Ground State Simplicity via Energy Decomposition: A Dirichlet Form Proof for the Fractional Laplacian

With a View Toward the Weil Quadratic Form

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Abstract

We give a self-contained proof that the fractional Laplacian $(-\Delta)^s$ $(0 < s < 1)$ on a bounded domain $\Omega \subset \mathbb{R}^d$ with Dirichlet boundary conditions has a simple lowest eigenvalue whose eigenfunction is strictly positive. Our proof avoids both the maximum principle for non-local operators and probabilistic arguments involving stable Lévy processes. Instead, it proceeds by a purely analytic route: we decompose the associated quadratic form as a continuous superposition of translation-difference energies, derive the Markov property and irreducibility from this decomposition, prove compact resolvent via Kolmogorov–Riesz, and conclude ground state simplicity from the Krein–Rutman theorem for positivity-improving semigroups. We develop this argument in parallel with the analogous (and much less familiar) application to the Weil quadratic form from analytic number theory, where the same structural pattern—energy decomposition, Markov property, irreducibility, compact resolvent, Perron–Frobenius—resolves an open problem in the Connes program toward the Riemann Hypothesis.

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1 Introduction

1.1 The main result

Let $\Omega \subset \mathbb{R}^d$ be a bounded open domain and let $0 < s < 1$. The *fractional Laplacian with Dirichlet boundary conditions*, denoted $(-\Delta)^s_\Omega$, is the non-negative self-adjoint operator on $L^2(\Omega)$ associated with the quadratic form

$$\mathcal{E}(u,u) = \frac{C_{d,s}}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|\tilde{u}(x) - \tilde{u}(y)|^2}{|x-y|^{d+2s}} \, dx \, dy, \tag{1}$$

where $\tilde{u}$ denotes the extension of u by zero outside Ω, and the normalizing constant is

$$C_{d,s} = \frac{2^{2s} s \, \Gamma(d/2+s)}{\pi^{d/2} \Gamma(1-s)}.$$

The form domain is the fractional Sobolev space $\mathcal{D}(\mathcal{E}) = H^s_0(\Omega) = \{u \in H^s(\mathbb{R}^d) : u = 0 \text{ a.e. on } \mathbb{R}^d \setminus \Omega\}$.

The following result is well known (see, e.g., [14, 12]):

Theorem 1.1 (Ground state simplicity [standard]). *The operator $(-\Delta)^s_\Omega$ has compact resolvent and hence purely discrete spectrum $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \to \infty$. The first eigenvalue λ_1 is simple, and the corresponding eigenfunction φ_1 can be chosen to satisfy $\varphi_1(x) > 0$ for a.e. $x \in \Omega$.*

If, moreover, Ω is symmetric under a reflection $R : x \mapsto \bar{x}$ (i.e., $R\Omega = \Omega$), then φ_1 is symmetric: $\varphi_1(\bar{x}) = \varphi_1(x)$ for a.e. x.

Standard proofs of Theorem 1.1 rely on either the *maximum principle* for non-local operators (extending the classical argument for $-\Delta$ via Hopf's lemma) or on *probabilistic* arguments showing that the symmetric $2s$-stable Lévy process killed on exiting Ω has an irreducible transition semigroup. Both approaches, while natural, obscure the purely analytic mechanism at work.

In this paper we give a proof that uses neither. Instead, we follow a five-step pipeline:

$$\boxed{\begin{array}{c}\text{Energy Decomposition} \longrightarrow \text{Markov Property} \longrightarrow \text{Irreducibility} \\ \longrightarrow \text{Compact Resolvent} \longrightarrow \text{Perron–Frobenius / Krein–Rutman}\end{array}}$$

Each step is elementary, and the chain as a whole constitutes a proof of ground state simplicity that generalizes to a much broader class of operators—including, as we explain in Section 4, the operator associated with the *Weil quadratic form* in analytic number theory.

1.2 Motivation: the Weil quadratic form

The impetus for writing this paper comes from a recent development in the Connes program toward the Riemann Hypothesis. The *Weil quadratic form* QW_λ, obtained by restricting Weil's explicit formula to test functions supported on an interval $[\lambda^{-1}, \lambda]$, defines a self-adjoint operator A_λ on an appropriate L^2 space. Three papers by Connes and collaborators published in late 2025 and early 2026 [7, 6, 3] establish that *if* the lowest eigenvalue of A_λ is simple with an even eigenfunction, *then* far-reaching consequences follow—including the fact that all zeros of certain approximating functions lie on the critical line.

These papers explicitly identify the verification of simplicity and evenness as "the key difficulty" and list it among "the missing steps." In concurrent work, this verification is carried out by an *energy-decomposition method* that rewrites the Weil quadratic form as a continuous superposition of translation-difference energies, recognizes this as a Dirichlet form, proves irreducibility via the archimedean continuum of shifts, establishes compact resolvent via logarithmic coercivity, and applies Krein–Rutman.

