Differential Geometry

Transcript[1] of lectures by Marco Zambon at KU Leuven in 2019–20
Notes taken and typeset by Gilles Castel, proofread by the lecturer

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This is a transcript of the Differential Geometry lectures by Marco Zambon[1] at the KU Leuven Master program in Mathematics, in 2019–20.

The notes were taken and typeset by Gilles Castel, whom the lecturer gratefully acknowledges, and were proofread by the lecturer.

The lectures mainly drew from the following two classical references:

[Lee] Lee, John
Introduction to Smooth Manifolds
Graduate Texts in Mathematics, Volume 218, 2012 (Second edition).
https://sites.math.washington.edu/~lee/Books/ISM/

[Tu] Tu, Loring
An introduction to Manifolds
Universitext, 2010 (Second edition).
https://link.springer.com/book/10.1007/978-1-4419-7400-6

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Chapter 1

Differentiable manifolds

1.0 Topological spaces

Definition 1.1 (Topological space). A topological space is (X, σ) where X is a set and σ a family of subsets of X, called open sets, such that:

- $\emptyset, X \in \sigma$
- $\bigcup_{i \in I} U_i \in \sigma$ whenever $U_i \in \sigma$ for all i
- $\bigcap_{i<n} U_i \in \sigma$ whenever $U_i \in \sigma$ for all i (finite intersection)

Let (X, σ) be a topological space.

Definition 1.2 (Open neighbourhood). An open subset that contains $p \in X$ is called a (open) neighbourhood of p.

Definition 1.3 (Subspace topology). If $Y \subset X$ then (Y, σ_Y) is a topological space, where

$$\sigma_Y = \{U \cap Y \mid U \in \sigma\}.$$

We call σ_Y the subspace topology.

Example. Endowing $\mathbb{R}^2$ with the Euclidean topology, the subspace topology on $\mathbb{R} \times \{0\} \subset \mathbb{R}^2$ is also the Euclidean topology. ◇

Definition 1.4 (Quotient topology). Let $\sim$ be an equivalence relation on X. Consider $\pi : X \to X/\sim$. Then $X/\sim$ is a topological space, where the open sets are by definition the sets U such that $\pi^{-1}(U)$ is open in X.

Definition 1.5 (Continuous functions). A function $f : X_1 \to X_2$ is called continuous iff $\forall U \in \sigma_2 : f^{-1}(U) \in \sigma_1$.

Definition 1.6 (Hausdorff). A topological space is called Hausdorff iff $\forall x, y \in X$, there exist neighbourhoods U of x, V of y such that $U \cap V = \emptyset$.

Example. Endow $\mathbb{R}^2 \setminus \{0\}$ with the equivalence relation given by the thick lines and the two half lines in the following figure. That is:

$$(x, y) \sim (x', y') \Leftrightarrow \begin{cases} x = x' & \text{if } x \neq 0 \\ yy' > 0 & \text{if } x = 0. \end{cases}$$

Then the quotient topology on $(\mathbb{R}^2 \setminus \{0\})/\sim$ is not Hausdorff. ◇

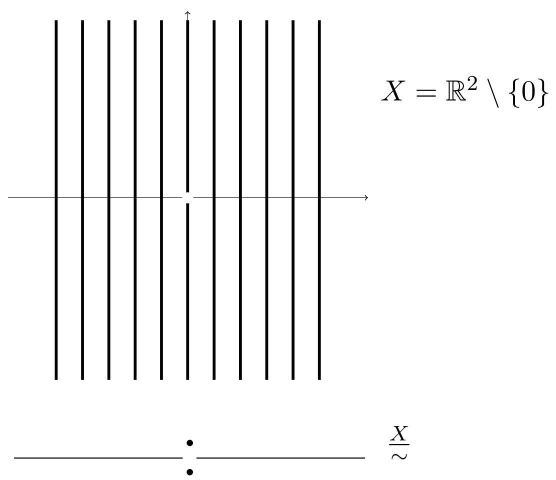


Figure 1.1: Example of a topology which is not Hausdorff

Definition 1.7 (Basis for a topology). A basis for the topology is $S \subset \sigma$ such that every open set of X is a union of elements of S.

Definition 1.8 (C2). A space (X, σ) is second countable if there exists a countable basis.

Example. $\mathbb{R}^n$ is second countable. Indeed $\{B_{\frac{1}{m}}(x) \mid x \in \mathbb{Q}^n, m \in \mathbb{N}\}$ is a countable basis for the topology. Here $B_r(x)$ is the open ball with radius r around x. ◇

1.1 Differentiable manifolds

Definition 1.9 (Topological manifold). A topological manifold M of dimension m is a second countable, Hausdorff topological space which is locally homeomorphic to $\mathbb{R}^m$.

Remark. 'Locally homeomorphic to $\mathbb{R}^m$' means that $\forall p \in M$, there exists a neighborhood U of p and a homeomorphism $\phi : U \to V \underset{\text{open}}{\subset} \mathbb{R}^m$. Recall that homeomorphism means: bijective map that is continuous in *both* directions.

Definition 1.10 (Chart). The pair (U, ϕ) is called a chart.

Remark.

- Any subset of a Hausdorff space is Hausdorff
- Any subset of a C2 space is C2.

Recall that a map between open subsets of $\mathbb{R}^m$ is a diffeomorphism if it is bijective, differentiable and the inverse is differentiable, where "differentiable" means that all partial derivatives exist.

Definition 1.11 (Compatibility). Two charts (U_1, ϕ_1) and (U_2, ϕ_2) are called smoothly compatible if

$$\phi_2 \circ (\phi_1)^{-1}|_{\phi_1(U_1 \cap U_2)} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$$

is a diffeomorphism.

Definition 1.12 (Smooth atlas). A smooth atlas for M is called a collection of charts $\{(U_\alpha, \phi_\alpha)\}$ such that $\bigcup_\alpha U_\alpha = M$ and any two charts are smoothly compatible.

Definition 1.13 (Maximal smooth atlas). A smooth atlas $\mathcal{A}$ is maximal if: whenever $\mathcal{B}$ is a smooth atlas and $\mathcal{A} \subset \mathcal{B}$ then $\mathcal{B} = \mathcal{A}$.

Definition 1.14 (Differentiable manifold). A differentiable manifold (also called a smooth manifold) is a topological manifold M together with a maximal smooth atlas.

Remark. Given a smooth atlas $\mathcal{A}$ on a topological manifold M, there exists a unique maximal smooth atlas containing it, namely

$$\{(V, \psi) \mid (V, \psi) \text{ is smoothly compatible with all charts of } \mathcal{A}\}.$$

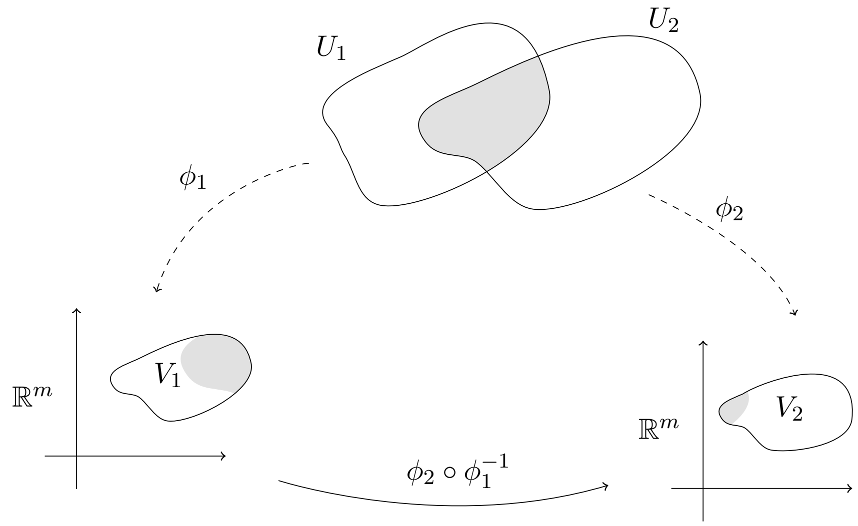


Figure 1.2: Compatible charts

Example. Let $U \subset \mathbb{R}^n$ be open. Then U is a smooth manifold: an atlas is $\{(U, \mathrm{Id})\}$. Take the maximal smooth atlas containing it. $\diamond$

Example. Let $S^n := \{\mathbf{x} \in \mathbb{R}^{n+1} \mid \|\mathbf{x}\| = 1\}$. The sphere S^n with the subspace topology is Hausdorff and C2, simply because $\mathbb{R}^{n+1}$ is. Two charts are given by the stereographic projections from the Northpole N and Southpole S:

$$\phi_N : S^n \setminus \{N\} \to \mathbb{R}^n : (x_1, \ldots, x_{n+1}) \mapsto \frac{(x_1, \ldots, x_n)}{1 - x_{n+1}}$$
$$\phi_S : S^n \setminus \{S\} \to \mathbb{R}^n : (x_1, \ldots, x_{n+1}) \mapsto \frac{(x_1, \ldots, x_n)}{1 + x_{n+1}}.$$

Now, ϕ_N and ϕ_S are homeomorphisms. Furthermore, $\|\phi_N(p)\| \cdot \|\phi_S(p)\| = 1$, which allows us to calculate the inverse of ϕ_N. Hence

$$(\phi_S \circ \phi_N^{-1})|_{\phi_N(S^n \setminus \{N,S\})} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\} : y \mapsto \frac{y}{\|y\|^2},$$

so ϕ_N and ϕ_S are smoothly compatible. Take the maximal smooth atlas containing ϕ_N and ϕ_S. $\diamond$

Remark. We could have started with other points $P, Q \in S^n$ instead of N, S. The smooth atlases $\{\phi_P, \phi_Q\}$ and $\{\phi_N, \phi_S\}$ would be different, but they define the same maximal smooth atlas.

1.2 Differentiable maps

Let M be a smooth manifold.

Definition 1.15 (Smooth function). A function $f : M \to \mathbb{R}$ is differentiable (or smooth) at $p \in M$ iff $\exists$ a chart (U, ϕ) around p such that $f \circ \phi^{-1}$ is differentiable in $\phi(p)$.

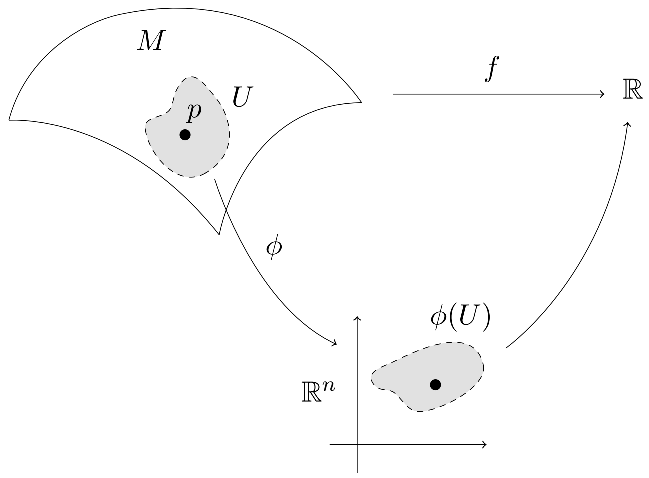


Figure 1.3: Smooth function from M to $\mathbb{R}$

Remark. If $f \circ \phi^{-1}$ is differentiable at $\phi(p)$ for a chart (U, ϕ), then $f \circ \psi^{-1}$ is also differentiable at $\psi(p)$, for any other chart (V, ψ) (in the maximal atlas of M).

Proof. We want to argue that $f \circ \psi^{-1}$ is smooth.

$$f \circ \psi^{-1} = \underbrace{(f \circ \phi^{-1})}_{C^\infty} \circ \underbrace{(\phi \circ \psi^{-1})}_{C^\infty}.$$

$\square$

Notation. We write $C^\infty(M)$ to denote all smooth functions $M \to \mathbb{R}$.

Definition 1.16 (Smooth map). $f : M \to N$ is differentiable at $p \in M$ iff

- it is continuous
- there exist charts (U_M, ϕ_M) around p and (U_N, ϕ_N) around $f(p)$ such that $\phi_N \circ f \circ \phi_M^{-1}$ is differentiable at $\phi_M(p)$

Remark. The map $\phi_N \circ f \circ \phi_M^{-1}$ is defined on $\phi_M(U_M \cap f^{-1}(U_N))$. The continuity of f ensures that this is an open neighborhood of $\phi_M(p)$ in $\mathbb{R}^m$, hence it makes sense to talk about the differentiability of the above map at $\phi_M(p)$.

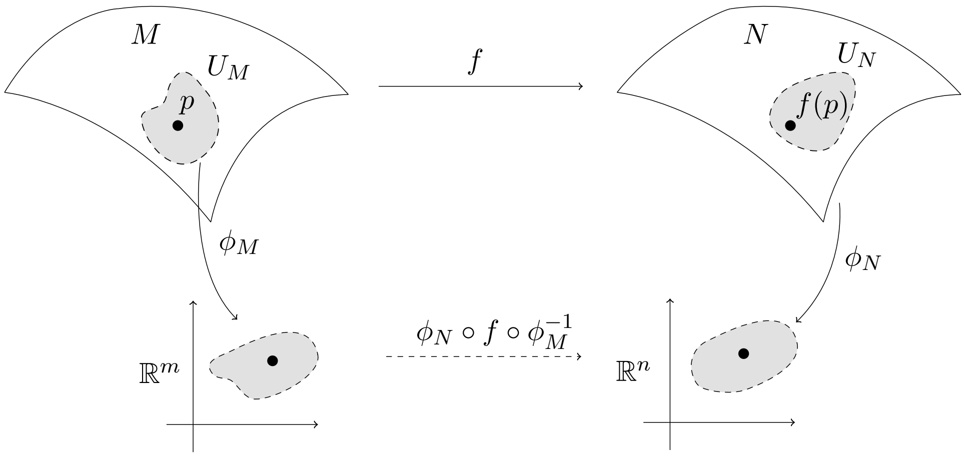


Figure 1.4: Smooth function from M to N

Remark. A map f being a differentiable and a homeomorphism does not imply that f is a diffeomorphism (which also include the differentiability of the inverse). For instance, $f\colon \mathbb{R} \to \mathbb{R}, x \mapsto x^3$ is not a diffeomorphism, because the inverse $x \mapsto \sqrt[3]{x}$ is not differentiable at zero.

1.3 Partition of unity

Definition 1.17 (Partition of unity). A partition of unity is a family $\{e_\alpha\}_{\alpha \in A}$ of smooth functions $e_\alpha : M \to [0, 1]$ such that

- For all $p \in M$, there exists a neighborhood U of p such that the set $\{\alpha \in A : e_\alpha|_U \not\equiv 0\}$ is finite.
- $\sum e_\alpha \equiv 1$

Definition 1.18 (Subordinate). Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of M. A partition of unity $\{e_\alpha\}_{\alpha\in A}$ is subordinate to the cover $\Leftrightarrow$ $\operatorname{supp}(e_\alpha)\subset U_\alpha$, where

$$\operatorname{supp}(e_\alpha)=\overline{\{p\in M : e_\alpha(p)\neq 0\}}.$$

Proposition 1.19. Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of M. Then there exists a partition of unity subordinate to it.

This is useful in the following way: we can define functions, or vector fields s_α on open subsets of M which locally look like $\mathbb{R}^n$. Then we create a partition of unity subordinate to this cover, and paste the functions smoothly together: $\sum e_\alpha\cdot s_\alpha$. This gives a smooth function (or vector field) on the whole of M. The proof uses that the topology of M is second countable, which is one of the reasons of requiring C2 in the definition of a manifold.

Proof. We present the idea of the proof, assuming M compact. For all $q\in M$, choose $\alpha\in A$ such that $q\in U_\alpha$. Let ψ_q be a function[a] such that $\psi_q(q)=1$ supported in U_α. This is always possible, since the manifold locally looks like $\mathbb{R}^n$.

Since M is compact, the open cover $\{(\psi_q)^{-1}(\mathbb{R}_{>0})\}_{q\in M}$ has a finite subcover. So $\exists q_1,\dots,q_m\in M$ such that $\psi=\sum_{i=1}^m\psi_{q_i}>0$ on M. Now, define $\phi_i=\frac{\psi_{q_i}}{\psi}$, which satisfies $\sum\phi_i=1$ and $\operatorname{supp}(\phi_i)=\operatorname{supp}(\psi_{q_i})\subset U_\alpha$ for some $\alpha\in A$. Now, one can rearrange these functions such that they are indexed by A. $\square$

[a] Such functions are sometime called bump functions.

1.4 Submanifolds

Let N be a smooth manifold of dimension n.

Definition 1.20 (Submanifold). A subset $M\subset N$ is a submanifold of dimension m if $\forall p\in M$ there exists a chart (U,ϕ) such that

$$\phi(U\cap M)=\phi(U)\cap(\mathbb{R}^m\times\{0\}).$$

Remark. A chart as above is called adapted to M.

