Energy-Decomposition and Perron–Frobenius Consequences for the Restricted Weil Quadratic Form

Abstract

We record a completely concrete and rigorous functional-analytic step that arises in the spectral approach to Weil's criterion when one restricts test functions to a compact multiplicative interval $[\lambda^{-1}, \lambda] \subset \mathbb{R}_+^*$. Starting from the explicit local distributions at the primes and at ∞, we derive an "energy decomposition" expressing the quadratic form (up to an additive constant multiple of $\|g\|_2^2$) as a positive combination of translation-difference energies $\|G - \tau_t G\|_2^2$ in logarithmic coordinates. We then prove the Markov (normal contraction) property and a translation-invariance lemma which yields irreducibility from the archimedean continuum of shifts. Assuming (as in the standard setup) that the associated selfadjoint operator has compact resolvent, we deduce that the ground-state eigenvalue is simple and its eigenfunction can be chosen strictly positive and, by inversion symmetry, even.

Contents

1 Setup on $\mathbb{R}_+^*$

Let $\mathbb{R}_+^* = (0,\infty)$ with multiplicative Haar measure

$$d^*x := \frac{dx}{x}.$$

For measurable g, h define multiplicative convolution

$$(g * h)(x) := \int_{\mathbb{R}_+^*} g(y)\, h(x/y)\, d^*y,$$

and involution

$$g^*(x) := \overline{g(x^{-1})}.$$

If $g \in L^2(\mathbb{R}_+^*, d^*x)$, define the unitary dilation operator

$$(U_a g)(x) := g(x/a) \qquad (a > 0). \tag{1}$$

Then $\|U_a g\|_2 = \|g\|_2$ and $\langle g, U_a g\rangle$ is well-defined.

Lemma 1 (Convolution inner-product identity)**.** *Let $f = g * g^*$. Then for all $a > 0$,*

$$f(a) = \langle g, U_a g\rangle_{L^2(d^*x)} = \int_{\mathbb{R}_+^*} g(x)\,\overline{g(x/a)}\, d^*x, \qquad f(a^{-1}) = \overline{f(a)}.$$

In particular $f(a) + f(a^{-1}) = 2\Re\langle g, U_a g\rangle$ and $f(1) = \|g\|_2^2$.

Proof. By definition,

$$(g * g^*)(a) = \int g(y)\, g^*(a/y)\, d^*y = \int g(y)\, \overline{g((a/y)^{-1})}\, d^*y = \int g(y)\, \overline{g(y/a)}\, d^*y = \langle g, U_a g\rangle.$$

The relation $f(a^{-1}) = \overline{f(a)}$ follows by replacing a with a^{-1} and complex conjugating. □

Lemma 2 (A basic unitary identity)**.** *For any unitary U on a Hilbert space and any vector h,*

$$2\Re\langle h, Uh\rangle = 2\|h\|^2 - \|h - Uh\|^2.$$

Proof. Expand $\|h - Uh\|^2 = \|h\|^2 + \|Uh\|^2 - 2\Re\langle h, Uh\rangle$ and use $\|Uh\| = \|h\|$. □

2 Local explicit-formula terms

Fix $\lambda > 1$ and consider g supported in $[\lambda^{-1}, \lambda]$.

We record the two local distributions we use; these are the only "input formulas".

2.1 Prime terms

For a prime p define

$$W_p(f) := (\log p) \sum_{m \geq 1} p^{-m/2} \big(f(p^m) + f(p^{-m}) \big). \tag{2}$$

2.2 Archimedean term

Define

$$W_{\mathbb{R}}(f) := (\log 4\pi + \gamma)\, f(1) + \int_1^\infty \Big(f(x) + f(x^{-1}) - 2x^{-1/2} f(1) \Big) \frac{x^{1/2}}{x - x^{-1}}\, d^*x, \tag{3}$$

where γ is the Euler–Mascheroni constant.

Remark 3 (Restriction to a compact multiplicative interval)*.* If $\operatorname{supp}(g) \subset [\lambda^{-1}, \lambda]$, then for $a > \lambda^2$ the supports of g and $U_a g$ are disjoint, hence $\langle g, U_a g \rangle = 0$ and $f(a) = 0$. Consequently:

- in (2) only those (p, m) with $p^m \leq \lambda^2$ contribute;
- in (3), after the change of variables $x = e^t$, only $t \in [0, 2 \log \lambda]$ contributes to the term involving $f(e^t) + f(e^{-t})$.

This finiteness is crucial and is completely elementary.

3 Logarithmic coordinates and translations

Set $u = \log x$, so that $d^*x = du$ and the interval $[\lambda^{-1}, \lambda]$ becomes

$$I := (-L, L), \qquad L := \log \lambda.$$

For $G \in L^2(I)$ we denote by $\widetilde{G}$ its extension by 0 to $\mathbb{R}$. Let S_t be translation on $L^2(\mathbb{R})$:

$$(S_t \phi)(u) := \phi(u - t).$$

Then in logarithmic coordinates, the dilation U_{e^t} from (1) corresponds to translation: if $G(u) = g(e^u)$, then $(U_{e^t} g)(e^u) = g(e^{u-t})$, i.e. $\widetilde{G} \mapsto S_t \widetilde{G}$.

