Resolution of the Simplicity and Evenness Condition for the Ground State of the Restricted Weil Quadratic Form: Connecting the Energy-Decomposition Paper to Connes' Programme

Abstract

We analyse how the results of the paper *Energy-Decomposition and Perron–Frobenius Consequences for the Restricted Weil Quadratic Form* (hereafter **[ED]**) resolve the first of the two open conditions identified by Alain Connes in §6.6 of his 2026 survey *The Riemann Hypothesis: Past, Present and a Letter Through Time* (hereafter **[C26]**). Specifically, **[ED]** proves that the self-adjoint operator A_λ associated with the restricted Weil quadratic form has compact resolvent and that its lowest eigenvalue is simple with a strictly positive, even eigenfunction—exactly the hypothesis needed for Connes–van Suijlekom's Theorem 6.1 in **[C26]** to guarantee that the Fourier transform of the ground-state minimiser has all its zeros on the critical line. We give a detailed term-by-term comparison, discuss what remains unresolved, and place the contribution within the broader strategy towards the Riemann Hypothesis outlined in **[C26]**.

Contents

1 Context: Connes' strategy and the role of Theorem 6.1

1.1 The Weil positivity approach

Connes' survey **[C26]** describes a strategy towards the Riemann Hypothesis (RH) rooted in the Weil explicit formula. For $\lambda > 1$, one restricts test functions to the compact multiplicative interval $[\lambda^{-1}, \lambda] \subset \mathbb{R}_+^*$ and studies the quadratic form

$$QW_\lambda(g) := \sum_v W_v(g * g^*), \tag{1}$$

where W_v are the local distributions at each place v of $\mathbb{Q}$, evaluated on $f = g * g^*$ (multiplicative convolution with involution). By Weil's criterion, the positivity of QW_λ for all $\lambda > 1$ (subject to the vanishing condition $\hat{g}(\pm i/2) = 0$) is equivalent to RH.

1.2 Theorem 6.1 of Connes–van Suijlekom

A central tool in **[C26]** is the following theorem from joint work with van Suijlekom [32]:

Theorem 1 (Connes–van Suijlekom, [32])**.** *Let $L > 0$, let D be a real distribution on $[0, L]$, and let $\widetilde{D}$ be the associated even distribution on $[-L, L]$. Assume that the quadratic form with Schwartz kernel $\widetilde{D}(x - y)$ defines a lower-bounded selfadjoint operator on $L^2([-L/2, L/2])$, and that the minimum of its spectrum is a simple, isolated eigenvalue with even eigenfunction η. Then all zeros of the entire function $\hat{\eta}(z)$, $z \in \mathbb{C}$, lie on the real line.*

This theorem provides a mechanism for constructing entire functions whose zeros are provably on the critical line: if the ground-state eigenfunction of an appropriate operator is simple and even, its Fourier transform has only real zeros.

1.3 The two remaining steps (§6.6 of [C26])

In §6.6 of **[C26]**, Connes identifies two conditions that remain to be established in order to apply Theorem 1 to the Weil quadratic form QW_λ and ultimately connect the approximating zeros to those of the Riemann zeta function:

(I) **Simplicity and evenness:** Show that the smallest eigenvalue of QW_λ is simple and that the corresponding eigenfunction is even.

(II) **Approximation quality:** Show that the prolate-function ansatz k_λ (constructed via the summation map $\mathcal{E}$ from prolate spheroidal wave functions) is a sufficiently good approximation of the actual minimal eigenvector θ_x of QW_λ.

Connes notes that "the analogue of this property is known for the prolate wave operator" but that the result for QW_λ itself had not been established.

The central claim of this note is that condition (I) is fully resolved by **[ED]**.

2 What the energy-decomposition paper proves

We summarise the relevant results of **[ED]**, using its numbering.

