Potential Consequences of the Energy-Decomposition Method Beyond the Riemann Zeta Function:

An Assessment of Three Directions

February 2026

Abstract

The energy-decomposition method—which proves ground state simplicity for the restricted Weil quadratic form operator by establishing a Dirichlet form structure, irreducibility, and compact resolvent—raises a natural question: where else does this strategy apply, and what would the consequences be? We analyze three concentric extensions: (I) Dirichlet L-functions with real characters, (II) non-local operators with arithmetic kernels, and (III) operators on adelic and locally compact groups. For each, we describe the expected form of the extension, identify the principal mathematical consequences, and assess the technical obstacles. We argue that the three directions are not independent but form a coherent program whose full realization would establish a Perron–Frobenius theory for operators on adelic groups, connecting ground state simplicity to the Euler product structure of automorphic L-functions and to multiplicity-one theorems in the Langlands program.

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1 Introduction

1.1 Context

The energy-decomposition method, as developed for the restricted Weil quadratic form operator A_λ, establishes three properties: compact resolvent, simplicity of the lowest eigenvalue, and evenness of the ground state eigenfunction. The proof follows a five-step pipeline:

$$\boxed{\begin{array}{c}\text{Energy Decomposition} \longrightarrow \text{Markov Property} \longrightarrow \text{Irreducibility}\\ \longrightarrow \text{Compact Resolvent} \longrightarrow \text{Perron–Frobenius / Krein–Rutman}\end{array}}$$

These results resolve what Connes and collaborators identify as "the key difficulty" [5] and one of "the missing steps" [4] in their program toward the Riemann Hypothesis. The conditional theorems of Connes–van Suijlekom [5] and Connes–Consani–Moscovici [4] then imply that the approximate zeros of the associated entire functions lie on the critical line.

The method's key structural features—the decomposition of an arithmetic quadratic form into non-negative translation-difference energies, the role of the archimedean continuum in forcing irreducibility, and the use of Fourier-analytic coercivity for compactness—are not specific to the Riemann zeta function. This paper explores three natural directions in which the method might extend, and assesses the mathematical consequences of each.

1.2 The three directions

We consider three extensions of increasing generality and ambition:

Direction I. Extension to Dirichlet L-functions $L(s, \chi)$ with real characters χ.

Direction II. Application to non-local operators whose kernels are defined by arithmetic data (partial Euler products, Epstein zeta functions, Rankin–Selberg convolutions).

Direction III. Formulation on locally compact abelian groups and adelic groups, connecting to the Langlands program and non-commutative geometry.

These form concentric circles: Direction I validates the method across a family; Direction II transforms it into a criterion for a class of operators; Direction III embeds the framework in its natural algebraic home.

2 Direction I: Dirichlet L-functions with real characters

2.1 Setup

Let χ be a real primitive Dirichlet character modulo q (i.e., a Kronecker symbol $\chi = \left(\frac{d}{\cdot}\right)$ for a fundamental discriminant d). The Weil explicit formula for $L(s, \chi)$ takes the form

$$-\sum_v W_v^\chi(g * \tilde{g}) = \sum_\gamma h(\gamma) + (\text{polar terms}),$$

where the sum on the left runs over all places v (primes and the archimedean place), W_v^χ are the local distributions twisted by χ, and the sum on the right runs over zeros $\rho = \frac{1}{2} + i\gamma$ of $L(s, \chi)$.

Restricting to test functions supported on $[\lambda^{-1}, \lambda]$ (or equivalently on $[-L, L]$ in logarithmic coordinates, with $L = \log \lambda$) defines a quadratic form $\mathcal{E}_\lambda^\chi$ and an associated self-adjoint operator A_λ^χ on $L^2([-L, L])$.

2.2 How the energy decomposition transfers

The local distributions decompose as:

- **Unramified primes with** $\chi(p) = +1$: The contribution from p involves the same difference-energy structure as the $\zeta(s)$ case:

$$\mathcal{E}_p^{\chi}(G, G) = \sum_{k=1}^{\infty} \frac{1}{kp^k} \|\tilde{G} - S_{k\ell_p}\tilde{G}\|_{L^2}^2, \quad \ell_p = \log p.$$

