VNUHCM – INTERNATIONAL UNIVERSITY

Chapter 4: Integration

CALCULUS I

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Slides are adapted based on the lecture slides of Dr. Nguyen Minh Quan

Contents

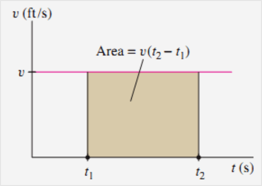
Introduction

- In Chapter 2 we learnt about differentiation. Given a total quantity, differentiation allows us to find its rate of change by taking its derivative.
- In this chapter, we study the reverse process: given a rate of change, integration allows us to find a total quantity.
- Do we have any connection between differentiation and integration?
- Integral and differential calculus are connected by The Fundamental Theorem of Calculus (FTC) which connects them.
- Integration has various of applications such as measuring area, volume,... It thus is related to measure the area under the curve.
- References: Chapter 4 and Chapter 7, Calculus 8th, by J. Stewart 2016.

1. Areas under Curves

- Assume that a car was moving with constant velocity v (m/s) during the time t_1 and t_2. Then the distance that this car traveled on $[t_1, t_2]$ is $S = v \times (t_2 - t_1)$.

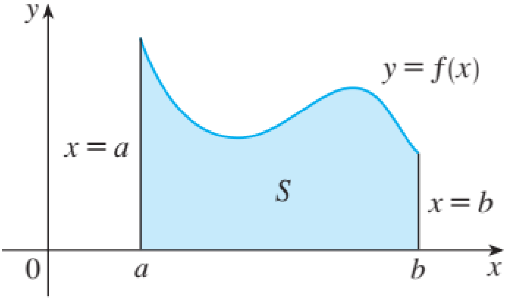
- This quantity equals to the area bounded by the horizontal line $y = v$ and the two vertical lines $x = t_1$, $x = t_2$.



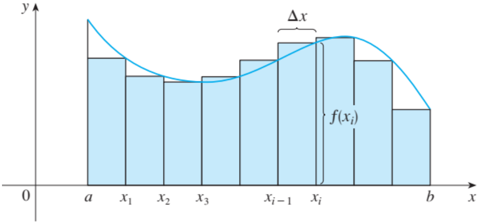
- When the velocity of the car is not constant; that is, v varies in time $v = v(t)$. What is the distance traveled by this car?

1. Areas under Curves

The previous example is related to a fundamental problem of calculus: given a nonnegative continuous function $y = f(x)$, find the area S of the region R lying under the graph of f, above the x-axis and between the vertical lines $x = a$ and $x = b$, where $a < b$.



1. Areas under Curves



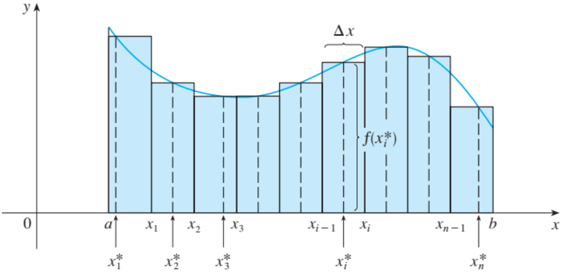
- The idea behind the computing this area is that we can effectively compute such quantities by breaking it into small pieces and then summing the contributions from each piece.
- We divide $[a, b]$ into N closed subinterval so that

$$a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_N = b$$

and

$$\Delta x := x_k - x_{k-1} = (b-a)/N \to 0 \text{ as } N \to \infty.$$

1. Areas under Curves



- The set $P = \{x_0, x_1, ..., x_{N-1}, x_N\}$ is called a partition of $[a, b]$.
- In each subinterval $[x_{i-1}, x_i]$, we select an arbitrary point x_i^*, $i = 1, .., n$. The area of a i^{th} rectangle with height $f(x_i^*)$ and width Δx is $f(x_i^*)\Delta x$.
- We approximate the area S by summing the areas of all the rectangles: $S_P = \sum_{i=1}^{N} f(x_i^*)\Delta x$. This sum is called Riemann sum. As $N \to \infty$, the area of the rectangles becomes closer to S.

1. Areas under Curves

Definition

The area S of the region under a nonnegative, continuous function f is the limit of the sum of the areas of approximating rectangles:

$$S = \lim_{N \to \infty} S_P = \lim_{n \to \infty} \sum_{i=1}^{N} f\left(x_i^*\right) \Delta x$$

Note: It can be shown that if f is continuous, this limit always exists.

2. The Definite Integral

Georg Friedrich Riemann (1826–1866)

Definition

The definite integral of a nonnegative, continuous function f over $[a, b]$ is the limit of Riemann sums (that is, the limit of the sum of the areas of approximating rectangles) and is denoted by the integral sign:

$$\int_a^b f(x)dx = \lim_{N\to\infty} S_P = \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i^*)\,\Delta x$$

2. The Definite Integral

Example

Set up an expression for $\int_0^1 x dx$ as a limit of sums and evaluate the limit.

Solution

Choose $x_i^* = x_i = i/N$, $\Delta x = 1/N$, $i = 1, .., N$ (x_i^* are right endpoints).