2.1 The quadratic form $\mathcal{E}_\lambda$ and its relation to QW_λ

Working in logarithmic coordinates $u = \log x$ on the interval $I = (-L, L)$ with $L = \log \lambda$, and writing $G(u) = g(e^u)$ with $\widetilde{G}$ denoting extension by zero to $\mathbb{R}$, paper **[ED]** defines

$$\mathcal{E}_\lambda(G) := \int_0^{2L} w(t)\, \|\widetilde{G} - S_t\widetilde{G}\|^2_{L^2(\mathbb{R})}\, dt + \sum_{\substack{p \text{ prime} \\ p^m \le \lambda^2}} (\log p)\, p^{-m/2}\, \|\widetilde{G} - S_{m\log p}\widetilde{G}\|^2_{L^2(\mathbb{R})}, \tag{2}$$

where $w(t) = e^{t/2}/(2\sinh t)$ and S_t is translation by t. Lemmas 3 and 4 of **[ED]** show that

$$-\sum_v W_v(g * g^*) = \mathcal{E}_\lambda(G) + c(\lambda)\, \|G\|_2^2, \tag{3}$$

where $c(\lambda) \in \mathbb{R}$ is a finite constant. The additive shift $c(\lambda)\|G\|_2^2$ merely translates the spectrum of the associated operator by a scalar, leaving simplicity, positivity of eigenfunctions, and evenness invariant.

2.2 Selfadjoint operator with compact resolvent

Theorem 2 (Theorem 18 of **[ED]**)**.** *There exists a unique selfadjoint operator $A_\lambda \ge 0$ on $L^2(I)$ associated with the closed form $\mathcal{E}_\lambda$. Moreover, A_λ has compact resolvent; equivalently, $(A_\lambda + 1)^{-1}$ is compact on $L^2(I)$.*

The proof proceeds in several self-contained steps:

1. *Fourier representation.* The ambient form $\mathcal{E}_\lambda^{\mathbb{R}}$ on $L^2(\mathbb{R})$ is identified via Plancherel as multiplication by a symbol $\psi_\lambda(\xi) \ge 0$ in Fourier space.
2. *Closedness.* The form domain, equipped with the graph norm, is isometric to a weighted L^2 space and hence complete. Restriction to the closed subspace $H_I = \{\phi \in L^2(\mathbb{R}) : \phi = 0 \text{ a.e. on } \mathbb{R} \setminus I\}$ preserves closedness.
3. *Logarithmic coercivity.* A lower bound $w(t) \ge c_0/t$ for small t yields $\psi_\lambda(\xi) \ge c_1 \log|\xi| - c_2$ for large $|\xi|$, via an interval-counting argument.
4. *Compact embedding.* The logarithmic growth of ψ_λ provides a log-frequency moment bound, which combined with the Kolmogorov–Riesz compactness criterion (tightness is automatic from bounded support) gives compactness of the embedding $\mathcal{D}(\mathcal{E}_\lambda) \hookrightarrow L^2(I)$.
5. *Compact resolvent.* Standard resolvent estimates reduce to compact embedding.

Compact resolvent implies the spectrum of A_λ is discrete, consisting of isolated eigenvalues of finite multiplicity accumulating only at $+\infty$.

2.3 Simplicity and strict positivity of the ground state

Proposition 3 (Proposition 20 of **[ED]**)**.** *The semigroup $T(t) = e^{-tA_\lambda}$ is positivity improving, and the lowest eigenvalue of A_λ is simple with a strictly positive a.e. eigenfunction ψ.*

This is established by verifying three properties:

1. **Positivity preservation (Markov property).** For every normal contraction Φ (i.e. $\Phi(0) = 0$, $|\Phi(a) - \Phi(b)| \le |a - b|$), the difference-energy structure of $\mathcal{E}_\lambda$ gives $\mathcal{E}_\lambda(\Phi \circ G) \le \mathcal{E}_\lambda(G)$ pointwise under the integral. This is the Beurling–Deny criterion for Markovian semigroups.
2. **Irreducibility.** The archimedean *continuum* of shifts—the integral over $t \in (0, 2L)$ with strictly positive weight $w(t) > 0$—is the decisive ingredient. If $\mathcal{E}_\lambda(\mathbf{1}_B) = 0$ for a measurable

$B \subset I$, then the non-negativity of all weights forces $\|\widetilde{\mathbf{1}_B} - S_t\widetilde{\mathbf{1}_B}\|_2 = 0$ for a.e. $t \in (0, 2L)$. Continuity of $t \mapsto \|\phi - S_t\phi\|_2^2$ upgrades this to *all* $t \in (0, 2L)$, and a mollifier argument (Lemma 7 of [**ED**]) forces B to be null or conull. Via the Beurling–Deny/Fukushima equivalence from Dirichlet form theory, this yields irreducibility of $T(t)$.

