VNUHCM – INTERNATIONAL UNIVERSITY

Chapter 5: Applications of Integration

CALCULUS I

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Slides are adapted based on the lecture slides of Dr. Nguyen Minh Quan

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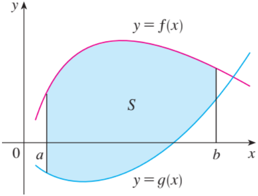
Motivation

In this chapter we explore some of the applications of the definite integral by using it to compute areas between curves, volumes of solids, the work done by a varying force, and other applications.

1. Areas Between Curves

- What is area of the region between the graphs of f and g?



Formula

The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$, $y = b$, where f and g are continuous and $f(x) \geq g(x)$ for all x in $[a, b]$, is:

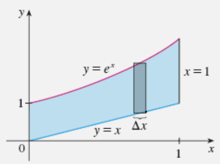
$$A = \int_a^b [f(x) - g(x)]dx$$

Note: In general, $A = \int_a^b |f(x) - g(x)|\, dx$

1. Areas Between Curves

Example 1

Find the area of the region bounded above by $y = e^x$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$.



Solution:

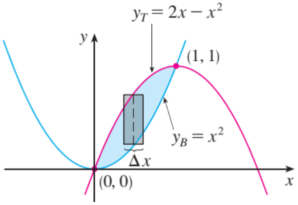
We use the previous area formula with $f(x) = e^x, g(x) = x, a = 0, b = 1$:

$$A = \int_0^1 (e^x - x)dx = e^x - \frac{x^2}{2}\Big|_0^1 = e - \frac{3}{2}$$

1. Areas Between Curves

Example 2

Find the area of the region bounded above by $y = x^2$, bounded below by $y = 2x - x^2$.



Solution:

We first find the points of intersection of the parabolas:

$$x^2 = 2x - x^2 \Leftrightarrow 2x(1 - x) = 0 \Leftrightarrow x = 0, x = 1$$

The points of intersection are $(0, 0)$ and $(1, 1)$.

1. Areas Between Curves

Example 2 (Solution continued)

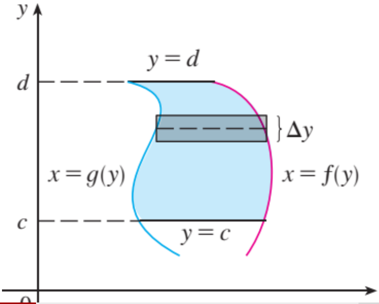
We use the area formula with $f(x) = 2x - x^2, g(x) = x^2, a = 0, b = 1$:

$$A = \int_0^1 \left(2x - 2x^2\right) dx = 2\int_0^1 \left(x - x^2\right) dx = 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]\Bigg|_0^1 = \frac{1}{3}$$

1. Areas Between Curves

Some regions are best treated by regarding x as a function of y. If a region is bounded by curves with equations $x = f(y) := x_R$, $x = g(y) := x_L$, $y = c$, and $y = d$, where f and g are continuous $(f(y) \geqslant g(y))$, then its area is

$$A = \int_c^d [f(y) - g(y)]\, dy$$

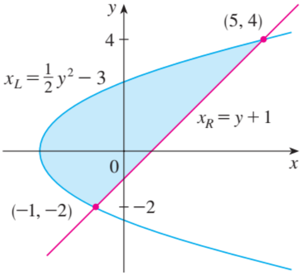


1. Areas Between Curves

Example 3

Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

Solution:



Intersections are $(-1, -2), (5, 4)$.

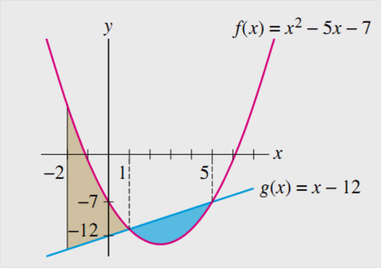
$$A = \int_{-2}^{4} [x_R - x_L]\, dy = \int_{-2}^{4} \left[(y+1) - \left(\frac{1}{2}y^2 - 3 \right) \right] dy = 18$$

1. Areas Between Curves

Example 4 (Calculate area by dividing the region)

Find the area the graphs of $f(x) = x^2 - 5x - 7$ the line $g(x) = x - 12$ over $[-2, 5]$.

