Introduction to PDEs, Fall 2024

Homework 5 solutions

Name:_______________

1. Let us revisit the example given in the lecture:

$$\begin{cases} u_t = Du_{xx}, & x \in (0, L), t > 0 \\ u(x, 0) = \phi(x), & x \in (0, L), \\ u(x, t) = 0, & x = 0, L. \end{cases} \tag{0.1}$$

We have already shown that the solution is unique and is explicitly given by the infinite series

$$u(x, t) = \sum_{k=1}^{\infty} C_k e^{-D(\frac{k\pi}{L})^2 t} \sin \frac{k\pi x}{L},$$

where

$$C_k = \frac{2}{L} \int_0^L \phi(x) \sin \frac{k\pi x}{L} dx$$

or $C_k = \frac{2}{L} \int_0^L \phi(y) \sin \frac{k\pi y}{L} dy$ to avoid confusion.

(1) Suppose that the initial data are $\phi(x) = x$, and $D = 1$ and $L = \pi$. Use a computer program to plot the sum of the first N–terms $u^{(N)}(x, t)$

$$u^{(N)}(x, t) := \sum_{n=1}^{N} C_n X_n(x) T_n(t)$$

of the series at time $t = 0.1$ by taking $N = 1, 2, 5, 10, 20$ (in different colors or lines such as dash, dot, etc) respectively. You shall observe that $u^{(N)}(x, 0.1)$ converges as N increases (well, for each fixed t indeed). You can also report your CPU time, and you should see that a larger N generally takes longer time to calculate. Therefore, though it is impossible to plot $u^{(\infty)}(x, t)$, one can, given applications, employ $u^{(N)}(x, t)$ to approximate the true solution by taking N large enough.

(2) Assume that $u^{(10)}(x, t)$ above is a good enough approximation* of the exact solution (i.e., the infinite series). Plot the graphs of $u^{(10)}(x, t)$ for $t = 0.1, 0.5, 1, 2, 5$. What is the limit as $t \to \infty$?

Solution 1. *(1) We first have, thanks to the IC and the BC, that the coefficients in the infinite series are*

$$C_k = \frac{2}{\pi} \int_0^\pi x \sin \frac{k\pi x}{L} dx = \frac{2}{\pi} \Big(\frac{\sin k\pi - k\pi \cos k\pi}{k^2} \Big) = \frac{2(-1)^{k+1}}{k}, k \in \mathbb{N}^+.$$

(2) If we choose the truncated solution with $N = 10$ to be the approximation, then

2. Solve the following IBVP by separation of variables and write its solution in terms of infinite series

$$\begin{cases} u_t = Du_{xx}, & x \in (-L, L), t > 0 \\ u(x, 0) = \phi(x), & x \in (-L, L), \\ u(-L, t) = u(L, t) = 0, & t > 0. \end{cases} \tag{0.2}$$

Remark: Ambitious and motivated students are encouraged to explore this problem by replacing $(-L, L)$ by (a, b), though I do not require you to do so. We shall see later in this course that by passing $L \to \infty$, we collect (0.2) in the whole space.

*so you can balance the computational time

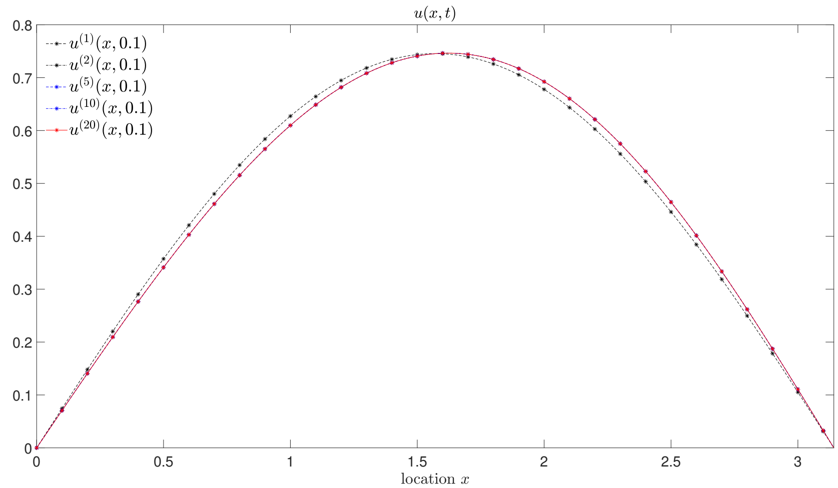


Figure 1: Plots of $u^N(x, 0.1)$ for $N = 1, 2, 5, 10$ and 20. It provides visual evidence that $N = 10$ is already a good approximation.

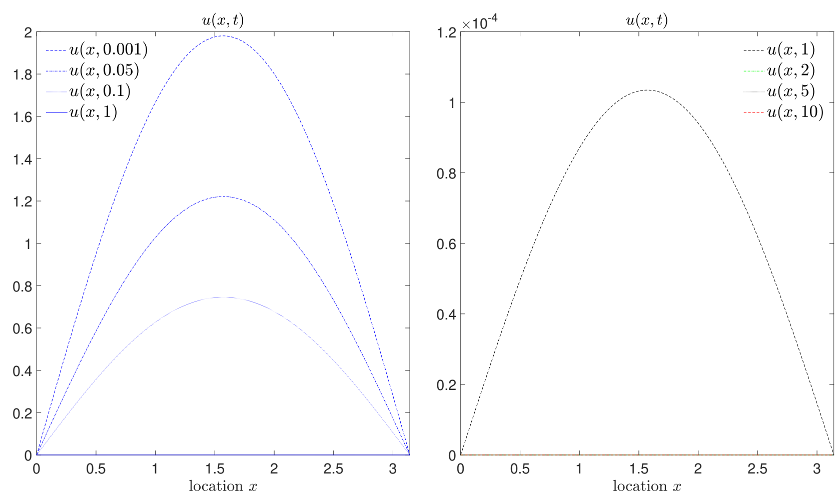


Figure 2: Left: Plots of $u^{(10)}(x, t)$ for $t = 0.001, 0.05, 0.1$ and 1. It provides visual evidence that the solution approximated by $N = 10$ converges to $u \equiv 0$ as $t \to \infty$. Right: We present additional evidence for such convergence, as $u(x, t)$ is already of order 10^{-4} for time $t > 1$.