Remark. The set M itself inherits the structure of a smooth manifold, with smooth atlas given by

$$\{(U\cap M,\phi|_{U\cap M}) : (U,\phi)\text{ is a chart of } N \text{ adapted to } M\}.$$

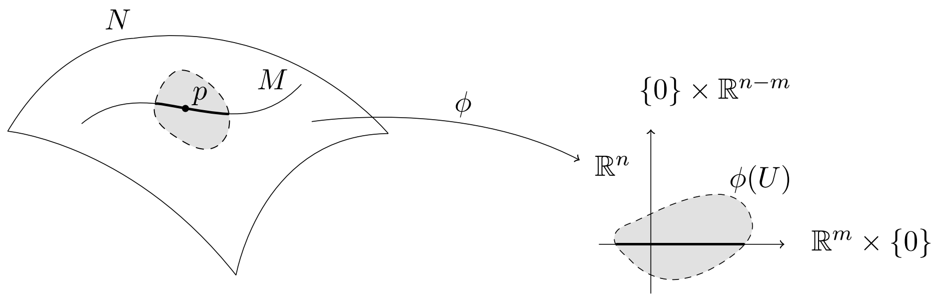


Figure 1.5: Definition of a submanifold

Example. The submanifolds of the same dimension as N are exactly the open sets of N. The (connected) submanifolds with dimension 0 are exactly the points. $\diamond$

Example. The union of the axes $\{0\} \times \mathbb{R} \cup \mathbb{R} \times \{0\}$ is not a submanifold of $\mathbb{R}^2$. The problematic point is the origin (the "cross"). $\diamond$

Example. The unit circle $S^1 \subset \mathbb{R}^2$ is a submanifold of $\mathbb{R}^2$. The idea is that, for each point of S^1, we need to find a chart that 'flattens' a part of the circle to a line. Define

$$\begin{aligned} \phi_a : \mathbb{R}^2 - (\mathbb{R}_{\geq 0} \times \{0\}) &\longrightarrow (0, 2\pi) \times \mathbb{R}_+ & \phi_b : \mathbb{R}^2 - (\mathbb{R}_{\leq 0} \times \{0\}) &\longrightarrow (-\pi, \pi) \times \mathbb{R}_+ \\ \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} &\longmapsto (\theta, r) & \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} &\longmapsto (\theta, r). \end{aligned}$$

Note that 0 is not included in either of the charts, but that is not a problem.

We have that $\phi_a(S^1 - \{(1,0)\}) = (0, 2\pi) \times \{1\}$ and $\phi_b(S^1 - \{(-1,0)\}) = (-\pi, \pi) \times \{1\}$. This is already flat, but per definition of submanifold, we need to move this to zero. So instead of $\{\phi_a, \phi_b\}$, use $\{\phi_a - (0,1), \phi_b - (0,1)\}$.

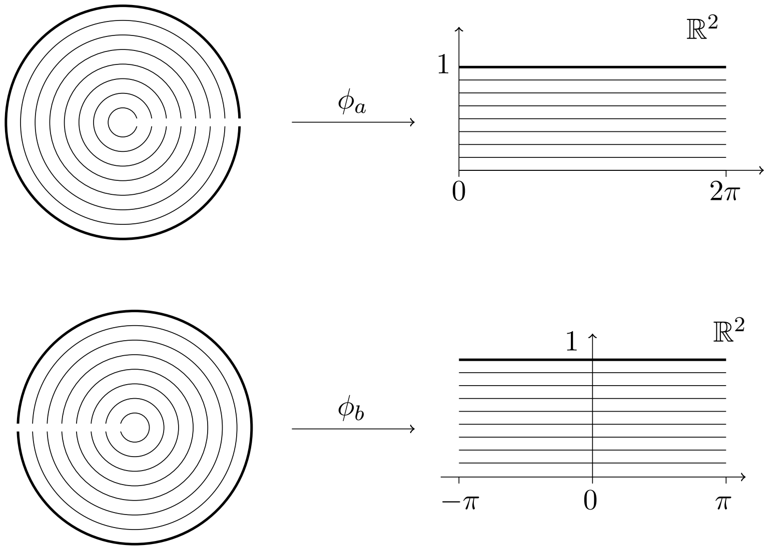


Figure 1.6: A circle is a submanifold of the plane.

$\diamond$

Example. Consider the torus, $S^1 \times S^1$, where $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Fix $\alpha \in \mathbb{R}$. Now consider

$$M = \{(e^{2\pi i t}, e^{2\pi i \alpha t}) : t \in \mathbb{R}\},$$

a subset of the torus. This is a submanifold of the torus iff $\alpha \in \mathbb{Q}$ (this happens exactly when the spiral closes up). When $\alpha \notin \mathbb{Q}$, M is dense in the torus. $\diamond$

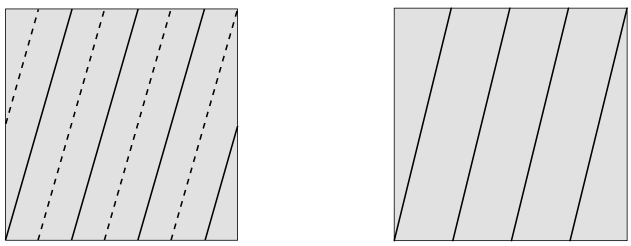


Figure 1.7: Two examples of submanifolds of the torus. On the left, $\alpha = \frac{7}{2}$ and on the right $\alpha = 4$.

Chapter 2

Tangent vectors

2.1 Tangent vectors, tangent spaces

Note. This and the next section do not follow neither Lee's nor Tu's book.

Remark. Given a submanifold M of $\mathbb{R}^n$, one can define tangent vectors to M at some point $p \in M$ as the collection of $\frac{d}{dt}\big|_0 \gamma(t)$ where $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ is smooth and $\operatorname{Im}\gamma \subset M$, $\gamma(0) = p$. This uses the ambient space $\mathbb{R}^n$, hence to define tangent vectors to manifolds we cannot proceed in the same way.

Let M be a smooth manifold of dimension m.

Definition 2.1 (Tangent vector)**.** A tangent vector of M at p is an equivalence class $[\gamma]$ of smooth curves $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = p$, where

$$\gamma_1 \sim \gamma_2 \Leftrightarrow \exists \text{ a chart } (U, \phi) \text{ containing } p \text{ s.t. } (\phi \circ \gamma_1)'(0) = (\phi \circ \gamma_2)'(0).$$

Lemma 2.2. If $(\phi \circ \gamma_1)'(0) = (\phi \circ \gamma_2)'(0)$ for a chart ϕ, then the same holds for all charts ψ.

Proof. The linear map $D_{\phi(p)}(\psi \circ \phi^{-1}) : \mathbb{R}^m \to \mathbb{R}^m$ sends $(\phi \circ \gamma_i)'(0)$ to $(\psi \circ \gamma_i)'(0)$ because of the chain rule, for $i = 1, 2$. Now, as $(\phi \circ \gamma_1)'(0) = (\phi \circ \gamma_2)'(0)$, the vectors obtained applying $D_{\phi(p)}(\psi \circ \phi^{-1})$ must also be the same. $\square$

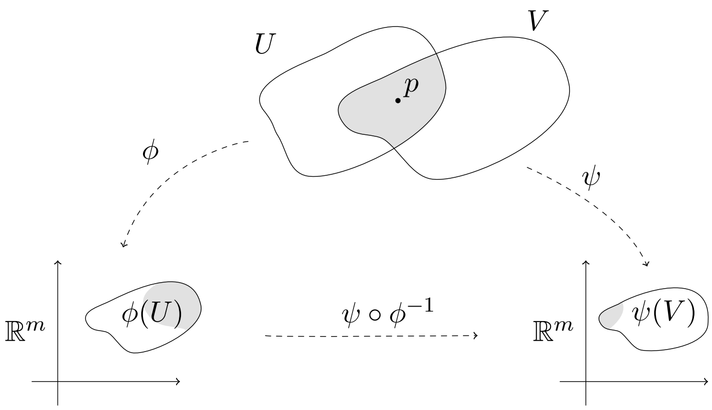


Figure 2.1: Charts in the proof of Lemma 2.2

Definition 2.3 (Tangent space)**.** $\forall p \in M$, the set of all tangent vectors at p is denoted T_pM, and called the tangent space at p.

Proposition 2.4. T_pM is a vector space, of dimension equal to $\dim(M)$.

Proof. Let (U, ϕ) be a chart. We obtain a map

$$T_pM \to \mathbb{R}^m, [\gamma] \mapsto (\phi \circ \gamma)'(0).$$

It's well-defined and injective by the definition of tangent vector. It is also surjective. Indeed, given $v \in \mathbb{R}^m$, take the straight line $t \mapsto \phi(p)+tv$ in $\mathbb{R}^m$ and then apply ϕ^{-1}, to obtain the following curve on M:

$$\gamma(t) = \phi^{-1}(\phi(p) + tv).$$

Now $[\gamma]$ maps to v.

Now that we've proved that this is a bijection, we immediately get a vector space structure on T_pM by "transporting" the one on $\mathbb{R}^m$. Suppose we had started with another chart (V, ψ), then we would have obtained the same vector space structure on T_pM, because

$$D_{\phi(p)}(\psi \circ \phi^{-1}) : \mathbb{R}^m \to \mathbb{R}^m$$

is a linear isomorphism making this diagram commute:

$$\begin{tikzcd}
 & {[\gamma] \in T_pM} \arrow[dl] \arrow[dr] & \\
(\phi \circ \gamma)'(0) \in \mathbb{R}^m \arrow[rr, "D_{\phi(p)}(\psi\circ\phi^{-1})"] & & (\psi \circ \gamma)'(0) \in \mathbb{R}^m
\end{tikzcd}$$

□

Remark. Let $W \subset \mathbb{R}^m$ open, $p \in W$. There is a canonical linear isomorphism $T_pW \to \mathbb{R}^m$, $[\gamma] \to \gamma'(0)$. To see this, take $\phi = \mathrm{Id}$ in the proof of the previous lemma.

When you choose a chart around $p \in M$, you get a basis of T_pM.

Definition 2.5 (Basis of T_pM induced by a chart)**.** Let $p \in M$, (ϕ, U) a chart with $p \in U$ whose components we denote by $x_1, \dots, x_m$ (hence $x_i : U \to \mathbb{R}$). By means of the isomorphism of vector spaces

$$T_pM \to \mathbb{R}^m : [\gamma] \mapsto (\phi \circ \gamma)'(0),$$

the standard basis of $\mathbb{R}^m$ induces a basis of T_pM, which we denote by $\frac{\partial}{\partial x_1}\Big|_p, \dots, \frac{\partial}{\partial x_m}\Big|_p$.

2.2 The derivative of a map

Definition 2.6 (Derivative of a smooth map)**.** If $f : M \to N$ is differentiable, then its derivative at $p \in M$ is

$$(f_*)_p : T_pM \to T_{f(p)}N : [\gamma] \mapsto [f \circ \gamma].$$

Proposition 2.7. $(f_*)_p$ is well defined and linear.

Proof. Let ϕ_M be a chart for M near p. Let ϕ_N be a chart for N near $f(p)$. Consider the commutative diagram

$$\begin{tikzcd}
{[\gamma] \in T_pM} \arrow[r, "(f_*)_p"] \arrow[d] & T_{f(p)}N \ni [f \circ \gamma] \arrow[d] \\
(\phi_M \circ \gamma)'(0) \in \mathbb{R}^m \arrow[r, dashed, "D_{\phi_M(p)}(\phi_N\circ f\circ\phi_M^{-1})"] & \mathbb{R}^n \ni (\phi_N \circ (f \circ \gamma))'(0)
\end{tikzcd}$$

As the two vertical maps are linear isomorphisms (as we saw in the proof of Proposition 2.4) and the horizontal map is linear (being the derivative of a smooth map between open subsets of Euclidean space), the composition $(f_*)_p$ is also linear. $\square$

Proposition 2.8 (Chain rule). If $M \xrightarrow{f} N \xrightarrow{g} L$ are smooth maps, then $\forall p \in M$ we have $(g \circ f)_{*p} = (g_*)_{f(p)} \circ (f_*)_p \colon T_pM \to T_{(g\circ f)(p)}L$.

Remark. Let $p \in M$ and (U, ϕ) a chart around p. One can show that $\phi : U \to \phi(U)$ is a diffeomorphism of manifolds. (Notice that since U is an open set of M and $\phi(U)$ is an open set of $\mathbb{R}^m$, both carry manifold structures). We have a commutative diagram of isomorphisms, where the bottom isomorphism is the one of the last remark.

$$\begin{tikzcd} & {[\gamma] \in T_pM} \arrow[dl] \arrow[dr, "(\phi_*)_p"] & \\ (\phi\circ\gamma)'(0) \in \mathbb{R}^m \arrow[rr, phantom, "\cong"] & & T_{\phi(p)}(\phi(U)) \end{tikzcd}$$

In other words, the map $T_pM \to \mathbb{R}^m$ from the proof of Proposition 2.4, under the identification given in the previous remark, is $(\phi_*)_p$. In particular

$$\left.\frac{\partial}{\partial x^i}\right|_p \in T_pM$$

and the i-th standard basis vector of $T_{\phi(p)}\phi(U) \cong \mathbb{R}^m$ correspond under $(\phi_*)_p$.

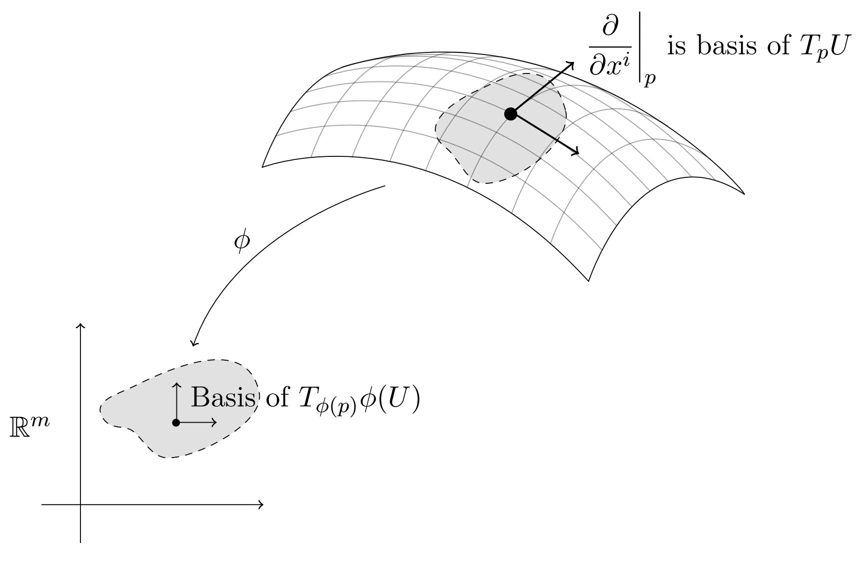


Figure 2.2: The i-th standard basis vector and $\left.\frac{\partial}{\partial x^i}\right|_p$ correspond under $(\phi_*)_p$.

2.3 The regular level set theorem

Remark. Let $f : U \to V$ be a diffeomorphism between open subsets of $\mathbb{R}^n$. Then $\forall q \in U$, $D_q f : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism, because its inverse is $D_{f(q)} f^{-1}$.

Conversely we have:

Lemma 2.9 (Inverse function theorem in $\mathbb{R}^n$). Let $U \subset \mathbb{R}^n$ open, $f : U \to \mathbb{R}^n$ smooth s.t. $D_q f : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism for some $q \in U$. Then there exists a neighbourhood $V \subset U$ of q such that $f|_V : V \to f(V)$ is a diffeomorphism.

Corollary 2.10 (Inverse function theorem for manifolds). Let $f : M \to N$ be a smooth map $p \in M$, such that $f_*(p) : T_pM \to T_{f(p)}N$ is an isomorphism. Then there exists a neighbourhood W of p s.t. the map $f|_W : W \to f(W)$ is a diffeomorphism.

Idea of the proof: this is a local statement, and by means of charts, locally every manifolds can be identified with an open subset of $\mathbb{R}^n$.

Example. Consider $f \colon \mathbb{R} \to S^1 \subset \mathbb{C}, t \mapsto e^{2\pi i t}$. This is not a diffeomorphism. But $f_*(t)$ is an isomorphism for all $t \in \mathbb{R}$. So locally f restricts to a diffeomorphism onto it image. $\diamond$

Given $k \geq n$, we denote

$$\pi : \mathbb{R}^n \times \mathbb{R}^{k-n} \to \mathbb{R}^n, (v, w) \mapsto v.$$

Lemma 2.11 (Submersion theorem in $\mathbb{R}^n$). Let U be a neighbourhood of the origin 0 in $\mathbb{R}^n \times \mathbb{R}^{k-n}$ and $f : U \to \mathbb{R}^n$ smooth such that

$$(D_0 f)|_{\mathbb{R}^n \times \{0\}} : \mathbb{R}^n \times \{0\} \to \mathbb{R}^n$$

is an isomorphism. Then there exists a diffeomorphism τ between neighbourhoods in $\mathbb{R}^n \times \mathbb{R}^{k-n}$ such that $f \circ \tau^{-1} = \pi$.

This theorem states that precomposing f with a diffeomorphism, we can arrange that it becomes the projection on the first components.

