VNUHCM – INTERNATIONAL UNIVERSITY

CHAPTER 1. FUNCTIONS, LIMITS AND CONTINUITY

CALCULUS I

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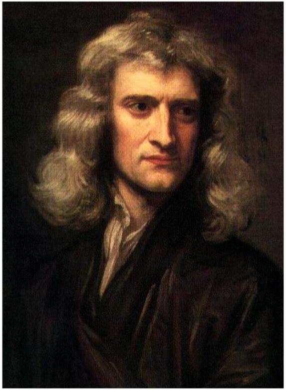
Slides are adapted based on the lecture slides of Dr. Nguyen Minh Quan

CONTENTS

1.1. An introduction to Calculus

Calculus generally considered to have been founded in the 17th century by Isaac Newton and Gottfried Leibniz.

Leibniz (1646-1716)

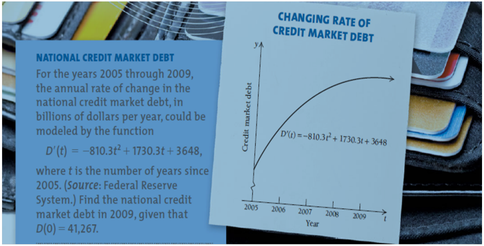


Newton (1642 -1727)

1.1. An introduction to Calculus

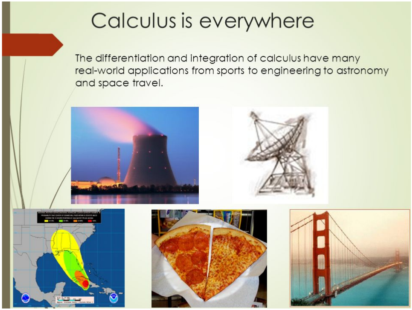
What is Calculus and why Calculus is useful?

- Calculus has two major branches, differential calculus (concerning rates of change and slopes of curves), and integral calculus (concerning accumulation of quantities and the areas under curves). Both are based on the concept of *limits*.
- Calculus are used in many fields: Mathematics, Physics, Engineering, Biotechnology, Computing science, Data science, and other sciences.



1.1. An introduction to Calculus

- Calculus give us a way to construct quantitative models in practice, and to deduce the predictions of such models.



In this course, we study: [a] basic concepts of functions and limits,

[b] techniques of differentiation and integration,

[c] applications to a wide range of practical situations.

1.2. Functions and graphs

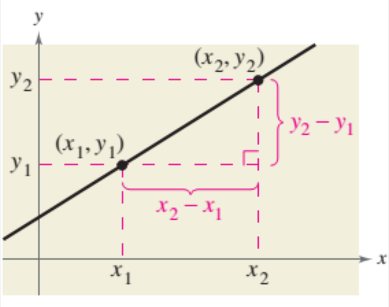
Functions arise whenever one quantity depends on another. There are four ways to represent a function:

- By a formula (algebraically).
- By a table of data (numerically).
- By a description in words (verbally).
- By a figure/graph (visually).

1.2 Functions and graphs: Straight Lines

An important characteristic of a straight line is its slope, a number that represents the "steepness" or the "slant" of the line. The slope of a non-vertical line is

$$m \equiv \tan\theta = \frac{rise}{run} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \left[\equiv \frac{y_1 - y_2}{x_1 - x_2} \right]$$



Furthermore, the slope of a horizontal line is 0, while the slope of a vertical line is undefined.

1.2 Functions and graphs: Straight Lines

The standard form equation of a straight line:

$$Ax + By = C \quad \text{with } A, B, C \in \mathbb{R}$$

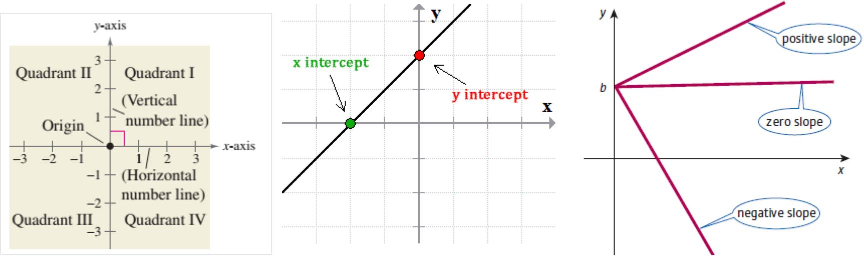
The slope-point equation of a straight line passing through (x_A, y_A) :

$$y = m(x - x_A) + y_A$$

The slope-intercept equation of a straight line:

$$y = mx + b$$

1.2 Functions and graphs: Straight Lines



By convention, $x-$intercept is the x coordinate of a point at which the graph passes through the $x-$axis, while $y-$intercept is the y coordinate of a point at which the graph passes through the $y-$axis.

1.2 Functions and graphs: Straight Lines

Sketching the graph of linear functions

In general, to sketch a straight line from its mathematical equation, it is sufficient to follow the basic steps. These steps may overlap in time.

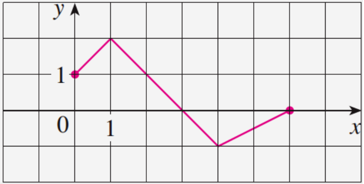
- calculate the coordinates of any two distinct points lying on the line. The easiest thing (in terms of the amount of arithmetic involved) is to put $x = 0$ and find y and then to put $y = 0$ and find x.
- sketch the rectangular coordinate system consisting 2 perpendicular real number lines. Horizontal line represents $x-$ values and vertical line represents $y-$ values.
- plot these two points on graph paper and use a ruler to draw the line passing through them.

