Differential Geometry 1: G0B08a

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Preface

This course seems to involve playing around with manifolds (which I'm given to understand are surfaces not situated *within* another space, but rather are the space of study themselves). In particular, this course seems to involve a lot of geometry (DUH) and building up a calculus on various weird shapes (so to speak). Over the course of the semester, we will see how my initial impressions change.

Contents

Chapter 1

Differentiable Manifolds

This chapter is supposed to cover all the basics that I will end up using this semester in my study of these strange things called *manifolds*, so we build up the required theory, starting from topological spaces, then on to topological manifolds, and then mapping, charting and atlas-ing our way to differentiable manifolds.

1.1 Topological Spaces

This just rehashes now what we've seen in all previous lectures (and is supposed to be prerequisite material) but once more unto the breach

Definition. A *topological space* (X, τ) is when you are given a set X and $\tau \subset P(X)$ such that the following axioms hold true

1. $\phi, X \in \tau$
2. For any set $\{U_i\}_{i \in I}$ of $U_i \in \tau \forall i \in I$, then
$$\bigcup_{i \in I} U_i \in \tau$$
where I is an arbitrary index set.
3. For a set of $\{U_i\}_{i=1}^{n}$, again, where each element is in τ, then
$$\bigcap_{i=1}^{n} U_i \in \tau$$

The elements of τ are called *open* sets.

Thoughts. Note that you don't really need to define an explicit "finite intersection closure", you could just work with intersection, and then proceed by induction to prove that it works for a finite number of sets.

Definition. Let $p \in X$. Then a *neighborhood* of p is an open subset $U \in \tau$ such that $p \in U \in \tau$

Finally we think about subsets in a topological space, and whether these arbitrary subsets themselves have a topological structure. It turns out that they do.

Definition. If $Y \subset X$, then (Y, τ_Y) is a topological space where

$$\tau_Y = \{U \cap Y | U \in \tau\}$$

This topology is called an induced topology (intersection of the parents open sets with the subset under scrutiny.)

We know pause and recall a certain fact about equivalence relations partition the sets they are used on.

Definition. Let $\sim$ be an equivalence relation on a topological space (X, τ) (going to ommit writing the τ in future). Now consider $\pi : X \to X/\sim$ being a function. Here, since we have a function from a set to it's quotient set (partitioned into equivalence classes by an equivalence relation), we can say $X/\sim$ is a topologial space, by saying $U \in X/\sim$ open whenever $\pi^{-1}(U)$ open in X. This definition also implicitly makes π a continuous function (because by definition, it pulls back open sets in the target to open sets in the domain.)

Definition. We can call a topological space *Hausdorff* if and only if

$$\forall x, y \in X x \neq y, \exists U_x, U_y \in \tau; U_x \cap U_y = \phi$$

Essentially, when you consider two distinct points in a space, then you can also find neighborhood's around those points that do not overlap.

Definition. A *basis* for a topology $B \subset \tau$ is a set such that each open set in τ is a union of elements of B. The topological space (X, τ) is considered to be *second-countable* if there is a countable basis for τ.