Energy-Decomposition and Perron–Frobenius Consequences for the Restricted Weil Quadratic Form

Abstract

We record a completely concrete and rigorous functional-analytic step that arises in the spectral approach to Weil's criterion when one restricts test functions to a compact multiplicative interval $[\lambda^{-1}, \lambda] \subset \mathbb{R}_+^*$. Starting from the explicit local distributions at the primes and at ∞, we derive an "energy decomposition" expressing the quadratic form (up to an additive constant multiple of $\|g\|_2^2$) as a positive combination of translation-difference energies $\|G - \tau_t G\|_2^2$ in logarithmic coordinates. We then prove the Markov (normal contraction) property and a translation-invariance lemma which yields irreducibility from the archimedean continuum of shifts. Assuming (as in the standard setup) that the associated selfadjoint operator has compact resolvent, we deduce that the ground-state eigenvalue is simple and its eigenfunction can be chosen strictly positive and, by inversion symmetry, even.

Contents

1 Setup on $\mathbb{R}_+^*$

Let $\mathbb{R}_+^* = (0, \infty)$ with multiplicative Haar measure

$$d^*x := \frac{dx}{x}.$$

For measurable g, h define multiplicative convolution

$$(g * h)(x) := \int_{\mathbb{R}_+^*} g(y)\, h(x/y)\, d^*y,$$

and involution

$$g^*(x) := \overline{g(x^{-1})}.$$

If $g \in L^2(\mathbb{R}_+^*, d^*x)$, define the unitary dilation operator

$$(U_a g)(x) := g(x/a) \qquad (a > 0). \tag{1}$$

Then $\|U_a g\|_2 = \|g\|_2$ and $\langle g, U_a g\rangle$ is well-defined.

Lemma 1 (Convolution inner-product identity)**.** *Let $f = g * g^*$. Then for all $a > 0$,*

$$f(a) = \langle g, U_a g\rangle_{L^2(d^*x)} = \int_{\mathbb{R}_+^*} g(x)\, \overline{g(x/a)}\, d^*x, \qquad f(a^{-1}) = \overline{f(a)}.$$

In particular $f(a) + f(a^{-1}) = 2\Re\langle g, U_a g\rangle$ and $f(1) = \|g\|_2^2$.

Proof. By definition,

$$(g * g^*)(a) = \int g(y)\, g^*(a/y)\, d^*y = \int g(y)\, \overline{g((a/y)^{-1})}\, d^*y = \int g(y)\, \overline{g(y/a)}\, d^*y = \langle g, U_a g\rangle.$$

The relation $f(a^{-1}) = \overline{f(a)}$ follows by replacing a with a^{-1} and complex conjugating. □

Lemma 2 (A basic unitary identity)**.** *For any unitary U on a Hilbert space and any vector h,*

$$2\Re\langle h, Uh\rangle = 2\|h\|^2 - \|h - Uh\|^2.$$

Proof. Expand $\|h - Uh\|^2 = \|h\|^2 + \|Uh\|^2 - 2\Re\langle h, Uh\rangle$ and use $\|Uh\| = \|h\|$. □

2 Local explicit-formula terms

Fix $\lambda > 1$ and consider g supported in $[\lambda^{-1}, \lambda]$.

We record the two local distributions we use; these are the only "input formulas".

2.1 Prime terms

For a prime p define

$$W_p(f) := (\log p) \sum_{m\geq 1} p^{-m/2}\big(f(p^m) + f(p^{-m})\big). \tag{2}$$

2.2 Archimedean term

Define

$$W_{\mathbb{R}}(f) := (\log 4\pi + \gamma)\, f(1) + \int_1^\infty \Big(f(x) + f(x^{-1}) - 2x^{-1/2} f(1)\Big) \frac{x^{1/2}}{x - x^{-1}}\, d^*x, \tag{3}$$

where γ is the Euler–Mascheroni constant.

Remark 3 (Restriction to a compact multiplicative interval)*.* If $\operatorname{supp}(g) \subset [\lambda^{-1}, \lambda]$, then for $a > \lambda^2$ the supports of g and $U_a g$ are disjoint, hence $\langle g, U_a g\rangle = 0$ and $f(a) = 0$. Consequently:

- in (2) only those (p, m) with $p^m \leq \lambda^2$ contribute;
- in (3), after the change of variables $x = e^t$, only $t \in [0, 2\log\lambda]$ contributes to the term involving $f(e^t) + f(e^{-t})$.

This finiteness is crucial and is completely elementary.

3 Logarithmic coordinates and translations

Set $u = \log x$, so that $d^*x = du$ and the interval $[\lambda^{-1}, \lambda]$ becomes

$$I := (-L, L), \qquad L := \log\lambda.$$

For $G \in L^2(I)$ we denote by $\widetilde{G}$ its extension by 0 to $\mathbb{R}$. Let S_t be translation on $L^2(\mathbb{R})$:

$$(S_t\phi)(u) := \phi(u - t).$$

Then in logarithmic coordinates, the dilation U_{e^t} from (1) corresponds to translation: if $G(u) = g(e^u)$, then $(U_{e^t} g)(e^u) = g(e^{u-t})$, i.e. $\widetilde{G} \mapsto S_t\widetilde{G}$.

4 Energy decomposition into translation differences

4.1 Prime contributions

Lemma 4 (Prime term as a difference energy plus a constant)**.** *Let $f = g * g^*$ with g supported in $[\lambda^{-1}, \lambda]$, and let $G(u) = g(e^u)$. Then*

$$-W_p(f) = \sum_{\substack{m\geq 1\\ p^m \leq \lambda^2}} (\log p)\, p^{-m/2}\, \|\widetilde{G} - S_{m\log p}\widetilde{G}\|^2_{L^2(\mathbb{R})} \;+\; c_p(\lambda)\, \|G\|^2_{L^2(I)},$$

where $c_p(\lambda) \in \mathbb{R}$ is a finite constant depending only on p and λ.