Proof. Consider $\tau := (f_1, \dots f_n, x_{n+1}, \dots, x_k) : U \to \mathbb{R}^n \times \mathbb{R}^{k-m}$, which we can write concisely as $(f, \mathrm{Id}_{\mathbb{R}^{k-n}})$. Consider its derivative at the

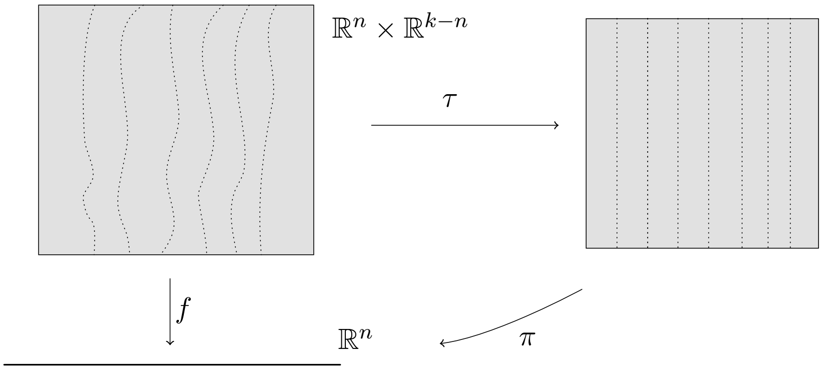


Figure 2.3: The submersion theorem. The dotted lines denote the preimages of points of $\mathbb{R}^n$ under f and π.

origin

$$D_0\tau = \begin{pmatrix} (D_0 f)|_{\mathbb{R}^n \times \{0\}} & * \\ 0 & \mathrm{Id}_{\mathbb{R}^{k-n}} \end{pmatrix}.$$

This matrix is invertible, because it is an upper block matrix, $(D_0 f)|_{\mathbb{R}^n \times \{0\}}$ is invertible, and $\mathrm{Id}_{\mathbb{R}^{k-n}}$ is invertible. This means that τ is a diffeomorphism near 0, by the inverse function theorem. We have $\pi \circ \tau = f$. $\square$

Definition 2.12 (Regular value)**.** Given a smooth map $f : M \to N$, a point $c \in N$ is a regular value iff $\forall p \in f^{-1}(c)$, the derivative $(f_*)_p : T_pM \to T_cN$ is *surjective.*

Theorem 2.13. Let $f : M^k \to N^n$ be a smooth, and let $c \in N$ be a regular value s.t. $f^{-1}(c) \neq \emptyset$. Then

- $f^{-1}(c)$ is a submanifold of M with dimension $k - n$
- $\forall p \in f^{-1}(c) : T_p(f^{-1}(c)) = \mathrm{Ker}(f_*(p))$

The first item states that the codimension is preserved when taking inverse images.

Example. Consider the "height function" $f : S^2 \to \mathbb{R} : (x, y, z) \mapsto z$. Note that -1 and 1 are not regular values. For all $c \in (-1, 1)$, the preimage $f^{-1}(c)$ is a submanifold of S^2, diffeomorphic to a circle. $\diamond$

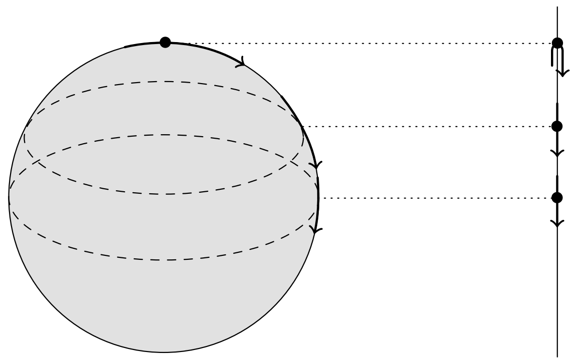


Figure 2.4: The “height function” $f : S^2 \to \mathbb{R}$. The derivative of f vanishes at the Northpole and Southpole.

Proof. Fix $p \in f^{-1}(c)$. Take charts (U, ϕ_M) near p, and (V, ϕ_N) near $f(p) = c$, chosen so that $\phi_N(c) = 0$. We know that $D_{\phi(p)}(\phi_N \circ f \circ \phi_M^{-1}) : \mathbb{R}^k \to \mathbb{R}^n$ is surjective (because c is a regular value). We can assume that

$$D_{\phi(p)}(\phi_N \circ f \circ \phi_M^{-1})\Big|_{\mathbb{R}^n \times \{0\}}$$

is an isomorphism. (When we restrict a surjective linear map to a subspace transverse to the kernel, it becomes an isomorphism. If $\mathbb{R}^n \times \{0\}$ is not transverse to the kernel of $D_{\phi(p)}(\phi_N \circ f \circ \phi_M^{-1})$, we can change the chart ϕ_N by composing it with e.g. a rotation in $\mathbb{R}^n$.)

By the last lemma (Submersion theorem in $\mathbb{R}^n$), there exists a diffeomorphism τ such that $(\phi_N \circ f \circ \phi_M^{-1}) \circ \tau^{-1} = \pi$. This can be rewritten as

$$\phi_N \circ f \circ (\tau \circ \phi_M)^{-1} = \pi.$$

The situation is summarized by the following diagram:

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\tau\circ\phi_M"] & V \ni c \arrow[d, "\phi_N"] \\ \text{open} \subset \mathbb{R}^k \arrow[r, dashed, "\pi"] & \text{open} \subset \mathbb{R}^n \ni 0 \end{tikzcd}.$$

Hence $\tau \circ \phi_M$ is a chart of M adapted to $f^{-1}(c)$.

For the second part, let $p \in f^{-1}(c)$, and take a path $\gamma : (-\varepsilon, \varepsilon) \to f^{-1}(c)$ with $\phi(0) = p$. Then

$$(f_*)_p[\gamma] = [f \circ \gamma] = 0 \in T_cN$$

as $f \circ \gamma \equiv c$. This shows that $T_p(f^{-1}(c)) \subset \mathrm{Ker}(f_*)_p$. Both vector spaces have the same dimension, which gives the equality. $\square$

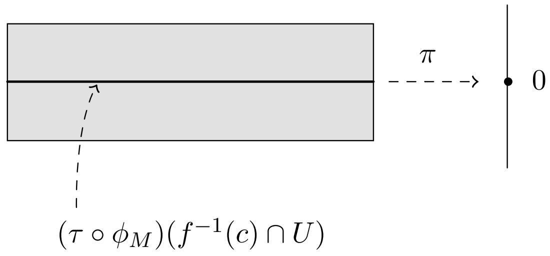


Figure 2.5: The chart $\tau \circ \phi_M$ straightens out $f^{-1}(c)$ (intersected with the domain of the chart).

2.4 Tangent vectors as derivations at a point

Fix $M = \mathbb{R}^n$. Recall that a tangent vector at p is an element of $T_p\mathbb{R}^n \cong \mathbb{R}^n$.

Definition 2.14 (Derivation at a point)**.** A derivation at a point $p \in \mathbb{R}^n$ is a linear map $D : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ satisfying the Leibniz rule:

$$D(fg) = D(f)g(p) + f(p)D(g).$$

Example. $\forall v \in \mathbb{R}^n$, the directional derivative

$$\begin{aligned} C^\infty(\mathbb{R}^n) &\longrightarrow \mathbb{R} \\ f &\longmapsto (d_p f)(v) = \sum_i v_i \frac{\partial f}{\partial x_i}(p) \end{aligned}$$

is a derivation at p. This follows from the fact that partial derivatives obey the Leibniz rule. $\diamond$

Remark. For a constant function c, we have $Dc = 0$. Because D is linear, it is enough to show this for $c = 1$. We have $D1 = (D1)1 + 1(D1) = 2(D1)$, so $D1 = 0$.

Proposition 2.15. The map

$$\begin{aligned} \phi : T_p\mathbb{R}^n &\longrightarrow \text{Derivations at } p \\ v &\longmapsto \Big(f \mapsto \sum_i v_i \frac{\partial f}{\partial x_i}(p)\Big) \end{aligned}$$

is an isomorphism of vector spaces.

Proof.

- This formula really defines a derivation at p, as we've showed in the previous example.
- It is clearly linear.
- To show that it's injective, we check that the kernel is 0. If for all f, $\sum_i v_i \frac{\partial f}{\partial x_i}(p) = 0$, then it is particular true for the functions x_j, so $v_j = 0$ for all j. In formulae: $v_j = \sum_i v_i \frac{\partial x_j}{\partial x_i}(p) = 0$.
- Surjectivity. Let D be a derivation at p. For all $f \in C^\infty(\mathbb{R}^n)$, we have
$$f(x) = f(p) + \sum_i (x_i - p_i) g_i(x),$$
where $g_i(x)$ is a smooth function with $g_i(p) = \frac{\partial f}{\partial x_i}(p)$. (This is a version of Taylor's theorem.) Then
$$\begin{aligned} Df &= 0 + \sum_i D(x_i) g_i(p) + 0 \\ &= \sum_i D(x_i) \frac{\partial f}{\partial x_i}(p). \end{aligned}$$
So $v = (D(x_1), \ldots, D(x_n))$ maps to D.

□

Chapter 3

Vector fields

3.1 Vector fields

Definition 3.1 (Vector field). A vector field on a manifold M is a map $X : M \to \bigcup_{p \in M} T_pM$, such that

- $X(p) \in T_pM$
- X satisfies the following smoothness condition: for any chart (U, ϕ), writing $X(p) = \sum_i a_i(p) \frac{\partial}{\partial x_i}\Big|_p$, all the coefficients $a_i : U \to \mathbb{R}$ are smooth.

Notation. We denote the set of all vector fields on M with $\mathfrak{X}(M)$.

Example. Let (U, ϕ) be a chart on M. Then $\frac{\partial}{\partial x_i}$ is a vector field on U, for all $i = 1, \ldots, \dim(M)$.

If $U \subset \mathbb{R}^n$ is open, using the chart Id, $\frac{\partial}{\partial x_1}$ is just the vector field with unit vectors pointing in the x_1 direction. $\diamond$

3.2 Integral curves

Definition 3.2 (Integral curve). Let $X \in \mathfrak{X}(M)$. A smooth curve $\gamma : (a, b) \to M$ is an integral curve of X iff

$$\dot{\gamma}(t) = X|_{\gamma(t)}$$

for all $t \in (a, b)$.

Remark. Here $\dot{\gamma}(t)$ is defined as $(\gamma_*)_t(1)$, where $(\gamma_*)_t : T_t(a, b) \to T_{\gamma(t)}M$, and $T_t(a, b) \cong \mathbb{R}$, so using 1 as an input is valid. Another way to look at it, in terms of tangent vectors as equivalence classes of curves: $\dot{\gamma}(t)$ equals $[s \mapsto \gamma(s + t)]$.

Proposition 3.3. Let X be a vector field, $p \in M$. Then there exists a neighborhood U of p, an $\varepsilon > 0$ and a unique smooth map

$$F : U \times (-\varepsilon, \varepsilon) \to M$$

s.t. for all $q \in U$, the curve γ_q defined by $\gamma_q(t) = F(q,t)$ is an integral curve of X with $\gamma_q(0) = q$.

Proof. Fix a chart (ϕ, V) near p. In these coordinates, we have $X = \sum_i f_i(x) \frac{\partial}{\partial x^i}$ for some smooth functions f_i. We need to show that there exists a neighboorhood $W \subset \phi(V)$ of $\phi(p)$, $\exists \varepsilon > 0$ and $\exists!\ \ y : W \times (-\varepsilon, \varepsilon) \to \phi(V)$ such that

$$\begin{cases} \frac{\partial}{\partial t} y(x,t) = f(y(x,t)) \\ y(x,0) = x \end{cases}$$

for all x. This holds by the fundamental theorem of ODE's. (It says that for each initial value x, there is a unique solution defined on a small interval $(-\varepsilon, \varepsilon)$, and the solution varies smoothly with x.) $\square$

Remark. The map F in the above proposition is called *flow*.

Remark. In particular, for all $p \in M$, there exists an $\varepsilon > 0$ and a unique integral curve γ_p of X defined on $(-\varepsilon, \varepsilon)$, starting at the point p.

Example. Sometimes, we cannot continue the curve. For example, look at $\mathbb{R}^2 \setminus \{(0,0)\}$ and $X = \frac{\partial}{\partial x_1}$. Then the integral curve starting at $(-2,0)$ is defined only up to time $t = 2$. $\diamond$

Assume, for the sake of simplicity, that the flow of X is defined on $M \times \mathbb{R}$. Then F is a 1-parameter group of diffeomorphisms, as a consequence of the uniqueness in the above proposition.

Definition 3.4 (1-parameter group of diffeomorphisms)**.** A 1-parameter group of diffeomorphisms on M is a smooth map $F \colon M \times \mathbb{R} \to M$ such that, using the notation $F_t(p) = F(p,t)$ (think of it as fixing t and varying the point), one has

- $F_s \circ F_t = F_{s+t} \ \ \forall s,t$
- $F_0 = \mathrm{Id}$

(It then follows that F_t is a diffeomorphism for all t.)

Remark. There is a bijection between

- vector fields on M whose flow is defined on $M \times \mathbb{R}$ (the biggest possible domain), and

- 1-parameter groups of diffeomorphisms on M.

The bijection reads:

$$\begin{aligned} X &\longmapsto \text{flow } F \text{ as above} \\ X \text{ given by } X(q) = \tfrac{d}{dt}\big|_0 F_t(q) &\longleftarrow F\colon M \times \mathbb{R} \to M. \end{aligned}$$

Example. On $\mathbb{R}^2$ take $X = \frac{\partial}{\partial x_1}$. Then the flow is $F_t(x_1, x_2) = (x_1 + t, x_2)$. ◇

Example. On $\mathbb{R}^2$ let $X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$. Then the images of integral curves are circles. The flow is given by $F_t(x, y) = R_t(x, y)$, where R_t denotes the rotation by the angle t. ◇

3.3 The Lie bracket of vector fields

Recall that $C^\infty(M) = \{\text{smooth functions from } M \text{ to } \mathbb{R}\}$.

Remark. $C^\infty(M)$ is an algebra: it is a vectorspace, but we can also multiply two functions.

Definition 3.5 (Derivation of $C^\infty(M)$)**.** A derivation of $C^\infty(M)$ is a linear map $D : C^\infty(M) \to C^\infty(M)$ such that

$$D(fg) = D(f)g + fD(g).$$

Remark. If D_1, D_2 are derivations of $C^\infty(M)$, then the commutator $D_1 \circ D_2 - D_2 \circ D_1$ is also a derivation. But $D_1 \circ D_2$ on its own is not a derivation!

Proposition 3.6. There is a linear map

$$\begin{aligned} \Phi : \mathfrak{X}(M) &\longrightarrow \text{Derivations of } C^\infty(M) \\ X &\longmapsto (f \mapsto f_*(X)). \end{aligned}$$

Here $f_*(X)$ is the function on M given by $(f_*(X))_p = (f_*)_p(X|_p) \in T_{f(p)}\mathbb{R} \cong \mathbb{R}$. We denote $f_*(X) =: X(f)$.

Proof. We'll show that $\forall X \in \mathfrak{X}(M)$, $\Phi(X)$ is a derivation. Let $f, g \in C^\infty(M)$, $p \in M$. Let γ be a curve in M such that $\gamma(0) = p, [\gamma] = X_p$.

Then

$$\begin{aligned}
(fg)_{*p}(X_p) &= [(fg)\circ\gamma] \in T_{(fg)(p)}\mathbb{R} \\
&= ((fg)\circ\gamma)'(0) \in \mathbb{R} \\
&= \big((f\circ\gamma)\cdot(g\circ\gamma)\big)'(0) \\
&= (f\circ\gamma)'(0)\cdot g(\gamma(0)) + f(\gamma(0))\cdot(g\circ\gamma)'(0) \\
&= (f_*)_p(X|_p)\cdot g(p) + f(p)\cdot(g_*)_p(X|_p).
\end{aligned}$$

In the second-last equality we applied the product rule of calculus. □

Remark. Φ is an isomorphism of vector spaces. This can be showed using the material in §2.4.

Definition 3.7 (Lie bracket of vector fields). The Lie bracket of two vector fields $X, Y \in \mathfrak{X}(M)$ is

$$[X, Y] := X \circ Y - Y \circ X,$$

using the identification between $\mathfrak{X}(M)$ and the derivations of $C^\infty(M)$.

Definition 3.8 (Lie algebra). A Lie algebra is a vector space $\mathfrak{g}$ with a bilinear map $[\cdot,\cdot] : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ such that

- $[X, Y] = -[Y, X]$ *Skew symmetry*
- $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ *Jacobi Identity*

Example. $(\mathfrak{X}(M), [\cdot,\cdot])$ is a Lie algebra. ◇

Example. The square matrices $M(n, \mathbb{R})$ with $[A, B] := AB - BA$ form a Lie algebra. The Jacobi identity holds as a consequence of the associativity of matrix multiplication. ◇

Remark. Let (U, ϕ) be a chart on M. Then we get vector fields $\frac{\partial}{\partial x_i}$ on $U \subset M$.

- The vector field $\frac{\partial}{\partial x_i}$, seen as a derivation, maps $x_j \in C^\infty(U)$ to δ_{ij}
- If $X = \sum_i a_i \frac{\partial}{\partial x_i}$, $Y = \sum_i b_i \frac{\partial}{\partial x_i}$, with $a_i, b_i \in C^\infty(U)$, then

$$[X, Y] = \sum_j \left(\sum_i \left(a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i}\right)\right) \frac{\partial}{\partial x_j}.$$

  This can be seen by applying $[X, Y]$ to the functions x_i. In particular $\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0$, because the a_i and b_i are constants here.

