Does the Writeup Discharge the First Requirement in §6.6? Simplicity and Evenness of the Lowest Eigenvector of QW_λ

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Abstract

Section 6.6 (Connes) states that to apply Theorem 6.1 one needs to show that the smallest eigenvalue of the Weil quadratic form QW_λ is simple with an even eigenvector, analogously to a known fact for the prolate wave operator. This note explains how the combined writeup establishes that property for the truncated/semi-local Weil form constructed from the local explicit-formula distributions, and it highlights which parts rely on standard external results.

1 Statement from §6.6

The relevant requirement in §6.6 is:

> "In order to apply Theorem 6.1 one needs to show that the smallest eigenvalue of the Weil quadratic form QW_λ is simple with even eigenvector. The analogue of this property is known for the prolate wave operator."

The combined writeup addresses precisely this *spectral simplicity and symmetry* property for the form it realizes (up to an additive constant shift).

2 What the writeup proves

Interpreting QW_λ as the truncated (semi-local) Weil quadratic form attached to test functions supported in $[\lambda^{-1}, \lambda]$ and defined using the local explicit-formula distributions W_p and $W_\mathbb{R}$, the writeup establishes:

- **Simplicity of the bottom eigenvalue:** the lowest eigenvalue is simple and has a strictly positive eigenfunction (a.e.).
- **Evenness of a lowest eigenvector:** the ground state can be chosen even (under the reflection symmetry in logarithmic coordinates).

Since adding a constant multiple of the identity to a selfadjoint operator shifts the spectrum without changing eigenvectors or multiplicities, the same simplicity/eigenfunction conclusions hold for the quadratic form before/after such a constant shift.

3 How the writeup achieves "simple + even"

The argument decomposes into three conceptual steps.

3.1 Step 1: Rewrite QW_λ as a Dirichlet-form-type energy (up to a constant)

The local terms are rewritten as nonnegative translation-difference energies plus constants.

- **Archimedean term:** after the change of variables $x = e^t$, the term $-W_{\mathbb{R}}(f)$ is written in the form

$$-W_{\mathbb{R}}(f) \;=\; \int w(t)\, \|\widetilde{G} - S_t\widetilde{G}\|^2\, dt \;+\; c_\infty(\lambda)\, \|G\|^2,$$

  where S_t denotes translation on the logarithmic variable, and the tail $t > 2L$ collapses to a constant multiple of $\|G\|^2$ by disjoint support.

- **Prime terms:** similarly, each $-W_p(f)$ becomes a (finite, by truncation) weighted sum of squared differences

$$-W_p(f) \;=\; \sum_m \alpha_{p,m}\, \|\widetilde{G} - S_{m\log p}\widetilde{G}\|^2 \;+\; c_p(\lambda)\, \|G\|^2,$$

  with only finitely many relevant m because the support truncation forces $\langle g, U_a g\rangle = 0$ for $a > \lambda^2$.

These formulas motivate the definition of the quadratic form $\mathcal{E}_\lambda$ as the sum of all translation-difference energies (prime and Archimedean), so that QW_λ agrees with $\mathcal{E}_\lambda$ up to an additive constant multiple of $\|G\|^2$.

3.2 Step 2: Show the ground state is simple (via positivity improving + compact resolvent)

The writeup then places $\mathcal{E}_\lambda$ in the framework of (symmetric) Dirichlet forms:

- **Markov property:** the form satisfies the normal contraction property $\mathcal{E}_\lambda(\Phi(\phi)) \le \mathcal{E}_\lambda(\phi)$ for standard contractions Φ. This yields a positive semigroup (standard Dirichlet form theory).

- **Irreducibility criterion:** the writeup proves a key measurable-set lemma: if $B \subset I$ and $\mathcal{E}_\lambda(\mathbf{1}_B) = 0$, then B is null or conull in I. Combined with standard equivalences in Dirichlet form theory, this gives *irreducibility* of the associated semigroup.

- **Compact resolvent:** the writeup realizes the form as a closed form and uses translation-control and the Kolmogorov–Riesz compactness criterion to show the embedding of the form domain into $L^2(I)$ is compact, hence the selfadjoint operator A_λ has compact resolvent.

- **Positivity improving:** invoking a standard theorem of the form

$$\text{positive} + \text{irreducible} + \text{holomorphic semigroup} \implies \text{positivity improving},$$

  the semigroup maps nonzero $f \ge 0$ to strictly positive functions for $t > 0$.

- **Perron–Frobenius/Krein–Rutman:** positivity improving together with compact resolvent implies the principal (lowest) eigenvalue is simple and admits a strictly positive eigenfunction.

This establishes the *simplicity* portion required in §6.6 (for the realized/truncated form).

3.3 Step 3: Force evenness by reflection symmetry

Finally, the writeup checks that the form is invariant under reflection in the logarithmic variable:

$$(RG)(u) := G(-u).$$

Form invariance implies the associated operator commutes with R. Since the lowest eigenspace is one-dimensional by Step 2, it is invariant under R and R acts by a scalar ± 1 on that eigenspace. Because the ground state is strictly positive a.e., it cannot be odd; hence it is even.

4 Does this resolve the first part of §6.6?

4.1 Answer

Yes, *for the truncated/semi-local Weil form QW_λ as realized in the writeup* (from local explicit-formula distributions and test functions supported in $[\lambda^{-1}, \lambda]$): the writeup proves the smallest eigenvalue is **simple** and its eigenvector can be chosen **even**.

4.2 What is still "input" rather than proved in-line

The conclusion relies on the following external or assumed items:

1. **Identification of QW_λ with the realized form:** one must match Connes's definition of QW_λ with the writeup's explicit-formula-based construction (up to an additive constant shift). The constant shift does not affect eigenvectors/multiplicity, but the equality of the underlying quadratic form must be checked at the level of definitions.

2. **Standard Dirichlet-form equivalences:** the step from the indicator-energy lemma to semigroup irreducibility uses standard theorems (not reproved in full detail).

3. **Positivity-improving and Perron–Frobenius consequences:** the implication "positive + irreducible + holomorphic $\Rightarrow$ positivity improving" and the resulting simplicity/positivity of the ground state with compact resolvent are standard inputs (e.g. Arendt–Batty–Hieber–Neubrander and Jentzsch/Krein–Rutman type results).

5 Important: §6.6 mentions an additional remaining step

Section 6.6 also states that it *still remains* to show an approximation property (e.g. that a certain k_λ approximates θ_x with $\lambda = \sqrt{x}$). The writeup under discussion addresses the *spectral simplicity/evenness* requirement, but it does *not* by itself resolve that separate approximation problem.

6 Conclusion

The writeup does discharge the first prerequisite singled out in §6.6: it establishes that the bottom of the spectrum of the realized/truncated Weil quadratic form is *simple* and its eigenfunction can be taken *even*. The remaining work in §6.6 (as stated there) concerns approximation of auxiliary kernels/functions, which is logically separate from the Perron–Frobenius/symmetry analysis.