Example. Sketch the line according to the equation $2x + 3y = 6$.

1.2 Functions and graphs: Straight Lines

Exercise.

(1) For the graph shown below, state the slope of each line segment.

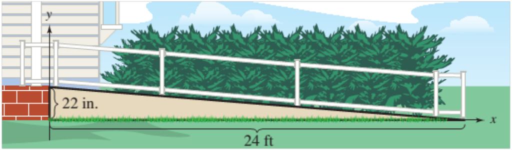


(2) The equation of the line passing through the point $(-2, 0)$ and $(3, 1)$.

(3) Find the slope-intercept forms of the equations of the lines that pass through the point $(2, -1)$ and are (a) parallel to and (b) perpendicular to the line $2x - 3y = 5$.

1.2 Functions and graphs: Straight Lines

Exercise. The maximum recommended slope of a wheelchair ramp is 1/12. A business is installing a wheelchair ramp that rises 22 inches over a horizontal length of 24 feet. Is the ramp steeper than recommended? Given 1 foot=12 inches.



Solution. The horizontal length of the ramp is 24 feet or 288 inches:

$$|Slope| = \frac{|vertical\ \ change|}{|horizontal\ \ change|} = \frac{22}{288} \approx 0.076 < \frac{1}{12}.$$

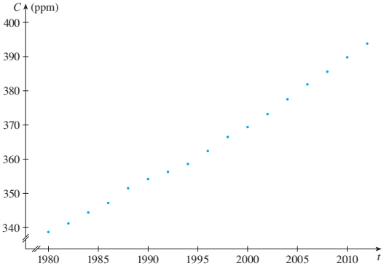
1.2 Functions and graphs: Linear model

Example. Table 1 lists the average carbon dioxide level in the atmosphere, measured in parts per million at Mauna Loa Observatory from 1980 to 2012. Use the data in Table 1 to find a model for the carbon dioxide level.

Table 1

Year	CO_2 level (in ppm)	Year	CO_2 level (in ppm)
1980	338.7	1998	366.5
1982	341.2	2000	369.4
1984	344.4	2002	373.2
1986	347.2	2004	377.5
1988	351.5	2006	381.9
1990	354.2	2008	385.6
1992	356.3	2010	389.9
1994	358.6	2012	393.8
1996	362.4		

1.2 Functions and graphs: Linear model



Solution. We use the data in Table 1 to make the scatter plot as in the figure above where t represents time (in years) and C represents the CO_2 level. We find the equation of the line that passes through the first and last data points.

The slope is $m = \dfrac{393.8 - 338.7}{2012 - 1980} = 1.722.$

Thus, $C - 338.7 = 1.722\,(t - 1980)$ or $C = 1.722t - 3070.86.$

1.2 Functions and graphs: Linear model

Exercise. At a certain place, the CO_2 concentration in the atmosphere was measured to be 338.7 ppm in the year 1980 and 393.8 ppm in 2012. Assume a linear model.

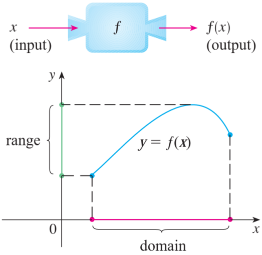
(a) Find an equation for the CO_2 concentration C (in ppm) as a function of time t (in years).

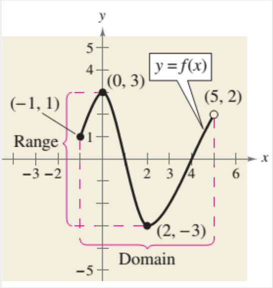
(b) Use your equation to predict the CO_2 concentration in 2021.

1.2 Functions and graphs: Domain and Range

Definition (Function)

A function of a variable x is a rule f that assigns to each value of x in a set D a **unique** number $f(x)$ in a set E, called the value of the function at x. [We read "f(x)" or "f of x".]





The set D is called the domain and the range is the set of all possible values of $f(x)$ as x varies throughout the domain.

1.2 Functions and graphs: Domain and Range

Example.

(1) The domain of the function $y = f(x) = \sqrt{x}$ is the set $D = \{x \in \mathbb{R} : x \geqslant 0\}$, and the range of this function is $[0, \infty)$

(2) [Piecewise Defined Functions] The functions in the following example is defined by different formulas in different parts of their domains. Such function is called piecewise defined function.

$$f(x) = \begin{cases} 1 - x & \text{if } \; x \leq 1 \\ x^2 & \text{if } \; x > 1 \end{cases}$$

$D = \mathbb{R}$ and the range is $[0, \infty)$.

1.2. Functions and graphs

Exercise.

(1) Express the area A of a rectangle as a function of the length x if the length of the rectangle is twice its width.

(2) A Boeing 747 crosses the Atlantic Ocean (3000 miles) with an airspeed of 500 miles per hour. The cost C (in dollars) per passenger is given by

$$C(x) = 100 + \frac{x}{10} + \frac{36,000}{x}$$

where x is the ground speed (airspeed $\pm$ wind).

(a) What is the cost per passenger for quiescent (no wind) conditions?