3. **Holomorphy.** Since A_λ is selfadjoint and lower bounded, e^{-zA_λ} is bounded and holomorphic on $\{\Re z > 0\}$ by the spectral theorem.

These three properties together invoke the theorem of Arendt et al.: positivity + irreducibility + holomorphy $\Rightarrow$ positivity improving. The Krein–Rutman / Perron–Frobenius theorem for compact positive operators then gives simplicity and strict positivity of the ground state.

2.4 Evenness of the ground state

Corollary 4 (Corollary 22 of [**ED**])**.** *The ground-state eigenfunction ψ satisfies $\psi(-u) = \psi(u)$ a.e.*

Proof sketch. The reflection operator $R\colon G(u) \mapsto G(-u)$ is unitary on $L^2(I)$ (since $I = (-L, L)$ is symmetric) and satisfies $RS_t = S_{-t}R$. Using $\|\phi - S_{-t}\phi\| = \|\phi - S_t\phi\|$ and the symmetry of all weights in $\mathcal{E}_\lambda$, one obtains $\mathcal{E}_\lambda(RG) = \mathcal{E}_\lambda(G)$, hence $A_\lambda R = RA_\lambda$.

Since $R\psi$ is then a strictly positive eigenfunction for the same (simple) eigenvalue, $R\psi = c\psi$ with $c > 0$. Unitarity of R forces $c = 1$, so ψ is even. □

3 Term-by-term comparison with Connes' requirements

The following table makes the correspondence explicit.

Requirement from §6.6 of [C26]	**Result in [ED]**
QW_λ defines a lower-bounded selfadjoint operator	Theorem 18: $A_\lambda \geq 0$ is selfadjoint, associated to the closed form $\mathcal{E}_\lambda$ via Kato's representation theorem
Spectrum is discrete with isolated eigenvalues	Theorem 18: A_λ has compact resolvent, hence discrete spectrum accumulating at $+\infty$
Smallest eigenvalue is simple	Proposition 20: Markov property + irreducibility + holomorphy $\Rightarrow$ positivity improving; then Krein–Rutman gives simplicity
Ground-state eigenfunction is even	Corollary 22: A_λ commutes with $R\colon G(u) \mapsto G(-u)$; simplicity + strict positivity forces $R\psi = \psi$

The relationship between $\mathcal{E}_\lambda$ in [**ED**] and QW_λ in [**C26**] is given by (3): they differ by an additive constant times $\|G\|_2^2$. Such a shift translates every eigenvalue by the same scalar $c(\lambda)$ without altering eigenvectors, simplicity, or eigenfunction symmetry. Paper [**ED**] notes this explicitly.

4 The role of the archimedean continuum

It is worth emphasising the structural reason why irreducibility holds for A_λ but would *not* hold for a purely "prime" version of the quadratic form.

The prime contributions to $\mathcal{E}_\lambda$ involve differences $S_{m\log p}$ at *discrete* shift values $\{m \log p : p^m \leq \lambda^2\}$. A measurable set $B \subset I$ could, in principle, be invariant under all these discrete shifts while having $0 < m(B) < m(I)$.

The archimedean term, by contrast, involves a *continuum* of shifts—an integral over all $t \in (0, 2L)$ with the strictly positive weight $w(t) > 0$. If $\mathcal{E}_\lambda(\mathbf{1}_B) = 0$, then $\mathbf{1}_B$ must be invariant under translation by *every* t in a full interval $(0, \varepsilon)$. The mollifier argument of Lemma 7 in **[ED]** then forces $\mathbf{1}_B$ to be a.e. constant—the set is null or conull.

This mirrors an important theme in Connes' programme: the archimedean place carries qualitatively different (and essential) information compared to the non-archimedean places. In the language of **[C26]**, the archimedean trace formula (§7.1) and the role of the Sonin space (§7.2) are manifestations of the same phenomenon.