Proof. By Lemma 1 and (2),

$$W_p(f) = (\log p) \sum_{m\geq 1} p^{-m/2}\, 2\Re\langle g, U_{p^m} g\rangle.$$

By Lemma 2 (with $U = U_{p^m}$),

$$2\Re\langle g, U_{p^m} g\rangle = 2\|g\|_2^2 - \|g - U_{p^m} g\|_2^2.$$

In logarithmic coordinates, $\|g - U_{p^m} g\|_2 = \|\widetilde{G} - S_{m\log p}\widetilde{G}\|_{L^2(\mathbb{R})}$. Moreover, if $p^m > \lambda^2$ then $\langle g, U_{p^m} g\rangle = 0$ by Remark 3, so those terms vanish. Collecting the $\|g\|_2^2$ contributions yields the constant $c_p(\lambda)$. □

4.2 Archimedean contribution

Lemma 5 (Archimedean term as a continuum of difference energies plus a constant)**.** *Let $f = g * g^*$ with g supported in $[\lambda^{-1}, \lambda]$, and let $G(u) = g(e^u)$. Define the strictly positive weight on $(0, \infty)$,*

$$w(t) := \frac{e^{t/2}}{e^t - e^{-t}} = \frac{e^{t/2}}{2\sinh t}.$$

Then

$$-W_{\mathbb{R}}(f) = \int_0^{2L} w(t)\, \|\widetilde{G} - S_t\widetilde{G}\|_{L^2(\mathbb{R})}^2\, dt \;+\; c_\infty(\lambda)\, \|G\|_{L^2(I)}^2,$$

where $c_\infty(\lambda) \in \mathbb{R}$ is a finite constant depending only on λ.

Proof. Start from (3). Substitute $x = e^t$ (so $d^*x = dt$) to obtain

$$W_{\mathbb{R}}(f) = (\log 4\pi + \gamma)\, f(1) + \int_0^\infty \Big(f(e^t) + f(e^{-t}) - 2e^{-t/2} f(1)\Big)\, w(t)\, dt.$$

Using Lemma 1, $f(1) = \|g\|_2^2$, and

$$f(e^t) + f(e^{-t}) = 2\Re\langle g, U_{e^t} g\rangle,$$

we get

$$-W_{\mathbb{R}}(f) = -(\log 4\pi + \gamma)\|g\|_2^2 + \int_0^\infty \Big(-2\Re\langle g, U_{e^t} g\rangle + 2e^{-t/2}\|g\|_2^2\Big)\, w(t)\, dt.$$

Apply Lemma 2 with $U = U_{e^t}$:

$$-2\Re\langle g, U_{e^t} g\rangle = \|g - U_{e^t} g\|_2^2 - 2\|g\|_2^2.$$

Thus the integrand equals

$$\|g - U_{e^t} g\|_2^2 + 2(e^{-t/2} - 1)\|g\|_2^2.$$

In logarithmic coordinates $\|g - U_{e^t} g\|_2 = \|\widetilde{G} - S_t\widetilde{G}\|_{L^2(\mathbb{R})}$.

Now we split the integral at $t = 2L$. By Remark 3, for $t > 2L$ the supports of $\widetilde{G}$ and $S_t\widetilde{G}$ are disjoint, so $\|\widetilde{G} - S_t\widetilde{G}\|_2^2 = 2\|G\|_2^2$ (not zero). Hence for $t > 2L$ the integrand becomes $2\|G\|_2^2 + 2(e^{-t/2} - 1)\|G\|_2^2 = 2e^{-t/2}\|G\|_2^2$. This tail integral $\int_{2L}^\infty 2e^{-t/2}\, w(t)\, dt$ converges (since $w(t) \sim e^{-t/2}$ as $t \to \infty$) and contributes a finite constant times $\|G\|_2^2$.

For $t \in [0, 2L]$ we retain the difference-energy term $w(t)\|\widetilde{G} - S_t\widetilde{G}\|_2^2$ and absorb the $2(e^{-t/2} - 1)\, w(t)\|G\|_2^2$ contribution into the constant. Combining all $\|G\|_2^2$ terms—from the $(\log 4\pi + \gamma)$ prefactor, the integral over $[0, 2L]$ of $2(e^{-t/2} - 1)\, w(t)$, and the tail integral over $(2L, \infty)$—yields the finite constant $c_\infty(\lambda)$. The integral of $w(t)(e^{-t/2} - 1)$ over $[0, 2L]$ converges absolutely (near 0, $w(t) \sim 1/(2t)$ and $e^{-t/2} - 1 \sim -t/2$, giving an integrable $O(1)$ contribution). □

4.3 Global quadratic form on the interval

Definition 6 (Difference-energy form). Fix $\lambda > 1$ and $L = \log \lambda$. For $G \in L^2(I)$ define

$$\mathcal{E}_\lambda(G) := \int_0^{2L} w(t)\, \|\widetilde{G} - S_t\widetilde{G}\|^2_{L^2(\mathbb{R})}\, dt + \sum_{\substack{p \text{ prime} \\ p \le \lambda^2}} \sum_{\substack{m \ge 1 \\ p^m \le \lambda^2}} (\log p) p^{-m/2}\, \|\widetilde{G} - S_{m\log p}\widetilde{G}\|^2_{L^2(\mathbb{R})}. \tag{4}$$

Remark 7 (What we have proved so far). Lemmas 4 and 5 show that for $f = g * g^*$ with $\operatorname{supp}(g) \subset [\lambda^{-1}, \lambda]$, the quantity

$$-\sum_{v \in \{\infty\} \cup \{p\}} W_v(f)$$

equals $\mathcal{E}_\lambda(G)$ plus an additive constant multiple of $\|G\|_2^2$. Since adding a constant multiple of $\|G\|_2^2$ only shifts the spectrum of the associated operator, it does not affect positivity/irreducibility properties of the semigroup and does not affect eigenfunction parity considerations.

5 Markov property (normal contractions)

Definition 8 (Normal contraction). A map $\Phi : \mathbb{R} \to \mathbb{R}$ is a normal contraction if $\Phi(0) = 0$ and $|\Phi(a) - \Phi(b)| \le |a - b|$ for all $a, b \in \mathbb{R}$.

Lemma 9 (Markov property)**.** *For every normal contraction Φ and every $G \in L^2(I)$,*

$$\mathcal{E}_\lambda(\Phi \circ G) \le \mathcal{E}_\lambda(G).$$

In particular, $\mathcal{E}_\lambda(|G|) \le \mathcal{E}_\lambda(G)$.

