Energy Decomposition, Compact Resolvent, and Perron–Frobenius Properties
of the Restricted Weil Quadratic Form

Abstract

We record a completely concrete and rigorous functional-analytic step that arises in the spectral approach to Weil's criterion when one restricts test functions to a compact multiplicative interval $[\lambda^{-1}, \lambda] \subset \mathbb{R}_+^*$. Starting from the explicit local distributions at the primes and at ∞, we derive an "energy decomposition" expressing the quadratic form (up to an additive constant multiple of $\|g\|_2^2$) as a positive combination of translation-difference energies $\|G - S_t G\|_2^2$ in logarithmic coordinates. We then prove the Markov (normal contraction) property and a translation-invariance lemma which yields irreducibility from the archimedean continuum of shifts. Finally, we show that the quadratic form is closed and that its associated selfadjoint operator has compact resolvent, using a logarithmic lower bound on the Fourier symbol together with the Kolmogorov–Riesz compactness criterion. From this we deduce that the ground-state eigenvalue is simple and its eigenfunction can be chosen strictly positive and, by inversion symmetry, even.

Note on proof style. Every proof in this document is presented in L. Lamport's hierarchical structured-proof format [3]. Each step states a claim and its justification, and sub-steps may be expanded for further detail. The intent is that any single step can be verified independently.

Contents

1 Setup on $\mathbb{R}_+^*$

Let $\mathbb{R}_+^* = (0, \infty)$ with multiplicative Haar measure

$$d^*x := \frac{dx}{x}.$$

For measurable g, h define multiplicative convolution

$$(g * h)(x) := \int_{\mathbb{R}_+^*} g(y)\, h(x/y)\, d^*y,$$

and involution

$$g^*(x) := \overline{g(x^{-1})}.$$

If $g \in L^2(\mathbb{R}_+^*, d^*x)$, define the unitary dilation operator

$$(U_a g)(x) := g(x/a) \qquad (a > 0). \tag{1}$$

Then $\|U_a g\|_2 = \|g\|_2$ and $\langle g, U_a g\rangle$ is well-defined.

Lemma 1 (Convolution inner-product identity)**.** *Let $f = g * g^*$. Then for all $a > 0$,*

$$f(a) = \langle g, U_a g\rangle_{L^2(d^*x)} = \int_{\mathbb{R}_+^*} g(x)\, \overline{g(x/a)}\, d^*x, \qquad f(a^{-1}) = \overline{f(a)}.$$

In particular $f(a) + f(a^{-1}) = 2\Re\langle g, U_a g\rangle$ and $f(1) = \|g\|_2^2$.

Structured Proof. **Step 1.** $f(a) = \langle g, U_a g\rangle$.

Step 1.1. Expand the definition of multiplicative convolution applied to $g * g^*$ at the point a.
Justification: By definition, $(g * g^*)(a) = \int_{\mathbb{R}_+^*} g(y)\, g^*(a/y)\, d^*y$.

Step 1.2. Apply the definition of involution: $g^*(a/y) = \overline{g((a/y)^{-1})} = \overline{g(y/a)}$.

Justification: The involution is defined as $g^*(x) = \overline{g(x^{-1})}$. Set $x = a/y$ to get $g^*(a/y) = \overline{g(y/a)}$.

Step 1.3. Substitute into the integral: $(g * g^*)(a) = \int_{\mathbb{R}_+^*} g(y)\,\overline{g(y/a)}\,d^*y$.

Justification: Combine Steps 1.1 and 1.2.

Step 1.4. Recognize this as $\langle g, U_a g\rangle_{L^2(d^*x)}$.

Justification: By definition of the dilation operator (1), $(U_a g)(y) = g(y/a)$, and the $L^2(d^*x)$ inner product is $\langle g, h\rangle = \int g(y)\overline{h(y)}\,d^*y$. Thus $\int g(y)\,\overline{g(y/a)}\,d^*y = \langle g, U_a g\rangle$.

Step 2. $f(a^{-1}) = \overline{f(a)}$.

Justification: Replace a by a^{-1} in the result of Step 1: $f(a^{-1}) = \langle g, U_{a^{-1}} g\rangle = \int g(y)\,\overline{g(ya)}\,d^*y$. Substituting $y \mapsto y/a$ (with $d^*(y/a) = d^*y$) gives $\int g(y/a)\,\overline{g(y)}\,d^*y = \overline{\int g(y)\,\overline{g(y/a)}\,d^*y} = \overline{f(a)}$.

Step 3. $f(a) + f(a^{-1}) = 2\Re\langle g, U_a g\rangle$ and $f(1) = \|g\|_2^2$.

Justification: From Steps 1 and 2, $f(a) + f(a^{-1}) = \langle g, U_a g\rangle + \overline{\langle g, U_a g\rangle} = 2\Re\langle g, U_a g\rangle$. Setting $a = 1$: $U_1 = \mathrm{Id}$, so $f(1) = \langle g, g\rangle = \|g\|_2^2$.

Q.E.D. □

Lemma 2 (A basic unitary identity). *For any unitary U on a Hilbert space and any vector h,*

$$2\Re\langle h, Uh\rangle = 2\|h\|^2 - \|h - Uh\|^2.$$

Structured Proof. **Step 1.** $\|h - Uh\|^2 = \|h\|^2 + \|Uh\|^2 - 2\Re\langle h, Uh\rangle$.

Justification: Expand the inner product: $\|h - Uh\|^2 = \langle h - Uh, h - Uh\rangle = \langle h, h\rangle - \langle h, Uh\rangle - \langle Uh, h\rangle + \langle Uh, Uh\rangle = \|h\|^2 + \|Uh\|^2 - 2\Re\langle h, Uh\rangle$.

Step 2. $\|Uh\|^2 = \|h\|^2$.

Justification: U is unitary, hence isometric.

Step 3. Substituting Step 2 into Step 1: $\|h - Uh\|^2 = 2\|h\|^2 - 2\Re\langle h, Uh\rangle$.

Justification: Replace $\|Uh\|^2$ by $\|h\|^2$ in Step 1.

Step 4. Rearrange: $2\Re\langle h, Uh\rangle = 2\|h\|^2 - \|h - Uh\|^2$.

Justification: Solve Step 3 for $2\Re\langle h, Uh\rangle$.

Q.E.D. □

2 Local explicit-formula terms

Fix $\lambda > 1$ and consider g supported in $[\lambda^{-1}, \lambda]$.

We record the two local distributions we use; these are the only "input formulas".

2.1 Prime terms

For a prime p define

$$W_p(f) := (\log p) \sum_{m \geq 1} p^{-m/2}\big(f(p^m) + f(p^{-m})\big). \tag{2}$$

2.2 Archimedean term

Define

$$W_{\mathbb{R}}(f) := (\log 4\pi + \gamma)\, f(1) + \int_1^\infty \Big(f(x) + f(x^{-1}) - 2x^{-1/2} f(1)\Big) \frac{x^{1/2}}{x - x^{-1}}\, d^*x, \tag{3}$$

where γ is the Euler–Mascheroni constant.

Remark 3 (Restriction to a compact multiplicative interval)*.* If $\mathrm{supp}(g) \subset [\lambda^{-1}, \lambda]$, then for $a > \lambda^2$ the supports of g and $U_a g$ are disjoint, hence $\langle g, U_a g\rangle = 0$ and $f(a) = 0$. Consequently:

- in (2) only those (p, m) with $p^m \le \lambda^2$ contribute;
- in (3), after the change of variables $x = e^t$, only $t \in [0, 2\log\lambda]$ contributes to the term involving $f(e^t) + f(e^{-t})$.

This finiteness is crucial and is completely elementary.

3 Logarithmic coordinates and translations

Set $u = \log x$, so that $d^*x = du$ and the interval $[\lambda^{-1}, \lambda]$ becomes

$$I := (-L, L), \qquad L := \log\lambda.$$

For $G \in L^2(I)$ we denote by $\widetilde{G}$ its extension by 0 to $\mathbb{R}$. Let S_t be translation on $L^2(\mathbb{R})$:

$$(S_t\phi)(u) := \phi(u - t).$$

Then in logarithmic coordinates, the dilation U_{e^t} from (1) corresponds to translation: if $G(u) = g(e^u)$, then $(U_{e^t} g)(e^u) = g(e^{u-t})$, i.e. $\widetilde{G} \mapsto S_t\widetilde{G}$.

4 Energy decomposition into translation differences

4.1 Prime contributions

Lemma 4 (Prime term as a difference energy plus a constant)**.** *Let $f = g * g^*$ with g supported in $[\lambda^{-1}, \lambda]$, and let $G(u) = g(e^u)$. Then*

$$-W_p(f) = \sum_{\substack{m \ge 1 \\ p^m \le \lambda^2}} (\log p)\, p^{-m/2}\, \|\widetilde{G} - S_{m\log p}\widetilde{G}\|^2_{L^2(\mathbb{R})} \;+\; c_p(\lambda)\, \|G\|^2_{L^2(I)},$$

where $c_p(\lambda) \in \mathbb{R}$ is a finite constant depending only on p and λ.

Structured Proof. **Step 1.** $W_p(f) = (\log p) \sum_{m \ge 1} p^{-m/2}\, 2\Re\langle g, U_{p^m} g\rangle$.

Step 1.1. By (2), $W_p(f) = (\log p) \sum_{m \ge 1} p^{-m/2} \big(f(p^m) + f(p^{-m})\big)$.
Justification: Definition of W_p.

Step 1.2. $f(p^m) + f(p^{-m}) = 2\Re\langle g, U_{p^m} g\rangle$.
Justification: By Lemma 1, $f(a) + f(a^{-1}) = 2\Re\langle g, U_a g\rangle$. Set $a = p^m$.

Step 1.3. Combine Steps 1.1 and 1.2.
Justification: Substitute the identity from Step 1.2 into the sum from Step 1.1.

Step 2. For each $m \geq 1$ with $p^m \leq \lambda^2$: $2\Re\langle g, U_{p^m} g\rangle = 2\|g\|_2^2 - \|g - U_{p^m} g\|_2^2$.

Justification: Lemma 2 applied with $U = U_{p^m}$, $h = g$.

Step 3. For $m \geq 1$ with $p^m > \lambda^2$: $\langle g, U_{p^m} g\rangle = 0$.

Justification: By Remark 3: when $p^m > \lambda^2$, the supports of g (in $[\lambda^{-1}, \lambda]$) and $U_{p^m} g$ (in $[p^m\lambda^{-1}, p^m\lambda]$) are disjoint.

Step 4. In logarithmic coordinates: $\|g - U_{p^m} g\|_2 = \|\widetilde{G} - S_{m\log p}\widetilde{G}\|_{L^2(\mathbb{R})}$.

Justification: The substitution $u = \log x$ converts $d^*x = du$, $g(x) \mapsto G(u)$, and $(U_{p^m} g)(x) = g(x/p^m) \mapsto G(u - m\log p) = (S_{m\log p}\widetilde{G})(u)$. The $L^2(\mathbb{R}_+^*, d^*x)$ norm becomes the $L^2(\mathbb{R}, du)$ norm.

Step 5. Assemble the formula for $-W_p(f)$.

Step 5.1. From Steps 1–3, only terms with $p^m \leq \lambda^2$ contribute, and for those terms:

$$W_p(f) = (\log p) \sum_{\substack{m\geq 1\\ p^m\leq\lambda^2}} p^{-m/2}\big(2\|g\|_2^2 - \|g - U_{p^m} g\|_2^2\big).$$

Justification: Combine Steps 1, 2, and 3: the terms with $p^m > \lambda^2$ vanish by Step 3; the remaining terms are rewritten using Step 2.

Step 5.2. Negate and use Step 4:

$$-W_p(f) = \sum_{\substack{m\geq 1\\ p^m\leq\lambda^2}} (\log p)\, p^{-m/2}\, \|\widetilde{G} - S_{m\log p}\widetilde{G}\|_{L^2(\mathbb{R})}^2 - 2(\log p)\Big(\sum_{\substack{m\geq 1\\ p^m\leq\lambda^2}} p^{-m/2}\Big)\|g\|_2^2.$$

Justification: Negate Step 5.1 and replace $\|g - U_{p^m} g\|_2$ by $\|\widetilde{G} - S_{m\log p}\widetilde{G}\|_{L^2(\mathbb{R})}$ using Step 4.

Step 5.3. Set $c_p(\lambda) := -2(\log p)\sum_{m\geq 1,\, p^m\leq\lambda^2} p^{-m/2}$. Then $c_p(\lambda) \in \mathbb{R}$ is finite (the sum has finitely many terms), and $\|g\|_2^2 = \|G\|_{L^2(I)}^2$.

Justification: The sum is finite because only finitely many integers m satisfy $p^m \leq \lambda^2$. The norm identity $\|g\|_2 = \|G\|_{L^2(I)}$ follows from the change of variables $u = \log x$.

