A proof of Gabrielov's rank Theorem

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Collaborators

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Preliminary

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

We consider germs of $\mathbb{K}$-analytic mapping :

$$\begin{array}{cccc} \varphi : & (\mathbb{K}^m_u, 0) & \longrightarrow & (\mathbb{K}^n_x, 0) \\ & u & \mapsto & \varphi(u) = (\varphi_1(u), \ldots, \varphi_n(u)) \end{array}$$

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φ induces a morphism of convergent power series:

$$\begin{array}{rccc} \varphi^* : & \mathbb{K}\{x\} & \longrightarrow & \mathbb{K}\{u\} \\ & f & \mapsto & f \circ \varphi \end{array}$$

where $u := (u_1, \ldots, u_m)$ and $x := (x_1, \ldots, x_n)$.

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Question: what can be said about $\mathrm{Im}(\varphi)$?

Generic and Analytic ranks

In general, $\mathrm{Im}(\varphi)$ is **not** an analytic subset of $\mathbb{K}^n$.

Definition

Let $\varphi : (\mathbb{K}^m_u, 0) \longrightarrow (\mathbb{K}^n_x, 0)$ be a $\mathbb{K}$-analytic map:

$$\text{the Generic rank:} \quad \mathrm{r}(\varphi) := \mathrm{rank}_{\mathrm{Frac}(\mathbb{K}\{u\})}(\mathrm{Jac}(\varphi)),$$

$$\text{the Analytic rank:} \quad \mathrm{r}^{\mathcal{A}}(\varphi) := \dim\left(\frac{\mathbb{K}\{x\}}{\mathrm{Ker}(\varphi^*)}\right)$$

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- $\mathrm{r}(\varphi)$ is the topological dimension of $\mathrm{Im}(\varphi)$ at a generic point (half if $\mathbb{K} = \mathbb{C}$).
- $\mathrm{r}^{\mathcal{A}}(\varphi)$ is the $\mathbb{K}$-dimension of the analytic closure of $\mathrm{Im}(\varphi)$.

Remark: $\mathrm{r}(\varphi) \leqslant \mathrm{r}^{\mathcal{A}}(\varphi)$.

Classical results

Theorem (Chevalley 43, $\mathbb{K} = \mathbb{C}$, Tarski 48, $\mathbb{K} = \mathbb{R}$)

If $\varphi : (\mathbb{K}^m, 0) \longrightarrow (\mathbb{K}^n, 0)$ *is polynomial or algebraic, then:*

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Theorem (Remmert's proper mapping, 58)

Let $\varphi : X \to Y$ be a proper analytic morphism between complex analytic spaces. Suppose that Y is reduced. Then the image $\varphi(X)$ is an analytic space.

Osgood's Example (1916)

Let

$$\begin{aligned} \varphi : (\mathbb{K}^2, 0) &\longrightarrow (\mathbb{K}^3, 0) \\ (u, v) &\mapsto (u, uv, uve^v) \end{aligned}$$

Then $\mathrm{r}(\varphi) = 2$, but $\mathrm{r}^{\mathcal{A}}(\varphi) = 3$ (due to the transcendance of e^v).

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This morphism is **not proper**: the whole v-axis is sent to the origin.

Formal rank and a question of Grothendieck (1960)

Definition

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Let $\widehat{\varphi}^* : \mathbb{K}[\![x]\!] \longrightarrow \mathbb{K}[\![u]\!]$ be the extension of φ^* to the completion.

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Gabrielov proves that the answer is yes (71). There exists a map

$$\psi : (\mathbb{C}^2, 0) \quad \longrightarrow \quad (\mathbb{C}^4, 0)$$

such that $\mathrm{r}(\psi) = 2$, $\mathrm{r}^{\mathcal{F}}(\psi) = 3$ and $\mathrm{r}^{\mathcal{A}}(\psi) = 4$.

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Let $\varphi : (\mathbb{K}^m, 0) \longrightarrow (\mathbb{K}^n, 0)$ *be a* $\mathbb{K}$*-analytic morphism germ.*

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1. The result holds true for complex analytic morphisms:

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2. We can reduce the real analytic statement to the complex analytic statement by considering a complexification.