Solution 2. *First of all, we have that the eigen–value problem associated with this system is*

$$\begin{cases} X'' + \lambda X = 0, & x \in (-L, L), \\ X(-L) = X(L) = 0. \end{cases}$$

Now we need to find the eigen–pairs to the problem above. While you can do it by straightforward calculations, an alternative way is to do as follows: let us denote

$$Y(x) := X(x - L),$$

then it is easy to see that the eigen–value problem now becomes

$$\begin{cases} Y'' + \lambda Y = 0, & x \in (0, 2L), \\ Y(0) = Y(2L) = 0. \end{cases}$$

It is well known that the eigen–pairs are

$$Y_n(x) = \sin \frac{n\pi x}{2L}, \lambda_n = \Big(\frac{n\pi}{2L}\Big)^2, n = 1, 2, ...$$

therefore we have that

$$X_n(x) = Y_n(x + L) = \sin \frac{n\pi(x + L)}{2L} = \sin \Big(\frac{n\pi x}{2L} + \frac{n\pi}{2}\Big), \lambda_n = \Big(\frac{n\pi}{2L}\Big)^2, n = 1, 2, ...$$

moreover, we can also find that

$$\int_{-L}^{L} X_n^2(x)dx = \int_0^{2L} Y_n^2(x)dx = \int_0^{2L} \sin^2 \frac{n\pi x}{2L} dx = L,$$

while

$$\int_{-L}^{L} X_m(x)X_n(x)dx = 0, m \neq n.$$

According to the Sturm–Liouville Theory, we are able to write the solution in terms of the infinite series

$$u(x, t) = \sum_{n=1}^{\infty} C_n(t) \sin \frac{n\pi(x + L)}{2L}.$$

Substituting this solution into the PDE gives us

$$\sum_{n=1}^{\infty} C_n'(t) \sin \frac{n\pi(x + L)}{2L} = -D \sum_{n=1}^{\infty} C_n(t) \Big(\frac{n\pi}{2L}\Big)^2 \sin \frac{n\pi(x + L)}{2L};$$

Multiplying BHS of the system above by $X_n(x)$ and then integrating it over $(-L, L)$, we obtain that

$$C_n'(t) = -D\Big(\frac{n\pi}{2L}\Big)^2 C_n(t);$$

solving this ODE gives us

$$C_n(t) = C_n(0)e^{-D(\frac{n\pi}{2L})^2 t},$$

where $C_n(0)$ can be evaluated by the initial condition as

$$C_n(0) = \frac{1}{L} \int_{-L}^{L} \phi(x) \sin \frac{n\pi(x + L)}{2L} dx.$$

3. Use the method of separation of variables to find the solution to the following problem in terms of infinite series

$$\begin{cases} u_t = Du_{xx}, & x \in (0, L), t > 0 \\ u(x, 0) = x, & x \in (0, L), \\ u_x(x, t) = 0, & x = 0, L. \end{cases} \tag{0.3}$$

(i) Without solving this problem, use physical intuition to predict/explain what is the limit of $u(x,t)$ as $t \to \infty$? Hint: think of $u(x,t)$ as the temperature.

(ii) Try a separable solution $U_n(x,t) = X_n(x)T_n(t)$ of the PDE and then find it by the boundary condition. You should notice that $n = 0$ should not be abandoned as in the DBC.

(iii) Let $u(x,t) = \sum_{n=0}^{\infty} C_n X_n(x) T_n(t)$ and then find C_n by fitting the initial condition;

(iv) Choose $D = 1$ and $L = \pi$. Use a computer program to plot the truncated sum

$$u^{(N)}(x,t) := \sum_{n=0}^{N} C_n X_n(x) T_n(t)$$

of the series at time $t = 0.1$ by taking $N = 1, 2, 5, 10, 20$ as above;

(v) Plot the graphs of $u^{(10)}(x,t)$ for $t = 0.1, 0.5, 1, 2, 5$. What is the limit of this curve as $t \to \infty$?

Solution 3. *(i). As we have discussed in class, the moral of the story is that we recognize this problem as one arising from the physical scenario that a homogeneous (no heat resource), well–insulated (NBC) bar with initial temperature distribution in the form of x. Then it is natural to expect that the heat will flow from the region of higher temperature to lower temperature, with the whole bar approaching a constant temperature eventually. On the other hand, there is no creation and degradation of the thermal energy within the bar and across the boundary (endpoints), therefore the final constant temperature is expected to be the average value of the initial temperature, thanks to the conservation of the thermal energy. The question of how the temperature converges to the average (constant) is the goal of this homework problem;*

(ii). If we try the separable solution $U = X(x)T(t)$, then X satisfies the corresponding EP with NBC (do it yourself) and it is explicitly given by

$$X_n = \cos \frac{n\pi x}{L}, \; n = 0, 1, 2, \cdots,$$

therefore we can write $u(x,t)$ through a linear combination of X_n as

$$u(x,t) = \sum_{n=0}^{\infty} C_n(t) \cos \frac{n\pi x}{L}.$$

I would like to note that, when you were doing the problem, the Sturm–Liouville theorem was not expected and you can work out the problem following the separation of variable routine, which leads to the same infinite series for the solution. Substitute $u(x,t)$ into the PDE and we will find

$$\sum_{n=0}^{\infty} C_n'(t) \cos \frac{n\pi x}{L} = -D \sum_{n=0}^{\infty} \left(\frac{n\pi}{L}\right)^2 C_n(t) \cos \frac{n\pi x}{L},$$

