A Five-Step Energy-Decomposition Proof of the Discrete Dirichlet Ground State Theorem

Abstract

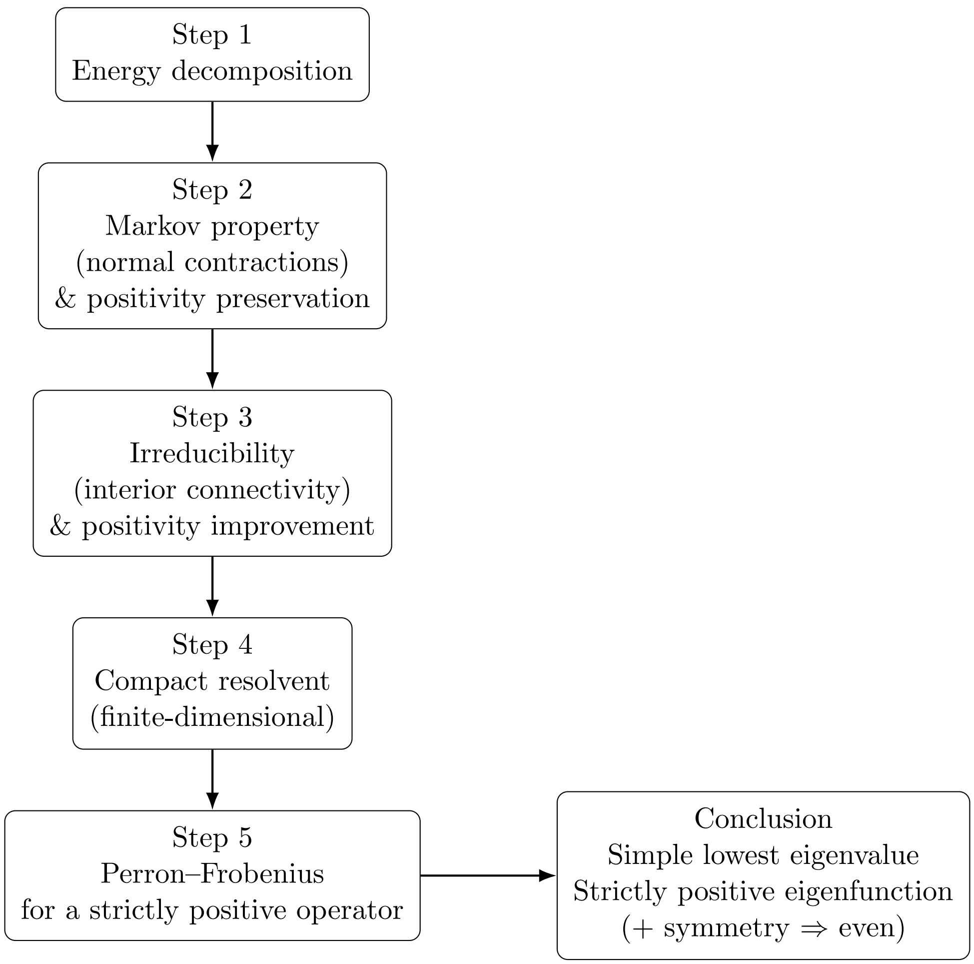
We give a completely rigorous and undergraduate-accessible proof of the classical "ground state" theorem for the Dirichlet graph Laplacian: on a connected interior, the smallest Dirichlet eigenvalue is simple and admits a strictly positive eigenfunction on the interior. The proof is organized to mirror a general methodology used in Dirichlet form theory: (1) energy decomposition, (2) Markov property, (3) irreducibility (connectivity of the interior), (4) compact resolvent (finite-dimensional), and (5) Perron–Frobenius/Krein–Rutman for a positive compact operator. Every step is proved concretely for finite graphs.

About standardness. We label each stated result as **(Standard)** when it is a well-known statement from linear algebra, spectral graph theory, discrete maximum principles, or Perron–Frobenius theory, and as **(New presentation)** when the content is primarily expository (e.g., the five-step organization or packaging of several standard facts). Proofs are included even for **(Standard)** results when they are short and illuminating.

