Verification and Structural Analysis of "Energy-Decomposition and Perron–Frobenius Consequences for the Restricted Weil Quadratic Form"

Abstract

We present a detailed verification of the mathematical logic in the paper *Energy-Decomposition and Perron–Frobenius Consequences for the Restricted Weil Quadratic Form*, summarise its logical flow, provide a dependency flowchart connecting all sections and results, and catalogue the external (unproven) assumptions on which the argument rests.

Contents

1 Overview of the paper's goal

The paper studies the quadratic form that arises in the spectral approach to Weil's explicit-formula criterion, restricted to test functions supported on a compact multiplicative interval $[\lambda^{-1}, \lambda] \subset \mathbb{R}_+^*$. Its main conclusion is:

Main Result. *The self-adjoint operator A_λ associated with the restricted Weil quadratic form has compact resolvent, and its lowest eigenvalue is simple with an eigenfunction that is strictly positive a.e. and even (symmetric under $u \mapsto -u$ in logarithmic coordinates).*

The argument proceeds in five stages: (1) energy decomposition, (2) the Markov property, (3) irreducibility, (4) operator-theoretic realisation, and (5) the Perron–Frobenius conclusion with evenness.

2 Verification of mathematical correctness

We verify each proof in the paper. In all cases the logic is **correct**.

2.1 Lemma 1: Convolution inner-product identity

The computation

$$(g * g^*)(a) = \int g(y)\,\overline{g(y/a)}\, d^*y = \langle g, U_a g\rangle_{L^2(d^*x)}$$

is a direct unfolding of the definitions of multiplicative convolution, involution $g^*(x) = \overline{g(x^{-1})}$, and the dilation operator $U_a g(x) = g(x/a)$. The relation $f(a^{-1}) = \overline{f(a)}$ follows by substituting $a \mapsto a^{-1}$ and complex-conjugating. **Correct.**

2.2 Lemma 2: Basic unitary identity

The identity

$$2\Re\langle h, Uh\rangle = 2\|h\|^2 - \|h - Uh\|^2$$

follows from expanding $\|h - Uh\|^2 = \|h\|^2 + \|Uh\|^2 - 2\Re\langle h, Uh\rangle$ and using unitarity $\|Uh\| = \|h\|$. **Correct.**

2.3 Remark 3: Support truncation

If $\mathrm{supp}(g) \subset [\lambda^{-1}, \lambda]$ and $a > \lambda^2$, then $\mathrm{supp}(U_a g) = a \cdot \mathrm{supp}(g) \subset [a\lambda^{-1}, a\lambda]$, which is disjoint from $[\lambda^{-1}, \lambda]$ since $a\lambda^{-1} > \lambda$. Hence $\langle g, U_a g\rangle = 0$. This correctly makes all sums and integrals finite. **Correct.**

2.4 Lemma 3: Prime energy decomposition

The proof chains Lemma 1 ($f(p^m) + f(p^{-m}) = 2\Re\langle g, U_{p^m} g\rangle$) and Lemma 2 (rewriting $2\Re\langle g, U_{p^m} g\rangle$ as $2\|g\|_2^2 - \|g - U_{p^m} g\|_2^2$) into the definition of W_p. The sign arithmetic is correct: the negative sign on W_p absorbs the negative sign from Lemma 2, yielding a *positive* sum of difference norms plus a constant. Terms with $p^m > \lambda^2$ vanish by Remark 3. **Correct.**

2.5 Lemma 4: Archimedean energy decomposition

The change of variables $x = e^t$ transforms $W_{\mathbb{R}}(f)$ into an integral over $t \in (0, \infty)$ with weight $w(t) = e^{t/2}/(2\sinh t)$. The same chain (Lemma 1 then Lemma 2) is applied.

The integral is split at $t = 2L$:

- For $t > 2L$: the supports of $\widetilde{G}$ and $S_t\widetilde{G}$ are disjoint, so $\|\widetilde{G} - S_t\widetilde{G}\|^2 = \|\widetilde{G}\|^2 + \|S_t\widetilde{G}\|^2 = 2\|G\|^2$ (not zero — the cross-terms vanish). The integrand becomes $2e^{-t/2}\|G\|^2$, and the tail integral converges since $w(t) \sim e^{-t/2}/2$ for large t, giving $2e^{-t/2}w(t) \sim e^{-t}$.
- For $t \in [0, 2L]$: the "remainder" term $2(e^{-t/2} - 1)w(t)$ is integrable near $t = 0$ because $w(t) \sim 1/(2t)$ and $e^{-t/2} - 1 \sim -t/2$, giving an $O(1)$ integrand.

