Introduction to PDEs, Fall 2024

Homework 6 solutions

Name:________________

1. Solve for $u(x,t)$ in infinite series for the following problem under DBC

$$\begin{cases} u_t = u_{xx} + e^{-t}\sin 2x, & x \in (0,1), t \in \mathbb{R}^+, \\ u(x,0) = x, & x \in (0,1), \\ u = 0, & x = 0,1, t \in \mathbb{R}^+. \end{cases} \tag{0.1}$$

Solution 1. *After identifying the boundary condition as a Dirichlet boundary condition (DBC), the Sturm–Liouville theory guarantees that the solution can be uniquely expressed as the following series*

$$u(x,t) = \sum_{n=1}^{\infty} C_n(t) \sin n\pi x.$$

Then the PDE implies that

$$\sum_{n=1}^{\infty} C_n'(t) \sin n\pi x = -\sum_{n=1}^{\infty} (n\pi)^2 C_n(t) \sin n\pi x + e^{-t} \sin 2x,$$

hence we have that

$$C_2'(t) = -(2\pi)^2 C_2(t) + e^{-t} \tag{0.2}$$

and

$$C_n'(t) = -(n\pi)^2 C_n(t), \forall n \neq 2. \tag{0.3}$$

Equation (0.3) is very simple and one immediately collects that

$$C_n(t) = C_n(0) e^{-(n\pi)^2 t}, \forall n \neq 2.$$

Equation (0.2) can be somewhat tricky unless you utilize a technique known as the integrating factor.

Indeed, one does not need to fully rely on this technique, and I will show you why. For notational simplicity, let us set $\lambda := 4\pi^2$. With this, we can rewrite (0.2) as $(e^{\lambda t} C_2(t))' e^{-\lambda t} = e^t$, which is equivalently expressed as:

$$\left(e^{\lambda t} C_2(t)\right)' = e^{(\lambda - 1)t}.$$

Integrating this identity over $(0,t)$ gives us

$$C_2(t) = C_2(0) e^{-\lambda t} + \frac{1}{\lambda - 1}\left(e^{-t} - e^{-\lambda t}\right) = C_2(0) e^{-4\pi^2 t} + \frac{1}{4\pi^2 - 1}\left(e^{-t} - e^{-4\pi^2 t}\right).$$

Then the solution can be rewritten into the following series

$$u(x,t) = C_2(t) \sin 2\pi x + \sum_{n \neq 2} C_n(t) \sin n\pi x,$$

where $C_n(0)$, $n \geq 1$, are determined by the initial condition as

$$C_n(0) = 2\int_0^1 x \sin n\pi x dx = \frac{2}{n\pi}(-1)^{n-1}, \forall n \geq 1.$$

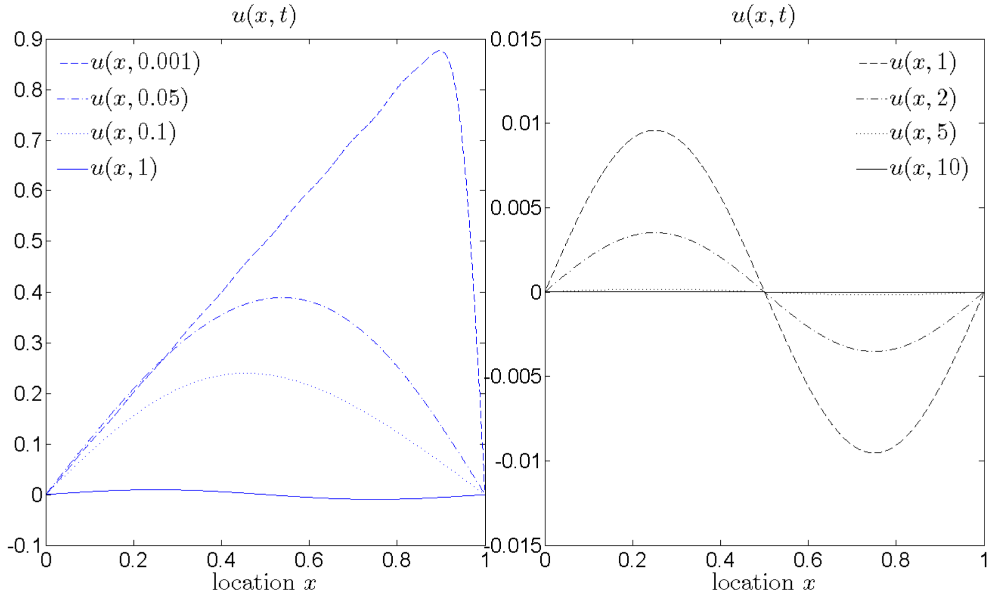

Figure 1: Illustration of the convergence to zero of the solution to Problem (0.1). **Left:** The solution is plotted at multiple time steps to demonstrate its stabilization. It is evident that the dynamics are primarily governed by diffusion and the boundary condition. **Right:** The solution is plotted at significantly larger time steps, highlighting the dominance of the kinetics, represented by $\sin 2\pi x$. At this stage, the diffusion effects have largely dissipated, and the PDE dynamics are effectively reduced to those of the ODE, with the solution aligning closely with its mode.

