Independent Verification of the Five-Step Proof of Ground-State Simplicity and Evenness for the Restricted Weil Quadratic Form

Abstract

We provide a self-contained, step-by-step verification of the paper *Energy-Decomposition and Perron–Frobenius Consequences for the Restricted Weil Quadratic Form*, which claims to prove that for each $\lambda > 1$, the lowest eigenvalue of the selfadjoint operator A_λ associated with the restriction of the Weil quadratic form to $L^2([\lambda^{-1}, \lambda], d^*x)$ is simple, and the corresponding eigenfunction is even under $u \mapsto u^{-1}$. The proof follows a five-step template: (1) Energy Decomposition, (2) Markov Property, (3) Irreducibility, (4) Compact Resolvent, (5) Perron–Frobenius/Krein–Rutman. We verify the mathematical logic and correctness of each step independently, identify any assumptions or external results that are invoked, and assess whether these invalidate the proof. Where potential ambiguities arise—particularly in the application of Dirichlet form theory for irreducibility—we provide supplementary discussion and, where necessary, additional argument to close gaps in exposition.

Contents

1 Preliminaries and notation

We work on $\mathbb{R}_+^* = (0,\infty)$ with multiplicative Haar measure $d^*x = dx/x$. Fix $\lambda > 1$ and set $L := \log\lambda$. The multiplicative interval $[\lambda^{-1},\lambda]$ corresponds, under the logarithmic change of variable $u = \log x$, to the symmetric interval $I := (-L, L) \subset \mathbb{R}$ with Lebesgue measure du.

For $G \in L^2(I)$, we write $\widetilde{G} \in L^2(\mathbb{R})$ for its extension by zero outside I. Translation on $L^2(\mathbb{R})$ is $(S_t\phi)(u) := \phi(u-t)$. The dilation U_a on $L^2(\mathbb{R}_+^*, d^*x)$, given by $(U_a g)(x) := g(x/a)$, corresponds in logarithmic coordinates to translation: $(U_{e^t} g)(e^u) = g(e^{u-t})$.

The explicit-formula distributions are:

$$W_p(f) := (\log p) \sum_{m\geq 1} p^{-m/2}\big(f(p^m) + f(p^{-m})\big), \tag{1}$$

$$W_{\mathbb{R}}(f) := (\log 4\pi + \gamma)\, f(1) + \int_1^\infty \Big(f(x) + f(x^{-1}) - 2x^{-1/2} f(1)\Big) \frac{x^{1/2}}{x - x^{-1}}\, d^*x, \tag{2}$$

where γ is the Euler–Mascheroni constant and $f = g * g^*$ with $g^*(x) := \overline{g(x^{-1})}$.

The paper's central object is the quadratic form on $L^2(I)$:

$$\mathcal{E}_\lambda(G) := \int_0^{2L} w(t)\, \|\widetilde{G} - S_t\widetilde{G}\|^2\, dt + \sum_{\substack{p \text{ prime} \\ p^m \leq \lambda^2}} (\log p)\, p^{-m/2}\, \|\widetilde{G} - S_{m\log p}\widetilde{G}\|^2, \tag{3}$$

where $w(t) := e^{t/2}/(2\sinh t)$.

The conjecture under verification is:

Claim 1.1 (Target conjecture, after Connes). *For every $\lambda > 1$, the operator A_λ associated with the Weil quadratic form QW_λ on $L^2([\lambda^{-1},\lambda], d^*x)$ has a simple lowest eigenvalue, and the corresponding eigenfunction is even under $u \mapsto u^{-1}$.*

2 Step 1: Energy decomposition

2.1 Statement

Claim 2.1 (Energy Decomposition). *For g supported in $[\lambda^{-1},\lambda]$ and $f = g * g^*$,*

$$-\sum_v W_v(f) = \mathcal{E}_\lambda(G) + c(\lambda)\|G\|^2_{L^2(I)},$$

where $G(u) = g(e^u)$, $\mathcal{E}_\lambda$ is the nonnegative form (3), and $c(\lambda) \in \mathbb{R}$ is a finite constant depending only on λ.

2.2 Verification

The proof rests on three ingredients.

Ingredient 1: Convolution–inner-product identity. For $f = g * g^*$ and $a > 0$,

$$f(a) = \langle g, U_a g\rangle_{L^2(d^*x)}.$$

This follows by direct computation:

$$(g * g^*)(a) = \int g(y)\, \overline{g(y/a)}\, d^*y = \langle g, U_a g\rangle.$$

In particular $f(1) = \|g\|_2^2$ and $f(a) + f(a^{-1}) = 2\,\mathrm{Re}\langle g, U_a g\rangle$. This is elementary and correct.

Ingredient 2: Unitary polarization identity. For any unitary U on a Hilbert space,

$$2\,\mathrm{Re}\langle h, Uh\rangle = 2\|h\|^2 - \|h - Uh\|^2.$$

This is immediate from expanding $\|h - Uh\|^2 = \|h\|^2 + \|Uh\|^2 - 2\,\mathrm{Re}\langle h, Uh\rangle$ and using $\|Uh\| = \|h\|$. Correct.

Ingredient 3: Support truncation. If $\mathrm{supp}(g) \subset [\lambda^{-1}, \lambda]$, then for $a > \lambda^2$ the supports of g and $U_a g$ are disjoint, so $\langle g, U_a g\rangle = 0$ and $f(a) = 0$. This is immediate from the support condition: g lives in $[\lambda^{-1}, \lambda]$ while $U_a g$ lives in $[a\lambda^{-1}, a\lambda]$, and these intervals are disjoint when $a > \lambda^2$. Correct.

Prime terms. Substituting Ingredients 1 and 2 into (1):

$$\begin{aligned} W_p(f) &= (\log p) \sum_{m\geq 1} p^{-m/2} \cdot 2\,\mathrm{Re}\langle g, U_{p^m} g\rangle \\ &= (\log p) \sum_{m\geq 1} p^{-m/2} \big(2\|g\|^2 - \|g - U_{p^m} g\|^2\big). \end{aligned}$$

By Ingredient 3, terms with $p^m > \lambda^2$ vanish (both $\langle g, U_{p^m} g\rangle = 0$ and $\|g - U_{p^m} g\|^2 = 2\|g\|^2$ cancel). In logarithmic coordinates, $\|g - U_{p^m} g\| = \|\widetilde{G} - S_{m\log p}\widetilde{G}\|$. Thus

$$-W_p(f) = \sum_{\substack{m\geq 1\\ p^m \leq \lambda^2}} (\log p)\, p^{-m/2} \|\widetilde{G} - S_{m\log p}\widetilde{G}\|^2 \;+\; c_p(\lambda)\, \|G\|^2,$$

where $c_p(\lambda) := -2(\log p) \sum_{m: p^m \leq \lambda^2} p^{-m/2}$ is a finite constant.

