Differential Geometry 1: G0B08a

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Preface

This course seems to involve playing around with manifolds (which I'm given to understand are surfaces not situated *within* another space, but rather are the space of study themselves). In particular, this course seems to involve a lot of geometry (DUH) and building up a calculus on various weird shapes (so to speak). Over the course of the semester, we will see how my initial impressions change.

Contents

Chapter 1

Differentiable Manifolds

This chapter is supposed to cover all the basics that I will end up using this semester in my study of these strange things called *manifolds*, so we build up the required theory, starting from topological spaces, then on to topological manifolds, and then mapping, charting and atlas-ing our way to differentiable manifolds.

1.1 Topological Spaces and Manifolds

This just rehashes now what we've seen in all previous lectures (and is supposed to be prerequisite material) but once more unto the breach

Definition. A *topological space* (X, τ) is when you are given a set X and $\tau \subset P(X)$ such that the following axioms hold true

1. $\phi, X \in \tau$
2. For any set $\{U_i\}_{i \in I}$ of $U_i \in \tau \forall i \in I$, then

   $$\bigcup_{i \in I} U_i \in \tau$$

   where I is an arbitrary index set.
3. For a set of $\{U_i\}_{i=1}^{n}$, again, where each element is in τ, then

   $$\bigcap_{i=1}^{n} U_i \in \tau$$

The elements of τ are called *open* sets.

Thoughts. Note that you don't really need to define an explicit "finite intersection closure", you could just work with intersection, and then proceed by induction to prove that it works for a finite number of sets.

Definition. Let $p \in X$. Then a *neighborhood* of p is an open subset $U \in \tau$ such that $p \in U \in \tau$

Finally we think about subsets in a topological space, and whether these arbitrary subsets themselves have a topological structure. It turns out that they do.

Definition. If $Y \subset X$, then (Y, τ_Y) is a topological space where

$$\tau_Y = \{U \cap Y | U \in \tau\}$$

This topology is called an induced topology (intersection of the parents open sets with the subset under scrutiny.)

We know pause and recall a certain fact about equivalence relations partition the sets they are used on.

Definition. Let $\sim$ be an equivalence relation on a topological space (X, τ) (going to ommit writing the τ in future). Now consider $\pi : X \to X/\sim$ being a function. Here, since we have a function from a set to it's quotient set (partitioned into equivalence classes by an equivalence relation), we can say $X/\sim$ is a topologial space, by saying $U \in X/\sim$ open whenever $\pi^{-1}(U)$ open in X. This definition also implicitly makes π a continuous function (because by definition, it pulls back open sets in the target to open sets in the domain.)

Definition. We can call a topological space *Hausdorff* if and only if

$$\forall x, y \in X x \neq y, \exists U_x, U_y \in \tau; U_x \cap U_y = \phi$$

Essentially, when you consider two distinct points in a space, then you can also find neighborhood's around those points that do not overlap.

Definition. A *basis* for a topology $B \subset \tau$ is a set such that each open set in τ is a union of elements of B. The topological space (X, τ) is considered to be *second-countable* if there is a countable basis for τ.

We now proceed to figure out what manifolds are (FINALLY)

Definition. A *topological manifold of dimension m* is a topological space which is Hausdorff and second-countable, AND locally homeomorphic to $\mathbb{R}^m$. Here a homeomorphism is a mapping that is continuous, bijective, and whose inverse is also continuous.

Remark. When we mean locally homeomorphic, we mean there is an open neighborhood $U \subset M$ and a homeomorphism $\phi : U \to V \subset \mathbb{R}^m$ such that V is an open subset of $\mathbb{R}^m$

Some other remarks will also need to be made, to better understand this preliminary concept, as well as lead to other observations that will require proofs

Remark. We consider (U, ϕ) to be a *chart*.

Remark. Any subset of a Hausdorff subspace is Hausdorff. All metric spaces are Hausdorff. The case is similar for the case of second-countability.

But the notion of "charts" means there could exist multiple homeomorphisms between the topological space and $\mathbb{R}^m$, when could both of these be compared? Or rather how can such similar maps be compared in the first place?

