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Subject: Linear Algebra	**Code:** MAC0338
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REVISION ON IMPORTANT LINEAR ALGEBRA

1 Determinants

Let $A \in \mathbb{R}^{n\times n}$ and $x \in \mathbb{R}^n$. Which of the following statements are true?
Prove or provide a counterexample:
(a) There exists $x^* \neq 0$ such that $Ax^* = 0$ if $\det(A) = 0$;
(b) There exists $x^* \neq 0$ such that $Ax^* = 0$ only if $\det(A) = 0$;
(c) There exists $x^* \neq 0$ such that $Ax^* = 0$ if and only if $\det(A) \neq 0$.

Solution

(a) True.
Given that the determinant of matrix **A** is zero, we can conclude that at least one column can be expressed as a linear combination of the others. Consequently, through elementary column operations, we can obtain an equivalent matrix **B**, where one of its columns is reduced to zeros, i.e., $A \equiv B$. It's important to note that these column operations do not alter the vector space generated by the matrix, so $Span(A) = Span(B)$. (The solution space of the linear system remains the same when we replace matrix A with matrix B.)

Now, we have the system $Bx = 0$, and since one of the columns of B is entirely zero, we have more variables than equations, indicating that the solution space for this linear system contains a free component that can take any nonzero real value. Therefore, there exists a non-zero vector $x^* \neq 0$ that satisfies the linear system $Ax = 0^n$.

(b) True.
Let's employ a counter-positive proof. Suppose that $det(A) \neq 0$, which implies that the matrix is invertible. Now, consider the equation $Ax = 0$. We can multiply both sides by A^{-1}, yielding $A^{-1} \times Ax = A^{-1} \times 0$. As a result, we find that $x = 0$ is the only solution to this equation.

Therefore, if we assume $det(A) \neq 0$, it implies that the only solution to $Ax = 0$ is $x = 0$. In other words, there are no nonzero solutions ($x^* \neq 0$) to this equation when $det(A) \neq 0^n$.

(c) False.
In reference to item (b), when $det(A) \neq 0$, it implies that the unique solution to the equation is $x = \mathbf{0}^n$, where $\mathbf{0}^n$ represents the zero vector in $\mathbb{R}^n$.

2 Rank

Let $A \in \mathbb{R}^{m\times n}$, where $m \geq n$ and $\text{rank}(A) = n$. Prove that $A^T A$ is non-singular.

Solution

We start by considering matrix A with dimensions $m \times n$, which implies that its transpose, A^T, is $n \times m$. Therefore, the product $A^T A$ results in a square matrix of size $m \times m$, as required for nonsingularity.

To complete the proof, we need to demonstrate that if $A^T A$ is nonsingular, then the null space of the matrix contains only the zero-vector.

We begin with the equation $A^T Ax = 0$, where $x \in \mathbb{R}^m$. By multiplying both sides by x^T, we get:

$$x^T A^T Ax = x^T \cdot 0.$$

Simplifying further:

$$(Ax)^T (Ax) = 0.$$

Now, applying the Euclidean norm (L_2 norm), we have:

$$||Ax||_2^2 = 0.$$

The Euclidean norm of a vector is only equal to zero if and only if the vector itself is the zero-vector, i.e., $||y||_2 = 0 \iff y = 0$. Thus:

$$||Ax||_2 = 0 \iff x = 0.$$

This result indicates that Ax can only be the zero vector when x is the zero vector.

Now, let's interpret this outcome. Ax represents the result of a linear combination of the columns of matrix A. Given that the rank of A is n, we can conclude that the columns of A are linearly independent. Consequently, the only solution for x that yields $Ax = 0$ is the zero-vector.

Returning to our original system:

$$A^T Ax = 0$$

Given that the only solution for x is the zero vector, we can confidently assert that $A^T A$ is indeed nonsingular.