Solution:



To determine where the graphs intersect, we solve $f(x) = g(x)$. The points of intersection are $x = 1, 5$.

1. Areas Between Curves

Solution (Continued)

$$\int_{-2}^{5} (y_{top} - y_{bot})\,dx = \int_{-2}^{1} (f(x) - g(x))\,dx + \int_{1}^{5} (g(x) - f(x))\,dx$$

$$= \int_{-2}^{1} \left(\left(x^2 - 5x - 7\right) - (x - 12)\right) dx + \int_{1}^{5} \left((x - 12) - \left(x^2 - 5x - 7\right)\right) dx.$$

$$= \int_{-2}^{1} \left(x^2 - 6x + 5\right) dx + \int_{1}^{5} \left(-x^2 + 6x - 5\right) dx = \frac{113}{3}.$$

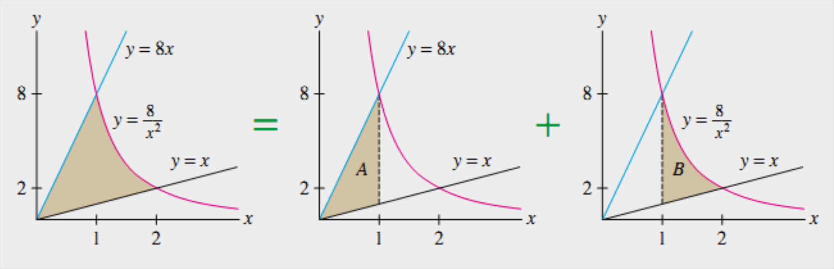
1. Areas Between Curves

Areas between three or more curves: we divide the area into different sections.

Example 5 (Calculate area by dividing the region)

Find the area enclosed by the graphs $y = 8/x^2, y = x$ and $y = 8x$.

Hint:



$$\text{area} = 7/2 + 5/2 = 6$$

2. Areas Enclosed by Parametric Curves

Suppose the curve is described by the parametric equations
$x = f(t)$, and $y = g(t)$, $\alpha \leqslant t \leqslant \beta$, then:

$$Area = \left| \int_{\alpha}^{\beta} g(t) f'(t)\, dt \right|$$

Example 1

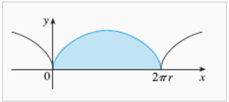
Find the area under one arch of the cycloid.

$$x = r(\theta - \sin\theta),\, y = r(1 - \cos\theta), \quad 0 \leqslant \theta \leqslant 2\pi$$

2. Areas Enclosed by Parametric Curves

Example 1

Solution:



One arch of the cycloid is given by $0 \leqslant \theta \leqslant 2\pi$. Using the Substitution Rule:

$$A = \int_0^{2\pi} r\,(1-\cos\theta) r\,(1-\cos\theta)\,d\theta = r^2 \int_0^{2\pi} (1-\cos\theta)^2 d\theta$$

$$\Rightarrow A = r^2 \int_0^{2\pi} \left(1 - 2\cos\theta + \frac{1}{2}(1+\cos 2\theta)\right) d\theta = 3\pi r^2$$

2. Areas Enclosed by Parametric Curves

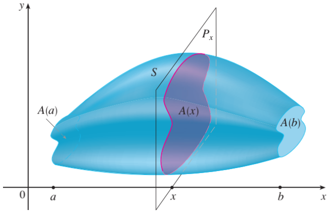
Example 2

Find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

3. Volume of a solid

- How to find the volume of a solid?