(b) What is the cost per passenger with a head wind of 50 miles per hour?

1.2. Functions and graphs

Note that not every curve in the coordinate plane can be the graph of a function!

The Vertical Line Test for a Function

A curve in the $xy-$plane is the graph of a function iff no vertical line intersects the curve more than once.

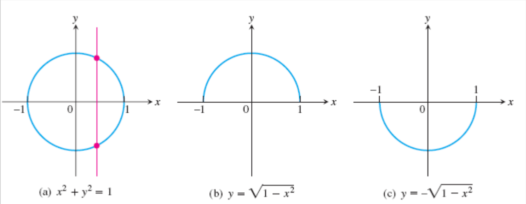


Figure: (a) The circle is not the graph of a function. (b) The upper semicircle is the graph of a function $y = \sqrt{1 - x^2}$ (c) The lower semicircle is the graph of a function $y = -\sqrt{1 - x^2}$.

1.2. Functions and graphs: increasing/decreasing functions

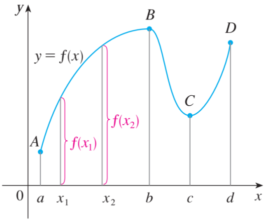
Definition (Increasing/decreasing function)

A function is called

- **increasing** on an interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$,
- **decreasing** on an interval I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$

for all x_1, x_2 in I.

The following function is said to be increasing on the interval $[a, b]$, decreasing on $[b, c]$, and increasing again on $[c, d]$.



1.2. Functions and graphs: symmetry

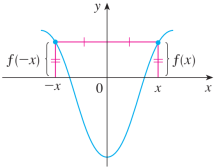
Definition (Even/old function)

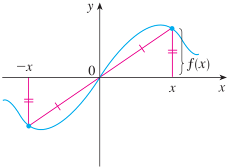
If f satisfies $f(-x) = f(x)$ for every number x in its domain, then f is called an even function. For example, the function $f(x) = x^2$ is even.

The graph of an even function is symmetric with respect to the y-axis.

If f satisfies $f(-x) = -f(x)$ for every number x in its domain, then f is called an odd function. For example, the function $f(x) = x^3$ is odd.

The graph of an odd function is symmetric about the origin.

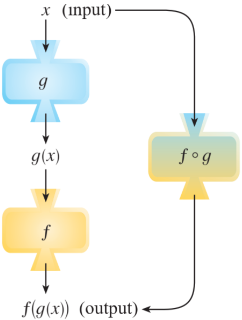




1.2. Functions and graphs: composite functions

Definition (Composite function)

Given two functions f and g, the composite function (also called the composition of f with g) is $f \circ g$ defined by $(f \circ g)(x) = f(g(x))$.



The $f \circ g$ machine is composed of the g machine (first) and then the f machine.

1.2. Functions and graphs: composite functions

Example. Given $f(x) = 3x^2, g(x) = x - 1$. Then

(a) $(f \circ g)(x) = f(g(x)) = f(x-1) = 3(x-1)^2$.

(b) $(g \circ f)(x) = g(f(x)) = g\left(3x^2\right) = 3x^2 - 1$.

Observe, in general, that $f \circ g \neq g \circ f$.

Exercise.

(1) Find $f \circ g \circ h$ where $f(x) = \sqrt{1-x}$, $g(x) = 1 - x^2$, and $h(x) = 1 + \sqrt{x}$.

(2) If $T(x) = \frac{1}{\sqrt{1+\sqrt{x}}}$, give an example for f, g, and h such that $f \circ g \circ h = T$.

1.2. Functions and graphs: composite functions

Exercise. The spread of a contaminant is increasing in a circular pattern on the surface of a lake. The radius of the contaminant can be modeled by $r\left(t\right) = 5.25\sqrt{t}$, where is the radius in meters and is the time in hours since contamination.

(a) Find a function that gives the area of the circular leak in terms of the time since the spread began.

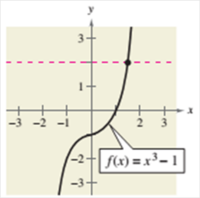
(b) Find the size of the contaminated area after 36 hours.

(c) Find when the size of the contaminated area is 6250 square meters.

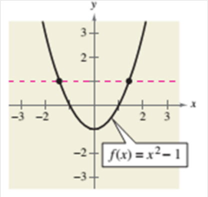
1.2. Functions and graphs: one-to-one functions

Definition (One-to-one function)

A function f is injective (or one-to-one, or an injection) if for every $y \in E$, there is **at most** one $x \in D$ such that $f(x) = y$.



One-to-one



Not one-to-one

Example. The function $f(x) = x^2, x \in \mathbb{R}$ is not one-to-one because both $f(-2) = 4$ and $f(2) = 4$. However, we can turn $f(x) = x^2$ into a one-to-one function if we restrict ourselves to $0 \leqslant x < \infty$.

1.2. Functions and graphs: inverse functions

Definition (Inverse function)

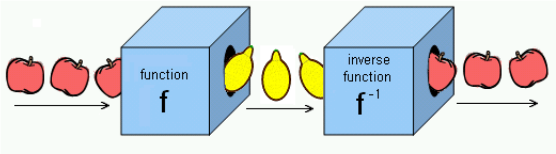
Given two one-to-one functions $f(x)$ and $g(x)$ if

$$(f \circ g)(x) = (g \circ f)(x) = x$$

then we say that $f(x)$ and $g(x)$ are inverses of each other.