hence we have

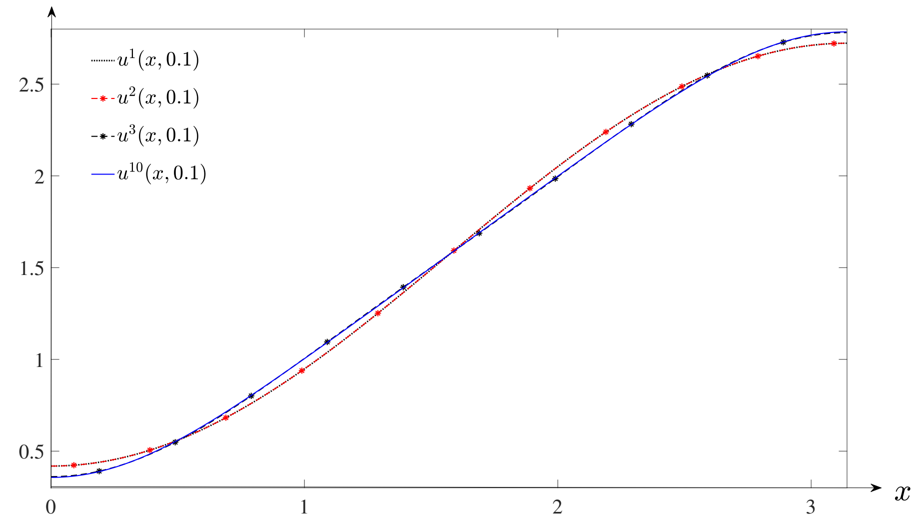
$$C_n(t) = C_n(0) e^{-D(\frac{n\pi}{L})^2 t}, n = 0, 1, 2, \cdots$$

(iii). Moreover, from the IC $u(x,0) = x$, we have that

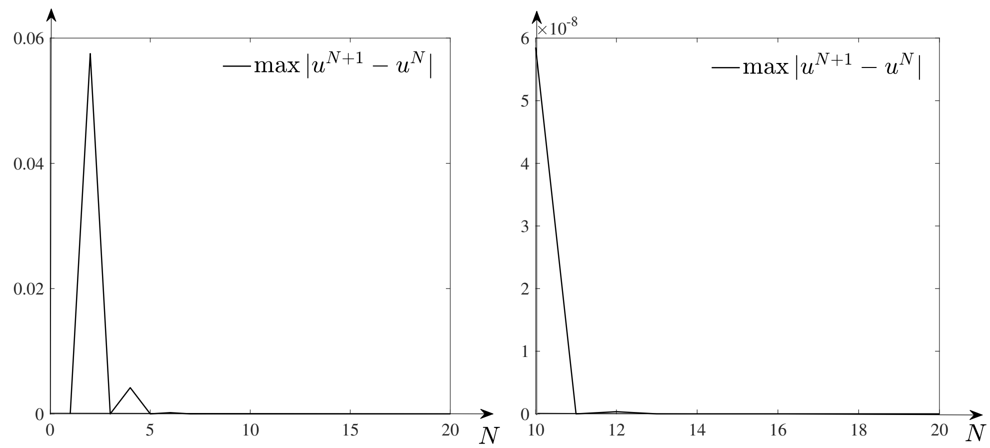
$$u(x,0) = \phi(x) = C_0(0) + \sum_{n=1}^{\infty} C_n(0) \cos \frac{n\pi x}{L},$$

For $n = 1, 2, ...$, we multiply BHS by $\cos \frac{n\pi x}{L}$ and integrate it over $(0, L)$ by parts to obtain

$$\begin{aligned}
C_n(0) &= \frac{2}{L} \int_0^L \phi(x) \cos \frac{n\pi x}{L} dx = \frac{2}{L} \int_0^L x \cos \frac{n\pi x}{L} dx \\
&= \frac{2}{L} \int_0^L \left(\frac{L}{n\pi}\right) x d \sin \frac{n\pi x}{L} = \frac{2}{n\pi} \left(x \sin \frac{n\pi x}{L} \Big|_0^L - \int_0^L \sin \frac{n\pi x}{L} dx \right) \\
&= \frac{2L}{n^2\pi^2} \cos \frac{n\pi x}{L} \Big|_0^L = \frac{2L}{n^2\pi^2} \Big((-1)^n - 1 \Big);
\end{aligned}$$



(a) Plots of $u^N(x, 0.1)$ for $N = 1, 2, 3$ and 10. It provides evidence that $N = 10$ is already a good approximation.



(b) Absolute errors of difference terms $u^{N+1} - u^N$ for several N. Left column: we observe that the error is negligible when N is larger than 5, hence $N = 10$ is a good approximation. Right column: we provide further evidence that the error is infinitesimal when $N \geq 10$ and this sequence is Cauchy as $N \to \infty$. Therefore u^N converges to some limiting function in a certain complete function space (complete means any Cauchy sequence converges to some limit in this space), which turns out to be our exact solution. Again, the moral of the story is that u^{10} serves as a nice approximation (to our understanding), and a large N requires a longer computation hence sometimes is not taken in practice.

to find $C_0(0)$, we just simply integrate BHS over $(0, L)$ and find

$$C_0(0) = \frac{\int_0^L x dx}{L} = \frac{L}{2},$$

therefore, we have

$$u(x,t) = \sum_{n=0}^{\infty} C_n(t) \cos\frac{n\pi x}{L} = \frac{L}{2} + \sum_{n=1}^{\infty} \frac{2L}{n^2\pi^2}\Big((-1)^n - 1\Big)e^{-D(\frac{n\pi}{L})^2 t} \cos\frac{n\pi x}{L}.$$

(iv)–(vi). When $D = 1$ and $L = \pi$, the solutions above is

$$u(x,t) = \sum_{n=0}^{\infty} C_n(t) \cos nx = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}\Big((-1)^n - 1\Big)e^{-n^2 t} \cos nx$$

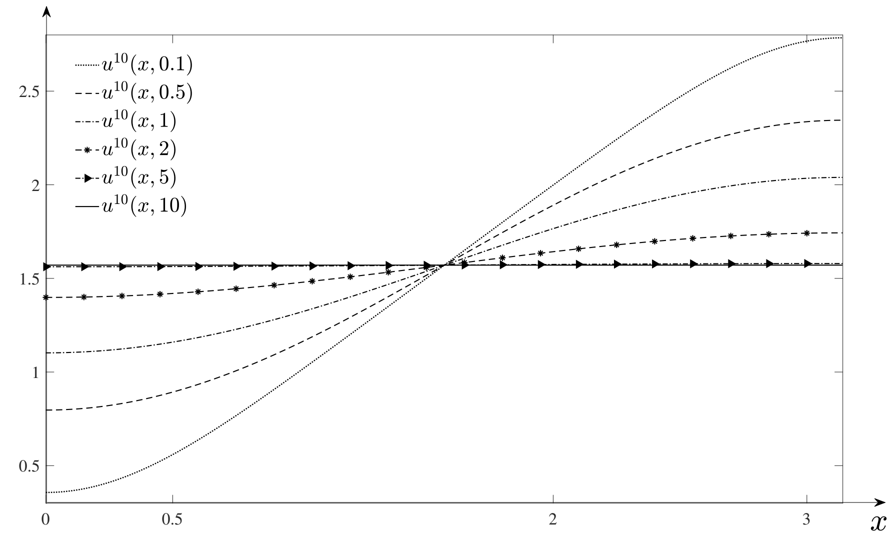


Figure 4: Evolution of the temperature as time increases. We observe that the left end increases and the right end decreases; apparently, this is due to the transfer of heat from the region of high to lower temperatures; moreover, in a large time $t \geq 5$, the temperature almost reaches a homogeneity, which agrees well with our intuition and physical expectation.