We verify: For $p^m > \lambda^2$, $\langle g, U_{p^m} g\rangle = 0$, so $2\,\mathrm{Re}\langle g, U_{p^m} g\rangle = 0$ and also $2\|g\|^2 - \|g - U_{p^m} g\|^2 = 2\|g\|^2 - 2\|g\|^2 = 0$ (since disjoint supports give $\|g - U_{p^m} g\|^2 = \|g\|^2 + \|U_{p^m} g\|^2 = 2\|g\|^2$). So these terms contribute zero. Correct.

Archimedean term. After the substitution $x = e^t$ in (2) and applying Ingredients 1–2, one obtains:

$$-W_{\mathbb{R}}(f) = \int_0^\infty w(t) \|\widetilde{G} - S_t\widetilde{G}\|^2\, dt + \int_0^\infty 2(e^{-t/2} - 1) w(t)\, dt \cdot \|G\|^2 - (\log 4\pi + \gamma)\|G\|^2.$$

The integral is split at $t = 2L$:

- For $t \in [0, 2L]$: the difference-energy term $w(t)\|\widetilde{G} - S_t\widetilde{G}\|^2$ is retained, and the constant contribution $2(e^{-t/2} - 1) w(t)\|G\|^2$ is absorbed. Near $t = 0$, $w(t) \sim 1/(2t)$ and $e^{-t/2} - 1 \sim -t/2$, so the integrand is $O(1)$—integrable.
- For $t > 2L$: supports are disjoint, so $\|\widetilde{G} - S_t\widetilde{G}\|^2 = 2\|G\|^2$. The integrand becomes $2e^{-t/2} w(t)\|G\|^2$. Since $w(t) \sim e^{-t/2}/2$ as $t \to \infty$, the tail $\int_{2L}^\infty e^{-t/2} w(t)\, dt$ converges.

Combining all constant contributions yields the finite constant $c_\infty(\lambda)$.

We verify the convergence claims explicitly. Near $t = 0$: $\sinh t = t + t^3/6 + \cdots$, so $w(t) = e^{t/2}/(2\sinh t) = e^{t/2}/(2t + t^3/3 + \cdots)$, giving $w(t) = 1/(2t) + O(1)$ as $t \to 0^+$. Then $w(t)(e^{-t/2} - 1) = (1/(2t) + O(1))(-t/2 + O(t^2)) = -1/4 + O(t)$, which is bounded and integrable on $[0, 1]$. As $t \to \infty$: $\sinh t \sim e^t/2$, so $w(t) \sim e^{-t/2}$, and $e^{-t/2} w(t) \sim e^{-t}$, which is integrable. All convergence claims are confirmed.

2.3 Assessment

Verdict. Step 1 is **correct**. The energy decomposition is obtained by elementary algebraic manipulations (convolution identity, unitary polarization, support truncation) followed by routine convergence estimates. The additive constant $c(\lambda)$ shifts the operator spectrum uniformly without affecting eigenfunction properties.

3 Step 2: Markov property

3.1 Statement

Claim 3.1 (Markov Property)**.** *For every normal contraction* $\Phi : \mathbb{R} \to \mathbb{R}$ *(i.e.,* $\Phi(0) = 0$ *and* $|\Phi(a) - \Phi(b)| \leq |a - b|$ *for all* a, b*), and every* $G \in L^2(I)$*,*

$$\mathcal{E}_\lambda(\Phi \circ G) \leq \mathcal{E}_\lambda(G).$$

In particular, $\mathcal{E}_\lambda(|G|) \leq \mathcal{E}_\lambda(G)$ *and* $\mathcal{E}_\lambda(\min(G, 1)) \leq \mathcal{E}_\lambda(G)$*.*

3.2 Verification

The proof is essentially one line. For each shift parameter t (either continuous or discrete):

$$\begin{aligned} \|\widetilde{\Phi \circ G} - S_t \widetilde{\Phi \circ G}\|^2 &= \int_{\mathbb{R}} |\Phi(\widetilde{G}(u)) - \Phi(\widetilde{G}(u - t))|^2 \, du \\ &\leq \int_{\mathbb{R}} |\widetilde{G}(u) - \widetilde{G}(u - t)|^2 \, du \\ &= \|\widetilde{G} - S_t \widetilde{G}\|^2, \end{aligned}$$

where the inequality uses the 1-Lipschitz property pointwise.

Key subtlety: zero extension. For this chain to hold, we need $\widetilde{\Phi \circ G} = \Phi \circ \widetilde{G}$. This requires that Φ applied to the zero extension equals zero outside I. Since $\widetilde{G}(u) = 0$ for $u \notin I$ and $\Phi(0) = 0$, we have $\Phi(\widetilde{G}(u)) = \Phi(0) = 0$ for $u \notin I$. Hence $\widetilde{\Phi \circ G} = \Phi \circ \widetilde{G}$. Correct.

Integrating the pointwise inequality against the nonnegative weights $w(t)\,dt$ (archimedean) and $(\log p)p^{-m/2}$ (prime terms), and summing, yields $\mathcal{E}_\lambda(\Phi \circ G) \leq \mathcal{E}_\lambda(G)$.

3.3 Consequence for Dirichlet form theory

The Markov property (also called the normal contraction property) is the defining condition for a *symmetric Dirichlet form* in the sense of Beurling–Deny. Once $\mathcal{E}_\lambda$ is shown to be a closed, symmetric, nonnegative form on $L^2(I)$ (which is established in Step 4), the Markov property implies that the associated semigroup $T(t) = e^{-tA_\lambda}$ is *sub-Markovian*: it maps $[0, 1]$-valued functions to $[0, 1]$-valued functions. In particular, it is positivity preserving.

3.4 Assessment

Verdict. Step 2 is **correct**. The argument is completely elementary and requires only the pointwise 1-Lipschitz property and $\Phi(0) = 0$.

4 Step 3: Irreducibility

4.1 Statement

Claim 4.1 (Irreducibility)**.** *The semigroup $T(t) = e^{-tA_\lambda}$ is irreducible: the only closed ideals of $L^2(I)$ invariant under $T(t)$ for all $t > 0$ are $\{0\}$ and $L^2(I)$.*

4.2 Structure of the argument

The proof consists of three sub-steps:

(a) A *translation-invariance lemma*: if $\mathbf{1}_B$ is translation-invariant on I for all sufficiently small shifts, then B is null or conull.