2. Solve for $u(x,t)$ in infinite series for the following problem under NBC

$$\begin{cases} u_t = u_{xx} + e^{-t}\sin 2x, & x \in (0,1), t \in \mathbb{R}^+, \\ u(x,0) = x, & x \in (0,1), \\ u_x = 0, & x = 0, 1, t \in \mathbb{R}^+. \end{cases} \tag{0.4}$$

Solution 2. *The approach is pretty much the same as above except that now the series becomes*

$$u(x,t) = \sum_{n=0}^{\infty} C_n(t)\cos n\pi x,$$

in light of the NBC, whence the PDE implies that

$$\sum_{n=0}^{\infty} C'_n(t)\cos n\pi x = -\sum_{n=0}^{\infty}(n\pi)^2 C_n(t)\cos n\pi x + e^{-t}\sin 2x.$$

Testing this ODE by $\cos n\pi x$ *over* $(0,1)$*, we find that*

$$C'_n(t) = -(n\pi)^2 C_n(t) + 2e^{-t}\int_0^1 \sin 2x \cos n\pi x dx, \forall n \geq 0.$$

To evaluate the integral, one can use the triangle formula $\sin a \cos b = \frac{\sin(a+b)-\sin(a-b)}{2}$*. My calculations tell that this integral is*

$$\int_0^1 \sin 2x \cos n\pi x dx = \frac{1-\cos(2+n\pi)}{2(2+n\pi)} + \frac{\cos(2-n\pi)-1}{2(2-n\pi)}, n \geq 0.$$

This ODE can be solved in the same manner as (0.2), with the coefficients $C_n(0)$ *explicitly determined by the initial condition as usual. I will omit the remaining calculations and plots. By now, however, you should be equipped to evaluate everything and assemble the desired series representation for this solution.*

3. We observe in class that the method of separation of variables can be applied to solve non-autonomous problems (i.e., the reaction term f does not depend on u), whereas the PDE $u_t = u_{xx} + f(u)$ is nonlinear and can not be solved explicitly in general. However, this method is applicable as long as the corresponding ODE is solvable. To see this, let us consider the following non-autonomous problem

$$\begin{cases} u_t = u_{xx} - \lambda u + \mu, & x \in (0, L), t \in \mathbb{R}^+, \\ u(x, 0) = \phi(x), & x \in (0, L), \\ u_x(0, t) = 0, u_x(L, t) = 0, & t \in \mathbb{R}^+, \end{cases} \tag{0.5}$$

where λ and μ are positive constants. Find the (pointwise) limit of $u(x, t)$ as $t \to \infty$? If you can not, do some numerical simulations by choosing $N = 10$ and choosing $\phi(x) = x$ or $1 + \cos \frac{\pi x}{L}$ to give you some intuitions.

Solution 3. *I assume that you are able to work this problem out by introducing w as suggested. Here in the following, I shall show that this is not necessary.*

The IBVP has an homogeneous NBC, therefore we can substitute its eigen–expansion

$$u(x, t) = \sum_{n=0}^{\infty} C_n(t) \cos \frac{n\pi x}{L}$$

into the PDE and collect that

$$\sum_{n=0}^{\infty} C_n'(t) \cos \frac{n\pi x}{L} = -\sum_{n=0}^{\infty} \left(\frac{n\pi}{L}\right)^2 C_n(t) \cos \frac{n\pi x}{L} - \lambda \sum_{n=0}^{\infty} C_n(t) \cos \frac{n\pi x}{L} + \mu.$$

Equating the coefficients gives us

$$C_n'(t) = -\Big(\big(\frac{n\pi}{L}\big)^2 + \lambda\Big) C_n(t) + \frac{2\mu}{L} \int_0^L \cos \frac{n\pi x}{L},$$

i.e.,

$$C_n'(t) = -\Big(\big(\frac{n\pi}{L}\big)^2 + \lambda\Big) C_n(t).$$

Solving this ODE with the initial condition, we have

$$C_0(t) = \frac{\mu}{\lambda}(1 - e^{-\lambda t}), C_n(t) = 0, n = 1, 2, ...$$

Therefore we have that

$$u(x, t) = C_0(t) = \frac{\mu}{\lambda}(1 - e^{-\lambda t}).$$

It is quick to see that $u(x, t) \to \frac{\mu}{\lambda}$ as $t \to \infty$.

Indeed, we can formally expect this solution without even solving the PDE. This IBVP describes a homogeneous well-insulated bar with zero temperature, therefore, there is no heat flux at all afterward, hence the PDE indeed should be an ODE. Solving the ODE gives the desired solution.

4. Solve the following non-autonomous problem with a different initial condition

$$\begin{cases} u_t = u_{xx} - \lambda u + \mu, & x \in (0, L), t \in \mathbb{R}^+, \\ u(x, 0) = \phi(x), & x \in (0, L), \\ u_x(0, t) = 0, u_x(L, t) = 0, & t \in \mathbb{R}^+, \end{cases} \tag{0.6}$$

where λ and μ are positive constants. Find the (pointwise) limit of $u(x, t)$ as $t \to \infty$? If you can not, do some numerical simulations by choosing $N = 10$ and choosing $\phi(x) = x$ or $1 + \cos \frac{\pi x}{L}$ to give you some intuitions.