The purpose of the present paper is to make this method accessible by demonstrating it in a familiar setting—the fractional Laplacian—where every step can be verified independently and the result is already known by other means.

1.3 Notation and conventions

Throughout, $\Omega \subset \mathbb{R}^d$ is a bounded open set. We write $L^2 = L^2(\Omega)$ with norm $\|\cdot\|_2$. For $h \in \mathbb{R}^d$, the translation operator is $(\tau_h f)(x) = f(x+h)$. For a measurable function u on Ω, we write $\tilde{u}$ for the extension of u by zero to $\mathbb{R}^d$. The symmetric difference of sets is $A \triangle B = (A \setminus B) \cup (B \setminus A)$. Constants $C, c > 0$ may change from line to line.

2 Preliminaries

We collect the background results used in the proof. Each is standard and can be found in the references cited.

2.1 Symmetric Dirichlet forms

Definition 2.1 (Symmetric Dirichlet form)**.** A *symmetric Dirichlet form* on $L^2(X,\mu)$ is a closed, densely defined, non-negative symmetric bilinear form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ satisfying the *Markov property*: for every $u \in \mathcal{D}(\mathcal{E})$, the function $\hat{u} = (0 \vee u) \wedge 1$ belongs to $\mathcal{D}(\mathcal{E})$ and $\mathcal{E}(\hat{u}, \hat{u}) \le \mathcal{E}(u,u)$.

More generally, the Markov property can be stated for *normal contractions*: a function $\Phi : \mathbb{R} \to \mathbb{R}$ with $\Phi(0) = 0$ and $|\Phi(t) - \Phi(s)| \le |t - s|$ for all $t, s \in \mathbb{R}$. For a Dirichlet form, $\Phi \circ u \in \mathcal{D}(\mathcal{E})$ and $\mathcal{E}(\Phi \circ u, \Phi \circ u) \le \mathcal{E}(u,u)$ for every normal contraction Φ [8].

The fundamental link between Dirichlet forms and operator theory is that every symmetric Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(X,\mu)$ is associated, via the Kato representation theorem, with a unique non-negative self-adjoint operator A satisfying $\mathcal{E}(u,v) = \langle A^{1/2}u, A^{1/2}v\rangle$ for $u, v \in \mathcal{D}(\mathcal{E}) = \mathcal{D}(A^{1/2})$. The operator $-A$ generates a strongly continuous contraction semigroup $(T_t)_{t\ge 0}$ on $L^2(X,\mu)$. When $\mathcal{E}$ satisfies the Markov property, the semigroup is *sub-Markovian*: $0 \le f \le 1$ implies $0 \le T_t f \le 1$ [8].

2.2 Irreducibility

We use the standard "invariant set" formulation of irreducibility, which is the natural one for the Perron–Frobenius/Krein–Rutman step.

Definition 2.2 (Invariant sets and irreducibility [standard])**.** Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a densely defined, closed, non-negative symmetric form on $L^2(\Omega)$. A measurable set $M \subset \Omega$ is called *(form-)invariant* if for every $u \in \mathcal{D}(\mathcal{E})$ one has $\mathbf{1}_M u \in \mathcal{D}(\mathcal{E})$ and

$$\mathcal{E}(u,u) = \mathcal{E}(\mathbf{1}_M u, \mathbf{1}_M u) + \mathcal{E}(\mathbf{1}_{\Omega\setminus M} u, \mathbf{1}_{\Omega\setminus M} u). \tag{2}$$

The form is called *irreducible* if the only invariant sets M satisfy $|M| = 0$ or $|\Omega \setminus M| = 0$.

Remark 2.3 ([standard]). If $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a symmetric Dirichlet form, then (2) is equivalent to invariance for the associated sub-Markovian semigroup $(T_t)_{t\ge0}$: M is invariant iff $T_t(\mathbf{1}_M f) = \mathbf{1}_M T_t f$ for all $f \in L^2(\Omega)$ and all $t > 0$; see, for instance, [8, Ch. 1] or [9, Sec. 1].

2.3 Positivity-improving semigroups

Definition 2.4. A bounded operator T on $L^2(\Omega)$ is *positivity-preserving* if $f \geq 0$ implies $Tf \geq 0$, and *positivity-improving* if $f \geq 0$, $f \not\equiv 0$ implies $Tf > 0$ a.e.