4 Energy decomposition into translation differences

4.1 Prime contributions

Lemma 4 (Prime term as a difference energy plus a constant)**.** *Let $f = g * g^*$ with g supported in $[\lambda^{-1}, \lambda]$, and let $G(u) = g(e^u)$. Then*

$$-W_p(f) = \sum_{\substack{m \geq 1 \\ p^m \leq \lambda^2}} (\log p)\, p^{-m/2}\, \|\widetilde{G} - S_{m \log p} \widetilde{G}\|^2_{L^2(\mathbb{R})} \; + \; c_p(\lambda)\, \|G\|^2_{L^2(I)},$$

where $c_p(\lambda) \in \mathbb{R}$ is a finite constant depending only on p and λ.

Proof. By Lemma 1 and (2),

$$W_p(f) = (\log p) \sum_{m\geq 1} p^{-m/2}\, 2\Re\langle g, U_{p^m} g\rangle.$$

By Lemma 2 (with $U = U_{p^m}$),

$$2\Re\langle g, U_{p^m} g\rangle = 2\|g\|_2^2 - \|g - U_{p^m} g\|_2^2.$$

In logarithmic coordinates, $\|g - U_{p^m} g\|_2 = \|\widetilde{G} - S_{m\log p}\widetilde{G}\|_{L^2(\mathbb{R})}$. Moreover, if $p^m > \lambda^2$ then $\langle g, U_{p^m} g\rangle = 0$ by Remark 3, so those terms vanish. Collecting the $\|g\|_2^2$ contributions yields the constant $c_p(\lambda)$. □

4.2 Archimedean contribution

Lemma 5 (Archimedean term as a continuum of difference energies plus a constant). *Let $f = g * g^*$ with g supported in $[\lambda^{-1}, \lambda]$, and let $G(u) = g(e^u)$. Define the strictly positive weight on $(0, \infty)$,*

$$w(t) := \frac{e^{t/2}}{e^t - e^{-t}} = \frac{e^{t/2}}{2\sinh t}.$$

Then

$$-W_{\mathbb{R}}(f) = \int_0^{2L} w(t)\, \|\widetilde{G} - S_t\widetilde{G}\|_{L^2(\mathbb{R})}^2\, dt \;+\; c_\infty(\lambda)\, \|G\|_{L^2(I)}^2,$$

where $c_\infty(\lambda) \in \mathbb{R}$ is a finite constant depending only on λ.

Proof. Start from (3). Substitute $x = e^t$ (so $d^*x = dt$) to obtain

$$W_{\mathbb{R}}(f) = (\log 4\pi + \gamma)\, f(1) + \int_0^\infty \Big(f(e^t) + f(e^{-t}) - 2e^{-t/2} f(1)\Big)\, w(t)\, dt.$$

Using Lemma 1, $f(1) = \|g\|_2^2$, and

$$f(e^t) + f(e^{-t}) = 2\Re\langle g, U_{e^t} g\rangle,$$

we get

$$-W_{\mathbb{R}}(f) = -(\log 4\pi + \gamma)\|g\|_2^2 + \int_0^\infty \Big(-2\Re\langle g, U_{e^t} g\rangle + 2e^{-t/2}\|g\|_2^2\Big)\, w(t)\, dt.$$

Apply Lemma 2 with $U = U_{e^t}$:

$$-2\Re\langle g, U_{e^t} g\rangle = \|g - U_{e^t} g\|_2^2 - 2\|g\|_2^2.$$

Thus the integrand equals

$$\|g - U_{e^t} g\|_2^2 + 2(e^{-t/2} - 1)\|g\|_2^2.$$

In logarithmic coordinates $\|g - U_{e^t} g\|_2 = \|\widetilde{G} - S_t\widetilde{G}\|_{L^2(\mathbb{R})}$.

Now we split the integral at $t = 2L$. By Remark 3, for $t > 2L$ the supports of $\widetilde{G}$ and $S_t\widetilde{G}$ are disjoint, so $\|\widetilde{G} - S_t\widetilde{G}\|_2^2 = 2\|G\|_2^2$ (not zero). Hence for $t > 2L$ the integrand becomes $2\|G\|_2^2 + 2(e^{-t/2} - 1)\|G\|_2^2 = 2e^{-t/2}\|G\|_2^2$. This tail integral $\int_{2L}^\infty 2e^{-t/2}\, w(t)\, dt$ converges (since $w(t) \sim e^{-t/2}$ as $t \to \infty$) and contributes a finite constant times $\|G\|_2^2$.

For $t \in [0, 2L]$ we retain the difference-energy term $w(t)\|\widetilde{G} - S_t\widetilde{G}\|_2^2$ and absorb the $2(e^{-t/2} - 1)\, w(t)\|G\|_2^2$ contribution into the constant. Combining all $\|G\|_2^2$ terms—from the $(\log 4\pi + \gamma)$ prefactor, the integral over $[0, 2L]$ of $2(e^{-t/2} - 1)\, w(t)$, and the tail integral over $(2L, \infty)$—yields the finite constant $c_\infty(\lambda)$. The integral of $w(t)(e^{-t/2} - 1)$ over $[0, 2L]$ converges absolutely (near 0, $w(t) \sim 1/(2t)$ and $e^{-t/2} - 1 \sim -t/2$, giving an integrable $O(1)$ contribution). □

4.3 Global quadratic form on the interval

Definition 6 (Difference-energy form). Fix $\lambda > 1$ and $L = \log \lambda$. For $G \in L^2(I)$ define

$$\mathcal{E}_\lambda(G) := \int_0^{2L} w(t)\, \|\widetilde{G} - S_t\widetilde{G}\|^2_{L^2(\mathbb{R})}\, dt + \sum_{\substack{p \text{ prime} \\ p \le \lambda^2}} \sum_{\substack{m \ge 1 \\ p^m \le \lambda^2}} (\log p) p^{-m/2}\, \|\widetilde{G} - S_{m \log p}\widetilde{G}\|^2_{L^2(\mathbb{R})}. \tag{4}$$

