Introduction to PDEs, Fall 2024

Homework 11 solutions

Name:______________

1. Find a fundamental solution of Laplacian Δ in 3D. Hint: $G(r) = -\frac{1}{4\pi}\frac{1}{|\mathbf{x}|} = -\frac{1}{4\pi}\frac{1}{r}$.

   Solution 1. *One can follow the approximation approach used in HW 9, but the details are omitted here.*

2. Find the Green's function over $\mathbb{R}^3_+ := \{(x, y, z) \in (-\infty, \infty) \times (-\infty, \infty) \times (0, \infty)\}$, and then solve

$$\begin{cases} \Delta u = 0, & x \in \mathbb{R}^3_+, \\ \frac{\partial u}{\partial \mathbf{n}} = g, & x \in \partial\mathbb{R}^3_+. \end{cases} \tag{0.1}$$

   Solution 2. *It is necessary to find G such that it satisfies the homogeneous Neumann boundary condition. This step is omitted.*

3. Verify that the Laplacian of $u(x, y)$ in the polar coordinates $x = r\cos\theta, y = r\sin\theta$ is

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

   Solution 3. *This follows from straightforward calculations and details skipped.*

4. Solve the following Poisson's equation for $u(r, \theta)$

$$\begin{cases} \Delta u = \cos\theta, & r \in (1, 2)\ \theta \in [0, 2\pi), \\ u|_{r=1} = 0,\ u|_{r=2} = 2. \end{cases} \tag{0.2}$$

   Hint: There are two methods you can tackle this problem. The first is to use the method of separation of variables. For each fixed $r \in (1, 2)$, $u(r, \theta)$ is a 2π–periodic function of θ. Show that it can expands into

$$u(r, \theta) = A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta + B_n \sin n\theta,$$

   where A_n and B_n are functions of r. Then substitute this into the polar coordinate of the PDE and collect the ODEs for A_n and B_n. You may need to solve some Euler–type ODE. Then find the coefficients by the boundary conditions. The second method is to find $u = v + w$ such that $\Delta v = 0$ and $\Delta w = \cos\theta$ and the main task is to find one specific w.

   Solution 4. *Since u is not a harmonic function, the general solution may not apply to this problem. However, we can adapt the approach used for solving inhomogeneous heat equations. Noting that $u(r, \theta)$ is periodic in θ, we can expand it into the following series:*

$$u(r, \theta) = A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta + B_n \sin n\theta,$$

   where $A_n = A_n(r)$ and $B_n = B_n(r)$ are functions of r. This result follows from the fact that $\{\cos n\theta, \sin n\theta\}$ form an orthogonal basis for L^2-functions with period 2π. Substituting the Fourier series expansion into the polar coordinate form of the Laplacian, given by

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2},$$

we derive the following equation from the partial differential equation (PDE):

$$A_0'' + \frac{1}{r}A_0' + \sum_{n=1}^{\infty} \left(A_n'' + \frac{1}{r}A_n' - \frac{n^2}{r^2}A_n\right)\cos n\theta + \left(B_n'' + \frac{1}{r}B_n' - \frac{n^2}{r^2}B_n\right)\sin n\theta = \cos\theta,$$

where the derivatives are taken with respect to r. Now, by comparing the coefficients, we obtain $A_n = B_n = 0$ for all $n \geq 1$ (this follows from the structure of the equation—why is this the case?), except for A_0 and A_1, which satisfy the following:

$$A_0'' + \frac{1}{r}A_0' = 0, \;\; A_1'' + \frac{1}{r}A_1' - \frac{1}{r^2}A_1 = 1.$$

Then solving these ODEs leads us to

$$A_0(r) = C_0 \ln r + D_0$$

and

$$A_1(r) = C_1 r + C_2 r^{-1} + r^2/3,$$

where C_0, C_1, D_0 and D_1 are constants independent of r and θ.

By applying the boundary conditions, we arrive at the following:

$$\begin{cases} C_0 \ln 1 + D_0 + (C_1 + C_2 + 1/3)\cos\theta = 0, \\ C_0 \ln 2 + D_0 + (C_1 + C_2/2 + 4/3)\cos\theta = 2. \end{cases}$$

Solving this algebraic system, we have that $C_0 = 2/\ln 2, D_0 = 0, C_1 - 7/9$ and $C_2 = 4/9$. Finally, we obtain that

$$u(r,\theta) = 2\ln r/\ln 2 + (-7r/9 + 4/(9r) + r^2/3)\cos\theta.$$

I would like to point out that an alternative approach to solving this problem is to introduce a new function that transforms the inhomogeneous PDE into a homogeneous one, i.e., a harmonic function. To this end, let us define $v = u + f(r,\theta)$, where $f(r,\theta) = -\frac{1}{3}r^2\cos\theta$. This allows us to derive the following:

$$\begin{aligned} \Delta w &= \Delta u + \Delta f(r,\theta) \\ &= \cos\theta + \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} \\ &= \cos\theta - \frac{2}{3}\cos\theta - \frac{2}{3}\cos\theta + \frac{1}{3}\cos\theta \\ &= 0 \end{aligned}$$

