VNUHCM – INTERNATIONAL UNIVERSITY

CHAPTER 2. DIFFERENTIATION

CALCULUS I

Lecturer: Nguyen Thi Thu Van, PhD

Slides are adapted based on the lecture slides of Dr. Nguyen Minh Quan

CONTENTS

1.1. Tangent – velocity problems

A major application of Calculus is determining how one quantity varies with another. For example,

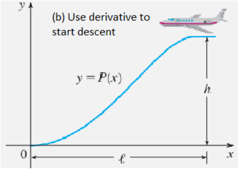
- How profit varies with amount spent on advertising?
- How the population of a colony of bacteria changes with time?
- How the energy loss of an electronic device changes with applied current, etc. ?

We need the concept "rate of change".

1.1. Tangent – velocity problems

What is the derivative of a function? This question has three equally important answers:

- the rate of change,
- the slope of a tangent line,
- and more formally, the limit of a difference quotient.



In this chapter, we explore these three facets of the derivative and develop the basic techniques for computing derivatives.

1.1. Tangent – velocity problems

Example

Suppose a car travels due north at a constant speed. After 3 hours the car has travelled 180 km.

(a) What is the speed of the car?

(b) Sketch a graph of displacement, s, as a function of time, t.

Solution.

(a) The velocity is velocity $= \dfrac{\text{distance travelled}}{\text{time taken}} = \dfrac{180 \text{ km}}{3 \text{ h}} = 60\text{km/h}$

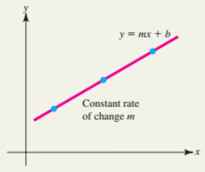
(b) The graph of $s(t)$ is a straight line. The velocity is the slope of the line:

$$\mathrm{v} = \frac{\Delta s}{\Delta t} = \frac{s_2 - s_1}{t_2 - t_1}.$$

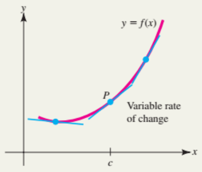
1.1. Tangent – velocity problems

Recall that a linear function $f(x) = mx + b$ changes at a constant rate m w.r.t. x, that is, the rate of change of $f(x)$ is the slope or the steepness of the line $y = mx + b$.

However if a function $f(x)$ is not linear, the rate of change is not a constant but varies with x. In particular, when $x = c$, the rate is given by the steepness of the graph of $f(x)$ at the point $P(c, f(c))$, which can be measured by the slope of the tangent line to the graph at P.



(a) A linear function $L(x) = mx + b$ changes at the constant rate m.



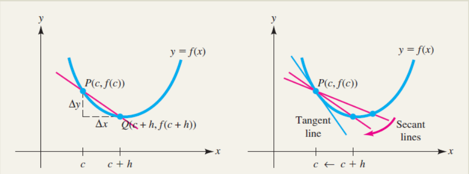
(b) If $f(x)$ is nonlinear, the rate of change at $x = c$ is given by the slope of the tangent line at $P(c, f(c))$.

1.1. Tangent – velocity problems

Tangent problem. Find the slope of a tangent line at $P(c, f(c))$ on the curve C of equation $y = f(x)$.

Strategy: consider a point $Q(c+h, f(c+h))$ nearby P $(Q \neq P)$. The slope of the secant line PQ is the difference quotient:

$$m_{PQ} = \frac{rise}{run} = \frac{\text{change in } f}{\text{change in } x} = \frac{f(c+h) - f(c)}{h}$$



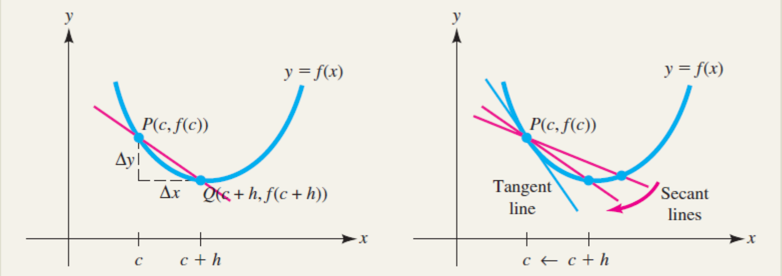
(a) The secant line from $P(c, f(c))$ to $Q(c+h, f(c+h))$