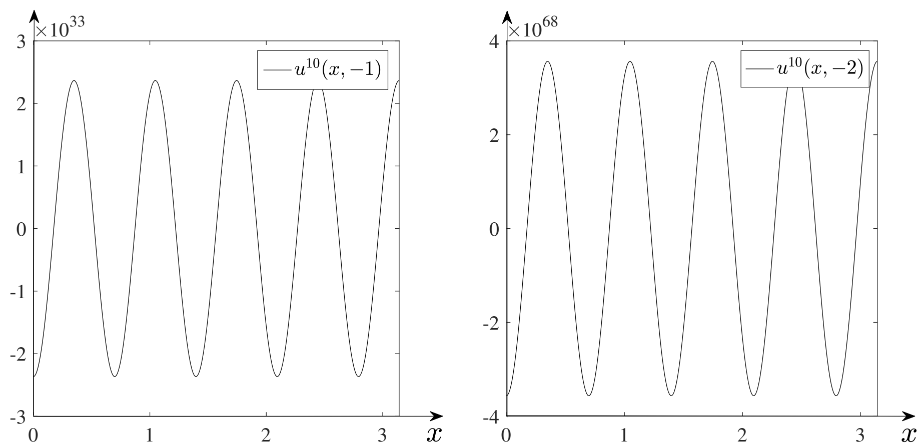


Figure 5: Ill-posedness of heat equation backward in time. We observe the blow-up of $u(x, t)$ back in time, which implies that the heat equation back in time does not make sense, though mathematically its solution is still unique as stated in HW.

with its finite/partial sum given by

$$u^N(x,t) = \sum_{n=0}^{N} C_n(t) \cos nx = \frac{\pi}{2} + \sum_{n=1}^{N} \frac{2}{n^2 \pi} \Big((-1)^n - 1 \Big) e^{-n^2 t} \cos nx.$$

We observe the aforementioned convergence to the average value as $t \to \infty$. One important observation is that the steady state is determined by the boundary condition, but not the initial data; moreover, for the problem backward in time, we can find that the solution ***blows up*** *in time, i.e., $u(x,t) \to \infty$ as $t \to -\infty$ (at least for a subsequence of such t) for each fixed $x \in (0, L)$.*

4. The method of separation of variables can also be used to solve wave–equation or hyperbolic equation

such as of the following form

$$\begin{cases} u_{tt} = Du_{xx}, & x \in (0, L), t \in \mathbb{R}_+, \\ u(x,0) = \phi(x), u_t(x,0) = 0, & x \in (0, L), \\ u(0,t) = 0, u(L,t) = 0, & t \in \mathbb{R}_+, \end{cases} \tag{0.4}$$

where

$$\phi(x) = \begin{cases} \frac{2h}{L}x, & x \in [0, \frac{L}{2}], \\ \frac{2h}{L}(L-x), & x \in [\frac{L}{2}, L], \end{cases}$$

wherein the left-hand side of the PDE, second-order derivative is taken concerning time. You can solve (0.4) by taking the following steps

(i) Try a separable solution of the form $U_n(x,t) = X_n(x)T_n(t)$; find the ODEs of X_n and T_n, then solve for X_n by the boundary condition thus $T_n(t)$. Now $T_n(t)$ satisfies second order ODE and it should take the form $T_n(t) = A_n \cos(...) + B_n \sin(...)$, where A_n and B_n are constants to be determined. Now you should have obtained $U_n(x,t) = X_n(x)T_n(t)$;

(ii) Let $u(x,t) = \sum_{n=1}^{\infty} X_n(x)T_n(t)$ and then find A_n and B_n by fitting the initial condition. *Remark: In the linear combination, the coefficient C_n is embedded into A_n and B_n.*

(iii) Choose $D = L = 1$. Use the first 10 terms as your approximate. Plot the graphs for $t = 1, 1.5, 2, 2.5, 3...$and the initial data on the same coordinate. What are your observations? Compare this with the heat equation.

(iv). You can even try to plot the graphs for $t = -1$, $t = -2$, $t = -3$. You may see that graphs propagate like a wave and this is why the PDE is called a wave equation. What is the speed of wave propagation?

Solution 4. *(i). Since the corresponding EP has homogeneous DBC, according to the Sturm–Liouville theorem we can write $u(x,t)$ into the following eigen–expansions*

$$u(x,t) = \sum_{n=1}^{\infty} C_n(t) \sin \frac{n\pi x}{L}.$$

Ok, I do not explain how and why sin *any more!!! Substituting this series into the PDE gives rise to*

$$C_n''(t) = -D\Big(\frac{n\pi}{L}\Big)^2 C_n(t),$$

which is a second–order linear ODE and its solution takes the form

$$C_n(t) = A_n \cos \frac{n\pi\sqrt{D}t}{L} + B_n \sin \frac{n\pi\sqrt{D}t}{L}, n = 1, 2, ...$$

Note that $n = 0$ is easily ignored thanks to the DBC above (however, it helps to always keep $n = 0$ for your solution in practice and then decide if it is necessary, at least when you are a PDE rookie). Now $u_n(x,t) = C_n(t)X_n(x)$ and the solution to (0.4) takes the form

$$u(x,t) = \sum_{n=1}^{\infty} \Big(A_n \cos \frac{n\pi\sqrt{D}t}{L} + B_n \sin \frac{n\pi\sqrt{D}t}{L}\Big) \sin \frac{n\pi x}{L},$$

where A_n and B_n, $n \in \mathbb{N}^+$ are constants to be determined. Note that the coefficients in the linear combination are embedded into A_n and B_n.

