CALCULUS 1 (MA001IU) – MIDTERM EXAMINATION

Semester 1, 2023-24 • Duration: 90 minutes • Date: November 13, 2023

SUBJECT: CALCULUS 1	
Department of Mathematics	Lecturers
Prof. Pham Huu Anh Ngoc	N. Dinh, P.T. Duong, T.V. Khanh, N.M. Quan, M.D. Thanh, N.T.T. Van

INSTRUCTIONS:

- Use of calculator is allowed. Each student is allowed one double-sided sheet of reference material (size A4 or similar). All other documents and electronic devices are forbidden.
- You must explain your answers in detail; no points will be given for the answer alone.
- There is a total of 10 (ten) questions. Each one carries 10 points.

1. Use the Intermediate Value Theorem to prove that there is a solution to the equation $x^3 \cos x = 4$.

2. Apply the Squeeze Theorem to evaluate $\lim\limits_{x\to 0} x^5 \sin\left(\dfrac{1}{\sqrt[3]{x}}\right)$.

3. Let $f(x) = x^2 - 1$ and $g(x) = \sqrt{x-1}$. Find the functions $f \circ g$ and $g \circ f$ and their domains.

4. Find a formula for the inverse of the function $f(x) = x^2 - 2x$ for $x \geq 1$ and find its domain.

5. Let
$$f(x) = \begin{cases} x & \text{if } x \leq 1 \\ (x-1)^2 + x & \text{if } x > 1. \end{cases}$$
(a) *[5 points]* Show that $f(x)$ is continuous at $x = 1$.
(b) *[5 points]* Is $f(x)$ differentiable at $x = 1$? Explain your answer.

6. Calculate $f'(x)$ where
$$f(x) = \ln\left(\sqrt{x^2 + 2x + 3}\right).$$

7. Let $f(x) = e^{2x-2} + \sqrt{x}$ where $x \geq 0$. Find $f'(x)$ and $(f^{-1})'(2)$.

8. Write the equation(s) of the tangent line(s) to the curve $\mathscr{C}$: $x^4 + y^4 = 25$ at the point(s) (x_0, y_0) on this curve where $x_0 = \sqrt{3}$.

9. An object moves on a straight road with a positive direction from left to right. The displacement (in kilometers) relative to an origin O is $s(t) = t^4 - 2t^3 + t^2$, where t is in hours ($0 \leq t \leq 3$).

   (a) *[5 points]* What is the direction that the object moves on the time interval $(\frac{1}{2}, 1)$?

   (b) *[5 points]* What is the distance that the object covers after 2 hours?

10. Use the fact that the world population was 6144 millions in 2000 and was 7821 millions in 2020 to model the population of the world in the first half of the 21th century. Assume that the growth rate is proportional to the population size. Use the model to estimate the population in 2023. At what rate is the world population increasing in 2023?
    Hint: Measure the time t in years and let $t = 0$ in the year 2000.

—END OF THE QUESTION PAPER—

ANSWERS

1. The given equation can be re-written as

$$x^3 \cos x - 4 = 0.$$

Let $f(x) = x^3 \cos x - 4$. This function is continuous on $\mathbb{R}$. [4pts]
Note that $f(0) \times f(2\pi) = (-4) \times [(2\pi)^3 - 4] < 0$. [4pts]
Therefore, by the IVT, there exists $c \in (0, 2\pi)$ such that $f(c) = 0$. [2pts]

2. Observe that $|\sin\left(\frac{1}{\sqrt[3]{x}}\right)| \leq 1$. Thus,

$$0 \leq |x^5 \sin\left(\frac{1}{\sqrt[3]{x}}\right)| = |x^5| \times |\sin\left(\frac{1}{\sqrt[3]{x}}\right)| \leq |x^5|, \quad \forall x \neq 0.$$

Therefore, $-|x^5| \leq x^5 \sin\left(\frac{1}{\sqrt[3]{x}}\right) \leq |x^5|, \quad \forall x \neq 0$. [6pts]
Since $\lim\limits_{x\to 0} -|x^5| = 0 = \lim\limits_{x\to 0} |x^5|$, by Squeeze theorem, it implies that $\lim\limits_{x\to 0} x^5 \sin\left(\frac{1}{\sqrt[3]{x}}\right) = 0$. [4pts]

3. By the definition of composite function, one obtains

$$(f \circ g)(x) = f(g(x)) = f(\sqrt{x-1}) = (\sqrt{x-1})^2 - 1 = x - 1 - 1 = x - 2 \quad [3\text{pts}]$$

and

$$(g \circ f)(x) = g(f(x)) = g(x^2 - 1) = \sqrt{x^2 - 2} \quad [3\text{pts}]$$

Note that the domain of f is $D_f = \mathbb{R}$ and the domain of g is $D_g = [1, \infty)$. It implies that the domain of $f \circ g$ is $D_{f\circ g} = [1, \infty)$ and the domain of $g \circ f$ is $D_{g\circ f} = \{x \in \mathbb{R} : f(x) \geq 1\} = (-\infty, \sqrt{2}] \cup [\sqrt{2}, \infty)$. [2pts] + [2pts]

