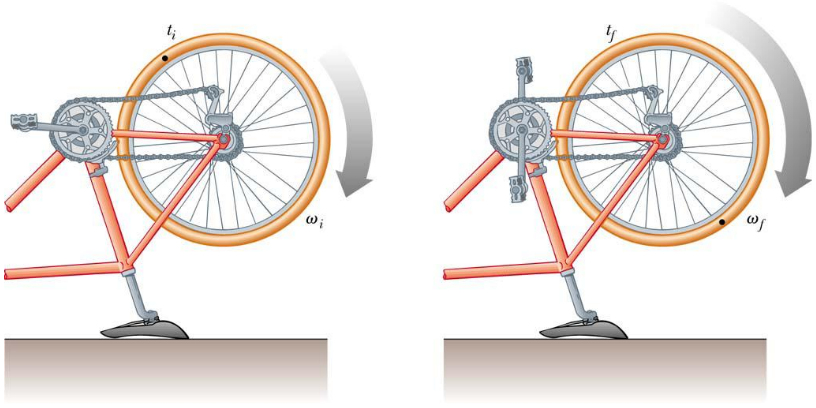
$\alpha.\omega < 0 \rightarrow$ **Decreasing speed**

Analogies Between Linear and Rotational Motion

Rotational Motion About a Fixed Axis with Constant Acceleration	Linear Motion with Constant Acceleration
$\omega = \omega_0 + \alpha t$	$v = v_0 + at$
$\Delta\theta = \omega_0 t + \frac{1}{2}\alpha t^2$	$\Delta x = v_0 t + \frac{1}{2}at^2$
$\omega^2 = \omega_0^2 + 2\alpha\Delta\theta$	$v^2 = v_0^2 + 2a\Delta x$

EXAMPLE 1

1. Bicycle wheel turns 240 revolutions/min. What is its angular velocity in radians/second?



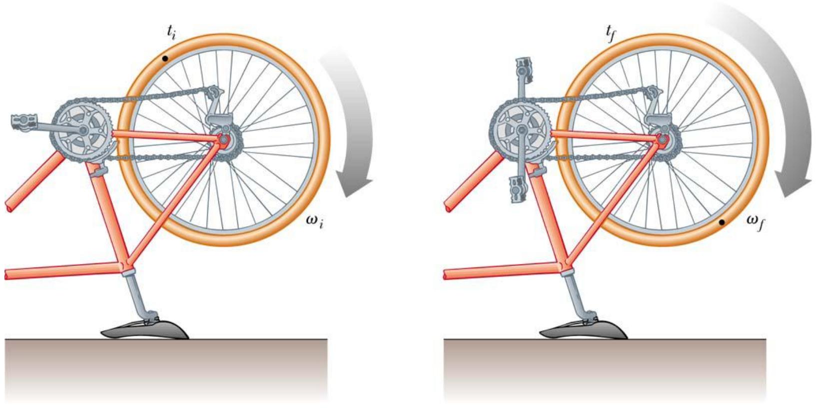
$$\omega = 240\,\frac{\text{rev}}{\text{min}} \times \frac{1\,\text{min}}{60\,\text{s}} \times \frac{2\pi\ \text{rads}}{1\,\text{rev}} = 8\pi\,\text{rad/s} \approx 25.1\,\text{rad/s}$$

2. If wheel slows down uniformly to rest in 5 seconds, what is the angular acceleration?

$$\alpha = \frac{\omega_f - \omega_i}{\Delta t} = \frac{0 - 25\,\text{rad/s}}{5\,\text{s}} = -5\,\text{rad/s}^2$$

EXAMPLE 1

3. How many revolution does it turn in those 5 sec?



$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$= 25\,\text{rad/sec}(5\,\text{sec}) + \frac{1}{2}(-5\,\text{rad/sec})(5\,\text{sec})^2 = 62.5\,\text{rad}$$

$$\theta(rev) = 62.5\,\text{rad} \times \frac{1\,\text{rev}}{2\pi} = 10\ \text{revolutions}$$

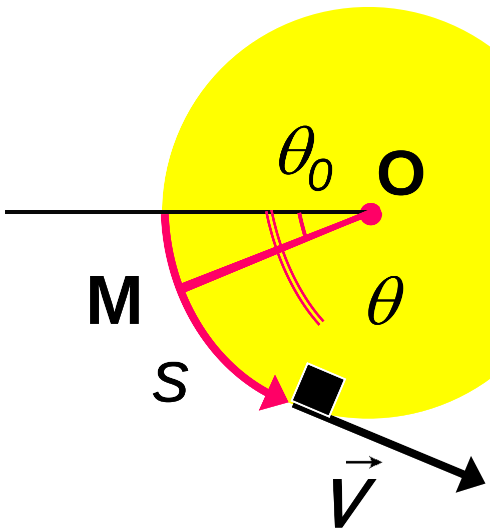
Relationship between linear and angular quantities

- **Linear and angular position**

$$s = R\theta$$

- **Linear (tangential speed)**

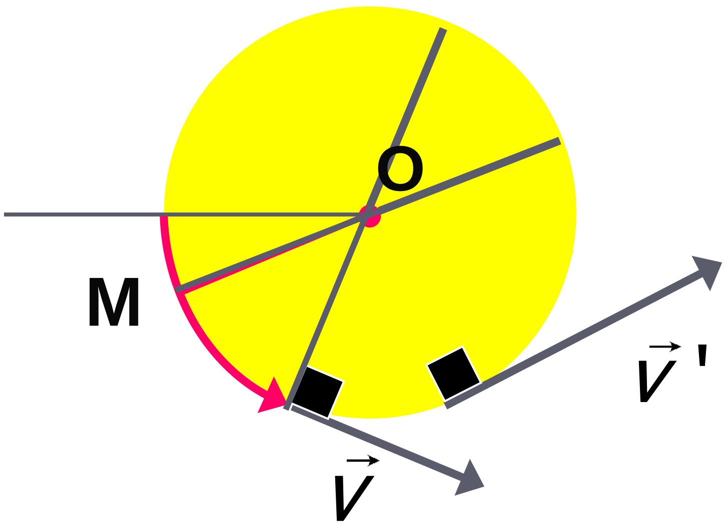
$$\omega = \theta' = \left(\frac{s}{R}\right)' = \frac{v}{R}$$

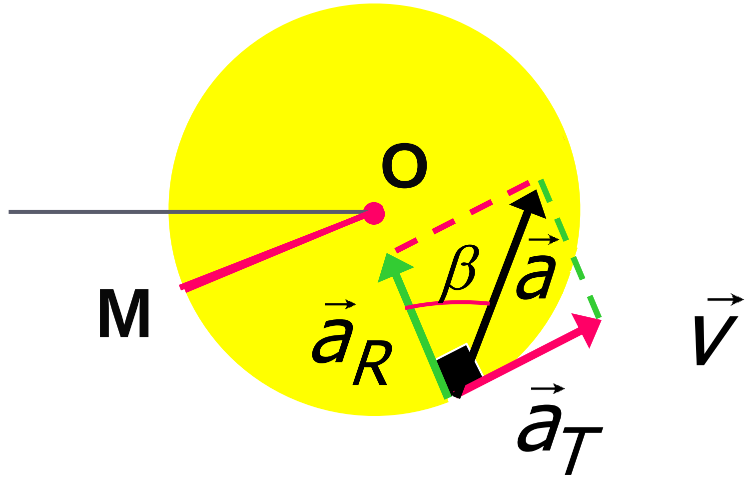


Linear speed vector :

$$\vec{V} \begin{cases} \textbf{Tangential to the trajectory} \\ \textbf{Direction of motion} \\ \textbf{Magnitude : } v = R\omega \end{cases}$$

• Linear acceleration





$$\vec{a} = \vec{a}_R + \vec{a}_T$$

Radial component : $a_R = \dfrac{v^2}{R} = R\omega^2$

$\rightarrow$ **Change of direction of** $\vec{V}$

Tangential component $a_T = R\alpha$

$\rightarrow$ **Change of magnitude of** $\vec{V}$

$$a = \sqrt{a_R^2 + a_T^2} \qquad \tan\beta = \frac{a_T}{a_R} = \frac{\alpha}{\omega^2}$$

PROBLEM 1

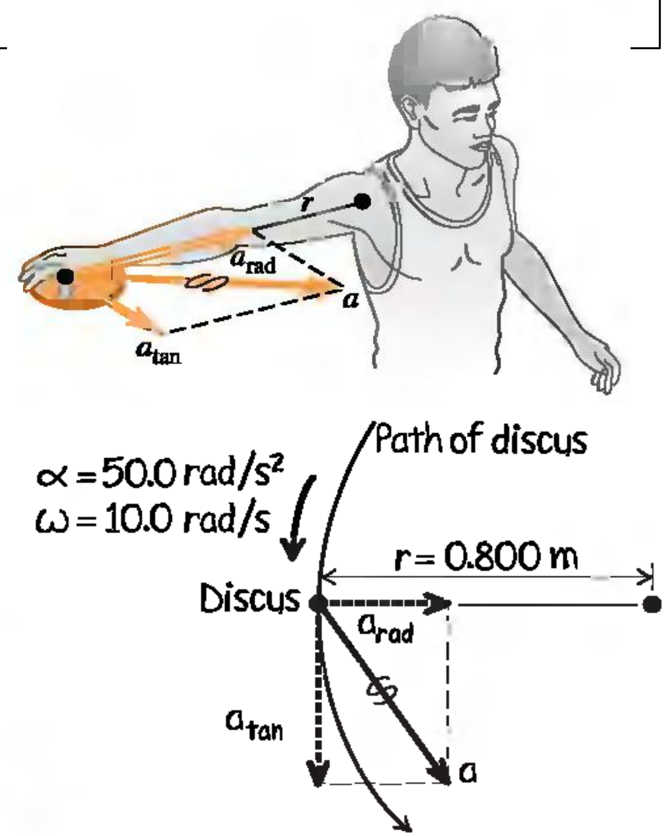
A discus thrower moves the discus in a circle of radius 80.0 cm. At a certain instant, the thrower is spinning at an angular speed of 10.0 rad/s and the angular speed is increasing at 50.0 rad/s^2 At this instant, find the tangential and centripetal components of the acceleration of the discus and the magnitude of the acceleration.