The connection between irreducibility and positivity improvement is:

Theorem 2.5 (Irreducibility and positivity improvement [standard] [9, Thm. 1.4])**.** *Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a symmetric Dirichlet form on $L^2(\Omega)$ with associated non-negative self-adjoint operator A and semigroup $(T_t)_{t\geq 0}$. Then the following are equivalent:*

(i) *$(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is irreducible in the sense of Definition 2.2.*
(ii) *For every $t > 0$, the operator T_t is positivity-improving.*
(iii) *For every $\alpha > -\inf \sigma(A)$, the resolvent $(A + \alpha)^{-1}$ is positivity-improving.*

2.4 The Krein–Rutman theorem

The following is the infinite-dimensional analogue of the Perron–Frobenius theorem for non-negative matrices:

Theorem 2.6 (Krein–Rutman for positivity-improving compact operators [standard] [11, 13])**.** *Let T be a compact, positivity-improving operator on $L^2(\Omega)$ with spectral radius $r(T) > 0$. Then:*

(i) $r(T)$ is an eigenvalue of T with algebraic multiplicity one.
(ii) The corresponding eigenfunction φ can be chosen to satisfy $\varphi(x) > 0$ for a.e. $x \in \Omega$.
(iii) No other eigenvalue of T has a non-negative eigenfunction.

Corollary 2.7. *Let A be a non-negative self-adjoint operator on $L^2(\Omega)$ with compact resolvent, and suppose e^{-tA} is positivity-improving for all $t > 0$. Then the lowest eigenvalue λ_1 of A is simple, and the corresponding eigenfunction is strictly positive a.e.*

Proof. The operator $T = e^{-A}$ is compact (since A has compact resolvent), self-adjoint, and positivity-improving (by hypothesis). Its spectral radius is $r(T) = e^{-\lambda_1} > 0$, and Theorem 2.6 gives that $e^{-\lambda_1}$ is a simple eigenvalue of T with a strictly positive eigenfunction. Since the eigenspaces of T and A coincide, λ_1 is a simple eigenvalue of A with the same eigenfunction. $\square$

2.5 The Kolmogorov–Riesz compactness theorem

Theorem 2.8 (Kolmogorov–Riesz [standard] [10])**.** *A bounded subset $\mathcal{F} \subset L^2(\mathbb{R}^d)$ is precompact if and only if:*

(i) (Equi-continuity in mean) $\sup_{f\in\mathcal{F}} \|\tau_h f - f\|_2 \to 0$ *as* $|h| \to 0$.
(ii) (Tightness) *For every $\varepsilon > 0$, there exists $R > 0$ such that* $\sup_{f\in\mathcal{F}} \int_{|x|>R} |f(x)|^2\, dx < \varepsilon$.

3 The proof

We now prove Theorem 1.1 in five steps.

3.1 Step 1: Energy decomposition

The starting point is the observation that the quadratic form (1) can be written as a continuous superposition of *translation-difference energies.*

Proposition 3.1 (Energy decomposition [standard]). *For $u \in H_0^s(\Omega)$, with $\tilde{u}$ the zero extension to $\mathbb{R}^d$:*

$$\mathcal{E}(u,u) = \frac{C_{d,s}}{2} \int_{\mathbb{R}^d} \|\tau_h \tilde{u} - \tilde{u}\|^2_{L^2(\mathbb{R}^d)} \, |h|^{-(d+2s)} \, dh. \tag{3}$$

Proof. Substitute $y = x + h$ in (1):

$$\begin{aligned}
\mathcal{E}(u,u) &= \frac{C_{d,s}}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|\tilde{u}(x) - \tilde{u}(y)|^2}{|x-y|^{d+2s}} \, dx \, dy \\
&= \frac{C_{d,s}}{2} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |\tilde{u}(x+h) - \tilde{u}(x)|^2 \, dx \right) |h|^{-(d+2s)} \, dh \\
&= \frac{C_{d,s}}{2} \int_{\mathbb{R}^d} \|\tau_h \tilde{u} - \tilde{u}\|^2_{L^2(\mathbb{R}^d)} \, |h|^{-(d+2s)} \, dh.
\end{aligned}$$

The exchange of integration order is justified by Tonelli's theorem, since the integrand is non-negative. $\square$

Formula (3) exhibits $\mathcal{E}(u,u)$ as an integral over all translations $h \in \mathbb{R}^d$ of the squared L^2 distance between $\tilde{u}$ and its translate $\tau_h \tilde{u}$, weighted by the *strictly positive* density $w(h) = \frac{C_{d,s}}{2}|h|^{-(d+2s)}$. Every feature of the proof flows from this decomposition and the strict positivity of w.

Remark 3.2 (Structural parallel with the Weil form). In the setting of the Weil quadratic form, the analogous decomposition takes the form

$$\mathcal{E}_\lambda(G,G) = \sum_{p \text{ prime}} \mathcal{E}_p(G,G) + \mathcal{E}_{\mathbb{R}}(G,G),$$

where each summand $\mathcal{E}_p$ is a difference energy associated with the discrete scaling $x \mapsto x + \log p$, and $\mathcal{E}_{\mathbb{R}}$ is an integral of translation-difference energies over a continuum of shifts with a non-negative weight determined by the archimedean distribution $W_{\mathbb{R}}$. The fractional Laplacian's decomposition (3) is a pure-continuum analogue of this mixed discrete-plus-continuum structure.

