# Introduction to PDEs, Fall 2024

**Mid-term 1** Make-up

2 Hours and 30 Minutes

**Name**(Print):________________

**Student No**:________________

**Signature**:________________

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***There are 10 problems, 10 points each, 100 points in total.***

***Show `details` to get full credits. Make your justifications `clear` and `direct`.***

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**Leave the following table blank**

| Score Table | | |
|---|---|---|
| Problem | Points | Score |
| 1 | 10 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 10 | |
| 5 | 10 | |
| 6 | 10 | |
| 7 | 10 | |
| 8 | 10 | |
| 9 | 10 | |
| 10 | 10 | |
| | Total score | |

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1. Suppose that $u(x,t)$ moves to $x \pm \Delta x$ at any time $t$ with a probability $\frac{1}{2}$, i.e., $p(x \to x \pm \Delta x, t) = \frac{1}{2}$. Denote $D = \frac{\Delta x^2}{\Delta t}$ as $\Delta t \to 0^+$. Derive the PDE for $u(x,t)$.

2. Suppose that $u(x,t)$ moves to $x \pm \Delta x$ at any time $t$ with a probability $p$ which depends location $x$ but not time, i.e., $p(x \to x \pm \Delta x, t) = \rho(x)$. Denote $D = \frac{\Delta x^2}{\Delta t}$ as $\Delta t \to 0^+$. Derive the PDE for $u(x,t)$.

3. Let us consider a similar scenario in a 2D lattice with mesh size $\Delta x = \Delta y$. Let $u(x, y, t)$ be the number of particles at location $(x, y) \in \mathbb{R}^2$ and time $t > 0$. Suppose that each particle, at the next time $t + \Delta t$, moves northwards, southwards, westwards or eastwards with probability $\frac{1}{4}$, i.e., $p((x, y) \to (x \pm \Delta x, y), t) = p((x, y) \to (x, y \pm \Delta y), t) = \frac{1}{4}$. Assume that $D = \frac{\Delta x^2}{\Delta t}$ as $\Delta t \to 0^+ > 0$. Derive the PDE for $u(x, y, t)$.

4. 1. Let $f(\mathbf{x})$ be a differentiable function defined over $\Omega \subset \mathbb{R}^n$, $n \geq 1$. Write down the definition of

   1). the gradient operator $\nabla f$;

   2). the Laplace operator $\Delta f$;

   2. Consider the following reaction-diffusion equation for $u(\mathbf{x}, t)$

$$u_t = D\Delta u + f(u, \mathbf{x}, t), \quad x \in \Omega, t > 0.$$

   1). Suppose that $u$ denotes the temperature at location-time $(\mathbf{x}, t)$. What does it mean or model if $f > 0$?

   2). Suppose that $u$ denotes the population at location-time $(\mathbf{x}, t)$. What does it mean or model if $f > 0$?

5. Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, $N \geq 1$.

   (i) Write down the definitions of Dirichlet, Neumann, Robin boundary conditions.

   (ii) What are the physical interpretations of DBC and NBC if $u(x,t)$ represents the temperature?

   (iii) What are the biological interpretations of DBC and NBC if $u(x,t)$ represents the population density?

6. Consider the following initial boundary value problem

$$\begin{cases} u_t = D\Delta u + f(x,t), & x \in \Omega, t > 0, \\ u(x,0) = \phi(x), & x \in \Omega, \\ u(x,t) = \psi(x), & x \in \partial\Omega, t > 0. \end{cases} \tag{1}$$

Use the energy method to prove the uniqueness of (1).

7. Let us reconsider the following reaction-diffusion system with mixed boundary condition

$$\begin{cases} u_t = D\Delta u + f(x,t), & x \in \Omega, t > 0, \\ u(x,0) = \phi(x), & x \in \Omega, \\ u(x,t) = \psi_1(x,t), & x \in \partial\Omega_1, t > 0, \\ \partial_{\mathbf{n}} u(x,t) = \psi_2(x,t), & x \in \partial\Omega_2, t > 0, \end{cases} \tag{2}$$

where $\partial\Omega_1$ denotes part of the boundary $\Omega$ and $\partial\Omega_2$ denotes the rest part. Prove the uniqueness of (2).

8. Denote $u_n(x,t) := e^{-D(\frac{n\pi}{L})^2 t} \sin \frac{n\pi x}{L}$.

i) show that, by formal calculation, for each $N < \infty$, the series $\sum_{n=1}^{N} c_n u_n(x,t)$ is a solution of the following baby heat equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2},$$

where $c_n$ are constants.

ii) Fixed $L$ and $D$, for example, by assuming that $L = \pi$ and $D = 0.05$. Sketch $u_1(x,t)$ over $x \in (0,\pi)$ with $t = 0$, $t = 0.5$, $t = 2$ in the same coordinate.

9. 1) Write down the definition of $L^2(\Omega)$ and $L^2$-norm;

   2) Write down the definition that functions $f$ and $g$ are orthogonal in $L^2(\Omega)$;

   3) Perform straightforward calculations to verify that $\sin\frac{m\pi x}{L}$ and $\sin\frac{n\pi x}{L}$ are orthogonal in $L^2(0, L)$ as

$$\int_0^L \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L} = \frac{L}{2}\delta_{mn} = \begin{cases} \frac{L}{2}, & \text{if } m = n, \\ 0, & \text{if } m \neq n; \end{cases}$$

   here $\delta$ is the so-called Kronecker delta function.

10. 1). Show that $f(x) = \frac{1}{\sqrt{x}} \in L^1(0,1)$, but not in $L^2(0,1)$.

2). What would be the general conditions for a function of the form $f(x) = x^\alpha$ to belong to $L^p(0,1)$, but not $L^q(0,1)$, assuming $p, q \in (1, \infty)$ for simplicity?