The Riemann sum is $\sum_{i=1}^{N} f\left(\frac{i}{N}\right)\frac{1}{N}$. Therefore,

$$\int_0^1 x dx = \lim_{N\to\infty} \sum_{i=1}^{N} f\left(\frac{i}{N}\right)\frac{1}{N} = \lim_{N\to\infty} \sum_{i=1}^{N} \frac{i}{N}\frac{1}{N}$$

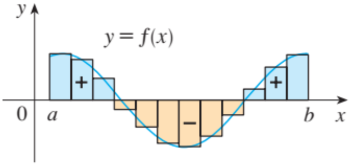
Thus,

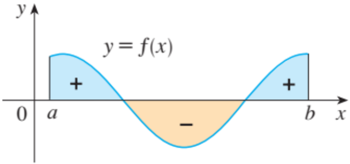
$$\int_0^1 x dx = \lim_{N\to\infty} \frac{1}{N^2}\sum_{i=1}^{N} i = \lim_{N\to\infty} \frac{N(N+1)}{2N^2} = \frac{1}{2}$$

2. The Definite Integral

Now we generalize further, removing the restrictions that f must be nonnegative. Assume that f is continuous on the closed, finite interval $[a; b]$. If $f(x)$ is nonnegative, then the definite integral is not equal to an area in the usual sense, but we may interpret it as the "signed area" between the graph and the x-axis.

Signed area (net area) is defined as the difference of the area above x-axis and the area below x-axis.





$$\int_a^b f(x)dx = \text{ net area}$$

2. The Definite Integral

Theorem

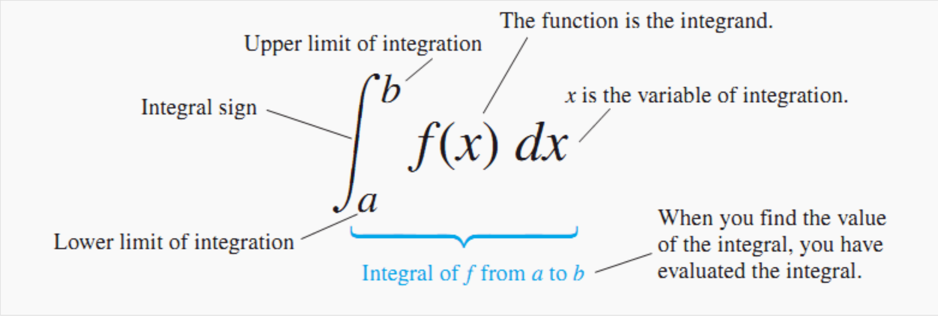
If f is continuous on $[a, b]$ then f is integrable on $[a, b]$.

Definition

The definite integral of an integrable function f over $[a, b]$ is the limit of Riemann sums (that is, the limit of the sum of the areas of approximating rectangles) and is denoted by the integral sign:

$$\int_a^b f(x)dx = \lim_{N\to\infty} S_P = \lim_{N\to\infty} \sum_{i=1}^{N} f(c_i)\,\Delta x_i$$

2. The Definite Integral



Properties of the Definite Integral

If f and g are integrable on an interval containing a, b and c.

- $\int_a^a f(x)dx = 0$.
- $\int_a^b f(x)dx = -\int_b^a f(x)dx$.
- $\int_a^b [\alpha f(x) + \beta g(x)]dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx$.
- $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$.
- If $f(x) \leqslant g(x)$ and $a \leqslant b$ then $\int_a^b f(x)dx \leqslant \int_a^b g(x)dx$.
- If $a \leqslant b$ then $\left|\int_a^b f(x)dx\right| \leqslant \int_a^b |f(x)|dx$
- If $\mathrm{f_{odd}}$ is an odd function then $\int_{-a}^a f_{odd}(x)dx = 0$.
- If $\mathrm{f_{even}}$ is an even function then $\int_{-a}^a f_{even}(x)dx = 2\int_0^a f_{even}(x)dx$.

Properties of the Definite Integral

Example

Use the properties of integrals to evaluate

$$\int_{-3}^{3} \left(1 + x^3\right) dx$$

Solution

$$\int_{-3}^{3} \left(1 + x^3\right) dx = \int_{-3}^{3} 1dx + \int_{-3}^{3} x^3 dx = 6$$

Since the first integral is the area of the rectangle of the sides 6 and 1. the second integral is zero due to the the fact that x^3 is an odd function.

The Mean Value Theorem for Integrals

Theorem

If f is continuous on [a, b] then there exists c $\in$ [a, b] such that

$$\int_a^b f(x)dx = f(c)(b-a)$$

Definition

The value $f(c)$ is called the average value or mean value of f on $[a, b]$,

$$f_{av} = f(c) = \frac{1}{b-a}\int_a^b f(x)dx.$$

The Mean Value Theorem for Integrals

Example

Find the mean (average) value of the function $f(x) = 1 + x^3$ on the interval $[-3, 3]$.

Solution

The average value is

$$\frac{1}{3-(-3)} \int_{-3}^{3} \left(1 + x^3\right) dx = 1$$

The Fundamental Theorem of Calculus

- Calculating integrals by taking the limit of Riemann sums is often long and difficult.
- There is a more effective way to calculate integrals. This effective way is based on the relationship between integration and differentiation which is called the Fundamental Theorem of Calculus (FTC).

The Fundamental Theorem of Calculus

Suppose that f is continuous on an interval I containing the point a.
(a) Let a function $F(x)$ be defined on I by

$$F(x) = \int_a^x f(t)dt$$

Then F is an antiderivative of f, i.e., $F'(x) = f(x)$.

(b) Newton-Leibniz formula. If G is any antiderivative of f on I, then for any b in I we have

$$\int_a^b f(x)dx = G(b) - G(a) := G(x)\big|_a^b$$

The Fundamental Theorem of Calculus

Example

(a) Find the average value of $f(x) = 1 + x$ on $[-3, 3]$.