(ii) By matching the initial datum $u(x,0) = \phi(x)$, we have that

$$\begin{aligned} A_n =& \frac{2}{L}\int_0^L \phi(x) \sin \frac{n\pi x}{L} dx \\ =& \frac{2}{L}\int_0^{\frac{L}{2}} \frac{2h}{L} x \sin \frac{n\pi x}{L} dx + \frac{2}{L}\int_{\frac{L}{2}}^L \frac{2h}{L}(L-x) \sin \frac{n\pi x}{L} dx \\ =& \frac{8h}{n^2\pi^2} \sin \frac{n\pi}{2} \\ =& \begin{cases} \frac{8h(-1)^k}{(2k-1)^2\pi^2}, & n = 2k-1, k = 1, 2, ... \\ 0 & n = 2k, k = 1, 2, ...; \end{cases} \end{aligned}$$

by matching the initial datum $u_t(x,0)=0$, we can easily find that $B_n=0$. Therefore we have that the solution is given by

$$u(x,t)=\sum_{k=1}^{\infty}\frac{8h(-1)^k}{(2k-1)^2\pi^2}\cos\frac{(2k-1)\pi\sqrt{D}t}{L}\sin\frac{(2k-1)\pi x}{L}.$$

(iii)-(iv). When $D=L=h=1$ (should have assumed $h=1$), the approximate solution takes the form

$$u^N(x,t)=\sum_{k=1}^{N}\frac{8(-1)^k}{(2k-1)^2\pi^2}\big(\cos(2k-1)\pi t\big)\big(\sin(2k-1)\pi x\big).$$

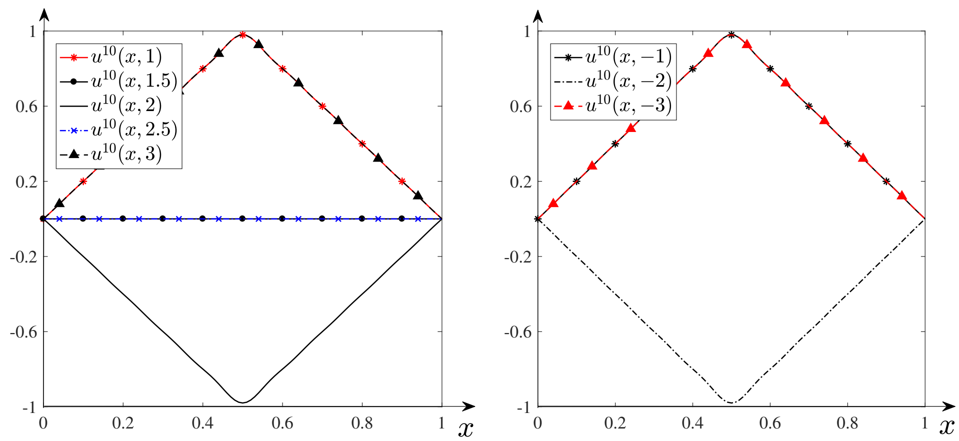


Figure 6: Graphes of $u^{10}(x,t)$ at different times. Note that all satisfy the DBC and there are certain periodicity in these solutions.

5. This is another example that applies the method of separation of variables. Consider the scenario that an agent starts with a total wealth x at time t, and can invest the total wealth X_s into a bond (risk-free) with a portion α_s and a stock (risky) with the rest portion $1-\alpha_s$. Then idealized modeling of the evolution of the total wealth is the following differential equation

$$dX_s=X_s\big(r+(\mu-r)\alpha_s\big)dt+\sigma\alpha_sX_sdW_s,$$

where r is the constant interest rate of the bond, and constants μ and σ are the interest rate and volatility of the stock. W_s is the Brownian motion and it is a user-friendly choice that models the uncertainty or the "risk" when investing in the stock. Then an optimization problem arises when this agent opts to maximize the "benefit" from the total wealth by altering the allocation α_s. That is, one aims in finding the maximum value function from the investment

$$u(x,t):=\sup_{\alpha_s\in\mathcal{A}}E[\mathbb{U}(X_T^{x,t})].$$

Here $\mathbb{U}(\cdot)$ is the so-called utility function, and t,x on the shoulder are included to highlight the effects endowment x at time t. Note that: i) X_t is a stochastic process, hence the expectation is taken; ii) one may wonder why not to maximize $E[X_T^{x,t}]$ but $E[\mathbb{U}(X_T^{x,t})]$. This utility function describes the well-accepted belief that utility or "satisfaction" should not be linear in wealth, but a concave function. Imagine that eating two apples is less than twice satisfying as eating one. Some may argue it might be more than twice in a certain situation which I agree with, however, this implies that a nonlinear function should be considered here anyhow, and that is the utility function. You do not have to understand everything above to do this problem, but I explain to them here to give you a motivation why the optimal value function $u(x,t)$ above is of interest.

By the standard dynamic programming principle (or Bellman's optimality condition), one can show that this function solves the following

$$\begin{cases} u_t + rxu_x + \sup_{\alpha\in\mathbb{R}}\left[\alpha(\mu-r)xu_x + \frac{\alpha^2\sigma^2}{2}x^2u_{xx}\right] = 0, & x\in(0,\infty), t\in(0,T),\\ u(x,T) = \mathbb{U}(x) = \frac{x^p}{p}, & x\in(0,\infty), \end{cases} \tag{0.5}$$

where for simplicity we assume that α is constant, and choose the so-called CRRV utility $\mathbb{U}(x)$ with $p\in(0,1)$. Use the method of separation of variables to solve for the optimal α^* and the value function of (0.5). Suggest answer: $u(x,t) = e^{\lambda(T-t)}x^p/p$, where $\lambda = \frac{p(\mu-r)^2}{2(1-p)\sigma^2} + pr$, and the optimal control is $\alpha^* = \frac{\mu-r}{(1-p)\sigma^2}$.