(b) As h tends to 0, the secant line approaches the tangent line at P

1.1. Tangent – velocity problems

Observe that if we let Q approach P by letting $c+h$ approach c then the pink lines PQ approach the blue line at P. In other words, the blue line is considered as the limiting line of the pink lines. As a result, the slope of the tangent line at P can be calculated as

$$m = \lim_{Q\to P} m_{PQ} = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$$



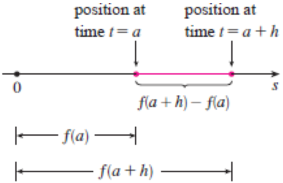
(a) The secant line from $P(c, f(c))$ to $Q(c+h, f(c+h))$

(b) As h tends to 0, the secant line approaches the tangent line at P

1.1. Tangent – velocity problems

Velocity problem. If an object is moving along a straight line according to an equation $s = f(t)$, where s is the displacement of the object from the origin at time t, then the average velocity denoted v_a of the object moving in the time interval from time a to time $a + h$ is the following difference quotient

$$v_a = \frac{displacement}{elapsed\ time} = \frac{f(a+h) - f(a)}{h}$$



1.2. Rate of change

If we consider the movement of the object in a shorter and shorter time interval, the average velocity becomes the instantaneous velocity

$$v(a) = lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

Note that we are not concerned with the direction in which the movement occurs, but displacement and velocity. The speed of the movement is $|velocity|$.

Both problems lead to finding limit

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \equiv \lim_{\Delta x\to 0}\frac{f(a+\Delta x)-f(a)}{\Delta x}.$$

This limit arises actually not only in geometry and physics but in many other practical situations to measure the rate of change, so it is given a special name: DERIVATIVE!

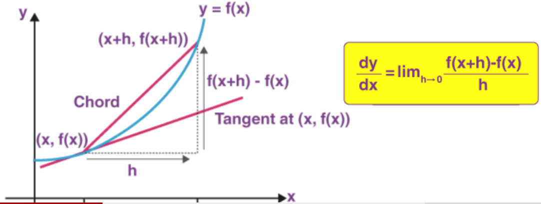
2.1. Derivatives

Definition

Given $y = f(x)$. The derivative of a function at the number a, denoted by $f'(x = a)$ (followed by Newton's notation) which is read 'f dashed of x' or denoted by $\frac{df}{dx}(x = a) \equiv \frac{dy}{dx}(x = a)$ (followed by Leibnitz's), is

$$f'(a) = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \frac{df}{dx}(x = a)$$

if the limit exists and, in this case, f is said to be derivable (or also called differentiable) at a.



2.1. Derivatives

Now we start with the simplest function: $f(x) = c$. The graph is the horizontal line $y = c$, which has slope 0. So, we must have $f'(x) = 0$.

- This is also easily shown from the definition:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = 0$$

- Therefore, we obtain the first rule of differentiation

$$\frac{d}{dx}(c) = 0$$

2.1. Derivatives

Every polynomial $P(x)$ is differentiable at every point. Every rational function $\dfrac{P(x)}{Q(x)}$ is also differentiable at *almost* every point, except where $Q(x) = 0$.

Example.

(1) If $f(x) = x^2 + x$, then

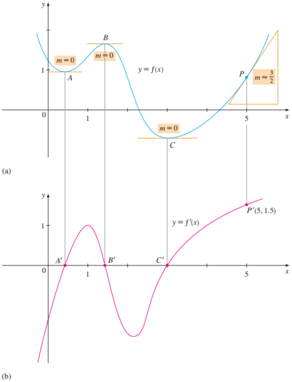
$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0}(2x + 1 + h) = 2x + 1 \end{aligned}$$

(2) If $f(x) = \frac{x}{x-1}$, for $x \neq 1$, then

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{-1}{(x+h-1)(x-1)} \\ &= \frac{-1}{(x-1)^2} \end{aligned}$$

2.1. Derivatives: Relation between function and its derivative

The graph of a function f is given in the figure. Use it to sketch the graph of the derivative f'.



We can find an approximate value for $f'(x)$ at any x by drawing a tangent to the graph $f(x)$ at that x and estimating its slope. We particularly notice that the slope is zero at three points: A, B and C.