Contents

1 The five-step spine and the main theorem

1.1 Flowchart



1.2 Main statement

Definition 1 (Weighted graph with boundary and connected interior)**.** Let $G = (V, E, w)$ be a finite undirected graph with vertex set V, edge set $E \subseteq \{\{x, y\} : x \neq y\}$, and weights $w_{xy} = w_{yx} > 0$ for $\{x, y\} \in E$ (and $w_{xy} = 0$ otherwise). Fix a nonempty boundary set $B \subset V$, and write $I := V \setminus B$ for the interior. Assume:

(i) G is connected (as an undirected graph),

(ii) $B \neq \emptyset$ and $I \neq \emptyset$,

(iii) the *interior induced graph* $G_I := (I, E_I, w)$ is connected, where

$$E_I := \big\{\{x, y\} \in E : \; x \in I, \; y \in I\big\}.$$

Remark 2 (Why interior connectivity is the right hypothesis). If G_I is not connected, then the Dirichlet operator L_B on $\mathbb{R}^I$ decomposes into a direct sum over the connected components of G_I, and the global smallest eigenvalue need not be simple. In particular, the conclusion "$\varphi_1 > 0$ on all of I" can fail. Theorem 4 therefore assumes the standard analogue of "connected domain": connectivity of the interior. A fully general componentwise statement is recorded in Remark 23 below.

Definition 3 (Dirichlet Laplacian)**.** For a function $f : V \to \mathbb{R}$ with $f|_B = 0$, define for $x \in I$

$$(L_B f)(x) := \sum_{y \in V} w_{xy} \big(f(x) - f(y)\big), \tag{1}$$

where $f(y) = 0$ for $y \in B$ (Dirichlet condition). Equivalently, L_B is the principal submatrix of the full Laplacian corresponding to I. We view L_B as a linear operator on $\mathbb{R}^I$ by identifying $f \in \mathbb{R}^I$ with its zero extension to V.

Theorem 4 (Discrete Dirichlet ground state theorem **(Standard)**)**.** *Let (G, w) be a finite connected weighted graph and $B \subset V$ be a nonempty boundary set, with interior $I = V \setminus B$ nonempty, and assume the induced interior graph G_I is connected. Then:*

(a) The smallest eigenvalue λ_1 of L_B (acting on $\mathbb{R}^I$) is strictly positive.

(b) λ_1 is simple (geometric and algebraic multiplicity one).

(c) An eigenfunction φ_1 for λ_1 can be chosen strictly positive on I:

$$\varphi_1(x) > 0 \quad \textit{for all } x \in I.$$

(d) (Optional symmetry corollary) If $R : V \to V$ is an involution ($R^2 = \mathrm{id}$) preserving E, w and preserving B, then φ_1 can be chosen R-even: $\varphi_1 \circ R = \varphi_1$ on I.

Remark 5 (What is "known" versus what is "methodology" **(New presentation)**). Theorem 4 is classical in spectral graph theory (see, e.g., [4, 3]); what is emphasized here is the organization into the five-step energy-decomposition spine and the explicit, reusable pattern of the proof.

2 Step 0: The Dirichlet energy form and the variational viewpoint

We work in the finite-dimensional Hilbert space $\ell^2(I)$ with inner product $\langle f, g\rangle := \sum_{x\in I} f(x)g(x)$.

Definition 6 (Dirichlet energy form)**.** For $f, g \in \mathbb{R}^I$, let $\tilde{f}, \tilde{g} : V \to \mathbb{R}$ be their zero extensions: $\tilde{f}|_I = f$ and $\tilde{f}|_B = 0$. Define the symmetric bilinear form

$$\mathcal{E}(f, g) := \frac{1}{2} \sum_{\{x,y\}\in E} w_{xy} \big(\tilde{f}(x) - \tilde{f}(y)\big)\big(\tilde{g}(x) - \tilde{g}(y)\big), \tag{2}$$

and write $\mathcal{E}(f) := \mathcal{E}(f, f)$.

Lemma 7 (Green's identity: the generator of the form **(Standard)**)**.** *For all $f, g \in \mathbb{R}^I$,*

$$\mathcal{E}(f, g) = \langle L_B f, g\rangle.$$

In particular, L_B is self-adjoint and positive semidefinite.