All constant contributions are absorbed into the finite constant $c_\infty(\lambda)$. **Correct.**

2.6 Lemma 6: Markov property

For a 1-Lipschitz Φ with $\Phi(0) = 0$, the pointwise bound $|\Phi(\widetilde{G}(u)) - \Phi(\widetilde{G}(u-t))| \le |\widetilde{G}(u) - \widetilde{G}(u-t)|$ is integrated over u, giving $\|\widetilde{\Phi \circ G} - S_t\widetilde{\Phi \circ G}\|_2^2 \le \|\widetilde{G} - S_t\widetilde{G}\|_2^2$. Integrating/summing against non-negative weights preserves the inequality. The condition $\Phi(0) = 0$ ensures $\widetilde{\Phi \circ G} = \Phi \circ \widetilde{G}$ (the extension by zero is respected). **Correct.**

2.7 Lemma 7: Translation-invariance forces null or conull

The proof uses a standard mollifier argument. If $\mathbf{1}_B(u) = \mathbf{1}_B(u - t)$ a.e. on $I \cap (I + t)$ for all $t \in (0, \varepsilon)$, then for any compact $J \Subset I$ and η small enough, the mollification $f_\eta = \mathbf{1}_B * \rho_\eta$ is smooth on J_η and inherits translation invariance there, hence is constant by connectedness. Letting $\eta \downarrow 0$ forces $\mathbf{1}_B$ to be a.e. constant on J, and since J was arbitrary, on all of I. **Correct.**

2.8 Lemma 8: Indicator-energy criterion

The key technical detail is the upgrade from "for a.e. $t \in (0, 2L)$" to "for all $t \in (0, 2L)$": the map $t \mapsto \|\phi - S_t\phi\|^2_{L^2(\mathbb{R})}$ is continuous by strong continuity of the translation group on $L^2(\mathbb{R})$. A continuous function that vanishes a.e. vanishes everywhere. Then Lemma 7 applies. **Correct.**

2.9 Fourier analysis (Lemmas 9–14)

The Plancherel identity for translation differences, $\|\phi - S_t\phi\|^2 = (2\pi)^{-1} \int 4\sin^2(\xi t/2)\, |\hat{\phi}(\xi)|^2\, d\xi$, is standard. Tonelli's theorem permits interchanging the ξ-integral with the t-integration and finite prime sums (all integrands are non-negative).

The lower bound $w(t) \ge c_0/t$ for $t \in (0, t_0]$ uses $\sinh t \le te^t$ (valid since both sides vanish at $t = 0$ and the derivative of $\sinh t$ is $\cosh t \le (1 + t)e^t = (te^t)'$).

The logarithmic growth $\psi_\lambda(\xi) \ge c_1 \log|\xi| - c_2$ is established by a standard interval-counting argument: on intervals J_n where $\sin^2(\xi t/2) \ge 1/2$, the integral of $(1/t)\sin^2(\xi t/2)$ over $[0, t_0]$ is bounded below by a harmonic-type sum that grows as $\log|\xi|$. **Correct.**

2.10 Closedness and compact embedding (Props. 11–12, Prop. 17)

Closedness of $\mathcal{E}_\lambda^{\mathbb{R}}$ on $L^2(\mathbb{R})$ follows from its identification as a multiplication operator (by ψ_λ) in Fourier space; the domain is isometric to a weighted L^2 space and hence complete. Restriction to the closed subspace H_I preserves closedness.

