Energy Decomposition, Compact Resolvent, and Perron–Frobenius Properties
of the Restricted Weil Quadratic Form

Abstract

We record a completely concrete and rigorous functional-analytic step that arises in the spectral approach to Weil's criterion when one restricts test functions to a compact multiplicative interval $[\lambda^{-1}, \lambda] \subset \mathbb{R}_+^*$. Starting from the explicit local distributions at the primes and at ∞, we derive an "energy decomposition" expressing the quadratic form (up to an additive constant multiple of $\|g\|_2^2$) as a positive combination of translation-difference energies $\|G - S_t G\|_2^2$ in logarithmic coordinates. We then prove the Markov (normal contraction) property and a translation-invariance lemma which yields irreducibility from the archimedean continuum of shifts. Finally, we show that the quadratic form is closed and that its associated selfadjoint operator has compact resolvent, using a logarithmic lower bound on the Fourier symbol together with the Kolmogorov–Riesz compactness criterion. From this we deduce that the ground-state eigenvalue is simple and its eigenfunction can be chosen strictly positive and, by inversion symmetry, even.

Note on proof style. Every proof in this document is presented in L. Lamport's hierarchical structured-proof format [3]. Each step states a claim and its justification, and sub-steps may be expanded for further detail. The intent is that any single step can be verified independently.

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1 Setup on $\mathbb{R}_+^*$

Let $\mathbb{R}_+^* = (0, \infty)$ with multiplicative Haar measure

$$d^*x := \frac{dx}{x}.$$

For measurable g, h define multiplicative convolution

$$(g * h)(x) := \int_{\mathbb{R}_+^*} g(y)\, h(x/y)\, d^*y,$$

and involution

$$g^*(x) := \overline{g(x^{-1})}.$$

If $g \in L^2(\mathbb{R}_+^*, d^*x)$, define the dilation operator

$$(U_a g)(x) := g(x/a) \qquad (a > 0). \tag{1}$$

Then U_a is unitary on $L^2(\mathbb{R}_+^*, d^*x)$: it is isometric ($\|U_a g\|_2 = \|g\|_2$ by Haar invariance $d^*(x/a) = d^*x$) and surjective ($U_a^{-1} = U_{a^{-1}}$). In particular, $\langle g, U_a g\rangle$ is well-defined and finite by Cauchy–Schwarz: $|\langle g, U_a g\rangle| \le \|g\|_2 \|U_a g\|_2 = \|g\|_2^2$.

Lemma 1 (Convolution inner-product identity)**.** *Let $f = g * g^*$. Then for all $a > 0$,*

$$f(a) = \langle g, U_a g\rangle_{L^2(d^*x)} = \int_{\mathbb{R}_+^*} g(x)\, \overline{g(x/a)}\, d^*x, \qquad f(a^{-1}) = \overline{f(a)}.$$

In particular $f(a) + f(a^{-1}) = 2\Re\langle g, U_a g\rangle$ and $f(1) = \|g\|_2^2$.

Structured Proof. **Step 1.** $f(a) = \langle g, U_a g\rangle$.

Step 1.1. Expand the definition of multiplicative convolution applied to $g * g^*$ at the point a. *Justification:* By definition, $(g * g^*)(a) = \int_{\mathbb{R}_+^*} g(y)\, g^*(a/y)\, d^*y$.

Step 1.2. Apply the definition of involution: $g^*(a/y) = \overline{g((a/y)^{-1})} = \overline{g(y/a)}$.

Justification: The involution is defined as $g^*(x) = \overline{g(x^{-1})}$. Set $x = a/y$ to get $g^*(a/y) = \overline{g(y/a)}$.

Step 1.3. Substitute into the integral: $(g * g^*)(a) = \int_{\mathbb{R}_+^*} g(y)\,\overline{g(y/a)}\,d^*y$.

Justification: Combine Steps 1.1 and 1.2.

Step 1.4. Recognize this as $\langle g, U_a g\rangle_{L^2(d^*x)}$.

Justification: By definition of the dilation operator (1), $(U_a g)(y) = g(y/a)$, and the $L^2(d^*x)$ inner product is $\langle g, h\rangle = \int g(y)\overline{h(y)}\,d^*y$. Thus $\int g(y)\,\overline{g(y/a)}\,d^*y = \langle g, U_a g\rangle$.

Step 2. $f(a^{-1}) = \overline{f(a)}$.

Justification: Replace a by a^{-1} in the result of Step 1: $f(a^{-1}) = \langle g, U_{a^{-1}} g\rangle = \int g(y)\,\overline{g(ya)}\,d^*y$. Substituting $y' = y/a$ (so $y = ay'$ and $d^*y = d^*y'$ by Haar invariance) gives $\int g(y/a)\,\overline{g(y)}\,d^*y = \overline{\int g(y)\,\overline{g(y/a)}\,d^*y} = \overline{f(a)}$.

Step 3. $f(a) + f(a^{-1}) = 2\Re\langle g, U_a g\rangle$ and $f(1) = \|g\|_2^2$.

Justification: From Steps 1 and 2, $f(a) + f(a^{-1}) = \langle g, U_a g\rangle + \overline{\langle g, U_a g\rangle} = 2\Re\langle g, U_a g\rangle$. Setting $a = 1$: $U_1 = \mathrm{Id}$, so $f(1) = \langle g, g\rangle = \|g\|_2^2$.

Q.E.D. □

Lemma 2 (A basic unitary identity)**.** *For any unitary U on a Hilbert space and any vector h,*

$$2\Re\langle h, Uh\rangle = 2\|h\|^2 - \|h - Uh\|^2.$$

Structured Proof. **Step 1.** $\|h - Uh\|^2 = \|h\|^2 + \|Uh\|^2 - 2\Re\langle h, Uh\rangle$.

Justification: Expand the inner product: $\|h - Uh\|^2 = \langle h - Uh, h - Uh\rangle = \langle h, h\rangle - \langle h, Uh\rangle - \langle Uh, h\rangle + \langle Uh, Uh\rangle = \|h\|^2 + \|Uh\|^2 - 2\Re\langle h, Uh\rangle$.

Step 2. $\|Uh\|^2 = \|h\|^2$.

Justification: U is unitary, hence isometric.

Step 3. Substituting Step 2 into Step 1: $\|h - Uh\|^2 = 2\|h\|^2 - 2\Re\langle h, Uh\rangle$.

Justification: Replace $\|Uh\|^2$ by $\|h\|^2$ in Step 1.

Step 4. Rearrange: $2\Re\langle h, Uh\rangle = 2\|h\|^2 - \|h - Uh\|^2$.

Justification: Solve Step 3 for $2\Re\langle h, Uh\rangle$.

Q.E.D. □

2 Local explicit-formula terms

Fix $\lambda > 1$ and consider $g \in C_c^\infty([\lambda^{-1}, \lambda])$. (This regularity ensures that $f = g * g^*$ is smooth and compactly supported in $[\lambda^{-2}, \lambda^2]$, so all integrals below converge absolutely. The quadratic form $\mathcal{E}_\lambda$ defined in Section 4.3 makes sense for arbitrary $G \in L^2(I)$ as an extended-real-valued form, and the subsequent functional-analytic results depend only on that abstract definition.)

We record the two local distributions we use; these are the only "input formulas".

2.1 Prime terms

For a prime p define

$$W_p(f) := (\log p) \sum_{m \geq 1} p^{-m/2}\big(f(p^m) + f(p^{-m})\big). \tag{2}$$

2.2 Archimedean term

Define

$$W_{\mathbb{R}}(f) := (\log 4\pi + \gamma)\, f(1) + \int_1^\infty \Big(f(x) + f(x^{-1}) - 2x^{-1/2} f(1)\Big) \frac{x^{1/2}}{x - x^{-1}}\, d^*x, \tag{3}$$

where γ is the Euler–Mascheroni constant.

Remark 3 (Restriction to a compact multiplicative interval)*.* If $\operatorname{supp}(g) \subset [\lambda^{-1}, \lambda]$, then for $a > \lambda^2$ the supports of g and $U_a g$ are disjoint, hence $\langle g, U_a g\rangle = 0$ and $f(a) = 0$. Consequently:

- in (2) only those (p, m) with $p^m \le \lambda^2$ contribute;
- in (3), after the change of variables $x = e^t$, only $t \in [0, 2\log\lambda]$ contributes to the term involving $f(e^t) + f(e^{-t})$.

This finiteness is crucial and is completely elementary.

3 Logarithmic coordinates and translations

Set $u = \log x$, so that $d^*x = du$ and the interval $[\lambda^{-1}, \lambda]$ becomes

$$I := (-L, L), \qquad L := \log\lambda.$$

For $G \in L^2(I)$ we denote by $\widetilde{G}$ its extension by 0 to $\mathbb{R}$. Let S_t be translation on $L^2(\mathbb{R})$:

$$(S_t\phi)(u) := \phi(u - t).$$

Then in logarithmic coordinates, the dilation U_{e^t} from (1) corresponds to translation: if $G(u) = g(e^u)$, then $(U_{e^t} g)(e^u) = g(e^{u-t})$, i.e. $\widetilde{G} \mapsto S_t\widetilde{G}$.

4 Energy decomposition into translation differences

4.1 Prime contributions

Lemma 4 (Prime term as a difference energy plus a constant)**.** *Let $f = g * g^*$ with $g \in C_c^\infty([\lambda^{-1}, \lambda])$, and let $G(u) = g(e^u)$. Then*

$$-W_p(f) = \sum_{\substack{m \ge 1 \\ p^m \le \lambda^2}} (\log p)\, p^{-m/2}\, \|\widetilde{G} - S_{m\log p}\widetilde{G}\|^2_{L^2(\mathbb{R})} \;+\; c_p(\lambda)\, \|G\|^2_{L^2(I)},$$

where $c_p(\lambda) \in \mathbb{R}$ is a finite constant depending only on p and λ.

Structured Proof. **Step 1.** $W_p(f) = (\log p) \sum_{m \ge 1} p^{-m/2}\, 2\Re\langle g, U_{p^m} g\rangle$.

Step 1.1. By (2), $W_p(f) = (\log p) \sum_{m \ge 1} p^{-m/2} \big(f(p^m) + f(p^{-m})\big)$.

Justification: Definition of W_p. (The sum is in fact finite: Step 3 below shows $\langle g, U_{p^m} g\rangle = 0$ for $p^m > \lambda^2$, so only finitely many terms contribute. Steps 1–2 are therefore a finite computation.)

Step 1.2. $f(p^m) + f(p^{-m}) = 2\Re\langle g, U_{p^m} g\rangle$.

Justification: By Lemma 1, $f(a) + f(a^{-1}) = 2\Re\langle g, U_a g\rangle$. Set $a = p^m$.

Step 1.3. Combine Steps 1.1 and 1.2.

Justification: Substitute the identity from Step 1.2 into the sum from Step 1.1.

Step 2. For each $m \geq 1$ with $p^m \leq \lambda^2$: $2\Re\langle g, U_{p^m} g\rangle = 2\|g\|_2^2 - \|g - U_{p^m} g\|_2^2$.
Justification: Lemma 2 applied with $U = U_{p^m}$, $h = g$.

Step 3. For $m \geq 1$ with $p^m > \lambda^2$: $\langle g, U_{p^m} g\rangle = 0$.
Justification: By Remark 3: when $p^m > \lambda^2$, the supports of g (in $[\lambda^{-1}, \lambda]$) and $U_{p^m} g$ (in $[p^m\lambda^{-1}, p^m\lambda]$) are disjoint.

Step 4. In logarithmic coordinates: $\|g - U_{p^m} g\|_2 = \|\widetilde{G} - S_{m \log p}\widetilde{G}\|_{L^2(\mathbb{R})}$.
Justification: The substitution $u = \log x$ converts $d^*x = du$, $g(x) \mapsto G(u)$, and $(U_{p^m} g)(x) = g(x/p^m) \mapsto G(u - m\log p) = (S_{m \log p}\widetilde{G})(u)$. The $L^2(\mathbb{R}_+^*, d^*x)$ norm becomes the $L^2(\mathbb{R}, du)$ norm. (Here $\widetilde{G}$ denotes the zero-extension of G to $\mathbb{R}$; the identity holds because g is supported in $[\lambda^{-1}, \lambda]$, so G is supported in $I = [-L, L]$, and the substitution $u = \log x$ applies simultaneously to both terms.)

Step 5. Assemble the formula for $-W_p(f)$.

Step 5.1. From Steps 1–3, only terms with $p^m \leq \lambda^2$ contribute, and for those terms:

$$W_p(f) = (\log p) \sum_{\substack{m \geq 1 \\ p^m \leq \lambda^2}} p^{-m/2} \big(2\|g\|_2^2 - \|g - U_{p^m} g\|_2^2\big).$$

Justification: Combine Steps 1, 2, and 3: the terms with $p^m > \lambda^2$ vanish by Step 3; the remaining terms are rewritten using Step 2.

Step 5.2. Negate and use Step 4:

$$-W_p(f) = \sum_{\substack{m \geq 1 \\ p^m \leq \lambda^2}} (\log p)\, p^{-m/2} \, \|\widetilde{G} - S_{m \log p}\widetilde{G}\|_{L^2(\mathbb{R})}^2 - 2(\log p)\Big(\sum_{\substack{m \geq 1 \\ p^m \leq \lambda^2}} p^{-m/2}\Big)\|g\|_2^2.$$

Justification: Negate Step 5.1 and replace $\|g - U_{p^m} g\|_2$ by $\|\widetilde{G} - S_{m \log p}\widetilde{G}\|_{L^2(\mathbb{R})}$ using Step 4.

Step 5.3. Set $c_p(\lambda) := -2(\log p)\sum_{m \geq 1,\, p^m \leq \lambda^2} p^{-m/2}$. Then $c_p(\lambda) \in \mathbb{R}$ is finite (the sum has finitely many terms), and $\|g\|_2^2 = \|G\|_{L^2(I)}^2$.

Justification: The sum is finite because only finitely many integers m satisfy $p^m \leq \lambda^2$. The norm identity $\|g\|_2 = \|G\|_{L^2(I)}$ follows from the change of variables $u = \log x$.

Q.E.D. $\square$

4.2 Archimedean contribution

Lemma 5 (Archimedean term as a continuum of difference energies plus a constant)**.** *Let $f = g * g^*$ with $g \in C_c^\infty([\lambda^{-1}, \lambda])$, and let $G(u) = g(e^u)$. Define the strictly positive weight on $(0, \infty)$,*

$$w(t) := \frac{e^{t/2}}{e^t - e^{-t}} = \frac{e^{t/2}}{2\sinh t}.$$

Then

$$-W_{\mathbb{R}}(f) = \int_0^{2L} w(t)\, \|\widetilde{G} - S_t\widetilde{G}\|_{L^2(\mathbb{R})}^2 \, dt \; + \; c_\infty(\lambda)\, \|G\|_{L^2(I)}^2,$$

where $c_\infty(\lambda) \in \mathbb{R}$ is a finite constant depending only on λ.