Definition 3.9 (F-related vector fields)**.** Let $F : M \to N$ be a smooth map and $X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(N)$. We say that X and Y are F-related iff

$$(F_*)_p(X_p) = Y_{F(p)}$$

for all points $p \in M$.

Remark. Equivalently: X and Y are F-related iff $\forall g \in C^\infty(N)$ we have $X(F^*g) = F^*(Y(g))$. Here F^* is the pullback of functions, i.e. $F^*(g) = g \circ F$.

Proposition 3.10. Suppose X_i is F-related to Y_i for $i = 1, 2$. Then $[X_1, X_2]$ is F-related to $[Y_1, Y_2]$.

Proof. Use $X_1(X_2(F^*(g)) = X_1(F^*(Y_2(g))) = F^*(Y_1(Y_2(g)))$. $\square$

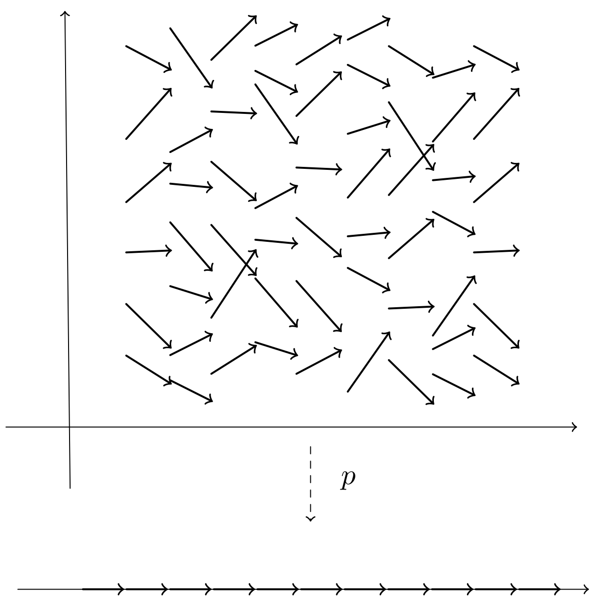


Figure 3.1: Two p-related vector fields, where $p \colon \mathbb{R}^2 \to \mathbb{R}$ is the first projection.

3.4 Interpretation of the Lie bracket

Definition 3.11 (Pushforward of a vector field by a diffeomorphism)**.** Given a diffeomorphism $\phi : M \to N$ and $X \in \mathfrak{X}(M)$, we denote by $\phi_* X$ the unique vector field on N such that X is ϕ-related to $\phi_* X$.

Explicitly, we have $(\phi_* X)_{\phi(p)} = (\phi_*)_p(X_p)$ for all $p \in M$.

Lemma 3.12. Let $\phi : M \to N$ be a diffeomorphism, let $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$. Then $Y = \phi_*(X)$ iff

$$F_t^Y \circ \phi = \phi \circ F_t^X \qquad \text{for all } t \text{ s.t. } F_t^X \text{ is defined.}$$

Here F^X denotes the flow of X, and similarly for Y.

Definition 3.13 (Lie derivative)**.** Let $X, Y \in \mathfrak{X}(M)$. The Lie derivative of Y in the direction of X is the vector field

$$\mathcal{L}_X Y = \frac{d}{dt}\Big|_{t=0} (F_{-t}^X)_* Y.$$

Remark. At every point p,

$$(\mathcal{L}_X Y)_p = \frac{d}{dt}\Big|_{t=0} [(F_{-t}^X)_* Y]_p = \frac{d}{dt}\Big|_{t=0} [(F_{-t}^X)_* Y_{F_t(p)}].$$

These are all tangent vectors in $T_p M$.

Remark. One can show that $\mathcal{L}_X Y = [X, Y]$.

Proposition 3.14. Let $X, Y \in \mathfrak{X}(M)$. The following are equivalent:

a) $\mathcal{L}_X Y = 0$

b) $(F_t^X)_* Y = Y$ for all t

c) The flows of X and Y commute: $F_t^X \circ F_s^Y = F_s^Y \circ F_t^X$ for all t, s.

Proof.
- *b)* $\Rightarrow$ *a)*. $\mathcal{L}_X Y = \frac{d}{dt}|_0 (F_{-t}^X)_* Y = \frac{d}{dt}|_0 Y = 0$.
- *a)* $\Rightarrow$ *b)*. Fix a point p. Consider $t \mapsto (F_{-t}^X)_* Y_{F_t^X(p)}$. This is a curve in $T_p M$, which at $t = 0$ equals Y_p. We will show that this is

a constant curve by taking the derivative. For all t_0:

$$\begin{aligned}\frac{d}{dt}\Big|_{t_0}(F^X_{-t})_*(Y_{F^X_t(p)}) &= \frac{d}{ds}\Big|_{s=0}(F^X_{-t_0-s})_*(Y_{F^X_{t_0+s}(p)})\\ &= (F^X_{-t_0})_*\Big(\frac{d}{ds}\Big|_{s=0}(F^X_{-s})_*Y_{F^X_s(F^X_{t_0}(p))}\Big)\\ &= (F^X_{-t_0})_*\Big((\mathcal{L}_XY)_{F^X_{t_0}(p)}\Big) = 0,\end{aligned}$$

where in the first equality we set $t = t_0 + s$.

- *b)* $\Leftrightarrow$ *c)*. For all t, apply the last lemma to $F^X_t : M \to M$.

□

Corollary 3.15. $[X, Y] = 0$ iff the flows commute.

Example. On $\mathbb{R}^2$, we know $\left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right] = 0$. Indeed, the flows given by $(\mathbf{x}, t) \mapsto \mathbf{x} + (t, 0)$ and $(\mathbf{x}, t) \mapsto \mathbf{x} + (0, t)$ commute. ◇

Proposition 3.16. Let $V_1, \dots, V_k$ be pointwise linearly independent vector fields on M such that $[V_i, V_j] = 0$. Then for every $p \in M$ there is a chart $(U, (s_1, \dots, s_n))$ centered at p such that $V_i = \frac{\partial}{\partial s_i}$ for $i = 1, \dots, k$.

Proof. Since this is a local statement, let's assume that we're working in $\mathbb{R}^n$, and $p = 0$. Assume that $\operatorname{span}_i\{V_i|_0\} \oplus (\{0\} \times \mathbb{R}^{n-k}) = \mathbb{R}^n$. Denote by θ_i the flow of V_i and let Ω be a small neighborhood of the origin in $\{0\} \times \mathbb{R}^{n-k}$. Define

$$\begin{aligned}\Phi : (-\varepsilon, \varepsilon)^k \times \Omega &\longrightarrow \mathbb{R}^n\\ (s_1, \dots, s_k, s_{k+1}, \dots, s_n) &\longmapsto (\theta_1)_{s_1} \circ \cdots \circ (\theta_k)_{s_k}(0, \dots, 0, s_{k+1}, \dots, s_n)^T.\end{aligned}$$

Then $\frac{\partial}{\partial s_i}$ is Φ-related to V_i for $i \le k$. Indeed for $i \le k$ you can compute that

$$(D_{(s_1,\dots,s_n)}\Phi)\Big(\frac{\partial}{\partial s_i}\Big) = V_i|_{\Phi(s_1,\dots,s_n)}$$

by using the curve $\varepsilon \mapsto (s_1, \dots, s_i + \varepsilon, \dots, s_n)$ and using the fact that the flows θ_j commute by the last corollary. At the point $p = 0$ we clearly have $(D_0\Phi)(\frac{\partial}{\partial s_i}) = \frac{\partial}{\partial x_i}|_0$ for $i > k$. Hence $d_0\Phi$ is an isomorphism.

By the inverse function theorem, there exists a neighbourhood V of 0 such that $\Phi|_V : V \to \Phi(V) =: U$ is a diffeomorphism. The desired chart is the inverse of this diffeomorphism. □

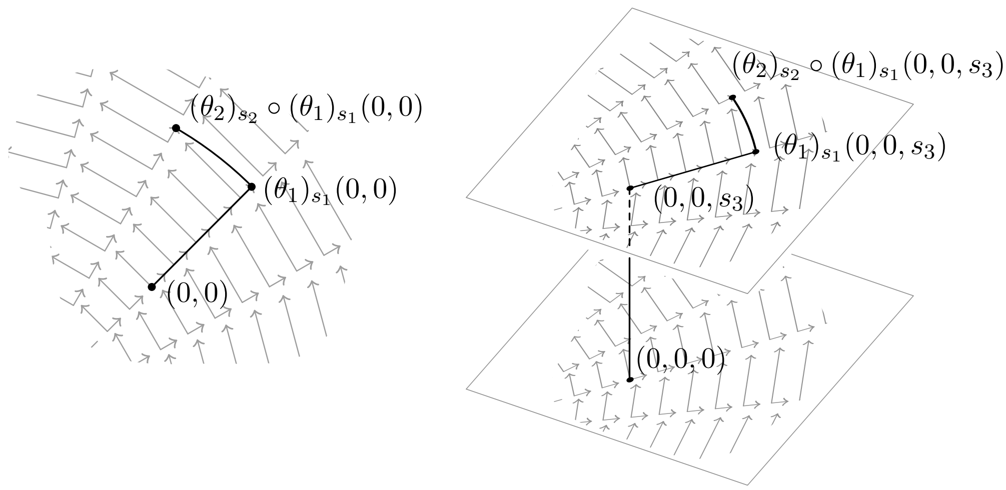


Figure 3.2: Two examples illustrating the proof. On the left, $n = k = 2$. On the right, $n = 3$ and $k = 2$.

Chapter 4

Bundles

4.1 Fiber bundles

Definition 4.1 (Fiber bundle). Let F be a manifold. A fiber bundle with typical fiber F is a smooth surjective map

$$\pi : E \to B$$

between manifolds s.t. for all $x \in B$, there exists an open neighbourhood U and a diffeomorphism $\psi : \pi^{-1}(U) \to U \times F$ making this diagram commute:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\psi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd}$$

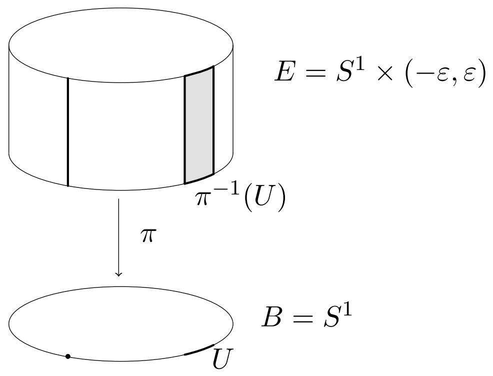


Figure 4.1: Example of a product fiber bundle

Remark. The map $\pi : E \to B$ is a submersion, i.e. $(\pi_*)_e$ is surjective in all points e of E. Indeed: locally, the map π looks like the projection onto the first factor π_1, as ψ is a diffeomorphism. So we can work with π_1, which is a submersion.

Remark. For all $x \in B$, the *fiber* $\pi^{-1}(x)$ is diffeomorphic to F. Indeed, $\pi^{-1}(x) \approx \pi_1^{-1}(x) = \{x\} \times F \approx F$. So we have a family of manifolds (the fibers) which all are diffeomorphic to F, and this family is parametrized by B.

E is called the *total space*, B the *base space* and ψ a *local trivilization*. A *section* is a smooth map $f : B \to E$ such that $\pi \circ f = Id_B$.

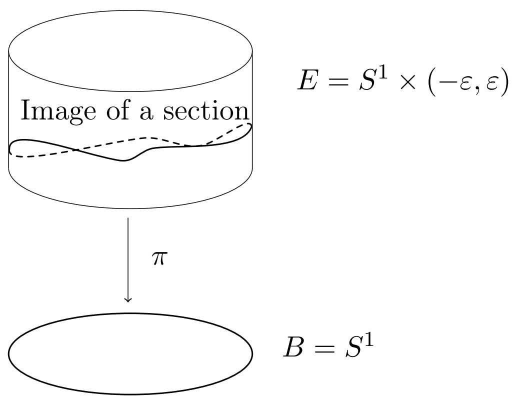


Figure 4.2: Definition of a section

Example. Given manifolds F and B, the product bundle is $B \times F \xrightarrow{\pi_1} B$. $\diamond$

Example. Let F be a manifold, $\phi : F \to F$ a diffeomorphism. Define the manifold $E := ([0,1] \times F)/\sim$, where $\sim$ is given by

$$(0, p) \sim (1, \phi(p)).$$

Let $B = S^1$. Then $\pi : E \to B$ is a fiber bundle with typical fiber F.

For instance, take $F = (-\varepsilon, \varepsilon)$, $\phi = -\operatorname{Id}$. Then E is the Möbius strip.

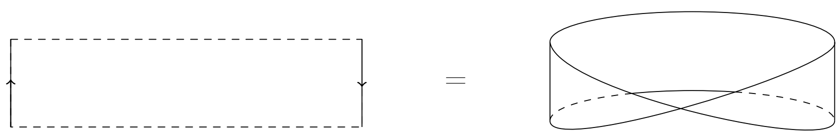


Figure 4.3: Möbius strip

◇

Example. The first projection $\pi_1 : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}, (x, y) \mapsto x$ is *not* a fiber bundle. A reason is that not all fibers are diffeomorphic. ◇

4.2 Vector bundles

Definition 4.2 (Vector bundle). Fix $n \in \mathbb{N}_{\geq 0}$. A vector bundle of rank n is a smooth surjective map $\pi : E \to B$ between manifolds E, B such that

- $E_p = \pi^{-1}(p)$ is a n-dimensional vector space for all $p \in B$
- For all $p \in B$, there exist a neighborhood U of p and a diffeomorphism $\psi : E|_U := \pi^{-1}(U) \to U \times \mathbb{R}^n$ such that the following diagram commutes

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\psi"] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^n \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd}$$

  and $\psi|_{E_q} : E_q \to \{q\} \times \mathbb{R}^n$ is a linear isomorphism, for all $q \in U$.

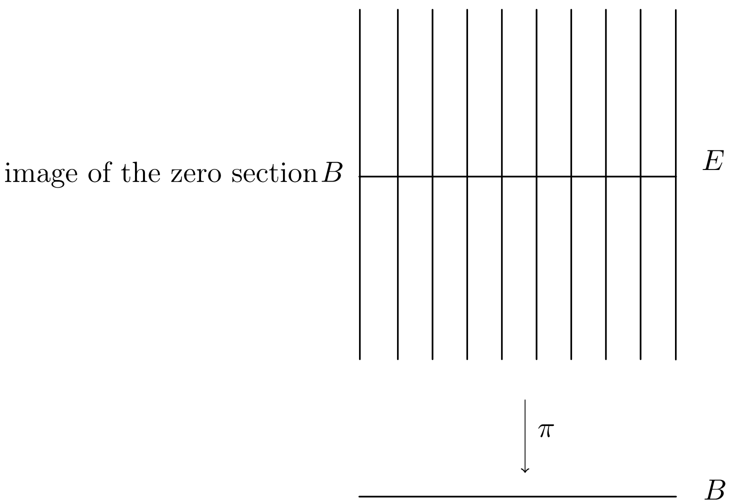


Figure 4.4: Definition of a vector bundle

Remark. Equivalently, a vector bundle of rank n is a fiber bundle s.t. each fiber is an n-dimensional vector space and trivializations are linear in each fiber.

Remark.
- There exists a canonical section $B \to E, q \mapsto$ (zero vector in E_q). It is called zero section.
- Denote by $\Gamma(E)$ the set of all sections of E, it is a vector space. Note that we can also multiply a section $B \to E$ with a function $B \to \mathbb{R}$. Hence $\Gamma(E)$ is a module over the algebra $C^\infty(B)$.

Example. Given a manifold B, take $B \times \mathbb{R}^n \xrightarrow{\pi_1} B$. Notice $\Gamma(B \times \mathbb{R}^n) = C^\infty(B, \mathbb{R}^n)$. ◇

Proposition 4.3. Let B be a manifold of dimension n. Then

$$TB := \bigsqcup_{p \in B} T_pB$$

is naturally a vector bundle of rank n, called the *tangent bundle of B*

Proof. Define $\pi : TB \to B, v \in T_pB \mapsto p$. Notice that $\pi^{-1}(p) = T_pB$ is a vector space.

We show that TB is a manifold. Consider a chart $\phi : U \to \phi(U)$ of B, denote its components by $(x_1, \ldots, x_n)$. It induces a bijection

$$\begin{aligned} \psi : (TB)|_U &\longrightarrow U \times \mathbb{R}^n \\ \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}\Big|_p &\longmapsto (p, a_i). \end{aligned}$$

Now take a cover of B by charts $(U_\alpha, \phi_\alpha)_{\alpha \in A}$. Consider the topology on TB with basis $\psi_\alpha^{-1}(\sigma)$ for $\sigma \subset U_\alpha \times \mathbb{R}^n$ open and $\alpha \in A$. Notice that $\psi_\beta \circ \psi_\alpha^{-1}$ is a smooth map from $(U_\alpha \cap U_\beta) \times \mathbb{R}^n$ to itself. Then $\{((TB)|_{U_\alpha}), \psi_\alpha\}_{\alpha \in A}$ is a[a] smooth atlas. Hence TB is a smooth manifold. For all $\alpha \in A$, the chart $\psi_\alpha : (TB)|_{U_\alpha} \to U_\alpha \times \mathbb{R}^n$ is a diffeomorphism and linear on every fiber. □

[a] Strictly speaking, ψ_α is not a chart, because $U_\alpha \times \mathbb{R}^n$ is not an open subset of $\mathbb{R}^n \times \mathbb{R}^n$. But we can identify U_α with $\phi_\alpha(U_\alpha)$, which is an open subset of $\mathbb{R}^n$.