We assume, from now on, that $\mathbb{K} = \mathbb{C}$.

History and Interest

Proofs in the literature:

1. Gabrielov, Izv. Akad. Naut. SSSR. (1973);
2. Tougeron, Lectures Notes in Math. Trento (1990);
3. Belotto, Curmi, Rond, pre-print (2020).

Applications and/or connected works:

1. **Study of map germs**:
   Eakin, Harris (1977); Izumi (1986, 1989);
2. **Foliation Theory**:
   Malgrange (1977), Cerveau, Mattei (1982);
3. **Subanalytic geometry**:
   Bierstone, Schwarz (1982), Bierstone, Milman (1982), Pawlucki (1990, 1992).
4. **Counter-examples in real-analytic geometry**:
   Pawlucki (1989), Bierstone, Parusinski (2020), Belotto, Bierstone (preprint).

Reduction to the low-dimensional case

Proposition (Reduction by contradiction)

Let $\varphi\colon (\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0)$ *be an analytic morphism such that*

$$2 \leqslant \mathrm{r}(\varphi) = \mathrm{r}^{\mathcal{F}}(\varphi) < \mathrm{r}^{\mathcal{A}}(\varphi).$$

Then there is $\varphi\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ *such that*

$$\mathrm{r}(\varphi) = \mathrm{r}^{\mathcal{F}}(\varphi) = 2 \textit{ and } \mathrm{r}^{\mathcal{F}}(\varphi) = 3.$$

To prove this Proposition, we use a certain number of "allowed operations", building the new morphism step by step.

Reduction: first step

Lemma (Blow-ups and power substitutions)

Let $\varphi : (\mathbb{C}^m, 0) \longrightarrow (\mathbb{C}^n, 0)$ *be a* $\mathbb{C}$*-analytic morphism germ.*

1. *Let* $\sigma : (\mathbb{C}^m, 0) \to (\mathbb{C}^m, 0)$ *be a (chart of a) blow-up or a power substitution;*
2. *Let* $\tau : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ *be a power substitution;*

Then the ranks of $\tau \circ \varphi \circ \sigma$ *coincide with the ranks of* φ*.*

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Then the ranks of $\tau \circ \varphi \circ \sigma$ *coincide with the ranks of* φ*.*

Warning: Blow-ups in the target may change the ranks!

Using this Lemma and some classical algebra tools, we build a morphism $\varphi \colon (\mathbb{C}^m, 0) \to (\mathbb{C}^{m+1}, 0)$ such that

$$\mathrm{r}(\varphi) = \mathrm{r}^{\mathcal{F}}(\varphi) = m, \text{ and } \mathrm{r}^{\mathcal{A}}(\varphi) = m + 1.$$

Reduction of dimension (restriction to hyperplanes)

Assume that:

$$\varphi : (\mathbb{C}^m, 0) \longrightarrow (\mathbb{C}^{m+1}_{x_1,\cdots,x_m,y}, 0)$$

is such that

$$\mathrm{r}(\varphi) = \mathrm{r}^{\mathcal{F}}(\varphi) = m,$$

$\mathrm{r}^{\mathcal{A}}(\varphi) = m + 1$ and $P(x, y) \in \mathbb{C}[\![x_1, \ldots, x_n]\!][y]$ an irreducible polynomial which generates $\ker(\widehat{\varphi}^*)$.

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Reduction ($m > 2$)**:** We restrict the morphism to a sufficiently generic hyperplane H (containing the y-axis) on the target:

$$\psi := \varphi|_{\varphi^{-1}(H)} : (\varphi^{-1}(H), 0) \to (H, 0)$$

such that $\varphi^{-1}(H)$ is a **smooth hypersurface** and $\mathrm{r}(\psi) = m - 1$.

Reduction of dimension (Main tools)

Let H be a sufficiently generic hyperplane (in x):

Theorem (Abhyankar-Moh, 70)

If $P \in \mathbb{C}[\![x]\!][y]$ is divergent, then $P|_H$ is divergent.