Q.E.D. □

4.2 Archimedean contribution

Lemma 5 (Archimedean term as a continuum of difference energies plus a constant)**.** *Let $f = g * g^*$ with g supported in $[\lambda^{-1}, \lambda]$, and let $G(u) = g(e^u)$. Define the strictly positive weight on $(0, \infty)$,*

$$w(t) := \frac{e^{t/2}}{e^t - e^{-t}} = \frac{e^{t/2}}{2\sinh t}.$$

Then

$$-W_{\mathbb{R}}(f) = \int_0^{2L} w(t)\, \|\widetilde{G} - S_t\widetilde{G}\|_{L^2(\mathbb{R})}^2\, dt \;+\; c_\infty(\lambda)\, \|G\|_{L^2(I)}^2,$$

where $c_\infty(\lambda) \in \mathbb{R}$ is a finite constant depending only on λ.

Structured Proof. **Step 1.** Rewrite $W_{\mathbb{R}}(f)$ by substituting $x = e^t$:

$$W_{\mathbb{R}}(f) = (\log 4\pi + \gamma)\, f(1) + \int_0^\infty \Big(f(e^t) + f(e^{-t}) - 2e^{-t/2} f(1)\Big)\, w(t)\, dt.$$

Justification: Starting from (3), set $x = e^t$ so that $d^*x = dt$. Then $x^{1/2}/(x - x^{-1}) = e^{t/2}/(e^t - e^{-t}) = w(t)$, and the integration range $x \in [1, \infty)$ becomes $t \in [0, \infty)$.

Step 2. $f(1) = \|g\|_2^2$ and $f(e^t) + f(e^{-t}) = 2\Re\langle g, U_{e^t} g\rangle$.

Justification: The first identity is Lemma 1 with $a = 1$. The second is Lemma 1: $f(a) + f(a^{-1}) = 2\Re\langle g, U_a g\rangle$ with $a = e^t$.

Step 3. $-2\Re\langle g, U_{e^t} g\rangle = \|g - U_{e^t} g\|_2^2 - 2\|g\|_2^2$.

Justification: Lemma 2 with $U = U_{e^t}$, $h = g$: $2\Re\langle g, U_{e^t} g\rangle = 2\|g\|_2^2 - \|g - U_{e^t} g\|_2^2$. Negate both sides.

Step 4. Substituting Steps 2 and 3 into the integral from Step 1, the integrand of $-W_{\mathbb{R}}(f)$ (inside $\int_0^\infty$) equals

$$w(t)\big(\|g - U_{e^t} g\|_2^2 + 2(e^{-t/2} - 1)\|g\|_2^2\big).$$

Step 4.1. From Step 1 (negated), $-W_{\mathbb{R}}(f) = -(\log 4\pi + \gamma)\|g\|_2^2 + \int_0^\infty \big(-2\Re\langle g, U_{e^t} g\rangle + 2e^{-t/2}\|g\|_2^2\big) w(t)\, dt$.

Justification: Negate Step 1 and use $f(1) = \|g\|_2^2$ from Step 2.

Step 4.2. Replace $-2\Re\langle g, U_{e^t} g\rangle$ by $\|g - U_{e^t} g\|_2^2 - 2\|g\|_2^2$ (Step 3): the integrand becomes $(\|g - U_{e^t} g\|_2^2 - 2\|g\|_2^2 + 2e^{-t/2}\|g\|_2^2) w(t) = w(t)(\|g - U_{e^t} g\|_2^2 + 2(e^{-t/2} - 1)\|g\|_2^2)$.

Justification: Algebra: $-2 + 2e^{-t/2} = 2(e^{-t/2} - 1)$.

Step 5. In logarithmic coordinates: $\|g - U_{e^t} g\|_2 = \|\widetilde{G} - S_t\widetilde{G}\|_{L^2(\mathbb{R})}$.

Justification: Same argument as in Lemma 4, Step 4.

Step 6. Split the integral at $t = 2L$. For $t > 2L$, $\|\widetilde{G} - S_t\widetilde{G}\|_2^2 = 2\|G\|_2^2$.

Justification: By Remark 3, when $t > 2L = 2\log\lambda$, the supports of $\widetilde{G}$ (contained in $[-L, L]$) and $S_t\widetilde{G}$ (contained in $[-L + t, L + t]$, with $-L + t > L$) are disjoint. Hence $\|\widetilde{G} - S_t\widetilde{G}\|_2^2 = \|\widetilde{G}\|_2^2 + \|S_t\widetilde{G}\|_2^2 = 2\|G\|_2^2$.

Step 7. The tail integral over $(2L, \infty)$ is a finite constant times $\|G\|_2^2$.

Step 7.1. For $t > 2L$, the integrand from Step 4 becomes $w(t)(2\|G\|_2^2 + 2(e^{-t/2} - 1)\|G\|_2^2) = 2e^{-t/2} w(t)\|G\|_2^2$.

Justification: Substitute $\|\widetilde{G} - S_t\widetilde{G}\|_2^2 = 2\|G\|_2^2$ from Step 6 into the integrand from Step 4 (after applying Step 5): $2\|G\|_2^2 + 2(e^{-t/2} - 1)\|G\|_2^2 = 2e^{-t/2}\|G\|_2^2$.

Step 7.2. $\int_{2L}^\infty 2e^{-t/2} w(t)\, dt < \infty$.

Justification: As $t \to \infty$, $w(t) = e^{t/2}/(2\sinh t) \sim e^{-t/2}$, so $2e^{-t/2} w(t) \sim 2e^{-t}$, which is integrable.

Step 8. The integral over $[0, 2L]$ contributes the main term plus a finite constant.

Step 8.1. For $t \in [0, 2L]$, the integrand (Step 4 with Step 5) splits as $w(t)\|\widetilde{G} - S_t\widetilde{G}\|_2^2 + 2(e^{-t/2} - 1) w(t)\|G\|_2^2$.

Justification: Rewrite Step 4 using Step 5.

Step 8.2. $\int_0^{2L} 2(e^{-t/2} - 1) w(t)\, dt$ is finite.

Justification: Near $t = 0$: $w(t) \sim (2t)^{-1}$ and $e^{-t/2} - 1 \sim -t/2$, so the integrand is $\sim -1/2$, which is bounded. On the compact interval $[0, 2L]$ with this bounded behavior, the integral converges absolutely.

Step 9. Define $c_\infty(\lambda) := -(\log 4\pi + \gamma) + \int_0^{2L} 2(e^{-t/2} - 1) w(t)\, dt + \int_{2L}^\infty 2e^{-t/2} w(t)\, dt$.

Justification: Collect all $\|G\|_2^2$-proportional terms: the $-(\log 4\pi + \gamma)$ from Step 4.1, the integral $\int_0^{2L} 2(e^{-t/2} - 1) w(t)\, dt$ from Step 8.2, and the tail integral from Step 7.2. Each is finite, so $c_\infty(\lambda) \in \mathbb{R}$.

Step 10. Conclusion: $-W_{\mathbb{R}}(f) = \int_0^{2L} w(t)\|\widetilde{G} - S_t\widetilde{G}\|_2^2\,dt + c_\infty(\lambda)\|G\|_{L^2(I)}^2$.

Justification: Combine Steps 8.1, 7, and 9, noting $\|g\|_2^2 = \|G\|_{L^2(I)}^2$ by the change of variables.

Q.E.D. □

4.3 Global quadratic form on the interval

Definition 6 (Difference-energy form). Fix $\lambda > 1$ and $L = \log\lambda$. For $G \in L^2(I)$ define

$$\mathcal{E}_\lambda(G) := \int_0^{2L} w(t)\,\|\widetilde{G} - S_t\widetilde{G}\|_{L^2(\mathbb{R})}^2\,dt + \sum_{\substack{p\text{ prime}\\ p\le\lambda^2}}\ \sum_{\substack{m\ge1\\ p^m\le\lambda^2}} (\log p)p^{-m/2}\,\|\widetilde{G} - S_{m\log p}\widetilde{G}\|_{L^2(\mathbb{R})}^2. \quad (4)$$

Remark 7 (What we have proved so far). Lemmas 4 and 5 show that for $f = g * g^*$ with $\operatorname{supp}(g) \subset [\lambda^{-1}, \lambda]$, the quantity

$$-\sum_{v\in\{\infty\}\cup\{p\}} W_v(f)$$

equals $\mathcal{E}_\lambda(G)$ plus an additive constant multiple of $\|G\|_2^2$. Since adding a constant multiple of $\|G\|_2^2$ only shifts the spectrum of the associated operator, it does not affect positivity/irreducibility properties of the semigroup and does not affect eigenfunction parity considerations.

5 Markov property (normal contractions)

Definition 8 (Normal contraction). A map $\Phi : \mathbb{R} \to \mathbb{R}$ is a normal contraction if $\Phi(0) = 0$ and $|\Phi(a) - \Phi(b)| \le |a - b|$ for all $a, b \in \mathbb{R}$.

Lemma 9 (Markov property). *For every normal contraction Φ and every $G \in L^2(I)$,*

$$\mathcal{E}_\lambda(\Phi \circ G) \le \mathcal{E}_\lambda(G).$$

In particular, $\mathcal{E}_\lambda(|G|) \le \mathcal{E}_\lambda(G)$.

Structured Proof. **Step 1.** For each fixed $t \in \mathbb{R}$: $\|\widetilde{\Phi\circ G} - S_t\widetilde{\Phi\circ G}\|_2^2 \le \|\widetilde{G} - S_t\widetilde{G}\|_2^2$.

Step 1.1. $\widetilde{\Phi\circ G} = \Phi\circ\widetilde{G}$.

Justification: Since $\Phi(0) = 0$, the extension by zero commutes with composition by Φ: for $u \notin I$, $\widetilde{G}(u) = 0$, so $\Phi(\widetilde{G}(u)) = \Phi(0) = 0 = \widetilde{\Phi\circ G}(u)$; for $u \in I$, both sides equal $\Phi(G(u))$.

Step 1.2. $|\Phi(\widetilde{G}(u)) - \Phi(\widetilde{G}(u-t))|^2 \le |\widetilde{G}(u) - \widetilde{G}(u-t)|^2$ for every $u \in \mathbb{R}$.

Justification: Φ is 1-Lipschitz by hypothesis (the "normal contraction" condition $|\Phi(a) - \Phi(b)| \le |a-b|$). Apply this pointwise with $a = \widetilde{G}(u)$, $b = \widetilde{G}(u-t)$, and square.

Step 1.3. Integrate Step 1.2 over $\mathbb{R}$ with respect to du:

$$\int_{\mathbb{R}} |\Phi(\widetilde{G}(u)) - \Phi(\widetilde{G}(u-t))|^2\,du \le \int_{\mathbb{R}} |\widetilde{G}(u) - \widetilde{G}(u-t)|^2\,du.$$

Justification: Integrate both sides of the pointwise inequality from Step 1.2.

Step 1.4. Rewrite using Step 1.1 and the definition of S_t: this is precisely $\|\widetilde{\Phi\circ G} - S_t\widetilde{\Phi\circ G}\|_2^2 \le \|\widetilde{G} - S_t\widetilde{G}\|_2^2$.

Justification: $(S_t\widetilde{\Phi\circ G})(u) = (\Phi\circ\widetilde{G})(u-t) = \Phi(\widetilde{G}(u-t))$.

Step 2. $\mathcal{E}_\lambda(\Phi \circ G) \le \mathcal{E}_\lambda(G)$.

Justification: By Definition 6, $\mathcal{E}_\lambda(G)$ is the integral of $w(t)\|\widetilde{G} - S_t\widetilde{G}\|_2^2$ over $[0, 2L]$ (with weight $w(t) \ge 0$) plus a finite sum of terms $(\log p)p^{-m/2}\|\widetilde{G} - S_{m\log p}\widetilde{G}\|_2^2$ (all coefficients ≥ 0). Step 1 shows each summand decreases (or stays the same) when G is replaced by $\Phi \circ G$. Since all weights are nonnegative, the integral and sum are each $\le$ the corresponding quantity for G.

Step 3. In particular, $\mathcal{E}_\lambda(|G|) \le \mathcal{E}_\lambda(G)$.

Justification: $\Phi(x) = |x|$ is a normal contraction: $\Phi(0) = 0$ and $\big||a| - |b|\big| \le |a - b|$ by the reverse triangle inequality. Apply Step 2.

Q.E.D. □

6 A translation-invariance lemma on an interval

Lemma 10 (Local translation invariance forces null or conull)**.** *Let $I \subset \mathbb{R}$ be a nontrivial open interval and let $B \subset I$ be measurable. Assume that there exists $\varepsilon > 0$ such that for every $t \in (0, \varepsilon)$,*

$$\mathbf{1}_B(u) = \mathbf{1}_B(u - t) \quad \textit{for a.e. } u \in I \cap (I + t). \tag{5}$$

Then either $m(B) = 0$ or $m(I \setminus B) = 0$.

Structured Proof. **Step 1.** Set $f := \mathbf{1}_B \in L^1_{\mathrm{loc}}(I)$. Fix a compact subinterval $J \Subset I$. Choose $0 < \delta < \min\{\varepsilon, \mathrm{dist}(J, \partial I)\}$.