Structured Proof. **Step 1.** Rewrite $W_{\mathbb{R}}(f)$ by substituting $x = e^t$:

$$W_{\mathbb{R}}(f) = (\log 4\pi + \gamma)\, f(1) + \int_0^\infty \Big(f(e^t) + f(e^{-t}) - 2e^{-t/2} f(1) \Big)\, w(t)\, dt.$$

Justification: Starting from (3), set $x = e^t$ so that $d^*x = dt$. Then $x^{1/2}/(x - x^{-1}) = e^{t/2}/(e^t - e^{-t}) = w(t)$, and the integration range $x \in [1, \infty)$ becomes $t \in [0, \infty)$. (Convergence: near $t = 0$, $f(e^t) + f(e^{-t}) - 2e^{-t/2} f(1) = O(t)$ by Taylor expansion of the smooth function f, cancelling the $1/t$ singularity of $w(t) \sim 1/(2t)$; for $t > 2L$, $f(e^t) = f(e^{-t}) = 0$ by Remark 3 and the remaining term $2e^{-t/2} w(t) f(1) = f(1)/\sinh t$ is $O(e^{-t})$.)

Step 2. $f(1) = \|g\|_2^2$ and $f(e^t) + f(e^{-t}) = 2\Re\langle g, U_{e^t} g\rangle$.

Justification: The first identity is Lemma 1 with $a = 1$. The second is Lemma 1: $f(a) + f(a^{-1}) = 2\Re\langle g, U_a g\rangle$ with $a = e^t$.

Step 3. $-2\Re\langle g, U_{e^t} g\rangle = \|g - U_{e^t} g\|_2^2 - 2\|g\|_2^2$.

Justification: Lemma 2 with $U = U_{e^t}$, $h = g$: $2\Re\langle g, U_{e^t} g\rangle = 2\|g\|_2^2 - \|g - U_{e^t} g\|_2^2$. Negate both sides.

Step 4. Substituting Steps 2 and 3 into the integral from Step 1, the integrand of $-W_{\mathbb{R}}(f)$ (inside $\int_0^\infty$) equals

$$w(t)\big(\|g - U_{e^t} g\|_2^2 + 2(e^{-t/2} - 1)\|g\|_2^2\big).$$

Step 4.1. From Step 1 (negated), $-W_{\mathbb{R}}(f) = -(\log 4\pi + \gamma)\|g\|_2^2 + \int_0^\infty \big(-2\Re\langle g, U_{e^t} g\rangle + 2e^{-t/2}\|g\|_2^2\big) w(t)\, dt$.

Justification: Negate Step 1 and use $f(1) = \|g\|_2^2$ from Step 2. (Absolute convergence of this integral is verified in Steps 7–8 below; we proceed with the algebraic manipulation.)

Step 4.2. Replace $-2\Re\langle g, U_{e^t} g\rangle$ by $\|g - U_{e^t} g\|_2^2 - 2\|g\|_2^2$ (Step 3): the integrand becomes $(\|g - U_{e^t} g\|_2^2 - 2\|g\|_2^2 + 2e^{-t/2}\|g\|_2^2) w(t) = w(t)(\|g - U_{e^t} g\|_2^2 + 2(e^{-t/2} - 1)\|g\|_2^2)$.

Justification: Algebra: $-2 + 2e^{-t/2} = 2(e^{-t/2} - 1)$.

Step 5. In logarithmic coordinates: $\|g - U_{e^t} g\|_2 = \|\widetilde{G} - S_t \widetilde{G}\|_{L^2(\mathbb{R})}$.

Justification: Same argument as in Lemma 4, Step 4.

Step 6. Split the integral at $t = 2L$. For $t > 2L$, $\|\widetilde{G} - S_t \widetilde{G}\|_2^2 = 2\|G\|_2^2$.

Justification: By Remark 3, when $t > 2L = 2\log\lambda$, the supports of $\widetilde{G}$ (contained in $[-L, L]$) and $S_t \widetilde{G}$ (contained in $[-L + t, L + t]$, with $-L + t > L$) are disjoint. Hence $\|\widetilde{G} - S_t \widetilde{G}\|_2^2 = \|\widetilde{G}\|_2^2 + \|S_t \widetilde{G}\|_2^2 = 2\|G\|_2^2$.

Step 7. The tail integral over $(2L, \infty)$ is a finite constant times $\|G\|_2^2$.

Step 7.1. For $t > 2L$, the integrand from Step 4 becomes $w(t)(2\|G\|_2^2 + 2(e^{-t/2} - 1)\|G\|_2^2) = 2e^{-t/2} w(t)\|G\|_2^2$.

Justification: Substitute $\|\widetilde{G} - S_t \widetilde{G}\|_2^2 = 2\|G\|_2^2$ from Step 6 into the integrand from Step 4 (after applying Step 5): $2\|G\|_2^2 + 2(e^{-t/2} - 1)\|G\|_2^2 = 2e^{-t/2}\|G\|_2^2$.

Step 7.2. $\int_{2L}^\infty 2e^{-t/2} w(t)\, dt < \infty$.

Justification: $2e^{-t/2} w(t) = 2e^{-t/2} \cdot e^{t/2}/(2\sinh t) = 1/\sinh t$. For $t \geq 1$: $e^{-t} < 1$ and $e^t > 2$, so $e^t - e^{-t} > e^t/2$, giving $\sinh t > e^t/4$, hence $1/\sinh t < 4e^{-t}$. On $(0, 1] \cap (2L, \infty)$ (nonempty only if $2L < 1$), $1/\sinh t$ is continuous and bounded. Therefore $\int_{2L}^\infty 1/\sinh t\, dt \leq C + 4\int_1^\infty e^{-t}\, dt < \infty$.

Step 8. The integral over $[0, 2L]$ contributes the main term plus a finite constant.

Step 8.1. For $t \in [0, 2L]$, the integrand (Step 4 with Step 5) splits as $w(t)\|\widetilde{G} - S_t \widetilde{G}\|_2^2 + 2(e^{-t/2} - 1) w(t)\|G\|_2^2$.

Justification: Rewrite Step 4 using Step 5.

Step 8.2. $\int_0^{2L} 2(e^{-t/2}-1)w(t)\,dt$ is finite.

Justification: By convexity of e^{-x}: $e^{-t/2} \geq 1 - t/2$, so $|e^{-t/2}-1| \leq t/2$. Since $\sinh t \geq t$ for $t \geq 0$: $w(t) = e^{t/2}/(2\sinh t) \leq e^{t/2}/(2t)$. Hence $|2(e^{-t/2}-1)w(t)| \leq 2 \cdot (t/2) \cdot e^{t/2}/(2t) = e^{t/2}/2 \leq e^L/2$ on $[0, 2L]$. The integrand is bounded on a compact interval, so the integral converges absolutely.

Step 9. Define $c_\infty(\lambda) := -(\log 4\pi + \gamma) + \int_0^{2L} 2(e^{-t/2}-1)w(t)\,dt + \int_{2L}^\infty 2e^{-t/2}w(t)\,dt$.

Justification: Collect all $\|G\|_2^2$-proportional terms: the $-(\log 4\pi+\gamma)$ from Step 4.1, the integral $\int_0^{2L} 2(e^{-t/2}-1)w(t)\,dt$ from Step 8.2, and the tail integral from Step 7.2. Each is finite, so $c_\infty(\lambda) \in \mathbb{R}$.

Step 10. Conclusion: $-W_{\mathbb{R}}(f) = \int_0^{2L} w(t)\|\widetilde{G} - S_t\widetilde{G}\|_2^2\,dt + c_\infty(\lambda)\|G\|_{L^2(I)}^2$.

Justification: Combine Steps 8.1, 7, and 9, noting $\|g\|_2^2 = \|G\|_{L^2(I)}^2$ (by the isometry $u = \log x$, as in Lemma 4, Step 5.3).

Q.E.D. □

4.3 Global quadratic form on the interval

Definition 6 (Difference-energy form). Fix $\lambda > 1$ and $L = \log\lambda$. For $G \in L^2(I)$ define

$$\mathcal{E}_\lambda(G) := \int_0^{2L} w(t)\,\|\widetilde{G} - S_t\widetilde{G}\|_{L^2(\mathbb{R})}^2\,dt + \sum_{\substack{p \text{ prime} \\ p \leq \lambda^2}} \sum_{\substack{m \geq 1 \\ p^m \leq \lambda^2}} (\log p)p^{-m/2}\,\|\widetilde{G} - S_{m\log p}\widetilde{G}\|_{L^2(\mathbb{R})}^2. \tag{4}$$

Remark 7 (What we have proved so far). Lemmas 4 and 5 show that for $f = g * g^*$ with $g \in C_c^\infty([\lambda^{-1}, \lambda])$, the quantity

$$-\sum_{v \in \{\infty\} \cup \{p\}} W_v(f)$$

equals $\mathcal{E}_\lambda(G)$ plus an additive constant multiple of $\|G\|_2^2$. The form $\mathcal{E}_\lambda$ (Definition 6) is then defined for all $G \in L^2(I)$ as an extended-real-valued quadratic form; from this point onward, all arguments use only the abstract properties of $\mathcal{E}_\lambda$ and do not depend on the explicit-formula derivation. Since adding a constant multiple of $\|G\|_2^2$ only shifts the spectrum of the associated operator, it does not affect positivity/irreducibility properties of the semigroup and does not affect eigenfunction parity considerations.

Warning (positivity vs. spectral shape). Note that $c_p(\lambda) < 0$ in Lemma 4, so the sign of the total additive constant C_λ is not controlled here. This means the decomposition does *not* by itself establish nonnegativity of the full explicit-formula quadratic form ("Weil positivity"). What it does establish is the Markov, irreducible, and compact-resolvent structure of the nonnegative difference-energy part $\mathcal{E}_\lambda$, and hence the simplicity, strict positivity, and evenness of the ground-state eigenfunction—properties that are unaffected by adding a scalar multiple of $\|G\|_2^2$.

5 Markov property (normal contractions)

Definition 8 (Normal contraction). A map $\Phi : \mathbb{R} \to \mathbb{R}$ is a normal contraction if $\Phi(0) = 0$ and $|\Phi(a) - \Phi(b)| \leq |a - b|$ for all $a, b \in \mathbb{R}$.

Lemma 9 (Markov property)**.** *For every normal contraction Φ and every $G \in L^2(I)$,*

$$\mathcal{E}_\lambda(\Phi \circ G) \le \mathcal{E}_\lambda(G).$$

In particular, $\mathcal{E}_\lambda(|G|) \le \mathcal{E}_\lambda(G)$.

Structured Proof. **Step 1.** $\Phi \circ G \in L^2(I)$.

Justification: Since $\Phi(0) = 0$ and $|\Phi(a) - \Phi(b)| \le |a - b|$, setting $b = 0$ gives $|\Phi(G(u))| \le |G(u)|$ pointwise. Hence $\|\Phi \circ G\|_{L^2(I)} \le \|G\|_{L^2(I)} < \infty$.

Step 2. For each fixed $t \in \mathbb{R}$: $\|\widetilde{\Phi \circ G} - S_t\widetilde{\Phi \circ G}\|_2^2 \le \|\widetilde{G} - S_t\widetilde{G}\|_2^2$.

Step 2.1. $\widetilde{\Phi \circ G} = \Phi \circ \widetilde{G}$.

Justification: Since $\Phi(0) = 0$, the extension by zero commutes with composition by Φ: for $u \notin I$, $\widetilde{G}(u) = 0$, so $\Phi(\widetilde{G}(u)) = \Phi(0) = 0 = \widetilde{\Phi \circ G}(u)$; for $u \in I$, both sides equal $\Phi(G(u))$.

Step 2.2. $|\Phi(\widetilde{G}(u)) - \Phi(\widetilde{G}(u-t))|^2 \le |\widetilde{G}(u) - \widetilde{G}(u-t)|^2$ for every $u \in \mathbb{R}$.

Justification: Φ is 1-Lipschitz by hypothesis (the "normal contraction" condition $|\Phi(a) - \Phi(b)| \le |a - b|$). Apply this pointwise with $a = \widetilde{G}(u)$, $b = \widetilde{G}(u-t)$, and square.

Step 2.3. Integrate Step 2.2 over $\mathbb{R}$ with respect to du:

$$\int_{\mathbb{R}} |\Phi(\widetilde{G}(u)) - \Phi(\widetilde{G}(u-t))|^2 \, du \le \int_{\mathbb{R}} |\widetilde{G}(u) - \widetilde{G}(u-t)|^2 \, du.$$

Justification: Integrate both sides of the pointwise inequality from Step 2.2.

Step 2.4. Rewrite using Step 2.1 and the definition of S_t: this is precisely $\|\widetilde{\Phi \circ G} - S_t\widetilde{\Phi \circ G}\|_2^2 \le \|\widetilde{G} - S_t\widetilde{G}\|_2^2$.

Justification: $(S_t\widetilde{\Phi \circ G})(u) = (\Phi \circ \widetilde{G})(u-t) = \Phi(\widetilde{G}(u-t))$.

Step 3. $\mathcal{E}_\lambda(\Phi \circ G) \le \mathcal{E}_\lambda(G)$.

Justification: When $\mathcal{E}_\lambda(G) = +\infty$ the inequality is trivial. Otherwise, by Definition 6, $\mathcal{E}_\lambda(G)$ is the integral of $w(t)\|\widetilde{G} - S_t\widetilde{G}\|_2^2$ over $[0, 2L]$ (with weight $w(t) \ge 0$) plus a finite sum of terms $(\log p)p^{-m/2}\|\widetilde{G} - S_{m\log p}\widetilde{G}\|_2^2$ (all coefficients ≥ 0). Step 2 shows each summand decreases (or stays the same) when G is replaced by $\Phi \circ G$. Since all weights are nonnegative, the integral and sum are each $\le$ the corresponding quantity for G.

Step 4. In particular, $\mathcal{E}_\lambda(|G|) \le \mathcal{E}_\lambda(G)$.

Justification: $\Phi(x) = |x|$ is a normal contraction: $\Phi(0) = 0$ and $\big||a| - |b|\big| \le |a - b|$ by the reverse triangle inequality. Apply Step 3.

Q.E.D. □

6 A translation-invariance lemma on an interval

Lemma 10 (Local translation invariance forces null or conull)**.** *Let $I \subset \mathbb{R}$ be a nontrivial open interval and let $B \subset I$ be measurable. Assume that there exists $\varepsilon > 0$ such that for every $t \in (0, \varepsilon)$,*

$$\mathbf{1}_B(u) = \mathbf{1}_B(u-t) \quad \textit{for a.e. } u \in I \cap (I + t). \tag{5}$$

Then either $m(B) = 0$ or $m(I \setminus B) = 0$.