  These terms are manifestly non-negative.
- **Unramified primes with** $\chi(p) = -1$: The character twist replaces the shift $S_{k\ell_p}$ with a *sign-alternating* shift. For odd powers of p, the term becomes:

$$\frac{1}{kp^k} \|\tilde{G} + S_{k\ell_p}\tilde{G}\|_{L^2}^2,$$

  which is the difference energy for the anti-symmetric combination. This is still a non-negative quadratic form, since $\|f + g\|^2 = \|f - (-g)\|^2$ is a difference energy with reflected shift. The key point is that $\chi(p) = -1$ changes which combination appears but does not introduce negative signs in the energy.
- **Ramified primes** ($p \mid q$, $\chi(p) = 0$): The local contribution $\mathcal{E}_p^{\chi}$ vanishes. These primes provide no energy and no shifts, but they do not obstruct the argument either.
- **Archimedean place**: For even characters $\chi(-1) = +1$, the archimedean distribution $W_{\mathbb{R}}^{\chi}$ is identical to $W_{\mathbb{R}}$ for $\zeta(s)$. For odd characters $\chi(-1) = -1$, there is a modification involving the digamma function ψ evaluated at a shifted argument, but the resulting distribution still gives rise to a non-negative continuous superposition of translation-difference energies over $t \in (0, 2L)$ with a strictly positive weight.

Proposition 2.1 (Expected)**.** *For every real primitive character χ and every $\lambda > 1$, the quadratic form $\mathcal{E}_{\lambda}^{\chi}$ satisfies the Markov property, is irreducible (via the archimedean continuum), and has a coercive Fourier symbol with logarithmic growth. Consequently, the operator A_{λ}^{χ} has compact resolvent, simple lowest eigenvalue, and a strictly positive eigenfunction with the appropriate symmetry.*

2.3 Mathematical consequences

2.3.1 Uniform verification of the Connes–van Suijlekom hypotheses

The conditional theorem of Connes–van Suijlekom [5] is stated in sufficient generality to cover operators beyond $\zeta(s)$. Proposition 2.1 would verify the hypotheses—simplicity and appropriate symmetry—for the restricted operator associated with every real Dirichlet L-function simultaneously. This gives:

Corollary 2.2 (Expected)**.** *For every real primitive character χ, every $\lambda > 1$, and the ground state eigenfunction ξ_{λ}^{χ} of A_{λ}^{χ}: all zeros of the entire function $\hat{\xi}_{\lambda}^{\chi}(z)$ lie on the real line (equivalently, on the critical line* $\mathrm{Re}(s) = \frac{1}{2}$ *after the usual change of variables).*

This would be a uniform statement across an infinite family of L-functions, established by a single structural argument rather than case-by-case analysis.

2.3.2 Implications for Siegel zeros

Real Dirichlet L-functions are precisely the family where *Siegel zeros*—hypothetical zeros very close to $s = 1$—are most dangerous and hardest to exclude. The existence of a Siegel zero for $L(s, \chi)$ would have dramatic consequences for the distribution of primes in arithmetic progressions.

The energy-decomposition result at finite λ does not directly exclude Siegel zeros, since that would require the $\lambda \to \infty$ convergence. However, it establishes that the finite-approximation framework is well-posed for every real character: the approximate zeros lie on the critical line, and the spectral gap $\lambda_2 - \lambda_1$ is strictly positive. If quantitative lower bounds on the spectral gap could be established (uniformly in χ), these would translate into zero-free regions for $L(s, \chi)$ near $s = 1$, providing a new analytic approach to the Siegel zero problem.

2.3.3 Robustness of the Connes program

Perhaps most importantly, extending the result to real characters would demonstrate that the energy-decomposition method is not an artifact of the special structure of $\zeta(s)$ but is robust across the family of L-functions with Euler products. This matters for the broader Connes program, which is naturally formulated for all automorphic L-functions via the adèle class space and the semi-local trace formula. Showing that the analytic step (ground state simplicity) works uniformly across the real family would be a meaningful indication that the method has the right level of generality.

2.4 Assessment of difficulty

Low to moderate. The character twist does not fundamentally alter the energy decomposition. The main work is bookkeeping: tracking the sign of $\chi(p)$ through the local computations and verifying that the archimedean term for odd characters retains a non-negative decomposition. No new ideas appear to be required beyond those in the original argument.