Since $\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta}$, we can choose $w = R(r)\Theta(\theta)$ and obtain

$$\Delta w = R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0.$$

Since the trivial solution is not meaningful, we can derive

$$\frac{R''}{R} + \frac{R'}{rR} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = 0,$$

or equivalently,

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda.$$

Since $\Theta(\theta)$ is a 2π-periodic function of θ, we can express $\Theta(\theta)$ as

$$\Theta(\theta) = \sum_{n=0}^{\infty} A_n \cos n\theta + B_n \sin n\theta.$$

moreover we can get

$$r^2\frac{R^{''}}{R} + r\frac{R^{'}}{R} = n^2$$

Let $R = Cr^\alpha$, then we have

$$C\alpha(\alpha-1)r^\alpha + C\alpha r^\alpha = Cn^2 r^\alpha,$$

i.e., $\alpha^2 = n^2$. *If* $n \neq 0$, *we have* $R_n(r) = C_n r^n$. *If* $n = 0$, *we have* $r^2 R'' + rR' = 0$, *solving the equation, we can get* $R(r) = C_0 + D_0 \ln r$. *Therefore we find that*

$$w(r,\theta) = (C_0 + D_0 \ln r) + \sum_{n=1}^{\infty} (C_n r^n + D_n r^{-n})(A_n \cos n\theta + B_n \sin n\theta)$$

Moreover we can get

$$u(r,\theta) = w(r,\theta) - f(r,\theta) = (C_0 + D_0 \ln r) + \frac{1}{3} r^2 \cos\theta + \sum_{n=1}^{\infty} (C_n r^n + D_n r^{-n})(A_n \cos n\theta + B_n \sin n\theta)$$

it is obvious that we can write $u(r,\theta)$ *as follows*

$$u(r,\theta) = A_0(r) + \sum_{n=1}^{\infty} A_n(r) \cos n\theta + B_n(r) \sin n\theta$$

Now substituting $u(r,\theta)$ *into PDE, we can get*

$$A_0''(r) + \frac{1}{r} A_0'(r) + \sum_{n=1}^{\infty} (A_n''(r) + \frac{1}{r} A_n'(r) - \frac{n^2}{r^2} A_n(r)) \cos n\theta + (B_n''(r) + \frac{1}{r} B_n'(r) - \frac{n^2}{r^2} B_n(r)) \sin n\theta = \cos\theta$$

Multiplying $\cos n\theta$ *and* $\sin n\theta$ *respectively and integrating over* $(0, L)$, *we can get*

$$\begin{cases} A_1'' + \frac{1}{r} A_1' - \frac{1}{r^2} A_1 = 1 & (1) \\ A_n'' + \frac{1}{r} A_n' - \frac{n^2}{r^2} A_n = 0 \quad (n \neq 1) & (2) \\ B_n'' + \frac{1}{r} B_n' - \frac{n^2}{r^2} B_n = 0 & (3) \end{cases}$$

Moreover, according to the BC, we have

$$\begin{cases} A_0(1) + \sum_{n=1}^{\infty} A_n(1) \cos n\theta + B_n(1) \sin n\theta = 0 \\ A_0(2) + \sum_{n=1}^{\infty} A_n(2) \cos n\theta + B_n(2) \sin n\theta = 2 \end{cases}$$

i.e.

$$A_0(1) = 0 \qquad A_0(2) = 2$$

and

$$A_n(1) = A_n(2) = 0, (n \geq 1) \qquad B_n(1) = B_n(2) = 0, (n \geq 0)$$

Solving the equation (2), *we can get*

$$A_n(r) = c_n r^n + d_n r^{-n}, (n \geq 1) \qquad A_0(r) = c_0 + d_0 \ln r$$

Solving the equation (3), *we can get*

$$B_n(r) = c_n' r^n + d_n' r^{-n}, (n \geq 1) \qquad B_0(r) = c_0' + d_0' \ln r$$

Then applying the BC, we can get

$$c_0 = 0,\ d_0 = \frac{2}{\ln 2};\ c_n = d_n = 0, n > 1;\ \text{and}\ c_n' = d_n' = 0, n \geq 0$$

Now, let us consider the value of equation (1) *withe the BC as follows*

$$\begin{cases} A_1'' + \frac{1}{r} A_1' - \frac{1}{r^2} A_1 = 1 \\ A_1(1) = A_1(2) = 0 \end{cases}$$

i.e.,

$$\begin{cases} r^2 A_1'' + r A_1' - A_1 = 1 & (4) \\ A_1(1) = A_1(2) = 0 \end{cases}$$

Let $r = e^x$, then we have the equation

$$\frac{d^2 A_1}{dr^2} - A_1 = e^{2x}.$$

First, we solve the homogeneous equation

$$\frac{d^2 A_1}{dr^2} - A_1 = 0.$$

The general solution for $A_1(r)$ is $m_1 e^x + m_2 e^{-x}$. Additionally, we can find a particular solution $A_1^*(r) = be^{2x} = br^2$. Substituting $A_1^*(r)$ into equation (4), we obtain $3b = 1$, which gives $b = \frac{1}{3}$.

Finally, the general solution for $A_1(r)$ is thus given by

$$A_1(r) = m_1 e^x + m_2 e^{-x} + \frac{1}{3} r^2.$$

$$A_1(r) = m_1 r + m_2 \frac{1}{r} + \frac{1}{3} r^2$$

since $A_1(1) = A_1(2) = 0$, i.e.,

$$\begin{cases} m_1 + m_2 + \frac{1}{3} = 0 \\ 2m_1 + \frac{1}{2} m_2 + \frac{4}{3} = 0 \end{cases}$$

Solving the system, we can get $m_1 = -\frac{7}{9}$ and $m_2 = \frac{4}{9}$. i.e.,

$$A_1(r) = \frac{1}{3} r^2 - \frac{7}{9} r + \frac{4}{9} \cdot \frac{1}{r}$$

Therefore, we can get

$$\begin{aligned} u(r, \theta) &= A_0(r) + \sum_{n=1}^{\infty} A_n(r) \cos n\theta + B_n(r) \sin n\theta \\ &= \frac{2}{\ln 2} \ln r + (\frac{1}{3} r^2 - \frac{7}{9} r + \frac{4}{9} \cdot \frac{1}{r}) \cos \theta \end{aligned}$$