Structured Proof. **Step 1.** Set $f := \mathbf{1}_B \in L^1_{\mathrm{loc}}(I)$. Fix a compact subinterval $J \Subset I$. Choose $0 < \delta < \min\{\varepsilon, \mathrm{dist}(J, \partial I)\}$.

Justification: Since $J \Subset I$, we have $\mathrm{dist}(J, \partial I) > 0$, so δ as described exists.

Step 2. For every $t \in (0, \delta)$: $f(u + t) = f(u)$ for a.e. $u \in J$.

Step 2.1. From (5): for $t \in (0, \varepsilon)$, $f(u) = f(u - t)$ for a.e. $u \in I \cap (I + t)$.

Justification: Hypothesis of the lemma, since $0 < t < \delta < \varepsilon$.

Step 2.2. Substitute $u \mapsto u + t$: $f(u + t) = f(u)$ for a.e. $u \in (I - t) \cap I$.

Justification: The set $I \cap (I + t)$ becomes $(I - t) \cap I$ after the shift.

Step 2.3. $J \subset (I - t) \cap I$.

Justification: For $u \in J$: since $\delta < \mathrm{dist}(J, \partial I)$, we have $u \in I$ and $u + t \in I$ (as $|t| < \delta$), so $u \in I$ and $u \in I - t$, giving $u \in (I - t) \cap I$.

Step 2.4. Combine: $f(u + t) = f(u)$ for a.e. $u \in J$.

Justification: Restrict the a.e. identity from Step 2.2 to the subset $J \subset (I - t) \cap I$ (Step 2.3).

Step 3. Summary: $f(u+t) = f(u)$ for a.e. $u \in J$, for all $t \in (0, \delta)$. At $t = 0$ the identity is trivial.

Justification: This is Step 2 restated for emphasis. (The downstream mollification argument, Steps 4–9, uses only $t \in (0, \delta/2)$, so no negative-t extension is required.)

Step 4. Fix the compact interval $K := [a + \delta, b - \delta]$ where $J = [a, b]$. Shrink J at the outset so that K is nonempty.

Justification: Choosing J with length $> 2\delta$ ensures $K \neq \emptyset$. Working on the fixed domain $K \Subset J$ will let us avoid any dependence on varying domains in the subsequent mollification argument.

Step 5. Let ρ be a standard nonneg. mollifier supported in $(-1, 1)$ with $\int \rho = 1$. Set $\rho_\eta(s) = \eta^{-1}\rho(s/\eta)$ and $f_\eta := f * \rho_\eta$ for $0 < \eta < \delta/4$.

Justification: Standard construction; $f_\eta \in C^\infty(\mathbb{R})$. (Extend $f = \mathbf{1}_B$ to $\mathbb{R}$ by zero outside I. For $u \in K$ the convolution samples f only at points $u - s$ with $|s| < \eta < \delta/4$, so $u - s \in J \subset I$ and the extension choice is immaterial.)

Step 6. For every $0 < \eta < \delta/4$ and every $t \in (0, \delta/2)$: $f_\eta(u + t) = f_\eta(u)$ for all $u \in K$.

Justification: Fix $u \in K$ and $|s| < \eta$. Then $u - s \in [a + \delta - \eta,\, b - \delta + \eta] \subset J$ (since $\eta < \delta/4$). Let $h_t := f(\cdot + t) - f(\cdot)$. By Step 2 (with $t \in (0, \delta/2) \subset (0, \delta)$), $h_t = 0$ a.e. on J, so $h_t = 0$ in $L^1(J)$. Hence

$$f_\eta(u + t) - f_\eta(u) = (h_t * \rho_\eta)(u) = \int h_t(u - s)\rho_\eta(s)\, ds = 0,$$

because $u - s \in J$ on the support of ρ_η.

Step 7. For each $\eta \in (0, \delta/4)$, f_η is constant on K; call this constant c_η.

Justification: $f_\eta \in C^\infty$ and Step 6 gives $f_\eta(u + t) = f_\eta(u)$ for all $u \in K$ and all $t \in (0, \delta/2)$. For any $u \in K^\circ$:

$$f'_\eta(u) = \lim_{t \downarrow 0} \frac{f_\eta(u + t) - f_\eta(u)}{t} = 0.$$

Hence f_η is constant on the connected interval K.

Step 8. $(c_\eta)_{\eta \downarrow 0}$ is Cauchy, hence converges to some $c \in \mathbb{R}$.

Justification: Since $f \in L^1(K)$ and mollification converges in $L^1(K)$:

$$m(K)|c_\eta - c_{\eta'}| = \|f_\eta - f_{\eta'}\|_{L^1(K)} \leq \|f_\eta - f\|_{L^1(K)} + \|f - f_{\eta'}\|_{L^1(K)} \to 0.$$

Hence (c_η) is Cauchy, so convergent.

Step 9. $f = \mathbf{1}_B$ equals c a.e. on K, hence $c \in \{0, 1\}$.

Justification:

$$\|f - c\|_{L^1(K)} \le \|f - f_\eta\|_{L^1(K)} + \|f_\eta - c\|_{L^1(K)} = \|f - f_\eta\|_{L^1(K)} + m(K)|c_\eta - c| \to 0.$$

Thus $f = c$ a.e. on K. Since $f = \mathbf{1}_B$ takes only values 0 and 1 a.e., we have $c \in \{0, 1\}$.

Step 10. $f = \mathbf{1}_B$ is a.e. constant on I, hence $m(B) = 0$ or $m(I \setminus B) = 0$.

Justification: Write $I = (\alpha, \beta)$. For each integer $n \ge 1$ set $J_n := [\alpha + 1/n,\ \beta - 1/n]$ (nonempty for n large) and $\delta_n := \min(\varepsilon, 1/n)/2$. Then $|J_n| = \beta - \alpha - 2/n > 2\delta_n$ for all sufficiently large n (since $\delta_n \le 1/(2n)$). Apply Steps 1–9 with $J = J_n$, $\delta = \delta_n$: the function $\mathbf{1}_B$ equals a constant $c_n \in \{0, 1\}$ a.e. on $K_n := [\alpha + 1/n + \delta_n,\ \beta - 1/n - \delta_n]$. For n large, $K_n \subset K_{n+1}^\circ$ (since $1/(n+1) + \delta_{n+1} < 1/n + \delta_n$ eventually), so $K_n \cap K_{n+1}$ has positive measure and $c_n = c_{n+1}$. Hence all c_n agree for n large, and $\bigcup_n K_n = I$ up to measure zero. Thus $\mathbf{1}_B$ is a.e. constant on I.

Q.E.D. □

7 Irreducibility from the archimedean continuum

7.1 A concrete criterion

Lemma 11 (Indicator-energy vanishes only for null/conull sets)**.** *Let $B \subset I$ be measurable. If $\mathcal{E}_\lambda(\mathbf{1}_B) = 0$, then $m(B) = 0$ or $m(I \setminus B) = 0$.*

Structured Proof. **Step 1.** The archimedean contribution to $\mathcal{E}_\lambda(\mathbf{1}_B)$ vanishes: $\int_0^{2L} w(t)\|\widetilde{\mathbf{1}_B} - S_t\widetilde{\mathbf{1}_B}\|_2^2\, dt = 0$.

Justification: By Definition 6, $\mathcal{E}_\lambda(\mathbf{1}_B)$ is a sum of nonneg. terms (the archimedean integral plus the prime sums). If the total is 0, each nonneg. summand must be 0. In particular, the archimedean integral (whose integrand is nonneg.) equals 0.

Step 2. $\|\widetilde{\mathbf{1}_B} - S_t\widetilde{\mathbf{1}_B}\|_2^2 = 0$ for a.e. $t \in (0, 2L)$.

Justification: The integrand $w(t)\|\widetilde{\mathbf{1}_B} - S_t\widetilde{\mathbf{1}_B}\|_2^2$ is nonneg. and $w(t) > 0$ for all $t > 0$ (since $w(t) = e^{t/2}/(2\sinh t)$ with numerator and denominator both positive for $t > 0$). A nonneg. integral vanishing (Step 1) with a strictly positive weight implies the other factor vanishes a.e.

Step 3. Upgrade to *all* $t \in (0, 2L)$: $\|\widetilde{\mathbf{1}_B} - S_t\widetilde{\mathbf{1}_B}\|_2^2 = 0$ for every $t \in (0, 2L)$.

Step 3.1. The function $t \mapsto \|\phi - S_t\phi\|_2^2$ is continuous for any $\phi \in L^2(\mathbb{R})$.

Justification: By strong continuity of the translation group $(S_t)_{t\in\mathbb{R}}$ on $L^2(\mathbb{R})$ (which follows from dominated convergence: if $t_n \to t$ then $S_{t_n}\phi \to S_t\phi$ in L^2), the map $t \mapsto S_t\phi$ is continuous $\mathbb{R} \to L^2(\mathbb{R})$, and the squared norm is a continuous function of its argument.

Step 3.2. A continuous function that vanishes a.e. on an interval vanishes everywhere on that interval.

Justification: Let $h : (0, 2L) \to [0, \infty)$ be continuous with $h = 0$ a.e. If $h(t_0) > 0$ for some t_0, then by continuity $h > 0$ on an open neighborhood of t_0, which has positive Lebesgue measure—contradicting $h = 0$ a.e.

Step 3.3. Apply Steps 3.1 and 3.2 with $\phi = \widetilde{\mathbf{1}_B}$.

Justification: Step 2 says $\|\widetilde{\mathbf{1}_B} - S_t\widetilde{\mathbf{1}_B}\|_2^2 = 0$ for a.e. $t \in (0, 2L)$; Step 3.1 says this function of t is continuous; Step 3.2 upgrades "a.e." to "all."

Step 4. For every $t \in (0, 2L)$: $\mathbf{1}_B(u) = \mathbf{1}_B(u - t)$ for a.e. $u \in I \cap (I + t)$.

Justification: $\|\widetilde{\mathbf{1}_B} - S_t\widetilde{\mathbf{1}_B}\|_2^2 = 0$ (Step 3) means $\widetilde{\mathbf{1}_B}(u) = \widetilde{\mathbf{1}_B}(u-t)$ for a.e. $u \in \mathbb{R}$. Restricting to $u \in I \cap (I+t)$: both $u \in I$ and $u - t \in I$, so $\widetilde{\mathbf{1}_B}(u) = \mathbf{1}_B(u)$ and $\widetilde{\mathbf{1}_B}(u-t) = \mathbf{1}_B(u-t)$.

Step 5. $m(B) = 0$ or $m(I \setminus B) = 0$.

Justification: Step 4 holds for all $t \in (0, 2L)$, which contains an interval $(0, \varepsilon)$ for any $\varepsilon \leq 2L$. Lemma 10 applies (with $\varepsilon = 2L$) and yields the conclusion.

Q.E.D. □

Remark 12 (In the nonconservative (killing) setting, only the null case can occur)**.** Since $\mathcal{E}_\lambda$ is defined by zero-extension (Definition 6), one has $\mathcal{E}_\lambda(1) > 0$, so the conull alternative $m(I \setminus B) = 0$ cannot occur under the hypothesis $\mathcal{E}_\lambda(\mathbf{1}_B) = 0$. Hence that hypothesis actually forces $m(B) = 0$.

Structured Proof. [Proof of Remark 12] **Step 1.** $\mathcal{E}_\lambda(1) > 0$.

Justification: For $0 < t < 2L$, $\widetilde{1} = \mathbf{1}_{(-L,L)}$ and $S_t\widetilde{1} = \mathbf{1}_{(-L+t,L+t)}$, so $\|\widetilde{1} - S_t\widetilde{1}\|_2^2 = m(I \,\Delta\, (I + t)) = 2t > 0$ (the symmetric difference consists of $(-L, -L+t)$ and $(L, L+t)$, each of measure t). Since $w(t) > 0$ on $(0, 2L)$:

$$\mathcal{E}_\lambda(1) \geq \int_0^{2L} w(t) \cdot 2t \, dt > 0.$$

Step 2. If $m(I \setminus B) = 0$, then $\mathcal{E}_\lambda(\mathbf{1}_B) = \mathcal{E}_\lambda(1) > 0$.

Justification: $m(I \setminus B) = 0$ implies $\mathbf{1}_B = 1$ in $L^2(I)$, hence $\mathcal{E}_\lambda(\mathbf{1}_B) = \mathcal{E}_\lambda(1) > 0$ by Step 1.

Step 3. Therefore $\mathcal{E}_\lambda(\mathbf{1}_B) = 0$ forces $m(B) = 0$.

Justification: Contrapositive of Step 2 combined with Lemma 11.

Q.E.D. □

7.2 Operator realization: closedness and compact resolvent

7.2.1 Ambient form on $L^2(\mathbb{R})$ and Fourier representation

Let $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ denote the unitary Fourier transform

$$\widehat{\phi}(\xi) := \int_{\mathbb{R}} \phi(u) e^{-iu\xi} \, du, \qquad \phi(u) = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{\phi}(\xi) e^{iu\xi} \, d\xi,$$

so that Plancherel reads $\|\phi\|_{L^2(\mathbb{R})}^2 = \frac{1}{2\pi} \int_{\mathbb{R}} |\widehat{\phi}(\xi)|^2 \, d\xi$.

Define the "ambient" quadratic form on $L^2(\mathbb{R})$ by

$$\begin{aligned} \mathcal{E}_\lambda^{\mathbb{R}}(\phi) := &\int_0^{2L} w(t) \, \|\phi - S_t\phi\|_{L^2(\mathbb{R})}^2 \, dt \\ &+ \sum_{\substack{p \text{ prime} \\ p \leq \lambda^2}} \sum_{\substack{m \geq 1 \\ p^m \leq \lambda^2}} (\log p) p^{-m/2} \, \|\phi - S_{m \log p}\phi\|_{L^2(\mathbb{R})}^2, \end{aligned}$$

with domain $\mathcal{D}(\mathcal{E}_\lambda^{\mathbb{R}}) := \{\phi \in L^2(\mathbb{R}) : \mathcal{E}_\lambda^{\mathbb{R}}(\phi) < \infty\}$. By definition, for $G \in L^2(I)$,

$$\mathcal{E}_\lambda(G) = \mathcal{E}_\lambda^{\mathbb{R}}(\widetilde{G}).$$

Lemma 13 (Plancherel identity for translation differences)**.** *For $\phi \in L^2(\mathbb{R})$ and $t \in \mathbb{R}$,*

$$\|\phi - S_t\phi\|_{L^2(\mathbb{R})}^2 = \frac{1}{2\pi} \int_{\mathbb{R}} |1 - e^{-i\xi t}|^2 \, |\widehat{\phi}(\xi)|^2 \, d\xi = \frac{1}{2\pi} \int_{\mathbb{R}} 4 \sin^2\!\Big(\frac{\xi t}{2}\Big) \, |\widehat{\phi}(\xi)|^2 \, d\xi.$$

Structured Proof. **Step 1.** $\widehat{S_t\phi}(\xi) = e^{-i\xi t}\widehat{\phi}(\xi)$.