Solution 5. *Let us try $u(x,t) = X(x)T(t)$ which is assumed nonzero as usual, then the PDE in (0.5) implies that*

$$\frac{T'}{T} + rx\frac{X'}{X} + \sup_{\alpha\in\mathbb{R}}\left[\alpha(\mu-r)x\frac{X'}{X} + \frac{\alpha^2\sigma^2x^2}{2}\frac{X''}{X}\right] = 0,$$

where prime $'$ denotes the derivative concerning the intrinsic variable. Then one concludes that $\frac{T'}{T} = \lambda$ must be a constant and accordingly

$$rx\frac{X'}{X} + \sup_{\alpha\in\mathbb{R}}\left[\alpha(\mu-r)x\frac{X'}{X} + \frac{\alpha^2\sigma^2x^2}{2}\frac{X''}{X}\right] = -\lambda. \tag{0.6}$$

We recognize that (0.6) has the pattern that differentiating X once balances out x in the coefficient and twice x^2 in the coefficient (when I was learning ODE, this was called an Euler's equation, and I do not know your story here). Therefore we can easily guess that X is a power function of x; on the other hand, the terminal condition is $u(x,T) = \frac{x^p}{p}$, then this power index must be p hence $X(x) = Cx^p$ for some constant C to be determined.

Bearing these facts in mind, let us proceed to find the optimal control α^*. To this, we find that $\alpha(\mu-r)x\frac{X'}{X} + \frac{\alpha^2\sigma^2x^2}{2}\frac{X''}{X}$, as a function of α, is a parabola, and therefore one can easily find that its optimal value is attained at

$$\alpha^* = -\frac{(\mu-r)x\frac{X'}{X}}{\sigma^2x^2\frac{X''}{X}} = -\frac{(\mu-r)}{\sigma^2(p-1)}$$

as expected. One can further find that λ is the value given above.

I would like to comment that this explicit solution is available for this particular utility function, and in general, the PDE or HJB system can not be solved explicitly, at least NOT by the method of separation of variables.

6. Let us consider the following problem under RBC

$$\begin{cases} u_t = u_{xx}, & x\in(0,1), t\in\mathbb{R}^+,\\ u(x,0) = x, & x\in(0,1),\\ u + u_x = 0, & x = 0,1, t\in\mathbb{R}^+. \end{cases} \tag{0.7}$$

(i) solve this problem in terms of infinite series;

(ii) use computer program to plot the sum of first N–terms $u^{(N)}(x,t)$

$$u^{(N)}(x,t) := \sum C_nX_n(x)T_n(t)$$

of the series at time $t = 0.1$ by taking $N = 1,2,3,10$ (in different colors or lines such as dash, dot, etc) respectively. Then again we shall observe that $u^{(N)}(x,t)$ converges as N increases and $u^{(N)}(x,t)$ to approximate the true solution if N is large enough;

(iii) assume that $u^{(10)}(x,t)$ is a good enough approximation of the exact solution (i.e., the infinite series)–this applies in the sequel. Plot the graphs of $u^{(10)}(x,t)$ for $t = 0.01, 0.05, 01, 1, 2, 5, \dots$. What are your observation of $u(x,t)$ when t is large?

Solution 6. *First of all, we recognize the corresponding eigen-value problem the eigen-functions of which form an orthogonal basis of* L^2 *is*

$$\begin{cases} X'' + \lambda X = 0, & x \in (0,1), \\ X + X' = 0, & x = 0, 1. \end{cases}$$

For $\lambda > 0$, *one finds that the eigen-pairs are*

$$(X_k, \lambda_k) = \Big\{ \Big(\sin \frac{k\pi x}{L} - \frac{k\pi}{L} \cos \frac{k\pi x}{L}, (\frac{k\pi}{L})^2 \Big\}_{k=1}^{\infty}$$

For $\lambda = 0$, *the eigen-function is trivial hence ignored. For* $\lambda < 0$, *one can show that* $\{(e^{-x}, -1)\}$ *is the eigen-pair for this case. Therefore, if we further denote*

$$X_0(x) := e^{-x},$$

then one invokes the Sturm–Liouville Theorem and write the solution in terms of infinite series as

$$u(x,t) = \sum_{k=0}^{\infty} C_k e^{-(k\pi/L)^2 t} X_k(x)^{\dagger}.$$

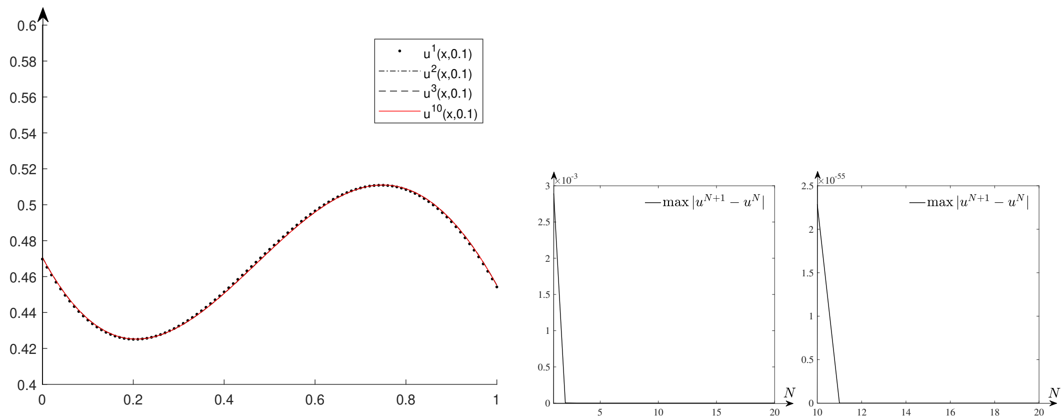


Figure 7: **Left Column**: the first N-th sum at time $t = 0.1$ for $N = 1, 2, 3$ and 10. **Right Column**: the error in L^∞ for each N large. Similar as above, we observe that when $N > 10$ the finite sum is an approximation with error tolerance of $O(10^{-55})$.

