External Theorem Audit

Energy Decomposition, Compact Resolvent, and Perron–Frobenius Properties of the Restricted Weil Quadratic Form

Abstract

This document lists every external mathematical result invoked in the paper, together with (i) the precise statement assumed, (ii) the exact bibliographic reference cited, and (iii) notes for the human verifier. Its purpose is to make cross-checking against the original sources as mechanical as possible. The audit has been conducted in three rounds: initial identification; a first literature search locating specific theorem numbers; and a second deep literature search with extended source analysis, including direct quotation from secondary sources. Items marked [!] still require physical confirmation against the book: the theorem number has been located by search but not verified against the actual page. Items marked [✓] have been confirmed by multiple independent sources (some with direct quotation); a physical spot-check is still advisable but the result is expected correct.

Bibliography keys used below.

- `KatoPerturbation`: T. Kato, *Perturbation Theory for Linear Operators*, Classics in Mathematics, Springer, 1995. DOI: 10.1007/978-3-642-66282-9.

  Note: the 1995 printing is a reprint of the 1980 second edition; theorem numbers VI.2.1, VI.2.23, IX.1.24, and Ex. IX.1.25 are consistent with this edition. **Correction**: "IX.1.4" cited in earlier audit versions is Remark 1.4 (unrelated); the correct holomorphic semigroup references are Thm. IX.1.24 and Ex. IX.1.25. Numbering differs from the 1966 first edition.
- `FOT`: M. Fukushima, Y. Oshima, M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*, 2nd revised ed., De Gruyter Studies in Math. vol. 19, De Gruyter, 2011. DOI: 10.1515/9783110218091.

  Note: publication year is 2011 per De Gruyter catalogue; some sources cite 2010.
- `ATGStrictPos2020`: W. Arendt, A. F. M. ter Elst, J. Glück, "Strict positivity for the principal eigenfunction...", *Adv. Nonlinear Stud.* **20** (2020), no. 3, 633–650. DOI: 10.1515/ans-2020-2091.

  Use the published DOI (not the arXiv preprint 1909.12194).
- `OuhabazHeatEq`: E.-M. Ouhabaz, *Analysis of Heat Equations on Domains*, LMS Monographs vol. 31, Princeton UP, 2005. ISBN: 978-0-691-12016-4.
- `BrezisFA`: H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Universitext, Springer, 2011. ISBN: 978-0-387-70913-0. DOI: 10.1007/978-0-387-70914-7.

  Note: cover reads "2010" but copyright/print year is 2011. This is the correct source for T2. *(New entry; add to bibliography.)*
- `LiebLossAnalysis`: E. H. Lieb, M. Loss, *Analysis*, 2nd ed., Grad. Studies in Math. vol. 14, AMS, 2001. ISBN: 978-0-8218-2783-3. DOI: 10.1090/gsm/014.

  Does not contain the Kolmogorov–Riesz theorem. The "Thm. 2.13" number was a citation error: that number belongs to *Ground States of Quantum Field Models* (Hiroshima 2019), a secondary source which internally numbers a version of the result as 2.13. Remove `LiebLossAnalysis` from the T2 citation.
- `SchaeferBanachLattices`: H. H. Schaefer, *Banach Lattices and Positive Operators*, Grundlehren vol. 215, Springer, 1974. ISBN: 978-3-540-06936-2. DOI: 10.1007/978-3-642-65970-6. *(New entry; add to bibliography.)*

#	Internal label / where used	Statement assumed in this paper	Cited reference	Notes for verifier
FORM / OPERATOR CORRESPONDENCE				
T1 [✓]	**Representation theorem for closed forms** *Used in:* `thm:operator` Step 1 (first occurrence)	If $\mathcal{E}$ is a densely defined, closed, lower-bounded, symmetric form on a Hilbert space H, then there exists a unique selfadjoint operator $A \geq 0$ such that $\mathcal{D}(\mathcal{E}) = \mathcal{D}(A^{1/2})$ and $\mathcal{E}(u,v) = \langle A^{1/2}u, A^{1/2}v\rangle$ for all $u, v \in \mathcal{D}(\mathcal{E})$.	`KatoPerturbation` *Thm. VI.2.1* Also: `KatoPerturbation` *Thm. VI.2.23* Also: `FOT` *Thm. 1.3.1*	[✓] **Physically confirmed** from uploaded Kato and FOT. Kato VI.2.1 (*First* Representation Theorem): existence of unique m-sectorial T with $t[u,v] = (Tu,v)$ for $u \in D(T)$, $v \in D(t)$. Kato VI.2.23 (*Second* Representation Theorem): $D(H^{1/2}) = D(\mathfrak{h})$ and $\mathfrak{h}[u,v] = (H^{1/2}u, H^{1/2}v)$. Both theorems together give the full statement. FOT Thm. 1.3.1 (physically confirmed): one-to-one correspondence between closed symmetric forms and non-positive definite selfadjoint operators on H; Corollary 1.3.1 gives the domain characterisation $\mathcal{E}(u,v) = (-Au, v)$ for $u \in D(A)$.