Remark 7 (What we have proved so far). Lemmas 4 and 5 show that for $f = g * g^*$ with $\operatorname{supp}(g) \subset [\lambda^{-1}, \lambda]$, the quantity

$$-\sum_{v \in \{\infty\} \cup \{p\}} W_v(f)$$

equals $\mathcal{E}_\lambda(G)$ plus an additive constant multiple of $\|G\|_2^2$. Since adding a constant multiple of $\|G\|_2^2$ only shifts the spectrum of the associated operator, it does not affect positivity/irreducibility properties of the semigroup and does not affect eigenfunction parity considerations.

5 Markov property (normal contractions)

Definition 8 (Normal contraction). A map $\Phi : \mathbb{R} \to \mathbb{R}$ is a normal contraction if $\Phi(0) = 0$ and $|\Phi(a) - \Phi(b)| \le |a - b|$ for all $a, b \in \mathbb{R}$.

Lemma 9 (Markov property)**.** *For every normal contraction Φ and every $G \in L^2(I)$,*

$$\mathcal{E}_\lambda(\Phi \circ G) \le \mathcal{E}_\lambda(G).$$

In particular, $\mathcal{E}_\lambda(|G|) \le \mathcal{E}_\lambda(G)$.

Proof. For each shift parameter t,

$$\|\widetilde{\Phi \circ G} - S_t\widetilde{\Phi \circ G}\|_2^2 = \int_{\mathbb{R}} |\Phi(\widetilde{G}(u)) - \Phi(\widetilde{G}(u-t))|^2\, du \le \int_{\mathbb{R}} |\widetilde{G}(u) - \widetilde{G}(u-t)|^2\, du = \|\widetilde{G} - S_t\widetilde{G}\|_2^2,$$

by the 1-Lipschitz property of Φ. Integrating against the nonnegative weights and summing proves the claim. $\square$

6 A translation-invariance lemma on an interval

Lemma 10 (Local translation invariance forces null or conull)**.** *Let $I \subset \mathbb{R}$ be a nontrivial open interval and let $B \subset I$ be measurable. Assume that there exists $\varepsilon > 0$ such that for every $t \in (0, \varepsilon)$,*

$$\mathbf{1}_B(u) = \mathbf{1}_B(u - t) \quad \text{for a.e. } u \in I \cap (I + t). \tag{5}$$

Then either $m(B) = 0$ or $m(I \setminus B) = 0$. Equivalently: if $0 < m(B) < m(I)$ then for every $\varepsilon > 0$ there exists $t \in (0, \varepsilon)$ with $m(B \cap (B + t)^c) > 0$.

Proof. Write $f := \mathbf{1}_B \in L^1_{\text{loc}}(I)$. Fix a compact subinterval $J \Subset I$ (so $\operatorname{dist}(J, \partial I) > 0$), and choose $0 < \delta < \min\{\varepsilon, \operatorname{dist}(J, \partial I)\}$. From (5) and the substitution $u \mapsto u + t$ we obtain: for every $t \in (0, \delta)$,

$$f(u + t) = f(u) \quad \text{for a.e. } u \in J.$$

Thus for every $t \in (-\delta, \delta)$ we have $f(u + t) = f(u)$ for a.e. $u \in J$ (replace t by $-t$).

Let $\rho \in C_c^\infty(\mathbb{R})$ be a standard mollifier with $\rho \geq 0$, $\int \rho = 1$ and $\operatorname{supp} \rho \subset (-1, 1)$, and set $\rho_\eta(s) := \eta^{-1}\rho(s/\eta)$ for $0 < \eta < \delta/2$. Define $f_\eta := f * \rho_\eta$ on the slightly smaller interval

$$J_\eta := \{u \in J : \operatorname{dist}(u, \mathbb{R} \setminus J) > \eta\}.$$

Then $f_\eta \in C^\infty(J_\eta)$, and for $u \in J_\eta$ and $|t| < \delta/2$ we may compute (using Fubini)

$$f_\eta(u+t) = \int_{\mathbb{R}} f(u+t-s)\rho_\eta(s)\, ds = \int_{\mathbb{R}} f(u-s)\rho_\eta(s)\, ds = f_\eta(u),$$

because $u - s \in J$ for $u \in J_\eta$ and $s \in \operatorname{supp} \rho_\eta$, and $f(\cdot + t) = f(\cdot)$ a.e. on J. Hence f_η is translation-invariant on the connected open interval J_η, so f_η is constant on J_η.

Letting $\eta \downarrow 0$, we have $f_\eta \to f$ in $L^1(J)$, so f is a.e. equal to a constant on J. Since $J \Subset I$ was arbitrary, f is a.e. constant on I, i.e. $\mathbf{1}_B$ is a.e. either 0 or 1 on I. Thus $m(B) = 0$ or $m(I \setminus B) = 0$. □

7 Irreducibility from the archimedean continuum

7.1 A concrete criterion

Lemma 11 (Indicator-energy vanishes only for null/conull sets)**.** *Let $B \subset I$ be measurable. If $\mathcal{E}_\lambda(\mathbf{1}_B) = 0$, then $m(B) = 0$ or $m(I \setminus B) = 0$.*