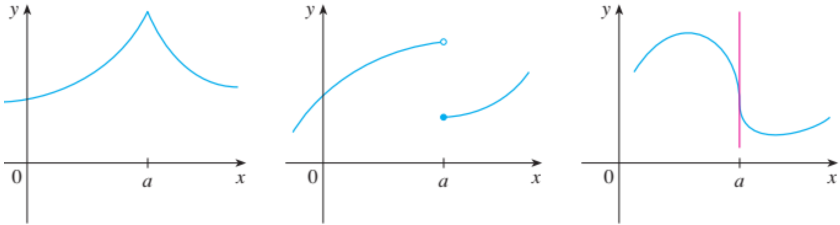
2.1. Derivatives: Differentiability and continuity

Theorem

If f is differentiable at a then f is continuous at a.

When does a function FAIL to be differentiable?

- Having a "corner" or "kink" (So the left and right hand limits are different, and the curve has no tangent at that point.)
- Having discontinuity (removable, jump or infinite).
- Having a vertical tangent (f is continuous, but $\lim\limits_{x \to a} |f'(x)| = \infty$).



2.1. Derivatives

Example. The function $f(x) = |x|$ is not differentiable at $x = 0$. Indeed,

$$f'(x) = \begin{cases} 1, & if \quad x > 0 \\ DNE, & if \quad x = 0 \\ -1, & if \quad x < 0 \end{cases}$$

Recall the right derivative and the left derivative at x = a is defined by

$$f'_+(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}, f'_-(a) = \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h}$$

Note that

$$f'(a) \text{ exists} \quad \Leftrightarrow \quad f'_+(a), f'_-(a) \text{ exist and equal.}$$

2.1. Derivatives

Exercise

Determine whether $f'(0)$ exists.

$$\text{(a)} \quad f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0\ , & \text{if } x = 0 \end{cases}$$

$$\text{(b)} \quad f(x) = \begin{cases} x - 1, & \text{if } x < 0 \\ x^2 - 1, & \text{if } x \geqslant 0 \end{cases}$$

Hint: Consider $\lim\limits_{x \to 0} \dfrac{f(x) - f(0)}{x - 0}$ and show that
(a) $f'(0) = 0$; (b) DNE.

2.1. Derivatives: New derivatives from old

- Constant multiple rule:

$$\frac{d}{dx}\left[cf(x)\right] = c\frac{d}{dx}f(x)$$

- Sum/difference rule: If f and g are both differentiable, then

$$\frac{d}{dx}\left[f(x) \pm g(x)\right] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$$

Example

$$\frac{d}{dx}(2x^7 - 5x^4 - 8x) = 2\frac{d}{dx}(x^7) - 5\frac{d}{dx}(x^4) - 8\frac{d}{dx}(x)$$
$$= 2(7x^6) - 5(4x^3) - 8(1) = 14x^6 - 20x^3 - 8$$

2.1. Derivatives: Product–Quotient Rule

- Product Rule: If f and g are both differentiable, then

$$\frac{d}{dx}\left[f(x)g(x)\right] = f(x)\frac{d}{dx}\left[g(x)\right] + g(x)\frac{d}{dx}\left[f(x)\right]$$

or $(fg)' = f.g' + f'.g$

- Quotient Rule: If f and g are both differentiable, then

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\,f' - f\,g'}{g^2}$$

Example

Find the derivatives of the given functions
(a) $f(t) = \sqrt{t}(1-3t)$
(b) $f(x) = \dfrac{x^2-1}{\sqrt{x}+2}$

2.2. Higher-order derivatives

If f is a differentiable function, then f' is also a function. So, f' may have a derivative of its own, $(f')'$. This is called the second derivative of f and denoted f".

- In Leibniz notation, the second derivative of $y = f(x)$ is

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

- The most familiar example is acceleration. If the displacement of a particle at time t is $s(t)$ Then it has velocity $v(t) = \frac{ds}{dt}$ and acceleration $a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$.