Theorem (Formal Bertini Theorem, Chow 58)

Let $m \geqslant 3$. If $P \in \mathbb{C}[\![x]\!][y]$ is irreducible, then $P|_H$ is irreducible.

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Let $m \geqslant 3$. If $P \in \mathbb{C}[\![x]\!][y]$ is irreducible, then $P|_H$ is irreducible.

Then we get that $P|_H$ is a non convergent irreducible polynomial in $\ker(\widehat{\psi}^*)$. Therefore

$$m-1 = r(\psi) \leqslant \mathrm{r}^{\mathcal{F}}(\psi) \leqslant m-1.$$

The fact that P is irreducible and non convergent implies that $\ker(\psi^*) = (0)$, therefore

$$\mathrm{r}(\psi) = \mathrm{r}^{\mathcal{F}}(\psi) = m-1, \text{ and } \mathrm{r}^{\mathcal{A}}(\psi) = m.$$

No more reductions!

Warning: The Formal Bertini Theorem fails if $m = 2$, e.g.:

$$P(x_1, x_2, y) = y^2 - (x_1^2 + x_2^2)$$

is irreducible in $\mathbb{C}[[x_1, x_2]][y]$.

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$$\forall \lambda \in \mathbb{C}, P(\lambda x_2, x_2, y) = y^2 - x_2^2(\lambda^2 + 1)$$

is not irreducible in $\mathbb{C}[\![x_2]\!][y]$.

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is not irreducible in $\mathbb{C}[\![x_2]\!][y]$.
Moral: If $n = 2$, a priori, it could happen that:

$$P|_H = Q_1(x, y) \cdot Q_2(x, y)$$

is divergent, while Q_1 is convergent (and Q_2 is divergent). This in turn could allow $\ker(\psi^*) \neq (0)$, and our argument of reduction fails.

The "difficult case" : Low dimension rank Theorem

Theorem (Low dimension Gabrielov's rank Theorem)

Let $\varphi : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^3, 0)$ be a $\mathbb{C}$-analytic morphism germ.

$$\mathrm{r}(\varphi) = \mathrm{r}^{\mathcal{F}}(\varphi) = 2 \implies \mathrm{r}^{\mathcal{A}}(\varphi) = 2.$$

By formal Weierstrass Preparation, we can distinguish a variable

$$(x_1, x_2, y)$$

so that $\ker(\widehat{\varphi}^*)$ is generated by an irreducible polynomial:

$$P(x, y) = y^d + \sum_{i=0}^{d-1} A_i(x) y^i, \quad A_i(x) \in \mathbb{C}[\![x_1, x_2]\!].$$

Goal: Prove that $P(x, y)$ is convergent.

Basic case: Quasi-ordinary polynomial

Now, suppose that the discriminant $\Delta(P)$ is monomial, that is:

$$\Delta(P) = x_1^{\alpha_1} x_2^{\alpha_2} \cdot \text{unit}$$

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By the **Abhyankar-Jung Theorem**, there exists $k \in \mathbb{N}$ such that:

$$P(x,y) = \prod_{i=1}^{d} \left(y - \xi_i \left(x_1^{1/k}, x_2^{1/k} \right) \right), \quad \xi_i \text{ formal power series,}$$

and ξ_i convergent $\Leftrightarrow$ ξ_j convergent, because P is irreducible.

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$$\prod_{i=1}^{d} \left(\varphi_3 - \xi_i \left(\varphi_1^{1/k}, \varphi_2^{1/k} \right) \right) = 0$$

and we conclude that one of the factors is **convergent** because **up to transforming** φ, we can assume $\varphi_1^{1/k} = u$ and $\varphi_2^{1/k} = uv$.

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and we conclude that one of the factors is **convergent** because **up to transforming** φ, we can assume $\varphi_1^{1/k} = u$ and $\varphi_2^{1/k} = uv$. Finally, one of the $\xi_i(u, uv)$ is convergent, therefore ξ_i is convergent, and P has convergent coefficients.