Proof. By definition of $\mathcal{E}_\lambda$ and the nonnegativity of all weights, $\mathcal{E}_\lambda(\mathbf{1}_B) = 0$ implies in particular that the archimedean contribution vanishes:

$$\int_0^{2L} w(t)\, \|\widetilde{\mathbf{1}_B} - S_t\widetilde{\mathbf{1}_B}\|_2^2\, dt = 0.$$

Since $w(t) > 0$ for every $t > 0$, it follows that

$$\|\widetilde{\mathbf{1}_B} - S_t\widetilde{\mathbf{1}_B}\|_2^2 = 0 \quad \text{for a.e. } t \in (0, 2L).$$

We now upgrade "a.e." to "all": for any $\phi \in L^2(\mathbb{R})$, the map $t \mapsto \|\phi - S_t\phi\|_2^2$ is continuous (by strong continuity of the translation group on $L^2(\mathbb{R})$, which follows from dominated convergence). Applying this to $\phi = \widetilde{\mathbf{1}_B} \in L^2(\mathbb{R})$, the function $t \mapsto \|\widetilde{\mathbf{1}_B} - S_t\widetilde{\mathbf{1}_B}\|_2^2$ is continuous, vanishes a.e. on $(0, 2L)$, and hence vanishes *everywhere* on $(0, 2L)$. In particular, for every $t \in (0, 2L)$,

$$\mathbf{1}_B(u) = \mathbf{1}_B(u-t) \quad \text{for a.e. } u \in I \cap (I + t).$$

Since this holds for all t in the interval $(0, 2L)$, which contains $(0, \varepsilon)$ for any $\varepsilon \leq 2L$, Lemma 10 applies and yields $m(B) = 0$ or $m(I \setminus B) = 0$. □

7.2 Operator realization: closedness and compact resolvent

In this subsection we show that the concrete form $\mathcal{E}_\lambda$ of Definition 6 is closed and yields a selfadjoint operator with compact resolvent. This replaces the abstract assumption previously made on the operator.

7.2.1 Ambient form on $L^2(\mathbb{R})$ and Fourier representation

Let $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ denote the unitary Fourier transform

$$\widehat{\phi}(\xi) := \int_{\mathbb{R}} \phi(u) e^{-iu\xi}\, du, \qquad \phi(u) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{\phi}(\xi) e^{iu\xi}\, d\xi,$$

so that Plancherel reads $\|\phi\|^2_{L^2(\mathbb{R})} = \frac{1}{2\pi} \int_{\mathbb{R}} |\widehat{\phi}(\xi)|^2\, d\xi$.

Define the "ambient" quadratic form on $L^2(\mathbb{R})$ by

$$\mathcal{E}^{\mathbb{R}}_\lambda(\phi) := \int_0^{2L} w(t)\, \|\phi - S_t\phi\|^2_{L^2(\mathbb{R})}\, dt + \sum_{\substack{p \text{ prime} \\ p \le \lambda^2}} \sum_{\substack{m \ge 1 \\ p^m \le \lambda^2}} (\log p) p^{-m/2}\, \|\phi - S_{m \log p}\phi\|^2_{L^2(\mathbb{R})},$$

with domain $\mathcal{D}(\mathcal{E}^{\mathbb{R}}_\lambda) := \{\phi \in L^2(\mathbb{R}) : \mathcal{E}^{\mathbb{R}}_\lambda(\phi) < \infty\}$. By definition, for $G \in L^2(I)$,

$$\mathcal{E}_\lambda(G) = \mathcal{E}^{\mathbb{R}}_\lambda(\widetilde{G}).$$

Lemma 12 (Plancherel identity for translation differences)**.** *For $\phi \in L^2(\mathbb{R})$ and $t \in \mathbb{R}$,*

$$\|\phi - S_t\phi\|^2_{L^2(\mathbb{R})} = \frac{1}{2\pi} \int_{\mathbb{R}} |1 - e^{-i\xi t}|^2\, |\widehat{\phi}(\xi)|^2\, d\xi = \frac{1}{2\pi} \int_{\mathbb{R}} 4 \sin^2\!\Big(\frac{\xi t}{2}\Big)\, |\widehat{\phi}(\xi)|^2\, d\xi.$$

Proof. Since $\widehat{S_t\phi}(\xi) = e^{-i\xi t}\widehat{\phi}(\xi)$, Plancherel gives the first identity. The second follows from $|1 - e^{-i\eta}|^2 = 4\sin^2(\eta/2)$. □

Lemma 13 (Fourier representation)**.** *For $\phi \in L^2(\mathbb{R})$,*

$$\mathcal{E}^{\mathbb{R}}_\lambda(\phi) = \frac{1}{2\pi} \int_{\mathbb{R}} \psi_\lambda(\xi)\, |\widehat{\phi}(\xi)|^2\, d\xi \quad \text{in } [0, \infty],$$

where

$$\psi_\lambda(\xi) := 4 \int_0^{2L} w(t)\, \sin^2\!\Big(\frac{\xi t}{2}\Big)\, dt + 4 \sum_{\substack{p \text{ prime} \\ p \le \lambda^2}} \sum_{\substack{m \ge 1 \\ p^m \le \lambda^2}} (\log p) p^{-m/2}\, \sin^2\!\Big(\frac{\xi m \log p}{2}\Big). \tag{6}$$

In particular ψ_λ is measurable, even, finite for each ξ, and $\psi_\lambda(\xi) \ge 0$.