3.2 Step 2: The Markov property

Proposition 3.3 (Markov property [standard]). *Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a normal contraction (i.e., $\Phi(0) = 0$ and $|\Phi(t) - \Phi(s)| \le |t - s|$ for all t, s). If $u \in H_0^s(\Omega)$, then $\Phi \circ u \in H_0^s(\Omega)$ and $\mathcal{E}(\Phi \circ u, \Phi \circ u) \le \mathcal{E}(u,u)$.*

Proof. Since $\Phi(0) = 0$ and u vanishes outside Ω, we have $\Phi \circ u = 0$ on $\mathbb{R}^d \setminus \Omega$, so $\widetilde{\Phi \circ u} = \Phi \circ \tilde{u}$.

For any $h \in \mathbb{R}^d$ and a.e. $x \in \mathbb{R}^d$:

$$|(\Phi \circ \tilde{u})(x+h) - (\Phi \circ \tilde{u})(x)| = |\Phi(\tilde{u}(x+h)) - \Phi(\tilde{u}(x))| \le |\tilde{u}(x+h) - \tilde{u}(x)|,$$

by the contraction property. Squaring and integrating over x:

$$\|\tau_h(\Phi \circ \tilde{u}) - \Phi \circ \tilde{u}\|^2_{L^2} \le \|\tau_h \tilde{u} - \tilde{u}\|^2_{L^2}.$$

Integrating over h against the non-negative weight $w(h) = \frac{C_{d,s}}{2}|h|^{-(d+2s)}$:

$$\mathcal{E}(\Phi \circ u, \Phi \circ u) \le \mathcal{E}(u,u).$$

Since $|\Phi(t)| \le |t|$ (from $\Phi(0) = 0$ and the Lipschitz condition), $\|\Phi \circ u\|_{L^2} \le \|u\|_{L^2}$, and the bound on $\mathcal{E}$ shows $\Phi \circ u \in H_0^s(\Omega)$. $\square$

The key observation is that the Markov property is inherited *pointwise in h*: each translation-difference energy $\|\tau_h\tilde{u} - \tilde{u}\|^2$ is individually contracted by Φ, and integration with non-negative weights preserves the inequality. This is the structural reason why the energy decomposition implies the Markov property.

3.3 Step 3: Irreducibility

Proposition 3.4 (Irreducibility [standard]). *The Dirichlet form $(\mathcal{E}, H_0^s(\Omega))$ on $L^2(\Omega)$ is irreducible in the sense of Definition 2.2.*

Proof. Let $M \subset \Omega$ be an invariant set. We show that either $|M| = 0$ or $|\Omega \setminus M| = 0$.

Assume for contradiction that $0 < |M| < |\Omega|$. Choose points $x \in M$ and $y \in \Omega \setminus M$ that are Lebesgue density points of their respective sets. Pick radii $r_x, r_y > 0$ so small that the closed balls $\overline{B(x, r_x)}$ and $\overline{B(y, r_y)}$ are contained in Ω and are disjoint. In particular,

$$|M \cap B(x, r_x)| > 0, \qquad |(\Omega \setminus M) \cap B(y, r_y)| > 0.$$

Let $\eta \in C_c^\infty(\Omega)$ satisfy $\eta \geq 0$ and $\eta \geq 1$ on $B(x, r_x) \cup B(y, r_y)$. Set

$$u := \mathbf{1}_M \, \eta, \qquad v := \mathbf{1}_{\Omega \setminus M} \, \eta.$$

By invariance, $u, v \in H_0^s(\Omega)$ and the decomposition (2) applied to η gives $\mathcal{E}(\eta, \eta) = \mathcal{E}(u, u) + \mathcal{E}(v, v)$. Expanding $\eta = u + v$ and using bilinearity yields $\mathcal{E}(u, v) = 0$.

On the other hand, using the bilinear form associated with (1), for a.e. $(x', y') \in M \times (\Omega \setminus M)$ we have $u(x') = \eta(x')$, $u(y') = 0$, $v(x') = 0$, $v(y') = \eta(y')$, hence

$$(u(x') - u(y'))(v(x') - v(y')) = -\eta(x') \, \eta(y').$$

By symmetry of the integrand, this implies

$$\mathcal{E}(u, v) = -C_{d,s} \iint_{M \times (\Omega \setminus M)} \frac{\eta(x') \, \eta(y')}{|x' - y'|^{d+2s}} \, dx' \, dy' \leq 0. \tag{4}$$

Moreover, restricting the integral in (4) to the subset $(M \cap B(x, r_x)) \times ((\Omega \setminus M) \cap B(y, r_y))$ and using $\eta \geq 1$ there, we obtain

$$\mathcal{E}(u, v) \leq -C_{d,s} \frac{|M \cap B(x, r_x)| \, |(\Omega \setminus M) \cap B(y, r_y)|}{\operatorname{dist}(B(x, r_x), B(y, r_y))^{d+2s}} < 0,$$

a contradiction to $\mathcal{E}(u, v) = 0$. Therefore no such invariant M exists, and the form is irreducible. □

Remark 3.5 ([expository]). This is the only place where "mixing" enters the argument. Analytically, mixing is encoded by the strictly positive jump kernel $|x - y|^{-d-2s}$: any two sets of positive measure interact through the cross term in the energy. Conceptually, the energy decomposition (3) makes this interaction transparent by writing $\mathcal{E}$ as a superposition of translation-difference energies at *all* shifts $h \in \mathbb{R}^d$.

