0. A crash course on quantum information

0.1 Classical vs. Quantum bits

$\boxed{\text{Lecture 1: The qubit}}$

classical bits are just 0 or 1

→ let's associate those bits w/ a vector

$$0 \longrightarrow |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

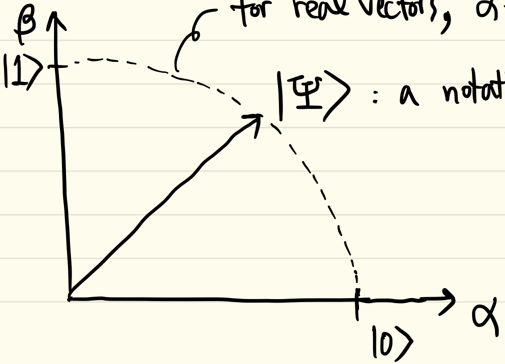
$$1 \longrightarrow |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

these two vectors are orthogonal

⇓

we could assume two axis

qubits



for real vectors, $\alpha^2 + \beta^2 = 1$

$|\Psi\rangle$: a notation of vector in quantum information

$$|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

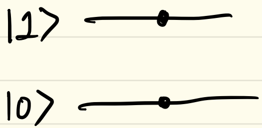
"a qubit has a little of the zero and a little of one bit at the same time"

⇒ Qubit can be a superposition of 0 & 1

what do qubits look like? what does superposition really means?

Quantumly, we could send a particle to left & right at the same time.
i.e. the particle would be in a superposition of being on the left & right.

qubits could be in the ground and the excited state at the same time.



standard basis

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

to construct the qubits, we started from the classical bits as vectors.

"standard basis"

$$|\Psi\rangle = \underbrace{\alpha |0\rangle + \beta}_{\text{amplitude}} |1\rangle \in \mathbb{C}^2$$

(amplitude can be complex numbers too)

Kets and Bras

Ket : $|\Psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

Bra : $\langle\Psi| = \big(\,|\Psi\rangle^{\overbrace{*}^{\text{conjugate}}}\big)^{\overbrace{T}^{\text{transpose}}} = \begin{pmatrix} \alpha^* \\ \beta^* \end{pmatrix}^T = (\alpha^* \;\; \beta^*)$

Inner product : $\langle\Psi||\Psi\rangle = \langle\Psi|\Psi\rangle$

$$= (\alpha^* \;\beta^*)\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha^*\alpha + \beta^*\beta = |\alpha|^2 + |\beta|^2 = 1$$

Lecture 2: More than one qubit

for classical bits: if we have n bits

$$\longrightarrow x = x_1, \cdots, x_n \in \{0,1\}^n$$

in total there are $d = 2^n$ possible strings

let's once again associate a vector w/ every possible classical string

$$x \to |x\rangle$$

: make the vector zero everywhere except for the index x $\begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$

$$n=1: \quad |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Quantum state of n qubits

$$|\Psi\rangle = \sum_{x \in \{0,1\}^n} \alpha_x |x\rangle \qquad \sum_{x \in \{0,1\}^n} |\alpha_x|^2 = 1$$

complex numbers

length = 1

$|\Psi\rangle \in \mathbb{C}^d$ with $d = 2^n$

$$\langle \Psi | \Psi \rangle = 1$$

"the state of n qubits is a vector in $\mathbb{C}^d$ ($\mathbb{C}^d$: complex vector space of dimension d), and it has length one"

ex: two qubits in equal superposition

standard basis for two qubits

$$|00\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad |01\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad |10\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad |11\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

equal superposition

$$|\Psi\rangle = \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$$

$$= \frac{1}{2}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

inner product of this vector shows length 1

$\Rightarrow$ valid two qubit quantum state ex: two qubits in an EPR pair

↳ Einstein, Podolsky, Rosen

$$|\Psi\rangle = |EPR\rangle = \underbrace{\frac{1}{\sqrt{2}}}(|00\rangle + |11\rangle)$$

↳ to normalize so that the vector has length one

$$= \frac{1}{\sqrt{2}}\left(\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$$

Let's check whether it is a valid quantum state: compute inner product

$$\langle\Psi|\Psi\rangle = \left(\frac{1}{\sqrt{2}}(\langle 00| + \langle 11|)\right)\left(\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\right)$$

$$= \frac{1}{2}\left(\underbrace{\langle 00|00\rangle}_{=1} + \underbrace{\langle 00|11\rangle}_{=0} + \underbrace{\langle 11|00\rangle}_{=0} + \underbrace{\langle 11|11\rangle}_{=1}\right)$$

$$= 1$$

elements of standard basis are orthogonal ex: another two qubit state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + |11\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$$

0.1 Quiz

1) No
2) YES
3) 0
4) No