4. To find $f^{-1}(x)$ of $y = f(x) = x^2 - 2x$ $(x \geq 1)$, one solves for x the equation $y = x^2 - 2x$. [3pts] That is, $(x-1)^2 = y + 1$. Thus, $x = 1 \pm \sqrt{y+1}$ and $y \geq -1$. With the condition of $x \geq 1$, we obtain the only solution of $x = 1 + \sqrt{1+y}, \quad y \geq -1$ [4pts]. Therefore,

$$f^{-1}(x) = 1 + \sqrt{1+x}$$

and the domain of $f^{-1}(x)$ is $D_{f^{-1}} = R_f = [-1, \infty)$. [3pts]

5. **(a)** [5pts] We have $f(1) = 1$ and

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x = 1$$
$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} [(x-1)^2 + x] = 1.$$

Hence, $\lim\limits_{x\to 1} f(x) = 1 = f(1)$. Thus $f(x)$ is continuous at $x = 1$.

(b) [5pts] We have

$$\lim_{x\to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^-} \frac{x-1}{x-1} = 1$$
$$\lim_{x\to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^+} \frac{(x-1)^2 + x - 1}{x - 1} = \lim_{x\to 1^+} \frac{(x-1)(x-1+1)}{x-1} = 1.$$

Hence,

$$= \lim_{x\to 1} \frac{f(x) - f(1)}{x - 1} = 1.$$

Thus, f is differentiable at $x = 1$ and $f'(1) = 1$.

6. By the chain rule, the derivative $f'(x)$ is

$$f'(x) = \frac{\left(\sqrt{x^2+2x+3}\right)'}{\sqrt{x^2+2x+3}} = \frac{\frac{2x+2}{2\sqrt{x^2+2x+3}}}{\sqrt{x^2+2x+3}} = \frac{x+1}{x^2+2x+3}.$$

[5+5=10pts]

7. We have $f'(x) = 2e^{2x-2} + \dfrac{1}{2\sqrt{x}}$ [3pts]. By the derivative of the inverse, one has

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))}. \quad \text{[4pts]}$$

Also, observe that $f(x)$ is one-to-one due to $f'(x) > 0$ on $(0,\infty)$ and $f(1) = 2$ so $f^{-1}(2) = 1$. Therefore,

$$(f^{-1})'(2) = \frac{1}{f'(1)} = \frac{2}{5}. \quad \text{[3pts]}$$

8. • The point $(\sqrt{3}, y_0)$ is on $\mathscr{C}$ if $9 + y_0^4 = 25$. This gives $y_0 = \pm 2$. Therefore, there are two points on $\mathscr{C}$: $(\sqrt{3}, 2)$ and $(\sqrt{3}, -2)$. [2pts]

   • Implicitly differentiating the equation of the curve $\mathscr{C}$, one has

$$y'_{|(x_0,y_0)} = -\frac{x_0^3}{y_0^3},$$

   and so, at $(\sqrt{3}, 2)$, $y'_{|(\sqrt{3},2)} = -\frac{3\sqrt{3}}{8}$ and at $(\sqrt{3}, -2)$, $y'_{|(\sqrt{3},-2)} = \frac{3\sqrt{3}}{8}$. [4pts]

   • The equations of the tangent lines at the two points $(\sqrt{3}, 2)$ and $(\sqrt{3}, -2)$ to the curve $\mathscr{C}$ are, respectively

$$y = -\frac{3\sqrt{3}}{8}x + \frac{25}{8}, \quad \text{and} \quad y = \frac{3\sqrt{3}}{8}x - \frac{25}{8}. \quad \text{[4pts]}$$

9. • We have $s'(t) = 2t(2t^2 - 3t + 1) = 2t(t-1)(t-1/2)$.

   • So the sign of s' (also the velocity):

   $s' > 0$ on $(0, 1/2)$,

   $s' < 0$ on $(1/2, 1)$,

   $s' > 0$ on $(1, 2)$.

   (a) On the interval (1/2, 1) the object moves following the opposite of the positive direction of the road (from right to left). [5pts]

   (b) As $s(0) = 0, s(1/2) = 1/16, s(1) = 0, s(2) = 4$, the distance that the object covers after 2 hours is:

$$[s(1/2) - s(0)] + [s(1/2) - s(1)] + s(2) - s(1)] = 1/(16) + 1/(16) + 4 = 33/8. \quad \text{[5pts]}$$

10. Let $P(t)$ be the world population after t years since 2000. Since we are assuming that $dP/dt = kP$, we have

$$P(t) = Ce^{kt}.$$

   where $C = P(0) = 6144$. Also, the population in 2020 is $P(20) = 6144e^{20k} = 7821$. Therefore, $k = \frac{1}{20}\ln\frac{7821}{6144} = 0.012067$ and the model is

$$P(t) = 6144e^{0.012067\,t}. \quad \text{[6pts]}$$

The estimated world population in 2023 (after 23 years since 2000) is

$$P(23) = 6144e^{0.012067\times(23)} = 8109 \text{ millions.} \quad [2\text{pts}]$$

The (estimated) rate of world population in 2023 is $P'(23) = kP(23) = 97.9$ millions people per year. [2pts]