2.2. Higher-order derivatives

- Similarly, the third derivative is:

$$y''' = f'''(x) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$$

- The nth derivative of f is denoted by $f^{(n)}$ and is obtained from f by differentiating n times,

$$y^{(n)} = f^{(n)}(x) = \frac{d^ny}{dx^n}.$$

2.2. Higher-order derivatives: Power rule

Theorem

For all exponents $\alpha \in \mathbb{R}$:

$$\frac{d}{dx}(x^\alpha) = \alpha x^{\alpha-1}$$

Example. Find the derivatives of the following functions:

(a) $f(x) = x^{7.2}$

(b) $g(x) = \dfrac{1}{x^2}$

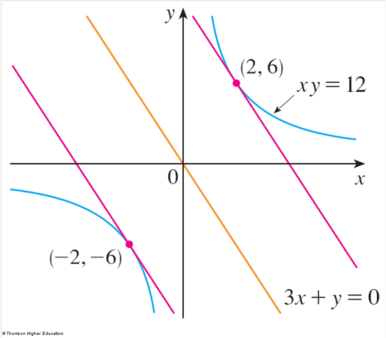
(c) $x(t) = \dfrac{1}{t\sqrt{t}}$.

2.2. Higher-order derivatives: Power rule

Example

Find the points on the hyperbola $y = \dfrac{12}{x}$ where the tangent is parallel to the line $3x + y = 0$.

Hint: $y = \dfrac{12}{x} \quad \to y' = -\dfrac{12}{x^2} = -3$. Hence $x = \pm 2$.



2.3. The chain rule

How to calculate $F'(x)$ where $\mathrm{F}(\mathrm{x}) = \sqrt{x^2+1}$? Note that F is a composite function of the two simpler functions $f(x) = \sqrt{x}$ and $g(x) = x^2 + 1$, that is, $F = f \circ g$. We already knew the derivatives of f and g, can we calculate the derivative of F? The following theorem give you the answer.

Theorem

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ (defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product:

$$F'(x) = f'(g(x)) \times g'(x)$$

In Leibniz notation,

$$\frac{dF}{dx} = \frac{df}{dg} \times \frac{dg}{dx}$$

2.3. The chain rule

Exercise

Find $F'(x)$ if $F(x) = \sqrt{x^2+1}$.

Solution. We have

$$F(x) = (f \circ g)(x) = f(g(x)),$$

where $f(u) = \sqrt{u}, g(x) = x^2 + 1$.

$$f'(u) = \frac{1}{2\sqrt{u}}, g'(x) = 2x.$$

By chain rule,

$$F'(x) = f'(g(x))\,g'(x) = \frac{1}{2\sqrt{x^2+1}}(2x) = \frac{x}{\sqrt{x^2+1}}.$$

2.3. The chain rule

Example. Find $F'(x)$

(a) $F(x) = \cos\left(x^2\right)$

(b) $F(x) = \sin\left(\frac{x}{x+1}\right)$

(c) $F(x) = \sqrt{x + \sqrt{x^2+1}}$

2.3. The chain rule

Corollary: General Power and Exponential Rules

(1) $\dfrac{d}{dx}\left(u^n\right) = nu^{n-1}\dfrac{du}{dx}$, or $\dfrac{d}{dx}\left(\left[g\left(x\right)\right]^n\right) = n\left[g\left(x\right)\right]^{n-1} g'\left(x\right)$,

(2) $\dfrac{d}{dx}e^{g(x)} = g'\left(x\right)e^{g(x)}$,

(3) $\dfrac{d}{dx}\left(a^x\right) = a^x \ln a$.

Example. Find the derivatives:

(a) $f\left(x\right) = \left(x^3 + 9x + 2\right)^{-1/3}$,

(b) $f\left(x\right) = e^{\cos x}$,

(c) $f\left(x\right) = \left(x^2 + \sqrt{x}e^{\cos x}\right)^3$.

Derivatives of Logarithmic Functions

Theorem

$$\frac{d}{dx}(\log_a) x = \frac{1}{x \ln a}, \quad \frac{d}{dx}(\ln x) = \frac{1}{x}.$$

The calculation of derivatives of complicated functions involving products, quotients, or powers can often be simplified by the method of logarithmic differentiation.