3 Direction II: Non-local operators with arithmetic kernels

3.1 Setup

We consider integral operators on $L^2([-L, L])$ (or on bounded domains in $\mathbb{R}^d$) whose kernels or Fourier symbols are defined by arithmetic data. Three natural classes present themselves:

Definition 3.1 (Partial Euler product operators)**.** For a finite set of primes S and $s_0 \in \mathbb{C}$, define the Fourier symbol

$$\psi_S(\xi) = -\sum_{p \in S} \log |1 - p^{-s_0 - i\xi}|^2 = -\sum_{p \in S} \sum_{k=1}^{\infty} \frac{2\cos(k\xi \log p)}{kp^{k\mathrm{Re}(s_0)}}.$$

The associated operator A_S on $L^2([-L, L])$ has this function as its Fourier symbol (up to boundary corrections from the restriction to a finite interval).

Definition 3.2 (Epstein zeta operators)**.** For a positive definite quadratic form Q in n variables, the Epstein zeta function $Z_Q(s) = \sum'_{\mathbf{m} \in \mathbb{Z}^n} Q(\mathbf{m})^{-s}$ satisfies a functional equation and defines a quadratic form via the explicit formula, analogous to the Weil form. The restricted operator A_Q^λ acts on L^2 of a bounded domain.

Definition 3.3 (Rankin–Selberg operators)**.** For two cuspidal automorphic representations π, π' on $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$, the Rankin–Selberg L-function $L(s, \pi \times \tilde{\pi}')$ has an Euler product and an explicit formula. The associated restricted operator $A_{\pi,\pi'}^\lambda$ is defined by the corresponding quadratic form.

3.2 The energy-decomposition criterion

The abstract framework (Section 5 of the companion paper [8]) gives a clean criterion: an operator with Fourier symbol ψ on a bounded domain has a simple ground state if:

(H1) The symbol admits a representation $\psi(\xi) = \int |e^{2\pi i h\xi} - 1|^2 \, d\mu(h)$ for a non-negative measure μ.

(H2) The support of μ is dense (irreducibility).

(H3) $\psi(\xi) \to \infty$ as $|\xi| \to \infty$ (coercivity).

The question for each class of arithmetic operators is: which of these hypotheses hold?

3.3 Analysis by class

3.3.1 Partial Euler product operators

For partial Euler products at $s_0 = \frac{1}{2}$ (the case relevant to the Riemann Hypothesis), the symbol $\psi_S(\xi)$ is a *finite* sum of cosines. Writing out:

$$\psi_S(\xi) = \sum_{p \in S} \sum_{k=1}^{\infty} \frac{2(1 - \cos(k\xi \log p))}{kp^{k/2}} = \int |e^{2\pi i h\xi} - 1|^2 \, d\mu_S(h),$$

where μ_S is a discrete measure supported on $\{k \log p : p \in S,\ k \geq 1\}$. The measure is non-negative (H1 holds), but its support is discrete and does not generate a dense subgroup of $\mathbb{R}$ unless S contains at least two primes with $\log p_1 / \log p_2$ irrational (which is the case for any two distinct primes, since $\log 2 / \log 3$ is irrational). However, with only finitely many primes, the support is contained in a lattice-like set, and irreducibility requires careful analysis.

Moreover, $\psi_S(\xi)$ is *bounded* (it is a finite sum of bounded terms), so $\psi_S(\xi) \not\to \infty$. **Coercivity fails.** This means that for finite Euler products without the archimedean contribution, the operator does not have compact resolvent, and the energy-decomposition pipeline breaks at Step 4.

This is consistent with the expectation that the archimedean place plays an essential role. The Weil form includes both the prime contributions *and* the archimedean term $W_{\mathbb{R}}$, and it is the latter that provides logarithmic coercivity. Partial Euler products alone are insufficient.

3.3.2 Epstein zeta operators

Epstein zeta functions $Z_Q(s)$ do *not* in general satisfy the Riemann Hypothesis—this has been known since Davenport and Heilbronn [6]. The energy-decomposition method must therefore fail somewhere for Epstein operators. Identifying *where* it fails is diagnostically valuable.

The explicit formula for Z_Q gives local contributions that, unlike the Weil form, do not decompose into an Euler product. The quadratic form associated to Z_Q has a "global" structure without a natural decomposition into independent local pieces. This means:

- Hypothesis (H1) (non-negative energy decomposition) is *not guaranteed*. Without an Euler product, there is no reason for the quadratic form to decompose as a sum of non-negative translation-difference energies. The Markov property may fail.
- Even if (H1) holds, the "shifts" arising from the lattice $\mathbb{Z}^n$ of summation in Z_Q are discrete, and irreducibility (H2) depends on the geometry of the lattice relative to the domain of restriction.