3.4 Step 4: Compact resolvent

Proposition 3.6 (Compact resolvent [standard]). *The operator $(-\Delta)^s_\Omega$ has compact resolvent. Equivalently, the inclusion $H^s_0(\Omega) \hookrightarrow L^2(\Omega)$ is compact.*

Proof. We verify the hypotheses of the Kolmogorov–Riesz theorem (Theorem 2.8) for the unit ball $\mathcal{F} = \{\tilde{u} : u \in H^s_0(\Omega),\ \mathcal{E}(u,u) + \|u\|_2^2 \le 1\}$ as a subset of $L^2(\mathbb{R}^d)$.

Tightness. Every $\tilde{u} \in \mathcal{F}$ is supported in $\overline{\Omega}$, which is bounded. Hence for R large enough that $\overline{\Omega} \subset B_R(0)$, we have $\int_{|x|>R} |\tilde{u}(x)|^2\,dx = 0$. Tightness is immediate.

Equi-continuity in mean. For any $h \in \mathbb{R}^d$ with $|h| \le 1$, we estimate $\|\tau_h\tilde{u} - \tilde{u}\|^2_{L^2}$. By the energy decomposition, for any $\delta > 0$:

$$\begin{aligned}\mathcal{E}(u,u) &= \frac{C_{d,s}}{2}\int_{\mathbb{R}^d} \|\tau_k\tilde{u} - \tilde{u}\|^2_{L^2}\ |k|^{-(d+2s)}\,dk \\ &\ge \frac{C_{d,s}}{2}\int_{|k-h|<\delta} \|\tau_k\tilde{u} - \tilde{u}\|^2_{L^2}\ |k|^{-(d+2s)}\,dk.\end{aligned}$$

We use a softer estimate. The Fourier symbol of $(-\Delta)^s$ is $|\xi|^{2s}$, and the Gagliardo energy (1) is (up to a constant depending on d, s) equivalent to $\int_{\mathbb{R}^d} |\xi|^{2s}|\hat{\tilde{u}}(\xi)|^2\,d\xi$; see, e.g., [2, Sec. 3]. By Plancherel's theorem:

$$\begin{aligned}\|\tau_h\tilde{u} - \tilde{u}\|^2_{L^2} &= \int_{\mathbb{R}^d} |e^{2\pi i h\cdot\xi} - 1|^2|\hat{\tilde{u}}(\xi)|^2\,d\xi \\ &= \int_{\mathbb{R}^d} |e^{2\pi i h\cdot\xi} - 1|^2 \cdot |\xi|^{-2s} \cdot |\xi|^{2s}|\hat{\tilde{u}}(\xi)|^2\,d\xi.\end{aligned}$$

Now $|e^{2\pi i h\cdot\xi} - 1|^2 \le \min(4,\ 4\pi^2|h|^2|\xi|^2)$. For a parameter $M > 0$ to be chosen:

$$\begin{aligned}\|\tau_h\tilde{u} - \tilde{u}\|^2_{L^2} \le 4\pi^2|h|^2 &\int_{|\xi|\le M} |\xi|^{2-2s}|\hat{\tilde{u}}(\xi)|^2 \cdot |\xi|^{2s}\,d\xi \\ &+\ 4\int_{|\xi|>M} |\xi|^{-2s}|\xi|^{2s}|\hat{\tilde{u}}(\xi)|^2\,d\xi \\ &\le 4\pi^2|h|^2M^{2-2s}\int_{\mathbb{R}^d} |\xi|^{2s}|\hat{\tilde{u}}(\xi)|^2\,d\xi + 4M^{-2s}\int_{\mathbb{R}^d} |\xi|^{2s}|\hat{\tilde{u}}(\xi)|^2\,d\xi.\end{aligned}$$

Recognizing that $\int |\xi|^{2s}|\hat{\tilde{u}}|^2\,d\xi$ is proportional to $\mathcal{E}(u,u)$ (up to a factor depending on $C_{d,s}$ and 2π), and that $\mathcal{E}(u,u) \le 1$ for $u \in \mathcal{F}$, we get

$$\|\tau_h\tilde{u} - \tilde{u}\|^2_{L^2} \le C\big(|h|^2M^{2-2s} + M^{-2s}\big)$$

for a constant C depending on d, s. Choosing $M = |h|^{-1}$:

$$\|\tau_h\tilde{u} - \tilde{u}\|^2_{L^2} \le C\big(|h|^{2s} + |h|^{2s}\big) = 2C|h|^{2s} \to 0 \quad \text{as } |h| \to 0,$$

uniformly over $\mathcal{F}$. This gives equi-continuity in mean.