Proof. For each shift parameter t,

$$\|\widetilde{\Phi \circ G} - S_t\widetilde{\Phi \circ G}\|_2^2 = \int_{\mathbb{R}} |\Phi(\widetilde{G}(u)) - \Phi(\widetilde{G}(u-t))|^2\, du \le \int_{\mathbb{R}} |\widetilde{G}(u) - \widetilde{G}(u-t)|^2\, du = \|\widetilde{G} - S_t\widetilde{G}\|_2^2,$$

by the 1-Lipschitz property of Φ. Integrating against the nonnegative weights and summing proves the claim. $\square$

6 A translation-invariance lemma on an interval

Lemma 10 (Local translation invariance forces null or conull)**.** *Let $I \subset \mathbb{R}$ be a nontrivial open interval and let $B \subset I$ be measurable. Assume that there exists $\varepsilon > 0$ such that for every $t \in (0, \varepsilon)$,*

$$\mathbf{1}_B(u) = \mathbf{1}_B(u - t) \quad \text{for a.e. } u \in I \cap (I + t). \tag{5}$$

Then either $m(B) = 0$ or $m(I \setminus B) = 0$. Equivalently: if $0 < m(B) < m(I)$ then for every $\varepsilon > 0$ there exists $t \in (0, \varepsilon)$ with $m(B \cap (B + t)^c) > 0$.

Proof. Write $f := \mathbf{1}_B \in L^1_{\text{loc}}(I)$. Fix a compact subinterval $J \Subset I$ (so $\operatorname{dist}(J, \partial I) > 0$), and choose $0 < \delta < \min\{\varepsilon, \operatorname{dist}(J, \partial I)\}$. From (5) and the substitution $u \mapsto u + t$ we obtain: for every $t \in (0, \delta)$,

$$f(u + t) = f(u) \quad \text{for a.e. } u \in J.$$

Thus for every $t \in (-\delta, \delta)$ we have $f(u + t) = f(u)$ for a.e. $u \in J$ (replace t by $-t$).

Let $\rho \in C_c^\infty(\mathbb{R})$ be a standard mollifier with $\rho \geq 0$, $\int \rho = 1$ and $\operatorname{supp} \rho \subset (-1,1)$, and set $\rho_\eta(s) := \eta^{-1}\rho(s/\eta)$ for $0 < \eta < \delta/2$. Define $f_\eta := f * \rho_\eta$ on the slightly smaller interval

$$J_\eta := \{u \in J : \operatorname{dist}(u, \mathbb{R} \setminus J) > \eta\}.$$

Then $f_\eta \in C^\infty(J_\eta)$, and for $u \in J_\eta$ and $|t| < \delta/2$ we may compute (using Fubini)

$$f_\eta(u+t) = \int_{\mathbb{R}} f(u+t-s)\rho_\eta(s)\,ds = \int_{\mathbb{R}} f(u-s)\rho_\eta(s)\,ds = f_\eta(u),$$

because $u - s \in J$ for $u \in J_\eta$ and $s \in \operatorname{supp}\rho_\eta$, and $f(\cdot + t) = f(\cdot)$ a.e. on J. Hence f_η is translation-invariant on the connected open interval J_η, so f_η is constant on J_η.

Letting $\eta \downarrow 0$, we have $f_\eta \to f$ in $L^1(J)$, so f is a.e. equal to a constant on J. Since $J \Subset I$ was arbitrary, f is a.e. constant on I, i.e. $\mathbf{1}_B$ is a.e. either 0 or 1 on I. Thus $m(B) = 0$ or $m(I \setminus B) = 0$. $\square$

7 Irreducibility from the archimedean continuum

7.1 A concrete criterion

Lemma 11 (Indicator-energy vanishes only for null/conull sets)**.** *Let $B \subset I$ be measurable. If $\mathcal{E}_\lambda(\mathbf{1}_B) = 0$, then $m(B) = 0$ or $m(I \setminus B) = 0$.*

Proof. By definition of $\mathcal{E}_\lambda$ and the nonnegativity of all weights, $\mathcal{E}_\lambda(\mathbf{1}_B) = 0$ implies in particular that the archimedean contribution vanishes:

$$\int_0^{2L} w(t)\,\|\widetilde{\mathbf{1}_B} - S_t\widetilde{\mathbf{1}_B}\|_2^2\,dt = 0.$$

Since $w(t) > 0$ for every $t > 0$, it follows that

$$\|\widetilde{\mathbf{1}_B} - S_t\widetilde{\mathbf{1}_B}\|_2^2 = 0 \quad \text{for a.e. } t \in (0, 2L).$$

We now upgrade "a.e." to "all": for any $\phi \in L^2(\mathbb{R})$, the map $t \mapsto \|\phi - S_t\phi\|_2^2$ is continuous (by strong continuity of the translation group on $L^2(\mathbb{R})$, which follows from dominated convergence). Applying this to $\phi = \widetilde{\mathbf{1}_B} \in L^2(\mathbb{R})$, the function $t \mapsto \|\widetilde{\mathbf{1}_B} - S_t\widetilde{\mathbf{1}_B}\|_2^2$ is continuous, vanishes a.e. on $(0, 2L)$, and hence vanishes *everywhere* on $(0, 2L)$. In particular, for every $t \in (0, 2L)$,

$$\mathbf{1}_B(u) = \mathbf{1}_B(u-t) \quad \text{for a.e. } u \in I \cap (I+t).$$

Since this holds for all t in the interval $(0, 2L)$, which contains $(0, \varepsilon)$ for any $\varepsilon \leq 2L$, Lemma 10 applies and yields $m(B) = 0$ or $m(I \setminus B) = 0$. $\square$

7.2 Operator realization: closedness and compact resolvent

In this subsection we show that the concrete form $\mathcal{E}_\lambda$ of Definition 6 is closed and yields a selfadjoint operator with compact resolvent. This replaces the abstract assumption previously made on the operator.

7.2.1 Ambient form on $L^2(\mathbb{R})$ and Fourier representation

Let $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ denote the unitary Fourier transform

$$\widehat{\phi}(\xi) := \int_{\mathbb{R}} \phi(u) e^{-iu\xi}\, du, \qquad \phi(u) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{\phi}(\xi) e^{iu\xi}\, d\xi,$$

so that Plancherel reads $\|\phi\|^2_{L^2(\mathbb{R})} = \frac{1}{2\pi} \int_{\mathbb{R}} |\widehat{\phi}(\xi)|^2\, d\xi$.