Justification: By definition, $\widehat{S_t\phi}(\xi) = \int_{\mathbb{R}} \phi(u-t)e^{-iu\xi}\,du$. Substituting $v = u - t$, $du = dv$: $= \int_{\mathbb{R}} \phi(v)e^{-i(v+t)\xi}\,dv = e^{-it\xi}\widehat{\phi}(\xi)$.

Step 2. $\|\phi - S_t\phi\|_2^2 = \frac{1}{2\pi}\int_{\mathbb{R}} |1 - e^{-i\xi t}|^2 |\widehat{\phi}(\xi)|^2\,d\xi$.

Justification: By Plancherel, $\|\phi - S_t\phi\|_2^2 = \frac{1}{2\pi}\int_{\mathbb{R}} |\widehat{\phi}(\xi) - \widehat{S_t\phi}(\xi)|^2\,d\xi$. Step 1 gives $\widehat{\phi}(\xi) - \widehat{S_t\phi}(\xi) = (1 - e^{-i\xi t})\widehat{\phi}(\xi)$.

Step 3. $|1 - e^{-i\eta}|^2 = 4\sin^2(\eta/2)$.

Justification: $1 - e^{-i\eta} = 1 - \cos\eta + i\sin\eta$, so $|1 - e^{-i\eta}|^2 = (1-\cos\eta)^2 + \sin^2\eta = 2 - 2\cos\eta = 4\sin^2(\eta/2)$ by the double-angle formula $\cos\eta = 1 - 2\sin^2(\eta/2)$.

Step 4. Set $\eta = \xi t$ in Step 3 and substitute into Step 2.

Justification: $|1 - e^{-i\xi t}|^2 = 4\sin^2(\xi t/2)$. This gives the second equality.

Q.E.D. □

Lemma 14 (Fourier representation)**.** *For $\phi \in L^2(\mathbb{R})$,*

$$\mathcal{E}_\lambda^{\mathbb{R}}(\phi) = \frac{1}{2\pi}\int_{\mathbb{R}} \psi_\lambda(\xi)\,|\widehat{\phi}(\xi)|^2\,d\xi \quad \text{in } [0,\infty],$$

where

$$\begin{aligned}\psi_\lambda(\xi) := 4\int_0^{2L} w(t)\,\sin^2\Big(\frac{\xi t}{2}\Big)\,dt \\ + 4\sum_{\substack{p\text{ prime}\\ p\le\lambda^2}}\ \sum_{\substack{m\ge1\\ p^m\le\lambda^2}} (\log p)p^{-m/2}\,\sin^2\Big(\frac{\xi m\log p}{2}\Big).\end{aligned} \tag{6}$$

In particular ψ_λ is measurable, even, finite for each ξ, and $\psi_\lambda(\xi) \ge 0$.

Structured Proof. **Step 1.** Apply Lemma 13 to each translation-difference norm in $\mathcal{E}_\lambda^{\mathbb{R}}(\phi)$.

Justification: Each term $\|\phi - S_t\phi\|_2^2$ or $\|\phi - S_{m\log p}\phi\|_2^2$ in Definition 6 equals $\frac{1}{2\pi}\int_{\mathbb{R}} 4\sin^2(\xi t/2)|\widehat{\phi}(\xi)|^2\,d\xi$ (resp. with t replaced by $m\log p$).

Step 2. Interchange integration/summation order by Tonelli's theorem:

$$\mathcal{E}_\lambda^{\mathbb{R}}(\phi) = \frac{1}{2\pi}\int_{\mathbb{R}} \psi_\lambda(\xi)|\widehat{\phi}(\xi)|^2\,d\xi.$$

Justification: The integrand $w(t)\cdot 4\sin^2(\xi t/2)\cdot|\widehat{\phi}(\xi)|^2$ is jointly measurable in (t,ξ): $w(t)$ is Borel on $(0,2L]$, $\sin^2(\xi t/2)$ is jointly continuous, and $|\widehat{\phi}|^2$ is measurable. All factors are nonneg. ($w(t) \ge 0$, $\sin^2 \ge 0$, $|\widehat{\phi}|^2 \ge 0$, $(\log p)p^{-m/2} \ge 0$). Tonelli's theorem permits interchange of the ξ-integral with the t-integral and the finite sums. The resulting multiplier of $|\widehat{\phi}(\xi)|^2$ is precisely $\psi_\lambda(\xi)$.

Step 3. ψ_λ is measurable, even, finite for each ξ, and ≥ 0.

Step 3.1. Measurability: ψ_λ is a sum of continuous functions of ξ.

Justification: Each $\sin^2(\xi t/2)$ is continuous in ξ. The integral $\int_0^{2L} w(t)\sin^2(\xi t/2)\,dt$ is continuous in ξ by dominated convergence: for $|\xi| \le M$, the integrand is bounded by $w(t)\min\big(1, (Mt/2)^2\big)$, which is integrable on $(0, 2L)$ (near $t = 0$: $w(t)(Mt/2)^2 \le M^2 t^2 e^{t/2}/(8t) = M^2 t e^{t/2}/8$, integrable; away from 0: w is bounded on $[1, 2L]$). The prime sum is a finite sum of continuous functions.

Step 3.2. Evenness: $\psi_\lambda(-\xi) = \psi_\lambda(\xi)$.

Justification: $\sin^2((-\xi)t/2) = \sin^2(\xi t/2)$.

Step 3.3. Finiteness: for each fixed ξ, $\psi_\lambda(\xi) < \infty$.

Justification: Split $(0, 2L) = (0, 1) \cup [1, 2L)$. On $[1, 2L)$, w is bounded and $\sin^2 \le 1$, so the integral over $[1, 2L)$ is finite. On $(0, 1)$, use $\sin^2(\xi t/2) \le (\xi t/2)^2$, giving $w(t)\sin^2(\xi t/2) \le \xi^2 t^2 w(t)/4 \le \xi^2 t e^{t/2}/8$ (using $w(t) \le e^{t/2}/(2t)$ from $\sinh t \ge t$), which is integrable near 0. The prime sum is finite (finitely many terms, each finite).

Step 3.4. Nonnegativity: $\psi_\lambda(\xi) \ge 0$.

Justification: Every summand is a product of nonneg. factors.

Q.E.D. □

Proposition 15 (Closedness on $L^2(\mathbb{R})$)**.** *The form $\mathcal{E}_\lambda^{\mathbb{R}}$ is densely defined, symmetric, nonneg., and closed on $L^2(\mathbb{R})$. Moreover,*

$$\mathcal{D}(\mathcal{E}_\lambda^{\mathbb{R}}) = \Big\{ \phi \in L^2(\mathbb{R}) : \int_{\mathbb{R}} \psi_\lambda(\xi)\, |\widehat{\phi}(\xi)|^2 \, d\xi < \infty \Big\},$$

and $\mathcal{D}(\mathcal{E}_\lambda^{\mathbb{R}})$ is a Hilbert space for the norm $\|\phi\|_{\mathcal{D}}^2 := \|\phi\|_{L^2(\mathbb{R})}^2 + \mathcal{E}_\lambda^{\mathbb{R}}(\phi)$.

Structured Proof. **Step 1.** $\mathcal{E}_\lambda^{\mathbb{R}}$ is the quadratic form of multiplication by ψ_λ in Fourier space.

Justification: Lemma 14: $\mathcal{E}_\lambda^{\mathbb{R}}(\phi) = \frac{1}{2\pi} \int_{\mathbb{R}} \psi_\lambda(\xi) |\widehat{\phi}(\xi)|^2 \, d\xi$.

Step 2. $\mathcal{D}(\mathcal{E}_\lambda^{\mathbb{R}}) = \{\phi \in L^2(\mathbb{R}) : \int \psi_\lambda(\xi)|\widehat{\phi}(\xi)|^2 \, d\xi < \infty\}$.

Justification: Immediate from Step 1: $\phi \in \mathcal{D}(\mathcal{E}_\lambda^{\mathbb{R}})$ iff $\mathcal{E}_\lambda^{\mathbb{R}}(\phi) < \infty$ iff $\int \psi_\lambda |\widehat{\phi}|^2 < \infty$.

Step 3. $\mathcal{D}(\mathcal{E}_\lambda^{\mathbb{R}})$ with the norm $\|\phi\|_{\mathcal{D}}^2 = \frac{1}{2\pi} \int (1 + \psi_\lambda(\xi)) |\widehat{\phi}(\xi)|^2 \, d\xi$ is a Hilbert space.

Justification: Via $\phi \mapsto \widehat{\phi}$, this norm space is isometrically isomorphic to $L^2(\mathbb{R}, (1 + \psi_\lambda(\xi)) \frac{d\xi}{2\pi})$, which is a weighted L^2 space with a nonneg. measurable weight, hence complete.

Step 4. The form is closed.

Justification: By Kato [5, Thm. VI.1.17], a nonneg. symmetric form is closed iff its domain equipped with the graph norm $\|\cdot\|_{\mathcal{D}}$ is complete. Step 3 verifies this.

Step 5. Nonnegativity and symmetry are immediate.

Justification: Nonnegativity: $\psi_\lambda \ge 0$ implies $\mathcal{E}_\lambda^{\mathbb{R}}(\phi) \ge 0$. Symmetry: $\mathcal{E}_\lambda^{\mathbb{R}}$ is defined on real-valued or complex-valued functions via a real nonneg. multiplier; the associated bilinear form inherits symmetry from the pointwise identity.

Step 6. $\mathcal{E}_\lambda^{\mathbb{R}}$ is densely defined: $C_c^\infty(\mathbb{R}) \subset \mathcal{D}(\mathcal{E}_\lambda^{\mathbb{R}})$.

Step 6.1. For $\phi \in C_c^\infty(\mathbb{R})$: $\|\phi - S_t\phi\|_2 \le |t| \|\phi'\|_2$.

Justification: By Plancherel: $\|\phi - S_t\phi\|_2^2 = \frac{1}{2\pi} \int 4 \sin^2(\xi t/2) |\widehat{\phi}(\xi)|^2 \, d\xi \le t^2 \frac{1}{2\pi} \int \xi^2 |\widehat{\phi}(\xi)|^2 \, d\xi = t^2 \|\phi'\|_2^2$, using $\sin^2 x \le x^2$ and $\frac{1}{2\pi} \int \xi^2 |\widehat{\phi}|^2 \, d\xi = \|\phi'\|_2^2$.

Step 6.2. $\int_0^{2L} w(t) t^2 \, dt < \infty$.

Justification: Since $\sinh t \ge t$: $w(t) = e^{t/2}/(2\sinh t) \le e^{t/2}/(2t)$, so $w(t)t^2 \le t e^{t/2}/2$, which is bounded and integrable on $[0, 2L]$.

Step 6.3. $\mathcal{E}_\lambda^{\mathbb{R}}(\phi) \le \|\phi'\|_2^2 \big(\int_0^{2L} w(t) t^2 \, dt + \sum_{p,m} (\log p) p^{-m/2} (m \log p)^2 \big) < \infty$.

Justification: From Step 6.1: each $\|\phi - S_t\phi\|_2^2 \le t^2\|\phi'\|_2^2$. Apply this to the archimedean integral and each prime term. The prime sum is finite (finitely many terms, each $\le (\log p)p^{-m/2}(m\log p)^2\|\phi'\|_2^2$).

Step 6.4. $C_c^\infty(\mathbb{R})$ is dense in $L^2(\mathbb{R})$.

Justification: Standard fact in functional analysis.

Q.E.D. $\square$

Proposition 16 (Closedness on $L^2(I)$)**.** *The form $\mathcal{E}_\lambda$ on $H = L^2(I)$ is densely defined, symmetric, nonneg., and closed.*

Structured Proof. **Step 1.** $G \mapsto \widetilde{G}$ is an isometry from $L^2(I)$ onto $H_I := \{\phi \in L^2(\mathbb{R}) : \phi = 0 \text{ a.e. on } \mathbb{R} \setminus I\}$.

Justification: $\|\widetilde{G}\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} |\widetilde{G}(u)|^2\,du = \int_I |G(u)|^2\,du = \|G\|_{L^2(I)}^2$. Surjectivity: any $\phi \in H_I$ is the zero-extension of $G := \phi|_I$.

Step 2. $\mathcal{E}_\lambda(G) = \mathcal{E}_\lambda^{\mathbb{R}}(\widetilde{G})$.

Justification: By definition (Definition 6 and the definition of $\mathcal{E}_\lambda^{\mathbb{R}}$).

Step 3. $\mathcal{E}_\lambda$ is the restriction of $\mathcal{E}_\lambda^{\mathbb{R}}$ to H_I.

Justification: Steps 1 and 2: H_I is a closed subspace of $L^2(\mathbb{R})$, and $\mathcal{E}_\lambda$ on $L^2(I)$ corresponds (via the isometry) to $\mathcal{E}_\lambda^{\mathbb{R}}$ restricted to H_I.

Step 4. $\mathcal{E}_\lambda$ is closed.

Justification: Suppose $G_n \to G$ in $L^2(I)$ and $\mathcal{E}_\lambda(G_n - G_m) \to 0$. Then $\widetilde{G}_n \to \widetilde{G}$ in $L^2(\mathbb{R})$ (Step 1) and $\mathcal{E}_\lambda^{\mathbb{R}}(\widetilde{G}_n - \widetilde{G}_m) \to 0$ (Step 2), so closedness of $\mathcal{E}_\lambda^{\mathbb{R}}$ (Prop. 15) gives $\widetilde{G} \in \mathcal{D}(\mathcal{E}_\lambda^{\mathbb{R}})$ and $\mathcal{E}_\lambda^{\mathbb{R}}(\widetilde{G}_n - \widetilde{G}) \to 0$. Since H_I is closed in $L^2(\mathbb{R})$ and each $\widetilde{G}_n \in H_I$, the limit $\widetilde{G} \in H_I$, whence $G \in \mathcal{D}(\mathcal{E}_\lambda)$ and $\mathcal{E}_\lambda(G_n - G) \to 0$.

Step 5. $\mathcal{E}_\lambda$ is densely defined.

Justification: $C_c^\infty(I) \subset \mathcal{D}(\mathcal{E}_\lambda)$: for $G \in C_c^\infty(I)$, $\widetilde{G} \in C_c^\infty(\mathbb{R}) \subset \mathcal{D}(\mathcal{E}_\lambda^{\mathbb{R}})$ (Proposition 15, Step 6), so $G \in \mathcal{D}(\mathcal{E}_\lambda)$. Since $C_c^\infty(I)$ is dense in $L^2(I)$, the form is densely defined.

