Energy decomposition (Step 1): a self-contained, rigorous formulation

1. Basic setup

Let $\mathbb{R}_+^* = (0, \infty)$ with multiplicative Haar measure $d^*x := dx/x$. Let $L^2(\mathbb{R}_+^*) := L^2(\mathbb{R}_+^*, d^*x)$ and $\langle \cdot, \cdot \rangle$ its inner product. For $a > 0$ define the unitary dilation $(U_a g)(x) := g(x/a)$ on $L^2(\mathbb{R}_+^*)$.

When $g, h \in L^1(\mathbb{R}_+^*, d^*x)$, the multiplicative convolution is

$$(g * h)(a) := \int_{\mathbb{R}_+^*} g(y)\, h(a/y)\, d^*y, \qquad a > 0,$$

and the involution is $g^*(x) := \overline{g(x^{-1})}$.

Definition 1 (Correlation function)**.** *For $g \in L^2(\mathbb{R}_+^*)$ define its correlation function $f : \mathbb{R}_+^* \to \mathbb{C}$ by*

$$f(a) := \langle g, U_a g \rangle = \int_{\mathbb{R}_+^*} g(x)\, \overline{g(x/a)}\, d^*x.$$

If in addition $g \in L^1(\mathbb{R}_+^*)$, then $f = g * g^*$ almost everywhere and the above integral coincides with the convolution formula.

Lemma 2 (Elementary properties of f)**.** *Let $g \in L^2(\mathbb{R}_+^*)$ and f as in Definition 1. Then for all $a > 0$,*

$$f(a^{-1}) = \overline{f(a)}, \qquad f(1) = \|g\|_2^2, \qquad |f(a)| \le \|g\|_2^2.$$

In particular $f(a) + f(a^{-1}) = 2\Re\langle g, U_a g\rangle$.

Proof. The identity $f(a^{-1}) = \overline{f(a)}$ follows by substituting $x \mapsto x/a$ and taking complex conjugates. The bound $|f(a)| \le \|g\|_2 \|U_a g\|_2 = \|g\|_2^2$ is Cauchy–Schwarz and unitarity of U_a. Finally $f(1) = \langle g, g\rangle$. □

Lemma 3 (Completion of squares for unitaries)**.** *For any unitary U on a Hilbert space and any vector h,*

$$2\Re\langle h, Uh\rangle = 2\|h\|^2 - \|h - Uh\|^2.$$

Proof. Expand $\|h - Uh\|^2 = \|h\|^2 + \|Uh\|^2 - 2\Re\langle h, Uh\rangle$ and use $\|Uh\| = \|h\|$. □

Remark 4 (Support truncation)**.** *Fix $\lambda > 1$ and assume* $\operatorname{supp}(g) \subset [\lambda^{-1}, \lambda]$. *Then $f(a) = 0$ whenever $a \notin [\lambda^{-2}, \lambda^2]$. Indeed,* $\operatorname{supp}(U_a g) = a \cdot \operatorname{supp}(g) \subset [a\lambda^{-1}, a\lambda]$. *If $a > \lambda^2$ then $a\lambda^{-1} > \lambda$, hence* $\operatorname{supp}(g) \cap \operatorname{supp}(U_a g) = \varnothing$ *and $\langle g, U_a g\rangle = 0$. If $0 < a < \lambda^{-2}$ then $a\lambda < \lambda^{-1}$ and the supports are again disjoint.*

2. Logarithmic coordinates

Set $u = \log x$, so that $d^*x = du$. Write $L := \log \lambda$ and $I := (-L, L)$. Define $G(u) := g(e^u) \in L^2(I)$ and let $\widetilde{G}$ be its extension by 0 to $\mathbb{R}$. Let S_t be translation on $L^2(\mathbb{R})$: $(S_t \phi)(u) := \phi(u - t)$.

