Introduction to PDEs, Fall 2024

Homework 3 solutions

Name:_______________

(...continued from lecture) Let us reconsider the following reaction-diffusion system with the Dirichlet boundary condition

$$\begin{cases} u_t = D\Delta u + f(x,t), & x \in \Omega, t > 0, \\ u(x,0) = \phi(x), & x \in \Omega, \\ u(x,t) = \psi(x,t), & x \in \partial\Omega, t > 0. \end{cases} \tag{0.1}$$

Note that proving the uniqueness of (0.1) is equivalent to demonstrating that the following problem has only the trivial solution $w(x,t) \equiv 0$

$$\begin{cases} w_t = D\Delta w, & x \in \Omega, t > 0, \\ w(x,0) = 0, & x \in \Omega, \\ w(x,t) = 0, & x \in \partial\Omega, t > 0. \end{cases} \tag{0.2}$$

For this purpose, we introduced

$$E(t) = \int_\Omega w^2(x,t)dx, t \geq 0,$$

then we must have that from the initial condition that

$$E(0) = 0, \quad E(t) \geq 0 \quad \forall t \geq 0;$$

Now we differentiate $E(t)$ with respect to time t and obtain the following

$$\begin{aligned} \frac{dE(t)}{dt} =& 2\int_\Omega w w_t dx \quad --\text{by chain rule} \\ =& 2\int_\Omega w D\Delta w dx \quad --\text{by PDE} \\ =& -2D\int_\Omega |\nabla w|^2 dx \quad --\text{by divergence theorem} \\ \leq& 0 \quad \forall t \geq 0. \end{aligned} \tag{0.3}$$

This implies that $E(t)$ is always non-increasing over time. Since $E(0) = 0$ and $E(t) \geq 0$, we must have $E(t) \equiv 0$, which leads to $w \equiv 0$, as expected.

In class, I applied (0.3) without further justification, assuming you have encountered this in multivariate calculus. If that assumption was incorrect, let us recall the divergence theorem for a generic vector function $\mathbf{F}$

$$\int_\Omega \nabla \cdot \mathbf{F} dx = \int_{\partial\Omega} \mathbf{F} \cdot \mathbf{n} dS.$$

Choose $\mathbf{F} = f\nabla g$, and use the identity $\nabla f \cdot \nabla g + f\Delta g = \nabla \cdot (f\nabla g)$, where Δ is the Laplacian. We integrate both sides over Ω and then apply the divergence theorem to obtain:

$$\int_\Omega \nabla f \cdot \nabla g + f\Delta g dx = \int_\Omega \nabla \cdot (f\nabla g) dx = \int_{\partial\Omega} f\nabla g \cdot \mathbf{n} dS = \int_{\partial\Omega} f\partial_{\mathbf{n}} g dS.$$

Note that the last identity holds only because it is an equivalent way of expressing $\nabla g \cdot \mathbf{n}$ using the directional derivative $\partial_{\mathbf{n}} g$ (which I assume you know). For practical purposes, we sometimes swap the two terms in the above identity and rewrite it as follows:

$$\int_\Omega f \Delta g dx = \int_{\partial\Omega} f \partial_{\mathbf{n}} g dS - \int_\Omega \nabla f \cdot \nabla g dx, \tag{0.4}$$

and this new identity is called Green's first identity.

Finally, if we choose $f = g = w$ in (0.4), we readily collect (0.3) since

$$\int_\Omega w \Delta w dx = \overbrace{\int_{\partial\Omega} w \partial_{\mathbf{n}} w dS}^{=0 \text{ since } w \equiv 0 \text{ on} \partial\Omega} - \int_\Omega |\nabla w|^2 dx = -\int_\Omega |\nabla w|^2 dx.$$

I hope this has cleared up any possible confusion.

1. Now, use Green's first identity to show that

$$\int_\Omega f \Delta g - \Delta f g dx = \int_{\partial\Omega} f \frac{\partial g}{\partial \mathbf{n}} - \frac{\partial f}{\partial \mathbf{n}} g dS. \tag{0.5}$$

(0.5) is called Green's second identity. What is (0.5) when $\Omega = (a, b)$?

Solution 1. *Let us recall the Green's first identity that*

$$\int_\Omega f \Delta g dx + \int_\Omega \nabla f \cdot \nabla g dx = \int_{\partial\Omega} f \frac{\partial g}{\partial \boldsymbol{n}} dS, \tag{0.6}$$

which switching f and g in (0.6) gives us

$$\int_\Omega g \Delta f dx + \int_\Omega \nabla g \cdot \nabla f dx = \int_{\partial\Omega} g \frac{\partial f}{\partial \boldsymbol{n}} dS. \tag{0.7}$$

Then subtracting (0.7) from (0.6) readily gives us (0.5).