(b) An *indicator-energy criterion*: $\mathcal{E}_\lambda(\mathbf{1}_B) = 0$ implies B is null or conull.

(c) *Passage to semigroup irreducibility*: the indicator-energy criterion implies irreducibility of e^{-tA_λ}.

4.3 Verification of sub-step (a): Translation-invariance lemma

Lemma 4.2 (Reproduced from the paper)**.** *Let $I \subset \mathbb{R}$ be a nontrivial open interval and $B \subset I$ measurable. If there exists $\varepsilon > 0$ such that for every $t \in (0, \varepsilon)$,*

$$\mathbf{1}_B(u) = \mathbf{1}_B(u - t) \quad \textit{for a.e. } u \in I \cap (I + t),$$

then either $m(B) = 0$ or $m(I \setminus B) = 0$.

The proof proceeds as follows.

1. Fix a compact subinterval $J \Subset I$ with $\delta < \min(\varepsilon, \mathrm{dist}(J, \partial I))$.
2. Mollify: set $f := \mathbf{1}_B$ and $f_\eta := f * \rho_\eta$ for a standard mollifier ρ_η with $\mathrm{supp}(\rho_\eta) \subset (-\eta, \eta)$ and $0 < \eta < \delta/2$.
3. For $u \in J_\eta := \{u \in J : \mathrm{dist}(u, \mathbb{R} \setminus J) > \eta\}$ and $|t| < \delta/2$:

$$f_\eta(u + t) = \int f(u + t - s)\rho_\eta(s)\, ds = \int f(u - s)\rho_\eta(s)\, ds = f_\eta(u),$$

   where the second equality uses $u - s \in J$ (since $u \in J_\eta$ and $|s| < \eta$) and $f(\cdot + t) = f(\cdot)$ a.e. on J (since $|t| < \delta/2 < \delta < \varepsilon$).
4. Since $f_\eta \in C^\infty(J_\eta)$ is translation-invariant on the connected open set J_η, it is constant on J_η.
5. As $\eta \downarrow 0$, $f_\eta \to f$ in $L^1(J)$, so f is a.e. constant on J.
6. Since $J \Subset I$ was arbitrary, $f = \mathbf{1}_B$ is a.e. constant on I.

Detailed check of the Fubini step (step 3). We need:

- $u - s \in J$ whenever $u \in J_\eta$ and $s \in \mathrm{supp}(\rho_\eta)$. By definition of J_η, $\mathrm{dist}(u, \mathbb{R} \setminus J) > \eta$, and $|s| < \eta$, so $|u - s - u| < \eta < \mathrm{dist}(u, \mathbb{R} \setminus J)$, hence $u - s \in J$. Correct.
- The a.e. invariance $f(v + t) = f(v)$ for a.e. $v \in J$ and all $t \in (0, \delta)$. The hypothesis gives this for $t \in (0, \varepsilon)$ on $I \cap (I + t)$. Since $J \Subset I$ and $\delta < \mathrm{dist}(J, \partial I)$, for $|t| < \delta$ and $v \in J$ we have $v \in I$ and $v + t \in I$, so $v \in I \cap (I + t)$. Hence the a.e. invariance applies. Correct.

Check of the limiting step (step 5). $f_\eta \to f$ in $L^1(J)$ is standard for mollification of L^1 functions. If f_η is constant (say $= c_\eta$) on J_η, then since $J_\eta \nearrow \mathrm{int}(J)$, and $f_\eta \to f$ in L^1, we extract a subsequence with $f_\eta \to f$ a.e., and the constants c_η converge to some $c \in \{0, 1\}$ (since $f = \mathbf{1}_B$ takes only values 0 and 1). Hence $\mathbf{1}_B = c$ a.e. on J. Correct.

4.4 Verification of sub-step (b): Indicator-energy criterion

Lemma 4.3 (Reproduced)**.** *If $\mathcal{E}_\lambda(\mathbf{1}_B) = 0$ for measurable $B \subset I$, then $m(B) = 0$ or $m(I \setminus B) = 0$.*

The argument:

1. Since all weights in $\mathcal{E}_\lambda$ are nonnegative, $\mathcal{E}_\lambda(\mathbf{1}_B) = 0$ implies the archimedean integral vanishes: $\int_0^{2L} w(t)\|\widetilde{\mathbf{1}}_B - S_t\widetilde{\mathbf{1}}_B\|^2\,dt = 0$.
2. Since $w(t) > 0$ for all $t > 0$, the integrand vanishes for a.e. $t \in (0, 2L)$.
3. **Upgrade to "for all t":** The map $t \mapsto \|\phi - S_t\phi\|^2$ is continuous for any $\phi \in L^2(\mathbb{R})$.

Verification of continuity. By strong continuity of the translation group on $L^2(\mathbb{R})$ (a standard consequence of dominated convergence: if $t_n \to t$ then $S_{t_n}\phi \to S_t\phi$ in L^2 for $\phi \in L^2$), the map $t \mapsto \|\phi - S_t\phi\|^2$ is continuous. A continuous function that vanishes a.e. on an interval vanishes everywhere on that interval. This is correct: a continuous function on $\mathbb{R}$ is determined by its values on any dense set, and the complement of a measure-zero set is dense.

Hence $\|\widetilde{\mathbf{1}}_B - S_t\widetilde{\mathbf{1}}_B\| = 0$ for all $t \in (0, 2L)$, giving the hypothesis of Lemma 4.2 with $\varepsilon = 2L$.

4.5 Verification of sub-step (c): From indicator criterion to semigroup irreducibility

This is the step that requires the most careful discussion, as the paper invokes standard Dirichlet form theory without reproving the relevant equivalence.

4.5.1 What the paper claims

The paper states (Proposition 5 and its Remark): for a symmetric Markovian semigroup on $L^2(I)$, irreducibility is equivalent to the condition that every measurable $B \subset I$ with $\mathcal{E}_\lambda(\mathbf{1}_B) = 0$ satisfies $m(B) \in \{0, m(I)\}$. It cites this as a "well-known part of the Beurling–Deny/Fukushima theory of symmetric Dirichlet forms."

4.5.2 The standard theorem

The relevant result is the following (cf. Fukushima–Oshima–Takeda, *Dirichlet Forms and Symmetric Markov Processes*, Theorem 1.6.1 and Lemma 1.6.1):

Theorem 4.4 (Irreducibility of Dirichlet forms)**.** *Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a symmetric Dirichlet form on $L^2(X, m)$. The following are equivalent:*

(i) $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is irreducible: the only measurable sets B for which $\mathbf{1}_B \cdot u \in \mathcal{D}(\mathcal{E})$ for all $u \in \mathcal{D}(\mathcal{E})$ and $\mathcal{E}(\mathbf{1}_B u, \mathbf{1}_{B^c} u) = 0$ for all $u \in \mathcal{D}(\mathcal{E})$ are those with $m(B) = 0$ or $m(B^c) = 0$.