Theorem 1

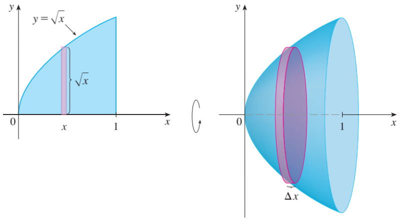
Let be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of in the plane, through x and perpendicular to the x-axis, is $A(x)$, where A is a continuous function, then the volume V of S is

$$V = \int_a^b A(x)\, dx$$

Solids of Revolution: Volumes found by Slicing

Example 1 (rotating about the x-axis)

Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.



Solution:

The area of the cross-section through the point x: $A(x) = \pi\left(\sqrt{x}\right)^2 = \pi x$

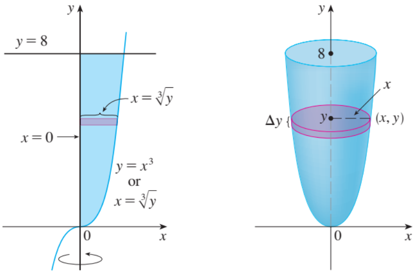
The solid lies between $x = 0$ and $x = 1$, so its volume is

$$V = \int_a^b A(x)\,dx = \int_0^1 \pi x dx = \pi \frac{x^2}{2}\Big|_0^1 = \frac{\pi}{2}$$

Solids of Revolution: Volumes found by Slicing

Example 2 (rotating about the y-axis)

Find the volume of the solid obtained by rotating the region bounded by $y = x^3$,$y = 8$, and $x = 0$ about the y-axis.



$$A(y) = \pi x^2 = \pi y^{2/3} \Rightarrow V = \int_0^8 A(y)\,dy = \int_0^8 \pi y^{2/3} dy = \frac{96\pi}{5}$$

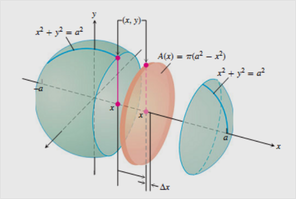
Solids of Revolution: Volumes found by Slicing

Example 3

The circle $x^2 + y^2 = a^2$ is rotated about the $x-$axis to generate a sphere. Find its volume.

Hint:

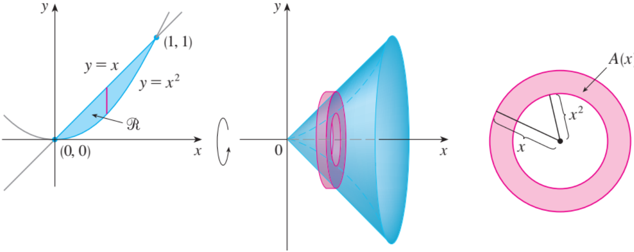
$$V = \int_{-a}^{a} A(x)\,dx = \pi \int_{-a}^{a} \left(a^2 - x^2\right) dx = \frac{4}{3}\pi a^3$$



Solids of Revolution: Volumes found by Washers

Example 4 (Using the washer method)

Find the volume of the solid obtained by rotating the region R enclosed by the curves $y = x$ and $y = x^2$ about the x-axis.



Hint:

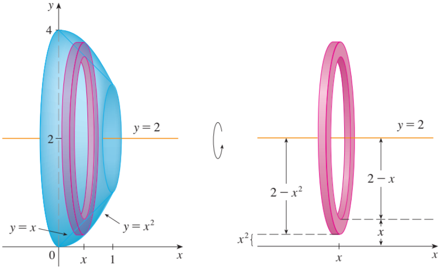
$$A(x) = \pi\left((\text{outer radius})^2 - (\text{inner radius})^2\right) = \pi\left(x^2 - x^4\right).$$

$$\Rightarrow V = \int_0^1 A(x)\,dx = \int_0^1 \pi\left(x^2 - x^4\right)dx = \frac{2\pi}{15}$$

3. Volume of a solid

Example 5. Rotating about a horizontal line (Additional reading)

Find the volume of the solid obtained by rotating the region in previous example (Example 4) about the line $y = 2$.