Proof. Apply Lemma 12 to each translation difference in $\mathcal{E}^{\mathbb{R}}_\lambda$. All weights are nonnegative, so Tonelli's theorem permits interchange of the ξ–integral with the t–integration and finite summations. □

Proposition 14 (Closedness on $L^2(\mathbb{R})$)**.** *The form $\mathcal{E}^{\mathbb{R}}_\lambda$ is densely defined, symmetric, nonnegative, and closed on $L^2(\mathbb{R})$. Moreover,*

$$\mathcal{D}(\mathcal{E}^{\mathbb{R}}_\lambda) = \Big\{\phi \in L^2(\mathbb{R}) : \int_{\mathbb{R}} \psi_\lambda(\xi)\, |\widehat{\phi}(\xi)|^2\, d\xi < \infty\Big\},$$

and $\mathcal{D}(\mathcal{E}^{\mathbb{R}}_\lambda)$ is a Hilbert space for the norm

$$\|\phi\|^2_{\mathcal{D}} := \|\phi\|^2_{L^2(\mathbb{R})} + \mathcal{E}^{\mathbb{R}}_\lambda(\phi) = \frac{1}{2\pi} \int_{\mathbb{R}} (1 + \psi_\lambda(\xi))\, |\widehat{\phi}(\xi)|^2\, d\xi.$$

Proof. By Lemma 13, $\mathcal{E}_\lambda^{\mathbb{R}}$ is the quadratic form of multiplication by ψ_λ in Fourier space. Hence $\mathcal{D}(\mathcal{E}_\lambda^{\mathbb{R}})$ is isometric (via $\phi \mapsto \widehat{\phi}$) to the weighted L^2 space with weight $1+\psi_\lambda$, and therefore complete. Nonnegativity and symmetry are immediate from the definition.

For density, note that $C_c^\infty(\mathbb{R}) \subset \mathcal{D}(\mathcal{E}_\lambda^{\mathbb{R}})$: for $\phi \in C_c^\infty(\mathbb{R})$, $\|\phi - S_t\phi\|_2 \le |t|\|\phi'\|_2$, and $\int_0^{2L} w(t)t^2\,dt < \infty$ (since $w(t) \sim (2t)^{-1}$ as $t \downarrow 0$ and the upper limit is finite); the prime sum in $\mathcal{E}_\lambda^{\mathbb{R}}$ is finite. Since $C_c^\infty(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, the form is densely defined. □

Proposition 15 (Closedness on $L^2(I)$). *The form $\mathcal{E}_\lambda$ on $H = L^2(I)$ is densely defined, symmetric, nonnegative, and closed.*

Proof. The map $G \mapsto \widetilde{G}$ is an isometry from $L^2(I)$ onto the closed subspace $H_I = \{\phi \in L^2(\mathbb{R}) : \phi = 0 \text{ a.e. on } \mathbb{R} \setminus I\}$. Moreover $\mathcal{E}_\lambda(G) = \mathcal{E}_\lambda^{\mathbb{R}}(\widetilde{G})$. Thus $\mathcal{E}_\lambda$ is the restriction of the closed form $\mathcal{E}_\lambda^{\mathbb{R}}$ (Proposition 14) to the closed subspace H_I, and therefore is closed. Density follows because $C_c^\infty(I) \subset \mathcal{D}(\mathcal{E}_\lambda)$ and is dense in $L^2(I)$. □

7.2.2 A coercive lower bound for the symbol ψ_λ

Lemma 16 (A lower bound for $w(t)$). *Let $t_0 := \min(1, 2L)$. There exists $c_0 = c_0(L) > 0$ such that for all $t \in (0, t_0]$,*

$$w(t) = \frac{e^{t/2}}{2\sinh t} \ge \frac{c_0}{t}.$$

Proof. For $t > 0$ one has $\sinh t \le te^t$, hence

$$w(t) = \frac{e^{t/2}}{2\sinh t} \ge \frac{e^{t/2}}{2te^t} = \frac{e^{-t/2}}{2t}.$$

For $t \in (0, 1]$, $e^{-t/2} \ge e^{-1/2}$, so we may take $c_0 := e^{-1/2}/2$ (or any smaller positive constant). □

Lemma 17 (Logarithmic growth of ψ_λ). *There exist constants $c_1, c_2 > 0$ and $\xi_0 \ge 2$ (depending only on L) such that for all $|\xi| \ge \xi_0$,*

$$\psi_\lambda(\xi) \ge c_1 \log|\xi| - c_2.$$

In particular $\psi_\lambda(\xi) \to \infty$ as $|\xi| \to \infty$.

Proof. Drop the nonnegative prime sum in (6):

$$\psi_\lambda(\xi) \ge 4\int_0^{2L} w(t)\sin^2\Big(\frac{\xi t}{2}\Big)\,dt \ge 4\int_0^{t_0} w(t)\sin^2\Big(\frac{\xi t}{2}\Big)\,dt.$$

By Lemma 16, for $t \in (0, t_0]$,

$$\psi_\lambda(\xi) \ge 4c_0 \int_0^{t_0} \frac{1}{t}\sin^2\Big(\frac{\xi t}{2}\Big)\,dt.$$

Assume $|\xi| \ge \frac{4\pi}{t_0}$ (this fixes ξ_0). Define intervals

$$J_n := \left[\frac{2\pi n + \pi/2}{|\xi|}, \frac{2\pi n + 3\pi/2}{|\xi|}\right], \qquad n \ge 0.$$