Step 6. Nonnegativity and symmetry follow from those of $\mathcal{E}_\lambda^{\mathbb{R}}$.

Justification: $\mathcal{E}_\lambda(G) = \mathcal{E}_\lambda^{\mathbb{R}}(\widetilde{G}) \ge 0$ (Prop. 15).

Q.E.D. $\square$

7.2.2 A coercive lower bound for the symbol ψ_λ

Lemma 17 (A lower bound for $w(t)$)**.** *Let $t_0 := \min(1, 2L)$. There exists $c_0 = c_0(L) > 0$ such that for all $t \in (0, t_0]$,*

$$w(t) = \frac{e^{t/2}}{2\sinh t} \ge \frac{c_0}{t}.$$

Structured Proof. **Step 1.** $\sinh t \le te^t$ for all $t > 0$.

Justification: $\sinh t = \frac{e^t - e^{-t}}{2} \le \frac{e^t}{2} \cdot 2t/(2t/(e^t - e^{-t}) \cdot 2)$—more directly, $\sinh t = \sum_{n=0}^\infty \frac{t^{2n+1}}{(2n+1)!} \le t\sum_{n=0}^\infty \frac{t^{2n}}{(2n)!} = t\cosh t \le te^t$.

Step 2. $w(t) \ge \frac{e^{-t/2}}{2t}$ for $t > 0$.

Justification: $w(t) = \frac{e^{t/2}}{2\sinh t} \ge \frac{e^{t/2}}{2te^t} = \frac{e^{-t/2}}{2t}$, using Step 1.

Step 3. For $t \in (0, 1]$: $e^{-t/2} \ge e^{-1/2}$.

Justification: $e^{-t/2}$ is decreasing; its minimum on $(0,1]$ is at $t=1$.

Step 4. Set $c_0 := e^{-1/2}/2$. Then for $t \in (0,t_0] \subset (0,1]$: $w(t) \geq \frac{c_0}{t}$.

Justification: $w(t) \geq \frac{e^{-t/2}}{2t} \geq \frac{e^{-1/2}}{2t} = \frac{c_0}{t}$, combining Steps 2 and 3.

Q.E.D. $\square$

Lemma 18 (Logarithmic growth of ψ_λ)**.** *There exist constants $c_1, c_2 > 0$ and $\xi_0 \geq 2$ (depending only on L) such that for all $|\xi| \geq \xi_0$,*

$$\psi_\lambda(\xi) \geq c_1 \log|\xi| - c_2.$$

In particular $\psi_\lambda(\xi) \to \infty$ as $|\xi| \to \infty$.

Structured Proof. **Step 1.** Drop the nonneg. prime sum: $\psi_\lambda(\xi) \geq 4\int_0^{t_0} w(t)\sin^2(\xi t/2)\,dt$.

Justification: The prime sum in (6) is ≥ 0, and we restrict the archimedean integral from $[0,2L]$ to $[0,t_0] \subset [0,2L]$ (the integrand is nonneg.).

Step 2. Apply Lemma 17: $\psi_\lambda(\xi) \geq 4c_0 \int_0^{t_0} \frac{1}{t}\sin^2(\xi t/2)\,dt$.

Justification: For $t \in (0,t_0]$, $w(t) \geq c_0/t$ by Lemma 17.

Step 3. For $|\xi| \geq \xi_0 := 4\pi/t_0$, define intervals $J_n := \left[\frac{2\pi n+\pi/2}{|\xi|}, \frac{2\pi n+3\pi/2}{|\xi|}\right]$ for $n \geq 0$. Then $\sin^2(\xi t/2) \geq 1/2$ for $t \in J_n$.

Justification: Set $\theta = |\xi|t/2$. For $t \in J_n$, $\theta \in [\pi n + \pi/4, \pi n + 3\pi/4]$. On this interval, $|\sin\theta| \geq \sin(\pi/4) = 1/\sqrt{2}$, so $\sin^2\theta \geq 1/2$. Since $\sin^2(\xi t/2) = \sin^2(|\xi|t/2) = \sin^2\theta$, the claim follows. The J_n are pairwise disjoint: the left endpoint of J_{n+1} exceeds the right endpoint of J_n by $(2\pi(n+1)+\pi/2-2\pi n-3\pi/2)/|\xi| = \pi/(2|\xi|) > 0$.

Step 4. Let N be the largest integer with $J_{N-1} \subset (0,t_0]$. Then $N \asymp |\xi|$.

Justification: The right endpoint of J_{N-1} is $\frac{2\pi(N-1)+3\pi/2}{|\xi|} \leq t_0$, giving $N \leq \frac{t_0|\xi|}{2\pi} + \frac{1}{4}$. The condition $|\xi| \geq 4\pi/t_0$ ensures $N \geq 1$. Thus N is of order $|\xi|$ with constants depending on t_0.

Step 5. $\int_0^{t_0} \frac{1}{t}\sin^2(\xi t/2)\,dt \geq \frac{1}{2}\sum_{n=0}^{N-1} \log\frac{2\pi n+3\pi/2}{2\pi n+\pi/2}$.

Justification: Restrict the integral to $\bigcup_{n=0}^{N-1} J_n \subset (0,t_0]$. On J_n, $\sin^2(\xi t/2) \geq 1/2$ (Step 3), so $\int_{J_n} \frac{1}{t}\cdot\frac{1}{2}\,dt = \frac{1}{2}\log\frac{2\pi n+3\pi/2}{2\pi n+\pi/2}$.

Step 6. $\sum_{n=0}^{N-1} \log\frac{2\pi n+3\pi/2}{2\pi n+\pi/2} \geq c'\log N$ for an absolute constant $c' > 0$.

Step 6.1. $\log\frac{2\pi n+3\pi/2}{2\pi n+\pi/2} = \log\left(1+\frac{\pi}{2\pi n+\pi/2}\right) \geq \frac{c}{n+1}$ for some absolute $c > 0$.

Justification: Using $\log(1+x) \geq x/(1+x)$: with $x = \pi/(2\pi n+\pi/2)$, $\log(1+x) \geq \frac{\pi/(2\pi n+\pi/2)}{1+\pi/(2\pi n+\pi/2)} = \frac{\pi}{2\pi n+3\pi/2} \geq \frac{\pi}{2\pi(n+1)+3\pi/2} \geq \frac{c}{n+1}$.

Step 6.2. $\sum_{n=0}^{N-1}\frac{1}{n+1} = H_N \geq \log N$ (where H_N is the N-th harmonic number).

Justification: Standard lower bound for harmonic numbers: $H_N \geq \log N$ for $N \geq 1$.

Step 6.3. Combine: the sum $\geq c\cdot\log N$.

Justification: Multiply Step 6.1 by 1 and sum, then apply Step 6.2.

Step 7. Since $N \asymp |\xi|$ (Step 4): $\log N = \log|\xi| + O(1)$.

Justification: $N = \Theta(|\xi|)$ implies $\log N = \log|\xi| + \log(N/|\xi|) = \log|\xi| + O(1)$.

Step 8. Conclusion: $\psi_\lambda(\xi) \geq 4c_0\cdot\frac{1}{2}\cdot c'\cdot(\log|\xi| - C') = c_1\log|\xi| - c_2$.

Justification: Chain Steps 2, 5, 6, and 7: $\psi_\lambda(\xi) \geq 4c_0\cdot\frac{1}{2}\cdot c'\log N = 2c_0c'\log N = 2c_0c'(\log|\xi| + O(1))$. Set $c_1 := 2c_0c'$ and absorb the $O(1)$ into c_2.

Q.E.D. $\square$

Corollary 19 (Energy controls a logarithmic frequency moment)**.** *There exist constants $a, b > 0$ (depending only on L) such that for every $\phi \in \mathcal{D}(\mathcal{E}_\lambda^{\mathbb{R}})$,*

$$\int_{\mathbb{R}} \log(2 + |\xi|)\, |\widehat{\phi}(\xi)|^2 \, d\xi \le a \, \|\phi\|^2_{L^2(\mathbb{R})} + b \int_{\mathbb{R}} \psi_\lambda(\xi)\, |\widehat{\phi}(\xi)|^2 \, d\xi.$$

Structured Proof. **Step 1.** For all $\xi \in \mathbb{R}$: $\log(2 + |\xi|) \le a' + b'\psi_\lambda(\xi)$ for suitable $a', b' > 0$.

Step 1.1. For $|\xi| \ge \xi_0$: $\psi_\lambda(\xi) \ge c_1 \log |\xi| - c_2$ (Lemma 18).
Justification: Direct application of Lemma 18.

Step 1.2. Hence $\log(2 + |\xi|) \le \log |\xi| + \log 3 \le \frac{1}{c_1}(\psi_\lambda(\xi) + c_2) + \log 3$ for $|\xi| \ge \xi_0$.
Justification: $\log(2 + |\xi|) \le \log(3|\xi|) = \log 3 + \log |\xi|$, and from Step 1.1: $\log |\xi| \le (\psi_\lambda(\xi) + c_2)/c_1$.

Step 1.3. For $|\xi| < \xi_0$: $\log(2 + |\xi|) \le \log(2 + \xi_0)$, a finite constant.
Justification: $\log(2 + |\xi|)$ is bounded on bounded sets.

Step 1.4. Combine: set $b' := 1/c_1$ and $a' := c_2/c_1 + \log 3 + \log(2 + \xi_0)$. Then $\log(2 + |\xi|) \le a' + b'\psi_\lambda(\xi)$ for all ξ.
Justification: For $|\xi| \ge \xi_0$, use Step 1.2. For $|\xi| < \xi_0$, $a' \ge \log(2 + \xi_0) \ge \log(2 + |\xi|)$ and $b'\psi_\lambda(\xi) \ge 0$.

Step 2. Multiply by $|\widehat{\phi}(\xi)|^2$ and integrate: $\int \log(2+|\xi|)|\widehat{\phi}(\xi)|^2 \, d\xi \le a' \int |\widehat{\phi}(\xi)|^2 \, d\xi + b' \int \psi_\lambda(\xi)|\widehat{\phi}(\xi)|^2 \, d\xi$.
Justification: Pointwise inequality from Step 1 times $|\widehat{\phi}(\xi)|^2 \ge 0$, integrated.

Step 3. By Plancherel, $\frac{1}{2\pi} \int |\widehat{\phi}|^2 \, d\xi = \|\phi\|_2^2$. Set $a := 2\pi a'$, $b := b'$.
Justification: Rewrite Step 2 in terms of $\|\phi\|_2^2$ and $\mathcal{E}_\lambda^{\mathbb{R}}(\phi)$ (via Lemma 14).

Q.E.D. □

7.2.3 Compact embedding and compact resolvent

Theorem 20 (Kolmogorov–Riesz compactness criterion in $L^2(\mathbb{R})$)**.** *A set $\mathcal{K} \subset L^2(\mathbb{R})$ is relatively compact if and only if:*

- *(i) (*tightness*) for every $\varepsilon > 0$ there exists $R > 0$ such that $\int_{|u|>R} |\phi(u)|^2 \, du < \varepsilon^2$ for all $\phi \in \mathcal{K}$;*
- *(ii) (*translation equicontinuity*) for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|\phi - S_h\phi\|_2 < \varepsilon$ for all $\phi \in \mathcal{K}$ and all $|h| < \delta$.*

Remark 21. See Brezis [6, Cor. 4.27] for a proof of Theorem 20. (Brezis Thm. 4.26 covers the bounded-domain case; Cor. 4.27 extends to full $L^2(\mathbb{R})$ and requires the tightness condition (i), which in Proposition 23 is supplied by compact support of all ϕ_n in $\overline{I}$.)

Lemma 22 (Uniform translation control from the form norm)**.** *Fix $M > 0$ and define $\mathcal{K}_M := \{\phi \in H_I : \|\phi\|_2^2 + \mathcal{E}_\lambda^{\mathbb{R}}(\phi) \le M\}$. Then $\mathcal{K}_M$ satisfies condition (ii) in Theorem 20.*

Structured Proof. **Step 1.** For $\phi \in \mathcal{K}_M$ and $h \in \mathbb{R}$: $\|\phi - S_h\phi\|_2^2 = \frac{1}{2\pi} \int_{\mathbb{R}} 4 \sin^2(\xi h/2)|\widehat{\phi}(\xi)|^2 \, d\xi$.
Justification: Lemma 13.

Step 2. Split the integral at a parameter $R \ge 1$ into low and high frequencies.

Step 2.1. Low-frequency bound ($|\xi| \le R$): $\int_{|\xi|\le R} 4 \sin^2(\xi h/2)|\widehat{\phi}(\xi)|^2 \, d\xi \le (Rh)^2 \cdot 2\pi\|\phi\|_2^2$.
Justification: $\sin^2(x) \le x^2$ gives $4 \sin^2(\xi h/2) \le (\xi h)^2 \le (Rh)^2$ for $|\xi| \le R$. Then $\int_{|\xi|\le R}(Rh)^2|\widehat{\phi}|^2 \le (Rh)^2 \int_{\mathbb{R}} |\widehat{\phi}|^2 = (Rh)^2 \cdot 2\pi\|\phi\|_2^2$.

Step 2.2. High-frequency bound ($|\xi| > R$): $\int_{|\xi|>R} 4\sin^2(\xi h/2)|\widehat{\phi}(\xi)|^2\,d\xi \le \frac{4}{\log(2+R)} \int_{\mathbb{R}} \log(2 + |\xi|)|\widehat{\phi}(\xi)|^2\,d\xi$.

Justification: $\sin^2 \le 1$ gives the left side $\le 4\int_{|\xi|>R} |\widehat{\phi}|^2\,d\xi$. For $|\xi| > R$, $1 \le \frac{\log(2+|\xi|)}{\log(2+R)}$, so $\int_{|\xi|>R} |\widehat{\phi}|^2 \le \frac{1}{\log(2+R)} \int_{|\xi|>R} \log(2 + |\xi|)|\widehat{\phi}|^2 \le \frac{1}{\log(2+R)} \int_{\mathbb{R}} \log(2 + |\xi|)|\widehat{\phi}|^2$.

Step 3. By Corollary 19, $\int_{\mathbb{R}} \log(2 + |\xi|)|\widehat{\phi}|^2 \le C(M, L)$ uniformly over $\phi \in \mathcal{K}_M$.

Justification: $\|\phi\|_2^2 + \mathcal{E}_\lambda^{\mathbb{R}}(\phi) \le M$ by definition of $\mathcal{K}_M$. Corollary 19 bounds $\int \log(2 + |\xi|)|\widehat{\phi}|^2$ by $a\|\phi\|_2^2 + b\int \psi_\lambda|\widehat{\phi}|^2\,d\xi = a\|\phi\|_2^2 + 2\pi b\,\mathcal{E}_\lambda^{\mathbb{R}}(\phi) \le (a + 2\pi b)M =: C(M, L)$.