(b) Find the derivatives of $A(x) = \int\limits_2^x e^{-t^2} dt$.

(c) Find the derivatives of $B(x) = \int\limits_0^{x^3} e^{-t^2} dt$.

Solution

(a) The average value is

$$\frac{1}{3-(-3)} \int\limits_{-3}^{3} (1+x)\, dx = 1$$

The Fundamental Theorem of Calculus

Solution (Cont.)

(b) Using FTC for $f(t) = e^{-t^2}$, we have $A'(x) = e^{-x^2}$.

(c) Let F be an antiderivative of e^{-t^2}, i.e, $F(x) = \int\limits_a^x e^{-t^2} dt$ for any constant a. We have $F'(x) = e^{-x^2}$ and

$$B(x) = \int\limits_0^{x^3} e^{-t^2} dt = F(x^3).$$

Thus, $\dfrac{d}{dx}B(x) = \dfrac{d}{dx}\left(F\left(x^3\right)\right) = 3x^2 F'\left(x^3\right) = 3x^2 e^{-x^6}$.

The Fundamental Theorem of Calculus

Example

(a) Find $\int_0^1 (2x^5 + 2\sqrt{x} - 1)\, dx$.

(b) Find the area under the curve $y = 1/x^2$ and above $y = 0$ between $x = 1$ and $x = 2$.

Table of indefinite integral

Table of antiderivatives or indefinite integral

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int a^x\,dx = \frac{a^x}{\ln a} + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$$

Practical Application: Finding Total Change

Theorem

If $F(x)$ is a total quantity and $f = dF/dx$ is its rate of change, then the integral of a rate of change is the total change

$$\int_a^b f(x)dx = \int_a^b \frac{dF}{dx}dx = F(b) - F(a)$$

Example of practical applications

1. If the rate of growth of a population is dP/dt, then the increase in population during the time period from t_1 to t_2 is

$$\int_{t_1}^{t_2} \frac{dP}{dt}dt = P(t_2) - P(t_1)$$

Practical Application: Finding Total Change

Example of practical applications

2. If $C(x)$ is the cost of producing x units of a commodity, then the cost of increasing production from x_1 to x_2 is

$$\int_{x_1}^{x_2} \mathrm{C'}\left(\mathrm{x}\right) dx = C\left(x_2\right) - C\left(x_1\right)$$

3. If an object move along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$, so the change of position (displacement) of the object during the time period from t_1 to t_2 is

$$\int_{t_1}^{t_2} v\left(t\right) dt = s\left(t_2\right) - s\left(t_1\right)$$

Practical Application: Finding Total Change

Note that the total distance traveled during t_1 and t_2 is $\int_{t_1}^{t_2} |v(t)|dt$.

Example of practical applications

The marginal cost in producing x computer chips (in units of 1000) is $C'(x) = 150x^2 - 3000x + 17500$ ($ per 1000 chips).
(a) Find the cost of increasing production from 10,000 to 15,000 chips.
(b) Assuming C(0) = $35,000 (that is, set up costs were $35,000), find the total cost of producing 15,000 chips.

Practical Application: Finding Total Change

Example of practical applications

A particle moves along a line so that its velocity at time t is
$v(t) = t^2 - t - 6$ (m/s).
(a) Find the displacement of the particle during the time period $1 \leq t \leq 4$.
(b) Find the distance traveled during this time period.

Practical Application: Newton's Law of Cooling

- **Newton's Law of Cooling**: Let $T(t)$ be the temperature of the object at time t and T_s be the temperature of the surroundings $(T_s < (T(t))$, then
$$\frac{dT}{dt} = -k\,(T - T_s)$$
where k is a positive constant.
That is, the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings.

- Find $T(t)$? Integrating the (differential) equation,
$$\int \frac{dT}{T - T_s} = k \int dt \Rightarrow \ln(T - T_s) = -kt + \ln(T(0) - T_s)$$
This implies
$$T(t) = T_s + (T(0) - T_s)\, e^{-kt}$$

Practical Application: Newton's Law of Cooling

Example: Newton's Law of Cooling

A detective found a corpse in a motel room at midnight and its temperature was 80^o F. The temperature of the room is kept constant at 60^o F. Two hours later the temperature of the corpse dropped to 75^o F. Find the time of death. Note that the temperature of a normal person is 98.6^oF (37^oC).

Solution

Let $T(t)$ be the temperature of the corpse at time t. We have $T(t) = T_s + (T(0) - T_s)\, e^{-kt}$. Thus, at $t = 2$:

$$T(2) = 60 + (80 - 60)\, e^{-2k} = 75 \Rightarrow k = -\frac{1}{2} \ln\left(\frac{75 - 60}{80 - 60}\right) = 0.1438$$

At time of death: $T(t_d) = 60 + (80 - 60)\, e^{-kt_d} = 98.6$.

Therefore, $t_d = -\frac{1}{k} \ln\left(\dfrac{98.6 - 60}{80 - 60}\right) = -4.57$ hours.

Substitution Method

Substitution Rule

$$\int f(u(x))\, u'(x)dx = \int f(u)\, du = F(u(x)) + C$$

where F is an anti-derivative of f, that is, $F'(x) = f(x)$.

Example

Evaluate $\int 3x^2 \cos(x^3)\, dx$.