More specifically we will say that $g(x)$ is the inverse of $f(x)$ and denote it by $g = f^{-1}$, that is,

$$(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$$



1.2. Functions and graphs: inverse functions

Given $f(x)$, how to find $f^{-1}(x)$?

Method.

(1) From $y = f(x)$, we solve for x to find $f^{-1}(y)$.

(2) We interchange x and y. Hence, $f^{-1}(y) = x \equiv x(y)$.

Example. Given $f(x) = 3x - 2$. Find $f^{-1}(x)$.

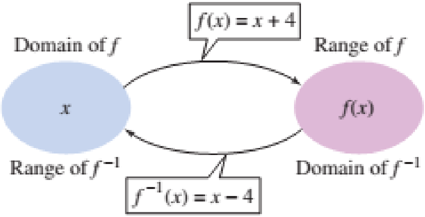
Solution. We have $y = f(x) = 3x - 2$. We solve for x: $x = \frac{1}{3}(y+2)$.

Thus, $f^{-1}(y) = \frac{1}{3}(y+2)$. We then interchange x and y to obtain

$$f^{-1}(x) = \frac{1}{3}(x+2).$$

Re-check: $f\left(f^{-1}(x)\right) = f\left(\frac{1}{3}(x+2)\right) = 3\left[\frac{1}{3}(x+2)\right] - 2 = x$.

1.2. Functions and graphs: inverse functions



Exercise.

(1) Let $f(x) = x + 4$. Find $f^{-1}(x)$.

(2) Let $f(x) = \sqrt{x-3}$. Find $f^{-1}(x)$ and the domain of f^{-1}.

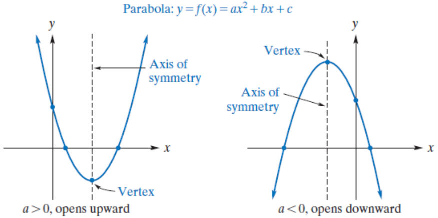
1.2. Functions and graphs: common functions

Polynomial function of degree n:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where n is non-negative integer and the coefficients $a_0, a_1, \ldots, a_n$ are constants and $a_n \neq 0$. Particularly,

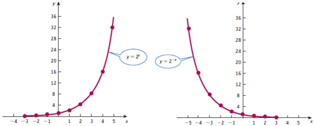
- $n = 0$: Constant functions $f(x) = c$, i.e. the value of f is always c whatever x is.
- $n = 1$: Linear functions $f(x) = ax + b$
- $n = 2$: Quadratic function $f(x) = ax^2 + bx + c$

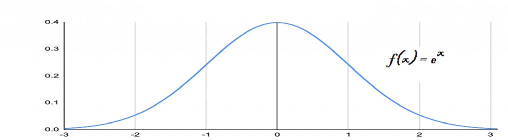


1.2. Functions and graphs: common functions

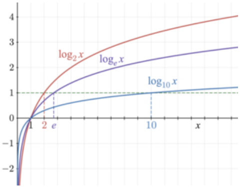
Suppose that $0 < a \neq 1$.

- Exponential function: $f(x) = a^x$



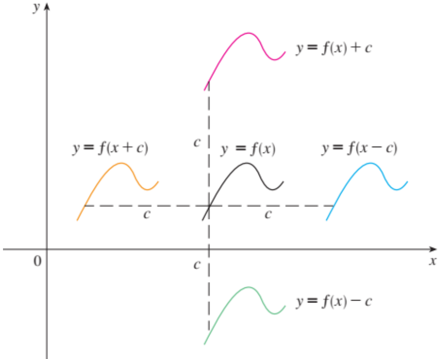


- Logarithmic functions: $f(x) = log_a x$

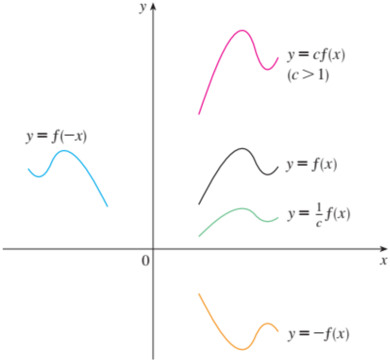


1.2. Functions and graphs: transformations

Suppose we know the graph of a certain function. By some simple transformations, we can quickly obtain the graphs of some related functions.



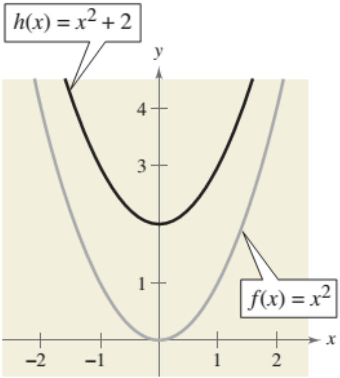
Translating the graph of f

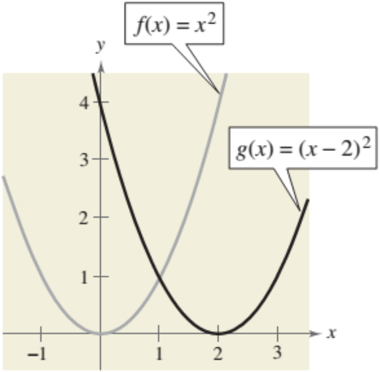


Stretching and reflecting the graph of f

1.2. Functions and graphs: transformations

- $y = f(x) + c$, shift the graph of $y = f(x)$ up by c units.
- $y = f(x) - c$, shift the graph of $y = f(x)$ down c units.
- $y = f(x + c)$, shift the graph of $y = f(x)$ left c units.
- $y = f(x - c)$, shift the graph of $y = f(x)$ right c units