Now we are left to determine the coefficients C_k, $k = 0, 1, ...$. *To this end, we evaluate the series at* $t = 0$ *and find that*

$$u(x, 0) = x = \sum_{k=0}^{\infty} C_k X_k(x).$$

Note we know from the proof that $X_k(x)$ *are orthogonal to each other in* L^2, *therefore we test both hand side of the equation above by* X_k *and collect*

$$\begin{aligned} C_k =& \frac{\int_0^1 x X_k(x) dx}{\int_0^1 X_k^2(x) dx} \\ =& \frac{2}{(k\pi)^2 + 1} \left(\frac{\sin k\pi x - k\pi x \cos k\pi x}{(k\pi)^2} \Big|_0^1 - \frac{k\pi x \sin k\pi x + \cos k\pi x}{k\pi} \Big|_0^1 \right) \\ =& \frac{2}{(k\pi)^2 + 1} \Big(\frac{-k\pi \cos k\pi}{(k\pi)^2} - \frac{\cos k\pi - 1}{k\pi} \Big) \\ =& \frac{2}{(k\pi)^2 + 1} \frac{1 + 2(-1)^k}{k\pi}, k \in \mathbb{N}^+. \end{aligned}$$

[†] Note that we only assume that k starts from 1 for the DBC, while k can be any (not necessarily negative) integer in general, one should be able to verify that X_0 and X_k are orthogonal in L^2 by showing that $\int_0^1 e^{-x} \big(\sin \frac{k\pi x}{L} - \frac{k\pi}{L} \cos \frac{k\pi x}{L} \big) dx = 0$ for each $k = 1, 2, ...$ by direct computation.

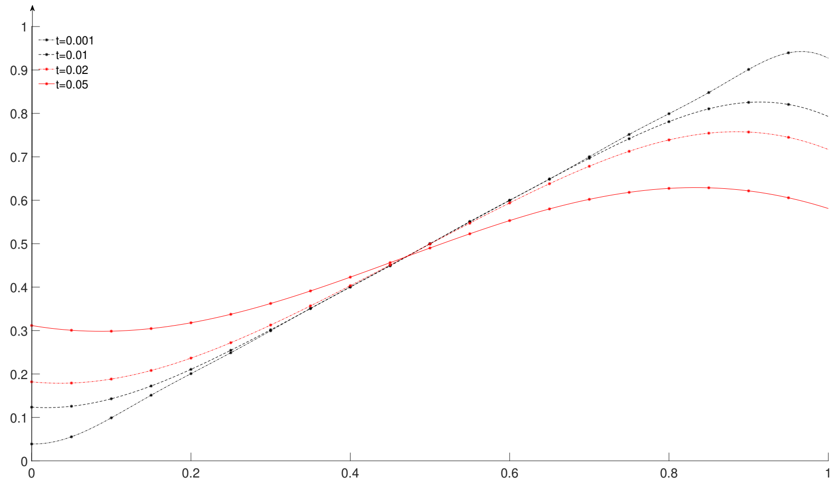


Figure 8: Local in time behavior of the solution. Here we plot the profiles of $u(x,t)$ for time t small so we can observe the local in time dynamics of the problem. Note $u+u_x=0$ implies that $u_x=-u$ at the end points by the boundary conditions, hence $u_x<0$ whenever $u>0$.

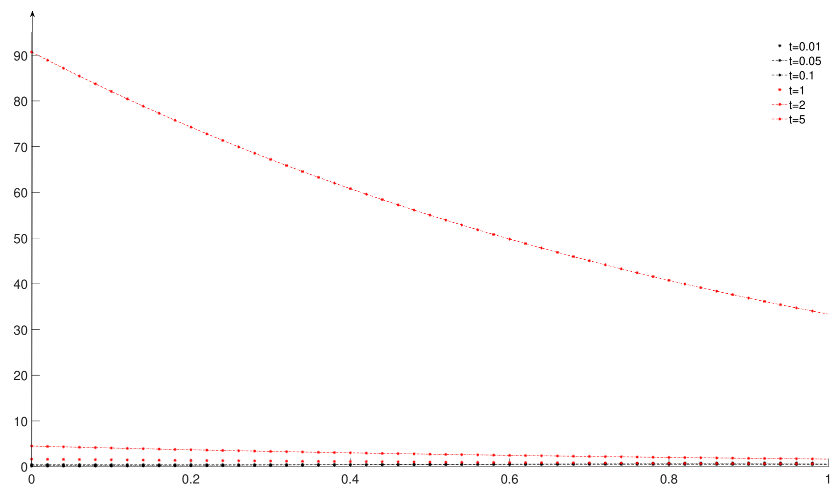


Figure 9: Evolution of approximated solutions $u^(N)(x,t)$ with $N=10$. We note that the solution "blows up " to infinite as t increases. Indeed, this is the case since the first eigen-value $\lambda_0>0$, hence the solution explodes according to the profile of the first eigen-function e^{-x}.

while

$$C_0=\frac{\displaystyle\int_0^1 xe^{-x}dx}{\displaystyle\int_0^1 (e^{-x})^2dx}=\frac{2-4e^{-1}}{1-e^{-2}}.$$

7. Separation of variables can also be applied to tackle some (most likely linear) PDEs in higher dimensions. Consider

$$\begin{cases} u_t=D\Delta u, & x\in\Omega, t\in\mathbb{R}^+,\\ u(x,0)=\phi(x), & x\in\Omega,\\ u(x,t)=0, & x\in\partial\Omega, t\in\mathbb{R}^+. \end{cases} \tag{0.8}$$

Write down $u(x,t)$ in terms of an infinite series by mimicking the approaches for 1D IBVP. You can assume similar properties of the eigen-value problem that you encounter.

Solution 7. *We write $u(x,t)$ into*

$$u(x,t)=\sum_{n=1}^{\infty}C_n(t)w_n(x),$$

where w_n is an eigen–function to the following problem

$$\begin{cases} \Delta w+\lambda w=0, & x\in\Omega \\ w=0, & x\in\partial\Omega. \end{cases} \tag{0.9}$$

Similarly as Sturm–Liouville theory, one has that $\{w_n\}_{n=1}^{\infty}$ form an orthogonal basis of $L^2(\Omega)$. Then encoding the initial data gives us that

$$C_n(t)=C_n(0)e^{-D\lambda_n t},$$

λ_n the eigen–value and

$$C_n(0)=\frac{\int_\Omega \phi(x)w_n(x)dx}{\int_\Omega w_n^2(x)dx},$$

therefore we have that

$$u(x,t)=\sum_{n=0}^{\infty}C_n(t)w_n(x).$$

I would like to mention that in general $w_0\equiv 0$, however a rigorous verification requires some advanced theories/studies about the eigen–value problem.