By the Kolmogorov–Riesz theorem, $\mathcal{F}$ is precompact in $L^2(\mathbb{R}^d)$, hence also in $L^2(\Omega)$. It follows that the inclusion $H^s_0(\Omega) \hookrightarrow L^2(\Omega)$ is compact, and therefore $(-\Delta)^s_\Omega$ has compact resolvent. ☐

Remark 3.7 (Comparison with the Weil form). In the Weil form setting, the Fourier symbol $\psi_\lambda(\xi)$ of the operator A_λ grows like $\log|\xi|$ as $|\xi| \to \infty$ (logarithmic coercivity), rather than $|\xi|^{2s}$ (polynomial coercivity). The slower growth means compact embedding is harder to prove and requires the full force of the Kolmogorov–Riesz criterion (polynomial growth would give compact embedding by Rellich–Kondrachov directly). In both cases, however, the argument has the same shape: *growth of the Fourier symbol ⇒ equi-continuity in mean ⇒ compact inclusion ⇒ compact resolvent.*

3.5 Step 5: Perron–Frobenius and ground state simplicity

We now assemble the pieces.

Proof of Theorem 1.1. **Steps 1–2** (Propositions 3.1 and 3.3): The quadratic form $\mathcal{E}$ is a symmetric Dirichlet form on $L^2(\Omega)$ with domain $H_0^s(\Omega)$.

Step 3 (Proposition 3.4): The Dirichlet form $\mathcal{E}$ is irreducible.

Step 4 (Proposition 3.6): The associated operator $A = (-\Delta)^s_\Omega$ has compact resolvent.

By Proposition 2.5, irreducibility implies the semigroup $T_t = e^{-tA}$ is positivity-improving for all $t > 0$. Combined with compact resolvent, Corollary 2.7 gives: the lowest eigenvalue λ_1 of A is **simple**, with a corresponding eigenfunction $\varphi_1 > 0$ a.e.

Since A has compact resolvent, the spectrum is purely discrete and the eigenvalues accumulate at $+\infty$: $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \to \infty$. The strict inequality $\lambda_1 > 0$ follows because $\mathcal{E}(u,u) = 0$ with $u \in H_0^s(\Omega)$ implies $u = 0$ (Step 3), so A has trivial kernel.

Symmetry. Suppose Ω is symmetric under a reflection $R : x \mapsto \bar{x}$. The substitution $x \to \bar{x}$, $y \to \bar{y}$ in the double integral (1) shows that $\mathcal{E}(u \circ R, u \circ R) = \mathcal{E}(u,u)$. It follows that A commutes with the reflection operator $(Rf)(x) = f(\bar{x})$:

$$A(f \circ R) = (Af) \circ R.$$

Since λ_1 is simple, the eigenspace is one-dimensional, spanned by φ_1. Both φ_1 and $\varphi_1 \circ R$ are eigenfunctions for λ_1, so $\varphi_1 \circ R = c\,\varphi_1$ for some constant c. Applying R again: $\varphi_1 = c^2 \varphi_1$, so $c = \pm 1$. Since $\varphi_1 > 0$ and $\varphi_1 \circ R > 0$, the case $c = -1$ is impossible. Therefore $\varphi_1 \circ R = \varphi_1$: the ground state is symmetric under R. □

4 The parallel with the Weil quadratic form

We now make explicit the structural analogy between the fractional Laplacian and the Weil quadratic form. The purpose is to show that the five-step pipeline is not specific to either setting but is a general strategy applicable to any operator whose quadratic form admits an energy decomposition with strictly positive weights.

4.1 The setup

For a finite set S of places including the archimedean place $\mathbb{R}$, and a cutoff parameter $\lambda > 1$, the Weil quadratic form restricted to test functions supported on $[\lambda^{-1}, \lambda]$ (in multiplicative coordinates) defines a quadratic form $\mathcal{E}_\lambda$ on $L^2([-L, L])$ where $L = \log \lambda$. The Weil explicit formula gives

$$\mathcal{E}_\lambda(G,G) = \sum_{p \in S \setminus \{\mathbb{R}\}} \mathcal{E}_p(G,G) \; + \; \mathcal{E}_\mathbb{R}(G,G),$$

where:

- Each *prime contribution* $\mathcal{E}_p(G,G)$ is a discrete difference energy. Writing $\ell_p = \log p$, the contribution from each power p^k involves terms $\|G - S_{\pm k\ell_p} G\|^2$ where S_t is the shift operator $(S_t G)(u) = G(u - t)$ restricted to $[-L, L]$.
- The *archimedean contribution* $\mathcal{E}_\mathbb{R}(G,G)$ involves a continuous integral of translation-difference energies:

$$\mathcal{E}_\mathbb{R}(G,G) = \int_0^{2L} \|S_t \tilde{G} - \tilde{G}\|^2_{L^2} \, w_\mathbb{R}(t)\, dt,$$

where $w_\mathbb{R}(t) \geq 0$ is determined by the archimedean local distribution $W_\mathbb{R}$.