Solution

$$\text{Let } u = x^3 \Rightarrow du = 3x^2 dx.$$

Applying the substitution rule:

$$\int 3x^2 \cos(x^3)\, dx = \int \cos(u)\, du = \sin u + C = \sin(x^3) + C.$$

Substitution Method

Example

Evaluate $\int 2x\left(x^2+9\right)^5 dx$.

Solution

Step 1. Choose the function u and compute du.
Let $u = x^2 + 9$ then $du = 2xdx$.
Step 2. Rewrite the integral in terms of u and du , and evaluate

$$\int 2x\left(x^2+9\right)^5 dx = \int u^5 du = \frac{1}{6}u^6 + C$$

Step 3. Express the final answer in terms of x

$$\int 2x\left(x^2+9\right)^5 dx = \frac{1}{6}\left(x^2+9\right)^6 + C.$$

Substitution Method

Example

Evaluate $\int \frac{x}{\sqrt{1-4x^2}}dx$.

Solution

$$\text{Let } u = 1 - 4x^2. \text{ Thus } du = -8xdx, \text{ so } xdx = -\frac{1}{8}du$$

$$\int \frac{x}{\sqrt{1-4x^2}}dx = -\frac{1}{8}\int \frac{1}{\sqrt{u}}du = -\frac{1}{8}\int u^{-1/2}du = -\frac{1}{8}\left(2\sqrt{u}\right) + C$$

Therefore,

$$\int \frac{x}{\sqrt{1-4x^2}}dx = -\frac{1}{4}\sqrt{1-4x^2} + C.$$

Substitution Method

Exercises

Using substitution method to evaluate

1. $\int \dfrac{3x^2+6x}{\left(x^3+3x^2+9\right)^4}dx$
2. $\int \left(x^2+2x\right)\left(x^3+3x^2+1\right)^4 dx$
3. $\int e^{3x}dx$
4. $\int xe^{x^2+1}dx$
5. $\int x^5\sqrt{1+x^2}dx$
6. $\int \cos^3 x \sin x\, dx$
7. $\int \dfrac{\tan^{-1} x}{1+x^2}dx$
8. $\int \dfrac{(\ln x)^2}{\ldots}dx, \quad [u=\ln x]$

Substitution Method For Definite Integrals

Change of Variables Formula for Definite Integrals

If $u(x)$ is differentiable on $[a, b]$ and $f(x)$ is integrable on the range of $u(x)$, then

$$\int_a^b f(u(x))\, u'(x)\, dx = \int_{u(a)}^{u(b)} f(u)\, du$$

Substitution Method For Definite Integrals

Example

Evaluate $\int_0^2 x^2\sqrt{x^3+1}dx$.

Solution

Let $u = x^3 + 1$, thus $du = 3x^2 dx$.
We have $x^2\sqrt{x^3+1} = \frac{1}{3}\sqrt{u}du$.

$$x = 0 \Rightarrow u(0) = 1$$

$$x = 2 \Rightarrow u(2) = 9$$

Therefore,

$$\int_0^2 x^2\sqrt{x^3+1}dx = \frac{1}{3}\int_1^9 \sqrt{u}du = \frac{2}{9}u^{3/2}\Big|_1^9 = \frac{52}{9}$$

Exercises (Change of Variables)

Use the Change of Variables Formula to evaluate the definite integral 1-4.

1. $\int\limits_{1}^{6} \sqrt{x+3}\,dx$.

2. $\int\limits_{0}^{4} x\sqrt{x^2+9}\,dx$.

3. $\int\limits_{0}^{\pi/2} \cos^3 x \sin x\,dx$.

4. $\int\limits_{e}^{e^4} \frac{dx}{x\sqrt{\ln x}}$.

5. Prove that $\int\limits_{a}^{1} \frac{dt}{1+t^2} = \int\limits_{1}^{\frac{1}{a}} \frac{dt}{1+t^2}$.

Hint: Let $u = 1/t \Rightarrow \int^{1} dt \ldots = \int^{\frac{1}{a}} du \ldots$

Integration by parts

Integration by parts

$$\int u(x)\, v'(x)\, dx = u(x)\, v(x) - \int u'(x)\, v(x)\, dx$$

Or

$$\int u dv = uv - \int v du$$

Integration by parts

Example

Evaluate $\int x \sin x dx$

Solution

$$\text{Let } \begin{cases} u = x \\ dv = \sin x dx \end{cases} \Rightarrow \begin{cases} du = dx \\ v = -\cos x \end{cases}$$

$$\int x \sin x dx = \int u dv = uv - \int v du$$

Thus,

$$\int x \sin x dx = -x \cos x + \int \cos x dx = -x \cos x + \sin x + C$$

Integration by parts

Example

Evaluate $\int \ln x dx$

Solution

$$\text{Let} \begin{cases} u = \ln x \\ dv = dx \end{cases} \Rightarrow \begin{cases} du = \frac{1}{x} dx \\ v = x \end{cases}$$

$$\int \ln x dx = \int u dv = uv - \int v du$$

Thus,

$$\int \ln x dx = x \ln x - \int dx = x \ln x - x + C$$

Integration by parts

Example

Evaluate $\int x^2 e^x dx$

Solution

$$\text{Let } \begin{cases} u = x^2 \\ dv = e^x dx \end{cases} \Rightarrow \begin{cases} du = 2x dx \\ v = e^x \end{cases}$$

$$\int x^2 e^x dx = \int u dv = uv - \int v du$$

$$\int x^2 e^x dx = x^2 e^x - 2 \int x e^x dx$$

Integration by parts

Solution(Cont.)