Proof. Write f, g for their zero extensions on V to simplify notation. Expand:

$$\mathcal{E}(f,g) = \frac{1}{2} \sum_{\{x,y\}\in E} w_{xy}\,(f(x)-f(y))(g(x)-g(y)).$$

Regroup by vertices $x \in I$ (note $g|_B = 0$):

$$\mathcal{E}(f,g) = \sum_{x\in I} g(x) \sum_{y\in V} w_{xy}(f(x)-f(y)) = \sum_{x\in I} g(x)\,(L_B f)(x) = \langle L_B f, g\rangle.$$

Self-adjointness follows by symmetry of $\mathcal{E}$, and $\mathcal{E}(f) = \langle L_B f, f\rangle \geq 0$ gives positivity. ☐

Remark 8 (Background: spectral theorem for symmetric matrices **(Standard)**). We repeatedly use that real symmetric matrices have an orthonormal eigenbasis and real eigenvalues (the spectral theorem); see, e.g., [5, 6].

Remark 9 (Variational characterization (optional background)). Since L_B is real symmetric, its eigenvalues are real and its smallest eigenvalue satisfies

$$\lambda_1 = \min_{f\neq 0} \frac{\langle L_B f, f\rangle}{\langle f, f\rangle} = \min_{f\neq 0} \frac{\mathcal{E}(f)}{\|f\|_2^2}.$$

We will not use calculus of variations for the positivity/simplicity conclusions; instead we follow the five-step spine.

3 Step 1: Energy decomposition

Proposition 10 (Energy decomposition **(Standard)**)**.** *For every $f \in \mathbb{R}^I$,*

$$\mathcal{E}(f) = \frac{1}{2} \sum_{\{x,y\}\in E} w_{xy} \left(\tilde{f}(x) - \tilde{f}(y)\right)^2. \tag{3}$$

In particular, $\mathcal{E}(f) = 0$ if and only if $\tilde{f}(x) = \tilde{f}(y)$ for every edge $\{x,y\} \in E$.

Proof. This is immediate from Definition 6 with $g = f$, and each summand is nonnegative. ☐

4 Step 2: Markov property (normal contractions) and positivity preservation

Definition 11 (Normal contraction)**.** A function $\Phi : \mathbb{R} \to \mathbb{R}$ is a *normal contraction* if $\Phi(0) = 0$ and

$$|\Phi(a) - \Phi(b)| \leq |a-b| \quad \text{for all } a, b \in \mathbb{R}.$$

Lemma 12 (Markov property **(Standard)**)**.** *For any normal contraction Φ and any $f \in \mathbb{R}^I$,*

$$\mathcal{E}(\Phi \circ f) \leq \mathcal{E}(f),$$

where $(\Phi \circ f)(x) := \Phi(f(x))$ on I and is extended by 0 on B (consistent with $\Phi(0) = 0$).

Proof. For each edge $\{x, y\} \in E$,

$$\big(\Phi(\tilde{f}(x)) - \Phi(\tilde{f}(y))\big)^2 \leq \big(\tilde{f}(x) - \tilde{f}(y)\big)^2$$

by the 1-Lipschitz property. Multiply by w_{xy}, sum over edges, and use (3). □

Remark 13 (How Step 2 is used **(New presentation)**). Lemma 12 is a standard "Markovianity" property of the Dirichlet form (see, e.g., [9, 3] for general Dirichlet forms and for graph Laplacians). In the present finite-graph proof, we will not use Lemma 12 directly; instead we use the closely related and equally standard consequence that the resolvent $(L_B + \mu \mathrm{Id})^{-1}$ is positivity preserving (Lemma 16).

Definition 14 (Positive and negative parts)**.** For $u \in \mathbb{R}^I$, define $u^+(x) := \max\{u(x), 0\}$ and $u^-(x) := \max\{-u(x), 0\}$ so that $u = u^+ - u^-$ and $u^+u^- = 0$ pointwise.

Lemma 15 (Scalar inequality for the negative part **(Standard)**)**.** *For all $a, b \in \mathbb{R}$, writing $t^- := \max\{-t, 0\}$, one has*

$$(a - b)\big(a^- - b^-\big) \leq -\big(a^- - b^-\big)^2. \tag{4}$$

Proof. We distinguish cases.