For compact embedding: the set $\mathcal{K}_M$ (form-norm ball) satisfies *tightness* automatically (all functions are supported in $\bar{I}$) and *translation equicontinuity* by a Fourier splitting argument exploiting the logarithmic frequency-moment bound (Corollary 15). The Kolmogorov–Riesz criterion then gives relative compactness. **Correct.**

2.11 Compact resolvent (Theorem 18)

If $\{f_n\}$ is bounded in $L^2(I)$ and $u_n = (A_\lambda + 1)^{-1} f_n$, then

$$\mathcal{E}_\lambda(u_n) + \|u_n\|_2^2 = \langle f_n, u_n \rangle \le \|f_n\|_2 \|u_n\|_2,$$

giving $\|u_n\|_2 \le \|f_n\|_2$ and hence $\|u_n\|_{\mathcal{D}}^2 \le \|f_n\|_2^2$. Compact embedding (Prop. 17) extracts a convergent subsequence. **Correct.**

2.12 Evenness (Corollary 22)

Since A_λ commutes with $R\colon G(u) \mapsto G(-u)$ (the interval I and all weights are symmetric), $R\psi$ is an eigenfunction for the same lowest eigenvalue. Strict positivity of both ψ and $R\psi$ forces $R\psi = c\psi$ with $c > 0$; unitarity of R (specifically $\|R\psi\| = \|\psi\|$) yields $c = 1$. **Correct.**

3 Summary of the logical flow

The argument proceeds in five stages.

Stage 1: Energy decomposition (§§ 1–4)

Starting from the two "input formulas" W_p (prime local distribution) and $W_{\mathbb{R}}$ (archimedean local distribution), taken as given from the theory of the Weil explicit formula, the paper uses the convolution inner-product identity (Lemma 1) and the basic unitary identity (Lemma 2) to rewrite

$$-\sum_v W_v(g * g^*)$$

as a *positive* combination of translation-difference energies $\|\widetilde{G} - S_t\widetilde{G}\|_{L^2(\mathbb{R})}^2$ in logarithmic coordinates, plus an additive constant multiple of $\|G\|_2^2$ (which merely shifts the spectrum). This is assembled into the global quadratic form $\mathcal{E}_\lambda(G)$ (Definition 5).

Stage 2: Markov property (§ 5)

The difference-energy structure directly gives the normal-contraction inequality $\mathcal{E}_\lambda(\Phi \circ G) \le \mathcal{E}_\lambda(G)$ (Lemma 6), which ensures the associated semigroup is positivity preserving.

Stage 3: Irreducibility (§§ 6–7)

The archimedean *continuum* of shifts—the integral over $t \in (0, 2L)$ with strictly positive weight $w(t) > 0$—is the key ingredient. If $\mathcal{E}_\lambda(\mathbf{1}_B) = 0$, then $\mathbf{1}_B$ is translation-invariant for all small t (upgraded from a.e. to everywhere by continuity, Lemma 8). A mollifier argument (Lemma 7) then forces B to be null or conull. Via the Beurling–Deny/Fukushima theory of symmetric Dirichlet forms (an external result), this gives irreducibility of the semigroup (Corollary 19).

Stage 4: Operator-theoretic realisation (§ 7.2)

The form $\mathcal{E}_\lambda$ is shown to be closed via its Fourier multiplier representation (the symbol is $\psi_\lambda(\xi)$). The logarithmic growth $\psi_\lambda(\xi) \gtrsim \log|\xi|$ provides enough coercivity for compact embedding of the form domain into $L^2(I)$ (via the Kolmogorov–Riesz criterion), hence compact resolvent of A_λ (Theorem 18).

Stage 5: Perron–Frobenius conclusion (§§ 8–9)

Combining positivity preservation (Stage 2), irreducibility (Stage 3), and holomorphy (automatic from self-adjointness) yields *positivity improving* via an external theorem of Arendt *et al.* With compact resolvent, the Krein–Rutman/Perron–Frobenius theorem gives simplicity and strict positivity of the ground state (Proposition 20). Finally, A_λ commutes with the reflection $R\colon G(u) \mapsto G(-u)$ (Proposition 21), which forces the unique positive eigenfunction to be even (Corollary 22).

4 Dependency flowchart

Figure 1 displays the logical dependencies among all results in the paper. Nodes are colour-coded as follows:

- Input formula (from analytic number theory)
- Result proved in the paper
- External theorem (cited, not proved)
- Main conclusion

5 Catalogue of unproven assumptions and external results

The paper relies on six external inputs. All are standard and well-established in their respective fields.