Define the "ambient" quadratic form on $L^2(\mathbb{R})$ by

$$\begin{aligned}\mathcal{E}^{\mathbb{R}}_\lambda(\phi) := \int_0^{2L} & w(t)\, \|\phi - S_t\phi\|^2_{L^2(\mathbb{R})}\, dt \\ & + \sum_{\substack{p \text{ prime} \\ p \le \lambda^2}} \sum_{\substack{m \ge 1 \\ p^m \le \lambda^2}} (\log p) p^{-m/2}\, \|\phi - S_{m\log p}\phi\|^2_{L^2(\mathbb{R})},\end{aligned}$$

with domain $\mathcal{D}(\mathcal{E}^{\mathbb{R}}_\lambda) := \{\phi \in L^2(\mathbb{R}) : \mathcal{E}^{\mathbb{R}}_\lambda(\phi) < \infty\}$. By definition, for $G \in L^2(I)$,

$$\mathcal{E}_\lambda(G) = \mathcal{E}^{\mathbb{R}}_\lambda(\widetilde{G}).$$

Lemma 12 (Plancherel identity for translation differences)**.** *For $\phi \in L^2(\mathbb{R})$ and $t \in \mathbb{R}$,*

$$\|\phi - S_t\phi\|^2_{L^2(\mathbb{R})} = \frac{1}{2\pi} \int_{\mathbb{R}} |1 - e^{-i\xi t}|^2\, |\widehat{\phi}(\xi)|^2\, d\xi = \frac{1}{2\pi} \int_{\mathbb{R}} 4 \sin^2\!\Big(\frac{\xi t}{2}\Big)\, |\widehat{\phi}(\xi)|^2\, d\xi.$$

Proof. Since $\widehat{S_t\phi}(\xi) = e^{-i\xi t}\widehat{\phi}(\xi)$, Plancherel gives the first identity. The second follows from $|1 - e^{-i\eta}|^2 = 4\sin^2(\eta/2)$. □

Lemma 13 (Fourier representation)**.** *For $\phi \in L^2(\mathbb{R})$,*

$$\mathcal{E}^{\mathbb{R}}_\lambda(\phi) = \frac{1}{2\pi} \int_{\mathbb{R}} \psi_\lambda(\xi)\, |\widehat{\phi}(\xi)|^2\, d\xi \quad \textit{in } [0, \infty],$$

where

$$\begin{aligned}\psi_\lambda(\xi) := 4 \int_0^{2L} & w(t)\, \sin^2\!\Big(\frac{\xi t}{2}\Big)\, dt \\ & + 4 \sum_{\substack{p \text{ prime} \\ p \le \lambda^2}} \sum_{\substack{m \ge 1 \\ p^m \le \lambda^2}} (\log p) p^{-m/2}\, \sin^2\!\Big(\frac{\xi m \log p}{2}\Big).\end{aligned} \tag{6}$$

In particular ψ_λ is measurable, even, finite for each ξ, and $\psi_\lambda(\xi) \ge 0$.

Proof. Apply Lemma 12 to each translation difference in $\mathcal{E}^{\mathbb{R}}_\lambda$. All weights are nonnegative, so Tonelli's theorem permits interchange of the ξ–integral with the t–integration and finite summations. □

Proposition 14 (Closedness on $L^2(\mathbb{R})$)**.** *The form $\mathcal{E}^{\mathbb{R}}_\lambda$ is densely defined, symmetric, nonnegative, and closed on $L^2(\mathbb{R})$. Moreover,*

$$\mathcal{D}(\mathcal{E}^{\mathbb{R}}_\lambda) = \Big\{\phi \in L^2(\mathbb{R}) : \int_{\mathbb{R}} \psi_\lambda(\xi)\, |\widehat{\phi}(\xi)|^2\, d\xi < \infty\Big\},$$

and $\mathcal{D}(\mathcal{E}^{\mathbb{R}}_\lambda)$ is a Hilbert space for the norm

$$\|\phi\|^2_{\mathcal{D}} := \|\phi\|^2_{L^2(\mathbb{R})} + \mathcal{E}^{\mathbb{R}}_\lambda(\phi) = \frac{1}{2\pi} \int_{\mathbb{R}} (1 + \psi_\lambda(\xi))\, |\widehat{\phi}(\xi)|^2\, d\xi.$$

Proof. By Lemma 13, $\mathcal{E}_\lambda^{\mathbb{R}}$ is the quadratic form of multiplication by ψ_λ in Fourier space. Hence $\mathcal{D}(\mathcal{E}_\lambda^{\mathbb{R}})$ is isometric (via $\phi \mapsto \widehat{\phi}$) to the weighted L^2 space with weight $1+\psi_\lambda$, and therefore complete. Nonnegativity and symmetry are immediate from the definition.

For density, note that $C_c^\infty(\mathbb{R}) \subset \mathcal{D}(\mathcal{E}_\lambda^{\mathbb{R}})$: for $\phi \in C_c^\infty(\mathbb{R})$, $\|\phi - S_t\phi\|_2 \le |t|\|\phi'\|_2$, and $\int_0^{2L} w(t)t^2\, dt < \infty$ (since $w(t) \sim (2t)^{-1}$ as $t \downarrow 0$ and the upper limit is finite); the prime sum in $\mathcal{E}_\lambda^{\mathbb{R}}$ is finite. Since $C_c^\infty(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, the form is densely defined. □

Proposition 15 (Closedness on $L^2(I)$)**.** *The form $\mathcal{E}_\lambda$ on $H = L^2(I)$ is densely defined, symmetric, nonnegative, and closed.*

Proof. The map $G \mapsto \widetilde{G}$ is an isometry from $L^2(I)$ onto the closed subspace $H_I = \{\phi \in L^2(\mathbb{R}) : \phi = 0 \text{ a.e. on } \mathbb{R} \setminus I\}$. Moreover $\mathcal{E}_\lambda(G) = \mathcal{E}_\lambda^{\mathbb{R}}(\widetilde{G})$. Thus $\mathcal{E}_\lambda$ is the restriction of the closed form $\mathcal{E}_\lambda^{\mathbb{R}}$ (Proposition 14) to the closed subspace H_I, and therefore is closed. Density follows because $C_c^\infty(I) \subset \mathcal{D}(\mathcal{E}_\lambda)$ and is dense in $L^2(I)$. □

7.2.2 A coercive lower bound for the symbol ψ_λ

Lemma 16 (A lower bound for $w(t)$)**.** *Let $t_0 := \min(1, 2L)$. There exists $c_0 = c_0(L) > 0$ such that for all $t \in (0, t_0]$,*

$$w(t) = \frac{e^{t/2}}{2\sinh t} \ge \frac{c_0}{t}.$$

Proof. For $t > 0$ one has $\sinh t \le te^t$, hence

$$w(t) = \frac{e^{t/2}}{2\sinh t} \ge \frac{e^{t/2}}{2te^t} = \frac{e^{-t/2}}{2t}.$$

For $t \in (0, 1]$, $e^{-t/2} \ge e^{-1/2}$, so we may take $c_0 := e^{-1/2}/2$ (or any smaller positive constant). □

Lemma 17 (Logarithmic growth of ψ_λ)**.** *There exist constants $c_1, c_2 > 0$ and $\xi_0 \ge 2$ (depending only on L) such that for all $|\xi| \ge \xi_0$,*

$$\psi_\lambda(\xi) \ge c_1 \log|\xi| - c_2.$$

In particular $\psi_\lambda(\xi) \to \infty$ as $|\xi| \to \infty$.