(i) If $a \geq 0$ and $b \geq 0$, then $a^- = b^- = 0$, so both sides are 0.

(ii) If $a \leq 0$ and $b \leq 0$, then $a^- = -a$ and $b^- = -b$, so the left-hand side equals

$$(a - b)\big((-a) - (-b)\big) = (a - b)(-a + b) = -(a - b)^2 = -(a^- - b^-)^2.$$

(iii) If $a \geq 0$ and $b \leq 0$, then $a^- = 0$ and $b^- = -b$, so $a^- - b^- = b$ and

$$(a - b)(a^- - b^-) = (a - b)b = ab - b^2 \leq -b^2 = -(a^- - b^-)^2,$$

since $ab \leq 0$. The case $a \leq 0 \leq b$ is symmetric.

□

Lemma 16 (Resolvent exists and is positivity preserving **(Standard)**)**.** *Fix $\mu > 0$. For every $f \in \mathbb{R}^I$ there is a unique $u \in \mathbb{R}^I$ solving*

$$(L_B + \mu \mathrm{Id})u = f. \tag{5}$$

Moreover, if $f \geq 0$ (pointwise on I), then $u \geq 0$.

Proof. Existence and uniqueness follow since $L_B + \mu \mathrm{Id}$ is a real symmetric matrix with

$$\langle (L_B + \mu \mathrm{Id})u, u \rangle = \mathcal{E}(u) + \mu \|u\|_2^2 \geq \mu \|u\|_2^2,$$

so it is positive definite and hence invertible.

Assume $f \geq 0$ and let u solve (5). Take inner product with u^-:

$$\langle (L_B + \mu \mathrm{Id})u, u^- \rangle = \langle f, u^- \rangle \geq 0.$$

Expand the left side using $u = u^+ - u^-$ and Lemma 7:

$$\langle L_B u, u^- \rangle + \mu \langle u, u^- \rangle = \mathcal{E}(u, u^-) + \mu(\langle u^+, u^- \rangle - \|u^-\|_2^2) = \mathcal{E}(u, u^-) - \mu \|u^-\|_2^2$$

since $\langle u^+, u^- \rangle = 0$.

We claim $\mathcal{E}(u, u^-) \le 0$. Write the edge expansion:

$$\mathcal{E}(u, u^-) = \frac{1}{2} \sum_{\{x,y\} \in E} w_{xy}\, (\tilde{u}(x) - \tilde{u}(y))(\widetilde{u^-}(x) - \widetilde{u^-}(y)).$$

By Lemma 15, for scalars $a, b \in \mathbb{R}$ the inequality (4) holds. Applying (4) with $a = \tilde{u}(x)$, $b = \tilde{u}(y)$, multiplying by w_{xy}, and summing gives

$$\mathcal{E}(u, u^-) \le -\mathcal{E}(u^-, u^-) \le 0.$$

Therefore,

$$0 \le \langle (L_B + \mu \mathrm{Id})u, u^- \rangle \le -\mathcal{E}(u^-) - \mu \|u^-\|_2^2 \le -\mu \|u^-\|_2^2,$$

which forces $\|u^-\|_2 = 0$, hence $u^- \equiv 0$ and $u \ge 0$. $\square$

5 A coercivity lemma (positivity of the first eigenvalue)

The following is the Dirichlet analogue of “constants are the only zero-energy functions, and the boundary kills constants.”

Proposition 17 (Zero energy forces constancy on components **(Standard)**)**.** *If $f \in \mathbb{R}^I$ satisfies $\mathcal{E}(f) = 0$, then $\tilde{f}$ is constant on each connected component of G.*

Proof. If $\mathcal{E}(f) = 0$, then by Proposition 10 every edge term satisfies $\tilde{f}(x) = \tilde{f}(y)$. Along any path, values propagate, so $\tilde{f}$ is constant on each connected component. $\square$

Corollary 18 (Dirichlet coercivity; L_B is positive definite **(Standard)**)**.** *Assume G is connected and $B \ne \emptyset$ and $I \ne \emptyset$. If $f \in \mathbb{R}^I$ satisfies $\mathcal{E}(f) = 0$, then $f \equiv 0$. Equivalently, L_B is positive definite and all its eigenvalues are strictly positive.*