SOLUTION

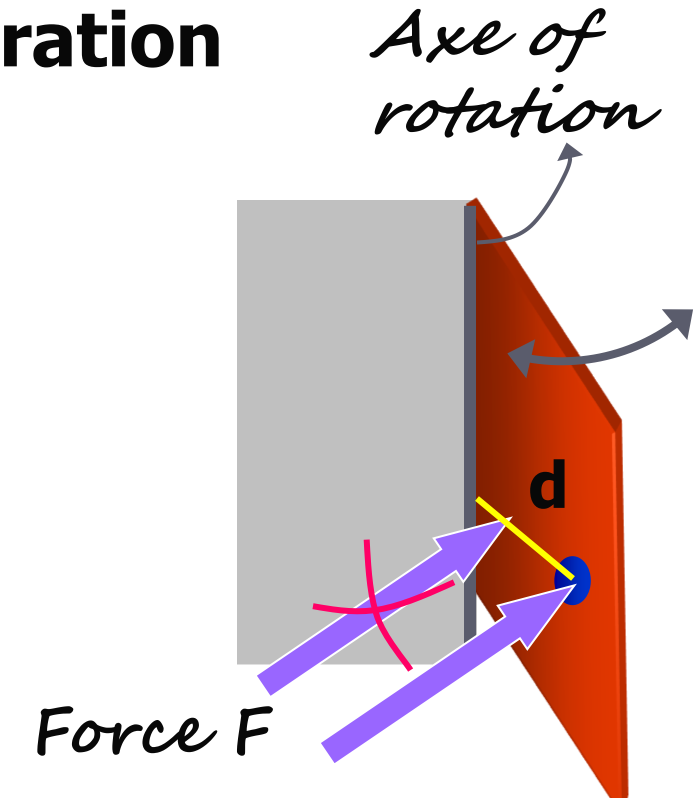
$$a_{tan} = r\alpha = (0.800\,\mathrm{m})(50.0\,\mathrm{rad/s^2}) = 40.0\,\mathrm{m/s^2}$$

$$a_{rad} = \omega^2 r = (10.0\,\mathrm{rad/s})^2(0.800\,\mathrm{m}) = 80.0\,\mathrm{m/s^2}$$

$$a = \sqrt{a_{tan}{}^2 + a_{rad}{}^2} = 89.4\,\mathrm{m/s^2}$$



2 Torque and Angular Acceleration



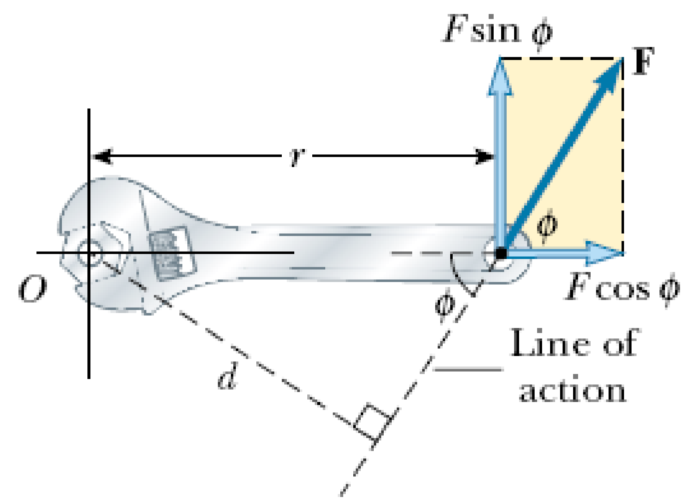
a. Torque

Torque characterizes the tendency of a force to rotate an object about some axis

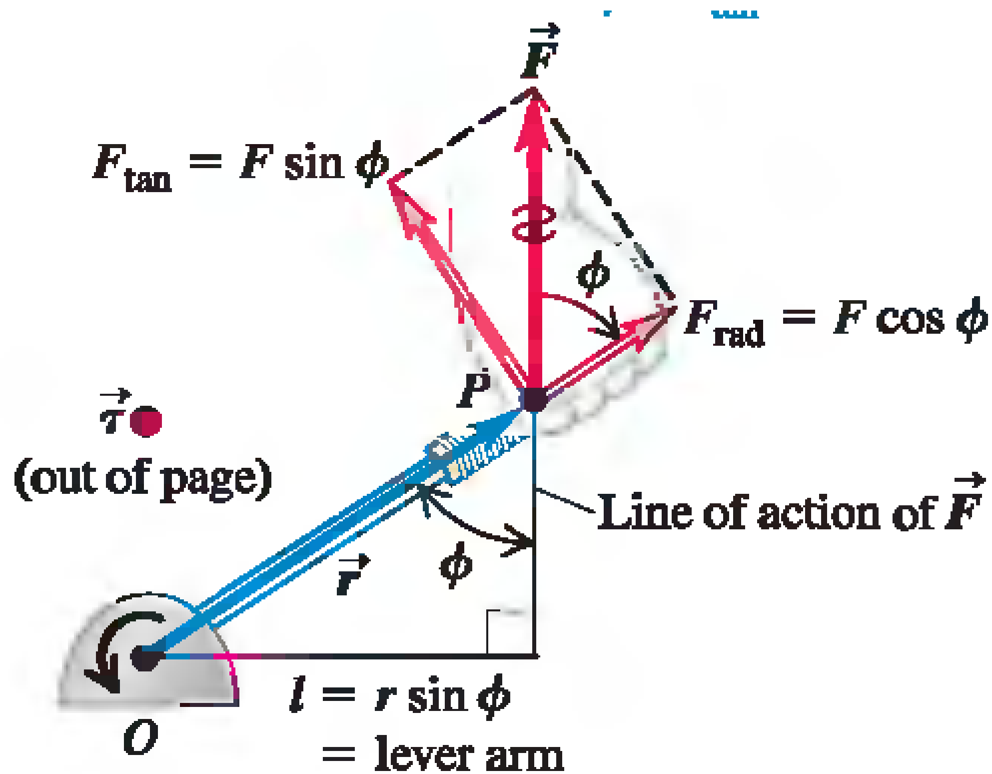
d* (moment arm or *lever arm*):** **the perpendicular distance from the pivot point to the line of action of *F

$$\boxed{\tau = Fd}$$

N.m N m



Torque as a Vector

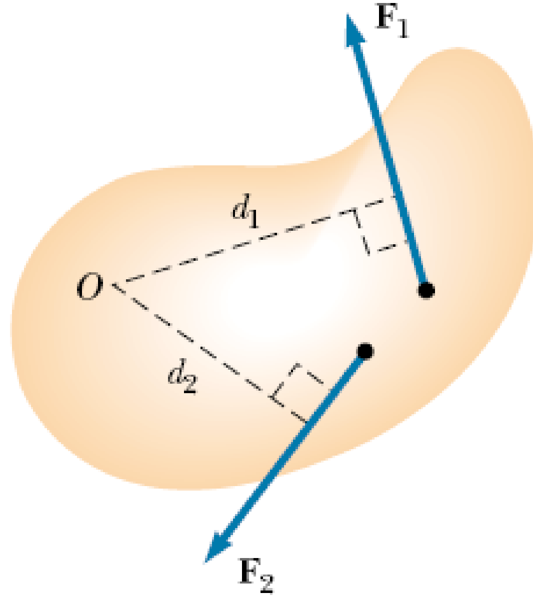


$$\boxed{\tau = Fd}$$

$$\boxed{\vec{\tau} = \vec{F} \times \vec{r} \equiv \vec{F} \wedge \vec{r}}$$

r **: the distance between the pivot point and the point of application of F**

If two or more forces are acting on a rigid object ?



Convention : the sign of the torque is positive if the turning tendency of the force is counterclockwise and is negative if the turning tendency is clockwise

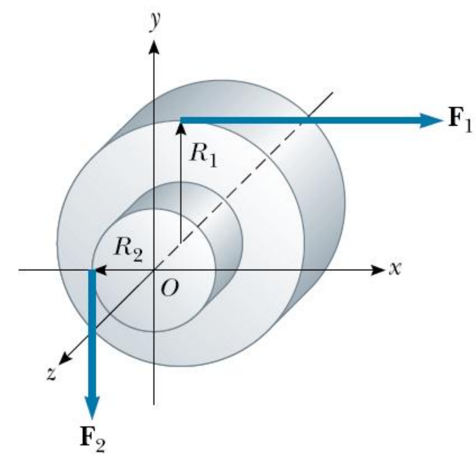
$$\sum \tau = \tau_1 + \tau_2 = F_1 d_1 - F_2 d_2$$

EXAMPLE 2

A one-piece cylinder is shaped with a core section protruding from the larger drum. The cylinder is free to rotate around the central axis shown in the drawing. A rope wrapped around the drum, which has radius R_1 = 1.0 m , exerts a force F_1 = 5.0 N to the right on the cylinder. A rope wrapped around the core, which has radius R_2 = 0.50 m, exerts a force F_2 = 15.0 N downward on the cylinder.
What is the net torque about the rotation axis, and which way does the cylinder rotate from rest?

$$\sum \tau = \tau_1 + \tau_2 = \boxed{-R_1F_1 + R_2F_2}$$

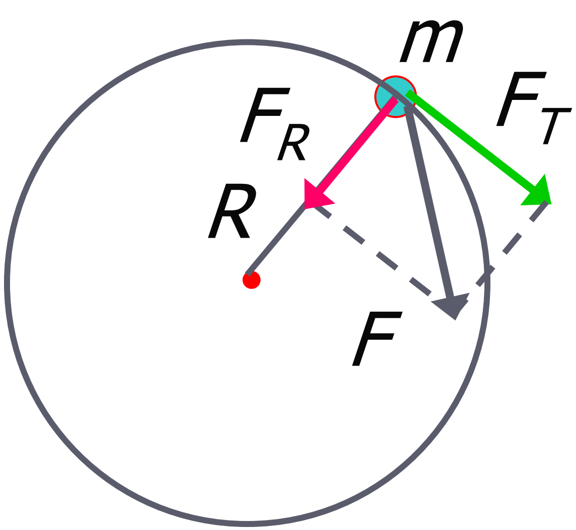
$$\sum \tau = -(5.0\ \mathrm{N})(1.0\ \mathrm{m}) + (15.0\ \mathrm{N})(0.50\ \mathrm{m}) = \boxed{2.5\ \mathrm{N \cdot m}}$$



The net torque is positive, if the cylinder starts from rest, it will commence rotating counterclockwise with increasing angular velocity.