3 Kernel & Image

Let $A \in \mathbb{R}^{m \times n}$, where $m \leq n$ and rank$(A) = k$. We define the subspaces:
Kernel of A: $N_u(A) = \{x \in \mathbb{R}^n | Ax = 0\}$;
Image of A: $Im(A) = \{y \in \mathbb{R}^m | \exists x \in \mathbb{R}^n | y = Ax\}$;
Prove that:
(a) $N_u(A) \perp Im(A^t)$;
(b) $\dim(N_u(A)) = n - k$;
(c) $\mathbb{R}^n = N_u(A) \oplus Im(A^t)$.

Solution

(a) To demonstrate the orthogonality of $N_u(A)$ and $Im(A^T)$, consider the following:

Let $u \in N_u(A)$ and $y \in Im(A^T)$. This means that there exists an x such that $y = Ax$. Now, we can proceed as follows:

$$\begin{aligned} u^T \cdot y &= u^T \cdot Ax \\ &= (A \cdot u) \cdot x \\ &= 0 \cdot x \\ &= 0 \end{aligned}$$

Therefore, we conclude that $N_u(A)$ is orthogonal to $Im(A^T)$, denoted as $N_u(A) \perp Im(A^T)$.

(b) To demonstrate that the dimension of the kernel of matrix A is equal to the number of columns, denoted as n, where $m > n$, subtracted by the rank of A, $Rank(A) = k$, consider the following argument:

We can perform elementary column operations on matrix A to transform it into a new column-echelon matrix R. This transformation ensures that the span of the columns of A remains the same, as column operations do not alter the vector space generated. Since $Rank(A) = k$, it follows that $Rank(R) = k$ because they share the same vector space.

Considering that the rank of a matrix represents the number of non-zero columns, we deduce that this new matrix R possesses k non-zero columns. Furthermore, the kernel of the column space of a matrix corresponds to the number of zero columns. Since matrix R is in echelon form, we have that there are $n-k$ zero columns, which precisely corresponds to the dimension of the null space of A.

(c) We aim to prove that the direct sum of the Null-space of matrix A and the Image Space of A forms the entire vector space of the columns of A.

To establish this, we can approach the proof as follows:

Firstly, let's establish that the intersection of $Nu(A)$ and $Im(A)$ is trivial, containing only the zero vector. Suppose $u \in Nu(A) \cap Im(A)$. As shown in item (a), we have previously established that $u^T u = 0$ to demonstrate the orthogonality of the kernel and the image space. Consequently, this intersection has a dimension of zero.

The rank of a matrix represents the dimension of its image space: $dim(Im(A)) = rank(A) = k$.

From item (b), we know that the dimension of the null-space is $dim(Nu(A)) = n-k$.

Therefore, the dimension of $Nu(A) + Im(A)$ is:

$$dim(Nu(A)+Im(A)) = dim(Nu(A)) + dim(Im(A)) - dim(Nu(A) \cap Im(A)) = (n-k) + k - 0 = n = dim(\mathbb{R}^n).$$

Since $Nu(A)$ and $Im(A)$ are both subsets of $\mathbb{R}^n$, we can conclude that $Nu(A) \oplus Im(A) \subset \mathbb{R}^n$.

The equality of dimensions and the fact that $Nu(A) \oplus Im(A) \subset \mathbb{R}^n$ are enough to establish that:

$$\mathbb{R}^n = Nu(A) \oplus Im(A).$$

This demonstrates that the direct sum of the Null-space of A and the Image Space of A indeed forms the entire vector space of the columns of A.

4 Intersection of Hyperplanes / Linear Systems / Equation of the Line

Consider the equations

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = 1, \ldots, n-1,$$

or equivalently, $Ax = b$ with $A \in \mathbb{R}^{(n-1)\times n}$, $b \in \mathbb{R}^{n-1}$, and $x \in \mathbb{R}^n$, corresponding to $n-1$ "linearly independent"hyperplanes. The intersection of these hyperplanes determines a line in $\mathbb{R}^n$. We can represent this line in the form

$$y = x + \lambda d$$

with $\lambda \in \mathbb{R}$ and $x, d \in \mathbb{R}^n$. Discuss how to choose x and d.