Proof. By Proposition 17, $\tilde{f}$ is constant on connected G. Since $\tilde{f}|_B = 0$ and $B \ne \emptyset$, the constant must be 0, hence $f \equiv 0$ on I. Now $\langle L_B f, f \rangle = \mathcal{E}(f)$ (Lemma 7) implies $\langle L_B f, f \rangle > 0$ for all $f \ne 0$, so L_B is positive definite. $\square$

Remark 19 (Where connectivity is used **(New presentation)**). Corollary 18 (and thus Theorem 4(a)) uses only that G is connected and that the boundary is nonempty: it ensures that the only globally constant zero-extension is the trivial one. By contrast, the *interior connectivity* hypothesis (Definition 1(iii)) is used only in Step 3 to obtain *positivity improvement* (Proposition 20) and hence simplicity/strict positivity of the ground state.

6 Step 3: Irreducibility (connected interior) and positivity improvement

6.1 A strong maximum principle adapted to the interior

Proposition 20 (Strong maximum principle for the resolvent on a connected interior **(Standard)**)**.** *Assume the induced interior graph G_I is connected. Fix $\mu > 0$ and let u solve $(L_B + \mu \mathrm{Id})u = f$. If $f \ge 0$ and $f \not\equiv 0$, then $u(x) > 0$ for every $x \in I$.*

Proof. By Lemma 16, $u \geq 0$. Suppose for contradiction that there exists $x_0 \in I$ with $u(x_0) = 0$. Evaluate the equation at x_0:

$$f(x_0) = (L_B u)(x_0) + \mu u(x_0) = \sum_{y \in V} w_{x_0 y}\big(u(x_0) - u(y)\big) = -\sum_{y \in V} w_{x_0 y} u(y) \leq 0,$$

since all $u(y) \geq 0$ and weights are nonnegative. But $f(x_0) \geq 0$, hence $f(x_0) = 0$ and

$$\sum_{y \in V} w_{x_0 y} u(y) = 0.$$

Every term in the sum is nonnegative, and $w_{x_0 y} > 0$ whenever $\{x_0, y\} \in E$, so we deduce

$$u(y) = 0 \quad \text{for every neighbor } y \text{ of } x_0.$$

Now restrict to neighbors $y \in I$ (interior neighbors). The same argument shows that every interior neighbor of x_0 also has value 0. Iterating along any path in the *interior graph* G_I shows that $u \equiv 0$ on the entire connected interior I.

Finally, $(L_B + \mu \mathrm{Id})u = f$ then forces $f \equiv 0$, contradicting $f \not\equiv 0$. Therefore no such x_0 exists and $u(x) > 0$ for all $x \in I$. □

Remark 21 (Background: discrete maximum principles **(Standard)**)*.* The argument in Proposition 20 is the standard discrete maximum principle for graph Laplacians with Dirichlet boundary; see, for example, [4, Ch. 1] or [3, Ch. 3].

Corollary 22 (The resolvent matrix is strictly positive **(Standard)**)**.** *Assume G_I is connected. Let $I = \{1, \dots, n\}$ index interior vertices. For $\mu > 0$, define*

$$R_\mu := (L_B + \mu \mathrm{Id})^{-1} \in \mathbb{R}^{n \times n}.$$

Then every entry of R_μ is strictly positive: $(R_\mu)_{ij} > 0$ for all i, j.

Proof. The jth column of R_μ is $u = R_\mu e_j$, where $e_j \geq 0$ and $e_j \not\equiv 0$. By Proposition 20, $u(i) > 0$ for all i, i.e. $(R_\mu)_{ij} > 0$ for all i. □

Remark 23 (Componentwise version without interior connectivity)*.* If G_I has connected components $I = \bigsqcup_\alpha I_\alpha$, then the same proof shows: for $f \geq 0$ and $u = (L_B + \mu \mathrm{Id})^{-1} f$, one has $u \geq 0$ and $u > 0$ on each component I_α where $f \not\equiv 0$. Equivalently, after reordering vertices, R_μ is block diagonal with strictly positive blocks corresponding to the I_α. This is the correct general form of "positivity improvement."