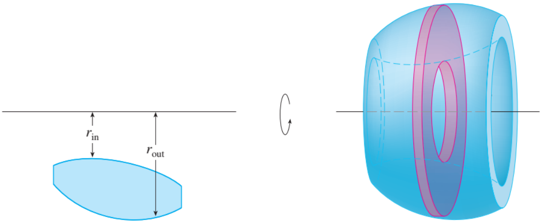
Hint:

$$A(x) = \pi\left(\left(2 - x^2\right)^2 - (2 - x)^2\right) \Rightarrow V = \int_0^1 A(x)\,dx = \frac{8\pi}{15}$$

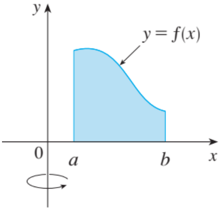
3. Volume of a solid

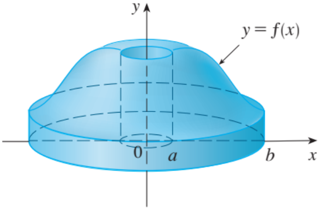
- In summary, the solids in Examples $1-5$ are all called solids of revolution because they are obtained by revolving a region about a line.
- Formula 1: $V = \int_a^b A(x)\,dx,$ or $V = \int_c^d A(y)\,dy$
- How to find A? Based on the cross-section is a disk or a washer.

$$A = \pi\,(radius)^2\,,\ \text{or } A = \pi\,(outer\ \ radius)^2 - \pi\,(inner\ \ radius)^2$$



3. Volume of a solid: the method of cylinders





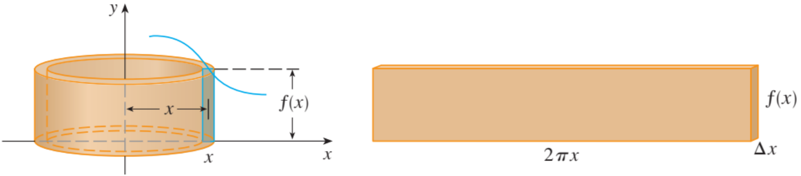
- Consider the problem of finding the volume of the solid obtained by rotating about the y-axis the region bounded by $y = f(x)$. What should we do if it is hard to solve $y = f(x)$ for x in term of y.
- **Formula 2 (method of cylinders/cylindrical shells):** The volume of the solid obtained by rotating about the y-axis the region under the curve from a to b, is

$$V = \int_a^b 2\pi x f(x)\, dx, \quad 0 \leqslant a < b$$

3. Volume of a solid

Elaborating: The best way to remember Formula 2 is to think of a typical shell, cut and flattened as in the following figure, with radius x, circumference $2\pi x$, height $f(x)$, and thickness Δx or dx:

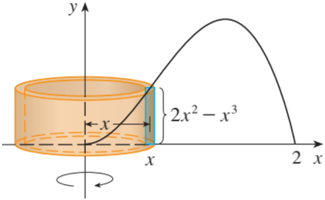
$$\int_a^b \underbrace{(2\pi x)}_{\text{circumference}} \underbrace{[f(x)]}_{\text{height}} \underbrace{dx}_{\text{thickness}}$$

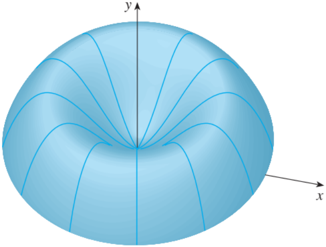


3. Volume of a solid

Example 5 (Using the method of cylinders)

Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.





Answer:

$$V = \int_a^b 2\pi x f(x)\, dx = \int_0^2 2\pi x \left(2x^2 - x^3\right) dx = \frac{16}{5}\pi$$

3. Volume of a solid

Example 6 (Using the method of cylinders)

Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2 - x^2$, $x = 0$ and $y = x$.

