Introduction to PDEs, Fall 2024

Homework 8 due Dec 5

Name:________________

1. We know from class that the solution to the following problem
$$\begin{cases} u_t = Du_{xx}, & x \in (0,\infty), t \in \mathbb{R}^+, \\ u(x,0) = \phi(x), & x \in (0,\infty), \\ u(0,t) = u(\infty,t) = 0, & t \in \mathbb{R}^+. \end{cases} \tag{0.1}$$
is given in the following form*
$$u(x,t) = \int_0^\infty \big(G(\xi;x,t) - G(\xi;-x,t)\big)\phi(\xi)d\xi.$$
Note that the integral mentioned above can be evaluated symbolically. Set $D = 1$ and choose the initial data as $\phi(x) \equiv 1$ for $x \in (0,1) \cup (2,3)$, and $\phi(x) \equiv 0$ otherwise. Plot the solution of (0.1) at times $t = 10^{-4}, 10^{-3}, 0.1, 0.5, 1,$ and 5. Keep in mind that the integral over $(0,\infty)$ must be truncated to $(0,L)$ for a sufficiently large L. Select L appropriately, ensuring the truncation error is within an acceptable range. (By now, you should be familiar with how to determine such an L based on the desired level of accuracy.)

2. Let us consider the following IBVP over half line $(0,\infty)$ with Neumann boundary condition
$$\begin{cases} u_t = Du_{xx}, & x \in (0,\infty), t \in \mathbb{R}^+, \\ u(x,0) = \phi(x), & x \in (0,\infty), \\ u_x(0,t) = 0, & t \in (0,\infty), \end{cases} \tag{0.2}$$
Similar as in class, tackle this problem by first solving its counterpart in $(0,L)$ and then sending $L \to \infty$. Hint: the suggested solution is
$$u(x,t) = \int_0^\infty \Big(G(\xi;x,t) + G(\xi;-x,t)\Big)\phi(\xi)d\xi.$$

3. Let us consider the following Cauchy problem
$$\begin{cases} u_t = Du_{xx}, & x \in (-\infty,\infty), t \in \mathbb{R}^+, \\ u(x,0) = \phi(x), & x \in (-\infty,\infty). \end{cases} \tag{0.3}$$
We can approximate the solution to this problem by first solving its counterpart in $(-L,L)$, which has been in a previous homework, and then sending $L \to \infty$.
Consider
$$\begin{cases} u_t = Du_{xx}, & x \in (-L,L), t \in \mathbb{R}^+, \\ u(x,0) = \phi(x), & x \in (-L,L), \\ u(-L,t) = u(L,t) = 0, & t \in \mathbb{R}^+. \end{cases} \tag{0.4}$$
(i). write the solution to (0.4) in terms of infinite series; you just present your final results, no need to show the details here;

*Throughout this homework, and probably the whole course, $G(\xi;x,t)$ is the heat kernel and it is explicitly given by
$$G(\xi;x,t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{|x-\xi|^2}{4Dt}}.$$
Sometimes we write $G(x,t;\xi)$ to highlight the variables or parameters x and t.

(ii). write the series above into an integral and then evaluate this integral by sending $L \to \infty$. Suggested answer:

$$u(x,t) = \int_{\mathbb{R}} G(\xi; x, t)\phi(\xi)d\xi, \tag{0.5}$$

We shall see several important applications of solution (0.5) in the future.

4. The heat kernel $G(\xi; x, t)$ is sometimes called fundamental solution of heat equation

$$G(\xi; x, t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{(x-\xi)^2}{4Dt}}.$$

Prove that

(i) $|\frac{\partial G}{\partial x}| \to 0$ as $|x| \to \infty$ for each t and ξ. Prove the same for $\frac{\partial^m G}{\partial x^m}$ for each $m \in \mathbb{N}^+$;

(ii) $G_t = DG_{xx}$, $x \in \mathbb{R}, t \in \mathbb{R}^+$; this should then imply that $u_t = Du_{xx}$;

(iii) $\int_{\mathbb{R}} G(\xi; x, t)dx = 1$.

Remark: I would like to note that we write the kernel $G(x, t; \xi)$ and $G(\xi; x, t)$ interchangeably. The former is to highlight the eventual solution as a function of x and t, whereas the latter is to focus on treating ξ as the integration variable whenever applicable.

5. To give yourself some physical intuitions on the heat kernel, let us consider the following situation in $\mathbb{R}$: put two separate units of thermal heat at locations $\xi = -1$ and $\xi = 1$ respectively at time $t = 0$. Suppose that the temperature $u(x,t)$ satisfies the heat equation with diffusion rate $D = 1$, then it is given by the following explicit form

$$u(x,t) = G(x,t;-1) + G(x,t;1) = \frac{1}{\sqrt{4\pi t}}\Big(e^{-\frac{(x+1)^2}{4t}} + e^{-\frac{(x-1)^2}{4t}}\Big).$$

Plot $u(x,t)$ over $x \in (-5,5)$ with $t = 0.01, 0.02, 0.05,$ 0.1 and 1 *on the same coordinate* in $(-R, R)$ (if R is large, then it approximates the whole line) to illustrate your results–please use different colors and/or line styles for better effects. We will know more about physical intuition in the future; indeed you should already have an intuition about: i) the evolution of thermal energy; ii) the connection between diffusion and Brownian motion or normal distribution.)

6. Consider the following problem

$$\begin{cases} u_t = Du_{xx} - \alpha u_x - ru, & x \in (-\infty, 0), t \in \mathbb{R}^+, \\ u(x,0) = \phi(x) \geq, \neq 0, & x \in (\infty, 0), \\ u(-\infty, t) = e^{-rt}K > 0, u(0,t) = 0, & t \in \mathbb{R}^+, \end{cases} \tag{0.6}$$

where D, α, r and K are positive constants.

Let us visit its truncated problem

$$\begin{cases} u_t = Du_{xx} - \alpha u_x - ru, & x \in (-L, 0), t \in \mathbb{R}^+, \\ u(x,0) = \phi(x) \geq, \neq 0, & x \in (\infty, 0), \\ u(-L, t) = e^{-rt}K > 0, u(0,t) = 0, & t \in \mathbb{R}^+. \end{cases} \tag{0.7}$$

(i) Solve (0.7) in terms of infinite series. Hint: its boundary condition is inhomogeneous; one can choose $v = e^{\theta x}u$ to "remove" the α term;

(ii) Send L to infinity and then find the limiting solution in terms of an integral.