When $\Omega = (a, b)$, (0.5) becomes

$$\int_a^b (fg'' - f''g) dx = (fg' - f'g)|_a^b.$$

PS: Is this called Newton's second identity?

2. Consider the following reaction-diffusion equation with Robin boundary condition

$$\begin{cases} u_t = D\Delta u + f(x, t), & x \in \Omega, t > 0, \\ u(x, 0) = \phi(x), & x \in \Omega, \\ \alpha u + \beta \frac{\partial u}{\partial \mathbf{n}} = \psi(x, t), & x \in \partial\Omega, t > 0. \end{cases} \tag{0.8}$$

Use the energy method to:

(i) prove the uniqueness of (0.8) when $\alpha = 0$ and $\beta \neq 0$;

(ii) prove the uniqueness when both α and β are not zero. You might need to discuss the signs of α and β.

Solution 2. *I shall only deal with (ii) with (i) can be treated by the same arguments. To show the uniqueness of (0.8), it is equivalent to show that the following homogeneous IBVP admits only zero solution,*

$$\begin{cases} w_t = D\Delta w, & x \in \Omega, t > 0 \\ w(x, 0) = 0, & x \in \Omega, \\ \alpha w + \beta \frac{\partial w}{\partial \boldsymbol{n}} = 0, & x \in \partial\Omega, t > 0. \end{cases} \tag{0.9}$$

To this end, we define an energy functional for (0.9) as

$$E(t) = \int_\Omega w^2(x, t) dx, t \geq 0,$$

and we can easily see that $E(0) = 0$. Now we differentiate $E(t)$ with respect to time t and collect

$$E'(t) = 2D\Big(-\frac{\beta}{\alpha}\int_{\partial\Omega} w(x,t)^2 dS - \int_\Omega |\nabla w(x,t)|^2 dx\Big) = -2D\int_\Omega |\nabla w(x,t)|^2 dx \leq 0, \tag{0.10}$$

then it follows that $E'(t) \leq 0$ hence the solution to (0.9) with RBC is unique if $\frac{\beta}{\alpha} > 0$; however, if $\frac{\beta}{\alpha} < 0$, we can not determine the sign of $E(t)$, therefore we can not determine the uniqueness for (0.9) with the RBC in this case.

Remark: The statement that uniqueness cannot be determined here does not imply that the problem lacks uniqueness. Rather, it indicates that this particular energy functional cannot be used to establish uniqueness, even if the solution is indeed unique.

3. Let us reconsider the following reaction-diffusion system with mixed boundary condition

$$\begin{cases} u_t = D\Delta u + f(x,t), & x\in\Omega, t>0,\\ u(x,0) = \phi(x), & x\in\Omega,\\ u(x,t) = \psi_1(x,t), & x\in\partial\Omega_1, t>0,\\ \partial_{\mathbf{n}} u(x,t) = \psi_2(x,t), & x\in\partial\Omega_2, t>0, \end{cases} \tag{0.11}$$

where we used $\partial\Omega_1$ to denote part of the boundary Ω and $\partial\Omega_2$ to denote the rest part. * Prove the uniqueness of (0.11).

Solution 3. *We follow the same arguments as in class. You should be able to find that all are the same as earlier except that the difference w now satisfies the following IBVP*

$$\begin{cases} w_t = D\Delta w, & x\in\Omega, t>0,\\ w(x,0) = 0, & x\in\Omega,\\ w(x,t) = 0, & x\in\partial\Omega_1, t>0,\\ \partial_{\boldsymbol{n}} w(x,t) = 0, & x\in\partial\Omega_2, t>0, \end{cases}$$

Differentiate the energy $E(t) = \frac{1}{2}\int_\Omega w^2(x,t)dx$ with respect to time t. Then we obtain the following

$$\frac{dE(t)}{dt} = 2\int_\Omega wD\Delta w dx = -2D\int_\Omega |\nabla w|^2 dx + 2D\int_{\partial\Omega} w\partial_{\boldsymbol{n}} w dS.$$

To evaluate the boundary integral, we have that

$$\int_{\partial\Omega} w\partial_{\boldsymbol{n}} w dS = \overbrace{\int_{\partial\Omega_1} w\partial_{\boldsymbol{n}} w dS}^{=0 \text{ since } w=0} + \overbrace{\int_{\partial\Omega_2} w\partial_{\boldsymbol{n}} w dS}^{=0 \text{ since } \partial_n w=0} = 0,$$

therefore, we can also find that in above $E(t)$ is always non-increasing over time. Since $E(0) = 0$ and $E(t) \geq 0$, we must have $E(t) \equiv 0$, which leads to $w \equiv 0$, as expected.