Geometrical Framework

Idea: we want to "make $\Delta(P)$ monomial"!

From now on, it is convenient to use geometrical notations:

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$$\sigma : (N, F) \to (\mathbb{C}^2, \mathfrak{a}), \quad F = \sigma^{-1}(\mathfrak{a}), \quad \mathfrak{b} \in F.$$

We consider the "pull-back of P by σ at $\mathfrak{b}$". More precisely

$$P_{\mathfrak{b}} = \widehat{\sigma}^*_{\mathfrak{b}}(P) \quad \text{where} \quad \widehat{\sigma}^*_{\mathfrak{b}} : \widehat{\mathcal{O}}_{\mathfrak{a}} \to \widehat{\mathcal{O}}_{\mathfrak{b}}.$$

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Definition

We say that $P_{\mathfrak{b}}$ **has a convergent factor** if there is $Q_{\mathfrak{b}} \in \mathcal{O}_{\mathfrak{b}}[y]$ which is a factor of $P_{\mathfrak{b}}$.

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$P_{\mathfrak{b}}$ is not irreducible in $\mathbb{C}[\![u, v]\!][y]$: if $\varphi^2 = 1 + v^2$, then

$$P_{\mathfrak{b}} = (y - u\varphi)(y + u\varphi).$$

Overarching inductive framework

Overarching framework $(*)$:
Let $\mathfrak{a} \in \mathbb{C}^2$ and $P \in \widehat{\mathcal{O}}_{\mathfrak{a}}[y]$ be non-constant reduced and monic. Consider a sequence of point blow-up

$$(\mathbb{C}^2, \mathfrak{a}) = (N_0, \mathfrak{a}) \xleftarrow[\sigma_1]{} (N_1, F_1) \xleftarrow[\sigma_2]{} \cdots \xleftarrow[\sigma_r]{} (N_r, F_r)$$

set $\sigma := \sigma_1 \circ \cdots \circ \sigma_r$, and assume that :

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For the induction scheme, it is useful to keep track of the history:

$$F_r = F_r^{(1)} \cup \cdots \cup F_r^{(r)}$$

where $F_r^{(j)}$ is the exceptional divisor which appeared at time j.

Inductive Scheme

Proposition (Inductive scheme)

Under framework $(*)$, assume that $\exists \mathfrak{c} \in F_r$ such that $P_{\mathfrak{c}}$ has a convergent factor. Then P admits a convergent factor.

Proof of Low-dimensional Gabrielov:

$$(\mathbb{C}^2, 0) \xrightarrow{(\varphi_1, \varphi_2)} \bullet \; a \in \mathbb{C}^2$$

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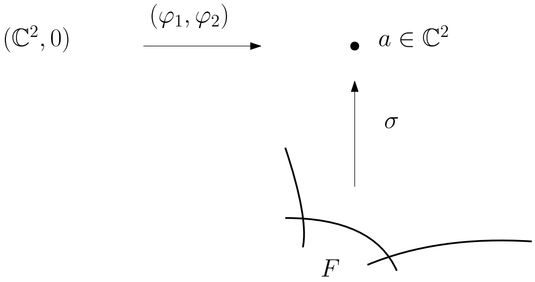

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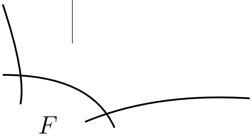

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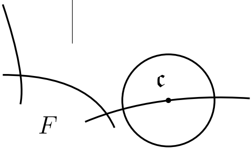

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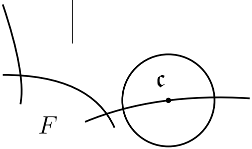

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It is enough to use the Quasi-ordinary case and the Proposition.