Solution 4. *We have that, all the calculations leading to*

$$u(x,t)=\sum_{n=0}^{\infty} C_n(t)\cos\frac{n\pi x}{L}$$

hold with

$$C_n(t)=C_n(0)e^{-((\frac{n\pi}{L})^2+\lambda)t}, n=1,2,...$$

and $C_0(t)$ satisfies the ODE, except that we have to find C_n to cope with the general initial data $\phi(x)$. This easily implies that

$$C_n(0)=\frac{2}{L}\int_0^L \phi(x)\cos\frac{n\pi x}{L}dx.$$

Finally, one has that

$$u(x,t)=C_0(t)+\sum_{n=1}^{\infty} C_n(0)e^{-((\frac{n\pi}{L})^2+\lambda)t}\cos\frac{n\pi x}{L}$$

or precisely

$$u(x,t)=C_0(t)+\frac{2}{L}\sum_{n=1}^{\infty}\Big(\int_0^L \phi(x)\cos\frac{n\pi\xi}{L}\cos\frac{n\pi x}{L}d\xi\Big)e^{-((\frac{n\pi}{L})^2+\lambda)t},$$

Then, it is straightforward to show that the series converges and decays to zero exponentially. Consequently, $u(x,t)$ asymptotically behaves like $C_0(t)$ for large t. Solving the ODE for $C_0(t)$ yields the desired conclusion that $u(x,t)\rightarrow\frac{\mu}{\lambda}$, consistent with the result above. I will skip the plots for the case when $\phi\not\equiv 0$ here. However, you should perform some numerical experiments to verify that $u\rightarrow\frac{\mu}{\lambda}$ as expected.

5. Consider the following problem

$$\begin{cases}\Delta w+\lambda w=0, & x\in\Omega,\\ \alpha\frac{\partial u}{\partial \mathbf{n}}+\beta u=0, & x\in\partial\Omega,\end{cases}\tag{0.7}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n\geq 2$, and $\alpha^2+\beta^2\neq 0$. Prove that w_m and w_n, corresponding to λ_m and λ_n respectively, are orthogonal in $L^2(\Omega)$, whenever $\lambda_m\neq\lambda_n$.

Solution 5. *The proof is the same as that for its one-dimensional counterpart, except that one applies the divergence theorem (or Green's identities) here. I skip typing the details.*

6. Let us recall in class the following approach to solving a problem with inhomogeneous boundary conditions: to take care of the boundary conditions, one must first take care of the corresponding eigenvalue problem and find the corresponding eigenfunctions. In almost all applications we can see in a bounded domain, homogeneous DBC, NBC, and RBC are the three main types, whose EPs you have already studied in a previous HW problem. This is enough for this course, and it helps a lot if you know these eigenfunctions by heart, e.g. $\sin\frac{k\pi x}{L}$ for DBC, $\cos\frac{k\pi x}{L}$ for NBC, etc.

However, when the boundary conditions are not homogeneous, we need to convert the problem into one with homogeneous BC as mentioned in class. Consider

$$\begin{cases}u_t=Du_{xx}, & x\in(0,L),t\in\mathbb{R}^+,\\ u(x,0)=\phi(x), & x\in(0,L),\\ u(0,t)=\mu_1(t),u(L,t)=\mu_2(t), & t\in\mathbb{R}^+.\end{cases}\tag{0.8}$$

Note that the BC is DBC, but we can not simply write $u=\sum C_k(t)\sin\frac{k\pi x}{L}$ since the BC is not homogeneous (it is easy to check that this series can not satisfy the BC for general functions $\mu_i(t)$). Therefore, we introduce $\tilde{u}(x,t)=u(x,t)+w(x,t)$ for some specific w to be chosen such that $\tilde{u}$ satisfies the homogeneous DBC, i.e., $\tilde{u}(0,t)=\tilde{u}(L,t)=0$ for any t. It is easy to see that we must restrict w such that $w(0,t) = -\mu_1(t)$ and $w(L,t) = -\mu_2(t)$. A natural choice of such w is $w(x,t) = \frac{x-L}{L}\mu_1(t) - \frac{x}{L}\mu_2(t)$.

i) work the problem for $\tilde{u}(x,t)$ as in class and then finish solving for $u(x,t)$ in terms of infinite series;

ii) again, the choice of such w is not unique; can you give an explicit form of another such $w(x,t)$? The motivation behind this is that there are a lot of such choices, but it seems the one that I proposed is the simplest (I would be happy to see that I am wrong).

Solution 6. *I believe I covered almost all the calculations in class. To complete the work, you need to recover u from $\tilde{u}$.*

7. Let us revisit the following IBVP

$$\begin{cases} u_t = u_{xx}, & x \in (0,\pi), t \in \mathbb{R}^+, \\ u(x,0) = x, & x \in (0,\pi), \\ u(0,t) = \sin t, u(L,t) = \cos t, & t \in \mathbb{R}^+. \end{cases} \tag{0.9}$$

This problem is to numerically test that the solution is independent of the choice of $w(x,t)$.

1) choose the first transformation function as above $w^{(1)}(x,t) = \frac{\pi-x}{\pi}\sin t + \frac{x}{\pi}\cos t$, and then find the solution, denoted by $u^{(1)}(x,t)$, in terms of infinite series;

2) pick an alternative $w^{(2)}(x,t)$ of your own choice and then find the corresponding $u^{(2)}(x,t)$;

3) plot $u^{(1)}(x,t)$ and $u^{(2)}(x,t)$ with truncated N for several t, say $N = 10$, probably you want to test first that $N = 10$ is large enough as previous HWs. Then show that $u^{(1)}(x,t)$ and $u^{(2)}(x,t)$ are the same for all time;

4) try to prove that $u^{(1)}(x,t)$ and $u^{(2)}(x,t)$ equal analytically.