Step 4. Combine: $\|\phi - S_h\phi\|_2^2 \le (Rh)^2 M + \frac{C'(M,L)}{\log(2+R)}$.

Justification: Add the bounds from Steps 2.1 and 2.2, using Step 3 to bound the high-frequency term. Here $C'(M, L) = 4C(M, L)/(2\pi)$ (absorbing constants).

Step 5. Given $\varepsilon > 0$, choose R then δ to make each term $\le \varepsilon^2/2$.

Step 5.1. Choose R so that $\frac{C'(M,L)}{\log(2+R)} \le \varepsilon^2/2$.

Justification: $\log(2 + R) \to \infty$ as $R \to \infty$, so such R exists.

Step 5.2. Then choose $\delta > 0$ so that $(R\delta)^2 M \le \varepsilon^2/2$.

Justification: Take $\delta := \varepsilon/(R\sqrt{2M})$ (assuming $M > 0$; if $M = 0$ then $\mathcal{K}_M = \{0\}$ and the result is trivial).

Step 5.3. For $|h| < \delta$: $\|\phi - S_h\phi\|_2^2 \le \varepsilon^2/2 + \varepsilon^2/2 = \varepsilon^2$.

Justification: Substitute into Step 4.

Q.E.D. □

Proposition 23 (Compact embedding of the form domain)**.** *The embedding $(\mathcal{D}(\mathcal{E}_\lambda), \|\cdot\|_{\mathcal{D}}) \hookrightarrow L^2(I)$ is compact.*

Structured Proof. **Step 1.** Let $\{G_n\} \subset \mathcal{D}(\mathcal{E}_\lambda)$ with $\|G_n\|_2^2 + \mathcal{E}_\lambda(G_n) \le M$. Set $\phi_n := \widetilde{G}_n \in H_I$.

Justification: Setup. Then $\|\phi_n\|_2^2 + \mathcal{E}_\lambda^{\mathbb{R}}(\phi_n) = \|G_n\|_2^2 + \mathcal{E}_\lambda(G_n) \le M$, so $\phi_n \in \mathcal{K}_M$.

Step 2. $\{\phi_n\}$ satisfies tightness condition (i) in Theorem 20.

Justification: Each ϕ_n is supported in $\overline{I} = [-L, L]$. For any $R > L$, $\int_{|u|>R} |\phi_n(u)|^2\,du = 0 < \varepsilon^2$.

Step 3. $\{\phi_n\}$ satisfies translation equicontinuity (ii) in Theorem 20.

Justification: Lemma 22: $\mathcal{K}_M$ satisfies (ii).

Step 4. $\{\phi_n\}$ is relatively compact in $L^2(\mathbb{R})$.

Justification: Theorem 20 applied to $\mathcal{K} := \{\phi_n : n \ge 1\}$, using Steps 2 and 3. (The required L^2-boundedness holds: $\|\phi_n\|_2^2 \le M$ by Step 1.)

Step 5. $\{G_n\}$ is relatively compact in $L^2(I)$.

Justification: Since H_I is closed in $L^2(\mathbb{R})$, every subsequential $L^2(\mathbb{R})$-limit of $\{\phi_n\}$ lies in H_I. The map $\phi \mapsto \phi|_I$ is a continuous surjection (indeed, isometry) $H_I \to L^2(I)$. Continuous images of relatively compact sets are relatively compact. So from Step 4, $\{G_n = \phi_n|_I\}$ is relatively compact in $L^2(I)$.

Q.E.D. □

Theorem 24 (Closed form, associated operator, and compact resolvent)**.** *There exists a unique selfadjoint operator $A_\lambda \ge 0$ on $L^2(I)$ associated to the closed form $\mathcal{E}_\lambda$ (Proposition 16) in the sense of the representation theorem for closed forms. Moreover, A_λ has compact resolvent.*

Structured Proof. **Step 1.** There exists a unique selfadjoint operator $A_\lambda \geq 0$ with $\mathcal{D}(\mathcal{E}_\lambda) = \mathcal{D}(A_\lambda^{1/2})$ and $\mathcal{E}_\lambda(G) = \|A_\lambda^{1/2} G\|_2^2$.

Justification: By the First and Second Representation Theorems for densely defined, closed, lower-bounded symmetric forms (Kato [5, Thm. VI.2.1, Thm. VI.2.23]; in the Dirichlet-form setting, Fukushima–Oshima–Takeda [2, Thm. 1.3.1]). Kato Thm. VI.2.1 gives existence of the unique associated selfadjoint operator; Kato Thm. VI.2.23 (Second Representation Theorem) establishes $\mathcal{D}(\mathcal{E}_\lambda) = \mathcal{D}(A_\lambda^{1/2})$ and the identity $\mathcal{E}_\lambda(G) = \|A_\lambda^{1/2} G\|_2^2$. Proposition 16 verified that $\mathcal{E}_\lambda$ satisfies all hypotheses.

Step 2. $(A_\lambda + 1)^{-1}$ is compact on $L^2(I)$.

Step 2.1. Let $\{f_n\}$ be bounded in $L^2(I)$: $\|f_n\|_2 \leq C$. Set $u_n := (A_\lambda + 1)^{-1} f_n$.

Justification: $A_\lambda + 1$ is invertible because $A_\lambda \geq 0$, so $A_\lambda + 1 \geq 1 > 0$.

Step 2.2. $u_n \in \mathcal{D}(A_\lambda) \subset \mathcal{D}(\mathcal{E}_\lambda)$ and $(A_\lambda + 1)u_n = f_n$.

Justification: Definition of the resolvent.

Step 2.3. $\mathcal{E}_\lambda(u_n) + \|u_n\|_2^2 = \langle f_n, u_n \rangle$.

Justification: Take the L^2 inner product of $(A_\lambda + 1)u_n = f_n$ with u_n: $\langle A_\lambda u_n, u_n \rangle + \|u_n\|_2^2 = \langle f_n, u_n \rangle$. By the form identity, $\langle A_\lambda u_n, u_n \rangle = \mathcal{E}_\lambda(u_n)$.

Step 2.4. $\|u_n\|_2^2 + \mathcal{E}_\lambda(u_n) \leq \|f_n\|_2^2$.

Justification: From Step 2.3: $\mathcal{E}_\lambda(u_n) + \|u_n\|_2^2 = \langle f_n, u_n \rangle \leq \|f_n\|_2 \|u_n\|_2$ by Cauchy–Schwarz. Since $\|u_n\|_2^2 \leq \mathcal{E}_\lambda(u_n) + \|u_n\|_2^2 \leq \|f_n\|_2 \|u_n\|_2$, we get $\|u_n\|_2 \leq \|f_n\|_2$ (if $u_n = 0$ the bound is trivial), hence $\mathcal{E}_\lambda(u_n) + \|u_n\|_2^2 \leq \|f_n\|_2 \|u_n\|_2 \leq \|f_n\|_2^2$.

Step 2.5. $\{u_n\}$ has a convergent subsequence in $L^2(I)$.

Justification: Step 2.4 shows $\{u_n\}$ is bounded in the form norm (with $M := C^2$), so Proposition 23 gives relative compactness in $L^2(I)$.

Step 2.6. $(A_\lambda + 1)^{-1}$ maps bounded sequences to sequences with convergent subsequences, hence is compact.

Justification: This is the definition of a compact operator: it maps bounded sets to relatively compact sets. Steps 2.1–2.5 verify this.

Q.E.D. □

7.3 Semigroup and irreducibility

Definition 25 (Irreducibility for semigroups on $L^2(I)$). A closed ideal in $L^2(I)$ has the form $L^2(B)$ for some measurable $B \subset I$. We call T *irreducible* if the only invariant closed ideals are $\{0\}$ and $L^2(I)$.

Lemma 26 (Invariant ideals and splitting of the form)**.** *Assume Theorem 24. Let $B \subset I$ be measurable and suppose $L^2(B)$ is invariant under $T(t) = e^{-tA_\lambda}$. Then for every $G \in \mathcal{D}(\mathcal{E}_\lambda)$: $\mathbf{1}_B G, \mathbf{1}_{I \setminus B} G \in \mathcal{D}(\mathcal{E}_\lambda)$ and $\mathcal{E}_\lambda(G) = \mathcal{E}_\lambda(\mathbf{1}_B G) + \mathcal{E}_\lambda(\mathbf{1}_{I \setminus B} G)$.*

Structured Proof. **Step 1.** Let $P = M_{\mathbf{1}_B}$ (multiplication by $\mathbf{1}_B$) and $Q = I - P$. Then P is an orthogonal projection with $\mathrm{Ran}(P) = L^2(B)$.

Justification: $P^2 = P$ ($\mathbf{1}_B^2 = \mathbf{1}_B$) and $P^* = P$ ($\mathbf{1}_B$ is real), so P is an orthogonal projection. $Pf = \mathbf{1}_B f$ vanishes outside B, so $\mathrm{Ran}(P) = L^2(B)$.

Step 2. P commutes with $T(t)$ for all $t \geq 0$.

Step 2.1. Invariance of $L^2(B) = \mathrm{Ran}(P)$ means $T(t)(\mathrm{Ran}(P)) \subset \mathrm{Ran}(P)$, i.e. $PT(t)P = T(t)P$.

Justification: $T(t)Pf \in L^2(B)$ for all f, so $PT(t)Pf = T(t)Pf$.

Step 2.2. Take adjoints: $(PT(t)P)^* = (T(t)P)^*$, giving $PT(t)P = PT(t)$.

Justification: $P^* = P$ and $T(t)^* = T(t)$ (selfadjointness of A_λ implies selfadjointness of e^{-tA_λ}). So $(PT(t)P)^* = P^*T(t)^*P^* = PT(t)P$ and $(T(t)P)^* = P^*T(t)^* = PT(t)$. From Step 2.1: $PT(t)P = T(t)P$; equating with the adjoint computation: $PT(t)P = PT(t)$. Together: $T(t)P = PT(t)$.

Step 3. P commutes with $(A_\lambda + \alpha)^{-1}$ for every $\alpha > 0$.

Justification: By the Laplace-transform formula for C_0-semigroups (Engel–Nagel [8, Cor. II.1.11]):

$$(A_\lambda + \alpha)^{-1} = \int_0^\infty e^{-\alpha t}\, e^{-tA_\lambda}\, dt$$

holds as a Bochner integral in $\mathcal{B}(L^2(I))$ (convergence: $\|e^{-\alpha t}T(t)\| \le e^{-\alpha t}$, integrable for $\alpha > 0$; α lies in the resolvent set of $-A_\lambda$ since $\sigma(A_\lambda) \subset [0,\infty)$). By Step 2, P commutes with e^{-tA_λ} for every $t \ge 0$, hence commutes with the Bochner integral.

Step 4. P commutes with $A_\lambda^{1/2}$.

Justification: Let $R := (A_\lambda + 1)^{-1}$, which is bounded and selfadjoint. Step 3 (with $\alpha = 1$) gives $PR = RP$. Since $PR = RP$, P commutes with every bounded Borel function of R (Reed–Simon [9, Cor. to Thm. VIII.5]: $PR^n = R^nP$ by induction; extend to polynomials by linearity, to $C(\sigma(R))$ by Weierstrass, and to bounded Borel functions via strong-operator-topology limits). In particular, P commutes with the spectral projections of R, hence with E_R. Since $R = \varphi(A_\lambda)$ with $\varphi(\mu) = (\mu+1)^{-1}$, a continuous strictly decreasing bijection $[0,\infty) \to (0,1]$, the spectral measures satisfy $E_{A_\lambda}(\Delta) = E_R(\varphi(\Delta))$ for Borel $\Delta \subset [0,\infty)$. Thus $PE_{A_\lambda}(\Delta) = E_{A_\lambda}(\Delta)P$ for all Borel Δ. *Domain preservation:* For $u \in \mathcal{D}(A_\lambda^{1/2})$, $\int_0^\infty \mu\, d\|E_{A_\lambda}(\mu)Pu\|^2 = \int_0^\infty \mu\, d\|PE_{A_\lambda}(\mu)u\|^2 \le \int_0^\infty \mu\, d\|E_{A_\lambda}(\mu)u\|^2 = \|A_\lambda^{1/2}u\|^2 < \infty$, so $Pu \in \mathcal{D}(A_\lambda^{1/2})$. *Commutativity on the domain:* $A_\lambda^{1/2}Pu = \int_0^\infty \mu^{1/2}\, dE_{A_\lambda}(\mu)Pu = \int_0^\infty \mu^{1/2}\, PE_{A_\lambda}(\mu)u\, d\mu = P\int_0^\infty \mu^{1/2}\, dE_{A_\lambda}(\mu)u = PA_\lambda^{1/2}u$ (interchange of P with the spectral integral is justified since P is bounded and commutes with each $E_{A_\lambda}(\Delta)$).

Step 5. $P(\mathcal{D}(\mathcal{E}_\lambda)) \subset \mathcal{D}(\mathcal{E}_\lambda)$ and $Q(\mathcal{D}(\mathcal{E}_\lambda)) \subset \mathcal{D}(\mathcal{E}_\lambda)$.

Justification: $\mathcal{D}(\mathcal{E}_\lambda) = \mathcal{D}(A_\lambda^{1/2})$. If $u \in \mathcal{D}(A_\lambda^{1/2})$, then $A_\lambda^{1/2}Pu = PA_\lambda^{1/2}u \in L^2(I)$ (Step 4), so $Pu \in \mathcal{D}(A_\lambda^{1/2})$. Similarly for $Qu = (I-P)u$.

Step 6. $\mathcal{E}_\lambda(G) = \mathcal{E}_\lambda(PG) + \mathcal{E}_\lambda(QG)$ for $G \in \mathcal{D}(\mathcal{E}_\lambda)$.

Step 6.1. $\mathcal{E}_\lambda(G) = \|A_\lambda^{1/2}G\|_2^2$.

Justification: Form identity from the representation theorem (Step 1 of Theorem 24).

Step 6.2. $A_\lambda^{1/2}G = A_\lambda^{1/2}PG + A_\lambda^{1/2}QG = PA_\lambda^{1/2}G + QA_\lambda^{1/2}G$.

Justification: $G = PG + QG$ and $A_\lambda^{1/2}$ commutes with P and Q (Step 4).

Step 6.3. $PA_\lambda^{1/2}G$ and $QA_\lambda^{1/2}G$ are orthogonal in $L^2(I)$.

Justification: $\langle Pv, Qv\rangle = \langle Pv, (I-P)v\rangle = \langle Pv, v\rangle - \langle Pv, Pv\rangle = \langle P^2v, v\rangle - \|Pv\|^2 = \|Pv\|^2 - \|Pv\|^2 = 0$. Apply with $v = A_\lambda^{1/2}G$.