Remark. $\Gamma(TB) = \mathfrak{X}(B)$

Remark. Let $\pi_1 : E_1 \to B$ and $\pi_2 : E_2 \to B$ be a vector bundle with the same base. Their direct sum (Whitney sum) is a vector bundle

$$E_1 \oplus E_2 \to B$$

with fiber over $p \in B$ given by $(E_1 \oplus E_2)_p = (E_1)_p \oplus (E_2)_p$. Trivialisations are given by taking the direct sum of the trivializations of E_1 and of E_2.

Definition 4.4 (Vector subbundle). Let $\pi : E \to B$ be a vector bundle. A vector subbundle is a subset $D \subset E$ s.t. $\pi|_D : D \to B$, with the induced smooth structure and vector space structure on the fibers, is a vector bundle.

Definition 4.5 (Vector bundle morphism). Let $\pi_i : E_i \to B_i$ be vector bundles for $i = 1, 2$. A smooth map $F : E_1 \to E_2$ is a vector bundle morphism

- if there exists a smooth map $f : B_1 \to B_2$ such that the following diagram commutes

$$\begin{tikzcd} E_1 \arrow[r, "F"] \arrow[d, "\pi_1"] & E_2 \arrow[d, "\pi_2"] \\ B_1 \arrow[r, dashed, "f"] & B_2 \end{tikzcd}$$

- and if for all $p \in B_1$, the map $F|_{(E_1)_p} : (E_1)_p \to (E_2)_{f(p)}$ is linear.

We call F an isomorphism if F is invertible and if F, F^{-1} are both vector bundle morphisms (or equivalently, if F is a diffeomorphism).

Definition 4.6 (Frame). Given a vector bundle $E \to B$, a frame is a collection of sections that form a basis at every point.

Remark. E admits a frame iff E is isomorphic to the product vector bundle $B \times \mathbb{R}^n$. In this case we call E a trivial vector bundle.

Example. Let $S^n \subset \mathbb{R}^{n+1}$ be the unit sphere. The tangent bundle TS^n and its orthogonal $(TS^n)^\perp$ are vector bundles over S^n.

- $(TS^n)^\perp$ is a trivial bundle. Indeed, we can choose a unit normal vector at each point as a frame.
- TS^n is not a trivial bundle in general. (TS^n only admits a frame when $n = 1, 3$. The fact that TS^2 does not admit a frame is a consequence of the Hairy ball theorem.)
- The Whitney sum $TS^n \oplus (TS^n)^\perp$ is isomorphic to $\mathbb{R}^{n+1} \times S^n$, which is a trivial vector bundle. So this is an example of a sum of a trivial and a non-trivial vector bundle being trivial.

◇

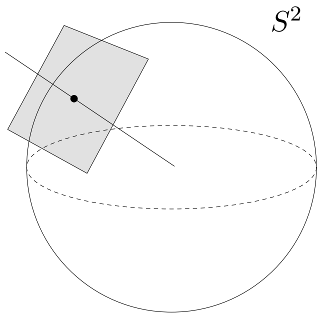


Figure 4.5: The Whitney sum of $TS^2 \oplus (TS^2)^\perp$ is a trivial vector bundle.

Chapter 5

Differential forms and integration

5.1 Forms on vector spaces

Let V be a finite dimensional real vector space.

Definition 5.1. For every $k \geq 1$, we define

$$\bigwedge^k V^* := \{\omega : \underbrace{V \times \cdots \times V}_{k \text{ times}} \to \mathbb{R} \mid \text{multilinear and skew symmetric}\}$$

Skew symmetric means: $\omega(\ldots, v, \ldots, w, \ldots) = -\omega(\ldots, w, \ldots, v, \ldots)$. We define $\bigwedge^0 V^* = \mathbb{R}$

Example. $\bigwedge^1 V^* = V^*$ ◇

Example. $\bigwedge^2 V^* =$ bilinear maps $V \times V \to \mathbb{R}$ that are skew symmetric. ◇

Definition 5.2 (Wedge product)**.** $\forall \ell, k \geq 0$, the wedge product is

$$\wedge : \bigwedge^k V^* \times \bigwedge^\ell V^* \longrightarrow \bigwedge^{\ell+k} V^*$$

$$(\omega \wedge \tau)(v_1, \ldots, v_{k+\ell}) \longmapsto \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} (-1)^\sigma \omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \cdot \tau(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}),$$

where $S_{k+\ell}$ denotes the permutation group.

Remark. $\omega \wedge \tau = (-1)^{k\ell} \tau \wedge \omega$, where $\omega \in \bigwedge^k V^*$ and $\tau \in \bigwedge^\ell V^*$.

Lemma 5.3. Suppose $\theta_1, \cdots, \theta_k \in V^*$ and $v_1, \ldots, v_k \in V$. We have

$$(\theta_1 \wedge \cdots \wedge \theta_k)(v_1, \ldots, v_k) = \det(\theta_i(v_j)).$$

Lemma 5.4. Let $m = \dim V$. Suppose $\theta_1, \ldots, \theta_m$ is a basis of V^*. Then $\forall k \geq 1$,

$$\{\theta_{i_1} \wedge \cdots \wedge \theta_{i_k} \mid 1 \leq i_1 < \ldots < i_k \leq m\}$$

is a basis of $\bigwedge^k V^*$.

Remark. If $k > m$, the dimension of the space, then $\bigwedge^k V^*$ is the zero vector space.

Example. Suppose $\dim V = 3$ and $(\theta_1, \theta_2, \theta_3)$ is a basis of V^*, then

- 1 is a basis of $\bigwedge^0 V^* = \mathbb{R}$
- $\{\theta_1, \theta_2, \theta_2\}$ is basis of $\bigwedge^1 V^* = V^*$
- $\{\theta_1 \wedge \theta_2, \theta_2 \wedge \theta_3, \theta_1 \wedge \theta_3\}$ is a basis of $\bigwedge^2 V^*$
- $\theta_1 \wedge \theta_2 \wedge \theta_3$ is a basis of $\bigwedge^3 V^*$.

◇

Definition 5.5 (Pullback)**.** Let $f : V \to W$ be a linear map. Then the dual map is $f^* : W^* \to V^*$, given by $(f^*\theta)(v) = \theta(f(v))$. More generally, $\forall k \geq 1$, the pullback by f is

$$\begin{aligned} f^* : \bigwedge^k W^* &\longrightarrow \bigwedge^k V^* \\ \omega &\longmapsto f^*\omega \end{aligned}$$

where

$$(f^*\omega)(v_1, \ldots, v_k) = \omega(f(v_1), \ldots, f(v_k)).$$

5.2 Differential forms on manifolds

Let M be a manifold, $f \in C^\infty(M)$. For all $p \in M$, define

$$(df)_p := (f_*)_p : T_pM \to T_{f(p)}\mathbb{R} = \mathbb{R}$$

This is a linear map, i.e. $(df)_p \in (T_pM)^* =: T_p^*M$

Let $(U, \phi = (x_1, \ldots, x_n))$ be a chart.

Lemma 5.6. The set $\{dx_1|_p, dx_2|_p, \ldots, dx_n|_p\}$ is a basis of T_p^*M. It is the basis dual to $\left\{\frac{\partial}{\partial x_i}\Big|_p\right\}$

Proof. We have that $\frac{\partial}{\partial x^i}\Big|_p = (\phi^{-1})_{*,\phi(p)}e_i$, where e_i is the i-th standard basis vector. Hence, for every j,

$$dx_j|_p\left(\frac{\partial}{\partial x^i}\Big|_p\right) = dx_j|_p\left((\phi^{-1})_{*,\phi(p)}e_i\right) = (x_j \circ \phi^{-1})_{*,\phi(p)}e_i.$$

Since $x_j \circ \phi^{-1}$ is the jth projection, this expression is the jth component of e_i, which is δ_{ij}. □

Example. Consider $\mathbb{R}^2$ with standard coordinates x_1, x_2. Then $dx_1|_p \colon T_p\mathbb{R}^2 \to \mathbb{R}$ is given by $\frac{\partial}{\partial x_1}|_p \mapsto 1$ and $\frac{\partial}{\partial x_2}|_p \mapsto 0$. ◇

Let $k \geq 0$. For every $p \in M$, consider the vector space $\bigwedge^k T_p^*M$.

Definition 5.7 (Differential form)**.** A k-form on M is a map

$$\alpha : M \to \bigsqcup_{p\in M} \bigwedge^k T_p^*M$$

such that

- $\alpha(p) \in \bigwedge^k T_p^*M$ for each p
- For any chart (U, ϕ), writing

  $$\alpha(p) = \sum_{1\leq i_1<\ldots<i_k\leq m} a_{i_1\ldots i_k} dx_{i_1}|_p \wedge \cdots \wedge dx_{i_k}|_p,$$

  the coefficients $a_{i_1\ldots i_k} : U \to \mathbb{R}$ are smooth.

Remark. In other words: a k-form is a section of the vector bundle $\bigwedge^k T^*M$. We denote

$$\Omega^k(M) = \{k\text{-forms on } M\}.$$

Notice that $\Omega^0(M) = C^\infty(M)$.

Remark. Given $\alpha \in \Omega^k(M)$ and vector fields $X_1, \ldots, X_k \in \mathfrak{X}(M)$, we define $\alpha(X_1, \ldots, X_k) \in C^\infty(M)$ by

$$(\alpha(X_1, \ldots, X_k))(p) := \alpha(p)(X_1(p), \ldots, X_k(p)).$$

Notice that for all $f \in C^\infty(M)$, $\alpha(fX_1, \ldots X_k) = f\alpha(X_1, \ldots, X_k)$.

Definition 5.8. Let $F : M \to N$ be a smooth map. The pullback of k-forms is $F^* : \Omega^k(N) \to \Omega^k(M)$:

$$(F^*\omega)(p)(v_1, \dots v_k) = \omega(F(p))((F_*)_p v_1, \dots, (F_*)_p v_k)$$

where $p \in M$, $v_i \in T_pM$.

We now study the pullback of differential forms on $\mathbb{R}^n$.

Proposition 5.9. Let $U \subset \mathbb{R}^m$ open, $V \subset \mathbb{R}^n$ open, $G : U \to V$ smooth. Denote with x_j the standard coordinates on $\mathbb{R}^m$ and by y_i those on $\mathbb{R}^n$. Then

1) $G^*(dy_i) = \sum_{j=1}^m \dfrac{\partial G_i}{\partial x_j} dx_j \in \Omega^1(U)$

2) If $m = n$, then

$$G^*(f\ dy_1 \wedge \ldots \wedge dy_m) = (f \circ G) \det(\operatorname{Jac} G)\ dx_1 \wedge \ldots \wedge dx_m.$$

Above $\operatorname{Jac} G$ denotes the Jacobian of G, i.e. the matrix representing the derivative DG of G.

Proof. 1) We have

$$G^*(dy_i)\left(\frac{\partial}{\partial x_j}\right) = dy_i\left(G_*\frac{\partial}{\partial x_j}\right) = dy_i\left(\sum_k \frac{\partial G_k}{\partial x_j}\frac{\partial}{\partial y_k}\right) = \frac{\partial G_i}{\partial x_j}.$$

2) We have $G^*(f dy_1 \wedge \ldots \wedge dy_m) = (G^*f)(G^*dy_1 \wedge \ldots \wedge G^*dy_m)$. Evaluating on $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}$, we get

$$(G^*f)\det\underbrace{\left(G^*(dy_i)\left(\frac{\partial}{\partial x_j}\right)\right)}_{=\frac{\partial G_i}{\partial x_j}\text{ by above}} = (f \circ G)\det(\operatorname{Jac} G).$$

$\square$

5.3 Orientation and volume forms

Definition 5.10 (Oriented atlas). An oriented atlas for M is a smooth atlas $\{(U_\alpha, \phi_\alpha)\}$ s.t.

$$\det\left(D(\phi_\beta \circ \phi_\alpha^{-1})\right) > 0$$

for all α, β.

Definition 5.11 (Orientation). An orientation is a choice of maximal oriented atlas.

Remark. Not all manifolds are orientable, e.g. the Möbius band and $\mathbb{RP}^2$ are not.

Remark. Let V be vector space. On $\{$ordered bases of $V\}$, there is an equivalence relation, given by: $(v_1, \ldots, v_m) \sim (w_1, \ldots, w_m)$ if the change of basis has $\det > 0$. An *orientation of* V is by definition a choice of one of the two equivalence classes. An ordered basis of an oriented vector space is *positive* if it belongs to the equivalence class that gives the orientation.

Now let M be a manifold. An orientation on M induces an orientation on each tangent space T_pM.

Definition 5.12 (Volume form). A volume form on M^m is an m-form Ω such that $\Omega(p) \neq 0$ for all $p \in M$.

Remark. If Ω is a volume form, then any other volume form looks like $f\Omega$, where $f : M \to \mathbb{R} \setminus \{0\}$.

Proposition 5.13. M is orientable iff there exists a volume form Ω

Proof. We denote by $\mathrm{Vol} := dx_1 \wedge \ldots \wedge dx_m$ the standard volume form on $\mathbb{R}^m$.

$\Leftarrow$: choose an atlas consisting of charts such that $(\phi_\alpha^{-1})^*\Omega = f_\alpha \mathrm{Vol}$, where f_α is a positive function. Then $\det D(\phi_\beta \circ \phi_\alpha^{-1}) > 0$ because of item 2 of the last proposition. So $\{(U_\alpha, \phi_\alpha)\}$ is an oriented atlas.

$\Rightarrow$: given an oriented atlas $\{(U_\alpha, \phi_\alpha)\}$, choose a partition of unity $\{e_\alpha\}$ subordinate to it, and define

$$\Omega = \sum_\alpha e_\alpha \cdot (\phi_\alpha)^*\mathrm{Vol}.$$

Then Ω is a volume form, because $e_\alpha \geq 0$ for all α and on $U_\alpha \cap U_\beta$ we have

$$(\phi_\beta)^*\mathrm{Vol} = (\phi_\alpha)^* \underbrace{(\phi_\beta \circ \phi_\alpha^{-1})^*\mathrm{Vol}}_{\det(D(\phi_\beta\circ\phi_\alpha^{-1}))\cdot\mathrm{Vol}} = (\text{a positive function}) \cdot (\phi_\alpha)^*\mathrm{Vol}.$$

□

5.4 Integration on manifolds

Let M^m be a oriented manifold. We want to define $\int_M \omega$, where $\omega \in \Omega^m(M)$ has compact support.

Step 1. Assume the support $\mathrm{supp}(\omega) := \overline{\{p \in M : \omega(p) \neq 0\}}$ is contained in one chart (U, ϕ) of the oriented atlas. Write $(\phi^{-1})^*\omega = f dx_1 \wedge \ldots \wedge dx_m$ for some $f \in C^\infty(\phi(U))$.

Definition 5.14 (Integral of top form supported in a chart).

$$\int_M \omega := \int_{\phi(U)} (\phi^{-1})^*\omega := \int_{\phi(U)} f(x)\, dx_1 dx_2 \cdots dx_m,$$

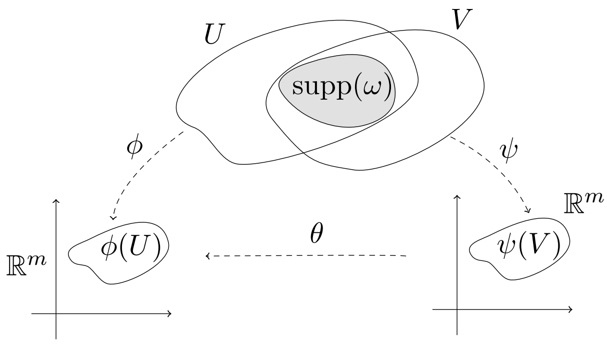
where the right hand side is the multiple Riemann integral of the function f over $\phi(U)$.

Remark. Recall the transformation rule in $\mathbb{R}^m$. Let $U, V \subset \mathbb{R}^m$ open and $\theta : V \to U$ a diffeomorphism. Let $f \in C^\infty(U)$ be integrable. Then

$$\int_U f(x)\, dx_1 \cdots dx_m = \int_V (f \circ \theta)(y) \big|\det(\mathrm{Jac}\,\theta)\big|\, dy_1 \cdots dy_m.$$

Lemma 5.15. $\int_M \omega$ is independent of the choice of chart in the oriented atlas.

Proof. Let (V, ψ) be another chart as above, and denote $\theta := \phi \circ \psi^{-1}$.