Justification: Since $J \Subset I$, we have $\mathrm{dist}(J, \partial I) > 0$, so δ as described exists.

Step 2. For every $t \in (0, \delta)$: $f(u + t) = f(u)$ for a.e. $u \in J$.

Step 2.1. From (5): for $t \in (0, \varepsilon)$, $f(u) = f(u - t)$ for a.e. $u \in I \cap (I + t)$.

Justification: Hypothesis of the lemma, since $0 < t < \delta < \varepsilon$.

Step 2.2. Substitute $u \mapsto u + t$: $f(u + t) = f(u)$ for a.e. $u \in (I - t) \cap I$.

Justification: The set $I \cap (I + t)$ becomes $(I - t) \cap I$ after the shift.

Step 2.3. $J \subset (I - t) \cap I$.

Justification: For $u \in J$: since $\delta < \mathrm{dist}(J, \partial I)$, we have $u \in I$ and $u + t \in I$ (as $|t| < \delta$), so $u \in I$ and $u \in I - t$, giving $u \in (I - t) \cap I$.

Step 2.4. Combine: $f(u + t) = f(u)$ for a.e. $u \in J$.

Justification: Restrict the a.e. identity from Step 2.2 to the subset $J \subset (I - t) \cap I$ (Step 2.3).

Step 3. The identity extends to $|t| < \delta$: $f(u + t) = f(u)$ for a.e. $u \in J$, for all $t \in (-\delta, \delta)$.

Justification: Step 2 handles $t \in (0, \delta)$. For $t \in (-\delta, 0)$, replace t by $-t \in (0, \delta)$ in Step 2 to get $f(u - t) = f(u)$ a.e. on J, which after the substitution $u \mapsto u + t$ (and noting $J + t \subset I$ for $|t| < \delta$, same argument as Step 2.3) yields $f(u) = f(u + t)$ a.e. on appropriate sets. At $t = 0$ the identity is trivial.

Step 4. Define the mollification $f_\eta := f * \rho_\eta$ on $J_\eta := \{u \in J : \mathrm{dist}(u, \mathbb{R} \setminus J) > \eta\}$, where $\rho_\eta(s) = \eta^{-1}\rho(s/\eta)$ for a standard nonneg. mollifier ρ supported in $(-1, 1)$ with $\int \rho = 1$, and $0 < \eta < \delta/2$.

Justification: Standard construction. $f_\eta \in C^\infty(J_\eta)$ by standard properties of convolution with C_c^∞ kernels.

Step 5. f_η is translation-invariant on J_η for shifts $|t| < \delta/2$: $f_\eta(u + t) = f_\eta(u)$ for $u \in J_\eta$.

Step 5.1. For $u \in J_\eta$ and $s \in \operatorname{supp}\rho_\eta \subset (-\eta, \eta)$: $u - s \in J$.

Justification: $|u - s - u| = |s| < \eta < \operatorname{dist}(u, \mathbb{R} \setminus J)$ by definition of J_η.

Step 5.2. $f_\eta(u+t) = \int_{\mathbb{R}} f(u+t-s)\rho_\eta(s)\,ds = \int_{\mathbb{R}} f(u-s)\rho_\eta(s)\,ds = f_\eta(u)$.

Justification: By Fubini and Step 5.1, for each $s \in \operatorname{supp} \rho_\eta$, $u - s \in J$ and $|t| < \delta/2 < \delta$, so Step 3 gives $f((u-s)+t) = f(u-s)$ for a.e. such $u-s$. Hence $f(u+t-s) = f(u-s)$ a.e., and integration against $\rho_\eta(s)\,ds$ yields the identity.

Step 6. f_η is constant on J_η.

Justification: J_η is a connected open interval (nonempty for small η) and $f_\eta \in C^\infty(J_\eta)$ is invariant under all translations of size $< \delta/2$ (Step 5). A smooth function on a connected open subset of $\mathbb{R}$ that is invariant under arbitrarily small translations has zero derivative everywhere: for any $u \in J_\eta$, $f_\eta'(u) = \lim_{t\to 0}(f_\eta(u+t) - f_\eta(u))/t = 0$. Hence f_η is constant.

Step 7. $f = \mathbf{1}_B$ is a.e. constant on J.

Justification: As $\eta \downarrow 0$, $f_\eta \to f$ in $L^1(J)$ (standard mollification convergence theorem). Since each f_η is constant on J_η and $J_\eta \nearrow J^\circ = J$, the limit f is a.e. equal to a constant on J.

Step 8. $f = \mathbf{1}_B$ is a.e. constant on I.

Justification: The compact subinterval $J \Subset I$ was arbitrary (Step 1). Cover I by a countable increasing sequence $J_1 \subset J_2 \subset \cdots$ of compact subintervals with $\bigcup_n J_n = I$. By Step 7, $\mathbf{1}_B$ is a.e. constant on each J_n. Since successive J_n's overlap, the constant must be the same on all of them. Hence $\mathbf{1}_B$ is a.e. constant on I.

Step 9. $m(B) = 0$ or $m(I \setminus B) = 0$.

Justification: Since $\mathbf{1}_B \in \{0, 1\}$ a.e., the only possible constants are 0 and 1. If $\mathbf{1}_B = 0$ a.e. on I, then $m(B) = 0$. If $\mathbf{1}_B = 1$ a.e. on I, then $m(I \setminus B) = 0$.

Q.E.D. □

7 Irreducibility from the archimedean continuum

7.1 A concrete criterion

Lemma 11 (Indicator-energy vanishes only for null/conull sets)**.** *Let $B \subset I$ be measurable. If $\mathcal{E}_\lambda(\mathbf{1}_B) = 0$, then $m(B) = 0$ or $m(I \setminus B) = 0$.*

Structured Proof. **Step 1.** The archimedean contribution to $\mathcal{E}_\lambda(\mathbf{1}_B)$ vanishes: $\int_0^{2L} w(t)\|\widetilde{\mathbf{1}_B} - S_t\widetilde{\mathbf{1}_B}\|_2^2\,dt = 0$.

Justification: By Definition 6, $\mathcal{E}_\lambda(\mathbf{1}_B)$ is a sum of nonneg. terms (the archimedean integral plus the prime sums). If the total is 0, each nonneg. summand must be 0. In particular, the archimedean integral (whose integrand is nonneg.) equals 0.

Step 2. $\|\widetilde{\mathbf{1}_B} - S_t\widetilde{\mathbf{1}_B}\|_2^2 = 0$ for a.e. $t \in (0, 2L)$.

Justification: The integrand $w(t)\|\widetilde{\mathbf{1}_B} - S_t\widetilde{\mathbf{1}_B}\|_2^2$ is nonneg. and $w(t) > 0$ for all $t > 0$ (since $w(t) = e^{t/2}/(2\sinh t)$ with numerator and denominator both positive for $t > 0$). A nonneg. integral vanishing (Step 1) with a strictly positive weight implies the other factor vanishes a.e.

Step 3. Upgrade to *all* $t \in (0, 2L)$: $\|\widetilde{\mathbf{1}_B} - S_t\widetilde{\mathbf{1}_B}\|_2^2 = 0$ for every $t \in (0, 2L)$.

Step 3.1. The function $t \mapsto \|\phi - S_t\phi\|_2^2$ is continuous for any $\phi \in L^2(\mathbb{R})$.

Justification: By strong continuity of the translation group $(S_t)_{t\in\mathbb{R}}$ on $L^2(\mathbb{R})$ (which follows from dominated convergence: if $t_n \to t$ then $S_{t_n}\phi \to S_t\phi$ in L^2), the map $t \mapsto S_t\phi$ is continuous $\mathbb{R} \to L^2(\mathbb{R})$, and the squared norm is a continuous function of its argument.

Step 3.2. A continuous function that vanishes a.e. on an interval vanishes everywhere on that interval.

Justification: Let $h : (0, 2L) \to [0, \infty)$ be continuous with $h = 0$ a.e. If $h(t_0) > 0$ for some t_0, then by continuity $h > 0$ on an open neighborhood of t_0, which has positive Lebesgue measure—contradicting $h = 0$ a.e.

Step 3.3. Apply Steps 3.1 and 3.2 with $\phi = \widetilde{\mathbf{1}_B}$.

Justification: Step 2 says $\|\widetilde{\mathbf{1}_B} - S_t\widetilde{\mathbf{1}_B}\|_2^2 = 0$ for a.e. $t \in (0, 2L)$; Step 3.1 says this function of t is continuous; Step 3.2 upgrades "a.e." to "all."

Step 4. For every $t \in (0, 2L)$: $\mathbf{1}_B(u) = \mathbf{1}_B(u - t)$ for a.e. $u \in I \cap (I + t)$.

Justification: $\|\widetilde{\mathbf{1}_B} - S_t\widetilde{\mathbf{1}_B}\|_2^2 = 0$ (Step 3) means $\widetilde{\mathbf{1}_B}(u) = \widetilde{\mathbf{1}_B}(u - t)$ for a.e. $u \in \mathbb{R}$. Restricting to $u \in I \cap (I + t)$: both $u \in I$ and $u - t \in I$, so $\widetilde{\mathbf{1}_B}(u) = \mathbf{1}_B(u)$ and $\widetilde{\mathbf{1}_B}(u - t) = \mathbf{1}_B(u - t)$.

Step 5. $m(B) = 0$ or $m(I \setminus B) = 0$.

Justification: Step 4 holds for all $t \in (0, 2L)$, which contains an interval $(0, \varepsilon)$ for any $\varepsilon \leq 2L$. Lemma 10 applies (with $\varepsilon = 2L$) and yields the conclusion.

Q.E.D. □

7.2 Operator realization: closedness and compact resolvent

7.2.1 Ambient form on $L^2(\mathbb{R})$ and Fourier representation

Let $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ denote the unitary Fourier transform

$$\widehat{\phi}(\xi) := \int_{\mathbb{R}} \phi(u)e^{-iu\xi}\, du, \qquad \phi(u) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\phi}(\xi)e^{iu\xi}\, d\xi,$$

so that Plancherel reads $\|\phi\|_{L^2(\mathbb{R})}^2 = \frac{1}{2\pi}\int_{\mathbb{R}} |\widehat{\phi}(\xi)|^2\, d\xi$.

Define the "ambient" quadratic form on $L^2(\mathbb{R})$ by

$$\mathcal{E}_\lambda^{\mathbb{R}}(\phi) := \int_0^{2L} w(t)\, \|\phi - S_t\phi\|_{L^2(\mathbb{R})}^2\, dt \\ + \sum_{\substack{p \text{ prime} \\ p \leq \lambda^2}} \sum_{\substack{m \geq 1 \\ p^m \leq \lambda^2}} (\log p)p^{-m/2}\, \|\phi - S_{m\log p}\phi\|_{L^2(\mathbb{R})}^2,$$

with domain $\mathcal{D}(\mathcal{E}_\lambda^{\mathbb{R}}) := \{\phi \in L^2(\mathbb{R}) : \mathcal{E}_\lambda^{\mathbb{R}}(\phi) < \infty\}$. By definition, for $G \in L^2(I)$,

$$\mathcal{E}_\lambda(G) = \mathcal{E}_\lambda^{\mathbb{R}}(\widetilde{G}).$$

Lemma 12 (Plancherel identity for translation differences)**.** *For $\phi \in L^2(\mathbb{R})$ and $t \in \mathbb{R}$,*

$$\|\phi - S_t\phi\|_{L^2(\mathbb{R})}^2 = \frac{1}{2\pi}\int_{\mathbb{R}} |1 - e^{-i\xi t}|^2\, |\widehat{\phi}(\xi)|^2\, d\xi = \frac{1}{2\pi}\int_{\mathbb{R}} 4\sin^2\!\Big(\frac{\xi t}{2}\Big)\, |\widehat{\phi}(\xi)|^2\, d\xi.$$

Structured Proof. **Step 1.** $\widehat{S_t\phi}(\xi) = e^{-i\xi t}\widehat{\phi}(\xi)$.

Justification: By definition, $\widehat{S_t\phi}(\xi) = \int_{\mathbb{R}} \phi(u - t)e^{-iu\xi}\, du$. Substituting $v = u - t$, $du = dv$: $= \int_{\mathbb{R}} \phi(v)e^{-i(v+t)\xi}\, dv = e^{-it\xi}\widehat{\phi}(\xi)$.

Step 2. $\|\phi - S_t\phi\|_2^2 = \frac{1}{2\pi}\int_{\mathbb{R}} |1 - e^{-i\xi t}|^2 |\widehat{\phi}(\xi)|^2\, d\xi$.

Justification: By Plancherel, $\|\phi - S_t\phi\|_2^2 = \frac{1}{2\pi}\int_{\mathbb{R}} |\widehat{\phi}(\xi) - \widehat{S_t\phi}(\xi)|^2\, d\xi$. Step 1 gives $\widehat{\phi}(\xi) - \widehat{S_t\phi}(\xi) = (1 - e^{-i\xi t})\widehat{\phi}(\xi)$.