Question 3.4. For which positive definite forms Q does the associated quadratic form admit a non-negative energy decomposition? Is the failure of such a decomposition equivalent to (or correlated with) the existence of zeros off the critical line?

An affirmative answer to the second part of Question 3.4 would give a new structural explanation for *why* certain zeta functions satisfy the Riemann Hypothesis and others do not: the dividing line would be the Markov property of the associated Dirichlet form, which in turn reflects the presence or absence of an Euler product.

3.3.3 Rankin–Selberg operators

Rankin–Selberg L-functions $L(s, \pi \times \tilde{\pi}')$ have Euler products:

$$L(s, \pi \times \tilde{\pi}') = \prod_p L_p(s, \pi_p \times \tilde{\pi}'_p),$$

where each local factor L_p is an inverse polynomial in p^{-s}. The explicit formula therefore decomposes into local contributions $W_p^{\pi,\pi'}$, and the energy decomposition should proceed place-by-place, just as for $\zeta(s)$ and Dirichlet L-functions.

The key subtlety is at *ramified primes*: when π or π' is ramified at p, the local factor L_p has a different structure (fewer terms, modified coefficients), and the corresponding local energy $\mathcal{E}_p^{\pi,\pi'}$ may have a more complex form. The question is whether these local energies remain non-negative quadratic forms.

For GL_2 (classical modular forms), the local factors at ramified primes are either 1 (supercuspidal) or $(1 - \alpha_p p^{-s})^{-1}$ (Steinberg), both of which give non-negative local energies (the former trivially, the latter by the same argument as for unramified primes with one fewer term). For GL_n with $n \geq 3$, the ramified local factors can be more complex, and a case-by-case analysis is needed.

3.4 Mathematical consequences

3.4.1 Functoriality and the Langlands program

The Langlands functoriality conjectures predict that for a morphism of L-groups ${}^L G \to {}^L G'$, there is a "transfer" of automorphic representations from G to G', compatible with L-functions. These transfers are often established by analyzing the analytic properties of Rankin–Selberg L-functions.

If the energy-decomposition method verifies ground state simplicity for the restricted operators associated with Rankin–Selberg L-functions $L(s, \pi \times \tilde{\pi}')$, this would establish that the approximate zeros of these L-functions lie on the critical line. While this is a finite-λ statement (and hence does not prove GRH for Rankin–Selberg L-functions), it would provide structural constraints on the analytic behavior of these L-functions that could serve as inputs to functoriality arguments.

More concretely, the spectral gap $\lambda_2 - \lambda_1$ of the restricted operator encodes information about the spacing of zeros. Quantitative lower bounds on this spectral gap would give *zero-density estimates* for Rankin–Selberg L-functions, which are among the most useful analytic inputs to the Langlands program.

3.4.2 Subconvexity bounds

Subconvexity bounds—estimates of the form $|L(\frac{1}{2}, \pi)| \ll C(\pi)^{1/4-\delta}$ for some $\delta > 0$, where $C(\pi)$ is the analytic conductor—are central tools in modern analytic number theory. They have applications to equidistribution of lattice points, quantum unique ergodicity, and mass equidistribution on arithmetic surfaces.

The spectral gap of the restricted Weil-type operator is related to the rate at which approximate zeros separate from the critical line. If the energy-decomposition method could provide *quantitative* lower bounds on $\lambda_2 - \lambda_1$ (uniformly in the representation π), these would translate into subconvexity-type estimates for the associated L-functions.

This connection is speculative but structurally motivated: the spectral gap controls the decay rate of correlations in the associated Markov semigroup, and this decay rate is directly related to the analytic behavior of $L(s, \pi)$ near the critical line.

3.4.3 Diagnostic value for Epstein zeta functions

As noted in Section 3.3.2, the energy-decomposition method is expected to *fail* for Epstein zeta functions (which violate the Riemann Hypothesis). Identifying the precise failure mode—does the Markov property break down? does irreducibility fail? does coercivity fail?—would give a new structural characterization of the Euler product as the arithmetic feature that "protects" the Riemann Hypothesis.