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#	Internal label / where used	Statement assumed in this paper	Cited reference	Notes for verifier
T1′ [✓]	**Representation theorem (domain characterisation)** *Used in:* `prop:reflection` Step 5.4	$w \in \mathcal{D}(A)$ if and only if there exists $h \in H$ such that $\mathcal{E}(w, v) = \langle h, v\rangle$ for all $v \in \mathcal{D}(\mathcal{E})$; in that case $Aw = h$. (Applied to conclude $Ru \in \mathcal{D}(A_\lambda)$ and $A_\lambda(Ru) = R(A_\lambda u)$.)	`KatoPerturbation` *Thm. VI.2.1 (iii)* (Same reference as T1)	[✓] **Physically confirmed** from uploaded Kato. Kato Thm. VI.2.1 condition (iii) states explicitly: "if $u \in D(t)$, $w \in H$, and $t[u, v] = (w, v)$ for every v in a core of t, then $u \in D(T)$ and $Tu = w$." This is the "if" direction. The "only if" direction follows from condition (i): $t[u, v] = (Tu, v)$ for $u \in D(T)$. Together conditions (i) and (iii) give the full "iff", validating Step 5.4.
Compactness				
T2 [✓]	**Kolmogorov–Riesz–Fréchet compactness criterion** *Stated as:* `thm:KR` *Applied in:* `prop:compactEmbed` Step 4	**Brezis Thm. 4.26** (bounded Ω, no tightness needed): F bounded in $L^p(\mathbb{R}^N)$ + translation equicontinuity $\Rightarrow F\|_\Omega$ has compact closure in $L^p(\Omega)$ for any measurable Ω with finite measure. **Brezis Cor. 4.27** (full $\mathbb{R}^N$, tightness required): Adds condition $\forall \varepsilon > 0\ \exists \Omega \subset \mathbb{R}^N$ bounded such that $\\|f\\|_{L^p(\mathbb{R}^N \setminus \Omega)} < \varepsilon\ \forall f \in F \Rightarrow F$ compact in $L^p(\mathbb{R}^N)$. (Remark 13 confirms the converse holds; this is the complete characterization.)	`BrezisFA` *Thm. 4.26 & Cor. 4.27, p. 111–113* (physically confirmed from uploaded book pages)	[✓] **Physically confirmed** from uploaded book pages. **Critical distinction:** If the compactness argument is on a bounded domain (finite measure), Thm. 4.26 alone suffices. If on $L^2(\mathbb{R})$ (full line), Cor. 4.27 is required; the proof must separately establish tightness (uniform L^2-tail decay). **Citation correction:** `LiebLossAnalysis` does not contain this theorem. "Thm. 2.13" originated in *Ground States of Quantum Field Models* (Hiroshima 2019) with its own internal numbering. Replace with: `\cite[Cor.~4.27]{BrezisFA}`.

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MARKOV PROPERTY AND POSITIVITY				
T3 [✓]	**Markov form ⇒ positivity-preserving semigroup** *Used in:* `prop:groundstate` Step 1	If a closed symmetric form $\mathcal{E}$ satisfies the normal contraction (Markov) property $\mathcal{E}(\Phi \circ G) \leq \mathcal{E}(G)$ for every normal contraction Φ, then the associated C_0-semigroup $T(t) = e^{-tA}$ is positivity preserving (i.e. $G \geq 0$ a.e. implies $T(t)G \geq 0$ a.e.).	`FOT` *Thm. 1.4.1* Also: `OuhabazHeatEq` *Thm. 1.4.1*	[✓] **Physically confirmed** from uploaded FOT. FOT Thm. 1.4.1 (physically verified): lists five equivalent conditions (a)–(e) for the Markovian property, including (a) T_t Markovian $\forall t > 0$; (c) $\mathcal{E}$ Markovian; (e) every normal contraction operates on $\mathcal{E}$. The five conditions are mutually equivalent; the implication used in the manuscript is (e)⇒(a) [or (c)⇒(a)], which is direct. Ouhabaz Thm. 1.4.1 treats the more general sectorial setting.
T4 [✓]	**Positivity improving from positivity + irreducibility + holomorphy** *Stated as:* `thm:ABHN` *Applied in:* `prop:groundstate` Step 4	Let E be a Banach lattice and S a positive, irreducible, holomorphic C_0-semigroup on E. Then S is *positivity improving*: for each $t > 0$ and each $0 \leq f \in E$, $f \neq 0$, one has $S(t)f > 0$ (on L^2 this means > 0 a.e.).	`ATGStrictPos2020` *Thm. 2.3*	[✓] Confirmed by deep literature search against published paper. ATG Thm. 2.3 is stated for general Banach lattice C_0-semigroups (not restricted to elliptic generators). **Holomorphy dependency chain**: selfadjoint $A_\lambda \geq 0$ ⇒ m-sectorial (T6, Kato IX.1.4) ⇒ holomorphic semigroup ⇒ hypothesis of T4 satisfied. Without T6, this application would be unjustified.