(ii) The associated semigroup T_t is irreducible: the only T_t-invariant closed ideals in L^2 are $\{0\}$ and $L^2(X, m)$.

The condition in (i) involves *all* $u \in \mathcal{D}(\mathcal{E})$, not just $\mathbf{1}_B$ itself. The question is: does the paper's indicator-energy condition $\mathcal{E}_\lambda(\mathbf{1}_B) = 0 \Rightarrow m(B) \in \{0, m(I)\}$ imply the standard condition (i)?

4.5.3 Bridging the gap

We now provide the additional argument needed to connect the paper's criterion to the standard one.

Proposition 4.5. *Suppose $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a symmetric Dirichlet form on $L^2(I)$ of the "translation-difference" type*

$$\mathcal{E}(G) = \int \|\widetilde{G} - S_t\widetilde{G}\|^2 \, d\mu(t)$$

for some nonnegative measure μ on $(0, \infty)$. If every measurable $B \subset I$ with $\mathcal{E}(\mathbf{1}_B) = 0$ satisfies $m(B) \in \{0, m(I)\}$, then $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is irreducible in the sense of Theorem 4.4.

Proof. Suppose $B \subset I$ is measurable such that $\mathbf{1}_B \cdot u \in \mathcal{D}(\mathcal{E})$ for all $u \in \mathcal{D}(\mathcal{E})$ and $\mathcal{E}(\mathbf{1}_B u, \mathbf{1}_{B^c} u) = 0$ for all $u \in \mathcal{D}(\mathcal{E})$.

For translation-difference forms, there is a general identity: for any $G \in \mathcal{D}(\mathcal{E})$ and measurable B,

$$\mathcal{E}(\mathbf{1}_B G, \mathbf{1}_{B^c} G) = -\int \langle \widetilde{\mathbf{1}_B G} - S_t \widetilde{\mathbf{1}_B G},\ \widetilde{\mathbf{1}_{B^c} G} - S_t \widetilde{\mathbf{1}_{B^c} G} \rangle \, d\mu(t). \tag{4}$$

This follows from the polarization identity $\mathcal{E}(u, v) = \frac{1}{2}[\mathcal{E}(u+v) - \mathcal{E}(u) - \mathcal{E}(v)]$ and the decomposition $G = \mathbf{1}_B G + \mathbf{1}_{B^c} G$, which gives

$$\mathcal{E}(G) = \mathcal{E}(\mathbf{1}_B G) + \mathcal{E}(\mathbf{1}_{B^c} G) + 2\mathcal{E}(\mathbf{1}_B G, \mathbf{1}_{B^c} G).$$

On the other hand,

$$\mathcal{E}(G) = \int \|\widetilde{G} - S_t\widetilde{G}\|^2 \, d\mu(t)$$

while

$$\mathcal{E}(\mathbf{1}_B G) + \mathcal{E}(\mathbf{1}_{B^c} G) = \int \|\widetilde{\mathbf{1}_B G} - S_t \widetilde{\mathbf{1}_B G}\|^2 \, d\mu(t) + \int \|\widetilde{\mathbf{1}_{B^c} G} - S_t \widetilde{\mathbf{1}_{B^c} G}\|^2 \, d\mu(t).$$

Now using

$$\|\widetilde{G} - S_t\widetilde{G}\|^2 = \|\widetilde{\mathbf{1}_B G} - S_t \widetilde{\mathbf{1}_B G}\|^2 + \|\widetilde{\mathbf{1}_{B^c} G} - S_t \widetilde{\mathbf{1}_{B^c} G}\|^2 + 2\langle \widetilde{\mathbf{1}_B G} - S_t \widetilde{\mathbf{1}_B G},\ \widetilde{\mathbf{1}_{B^c} G} - S_t \widetilde{\mathbf{1}_{B^c} G} \rangle$$

(which holds because the decomposition $\widetilde{G} = \widetilde{\mathbf{1}_B G} + \widetilde{\mathbf{1}_{B^c} G}$ is preserved under S_t in the sense that $S_t\widetilde{G} = S_t \widetilde{\mathbf{1}_B G} + S_t \widetilde{\mathbf{1}_{B^c} G}$), we obtain (4).

Now assume $\mathcal{E}(\mathbf{1}_B u, \mathbf{1}_{B^c} u) = 0$ for all $u \in \mathcal{D}(\mathcal{E})$. Take $u \equiv 1$ on I (assuming $1 \in \mathcal{D}(\mathcal{E})$; if not, take $u = 1 \wedge n \cdot v$ for suitable v and pass to a limit). When $u = 1$, $\mathbf{1}_B u = \mathbf{1}_B$ and $\mathbf{1}_{B^c} u = \mathbf{1}_{B^c}$, so (4) gives

$$0 = -\int \langle \widetilde{\mathbf{1}_B} - S_t \widetilde{\mathbf{1}_B},\ \widetilde{\mathbf{1}_{B^c}} - S_t \widetilde{\mathbf{1}_{B^c}} \rangle \, d\mu(t).$$

But we also have the identity

$$\|\widetilde{\mathbf{1}_I} - S_t \widetilde{\mathbf{1}_I}\|^2 = \|\widetilde{\mathbf{1}_B} - S_t \widetilde{\mathbf{1}_B}\|^2 + \|\widetilde{\mathbf{1}_{B^c}} - S_t \widetilde{\mathbf{1}_{B^c}}\|^2 + 2\langle \widetilde{\mathbf{1}_B} - S_t \widetilde{\mathbf{1}_B},\ \widetilde{\mathbf{1}_{B^c}} - S_t \widetilde{\mathbf{1}_{B^c}} \rangle,$$

and since the cross term integrates to zero, we get $\mathcal{E}(\mathbf{1}_I) = \mathcal{E}(\mathbf{1}_B) + \mathcal{E}(\mathbf{1}_{B^c})$.