b. The rotational analog of Newton's second law

- Consider a particle of mass m rotating in a circle of radius r under the influence of the force $\boldsymbol{F}$



Torque due to $\boldsymbol{F}$:

$$\tau_F = \tau_{FT} + \tau_{FR}$$
$$= RF_T + 0 = RF_T$$

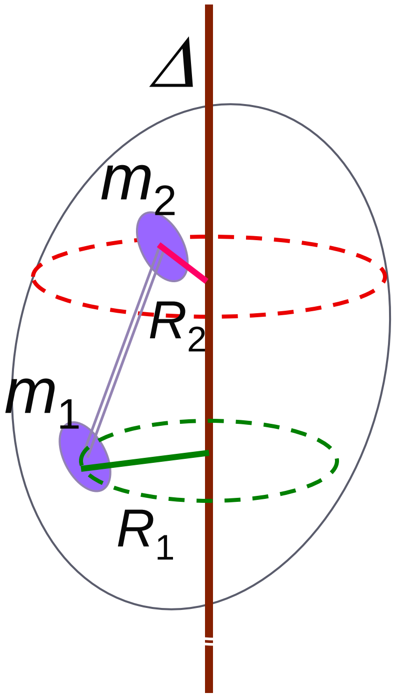
Newton's second law :

$$F_T = ma_T = m(R\alpha)$$

$$\rightarrow \quad \tau_F = R.m(R\gamma)\,;$$

$$\boxed{\tau_F = (mR^2)\alpha}$$

- A rigid object of arbitrary shape rotating about a fixed axis



The net torque on a rigid body :

$$\sum_i \tau_i = \sum_i \left[(m_i R_i^2) \alpha \right]$$

$$= \left(\sum_i m_i R_i^2 \right) \alpha$$

We put : $\boxed{I = \sum_i m_i R_i^2}$

($kg.m^2$)

I : the **moment of inertia** of the rotating particle about the Δ axis

$$\rightarrow \quad \boxed{\tau = I\alpha} \quad \Leftrightarrow \quad \vec{F} = m\vec{a}$$

(rotational analog of Newton's second law for a rigid body)

c. Moments of inertia

- Moment of inertia of **discrete mass points** :

$$\boxed{I = \sum_i m_i R_i^2}$$

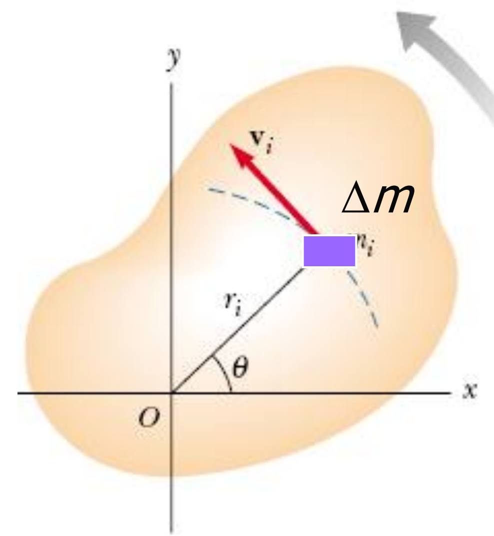
CAUTION : Moment of inertia depends on the choice of axis

- Moment of inertia of **continuous mass distribution** :

$$I = \sum_i \Delta m_i R_i^2$$

$$I = \lim_{\Delta m_i \to 0} \sum_i \Delta m_i R_i^2$$

$$\boxed{I = \int_M R^2 dm} \qquad \boxed{I = \int_V R^2 \rho dV}$$



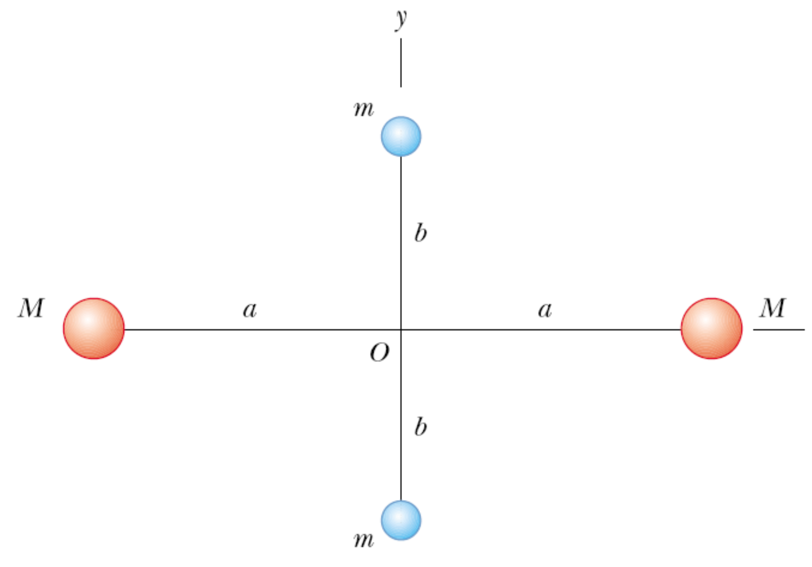
(ρ : mass density ; V : volume of the object)

EXAMPLE 3

Four very tiny spheres are fastened to the corners of a frame of negligible mass lying in the *xy* plane.

(a) If the system rotates about the *y* axis with an angular speed find the moment of inertia about this axis.

(b) Suppose the system rotates in the *xy* plane about an axis through *O* (the *z* axis). Calculate the moment of inertia about this axis.



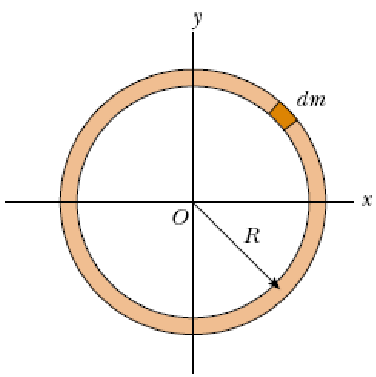
(a)

$$I_y = \sum_i m_i r_i^2 = Ma^2 + Ma^2 = \boxed{2Ma^2}$$

(b)

$$I_z = \sum_i m_i r_i^2 = Ma^2 + Ma^2 + mb^2 + mb^2 = \boxed{2Ma^2 + 2mb^2}$$

The moment of inertia of a uniform hoop of mass M and radius R about an axis perpendicular to the plane of the hoop and passing through its center

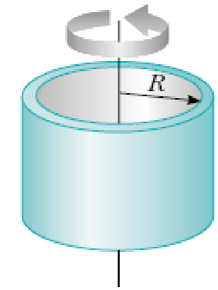


$$I = \int_M R^2 dm = R^2 \int_M dm$$

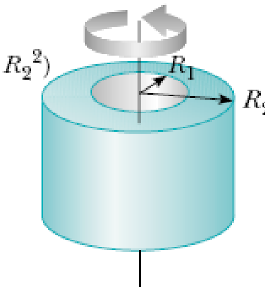
$$\boxed{I = MR^2}$$

TABLE 10.2 **Moments of Inertia of Homogeneous Rigid Bodies with Different Geometries**

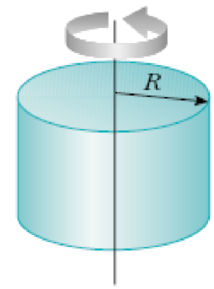
Hoop or cylindrical shell
$I_{CM} = MR^2$



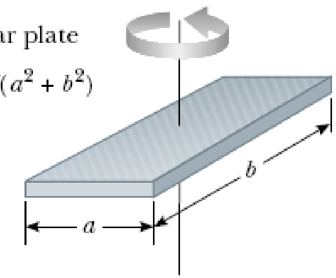
Hollow cylinder
$I_{CM} = \frac{1}{2} M(R_1^2 + R_2^2)$



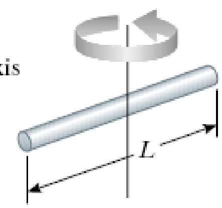
Solid cylinder or disk
$I_{CM} = \frac{1}{2} MR^2$



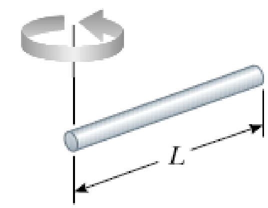
Rectangular plate
$I_{CM} = \frac{1}{12} M(a^2 + b^2)$



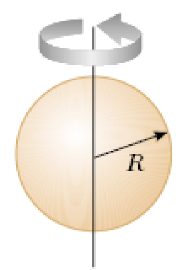
Long thin rod with rotation axis through center
$I_{CM} = \frac{1}{12} ML^2$



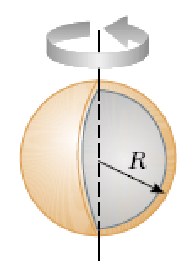
Long thin rod with rotation axis through end
$I = \frac{1}{3} ML^2$



Solid sphere
$I_{CM} = \frac{2}{5} MR^2$



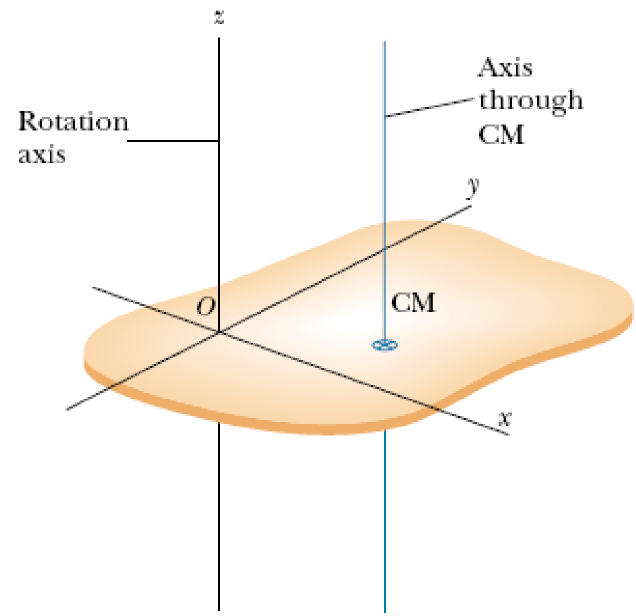
Thin spherical shell
$I_{CM} = \frac{2}{3} MR^2$



The parallel-axis theorem

Suppose the moment of inertia about an axis through the center of mass of an object is I_{CM}.