We want to find $g \in C^\infty(\psi(V))$ such that $(\psi^{-1})^*\omega = g(y)\, dy_1 \wedge dy_2 \cdots \wedge dy_m$. To do so we compute

$$(\psi^{-1})^*\omega = (\phi^{-1} \circ \theta)^*\omega = \theta^*((\phi^{-1})^*\omega) = \underbrace{(f \circ \theta) \det(\operatorname{Jac}\theta)}_{\text{so this is } g} dy_1 \wedge \ldots \wedge dy_m$$

using a Proposition from §5.2 in the last equality. By the transformation rule

$$\int_{\phi(U)} f(x)\, dx_1 dx_2 \cdots dx_m = \int_{\psi(V)} \underbrace{(f \circ \theta)(y) |\det(\operatorname{Jac}\theta)|}_{=g}\, dy_1 \cdots dy_m,$$

finishing the proof. (The function on the right hand side equals g since the absolute values can be removed, due to the fact that ψ and ϕ lie in the oriented atlas of M.) □

Step 2. For any $\omega \in \Omega^m(M)$ with compact support, let $\{(U_\alpha, \phi_\alpha)\}$ be an oriented atlas such that $\{\alpha : \operatorname{supp}(\omega) \cap U_\alpha \neq \emptyset\}$ is finite. Let e_α be a partition of unity subordinate to this cover. Notice that $\omega = (\sum e_\alpha)\omega = \sum(e_\alpha \omega)$, and the sum on the right is finite.

Definition 5.16 (Integral of top form).

$$\int_M \omega = \sum_\alpha \int e_\alpha \omega.$$

Notice that each summand $\int e_\alpha \omega$ was defined in Step 1, since $\operatorname{supp}(e_\alpha \omega) \subset U_\alpha$.

Remark. One can show: this definition is independent of the choice of oriented atlas and partition of unity.

Chapter 6

The exterior derivative and Stokes theorem

6.1 The exterior derivative in $\mathbb{R}^m$

Let $U \subset \mathbb{R}^m$ be open. Denote by $x_1, \ldots, x_m$ the standard coordinates on $\mathbb{R}^m$.

Remark. If $f \in C^\infty(U)$, then $df = f_*$ lies in $\Omega^1(U)$, because for all $p \in U$ we have $(f_*)_p \in T_p^* M$. Expressing df in terms of the dx_i, we have

$$df = \sum_i \frac{\partial f}{\partial x_i} dx_i,$$

since $(f_*)_p\left(\frac{\partial}{\partial x_i}\Big|_p\right) = (D_p f)(e_i) = \frac{\partial f}{\partial x_i}(p)$.

Example. On $\mathbb{R}^2$, we have $d(xy) = ydx + xdy$. ◇

Definition 6.1 (Exterior derivative on $\mathbb{R}^m$)**.** Let $U \subset \mathbb{R}^m$ be open. The exterior derivative (or de Rham differential)

$$d : \Omega^k(U) \longrightarrow \Omega^{k+1}(U)$$

is defined as follows. For $k = 0$: as above. For $k > 0$:

$$d\left(\sum_{1 \leq i_1 < \ldots < i_k \leq m} a_{i_1 \ldots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}\right) := \sum (da_{i_1 \ldots i_k}) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

Notice that here $a_{i_1 \ldots i_k} \in C^\infty(U)$.

Example. On $\mathbb{R}^3$ consider $\omega = (x_1)^3 dx_2 \wedge dx_3$, then

$$d\omega = 3(x_1)^2 dx_1 \wedge dx_2 \wedge dx_3.$$

◇

Example.

$$d\left((x_1)^2 dx_1 \wedge dx_2\right) = 0$$

◇

Proposition 6.2. The exterior derivative satisfies the following:

i) d is $\mathbb{R}$-linear

ii) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$, where $\alpha \in \Omega^k(U)$.

iii) $d^2 = 0$.

iv) If $F : U \to V$ smooth, then $d(F^*\omega) = F^*(d\omega)$.

Proof. iii) We prove it only for $g \in C^\infty(U) = \Omega^0(U)$.

$$\begin{aligned} d(dg) &= d\Big(\sum_j \frac{\partial g}{\partial x_j} dx_j\Big) \\ &= \sum_j d\Big(\frac{\partial g}{\partial x_j}\Big) \wedge dx_j \\ &= \sum_{j,i} \frac{\partial^2 g}{\partial x_i \partial x_j} dx_i \wedge dx_j. \end{aligned}$$

Now $dx_i \wedge dx_j$ is anti-symmetric in i and j, while $\frac{\partial^2 g}{\partial x_i \partial x_j}$ is symmetric in i and j (partial derivatives commute!), so $d^2 = 0$.

iv) We prove it only for $g \in C^\infty(U)$. Let $p \in U$, $v \in T_pU = \mathbb{R}^m$. Then

$$(F^*(dg))v = dg((F_*)_p v) = ((g \circ F)_*)_p v = (F^*g)_* v = d(F^*g)v,$$

using the chain rule in the second equality.

□

6.2 The exterior derivative on manifolds

Let M be a manifold.

Definition 6.3 (Exterior derivative on manifolds). Let $\omega \in \Omega^k(M)$, then $d\omega \in \Omega^{k+1}(M)$ is defined as follows: for all charts (U, ϕ),

$$(d\omega)|_U = \phi^*\Big(d\Big((\phi^{-1})^*\omega\Big)\Big).$$

Notice that $(\phi^{-1})^*\omega$ is a k-form on an open subset of $\mathbb{R}^m$.

Remark. The above is well defined because if (V, ψ) is another chart, then the map $(\psi \circ \phi^{-1})^*$ commutes with d by part iv) of the previous proposition.

6.3 Manifolds with boundary

Definition 6.4. $\mathbb{H}^m = \{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_1 \leq 0\}$

Definition 6.5 (Differentiable map on a open subset of $\mathbb{H}^m$). Let $U \subset \mathbb{H}^m$ be open. Then $f : U \to \mathbb{R}^k$ is differentiable iff there exists an open $\tilde{U} \subset \mathbb{R}^m$ such that $U = \tilde{U} \cap \mathbb{H}^m$, and there exists $\tilde{f} : \tilde{U} \to \mathbb{R}^k$ differentiable such that $\tilde{f}|_U = f$. In that case, for all $p \in U$, $D_p f = D_p \tilde{f}$.

Definition 6.6 (Manifold with boundary). A manifold with boundary of dimension m consists of topological space M that is second countable and Hausdorff, together with a maximal smooth atlas.

Here by smooth atlas we mean: an open cover $\{U_\alpha\}$ on M and homeomorphisms ϕ_α from U_α to open subsets of $\mathbb{H}^m$, such that $\phi_\beta \circ \phi_\alpha^{-1}$ is differentiable for all α, β.

Remark. For $m = 1$, additionally we allow homeomorphisms ϕ_α from U_α to open subsets of $\{x \in \mathbb{R} \mid x \geq 0\}$.

Remark. In particular, all manifolds are manifolds with boundary.

Definition 6.7 (Boundary). The boundary of M is

$$\partial M = \{p \in M : \exists \text{ chart } (U, \phi) : \phi(p) \in \partial \mathbb{H}^m\},$$

where $\partial \mathbb{H}^m = \{0\} \times \mathbb{R}^{m-1}$.

Remark. If one charts satisfies this property, all charts satisfy this property.

Example. Consider $M = \{v \in \mathbb{R}^m : \|v\| \leq 1\}$. Then the boundary of M is $\partial M = \{v \in \mathbb{R}^m : \|v\| = 1\}$. ◇

Remark. For all $p \in \partial M$, the tangent space T_pM is defined similarly to the case of manifolds (without boundary), and has dimension m.

Remark. The integration of differential forms on manifolds with boundary is analog to the one for manifolds.

One can show the following:

Proposition 6.8. ∂M is a manifold, of one dimension less than M.

Proposition 6.9. An orientation on M induces an orientation on ∂M, as follows:

$$\begin{aligned} \forall p \in \partial M: \qquad & (v_1, \ldots, v_{m-1}) \text{ is a positive basis of } T_p \partial M \\ & \Leftrightarrow \quad (e, v_1, \ldots, v_{m-1}) \text{ is a positive basis of } T_p M, \end{aligned}$$

where $e \in T_p M$ is "outward pointing".

Example. Note that $\mathbb{R}^2$ has a standard orientation (the one determined by the ordered basis (e_1, e_2)). Hence the unit disk $D \subset \mathbb{R}^2$ too. The induced orientation on the circle ∂D is the "anticlockwise" one.

To see this: (e, v_1) as in the figure is a positive basis of $\mathbb{R}^2$. Hence the orientation that D induces on ∂D is the one for which v_1 is a positive basis.

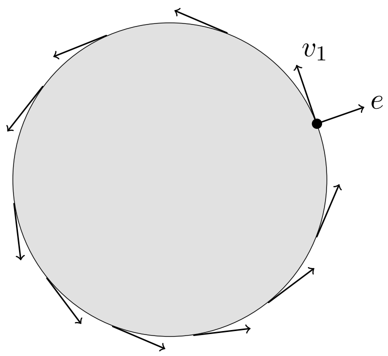


Figure 6.1: Orientation of the boundary of a disk

◇

6.4 Stokes' theorem

We generalize the fundamental theorem of calculus: $\int_a^b f'(x)dx = f|_a^b$.

Theorem 6.10 (Stokes). Let M^m be an oriented manifold with boundary and $\omega \in \Omega^{m-1}(M)$ with compact support. Then

$$\int_M d\omega = \int_{\partial M} i^*\omega,$$

where $i : \partial M \to M$ is the inclusion and ∂M has the induced orientation.

Remark. Note that since ω has compact support, so does $d\omega$.

The idea of the proof is to use a partition of unity to reduce this to the case where $\mathrm{supp}(\omega)$ is contained in a chart, i.e. to reduce to $M = \mathbb{H}^m$. There one can apply suitably the fundamental theorem of calculus.

Example. Let $D = \{v \in \mathbb{R}^2 : \|v\| \leq 1\}$. We have

$$\int_{S^1} i^*(xdy - ydx) = \int_D d(xdy - ydx) = 2\int_D dx \wedge dy = 2\pi.$$

◇

Corollary 6.11. Let M be a manifold (without boundary) and $\omega \in \Omega^{m-1}(M)$ with compact support. Then

$$\int_M d\omega = 0.$$

Corollary 6.12. If M is compact, orientable manifold with boundary, then there is no smooth $f : M \to \partial M$ such that $f|_{\partial M} = \mathrm{Id}_{\partial M}$.

Proof. Since M is orientable, ∂M is too. Therefore, there exists a volume form on the boundary, call it ω. Suppose f exists. Then $d(f^*\omega) = f^*d\omega = 0$, because $d\omega = 0$ by degree reasons. So

$$0 = \int_M d(f^*\omega) = \int_{\partial M} i^*f^*\omega = \int_{\partial M} (f \circ i)^*\omega = \int_{\partial M} \omega \neq 0,$$

using Stokes' theorem in the second equality and the fact that ω is a volume form in the inequality. □

Chapter 7

De Rham Cohomology

7.1 Basic definitions

Let M be a manifold of dimension m.

Definition 7.1 (Closed forms). $\omega \in \Omega^k(M)$ is called closed iff $d\omega = 0$.

Definition 7.2 (Exact forms). $\omega \in \Omega^k(M)$ is called exact iff $d\alpha = \omega$ for some $\alpha \in \Omega^{k-1}(M)$.

We have

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} \cdots \xrightarrow{d_{m-1}} \Omega^m(M) \longrightarrow 0$$

with $d_k \circ d_{k-1} = 0$, so in particular the image of d_{k-1} is included in the kernel of d_k. This is an example of a cochain complex. *Co*chain because d increases the degree.

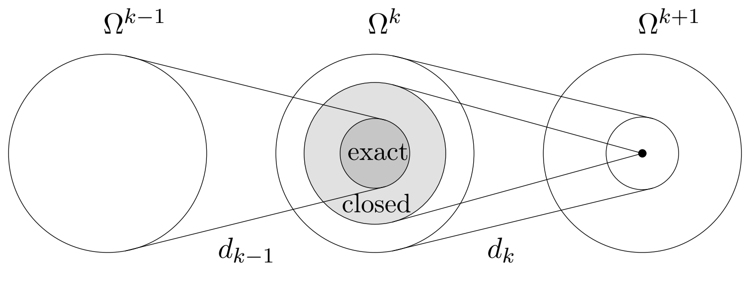


Figure 7.1: Visualization of the de Rham cochain complex. Exact k-forms lay per definition in the image of d_{k-1}, and closed k-forms in the kernel of d_k. As $d^2 = 0$, the exact k-forms form a subspace of the closed k-forms.

Definition 7.3 (De Rham cohomology). The k-th de Rham cohomology group of M is

$$H^k(M) := H^k_{dR}(M) := \frac{\operatorname{Ker} d_k}{\operatorname{Im} d_{k-1}} = \frac{\text{closed } k\text{-forms}}{\text{exact } k\text{-forms}}.$$

If $\omega \in \Omega^k(M)$ is closed, we denote by $[\omega] \in H^k(M)$ its class.

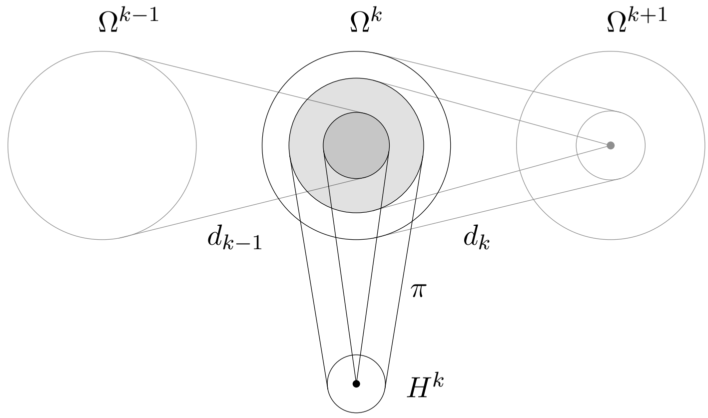


Figure 7.2: Definition of the De Rham cohomology.

Remark. Each $H^k(M)$ is a vector space. $H^k(M)$ can be different from zero only when $0 \leq k \leq m$. If M is compact, one can show that $H^k(M)$ is finite dimensional for all k. This is not trivial, because the dimension of $\operatorname{Ker} d_k$ is usually infinite-dimensional.

Proposition 7.4. If M is connected, then $H^0(M) \cong \mathbb{R}$.

Proof. For all $f \in \Omega^0(M)$, f is closed if $df = 0$, so f is constant (M is connected and f is smooth). f is exact iff it is zero, since $\Omega^{-1}(M) = \{0\}$. □

Remark. If M is not connected, we get $\mathbb{R}^k$, with k being the number of connected components.

Theorem 7.5. If M^m is *compact*, connected and orientable, then $H^m(M) \cong \mathbb{R}$.

Proof. Forms of 'top degree' are always closed. The idea is that we construct a surjective map from the m-forms to $\mathbb{R}$ and then show that the kernel is given precisely by the exact m-forms. Consider

$$\begin{aligned} I : \Omega^m(M) &\longrightarrow \mathbb{R} \\ \omega &\longmapsto \int_M \omega. \end{aligned}$$

Then I is surjective: since M is orientable, there exists a volume form, Ω, and $\int_M \Omega \neq 0$. We now argue that $\ker(I) = \{\text{exact } m\text{-forms}\}$.

For the inclusion "$\supset$": consider an exact m-form $\omega = d\alpha$. Then by Stokes' theorem

$$\int_M d\alpha = \int_{\partial M} \alpha = 0.$$

Idea for the inclusion "$\subset$": Assume $\int_M \omega = 0$, we need to show that $\omega = d\alpha$ for some α. Reduce to forms with support (necessarily compact) contained in a chart. Then this reduces to the following: if f is a function on $\mathbb{R}$ with compact support and $\int_{\mathbb{R}} f(x)dx = 0$, then $F(x) = \int_{-\infty}^{x} f(t)dt$ is a primitive of f with compact support (because when x is really small, this integral is zero, and when x is large enough, $F(x) = \int_{\mathbb{R}} f(x)dx = 0$).

This shows that

$$\frac{\Omega^m(M)}{\operatorname{Ker}(I)} = H^m(M).$$

But this quotient is isomorphic to the image of I, which is $\mathbb{R}$. $\square$

7.2 Homotopic maps and cohomology

Lemma 7.6. Let $f : N \to M$ be a smooth map. Then

- If $\omega \in \Omega(M)$ is closed, then $f^*\omega$ is also closed.
- If $\omega \in \Omega(M)$ is exact, then $f^*\omega$ is also exact.

Proof. We only prove the first part. $d(f^*\omega) = f^*d\omega = f^*0 = 0$. $\square$

Given $f : N \to M$, consider the pullback of differential forms $f^* : \Omega^k(M) \to \Omega^k(N)$. This restricts to a map between closed forms, and also between exact forms, so it induces a map on the level of cohomology groups:

$$\begin{aligned} H(f) : H^k(M) &\longrightarrow H^k(N) \\ [\omega] &\longmapsto [f^*\omega]. \end{aligned}$$

Remark. If f is a diffeomorphism, then it's clear that $H(f)$ is an isomorphism. But there are more general maps that induces an isomorphism in cohomology, as we now explain.

Definition 7.7 (Smooth homotopy)**.** Let $f, g : N \to M$ be smooth maps. A smooth homotopy between f and g is a smooth map $h : N \times [0,1] \to M$ such that $h|_{N\times\{0\}} = f$ and $h_{N\times\{1\}} = g$.