8. The multi-dimensional eigen-value problem over special geometries can be solved explicitly. For example, choose $\Omega=(0,a)\times(0,b)$ and consider the Dirichlet eigen-value problem (**??**). Find its eigen-pairs by starting with $u(x,y)=X(x)Y(y)$. Hint: your solution should be of the form $u_{mn}(x,y)=X_m(x)Y_n(y)$ and λ_{mn}, $m,n\in\mathbb{N}$.

   Solution 8. *We write the solution to (**??**) as $w(x,y)=X(x)Y(y)$ and substitute it into the PDE to collect that*

   $$X''Y+XY''+\lambda XY=0,$$

   since $XY\neq 0$, we have that

   $$\frac{X''}{X}+\frac{Y''}{Y}+\lambda=0,$$

   hence both $\frac{X''}{X}$ and $\frac{Y''}{Y}$ must be constants, which we shall denote by α and β respectively. Now I assume that you are able to show that $X(x)$ should take the $X_m=\sin\frac{m\pi x}{a}$ and $Y_n=\sin\frac{n\pi y}{b}$. The key point is that X and Y are independent hence each admits its sub-index. Finally, you should be able to find that $u_{mn}=\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$, $m,n\in\mathbb{N}^+$; moreover, the eigen-values are $\lambda_{mn}=(m\pi/a)^2+(n\pi/b)^2$, $m,n\in\mathbb{N}^+$.

9. One can also employ the method of separation of variables to solve other types of (multi-dimensional) PDEs. For example, consider the following problem over a 2D square $\Omega=(0,1)\times(0,1)$

   $$\begin{cases} \Delta u=0, & x\in(0,1)\times(0,1) \\ u_x(0,y)=u_x(1,y)=0, & y\in(0,1), \\ u(x,0)=0, u(x,1)=x. \end{cases} \tag{0.10}$$

   Find $u(x,y)$ in terms of infinite series by starting with $u(x,y)=X(x)Y(y)$. Suggested answer:

   $$u(x,y)=\frac{y}{2}+\sum_{n=1}^{\infty}\frac{2((-1)^n-1)}{(n\pi)^2}\frac{e^{n\pi y}-e^{-n\pi y}}{e^{n\pi}-e^{-n\pi}}\cos n\pi x.$$

   Plot $u(x,y)$ by choosing N large to see the graph yourself if it matches the boundary conditions. *Remark: If $\Delta u=0$, we say that u is a harmonic function. More about harmonic functions will be discussed with details in coming lectures.*

Solution 9. *Using the trial solution of the form* $u(x,y) = X(x)Y(y)$ *and invoking the PDE, we have that for some constant* λ

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda;$$

X *satisfying the Neumann boundary condition gives us*

$$X_n(x) = \cos n\pi x, \lambda_n = (n\pi)^2, n = 0, 1, 2, ...$$

On the other hand, $Y(y)$ *satisfies the ODE*

$$Y'' = (n\pi)^2 Y$$

hence $Y_n(y)$ *takes the form* $Y_n(y) = a_n e^{n\pi y} + b_n e^{-n\pi y}$ *for* $n = 1, 2, ...$ *and* $Y_0(y) = a_0 y + b_0$ *for* $n = 0$*; moreover, encoding the boundary condition gives us* $Y(0) = 0$ *hence*

$$Y_0(y) = a_0 y, Y_n(y) = a_n(e^{n\pi y} - e^{-n\pi y}), n = 1, 2, ...$$

therefore we have

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y) = a_0 y + \sum_{n=1}^{\infty} a_n(e^{n\pi y} - e^{-n\pi y}) \cos n\pi x,$$

which satisfies the whole problem except the boundary condition at the top of the square. To this end, we equate

$$u(x,1) = \sum_{n=0}^{\infty} u_n(x,1) = a_0 + \sum_{n=1}^{\infty} a_n(e^{n\pi} - e^{-n\pi}) \cos n\pi x = x$$

from which we obtain

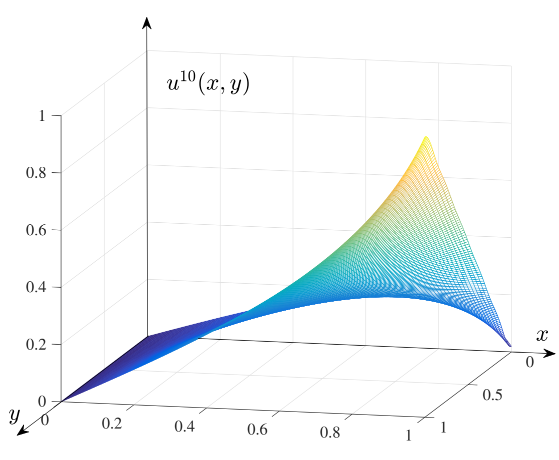
$$a_0 = \int_0^2 x dx = \frac{1}{2}$$

and

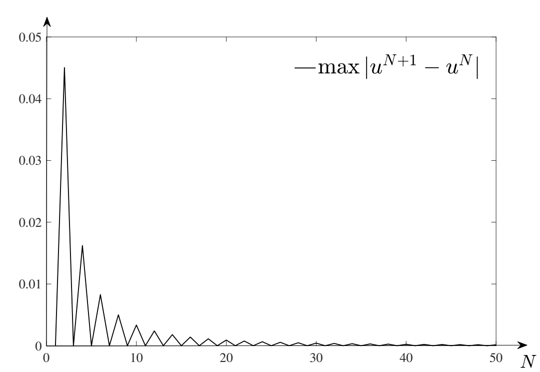
$$a_n(e^{n\pi} - e^{-n\pi}) = 2\int_0^1 x \cos nx dx = \frac{2}{(n\pi)^2}(\cos n\pi - 1), n = 1, 2, ...$$

finally, we collect the following solution as suggested

$$u(x,y) = \frac{y}{2} + \sum_{n=1}^{\infty} \frac{2((-1)^n - 1)}{(n\pi)^2} \frac{e^{n\pi y} - e^{-n\pi y}}{e^{n\pi} - e^{-n\pi}} \cos n\pi x$$



(a) Plots of the approximation $u^{10}(x,y)$.



(b) Absolute error for different N. It provides evidence that $N = 10$ is already a good approximation if an upper bound of error $\varepsilon \leq 0.01$ is acceptable.