Main technical tool : Semi-global extension

To prove the inductive scheme, we use:

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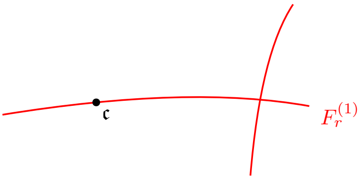

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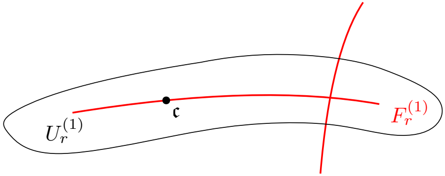

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We prove the inductive scheme by induction on the lexicographical order of (r, k).

Note that if $r = 1$, we simply need the following classical

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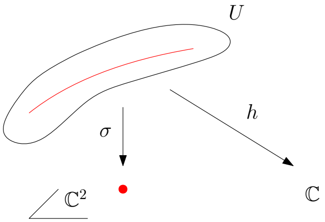
Let $\sigma\colon (N, F) \to (\mathbb{C}^2, 0)$ be the blow up of the origin, and let $h\colon U \to \mathbb{C}$ be an analytic function, where U is a neighbourhood of F.



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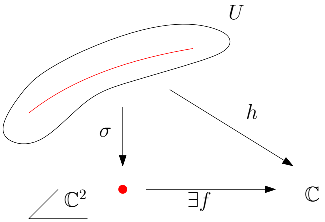

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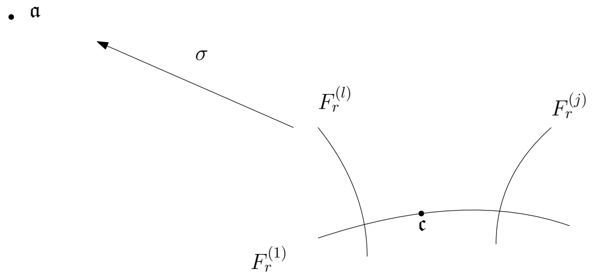

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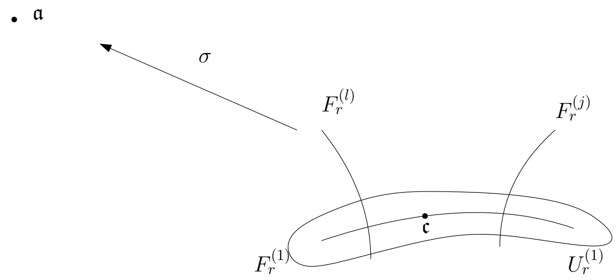

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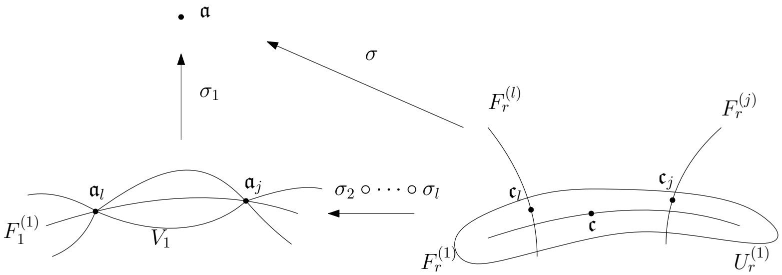

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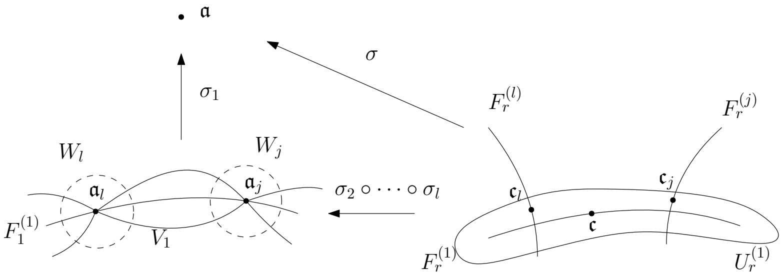

- $P_j := (\widehat{\sigma_1})^*_{\mathfrak{a}_j}(P)$ verifies our conditions after $r-1$ blow ups.
- We get a convergent factor of $(\sigma_1)^*(P)$ on a neighbourhood of $F_1^{(1)}$, hence a convergent factor of P.