Solution 7. *Skipped.*

8. Let us work on the following cousin problem of (0.8)

$$\begin{cases} u_t = Du_{xx}, & x \in (0,L), t \in \mathbb{R}^+, \\ u(x,0) = \phi(x), & x \in (0,L), \\ u_x(0,t) = \mu_1(t), u_x(L,t) = \mu_2(t), & t \in \mathbb{R}^+. \end{cases} \tag{0.10}$$

First convert this problem into one with homogeneous NBC. Then solve the resulting problem and then write $u(x,t)$ in infinite series.

Solution 8. *Skipped.*

9. Solve the following IBVP

$$\begin{cases} u_t = Du_{xx} + x - \pi, & x \in (0,\pi), t \in \mathbb{R}^+, \\ u(x,0) = \pi - x, & x \in (0,\pi), \\ u(0,t) = \pi, u(\pi,t) = 0, & t \in \mathbb{R}^+; \end{cases} \tag{0.11}$$

Let $D = 1$. Then plot u^{10} at $t = 1, 2, 5, 10$ etc to illustrate the large time behavior of the solution.

Solution 9. *First of all, we recognize this as an IBVP with inhomogeneous BC, hence we convert the inhomogeneous boundary condition into homogeneous by letting*

$$u(x,t) = U(x,t) + w(x,t), w(x,t) = \pi - x.$$

Then we can find that $U(x,t)$ satisfies the following system

$$\begin{cases} U_t = DU_{xx} + x - \pi, & x \in (0,\pi), t > 0 \\ U(x,0) = 0, & x \in (0,\pi), \\ U(0,t) = 0, U(\pi,t) = 0, & t > 0 \end{cases} \tag{0.12}$$

Let $U_n(x,t) = X_n(x)T_n(t)$. Then we have that X_n is an eigen-function of the following problem

$$\begin{cases} X'' + \lambda X = 0, & x \in (0,\pi), \\ X(x) = 0, & X = 0, L. \end{cases} \tag{0.13}$$

and we already know that

$$X_n = \sin\frac{n\pi x}{L} = \sin nx, \lambda_n = \left(\frac{n\pi}{L}\right)^2, n = 1,2,...$$

Then the solution U has the form

$$U(x,t) = \sum_{n=1}^{\infty} C_n(t)\sin nx.$$

Substituting it into the PDE gives us

$$\sum_{n=1}^{\infty} C_n'(t)\sin nx = -\sum_{n=1}^{\infty} Dn^2 C_n(t)\sin nx + (x-\pi).$$

Multiplying BHS by $\sin nx$ and then integrating it over $(0,\pi)$, we have

$$C_n'(t) = -Dn^2C_n(t) + \frac{2}{\pi}\int_0^{\pi}(x-\pi)\sin nx dx$$

and straightforward calculations give us

$$\frac{2}{\pi}\int_0^{\pi}(x-\pi)\sin nx dx = -\frac{2}{n};$$

on the other hand, we can also find from the initial condition that $C_n(0) = 0$, therefore we collect

$$\begin{cases} C_n'(t) = -Dn^2C_n(t) - \frac{2}{n}, & t > 0, \\ C_n(0) = 0 & . \end{cases} \tag{0.14}$$

Solving this ODE gives us

$$C_n(t) = \frac{2}{Dn^3}\left(e^{-n^2Dt} - 1\right), n = 1,2,..$$

hence

$$U(x,t) = \sum_{n=1}^{\infty}\frac{2}{Dn^3}\left(e^{-n^2Dt} - 1\right)\sin nx.$$

Finally we obtain that

$$u(x,t) = U(x,t) + w(x,t) = \pi - x + \sum_{n=1}^{\infty}\frac{2}{Dn^3}\left(e^{-n^2Dt} - 1\right)\sin nx.$$

I would like to add a few more remarks here. As $t \to \infty$, one expects that $u(x,t)$ converges to a function that depends on x but no longer on time t (since the limit of t is taken). This function, or solution, is called the stationary or steady state of the problem, as it remains unchanged over time. To study the steady state, we set $u_t = 0$ and obtain the equation $u_{xx} + x - \pi = 0$ in the interval $(0,\pi)$. Solving this problem gives us the following solution:

$$u(x) = -\frac{x^3}{6} + \frac{\pi x^2}{2} + C_1x + C_2$$

for some constants C_i to be determined, we apply the boundary conditions to solve for these constants. After solving, we find that $C_1 = -\left(\frac{\pi^2}{3} + 1\right)$ and $C_2 = \pi$. It is important to note that the initial condition does not matter in this case, because regardless of the initial data, the system will eventually converge to the steady state $u(x)$ as described above.

I would also like to mention that for this particular problem, the steady state is quite simple, as it is unique and can be explicitly obtained. However, for more general equations, particularly in systems of equations, the steady state may not be unique. Different initial data could lead to different steady states.