In other words, h is a smooth family of maps that interpolate between f and g. One can prove:

Proposition 7.8. If $f, g : N \to M$ are smoothly homotopic, then f^* and g^* are cochain homotopic, i.e. there exists a linear map

$$K : \Omega^\bullet(M) \to \Omega^{\bullet-1}(N),$$

such that $d \circ K + K \circ d = f^* - g^*$.

$$\begin{tikzcd}
\Omega^{k-1}(M) \arrow[r, "d"] & \Omega^k(M) \arrow[r, "d"] \arrow[dl, "K"'] \arrow[d, "f^*-g^*"] & \Omega^{k+1}(M) \arrow[dl, "K"] \\
\Omega^{k-1}(N) \arrow[r, "d"] & \Omega^k(N) \arrow[r, "d"] & \Omega^{k+1}(N)
\end{tikzcd}$$

Theorem 7.9. If f, g are smoothly homotopic, then $H(f) = H(g)$.

Proof. Let $K = \Omega^\bullet(M) \to \Omega^{\bullet-1}(M)$ a cochain homotopy between f^* and g^*. Let $\omega \in \Omega^k(M)$ be closed. Then

$$f^*\omega - g^*\omega = d(K\omega) + K(\underbrace{d\omega}_{=0})$$

is exact. Hence $[f^*\omega] = [g^*\omega]$. □

Corollary 7.10. Let M and N be homotopy equivalent, i.e. $\exists \alpha : N \to M$ and $\beta : M \to N$ such that $\alpha \circ \beta \simeq Id_M$ and $\beta \circ \alpha \simeq Id_N$, where $\simeq$ means smoothly homotopic. Then $H(\alpha) : H(M) \to H(N)$ is an isomorphism with inverse $H(\beta)$.

Proof. $H(\alpha) \circ H(\beta) = H(\beta \circ \alpha) = H(Id_N)$, because of the previous theorem, and $H(Id_N) = Id_{H(N)}$. You can do the same for the reverse order. $\square$

Remark (Poincaré Lemma). Let $U \subset \mathbb{R}^n$ be an open subset which is star shaped[1].

Then U is homotopy equivalent to a point $\{p\}$, via the inclusion $\alpha : \{p\} \to U$ and the constant map $\beta : U \to \{p\}$. Hence,

$$H^k(U) \cong H^k(\{p\}) \cong \begin{cases} \mathbb{R} & \text{if } k = 0 \\ 0 & \text{if } k > 0 \end{cases}$$

Notice that $\mathbb{R}^n$ is star-shaped, and every point of a manifold has a star-shaped neighborhood.

Example. If $E \to M$ is a vector bundle, then E is homotopic equivalent to M, as we can shrink the fibers (which are vector spaces) to the zero section. So the cohomology of M is the same as that of E: $H(E) \cong H(M)$. $\diamond$

Remark. Given manifolds M, N, if $H(M) \ncong H(N)$, then the two manifolds are not homotopic equivalent.

7.3 The Mayer–Vietoris theorem

Theorem 7.11. Suppose a manifold admits a cover by 2 open sets U and V, i.e $M = U \cup V$. Then

$$\begin{tikzcd} & \cdots & H^{k-1}(U \cap V) \arrow[dll, bend left] \\ H^k(M) \arrow[r, "A_k"] & H^k(U) \oplus H^k(V) \arrow[r, "B_k"] & H^k(U \cap V) \arrow[dll, bend left, "\delta_k"'] \\ H^{k+1}(M) \arrow[r] & \cdots & \end{tikzcd}$$

is a long exact sequence, i.e. the image of any arrow is the kernel of the next one. Here,

$$A_k[\omega] := [\omega|_U] \oplus [\omega|_V]$$
$$B_k(\alpha \oplus \beta) := \beta|_{U \cap V} - \alpha|_{U \cap V}.$$

Remark. The proof uses the smooth maps

$$U \cap V \underset{i_{U \cap V, U}}{\xrightarrow{i_{U \cap V, V}}} U \sqcup V \to M$$

[1] $\exists p$ such that for all $u \in U$ the line segment between u and p lies in U.

induced by inclusions.

Example. Let's consider the sphere S^m with $m \geq 2$. Write

$$S^m = U \cup V$$

where $U = S^m \setminus \{\text{North pole}\}$, $V = S^m \setminus \{\text{South pole}\}$. Notice that

- $U \simeq \{p\}$,
- $V \simeq \{p\}$,
- $U \cap V \simeq S^{m-1}$,

where $\simeq$ means homotopic equivalent. Applying the theorem and doing some diagram-chasing, we get

$$H^k(S^m) \cong \begin{cases} \mathbb{R} & \text{if } k = 0, m \\ 0 & \text{otherwise.} \end{cases}$$

$\diamond$

Chapter 8

Foliations

8.1 Immersed submanifolds

Definition 8.1 (Immersion). An immersion is a smooth map $F : H \to M$ such that $(F_*)_p$ is injective for all $p \in H$.

Definition 8.2 (Immersed submanifold). An immersed submanifold of M is a subset $H \subset M$ with a topology (not necessarily the subspace topology) and a smooth manifold structure such that the inclusion $i\colon H \to M$ is an immersion.

Remark. In that case, for all $p \in H$, the map $(i_*)_p$ is an isomorphism from T_pH to $(i_*)_p(T_pH) \subset T_pM$.

Remark. All submanifolds are immersed submanifolds. Immersed submanifolds are not necessarily submanifolds.

Remark. If H is an immersed submanifold, then for all $p \in H$, there exists an open neighbourhood U in H ("open" w.r.t. the topology of H) such that U is a submanifold of M.

Example. For all $\lambda \in \mathbb{R}$ consider $F : \mathbb{R} \to S^1 \times S^1 : t \mapsto (e^{2\pi i t}, e^{2\pi i \lambda t})$. If $\lambda \in \mathbb{Q}$, then $F(\mathbb{R})$ is a submanifold of $S^1 \times S^1$. If $\lambda \notin \mathbb{Q}$, then $F(\mathbb{R})$ is just an immersed submanifold with the smooth structure given by the bijection $\mathbb{R} \cong F(\mathbb{R})$ obtained from F. The topology on $F(\mathbb{R})$ differs from the subspace topology induced by $S^1 \times S^1$.

$$\begin{tikzcd} & S^1 \times S^1 \\ \mathbb{R} \arrow[ur, "F"] \arrow[r] & F(\mathbb{R}) \arrow[u, "i"'] \end{tikzcd}$$

◇

8.2 Distributions and involutivity

Definition 8.3 (Distribution). A distribution D on M is a subbundle of TM.

For all $p \in M$, we get a subspace $D_p \subset T_pM$ of constant dimension varying smoothly with p.

Definition 8.4 (Involutive distribution). A distribution D is involutive iff for all $X, Y \in \Gamma(D)$, we have $[X, Y] \in \Gamma(D)$

Remark. D is involutive iff for all $p \in M$, there exists a neighborhood $U \subset M$ and a frame $X_1, \ldots, X_k \in \Gamma(D|_U)$ such that $[X_i, X_j] \in \Gamma(D|_U)$.

Definition 8.5 (Integral manifold). An integral manifold of D is a (non-empty) immersed submanifold H of M such that $T_pH = D_p$ for all $p \in H$.

Example. Let X be a nowhere vanishing vector field on M. Then D given by $D_p = \operatorname{span} X_p$ is a rank 1 distribution, thus involutive. The image of any integral curve of X is an integral manifold of D. $\diamond$

Example. On $\mathbb{R}^3$, $D = \operatorname{span}\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}$ is an involutive rank-2 distribution. The planes $\{z = c\} \subset \mathbb{R}^3$ are integral manifolds. $\diamond$

Example. On $\mathbb{R}^3$, $D = \operatorname{span}\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}\right\}$ is not involutive. Indeed,

$$\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}\right] = -\frac{\partial}{\partial z} \notin D.$$

$\diamond$

Proposition 8.6. Let D be a distribution. Suppose that every point of M is contained in an integral manifold of D. Then D is involutive.

Proof. Let $X, Y \in \Gamma(D)$, $p \in M$. Let H be an integral manifold through p. Then $X|_H, Y|_H$ are tangent to H, so $[X, Y]|_H = [X|_H, Y|_H]$ is tangent to H. (To see this equality, use the naturality of the Lie bracket applied to the smooth map $i\colon H \to M$.) In particular, $[X, Y]_p \in T_pH = D_p$. $\square$

8.3 The Frobenius theorem

Let D be a rank k distribution on M^m.

Definition 8.7 (Flat chart for D). A chart (U, ϕ) is flat for D if $\phi(U) = I_1 \times \cdots \times I_n \subset \mathbb{R}^n$ and $D = \operatorname{span}\left\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^k}\right\}$ on U. Here, I_i are open intervals.

Definition 8.8 (Completely integrable distribution). D is completely integrable if for all $p \in M$, there exists a flat chart containing p.

Theorem 8.9 (Frobenius). D is completely integrable iff D is involutive.

Notice that the condition on the left is a local one, while the condition on the right is an infinitesimal one (thus easier to check).

Proof. $\boxed{\Rightarrow}$ Apply the last proposition.

Alternatively: At each $p \in M$, take a flat chart. Then $D = \operatorname{span}\left\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^k}\right\}$ and $\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0 \in \Gamma(D)$.

$\boxed{\Leftarrow}$ One can show that near every point $p \in M$, D has a frame of vector fields $X_1, \ldots, X_k$ s.t. $[X_i, X_j] = 0$. Then apply Prop. 3.16. (The last proposition in §3.4, stating that there is a chart $(U, (s_1, \ldots, s_n))$ centered at p such that $X_i = \frac{\partial}{\partial s_i}$ for $i = 1, \ldots, k$.)

□

One can prove:

Proposition 8.10. Let D be an involutive rank-k distribution, H an integral manifold. For any flat chart $(U, (x_1, \ldots, x_n))$ for D, $H \cap U$ is the union of countably many open subsets of slices $\{x_i = \text{const}\}$ $(i > k)$.

Example. On the torus $S^1 \times S^1$, let $D = \operatorname{span}\{\frac{\partial}{\partial x_1} + \lambda \frac{\partial}{\partial x_2}\}$, where $\lambda \in \mathbb{R} \setminus \mathbb{Q}$. Every line with slope λ in $S^1 \times S^1$ is an integral manifold, and the countable condition holds. ◇

8.4 Foliations

Let M^n be a smooth manifold.

Definition 8.11 (Foliation). A rank k foliation is a collection $\{L_\alpha\}_{\alpha\in A}$ of k-dimensional, connected, immersed submanifolds of M, called *leaves*, such that

- $M = \bigsqcup_{\alpha\in A} L_\alpha$ (disjoint union)
- For all $p \in M$, there exists a chart (U, ϕ) around p such that $\phi(U) = I_1 \times \cdots \times I_n$ and for all $\alpha \in A$ the following holds: $U \cap L_\alpha$ is a countable union of slices $\{x_{k+1} = \text{const}, \ldots, x_n = \text{const}\}$, or empty.

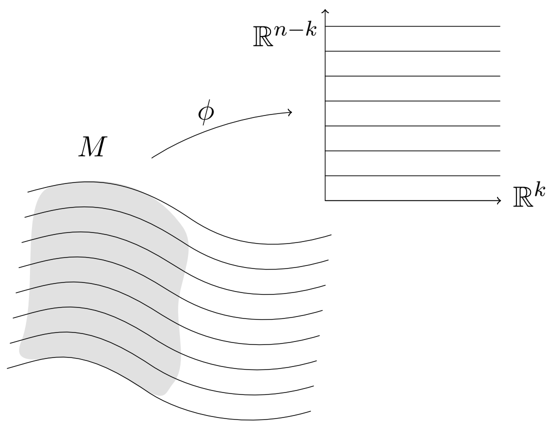


Figure 8.1: Definition of a foliation

Example. On $\mathbb{R}^2 \setminus \{0\}$, circles with radius $r > 0$ form a foliation. ⋄

Example. On $\mathbb{R}^2$, the following is not a foliation. (It's impossible to straighten all leaves nearby the x-axis.)

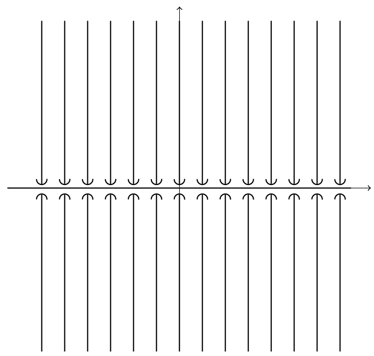


Figure 8.2: Not a foliation.

◇

The following theorem (sometimes called "Global Frobenius Theorem") gives a bijection between global objects on M and infinitesimal objects.

Theorem 8.12. Let M be a manifold. There is a bijection

$$\begin{aligned} \text{Foliations} &\leftrightarrow \text{Involutive distributions} \\ \{L_\alpha\} &\mapsto D \text{ s.t. } D_p = T_p(\text{leaf through } p) \end{aligned}$$

The inverse map reads

$$D \mapsto \{\text{maximal connected integral manifolds of } D\}.$$

Proof. We show that both maps are well-defined. It is easy to see that they are inverses of each other.

- $\rightarrow$: D, as defined above, is a distribution. Through every point of M there passes an integral manifold, hence D is involutive by Proposition 8.6.
- $\leftarrow$: By the Frobenius theorem, D is completely integrable, in particular for every point p of M there is an integral manifold through it. One can show that there is a *maximal* connected integral manifold through it (because the union of all integral manifolds through p is again an integral manifold). The decomposition of M into the above maximal connected integral manifolds is a foliation, since D is completely integrable and by Proposition 8.10.

$\square$

Chapter 9

Lie groups and Lie algebras

9.1 Lie groups, Lie subgroups

Definition 9.1 (Lie group)**.** A Lie group is a group which is a manifold, such that the multiplication $m : G \times G \to G$ and the inversion $i : G \to G, g \mapsto g^{-1}$ are differentiable.

Definition 9.2 (Lie group morphism)**.** A Lie group morphism is a group morphism which is smooth.

Example. $(\mathbb{R}^n, +)$ is a Lie group. Check, for instance, that $(x, y) \mapsto x + y$ is smooth. ◇

Example. We check that

$$\mathrm{GL}(n, \mathbb{R}) = \{A \in \mathrm{Mat}(n, \mathbb{R}) : A \text{ invertible}\}$$

is a Lie group. Here $\mathrm{Mat}(n, \mathbb{R})$ denotes the real $n \times n$ matrices. The set $\mathrm{GL}(n, \mathbb{R})$ is open in the vector space $\mathrm{Mat}(n, \mathbb{R}) \cong \mathbb{R}^{n^2}$, so it's a manifold. It is also a group, and the multiplication is smooth, because

$$(AB)_{ij} = \sum_k A_{jk} B_{ki}$$

is a polynomial in the entries of A and B. The inversion is also smooth: for $n = 2$, for instance,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

and similarly for arbitrary n. ◇

Remark. $\mathrm{GL}(n, \mathbb{R})$ has 2 connected components: $\det > 0$ and $\det < 0$.

Example. We check that

$$\mathrm{SL}(n,\mathbb{R}) = \{A \in \mathrm{Mat}(n,\mathbb{R}) : \det(A) = 1\}$$

is a Lie group.

The map $\det : \mathrm{Mat}(n,\mathbb{R}) \to \mathbb{R}$ is smooth, because $\det(A)$ is a polynomial in the entries of A. We check that $1 \in \mathbb{R}$ is a regular value of det, i.e. for all $A \in \mathrm{SL}(n,\mathbb{R})$, the derivative $d_A \det \colon T_A\mathrm{Mat}(n,\mathbb{R}) = \mathrm{Mat}(n,\mathbb{R}) \to T_1\mathbb{R} = \mathbb{R}$ is surjective:

$$\begin{aligned}(d_A \det)\left(\frac{d}{dt}\Big|_0 (t+1)A\right) &= \frac{d}{dt}\Big|_0 \det((t+1)A) \\ &= \frac{d}{dt}\Big|_0 (t+1)^n \det A \\ &= n\det(A) = n \neq 0.\end{aligned}$$

Hence $\mathrm{SL}(n,\mathbb{R}) = \det^{-1}(1)$ is a submanifold of $\mathrm{Mat}(n,\mathbb{R})$, by the regular value theorem, and thus a submanifold of its open set $\mathrm{GL}(n,\mathbb{R})$. The multiplication and inversion of $\mathrm{SL}(n,\mathbb{R})$ are smooth, being the restrictions of those of $\mathrm{GL}(n,\mathbb{R})$. $\diamond$

Example. The following are all examples of Lie groups:

- $\mathrm{O}(n)$, defined as matrices with $A^{-1} = A^T$, which has two components.
- $\mathrm{SO}(n)$, defined as matrices with $A^{-1} = A^T$ and $\det(A) = 1$. It is the connected component of the identity of $\mathrm{O}(n)$.
- $\mathrm{U}(n)$, defined as matrices $A \in \mathrm{Mat}(n,\mathbb{C})$ such that $A\overline{A}^T = 1$.
- $\mathrm{SU}(n)$, defined as matrices $A \in \mathrm{Mat}(n,\mathbb{C})$ such that $A\overline{A}^T = 1$ and $\det A = 1$.