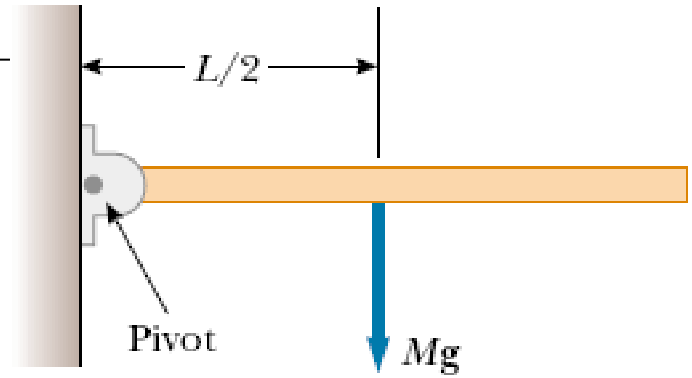
The moment of inertia about any axis parallel to and a distance D away from this axis is



$$\boxed{I = I_{CM} + MD^2}$$

PROBLEM 2

A uniform rod of length *L* and mass *M* is attached at one end to a frictionless pivot and is free to rotate about the pivot in the vertical plane. The rod is released from rest in the horizontal position. What is the initial angular acceleration of the rod and the initial linear acceleration of its right end?



SOLUTION

$$\tau = Mg\left(\frac{L}{2}\right)$$

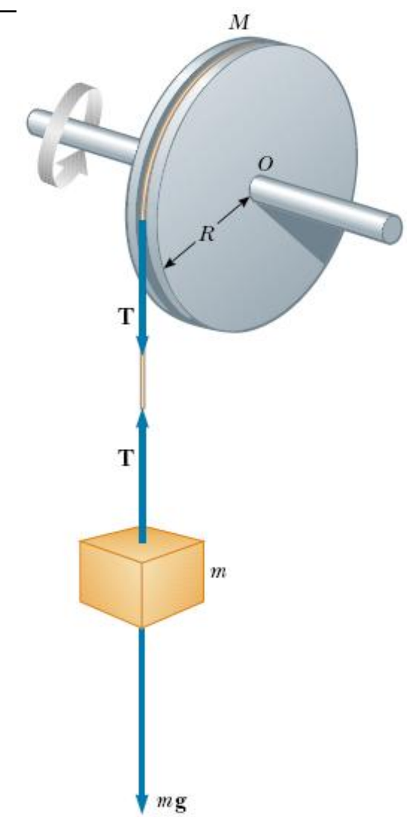
$$\alpha = \frac{\tau}{I} = \frac{Mg(L/2)}{1/3\,ML^2} = \boxed{\frac{3g}{2L}}$$

$$a_t = L\alpha = \boxed{\frac{3}{2}g}$$

PROBLEM 3

A wheel of radius R, mass M, and moment of inertia I is mounted on a frictionless, horizontal axle. A light cord wrapped around the wheel supports an object of mass m. Calculate the angular acceleration of the wheel, the linear acceleration of the object, and the tension in the cord.

SOLUTION



$$\sum \tau = I\alpha = TR$$

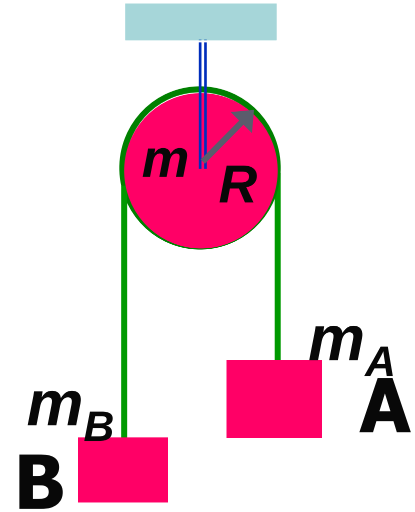
$$a = R\alpha \longrightarrow I\frac{a}{R} = TR\ ;$$

$$\sum F_y = mg - T = ma \longrightarrow mg - I\frac{a}{R^2} = ma\ ;$$

$$a = \boxed{\frac{g}{1 + I/mR^2}} \qquad T = \boxed{\frac{mg}{1 + \dfrac{mR^2}{I}}}$$

PROBLEM 4

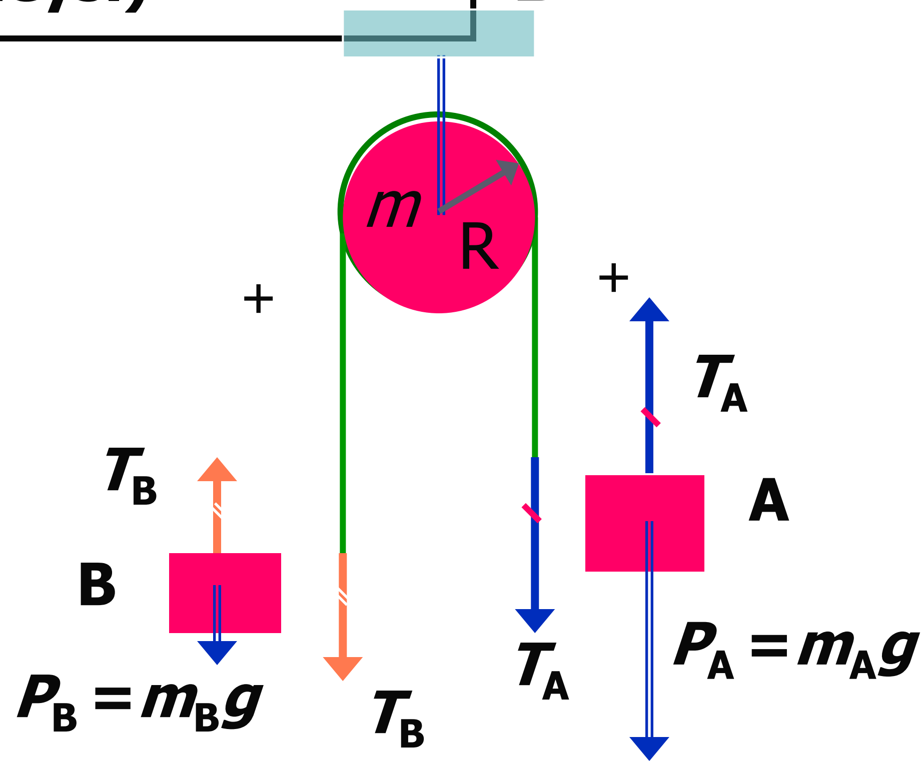
Two blocks having masses m_1 and m_2 are connected to each other by a light cord that passes over one frictionless pulley, having a moment of inertia I and radius R.
Find the acceleration of each block and the tensions T_1 , T_2 in the cord. (Assume no slipping between cord and pulleys.)



SOLUTION

- **For A :** $m_A g - T_A = m_A a$ **(1)**
- **For B :** $T_B - m_B g = m_B a$ **(2)**
- **For the pulley :**

$$\tau = T_A R - T_B R = I\alpha \quad \textbf{(3)}$$



- **For A :** $m_A g - T_A = m_A a$ **(1)**
- **For B :** $T_B - m_B g = m_B a$ **(2)**
- **For the pulley :** $\tau = T_A R - T_B R = I\alpha$ **(3)**

$$\alpha = \frac{a_T}{R} \equiv \frac{a}{R} \quad \textbf{(3)} \;\rightarrow\; (T_A - T_B)R = I\,\frac{a}{R}$$

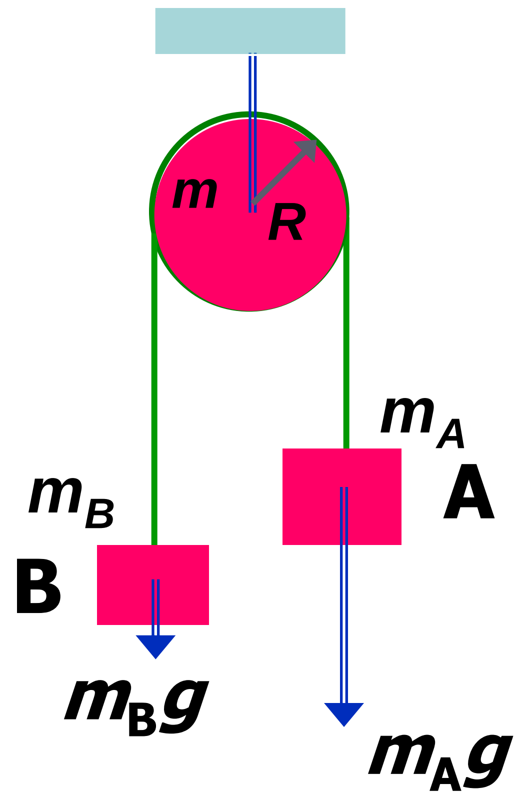
$$T_A - T_B = \frac{I}{R^2}a \quad \textbf{(3')}$$

$$\textbf{(1) + (2)} \;\rightarrow\; m_A g - m_B g - (T_A - T_B) = (m_A + m_B)a$$

$$\textbf{(3')} \;\rightarrow\; m_A g - m_B g - \frac{I}{R^2}a = (m_A + m_B)a$$

$$m_A g - m_A g - \frac{I}{R^2}a = (m_A + m_B)a \qquad \boxed{a = \frac{m_A g - m_B g}{m_A + m_B + \frac{I}{R^2}}}$$

$$\boxed{a = \frac{m_A g - m_B g}{m_A + m_B + \dfrac{I}{R^2}}}$$



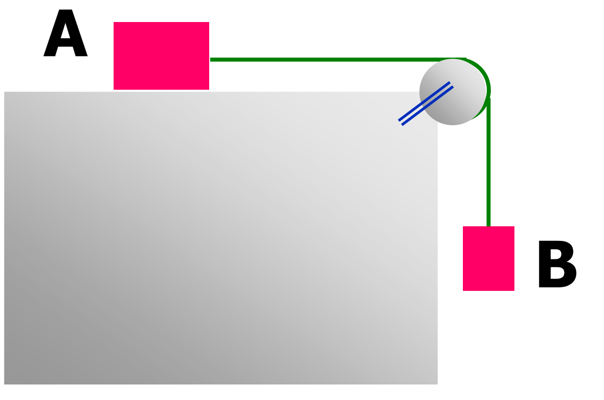
- **Notes :**

$$\textbf{Acceleration} = \frac{\textbf{Acting force}}{\textbf{System's Inertia}}$$

- **With :** $I = \frac{1}{2} m R^2 \rightarrow a = \frac{m_A g - m_B g}{m_A + m_B + \frac{1}{2} m}$

PROBLEM 5

Two blocks having masses m_A and m_B = 5.5 kg are connected to each other by a light cord that passes over one frictionless pulley, which is a thin-walled hollow cylinder and has a mass of 1.0 kg. The system begins to move from rest. After 2.0 s, the speed of A and B is 10 m/s Find m_A and the tensions T_A , T_B in the cord.