Conjecture 3.5 (Informal)**.** *An L-function $L(s)$ with functional equation and analytic continuation satisfies a Riemann Hypothesis if and only if the associated restricted quadratic form admits a non-negative energy decomposition (Markov property) with an irreducible shift structure. The Euler product is the arithmetic source of both properties.*

This conjecture is deliberately informal; making it precise would require specifying the class of L-functions under consideration and the sense in which the energy decomposition is "associated" to the L-function. But even as a heuristic, it suggests a fundamentally new perspective: the Riemann Hypothesis as a *positivity and irreducibility* statement about Dirichlet forms, rather than a statement about the location of zeros.

3.5 Assessment of difficulty

Moderate to high. The main challenge is that for general arithmetic kernels, the energy decomposition into non-negative pieces may not exist. The Euler product structure of L-functions is what makes the Weil form decompose nicely; without it, the local terms may lack definite sign. Each class of operators requires individual analysis. The Rankin–Selberg case for GL_2 is likely tractable; the general GL_n case and the Epstein diagnostic problem are substantially harder.

4 Direction III: Operators on groups

4.1 Setup

The translation-difference structure at the heart of the energy-decomposition method generalizes naturally to locally compact abelian groups. Let G be such a group with Haar measure μ, Pontryagin dual $\hat{G}$, and let $K \subset G$ be a compact subset with $\mu(K) > 0$.

Definition 4.1 (Energy form on a group)**.** Given a non-negative Lévy-type measure ν on G (i.e., a Radon measure with $\nu(\{0\}) = 0$ and $\int_G \min(1, d(g,0)^2)\, d\nu(g) < \infty$ for some metric d), define

$$\mathcal{E}_\nu(u,u) = \int_G \|\tau_g \tilde{u} - \tilde{u}\|^2_{L^2(G)}\, d\nu(g),$$

where $\tilde{u}$ is the extension of $u \in L^2(K)$ by zero and τ_g is translation by g.

The associated Fourier symbol on $\hat{G}$ is the continuous negative definite function

$$\psi(\hat{g}) = \int_G |1 - \hat{g}(g)|^2 \, d\nu(g), \quad \hat{g} \in \hat{G}.$$

The abstract criterion (hypotheses H1–H3 of Section 3.2) becomes:

(H1′) ν is a non-negative measure (automatic from the definition).

(H2′) The support of ν generates a dense subgroup of G (irreducibility).

(H3′) $\psi(\hat{g}) \to \infty$ as $\hat{g} \to \infty$ in $\hat{G}$ (coercivity).

4.2 Natural settings

4.2.1 The idèle class group

The most natural home for the Weil distribution is the idèle class group $C_{\mathbb{Q}} = \mathbb{A}_{\mathbb{Q}}^*/\mathbb{Q}^*$, where $\mathbb{A}_{\mathbb{Q}}$ is the adèle ring of $\mathbb{Q}$. Via the logarithmic map on the archimedean component, $C_{\mathbb{Q}} \cong \mathbb{R}_{>0} \times \prod_p \mathbb{Z}_p^*$ (as a topological group modulo the discrete action of $\mathbb{Q}^*$). The Weil distribution is naturally a distribution on $C_{\mathbb{Q}}$, and the local decomposition

$$W = \sum_p W_p + W_{\mathbb{R}}$$

reflects the factorization of $C_{\mathbb{Q}}$ into local components.

Working directly on $C_{\mathbb{Q}}$ rather than projecting to $\mathbb{R}$ would give a more canonical formulation of the energy decomposition. The "shifts" by $\log p$ and by the archimedean continuum become translations within $C_{\mathbb{Q}}$, and the irreducibility question becomes: does the support of the Weil distribution generate a dense subgroup of $C_{\mathbb{Q}}$?

The answer is yes: the elements $\{p\}_{p \text{ prime}}$, viewed in $C_{\mathbb{Q}}$, generate a dense subgroup (this is essentially the content of the Artin reciprocity law for $\mathbb{Q}$). Combined with the archimedean component, the full support of the Weil distribution generates all of $C_{\mathbb{Q}}$.

4.2.2 Adelic groups

For automorphic L-functions on GL_n, the natural setting is the quotient $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})/\mathrm{GL}_n(\mathbb{Q})$. Automorphic forms are functions on this quotient, and the trace formula—the principal tool for studying their spectral properties—is fundamentally a statement about operators on this space.

The energy-decomposition framework would need to be reformulated for *non-abelian* groups, since GL_n for $n \geq 2$ is not abelian. The translations τ_g would be replaced by the regular representation, and the "Fourier symbol" would be replaced by the spectral decomposition into automorphic representations. The Markov property, irreducibility, and coercivity would need non-commutative analogues.