Evaluate $\int xe^x dx$

$$\text{Let } \begin{cases} u = x \\ dv = e^x dx \end{cases} \Rightarrow \begin{cases} du = dx \\ v = e^x \end{cases}$$

$$\int xe^x dx = \int udv = uv - \int vdu$$

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C$$

Therefore,

$$\int x^2 e^x = x^2 e^x - 2xe^x + 2e^x + C_1$$

Integration by parts

Integration by parts for definite integrals

$$\int_a^b u(x)\, v'(x)\, dx = u(x)\, v(x)\big|_a^b - \int_a^b u'(x)\, v(x)\, dx$$

Example

Find the area of the region bounded by the curve $y = \tan^{-1} x$, the x-axis, and the lines x = 0 and x = 1.

Trigonometric Integrals

Trigonometric Substitution

Evaluate

$$\int \sin^m x \cos^n x dx$$

Method:

- If either m or n is odd, positive integer, the integral can be done easily by substitution.
- If the power of $\sin x$ and $\cos x$ are both even, then we can make use of the double-angle formulae

$$\cos^2 x = \frac{1+\cos 2x}{2}, \text{ and } \sin^2 x = \frac{1-\cos 2x}{2}$$

Trigonometric Integrals

Examples

Evaluate the following integrals:

1. $\displaystyle\int \sin^4 x \cos^3 x dx$
2. $\displaystyle\int \sin^4 x \ dx$
3. $\displaystyle\int \tan x dx$
4. $\displaystyle\int \frac{\cos x + \sin 2x}{\sin x} dx$
5. $\displaystyle\int \frac{\cos^5 x}{\sqrt{\sin x}} dx$
6. $\displaystyle\int \frac{\sin^3\left(\sqrt{x}\right)}{\sqrt{x}} dx$

Trigonometric Integrals

Example: Trigonometric Inverse Substitutions

Expression	Substitution	Identtity
$a^2 - x^2$	$x = a\sin\theta, \quad \frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$a^2 + x^2$	$x = a\tan\theta, \quad \frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$x^2 - a^2$	$x = a\sec\theta, \quad 0 \le \theta < \frac{\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

Example

1. Evaluate

$$\int_0^{1/2} x^2\sqrt{1-x^2}\,dx$$

2.

$$\int \frac{dx}{\sqrt{4+x^2}}$$

The Method of Partial Fractions

Find $\int \frac{P(x)}{Q(x)} dx$ where $P(x)$ and $Q(x)$ are polynomials (a ratio of polynomials)? We rewrite $\frac{P(x)}{Q(x)}$ as

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

where $deg(R) < deg(Q)$.

Example

$$\int \frac{x^3+x}{x-1} dx = \int \left(x^2 + x + 2 + \frac{2}{x-1} \right) dx$$
$$= \frac{x^3}{3} + \frac{x^2}{2} + 2x + 2\ln|x-1| + C$$

The Method of Partial Fractions

Case 1: The denominator $Q(x)$ is a product of **distinct** linear factors (no factor is repeated).

$$Q(x) = (a_1 x + b_1)(a_2 x + b_2) \dots (a_k x + b_k)$$

In this case the partial fraction theorem states that there exist constants $A_1, .., A_k$ such that

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1 x + b_1} + \frac{A_2}{a_2 x + b_2} + \dots + \frac{A_k}{a_k x + b_k}$$

Example

Evaluate $\displaystyle\int \frac{x+5}{x^2 + x - 2}$

The Method of Partial Fractions

Solution

Note that

$$\frac{x+5}{x^2+x-2}=\frac{x+5}{(x-1)(x+2)}:=\frac{A}{x-1}+\frac{B}{x+2}$$

$$x+5=A(x+2)+B(x-1)\,.\text{This implies }\begin{cases} A+B=1\\ 2A-B=5\end{cases}$$

$$\Rightarrow A=2, B=-1$$

Thus,

$$\int\frac{x+5}{x^2+x-2}dx=\int\left(\frac{2}{x-1}-\frac{1}{x+2}\right)dx=$$

$$=2\ln|x-1|-\ln|x+2|+C.$$

The Method of Partial Fractions. Case 1.

Exercises

Evaluate

$$\int \frac{\cos x dx}{\sin^2 x - 3\sin x + 2}$$

The Method of Partial Fractions

Case 2: $Q(x)$ is a product of linear factors, some of which are repeated.

Example: $\dfrac{x^3 - x + 1}{x^2(x-1)^2} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-1} + \dfrac{D}{(x-1)^2}$.

Example

Evaluate $\displaystyle\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}dx$

Hint:

$$\frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} = x + 1 + \frac{4x}{x^3 - x^2 - x + 1}$$

$$\frac{4x}{x^3 - x^2 - x + 1} = \frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

$$A = 1, B = 2, C = -1$$

The Method of Partial Fractions

Case 3: $Q(x)$contains irreducible quadratic factors, none of which is repeated.

If $Q(x)$ has the factor $ax^2 + bx + c$, where $b^2 - 4ac < 0$, then the expression for $R(x)/Q(x)$ will have a term of the form $\dfrac{Ax + B}{ax^2 + bx + c}$ where A and B are constants to be determined.