7 Step 4: Compact resolvent (finite-dimensional)

Lemma 24 (Compactness is automatic **(Standard)**)**.** *In finite dimensions, every linear operator is bounded and maps bounded sets to relatively compact sets. In particular, the resolvent $R_\mu = (L_B + \mu \mathrm{Id})^{-1}$ is a compact operator on $\ell^2(I)$.*

Proof. In $\mathbb{R}^n$, bounded sets have compact closure (Heine–Borel; see, e.g., [8]). Since R_μ is linear and continuous, it maps bounded sets to bounded sets, hence to relatively compact sets. □

8 Step 5: Perron–Frobenius and the ground state

We apply Perron–Frobenius to the strictly positive, compact operator R_μ. Because L_B is symmetric, R_μ is also symmetric, and eigenvalues are diagonalizable; thus geometric simplicity implies algebraic simplicity automatically in our application.

8.1 Perron–Frobenius for strictly positive matrices (finite-dimensional)

Theorem 25 (Perron–Frobenius for strictly positive matrices **(Standard)**)**.** *Let $A \in \mathbb{R}^{n\times n}$ have strictly positive entries: $A_{ij} > 0$ for all i, j. Then:*

(i) The spectral radius $\rho(A)$ is an eigenvalue of A.

(ii) There exists an eigenvector $v \in \mathbb{R}^n$ with $v_i > 0$ for all i and $Av = \rho(A)v$.

(iii) The eigenspace for $\rho(A)$ is one-dimensional (geometric simplicity).

(iv) If $w \geq 0$ and $w \not\equiv 0$, then Aw has strictly positive entries.

Remark 26. Part (iv) is immediate from $A_{ij} > 0$. Parts (i)–(iii) are proved in Appendix A. Standard references include [2, 1, 6].

8.2 From Perron–Frobenius to the Dirichlet ground state

Proposition 27 (Ground state simplicity and positivity **(Standard)**)**.** *Assume G_I is connected and fix $\mu > 0$. Let $R_\mu = (L_B + \mu\mathrm{Id})^{-1}$. Then R_μ has a unique (up to scaling) strictly positive eigenvector φ associated to its spectral radius $\rho(R_\mu)$. Moreover, φ is an eigenvector of L_B for the smallest eigenvalue*

$$\lambda_1 = \rho(R_\mu)^{-1} - \mu,$$

and λ_1 is simple.

Proof. By Corollary 22, R_μ has strictly positive entries, so Theorem 25 applies. Hence $\rho(R_\mu)$ is an eigenvalue with a strictly positive eigenvector φ, and its eigenspace is one-dimensional.

If $L_B\varphi = \lambda\varphi$, then $(L_B + \mu\mathrm{Id})\varphi = (\lambda + \mu)\varphi$, hence $R_\mu\varphi = (\lambda + \mu)^{-1}\varphi$. Thus eigenvalues of R_μ are exactly $(\lambda+\mu)^{-1}$ where λ ranges over eigenvalues of L_B. The largest eigenvalue of R_μ is $\rho(R_\mu)$, so it corresponds to the smallest eigenvalue λ_1 of L_B via $\rho(R_\mu) = (\lambda_1+\mu)^{-1}$, i.e. $\lambda_1 = \rho(R_\mu)^{-1} - \mu$.

Since R_μ is symmetric, it is diagonalizable. Therefore one-dimensionality of the $\rho(R_\mu)$-eigenspace implies that $\rho(R_\mu)$ has algebraic multiplicity one. The eigenspace correspondence between L_B and R_μ preserves dimensions, so λ_1 is also simple (geometric and algebraic multiplicity one). □

Proof of Theorem 4. (a) By Corollary 18, L_B is positive definite, hence $\lambda_1 > 0$.

(b)–(c) By Proposition 27, λ_1 is simple and has an eigenfunction $\varphi_1 > 0$ on I.