4.2.3 p-adic groups

The local distributions W_p are naturally defined on $\mathbb{Q}_p^*$ (the multiplicative group of p-adic numbers), and the local energy $\mathcal{E}_p$ is a Dirichlet form on $L^2(\mathbb{Q}_p^*)$. Studying these local forms individually, before taking the product over places, could provide insight into the local structure of the energy decomposition.

The p-adic groups $\mathbb{Q}_p^*$ are totally disconnected, which makes the analysis of Dirichlet forms rather different from the archimedean case. The "shifts" are multiplication by powers of p, which act on the ultrametric space $\mathbb{Q}_p^*$ in a way that has no Euclidean analogue. Nevertheless, the theory of Dirichlet forms on ultrametric spaces has been developed (see [1] and references therein), and could provide the necessary tools.

4.3 Mathematical consequences

4.3.1 Adelic Perron–Frobenius theory

If the semigroup generated by the Weil form on the idèle class group (or more generally on $\mathrm{GL}_n(\mathbb{A})/\mathrm{GL}_n(\mathbb{Q})$) is positivity-improving in an appropriate sense, this would give a *representation-theoretic* proof of ground state simplicity. The ground state would be the spherical (unramified) vector in the trivial automorphic representation, and simplicity would reflect the *multiplicity-one theorem* for automorphic representations of GL_n [11, 10].

This would be a deep connection: the multiplicity-one theorem is proved by purely algebraic/representation-theoretic methods (local uniqueness of Whittaker models), while ground state simplicity is proved by analytic methods (Dirichlet forms and Perron–Frobenius). If these two results could be identified as different manifestations of the same underlying structure, it would unify two of the most important tools in the theory of automorphic forms.

Conjecture 4.2 (Adelic Perron–Frobenius)**.** *Let π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_\mathbb{Q})$, and let $\mathcal{E}_\lambda^\pi$ be the quadratic form obtained from the explicit formula for $L(s, \pi)$ restricted to a cutoff λ. Then $\mathcal{E}_\lambda^\pi$ is an irreducible Dirichlet form, and the corresponding operator has a simple ground state if and only if π satisfies multiplicity one.*

4.3.2 Trace formula interpretation

The Arthur–Selberg trace formula equates spectral data (automorphic representations) with geometric data (orbital integrals):

$$\sum_\pi m(\pi)\,\mathrm{tr}\,\pi(f) = \sum_{\{\gamma\}} \mathrm{vol}(G_\gamma(\mathbb{Q})\backslash G_\gamma(\mathbb{A})) \cdot \mathcal{O}_\gamma(f),$$

where the left side sums over automorphic representations with multiplicity $m(\pi)$ and the right side sums over conjugacy classes $\{\gamma\}$ with orbital integrals $\mathcal{O}_\gamma(f)$.

The energy decomposition of the Weil form, lifted to the adelic setting, would give a new interpretation of the *spectral side* of the trace formula:

- The ground state simplicity at finite cutoff corresponds to the **isolation of the trivial representation** in the spectral decomposition.
- The spectral gap $\lambda_2 - \lambda_1$ encodes information about the **first non-trivial automorphic representation** (the one closest to the trivial representation in the spectral ordering).
- The rate of convergence of the spectral gap as $\lambda \to \infty$ would be controlled by the **Ramanujan conjecture** (which bounds the local components of cuspidal representations).

This perspective suggests that the energy-decomposition method could provide a new approach to the Ramanujan conjecture, at least in the form of bounds on the spectral gap. The Ramanujan conjecture for GL_2 over $\mathbb{Q}$ is known (by the work of Deligne on the Weil conjectures for varieties over finite fields [7]), but for GL_n with $n \geq 3$ it remains open in general.

4.3.3 Non-commutative geometry

Connes' broader program places the Riemann Hypothesis within the framework of non-commutative geometry, where the adèle class space $\mathbb{A}_\mathbb{Q}/\mathbb{Q}^*$ is treated as a "non-commutative space" and the zeros of $\zeta(s)$ are realized as the spectrum of a suitable operator.

The Dirichlet form framework has non-commutative analogues: the Cipriani–Sauvageot theory of *non-commutative Dirichlet forms* [2] extends the classical theory to the setting of von Neumann algebras and C^*-algebras. Extending the energy-decomposition method to this setting would connect to Connes' spectral realization of zeros on the non-commutative adèle class space.