$\diamond$

Definition 9.3 (Left translation, left invariant vector field). Let G be a Lie group. For all $g \in G$, the diffeomorphism

$$L_g : G \to G : h \mapsto gh$$

is called left translation. A vector field $X \in \mathfrak{X}(G)$ is left invariant iff

$$\forall g \in G : (L_g)_* X = X.$$

Notice that L_g is the restriction to $\{g\} \times G$ of the multiplication map $m : G \times G \to G$.

Remark. There is a linear isomorphism

$$\begin{aligned} T_eG &\longrightarrow \mathfrak{X}(G)^L := \{\text{left-invariant vector fields}\} \\ v &\longmapsto \overleftarrow{v} \text{ where } (\overleftarrow{v})_g = ((L_g)_*)_e v. \end{aligned}$$

The idea is that if you have a left invariant vector field, then it is determined by its value at any point, for example e, so that $X_g = [(L_g)_* X]_g = ((L_g)_*)_e(X_e) \in T_gG$.

It follows that:

- The tangent bundle TG is a trivial vector bundle. Indeed,

$$\begin{aligned} G \times T_eG &\longrightarrow TG \\ (g, v) &\longmapsto ((L_g)_*)_e v. \end{aligned}$$

  is a vector bundle isomorphism.

- G admits a volume form, hence it's orientable.

Example. Of all the spheres, only S^0, S^1, S^3 are Lie groups. (S^2 is not a Lie group; for instance, the Hairy ball theorem implies that the tangent bundle is not trivial.) $\diamond$

Proposition 9.4. A *connected* Lie group G is generated (as a group) by any open neighbourhood W of identity.

Proof. Let H be the subgroup generated by W, i.e. finite products of elements of W and W^{-1}.

H is open. Indeed $W_1 = W \cup W^{-1}$ is open. $W_2 := W_1 \cdot W_1 = \bigcup_{g \in W_1} gW_1$ is open as the left translation is smooth and any union of open subsets is open. Similarly, for all k we get that $W_k := W_1 \cdot W_{k-1}$ is open. Thus $H = \cup_{k \geq 1} W_k$ is open.

H is non-empty, as $e \in H$.

The complement $G - H$ is open. Indeed,

$$G - H = \bigcup_{g \notin H} gH$$

is also open, as the union of open sets.

Since G is connected, it follows that $G - H$ is empty, i.e. $H = G$. $\square$

Definition 9.5 (Lie subgroup). Let G be a Lie group. A Lie subgroup H is

- an (abstract) subgroup of G
- which is an immersed submanifold

such that H becomes a Lie group with the induced group and manifold structures.

Remark. H might not be a submanifold.

Example. $G = S^1 \times S^1$ is a Lie group, since $S^1 = U(1)$, and the product of two Lie groups is again a Lie group.

Let $\lambda \in \mathbb{R}$, and let

$$H = \{(e^{2\pi it}, e^{2\pi i\lambda t}) : t \in \mathbb{R}\}.$$

We check that H is a Lie subgroup of G.

Indeed: H is a subgroup of G. It is also an immersed submanifold (as we saw earlier). The multiplication and inversion are smooth. (To check this: the smooth structure on H is obtained from the one on $\mathbb{R}$, and the induced multiplication is the addition on $\mathbb{R}$.) $\diamond$

Proposition 9.6. Let G be a Lie group, let H be a subgroup and also a submanifold. Then H is a Lie subgroup.

Proof. There is an obvious manifold structure on H, since it is a submanifold of G. We have to check that the multiplication $m : H \times H \to H$ and the inversion $i : H \to H$ are smooth. We just do the latter: the inversion of G is smooth, so restricting it to a submanifold, we again get a smooth map. $\square$

One can show:

Theorem 9.7. Let G be a Lie group, H a subgroup. If $H \subset G$ is closed in the topological sense, then H is a submanifold, and therefore a Lie subgroup.

9.2 From Lie groups to Lie algebras

Recall:

Definition 9.8 (Lie algebra). A Lie algebra is a vector space $\mathfrak{g}$ equipped with a bilinear, skew symmetric map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, such that

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0.$$

We now define morphisms and the "subobjects" of Lie algebras:

Definition 9.9 (Lie algebra morphism). A map $\phi : (\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}) \to (\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{h}})$ is a Lie algebra morphism if it preserves all the structure:

- the map ϕ is linear
- $\phi([x, y]_{\mathfrak{g}}) = [\phi(x), \phi(y)]_{\mathfrak{h}} \quad \forall x, y \in \mathfrak{g}$.

Definition 9.10 (Lie subalgebra). A Lie subalgebra of $\mathfrak{g}$ is a vector subspace $\mathfrak{h}$, such that

$$[x, y] \in \mathfrak{h} \quad \forall x, y \in \mathfrak{h}.$$

We now show three propositions, labelled (A), (B), (C), about

(A) Lie groups

(B) Lie group morphisms

(C) Lie subgroups.

Later we will encounter their "converses".

Proposition 9.11 (A). Let G be a Lie group, then $\mathfrak{g} := T_eG$ is a Lie algebra, with bracket

$$[v, w] := [\overleftarrow{v}, \overleftarrow{w}]|_e.$$

Proof. Recall the linear isomorphism $T_eG \to \mathfrak{X}(G)^L, v \mapsto \overleftarrow{v}$. The Lie bracket of two left invariant vector fields is again a left invariant vector field:

$$(L_g)_*[\overleftarrow{v}, \overleftarrow{w}] = [(L_g)_*\overleftarrow{v}, (L_g)_*\overleftarrow{w}] = [\overleftarrow{v}, \overleftarrow{w}],$$

where in the first equation we used the naturality of the Lie bracket. So $\mathfrak{X}(G)^L$ is a Lie algebra. Now use the above linear isomorphism to transport this Lie algebra structure to T_eG. $\square$

Example. Consider $\mathrm{GL}(n, \mathbb{R})$. Then $T_e(\mathrm{GL}(n, \mathbb{R}))$ is all $n \times n$ matrices, i.e. $\mathrm{Mat}(n, \mathbb{R})$. So $\mathrm{Mat}(n, \mathbb{R})$ has an induced Lie algebra structure. Its bracket is

$$[A, B] = AB - BA.$$

(This is not at all obvious, it requires a bit of computation) $\diamond$

Proposition 9.12 (B)**.** Let $\Phi : G \to H$ be a Lie group morphism (i.e. a smooth group homomorphism). Then

$$d_e\Phi := (\Phi_*)_e : T_eG \to T_eH$$

is a Lie algebra morphism.

Proof. For all $v \in T_eG$, the vector field $\overleftarrow{v}$ is Φ-related to $\overleftarrow{(d_e\Phi)v}$. Indeed,

$$(d_g\Phi)((\overleftarrow{v})_g) = (d_g\Phi)((d_eL_g)(v)) = d_e(\Phi \circ L_g)(v) = d_e(L_{\Phi(g)} \circ \Phi)(v),$$

where the last equation holds because Φ is a group homomorphism. Now, pulling derivatives apart again, we see that the above equals

$$(L_{\Phi(g)})_*((d_e\Phi)v) = (\overleftarrow{(d_e\Phi)v})_{\Phi(g)}.$$

Take $v_1, v_2 \in T_eG$. Applying the previous statement and the naturality of the Lie bracket we obtain that $[\overleftarrow{v_1}, \overleftarrow{v_2}]$ is Φ-related to $[\overleftarrow{(d_e\Phi)v_1}, \overleftarrow{(d_e\Phi)v_2}]$. In particular, at $g = e$,

$$(d_e\Phi)([\overleftarrow{v_1}, \overleftarrow{v_2}]_e) = \left(\left[\overleftarrow{(d_e\Phi)v_1}, \overleftarrow{(d_e\Phi)v_2}\right]\right)_e.$$

$\square$

Proposition 9.13 (C)**.** Let H be a Lie subgroup of G. Then T_eH is a Lie subalgebra of T_eG.

Proof. We know that the inclusion is Lie group morphism. Therefore, its derivative $T_eH \to T_eG$ is a Lie algebra morphism by the last proposition. $\square$

Example. $\mathrm{SL}(n, \mathbb{R})$ is a Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$, hence the Lie algebra of $\mathrm{SL}(n, \mathbb{R})$ is a Lie subalgebra of $\mathrm{GL}(n, \mathbb{R})$. It turns out that the Lie algebra of $\mathrm{SL}(n, \mathbb{R})$ are exactly the traceless matrices. $\diamond$

9.3 The exponential map

Let G be a Lie group, $\mathfrak{g} = T_eG$.

Lemma 9.14. For all $v \in T_eG$, there is a unique morphism of Lie groups $\gamma_v : (\mathbb{R}, +) \to G$ with the property that $\gamma_v'(0) = v$.

Notice: there are lots of curves γ with velocity $\gamma'(0) = v$, but by the lemma there exists *only one* which is also a group homomorphism, i.e. $\gamma_v(s) \cdot \gamma_v(t) = \gamma_v(s+t)$.

Proof. Existence: The left invariant vector field $\overleftarrow{v}$ is complete, i.e. integral curves are defined for all times. Let

$$\gamma : \mathbb{R} \to G$$

be the integral curve of $\overleftarrow{v}$ starting at e. Clearly $\gamma'(0) = v$. Further γ is a group homomorphism, because for every fixed s, we have that the curves $t \mapsto L_{\gamma(s)}\gamma(t)$ and $t \mapsto \gamma(s+t)$ agree. (At $t = 0$, both of these go through $\gamma(s)$, and they are also integral curves[a] of $\overleftarrow{v}$.)

Uniqueness: let $\gamma : \mathbb{R} \to G$ be a Lie group morphism with $\gamma'(0) = v$. Since $\frac{\partial}{\partial t}$ is a left-invariant vector field on $\mathbb{R}$, by the proof of Proposition 9.12 it is γ-related to $\overleftarrow{(d_0\gamma)(\frac{\partial}{\partial t}|_0)} = \overleftarrow{\gamma'(0)} = \overleftarrow{v}$. Hence for every $t_0 \in \mathbb{R}$:

$$(d_{t_0}\gamma)\Big(\frac{\partial}{\partial t}\Big|_{t_0}\Big) = \overleftarrow{v}|_{\gamma(t_0)}.$$

The left hand side is just $\gamma'(t_0)$. Hence γ is the (unique) integral curve of $\overleftarrow{v}$ starting at e. $\square$

[a]For $t \mapsto \gamma(s+t)$, it's trivial. For $t \mapsto L_{\gamma(s)}\gamma(t)$, we have that this is an integral curve of $(L_{\gamma(s)})_*\overleftarrow{v}$, which is the same as $\overleftarrow{v}$.

Definition 9.15 (Exponential map)**.** Let G be a Lie group with Lie algebra $\mathfrak{g}$. The exponential map is

$$\exp : \mathfrak{g} \to G, v \mapsto \gamma_v(1).$$

Proposition 9.16. a) $\exp(tv) = \gamma_v(t)$ for all $t \in \mathbb{R}$ and $v \in \mathfrak{g}$.

b) There exists a neighborhood U of $0 \in \mathfrak{g}$ such that

$$\exp|_U : U \to \exp(U)$$

is a diffeomorphism onto an open subset of G. (This allows to study G close to the identity element e by studying the Lie algebra. This also defines a chart near e.)

c) If $H \subset G$ is a Lie subgroup. Then $\exp^H : T_eH \to H$ is the restriction of $\exp^G : T_eG \to G$.

Proof. a) The curves $s \mapsto \gamma_{tv}(s)$ and $s \mapsto \gamma_v(st)$ agree, as they are both Lie group morphism $(\mathbb{R}, +) \to G$ with the same velocity tv at $s = 0$. Take $s = 1$.

b) Idea: check that $d_0 \exp : T_0\mathfrak{g} = \mathfrak{g} \to T_eG = \mathfrak{g}$ is the identity on $\mathfrak{g}$. Then apply the inverse function theorem.

□

Example. For $\mathrm{GL}(n, \mathbb{R})$, we have

$$\begin{aligned} \exp : T_e \mathrm{GL}(n, \mathbb{R}) = \mathrm{Mat}(n, \mathbb{R}) &\longrightarrow \mathrm{GL}(n, \mathbb{R}) \\ A &\longmapsto e^A := \sum_{n=0}^{\infty} \frac{A^n}{n!}. \end{aligned}$$

◇

Proof. For all $A \in \mathrm{Mat}(n, \mathbb{R})$, the curve

$$(\mathbb{R}, +) \to \mathrm{GL}(n, \mathbb{R}), t \mapsto e^{tA}$$

is a group morphism, since $e^{tA} \cdot e^{sA} = e^{(t+s)A}$. Furthermore,

$$\frac{d}{dt}\Big|_0 e^{tA} = \frac{d}{dt}\Big|_0 (I + tA + O(t^2)) = A.$$

So this curve is γ_A.

9.4 From Lie algebras to Lie groups

Let $(\mathfrak{g}, [\cdot, \cdot])$ be a (finite dimensional) Lie algebra.

Definition 9.17 (Lie group integrating a given Lie algebra). A Lie group G integrates $\mathfrak{g}$ iff T_eG is isomorphic to $\mathfrak{g}$ (as Lie algebras).

We will show:

Theorem 9.18 (A). Up to isomorphism, there exists a unique *simply connected* Lie group G_{SC} integrating $\mathfrak{g}$.

Remark. All other connected Lie groups integrating $\mathfrak{g}$ are quotients of G_{SC} by discrete normal subgroups.

Example. Let $\mathfrak{g} = (\mathbb{R}, [\cdot,\cdot] = 0)$. Then $G_{SC} = (\mathbb{R}, +)$. Note that $U(1) = S^1 = \mathbb{R}/\mathbb{Z}$ is also a Lie group that integrates $\mathfrak{g}$. ◇

Example. Let $\mathfrak{g} = \{A \in \mathrm{Mat}(n, \mathbb{R}) : A + A^T = 0\}$. The Lie group $\mathrm{SO}(n)$ integrates $\mathfrak{g}$. ◇

Proposition 9.19 (C). Let G be a Lie group and $\mathfrak{h}$ a Lie subalgebra of $\mathfrak{g} = T_eG$. There exists a unique connected Lie subgroup H whose Lie algebra is $\mathfrak{h}$.

Example. Let $G = U(1) \times U(1)$, $\mathfrak{h} = \mathrm{span}(1, \lambda) \subset \mathbb{R}^2 = T_eG$, where $\lambda \in \mathbb{R}$. Then $H = \{(e^{it}, e^{i\lambda t}) : t \in \mathbb{R}\}$. ◇

Proof. Sketch: Denote by D the distribution on G given by

$$D_g := (L_g)_* \mathfrak{h}.$$

It is involutive because it is spanned by left-invariant vector fields and $\mathfrak{h}$ is a Lie subalgebra.

By the global Frobenius theorem, there is a (unique) foliation of G whose leaves are tangent to D. Notice that the foliation is invariant under left-translation: $(L_g)(S_{g'}) = S_{gg'}$, where $S_{g'}$ denotes the leaf of D through g'.

One can show that $H := S_e$, the leaf of the foliation through e, is a Lie subgroup with Lie algebra $\mathfrak{h}$. (Clearly S_e is an immersed manifold with $T_eS_e = \mathfrak{h}$. To show that it is a subgroup, use the left-invariance of the foliation. One can show that multiplication and inversion are smooth.) Further, one can show uniqueness. □

Corollary 9.20. Given a Lie group G, there is a bijection

$$\{\text{Connected Lie subgroups of } G\} \leftrightarrow \{\text{Lie subalgebras of } T_eG\}.$$

Remark. The nice bijection in this corollary justifies the (slightly involved) definition of Lie subgroup.

Once can show:

Proposition 9.21 (B)**.** Let G be a *simply connected* Lie group, H a Lie group, and $\Psi : T_eG \to T_eH$ a Lie algebra morphism. Then there is a unique Lie group morphism $\Phi : G \to H$ such that $d_e\Phi = \Psi$.

This proposition allows us to easily prove the uniqueness in Theorem (A):

Proof. Let G_1, G_2 be simply connected Lie groups and $\Psi : T_eG_1 \to T_eG_2$ a Lie algebra isomorphism. Then there exist

- a Lie group morphism $\Phi\colon G_1 \to G_2$ s.t. $d_e\Phi = \Psi$,
- a Lie group morphism $\chi\colon G_2 \to G_1$ s.t. $d_e\chi = \Psi^{-1}$.

Since $\chi \circ \Phi$ and Id_{G_1} are both Lie group morphisms with derivative $\mathrm{Id}_{T_eG_1}$, they must agree. Similarly, $\Phi\circ\chi = \mathrm{Id}_{G_2}$. So Φ is an isomorphism. □

We sketch the proof of the existence in Theorem (A), i.e.: given a Lie algebra $\mathfrak{g}$, there exists a simply connected Lie group integrating it.

Proof. Idea of proof: $\mathfrak{g}$ is isomorphic to a Lie subalgebra $\mathfrak{g}_0$ of $\mathrm{Mat}(n, \mathbb{R})$ for some n (Ado's theorem).

Let G_0 be the unique connected Lie subgroup of of $\mathrm{GL}(n, \mathbb{R})$ with Lie algebra $\mathfrak{g}_0$. Take G_{SC} to be the universal cover of G_0. (The universal cover is again a Lie group, and simply connected.) □