(d) Suppose $R : V \to V$ is an involution preserving E, w and B. Then R maps I to itself and induces a linear operator $(Uf)(x) := f(Rx)$ on $\mathbb{R}^I$. A direct check from (1) shows $UL_B = L_BU$, hence also $UR_\mu = R_\mu U$. Let φ span the $\rho(R_\mu)$-eigenspace and satisfy $\varphi > 0$ on I. Then $U\varphi$ is also a $\rho(R_\mu)$-eigenvector. By simplicity, $U\varphi = c\varphi$ for some scalar c. Since $U^2 = \mathrm{Id}$, we have $c^2 = 1$, so $c = \pm 1$. But $\varphi > 0$ and $U\varphi > 0$, so $c = -1$ is impossible. Therefore $c = +1$, i.e. $\varphi \circ R = \varphi$ on I. □

9 Worked example (optional for exposition)

Remark 28 (Path graph). Take the path graph on vertices $\{0, 1, \dots, m\}$ with boundary $B = \{0, m\}$ and unit weights. Then $I = \{1, \dots, m-1\}$ is connected, and L_B is the familiar tridiagonal matrix with 2 on the diagonal and -1 on off-diagonals. Theorem 4 recovers the classical fact that the first Dirichlet mode is strictly positive and unique up to scale.

A A proof of Perron–Frobenius for strictly positive matrices

This appendix proves Theorem 25(i)–(iii) in a form sufficient for the main text. Throughout, $A \in \mathbb{R}^{n\times n}$ has $A_{ij} > 0$ for all i, j.

A.1 Existence of a positive eigenvector

Let

$$\Delta := \Big\{ x \in \mathbb{R}^n : x_i \ge 0,\ \sum_{i=1}^{n} x_i = 1 \Big\}$$

be the standard simplex (compact and convex). Define the continuous map $T : \Delta \to \Delta$ by

$$T(x) := \frac{Ax}{\mathbf{1}^\top Ax}.$$

Since A has strictly positive entries, Ax has strictly positive coordinates for every $x \in \Delta$, and $\mathbf{1}^\top Ax > 0$. Thus T is well-defined and continuous, and indeed $T(\Delta) \subseteq \Delta$.

Lemma 29 (Brouwer fixed point $\Rightarrow$ positive eigenvector **(Standard)**)**.** *There exists $v \in \Delta$ and $\lambda > 0$ such that $Av = \lambda v$ and $v_i > 0$ for all i.*

Proof. By Brouwer's fixed point theorem (see, e.g., [7]), T has a fixed point $v \in \Delta$: $T(v) = v$. Then $Av = (\mathbf{1}^\top Av)\, v$, so $Av = \lambda v$ with $\lambda := \mathbf{1}^\top Av > 0$. Since $A > 0$ and $v \neq 0$, all coordinates of Av are positive, hence all coordinates of v are positive as well. □

A.2 The eigenvalue equals the spectral radius

For a fixed vector $v > 0$, define the weighted sup norm

$$\|x\|_v := \max_{1\le i\le n} \frac{|x_i|}{v_i}.$$

This is a genuine norm on $\mathbb{R}^n$.

Lemma 30 (If $Av = \lambda v$ with $v > 0$, then $\rho(A) = \lambda$ **(Standard)**)**.** *Let $v > 0$ and $\lambda > 0$ satisfy $Av = \lambda v$. Then $\rho(A) = \lambda$.*

Proof. First, $\rho(A) \ge |\lambda| = \lambda$ since λ is an eigenvalue.

For the reverse inequality, let $x \in \mathbb{R}^n$ and set $t := \|x\|_v$. Then $|x_i| \le t v_i$ for all i, i.e. $|x| \le tv$ entrywise. Since A has nonnegative entries, this implies

$$|Ax| \le A|x| \le A(tv) = tAv = t\lambda v$$

entrywise. Dividing by v_i and taking maxima gives $\|Ax\|_v \le \lambda \|x\|_v$, hence $\|A\|_{v\to v} \le \lambda$ as an operator norm. A standard fact from linear algebra (see, e.g., [6, 5]) is that for any induced operator norm $\|\cdot\|$, every eigenvalue μ of A satisfies $|\mu| \le \|A\|$ (apply $\|Ax\| \le \|A\|\,\|x\|$ to an eigenvector). Consequently,

$$\rho(A) \le \|A\|_{v\to v} \le \lambda.$$

Therefore $\rho(A) = \lambda$. □

Together, Lemmas 29 and 30 give Theorem 25(i)–(ii).

