CALCULUS 1 (MA001IU) – FINAL EXAMINATION

Semester 3, 2022-23 • Duration: 120 minutes • Date: August 7, 2023

SUBJECT: CALCULUS 1	
Department of Mathematics	Lecturer
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INSTRUCTIONS:

- Use of calculator is allowed. Each student is allowed two double-sided sheets of notes (size A4 or similar). All other documents and electronic devices are forbidden.
- Write the steps you use to arrive at the answers to each question. No marks will be given for the answer alone.
- There are a total of 10 (ten) questions. Each one carries 10 points.

1. Evaluate the following limit

$$\lim_{x\to 8}\frac{\sqrt[3]{x}-2}{\ln(x-7)}.$$

Ans We have

$$\lim_{x\to 8}\frac{\sqrt[3]{x}-2}{\ln(x-7)}=\lim_{x\to 8}\frac{(\sqrt[3]{x}-2)'}{[\ln(x-7)]'}=\lim_{x\to 8}\frac{\frac{1}{3}x^{-2/3}}{\frac{1}{x-7}}=\frac{1}{12}$$

2. Find the point on the hyperbola $y=\dfrac{1}{2x}$ in the first quadrant that is closest to the point $(0,0)$.

Ans 1. Let d be the distance between the points $(0,0)$ and $(x,\frac{1}{2x})$. We minimize $f(x):=d^2=x^2+\dfrac{1}{4x^2}$, for $x>0$. One get $f'(x)=2x-\dfrac{1}{2x^3}$. It implies that $f(x)$ attains its minimum at $x=\dfrac{1}{\sqrt{2}}$.

3. A particle moves in a straight line and its velocity is given by $v(t)=3t+2$ and its initial position is $s(0)=0$. Find its position function when $t=2$, i.e., find $s(2)$.

Ans. We have $s'(t)=v(t)$. Therefore

$$\begin{aligned}&\int_0^2 s'(t)\,dt=\int_0^2 v(t)\,dt=\int_0^2(3t+2)\,dt\\ \Longrightarrow\quad& s(2)-s(0)=10\\ \Longrightarrow\quad& s(2)=12.\end{aligned}$$

4. Let

$$F(x)=\int_1^{\pi x^2}\sqrt{t+\sin t}\,dt.$$

Find $F'(1)$.

Ans. We have

$$F'(x)=\sqrt{\pi x^2+\sin(\pi x^2)}\cdot(\pi x^2)'_x=2\pi x\sqrt{\pi x^2+\sin(\pi x^2)}.$$

Hence, $F'(1)=2\pi\sqrt{\pi}$.

5. Evaluate the integral $\int_0^1 x^2 e^{-x} dx$.

   Ans. Use the integration by parts twice to obtain: $\int_0^1 x^2 e^{-x} dx = 2 - \frac{5}{e}$.

6. Evaluate the integral

$$\int_0^1 \frac{x-1}{x^2+4x+3} dx.$$

   Ans. We find A and B satisfying

$$\begin{aligned} &\frac{x-1}{x^2+4x+3} = \frac{A}{x+1} + \frac{B}{x+3} \\ \Longrightarrow\ & x-1 = A(x+3) + B(x+1) = (A+B)x + (3A+B) \\ \Longrightarrow\ & \begin{cases} A+B=1 \\ 3A+B=-1 \end{cases} \Longrightarrow \begin{cases} A=-1 \\ B=2 \end{cases}. \end{aligned}$$

   Hence,

$$\int_0^1 \frac{x-1}{x^2+4x+3} dx = \int_0^1 \left(\frac{-1}{x+1} + \frac{2}{x+3} \right) dx = (-\ln|x+1| + 2\ln|x+3|)\,|_0^1 = 3\ln 2 - 2\ln 3 = \ln\frac{8}{9}.$$

7. Determine whether the improper integral $\int_1^\infty \frac{2x}{\sqrt{1+x^2}} dx$ is convergent or divergent. Explain.

   Ans. We have

$$\int_1^\infty \frac{2x}{\sqrt{1+x^2}} dx = \lim_{t\to\infty} \int_1^t \frac{2x}{\sqrt{1+x^2}} dx = \lim_{t\to\infty} \left(2\sqrt{1+x^2}\right)|_1^t = \lim_{t\to\infty} \left(2\sqrt{1+t^2} - 2\sqrt{2}\right) = \infty.$$

8. Use the Trapezoidal rule with 6 sub equal intervals (i.e., $n = 6$) to approximate the value of the integral $\int_0^2 \frac{1}{16+x^2} dx$

   Ans. By putting $f(x) = \frac{1}{16+x^2}$, we have

$$\int_0^2 \frac{1}{16+x^2} dx \approx (1/2)(1/3)\left[f(0) + 2f(1/3) + 2f(2/3) + 2f(1) + 2f(4/3) + 2f(5/3) + f(2)\right] = 0.116$$

9. Find the area of the region enclosed by the curves $y = 6x - x^2$ and $y = x$.

   Ans. The intersections are at $x = 0$ and $x = 5$. The area is $A = \int_0^5 (6x - x^2 - x)\, dx = \frac{125}{6}$.

10. Use Newton's method to approximate the positive root correct to six decimal places of the equation $2 - x^2 = \sin x$.

    Ans. Observe that $2 - x^2 = \sin x \Leftrightarrow f(x) \equiv 2 - x^2 - \sin x = 0$.
    Using Newton's method, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{2 - x_n^2 - \sin x_n}{-2x_n - \cos x_n}$$

    By choosing, for example, $x_1 = 1$, we can deduce from the latter equation that

$$\begin{aligned} x_2 &= 1.062405 \\ x_3 &= 1.061549 \\ x_4 &= 1.061549 \end{aligned}$$

    Since x_3 and x_4 agree to six decimal places, the approximate positive root needed to find is 1.061549.

—END OF THE QUESTION PAPER. GOOD LUCK!—