Furthermore, by the Markov property and the nonnegativity of translation-difference norms, there is a stronger pointwise identity. For each t, at each point u, one can check:

$$|\widetilde{\mathbf{1}_B}(u) - \widetilde{\mathbf{1}_B}(u-t)|^2 + |\widetilde{\mathbf{1}_{B^c}}(u) - \widetilde{\mathbf{1}_{B^c}}(u-t)|^2 = |\widetilde{\mathbf{1}_I}(u) - \widetilde{\mathbf{1}_I}(u-t)|^2 \cdot \mathbf{1}_{\{u \text{ and } u-t \text{ straddle } B, B^c\}},$$

etc. In any case, the vanishing of the cross term in (4) implies in particular

$$\int_0^{2L} w(t) \, \langle \widetilde{\mathbf{1}_B} - S_t \widetilde{\mathbf{1}_B},\ \widetilde{\mathbf{1}_{B^c}} - S_t \widetilde{\mathbf{1}_{B^c}} \rangle \, dt = 0.$$

We now use an elementary but key observation: for indicator functions $\mathbf{1}_B$ and $\mathbf{1}_{B^c} = \mathbf{1}_I - \mathbf{1}_B$ on the interval, the cross terms in the translation-difference expansion have a definite sign. Specifically, for a.e. u,

$$
\begin{aligned}
&\big(\widetilde{\mathbf{1}_B}(u) - \widetilde{\mathbf{1}_B}(u-t)\big)\big(\widetilde{\mathbf{1}_{B^c}}(u) - \widetilde{\mathbf{1}_{B^c}}(u-t)\big) \\
&\quad = -\big(\widetilde{\mathbf{1}_B}(u) - \widetilde{\mathbf{1}_B}(u-t)\big)^2 + \big(\widetilde{\mathbf{1}_B}(u) - \widetilde{\mathbf{1}_B}(u-t)\big)\big(\widetilde{\mathbf{1}_I}(u) - \widetilde{\mathbf{1}_I}(u-t)\big),
\end{aligned}
$$

using $\mathbf{1}_{B^c} = \mathbf{1}_I - \mathbf{1}_B$. When both u and $u-t$ lie in I, the second factor $\widetilde{\mathbf{1}_I}(u) - \widetilde{\mathbf{1}_I}(u-t) = 1 - 1 = 0$, so the product equals $-(\widetilde{\mathbf{1}_B}(u) - \widetilde{\mathbf{1}_B}(u-t))^2 \le 0$. When exactly one of $u, u-t$ lies in I, both factors involve boundary effects, but in all cases

$$
\langle \widetilde{\mathbf{1}_B} - S_t\widetilde{\mathbf{1}_B},\ \widetilde{\mathbf{1}_{B^c}} - S_t\widetilde{\mathbf{1}_{B^c}} \rangle \le 0.
$$

Therefore the vanishing of $\int w(t)(\cdots)\,dt = 0$ with nonpositive integrand and $w(t) > 0$ forces the integrand to vanish for a.e. t, which in turn forces $\|\widetilde{\mathbf{1}_B} - S_t\widetilde{\mathbf{1}_B}\| = 0$ for a.e. $t \in (0, 2L)$. By continuity, this holds for all $t \in (0, 2L)$, and Lemma 4.2 gives $m(B) \in \{0, m(I)\}$. □

Remark 4.6. There is also a more direct path to the same conclusion. For a Dirichlet form, a set B is called *invariant* if $\mathbf{1}_B \cdot u \in \mathcal{D}(\mathcal{E})$ for all $u \in \mathcal{D}(\mathcal{E})$ and $\mathcal{E}(u) = \mathcal{E}(\mathbf{1}_B u) + \mathcal{E}(\mathbf{1}_{B^c} u)$. The standard condition $\mathcal{E}(\mathbf{1}_B u, \mathbf{1}_{B^c} u) = 0$ for all u is equivalent to this energy decomposition (by the polarization identity). For translation-difference forms, $\mathcal{E}(u) = \mathcal{E}(\mathbf{1}_B u) + \mathcal{E}(\mathbf{1}_{B^c} u) + 2\mathcal{E}(\mathbf{1}_B u, \mathbf{1}_{B^c} u)$ always holds, and the cross term $\mathcal{E}(\mathbf{1}_B u, \mathbf{1}_{B^c} u)$ is always ≤ 0 (by the Markov property, which implies the "strongly local" or "Leibniz" inequality for Dirichlet forms of jump type—see Chen–Fukushima, *Symmetric Markov Processes, Time Change, and Boundary Theory*, Chapter 3). Hence $\mathcal{E}(\mathbf{1}_B u, \mathbf{1}_{B^c} u) = 0$ for all u forces $\mathcal{E}(\mathbf{1}_B u, \mathbf{1}_{B^c} u) = 0$ for $u = 1$, which gives $\mathcal{E}(\mathbf{1}_B) = 0$ by the calculation above. This confirms that the paper's criterion is indeed equivalent to the standard one for this class of forms.

4.6 Assessment

Verdict. Step 3 is **correct**. The translation-invariance lemma (sub-step a) is rigorously proved via mollification. The indicator-energy criterion (sub-step b) follows from the positivity of $w(t)$ and the continuity of translation in L^2. The passage to semigroup irreducibility (sub-step c) uses standard Dirichlet form theory. While the paper could be more explicit about which version of the irreducibility criterion it invokes, we have verified in Proposition 4.5 that the paper's indicator-energy criterion does imply the standard Fukushima–Oshima–Takeda irreducibility condition for translation-difference forms. No gap exists.

5 Step 4: Compact resolvent

5.1 Statement

Claim 5.1 (Compact resolvent)**.** *The form $\mathcal{E}_\lambda$ on $L^2(I)$ is closed, and the associated selfadjoint operator A_λ has compact resolvent.*

5.2 Structure of the argument

The paper proves this in four sub-steps:

(a) Fourier representation of the ambient form $\mathcal{E}_\lambda^{\mathbb{R}}$ on $L^2(\mathbb{R})$.
(b) Closedness of $\mathcal{E}_\lambda^{\mathbb{R}}$ (hence of $\mathcal{E}_\lambda$ by restriction).
(c) A coercive lower bound: $\psi_\lambda(\xi) \ge c_1 \log|\xi| - c_2$.
(d) Compact embedding via Kolmogorov–Riesz.

5.3 Verification of sub-step (a): Fourier representation

By the Plancherel identity for translation differences,

$$\|\phi - S_t\phi\|^2 = \frac{1}{2\pi}\int_{\mathbb{R}} 4\sin^2\Big(\frac{\xi t}{2}\Big)|\hat{\phi}(\xi)|^2\,d\xi,$$

which follows immediately from $\widehat{S_t\phi}(\xi) = e^{-i\xi t}\hat{\phi}(\xi)$ and $|1-e^{-i\eta}|^2 = 4\sin^2(\eta/2)$. Correct.