Lemma 5 (Dilation becomes translation)**.** *For all $t \in \mathbb{R}$,*

$$\|g - U_{e^t} g\|_{L^2(\mathbb{R}_+^*)} = \|\widetilde{G} - S_t \widetilde{G}\|_{L^2(\mathbb{R})}.$$

Proof. By $x = e^u$ and $d^*x = du$,

$$\|g - U_{e^t} g\|_2^2 = \int_{\mathbb{R}} |G(u) - G(u - t)|^2\, du = \int_{\mathbb{R}} |\widetilde{G}(u) - (S_t \widetilde{G})(u)|^2\, du.$$

□

3. Local distributions

Assume $\mathrm{supp}(g) \subset [\lambda^{-1}, \lambda]$ and let f be as in Definition 1.

Prime terms. For a prime p, define

$$W_p(f) := (\log p) \sum_{m \geq 1} p^{-m/2} \big(f(p^m) + f(p^{-m}) \big). \tag{1}$$

By Remark 4, the summand vanishes unless $p^m \leq \lambda^2$, so (1) is a finite sum for each fixed (p, λ).

Archimedean term. Let γ be Euler's constant and define the (a priori improper) integral

$$W_{\mathbb{R}}(f) := (\log 4\pi + \gamma)\, f(1) + \int_1^\infty \Big(f(x) + f(x^{-1}) - 2x^{-1/2} f(1) \Big) \frac{x^{1/2}}{x - x^{-1}}\, d^*x. \tag{2}$$

The integrand has a non-integrable weight at $x = 1$ unless there is sufficient cancellation; we will guarantee convergence by an explicit "energy" hypothesis below.

Define the strictly positive weight (for $t > 0$)

$$w(t) := \frac{e^{t/2}}{e^t - e^{-t}} = \frac{e^{t/2}}{2 \sinh t}. \tag{3}$$

Note the asymptotics $w(t) \sim (2t)^{-1}$ as $t \downarrow 0$ and $w(t) \sim e^{-t/2}$ as $t \to \infty$.

4. Archimedean energy and its natural form domain

Definition 6 (Archimedean difference-energy at scale λ)**.** *For $G \in L^2(I)$ (with zero extension $\widetilde{G}$) define*

$$\mathcal{E}_{\infty,\lambda}(G) := \int_0^{2L} w(t)\, \|\widetilde{G} - S_t \widetilde{G}\|^2_{L^2(\mathbb{R})}\, dt \ \in [0, \infty].$$

Let $\mathcal{D}_{\infty,\lambda}$ be the (maximal) form domain

$$\mathcal{D}_{\infty,\lambda} := \{\, G \in L^2(I) : \ \mathcal{E}_{\infty,\lambda}(G) < \infty \,\}.$$

Equivalently, $g \in L^2(\mathbb{R}_+^*)$ with $\mathrm{supp}(g) \subset [\lambda^{-1}, \lambda]$ belongs to the Archimedean form domain if and only if $\mathcal{E}_{\infty,\lambda}(G) < \infty$ for $G(u) = g(e^u)$.

Remark 7 (A closed form built from a smooth core)**.** *If one prefers a canonical Hilbert space structure, set $\mathcal{C}_\lambda := C_c^\infty(I)$ and equip it with $\|G\|^2_{\mathcal{D}} := \|G\|^2_{L^2(I)} + \mathcal{E}_{\infty,\lambda}(G)$. Let $\overline{\mathcal{D}}_{\infty,\lambda}$ be the completion of $\mathcal{C}_\lambda$ for this norm; then $\mathcal{E}_{\infty,\lambda}$ extends by continuity to $\overline{\mathcal{D}}_{\infty,\lambda}$. Everything below holds verbatim on $\mathcal{D}_{\infty,\lambda}$ (maximal domain) and in particular on $\overline{\mathcal{D}}_{\infty,\lambda}$.*

Remark 8 (Concrete sufficient conditions)**.** *If G is, say, C^1 on I (or $G \in H^1(I)$), then $\mathcal{E}_{\infty,\lambda}(G) < \infty$. Indeed, for small t one has the standard estimate $\|\widetilde{G} - S_t \widetilde{G}\|_2 \ll |t|^{1/2}(\|G'\|_2 + \|G\|_2)$, and for large t one has $\|\widetilde{G} - S_t \widetilde{G}\|_2 \leq 2\|G\|_2$ while $w(t)$ is integrable on $(1, 2L)$. The point of Definition 6 is that no such auxiliary regularity is needed: the finiteness of the energy is the only hypothesis.*