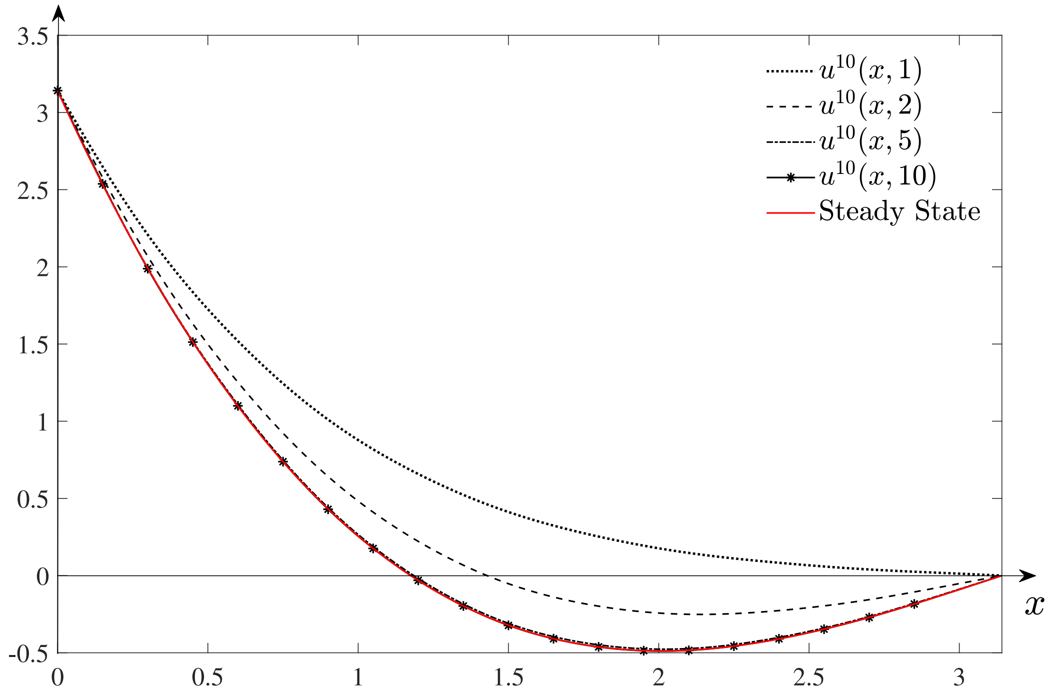

Figure 2: The evolution of the approximated solutions at $t = 1, 2, 5, 10$ is shown. We observe that $u(x,t)$ for long times ($t \geq 5$) agrees well with the explicit steady state obtained in the solution. It is important to verify that $N = 10$ is sufficiently large to extract a reasonably accurate solution for the IBVP.

10. Consider the following IBVP

$$\begin{cases} u_t = u_{xx}, & x \in (0,1), t > 0, \\ u(x,0) = 0, x \in (0, \frac{1}{2}); u(x,0) = 1, x \in [\frac{1}{2}, 1], & \\ u(0,t) = 1, u(1,t) = 2, & t > 0. \end{cases} \tag{0.15}$$

(i). Without solving the problem, state what is the solution or shape of $u(x,t)$ as $t \to \infty$. Hint: use your physical intuition. It should be a function independent of time t and we call it a *steady state*.

(ii). Solve (0.15) in terms of infinite series. Plot $u^{10}(x,t)$ for $t = 0, t = 0.01$, $t = 0.05$ and $t = 0.1$ over $(0,1)$ on the same graph. You should observe that the initially discontinuity at $x = 0$ is smeared out right away. Then plot $u^{10}(x,t)$ for $t = 10$ and the steady state in (i) on the same graph.

(iii). Send t to ∞ to rigorously confirm your observations in (ii).

Solution 10. *(i). We recognize this as a heat equation with no external source and Dirichlet boundary conditions (DBC). In the long time limit, the steady state $u(x)$ becomes time-independent, which means that $u_t = 0$. Thus, the equation reduces to $u_{xx} = 0$. The solution to this is a straight line connecting $u(0) = 1$ and $u(1) = 2$, which gives the steady state $u(x) = 1 + x$.*

(ii) first of all, we convert the inhomogeneous DBC into a homogeneous one by introduce the new notation

$$\tilde{u} := u - (x+1).$$

Then $\tilde{u}$ satisfies

$$\begin{cases} \tilde{u}_t = \tilde{u}_{xx}, & x \in (0,1), t > 0, \\ \tilde{u}(x,0) = -(x+1), x \in (0, \frac{1}{2}); \tilde{u}(x,0) = -x, x \in [\frac{1}{2}, 1], & \\ \tilde{u}(0,t) = \tilde{u}(1,t) = 0, & t > 0. \end{cases}$$

The solution takes the following form as we know in class

$$\tilde{u} = \sum_{k=1}^{\infty} C_k e^{-k^2 t} \sin k\pi x,$$

where the coefficients are determined by

$$C_k = 2\int_0^1 u(x,0)\sin k\pi x dx = 2\Big(\int_0^{\frac{1}{2}} (-(x{+}1))\sin k\pi x dx + \int_{\frac{1}{2}}^1 (-x)\sin k\pi x dx\Big) = \frac{2((-1)^k-1)}{k\pi}, k\in\mathbb{N}^+.$$