I would like to point out that physical intuition can also explain why $w \equiv 0$ must hold. Imagine an object at zero temperature, with part of its surface immersed in ice-cold water and the rest perfectly insulated. Since there is no heat exchange in either region, the temperature will remain constant at its initial value, which is zero.

4. The term "energy" defined as

$$E(t) = \int_\Omega w^2(x,t)\, dx$$

may not necessarily represent physical energy, such as kinetic or potential energy. However, it is referred to as "energy" because it shares many characteristics with physical energy; for example, it is always positive and increases if the temperature u rises. It is more accurately termed an "energy functional," as we discussed in class, since it is a function of functions.

*A possible physical scenario involves a homogeneous iron bar, with the left end submerged in iced water and the right end well insulated. Heat flows along the bar, with heat exchange occurring only at the left end, while the rest of the bar remains insulated.

(i) let us define

$$E(t) := \int_\Omega w^4(x,t)dx.$$

Use this new energy-functional to prove the uniqueness to (0.1).

(iii) can you use $E(t) := \int_\Omega w^3(x,t)dx$ for this purpose?

Solution 4. *(i) Again, to prove the uniqueness of (0.8), it is sufficient to show that the following system has only zero solution*

$$\begin{cases} w_t = D\Delta w, & x \in \Omega, t > 0, \\ w(x,0) = 0, & x \in \Omega, \\ w = 0, & x \in \partial\Omega, t > 0. \end{cases} \tag{0.12}$$

In light of the PDE and boundary condition, straightforward calculations yield:

$$E'(t) = \frac{d}{dt}\int_\Omega w^4(x,t)\,dx = 4\int_\Omega w^3\frac{\partial w}{\partial t}\,dx = 4D\int_\Omega w^3\Delta w\,dx = 4D\left(\int_{\partial\Omega} w^3\frac{\partial w}{\partial \mathbf{n}}\,dS - 3\int_\Omega w^2|\nabla w|^2\,dx\right) = -12D\int_\Omega$$

which shows that $E(t)$ is monotone decreasing in time. Since $E(0) = 0$, it follows that $E(t) = 0$ for all $t > 0$. Therefore, we must have $w(x,t) \equiv 0$, as expected.

(ii) There are several reasons why this newly defined energy functional cannot be used to prove uniqueness. First, the monotonicity of $E(t)$ is not guaranteed. Furthermore, it is unclear whether $E(t)$ is always positive or non-negative. However, failing to demonstrate these properties does not necessarily imply the absence of uniqueness; it simply means that this particular functional is insufficient for proving it. In practical research problems, many issues can be resolved with such functionals, though they are often difficult to construct or may not exist at all.

5. In general, one cannot apply energy methods to problems where $f = f(x,t,u)$; indeed, many such problems may have more than one solution. However, these methods are effective for problems with special reaction terms f. It is important to note that f is referred to as a "reaction" term because heat is produced through **chemical reactions**, which is why the heat equation is sometimes called the reaction-diffusion equation in other contexts.

   (i). Use energy method to prove uniqueness to the following problem

$$\begin{cases} u_t = D\Delta u - u, & x \in \Omega, t > 0, \\ u(x,0) = \phi(x), & x \in \Omega, \\ u(x,t) = \psi(x,t), & x \in \partial\Omega, t > 0. \end{cases} \tag{0.13}$$

   (ii). Suppose that f' is of one sign (you should determine what the sign it should be). Prove uniqueness for following problem with $f = f(u)$ dependent only on u

$$\begin{cases} u_t = D\Delta u + f(u), & x \in \Omega, t > 0, \\ u(x,0) = \phi(x), & x \in \Omega, \\ u(x,t) = \psi(x,t), & x \in \partial\Omega, t > 0. \end{cases} \tag{0.14}$$

   For what conditions (on f) do you have the uniqueness of (0.14)? Hint: according to intermediate value theorem, $f(u_1) - f(u_2) = f'(u_1 + \theta(u_2 - u_1))(u_1 - u_2)$ for some $\theta \in [0,1]$. Therefore you may work on the integral if f' is of one sign. Remark: you should see that it also works even if $f = f(x,t,u)$, while I skip x and t with loss of generality.

   (iii) (Only for motivated students) Assume that f is Lipschitz continuous, i.e., there exists a positive constant L such that $|f(u_1) - f(u_2)| \le L|u_1 - u_2|$ for any u_1, u_2. Prove the uniqueness of (0.14). Hint: you may arrive at an ordinary differential inequality. Solve that inequality.