Proof. Drop the nonnegative prime sum in (6):

$$\psi_\lambda(\xi) \ge 4\int_0^{2L} w(t)\sin^2\!\Big(\frac{\xi t}{2}\Big)\, dt \ge 4\int_0^{t_0} w(t)\sin^2\!\Big(\frac{\xi t}{2}\Big)\, dt.$$

By Lemma 16, for $t \in (0, t_0]$,

$$\psi_\lambda(\xi) \ge 4c_0 \int_0^{t_0} \frac{1}{t}\sin^2\!\Big(\frac{\xi t}{2}\Big)\, dt.$$

Assume $|\xi| \ge \frac{4\pi}{t_0}$ (this fixes ξ_0). Define intervals

$$J_n := \left[\frac{2\pi n + \pi/2}{|\xi|}, \frac{2\pi n + 3\pi/2}{|\xi|}\right], \qquad n \ge 0.$$

For $t \in J_n$, $\sin^2(\xi t/2) \ge 1/2$. Let $N \ge 1$ be the largest integer such that $J_{N-1} \subset (0, t_0]$. Then $N \asymp |\xi|$ (with constants depending only on t_0), and hence

$$\int_0^{t_0} \frac{1}{t}\sin^2\!\Big(\frac{\xi t}{2}\Big)\, dt \ge \sum_{n=0}^{N-1} \int_{J_n} \frac{1}{t}\cdot\frac{1}{2}\, dt = \frac{1}{2}\sum_{n=0}^{N-1} \log\frac{2\pi n + 3\pi/2}{2\pi n + \pi/2}.$$

Using $\log(1+x) \ge x/(1+x)$, one obtains

$$\log \frac{2\pi n + 3\pi/2}{2\pi n + \pi/2} = \log\left(1 + \frac{\pi}{2\pi n + \pi/2}\right) \ge \frac{c}{n+1}$$

for some absolute $c > 0$ and all $n \ge 0$. Therefore the sum is bounded below by $c' \sum_{n=0}^{N-1} \frac{1}{n+1} \ge c'' \log N - C$. Since $N \asymp |\xi|$, we have $\log N = \log|\xi| + O(1)$, giving the claim. $\square$

Corollary 18 (Energy controls a logarithmic frequency moment)**.** *There exist constants $a, b > 0$ (depending only on L) such that for every $\phi \in \mathcal{D}(\mathcal{E}_\lambda^{\mathbb{R}})$,*

$$\int_{\mathbb{R}} \log(2 + |\xi|)\, |\widehat{\phi}(\xi)|^2 \, d\xi \le a\, \|\phi\|_{L^2(\mathbb{R})}^2 + b \int_{\mathbb{R}} \psi_\lambda(\xi)\, |\widehat{\phi}(\xi)|^2 \, d\xi.$$

In particular, if $\|\phi\|_2^2 + \mathcal{E}_\lambda^{\mathbb{R}}(\phi) \le M$, then $\int \log(2+|\xi|)|\widehat{\phi}(\xi)|^2 \le C(M, L)$.

Proof. Lemma 17 implies $\log(2 + |\xi|) \le a' + b'\psi_\lambda(\xi)$ for suitable a', b' (after enlarging constants to handle bounded $|\xi|$). Multiply by $|\widehat{\phi}(\xi)|^2$ and integrate. $\square$

7.2.3 Compact embedding and compact resolvent

Theorem 19 (Kolmogorov–Riesz compactness criterion in $L^2(\mathbb{R})$)**.** *A set $\mathcal{K} \subset L^2(\mathbb{R})$ is relatively compact if and only if:*

*(i) (*tightness*) for every $\varepsilon > 0$ there exists $R > 0$ such that $\int_{|u|>R} |\phi(u)|^2 \, du < \varepsilon^2$ for all $\phi \in \mathcal{K}$;*

*(ii) (*translation equicontinuity*) for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|\phi - S_h\phi\|_2 < \varepsilon$ for all $\phi \in \mathcal{K}$ and all $|h| < \delta$.*

Remark 20. See, e.g., Lieb–Loss, *Analysis*, for a proof of Theorem 19.

Lemma 21 (Uniform translation control from the form norm)**.** *Fix $M > 0$ and define*

$$\mathcal{K}_M := \{\phi \in H_I : \ \|\phi\|_2^2 + \mathcal{E}_\lambda^{\mathbb{R}}(\phi) \le M\}.$$

Then $\mathcal{K}_M$ satisfies the translation equicontinuity condition (ii) in Theorem 19.

Proof. Let $\phi \in \mathcal{K}_M$ and $h \in \mathbb{R}$ with $|h| \le 1$. By Plancherel,

$$\|\phi - S_h\phi\|_2^2 = \frac{1}{2\pi} \int_{\mathbb{R}} 4\sin^2\left(\frac{\xi h}{2}\right) |\widehat{\phi}(\xi)|^2 \, d\xi.$$

Fix $R \ge 1$ and split the integral into $|\xi| \le R$ and $|\xi| > R$. Using $\sin^2(x) \le x^2$,

$$\int_{|\xi|\le R} 4\sin^2\left(\frac{\xi h}{2}\right) |\widehat{\phi}(\xi)|^2 \, d\xi \le \int_{|\xi|\le R} (\xi h)^2 \, |\widehat{\phi}(\xi)|^2 \, d\xi \le (Rh)^2 \int_{\mathbb{R}} |\widehat{\phi}(\xi)|^2 \, d\xi = (Rh)^2 (2\pi) \|\phi\|_2^2.$$

Also $\sin^2 \le 1$ gives

$$\int_{|\xi|>R} 4\sin^2\left(\frac{\xi h}{2}\right) |\widehat{\phi}(\xi)|^2 \, d\xi \le 4 \int_{|\xi|>R} |\widehat{\phi}(\xi)|^2 \, d\xi \le \frac{4}{\log(2+R)} \int_{\mathbb{R}} \log(2 + |\xi|)\, |\widehat{\phi}(\xi)|^2 \, d\xi.$$