Therefore we finally collect

$$u(x,t) = x+1+\sum_{k=1}^{\infty}\frac{2((-1)^k-1)}{k\pi}e^{-k^2 t}\sin k\pi x.$$

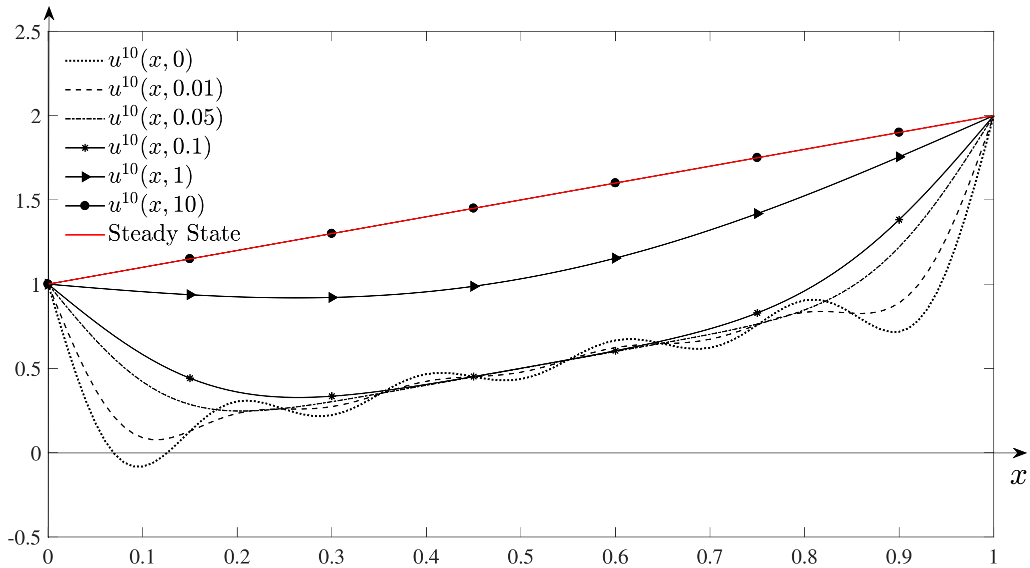

Figure 3: The evolution of the approximated solutions at $t = 0, 0.01, 0.05, 0.1$, and 10 is shown. We observe that $u(x,t)$ converges to the steady state $u(x) = x+1$ as time progresses. Advanced students might note that $u^{10}(x)$ is not the same as the initial condition $u_0(x)$, and this is because the series does not converge to the ”correct” step function. It's important to highlight that the convergence is only in the L^2 sense, not uniform convergence. This means that while the solution approaches the steady state in terms of its L^2 norm, the approximation may not converge uniformly to the exact step function.

I assume that you are able to verify that as $t\to\infty$

$$\Big|\sum_{k=1}^{\infty}\frac{2((-1)^k-1)}{k\pi}e^{-k^2t}\sin k\pi x\Big| \le \Big|\sum_{k=1}^{\infty}\frac{2((-1)^k-1)}{k\pi}e^{-k^2t}\Big| \to 0.$$

This readily implies that $u(x,t)\to x+1$ *as* $t\to\infty$*. However, we expect pointwise convergence, but not uniform convergence, to the initial data as* $t\to 0^+$*, as noted in the caption.*

11. Let us consider cooking a meatball in a boiling hot pot. Assumptions we make are:

    A1.) *the meatball is perfectly round with a radius* R;

    A2.) *the meatball is solid and homogeneous*;

    A3.) *the ball is well dipped into the water that is boiling at a constant temperature* (say 100°C;)

    Suppose that the meatball is of uniform temperature initially, say 25°C. We say that a meatball is cooked if the temperature of its center reaches a certain value, say 70 °C (the value itself does not matter much for this problem). Ask your parents or whoever that cooks the following questions (just use the gut feeling):

    i) suppose that it takes 10 mins to cook a meatball of weight 50g. How much time does it take to cook a meatball of weight 100g under the same condition? shorter than, is, or longer than 10 mins?

    ii) suppose that it takes 10 mins to cook a meatball of radius 1cm. How much time does it take to cook a meatball of radius 2cm under the same condition? shorter than, is, or longer than 10 mins?