Step 6.4. $\|A_\lambda^{1/2}G\|^2 = \|PA_\lambda^{1/2}G\|^2 + \|QA_\lambda^{1/2}G\|^2 = \|A_\lambda^{1/2}PG\|^2 + \|A_\lambda^{1/2}QG\|^2 = \mathcal{E}_\lambda(PG) + \mathcal{E}_\lambda(QG)$.

Justification: Pythagorean theorem (Step 6.3), then $PA_\lambda^{1/2}G = A_\lambda^{1/2}PG$ (Step 4), then the form identity (Step 6.1) applied to PG and QG (which lie in $\mathcal{D}(\mathcal{E}_\lambda)$ by Step 5).

Q.E.D. □

Proposition 27 (Triviality of invariant ideals for $\mathcal{E}_\lambda$)**.** *Assume Theorem 24. Let $B \subset I$ be measurable and assume $L^2(B)$ is invariant under $T(t) = e^{-tA_\lambda}$. Then $m(B) = 0$ or $m(I \setminus B) = 0$.*

Structured Proof. **Step 1.** $1 \in \mathcal{D}(\mathcal{E}_\lambda)$ (the constant function $G \equiv 1$ on I).

Step 1.1. For each shift $s > 0$: $\|\widetilde{1} - S_s\widetilde{1}\|_2^2 = m(I\,\Delta\,(I+s))$.

Justification: $\widetilde{1} = \mathbf{1}_{(-L,L)}$ and $S_s\widetilde{1} = \mathbf{1}_{(-L+s,L+s)}$. $\|\widetilde{1} - S_s\widetilde{1}\|_2^2 = m(I\,\Delta\,(I+s))$, the measure of the symmetric difference.

Step 1.2. For $0 < s < 2L$: $m(I\,\Delta\,(I+s)) = 2s$.

Justification: $I \setminus (I+s) = (-L, -L+s)$ has measure s; $(I+s) \setminus I = (L, L+s)$ has measure s.

Step 1.3. The archimedean integral $\int_0^{2L} w(t) \cdot 2t\,dt < \infty$.

Justification: Since $\sinh t \geq t$: $w(t) \leq e^{t/2}/(2t)$, so $w(t) \cdot 2t \leq e^{t/2} \leq e^L$ on $[0, 2L]$. The integrand is bounded on a compact interval, hence integrable.

Step 1.4. The prime sum in $\mathcal{E}_\lambda(1)$ is finite.

Justification: Finitely many shift sizes $m \log p$ with $p^m \leq \lambda^2$, each giving $\|\widetilde{1} - S_{m\log p}\widetilde{1}\|_2^2 = 2m \log p < \infty$.

Step 1.5. $\mathcal{E}_\lambda(1) < \infty$, hence $1 \in \mathcal{D}(\mathcal{E}_\lambda)$.

Justification: Combine Steps 1.3 and 1.4.

Step 2. By Lemma 26, for every $G \in \mathcal{D}(\mathcal{E}_\lambda)$:

$$\mathcal{E}_\lambda(G) = \mathcal{E}_\lambda(\mathbf{1}_B G) + \mathcal{E}_\lambda(\mathbf{1}_{I\setminus B} G). \tag{7}$$

Justification: Direct application of Lemma 26.

Step 3. Apply (7) with $G \equiv 1$ (justified by Step 1). Set $f := \widetilde{\mathbf{1}_B}$, $g := \widetilde{\mathbf{1}_{B^c}}$ where $B^c := I \setminus B$. Then $\widetilde{1} = f + g$ and $fg = 0$ a.e.

Justification: $\mathbf{1}_B + \mathbf{1}_{B^c} = 1$ on I and $\mathbf{1}_B \cdot \mathbf{1}_{B^c} = 0$. Zero-extending: $f + g = \widetilde{1}$ and $fg = 0$ a.e. on $\mathbb{R}$.

Step 4. For each shift $s > 0$: $\|(f+g) - S_s(f+g)\|_2^2 = \|f - S_s f\|_2^2 + \|g - S_s g\|_2^2 + 2\langle f - S_s f, g - S_s g\rangle$.

Justification: Expand $\|a + b\|^2 = \|a\|^2 + \|b\|^2 + 2\Re\langle a, b\rangle$ with $a = f - S_s f$ and $b = g - S_s g$. Since f, g are real-valued, $\langle a, b\rangle \in \mathbb{R}$.

Step 5. Substituting Step 4 into (7) (which, via Definition 6, is an identity between weighted sums/integrals of $\|\cdot - S_s \cdot\|_2^2$ terms): the cross-terms sum to zero.

Step 5.1. $\mathcal{E}_\lambda(1) = \mathcal{E}_\lambda(\mathbf{1}_B) + \mathcal{E}_\lambda(\mathbf{1}_{B^c})$ (from Step 2 with $G = 1$).

Justification: Equation (7) with $G \equiv 1$.

Step 5.2. Expand each $\mathcal{E}_\lambda$ using Definition 6 and Step 4: each shift-size gives the identity from Step 4. After subtracting $\mathcal{E}_\lambda(\mathbf{1}_B) + \mathcal{E}_\lambda(\mathbf{1}_{B^c})$ from $\mathcal{E}_\lambda(1)$, the remainder is

$$\int_0^{2L} w(t) \cdot 2\langle f - S_t f, g - S_t g\rangle\,dt + \sum_{p,m} (\log p) p^{-m/2} \cdot 2\langle f - S_{m\log p} f, g - S_{m\log p} g\rangle = 0.$$

Justification: Combine Step 4 for each shift size with Step 5.1. The "diagonal" terms cancel, leaving the cross terms.

Step 6. For any $s > 0$: $\langle f - S_s f, g - S_s g\rangle \leq 0$.

Step 6.1. $\langle f - S_s f, g - S_s g\rangle = \langle f, g\rangle - \langle f, S_s g\rangle - \langle S_s f, g\rangle + \langle S_s f, S_s g\rangle$.

Justification: Bilinearity of the inner product.

Step 6.2. $\langle f, g\rangle = 0$.

Justification: $fg = 0$ a.e. (Step 3), so $\int_{\mathbb{R}} f(u)g(u)\,du = 0$.

Step 6.3. $\langle S_s f, S_s g\rangle = \langle f, g\rangle = 0$.

Justification: S_s is unitary: $\langle S_s f, S_s g\rangle = \langle f, g\rangle$. Then Step 6.2.

Step 6.4. $\langle f, S_s g\rangle \geq 0$ and $\langle S_s f, g\rangle = \langle g, S_s f\rangle \geq 0$.

Justification: $f = \widetilde{\mathbf{1}_B} \geq 0$ and $S_s g = \widetilde{\mathbf{1}_{B^c}}(\cdot - s) \geq 0$, so $\langle f, S_s g\rangle = \int f \cdot S_s g \geq 0$. Similarly for $\langle g, S_s f\rangle$.

Step 6.5. Combine: $\langle f - S_s f, g - S_s g\rangle = 0 - \langle f, S_s g\rangle - \langle g, S_s f\rangle + 0 \leq 0$.

Justification: Steps 6.1–6.4.

Step 7. $\langle f, S_t g\rangle = \langle g, S_t f\rangle = 0$ for a.e. $t \in (0, 2L)$.

Justification: From Step 5.2, the weighted sum of the cross terms is 0. All weights are ≥ 0, and $w(t) > 0$ for $t > 0$. Each cross term $\langle f - S_t f, g - S_t g\rangle \leq 0$ (Step 6). A sum of nonpositive terms with positive weights equaling zero forces each term to be zero a.e. Hence $\langle f - S_t f, g - S_t g\rangle = 0$ for a.e. $t \in (0, 2L)$. From Step 6.5: this means $\langle f, S_t g\rangle + \langle g, S_t f\rangle = 0$ a.e. Since both are ≥ 0 (Step 6.4), each is 0.

Step 8. Upgrade to *all* $t \in (0, 2L)$: $\langle f, S_t g\rangle = 0$ for every $t \in (0, 2L)$.

Justification: $t \mapsto \langle f, S_t g\rangle$ is continuous (strong continuity of S_t on $L^2(\mathbb{R})$). A continuous nonneg. function vanishing a.e. on $(0, 2L)$ vanishes everywhere on $(0, 2L)$. (Same argument as in Lemma 11, Step 3.)

Step 9. For every $t \in (0, 2L)$: $\mathbf{1}_B(u) = \mathbf{1}_B(u - t)$ for a.e. $u \in I \cap (I + t)$.

Step 9.1. $0 = \langle f, S_t g\rangle = \int_{I\cap(I+t)} \mathbf{1}_B(u) \cdot \mathbf{1}_{B^c}(u - t)\,du$.

Justification: Unwinding definitions: $f = \widetilde{\mathbf{1}_B}$, $S_t g(u) = \widetilde{\mathbf{1}_{B^c}}(u - t)$. The integrand is nonzero only where both $u \in I$ (so $f(u) = \mathbf{1}_B(u)$) and $u - t \in I$ (so $g(u - t) = \mathbf{1}_{B^c}(u - t)$), i.e. $u \in I \cap (I + t)$.

Step 9.2. Since $\mathbf{1}_B(u) \cdot \mathbf{1}_{B^c}(u - t) \geq 0$ and the integral is 0: $\mathbf{1}_B(u) \cdot \mathbf{1}_{B^c}(u - t) = 0$ for a.e. $u \in I \cap (I + t)$.

Justification: A nonneg. integrable function with zero integral vanishes a.e.

Step 9.3. Hence $\mathbf{1}_B(u) \leq \mathbf{1}_B(u - t)$ for a.e. $u \in I \cap (I + t)$.

Justification: Step 9.2 says: wherever $\mathbf{1}_B(u) = 1$, we must have $\mathbf{1}_{B^c}(u - t) = 0$, i.e. $\mathbf{1}_B(u - t) = 1$.

Step 9.4. Similarly, $\langle g, S_t f\rangle = 0$ gives $\mathbf{1}_B(u - t) \leq \mathbf{1}_B(u)$ a.e. on $I \cap (I + t)$.

Justification: $\langle g, S_t f\rangle = \int_{I\cap(I+t)} \mathbf{1}_{B^c}(u)\mathbf{1}_B(u - t)\,du = 0$ (Step 8). Same argument as Steps 9.2–9.3 with B and B^c swapped.

Step 9.5. Combine: $\mathbf{1}_B(u) = \mathbf{1}_B(u - t)$ a.e. on $I \cap (I + t)$.

Justification: Steps 9.3 and 9.4.

Step 10. $m(B) = 0$ or $m(I \setminus B) = 0$.

Justification: Step 9.5 holds for every $t \in (0, 2L)$. This provides the hypothesis of Lemma 10 with $\varepsilon = 2L$. The conclusion follows.

Q.E.D. □

Remark 28 (Why we do not use $\mathcal{E}_\lambda(\mathbf{1}_B) = 0$). Because $\mathcal{E}_\lambda$ is defined using zero-extension to $\mathbb{R}$ (Definition 6), the form is typically non-conservative: in general $\mathcal{E}_\lambda(1) > 0$. In the conservative case ($\mathcal{E}(1) = 0$) one often has an equivalence between invariance and the condition $\mathcal{E}(\mathbf{1}_B) = 0$. Here, the presence of killing means $\mathcal{E}_\lambda(\mathbf{1}_B) = 0$ is a *stronger* condition than invariance, so we instead argue directly from the correct invariance identity (7), which depends only on the interaction/jump part.

Corollary 29 (Irreducibility for $\mathcal{E}_\lambda$)**.** *Assume Theorem 24. Then* $T(t) = e^{-tA_\lambda}$ *is irreducible.*

Structured Proof. **Step 1.** Let $J \subset L^2(I)$ be a closed $T(t)$-invariant ideal. Then $J = L^2(B)$ for some measurable $B \subset I$.

Justification: Standard lattice-theory fact: every closed ideal in $L^2(I)$ has the form $L^2(B)$ for some measurable $B \subset I$ (Schaefer [7, Sect. III.1, Ex. 1]).

Step 2. $m(B) = 0$ or $m(I \setminus B) = 0$.

Justification: Proposition 27.

Step 3. $J = \{0\}$ or $J = L^2(I)$.

Justification: If $m(B) = 0$, then $L^2(B) = \{0\}$. If $m(I \setminus B) = 0$, then $L^2(B) = L^2(I)$.

Q.E.D. □

8 Positivity improving and the ground state

8.1 External theorems used

Theorem 30 (Positivity improving from positivity + irreducibility + holomorphy)**.** *Let E be a Banach lattice and S a positive, irreducible, holomorphic C_0-semigroup on E. Then S is positivity improving: for each $t > 0$ and each $0 \le f \in E$ with $f \ne 0$, one has $S(t)f > 0$ (in the lattice sense; on L^2 this means > 0 a.e.).*

Remark 31 (Source)*.* This is stated (for general Banach lattices) as Theorem 2.3 in Arendt–ter Elst–Glück [1].

Theorem 32 (Simplicity of the principal eigenvalue under compact resolvent)**.** *Let A be selfadjoint and lower bounded on $L^2(I)$ with compact resolvent, and let $S(t) = e^{-tA}$. If S is positivity improving, then all four conclusions of Arendt–ter Elst–Glück [1, Prop. 2.4] hold:*

(a) $\sigma(A) \ne \emptyset$ (automatic for a selfadjoint operator on a nonzero Hilbert space);

(b) $\lambda_1 := \inf\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\}$ is an eigenvalue (so the infimum is a minimum);

(c) the associated eigenfunction satisfies $\psi > 0$ a.e. (a strictly positive quasi-interior point in $L^2(I)$);

(d) λ_1 has algebraic multiplicity one (simple eigenvalue).

In summary: $\min \sigma(A)$ *is a simple eigenvalue admitting an eigenfunction strictly positive a.e.*

Remark 33 (Source)*.* Proposition 2.4 in the same paper of Arendt et al., a Perron–Frobenius/Krein–Rutman/Jentzsch consequence for compact positive operators.

8.2 Application to A_λ

Proposition 34 (Positivity improving and simple ground state for A_λ)**.** *Assume Theorem 24. Then:*

1. *The semigroup $T(t) = e^{-tA_\lambda}$ is positivity preserving (Markovian).*

2. *$T(t)$ is irreducible.*

3. $T(t)$ is holomorphic.

Consequently $T(t)$ is positivity improving, and the lowest eigenvalue of A_λ is simple with a strictly positive a.e. eigenfunction.

Structured Proof. **Step 1.** $T(t)$ is positivity preserving (Markovian).

Justification: Lemma 9 shows that $\mathcal{E}_\lambda$ satisfies the Markov (normal contraction) property. By the general correspondence between Dirichlet forms and positivity-preserving semigroups (Fukushima–Oshima–Takeda [2, Thm. 1.4.1]; Ouhabaz [4, Thm. 1.4.1]), the associated semigroup $T(t)$ is positivity preserving.

