Data Structures & Algorithms
Assignment 1
Red-Black Trees

Chetan Moturi (CS21B017)
Mallela Vidhyabhushan (CS21B030)

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Chapter 1

Introduction

A binary tree search tree of height can support basic operations like search, insert, delete in logarithmic time, $O(\lg n)$, on average. However, the worst case time complexity of these operations is $O(n)$. This is because the tree can degenerate into a linked list. To avoid this, we can use a self-balancing binary search tree. A self-balancing binary search tree is a binary search tree in which the height of the tree is $O(\lg n)$. The height of a tree is the maximum number of edges from the root to a leaf.

One such self balancing tree is called the Red-Black Tree. A red-black tree is a binary search tree in which each node has an extra bit, and that bit is often interpreted as the color (red or black) of the node. By constraining how the colors are used in any particular tree and how the tree is modified, we can ensure that no path from root to leaf is more than twice as long as any other path, so that the tree is approximately balanced.

Fun Fact! The color "red" was chosen because it was the best-looking color produced by the color laser printer available to the authors while working at Xerox PARC.

Definition 1.0.1: Limit of Sequence in $\mathbb{R}$

Let $\{s_n\}$ be a sequence in $\mathbb{R}$. We say

$$\lim_{n \to \infty} s_n = s$$

where $s \in \mathbb{R}$ if $\forall$ real numbers $\epsilon > 0$ $\exists$ natural number N such that for $n > N$

$$s - \epsilon < s_n < s + \epsilon \text{ i.e. } |s - s_n| < \epsilon$$

Question 1

Is the set $x-$axis$\backslash\{$Origin$\}$ a closed set

Solution: We have to take its complement and check whether that set is a open set i.e. if it is a union of open balls

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Note:-

We will do topology in Normed Linear Space (Mainly $\mathbb{R}^n$ and occasionally $\mathbb{C}^n$)using the language of Metric Space
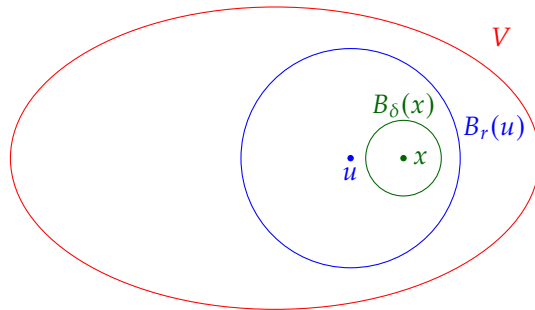

Claim 1.0.1 Topology

Topology is cool

Example 1.0.1 (Open Set and Close Set)

Open Set:
- ϕ
- $\bigcup_{x \in X} B_r(x)$ (Any $r > 0$ will do)
- $B_r(x)$ is open

Closed Set:
- $X, \ \phi$
- $\overline{B_r(x)}$

$x-$axis $\cup \ y-$axis

Theorem 1.0.1
If $x \in$ open set V then $\exists \ \delta > 0$ such that $B_\delta(x) \subset V$

Proof: By openness of V, $x \in B_r(u) \subset V$

Given $x \in B_r(u) \subset V$, we want $\delta > 0$ such that $x \in B_\delta(x) \subset B_r(u) \subset V$. Let $d = d(u,x)$. Choose δ such that $d + \delta < r$ (e.g. $\delta < \frac{r-d}{2}$)

If $y \in B_\delta(x)$ we will be done by showing that $d(u,y) < r$ but

$$d(u,y) \leqslant d(u,x) + d(x,y) < d + \delta < r$$

☺

Corollary 1.0.1
By the result of the proof, we can then show...

Lenma 1.0.1
Suppose $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^n$ is subspace of $\mathbb{R}^n$.

Proposition 1.0.1
$1 + 1 = 2$.

1.1 Random

> **Definition 1.1.1: Normed Linear Space and Norm $\| \cdot \|$**
>
> Let V be a vector space over $\mathbb{R}$ (or $\mathbb{C}$). A norm on V is function $\| \cdot \| \ V \to \mathbb{R}_{\geqslant 0}$ satisfying
>
> ① $\|x\| = 0 \iff x = 0 \ \forall \ x \in V$
>
> ② $\|\lambda x\| = |\lambda| \|x\| \ \forall \ \lambda \in \mathbb{R}(\text{or } \mathbb{C}), \ x \in V$
>
> ③ $\|x + y\| \leqslant \|x\| + \|y\| \ \forall \ x, y \in V$ (Triangle Inequality/Subadditivity)
>
> And V is called a normed linear space.
> • Same definition works with V a vector space over $\mathbb{C}$ (again $\|\cdot\| \to \mathbb{R}_{\geqslant 0}$) where ② becomes $\|\lambda x\| = |\lambda| \|x\|$
> $\forall \ \lambda \in \mathbb{C}, \ x \in V$, where for $\lambda = a + ib$, $|\lambda| = \sqrt{a^2 + b^2}$