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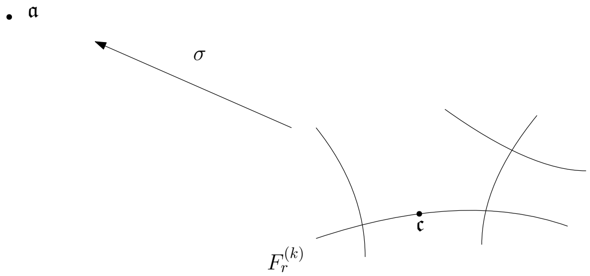

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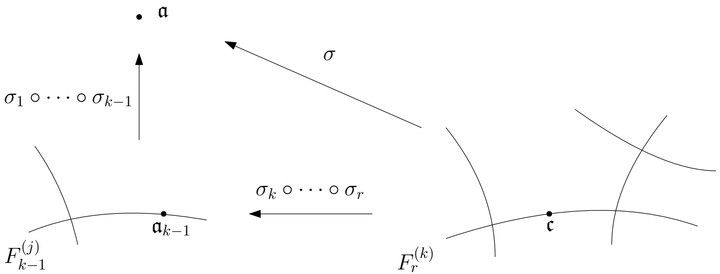

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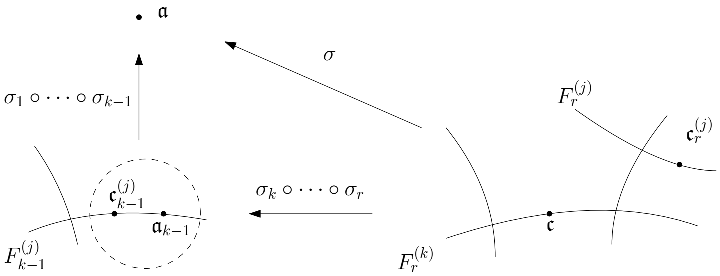

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- We obtain a convergent factor at a point $\mathfrak{c}_r^{(j)}$ of $F_r^{(j)}$, for some $j \leqslant k - 1$.

Semi-global extension: Recall

Gabrielov's rank
Theorem

$$\Uparrow$$

Low-dimension
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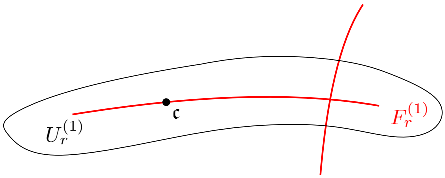

Overview

The proof has two main steps:

Newton-Puiseux-Eisenstein parametrization:

1. Projective rings;
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Local-to-Semi-global convergence of factors:

1. Projective convergent rings;
2. Local to Projective convergence of factors;
3. Semi-global formal extension.

Projective Ring: Motivation

We want to get a sub-ring $\mathbb{P}_h[\![x]\!]$ of $\overline{\mathbb{C}[\![x]\!]}$ such that:
If

$$P = \prod_{i=1}^{s} Q_i(x, y)$$

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$\Rightarrow$ If $P_\mathfrak{b}$ has a convergent factor for some $\mathfrak{b} \in F_r^{(1)}$ then one of the $Q_{i\mathfrak{b}}$ is convergent, and $Q_{i\mathfrak{c}}$ provides a convergent factor of $P_\mathfrak{c}$ at every $\mathfrak{c} \in F_r^{(1)}$.

$\square$

Denote by ν the (x)-adic valuation on $\mathbb{C}[\![x]\!]$. We consider the valuation ring V_ν associated to it (and its completion $\widehat{V}_\nu$), that is

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Remark: After one blow-up $\sigma(u, v) = (u, uv)$:

$$\frac{f}{g} \in V_\nu \implies \sigma^*\left(\frac{f}{g}\right) = \frac{u^{\nu(f)}\widetilde{f}}{u^{\nu(g)}\widetilde{g}}$$

and $\sigma^*\left(\frac{f}{g}\right)$ is well-defined outside the strict transform of $(g = 0)$.

Projective Ring

Let h be a homogeneous polynomial.