4.2 Step-by-step comparison

Step	Fractional Laplacian	Weil form
Energy decomposition	$\mathcal{E} = \int_{\mathbb{R}^d} \|\tau_h \tilde{u} - \tilde{u}\|^2 \, w(h)\, dh$ $w(h) = \frac{C_{d,s}}{2} \|h\|^{-(d+2s)}$	$\mathcal{E}_\lambda = \sum_p \mathcal{E}_p + \int_0^{2L} \| \cdot \|^2 \, w_{\mathbb{R}}\, dt$ Discrete (primes) + continuum (archim.)
Markov property	$\|\Phi(a) - \Phi(b)\| \leq \|a - b\|$ applied pointwise in h	Same pointwise argument applied to each $\mathcal{E}_p$ and to $\mathcal{E}_{\mathbb{R}}$
Irreducibility	$w(h) > 0$ for all $h \neq 0$; $\tilde{u}$ vanishes outside Ω	$w_{\mathbb{R}}(t) > 0$ for $t \in (0, 2L)$; $\tilde{G}$ vanishes outside $[-L, L]$
Compact resolvent	$\|\xi\|^{2s} \to \infty$ (polynomial) $\Rightarrow$ Rellich–Kondrachov	$\psi_\lambda(\xi) \gtrsim \log \|\xi\|$ (logarithmic) $\Rightarrow$ Kolmogorov–Riesz
Perron–Frobenius	Krein–Rutman $\Rightarrow$ λ_1 simple, $\varphi_1 > 0$	Krein–Rutman $\Rightarrow$ λ_1 simple, ground state > 0
Symmetry	Ω symmetric $\Rightarrow$ φ_1 symmetric	$[-L, L]$ symmetric $\Rightarrow$ ground state even

4.3 Where the proofs differ

Despite the structural parallelism, there are genuine differences:

(1) **Coercivity.** The fractional Laplacian has polynomial growth $|\xi|^{2s}$ of its Fourier symbol, which gives compact embedding by the standard fractional Rellich–Kondrachov theorem. The Weil form has only logarithmic growth, making compact embedding more delicate and requiring explicit use of the Kolmogorov–Riesz criterion. (In our proof of Proposition 3.6, we deliberately used Kolmogorov–Riesz to parallel the arithmetic setting, even though a simpler proof is available.)

(2) **Structure of shifts.** The fractional Laplacian integrates over *all* translations in $\mathbb{R}^d$, making irreducibility trivial. The Weil form has a discrete part (primes) and a continuous part (archimedean), and irreducibility depends on the archimedean contribution—without it, one would have only countably many shift directions, which is not enough.

(3) **Boundary conditions.** The fractional Laplacian uses zero extension to $\mathbb{R}^d$ (Dirichlet conditions), which kills the constant function. The Weil form uses restriction to $[-L, L]$ with similar effect: functions in the domain vanish outside $[-L, L]$, so the only function with zero energy is zero.

(4) **Context and significance.** Ground state simplicity for the fractional Laplacian is a well-known result with multiple proofs. For the Weil form, it resolves an open problem explicitly identified by Connes, Consani, Moscovici, and van Suijlekom as "the key difficulty" in their program.

5 An abstract framework

The argument naturally generalizes to the following class of operators.

Theorem 5.1 (Abstract ground state simplicity [expository])**.** *Let $X \subset \mathbb{R}^d$ be a bounded measurable set with $|X| > 0$, and let $w : \mathbb{R}^d \to [0, \infty)$ be a measurable weight satisfying:*
(H1) (Local integrability) $\int_{|h|<1} |h|^2 w(h)\, dh < \infty$.
(H2) (Coercivity) *The Fourier symbol* $\psi(\xi) = \int_{\mathbb{R}^d} |e^{2\pi i h\cdot\xi} - 1|^2 w(h)\, dh$ *satisfies* $\psi(\xi) \to \infty$ *as* $|\xi| \to \infty$.
(H3) (Irreducibility) *For every* $h_0 \in \mathbb{R}^d \setminus \{0\}$ *and every* $\varepsilon > 0$, $\int_{|h-h_0|<\varepsilon} w(h)\, dh > 0$.
Define

$$\mathcal{E}_w(u, u) = \int_{\mathbb{R}^d} \|\tau_h \tilde{u} - \tilde{u}\|_{L^2}^2\, w(h)\, dh$$

on $\mathcal{D}(\mathcal{E}_w) = \{u \in L^2(X) : \mathcal{E}_w(u, u) < \infty\}$, *where* $\tilde{u}$ *is the zero extension to* $\mathbb{R}^d$. *Then* $(\mathcal{E}_w, \mathcal{D}(\mathcal{E}_w))$ *is a symmetric Dirichlet form on* $L^2(X)$ *whose associated operator has compact resolvent, simple lowest eigenvalue, and a strictly positive eigenfunction.*