By Corollary 18, the last integral is $\le C(M, L)$ uniformly over $\phi \in \mathcal{K}_M$. Therefore

$$\|\phi - S_h\phi\|_2^2 \le (Rh)^2 \, M + \frac{C'(M, L)}{\log(2+R)}.$$

Given $\varepsilon > 0$, choose R so that $C'(M, L)/\log(2 + R) \le \varepsilon^2/2$, and then choose $\delta > 0$ so that $(R\delta)^2 M \le \varepsilon^2/2$. This gives $\|\phi - S_h\phi\|_2 < \varepsilon$ for all $\phi \in \mathcal{K}_M$ and $|h| < \delta$. $\square$

Proposition 22 (Compact embedding of the form domain)**.** *The embedding* $(\mathcal{D}(\mathcal{E}_\lambda), \|G\|_{\mathcal{D}}^2 := \|G\|_2^2 + \mathcal{E}_\lambda(G)) \hookrightarrow L^2(I)$ *is compact.*

Proof. Let $\{G_n\} \subset \mathcal{D}(\mathcal{E}_\lambda)$ be bounded in the form norm: $\|G_n\|_2^2 + \mathcal{E}_\lambda(G_n) \le M$. Put $\phi_n := \widetilde{G}_n \in H_I$. Then $\|\phi_n\|_2^2 + \mathcal{E}_\lambda^{\mathbb{R}}(\phi_n) \le M$, so $\phi_n \in \mathcal{K}_M$.

Since each ϕ_n is supported in the fixed bounded set $\overline{I}$, tightness (i) in Theorem 19 holds automatically. Translation equicontinuity (ii) holds by Lemma 21. Thus $\{\phi_n\}$ is relatively compact in $L^2(\mathbb{R})$ by Theorem 19; hence $\{G_n\}$ is relatively compact in $L^2(I)$. □

Theorem 23 (Closed form, associated operator, and compact resolvent)**.** *There exists a unique selfadjoint operator* $A_\lambda \ge 0$ *on* $L^2(I)$ *associated to the closed form* $\mathcal{E}_\lambda$ *(Proposition 15) in the sense of the representation theorem for closed forms. Moreover,* A_λ *has compact resolvent; equivalently,* $(A_\lambda + 1)^{-1}$ *is compact on* $L^2(I)$.

Proof. Existence and uniqueness of A_λ follow from the representation theorem for densely defined, closed, lower-bounded symmetric forms (see, e.g., Kato, *Perturbation Theory for Linear Operators*). To prove compact resolvent, let $\{f_n\}$ be bounded in $L^2(I)$ and set $u_n := (A_\lambda + 1)^{-1} f_n$. Then $u_n \in \mathcal{D}(A_\lambda) \subset \mathcal{D}(\mathcal{E}_\lambda)$ and $(A_\lambda + 1)u_n = f_n$. Taking the L^2 inner product with u_n and using the form identity gives

$$\mathcal{E}_\lambda(u_n) + \|u_n\|_2^2 = \langle f_n, u_n \rangle \le \|f_n\|_2 \, \|u_n\|_2.$$

Hence $\|u_n\|_2 \le \|f_n\|_2$, and therefore $\|u_n\|_2^2 + \mathcal{E}_\lambda(u_n) \le \|f_n\|_2^2$. Thus $\{u_n\}$ is bounded in the form norm, so by Proposition 22 it has a convergent subsequence in $L^2(I)$. This proves $(A_\lambda + 1)^{-1}$ is compact. □

7.3 Semigroup and irreducibility

Definition 24 (Irreducibility for semigroups on $L^2(I)$). A closed ideal in $L^2(I)$ has the form $L^2(B)$ for some measurable $B \subset I$. We call T *irreducible* if the only invariant closed ideals are $\{0\}$ and $L^2(I)$.

Lemma 25 (Invariant ideals and splitting of the form)**.** *Assume Theorem 23. Let* $B \subset I$ *be measurable and suppose the closed ideal* $L^2(B) \subset L^2(I)$ *is invariant under the semigroup* $T(t) = e^{-tA_\lambda}$*. Then for every* $G \in \mathcal{D}(\mathcal{E}_\lambda)$ *one has* $\mathbf{1}_B G, \mathbf{1}_{I \setminus B} G \in \mathcal{D}(\mathcal{E}_\lambda)$ *and*

$$\mathcal{E}_\lambda(G) = \mathcal{E}_\lambda(\mathbf{1}_B G) + \mathcal{E}_\lambda(\mathbf{1}_{I \setminus B} G).$$

This is standard for symmetric Dirichlet forms; see, e.g., Fukushima–Oshima–Takeda [3, Thm. 1.6.1]. We include a short proof for completeness.

Proof. Let $P : L^2(I) \to L^2(I)$ be the orthogonal projection $PG = \mathbf{1}_B G$, and set $Q = I - P$. Invariance of $L^2(B) = \mathrm{Ran}(P)$ implies $PT(t)P = T(t)P$ for all $t \ge 0$. Taking adjoints and using that $T(t)$ and P are selfadjoint gives $T(t)P = PT(t)$ for all $t \ge 0$. Hence P commutes with the functional calculus of A_λ, in particular with $A_\lambda^{1/2}$. Since $\mathcal{D}(\mathcal{E}_\lambda) = \mathcal{D}(A_\lambda^{1/2})$, it follows that $P\mathcal{D}(\mathcal{E}_\lambda) \subset \mathcal{D}(\mathcal{E}_\lambda)$ and likewise for Q.