5 What remains unresolved

Paper **[ED]** resolves condition (I) of §1 completely. It does **not** address condition (II): proving that the prolate-function ansatz k_λ (equation (17) of **[C26]**) is a sufficiently good approximation of the true ground-state eigenvector θ_x of QW_λ.

The full chain of Connes' strategy requires the following additional steps, none of which are provided by **[ED]**:

(a) Showing that the prolate-based approximation $k_\lambda = \mathcal{E}(h_\lambda)$ (where h_λ is the appropriate linear combination of $h_{0,\lambda}$ and $h_{4,\lambda}$ with vanishing integral) converges to the true minimiser θ_x in a suitable norm.

(b) Establishing that the Fourier transforms of the true minimisers θ_x converge, as $\lambda \to \infty$, to Riemann's Ξ-function uniformly on compact subsets of the open strip $|\mathrm{Im}(z)| < 1/2$.

(c) Applying Hurwitz's theorem to conclude that the zeros of the limit function (which are the nontrivial zeros of ζ) must lie on the real line, since the approximating functions have all their zeros there by Theorem 1.

Step (a) amounts to comparing two objects: the exact minimiser of QW_λ (whose existence, simplicity, and evenness are now established by **[ED]**) and the "near-radical" function k_λ constructed from prolate wave functions. As Connes discusses in §6.4 of **[C26]**, the conceptual justification is that the range of the summation map $\mathcal{E}$ lies in the radical of the *global* Weil form, so elements of this range restricted to $[\lambda^{-1}, \lambda]$ should be close to the radical of the *restricted* form QW_λ. The exponential smallness of the eigenvalue $\varepsilon(\lambda)$ (and its similarity to $1 - \chi_2(\lambda)$, as shown in Figure 1 of **[C26]**) provides strong numerical evidence for this, but a rigorous proof has not yet been given.

Step (b) is partially addressed by Fact 6.4 of **[C26]**, which establishes the convergence of the Fourier transforms of the prolate-based approximations $\hat{k}_\lambda$ towards the Ξ-function. The gap is in transferring this convergence from the ansatz k_λ to the true minimiser θ_x.

6 Summary

The results of **[ED]** supply the missing operator-theoretic foundation for condition (I) in Connes' strategy. The logical chain can be depicted as follows:

Energy decomposition (Lemmas 3–4 of [ED])

$\Downarrow$

Closed form $\mathcal{E}_\lambda$, compact resolvent (Props. 11–12, 17, Thm. 18 of [ED])

$\Downarrow$

Markov + Irreducibility + Holomorphy (Lemma 6, Cor. 19, spectral thm.)

$\Downarrow$

Positivity improving (Arendt et al.) $\Rightarrow$ **Simple ground state,** $\psi > 0$
(Krein–Rutman)

$\Downarrow$

Reflection symmetry (Prop. 21 of [ED]) $\Rightarrow$ ψ **is even** (Cor. 22 of [ED])

$\Downarrow$

Theorem 6.1 of [C26] applies:
All zeros of $\hat{\psi}(z)$ lie on the real line

The remaining challenge—condition (II)—is to show that the prolate-based approximation k_λ adequately represents the true ground state, and that the convergence $\hat{k}_\lambda \to \Xi$ can be transferred to the actual minimisers. This is the content of the semilocal trace formula approach described in §7.4 of **[C26]** and the subject of ongoing work by Connes, Consani, and Moscovici.

Step in Connes' programme	**Status**	**Source**
QW_λ has simple, even ground state	✓	[ED]: Prop. 20, Cor. 22
Zeros of $\hat{\psi}$ lie on the real line	✓	Follows from above + Thm. 6.1 of [C26]
$k_\lambda \approx \theta_x$ (prolate approximation)	Open	Numerical evidence in [C26] §6.4; rigorous proof pending
$\hat{k}_\lambda \to \Xi$ uniformly on compact sets	✓	Fact 6.4 of [C26] (for the ansatz)
Transfer convergence from ansatz to true minimiser	Open	Requires comparing k_λ and θ_x
Hurwitz's theorem $\Rightarrow$ zeros of Ξ on the line	Conditional	Follows if all preceding steps are completed

References

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