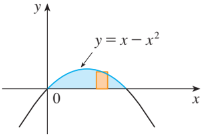
Hint:

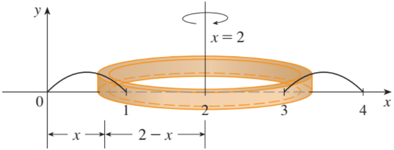
$$V = 2\pi \int_0^1 x\left(2 - x^2 - x\right) dx$$

3. Volume of a solid

Example 7 (Using the method of cylinders. Additional reading.)

Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ and about the line $x = 2$.





Answer:

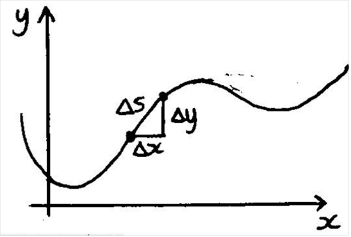
The region has radius $2 - x$, circumference $2\pi(2 - x)$ and height $x - x^2$.

$$V = \int_0^1 2\pi \left(2 - x\right)\left(x - x^2\right) dx = \frac{\pi}{2}$$

4. Lengths of curves

- **Arc Length Formula 1:** If a smooth curve with parametric equations $x = f(t), y = g(t), a \leqslant t \leqslant b$ is traversed exactly once as increases from a to b, then its length is

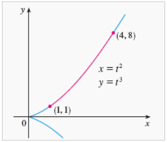
$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$



4. Lengths of curves

Example 1

Find the length of the arc of the curve $x = t^2$, $y = t^3$ that lies between the points $(1, 1)$ and $(4, 8)$.



$$L = \int_1^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_1^2 \sqrt{(2t)^2 + (3t^2)^2} dt$$

$$L = \int_1^2 t\sqrt{4 + 9t^2} dt = \frac{1}{27}\left(80\sqrt{10} - 13\sqrt{13}\right). \text{ (How?)}$$

4. Lengths of curves

- If we are given a curve with equation $y = f(x)$, $a \leqslant x \leqslant b$, then we can regard x as a parameter.

- The parametric equations are $x = x$, $y = f(x)$, and the Arc Length Formula 1 becomes:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

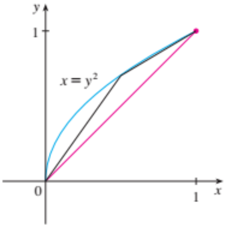
- Similarly, if $x = f(y)$, $a \leqslant y \leqslant b$, and the Arc Length Formula 1 becomes:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\, dy$$

4. Lengths of curves

Example 2

Find the length of the arc of the parabola $y^2 = x$ from $(0,0)$ to $(1,1)$.



Hint:

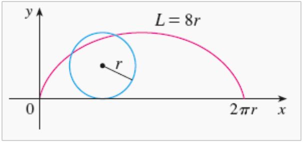
$$L = \int_a^b \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}dx = \int_0^1 \sqrt{4y^2+1}dx = \frac{\sqrt{5}}{2} + \ln\left(\frac{\sqrt{5}+2}{4}\right)$$

4. Lengths of curves

Example 3

Find the length of one arch of the cycloid:

$$x = r(\theta - \sin\theta),\, y = r(1 - \cos\theta)$$



Hint:

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = r\int_0^{2\pi} \sqrt{2(1-\cos\theta)}\, d\theta = 8r$$

4. Lengths of curves

Example 4

Consider the circle $x^2 + y^2 = R^2$.
(a) Write down parametric equations to traverse the circle once.
(b) Show that the length of the circumference is $2\pi R$.

5. The average value of a function

We define the average value of f on the interval $[a, b]$ as

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx$$

Example 1

Find the average value of the function $f(x) = 1 + x^2$ on the interval $[-1, 2]$.

Solution:

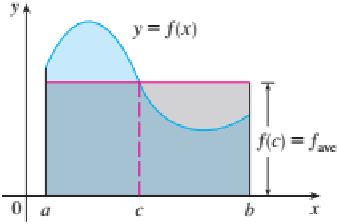
$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{2-(-1)}\int_{-1}^{2}\left(1+x^2\right)dx = 2$$

5. The average value of a function

The Mean Value Theorem for Integrals

If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that:

$$f(c) = f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx, \text{ Or, } f(c)(b-a) = \int_a^b f(x)dx$$



5. The average value of a function

How to find the value c of in the MVT for Integrals?