Specifically, the non-commutative Dirichlet form associated to the "Weil operator" on the adèle class space would encode both the spectral properties of $\zeta(s)$ and the positivity/irreducibility structure that implies ground state simplicity. The Markov property in the non-commutative setting is related to *complete positivity* of the associated semigroup, and irreducibility is related to *ergodicity* of the corresponding non-commutative dynamical system.

4.4 Assessment of difficulty

High to very high. The principal obstacles are:

1. **Compactness on groups.** The Kolmogorov–Riesz compactness criterion is specific to $\mathbb{R}^d$. Extending it to locally compact abelian groups (especially those with totally disconnected components, such as $\prod_p \mathbb{Z}_p^*$) requires developing new compactness criteria adapted to the group topology.
2. **Coercivity on non-Euclidean duals.** The "growth at infinity" of the Fourier symbol $\psi(\hat{g})$ must be formulated in terms of the topology of $\hat{G}$, which for adelic groups is a restricted direct product with a complicated structure at infinity.
3. **Non-commutativity.** For GL_n with $n \geq 2$, the group is non-abelian, and the Pontryagin dual is replaced by the unitary dual (the set of equivalence classes of irreducible unitary representations). The "Fourier symbol" becomes an operator-valued function on this dual, and the Dirichlet form framework must be replaced by its non-commutative analogue.
4. **Convergence as $\lambda \to \infty$.** Even in the abelian case, the convergence of ground states and eigenvalues as the cutoff $\lambda \to \infty$ is a hard problem that requires controlling the growth of the operator in the limit. This is the "second missing step" in the Connes program, independent of (and almost certainly harder than) the ground state simplicity question.

5 The three directions together

5.1 Concentric structure

The three directions form concentric circles of a single program:

Direction	Role	Key question
I. Real χ	Proof of concept: validates the method across a family	Does the character twist preserve non-negativity?
II. Arithmetic kernels	Criterion: characterizes which operators have simple ground states	Which arithmetic structures yield Markov property + irreducibility?
III. Groups	Foundation: embeds the framework in its natural algebraic home	Does Perron–Frobenius on adelic groups connect to multiplicity one?

5.2 The unified vision

If all three extensions were carried out, the combined result would amount to the following:

> *There exists a Perron–Frobenius theory for operators on adelic groups, in which ground state simplicity is equivalent to irreducibility of the associated Dirichlet form, and this irreducibility is guaranteed by the Euler product structure of automorphic L-functions.*

This would have four major consequences:

5.2.1 RH-type results as consequences of representation theory

The chain

$$\text{Euler product} \implies \text{non-negative energy decomposition} \implies \text{Markov property}$$
$$\implies \text{irreducibility} \implies \text{positivity-improving} \implies \text{simple ground state}$$
$$\implies \text{approximate zeros on critical line}$$

would establish that the location of approximate zeros is a *consequence of the multiplicative structure of L-functions*—that is, of the Euler product, which is the arithmetic incarnation of the local-global principle. Combined with the multiplicity-one connection (Conjecture 4.2), this would give a representation-theoretic explanation for why RH-type statements are true: they are shadow of the uniqueness of automorphic forms.

5.2.2 A diagnostic for the Riemann Hypothesis

For objects without Euler products (Epstein zeta functions, linear combinations of L-functions, Beurling zeta functions), the energy decomposition would fail, and the *specific failure mode*—which hypothesis (H1), (H2), or (H3) breaks down—would explain which zeros leave the critical line and why:

Failure mode	**Consequence**	**Example**
(H1) Markov fails: energy decomposition has indefinite terms	Ground state may not be positive; semigroup not positivity-preserving	Epstein zeta with no Euler product
(H2) Irreducibility fails: shifts too sparse	Ground state may not be unique; degenerate eigenspaces possible	Partial Euler product with too few primes
(H3) Coercivity fails: symbol bounded	Resolvent not compact; continuous spectrum possible	Finite Euler product without archimedean term

5.2.3 A new analytic foundation for the Connes program

Currently, the Connes program rests on two principal tools: the semi-local trace formula and the spectral realization of zeros. The energy-decomposition/Dirichlet-form approach would add a third component—*positivity and irreducibility in the sense of Markov processes*—that provides the "missing step" not just for $\zeta(s)$ but systematically across all L-functions with Euler products.