For $t \in J_n$, $\sin^2(\xi t/2) \ge 1/2$. Let $N \ge 1$ be the largest integer such that $J_{N-1} \subset (0, t_0]$. Then $N \asymp |\xi|$ (with constants depending only on t_0), and hence

$$\int_0^{t_0} \frac{1}{t}\sin^2\Big(\frac{\xi t}{2}\Big)\,dt \ge \sum_{n=0}^{N-1}\int_{J_n} \frac{1}{t}\cdot\frac{1}{2}\,dt = \frac{1}{2}\sum_{n=0}^{N-1}\log\frac{2\pi n + 3\pi/2}{2\pi n + \pi/2}.$$

Using $\log(1+x) \ge x/(1+x)$, one obtains

$$\log \frac{2\pi n + 3\pi/2}{2\pi n + \pi/2} = \log\left(1 + \frac{\pi}{2\pi n + \pi/2}\right) \ge \frac{c}{n+1}$$

for some absolute $c > 0$ and all $n \ge 0$. Therefore the sum is bounded below by $c' \sum_{n=0}^{N-1} \frac{1}{n+1} \ge c'' \log N - C$. Since $N \asymp |\xi|$, we have $\log N = \log|\xi| + O(1)$, giving the claim. □

Corollary 18 (Energy controls a logarithmic frequency moment)**.** *There exist constants $a, b > 0$ (depending only on L) such that for every $\phi \in \mathcal{D}(\mathcal{E}_\lambda^{\mathbb{R}})$,*

$$\int_{\mathbb{R}} \log(2+|\xi|)\,|\widehat{\phi}(\xi)|^2\,d\xi \le a\,\|\phi\|_{L^2(\mathbb{R})}^2 + b\int_{\mathbb{R}} \psi_\lambda(\xi)\,|\widehat{\phi}(\xi)|^2\,d\xi.$$

In particular, if $\|\phi\|_2^2 + \mathcal{E}_\lambda^{\mathbb{R}}(\phi) \le M$, then $\int \log(2+|\xi|)|\widehat{\phi}(\xi)|^2 \le C(M,L)$.

Proof. Lemma 17 implies $\log(2+|\xi|) \le a' + b'\psi_\lambda(\xi)$ for suitable a', b' (after enlarging constants to handle bounded $|\xi|$). Multiply by $|\widehat{\phi}(\xi)|^2$ and integrate. □

7.2.3 Compact embedding and compact resolvent

Theorem 19 (Kolmogorov–Riesz compactness criterion in $L^2(\mathbb{R})$)**.** *A set $\mathcal{K} \subset L^2(\mathbb{R})$ is relatively compact if and only if:*

*(i) (*tightness*) for every $\varepsilon > 0$ there exists $R > 0$ such that $\int_{|u|>R} |\phi(u)|^2\,du < \varepsilon^2$ for all $\phi \in \mathcal{K}$;*

*(ii) (*translation equicontinuity*) for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|\phi - S_h\phi\|_2 < \varepsilon$ for all $\phi \in \mathcal{K}$ and all $|h| < \delta$.*

Remark 20. See, e.g., Lieb–Loss, *Analysis*, for a proof of Theorem 19.

Lemma 21 (Uniform translation control from the form norm)**.** *Fix $M > 0$ and define*

$$\mathcal{K}_M := \{\phi \in H_I :\ \|\phi\|_2^2 + \mathcal{E}_\lambda^{\mathbb{R}}(\phi) \le M\}.$$

Then $\mathcal{K}_M$ satisfies the translation equicontinuity condition (ii) in Theorem 19.

Proof. Let $\phi \in \mathcal{K}_M$ and $h \in \mathbb{R}$ with $|h| \le 1$. By Plancherel,

$$\|\phi - S_h\phi\|_2^2 = \frac{1}{2\pi}\int_{\mathbb{R}} 4\sin^2\!\left(\frac{\xi h}{2}\right)|\widehat{\phi}(\xi)|^2\,d\xi.$$

Fix $R \ge 1$ and split the integral into $|\xi| \le R$ and $|\xi| > R$. Using $\sin^2(x) \le x^2$,

$$\int_{|\xi|\le R} 4\sin^2\!\left(\frac{\xi h}{2}\right)|\widehat{\phi}(\xi)|^2\,d\xi \le \int_{|\xi|\le R} (\xi h)^2\,|\widehat{\phi}(\xi)|^2\,d\xi \le (Rh)^2 \int_{\mathbb{R}} |\widehat{\phi}(\xi)|^2\,d\xi = (Rh)^2(2\pi)\|\phi\|_2^2.$$

Also $\sin^2 \le 1$ gives

$$\int_{|\xi|>R} 4\sin^2\!\left(\frac{\xi h}{2}\right)|\widehat{\phi}(\xi)|^2\,d\xi \le 4\int_{|\xi|>R} |\widehat{\phi}(\xi)|^2\,d\xi \le \frac{4}{\log(2+R)}\int_{\mathbb{R}} \log(2+|\xi|)\,|\widehat{\phi}(\xi)|^2\,d\xi.$$

By Corollary 18, the last integral is $\le C(M,L)$ uniformly over $\phi \in \mathcal{K}_M$. Therefore

$$\|\phi - S_h\phi\|_2^2 \le (Rh)^2\,M + \frac{C'(M,L)}{\log(2+R)}.$$

Given $\varepsilon > 0$, choose R so that $C'(M,L)/\log(2+R) \le \varepsilon^2/2$, and then choose $\delta > 0$ so that $(R\delta)^2 M \le \varepsilon^2/2$. This gives $\|\phi - S_h\phi\|_2 < \varepsilon$ for all $\phi \in \mathcal{K}_M$ and $|h| < \delta$. □

Proposition 22 (Compact embedding of the form domain)**.** *The embedding* $(\mathcal{D}(\mathcal{E}_\lambda), \|G\|_{\mathcal{D}}^2 := \|G\|_2^2 + \mathcal{E}_\lambda(G)) \hookrightarrow L^2(I)$ *is compact.*

Proof. Let $\{G_n\} \subset \mathcal{D}(\mathcal{E}_\lambda)$ be bounded in the form norm: $\|G_n\|_2^2 + \mathcal{E}_\lambda(G_n) \leq M$. Put $\phi_n := \widetilde{G}_n \in H_I$. Then $\|\phi_n\|_2^2 + \mathcal{E}_\lambda^{\mathbb{R}}(\phi_n) \leq M$, so $\phi_n \in \mathcal{K}_M$.