Definition (Projective ring)

We denote by $\mathbb{P}_h[\![x]\!]$ the subring of $\widehat{V}_\nu$ characterized as follows: $A \in \mathbb{P}_h[\![x]\!]$ if there exists α, $\beta \in \mathbb{N}$ and a sequence of polynomials $(a_k)_{k\in\mathbb{N}}$ so that:

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And we denote by $\mathbb{P}_h\{x\}$ the subring of $\mathbb{P}_h[\![x]\!]$ characterized by: $A \in \mathbb{P}_h\{x\}$ if

$$\sum_{k\geqslant 0} a_k(x) \in \mathbb{C}\{x\}.$$

Integral homogeneous elements

Remark: In order to describe the roots of

$$P(x, y) = y^2 - (x_1^3 + x_2^3)$$

we need to add the element:

$$\gamma = \sqrt{x_1^3 + x_2^3}, \text{ which is a root of } \Gamma(x, z) = z^2 - (x_1^3 + x_2^3)$$

Definition

An integral homogeneous element γ is an element of $\overline{\mathbb{C}(x)}$, satisfying a relation of the form

$$\Gamma(x, \gamma) = 0$$

where $\Gamma(x, z)$ is a **weighted** homogeneous polynomial monic in z.

Newton-Puiseux-Eisenstein Theorem

Theorem (Newton-Puiseux-Eisenstein factorization (simplified))

Let $P \in \mathbb{C}[\![x]\!][y]$ be a monic polynomial. There exists an integral homogeneous element γ, and a homogeneous polynomial $h(x)$, such that:

$$P(x, y) = \prod_{i=1}^{s} Q_i(x, y) \tag{1}$$

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1. *the $Q_i \in \mathbb{P}_h[\![x]\!][y]$ are irreducible in $\widehat{V}_\nu[y]$;*
2. *for fixed i, the $\xi_{ij} \in \mathbb{P}_h[\![x]\!][\gamma]$ can be obtained from one another by replacing γ by one of its conjugates.*

Semi-global formal extension (simplified)

Under framework $(*)$, let $A \in \mathbb{P}_h[\![x]\!]$. Fix $\mathfrak{b} \in F_r^{(1)}$.
We say that A *extends formally (resp. analytically)* at $\mathfrak{b}$ if the composition $A_{\mathfrak{b}} := \widehat{\sigma}_{\mathfrak{b}}^*(A)$ belongs to $\widehat{\mathcal{O}}_{\mathfrak{b}}$ (resp. $\mathcal{O}_{\mathfrak{b}}$).

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Theorem (Semi-global formal extension (simplified))

Under framework $(*)$, let $P(x,y) \in \mathbb{C}[\![x]\!][y]$ be a monic reduced polynomial, and consider the factorization given in (1):

$$P(x,y) = \prod_{i=1}^{s} Q_i(x,y)$$

The polynomials Q_i extend formally at every point $\mathfrak{b} \in F_r^{(1)}$. Furthermore, this extension is analytic if and only if $Q_i \in \mathbb{P}_h\{x\}[y]$.

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Under framework $(*)$, *suppose that there exists a point* $\mathfrak{c} \in F_r^{(1)}$ *such that* $P_{\mathfrak{c}}$ *admits a convergent factor. Then, there exists* i *such that* $Q_i \in \mathbb{P}_h\{x\}[y]$.

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Difficulty: In the above setting, we have that:

$$\sigma_{\mathfrak{c}}^*(Q_i) = \prod_{j=1}^{s_i} R_{ij}(x,y) \in \widehat{\mathcal{O}}_{\mathfrak{c}}[y]$$

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Then $\sigma_{\mathfrak{c}}^*(Q_i)$ is convergent, so $Q_i \in \mathbb{P}_h\{x\}[y]$, then **finally** $\sigma_{\mathfrak{c}}^*(Q_i)$ is a convergent factor of $P_{\mathfrak{c}}$ at every $\mathfrak{c} \in F_r^{(1)}$.

Thank you for your attention!