Introduction to PDEs, Fall 2024

Homework 6 due Nov 21

Name:_______________

1. Solve for $u(x,t)$ in infinite series for the following problem under DBC

$$\begin{cases} u_t = u_{xx} + e^{-t}\sin 2x, & x \in (0,1), t \in \mathbb{R}^+, \\ u(x,0) = x, & x \in (0,1), \\ u = 0, & x = 0,1, t \in \mathbb{R}^+. \end{cases} \tag{0.1}$$

2. Solve for $u(x,t)$ in infinite series for the following problem under NBC

$$\begin{cases} u_t = u_{xx} + e^{-t}\sin 2x, & x \in (0,1), t \in \mathbb{R}^+, \\ u(x,0) = x, & x \in (0,1), \\ u_x = 0, & x = 0,1, t \in \mathbb{R}^+. \end{cases} \tag{0.2}$$

3. We observe in class that the method of separation of variables can be applied to solve non-autonomous problems (i.e., the reaction term f does not depend on u), whereas the PDE $u_t = u_{xx} + f(u)$ is nonlinear and can not be solved explicitly in general. However, this method is applicable as long as the corresponding ODE is solvable. To see this, let us consider the following non-autonomous problem

$$\begin{cases} u_t = u_{xx} - \lambda u + \mu, & x \in (0,L), t \in \mathbb{R}^+, \\ u(x,0) = \phi(x), & x \in (0,L), \\ u_x(0,t) = 0, u_x(L,t) = 0, & t \in \mathbb{R}^+, \end{cases} \tag{0.3}$$

where λ and μ are positive constants. Find the (pointwise) limit of $u(x,t)$ as $t \to \infty$? If you can not, do some numerical simulations by choosing $N = 10$ and choosing $\phi(x) = x$ or $1 + \cos\frac{\pi x}{L}$ to give you some intuitions.

4. Consider the following problem

$$\begin{cases} \Delta w + \lambda w = 0, & x \in \Omega, \\ \alpha\frac{\partial u}{\partial \mathbf{n}} + \beta u = 0, & x \in \partial\Omega, \end{cases} \tag{0.4}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 2$, and $\alpha^2 + \beta^2 \neq 0$. Prove that w_m and w_n, corresponding to λ_m and λ_n respectively, are orthogonal in $L^2(\Omega)$, whenever $\lambda_m \neq \lambda_n$.

5. Let us recall in class the following approach to solving a problem with inhomogeneous boundary conditions: to take care of the boundary conditions, one must first take care of the corresponding eigenvalue problem and find the corresponding eigenfunctions. In almost all applications we can see in a bounded domain, homogeneous DBC, NBC, and RBC are the three main types, whose EPs you have already studied in a previous HW problem. This is enough for this course, and it helps a lot if you know these eigenfunctions by heart, e.g. $\sin\frac{k\pi x}{L}$ for DBC, $\cos\frac{k\pi x}{L}$ for NBC, etc.

   However, when the boundary conditions are not homogeneous, we need to convert the problem into one with homogeneous BC as mentioned in class. Consider

$$\begin{cases} u_t = Du_{xx}, & x \in (0,L), t \in \mathbb{R}^+, \\ u(x,0) = \phi(x), & x \in (0,L), \\ u(0,t) = \mu_1(t), u(L,t) = \mu_2(t), & t \in \mathbb{R}^+. \end{cases} \tag{0.5}$$

Note that the BC is DBC, but we can not simply write $u = \sum C_k(t) \sin \frac{k\pi x}{L}$ since the BC is not homogeneous (it is easy to check that this series can not satisfy the BC for general functions $\mu_i(t)$). Therefore, we introduce $\tilde{u}(x,t) = u(x,t) + w(x,t)$ for some specific w to be chosen such that $\tilde{u}$ satisfies the homogeneous DBC, i.e., $\tilde{u}(0,t) = \tilde{u}(L,t) = 0$ for any t. It is easy to see that we must restrict w such that $w(0,t) = -\mu_1(t)$ and $w(L,t) = -\mu_2(t)$. A natural choice of such w is $w(x,t) = \frac{x-L}{L}\mu_1(t) - \frac{x}{L}\mu_2(t)$.

i) work the problem for $\tilde{u}(x,t)$ as in class and then finish solving for $u(x,t)$ in terms of infinite series;

ii) again, the choice of such w is not unique; can you give an explicit form of another such $w(x,t)$? The motivation behind this is that there are a lot of such choices, but it seems the one that I proposed is the simplest (I would be happy to see that I am wrong).

6. Let us revisit the following IBVP

$$\begin{cases} u_t = u_{xx}, & x \in (0,\pi), t \in \mathbb{R}^+, \\ u(x,0) = x, & x \in (0,\pi), \\ u(0,t) = \sin t, u(L,t) = \cos t, & t \in \mathbb{R}^+. \end{cases} \tag{0.6}$$

This problem is to numerically test that the solution is independent of the choice of $w(x,t)$.

1) choose the first transformation function as above $w^{(1)}(x,t) = \frac{\pi - x}{\pi}\sin t + \frac{x}{\pi}\cos t$, and then find the solution, denoted by $u^{(1)}(x,t)$, in terms of infinite series;

2) pick an alternative $w^{(2)}(x,t)$ of your own choice and then find the corresponding $u^{(2)}(x,t)$;

3) plot $u^{(1)}(x,t)$ and $u^{(2)}(x,t)$ with truncated N for several t, say $N = 10$, probably you want to test first that $N = 10$ is large enough as previous HWs. Then show that $u^{(1)}(x,t)$ and $u^{(2)}(x,t)$ are the same for all time;

4) try to prove that $u^{(1)}(x,t)$ and $u^{(2)}(x,t)$ equal analytically.