Example

$$\int \frac{-2x + 4}{(x^2 + 1)(x - 1)} dx$$

Answer:

$$-\frac{1}{2} \ln \left(x^2 + 1\right) - 3 \arctan (x) + \ln |x - 1| + C$$

Approximate Integration

- How to evaluate the following integral

$$\int_0^1 \sin\left(x^2\right) dx = ?$$

- The antiderivatives of some functions, like $\sin(x^2)$, $1/\ln(x)$, and $\sqrt{1+x^4}$, have no elementary formulas. When we cannot find a workable antiderivative for a function f that we have to integrate. We can partition the interval of integration, replace f by a closely fitting polynomial on each subinterval, integrate the polynomials, and add the results to approximate the integral of f.

- This procedure is an example of numerical integration.

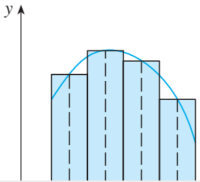
Midpoint Rule

Assume we want to approximate the following integral $\int_a^b f(x)dx$. We divide $[a, b]$ into n subintervals of equal length $\Delta x = \frac{b-a}{n}$. Denoting $x_k = a + i\Delta x, i = 0, 1, 2, .., n$.

Midpoint Rule

$$\int_a^b f(x)dx \approx M_n = \Delta x \left[f(\bar{x}_1) + f(\bar{x}_2) + ... + f(\bar{x}_n)\right]$$

where $\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i) =$ midpoint of $[x_{i-1}, x_i]$



Midpoint Rule

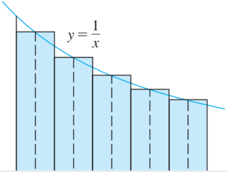
Example

Use the Midpoint Rule with $n = 5$ to approximate the integral $\int_1^2 \frac{dx}{x}$.

Solution

Noting that $a = 1, b = 2, n = 5, \Delta x = 0.2; x_0 = 1, x_1 = 1.2, x_2 = 1.4, x_3 = 1.6, x_4 = 1.8, x_5 = 2.$

$$\int_1^2 \frac{dx}{x} \approx \Delta x \left[f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9)\right] \approx 0.69191$$

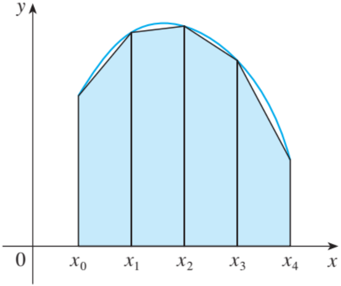


The Trapezoidal Rule

The n-subinterval Trapezoid Rule approximation to $\int_a^b f(x)dx$ is given by

$$\int_a^b f(x)dx \approx T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + \ldots + 2f(x_{n-1}) + f(x_n)\right]$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$.



The Trapezoidal Rule

Example

Use the Trapezoidal Rule with $n = 4$ to approximate the integral $\int_1^2 x^2 dx$.

Solution

Noting that $a = 1, b = 2, n = 4, \Delta x = 1/4; x_0 = 1, x_1 = 5/4, x_2 = 6/4, x_3 = 7/4, x_4 = 2.$

$$T_4 = \frac{1}{8}\left[f\left(x_0\right) + 2f\left(x_1\right) + 2f\left(x_2\right) + 2f\left(x_3\right) + f\left(x_4\right)\right]$$

$$T_4 = \frac{1}{8}\left[1 + 2\left(\frac{25}{16}\right) + 2\left(\frac{36}{16}\right) + 2\left(\frac{49}{16}\right) + 4\right] = \frac{75}{32} = 2.34375$$

The Simpson's Rule

The Simpson's Rule approximation to $\int_a^b f(x)dx$ based on an even number n of subintervals of equal length is

$$S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + .. + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

where $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\Delta x$.
In other words,

$$S_n = \frac{\Delta x}{3}\left[f(x_{end}) + 4f(x_{odd}) + 2f(x_{even})\right]$$

The Simpson's Rule

Example

Use the Simpson's Rule with $n = 4$ to approximate the integral $\int_0^2 5x^2 dx$.

Solution

$a = 0, b = 2, n = 4, \Delta x = 1/2; x_0 = 0, x_1 = 1/2, x_2 = 1, x_3 = 3/2, x_4 = 2.$

$$S_4 = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right]$$

$$S_4 = \frac{1}{6}\left[0 + 4\left(\frac{5}{4}\right) + 2(5) + 4\left(\frac{45}{4}\right) + 20\right] = \frac{40}{3}.$$

Approximate Integration

Exercises

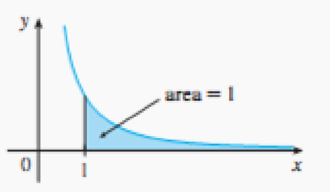
1. Calculate the Trapezoid Rule approximations T_4, T_8 and T_{16} for $\int_1^2 \frac{dx}{x}$.

2. Calculate the Simpson's Rule approximations S_4, S_8 and S_{16} for $\int_1^2 \frac{dx}{x}$.

Compute the exact errors if we know the value of the integral $\int_1^2 \frac{dx}{x} = \ln 2 = 0.69314718\ldots$

Improper Integrals

- Up to now, we have required definite integrals to have two properties. First, that the domain of integration $[a, b]$ be finite. Second, that the range of the integrand be finite on this domain.
- In practice, we may encounter problems that fail to meet one or both of these conditions. In either case the integral is called an improper integral.
- The integral for the area under the curve $y = 1/x^2$ from $x = 1$ to $x = \infty$ is an example for which the domain is infinite.