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#	Internal label / where used	Statement assumed in this paper	Cited reference	Notes for verifier
T5 [✓]	**Simplicity of principal eigenvalue under positivity improving + compact resolvent** *Stated as:* `thm:principal-simple` *Applied in:* `prop:groundstate` Step 5	Let A be selfadjoint, lower bounded, with compact resolvent on $L^2(I)$. If e^{-tA} is positivity improving, then $\min \sigma(A)$ is a simple eigenvalue admitting an eigenfunction strictly positive a.e.	`ATGStrictPos2020` *Prop. 2.4*	[✓] Deep audit confirms ATG Prop. 2.4 requires exactly positivity-improving + compact resolvent (no additional hypotheses). The three explicit conclusions stated in the paper are: (a) the spectral bound $s(-A)$ is an eigenvalue of $-A$; (b) the associated eigenfunction is strictly positive ($\psi \gg 0$); (c) the algebraic multiplicity is one. Compact resolvent $\Rightarrow e^{-tA}$ compact for $t > 0$ (spectral theorem), which supplies the compactness needed for the Perron–Frobenius/Krein–Rutman argument.
Standard results invoked without explicit citation				

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#	Internal label / where used	Statement assumed in this paper	Cited reference	Notes for verifier
T6 [✓]	**Selfadjoint $A \geq 0$ generates holomorphic semigroup** *Used in:* `prop:groundstate` Step 3	If A is selfadjoint with $A \geq 0$, then e^{-zA} is a bounded operator for all z with $\operatorname{Re} z > 0$, $\\|e^{-zA}\\| \leq 1$, and $z \mapsto e^{-zA}$ is holomorphic on $\{\operatorname{Re} z > 0\}$.	`KatoPerturbation` *Thm. IX.1.24* *Ex. IX.1.25* (**correction**: previous citation "IX.1.4" was wrong)	[✓] **Physically confirmed** from uploaded Kato. **Critical correction**: "Thm. IX.1.4" does **not exist** in Kato. Label IX.1.4 is *Remark 1.4* (about recovering a generator from a contraction semigroup — unrelated to holomorphic semigroups). **Correct references (both physically confirmed):** • **Kato Thm. IX.1.24**: Let T be m-sectorial in a Hilbert space with vertex 0 and half-angle $\omega \in (0, \frac{\pi}{2}]$. Then e^{-tT} is holomorphic for $\|\arg t\| < \omega$ and $\\|e^{-tT}\\| \leq 1$. • **Kato Ex. IX.1.25** (the directly applicable result): "If H is a nonneg. selfadjoint operator in a Hilbert space, e^{-tH} is holomorphic for $\operatorname{Re} t > 0$ and $\\|e^{-tH}\\| \leq 1$." Logical chain: selfadjoint $A \geq 0$ has numerical range in $[0, \infty)$, hence is m-sectorial with any $\omega < \frac{\pi}{2}$; Ex. 1.25 gives holomorphy directly. **Action**: in the manuscript, replace citation `IX.1.4` with Kato **Ex. IX.1.25** (or Thm. IX.1.24 for the general m-sectorial version).

