# QE with Cylindrical Algebraic Decomposition for real polynomials

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# Problem statement

- Solve system of eqns and ineqns of real polynomials with any number of variables:
  - $f_1(x_1, x_2, \ldots x_n) < 0$
  - $f_2(x_1, x_2, \ldots x_n) = 0$
  - $f_3(x_1, x_2, \ldots x_n) >= 0$
  - $\ldots$
- Build a choose() function that returns UNSAT or a valid assignment for variables.

# Basic idea

- Split $R^n$ in cells (what are they?)
- Signs of all polynomials (+, - , 0) are invariant on each cell.
- Each cell keeps representatives for variables $x_1 \ldots x_n$
- If we know all cells, we can build choose() immediatly.
- Demos !

# Examples

- First example (PDF):
  - $x_1^2 + x_1^2 - 1 < 0$
  - $x_1^3 - x_2^2 = 0$
- Example of cells : fig 5 and fig 6 - page 9
- Nice example : fig 8 page 26
- Projection: page 30

# Algorithm

- 3 phases: Projection , Base , Extension

# Projection

- We start with a family $F_n$ of our polynomials: $F_n \subset R[x_1, \dots x_n]$
- Idea: from a set $F_i \subset R[x_1, \dots x_i]$ obtain a set $F_{i-1} \subset R[x_1, \dots x_{i-1}]$ such that the all the real roots of polys from $F_{i-1}$ are all representative points of all polys from $F_i$
- We define $F_{i-1} = proj(F_i)$
- Example pag 30 and example 5.1 page 26
- At the end, $F_1$ will contain just univariate polys that correspond to all real representatives of variable $x_1$

# Projection construction

- How to build $F_{n-1}$ from $F_n = \{g_1(x_n), \ldots g_k(x_n)\} \subset R[x_1 \ldots x_{n-1}][x_n]$ ?
- We need 3 properties to happen (theorem page 19):
- 1) For a fix $\delta$, total number of complex roots of each $g_{i,\delta}(x_n)$ remains invariant. This means that degree of $g_i$ remains constant. So add all coefficients of $g_i(x_n)$
- 2) For a fix $\delta$, total number of distinct complex roots of each $g_{i,\delta}(x_n)$ remains invariant. This is equivalent with $gcd(g_i, g_i')$ is constant. Add its dominant coefficient: $psc_k(g_i, g_i')$
- 3) For a fix $\delta$, total number of common complex roots of each pair $(g_{i,\delta}(x_n), g_{j,\delta}(x_n))$ remains invariant. This is equivalent with $gcd(g_i, g_j)$ is constant. Add its dominant coefficient: $psc_k(g_i, g_j)$

# Base phase

- We start now have a family $F_1$ of univariate polynomials: $F_1 = \{g_1(x_1), \dots g_k(x_1)\} \subset R[x_1]$
- All roots of $g_i$'s and intermediary points will be sufficient for representatives of $x_1$
- Algorithm for finding roots of real polynomials
  - We have a procedure that gives us the number of roots of P from an interval (a,b)
  - Then what? Binary search starting from $(-\infty, \infty)$.

# Base phase: Problems in paradise

- Problems:
  - What is $\infty$ ? Is it MAXINT? Not at a second thought!
  - Try $max(1, \sum_{i=0}^{n-1} \frac{|a_i|}{|a_n|})$
  - When to stop? Errors if the interval is too small. Try Newton from now on.
  - What is $P(x) == 0$. $P(x) < 0$? $x == y$ ? How many epsilons? What values for them?
  - How to evaluate a P(x) ? (Horner)
  - Remainder computing errors: p%q
  - How to know if different polynomials return the same values (Redundant roots will propagate exponentially later)
  - Redundant polynomials $(x^2 = 0; 4x^5 = 0) \Longrightarrow$ Redundant computations

# Extension phase

- If we know all possible representatives for $x_1, x_2 \dots x_i$ (we know all cells for $R^i$) then we can find all representatives for $x_{i+1}$
- How? By substituting values from each partial cell to each polynomial of projection $F_{i+1}$
- Then we get univariate polynomials for each cell
- Find all their roots
- Finish in $R^n$ when we get all cells.

# Extension phase: Problems

- Explosion in number of cells
- Redundant cells (due to numeric errors or not)
- Big computations
- Theoretically good for any examples. Practically bad for big ones.

# Thank you!

- Questions ?
- Thank you!