Example 2

Find the value c satisfies the the MVT for Integrals of the function $f(x) = 1 + x^2$ on the interval $[-1, 2]$.

Solution:

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{2-(-1)}\int_{-1}^2 \left(1+x^2\right)dx = 2$$

$$f(c) = f_{ave} = 2 \Leftrightarrow 1 + c^2 = 2 \Leftrightarrow c = \pm 1$$

5. The average value of a function

Example 3

If a cup of coffee has temperature 95^o C in a room where the temperature is 20^o C, then, according to Newton's Law of Cooling, the temperature of the coffee after t minutes is $T(t) = 20 + 75e^{-t/50}$. What is the average temperature of the coffee during the first half hour?

6. Applications to Engineering and Economics

- **WORK DONE:** The work done in moving the object from a to b by a variable force $f(x)$ acts on the object, where f is a continuous function, is defined as

$$W = \int_a^b f(x)\,dx$$

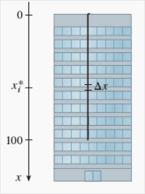
- Example 1: When a particle is located a distance x feet from the origin, a force of $x^2 + 2x$ pounds acts on it. How much work is done in moving it from $x = 1$ to $x = 3$? Answer:

$$W = \int_1^3 \left(x^2 + x\right) dx = \frac{50}{3}$$

6. Applications to Engineering and Economics

Example 2

A 200-lb cable is 100 ft long and hangs vertically from the top of a tall building. How much work is required to lift the cable to the top of the building?

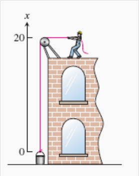


Hint: $W = \int_0^{100} 2\,(100 - x)\,dx = 10,000$ ft-lb.

6. Applications to Engineering and Economics

Example 3

A 5-lb bucket is lifted from the ground into the air by pulling in 20 ft of rope at a constant speed. The rope weighs 0.08 lb ft. How much work was spent lifting the bucket and rope?



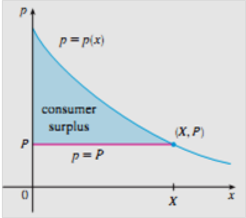
Hint: $W = 100 + \int_0^{20} (0.08)\,(20 - x)\,dx = 116$ ft-lb.

6. Applications to Engineering and Economics

We consider some applications of integration to economics.

Consumer Surplus

Recall that the demand function is the price that a company has to charge in order to sell units of a commodity. If X is the amount of the commodity that is currently available, then $P = p(X)$ is the current selling price. The graph of the demand function $y = p(x)$ is called the demand curve.



6. Applications to Engineering and Economics

Consumer Surplus

We define

$$\int_0^X [p(x) - P]dx$$

as the consumer surplus for the commodity. The consumer surplus represents the amount of money saved by consumers in purchasing the commodity at price P, corresponding to an amount demanded of X.

Consumer Surplus

Example

The demand for a product, in dollars, is

$$p = 1200 - 0.2x - 0.0001x^2.$$

Find the consumer surplus when the sales level is 500.

Solution

$$P = p(X) = p(500) = 1075$$

$$\Rightarrow \int_0^{500} [p(x) - P]dx = \int_0^{500} \left[1200 - 0.2x - 0.0001x^2 - 1075\right] dx$$

$$\Rightarrow \int_0^{500} [p(x) - P]dx = 33,333.33 \ (dollars).$$

The End. Thank you!

HOMEWORK

(1) Areas between curves: Exs. 1–28, page 362 (see 5.1)

(2) Volume: Exs. 1–18, page 374 (see 5.2)

(3) Arc length : Exs. 9–20, page 589 (see 8.1)

James Stewart: *Calculus*, 8th edition, Cengate learning (2016)