Step 3. $|1 - e^{-i\eta}|^2 = 4\sin^2(\eta/2)$.

Justification: $1 - e^{-i\eta} = 1 - \cos\eta + i\sin\eta$, so $|1 - e^{-i\eta}|^2 = (1-\cos\eta)^2 + \sin^2\eta = 2 - 2\cos\eta = 4\sin^2(\eta/2)$ by the double-angle formula $\cos\eta = 1 - 2\sin^2(\eta/2)$.

Step 4. Set $\eta = \xi t$ in Step 3 and substitute into Step 2.

Justification: $|1 - e^{-i\xi t}|^2 = 4\sin^2(\xi t/2)$. This gives the second equality.

Q.E.D. □

Lemma 13 (Fourier representation)**.** *For $\phi \in L^2(\mathbb{R})$,*

$$\mathcal{E}_\lambda^{\mathbb{R}}(\phi) = \frac{1}{2\pi}\int_{\mathbb{R}} \psi_\lambda(\xi)\, |\widehat{\phi}(\xi)|^2\, d\xi \quad \text{in } [0,\infty],$$

where

$$\psi_\lambda(\xi) := 4\int_0^{2L} w(t)\, \sin^2\Big(\frac{\xi t}{2}\Big)\, dt + 4\sum_{\substack{p \text{ prime} \\ p \le \lambda^2}} \sum_{\substack{m \ge 1 \\ p^m \le \lambda^2}} (\log p) p^{-m/2}\, \sin^2\Big(\frac{\xi m \log p}{2}\Big). \tag{6}$$

In particular ψ_λ is measurable, even, finite for each ξ, and $\psi_\lambda(\xi) \ge 0$.

Structured Proof. **Step 1.** Apply Lemma 12 to each translation-difference norm in $\mathcal{E}_\lambda^{\mathbb{R}}(\phi)$.

Justification: Each term $\|\phi - S_t\phi\|_2^2$ or $\|\phi - S_{m\log p}\phi\|_2^2$ in Definition 6 equals $\frac{1}{2\pi}\int_{\mathbb{R}} 4\sin^2(\xi t/2)|\widehat{\phi}(\xi)|^2\, d\xi$ (resp. with t replaced by $m\log p$).

Step 2. Interchange integration/summation order by Tonelli's theorem:

$$\mathcal{E}_\lambda^{\mathbb{R}}(\phi) = \frac{1}{2\pi}\int_{\mathbb{R}} \psi_\lambda(\xi)|\widehat{\phi}(\xi)|^2\, d\xi.$$

Justification: All integrands are nonneg. ($w(t) \ge 0$, $\sin^2 \ge 0$, $|\widehat{\phi}|^2 \ge 0$, $(\log p)p^{-m/2} \ge 0$). Tonelli's theorem permits interchange of the ξ-integral with the t-integral and the finite sums. The resulting multiplier of $|\widehat{\phi}(\xi)|^2$ is precisely $\psi_\lambda(\xi)$.

Step 3. ψ_λ is measurable, even, finite for each ξ, and ≥ 0.

Step 3.1. Measurability: ψ_λ is a sum of continuous functions of ξ.

Justification: Each $\sin^2(\xi t/2)$ is continuous in ξ. The integral $\int_0^{2L} w(t)\sin^2(\xi t/2)\, dt$ is continuous in ξ by dominated convergence (the integrand is bounded by $w(t)$, which is integrable). The prime sum is a finite sum of continuous functions.

Step 3.2. Evenness: $\psi_\lambda(-\xi) = \psi_\lambda(\xi)$.

Justification: $\sin^2((-\xi)t/2) = \sin^2(\xi t/2)$.

Step 3.3. Finiteness: for each fixed ξ, $\psi_\lambda(\xi) < \infty$.

Justification: The integral $\int_0^{2L} w(t)\sin^2(\xi t/2)\, dt \le \int_0^{2L} w(t)\, dt < \infty$ (since $w(t) \sim (2t)^{-1}$ near 0 and the domain is bounded). The prime sum is finite (finitely many terms, each finite).

Step 3.4. Nonnegativity: $\psi_\lambda(\xi) \ge 0$.

Justification: Every summand is a product of nonneg. factors.

Q.E.D. □

Proposition 14 (Closedness on $L^2(\mathbb{R})$)**.** *The form $\mathcal{E}_\lambda^{\mathbb{R}}$ is densely defined, symmetric, nonneg., and closed on $L^2(\mathbb{R})$. Moreover,*

$$\mathcal{D}(\mathcal{E}_\lambda^{\mathbb{R}}) = \Big\{\phi \in L^2(\mathbb{R}) : \int_{\mathbb{R}} \psi_\lambda(\xi)\, |\widehat{\phi}(\xi)|^2\, d\xi < \infty\Big\},$$

and $\mathcal{D}(\mathcal{E}_\lambda^{\mathbb{R}})$ is a Hilbert space for the norm $\|\phi\|_{\mathcal{D}}^2 := \|\phi\|_{L^2(\mathbb{R})}^2 + \mathcal{E}_\lambda^{\mathbb{R}}(\phi)$.

Structured Proof. **Step 1.** $\mathcal{E}_\lambda^{\mathbb{R}}$ is the quadratic form of multiplication by ψ_λ in Fourier space.

Justification: Lemma 13: $\mathcal{E}_\lambda^{\mathbb{R}}(\phi) = \frac{1}{2\pi}\int_{\mathbb{R}} \psi_\lambda(\xi)|\widehat{\phi}(\xi)|^2\, d\xi$.

Step 2. $\mathcal{D}(\mathcal{E}_\lambda^{\mathbb{R}}) = \{\phi \in L^2(\mathbb{R}) : \int \psi_\lambda(\xi)|\widehat{\phi}(\xi)|^2\, d\xi < \infty\}$.

Justification: Immediate from Step 1: $\phi \in \mathcal{D}(\mathcal{E}_\lambda^{\mathbb{R}})$ iff $\mathcal{E}_\lambda^{\mathbb{R}}(\phi) < \infty$ iff $\int \psi_\lambda|\widehat{\phi}|^2 < \infty$.

Step 3. $\mathcal{D}(\mathcal{E}_\lambda^{\mathbb{R}})$ with the norm $\|\phi\|_{\mathcal{D}}^2 = \frac{1}{2\pi}\int(1+\psi_\lambda(\xi))|\widehat{\phi}(\xi)|^2\, d\xi$ is a Hilbert space.

Justification: Via $\phi \mapsto \widehat{\phi}$, this norm space is isometrically isomorphic to $L^2(\mathbb{R}, (1+\psi_\lambda(\xi))\frac{d\xi}{2\pi})$, which is a weighted L^2 space with a nonneg. measurable weight, hence complete.

Step 4. The form is closed.

Justification: A nonneg. quadratic form is closed iff its domain equipped with the graph norm is complete. This is Step 3.

Step 5. Nonnegativity and symmetry are immediate.

Justification: Nonnegativity: $\psi_\lambda \geq 0$ implies $\mathcal{E}_\lambda^{\mathbb{R}}(\phi) \geq 0$. Symmetry: $\mathcal{E}_\lambda^{\mathbb{R}}$ is defined on real-valued or complex-valued functions via a real nonneg. multiplier; the associated bilinear form inherits symmetry from the pointwise identity.

Step 6. $\mathcal{E}_\lambda^{\mathbb{R}}$ is densely defined: $C_c^\infty(\mathbb{R}) \subset \mathcal{D}(\mathcal{E}_\lambda^{\mathbb{R}})$.

Step 6.1. For $\phi \in C_c^\infty(\mathbb{R})$: $\|\phi - S_t\phi\|_2 \leq |t|\|\phi'\|_2$.

Justification: By the mean value theorem applied pointwise: $|\phi(u)-\phi(u-t)| = |t\phi'(\xi_u)| \leq |t|\cdot\|\phi'\|_\infty$ for some ξ_u between $u-t$ and u. But we can be more precise: $\phi(u)-\phi(u-t) = \int_0^t \phi'(u-s)\, ds$, so $|\phi(u)-\phi(u-t)|^2 \leq t^2\|\phi'\|_\infty^2$. Integrating and using that ϕ has compact support: $\|\phi - S_t\phi\|_2^2 \leq t^2\|\phi'\|_2^2$ (by a Cauchy–Schwarz refinement, or simply note $\leq t^2 \cdot \mathrm{meas}(\mathrm{supp}(\phi) + [-|t|, |t|]) \cdot \|\phi'\|_\infty^2$; more directly, Plancherel gives $\|\phi - S_t\phi\|_2^2 = \frac{1}{2\pi}\int 4\sin^2(\xi t/2)|\widehat{\phi}(\xi)|^2\, d\xi \leq t^2\frac{1}{2\pi}\int \xi^2|\widehat{\phi}(\xi)|^2\, d\xi = t^2\|\phi'\|_2^2$).

Step 6.2. $\int_0^{2L} w(t)t^2\, dt < \infty$.

Justification: Near $t = 0$: $w(t) \sim (2t)^{-1}$, so $w(t)t^2 \sim t/2$, which is integrable. The domain $[0, 2L]$ is bounded, so the integral is finite.

Step 6.3. $\mathcal{E}_\lambda^{\mathbb{R}}(\phi) \leq \|\phi'\|_2^2\big(\int_0^{2L} w(t)t^2\, dt + \sum_{p,m}(\log p)p^{-m/2}(m\log p)^2\big) < \infty$.

Justification: From Step 6.1: each $\|\phi - S_t\phi\|_2^2 \leq t^2\|\phi'\|_2^2$. Apply this to the archimedean integral and each prime term. The prime sum is finite (finitely many terms, each $\leq (\log p)p^{-m/2}(m\log p)^2\|\phi'\|_2^2$).

Step 6.4. $C_c^\infty(\mathbb{R})$ is dense in $L^2(\mathbb{R})$.

Justification: Standard fact in functional analysis.

Q.E.D. □

Proposition 15 (Closedness on $L^2(I)$)**.** *The form $\mathcal{E}_\lambda$ on $H = L^2(I)$ is densely defined, symmetric, nonneg., and closed.*

Structured Proof. **Step 1.** $G \mapsto \widetilde{G}$ is an isometry from $L^2(I)$ onto $H_I := \{\phi \in L^2(\mathbb{R}) : \phi = 0 \text{ a.e. on } \mathbb{R} \setminus I\}$.

Justification: $\|\widetilde{G}\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} |\widetilde{G}(u)|^2\,du = \int_I |G(u)|^2\,du = \|G\|^2_{L^2(I)}$. Surjectivity: any $\phi \in H_I$ is the zero-extension of $G := \phi|_I$.

Step 2. $\mathcal{E}_\lambda(G) = \mathcal{E}^{\mathbb{R}}_\lambda(\widetilde{G})$.

Justification: By definition (Definition 6 and the definition of $\mathcal{E}^{\mathbb{R}}_\lambda$).

Step 3. $\mathcal{E}_\lambda$ is the restriction of $\mathcal{E}^{\mathbb{R}}_\lambda$ to H_I.

Justification: Steps 1 and 2: H_I is a closed subspace of $L^2(\mathbb{R})$, and $\mathcal{E}_\lambda$ on $L^2(I)$ corresponds (via the isometry) to $\mathcal{E}^{\mathbb{R}}_\lambda$ restricted to H_I.

Step 4. $\mathcal{E}_\lambda$ is closed.

Justification: Suppose $G_n \to G$ in $L^2(I)$ and $\mathcal{E}_\lambda(G_n - G_m) \to 0$. Then $\widetilde{G}_n \to \widetilde{G}$ in $L^2(\mathbb{R})$ (Step 1) and $\mathcal{E}^{\mathbb{R}}_\lambda(\widetilde{G}_n - \widetilde{G}_m) \to 0$ (Step 2), so closedness of $\mathcal{E}^{\mathbb{R}}_\lambda$ (Prop. 14) gives $\widetilde{G} \in \mathcal{D}(\mathcal{E}^{\mathbb{R}}_\lambda)$ and $\mathcal{E}^{\mathbb{R}}_\lambda(\widetilde{G}_n - \widetilde{G}) \to 0$. Since H_I is closed in $L^2(\mathbb{R})$ and each $\widetilde{G}_n \in H_I$, the limit $\widetilde{G} \in H_I$, whence $G \in \mathcal{D}(\mathcal{E}_\lambda)$ and $\mathcal{E}_\lambda(G_n - G) \to 0$.

Step 5. $\mathcal{E}_\lambda$ is densely defined.

Justification: $C^\infty_c(I) \subset \mathcal{D}(\mathcal{E}_\lambda)$: for $G \in C^\infty_c(I)$, $\widetilde{G} \in C^\infty_c(\mathbb{R}) \subset \mathcal{D}(\mathcal{E}^{\mathbb{R}}_\lambda)$ (Proposition 14, Step 6), so $G \in \mathcal{D}(\mathcal{E}_\lambda)$. Since $C^\infty_c(I)$ is dense in $L^2(I)$, the form is densely defined.

Step 6. Nonnegativity and symmetry follow from those of $\mathcal{E}^{\mathbb{R}}_\lambda$.