Definition. We call 2 charts, (U_1, ϕ_1) and (U_2, ϕ_2) to be smoothly compatible if the following function (with the given restricted domain) is a diffeomorphism.

$$\phi_2 \circ (\phi_1^{-1})|_{\phi_1(U_1 \cap U_2)} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$$

A *diffeomorphism* is a map that is differentiable, bijective, and whose inverse is also differentiable.

Now that we have a notion of a chart, and the notion of a manifold, what can we do about a collection of charts from a topological space?

Definition. We define a *smooth atlas* $\mathcal{A}$ for M to be a collection of charts $(U_i \phi_I), i \in I$ such that they cover the whole of M and that each the charts taken pairwise are smoothly compatible.

Remark. A smooth atlas $\mathcal{A}$ is said to be maximal, if for any other atlas $\mathcal{B}$ that exists such that $\mathcal{A} \subset \mathcal{B} \implies \mathcal{A} = \mathcal{B}$

Now

Definition. A *differentiable or smooth* manifold, is a topological manifold equipped with a maximal smooth atlas.

1.2 Differentiable Maps

Let M be a differentiable/smooth manifold (i.e. a topological space with a maximal smooth atlas, i.e. a topological space equipped with a collection of charts, which are pairwise smoothly compatible, i.e. a topological space that is hausdorff and second-countable, that has an an open cover, each element of which is locally homeomorphic to real euclidean space.)

Definition. Now we say a mapping $f : M \to \mathbb{R}$ is differentiable at $p \in M \iff \exists$ a chart (U, ϕ) aroudn p such that $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ is differentiable at $\phi(p)$

Definition. We now define a function/mapping between manifolds to be differentiable. $f : M \to N$ is smooth at $p \in M$ if

- f is continuous
- there are charts (U_m, ϕ_m) and (U_n, ϕ_n) around $p, f(p)$ respectively such that $\phi_n \circ f\phi_m^{-1}$ is differentiable at $\phi_m(p)$

Remark. This differentiability, if it holds for one chart, holds for all other charts given the same domain.

Remark. $f : M \to N$ where M, N are both differentiable manifolds, is a differentiable homeomorphism (NOTE: this does not mean it is a diffeomorphism, because f^{-1} may in fact not be differentiable.)

1.3 Partitions of Unity

What on earth are these things, and why are they being introduced?

Definition. Let M be a manifold. Then a *partition of unity* is a family of smooth functions $\{\mathcal{C}_a\}_{a\in A}, \mathcal{C}_\dashv : M \to [0, 1]$ such that

- $\forall p \in M, \exists U$ a neighborhood of $p; \{a \in A | \mathcal{C}_a|_U \neq 0$ is finite $\}$ (only finitely many non-zero maps inside the particular family, from a neighborhood around a point on the manifold (in the manifold?))
- $\sum_{a\in A} \mathcal{C}_a = 1$

Remark. This concept does give us a pretty interesting visualisation, of a family of functions that peaks finitely around a point, before gradually subsiding away from that point?

Definition. If we consider an open cover of M, then a partition of unity is said to be *subordinate* to the cover, if the support of $\mathcal{C}_a \subset U_a$. Here we define the support to be the closure of the set of points that the function maps to non-zero reals.

Remark. As a remark, it is possible to show that for every open cover of M, there will exist a partition of unity. The proof this statement involves the use of the fact that M is second-countable.

1.4 Submanifolds

We now come to the last section of this chapter, where we deal with smaller sections of the larger manifolds we've begun to construct.

Definition. Let N be a smooth manifold of dimension n. Then $M \subset N$ is a submanifold of dimension $m \iff$

$$\forall p \in M \exists (U, \phi), (p \in U); \phi(U \cap M) = \phi(U) \cap \mathbb{R}^m \times \{0\}$$

This is what we call a chart that has been adapted to M. Such a set inherits the structure of a manifold, with its atlas given by the correspondingly restricted charts from the parent manifold.