This three-legged structure (trace formula + spectral realization + Dirichlet form theory) would be more robust than any two components alone, because the Dirichlet form approach provides the *analytic* mechanism (positivity-improving semigroup) that converts the *algebraic* input (Euler product) into the *spectral* conclusion (ground state simplicity).

5.2.4 A bridge between analytic number theory and PDE

The fractional Laplacian paper [8] demonstrates that the same abstract machinery governs both the Weil quadratic form and the fractional Laplacian. A fully developed theory would mean that techniques from non-local PDE—regularity theory, heat kernel estimates, spectral gap bounds, Harnack inequalities—could be imported into number theory, and vice versa.

This kind of cross-pollination has historical precedent: the Selberg trace formula was inspired by the spectral theory of the Laplacian on hyperbolic surfaces; the Langlands program draws on the representation theory of Lie groups; and the proof of the Weil conjectures by Deligne used étale cohomology, a tool imported from algebraic topology. The energy-decomposition method suggests a new channel of communication: between the theory of non-local operators (fractional Laplacians, Lévy processes, Dirichlet forms on metric measure spaces) and the analytic theory of L-functions.

Specific techniques that could transfer include:

- **Spectral gap estimates** from the theory of Dirichlet forms on fractals and metric measure spaces, adapted to give quantitative bounds on $\lambda_2 - \lambda_1$ for arithmetic operators.
- **Heat kernel bounds** (Li–Yau type, or sub-Gaussian estimates) for the semigroup generated by the Weil operator, translating into zero-density estimates for L-functions.
- **Functional inequalities** (Poincaré, log-Sobolev, Nash) for arithmetic Dirichlet forms, giving new analytic inputs to the study of L-functions near the critical line.

5.3 Principal caveats

The vision described above is contingent on assumptions that are not yet verified. The most important caveats are:

1. **Non-negativity of the energy decomposition.** The entire framework depends on the quadratic form decomposing into non-negative pieces. For $\zeta(s)$ and real Dirichlet L-functions, this appears to hold. For general automorphic L-functions, especially at ramified primes, it is unproven and could fail. The boundary of where non-negativity holds would itself be an important discovery.
2. **The convergence problem.** Even if ground state simplicity is established for every finite cutoff λ, the full Riemann Hypothesis requires $\lambda \to \infty$. The convergence of approximate zeros to actual zeros is a separate and almost certainly harder problem, identified by Connes as the other "missing step." The energy-decomposition method, as currently formulated, does not address this.
3. **Non-commutative obstacles.** The extension to non-abelian groups (GL_n for $n \geq 2$) is genuinely difficult and may require new ideas beyond the scope of classical Dirichlet form theory. The Cipriani–Sauvageot framework provides a starting point, but substantial development would be needed.

4. **Quantitative vs. qualitative.** The current method gives *qualitative* results (the ground state is simple; the eigenfunction is positive) but not *quantitative* ones (how large is the spectral gap? how positive is the eigenfunction?). For the deeper applications (subconvexity bounds, Ramanujan conjecture, zero-density estimates), quantitative control is essential and would require new techniques beyond the abstract Perron–Frobenius framework.

6 Summary

The energy-decomposition method, in its current form, resolves a specific open problem (ground state simplicity for the restricted Weil operator) by a specific route (Dirichlet form theory and Perron–Frobenius). But the structural features of the argument—non-negative decomposition, irreducibility via rich shift structures, coercivity via Fourier growth—are not tied to the Riemann zeta function. They suggest a broader framework in which:

- Direction I (real characters) validates the method's robustness and provides the first uniform result across a family of L-functions, with potential implications for the Siegel zero problem.
- Direction II (arithmetic kernels) transforms the method into a diagnostic tool that characterizes which L-functions "should" satisfy the Riemann Hypothesis based on the Markov/irreducibility properties of their associated Dirichlet forms, with connections to functoriality and subconvexity.
- Direction III (operators on groups) embeds the framework in the Langlands program and non-commutative geometry, potentially connecting ground state simplicity to multiplicity-one theorems and providing a new analytic foundation for the Connes program.

The full vision—an adelic Perron–Frobenius theory connecting Euler products, Dirichlet form irreducibility, and automorphic multiplicity one—is ambitious and largely conjectural. But each step along the way (starting with Direction I, which appears to require minimal new ideas) would be a concrete mathematical contribution, and the overall direction suggests a genuine structural insight: the Riemann Hypothesis, at its core, may be a statement about positivity and irreducibility.

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