1.2. Functions and graphs: transformations

Example. Sketch the graphs

(a) $y = x^2$

(b) $y = x^2 - 1$

(c) $y = (x-1)^2$

(d) $y = (x-1)^2 - 3.$

1.2. Functions and graphs: transformations

To obtain the graph of

- $y = cf(x)$, stretch $y = f(x)$ vertically by a factor c.
- $y = f(cx)$, compress $y = f(x)$ horizontally by a factor c.
- $y = -f(x)$ reflect the graph of $y = f(x)$ about the x-axis.
- $y = f(-x)$ reflect the graph of $y = f(x)$ about the y-axis.

Example. Sketch the graphs

(1) $y = 2\sin x$.

(2) $y = \sin(\pi x)$.

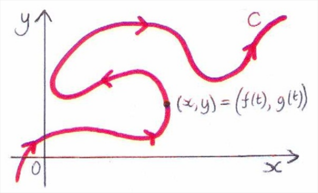
(3) $y = 2e^{-x}$.

1.2. Functions and graphs: parametric curves

So far we have described plane curves by giving as

- a function of x or as a function of y. Ex. $y = 2x + 5$.
- or by giving a relation between x and y that defines y implicitly as a function of x. Ex. $x^3 + y^2 = 2xy$.

However, some curves are best handled when both x and y are given in terms of a third variable called a parameter t.

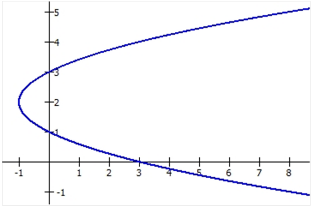


Each value of t determines a point $(x(t), y(t))$ in a coordinate plane. As t varies, the point moves and traces out a curve which is called parametric curve: $c(t) = (x = f(t), y = g(t))$.

1.2. Functions and graphs: parametric curves

Example. Sketch the curve defined by $x = t^2 - 2t$, $y = t + 1$.
We construct a table of values and thus plot the curve:

t	x	y
-2	8	-1
-1	3	0
0	0	1
1	-1	2
2	0	3
3	3	4
4	8	5



1.2. Functions and graphs: parametric Curves

Example.

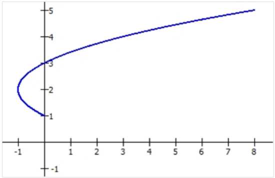
(1) What curve is represented by the following parametric equations?

$$x = \cos t, y = \sin t, 0 \le t \le \pi.$$

Hint: Use $\sin^2 t + \cos^2 t = 1$. Think about the equation of the unit circle.

(2) Eliminate the parameter to find a Cartesian equation of the following parametric equations $x = \sqrt[3]{t} - 1, y = 2t^2 + t + 1$.

(3) Sketch the curve defined by $x = t^2 - 2t, y = t + 1, 0 \le t \le 4$.



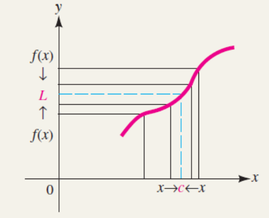
Note that the parametric equations not only describe the curve but also tell us how it is traced.

2.1. Limits

In mathematics, a limit is the value that a function "approaches" as the input "approaches" some value.

Definition (Limit)

Let f be a function defined in a neighborhood of a point a but not necessarily at a. We say loosely that the limit of f when x goes to a is L, and write $lim_{x \to a} f(x) = L$ if we can make $f(x)$ arbitrarily close to L by taking x sufficiently close but not equal to a.



2.1. Limits: One-sided limits

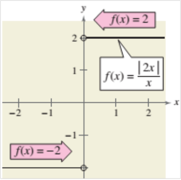
Definition (One-sided limit)

Let f be a function defined in a neighborhood of a point a but not necessarily at a. We say that the limit of f when x goes to a on the right (resp., on the left) is L, and write

$$lim_{x\to a^+} f(x) = L \quad (reps., lim_{x\to a^-} f(x) = L)$$

if we can make $f(x)$ arbitrarily close to L by taking x sufficiently close but bigger than a (reps., smaller than a).

Example. Find the limit as $x \to 0$ from the left and the right for $f(x) = \dfrac{|2x|}{x}$.