7. Let us work on the following cousin problem of (0.5)

$$\begin{cases} u_t = Du_{xx}, & x \in (0,L), t \in \mathbb{R}^+, \\ u(x,0) = \phi(x), & x \in (0,L), \\ u_x(0,t) = \mu_1(t), u_x(L,t) = \mu_2(t), & t \in \mathbb{R}^+. \end{cases} \tag{0.7}$$

First convert this problem into one with homogeneous NBC. Then solve the resulting problem and then write $u(x,t)$ in infinite series.

8. Solve the following IBVP

$$\begin{cases} u_t = Du_{xx} + x - \pi, & x \in (0,\pi), t \in \mathbb{R}^+, \\ u(x,0) = \pi - x, & x \in (0,\pi), \\ u(0,t) = \pi, u(\pi,t) = 0, & t \in \mathbb{R}^+; \end{cases} \tag{0.8}$$

Let $D = 1$. Then plot u^{10} at $t = 1, 2, 5, 10$ etc to illustrate the large time behavior of the solution.

9. Consider the following IBVP

$$\begin{cases} u_t = u_{xx}, & x \in (0,1), t > 0, \\ u(x,0) = 0, x \in (0, \frac{1}{2}); u(x,0) = 1, x \in [\frac{1}{2}, 1], & \\ u(0,t) = 1, u(1,t) = 2, & t > 0. \end{cases} \tag{0.9}$$

(i). Without solving the problem, state what is the solution or shape of $u(x,t)$ as $t \to \infty$. Hint: use your physical intuition. It should be a function independent of time t and we call it a *steady state*.

(ii). Solve (0.9) in terms of infinite series. Plot $u^{10}(x,t)$ for $t = 0, t = 0.01$, $t = 0.05$ and $t = 0.1$ over $(0,1)$ on the same graph. You should observe that the initially discontinuity at $x = 0$ is smeared out right away. Then plot $u^{10}(x,t)$ for $t = 10$ and the steady state in (i) on the same graph.

(iii). Send t to ∞ to rigorously confirm your observations in (ii).

10. Let us consider cooking a meatball in a boiling hot pot. Assumptions we make are:

    A1.) *the meatball is perfectly round with a radius* R;

    A2.) *the meatball is solid and homogeneous*;

    A3.) *the ball is well dipped into the water that is boiling at a constant temperature* (say 100°C;)

    Suppose that the meatball is of uniform temperature initially, say 25°C. We say that a meatball is cooked if the temperature of its center reaches a certain value, say 70 °C (the value itself does not matter much for this problem). Ask your parents or whoever that cooks the following questions (just use the gut feeling):

    i) suppose that it takes 10 mins to cook a meatball of weight 50g. How much time does it take to cook a meatball of weight 100g under the same condition? shorter than, is, or longer than 10 mins?

    ii) suppose that it takes 10 mins to cook a meatball of radius 1cm. How much time does it take to cook a meatball of radius 2cm under the same condition? shorter than, is, or longer than 10 mins?

11. Now we PDE the hotpot. Let $u(x,t)$ be the temperature of a meatball at location $x = (x_1, x_2, x_3)$ and time t, then its cooking follows

$$\begin{cases} u_t = D\Delta u, & x \in B_0(R), t \in \mathbb{R}^+, \\ u(x,0) = 25°\text{C}, & x \in B_0(R), \\ u(x,t) = 100°\text{C}, & x \in \partial B_0(R), t \in \mathbb{R}^+. \end{cases} \tag{0.10}$$

We recognize (0.10) as a homogeneous heat equation with inhomogeneous DBC. Let us now try to solve this problem. Without loss of generality, we fix the center of the meatball as the origin.

i) now that the meatball is homogeneous, it is not hard to imagine that each layer has the same temperature with $u(x,t) = u(r,t)$, $r = |x|$. Show that (0.10) becomes

$$\begin{cases} u_t = D(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}), & r \in (0,R), t \in \mathbb{R}^+, \\ u(x,0) = 25°\text{C}, & r \in [0,R), \\ u(x,t) = 100°\text{C}, & r = R, t \in \mathbb{R}^+. \end{cases} \tag{0.11}$$

ii) we can convert (0.11) into a problem with homogeneous DBC by introducing $\mathbb{U} := u - 100°\text{C}$

$$\begin{cases} \mathbb{U}_t = D(\frac{\partial^2 \mathbb{U}}{\partial r^2} + \frac{2}{r}\frac{\partial \mathbb{U}}{\partial r}), & r \in (0,R), t \in \mathbb{R}^+, \\ \mathbb{U}(x,0) = -75°\text{C}, & r \in [0,R), \\ \mathbb{U}(x,t) = 0°\text{C}, & r = R, t \in \mathbb{R}^+. \end{cases} \tag{0.12}$$

To solve (0.12), let us apply the separation of variables by first writing $\mathbb{U} = \mathbb{R}(r)\mathbb{T}(t)$. Denote $\tilde{\mathbb{R}} := r\mathbb{R}$. Show that $\tilde{\mathbb{R}} = \sin\frac{k\pi r}{R}$. Hint:$\mathbb{R}(0) < \infty$.

iii) proceed to solve for u. *Suggested answer*:

$$u = 100 + \sum_{k=1}^{\infty} \frac{C_k}{r} \sin\frac{k\pi r}{R} e^{-D(\frac{k\pi}{R})^2 t}, C_k =?$$

iv) choose $D = 0.01$ and $R = 1$, then plot the approximate temperature at the center $u^{10}(0,t)$ for $t \in (0,100)$; (we are sloppy with the units, but this gives you an idea how the temperature takes off after a certain time).