Justification: $\mathcal{E}_\lambda(G) = \mathcal{E}^{\mathbb{R}}_\lambda(\widetilde{G}) \geq 0$ (Prop. 14).

Q.E.D. □

7.2.2 A coercive lower bound for the symbol ψ_λ

Lemma 16 (A lower bound for $w(t)$)**.** *Let $t_0 := \min(1, 2L)$. There exists $c_0 = c_0(L) > 0$ such that for all $t \in (0, t_0]$,*

$$w(t) = \frac{e^{t/2}}{2\sinh t} \geq \frac{c_0}{t}.$$

Structured Proof. **Step 1.** $\sinh t \leq te^t$ for all $t > 0$.

Justification: $\sinh t = \frac{e^t - e^{-t}}{2} \leq \frac{e^t}{2} \cdot 2t/(2t/(e^t - e^{-t}) \cdot 2)$—more directly, $\sinh t = \sum_{n=0}^\infty \frac{t^{2n+1}}{(2n+1)!} \leq t \sum_{n=0}^\infty \frac{t^{2n}}{(2n)!} = t\cosh t \leq te^t$.

Step 2. $w(t) \geq \frac{e^{-t/2}}{2t}$ for $t > 0$.

Justification: $w(t) = \frac{e^{t/2}}{2\sinh t} \geq \frac{e^{t/2}}{2te^t} = \frac{e^{-t/2}}{2t}$, using Step 1.

Step 3. For $t \in (0, 1]$: $e^{-t/2} \geq e^{-1/2}$.

Justification: $e^{-t/2}$ is decreasing; its minimum on $(0, 1]$ is at $t = 1$.

Step 4. Set $c_0 := e^{-1/2}/2$. Then for $t \in (0, t_0] \subset (0, 1]$: $w(t) \geq \frac{c_0}{t}$.

Justification: $w(t) \geq \frac{e^{-t/2}}{2t} \geq \frac{e^{-1/2}}{2t} = \frac{c_0}{t}$, combining Steps 2 and 3.

Q.E.D. □

Lemma 17 (Logarithmic growth of ψ_λ)**.** *There exist constants $c_1, c_2 > 0$ and $\xi_0 \geq 2$ (depending only on L) such that for all $|\xi| \geq \xi_0$,*

$$\psi_\lambda(\xi) \geq c_1 \log|\xi| - c_2.$$

In particular $\psi_\lambda(\xi) \to \infty$ as $|\xi| \to \infty$.

Structured Proof. **Step 1.** Drop the nonneg. prime sum: $\psi_\lambda(\xi) \geq 4\int_0^{t_0} w(t)\sin^2(\xi t/2)\,dt$.

Justification: The prime sum in (6) is ≥ 0, and we restrict the archimedean integral from $[0, 2L]$ to $[0, t_0] \subset [0, 2L]$ (the integrand is nonneg.).

Step 2. Apply Lemma 16: $\psi_\lambda(\xi) \geq 4c_0 \int_0^{t_0} \frac{1}{t}\sin^2(\xi t/2)\,dt$.

Justification: For $t \in (0, t_0]$, $w(t) \geq c_0/t$ by Lemma 16.

Step 3. For $|\xi| \geq \xi_0 := 4\pi/t_0$, define intervals $J_n := \left[\frac{2\pi n+\pi/2}{|\xi|}, \frac{2\pi n+3\pi/2}{|\xi|}\right]$ for $n \geq 0$. Then $\sin^2(\xi t/2) \geq 1/2$ for $t \in J_n$.

Justification: Set $\theta = |\xi|t/2$. For $t \in J_n$, $\theta \in [\pi n + \pi/4, \pi n + 3\pi/4]$. On this interval, $|\sin\theta| \geq \sin(\pi/4) = 1/\sqrt{2}$, so $\sin^2\theta \geq 1/2$. Since $\sin^2(\xi t/2) = \sin^2(|\xi|t/2) = \sin^2\theta$, the claim follows.

Step 4. Let N be the largest integer with $J_{N-1} \subset (0, t_0]$. Then $N \asymp |\xi|$.

Justification: The right endpoint of J_{N-1} is $\frac{2\pi(N-1)+3\pi/2}{|\xi|} \leq t_0$, giving $N \leq \frac{t_0|\xi|}{2\pi} + \frac{1}{4} + 1$. The condition $|\xi| \geq 4\pi/t_0$ ensures $N \geq 1$. Thus N is of order $|\xi|$ with constants depending on t_0.

Step 5. $\int_0^{t_0} \frac{1}{t}\sin^2(\xi t/2)\,dt \geq \frac{1}{2}\sum_{n=0}^{N-1} \log\frac{2\pi n+3\pi/2}{2\pi n+\pi/2}$.

Justification: Restrict the integral to $\bigcup_{n=0}^{N-1} J_n \subset (0, t_0]$. On J_n, $\sin^2(\xi t/2) \geq 1/2$ (Step 3), so $\int_{J_n} \frac{1}{t}\cdot\frac{1}{2}\,dt = \frac{1}{2}\log\frac{2\pi n+3\pi/2}{2\pi n+\pi/2}$.

Step 6. $\sum_{n=0}^{N-1} \log\frac{2\pi n+3\pi/2}{2\pi n+\pi/2} \geq c'\log N - C$ for absolute constants $c', C > 0$.

Step 6.1. $\log\frac{2\pi n+3\pi/2}{2\pi n+\pi/2} = \log\left(1 + \frac{\pi}{2\pi n+\pi/2}\right) \geq \frac{c}{n+1}$ for some absolute $c > 0$.

Justification: Using $\log(1+x) \geq x/(1+x)$: with $x = \pi/(2\pi n + \pi/2)$, $\log(1+x) \geq \frac{\pi/(2\pi n+\pi/2)}{1+\pi/(2\pi n+\pi/2)} = \frac{\pi}{2\pi n+3\pi/2} \geq \frac{\pi}{2\pi(n+1)+3\pi/2} \geq \frac{c}{n+1}$.

Step 6.2. $\sum_{n=0}^{N-1} \frac{1}{n+1} = H_N \geq \log N$ (where H_N is the N-th harmonic number).

Justification: Standard lower bound for harmonic numbers: $H_N \geq \log N$ for $N \geq 1$.

Step 6.3. Combine: the sum $\geq c \cdot \log N$.

Justification: Multiply Step 6.1 by 1 and sum, then apply Step 6.2.

Step 7. Since $N \asymp |\xi|$ (Step 4): $\log N = \log|\xi| + O(1)$.

Justification: $N = \Theta(|\xi|)$ implies $\log N = \log|\xi| + \log(N/|\xi|) = \log|\xi| + O(1)$.

Step 8. Conclusion: $\psi_\lambda(\xi) \geq 4c_0 \cdot \frac{1}{2} \cdot c' \cdot (\log|\xi| - C') = c_1\log|\xi| - c_2$.

Justification: Chain Steps 2, 5, 6, and 7: $\psi_\lambda(\xi) \geq 4c_0 \cdot \frac{1}{2}\cdot(c'\log N - C) \geq 2c_0c'(\log|\xi| + O(1)) - 2c_0C$. Set $c_1 := 2c_0c'$ and absorb the $O(1)$ into c_2.

Q.E.D. □

Corollary 18 (Energy controls a logarithmic frequency moment)**.** *There exist constants $a, b > 0$ (depending only on L) such that for every $\phi \in \mathcal{D}(\mathcal{E}_\lambda^{\mathbb{R}})$,*

$$\int_{\mathbb{R}} \log(2+|\xi|)\,|\widehat{\phi}(\xi)|^2\,d\xi \leq a\,\|\phi\|_{L^2(\mathbb{R})}^2 + b\int_{\mathbb{R}} \psi_\lambda(\xi)\,|\widehat{\phi}(\xi)|^2\,d\xi.$$

Structured Proof. **Step 1.** For all $\xi \in \mathbb{R}$: $\log(2+|\xi|) \leq a' + b'\psi_\lambda(\xi)$ for suitable $a', b' > 0$.

Step 1.1. For $|\xi| \geq \xi_0$: $\psi_\lambda(\xi) \geq c_1\log|\xi| - c_2$ (Lemma 17).

Justification: Direct application of Lemma 17.

Step 1.2. Hence $\log(2+|\xi|) \leq \log|\xi| + \log 3 \leq \frac{1}{c_1}(\psi_\lambda(\xi) + c_2) + \log 3$ for $|\xi| \geq \xi_0$.

Justification: $\log(2+|\xi|) \le \log(3|\xi|) = \log 3 + \log|\xi|$, and from Step 1.1: $\log|\xi| \le (\psi_\lambda(\xi) + c_2)/c_1$.

Step 1.3. For $|\xi| < \xi_0$: $\log(2+|\xi|) \le \log(2+\xi_0)$, a finite constant.

Justification: $\log(2+|\xi|)$ is bounded on bounded sets.

Step 1.4. Combine: set $b' := 1/c_1$ and $a' := c_2/c_1 + \log 3 + \log(2+\xi_0)$. Then $\log(2+|\xi|) \le a' + b'\psi_\lambda(\xi)$ for all ξ.

Justification: For $|\xi| \ge \xi_0$, use Step 1.2. For $|\xi| < \xi_0$, $a' \ge \log(2+\xi_0) \ge \log(2+|\xi|)$ and $b'\psi_\lambda(\xi) \ge 0$.

Step 2. Multiply by $|\widehat{\phi}(\xi)|^2$ and integrate: $\int \log(2+|\xi|)|\widehat{\phi}(\xi)|^2\, d\xi \le a' \int |\widehat{\phi}(\xi)|^2\, d\xi + b' \int \psi_\lambda(\xi)|\widehat{\phi}(\xi)|^2\, d\xi$.

Justification: Pointwise inequality from Step 1 times $|\widehat{\phi}(\xi)|^2 \ge 0$, integrated.

Step 3. By Plancherel, $\frac{1}{2\pi}\int |\widehat{\phi}|^2\, d\xi = \|\phi\|_2^2$. Set $a := 2\pi a'$, $b := b'$.

Justification: Rewrite Step 2 in terms of $\|\phi\|_2^2$ and $\mathcal{E}_\lambda^{\mathbb{R}}(\phi)$ (via Lemma 13).

Q.E.D. □

7.2.3 Compact embedding and compact resolvent

Theorem 19 (Kolmogorov–Riesz compactness criterion in $L^2(\mathbb{R})$). *A set $\mathcal{K} \subset L^2(\mathbb{R})$ is relatively compact if and only if:*

*(i) (*tightness*) for every $\varepsilon > 0$ there exists $R > 0$ such that $\int_{|u|>R} |\phi(u)|^2\, du < \varepsilon^2$ for all $\phi \in \mathcal{K}$;*

*(ii) (*translation equicontinuity*) for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|\phi - S_h\phi\|_2 < \varepsilon$ for all $\phi \in \mathcal{K}$ and all $|h| < \delta$.*

Remark 20. See, e.g., Lieb–Loss [7] for a proof of Theorem 19.

Lemma 21 (Uniform translation control from the form norm). *Fix $M > 0$ and define $\mathcal{K}_M := \{\phi \in H_I : \|\phi\|_2^2 + \mathcal{E}_\lambda^{\mathbb{R}}(\phi) \le M\}$. Then $\mathcal{K}_M$ satisfies condition (ii) in Theorem 19.*

Structured Proof. **Step 1.** For $\phi \in \mathcal{K}_M$ and $|h| \le 1$: $\|\phi - S_h\phi\|_2^2 = \frac{1}{2\pi}\int_{\mathbb{R}} 4\sin^2(\xi h/2)|\widehat{\phi}(\xi)|^2\, d\xi$.

Justification: Lemma 12.

Step 2. Split the integral at a parameter $R \ge 1$ into low and high frequencies.

Step 2.1. Low-frequency bound ($|\xi| \le R$): $\int_{|\xi|\le R} 4\sin^2(\xi h/2)|\widehat{\phi}(\xi)|^2\, d\xi \le (Rh)^2 \cdot 2\pi\|\phi\|_2^2$.

Justification: $\sin^2(x) \le x^2$ gives $4\sin^2(\xi h/2) \le (\xi h)^2 \le (Rh)^2$ for $|\xi| \le R$. Then $\int_{|\xi|\le R}(Rh)^2|\widehat{\phi}|^2 \le (Rh)^2\int_{\mathbb{R}}|\widehat{\phi}|^2 = (Rh)^2 \cdot 2\pi\|\phi\|_2^2$.

Step 2.2. High-frequency bound ($|\xi| > R$): $\int_{|\xi|>R} 4\sin^2(\xi h/2)|\widehat{\phi}(\xi)|^2\, d\xi \le \frac{4}{\log(2+R)}\int_{\mathbb{R}}\log(2+|\xi|)|\widehat{\phi}(\xi)|^2\, d\xi$.