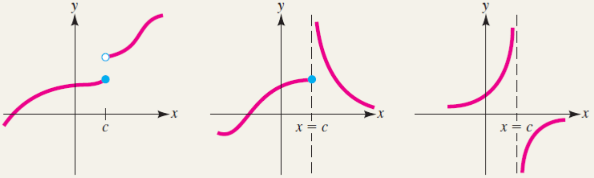


2.1. Limits: One-sided limits

Theorem (One-sided limit)

$\lim_{x\to a} f(x)$ *exists if and only if* $\lim_{x\to a^+} f(x)$ *and* $\lim_{x\to a^-} f(x)$ *exist and are equal. In that case* $\lim_{x\to a} f(x) = \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$

Consequently, if $\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x)$ *then* $\lim_{x\to a} f(x)$ *does not exist.*



(a) A finite gap: $\lim_{x\to c^-} f(x) \neq \lim_{x\to c^+} f(x)$

(b) An infinite gap: $\lim_{x\to c^-} f(x)$ is finite but $\lim_{x\to c^+} f(x) = +\infty$

(c) An infinite gap: $\lim_{x\to c^-} f(x) = +\infty$ and $\lim_{x\to c^+} f(x) = -\infty$

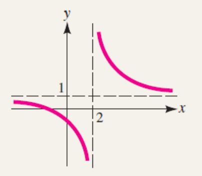
2.1. Limits: Limits as infinity

Definition (Limit as infinity)

Let f be a function defined in a neighborhood of a point a but not necessarily at a. We say that the limit of f when x goes to a is infinity, and write

$$lim_{x \to a} f(x) = \infty$$

if we can make $f(x)$ arbitrarily big by taking x sufficiently close but not equal to a.



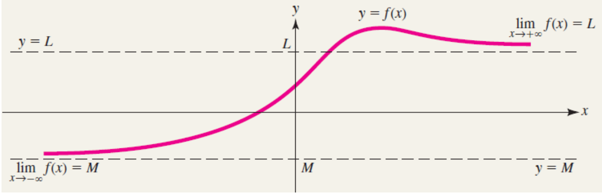
2.1. Limits: Limits at infinity

Definition (Limits at infinity)

If the values of the function $f(x)$ approach a number L as x increases without bound, we write

$$lim_{x\to\infty} f(x) = L$$

Similarly, we write $lim_{x\to-\infty} f(x) = M$ when the functional values of f approach a number M as x decreases without bound.

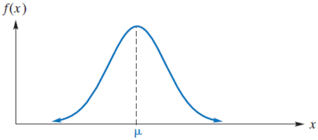


2.1. Limits: Limits at infinity

Example. The following well-known function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left[\frac{(x-\mu)}{\sigma}\right]^2}$$

is called the normal **[Gaussian]** density function and its graph is shaped like a bell-curve, in which the parameters μ and σ are the mean and the standard deviation of data values observed, respectively.



Normal curve

We have that $\lim_{x\to\pm\infty} f(x) = 0$.

The Gaussian density function is after the German mathematician **Carl Friedrich Gauss** [1777–1855]

2.1. Limits: limit laws

Theorem (Limit)

If $c \in \mathbb{R}$ then the constant function $f(x) \equiv c$ is often denoted c itself. And we have

$$\lim_{x \to a} c = c \quad \textit{and} \quad \lim_{x \to a} x = a$$

It'd be worth introducing one of the important theorems of Euler:

$$\lim_{x \to 0} (1+x)^{\frac{1}{x}} = e \quad \text{and} \quad \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{x} = e$$

The number e is named after the Swiss mathematician **Leonhard Euler** [1707–1783] and is $\lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^{m}$.

2.1. Limits: limit laws

Theorem (Limit laws)

If $lim_{x\to a}f(x)$ and $lim_{x\to a}g(x)$ exist then

(a) $lim_{x\to a}[f(x) \pm g(x)] = lim_{x\to a}f(x) \pm lim_{x\to a}g(x)$

(b) $lim_{x\to a}f(x)g(x) = [lim_{x\to a}f(x)][lim_{x\to a}g(x)]$

(c) $lim_{x\to a}\frac{f(x)}{g(x)} = \frac{lim_{x\to a}f(x)}{lim_{x\to a}g(x)}$ *if* $lim_{x\to a}g(x) \neq 0$.

The laws are still valid with the process $x \to \infty$ instead of $x \to a$.

2.1. Limits: limit laws

Example.

(1) Show that $\lim\limits_{t\to 2}(3t-5)=1$.

$$\begin{aligned}\lim_{t\to 2}(3t-5) &= \lim_{t\to 2}\{(3)(t)+(-5)\}\\ &= (\lim_{t\to 2}3)(\lim_{t\to 2}t)+(\lim_{t\to 2}(-5))\\ &= 3\cdot 2-5=1.\end{aligned}$$

(2) Evaluate $\lim\limits_{x\to -3}\frac{x^2+x-6}{x+3}$.

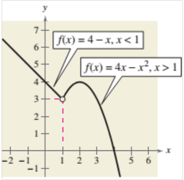
$$\lim_{x\to -3}\frac{x^2+x-6}{x+3}=\lim_{x\to -3}\frac{(x-2)(x+3)}{x+3}=\lim_{x\to -3}(x-2)=-5$$

(3) Evaluate $\lim\limits_{x\to -\infty}\dfrac{x}{\sqrt{x^2+1}}$.