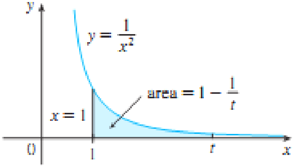
Improper Integrals of Type 1 (Infinite Intervals)

Let consider the area A under the curve $y = 1/x^2$ from $x = 1$ to $x = \infty$. The area of the part of that lies to the left of the line $x = t$ is

$$A(t) = \int_1^t \frac{dx}{x^2} = -\frac{1}{x}\bigg|_1^t = 1 - \frac{1}{t}$$

We define A as the limit of $A(t)$ as $t \to \infty$: $A = \lim\limits_{t\to\infty} A(t) = 1 \quad sq.unit.$

Thus, it is reasonable to define $\int\limits_1^\infty \frac{dx}{x^2} = \lim\limits_{t\to\infty} \int\limits_1^t \frac{dx}{x^2}$.



Improper Integrals of Type 1 (Infinite Intervals)

Definition

Integrals with infinite limits of integration are improper integrals of Type I.

- If f is continuous on $[a, \infty)$, then $\int\limits_a^{\infty} f(x)dx = \lim\limits_{b \to \infty} \int\limits_a^b f(x)dx$.
- If f is continuous on $[-\infty, b)$, then $\int\limits_{-\infty}^{b} f(x)dx = \lim\limits_{a \to -\infty} \int\limits_a^b f(x)dx$.
- If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int\limits_{-\infty}^{\infty} f(x)dx = \int\limits_{-\infty}^{c} f(x)dx + \int\limits_{c}^{\infty} f(x)dx.$$

Improper Integrals of Type 1 (Infinite Intervals)

Example

Evaluate $I = \int\limits_0^\infty e^{-x/2} dx$.

Solution

By definition

$$I = \lim_{b\to\infty} \int\limits_0^b e^{-x/2} dx = \lim_{b\to\infty} \left(-2e^{-x/2} \Big|_0^b \right) = \lim_{b\to\infty} \left(2 - 2e^{-b/2} \right) = 2.$$

Note that an improper integral is called convergent if the limit exists as a finite number, and divergent otherwise.

Improper Integrals of Type 1 (Infinite Intervals)

Example

Evaluate $I = \int\limits_0^\infty \frac{2x-1}{e^{3x}} dx$.

Solution

Applying the technique of integration by parts and by the definition of improper integral, we obtain:

$$\int\limits_0^\infty \frac{2x-1}{e^{3x}} dx = \lim_{t\to\infty} \int\limits_0^t (2x-1)e^{-3x} dx = \lim_{t\to\infty} \left(\frac{1-2t}{3e^{3t}} - \frac{2}{9e^{3t}} - \frac{1}{9} \right)$$

Therefore, $I = -\frac{1}{9}$.

Improper Integrals of Type 1 (Infinite Intervals)

Example

Evaluate $I = \int\limits_0^{+\infty} \frac{2\arctan 2x}{1+4x^2}dx$

Solution

Applying the technique of substitution, we obtain:

$$u = \arctan 2x, I = \lim_{t\to+\infty} \int\limits_0^t \frac{2\arctan 2x}{1+4x^2}dx = \lim_{t\to+\infty} \int\limits_0^{\arctan 2t} udu$$

$$= \lim_{t\to+\infty} \left\{ \frac{u^2}{2}\bigg|_0^{\arctan 2t} \right\} = \frac{\pi^2}{8}.$$

Improper Integrals of Type 1 (Infinite Intervals)

Example

Evaluate $\displaystyle\int_1^{\infty} \frac{\ln x}{x^3}\,dx.$

Hint Using the technique of integration by parts

Improper Integrals of Type 1 (Infinite Intervals)

Example

Evaluate $\int_{0}^{\infty} xe^{-x}\,dx$.

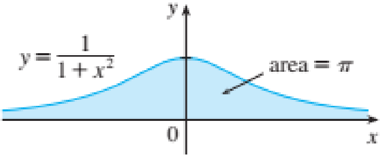
Hint Using the technique of integration by parts

Improper Integrals of Type 1 (Infinite Intervals)

Example

1. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$. Hint: $\displaystyle\lim_{t\to\infty} \arctan x\Big|_{-t}^{t} = \pi$.

2. Evaluate $\displaystyle\int_{1}^{\infty} \frac{dx}{\sqrt{x}}$.



Improper Integrals of Type 1 (Infinite Intervals)

Example

For what values of p is the integral $\int_1^{\infty} \frac{dx}{x^p}$ convergent?

Hint

$$\int_1^{\infty} \frac{dx}{x^p} = \begin{cases} \infty, \text{ if } p = 1 \\ \lim\limits_{t\to\infty} \dfrac{t^{1-p} - 1}{1-p}, \text{ if } p \neq 1 \end{cases}$$

Thus, $\int_1^{\infty} \frac{dx}{x^p}$ is convergent for $p > 1$ and divergent if $p \leq 1$.

Theorem

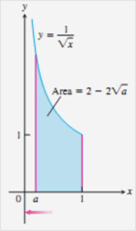
$$\int_1^{\infty} \frac{dx}{x^p} = \begin{cases} \dfrac{1}{p-1} \text{ if } p > 1 \\ \infty \text{ if } p \leq 1 \end{cases}$$

Improper Integrals of Type 2

Another type of improper integral arises when the integrand has a vertical asymptote at a limit of integration.

Example

Consider the region in the first quadrant that lies under the curve $y = 1/\sqrt{x}$ from $x = 0$ to $x = 1$.