Justification: $\sin^2 \le 1$ gives the left side $\le 4\int_{|\xi|>R}|\widehat{\phi}|^2\, d\xi$. For $|\xi| > R$, $1 \le \frac{\log(2+|\xi|)}{\log(2+R)}$, so $\int_{|\xi|>R}|\widehat{\phi}|^2 \le \frac{1}{\log(2+R)}\int_{|\xi|>R}\log(2+|\xi|)|\widehat{\phi}|^2 \le \frac{1}{\log(2+R)}\int_{\mathbb{R}}\log(2+|\xi|)|\widehat{\phi}|^2$.

Step 3. By Corollary 18, $\int_{\mathbb{R}}\log(2+|\xi|)|\widehat{\phi}|^2 \le C(M,L)$ uniformly over $\phi \in \mathcal{K}_M$.

Justification: $\|\phi\|_2^2 + \mathcal{E}_\lambda^{\mathbb{R}}(\phi) \le M$ by definition of $\mathcal{K}_M$. Corollary 18 bounds $\int \log(2+|\xi|)|\widehat{\phi}|^2$ by $a\|\phi\|_2^2 + b\int \psi_\lambda|\widehat{\phi}|^2\, d\xi = a\|\phi\|_2^2 + 2\pi b\,\mathcal{E}_\lambda^{\mathbb{R}}(\phi) \le (a+2\pi b)M =: C(M,L)$.

Step 4. Combine: $\|\phi - S_h\phi\|_2^2 \le (Rh)^2 M + \frac{C'(M,L)}{\log(2+R)}$.

Justification: Add the bounds from Steps 2.1 and 2.2, using Step 3 to bound the high-frequency term. Here $C'(M,L) = 4C(M,L)/(2\pi)$ (absorbing constants).

Step 5. Given $\varepsilon > 0$, choose R then δ to make each term $\leq \varepsilon^2/2$.

Step 5.1. Choose R so that $\frac{C'(M,L)}{\log(2+R)} \leq \varepsilon^2/2$.
Justification: $\log(2+R) \to \infty$ as $R \to \infty$, so such R exists.

Step 5.2. Then choose $\delta > 0$ so that $(R\delta)^2 M \leq \varepsilon^2/2$.
Justification: Take $\delta := \varepsilon/(R\sqrt{2M})$ (assuming $M > 0$; if $M = 0$ then $\mathcal{K}_M = \{0\}$ and the result is trivial).

Step 5.3. For $|h| < \delta$: $\|\phi - S_h\phi\|_2^2 \leq \varepsilon^2/2 + \varepsilon^2/2 = \varepsilon^2$.
Justification: Substitute into Step 4.

Q.E.D. □

Proposition 22 (Compact embedding of the form domain)**.** *The embedding* $(\mathcal{D}(\mathcal{E}_\lambda), \|\cdot\|_{\mathcal{D}}) \hookrightarrow L^2(I)$ *is compact.*

Structured Proof. **Step 1.** Let $\{G_n\} \subset \mathcal{D}(\mathcal{E}_\lambda)$ with $\|G_n\|_2^2 + \mathcal{E}_\lambda(G_n) \leq M$. Set $\phi_n := \widetilde{G}_n \in H_I$.
Justification: Setup. Then $\|\phi_n\|_2^2 + \mathcal{E}_\lambda^{\mathbb{R}}(\phi_n) = \|G_n\|_2^2 + \mathcal{E}_\lambda(G_n) \leq M$, so $\phi_n \in \mathcal{K}_M$.

Step 2. $\{\phi_n\}$ satisfies tightness condition (i) in Theorem 19.
Justification: Each ϕ_n is supported in $\overline{I} = [-L, L]$. For any $R > L$, $\int_{|u|>R} |\phi_n(u)|^2\, du = 0 < \varepsilon^2$.

Step 3. $\{\phi_n\}$ satisfies translation equicontinuity (ii) in Theorem 19.
Justification: Lemma 21: $\mathcal{K}_M$ satisfies (ii).

Step 4. $\{\phi_n\}$ is relatively compact in $L^2(\mathbb{R})$.
Justification: Theorem 19 applied to $\mathcal{K} := \{\phi_n : n \geq 1\}$, using Steps 2 and 3.

Step 5. $\{G_n\}$ is relatively compact in $L^2(I)$.
Justification: The map $\phi \mapsto \phi|_I$ is a continuous surjection $H_I \to L^2(I)$ (indeed, an isometry). Continuous images of relatively compact sets are relatively compact. So from Step 4, $\{G_n = \phi_n|_I\}$ is relatively compact in $L^2(I)$.

Q.E.D. □

Theorem 23 (Closed form, associated operator, and compact resolvent)**.** *There exists a unique selfadjoint operator $A_\lambda \geq 0$ on $L^2(I)$ associated to the closed form $\mathcal{E}_\lambda$ (Proposition 15) in the sense of the representation theorem for closed forms. Moreover, A_λ has compact resolvent.*

Structured Proof. **Step 1.** There exists a unique selfadjoint operator $A_\lambda \geq 0$ with $\mathcal{D}(\mathcal{E}_\lambda) = \mathcal{D}(A_\lambda^{1/2})$ and $\mathcal{E}_\lambda(G) = \|A_\lambda^{1/2} G\|_2^2$.
Justification: By the representation theorem for densely defined, closed, lower-bounded symmetric forms (Kato [6, Thm. VI.2.1]; in the Dirichlet-form setting, Fukushima–Oshima–Takeda [2, Thm. 1.3.1]). Proposition 15 verified that $\mathcal{E}_\lambda$ satisfies all hypotheses.

Step 2. $(A_\lambda + 1)^{-1}$ is compact on $L^2(I)$.

Step 2.1. Let $\{f_n\}$ be bounded in $L^2(I)$: $\|f_n\|_2 \leq C$. Set $u_n := (A_\lambda + 1)^{-1} f_n$.
Justification: $A_\lambda + 1$ is invertible because $A_\lambda \geq 0$, so $A_\lambda + 1 \geq 1 > 0$.

Step 2.2. $u_n \in \mathcal{D}(A_\lambda) \subset \mathcal{D}(\mathcal{E}_\lambda)$ and $(A_\lambda + 1)u_n = f_n$.
Justification: Definition of the resolvent.

Step 2.3. $\mathcal{E}_\lambda(u_n) + \|u_n\|_2^2 = \langle f_n, u_n \rangle$.

Justification: Take the L^2 inner product of $(A_\lambda + 1)u_n = f_n$ with u_n: $\langle A_\lambda u_n, u_n \rangle + \|u_n\|_2^2 = \langle f_n, u_n \rangle$. By the form identity, $\langle A_\lambda u_n, u_n \rangle = \mathcal{E}_\lambda(u_n)$.

Step 2.4. $\|u_n\|_2^2 + \mathcal{E}_\lambda(u_n) \leq \|f_n\|_2^2$.

Justification: From Step 2.3: $\mathcal{E}_\lambda(u_n) + \|u_n\|_2^2 = \langle f_n, u_n \rangle \leq \|f_n\|_2 \|u_n\|_2$ by Cauchy–Schwarz. Since $\|u_n\|_2^2 \leq \mathcal{E}_\lambda(u_n) + \|u_n\|_2^2 \leq \|f_n\|_2 \|u_n\|_2$, we get $\|u_n\|_2 \leq \|f_n\|_2$, hence $\mathcal{E}_\lambda(u_n) + \|u_n\|_2^2 \leq \|f_n\|_2 \|u_n\|_2 \leq \|f_n\|_2^2$.

Step 2.5. $\{u_n\}$ has a convergent subsequence in $L^2(I)$.

Justification: Step 2.4 shows $\{u_n\}$ is bounded in the form norm (with $M := C^2$), so Proposition 22 gives relative compactness in $L^2(I)$.

Step 2.6. $(A_\lambda + 1)^{-1}$ maps bounded sequences to sequences with convergent subsequences, hence is compact.

Justification: This is the definition of a compact operator: it maps bounded sets to relatively compact sets. Steps 2.1–2.5 verify this.

Q.E.D. □

7.3 Semigroup and irreducibility

Definition 24 (Irreducibility for semigroups on $L^2(I)$). A closed ideal in $L^2(I)$ has the form $L^2(B)$ for some measurable $B \subset I$. We call T *irreducible* if the only invariant closed ideals are $\{0\}$ and $L^2(I)$.

Lemma 25 (Invariant ideals and splitting of the form)**.** *Assume Theorem 23. Let $B \subset I$ be measurable and suppose $L^2(B)$ is invariant under $T(t) = e^{-tA_\lambda}$. Then for every $G \in \mathcal{D}(\mathcal{E}_\lambda)$: $\mathbf{1}_B G, \mathbf{1}_{I\setminus B} G \in \mathcal{D}(\mathcal{E}_\lambda)$ and $\mathcal{E}_\lambda(G) = \mathcal{E}_\lambda(\mathbf{1}_B G) + \mathcal{E}_\lambda(\mathbf{1}_{I\setminus B} G)$.*

Structured Proof. **Step 1.** Let $P = M_{\mathbf{1}_B}$ (multiplication by $\mathbf{1}_B$) and $Q = I - P$. Then P is an orthogonal projection with $\mathrm{Ran}(P) = L^2(B)$.

Justification: $P^2 = P$ ($\mathbf{1}_B^2 = \mathbf{1}_B$) and $P^* = P$ ($\mathbf{1}_B$ is real), so P is an orthogonal projection. $Pf = \mathbf{1}_B f$ vanishes outside B, so $\mathrm{Ran}(P) = L^2(B)$.

Step 2. P commutes with $T(t)$ for all $t \geq 0$.

Step 2.1. Invariance of $L^2(B) = \mathrm{Ran}(P)$ means $T(t)(\mathrm{Ran}(P)) \subset \mathrm{Ran}(P)$, i.e. $PT(t)P = T(t)P$.

Justification: $T(t)Pf \in L^2(B)$ for all f, so $PT(t)Pf = T(t)Pf$.

Step 2.2. Take adjoints: $(PT(t)P)^* = (T(t)P)^*$, giving $PT(t)P = PT(t)$.

Justification: $P^* = P$ and $T(t)^* = T(t)$ (selfadjointness of A_λ implies selfadjointness of e^{-tA_λ}). So $(PT(t)P)^* = P^*T(t)^*P^* = PT(t)P$ and $(T(t)P)^* = P^*T(t)^* = PT(t)$. From Step 2.1: $PT(t)P = T(t)P$; equating with the adjoint computation: $PT(t)P = PT(t)$. Together: $T(t)P = PT(t)$.

Step 3. P commutes with $A_\lambda^{1/2}$.

Justification: P commutes with $T(t) = e^{-tA_\lambda}$ for all $t > 0$ (Step 2). Since the family of functions $\{e^{-t\lambda}\}_{t>0}$ separates points on $[0, \infty)$, a bounded operator commuting with every e^{-tA_λ} must commute with the spectral measure $E(\cdot)$ of A_λ (by the spectral theorem and a monotone-class or Stone–Weierstrass argument), hence with every bounded Borel function of A_λ. In particular, P commutes with $A_\lambda^{1/2} = \int_0^\infty \lambda^{1/2}\, dE(\lambda)$.

Step 4. $P(\mathcal{D}(\mathcal{E}_\lambda)) \subset \mathcal{D}(\mathcal{E}_\lambda)$ and $Q(\mathcal{D}(\mathcal{E}_\lambda)) \subset \mathcal{D}(\mathcal{E}_\lambda)$.

Justification: $\mathcal{D}(\mathcal{E}_\lambda) = \mathcal{D}(A_\lambda^{1/2})$. If $u \in \mathcal{D}(A_\lambda^{1/2})$, then $A_\lambda^{1/2}Pu = PA_\lambda^{1/2}u \in L^2(I)$ (Step 3), so $Pu \in \mathcal{D}(A_\lambda^{1/2})$. Similarly for $Qu = (I-P)u$.

Step 5. $\mathcal{E}_\lambda(G) = \mathcal{E}_\lambda(PG) + \mathcal{E}_\lambda(QG)$ for $G \in \mathcal{D}(\mathcal{E}_\lambda)$.

Step 5.1. $\mathcal{E}_\lambda(G) = \|A_\lambda^{1/2}G\|_2^2$.

Justification: Form identity from the representation theorem (Step 1 of Theorem 23).

Step 5.2. $A_\lambda^{1/2}G = A_\lambda^{1/2}PG + A_\lambda^{1/2}QG = PA_\lambda^{1/2}G + QA_\lambda^{1/2}G$.

Justification: $G = PG + QG$ and $A_\lambda^{1/2}$ commutes with P and Q (Step 3).

Step 5.3. $PA_\lambda^{1/2}G$ and $QA_\lambda^{1/2}G$ are orthogonal in $L^2(I)$.

Justification: $\langle Pv, Qv\rangle = \langle Pv, (I-P)v\rangle = \langle Pv, v\rangle - \langle Pv, Pv\rangle = \langle P^2v, v\rangle - \|Pv\|^2 = \|Pv\|^2 - \|Pv\|^2 = 0$. Apply with $v = A_\lambda^{1/2}G$.