2.2. Evaluating limits

Example. Find the limit of $f(x)$ as x approaches 1.

$$f(x) = \begin{cases} 4 - x, & \text{if } x < 1 \\ 4x - x^2, & \text{if } x > 1 \end{cases}$$



Solution. Remember that you are concerned about the value of near $x = 1$ rather at $x = 1$. We have

$$\lim_{x\to 1^-} f(x) = 4 - 1 = 3; \quad \lim_{x\to 1^+} f(x) = 4(1) - 1^2 = 3.$$

Because the one-sided limits both exist and are equal to 3, it follows that

$$\lim_{x\to 1} f(x) = 3.$$

2.2. Evaluating limits

Example. [Multiplying by the Conjugate] Evaluate

$$\lim_{x\to 10} \frac{\sqrt{x-6}-2}{x-10}.$$

$$\begin{aligned}
\lim_{x\to 10} \frac{\sqrt{x-6}-2}{x-10} &= \lim_{x\to 10} \frac{(\sqrt{x-6}-2)(\sqrt{x-6}+2)}{(x-10)(\sqrt{x-6}+2)} \\
&= \lim_{x\to 10} \frac{(x-6)-4}{(x-10)(\sqrt{x-6}+2)} \\
&= \lim_{x\to 10} \frac{1}{\sqrt{x-6}+2} \\
&= \frac{1}{\sqrt{10-6}+2} = \frac{1}{4}.
\end{aligned}$$

2.2. Evaluating limits

Exercise. (1) Evaluate

$$\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}.$$

(2) Assume

$$\lim_{x\to -4} f(x) = 2, \lim_{x\to -4} g(x) = 3$$

Evaluate

(a) $\lim_{x\to -4} f(x)g(x)$,

(b) $\lim_{x\to -4} \left[2f(x) + 3g(x)\right]$,

(c) $\lim_{x\to -4} \frac{g(x)}{x^2}$,

(d) $\lim_{x\to -4} \dfrac{f(x)+1}{3g(x)-2}$

2.2. Evaluating limits

Exercise. (3) Find the following limits

(a) $\lim\limits_{x\to 1} \frac{\sqrt{x}-1}{x^3-1}$,

(b) $\lim\limits_{x\to\infty} (\sqrt{x+1} - \sqrt{x-1})$,

(c) $\lim\limits_{x\to\infty} \left(-\frac{2e^{ax}}{e^{3x}} + be^{-cx} \right)$, where a, b, c are constants, $0 < a < 3$, and $c > 0$.

(4) Let

$$f(x) = \begin{cases} x^2 - 2 \text{ for } x < 0 \\ 2 - x^2 \text{ for } x \geqslant 0 \end{cases}$$

Find $\lim\limits_{x\to 0} f(x)$ and $\lim\limits_{x\to -2} f(x)$.

2.3. The squeeze theorem

Sandwich Theorem or Squeeze Theorem

If for x is near a:

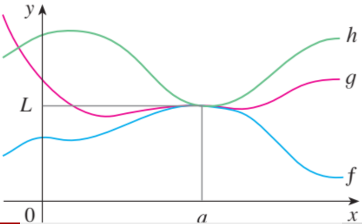
$$f(x) \leqslant g(x) \leqslant h(x)$$

and

$$\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x).$$

Then

$$\lim_{x \to a} g(x) = L.$$



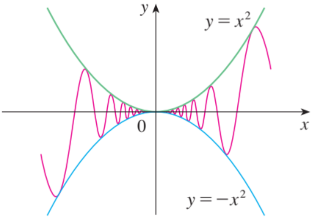
2.3. The squeeze theorem

Example. Show that $\lim\limits_{x\to 0} x^2 \sin \frac{1}{x} = 0$.

Solution. Applying the Sandwich Theorem, note that:

$$-x^2 \leqslant x^2 \sin \frac{1}{x} \leqslant x^2 \quad \forall x \neq 0$$

and $\lim\limits_{x\to 0} x^2 = \lim\limits_{x\to 0} -x^2 = 0$. Therefore $\lim\limits_{x\to 0} x^2 \sin \frac{1}{x} = 0$.



2.3. The squeeze theorem

Exercise. Find the following limits:

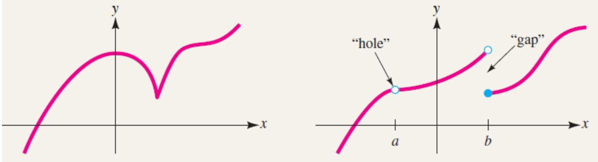
(a) $\lim\limits_{x \to 0} x \sin\left(\frac{1}{x^3}\right).$

(b) $\lim\limits_{x \to 0} x^2 e^{-\cos\left(\frac{1}{x}\right)}.$

3.1. Continuity: definitions and properties

Definition (Continuity)

Suppose f is defined in an open interval that contains a, then f is continuous at a if and only if $\lim_{x \to a} f(x) = f(a)$.



(a) A continuous graph

(b) A graph with "holes" or "gaps" is not continuous

Therefore, by definition, f is continuous at a if and only if all three things hold:

(1) $f(a)$ is defined (that is, a is in the domain of f)

(2) $\lim\limits_{x \to a} f(x)$ exists

(3) $\lim\limits_{x \to a} f(x) = f(a)$.

3.1. Continuity: definitions and properties

Example. Show that

$$f(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$$

is discontinuous at 0.

Solution. Note that $\lim\limits_{x \to 0^+} f(x) = 1$ and $\lim\limits_{x \to 0^-} f(x) = 0$, thus $\lim\limits_{x \to 0} f(x)$ does NOT exist! Therefore $f(x)$ is discontinuous at 0.

3.1. Continuity: definitions and properties

Example. Show that $f(x) = x$ and $g(x) = k$ (constant) are continuous everywhere.

Solution. At any point a

$$\lim_{x \to a} f(x) = \lim_{x \to a} x = a = f(a)$$

Therefore f is continuous everywhere.

$$\lim_{x \to a} g(x) = \lim_{x \to a} k = k = g(a)$$

Therefore g is continuous everywhere.

