4) Calcule los coeficientes A, B y C de One-sided - D_{+2} y diga por qué es de orden $O(h^2)$.

Los puntos x_i, x_{i+1}, x_{i+1}, x_{i+2}, x_{i-2} están igualment espaciados por la distancia h.

Estimación de $f'(x_i)$

$x_{i+1} = x_i + h$ $\quad$ $x_{i+2} = x_i + 2h$

$x_{i-1} = x_i - h$ $\quad$ $x_{i-2} = x_i - 2h$

$$f(x_i) = A_{x_{i+2}} + B_{x_{i+1}} + C_{x_i}$$

$$f(x_{i+2}) = f(x_i) + (2h) f'(x_i) + \frac{1}{2!}(2h)^2 f''(x_i) + \frac{1}{3!}(2h)^3 f'''(x_i) + \dots$$

$$f(x_{i+1}) = f(x_i) + h f'(y_i) + \frac{1}{2!} h^2 f''(x_i) + \frac{1}{3!} h^3 f'''(x_i) + \dots$$

$$f(x_i) = A\left[\cancel{f_i} + 2h\left(\cancel{f_i'} + 4\frac{h^2}{2} f_i'' + \frac{8h^3}{6} f''' + \dots\right)\right] + B\left[\cancel{f_i} + hf_i' + \frac{h^2}{2} f_i'' + \dots + \frac{h^3}{6}\cancel{f_i'''} + \dots)\right] + C\cancel{f_i}$$

$$f_i'' = (A+B+C) f_i + (2A+B) hf_i + \left(2A + \frac{B}{2}\right) h^2 f_i'' + O^3$$

$A + B + C = 0$

$2A + B = 0$

$4A + B = \frac{2}{h^2}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{2}{h^2} \end{bmatrix}$$

$A = \frac{1}{h^2}$

$B = -\frac{2}{h^2}$

$C = \frac{1}{h^2}$

$$\boxed{f''(x_i) = \frac{f_{x_{i+2}} - 2f_{x_{i+1}} + f_{x_i}}{h^2}}$$

b) Calcule los coeficientes A, B, C y D de One-sided-D_{-3} y diga porqué es de orden $O(h^3)$.

Tipo = One-sided-D_{-3} ; Orden = $O(h^3)$

Fórmula: $Au_{i+2} + Bu_{i+1} + Cu_i + Du_{i-1}$

$$f(x_{i+2}) = f(x_i) + 2h\, f'(x_i) + \frac{1}{2!}(2h^2)\, f''(x_i) + \frac{1}{3!}(2h)^3 f'''(x_i) + \dots$$

$$f(x_{i+1}) = f(x_i) + h\, f'(x_i) + \frac{1}{2!}h^2 f''(x_i) + \frac{1}{3!}h^3 f'''(x_i) + \dots$$

$$f(x_{i-1}) = f(x_i) - h\, f'(x_i) + \frac{(-h^2)}{2!} f_i'' + \frac{(-2h)^3}{3!} f'''(x_i) + \dots$$

$$f(x_i) = A\left[\cancel{f_i} + \cancel{2hf_i'} + \frac{4}{2}h^2 f_i'' + \frac{8}{6}h^3 f_i''' + \dots\right] + \dots$$

$$+ B\left[\cancel{f_i} + \cancel{hf_i'} + \frac{h^2}{2} f_i'' + \frac{h^3}{6} f_i''' + \dots\right] +$$

$$+ C\left[\cancel{f_i}\right] + D\left[\cancel{f_i} - \cancel{h f'(x)} + \left(\frac{-h^2}{2}\right) f_i'' + \frac{(-2h)^3}{6} f'''\right.$$

$$f_i = [A+B+C+D]\, f_i + [2A+B-D]\, h f_i' + \left[2A + \frac{B}{2} + \frac{D}{2}\right] h^2 f_i'' + \dots$$

$$\dots + \left[\frac{8}{6}A + \frac{B}{6} - \frac{D}{6}\right] h^3 f''' + O(h^3)$$

$A+B+C+D=0$

$2A+B \quad -D=0$

$2A+\frac{B}{2} \quad +\frac{D}{2}=0$

$8A+B \quad -D=\frac{6}{h^3}$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & -1 \\ 2 & 1/2 & 0 & 1/2 \\ 8 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{6}{h^3} \end{bmatrix}$$

$A = \frac{1}{h^3}$

$B = -\frac{3}{h^3}$

$C = \frac{3}{h^3}$

$D = -\frac{1}{h^3}$

$$\Rightarrow f'''(x) = \frac{f_{i+2} - 3f_{i+1} + 3f_i - f_{i-1}}{h^3}$$

c) Explique como se obtiene la fórmula de Centered - D_0^2.

Las diferencias finitas centradas son a menudo más exactas:

$$\frac{\partial F}{\partial x} \approx \frac{F(x+\frac{1}{2}\Delta x, y) - F(x-\frac{1}{2}\Delta x, y)}{\Delta x}$$

La segunda derivada es la derivada de la primera derivada, y si utilizamos una aproximación de diferencia finita central, obtenemos:

$$\frac{\partial^2 F}{\partial x^2} \approx \frac{F(x+\Delta x, y) - 2F(x,y) + F(x-\Delta x, y)}{(\Delta x)^2} =$$

$$= \frac{F_{i+1,j} - 2F_{i,j} + F_{i-1,j}}{(\Delta x)^2}$$