Step 5.4. $\|A_\lambda^{1/2}G\|^2 = \|PA_\lambda^{1/2}G\|^2 + \|QA_\lambda^{1/2}G\|^2 = \|A_\lambda^{1/2}PG\|^2 + \|A_\lambda^{1/2}QG\|^2 = \mathcal{E}_\lambda(PG) + \mathcal{E}_\lambda(QG)$.

Justification: Pythagorean theorem (Step 5.3), then $PA_\lambda^{1/2}G = A_\lambda^{1/2}PG$ (Step 3), then the form identity (Step 5.1) applied to PG and QG (which lie in $\mathcal{D}(\mathcal{E}_\lambda)$ by Step 4).

Q.E.D. □

Proposition 26 (Triviality of invariant ideals for $\mathcal{E}_\lambda$)**.** *Assume Theorem 23. Let $B \subset I$ be measurable and assume $L^2(B)$ is invariant under $T(t) = e^{-tA_\lambda}$. Then $m(B) = 0$ or $m(I \setminus B) = 0$.*

Structured Proof. **Step 1.** By Lemma 25, for every $G \in \mathcal{D}(\mathcal{E}_\lambda)$:

$$\mathcal{E}_\lambda(G) = \mathcal{E}_\lambda(\mathbf{1}_B G) + \mathcal{E}_\lambda(\mathbf{1}_{I\setminus B}G). \tag{7}$$

Justification: Direct application of Lemma 25.

Step 2. $1 \in \mathcal{D}(\mathcal{E}_\lambda)$ (the constant function $G \equiv 1$ on I).

Step 2.1. For each shift $s > 0$: $\|\widetilde{1} - S_s\widetilde{1}\|_2^2 = m(I \,\Delta\, (I+s))$.

Justification: $\widetilde{1} = \mathbf{1}_{(-L,L)}$ and $S_s\widetilde{1} = \mathbf{1}_{(-L+s,L+s)}$. $\|\widetilde{1} - S_s\widetilde{1}\|_2^2 = m(I \,\Delta\, (I+s))$, the measure of the symmetric difference.

Step 2.2. For $0 < s < 2L$: $m(I \,\Delta\, (I+s)) = 2s$.

Justification: $I \setminus (I+s) = (-L, -L+s)$ has measure s; $(I+s)\setminus I = (L, L+s)$ has measure s.

Step 2.3. The archimedean integral $\int_0^{2L} w(t)\cdot 2t\,dt < \infty$.

Justification: $w(t) \sim (2t)^{-1}$ near 0, so $w(t)\cdot 2t \sim 1$ near 0: integrable. On $[0, 2L]$ (bounded), the integral is finite.

Step 2.4. The prime sum in $\mathcal{E}_\lambda(1)$ is finite.

Justification: Finitely many shift sizes $m\log p$ with $p^m \le \lambda^2$, each giving $\|\widetilde{1} - S_{m\log p}\widetilde{1}\|_2^2 = 2m\log p < \infty$.

Step 2.5. $\mathcal{E}_\lambda(1) < \infty$, hence $1 \in \mathcal{D}(\mathcal{E}_\lambda)$.

Justification: Combine Steps 2.3 and 2.4.

Step 3. Apply (7) with $G \equiv 1$. Set $f := \widetilde{\mathbf{1}_B}$, $g := \widetilde{\mathbf{1}_{B^c}}$ where $B^c := I \setminus B$. Then $\widetilde{1} = f + g$ and $fg = 0$ a.e.

Justification: $\mathbf{1}_B + \mathbf{1}_{B^c} = 1$ on I and $\mathbf{1}_B \cdot \mathbf{1}_{B^c} = 0$. Zero-extending: $f + g = \widetilde{1}$ and $fg = 0$ a.e. on $\mathbb{R}$.

Step 4. For each shift $s > 0$: $\|(f+g)-S_s(f+g)\|_2^2 = \|f-S_sf\|_2^2+\|g-S_sg\|_2^2+2\langle f-S_sf, g-S_sg\rangle$.

Justification: Expand $\|a+b\|^2 = \|a\|^2 + \|b\|^2 + 2\Re\langle a, b\rangle$ with $a = f - S_sf$ and $b = g - S_sg$. Since f, g are real-valued, $\langle a, b\rangle \in \mathbb{R}$.

Step 5. Substituting Step 4 into (7) (which, via Definition 6, is an identity between weighted sums/integrals of $\| \cdot - S_s \cdot \|_2^2$ terms): the cross-terms sum to zero.

Step 5.1. $\mathcal{E}_\lambda(1) = \mathcal{E}_\lambda(\mathbf{1}_B) + \mathcal{E}_\lambda(\mathbf{1}_{B^c})$ (from Step 1 with $G = 1$).

Justification: Equation (7) with $G \equiv 1$.

Step 5.2. Expand each $\mathcal{E}_\lambda$ using Definition 6 and Step 4: each shift-size gives the identity from Step 4. After subtracting $\mathcal{E}_\lambda(\mathbf{1}_B) + \mathcal{E}_\lambda(\mathbf{1}_{B^c})$ from $\mathcal{E}_\lambda(1)$, the remainder is

$$\int_0^{2L} w(t) \cdot 2\langle f - S_tf, g - S_tg\rangle \, dt + \sum_{p,m} (\log p)p^{-m/2} \cdot 2\langle f - S_{m \log p}f, g - S_{m \log p}g\rangle = 0.$$

Justification: Combine Step 4 for each shift size with Step 5.1. The "diagonal" terms cancel, leaving the cross terms.

Step 6. For any $s > 0$: $\langle f - S_sf, g - S_sg\rangle \le 0$.

Step 6.1. $\langle f - S_sf, g - S_sg\rangle = \langle f, g\rangle - \langle f, S_sg\rangle - \langle S_sf, g\rangle + \langle S_sf, S_sg\rangle$.

Justification: Bilinearity of the inner product.

Step 6.2. $\langle f, g\rangle = 0$.

Justification: $fg = 0$ a.e. (Step 3), so $\int_{\mathbb{R}} f(u)g(u)\, du = 0$.

Step 6.3. $\langle S_sf, S_sg\rangle = \langle f, g\rangle = 0$.

Justification: S_s is unitary: $\langle S_sf, S_sg\rangle = \langle f, g\rangle$. Then Step 6.2.

Step 6.4. $\langle f, S_sg\rangle \ge 0$ and $\langle S_sf, g\rangle = \langle g, S_sf\rangle \ge 0$.

Justification: $f = \widetilde{\mathbf{1}_B} \ge 0$ and $S_sg = \widetilde{\mathbf{1}_{B^c}}(\cdot - s) \ge 0$, so $\langle f, S_sg\rangle = \int f \cdot S_sg \ge 0$. Similarly for $\langle g, S_sf\rangle$.

Step 6.5. Combine: $\langle f - S_sf, g - S_sg\rangle = 0 - \langle f, S_sg\rangle - \langle g, S_sf\rangle + 0 \le 0$.

Justification: Steps 6.1–6.4.

Step 7. $\langle f, S_tg\rangle = \langle g, S_tf\rangle = 0$ for a.e. $t \in (0, 2L)$.

Justification: From Step 5.2, the weighted sum of the cross terms is 0. All weights are ≥ 0, and $w(t) > 0$ for $t > 0$. Each cross term $\langle f - S_tf, g - S_tg\rangle \le 0$ (Step 6). A sum of nonpositive terms with positive weights equaling zero forces each term to be zero a.e. Hence $\langle f - S_tf, g - S_tg\rangle = 0$ for a.e. $t \in (0, 2L)$. From Step 6.5: this means $\langle f, S_tg\rangle + \langle g, S_tf\rangle = 0$ a.e. Since both are ≥ 0 (Step 6.4), each is 0.

Step 8. Upgrade to *all* $t \in (0, 2L)$: $\langle f, S_tg\rangle = 0$ for every $t \in (0, 2L)$.

Justification: $t \mapsto \langle f, S_tg\rangle$ is continuous (strong continuity of S_t on $L^2(\mathbb{R})$). A continuous nonneg. function vanishing a.e. on $(0, 2L)$ vanishes everywhere on $(0, 2L)$. (Same argument as in Lemma 11, Step 3.)

Step 9. For every $t \in (0, 2L)$: $\mathbf{1}_B(u) = \mathbf{1}_B(u - t)$ for a.e. $u \in I \cap (I + t)$.

Step 9.1. $0 = \langle f, S_tg\rangle = \int_{I\cap(I+t)} \mathbf{1}_B(u) \cdot \mathbf{1}_{B^c}(u - t)\, du$.

Justification: Unwinding definitions: $f = \widetilde{\mathbf{1}_B}$, $S_tg(u) = \widetilde{\mathbf{1}_{B^c}}(u-t)$. The integrand is nonzero only where both $u \in I$ (so $f(u) = \mathbf{1}_B(u)$) and $u-t \in I$ (so $g(u-t) = \mathbf{1}_{B^c}(u-t)$), i.e. $u \in I \cap (I+t)$.

Step 9.2. Since $\mathbf{1}_B(u) \cdot \mathbf{1}_{B^c}(u - t) \ge 0$ and the integral is 0: $\mathbf{1}_B(u) \cdot \mathbf{1}_{B^c}(u - t) = 0$ for a.e. $u \in I \cap (I + t)$.

Justification: A nonneg. integrable function with zero integral vanishes a.e.

Step 9.3. Hence $\mathbf{1}_B(u) \le \mathbf{1}_B(u - t)$ for a.e. $u \in I \cap (I + t)$.

Justification: Step 9.2 says: wherever $\mathbf{1}_B(u) = 1$, we must have $\mathbf{1}_{B^c}(u-t) = 0$, i.e. $\mathbf{1}_B(u-t) = 1$.

Step 9.4. Similarly, $\langle g, S_t f\rangle = 0$ gives $\mathbf{1}_B(u-t) \le \mathbf{1}_B(u)$ a.e. on $I \cap (I+t)$.

Justification: $\langle g, S_t f\rangle = \int_{I\cap(I+t)} \mathbf{1}_{B^c}(u)\mathbf{1}_B(u-t)\,du = 0$ (Step 8). Same argument as Steps 9.2–9.3 with B and B^c swapped.

Step 9.5. Combine: $\mathbf{1}_B(u) = \mathbf{1}_B(u-t)$ a.e. on $I \cap (I+t)$.

Justification: Steps 9.3 and 9.4.

Step 10. $m(B) = 0$ or $m(I \setminus B) = 0$.

Justification: Step 9.5 holds for every $t \in (0, 2L)$. This provides the hypothesis of Lemma 10 with $\varepsilon = 2L$. The conclusion follows.

Q.E.D. □

Remark 27 (Why we do not use $\mathcal{E}_\lambda(\mathbf{1}_B) = 0$). Because $\mathcal{E}_\lambda$ is defined using zero-extension to $\mathbb{R}$ (Definition 6), the form is typically non-conservative: in general $\mathcal{E}_\lambda(1) > 0$. In the conservative case ($\mathcal{E}(1) = 0$) one often has an equivalence between invariance and the condition $\mathcal{E}(\mathbf{1}_B) = 0$. Here, the presence of killing means $\mathcal{E}_\lambda(\mathbf{1}_B) = 0$ is a *stronger* condition than invariance, so we instead argue directly from the correct invariance identity (7), which depends only on the interaction/jump part.

Corollary 28 (Irreducibility for $\mathcal{E}_\lambda$)**.** *Assume Theorem 23. Then $T(t) = e^{-tA_\lambda}$ is irreducible.*

Structured Proof. **Step 1.** Let $J \subset L^2(I)$ be a closed $T(t)$-invariant ideal. Then $J = L^2(B)$ for some measurable $B \subset I$.

Justification: Standard lattice-theory fact: every closed ideal in $L^2(I)$ has the form $L^2(B)$ for some measurable $B \subset I$.

Step 2. $m(B) = 0$ or $m(I \setminus B) = 0$.

Justification: Proposition 26.

Step 3. $J = \{0\}$ or $J = L^2(I)$.

Justification: If $m(B) = 0$, then $L^2(B) = \{0\}$. If $m(I \setminus B) = 0$, then $L^2(B) = L^2(I)$.

Q.E.D. □

8 Positivity improving and the ground state

8.1 External theorems used

Theorem 29 (Positivity improving from positivity + irreducibility + holomorphy)**.** *Let E be a Banach lattice and S a positive, irreducible, holomorphic C_0-semigroup on E. Then S is positivity improving: for each $t > 0$ and each $0 \le f \in E$ with $f \ne 0$, one has $S(t)f > 0$ (in the lattice sense; on L^2 this means > 0 a.e.).*

Remark 30 (Source). This is stated (for general Banach lattices) as Theorem 2.3 in Arendt–ter Elst–Glück [1].

Theorem 31 (Simplicity of the principal eigenvalue under compact resolvent)**.** *Let A be selfadjoint and lower bounded on $L^2(I)$ with compact resolvent, and let $S(t) = e^{-tA}$. If S is positivity improving, then $\min \sigma(A)$ is a simple eigenvalue and admits an eigenfunction which is strictly positive a.e.*

Remark 32 (Source). Proposition 2.4 in the same paper of Arendt et al., a Perron–Frobenius/Krein–Rutman/Jentzsch consequence for compact positive operators.