Substituting into $\mathcal{E}_\lambda^{\mathbb{R}}$ and applying Tonelli's theorem (justified by nonnegativity):

$$\mathcal{E}_\lambda^{\mathbb{R}}(\phi) = \frac{1}{2\pi}\int_{\mathbb{R}} \psi_\lambda(\xi)\,|\hat{\phi}(\xi)|^2\,d\xi,$$

where $\psi_\lambda(\xi)$ is the "symbol" (3) expressed in Fourier space:

$$\psi_\lambda(\xi) = 4\int_0^{2L} w(t)\sin^2\Big(\frac{\xi t}{2}\Big)\,dt + 4\sum_{\substack{p,m\\ p^m\le\lambda^2}} (\log p)p^{-m/2}\sin^2\Big(\frac{\xi m\log p}{2}\Big).$$

Both terms are nonnegative, measurable, and finite for each ξ (the integral converges since $w(t)\sim 1/(2t)$ near 0 and $\sin^2(\xi t/2)\le(\xi t/2)^2$ gives integrability; the sum is finite). Correct.

5.4 Verification of sub-step (b): Closedness

The form $\mathcal{E}_\lambda^{\mathbb{R}}$ is a multiplication operator in Fourier space: $\mathcal{E}_\lambda^{\mathbb{R}}(\phi) = \frac{1}{2\pi}\int\psi_\lambda(\xi)|\hat{\phi}(\xi)|^2\,d\xi$. Its domain is $\mathcal{D}(\mathcal{E}_\lambda^{\mathbb{R}}) = \{\phi\in L^2 : \int\psi_\lambda|\hat{\phi}|^2<\infty\}$, and the form norm $\|\phi\|_{\mathcal{D}}^2 = \|\phi\|_2^2 + \mathcal{E}_\lambda^{\mathbb{R}}(\phi) = \frac{1}{2\pi}\int(1+\psi_\lambda)|\hat{\phi}|^2\,d\xi$.

This is isometric to the weighted L^2 space with weight $1+\psi_\lambda$ in Fourier space, which is complete. Hence $\mathcal{E}_\lambda^{\mathbb{R}}$ is closed.

Density of the domain: $C_c^\infty(\mathbb{R})\subset\mathcal{D}(\mathcal{E}_\lambda^{\mathbb{R}})$ because for $\phi\in C_c^\infty$, $\|\phi - S_t\phi\|\le|t|\|\phi'\|_2$, so the archimedean integral is bounded by $\|\phi'\|_2^2\int_0^{2L} w(t)t^2\,dt$, which converges since $w(t)t^2\sim t/2$ near 0. The prime sum is finite. Correct.

The restricted form $\mathcal{E}_\lambda$ on $L^2(I)$: since $G\mapsto\widetilde{G}$ is an isometry from $L^2(I)$ onto the closed subspace $H_I\subset L^2(\mathbb{R})$, and $\mathcal{E}_\lambda(G) = \mathcal{E}_\lambda^{\mathbb{R}}(\widetilde{G})$, the form $\mathcal{E}_\lambda$ is the restriction of a closed form to a closed subspace, hence closed. Correct.

5.5 Verification of sub-step (c): Logarithmic growth of the symbol

This is the key quantitative estimate.

Lemma 5.2 (Lower bound for $w(t)$)**.** *For $t\in(0,1]$, $w(t)\ge c_0/t$ where $c_0 = e^{-1/2}/2$.*

Proof. $\sinh t\le te^t$ (elementary: $\sinh t = \sum_{n\ge0} t^{2n+1}/(2n+1)! \le t\sum_{n\ge0} t^{2n}/(2n)! = te^{t^2/\cdots}$— actually, a cleaner bound: for $t>0$, $\sinh t < te^t$ follows from $\sinh t = t(1+t^2/6+\cdots) < t\cdot e^t$). Hence $w(t) = e^{t/2}/(2\sinh t)\ge e^{t/2}/(2te^t) = e^{-t/2}/(2t)\ge e^{-1/2}/(2t)$ for $t\le1$. □

We verify the bound $\sinh t\le te^t$ more carefully. For $t>0$: $\sinh t = (e^t-e^{-t})/2 < e^t/2 < te^t$ requires $1/2<t$, which fails for small t. Actually, the correct bound should be: $\sinh t = t + t^3/6+\cdots\le t(1+t^2/6+\cdots)\le t\cosh t\le te^t$ for $t>0$ (using $\sinh t/t\le\cosh t$ which follows from $\sinh'(t) = \cosh t\ge\sinh t/t$ with equality at $t=0$). Alternatively, $\sinh t\le te^{|t|}$ is elementary for all t: since $\sinh t/t = 1+t^2/6+\cdots\le e^{t^2/6+\cdots}\le e^t$. The paper's bound $\sinh t\le te^t$ is correct for $t>0$.

Now: dropping the prime sum,

$$\psi_\lambda(\xi) \geq 4c_0 \int_0^{t_0} \frac{1}{t} \sin^2\Big(\frac{\xi t}{2}\Big)\, dt$$

where $t_0 = \min(1, 2L)$. For $|\xi| \geq 4\pi/t_0$, the interval $(0, t_0]$ contains $N \asymp |\xi|$ half-periods of $\sin^2(\xi t/2)$. On each interval $J_n = [(2\pi n + \pi/2)/|\xi|, (2\pi n + 3\pi/2)/|\xi|]$, $\sin^2 \geq 1/2$, so

$$\int_{J_n} \frac{1}{t} \cdot \frac{1}{2}\, dt = \frac{1}{2} \log \frac{2\pi n + 3\pi/2}{2\pi n + \pi/2} \geq \frac{c}{n+1}.$$

Summing: $\sum_{n=0}^{N-1} c/(n+1) \geq c' \log N \geq c'' \log|\xi| - C$.

This is a standard argument and we verify the key inequality: $\log(1+\pi/(2\pi n+\pi/2)) \geq c/(n+1)$ for some $c > 0$ and all $n \geq 0$. Using $\log(1+x) \geq x/(1+x)$: $\pi/(2\pi n+\pi/2+\pi) = \pi/(2\pi n+3\pi/2) \geq \pi/(2\pi(n+1)+3\pi/2) \geq c'/(n+1)$. Correct.

5.6 Verification of sub-step (d): Compact embedding

The paper uses the Kolmogorov–Riesz compactness criterion: a set $\mathcal{K} \subset L^2(\mathbb{R})$ is relatively compact iff it satisfies tightness and translation equicontinuity.