5. Energy decomposition: primes and infinity

Lemma 9 (Prime term = discrete difference-energy + constant)**.** *With the above notation,*

$$-W_p(f) = \sum_{\substack{m\ge 1\\ p^m\le\lambda^2}} (\log p)p^{-m/2}\,\|\widetilde{G} - S_{m\log p}\widetilde{G}\|^2_{L^2(\mathbb{R})} \;+\; c_p(\lambda)\,\|G\|^2_{L^2(I)},$$

where

$$c_p(\lambda) := -2(\log p)\sum_{\substack{m\ge 1\\ p^m\le\lambda^2}} p^{-m/2} \in \mathbb{R}.$$

Proof. By Lemma 2,

$$f(p^m) + f(p^{-m}) = 2\Re\langle g, U_{p^m} g\rangle.$$

Hence, by (1),

$$W_p(f) = (\log p)\sum_{m\ge 1} p^{-m/2}\, 2\Re\langle g, U_{p^m} g\rangle.$$

Apply Lemma 3 with $U = U_{p^m}$:

$$2\Re\langle g, U_{p^m} g\rangle = 2\|g\|_2^2 - \|g - U_{p^m} g\|_2^2.$$

If $p^m > \lambda^2$ then $f(p^m) = 0$ by Remark 4, hence those terms contribute 0. Therefore

$$W_p(f) = \sum_{\substack{m\ge 1\\ p^m\le\lambda^2}} (\log p)p^{-m/2}\Big(2\|g\|_2^2 - \|g - U_{p^m} g\|_2^2\Big),$$

which rearranges to the stated formula. Finally $\|g\|_2^2 = \|G\|^2_{L^2(I)}$ and $\|g - U_{p^m} g\|_2 = \|\widetilde{G} - S_{m\log p}\widetilde{G}\|_2$ by Lemma 5. □

Lemma 10 (Archimedean term = continuum of difference energies + constant)**.** *Assume $G \in \mathcal{D}_{\infty,\lambda}$ (Definition 6). Then the improper integral in* (2) *converges, and one has*

$$-W_{\mathbb{R}}(f) = \int_0^{2L} w(t)\,\|\widetilde{G} - S_t\widetilde{G}\|^2_{L^2(\mathbb{R})}\,dt \;+\; c_\infty(\lambda)\,\|G\|^2_{L^2(I)},$$

where

$$c_\infty(\lambda) := -(\log 4\pi + \gamma) + \int_0^{2L} 2\big(e^{-t/2} - 1\big)w(t)\,dt + \int_{2L}^{\infty} 2e^{-t/2} w(t)\,dt \in \mathbb{R}.$$

Both integrals defining $c_\infty(\lambda)$ converge absolutely.

Proof. Start from (2) and substitute $x = e^t$ (so $d^*x = dt$). Using (3),

$$W_{\mathbb{R}}(f) = (\log 4\pi + \gamma)f(1) + \int_0^\infty \Big(f(e^t) + f(e^{-t}) - 2e^{-t/2}f(1)\Big)w(t)\,dt.$$

By Lemma 2, $f(1) = \|g\|_2^2$ and $f(e^t) + f(e^{-t}) = 2\Re\langle g, U_{e^t} g\rangle$. Thus

$$W_{\mathbb{R}}(f) = (\log 4\pi + \gamma)\|g\|_2^2 + \int_0^\infty \Big(2\Re\langle g, U_{e^t} g\rangle - 2e^{-t/2}\|g\|_2^2\Big)w(t)\,dt.$$