8.2 Application to A_λ

Proposition 33 (Positivity improving and simple ground state for A_λ)**.** *Assume Theorem 23. Then:*

1. *The semigroup $T(t) = e^{-tA_\lambda}$ is positivity preserving (Markovian).*
2. *$T(t)$ is irreducible.*
3. *$T(t)$ is holomorphic.*

Consequently $T(t)$ is positivity improving, and the lowest eigenvalue of A_λ is simple with a strictly positive a.e. eigenfunction.

Structured Proof. **Step 1.** $T(t)$ is positivity preserving (Markovian).

Justification: Lemma 9 shows that $\mathcal{E}_\lambda$ satisfies the Markov (normal contraction) property. By the general correspondence between Dirichlet forms and positivity-preserving semigroups (Fukushima–Oshima–Takeda [2, Thm. 1.4.1]; Ouhabaz [5, Ch. 1, §§1.4–1.5]), the associated semigroup $T(t)$ is positivity preserving.

Step 2. $T(t)$ is irreducible.

Justification: Corollary 28.

Step 3. $T(t)$ is holomorphic.

Justification: A_λ is selfadjoint and lower bounded ($A_\lambda \geq 0$). By the spectral theorem, e^{-zA_λ} is bounded and holomorphic on $\{z \in \mathbb{C} : \Re z > 0\}$. In particular, $T(t) = e^{-tA_\lambda}$ extends to a holomorphic semigroup.

Step 4. $T(t)$ is positivity improving.

Justification: Apply Theorem 29: $T(t)$ is positive (Step 1), irreducible (Step 2), and holomorphic (Step 3). Therefore it is positivity improving.

Step 5. The lowest eigenvalue of A_λ is simple, with a strictly positive a.e. eigenfunction.

Justification: Apply Theorem 31: A_λ is selfadjoint, lower bounded, has compact resolvent (Theorem 23), and $T(t) = e^{-tA_\lambda}$ is positivity improving (Step 4). The conclusion follows.

Q.E.D. □

9 Evenness of the ground state from inversion symmetry

Proposition 34 (Inversion (reflection) symmetry)**.** *Let $R : L^2(I) \to L^2(I)$ be the unitary involution $(RG)(u) := G(-u)$. Then $R(\mathcal{D}(\mathcal{E}_\lambda)) = \mathcal{D}(\mathcal{E}_\lambda)$ and $\mathcal{E}_\lambda(RG) = \mathcal{E}_\lambda(G)$ for all $G \in \mathcal{D}(\mathcal{E}_\lambda)$. Consequently, A_λ commutes with R.*

Structured Proof. **Step 1.** R is a well-defined unitary involution on $L^2(I)$, and R preserves $H_I \subset L^2(\mathbb{R})$.

Justification: $I = (-L, L)$ is symmetric about 0: if $u \in I$ then $-u \in I$. So $(RG)(u) = G(-u)$ maps $L^2(I)$ to itself. R is unitary ($\|RG\|_2 = \|G\|_2$ by substitution $u \mapsto -u$) and $R^2 = \mathrm{Id}$. If $\phi \in H_I$ (i.e. $\phi = 0$ outside I), then $R\phi(u) = \phi(-u)$ vanishes for $u \notin I$ (since $-u \notin I$), so $R\phi \in H_I$.

Step 2. $RS_t = S_{-t}R$ on $L^2(\mathbb{R})$.

Justification: $(RS_t\phi)(u) = (S_t\phi)(-u) = \phi(-u-t)$ and $(S_{-t}R\phi)(u) = (R\phi)(u-(-t)) = (R\phi)(u+t) = \phi(-u-t)$. They agree.

Step 3. $\|\widetilde{RG} - S_t\widetilde{RG}\|_2 = \|\widetilde{G} - S_t\widetilde{G}\|_2$ for every $t \in \mathbb{R}$.

Step 3.1. $\widetilde{RG} = R\widetilde{G}$ (extension by zero commutes with reflection, since I is symmetric).

Justification: For $u \in I$: $\widetilde{RG}(u) = (RG)(u) = G(-u) = (R\widetilde{G})(u)$. For $u \notin I$: $\widetilde{RG}(u) = 0$ and $(R\widetilde{G})(u) = \widetilde{G}(-u) = 0$ (since $-u \notin I$).

Step 3.2. $\|R\widetilde{G} - S_tR\widetilde{G}\|_2 = \|R(\widetilde{G} - S_{-t}\widetilde{G})\|_2$ (using Step 2: $S_tR = RS_{-t}$, so $S_tR\widetilde{G} = RS_{-t}\widetilde{G}$).

Justification: $R\widetilde{G} - S_tR\widetilde{G} = R\widetilde{G} - RS_{-t}\widetilde{G} = R(\widetilde{G} - S_{-t}\widetilde{G})$.

Step 3.3. $\|R(\widetilde{G} - S_{-t}\widetilde{G})\|_2 = \|\widetilde{G} - S_{-t}\widetilde{G}\|_2$ (R is unitary).

Justification: Unitarity of R.

Step 3.4. $\|\widetilde{G} - S_{-t}\widetilde{G}\|_2 = \|\widetilde{G} - S_t\widetilde{G}\|_2$.

Justification: $\|\phi - S_{-t}\phi\|_2 = \|S_t\phi - \phi\|_2 = \|\phi - S_t\phi\|_2$ (norm is symmetric). More formally: $\|\phi - S_{-t}\phi\|_2^2 = \int |\phi(u) - \phi(u+t)|^2\,du$; substituting $v = u+t$: $= \int |\phi(v-t) - \phi(v)|^2\,dv = \|\phi - S_t\phi\|_2^2$.

Step 3.5. Chain Steps 3.1–3.4: $\|\widetilde{RG} - S_t\widetilde{RG}\|_2 = \|R\widetilde{G} - S_tR\widetilde{G}\|_2 = \|\widetilde{G} - S_{-t}\widetilde{G}\|_2 = \|\widetilde{G} - S_t\widetilde{G}\|_2$.

Step 4. $\mathcal{E}_\lambda(RG) = \mathcal{E}_\lambda(G)$.

Justification: By Definition 6, $\mathcal{E}_\lambda$ is built from terms of the form $\|\widetilde{G} - S_t\widetilde{G}\|_2^2$ with nonneg. weights. Step 3 shows each such term is the same for RG as for G. Therefore $\mathcal{E}_\lambda(RG) = \mathcal{E}_\lambda(G)$. In particular, $RG \in \mathcal{D}(\mathcal{E}_\lambda)$ iff $G \in \mathcal{D}(\mathcal{E}_\lambda)$.

Step 5. A_λ commutes with R.

Step 5.1. For $u \in \mathcal{D}(A_\lambda)$ and $v \in \mathcal{D}(\mathcal{E}_\lambda)$: $\mathcal{E}_\lambda(Ru, v) = \mathcal{E}_\lambda(u, Rv)$.

Justification: $Ru \in \mathcal{D}(\mathcal{E}_\lambda)$ (since R preserves $\mathcal{D}(\mathcal{E}_\lambda)$ by Step 4). Invariance of $\mathcal{E}_\lambda$ under R (Step 4) implies, by polarization, $\mathcal{E}_\lambda(Ru, Rv) = \mathcal{E}_\lambda(u, v)$ for all $u, v \in \mathcal{D}(\mathcal{E}_\lambda)$. Set $v \mapsto Rv$: $\mathcal{E}_\lambda(Ru, R(Rv)) = \mathcal{E}_\lambda(Ru, v) = \mathcal{E}_\lambda(u, Rv)$ (using $R^2 = \mathrm{Id}$).

Step 5.2. $\mathcal{E}_\lambda(u, Rv) = \langle A_\lambda u, Rv\rangle = \langle RA_\lambda u, v\rangle$.

Justification: Form identity: $\mathcal{E}_\lambda(u, Rv) = \langle A_\lambda u, Rv\rangle$ (since $u \in \mathcal{D}(A_\lambda)$ and $Rv \in \mathcal{D}(\mathcal{E}_\lambda)$). Then $\langle A_\lambda u, Rv\rangle = \langle R^* A_\lambda u, v\rangle = \langle RA_\lambda u, v\rangle$ since $R^* = R$ (R is selfadjoint: $R = R^{-1} = R^*$).

Step 5.3. Combining: $\mathcal{E}_\lambda(Ru, v) = \langle RA_\lambda u, v\rangle$ for all $v \in \mathcal{D}(\mathcal{E}_\lambda)$.

Justification: Steps 5.1 and 5.2.

Step 5.4. $Ru \in \mathcal{D}(A_\lambda)$ and $A_\lambda Ru = RA_\lambda u$.

Justification: By the representation theorem (Kato [6, Thm. VI.2.1]), $w \in \mathcal{D}(A_\lambda)$ if and only if there exists $h \in L^2(I)$ such that $\mathcal{E}_\lambda(w, v) = \langle h, v\rangle$ for all $v \in \mathcal{D}(\mathcal{E}_\lambda)$, in which case $A_\lambda w = h$. Step 5.3 provides exactly this with $w = Ru$ and $h = RA_\lambda u \in L^2(I)$.

Q.E.D. □

Corollary 35 (Even ground state)**.** *Assume Theorem 23 and Proposition 34. Let ψ be the strictly positive ground-state eigenfunction from Proposition 33. Then ψ is even: $\psi(-u) = \psi(u)$ a.e.*

Structured Proof. **Step 1.** Define $\psi^\sharp := R\psi$. Then $\psi^\sharp$ is an eigenfunction of A_λ for the same eigenvalue $\mu_0 := \min \sigma(A_\lambda)$.

Justification: $A_\lambda\psi = \mu_0\psi$. Since $A_\lambda R = RA_\lambda$ (Proposition 34): $A_\lambda\psi^\sharp = A_\lambda R\psi = RA_\lambda\psi = R(\mu_0\psi) = \mu_0 R\psi = \mu_0\psi^\sharp$.

Step 2. $\psi^\sharp > 0$ a.e.

Justification: $\psi > 0$ a.e. (Proposition 33). $\psi^\sharp(u) = \psi(-u)$. Since I is symmetric and $\psi > 0$ a.e. on I, the reflection $\psi^\sharp > 0$ a.e. on I.

Step 3. $\psi^\sharp = c\psi$ for some $c \in \mathbb{R}$.

Justification: Proposition 33 says the eigenspace for μ_0 is one-dimensional (simple eigenvalue). Both ψ and $\psi^\sharp$ lie in this eigenspace (Steps 1 and the original eigenvalue equation). Hence $\psi^\sharp = c\psi$ for some scalar c. Since both are real-valued (reflection preserves real-valuedness), $c \in \mathbb{R}$.

Step 4. $c > 0$.

Justification: $\psi^\sharp > 0$ a.e. (Step 2) and $\psi > 0$ a.e. (Proposition 33). If $c \leq 0$, then $\psi^\sharp = c\psi \leq 0$ a.e., contradicting $\psi^\sharp > 0$ a.e.

Step 5. $c = 1$.

Justification: $\|\psi^\sharp\|_2 = \|R\psi\|_2 = \|\psi\|_2$ (R is unitary). From $\psi^\sharp = c\psi$: $\|c\psi\|_2 = |c|\|\psi\|_2 = \|\psi\|_2$. Since $\|\psi\|_2 > 0$ ($\psi \neq 0$), $|c| = 1$. Combined with $c > 0$ (Step 4): $c = 1$.

Step 6. $\psi(-u) = \psi(u)$ a.e.

Justification: $\psi^\sharp = \psi$ (Step 5), i.e. $R\psi = \psi$, i.e. $\psi(-u) = \psi(u)$ a.e.

Q.E.D. $\square$

10 Summary of concrete progress

- Starting solely from the explicit local formulas (2)–(3), we derived a representation of $-\sum_v W_v(g * g^*)$ (up to an additive constant multiple of $\|g\|_2^2$) as a positive combination of translation-difference energies in log-coordinates (Definition 6, Lemmas 4–5).
- We proved the Markov/normal contraction inequality for this form (Lemma 9).
- Using only measure theory (Lebesgue density), we proved that invariance under all sufficiently small translations forces a measurable subset of an interval to be null or conull (Lemma 10), and we used it to show that $\mathcal{E}_\lambda(\mathbf{1}_B) = 0$ implies B is null or conull (Lemma 11).
- We proved that the quadratic form is closed (Propositions 14–15), established a logarithmic lower bound on its Fourier symbol (Lemma 17), and used the Kolmogorov–Riesz compactness criterion to show that the associated selfadjoint operator has compact resolvent (Theorem 23).
- From this operator setup we obtained irreducibility and then (by a standard external theorem) positivity improving of the semigroup, hence simplicity and strict positivity of the ground state (Proposition 33).
- Finally, inversion symmetry forces that strictly positive simple ground state to be even (Corollary 35).

11 Bibliographic pointers

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