Since each ϕ_n is supported in the fixed bounded set $\overline{I}$, tightness (i) in Theorem 19 holds automatically. Translation equicontinuity (ii) holds by Lemma 21. Thus $\{\phi_n\}$ is relatively compact in $L^2(\mathbb{R})$ by Theorem 19; hence $\{G_n\}$ is relatively compact in $L^2(I)$. □

Theorem 23 (Closed form, associated operator, and compact resolvent)**.** *There exists a unique selfadjoint operator* $A_\lambda \geq 0$ *on* $L^2(I)$ *associated to the closed form* $\mathcal{E}_\lambda$ *(Proposition 15) in the sense of the representation theorem for closed forms. Moreover,* A_λ *has compact resolvent; equivalently,* $(A_\lambda + 1)^{-1}$ *is compact on* $L^2(I)$*.*

Proof. Existence and uniqueness of A_λ follow from the representation theorem for densely defined, closed, lower-bounded symmetric forms (see, e.g., Kato, *Perturbation Theory for Linear Operators*). To prove compact resolvent, let $\{f_n\}$ be bounded in $L^2(I)$ and set $u_n := (A_\lambda + 1)^{-1} f_n$. Then $u_n \in \mathcal{D}(A_\lambda) \subset \mathcal{D}(\mathcal{E}_\lambda)$ and $(A_\lambda + 1)u_n = f_n$. Taking the L^2 inner product with u_n and using the form identity gives

$$\mathcal{E}_\lambda(u_n) + \|u_n\|_2^2 = \langle f_n, u_n \rangle \leq \|f_n\|_2 \, \|u_n\|_2.$$

Hence $\|u_n\|_2 \leq \|f_n\|_2$, and therefore $\|u_n\|_2^2 + \mathcal{E}_\lambda(u_n) \leq \|f_n\|_2^2$. Thus $\{u_n\}$ is bounded in the form norm, so by Proposition 22 it has a convergent subsequence in $L^2(I)$. This proves $(A_\lambda + 1)^{-1}$ is compact. □

7.3 Semigroup and irreducibility

Definition 24 (Irreducibility for semigroups on $L^2(I)$). A closed ideal in $L^2(I)$ has the form $L^2(B)$ for some measurable $B \subset I$. We call T *irreducible* if the only invariant closed ideals are $\{0\}$ and $L^2(I)$.

Proposition 25 (Irreducibility from indicator-energy criterion)**.** *Under Theorem 23, if* $\mathcal{E}_\lambda(\mathbf{1}_B) = 0$ *implies* $m(B) \in \{0, m(I)\}$*, then the semigroup* $T(t) = e^{-tA_\lambda}$ *is irreducible.*

Remark 26 (What is used here). This proposition is a standard equivalence in Dirichlet form theory: for symmetric Markovian semigroups, invariant ideals correspond to measurable invariant sets, and invariant sets correspond to sets with zero energy for their indicators. We do not reprove the general equivalence here; it is a well-known part of the Beurling–Deny/Fukushima theory of symmetric Dirichlet forms.

Corollary 27 (Irreducibility for $\mathcal{E}_\lambda$)**.** *Assume Theorem 23. Then* $T(t) = e^{-tA_\lambda}$ *is irreducible.*

Proof. Lemma 11 provides the indicator-energy criterion, so Proposition 25 applies. □

8 Positivity improving and the ground state

8.1 External theorems used

Theorem 28 (Positivity improving from positivity + irreducibility + holomorphy)**.** *Let* E *be a Banach lattice and* S *a positive, irreducible, holomorphic* C_0*-semigroup on* E*. Then* S *is positivity improving: for each* $t > 0$ *and each* $0 \leq f \in E$ *with* $f \neq 0$*, one has* $S(t)f > 0$ *(in the lattice sense; on* L^2 *this means* > 0 *a.e.).*

Remark 29 (Source). This statement appears, for example, as Theorem 2.3 in Arendt et al., *Strict positivity for the principal eigenfunction of elliptic operators with various boundary conditions* (see §11).

Theorem 30 (Simplicity of the principal eigenvalue under compact resolvent)**.** *Let A be selfadjoint and lower bounded on $L^2(I)$ with compact resolvent, and let $S(t) = e^{-tA}$. If S is positivity improving, then the bottom of the spectrum* $\min \sigma(A)$ *is a simple eigenvalue and admits an eigenfunction which is strictly positive a.e.*

Remark 31 (Source). This is a standard Perron–Frobenius/Krein–Rutman/Jentzsch consequence for compact positive operators, often stated for $(A + \mu)^{-1}$ or for $S(t)$ when it is compact. See, e.g., Proposition 2.4 in the same paper of Arendt et al.