A.3 Geometric simplicity

Lemma 31 (Uniqueness of the positive eigenvector **(Standard)**). *If $Av = \rho(A)v$ and $Aw = \rho(A)w$ with $v > 0$ and $w > 0$, then w is a positive scalar multiple of v.*

Proof. Define

$$t_* := \inf\{t > 0 : \ w \le tv \text{ entrywise}\}.$$

Since $v > 0$ and $w > 0$, the set is nonempty and $t_* \in (0, \infty)$. By definition, $w \le t_* v$ and there exists an index i_0 with $w_{i_0} = t_* v_{i_0}$.

If $w \ne t_* v$, then there exists j with $w_j < t_* v_j$. Because $A > 0$, strict inequality in one coordinate forces strict inequality after applying A in *every* coordinate:

$$(Aw)_i = \sum_j A_{ij} w_j < \sum_j A_{ij}(t_* v_j) = (t_* Av)_i \quad \text{for all } i,$$

hence $Aw < t_* Av$ entrywise. Using $Aw = \rho(A)w$ and $Av = \rho(A)v$, this becomes $w < t_* v$ entrywise, contradicting the definition of t_* (we could then decrease t slightly and still have $w \le tv$). Therefore $w = t_* v$. □

Lemma 32 (Any $\rho(A)$-eigenvector has constant sign **(Standard)**). *If $Ay = \rho(A)y$ and $y \ne 0$, then either $y \ge 0$ entrywise or $y \le 0$ entrywise.*

Proof. Suppose y has both positive and negative coordinates, and set $x := |y| > 0$ entrywise. Then by positivity of A,

$$Ax = A|y| \ge |Ay| = |\rho(A)y| = \rho(A)|y| = \rho(A)x$$

entrywise. Because y has mixed signs and $A > 0$, the triangle inequality is strict in every coordinate: for each i,

$$(A|y|)_i = \sum_j A_{ij}|y_j| \ > \ \Big|\sum_j A_{ij} y_j\Big| = |(Ay)_i|,$$

so $Ax > \rho(A)x$ entrywise.

Now apply A once more: since $Ax - \rho(A)x \ge 0$ is nonzero and $A > 0$, we have

$$A(Ax - \rho(A)x) > 0 \quad \Rightarrow \quad A^2 x > \rho(A)Ax \ge \rho(A)\cdot\rho(A)x = \rho(A)^2 x$$

entrywise. In particular, letting

$$m := \min_{1\le i\le n} \frac{(A^2x)_i}{x_i},$$

we have $m > \rho(A)^2$.

On the other hand, for any matrix M and any $x > 0$, the inequality $Mx \ge mx$ implies $\rho(M) \ge m$ by iterating $Mx \ge mx$ and taking norms (a one-line argument identical to the standard lower Collatz–Wielandt bound). Applying this to $M = A^2$ yields

$$\rho(A^2) \ge m > \rho(A)^2,$$

contradicting $\rho(A^2) = \rho(A)^2$. Therefore y cannot have mixed signs, proving the claim. □

Lemma 33 (Geometric simplicity of $\rho(A)$ **(Standard)**). *The eigenspace for $\rho(A)$ is one-dimensional.*

Proof. By Lemma 29 and Lemma 30, there exists $v > 0$ with $Av = \rho(A)v$. Let y satisfy $Ay = \rho(A)y$. By Lemma 32, either $y \geq 0$ or $y \leq 0$. Replacing y by $-y$ if necessary, we may assume $y \geq 0$ and $y \neq 0$.

Because $A > 0$ and $y \geq 0$, we have $Ay > 0$, hence y must in fact satisfy $y > 0$ (otherwise a zero coordinate would force a zero coordinate in $Ay = \rho(A)y$, impossible since $Ay > 0$). Thus $y > 0$, and Lemma 31 implies y is a scalar multiple of v. Therefore the eigenspace is one-dimensional. $\square$

Lemmas 29, 30, and 33 yield Theorem 25(i)–(iii).

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