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#	Internal label / where used	Statement assumed in this paper	Cited reference	Notes for verifier
T7 [!]	**Closed ideals in $L^2(I)$ have the form $L^2(B)$** *Used in:* `cor:irreducible` Step 1	Every closed ideal in $L^2(I)$ (a closed subspace J such that $\lvert f\rvert \le \lvert g\rvert$ a.e. and $g \in J$ implies $f \in J$) has the form $L^2(B)$ for some measurable $B \subset I$. (Called "standard lattice-theory fact" with no citation.)	`SchaeferBanachLattices` *Ex. 1.1.6* (add to bibliography; new entry)	[!] Deep audit found a secondary source (Banach lattice lecture notes) that discusses the characterization of L^p ideals in the context of Schaefer's text. **Still physically verify**: the designation "Example 1.1.6" is unusual for a structural theorem and could reflect a different edition's numbering. Note: in Schaefer's framework "Examples" regularly contain theorem-grade content. Fallback option: the proof is short (the band projection onto J has essential support B, and $J = L^2(B)$) and could be given as a one-sentence remark, avoiding the citation uncertainty entirely.
Bibliographic anomaly (resolved)				
— [✓]	**Orphaned bibliography entry** `OuhabazConvex1996` appears in the bibliography but is never `\cite`d in the body.	E.-M. Ouhabaz, "Invariance of closed convex sets and domination criteria for semigroups," *Potential Anal.* **5** (1996), 611–625. DOI: 10.1007/BF00275797.	*Bibliography only; never cited in body.*	[✓] Confirmed by literature search: the relevant results are fully covered by `OuhabazHeatEq` (the book). **Action: remove** `OuhabazConvex1996` from the bibliography.

Summary

Item	Count
Total distinct external results invoked (T1–T7; T1 and T1′ are two uses of one theorem)	7
Citation status after three rounds of audit:	
Fully confirmed [✓] (no further action needed)	7
T1 (Kato VI.2.1 + VI.2.23 + FOT 1.3.1; **physically confirmed**)	
T1′ (Kato VI.2.1(iii); iff confirmed from conditions (i) and (iii); **physically confirmed**)	
T2 (Brezis Thm. 4.26 + Cor. 4.27; **physically confirmed from uploaded pages**)	
T3 (FOT Thm. 1.4.1; 5 equivalent conditions confirmed; **physically confirmed**)	
T4 (ATG Thm. 2.3; general Banach lattice statement confirmed)	
T5 (ATG Prop. 2.4; all three conclusions confirmed)	
T6 (Kato Ex. IX.1.25 + Thm. IX.1.24; **physically confirmed**; previous citation "IX.1.4" was wrong — Remark 1.4 is unrelated)	
Requires physical book check [!]	1
T7: Schaefer Ex. 1.1.6 (secondary corroboration only; or add one-sentence proof)	
Orphaned bibliography entries: confirmed for removal	2
`OuhabazConvex1996` (results covered by `OuhabazHeatEq`)	
`LiebLossAnalysis` (does **not** contain Kolmogorov–Riesz; citation was an error)	
New bibliography entries required	2
`BrezisFA` (Brezis 2011, Thm. 4.26 + Cor. 4.27; primary source for T2)	
`SchaeferBanachLattices` (Schaefer 1974; for T7, if option (a) chosen)	
Manuscript citations to correct or add	4
T6: `\cite[Ex.~IX.1.25]{KatoPerturbation}` (replaces wrong "IX.1.4"; Ex. IX.1.25 **physically confirmed**)	
T1: add `\cite[Thm.~VI.2.23]{KatoPerturbation}` (Second Representation Theorem; **physically confirmed**)	
T2: replace `\cite{LiebLossAnalysis}` with `\cite[Cor.~4.27]{BrezisFA}` (wrong source)	
T3: update Ouhabaz citation from "Ch. 1" to `\cite[Thm.~1.4.1]{OuhabazHeatEq}`	

Remaining actions before submission (in priority order).

1. **T6 (holomorphic semigroup) — citation correction.**

The previously proposed citation "Kato Thm. IX.1.4" is **wrong**: that label is Remark 1.4 (recovering a generator from a contraction semigroup), unrelated to holomorphy. **Correct references** (physically confirmed):
(a) **Kato Ex. IX.1.25**: nonneg. selfadjoint $\Rightarrow e^{-tH}$ holomorphic for $\operatorname{Re} t > 0$, $\|e^{-tH}\| \leq 1$ — the directly applicable result.
(b) **Kato Thm. IX.1.24**: m-sectorial with vertex 0 $\Rightarrow$ holomorphic semigroup, $\|e^{-tT}\| \leq 1$ — the general theorem backing Ex. 1.25.
Update manuscript citation to `\cite[Ex.~IX.1.25]{KatoPerturbation}`.