$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{a \to 0^+} \int_a^1 \frac{dx}{\sqrt{x}} = \lim_{a \to 0^+} \left(2 - 2\sqrt{a}\right) = 2$$

Definition of Improper Integrals of Type 2

Integrals of functions that become infinite at a point within the interval of integration are improper integrals of Type II.

- If f is continuous on $(a, b]$ and discontinuous at a, then $\int_a^b f(x)dx = \lim_{c \to a^+} \int_c^b f(x)dx$.
- If f is continuous on $[a, b)$ and discontinuous at b, then $\int_a^b f(x)dx = \lim_{c \to b^-} \int_a^c f(x)dx$.
- If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

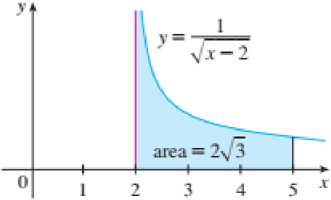
Improper Integrals of Type 2

Example

Evaluate $\int_2^5 \frac{1}{\sqrt{x-2}}dx$

Solution

$$\int_2^5 \frac{1}{\sqrt{x-2}}dx = \lim_{b\to 2^+} \int_b^5 \frac{1}{\sqrt{x-2}}dx = \lim_{b\to 2^+} \left(2\sqrt{x-2}\Big|_b^5 \right) = 2\sqrt{3}$$



Improper Integrals of Type 2

Example

Evaluate $\int\limits_0^3 \dfrac{dx}{x-1}$ if possible.

Solution

Observe that the integrand $\dfrac{1}{x-1}$ has an infinite discontinuity at $x = 1$.

By definition c: $\int\limits_0^3 \dfrac{dx}{x-1} = \int\limits_0^1 \dfrac{dx}{x-1} + \int\limits_1^3 \dfrac{dx}{x-1}$. On the other hand,

$\int\limits_0^1 \dfrac{dx}{x-1} = \lim\limits_{t\to 1^-} \int\limits_0^t \dfrac{dx}{x-1} = \lim\limits_{t\to 1^-} \ln|x-1|\Big|_0^t$

$\Rightarrow \int\limits_0^1 \dfrac{dx}{x-1} = \lim\limits_{t\to 1^-} (\ln|t-1| - \ln|-1|) = -\infty.$

Thus, $\int\limits_0^1 \dfrac{dx}{x-1}$ diverges and hence $\int\limits_0^3 \frac{dx}{x-1}$ diverges.

Improper Integrals of Type 2

Example (Cont.)

Warning The following "solution" is **NOT** correct.

$$\int_0^3 \frac{dx}{x-1} = \ln|x-1|\Big|_0^3 = \ln 2$$

The reason is that there is a infinite discontinuity at $x = 1$ (this leads to the fact that the Fundamental Theorem of Calculus is not satisfied).

Exercises (For Improper Integrals of Type 2)

1. Evaluate $\displaystyle\int_0^3 \frac{dx}{(x-1)^{2/3}}$.

2. Evaluate $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}}$.

3. Evaluate $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$.

4. Evaluate $\displaystyle\int_0^1 \frac{dx}{x^2}$.

5. Evaluate $\displaystyle\int_0^1 \ln x dx$

Comparison Test

Theorem

Suppose that f and g are continuous functions with $f(x) \geqslant g(x) \geqslant 0$ for $x \geqslant a$.

- If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.
- If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

Note: The Comparison Test is also valid for improper integrals of type 2.

Comparison Test

Example

Show that $\int_0^\infty e^{-x^2}dx$ is convergent.

Solution

$$\int_0^\infty e^{-x^2}dx = \int_0^1 e^{-x^2}dx + \int_1^\infty e^{-x^2}dx$$

$\int_0^1 e^{-x^2}dx \leqslant \int_0^1 e^0 dx = 1 \Rightarrow \int_0^1 e^{-x^2}dx$ is convergent.
We note that $e^{-x^2} \leqslant e^{-x} (\forall x \geqslant 1)$ and
$\int_1^\infty e^{-x}dx = \lim_{t\to\infty} \int_1^t e^{-x}dx = e^{-1}$: convergent.

By Comparison Theorem, $\int e^{-x^2}dx$ is convergent, so is $\int e^{-x^2}dx$.

Exercises

Determine whether each integral is convergent or divergent.

1. $\displaystyle\int_1^{\infty} \frac{\sin^2 x}{x^2+1}\,dx$. Hint: $0 \leqslant \dfrac{\sin^2(x)}{x^2+1} \leqslant \dfrac{1}{x^2}$.
2. $\displaystyle\int_0^{\infty} x e^{-x^2}\,dx$. Hint: Substitution.
3. $\displaystyle\int_1^{\infty} \frac{x+1}{x^2+2x}\,dx$. Hint: Divergent by the Comparison Test.
4. $\displaystyle\int_1^{\infty} \frac{\ln x}{x}\,dx$. Hint: Integration by parts.
5. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{4x^2+4x+5}$. Answer: $\pi/4$.

–End of Chapter 4. Thank you!–

HOMEWORK

(1) Integrals: Exs. 11–30, pages 349–350 (see 4.2–4.5)

(2) Numerical integration: Exs. 7–18, page 564 (see 7.7)

(3) Improper integrals: Exs. 5–46, pages 574–575 (see 7.8)

(4) Fundamental theorem of Calculus: Exs. 7–18, page 327 (see 4.3)

James Stewart: *Calculus*, 8th edition, Cengate learning (2016)