Solution

Considering that the system has n variables and $n-1$ equations, we have a free variable that may assume any real value.

Then, by defining an arbitrary value for this free variable and solving the $(n-1)\times(n-1)$ system, we get a single solution x_0 one they are linearly independent. The, fixing a new value for the free variable, and solving again the linear system, we will get another value x_1. Then, with these two values we are capable of defining a equation for the line formed by the intersection of the hyperplanes

$$y = x_1 + (x_1 - x_0) \cdot \lambda$$

5 Eigenvalues and Eigenvectors

Find the eigenvalues and eigenvectors of the matrix $A = uu^T$, where $u \in \mathbb{R}^n$.

Solution

We have define that the matrix A is equal the outer product uu^T. Then, every line of A is a linear combination of u^T, so the rank of A is equal to 1.

$$uu^T = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}$$

$$\begin{aligned} A &= uu^T \\ A \cdot u &= uu^T \cdot u = u \cdot < u^T, u > \\ A \cdot u &=< u^T, u > \cdot u \end{aligned}$$

Then, $\lambda =< u^T, u >$ is one eigenvalue associated with the eigenvector u.

NOTE: There are some other eigenvalues, and apparently we are supposed to prove that they are all zero eigeinvalues, I can't do it right now, I'll try to prove it afterwards

6 Eigenvalues and Eigenvectors and Orthogonality

Prove that the eigenvectors of a matrix associated with distinct eigenvalues are linearly independent, and if the matrix is symmetric, they are orthogonal.

7 Positive Definite Matrix and Eigenvalues

Prove that the eigenvalues of a symmetric matrix are positive if and only if the matrix is positive definite.

Solution

We are going to proceed with the technique for biconditional proof.

$\Rightarrow$ Let A be a symmetric matrix with all its eigenvalues positive. Then, we can decomposite this matrix in the following manner:

$$A = Q\Lambda Q^T$$

where:
Q is a orthogonal matrix *(i.e $QQ^T = I$)*
λ is a diagonal matrix with its eigenvalues.

Let $x \neq 0$, then

$$x^T Ax = x^T Q\Lambda Q^T X$$

Furthermore, we can have $\Lambda = \sqrt{\Lambda}\sqrt{\Lambda}$, then

$$\begin{aligned} x^T Ax &= x^T Q\sqrt{\Lambda}\sqrt{\Lambda}Q^T X \\ &= ||\sqrt{\Lambda}Q^T x||_2^2 \end{aligned}$$

Then, since $\sqrt{\Lambda}Q^T$ is non-singular, x is different from 0 and a norm is positive, we conclude that A have to be positive definite.

$\Leftarrow$ Let B be a positive definite matrix. Then,

$$
\begin{aligned}
Av &= \lambda v \\
v^T Av &= v^T \lambda v
\end{aligned}
$$

Considering A as positive definite, the left-hand side of the equation is positive, then λ must be positive. This concludes the proof that the eigenvalues must be positive.

8 Eigenvalues of Invertible Matrices

Prove that if λ is an eigenvalue of a non-singular matrix A, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

Solution

$$
\begin{aligned}
Av &= \lambda v \\
A^{-1}Av &= \lambda A^{-1}v \\
Iv &= \lambda A^{-1}v \\
\frac{1}{\lambda}v &= A^{-1}v
\end{aligned}
$$

Therefore, $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

9 Eigenvalues

Prove that $A \in \mathbb{R}^{n \times n}$ is singular if and only if 0 is an eigenvalue.

Solution

$\Rightarrow$ Let A be a singular matrix, then $det(A) = 0$. For the characteristic equation:

$$det(A - I\lambda) = 0$$

, where λ are the eigenvalues.