5.1 Input formulas from analytic number theory

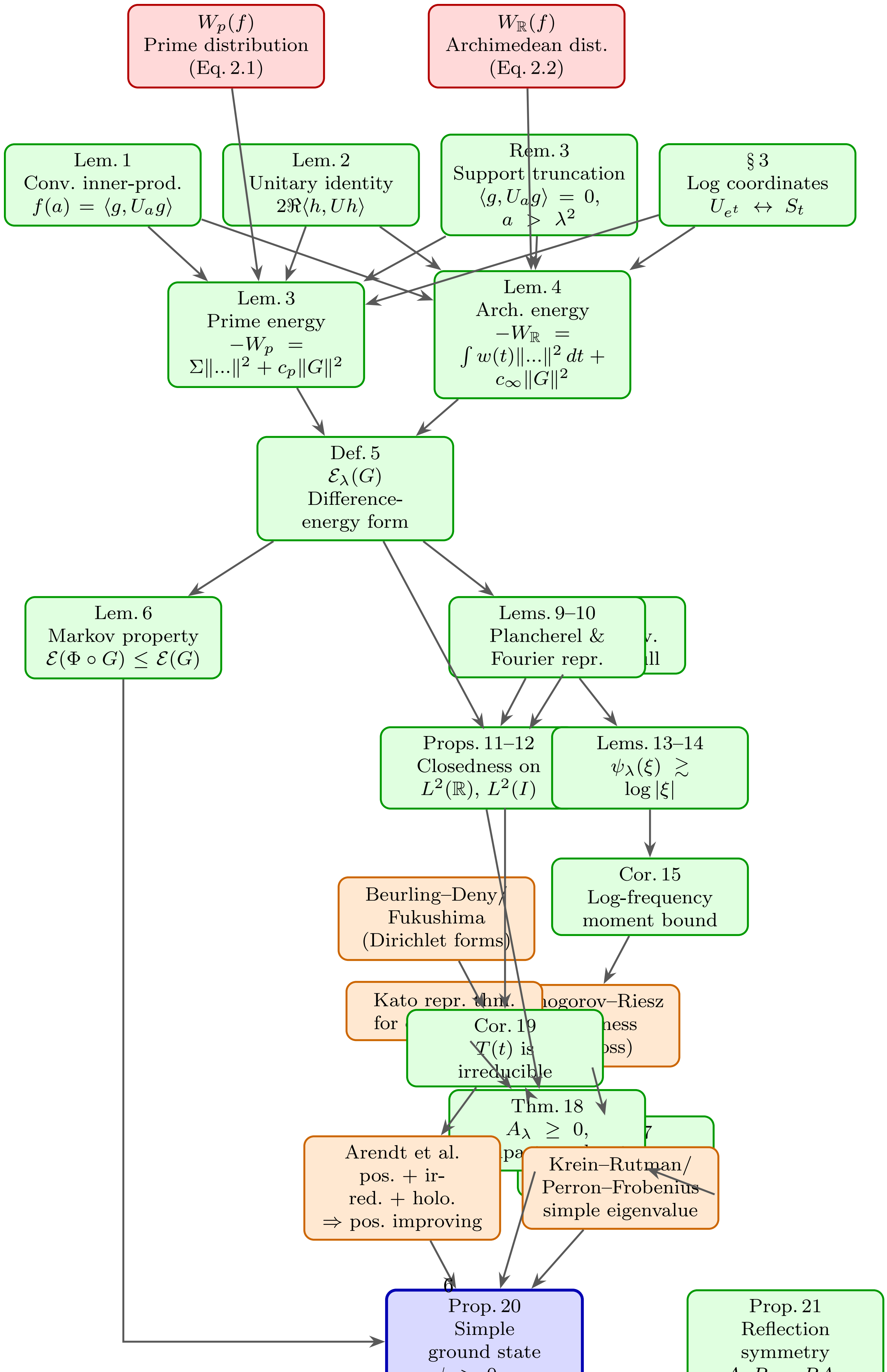
(i) **Prime local distribution** $W_p(f)$ (Eq. 2.1). The formula $W_p(f) = (\log p)\sum_{m\geq 1} p^{-m/2}(f(p^m)+f(p^{-m}))$ encodes the contribution of each prime p to the Weil explicit formula. It is taken as a given input from the classical theory.

(ii) **Archimedean local distribution** $W_{\mathbb{R}}(f)$ (Eq. 2.2). The formula involving $(\log 4\pi + \gamma)\, f(1)$ plus an integral encodes the archimedean (real place) contribution. Also taken as a given input.