> **Example 1.1.1 ($p-$Norm)**
> $V = \mathbb{R}^m$, $p \in \mathbb{R}_{\geqslant 0}$. Define for $x = (x_1, x_2, \cdots, x_m) \in \mathbb{R}^m$
>
> $$\|x\|_p = \left(|x_1|^p + |x_2|^p + \cdots + |x_m|^p \right)^{\frac{1}{p}}$$
>
> (In school $p = 2$)

    Special Case $p = 1$: $\|x\|_1 = |x_1| + |x_2| + \cdots + |x_m|$ is clearly a norm by usual triangle inequality.
    Special Case $p \to \infty$ ($\mathbb{R}^m$ with $\|\cdot\|_\infty$): $\|x\|_\infty = \max\{|x_1|, |x_2|, \cdots, |x_m|\}$
For $m = 1$ these $p-$norms are nothing but $|x|$. Now exercise

> **Question 2**
>
> Prove that triangle inequality is true if $p \geqslant 1$ for $p-$norms. (What goes wrong for $p < 1$?)

Solution: **For Property** ③ **for norm-2**

When field is $\mathbb{R}$:

We have to show

$$\sum_i (x_i + y_i)^2 \leqslant \left(\sqrt{\sum_i x_i^2} + \sqrt{\sum_i y_i^2} \right)^2$$

$$\implies \sum_i (x_i^2 + 2x_i y_i + y_i^2) \leqslant \sum_i x_i^2 + 2\sqrt{\left[\sum_i x_i^2 \right] \left[\sum_i y_i^2 \right]} + \sum_i y_i^2$$

$$\implies \left[\sum_i x_i y_i \right]^2 \leqslant \left[\sum_i x_i^2 \right] \left[\sum_i y_i^2 \right]$$

So in other words prove $\langle x, y \rangle^2 \leqslant \langle x, x \rangle \langle y, y \rangle$ where

$$\langle x, y \rangle = \sum_i x_i y_i$$

> **Note:-**
>
> • $\|x\|^2 = \langle x, x \rangle$
>
> • $\langle x, y \rangle = \langle y, x \rangle$

- $\langle \cdot, \cdot \rangle$ is $\mathbb{R}$−linear in each slot i.e.

$$\langle rx + x', y \rangle = r\langle x, y \rangle + \langle x', y \rangle \text{ and similarly for second slot}$$

Here in $\langle x, y \rangle$ x is in first slot and y is in second slot.

Now the statement is just the Cauchy-Schwartz Inequality. For proof

$$\langle x, y \rangle^2 \leqslant \langle x, x \rangle \langle y, y \rangle$$

expand everything of $\langle x - \lambda y, x - \lambda y \rangle$ which is going to give a quadratic equation in variable λ

$$\begin{aligned}
\langle x - \lambda y, x - \lambda y \rangle &= \langle x, x - \lambda y \rangle - \lambda \langle y, x - \lambda y \rangle \\
&= \langle x, x \rangle - \lambda \langle x, y \rangle - \lambda \langle y, x \rangle + \lambda^2 \langle y, y \rangle \\
&= \langle x, x \rangle - 2\lambda \langle x, y \rangle + \lambda^2 \langle y, y \rangle
\end{aligned}$$

Now unless $x = \lambda y$ we have $\langle x - \lambda y, x - \lambda y \rangle > 0$ Hence the quadratic equation has no root therefore the discriminant is greater than zero.

When field is $\mathbb{C}$:

Modify the definition by

$$\langle x, y \rangle = \sum_i \overline{x_i} y_i$$

Then we still have $\langle x, x \rangle \geqslant 0$

1.2   Algorithms

Algorithm 1: what

Input: This is some input
Output: This is some output
/* This is a comment */

1 some code here;
2 $x \leftarrow 0$;
3 $y \leftarrow 0$;
4 **if** $x > 5$ **then**
5 | x is greater than 5 ;   // This is also a comment
6 **else**
7 | x is less than or equal to 5;
8 **end**
9 **foreach** y *in 0..5* **do**
10 | $y \leftarrow y + 1$;
11 **end**
12 **for** y *in* 0..5 **do**
13 | $y \leftarrow y - 1$;
14 **end**
15 **while** $x > 5$ **do**
16 | $x \leftarrow x - 1$;
17 **end**
18 **return** Return something here;