Hence, $\lambda = 0$ is an eigenvalue of the singular matrix A.
$\Leftarrow$ Let $\lambda = 0$ be an eigenvalue of the square matrix A associated with the eigenvector v. Then, $Av = 0$. Eigenvectors are by definition non-zero, so the solution set for the system is not trivial so the matrix A is singular.

10 Limits of Sequences

Suppose $\lim_{k\to\infty} x_k = \alpha$. Prove that if $\alpha > \beta$, there exists $M > 0$ such that for all $k \geq M$, we have $x_k > \beta$.

Solution

Considering the fact that $\lim_{k\to\infty} x_k = \alpha$, then for every $\epsilon > 0$ there exists $k > k_0$ such that:

$$\begin{aligned}|x^k - \alpha| &< \epsilon \\ -\epsilon < x^k - \alpha &< \epsilon \\ \alpha - \epsilon < x^k &< \alpha + \epsilon \\ \beta - \epsilon < \alpha - \epsilon < x^k &< \alpha + \epsilon\end{aligned}$$

Since there is k for every $\epsilon > 0$, we can take $\epsilon = \alpha - \beta$. Then,

$$\begin{aligned}\alpha - (\alpha - \beta) &< x^k \\ \beta &< x^k\end{aligned}$$

11 Limit of Sequence

Prove que if $\lim_{k\to\infty} x_k = \alpha$ and for all $k \geq 0$, $x_k \geq \beta$, then $\alpha \geq \beta$. If we change the $\geq$ sign to $>$, does the statement still hold? Prove or provide a counterexample.

Solution

As the limit of x_k as k goes to the positive infinity, we have that for every ϵ greater than 0 we find a $k > k_0$ such that:

$$|x_k - \alpha| < \epsilon$$

Then,

$$\begin{aligned}|x_k - \alpha| &< \epsilon \\ -\epsilon < x_k - \alpha &< \epsilon \\ \alpha - \epsilon < x_k &< \alpha + \epsilon \\ \beta \leq x_k &< \alpha + \epsilon \\ \text{Then}& \\ \beta &< \alpha + \epsilon \\ \text{Taking: } \epsilon &= \alpha - \beta \\ \beta &< \alpha + \alpha - \beta \\ \text{Hence } \beta &< \alpha\end{aligned}$$

12 Convergence of Sequences

If $\{x_k\}$ is a convergent sequence, is this sequence bounded? Is the converse true?

13 Norms

Prove that the following functions are norms:
(a) $|\cdot|_\infty : \mathbb{R}^n \to \mathbb{R}$, $|x|_\infty = \max_{1\leq i\leq n} |x_i|$;
(b) $|\cdot|_1 : C([a,b]) \to \mathbb{R}$, $|f|_1 = \int_a^b |f(x)|\,dx$ (where $C([a,b])$ represents continuous functions on $[a,b] \to \mathbb{R}$).

14 Jocobians

Consider the functions $f : \mathbb{R}^m \to \mathbb{R}^p$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ with Jacobians $J_f \in \mathbb{R}^{p\times m}$ and $J_g \in \mathbb{R}^{m\times n}$, respectively. Find the Jacobian of the composite function $h : \mathbb{R}^n \to \mathbb{R}^p$ given by $h(x) = f(g(x))$.

15 Gradient and Hessian

Compute the gradient and Hessian of the functions $f : \mathbb{R}^n \to \mathbb{R}$ below:
(a) $f(x) = a^T x$;
(b) $f(x) = 2x^T Ax + b^T x + c$, where $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}$;
(c) $f(x) = g^T(x)g(x) = |g(x)|_2^2$, where $g : \mathbb{R}^n \to \mathbb{R}^m$.

16 Gradient and Hessian

Let $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$. For $x \in \mathbb{R}^n$, we define $q(x) = f(Ax + b)$ with $f : \mathbb{R}^m \to \mathbb{R}$. Calculate the gradient and Hessian of the function q.