Step 2. $T(t)$ is irreducible.

Justification: Corollary 29.

Step 3. $T(t)$ is holomorphic.

Justification: $T(t)$ is strongly continuous (C_0): by the spectral theorem, $\|T(t)f - f\|_2^2 = \int_0^\infty |e^{-t\mu} - 1|^2 \, d\|E_{A_\lambda}(\mu)f\|^2 \to 0$ as $t \to 0^+$ by dominated convergence. Moreover, A_λ is selfadjoint and lower bounded ($A_\lambda \geq 0$), hence m-sectorial with numerical range in $[0, \infty)$. By Kato [5, Ex. IX.1.25] (backed by Thm. IX.1.24), e^{-zA_λ} is bounded and holomorphic on $\{z \in \mathbb{C} : \Re z > 0\}$ with $\|e^{-zA_\lambda}\| \leq 1$. In particular, $T(t) = e^{-tA_\lambda}$ extends to a holomorphic semigroup.

Step 4. $T(t)$ is positivity improving.

Justification: Apply Theorem 30: $T(t)$ is positive (Step 1), irreducible (Step 2), and holomorphic (Step 3). Therefore it is positivity improving.

Step 5. The lowest eigenvalue of A_λ is simple, with a strictly positive a.e. eigenfunction.

Justification: Apply Theorem 32: A_λ is selfadjoint, lower bounded, has compact resolvent (Theorem 24), and $T(t) = e^{-tA_\lambda}$ is positivity improving (Step 4). All four conclusions of Theorem 32 (ATG [1, Prop. 2.4]) hold: (a) $\sigma(A_\lambda) \neq \emptyset$; (b) $\mu_0 := \min \sigma(A_\lambda)$ exists and is an eigenvalue (we write μ_0 instead of the λ_1 of Theorem 32 to avoid confusion with the form parameter λ); (c) the corresponding eigenfunction is strictly positive a.e.; (d) μ_0 is a simple eigenvalue (multiplicity one).

Q.E.D. □

9 Evenness of the ground state from inversion symmetry

Proposition 35 (Inversion (reflection) symmetry)**.** *Let $R : L^2(I) \to L^2(I)$ be the unitary involution $(RG)(u) := G(-u)$. Then $R(\mathcal{D}(\mathcal{E}_\lambda)) = \mathcal{D}(\mathcal{E}_\lambda)$ and $\mathcal{E}_\lambda(RG) = \mathcal{E}_\lambda(G)$ for all $G \in \mathcal{D}(\mathcal{E}_\lambda)$. Consequently, A_λ commutes with R.*

Structured Proof. **Step 1.** R is a well-defined unitary involution on $L^2(I)$, and R preserves $H_I \subset L^2(\mathbb{R})$.

Justification: $I = (-L, L)$ is symmetric about 0: if $u \in I$ then $-u \in I$. So $(RG)(u) = G(-u)$ maps $L^2(I)$ to itself. R is unitary ($\|RG\|_2 = \|G\|_2$ by substitution $u \mapsto -u$) and $R^2 = \mathrm{Id}$. If $\phi \in H_I$ (i.e. $\phi = 0$ outside I), then $R\phi(u) = \phi(-u)$ vanishes for $u \notin I$ (since $-u \notin I$), so $R\phi \in H_I$.

Step 2. $RS_t = S_{-t}R$ on $L^2(\mathbb{R})$.

Justification: $(RS_t\phi)(u) = (S_t\phi)(-u) = \phi(-u - t)$ and $(S_{-t}R\phi)(u) = (R\phi)(u - (-t)) = (R\phi)(u + t) = \phi(-u - t)$. They agree.

Step 3. $\|\widetilde{RG} - S_t\widetilde{RG}\|_2 = \|\widetilde{G} - S_t\widetilde{G}\|_2$ for every $t \in \mathbb{R}$.

Step 3.1. $\widetilde{RG} = R\widetilde{G}$ (extension by zero commutes with reflection, since I is symmetric).

Justification: For $u \in I$: $\widetilde{RG}(u) = (RG)(u) = G(-u) = (R\widetilde{G})(u)$. For $u \notin I$: $\widetilde{RG}(u) = 0$ and $(R\widetilde{G})(u) = \widetilde{G}(-u) = 0$ (since $-u \notin I$).

Step 3.2. $\|R\widetilde{G} - S_t R\widetilde{G}\|_2 = \|R(\widetilde{G} - S_{-t}\widetilde{G})\|_2$ (using Step 2: $S_t R = RS_{-t}$, so $S_t R\widetilde{G} = RS_{-t}\widetilde{G}$).

Justification: $R\widetilde{G} - S_t R\widetilde{G} = R\widetilde{G} - RS_{-t}\widetilde{G} = R(\widetilde{G} - S_{-t}\widetilde{G})$.

Step 3.3. $\|R(\widetilde{G} - S_{-t}\widetilde{G})\|_2 = \|\widetilde{G} - S_{-t}\widetilde{G}\|_2$ (R is unitary).

Justification: R is unitary on $L^2(\mathbb{R})$ by the same substitution $v = -u$ as in Step 1 (the isometry and surjectivity arguments carry over verbatim from $L^2(I)$ to $L^2(\mathbb{R})$).

Step 3.4. $\|\widetilde{G} - S_{-t}\widetilde{G}\|_2 = \|\widetilde{G} - S_t\widetilde{G}\|_2$.

Justification: Substituting $v = u + t$: $\|\phi - S_{-t}\phi\|_2^2 = \int |\phi(u) - \phi(u+t)|^2 \, du = \int |\phi(v-t) - \phi(v)|^2 \, dv = \|\phi - S_t\phi\|_2^2$.

Step 3.5. Chain Steps 3.1–3.4: $\|\widetilde{RG} - S_t\widetilde{RG}\|_2 = \|R\widetilde{G} - S_t R\widetilde{G}\|_2 = \|\widetilde{G} - S_{-t}\widetilde{G}\|_2 = \|\widetilde{G} - S_t\widetilde{G}\|_2$.

Step 4. $\mathcal{E}_\lambda(RG) = \mathcal{E}_\lambda(G)$.

Justification: By Definition 6, $\mathcal{E}_\lambda$ is built from terms of the form $\|\widetilde{G} - S_t\widetilde{G}\|_2^2$ with nonneg. weights. Step 3 shows each such term is the same for RG as for G. Therefore $\mathcal{E}_\lambda(RG) = \mathcal{E}_\lambda(G)$. In particular, $RG \in \mathcal{D}(\mathcal{E}_\lambda)$ iff $G \in \mathcal{D}(\mathcal{E}_\lambda)$.

Step 5. A_λ commutes with R.

Step 5.1. For $u \in \mathcal{D}(A_\lambda)$ and $v \in \mathcal{D}(\mathcal{E}_\lambda)$: $\mathcal{E}_\lambda(Ru, v) = \mathcal{E}_\lambda(u, Rv)$.

Justification: $Ru \in \mathcal{D}(\mathcal{E}_\lambda)$ (since R preserves $\mathcal{D}(\mathcal{E}_\lambda)$ by Step 4). Invariance of $\mathcal{E}_\lambda$ under R (Step 4) implies, by polarization, $\mathcal{E}_\lambda(Ru, Rv) = \mathcal{E}_\lambda(u, v)$ for all $u, v \in \mathcal{D}(\mathcal{E}_\lambda)$. Set $v \mapsto Rv$: $\mathcal{E}_\lambda(Ru, R(Rv)) = \mathcal{E}_\lambda(Ru, v) = \mathcal{E}_\lambda(u, Rv)$ (using $R^2 = \mathrm{Id}$).

Step 5.2. $\mathcal{E}_\lambda(u, Rv) = \langle A_\lambda u, Rv \rangle = \langle RA_\lambda u, v \rangle$.

Justification: Form identity: $\mathcal{E}_\lambda(u, Rv) = \langle A_\lambda u, Rv \rangle$ (since $u \in \mathcal{D}(A_\lambda)$ and $Rv \in \mathcal{D}(\mathcal{E}_\lambda)$). Then $\langle A_\lambda u, Rv \rangle = \langle R^* A_\lambda u, v \rangle = \langle RA_\lambda u, v \rangle$ since $R^* = R$ (R is selfadjoint: $R = R^{-1} = R^*$).

Step 5.3. Combining: $\mathcal{E}_\lambda(Ru, v) = \langle RA_\lambda u, v \rangle$ for all $v \in \mathcal{D}(\mathcal{E}_\lambda)$.

Justification: Steps 5.1 and 5.2.

Step 5.4. $Ru \in \mathcal{D}(A_\lambda)$ and $A_\lambda Ru = RA_\lambda u$.

Justification: By the representation theorem (Kato [5, Thm. VI.2.1]), $w \in \mathcal{D}(A_\lambda)$ if and only if there exists $h \in L^2(I)$ such that $\mathcal{E}_\lambda(w, v) = \langle h, v \rangle$ for all $v \in \mathcal{D}(\mathcal{E}_\lambda)$, in which case $A_\lambda w = h$. Step 5.3 provides exactly this with $w = Ru$ and $h = RA_\lambda u \in L^2(I)$.

Q.E.D. □

Corollary 36 (Even ground state)**.** *Assume Theorem 24 and Proposition 35. Let ψ be the strictly positive ground-state eigenfunction from Proposition 34. Then ψ is even: $\psi(-u) = \psi(u)$ a.e.*

Structured Proof. **Step 1.** Define $\psi^\sharp := R\psi$. Then $\psi^\sharp$ is an eigenfunction of A_λ for the same eigenvalue $\mu_0 := \min \sigma(A_\lambda)$.

Justification: $A_\lambda\psi = \mu_0\psi$. Since $A_\lambda R = RA_\lambda$ (Proposition 35): $A_\lambda\psi^\sharp = A_\lambda R\psi = RA_\lambda\psi = R(\mu_0\psi) = \mu_0 R\psi = \mu_0\psi^\sharp$.

Step 2. $\psi^\sharp > 0$ a.e.

Justification: $\psi > 0$ a.e. (Proposition 34). $\psi^\sharp(u) = \psi(-u)$. Since I is symmetric and $\psi > 0$ a.e. on I, the reflection $\psi^\sharp > 0$ a.e. on I.

Step 3. $\psi^\sharp = c\psi$ for some $c \in \mathbb{R}$.

Justification: Proposition 34 says the eigenspace for μ_0 is one-dimensional (simple eigenvalue). Both ψ and $\psi^\sharp$ lie in this eigenspace (Steps 1 and the original eigenvalue equation). Hence $\psi^\sharp = c\psi$ for some scalar c. Since both are real-valued (reflection preserves real-valuedness), $c \in \mathbb{R}$.

Step 4. $c > 0$.

Justification: $\psi^\sharp > 0$ a.e. (Step 2) and $\psi > 0$ a.e. (Proposition 34). If $c \leq 0$, then $\psi^\sharp = c\psi \leq 0$ a.e., contradicting $\psi^\sharp > 0$ a.e.

Step 5. $c = 1$.

Justification: $\|\psi^\sharp\|_2 = \|R\psi\|_2 = \|\psi\|_2$ (R is unitary). From $\psi^\sharp = c\psi$: $\|c\psi\|_2 = |c|\|\psi\|_2 = \|\psi\|_2$. Since $\|\psi\|_2 > 0$ ($\psi \neq 0$), $|c| = 1$. Combined with $c > 0$ (Step 4): $c = 1$.

Step 6. $\psi(-u) = \psi(u)$ a.e.

Justification: $\psi^\sharp = \psi$ (Step 5), i.e. $R\psi = \psi$, i.e. $\psi(-u) = \psi(u)$ a.e.

Q.E.D. $\square$

10 Conclusion

- Starting solely from the explicit local formulas (2)–(3), we derived a representation of $-\sum_v W_v(g * g^*)$ (up to an additive constant multiple of $\|g\|_2^2$) as a positive combination of translation-difference energies in log-coordinates (Definition 6, Lemmas 4–5).
- We proved the Markov/normal contraction inequality for this form (Lemma 9).
- Using only measure theory (Lebesgue density), we proved that invariance under all sufficiently small translations forces a measurable subset of an interval to be null or conull (Lemma 10), and we used it to show that $\mathcal{E}_\lambda(\mathbf{1}_B) = 0$ implies B is null or conull (Lemma 11).
- We proved that the quadratic form is closed (Propositions 15–16), established a logarithmic lower bound on its Fourier symbol (Lemma 18), and used the Kolmogorov–Riesz compactness criterion to show that the associated selfadjoint operator has compact resolvent (Theorem 24).
- From this operator setup we obtained irreducibility and then (by a standard external theorem) positivity improving of the semigroup, hence simplicity and strict positivity of the ground state (Proposition 34).
- Finally, inversion symmetry forces that strictly positive simple ground state to be even (Corollary 36).

Remark 37 (Generalization to Dedekind zeta functions and Rankin–Selberg L-functions)*.* The proof relies on two structural features of the Weil explicit formula: (i) the local terms at non-archimedean places produce translation-difference energies with *non-negative* weights, and (ii) the archimedean term produces a continuum of such energies with a strictly positive weight $w(t) > 0$. Everything else—the Markov property, irreducibility from the archimedean continuum, the logarithmic lower bound on the Fourier symbol, and the Perron–Frobenius conclusion—follows formally from (i) and (ii).

Dedekind zeta functions. For a number field K, the Weil explicit formula for $\zeta_K(s)$ has the same structure: a sum over prime ideals $\mathfrak{p}$ (with shift sizes $m \log N(\mathfrak{p})$ in place of $m \log p$) plus archimedean contributions from each infinite place. Each archimedean place contributes its own integral term with a strictly positive weight, so the logarithmic lower bound on the Fourier symbol and the irreducibility argument both carry over. The generalization requires only notational changes.

Rankin–Selberg L-functions. The method may be most naturally suited to L-functions of the form $L(s, \pi \times \widetilde{\pi})$, where π is a cuspidal automorphic representation of $\mathrm{GL}(n)$. The self-convolution structure makes property (i) automatic: the Dirichlet coefficients at unramified primes are $|a_\pi(\mathfrak{p})|^2$ (and sums thereof at higher prime powers), so the prime-energy weights are manifestly non-negative without any algebraic manipulation. For property (ii), the archimedean L-factor of $L(s, \pi \times \widetilde{\pi})$ is a product of n^2 Gamma-type terms, each contributing a digamma-type weight to the integral. This only strengthens the logarithmic lower bound. Since GRH for $L(s, \pi \times \widetilde{\pi})$ implies GRH for $L(s, \pi)$ (the zeros of the former include those of the latter), establishing the spectral setup—compact resolvent, simple ground state, Perron–Frobenius—for Rankin–Selberg L-functions would bring a wide class of automorphic L-functions under a single framework.

11 Bibliographic pointers

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