Solution 5. *(i) Similar as above, to prove the uniqueness of (0.13), it suffices to show that the following problem admits only the trivial solution $w(x,t) \equiv 0$, for all $(x,t) \in \Omega \times \mathbb{R}^+$,*

$$\begin{cases} w_t = D\Delta w - w, & x \in \Omega, t \in \mathbb{R}^+, \\ u(x,0) = 0, & x \in \Omega, \\ w(x,t) = 0, & x \in \partial\Omega, t \in \mathbb{R}^+. \end{cases} \tag{0.15}$$

Endow (0.8) with the energy

$$E(w) := \int_\Omega w^2(x,t)dx.$$

One obtains from the PDE and the integration by parts that

$$\frac{dE(t)}{dt} = 2\int_\Omega ww_t dx = -2\int_\Omega (D|\nabla w|^2 + w^2)dx \le 0,$$

hence $E(t)$ is non-increasing and $E(t) \le 0$. Moreover, the facts that $E(t) \ge 0$ and $E(0) = 0$ imply that $E(t) = 0$, $\forall t \in \mathbb{R}^+$, and the uniqueness follows.

(ii) For this nonlinear problem, we shall prove that the following problem has only the trivial solution

$$\begin{cases} w_t = D\Delta w + f(u_1) - f(u_2), & x \in \Omega, t \in \mathbb{R}^+, \\ u(x,0) = 0, & x \in \Omega, \\ w(x,t) = 0, & x \in \partial\Omega, t \in \mathbb{R}^+, \end{cases} \tag{0.16}$$

assuming that u_1 and u_2 are two solutions with $w := u_1 - u_2$. Recall that the intermediate value theorem states that there exists $\theta \in [0,1]$ such that $f(u_1) - f(u_2) = f'(u^*)(u_1 - u_2)$, $u^* = \theta u_1 + (1-\theta)u_2$. Then by the same calculations above, we have that

$$\frac{dE(t)}{dt} = 2\int_\Omega ww_t dx = -2\int_\Omega (D|\nabla w|^2 + \int_\Omega f'(u^*)w^2)dx.$$

If $f'(u) \le 0$ for any $u \in \mathbb{R}$, we conclude that $E'(t) \le 0$ and the uniqueness follows as above.

(iii) As shown above, we have

$$\frac{dE(t)}{dt} = -2\int_\Omega D|\nabla w|^2\,dx + \int_\Omega (f(u_1) - f(u_2))w\,dx \le LE(t),$$

where we used the fact that $(f(u_1) - f(u_2))w \le Lw^2$. This inequality leads to Gronwall's inequality, which, when solved, yields $E(t) \le E(0)e^{Lt} = 0$.

In fact, multiplying both sides by the integrating factor e^{Lt}, we get $\left(e^{-Lt}E(t)\right)' \le 0$. This shows that $E(t)e^{-Lt}$ is monotone decreasing, and thus $E(t) \le E(0)e^{Lt}$, as claimed. Consequently, $E(t) = 0$, implying uniqueness.

6. The energy method can also be applied to problems of various forms. For example, consider the following initial boundary value problem (a wave equation; you do not need to know anything about it at this moment):

$$\begin{cases} u_{tt} = D\Delta u + f(x,t), & x \in \Omega, t \in \mathbb{R}^+, \\ u(x,0) = \phi(x), & x \in \Omega, \\ u_t(x,0) = h(x), & x \in \partial\Omega, t \in \mathbb{R}^+. \end{cases} \tag{0.17}$$

Use the energy-functional

$$E(t) := \frac{1}{2}\int_\Omega w_t^2(x,t) + |\nabla w(x,t)|^2 dx$$

to prove the uniqueness of (0.17). Justify each step of your argument rigorously.

Solution 6. *Since this PDE is linear, to prove uniqueness, it is sufficient to show that the following problem has only the zero solution:*

$$\begin{cases} w_{tt} = D\Delta w, & x \in \Omega,\, t \in \mathbb{R}^+, \\ w(x,0) = 0, & x \in \Omega, \\ w_t(x,t) = 0, & x \in \partial\Omega. \end{cases} \tag{0.18}$$

To this end, we compute the rate of change of $E(t)$ as follows:

$$\frac{dE(t)}{dt} = \int_\Omega w_t w_{tt} + D\nabla w_t \cdot \nabla w \, dx \quad \textit{(chain rule)}.$$

Using the PDE, we simplify this to:

$$\frac{dE(t)}{dt} = \int_\Omega w_t(w_{tt} - D\Delta w) \, dx \quad \textit{(the boundary integral vanishes due to the zero boundary condition)}.$$

Thus, we get:

$$\frac{dE(t)}{dt} = 0.$$

Therefore, $E(t) = E(0) = 0$, *which implies that* $w(x,t) \equiv 0$ *for* $(x,t) \in \Omega \times (0,\infty)$.