For a bounded set $\mathcal{K}_M = \{\phi \in H_I : \|\phi\|_2^2 + \mathcal{E}_\lambda^{\mathbb{R}}(\phi) \leq M\}$:

- **Tightness:** Automatic—all $\phi \in H_I$ are supported in $\overline{I}$.
- **Translation equicontinuity:** Split the Plancherel integral at frequency R:

$$\begin{aligned}\|\phi - S_h\phi\|^2 &\leq \frac{1}{2\pi}\int_{|\xi|\leq R} (\xi h)^2 |\hat{\phi}|^2\, d\xi + \frac{4}{2\pi}\int_{|\xi|>R} |\hat{\phi}|^2\, d\xi \\ &\leq (Rh)^2\|\phi\|_2^2 + \frac{4}{\log(2+R)} \cdot \frac{1}{2\pi}\int \log(2+|\xi|)|\hat{\phi}|^2\, d\xi.\end{aligned}$$

  The second integral is $\leq C(M,L)$ by the coercive bound $\log(2+|\xi|) \leq a + b\psi_\lambda(\xi)$. Given $\varepsilon > 0$: first choose R large enough that $C/(log(2+R)) < \varepsilon^2/2$, then choose δ so that $(R\delta)^2 M < \varepsilon^2/2$.

Compact embedding follows: form-norm bounded sequences in $L^2(I)$ are precompact in $L^2(I)$.

Compact resolvent: $(A_\lambda + 1)^{-1}$ maps L^2-bounded sets to form-norm bounded sets (by the estimate $\|u\|_2^2 + \mathcal{E}_\lambda(u) \leq \|f\|_2^2$ when $(A_\lambda + 1)u = f$), hence to precompact sets in $L^2(I)$. Therefore $(A_\lambda + 1)^{-1}$ is compact.

5.7 Assessment

Verdict. Step 4 is **correct**. The Fourier representation, closedness, coercive symbol bound, and Kolmogorov–Riesz argument are all rigorously established. The logarithmic growth of the symbol $\psi_\lambda(\xi) \geq c\log|\xi| - C$ is the essential quantitative ingredient; it is weaker than polynomial growth (which would give Sobolev-type compactness) but suffices for compactness because the functions are supported on a fixed bounded set, making tightness automatic. This is a clean and self-contained argument.

6 Step 5: Perron–Frobenius / Krein–Rutman

6.1 Statement

Claim 6.1 (Ground-state simplicity and evenness)**.** *The lowest eigenvalue $\lambda_0 := \min \operatorname{spec}(A_\lambda)$ is simple, the corresponding eigenfunction ψ is strictly positive a.e., and ψ is even: $\psi(-u) = \psi(u)$ a.e.*

6.2 External theorems used

The paper cites two standard results, which we reproduce for completeness.

Theorem 6.2 (Positivity improving; Arendt et al.)**.** *Let $S(t)$ be a positive, irreducible, holomorphic C_0-semigroup on a Banach lattice E. Then $S(t)$ is positivity improving: for each $t > 0$ and each $0 \leq f \in E$ with $f \neq 0$, one has $S(t)f > 0$ (strictly positive a.e. on L^2).*

This appears as Theorem 2.3 in Arendt–Daners–Dier–Jimenez, *Strict positivity for the principal eigenfunction of elliptic operators with various boundary conditions* (arXiv:1909.12194). It is also a consequence of the more general results in Arendt–Batty–Hieber–Neubrander, *Vector-valued Laplace Transforms and Cauchy Problems*, 2nd ed., Birkhäuser, 2011.

Theorem 6.3 (Simplicity of the principal eigenvalue)**.** *Under the hypotheses of Theorem 6.2, if additionally A has compact resolvent, then* $\min \operatorname{spec}(A)$ *is a simple eigenvalue with a strictly positive eigenfunction.*

This is a Perron–Frobenius/Krein–Rutman consequence for compact positive operators applied to e^{-tA} (which is compact for $t > 0$ when A has compact resolvent). See Proposition 2.4 in the same paper, or Schaefer, *Banach Lattices and Positive Operators*, Chapter V.

6.3 Verification: Hypotheses are satisfied

We check the three hypotheses of Theorem 6.2:

(1) Positivity preserving (Markov). From Step 2, $\mathcal{E}_\lambda$ satisfies the normal contraction property. Combined with closedness (Step 4), this makes $(\mathcal{E}_\lambda, \mathcal{D}(\mathcal{E}_\lambda))$ a symmetric Dirichlet form. By the Beurling–Deny theory, the associated semigroup is sub-Markovian, hence positivity preserving.

Note: for the form to generate a sub-Markovian semigroup, we need the form to be a Dirichlet form in the standard sense, i.e., closed + symmetric + nonnegative + Markov. All four properties are established: closedness in Step 4, symmetry and nonnegativity are manifest from the definition (3), and the Markov property in Step 2. Correct.

(2) Irreducibility. From Step 3. Correct.

(3) Holomorphy. A_λ is selfadjoint and lower bounded (by the representation theorem for closed semibounded forms). By the spectral theorem, e^{-zA_λ} is bounded and holomorphic on $\{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$. This is standard functional analysis and correct.

Compact resolvent. From Step 4. This ensures that e^{-tA_λ} is compact for each $t > 0$ (since it is the composition of a bounded operator with $(A_\lambda + 1)^{-1}$, which is compact). Correct.

Therefore Theorems 6.2 and 6.3 apply, yielding:

- e^{-tA_λ} is positivity improving for all $t > 0$.
- $\lambda_0 = \min \operatorname{spec}(A_\lambda)$ is a simple eigenvalue.
- The eigenfunction ψ satisfies $\psi > 0$ a.e. (or $\psi < 0$ a.e.; we choose the positive normalization).

6.4 Verification: Evenness from inversion symmetry

Define the reflection operator $R : L^2(I) \to L^2(I)$ by $(RG)(u) := G(-u)$. Since $I = (-L, L)$ is symmetric, R preserves $L^2(I)$ and is a unitary involution ($R^2 = \mathrm{Id}$).

Lemma 6.4. *$\mathcal{E}_\lambda(RG) = \mathcal{E}_\lambda(G)$ for all $G \in \mathcal{D}(\mathcal{E}_\lambda)$.*

Proof. On $L^2(\mathbb{R})$, R satisfies $RS_t = S_{-t}R$. Hence for any $\phi \in L^2(\mathbb{R})$:

$$\|R\phi - S_tR\phi\| = \|R(\phi - S_{-t}\phi)\| = \|\phi - S_{-t}\phi\| = \|S_t\phi - \phi\| = \|\phi - S_t\phi\|,$$

where we used unitarity of R, the identity $RS_t = S_{-t}R$, and the fact that $\|\phi - S_{-t}\phi\| = \|S_t(\phi - S_{-t}\phi)\| = \|S_t\phi - \phi\| = \|\phi - S_t\phi\|$ (unitarity of S_t).

Since every weight in $\mathcal{E}_\lambda$ is nonnegative and the norms $\|\widetilde{G} - S_t\widetilde{G}\|$ are R-invariant, $\mathcal{E}_\lambda(RG) = \mathcal{E}_\lambda(G)$. □

Corollary 6.5. $A_\lambda R = RA_\lambda$.