For $u \in \mathcal{D}(\mathcal{E}_\lambda)$,

$$\mathcal{E}_\lambda(u) = \|A_\lambda^{1/2} u\|_2^2 = \|A_\lambda^{1/2} Pu\|_2^2 + \|A_\lambda^{1/2} Qu\|_2^2 = \mathcal{E}_\lambda(Pu) + \mathcal{E}_\lambda(Qu),$$

because $A_\lambda^{1/2} Pu = P(A_\lambda^{1/2} u)$ and $A_\lambda^{1/2} Qu = Q(A_\lambda^{1/2} u)$ are orthogonal in $L^2(I)$. Taking $u = G$ gives the claimed splitting. □

Proposition 26 (Triviality of invariant ideals for $\mathcal{E}_\lambda$)**.** *Assume Theorem 23. Let $B \subset I$ be measurable and assume that the closed ideal $L^2(B) \subset L^2(I)$ is invariant under the semigroup $T(t) = e^{-tA_\lambda}$. Then $m(B) = 0$ or $m(I \setminus B) = 0$.*

Proof. By Lemma 25 (cf. [3, Thm. 1.6.1]), invariance of the ideal $L^2(B)$ implies that for every $G \in \mathcal{D}(\mathcal{E}_\lambda)$ one has $\mathbf{1}_B G, \mathbf{1}_{I\setminus B} G \in \mathcal{D}(\mathcal{E}_\lambda)$ and

$$\mathcal{E}_\lambda(G) = \mathcal{E}_\lambda(\mathbf{1}_B G) + \mathcal{E}_\lambda(\mathbf{1}_{I\setminus B} G). \tag{7}$$

We apply (7) with $G \equiv 1$ on I. Note that $1 \in \mathcal{D}(\mathcal{E}_\lambda)$: indeed, for each shift size $s > 0$ one has $\|\widetilde{1} - S_s\widetilde{1}\|_2^2 = m(I\Delta(I+s)) = 2s$, so the archimedean integral $\int_0^{2L} w(t)\, 2t\, dt$ is finite (cf. the discussion after Lemma 5), and the prime sum in Definition 6 is finite because it contains only finitely many shift sizes.

Write $B^c := I \setminus B$ and set $f := \widetilde{\mathbf{1}_B}$ and $g := \widetilde{\mathbf{1}_{B^c}}$ in $L^2(\mathbb{R})$, so that $\widetilde{1} = f + g$ and $fg = 0$ a.e. For each shift size $s > 0$,

$$\|(f+g) - S_s(f+g)\|_2^2 = \|f - S_s f\|_2^2 + \|g - S_s g\|_2^2 + 2\langle f - S_s f,\ g - S_s g\rangle.$$

Using this identity in Definition 6 and (7) (with $G \equiv 1$) gives

$$\int_0^{2L} w(t)\,\langle f - S_t f,\ g - S_t g\rangle\, dt + \sum_{\substack{p \text{ prime} \\ p \le \lambda^2}} \sum_{\substack{m \ge 1 \\ p^m \le \lambda^2}} (\log p) p^{-m/2}\,\langle f - S_{m\log p} f,\ g - S_{m \log p} g\rangle = 0.$$

For any $s > 0$, since $fg = 0$ a.e. and translations are unitary on $L^2(\mathbb{R})$,

$$\begin{aligned}\langle f - S_s f,\ g - S_s g\rangle &= \langle f, g\rangle - \langle f, S_s g\rangle - \langle S_s f, g\rangle + \langle S_s f, S_s g\rangle \\ &= -\langle f, S_s g\rangle - \langle g, S_s f\rangle \ \le\ 0,\end{aligned}$$

because the inner products are nonnegative. Since all weights in the preceding display are nonnegative and $w(t) > 0$ for all $t > 0$, it follows that

$$\langle f, S_t g\rangle = \langle g, S_t f\rangle = 0 \quad \text{for a.e. } t \in (0, 2L).$$

For fixed $f, g \in L^2(\mathbb{R})$, the map $t \mapsto \langle f, S_t g\rangle$ is continuous (strong continuity of translations), hence the equalities hold for *all* $t \in (0, 2L)$.

Unwinding the definitions, for each $t \in (0, 2L)$,

$$0 = \langle f, S_t g\rangle = \int_{\mathbb{R}} \widetilde{\mathbf{1}_B}(u)\,\widetilde{\mathbf{1}_{B^c}}(u-t)\, du = \int_{I\cap(I+t)} \mathbf{1}_B(u)\,\mathbf{1}_{B^c}(u-t)\, du,$$

so $\mathbf{1}_B(u) \le \mathbf{1}_B(u-t)$ for a.e. $u \in I \cap (I+t)$. The same argument with $\langle g, S_t f\rangle = 0$ gives the reverse inequality, hence

$$\mathbf{1}_B(u) = \mathbf{1}_B(u-t) \quad \text{for a.e. } u \in I \cap (I+t) \ \text{ and every } t \in (0, 2L).$$

Lemma 10 now applies and yields $m(B) = 0$ or $m(I \setminus B) = 0$. ☐

Remark 27 (Why we do not use $\mathcal{E}_\lambda(\mathbf{1}_B) = 0$). Because $\mathcal{E}_\lambda$ is defined using zero-extension to $\mathbb{R}$ (Definition 6), the form is typically non-conservative: in general $\mathcal{E}_\lambda(1) > 0$. In the conservative case ($\mathcal{E}(1) = 0$) one often has an equivalence between invariance and the condition $\mathcal{E}(\mathbf{1}_B) = 0$. Here, the presence of killing means $\mathcal{E}_\lambda(\mathbf{1}_B) = 0$ is a *stronger* condition than invariance, so we instead argue directly from the correct invariance identity (7), which depends only on the interaction/jump part.

Corollary 28 (Irreducibility for $\mathcal{E}_\lambda$)**.** *Assume Theorem 23. Then $T(t) = e^{-tA_\lambda}$ is irreducible.*

Proof. If $J \subset L^2(I)$ is a closed $T(t)$-invariant ideal, then $J = L^2(B)$ for some measurable $B \subset I$. Proposition 26 forces $m(B) = 0$ or $m(I \setminus B) = 0$, hence $J = \{0\}$ or $J = L^2(I)$. ☐

8 Positivity improving and the ground state

8.1 External theorems used

Theorem 29 (Positivity improving from positivity + irreducibility + holomorphy)**.** *Let E be a Banach lattice and S a positive, irreducible, holomorphic C_0-semigroup on E. Then S is positivity improving: for each $t > 0$ and each $0 \leq f \in E$ with $f \neq 0$, one has $S(t)f > 0$ (in the lattice sense; on L^2 this means > 0 a.e.).*

Remark 30 (Source)*.* This statement appears, for example, as Theorem 2.3 in Arendt et al., *Strict positivity for the principal eigenfunction of elliptic operators with various boundary conditions* (see §11).