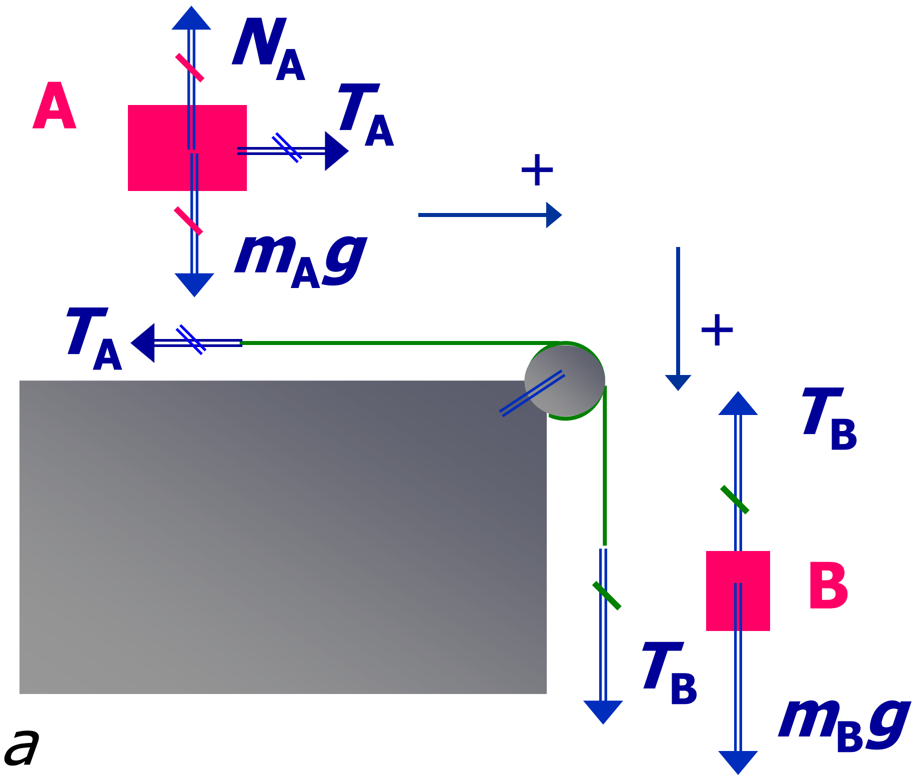


SOLUTION

- **For A :** $T_A = m_A a$
- **For B :** $m_B g - T_B = m_B a$
- **For the pulley :**

$$\tau = T_B R - T_A R = I\alpha$$

$$\alpha = a_T / R \equiv a / R$$



$$(T_B - T_A)R = I\frac{a}{R}\;; \quad T_B - T_A = \frac{I}{R^2}a$$

$$(m_B g - m_B a) - m_A a = \frac{I}{R^2}a\;; \quad a = \frac{m_B g}{m_A + m_B + \dfrac{I}{R^2}}$$

$$a = \frac{m_B g}{m_A + m_B + \dfrac{mR^2}{2R^2}} = \frac{m_B g}{m_A + m_B + \dfrac{m}{2}}$$

SOLUTION

$$a = \frac{m_B g}{m_A + m_B + \dfrac{mR^2}{2R^2}} = \frac{m_B g}{m_A + m_B + \dfrac{m}{2}} \; ; \quad m_A = m_B\left(\frac{g}{a} - 1\right) - \frac{m}{2}$$

$$v = at + v_0 = at \; ; \; a = \frac{v}{t} = \frac{10\ m/s}{2{,}0\ s} = 5{,}0\ m/s^2$$

$$m_A = 5{,}5\ kg \times \left(\frac{10}{5{,}0} - 1\right) - \frac{1{,}0}{2} \; ; \quad \boxed{m_A = 5{,}0\ kg}$$

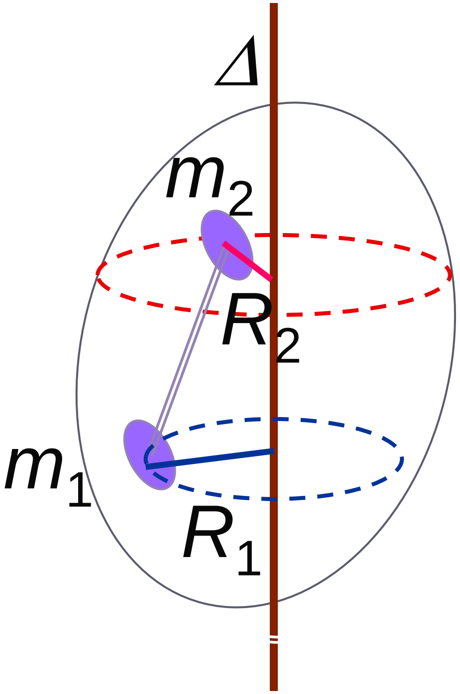
$$T_A = m_A a = 5{,}0\ kg \times 5{,}0\ m/s^2 \; ; \quad \boxed{T_A = 25\ N}$$

$$m_B g - T_B = m_B a \; ; \; T_B = m_B(g - a)$$

$$= 5{,}5\ kg \times (10 - 5{,}0)\ m/s^2$$

$$\boxed{T_B = 27{,}5\ N}$$

3 Rotational Kinetic Energy



10.3.1 The *total* kinetic energy of the rotating rigid object

$$K = \sum_i K_i = \sum_i \left(\frac{1}{2} m_i v_i^2 \right)$$

$$= \sum_i \frac{1}{2} m_i (R_i \omega)^2 = \frac{1}{2} \omega^2 \left(\sum_i m_i R_i^2 \right)$$

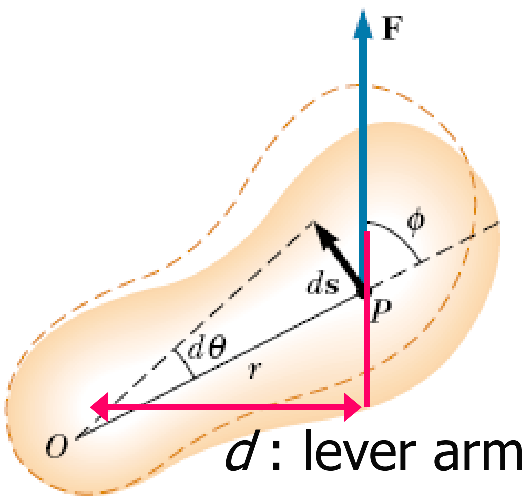
With the **moment of inertia** :

$$I = \sum_i m_i R_i^2$$

The **rotational kinetic energy** of a object : $\boxed{K = \frac{1}{2} I \omega^2}$

To compare with the linear motion : $K = \frac{1}{2} m v^2$

10.3.2 Work– kinetic energy theorem



The work done by external forces

$$dW = \vec{F}d\vec{s} = (F\sin\phi)rd\theta$$

$$r\sin\phi = d$$

$$\longrightarrow \quad dW = \tau d\theta$$

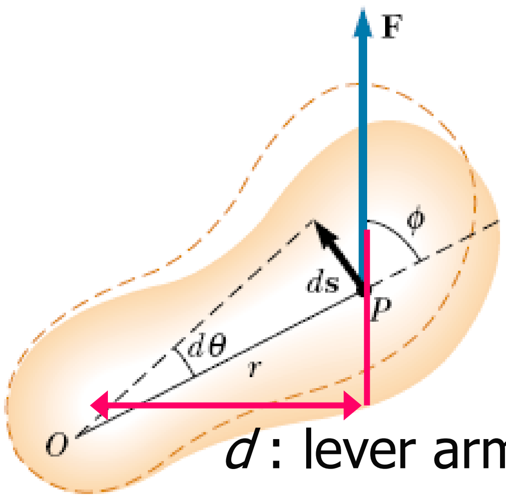
The **Newton's law** : $\tau = I\alpha = I\dfrac{d\omega}{dt}$ $\quad I\dfrac{d\omega}{d\theta}\dfrac{d\theta}{dt} = I\dfrac{d\omega}{d\theta}\omega$

$$dW = \sum \tau d\theta = I\omega d\omega$$

$$\sum W = \int dW = \int_{\omega_i}^{\omega_f} I\omega d\omega \,; \quad \boxed{\sum W = \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_i^2}$$

To compare with the linear motion : $\sum W = \dfrac{1}{2}mv_f^2 - \dfrac{1}{2}mv_i^2$

10.3.3 Work and Power



d : lever arm

The work done by external forces

$$dW = \vec{F}d\vec{s} \quad = (F\sin\phi)rd\theta$$

$$r\sin\phi = d$$

$$dW = \tau d\theta$$

$$\frac{dW}{dt} = \tau\frac{d\theta}{dt}$$

The power by external forces : P

Angular velocity : ω

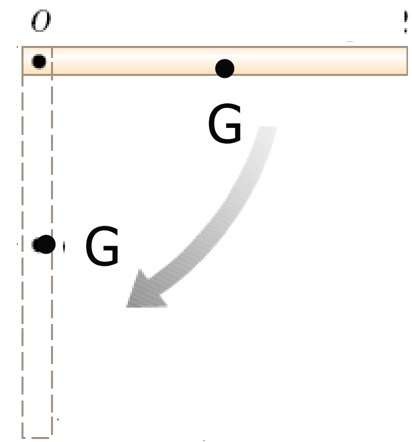
$$\boxed{P = \tau\omega}$$

PROBLEM 6

A uniform rod of length L and mass M is free to rotate on a frictionless pin passing through one end. The rod is released from rest in the horizontal position.

(a) What is its angular speed when it reaches its lowest position?