Proof sketch. The Markov property follows from the pointwise contraction argument (Step 2). Hypothesis (H3) guarantees irreducibility by the same argument as Step 3: if $\mathcal{E}_w(u, u) = 0$, then $\|\tau_h \tilde{u} - \tilde{u}\| = 0$ for h in the support of w, and (H3) ensures this support is dense in $\mathbb{R}^d$, forcing $\tilde{u}$ to be constant and hence zero. Hypothesis (H2) gives equi-continuity in mean: $\|\tau_h \tilde{u} - \tilde{u}\|^2 \le \int |\hat{\tilde{u}}|^2 |e^{2\pi i h\cdot\xi} - 1|^2\, d\xi$, and the growth of $\psi(\xi)$ controls this. Tightness follows from the bounded support of X. Kolmogorov–Riesz gives compact inclusion, and Krein–Rutman gives simplicity. □

The fractional Laplacian corresponds to $w(h) = \frac{C_{d,s}}{2} |h|^{-(d+2s)}$ (satisfying all three hypotheses for $0 < s < 1$). The Weil form corresponds to a weight that is a superposition of discrete masses at prime-logarithm multiples and a continuous archimedean density, with the Fourier symbol growing logarithmically.

Remark 5.2 (Scope and limitations [expository])*.* Hypothesis (H3) is a convenient *sufficient* condition for the irreducibility mechanism used in the proof of Theorem 5.1: it forces the translation-difference energy to see *all* small shifts. When w is supported on a discrete subgroup (e.g. a lattice), Hypothesis (H3) fails and reducibility can indeed occur.

A concrete example is $d = 1$, $w = \delta_2 + \delta_{-2}$ and $X = (0, 1) \cup (3, 4)$. Then the energy $\mathcal{E}_w(u, u)$ only couples values of $\tilde{u}$ across shifts of size 2, and the subspaces $L^2((0, 1))$ and $L^2((3, 4))$ are invariant: the form is not irreducible and the lowest eigenvalue of the associated operator has multiplicity 2 (one copy on each component). In contrast, for the fractional Laplacian the continuum of shifts encoded by (3) prevents such decoupling.

6 Discussion

6.1 Comparison with other proofs

The classical proof of ground state simplicity for the standard Laplacian $-\Delta$ on a bounded domain (the Courant–Hilbert argument) uses the variational characterization of λ_1 together with the observation that $|u|$ has the same Dirichlet energy as u, so minimizers can be taken non-negative, and then the strong maximum principle forces them to be strictly positive. Simplicity follows because two orthogonal positive functions cannot exist.

For the fractional Laplacian, this argument breaks down at the maximum principle step: the non-local maximum principle for $(-\Delta)^s$ is substantially harder to prove and requires regularity theory [12, 1].

The probabilistic proof (via the symmetric stable process) replaces the maximum principle with the observation that the Lévy jump kernel $|x-y|^{-(d+2s)}$ is strictly positive, so the process can jump from any region to any other, making the semigroup positivity-improving. This is elegant but requires probabilistic machinery.

Our Dirichlet form proof occupies a middle ground: it uses neither the maximum principle nor probability theory, and it makes the mechanism transparent. The ground state is simple because (a) the energy is a superposition of non-negative pieces (Markov property), (b) the pieces involve a rich enough family of shifts (irreducibility), and (c) the Fourier symbol grows (compact resolvent). Each condition is easy to verify, and together they are sufficient.

6.2 Further directions

The abstract framework of Theorem 5.1 suggests several natural extensions:

1. **Other L-functions.** The Weil explicit formula for Dirichlet L-functions $L(s, \chi)$ with real characters has local distributions W_p^χ that differ from the Riemann zeta case only by character twists. The energy-decomposition method should extend to verify the Connes–van Suijlekom hypotheses for these L-functions, giving simplicity and evenness for an infinite family of restricted operators.
2. **Non-local operators with arithmetic kernels.** Integral operators whose kernels are defined by Euler products or Epstein zeta functions may admit energy decompositions of the form considered here. The abstract theorem provides a systematic criterion for when such operators have simple ground states.
3. **Operators on groups.** The translation-difference structure generalizes naturally to locally compact abelian groups. The Weil form on the idèle class group and the fractional Laplacian on $\mathbb{R}^d$ are both instances of convolution-type operators on groups, and the energy-decomposition method should extend to this level of generality.

Acknowledgments

This paper is intended as an expository companion to the energy-decomposition approach to the Weil quadratic form. Its purpose is to demonstrate the method in a setting where the result is already known, thereby making the strategy accessible to readers from PDE and spectral theory who may not be familiar with the number-theoretic context.

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