

**CALCULUS 1 (MA001IU) – FINAL EXAMINATION**

Semester 2, 2022-23 • Duration: 120 minutes • Date: August 7, 2023

**SUBJECT: CALCULUS 1**

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Lecturer

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**INSTRUCTIONS:**

- Use of calculator is allowed. Each student is allowed two double-sided sheets of notes (size A4 or similar). All other documents and electronic devices are forbidden.
- Write the steps you use to arrive at the answers to each question. No marks will be given for the answer alone.
- There are a total of 10 (ten) questions. Each one carries 10 points.

1. Let  $f(x) = x^{\sin x}$ ,  $x > 0$ . Find  $f'(x)$ .**Ans 1.** We have

$$\begin{aligned}\ln f(x) &= \sin x \ln x \\ \Rightarrow \frac{f'(x)}{f(x)} &= \cos x \ln x + \frac{\sin x}{x} \\ \Rightarrow f'(x) &= \left( \cos x \ln x + \frac{\sin x}{x} \right) x^{\sin x}\end{aligned}$$

2. Evaluate the following limit

$$\lim_{x \rightarrow 8} \frac{\sqrt[3]{x} - 2}{\ln(x - 7)}.$$

**Ans** We have

$$\lim_{x \rightarrow 8} \frac{\sqrt[3]{x} - 2}{\ln(x - 7)} = \lim_{x \rightarrow 8} \frac{(\sqrt[3]{x} - 2)'}{[\ln(x - 7)]'} = \lim_{x \rightarrow 8} \frac{\frac{1}{3}x^{-2/3}}{\frac{1}{x-7}} = \frac{1}{12}$$

3. Let

$$F(x) = \int_1^{\pi x^2} \sqrt{t + \sin t} \, dt.$$

Find  $F'(1)$ .**Ans.** We have

$$F'(x) = \sqrt{\pi x^2 + \sin(\pi x^2)} \cdot (\pi x^2)'_x = 2\pi x \sqrt{\pi x^2 + \sin(\pi x^2)}.$$

Hence,  $F'(1) = 2\pi \sqrt{\pi}$ .

4. Evaluate the integral:

$$\int_0^1 \frac{x-1}{x^2+4x+3} dx.$$

**Ans.** We find  $A$  and  $B$  satisfying

$$\begin{aligned}\frac{x-1}{x^2+4x+3} &= \frac{A}{x+1} + \frac{B}{x+3} \\ \Rightarrow x-1 &= A(x+3) + B(x+1) = (A+B)x + (3A+B) \\ \Rightarrow \begin{cases} A+B=1 \\ 3A+B=-1 \end{cases} &\Rightarrow \begin{cases} A=-1 \\ B=2 \end{cases}.\end{aligned}$$

Hence,

$$\int_0^1 \frac{x-1}{x^2+4x+3} dx = \int_0^1 \left( \frac{-1}{x+1} + \frac{2}{x+3} \right) dx = (-\ln|x+1| + 2\ln|x+3|) \Big|_0^1 = 3\ln 2 - 2\ln 3 = \ln \frac{8}{9}.$$

5. Determine whether the improper integral  $\int_1^\infty \frac{2x}{\sqrt{1+x^2}} dx$  is convergent or divergent. Explain.

**Ans.** We have

$$\int_1^\infty \frac{2x}{\sqrt{1+x^2}} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{2x}{\sqrt{1+x^2}} dx = \lim_{t \rightarrow \infty} \left( 2\sqrt{1+x^2} \right) \Big|_1^t = \lim_{t \rightarrow \infty} \left( 2\sqrt{1+t^2} - 2\sqrt{2} \right) = \infty.$$

6. A particle moves in a straight line and its velocity is given by  $v(t) = 3t + 2$  and its initial position is  $s(0) = 0$ . Find its position function when  $t = 2$ , i.e., find  $s(2)$ .

**Ans.** We have  $s'(t) = v(t)$ . Therefore

$$\begin{aligned}\int_0^2 s'(t) dt &= \int_0^2 v(t) dt = \int_0^2 (3t+2) dt \\ \Rightarrow s(2) - s(0) &= 10 \\ \Rightarrow s(2) &= 12.\end{aligned}$$

7. Use the Trapezoidal rule with 6 sub equal intervals (i.e.,  $n = 6$ ) to approximate the value of the integral  $\int_0^2 \frac{1}{16+x^2} dx$

**Ans.** By putting  $f(x) = \frac{1}{16+x^2}$ , we have

$$\int_0^2 \frac{1}{16+x^2} dx \approx (1/2)(1/3) [f(0) + 2f(1/3) + 2f(2/3) + 2f(1) + 2f(4/3) + 2f(5/3) + f(2)] = 0.116$$

8. Determine the length of  $x = \frac{2}{3}(y-1)^{3/2}$  between  $1 \leq y \leq 4$ .

**Ans.** We have  $\frac{dx}{dy} = (y-1)^{1/2}$ . The length of the curve corresponding to  $y \in [1, 4]$  is defined by

$$\begin{aligned}L &= \int_1^4 \sqrt{1 + \left( \frac{dx}{dy} \right)^2} dy \\ &= \int_1^4 \sqrt{y} dy = 14/3.\end{aligned}$$

9. Let  $f(x)$  be continuous on  $[0, 3]$  and differentiable over  $(0, 3)$ . Suppose that  $f(0) = -3$  and that  $f'(x) \leq 2$  for all  $x \in (0, 3)$ . What is the largest possible value for  $f(3)$ ?

**Ans.** Because  $f(x)$  is continuous on  $[0, 3]$  and differentiable over  $(0, 3)$ , by the MVT, there exists  $c \in (0, 3)$  such that

$$\begin{aligned}f(3) - f(0) &= f'(c) \times (3-0) \\ \Rightarrow f(3) &= 3f'(c) + f(0) \leq 3 \times 2 - 3 = 3.\end{aligned}$$

10. Use Newton's method to approximate the positive root correct to six decimal places of the equation  $2 - x^2 = \sin x$ .

**Ans.** Observe that  $2 - x^2 = \sin x \Leftrightarrow f(x) \equiv 2 - x^2 - \sin x = 0$ .

Using Newton's method, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{2 - x_n^2 - \sin x_n}{-2x_n - \cos x_n}$$

By choosing, for example,  $x_1 = 1$ , we can deduce from the latter equation that

$$x_2 = 1.062405$$

$$x_3 = 1.061549$$

$$x_4 = 1.061549$$

Since  $x_3$  and  $x_4$  agree to six decimal places, the approximate positive root needed to find is 1.061549.

—END OF THE QUESTION PAPER. GOOD LUCK!—