SOLUTION



(a)

$$\sum W = \frac{1}{2} I\omega_f^2 - \frac{1}{2} I\omega_i^2 \ ; \quad \frac{1}{2} MgL = \frac{1}{2} I\omega_f^2 - 0$$

$$= \frac{1}{2}\left(\frac{1}{3} ML^2\right)\omega_f^2$$

$$\boxed{\omega_f = \sqrt{\frac{3g}{L}}}$$

PROBLEM 6

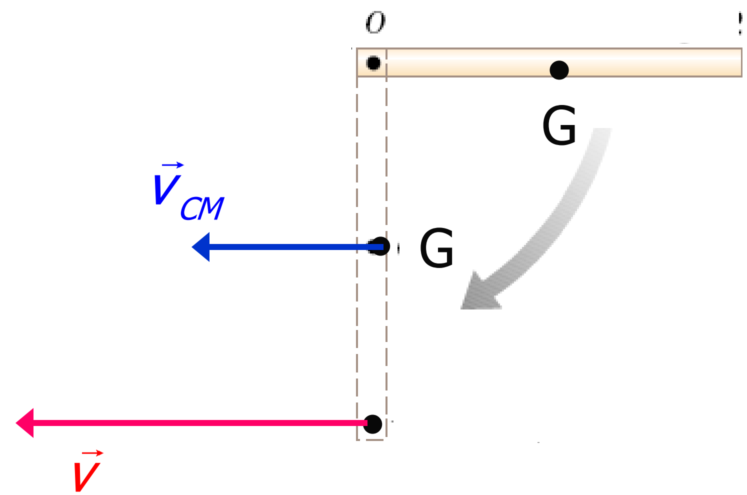
A uniform rod of length L and mass M is free to rotate on a frictionless pin passing through one end. The rod is released from rest in the horizontal position.
(b) Determine the linear speed of the center of mass and the linear speed of the lowest point on the rod when it is in the vertical position.

SOLUTION

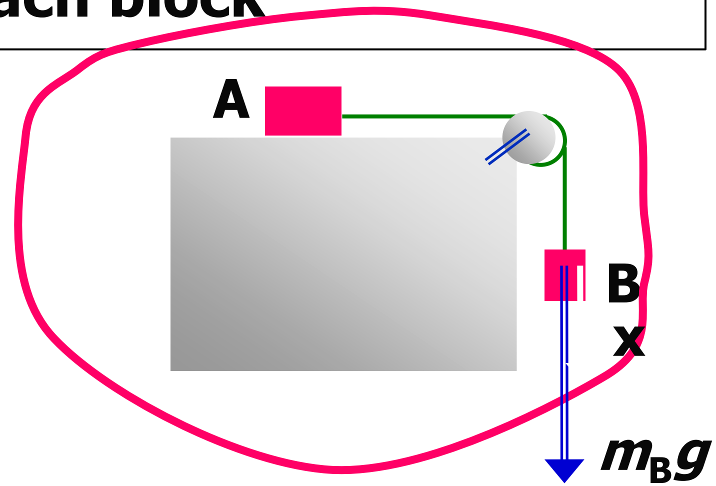
(a) $\omega_f = \sqrt{\dfrac{3g}{L}}$

(b) $v_{CM} = R\omega = \dfrac{L}{2}\sqrt{\dfrac{3g}{L}}$; $v_{CM} = \dfrac{1}{2}\sqrt{3gL}$

$v = R'\omega = L\sqrt{\dfrac{3g}{L}}$; $\boxed{v = \sqrt{3gL}}$



PROBLEM 7 **Two blocks having masses m_A and m_B are connected to each other by a light cord that passes over one frictionless pulley, which is a thin-walled hollow cylinder and has a mass m. The system begins to move from rest. Find the acceleration of each block**



SOLUTION

$$\frac{1}{2}I\omega^2 + \frac{1}{2}m_A v^2 + \frac{1}{2}m_B v^2 - 0 = m_B g x$$

$$\frac{1}{2}I\frac{v^2}{R^2} + \frac{1}{2}m_A v^2 + \frac{1}{2}m_B v^2 - 0 = m_B g x$$

$$\left(\frac{I}{R^2} + m_A + m_B\right)v^2 = 2m_B g x$$

Derive with respect to the time : $\left(\frac{I}{R^2} + m_A + m_B\right)2v'v = 2m_B g v$

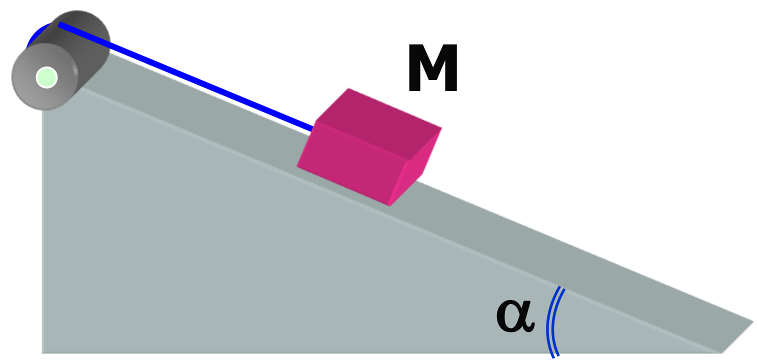
$$\left(\frac{I}{R^2} + m_A + m_B\right)a = m_B g \;;\quad \boxed{a = \frac{m_B g}{m_A + m_B + \frac{I}{R^2}} = \frac{m_B g}{m_A + m_B + m}}$$

PROBLEM 8 A block with mass m = 2.00 kg slides down a surface inclined 30^0 to the horizontal. A string attached to the block is wrapped around a flywheel on a fixed axis at O. The flywheel is a hollow cylinder and has mass m = 2.00 kg. The string pulls without slipping.

(a) What is the acceleration of the block down the plane?

(b) What is the tension in the string?

SOLUTION



(a)

$$\frac{1}{2}I\omega^2 + \frac{1}{2}mv^2 - 0 = mgx\sin\alpha$$

$$\frac{1}{2}(mR^2)\left(\frac{v}{R}\right)^2 + \frac{1}{2}mv^2 - 0 = mgx\sin\alpha$$

$$v^2 = gx\sin\alpha\ ; \quad 2v'v = gv\sin\alpha\ ;$$

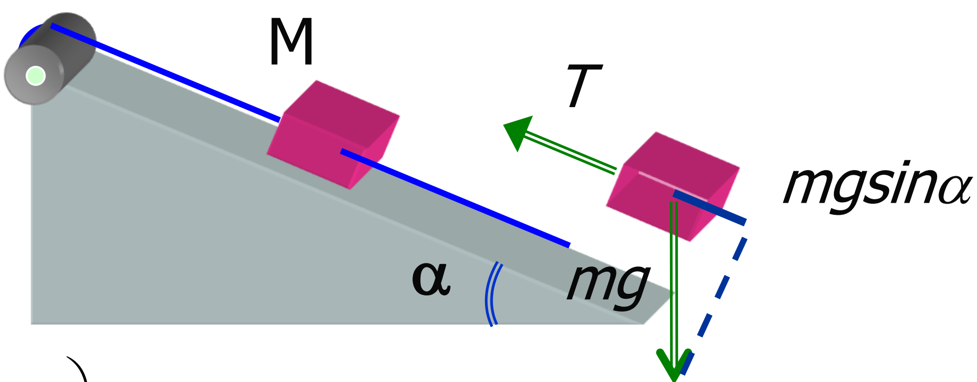
$$\boxed{a = \frac{1}{2}g\sin\alpha} \qquad a = \frac{1}{2}\times(9.81\ m/s^2).\sin 30^0 = 2.45\ \ m/s^2$$

PROBLEM 8 A block with mass m = 2.00 kg slides down a surface inclined 30^0 to the horizontal. A string attached to the block is wrapped around a flywheel on a fixed axis at O. The flywheel is a hollow cylinder and has mass m = 2.00 kg. The string pulls without slipping.

(a) What is the acceleration of the block down the plane?

(b) What is the tension in the string?

SOLUTION



(a) $\boxed{a = \frac{1}{2} g \sin\alpha}$

(b)

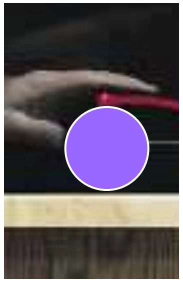
$$mg\sin\alpha - T = ma = m\left(\frac{1}{2} g \sin\alpha\right)$$

$$T = \frac{1}{2} mg \sin\alpha = \frac{1}{2} \times 2 \times (9.81\ m/s^2) \times \sin 30^0$$

$$\boxed{T = 4.90\ N}$$

4 Rolling Motion of a Rigid Object

Suppose a cylinder is rolling on a straight path :
The center of mass CM moves in a straight line.
Each point of the cylinder moves about this CM
$\rightarrow$ path called a *cycloid* $\rightarrow$ Combined Translation and Rotation

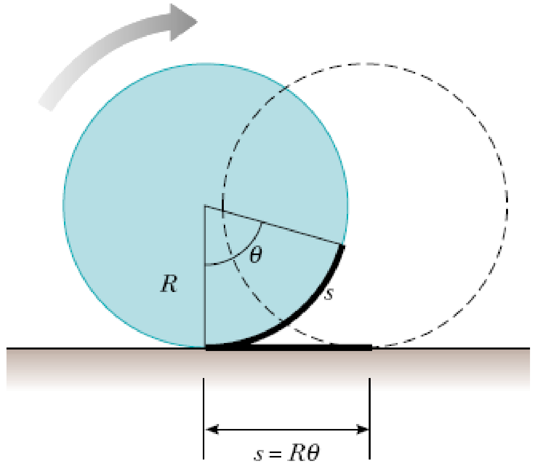


The total kinetic energy of a rigid object rolling on a rough surface without slipping equals the rotational kinetic energy about its center of mass, plus the translational kinetic energy of the center of mass :

$$\boxed{K = \frac{1}{2} I_{CM}\omega^2 + \frac{1}{2} M v_{CM}^2}$$

$$\boxed{K = \frac{1}{2} I_{CM}\omega^2 + \frac{1}{2} M v_{CM}^2}$$

Relationships between ω and v_{CM} ?