Proof. Form invariance under a unitary implies operator commutation: for $u \in \mathcal{D}(A_\lambda)$ and $v \in \mathcal{D}(\mathcal{E}_\lambda)$,

$$\langle A_\lambda Ru, v\rangle = \mathcal{E}_\lambda(Ru, v) = \mathcal{E}_\lambda(u, R^{-1}v) = \langle A_\lambda u, Rv\rangle = \langle RA_\lambda u, v\rangle.$$

(The second equality uses the sesquilinearity of $\mathcal{E}_\lambda$ and invariance under R: $\mathcal{E}_\lambda(Ru, Rv) = \mathcal{E}_\lambda(u, v)$ by polarization of $\mathcal{E}_\lambda(RG) = \mathcal{E}_\lambda(G)$.) Since this holds for all v in the dense set $\mathcal{D}(\mathcal{E}_\lambda)$, $A_\lambda Ru = RA_\lambda u$. □

Corollary 6.6 (Even ground state)**.** *The ground-state eigenfunction ψ satisfies $\psi(-u) = \psi(u)$ a.e.*

Proof. By Corollary 6.5, $R\psi$ is an eigenfunction of A_λ for the same eigenvalue λ_0. Since $\psi > 0$ a.e., also $R\psi > 0$ a.e. (reflection preserves strict positivity on a symmetric interval). By simplicity of λ_0, $R\psi = c\psi$ for some $c \in \mathbb{R}$. Strict positivity of both sides forces $c > 0$. Since R is unitary, $\|R\psi\| = \|\psi\|$, hence $|c| = 1$, so $c = 1$. Therefore $\psi(-u) = \psi(u)$ a.e. □

6.5 Assessment

Verdict. Step 5 is **correct**. The external theorems (Perron–Frobenius for positive semigroups) are correctly cited, and all hypotheses—positivity preservation (Step 2), irreducibility (Step 3), holomorphy (spectral theorem), compact resolvent (Step 4)—are verified. The evenness argument via inversion symmetry is clean: form invariance ⇒ operator commutation ⇒ the reflected ground state is a scalar multiple of the original ⇒ positivity forces the scalar to be 1.

7 Global assessment

7.1 Summary table

Step	Name	Status	Key observation
1	Energy decomposition	Correct	Direct algebra; unitary polarization identity
2	Markov property	Correct	One-line from 1-Lipschitz and $\Phi(0) = 0$
3	Irreducibility	Correct	Archimedean continuum of shifts drives irreducibility; Dirichlet form criterion verified in §4.5
4	Compact resolvent	Correct	Logarithmic growth of Fourier symbol; Kolmogorov–Riesz
5	Perron–Frobenius	Correct	Standard theorems; evenness from reflection symmetry

7.2 Points requiring additional discussion

7.2.1 The Dirichlet form irreducibility criterion (§4.5)

The paper's main expositional gap is in sub-step (c) of Step 3, where it invokes the equivalence between the indicator-energy condition and semigroup irreducibility as a "standard result" from Dirichlet form theory. While indeed standard, the precise formulation matters. In Proposition 4.5 above, we provided the additional argument needed: for translation-difference forms, the cross term $\mathcal{E}(\mathbf{1}_B u, \mathbf{1}_{B^c} u)$ is always nonpositive, so its vanishing for all u forces $\mathcal{E}(\mathbf{1}_B) = 0$, reducing the standard condition to the paper's criterion.

This additional argument does not invalidate the proof. It fills an expositional gap with a straightforward calculation.

7.2.2 Membership of $\mathbf{1}_B$ in the form domain

A related subtlety: does $\mathbf{1}_B \in \mathcal{D}(\mathcal{E}_\lambda)$ for every measurable $B \subset I$? The archimedean contribution to $\mathcal{E}_\lambda(\mathbf{1}_B)$ includes $\int_0^{2L} w(t)\|\widetilde{\mathbf{1}}_B - S_t\widetilde{\mathbf{1}}_B\|^2\,dt$. Near $t = 0$, $w(t) \sim 1/(2t)$ and $\|\widetilde{\mathbf{1}}_B - S_t\widetilde{\mathbf{1}}_B\|^2 \to 0$, but the rate of convergence depends on the regularity of B. For a general measurable set, the strong continuity of translation gives $\|\widetilde{\mathbf{1}}_B - S_t\widetilde{\mathbf{1}}_B\|^2 \to 0$ as $t \to 0$, but the integral $\int_0^{2L}(1/t)\|\widetilde{\mathbf{1}}_B - S_t\widetilde{\mathbf{1}}_B\|^2\,dt$ may diverge.

However, this subtlety does not affect the proof. The argument in sub-step (b) only needs to extract information from $\mathcal{E}_\lambda(\mathbf{1}_B) = 0$, which is a hypothesis (not something that needs to be verified for all B). If $\mathcal{E}_\lambda(\mathbf{1}_B) = 0$, then in particular the archimedean integral vanishes, and the argument proceeds. If $\mathcal{E}_\lambda(\mathbf{1}_B) = +\infty$, then the condition $\mathcal{E}_\lambda(\mathbf{1}_B) = 0$ simply cannot hold, and B poses no threat to irreducibility. The only sets that could witness reducibility are those with $\mathcal{E}_\lambda(\mathbf{1}_B) = 0$ (finite, in fact zero), and for those the argument is complete.

7.2.3 Relationship between $\mathcal{E}_\lambda$ and QW_λ

The paper's form $\mathcal{E}_\lambda$ and Connes's Weil quadratic form QW_λ differ by an additive constant $c(\lambda)\|G\|^2$. This means $A_\lambda^{\text{paper}} = A_\lambda^{\text{Connes}} + c(\lambda)\cdot\mathrm{Id}$. Such a spectral shift preserves:

- simplicity of eigenvalues (the eigenspaces are identical),
- the eigenfunctions (unchanged by adding a scalar to the operator),
- evenness of the ground state.

Hence the paper's result directly implies the conjecture for Connes's operator.

7.3 Conclusion

The paper provides a complete and correct proof that the ground-state eigenvalue of A_λ is simple and the corresponding eigenfunction is even, for every $\lambda > 1$. The five-step structure is logically clean: each step has well-defined inputs and outputs, and the external results invoked (Perron–Frobenius, Kolmogorov–Riesz, Dirichlet form theory) are standard and correctly applied. The one expositional gap—the precise form of the Dirichlet form irreducibility criterion—is closed by the supplementary argument in §4.5, which shows that the paper's indicator-energy criterion is equivalent to the standard one for this class of forms.