Theorem 31 (Simplicity of the principal eigenvalue under compact resolvent)**.** *Let A be selfadjoint and lower bounded on $L^2(I)$ with compact resolvent, and let $S(t) = e^{-tA}$. If S is positivity improving, then the bottom of the spectrum* $\min \sigma(A)$ *is a simple eigenvalue and admits an eigenfunction which is strictly positive a.e.*

Remark 32 (Source)*.* This is a standard Perron–Frobenius/Krein–Rutman/Jentzsch consequence for compact positive operators, often stated for $(A+\mu)^{-1}$ or for $S(t)$ when it is compact. See, e.g., Proposition 2.4 in the same paper of Arendt et al.

8.2 Application to A_λ

Proposition 33 (Positivity improving and simple ground state for A_λ)**.** *Assume Theorem 23. Then:*

1. *The semigroup $T(t) = e^{-tA_\lambda}$ is positivity preserving (Markovian).*
2. *$T(t)$ is irreducible.*
3. *$T(t)$ is holomorphic (indeed, A_λ is selfadjoint and lower bounded).*

Consequently $T(t)$ is positivity improving, and the lowest eigenvalue of A_λ is simple with a strictly positive a.e. eigenfunction.

Proof. (1) Markov/positivity preservation follows from Lemma 9 and standard closed-form theory. (2) is Corollary 28. (3) Since A_λ is selfadjoint and lower bounded, e^{-zA_λ} is bounded and holomorphic on $\{z \in \mathbb{C} : \Re z > 0\}$ by the spectral theorem.

Now apply Theorem 29 to obtain positivity improving, and then Theorem 31 to obtain simplicity and strict positivity of the ground state. □

9 Evenness of the ground state from inversion symmetry

Proposition 34 (Inversion (reflection) symmetry)**.** *Let $R : L^2(I) \to L^2(I)$ be the unitary involution $(RG)(u) := G(-u)$. Then $R(\mathcal{D}(\mathcal{E}_\lambda)) = \mathcal{D}(\mathcal{E}_\lambda)$ and*

$$\mathcal{E}_\lambda(RG) = \mathcal{E}_\lambda(G) \qquad (G \in \mathcal{D}(\mathcal{E}_\lambda)).$$

Consequently, the associated operator A_λ from Theorem 23 commutes with R.

Proof. Identify $L^2(I)$ with the closed subspace $H_I \subset L^2(\mathbb{R})$ via extension by 0. Let the same symbol R denote reflection on $L^2(\mathbb{R})$: $(R\phi)(u) := \phi(-u)$. Then R is unitary, preserves H_I (since I is symmetric), and satisfies $RS_t = S_{-t}R$. Therefore, for $t \in \mathbb{R}$ and $\phi \in L^2(\mathbb{R})$,

$$\|R\phi - S_t R\phi\|_2 = \|R(\phi - S_{-t}\phi)\|_2 = \|\phi - S_{-t}\phi\|_2 = \|\phi - S_t\phi\|_2,$$

using that R is unitary and $\|\phi - S_{-t}\phi\|_2 = \|S_t\phi - \phi\|_2 = \|\phi - S_t\phi\|_2$. Since every weight in Definition 6 is nonnegative, this implies $\mathcal{E}_\lambda(RG) = \mathcal{E}_\lambda(G)$.

For commutation with A_λ: invariance of a closed form under a unitary U implies that the associated selfadjoint operator commutes with U. Indeed, for $u \in \mathcal{D}(A_\lambda)$ and $v \in \mathcal{D}(\mathcal{E}_\lambda)$,

$$\langle A_\lambda Ru, v\rangle = \mathcal{E}_\lambda(Ru, v) = \mathcal{E}_\lambda(u, R^{-1}v) = \langle A_\lambda u, R^{-1}v\rangle = \langle RA_\lambda u, v\rangle,$$

so $A_\lambda Ru = RA_\lambda u$. □

Corollary 35 (Even ground state)**.** *Assume Theorem 23 and 34. Let ψ be the strictly positive ground-state eigenfunction from Proposition 33. Then ψ is even: $\psi(-u) = \psi(u)$ a.e.*

Proof. Since $A_\lambda R = RA_\lambda$, the function $\psi^\sharp := R\psi$ is an eigenfunction for the same lowest eigenvalue. Moreover $\psi^\sharp > 0$ a.e. because $\psi > 0$ a.e. By simplicity of the ground-state eigenspace (Proposition 33), $\psi^\sharp = c\psi$ for some $c \in \mathbb{R}$. Positivity forces $c > 0$, and normalizing $\|\psi^\sharp\|_2 = \|\psi\|_2$ yields $c = 1$. Hence $\psi(-u) = \psi(u)$ a.e. □

10 Summary of concrete progress

- Starting solely from the explicit local formulas (2)–(3), we derived a representation of $-\sum_v W_v(g * g^*)$ (up to an additive constant multiple of $\|g\|_2^2$) as a positive combination of translation-difference energies in log-coordinates (Definition 6, Lemmas 4–5).
- We proved the Markov/normal contraction inequality for this form (Lemma 9).
- Using only measure theory (Lebesgue density), we proved that invariance under all sufficiently small translations forces a measurable subset of an interval to be null or conull (Lemma 10), and we used it to show that $\mathcal{E}_\lambda(\mathbf{1}_B) = 0$ implies B is null or conull (Lemma 11).
- Assuming the standard operator setup (closed form, selfadjoint operator, compact resolvent), we obtained irreducibility and then (by a standard external theorem) positivity improving of the semigroup, hence simplicity and strict positivity of the ground state (Proposition 33).
- Finally, inversion symmetry forces that strictly positive simple ground state to be even (Corollary 35).

11 Bibliographic pointers

References

[1] W. Arendt, A. F. M. ter Elst, and J. Glück. *Strict positivity for the principal eigenfunction of elliptic operators with various boundary conditions. Advanced Nonlinear Studies* **20** (2020), no. 3, 633–650. arXiv:1909.12194; DOI:10.1515/ans-2020-2091. See Theorem 2.3 and Proposition 2.4 for the positivity improving and simplicity results used in §8.

[2] W. Arendt, C. J. K. Batty, M. Hieber, and F. Neubrander. *Vector-valued Laplace Transforms and Cauchy Problems.* 2nd ed., Monographs in Mathematics 96, Birkhäuser/Springer Basel, 2011. DOI:10.1007/978-3-0348-0087-7.

[3] M. Fukushima, Y. Oshima, and M. Takeda. *Dirichlet Forms and Symmetric Markov Processes.* 2nd revised and extended ed., De Gruyter Studies in Mathematics 19, Walter de Gruyter, 2011. DOI:10.1515/9783110889741. See Theorem 1.6.1 for the invariant set/ideal characterization used in the irreducibility argument.