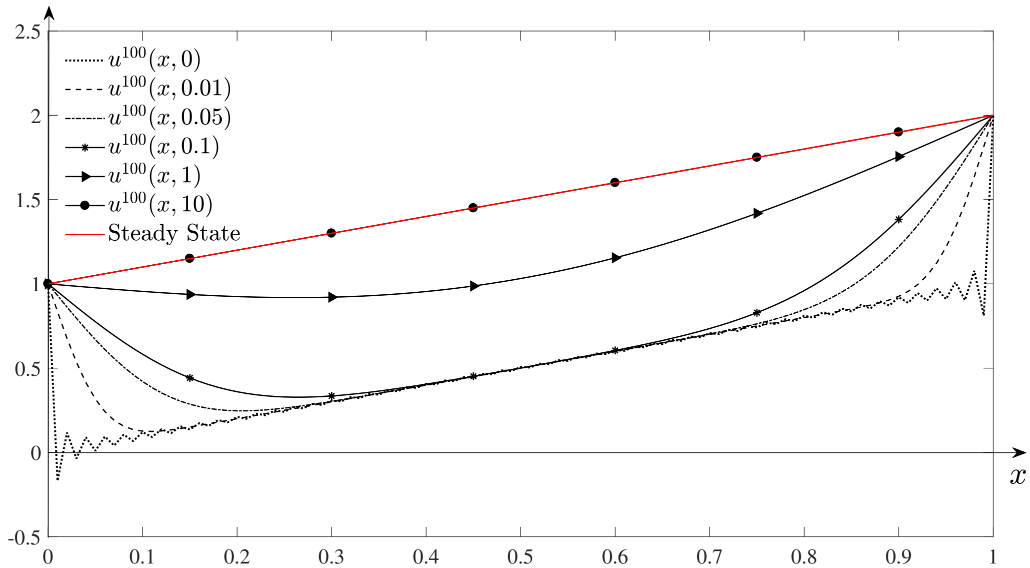

Figure 4: We note that $N = 10$ is not sufficient to recover the initial data. If we choose the truncation term $N = 100$, then we expect the same spatiotemporal dynamics as the IBVP, suggesting that $N = 10$ is a reasonable approximation. However, it does not recover the convergence to the initial data. Indeed, a Gibbs phenomenon is observed for $t = 0$, which again highlights that the series does not converge to the "correct" step function. It is important to note that the convergence is only in the L^2 sense, not uniform convergence. Nonetheless, as long as the long-time dynamics are of concern, the finite series provides a reasonably accurate approximation.

Solution 11. *It would indeed take a longer time than 10 minutes to cook the meatball in both scenarios. Intuitively, one might think of the meatball with a radius of 2 cm as consisting of an inner ball with a radius of 1 cm and its outer coating. It takes 10 minutes to cook the inner ball, while an additional time is required to cook the coating, which is 1 cm thick. Mathematically, the first eigenvalue of the differential operator* $\frac{d^2}{dx^2}$ *is inversely proportional to the length of the interval, L. In the case of a ball, this eigenvalue reflects the rate of heat diffusion, and for larger balls, the eigenvalue becomes smaller, indicating that the convergence to the outside temperature is slower. This conclusion holds true in higher dimensions as well, where the eigenvalue decreases with increasing radius, leading to slower thermal equilibration.*

12. Now we PDE the hotpot. Let $u(x,t)$ be the temperature of a meatball at location $x = (x_1, x_2, x_3)$ and time t, then its cooking follows

$$\begin{cases} u_t = D\Delta u, & x \in B_0(R), t \in \mathbb{R}^+, \\ u(x,0) = 25°\mathrm{C}, & x \in B_0(R), \\ u(x,t) = 100°\mathrm{C}, & x \in \partial B_0(R), t \in \mathbb{R}^+. \end{cases} \tag{0.16}$$

We recognize (0.16) as a homogeneous heat equation with inhomogeneous DBC. Let us now try to solve this problem. Without loss of generality, we fix the center of the meatball as the origin.

i) now that the meatball is homogeneous, it is not hard to imagine that each layer has the same temperature with $u(x,t) = u(r,t)$, $r = |x|$. Show that (0.16) becomes

$$\begin{cases} u_t = D(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}), & r \in (0,R), t \in \mathbb{R}^+, \\ u(x,0) = 25°\mathrm{C}, & r \in [0,R), \\ u(x,t) = 100°\mathrm{C}, & r = R, t \in \mathbb{R}^+. \end{cases} \tag{0.17}$$

ii) we can convert (0.17) into a problem with homogeneous DBC by introducing $\mathbb{U} := u - 100°\mathrm{C}$

$$\begin{cases} \mathbb{U}_t = D(\frac{\partial^2 \mathbb{U}}{\partial r^2} + \frac{2}{r}\frac{\partial \mathbb{U}}{\partial r}), & r \in (0,R), t \in \mathbb{R}^+, \\ \mathbb{U}(x,0) = -75°\mathrm{C}, & r \in [0,R), \\ \mathbb{U}(x,t) = 0°\mathrm{C}, & r = R, t \in \mathbb{R}^+. \end{cases} \tag{0.18}$$

To solve (0.18), let us apply the separation of variables by first writing $\mathbb{U} = \mathbb{R}(r)\mathbb{T}(t)$. Denote $\tilde{\mathbb{R}} := r\mathbb{R}$. Show that $\tilde{\mathbb{R}} = \sin\frac{k\pi r}{R}$. Hint:$\mathbb{R}(0) < \infty$.

iii) proceed to solve for u. *Suggested answer*:

$$u = 100 + \sum_{k=1}^{\infty} \frac{C_k}{r} \sin\frac{k\pi r}{R} e^{-D(\frac{k\pi}{R})^2 t}, C_k = ?$$

iv) choose $D = 0.01$ and $R = 1$, then plot the approximate temperature at the center $u^{10}(0,t)$ for $t \in (0,100)$; (we are sloppy with the units, but this gives you an idea how the temperature takes off after a certain time).