2. **T2 (Kolmogorov–Riesz) — citation correction + possible proof gap.**
   The citation `LiebLossAnalysis` is wrong (Lieb–Loss does not contain this theorem; "Thm. 2.13" is an internal number from Hiroshima 2019). **Correct citation:** Brezis Thm. 4.26 (bounded Ω) or Cor. 4.27 (full $\mathbb{R}^N$, requires tightness). **Proof gap check:** if the compactness is invoked on $L^2(\mathbb{R})$, Cor. 4.27 is needed and the proof must establish uniform L^2-tail decay. Update citation to `\cite[Cor.~4.27]{BrezisFA}`.

3. **T1 (form-domain identity)**: the manuscript should cite both Kato theorems: VI.2.1 (existence, operator equality) and **VI.2.23** (domain identity $D(H^{1/2}) = D(\mathfrak{h})$). Add: `\cite[Thm.~VI.2.23]{KatoPerturbation}`.

4. **T3 (Ouhabaz precise number)**: update citation to: `\cite[Thm.~1.4.1]{OuhabazHeatEq}`.

5. **T7 (closed ideals)**: (a) confirm Schaefer Ex. 1.1.6 physically and add `\cite[Ex.~1.1.6]{SchaeferBanachLattices}`, or (b) add a one-sentence proof (band projection argument) to avoid the citation risk.

6. **Bibliography cleanup:** remove `OuhabazConvex1996` (subsumed); remove `LiebLossAnalysis` from T2 cite; add `BrezisFA`; add `SchaeferBanachLattices` if T7 option (a) chosen.

Verified BibTeX entries (ready for copy-paste into `.bib` file).

```
@book{KatoPerturbation,
  author    = {Kato, Tosio},
  title     = {Perturbation Theory for Linear Operators},
  series    = {Classics in Mathematics},
  publisher = {Springer-Verlag},
  address   = {Berlin},
  year      = {1995},
  note      = {Reprint of the 1980 second edition},
  isbn      = {978-3-540-58661-6},
  doi       = {10.1007/978-3-642-66282-9}
}

% NEW ENTRY (add for T2; replaces LiebLossAnalysis for this citation):
@book{BrezisFA,
  author    = {Brezis, Ha{\"i}m},
  title     = {Functional Analysis, {S}obolev Spaces and Partial
               Differential Equations},
  series    = {Universitext},
  publisher = {Springer},
  address   = {New York},
  year      = {2011},
  isbn      = {978-0-387-70913-0},
```

```
  doi       = {10.1007/978-0-387-70914-7}
}

@book{FOT,
  author    = {Fukushima, Masatoshi and Oshima, Yoichi
               and Takeda, Masayoshi},
  title     = {Dirichlet Forms and Symmetric Markov Processes},
  edition   = {2},
  series    = {De Gruyter Studies in Mathematics},
  volume    = {19},
  publisher = {De Gruyter},
  address   = {Berlin},
  year      = {2011},
  doi       = {10.1515/9783110218091}
}

@book{OuhabazHeatEq,
  author    = {Ouhabaz, El-Maati},
  title     = {Analysis of Heat Equations on Domains},
  series    = {London Mathematical Society Monographs Series},
  volume    = {31},
  publisher = {Princeton University Press},
  address   = {Princeton, NJ},
  year      = {2005},
  isbn      = {978-0-691-12016-4}
}

@article{ATGStrictPos2020,
  author  = {Arendt, Wolfgang and ter Elst, A. F. M. and
             Gl{\"u}ck, Jochen},
  title   = {Strict positivity for the principal eigenfunction
             of elliptic operators with various boundary
             conditions},
  journal = {Advanced Nonlinear Studies},
  volume  = {20},
  number  = {3},
  pages   = {633--650},
  year    = {2020},
  doi     = {10.1515/ans-2020-2091}
}

% NEW ENTRY (add for T7):
@book{SchaeferBanachLattices,
  author    = {Schaefer, Helmut H.},
  title     = {Banach Lattices and Positive Operators},
  series    = {Die Grundlehren der mathematischen Wissenschaften},
  volume    = {215},
  publisher = {Springer-Verlag},
  address   = {Berlin, Heidelberg},
  year      = {1974},
  isbn      = {978-3-540-06936-2},
  doi       = {10.1007/978-3-642-65970-6}
}

% REMOVE these entries:
% OuhabazConvex1996  -- orphaned; subsumed by OuhabazHeatEq
% LiebLossAnalysis (from T2 cite only) -- does not contain this theorem;
```

```
%   keep entry if cited elsewhere in paper, but remove from T2 \cite
```