Apply Lemma 3 with $U = U_{e^t}$ to rewrite

$$2\Re\langle g, U_{e^t} g\rangle = 2\|g\|_2^2 - \|g - U_{e^t} g\|_2^2.$$

Hence the integrand becomes

$$\Big(2(1 - e^{-t/2})\|g\|_2^2 - \|g - U_{e^t} g\|_2^2\Big) w(t).$$

We now split at $t = 2L$. If $t > 2L$, then $e^t > \lambda^2$ and also $e^{-t} < \lambda^{-2}$, so by Remark 4

$$f(e^t) = f(e^{-t}) = 0.$$

Equivalently, g and $U_{e^t} g$ have disjoint supports, hence

$$\|g - U_{e^t} g\|_2^2 = \|g\|_2^2 + \|U_{e^t} g\|_2^2 = 2\|g\|_2^2.$$

Therefore for $t > 2L$ the contribution of the integrand is

$$\Big(2(1 - e^{-t/2})\|g\|_2^2 - 2\|g\|_2^2\Big) w(t) = -2e^{-t/2} w(t)\, \|g\|_2^2,$$

which yields the tail constant $\int_{2L}^{\infty} 2e^{-t/2} w(t)\, dt$ after moving to $-W_{\mathbb{R}}(f)$.

On $t \in [0, 2L]$ we keep the difference-energy term and absorb the remaining $2(e^{-t/2} - 1) w(t)\|g\|_2^2$ into the constant.

Finally, by Lemma 5, $\|g - U_{e^t} g\|_2 = \|\widetilde{G} - S_t \widetilde{G}\|_2$ and $\|g\|_2^2 = \|G\|_{L^2(I)}^2$.

Convergence: the tail constant converges since $e^{-t/2} w(t) \sim e^{-t}$ as $t \to \infty$. On $[0, 2L]$, the function $t \mapsto 2(e^{-t/2} - 1) w(t)$ is integrable at 0 because $e^{-t/2} - 1 \sim -t/2$ and $w(t) \sim (2t)^{-1}$. The remaining term $\int_0^{2L} w(t)\|g - U_{e^t} g\|_2^2\, dt$ is finite by the assumption $G \in \mathcal{D}_{\infty,\lambda}$. Thus the improper integral in (2) converges and the stated identity holds. □

Theorem 11 (Energy decomposition of the restricted Weil form)**.** *Let $\lambda > 1$ and let $g \in L^2(\mathbb{R}_+^*)$ satisfy* $\operatorname{supp}(g) \subset [\lambda^{-1}, \lambda]$*. Let $f(a) = \langle g, U_a g\rangle$ and define $G(u) = g(e^u)$ on $I = (-\log\lambda, \log\lambda)$, with extension $\widetilde{G}$ by 0 to $\mathbb{R}$. Assume $G \in \mathcal{D}_{\infty,\lambda}$ (equivalently $\mathcal{E}_{\infty,\lambda}(G) < \infty$). Then $W_{\mathbb{R}}(f)$ is well-defined by* (2) *and*

$$-\Big(W_{\mathbb{R}}(f) + \sum_{p \text{ prime}} W_p(f)\Big) = \mathcal{E}_\lambda(G) + C(\lambda)\, \|G\|_{L^2(I)}^2,$$

where the difference-energy form *is*

$$\mathcal{E}_\lambda(G) := \int_0^{2L} w(t)\, \|\widetilde{G} - S_t\widetilde{G}\|_{L^2(\mathbb{R})}^2\, dt + \sum_{\substack{p \text{ prime} \\ p \le \lambda^2}} \sum_{\substack{m \ge 1 \\ p^m \le \lambda^2}} (\log p) p^{-m/2}\, \|\widetilde{G} - S_{m\log p}\widetilde{G}\|_{L^2(\mathbb{R})}^2,$$

and the constant is

$$C(\lambda) := c_\infty(\lambda) + \sum_{\substack{p \text{ prime} \\ p \le \lambda^2}} c_p(\lambda) \in \mathbb{R}.$$

All sums are finite for each fixed λ.

Proof. Lemma 10 gives $-W_{\mathbb{R}}(f)$ as an Archimedean difference-energy plus a constant multiple of $\|G\|_2^2$. Summing Lemma 9 over primes $p \le \lambda^2$ (finitely many) gives the prime contribution. Adding the identities yields the claim. □