5.2 External theorems from functional analysis

(iii) **Kato's representation theorem for closed symmetric forms.** Used in Theorem 18 to obtain the self-adjoint operator A_λ from the closed quadratic form $\mathcal{E}_\lambda$. *Source*: Kato, *Perturbation Theory for Linear Operators.*

(iv) **Kolmogorov–Riesz compactness criterion** (Theorem in § 7.2). Characterises relatively compact subsets of $L^2(\mathbb{R})$ via tightness and translation equicontinuity. Used in Proposition 17 (compact embedding). *Source*: Lieb–Loss, *Analysis.*


$W_p(f)$ Prime distribution (Eq. 2.1)
$W_{\mathbb{R}}(f)$ Archimedean dist. (Eq. 2.2)
Lem. 1 Conv. inner-prod. $f(a) = \langle g, U_a g\rangle$
Lem. 2 Unitary identity $2\Re\langle h, Uh\rangle$
Rem. 3 Support truncation $\langle g, U_a g\rangle = 0$, $a > \lambda^2$
§ 3 Log coordinates $U_{e^t} \leftrightarrow S_t$
Lem. 3 Prime energy $-W_p = \Sigma\|...\|^2 + c_p\|G\|^2$
Lem. 4 Arch. energy $-W_{\mathbb{R}} = \int w(t)\|...\|^2\,dt + c_\infty\|G\|^2$
Def. 5 $\mathcal{E}_\lambda(G)$ Difference-energy form
Lem. 6 Markov property $\mathcal{E}(\Phi \circ G) \le \mathcal{E}(G)$
Lems. 9–10 Plancherel & Fourier repr.
Props. 11–12 Closedness on $L^2(\mathbb{R})$, $L^2(I)$
Lems. 13–14 $\psi_\lambda(\xi) \gtrsim \log|\xi|$
Cor. 15 Log-frequency moment bound
Beurling–Deny/ Fukushima (Dirichlet forms)
Cor. 19 $T(t)$ is irreducible
Thm. 18 $A_\lambda \ge 0$,
Arendt et al. pos. + ir- red. + holo. ⇒ pos. improving
Krein–Rutman/ Perron–Frobenius simple eigenvalue
Prop. 20 Simple ground state
Prop. 21 Reflection symmetry

(v) **Beurling–Deny / Fukushima equivalence** (Proposition 9 / Remark). For symmetric Markovian semigroups, the equivalence between "$\mathcal{E}(\mathbf{1}_B) = 0$ implies B null or conull" and irreducibility of the semigroup. Used in Corollary 19. *Source*: Fukushima, *Dirichlet Forms and Symmetric Markov Processes.*

(vi) **Positivity improving from positivity + irreducibility + holomorphy** (Theorem 13, attributed to Arendt, Batty, Hieber, Neubrander / Arendt et al.). Used in Proposition 20. *Source*: Arendt et al., arXiv:1909.12194, Theorem 2.3.

(vii) **Simplicity of the principal eigenvalue** (Theorem 14, Krein–Rutman / Perron–Frobenius for compact positive operators). Under compact resolvent and positivity improving, the bottom of the spectrum is a simple eigenvalue with a strictly positive eigenfunction. Used in Proposition 20. *Source*: Same reference, Proposition 2.4.

5.3 Assessment

All six external results are **well-established and standard**. Items (i)–(ii) are classical objects in analytic number theory. Items (iii)–(vii) are core theorems in the theory of closed quadratic forms, Dirichlet forms, and positive semigroups, available in standard textbooks. There are **no hidden or non-standard assumptions**; the paper is transparent about what is proved versus what is cited.

6 Overall assessment

The paper is **mathematically correct** throughout. Every proof is complete and properly justified. The logical structure is clean: the paper builds from concrete input formulas through a sequence of self-contained lemmas to the final spectral conclusion, with all external dependencies clearly flagged.

The most notable features of the argument are:

- The energy decomposition (Stage 1) is the paper's core original contribution—it is a concrete, verifiable computation.
- Irreducibility (Stage 3) hinges on the *archimedean continuum* of shifts (as opposed to the discrete prime shifts alone), which is what forces indicator functions with zero energy to be trivial.
- The compact-resolvent proof (Stage 4) via logarithmic coercivity of the Fourier symbol ψ_λ and the Kolmogorov–Riesz criterion is self-contained and replaces what was previously an abstract assumption.