Solution 12. *i) By straightforward calculations, we find that for any radially symmetric function f, its Laplacian is given by*

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r}.$$

Indeed, this identity holds in $\mathbb{R}^n$, with $\frac{2}{r}$ replaced by $\frac{n-1}{r}$. I will omit the detailed derivation here.

ii) It is straightforward to verify that $\mathbb{U}$, defined by the shift, satisfies the PDE, initial condition, and boundary condition. With the new boundary condition now homogeneous, we use the separation of variables method, writing $\mathbb{U} = \mathbb{R}(r)\mathbb{T}(t)$. Substituting this into the PDE yields:

$$\mathbb{R}\mathbb{T}' = D\left(\mathbb{R}''\mathbb{T} + \frac{2}{r}\mathbb{R}'\mathbb{T}\right).$$

Dividing through by $\mathbb{R}(r)\mathbb{T}(t)$, we obtain:

$$\frac{\mathbb{T}'}{\mathbb{T}} = D\frac{\mathbb{R}'' + \frac{2}{r}\mathbb{R}'}{\mathbb{R}} = \lambda,$$

where λ is a constant to be determined. For $\tilde{\mathbb{R}} = r\mathbb{R}$, straightforward calculations yield

$$\tilde{\mathbb{R}}'' = r\mathbb{R}'' + 2\mathbb{R}.$$

Using the earlier equation for $\mathbb{R}$, this implies

$$\tilde{\mathbb{R}}'' = \lambda\tilde{\mathbb{R}}, \quad r \in (0,R).$$

From the boundary conditions, it is clear that $\tilde{\mathbb{R}}(R) = 0$. Additionally, $\tilde{\mathbb{R}}(0) = 0$ because $\mathbb{R}$ is bounded and $r \to 0$. Thus, $\tilde{\mathbb{R}}$ satisfies a Dirichlet boundary condition (DBC) on $(0,R)$.

The solution must then take the form

$$\tilde{\mathbb{R}}(r) = \sin\left(\frac{k\pi r}{R}\right),$$

where $\lambda_k = -\left(\frac{k\pi}{R}\right)^2$ for $k = 1, 2, \ldots$.

iii) According to (ii), the solution can be expressed as:

$$\mathbb{U} = \sum_{k=1}^{\infty} \frac{C_k}{r} \sin\left(\frac{k\pi r}{R}\right) e^{-D(\frac{k\pi}{R})^2 t},$$

where the coefficients C_k are determined by the initial condition:

$$\sum_{k=1}^{\infty} \frac{C_k}{r} \sin\left(\frac{k\pi r}{R}\right) = -75.$$

Finding C_k directly is tricky because the term $\frac{1}{r}$ disrupts the usual orthogonality of sine functions. To address this, we rewrite the equation by multiplying through by r:

$$\sum_{k=1}^{\infty} C_k \sin\left(\frac{k\pi r}{R}\right) = -75r.$$

Now the coefficients C_k *can be determined using the standard orthogonality property of sine functions on the interval* $(0, R)$*:*

$$C_k = \frac{2}{R}\int_0^R (-75r)\sin\left(\frac{k\pi r}{R}\right)\, dr.$$

Testing this new identity by $\sin\frac{k\pi r}{R}$ *over* $(0, R)$ *gives us*

$$C_k = \frac{2}{R}\int_0^R (-75)r\sin\frac{k\pi r}{R}dr = \frac{150R}{k\pi}(-1)^k, k \in \mathbb{N}^+.$$

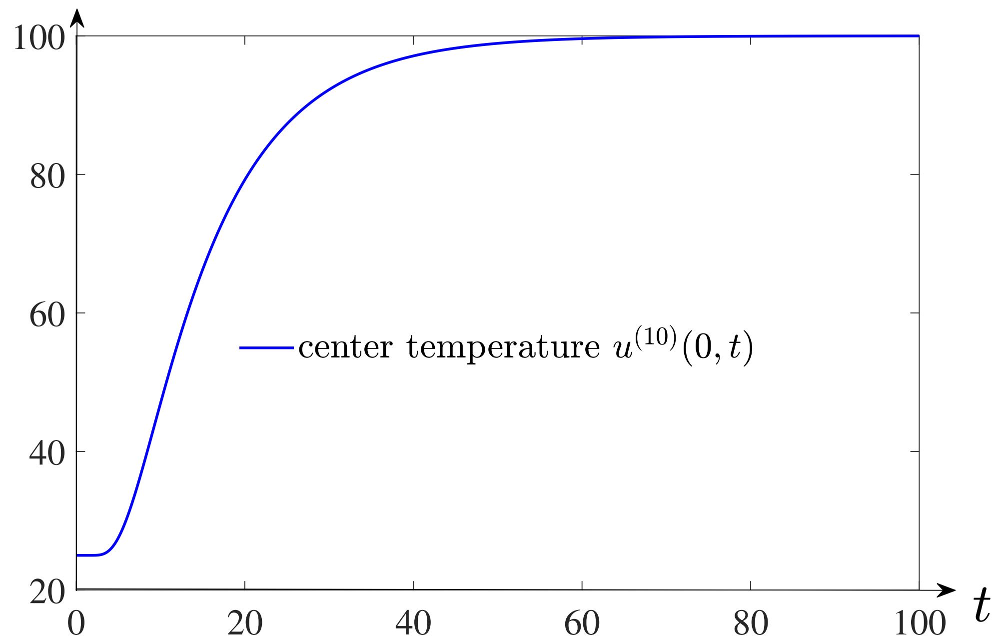

Figure 5: **Evolution of the Temperature at the Center**. This chart depicts the temperature changes at the center of the meatball over time. It can be used to estimate the time required to cook the center to a specific target temperature.