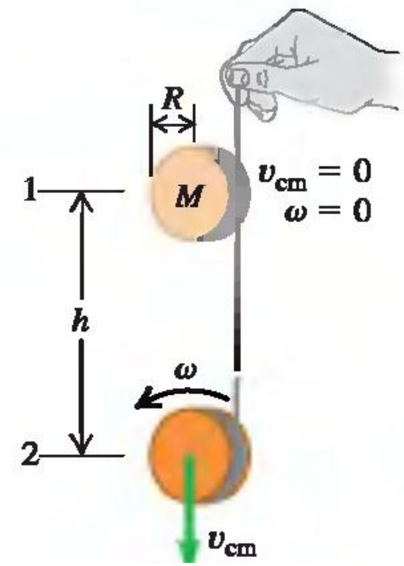


If cylinder or sphere rolls **without slipping** (pure rolling motion) :

$$v_{CM} = \frac{ds}{dt} = R\frac{d\theta}{dt} \longrightarrow v_{CM} = R\omega$$

PROBLEM 9 A primitive yo-yo is made by wrapping a string several times around a solid cylinder with mass M and radius R. You hold the end of the string stationary while releasing the cylinder with no initial motion. The string unwinds but does not slip or stretch as the cylinder drops and rotates. Use energy considerations to find the speed of the center of mass of the solid cylinder after it has dropped a distance *h*.

SOLUTION



The kinetic energy at point 2 :

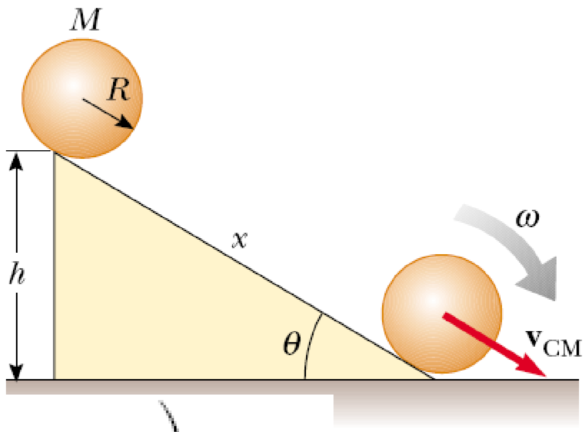
$$K_2 = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\left(\frac{v_{cm}}{R}\right)^2$$

$$= \frac{3}{4}Mv_{cm}^2$$

Conservation of energy : $K_1 + U_1 = K_2 + U_2$

$$0 + Mgh = \frac{3}{4}Mv_{cm}^2 + 0 \longrightarrow v_{cm} = \sqrt{\frac{4}{3}gh}$$

PROBLEM 10 For the solid sphere shown in the figure, calculate the linear speed of the center of mass at the bottom of the incline and the magnitude of the linear acceleration of the center of mass.



SOLUTION

$$\boxed{K = \frac{1}{2} I_{CM}\omega^2 + \frac{1}{2} M v_{CM}^2} \qquad \boxed{v_{CM} = R\omega}$$

$$K = \tfrac{1}{2} I_{CM}\left(\frac{v_{CM}}{R}\right)^2 + \tfrac{1}{2} M {v_{CM}}^2 \qquad K = \tfrac{1}{2}\left(\frac{I_{CM}}{R^2} + M\right){v_{CM}}^2$$

Work–kinetic energy theorem : $\frac{1}{2}\left(\frac{I_{CM}}{R^2} + M\right){v_{CM}}^2 = Mgh$

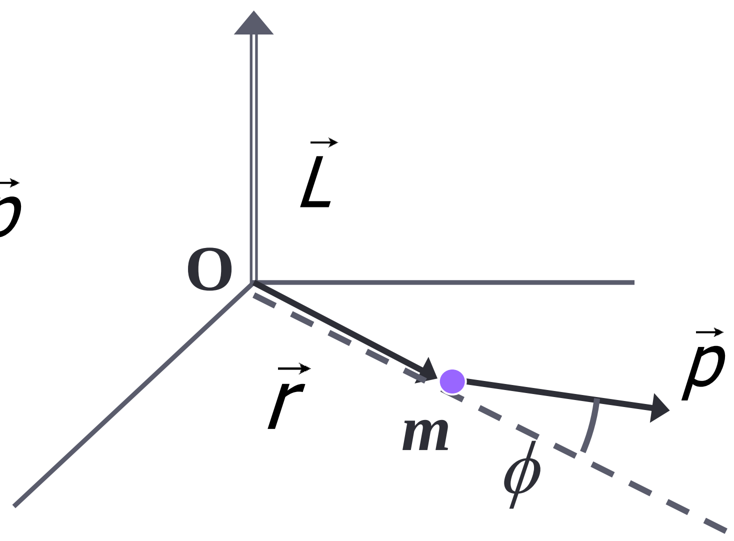
$$\longrightarrow \quad v_{CM} = \left(\frac{2gh}{1 + I_{CM}/MR^2}\right)^{1/2} \qquad v_{CM} = \left(\frac{2gh}{1 + \frac{2/5MR^2}{MR^2}}\right)^{1/2} = \left(\frac{10}{7}gh\right)^{1/2}$$

$${v_{CM}}^2 = 2a_{CM}x \quad \longrightarrow \quad a_{CM} = \frac{5}{7} g \sin\theta$$

5.1 Angular momentum of a particle

For a particle with constant mass *m*, velocity ***v***, momentum ***p*** = *m**v***, and position vector ***r*** relative to the origin 0 of an inertial frame, the **angular momentum** is defined as

$$\vec{L} = \vec{r} \times \vec{p}$$



is the **cross product** of $\vec{r}$ and $\vec{p}$

is perpendicular to the *rp*-plane.

its direction is upward, and its magnitude is

$$L = rp\sin\phi = mrv\sin\phi$$

$$kg.m^2 / s$$

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

$$= \vec{v} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{0} + \vec{r} \times \frac{d\vec{p}}{dt}$$

With : $\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times \frac{d\vec{p}}{dt}$ $\longrightarrow$ $\boxed{\vec{\tau} = \frac{d\vec{L}}{dt}}$

- **To compare with linear motion :** $\vec{F} = \frac{d\vec{p}}{dt}$

5.2 Angular momentum of a system of particles

The net external torque acting on the system :

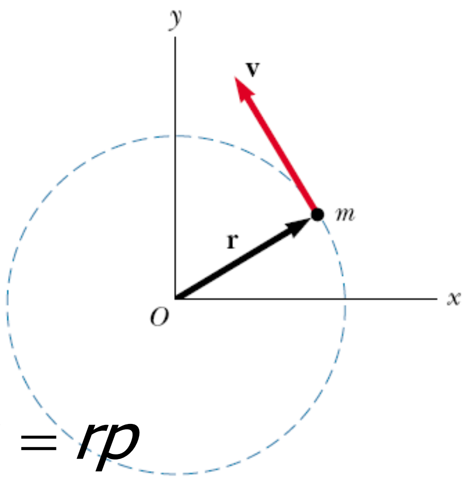
$$\sum \vec{\tau} = \sum_i \frac{d\vec{L}_i}{dt} = \frac{d}{dt} \sum_i \vec{L}_i = \frac{d\vec{L}}{dt}$$

with : $\vec{L} = \sum_i \vec{L}_i$ being the **total angular momentum** of the system

PROBLEM 11 A car of mass 1 500 kg moves with a linear speed of 40 m/s on a circular race track of radius 50 m.

(a) What is the magnitude of its angular momentum relative to the center O of the track?

(b) Find the moment of inertia of the car about O.



SOLUTION

(a) $\vec{L}$ is perpendicular to the *rp*-plane. Its direction is upward, and its magnitude is

$$L = rp\sin\phi = mrv\sin\phi = mrv\sin 90^0 = mrv = rp$$

$$L = 1500\ kg \times 50\ m \times 40\ m/s = 3.0 \times 10^6\ kg.m^2/s$$

(b) $L = mrv = mr(r\omega) = (mr^2)\omega = I\omega$;

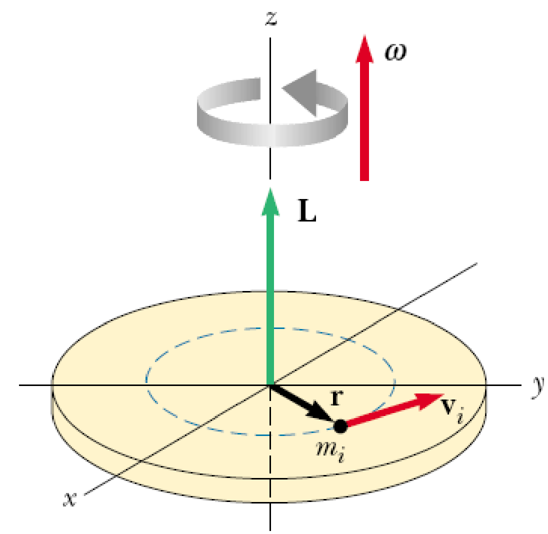
$$I = \frac{L}{\omega} = \frac{3.0 \times 10^6\ kg.m^2/s}{(40\ m/s)/(50\ m)} = 3.75\ kg.m^2$$

5.3 Angular Momentum of a Rotating Rigid Object

Consider a rigid object rotating about a fixed axis that coincides with the *z* axis of a coordinate system

The angular momentum of each particle:

$$L_i = m_i r_i v_i \sin 90^0 = m_i r_i (r_i \omega) = m_i r_i^2 \omega$$



The angular momentum of the whole object:

$$L_z = \sum_I m_i r_i^2 \omega = \left(\sum_I m_i r_i^2 \right) \omega$$

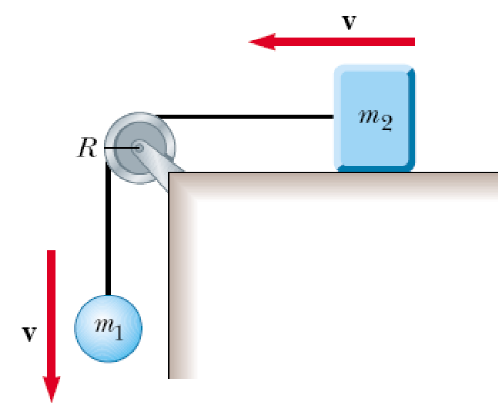
$$\boxed{L_z = I\omega}$$

Differentiate with respect to time : $\dfrac{dL_z}{dt} = I \dfrac{d\omega}{dt} = I\alpha$

$$\boxed{\tau = \frac{dL}{dt}}$$

PROBLEM 12 A sphere of mass m_1 and a block of mass m_2 are connected by a light cord that passes over a pulley. The radius of the pulley is R, and the moment of inertia about its axle is I. The block slides on a frictionless, horizontal surface. Find an expression for the linear acceleration of the two objects, using the concepts of angular momentum and torque.