Example. Differentiate

$$y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}, \quad (x > 0)$$

Logarithmic Differentiation

$$y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}, \quad (x > 0)$$

Solution. There are 3 steps:

Step 1. Taking logarithms of both sides of the equation

$$\ln y = \frac{3}{4}\ln x + \frac{1}{2}\ln\left(x^2+1\right) - 5\ln\left(3x+2\right).$$

Step 2. Differentiating implicitly with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{4}\frac{1}{x} + \frac{1}{2}\frac{2x}{x^2+1} - 5\frac{3}{3x+2}.$$

Step 3. Solving for $\frac{dy}{dx}$, we get

$$\frac{dy}{dx} = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right).$$

Logarithmic Differentiation

Exercise. Differentiate

(1) $f(x) = \frac{(x+1)^2\left(2x^2+3\right)}{\sqrt{x^2+1}}$,

(2) $f(x) = x^{\sin x} \quad (x > 0)$,

(3) $f(x) = (\sin x)^{\ln x}, \quad (0 < x < \pi)$

Calculating limits by differentiation

Example. Prove that

(1) $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$

(2) $\lim\limits_{x\to 0} \dfrac{\tan(x)}{x} = 1$

(3) $\lim\limits_{x\to 0} \dfrac{\arcsin(x)}{x} = 1$

(4) $\lim\limits_{x\to 0} \dfrac{\ln(x+1)}{x} = 1$

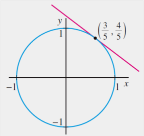
(5) $\lim\limits_{x\to 0} \dfrac{e^x - 1}{x} = 1$

Solution. (2) By the definition of derivative, we have

$$\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\tan x - \tan 0}{x - 0} = \tan'(0) = \frac{1}{\cos^2 0} = 1$$

3.1. Implicit function

How can we find the slope of the tangent line at $P\left(\frac{3}{5}, \frac{4}{5}\right)$?



Observe that the circle equation is given by $x^2 + y^2 = 1$.

To compute $\dfrac{dy}{dx}$, first differentiate both sides of the equation:

$$\frac{d}{dx}\left(x^2 + y^2\right) = \frac{d}{dx}(1) \Leftrightarrow 2x + \frac{d}{dx}\left(y^2\right) = 0 \Leftrightarrow 2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

Substitute $x = \frac{3}{5}, y = \frac{4}{5}$ into the equation $\frac{dy}{dx} = -\frac{x}{y}$, we obtain the slope

$$\left.\frac{dy}{dx}\right|_P = -\frac{3}{4}$$

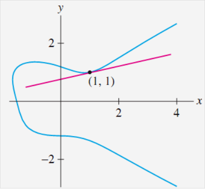
3.1. Implicit function

Method.

Step 1. Differentiate both sides of the equation with respect to x.

Step 2. Solve for y'.

Example. Find an equation of the tangent line at the point $P = (1, 1)$ on the curve $y^4 + xy = x^3 - x + 2$.



3.1. Implicit function

Solution. Differentiate both sides of the equation with respect to x

$$4y^3y' + \left(xy' + y\right) = 3x^2 - 1$$

Then factor out y'

$$y'\left(4y^3 + x\right) = 3x^2 - 1 - y$$

$$y' = \frac{3x^2 - 1 - y}{4y^3 + x}. \text{ Thus, } \left.\frac{dy}{dx}\right|_{(1,1)} = \frac{1}{5}$$

The equation of the tangent line can be written

$$y - 1 = \frac{1}{5}(x - 1) \text{ or } y = \frac{1}{5}x + \frac{4}{5}$$

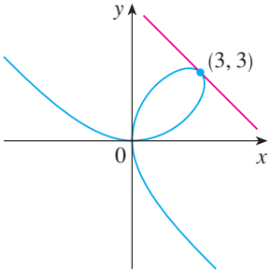
3.1. Implicit function

Exercise.

(a) Find y' if $x^3 + y^3 = 6xy$.

(b) Find the tangent to the curve (which is called folium of Descartes) $x^3 + y^3 = 6xy$ at the point $(3, 3)$.

(c) At what points in the first quadrant is the tangent line horizontal?



3.1. Implicit function

Exercise. Find y' if

(1) $2x^3 + x^2y - xy^3 = 2$

(2) $y^5 + x^2y^3 = 1 + ye^{x^2}$

(3) $\sin(xy) = \sin x + \sin y$

(4) $x^4 + y^4 = 16$. Also, find y''.

4.1. Inverse function

Theorem (Derivative of the inverse)

Assume that $f(x)$ is differentiable and one-to-one with inverse $g(x) = f^{-1}(x)$. If b belongs to the domain of $g(x)$ and $f'(g(b)) \neq 0$, then $g'(b)$ exists and

$$g'(b) = \frac{1}{f'(g(b))}$$

4.1. Inverse function

Example

Differentiate $g(x) = f^{-1}(x)$ where $g(x)$ is the inverse of

$$f(x) = x^2 + 4$$

on the domain $\{x : x \geqslant 0\}$.