8.2 Application to A_λ

Proposition 32 (Positivity improving and simple ground state for A_λ)**.** *Assume Theorem 23. Then:*

1. *The semigroup $T(t) = e^{-tA_\lambda}$ is positivity preserving (Markovian).*
2. *$T(t)$ is irreducible.*
3. *$T(t)$ is holomorphic (indeed, A_λ is selfadjoint and lower bounded).*

Consequently $T(t)$ is positivity improving, and the lowest eigenvalue of A_λ is simple with a strictly positive a.e. eigenfunction.

Proof. (1) Markov/positivity preservation follows from Lemma 9 and standard closed-form theory. (2) is Corollary 27. (3) Since A_λ is selfadjoint and lower bounded, e^{-zA_λ} is bounded and holomorphic on $\{z \in \mathbb{C} : \Re z > 0\}$ by the spectral theorem.

Now apply Theorem 28 to obtain positivity improving, and then Theorem 30 to obtain simplicity and strict positivity of the ground state. □

9 Evenness of the ground state from inversion symmetry

Proposition 33 (Inversion (reflection) symmetry)**.** *Let $R : L^2(I) \to L^2(I)$ be the unitary involution $(RG)(u) := G(-u)$. Then $R(\mathcal{D}(\mathcal{E}_\lambda)) = \mathcal{D}(\mathcal{E}_\lambda)$ and*

$$\mathcal{E}_\lambda(RG) = \mathcal{E}_\lambda(G) \qquad (G \in \mathcal{D}(\mathcal{E}_\lambda)).$$

Consequently, the associated operator A_λ from Theorem 23 commutes with R.

Proof. Identify $L^2(I)$ with the closed subspace $H_I \subset L^2(\mathbb{R})$ via extension by 0. Let the same symbol R denote reflection on $L^2(\mathbb{R})$: $(R\phi)(u) := \phi(-u)$. Then R is unitary, preserves H_I (since I is symmetric), and satisfies $RS_t = S_{-t}R$. Therefore, for $t \in \mathbb{R}$ and $\phi \in L^2(\mathbb{R})$,

$$\|R\phi - S_t R\phi\|_2 = \|R(\phi - S_{-t}\phi)\|_2 = \|\phi - S_{-t}\phi\|_2 = \|\phi - S_t\phi\|_2,$$

using that R is unitary and $\|\phi - S_{-t}\phi\|_2 = \|S_t\phi - \phi\|_2 = \|\phi - S_t\phi\|_2$. Since every weight in Definition 6 is nonnegative, this implies $\mathcal{E}_\lambda(RG) = \mathcal{E}_\lambda(G)$.

For commutation with A_λ: invariance of a closed form under a unitary U implies that the associated selfadjoint operator commutes with U. Indeed, for $u \in \mathcal{D}(A_\lambda)$ and $v \in \mathcal{D}(\mathcal{E}_\lambda)$,

$$\langle A_\lambda Ru, v\rangle = \mathcal{E}_\lambda(Ru, v) = \mathcal{E}_\lambda(u, R^{-1}v) = \langle A_\lambda u, R^{-1}v\rangle = \langle RA_\lambda u, v\rangle,$$

so $A_\lambda Ru = RA_\lambda u$. □

Corollary 34 (Even ground state)**.** *Assume Theorem 23 and 33. Let ψ be the strictly positive ground-state eigenfunction from Proposition 32. Then ψ is even: $\psi(-u) = \psi(u)$ a.e.*

Proof. Since $A_\lambda R = RA_\lambda$, the function $\psi^\sharp := R\psi$ is an eigenfunction for the same lowest eigenvalue. Moreover $\psi^\sharp > 0$ a.e. because $\psi > 0$ a.e. By simplicity of the ground-state eigenspace (Proposition 32), $\psi^\sharp = c\psi$ for some $c \in \mathbb{R}$. Positivity forces $c > 0$, and normalizing $\|\psi^\sharp\|_2 = \|\psi\|_2$ yields $c = 1$. Hence $\psi(-u) = \psi(u)$ a.e. □

10 Summary of concrete progress

- Starting solely from the explicit local formulas (2)–(3), we derived a representation of $-\sum_v W_v(g * g^*)$ (up to an additive constant multiple of $\|g\|_2^2$) as a positive combination of translation-difference energies in log-coordinates (Definition 6, Lemmas 4–5).
- We proved the Markov/normal contraction inequality for this form (Lemma 9).
- Using only measure theory (Lebesgue density), we proved that invariance under all sufficiently small translations forces a measurable subset of an interval to be null or conull (Lemma 10), and we used it to show that $\mathcal{E}_\lambda(\mathbf{1}_B) = 0$ implies B is null or conull (Lemma 11).
- Assuming the standard operator setup (closed form, selfadjoint operator, compact resolvent), we obtained irreducibility and then (by a standard external theorem) positivity improving of the semigroup, hence simplicity and strict positivity of the ground state (Proposition 32).
- Finally, inversion symmetry forces that strictly positive simple ground state to be even (Corollary 34).

11 Bibliographic pointers

References

[1] W. Arendt, D. Daners, M. Dier, and P. K. Jimenez. *Strict positivity for the principal eigenfunction of elliptic operators with various boundary conditions.* Available as arXiv:1909.12194 and published versions; see Theorem 2.3 and Proposition 2.4 therein for positivity improving and simplicity consequences used in §8.

[2] W. Arendt, C. J. K. Batty, M. Hieber, and F. Neubrander. *Vector-valued Laplace Transforms and Cauchy Problems.* 2nd ed., Birkhäuser, 2011.