SOLUTION



The angular momentum of the sphere and of the block about the axle of the pulley:

$$L_1 = m_1 vR \, ; \; L_2 = m_2 vR$$

The angular momentum of the pulley:

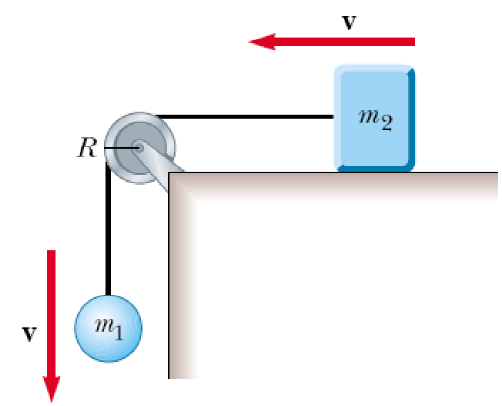
$$L_3 = I\omega = I\frac{V}{R}$$

The total angular momentum of the system:

$$L = m_1 vR + m_2 vR + I\frac{V}{R}$$

PROBLEM 12 A sphere of mass m_1 and a block of mass m_2 are connected by a light cord that passes over a pulley. The radius of the pulley is R, and the moment of inertia about its axle is I. The block slides on a frictionless, horizontal surface. Find an expression for the linear acceleration of the two objects, using the concepts of angular momentum and torque.

$$L = m_1 vR + m_2 vR + I\frac{v}{R}$$



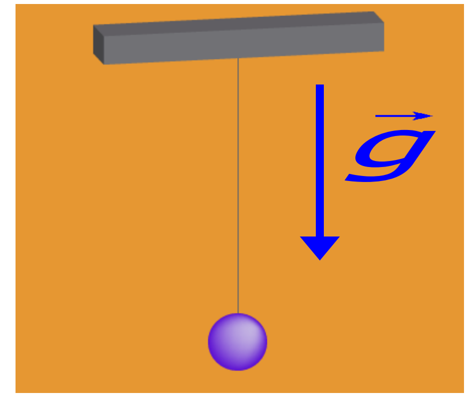
The total external torque acting on the system about the pulley axle: $\tau = m_1 gR$

$$\boxed{\tau = \frac{dL}{dt}}$$

$$m_1 gR = \frac{d}{dt}\left(m_1 vR + m_2 vR + I\frac{v}{R}\right) = (m_1 + m_2)Ra + \frac{I}{R}a$$

$$a = \frac{m_1 g}{m_1 + m_2 + \dfrac{I}{R^2}}$$

PROBLEM 13 By using the concepts of angular momentum and torque, find the equation of motion for a pendulum.



SOLUTION

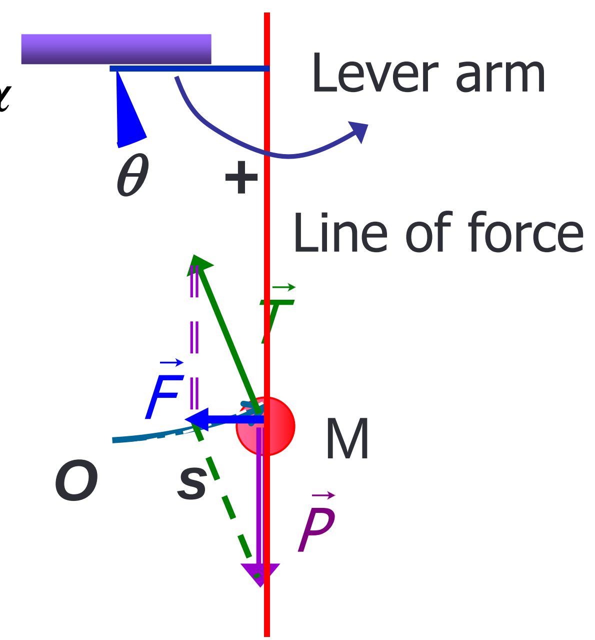
The angular momentum of the object about the axis of rotation: $L = I\omega$

The total external torque acting on the system: $\tau = mgd = mgl\sin\theta$

$$\boxed{\tau = \frac{dL}{dt}} \longrightarrow mgl\sin\alpha = \frac{d}{dt}(I\omega) = I\frac{d\omega}{dt} = I\alpha$$

With small θ: $mgl\theta = I\theta''$; $\boxed{\theta'' + \frac{mgl}{I}\theta = 0}$

Simple pendulum: $I = ml^2 \longrightarrow \boxed{\theta'' + \frac{g}{l}\theta = 0}$



5.4 Conservation of Angular Momentum

From : $$\boxed{\vec{\tau} = \frac{d\vec{L}}{dt}}$$

If: $\vec{\tau} = \vec{0} \longrightarrow \dfrac{d\vec{L}}{dt} = \vec{0}\ ;\ \vec{L} = \overrightarrow{const}$

The total angular momentum of a system is constant in both magnitude and direction if the resultant external torque acting on the system is zero.

$$\boxed{\vec{L}_i = \vec{L}_f}$$

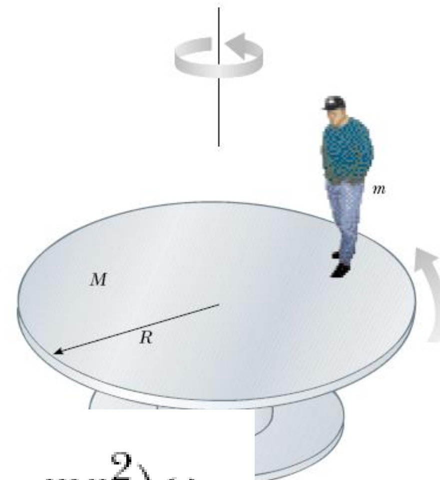
$$\boxed{I_i\omega_i = I_f\omega_f}$$

Three Conservation Laws for an Isolated System

- **Conservation of energy :** $\boxed{K_i + U_i = K_f + U_f}$
- **Conservation of linear momentum :** $\boxed{\vec{p}_i = \vec{p}_f}$
- **Conservation of angular momentum :** $\boxed{\vec{L}_i = \vec{L}_f}$

PROBLEM 14 A horizontal platform in the shape of a circular disk rotates in a horizontal plane about a frictionless vertical axle. The platform has a mass $M = 100$ kg and a radius $R = 2.0$ m. A student whose mass is $m = 60$ kg walks slowly from the rim of the disk toward its center. If the angular speed of the system is 2.0 rad/s when the student is at the rim, what is the angular speed when he has reached a point $r = 0.50$ m from the center?

SOLUTION



$$I_i = I_{pi} + I_{si} = \tfrac{1}{2}MR^2 + mR^2$$

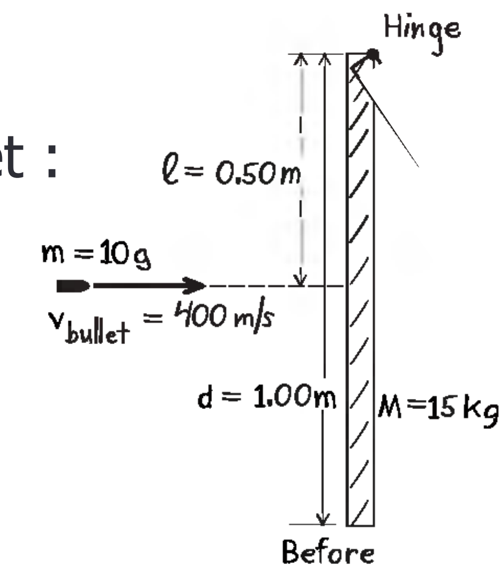
$$I_f = I_{pf} + I_{sf} = \tfrac{1}{2}MR^2 + mr^2$$

$$I_i\omega_i = I_f\omega_f \longrightarrow \left(\tfrac{1}{2}MR^2 + mR^2\right)\omega_i = \left(\tfrac{1}{2}MR^2 + mr^2\right)\omega_f$$

$$\omega_f = \left(\frac{\tfrac{1}{2}MR^2 + mR^2}{\tfrac{1}{2}MR^2 + mr^2}\right)\omega_i = \boxed{4.1\ \mathrm{rad/s}}$$

PROBLEM 15 A door 1.00 m wide, of mass 15 kg, is hinged at one side so that it can rotate without friction about a vertical axis. It is unlatched. A bullet with a mass of 10 g is fired at a speed of 400 m/ s into the exact center of the door, in a direction perpendicular to the plane of the door. Find the angular speed of the door just after the bullet embeds itself in the door. Is kinetic energy conserved?

SOLUTION



The initial angular momentum of the bullet :

$$L = mvl = (0.010\,\mathrm{kg})(400\,\mathrm{m/s})(0.50\,\mathrm{m})$$
$$= 2.0\,\mathrm{kg\cdot m^2/s}$$

The final angular momentum :

$$I\omega = (I_{door} + I_{bullet})\omega$$

$$I_{door} = \frac{Md^2}{3} = \frac{(15\,\mathrm{kg})(1.0\,\mathrm{m})^2}{3} = 5.0\,\mathrm{kg\cdot m^2}$$

$$I_{bullet} = ml^2 = (0.010\,\mathrm{kg})(0.50\,\mathrm{m})^2 = 0.0025\,\mathrm{kg\cdot m^2}$$

PROBLEM 15 A door 1.00 m wide, of mass 15 kg, is hinged at one side so that it can rotate without friction about a vertical axis. It is unlatched. A bullet with a mass of 10 g is fired at a speed of 400 m/ s into the exact center of the door, in a direction perpendicular to the plane of the door. Find the angular speed of the door just after the bullet embeds itself in the door. Is kinetic energy conserved?

SOLUTION

Conservation of angular momentum requires: $mvl = I\omega$

$$\omega = \frac{mvl}{I} = \frac{2.0\ \mathrm{kg \cdot m^2/s}}{5.0\ \mathrm{kg \cdot m^2} + 0.0025\ \mathrm{kg \cdot m^2}} = 0.40\ \mathrm{rad/s}$$

We calculate the initial and final kinetic energies:

$$\left.\begin{aligned} K_1 &= \frac{1}{2}mv^2 = \frac{1}{2}(0.010\ \mathrm{kg})(400\ \mathrm{m/s})^2 = 800\ \mathrm{J} \\ K_2 &= \frac{1}{2}I\omega^2 = \frac{1}{2}(5.0025\ \mathrm{kg \cdot m^2})(0.40\ \mathrm{rad/s})^2 = 0.40\ \mathrm{J} \end{aligned}\right\} \quad K_1 \neq K_2$$