3.1. Continuity: definitions and properties

Theorem (Continuity)

Suppose f, g are continuous at a. Then

$$f + g, f - g, fg$$

are continuous at a. So are the functions kf , if k is a constant, and f/g if $g(a) \neq 0$.

In particular, every polynomial $P(x)$ is defined and continuous at every point. If $Q(x)$ is another polynomial and $Q(a) \neq 0$, then $\dfrac{P(x)}{Q(x)}$ is also continuous at a.

Example. $f(x) = x(x+3)$; $f(x) = \dfrac{x-1}{x^2+1}$ are continuous at every point.

3.1. Continuity: definitions and properties

Theorem (Continuity)

If f is a continuous bijection from an interval I onto an interval J, then f^{-1} is continuous on J.

Example. The functions x^n are continuous bijections from $[0, \infty)$ onto itself. Therefore $\sqrt{x}, \sqrt[n]{x}$ are defined and continuous at every non-negative points. If n is odd, $\sqrt[n]{x}$ is continuous at every point.

Theorem (Continuity)

Let $F(x) = f(g(x))$ be a composite function. If g is continuous at a and f is continuous at $g(a)$, then $F(x)$ is continuous at a.

Example. $\sqrt{x^2+1}$ and $\sqrt[3]{x^5+4x^2-7x+3}$ are composite functions of continuous functions and therefore defined and continuous *everywhere*.

3.1. Continuity: definitions and properties

Theorem (Continuity)

The following types of functions are continuous at every number **in their domains***:*

- *Polynomials, rational functions, root functions,*
- *Trigonometric functions,*
- *Inverse trigonometric functions,*
- *Exponential functions, logarithmic functions.*

Example. The following functions are continuous in their domain:

$$f(x) = \sin(x^2 + x - 1);\ f(x) = e^{x^2-1};\ f(x) = \frac{\ln(\sqrt{x^4+1})}{\sin(x^2+x-1)}.$$

3.1. Continuity: definitions and properties

Exercise.

(1) Show that f is continuous everywhere.

$$f(x) = \begin{cases} -x+1 & x<0 \\ x^2+1 & x \geq 0 \end{cases}$$

(2) Show that f is continuous at 0.

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

(3) Find all values of a such that $f(x)$ is continuous on $\mathbb{R}$.

$$f(x) = \begin{cases} x+1 \text{ if } x \leqslant a \\ x^2 \text{ if } x > a \end{cases}$$

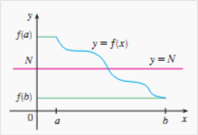
(4) Where f is discontinuous?

$$f(x) = \begin{cases} \sqrt{-x} & \text{if} \quad x<0 \\ 3-x & \text{if} \quad 0 \leqslant x < 3 \\ (x-3)^2 & \text{if} \quad x \geqslant 3 \end{cases}$$

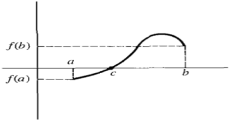
3.2. The Intermediate Value Theorem

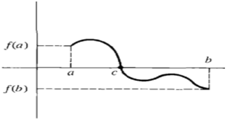
Theorem (The Intermediate Value Theorem)

Suppose f is continuous on an interval that contains two points a, b and $f(a) \neq f(b)$. Then for every value N between $f(a)$ and $f(b)$, there exists c between a and b such that $f(c) = N$.



In particular, if f is continuous on $[a, b]$ and $f(a) \times f(b) < 0$, the equation $f(x) = 0$ has a real root $c \in (a, b)$.





3.2. The Intermediate Value Theorem

Example. Using the IVT to show the existence of a root of the following equations

(1) $f(x) \equiv 5^x - 6x = 0$ between 0 and 1.

Solution. f is continuous on $[0, 1]$, $f(0) = 1 > 0$ and $f(1) = -1 < 0$.

Hence, by the IVT, there exists a real root $c \in (0, 1)$ of the equation $5^x - 6x = 0$.

(2) $x^4 + x^2 - x - 3 = 0$ on $(1, 2)$.

Solution.

3.2. The Intermediate Value Theorem

Exercise.

(1) Show that the equation

$$x^2 - x - 1 = \frac{1}{x+1}$$

has a real root in $(1, 2)$.

(2) Show that the equation $x^4 + x^2 - x - 3 = 0$ has at least two real roots.

(3) Show that there exists a value c between 1 and 2 such that $\sqrt{c} + \sqrt{c-1} = 2$.

–END OF CHAPTER 1. THANK YOU!–

HOMEWORK

(1) Functions: Exs. 5–8, 11–16, page 96 (see 1.2)

(2) Inverse Functions: Exs. 23–28, page 407 (see 6.1–6.3)

(3) Parametric Curves: Exs. 5–10, page 685 (see 10.1)

(4) Limits: Exs. 45–46, page 97 (see 1.5–1.6)

(5) The Squeeze Theorem: Exs. 36–40, page 71 (see 1.6)

(6) Continuity: Exs. 17–22, page 92 (see 1.8)

(7) The Intermediate Value Theorem: Exs. 53–56, page 93; Exs. 49–50, page 97 (see 1.8)

James Stewart: *Calculus*, 8th edition, Cengate learning (2016)