Solution.

By technique of finding the inverse function, we obtain

$$g(x) = \sqrt{x-4}$$

By the derivative of the inverse theorem

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{2g(x)} = \frac{1}{2\sqrt{x-4}}$$

4.1. Inverse function

Example: Calculating $g'(x)$ without solving for $g(x)$

Calculate $g'(1)$, where $g(x)$ is the inverse of $f(x) = x + e^x$.

Solution. By the derivative of the inverse theorem

$$g'(1) = \frac{1}{f'(g(1))} = \frac{1}{f'(c)} = \frac{1}{1 + e^c}$$

where $c = g(1) = f^{-1}(1)$. On the other hand $f(0)=1$, thus $c = f^{-1}(1) = 0$. Therefore, $g'(1) = \frac{1}{2}$.

4.1. Inverse function

Example

Differentiate

$$(a)\quad f(x) = \arcsin\sqrt{x}, \qquad (b)\; f(x) = \arctan(3x+1)$$

Solution. By the chain rule:

(a)

$$\left(\arcsin\sqrt{x}\right)' = \frac{\left(\sqrt{x}\right)'}{\sqrt{1-\left(\sqrt{x}\right)^2}} = \frac{1}{2\sqrt{x}\sqrt{1-x}}$$

(b)

$$\left(\arctan(3x+1)\right)' = \frac{(3x+1)'}{1+(3x+1)^2} = \frac{3}{1+(3x+1)^2}$$

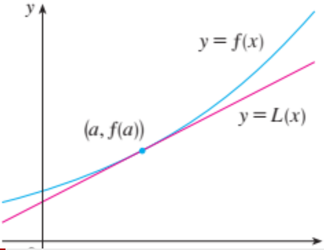
5.1. Linear approximation. Differentials

Definition

The approximation

$$f(x) \approx f(a) + f'(a)(x-a)$$

is called the linear approximation or tangent line approximation of f at $x=a$, and the function $L(x) = f(a) + f'(a)(x-a)$ is called the linearization of f at $x=a$ (when x is near a).



5.1. Linear approximation

Example. (a) Find the linearization of the function $f(x) = e^x$ at $a = 0$ and use it to approximate the number $e^{0.01}$.

(b) Find the linearization of the function $f(x) = \sqrt{x}$ at $a = 1$ and use it to approximate the number $\sqrt{1.001}$.

Solution. (a) We have $a = 0, f(x) = e^x \Rightarrow f'(x) = e^x, f'(0) = 1$.
By linear approximation,

$$f(x) \approx 1 + 1(x - 0) = x + 1$$

Thus, $e^{0.01} = f(0.01) \approx 1.01$.

5.1. Linear approximation

Exercise.

(1) Find the linearization of the function at $a = 0$

(a) $f(x) = \sin x$.

(b) $f(x) = \cos x$.

(2) Use the linearization to estimate $\tan\left(\frac{\pi}{4} + 0.02\right)$.

(3) Use the linearization to estimate $\sqrt{3.98}$.

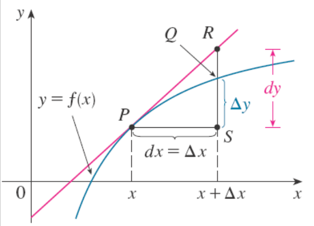
5.2. Differentials

Recall that

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \frac{df}{dx} \equiv \frac{dy}{dx}.$$

in which dy is called the differential of y and defined by

$$df \equiv dy = f'(a)\, dx$$



Example. If $y = x^3$ then $dy = 3x^2 dx$.

HOMEWORK

(1) Rates of change: Exs. 43–46, page 115 (see 2.1)

(2) Implicit differentiation: Exs. 5–24, page 166 (see 2.6)

(3) Differentiation of inverse functions: Exs. 39–46, page 407 (see 6.1)

(4) Linear approximation: Exs. 1–4, page 192; 11–30, page 193 (see 2.9)

James Stewart: *Calculus*, 8th edition, Cengate learning (2016)