Introduction to PDEs, Fall 2024

Homework 7 due Nov 28

Name:________________

1. The Sturm–Liouville theory applies to the linear ordinary differential equations of the general form

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y = -\lambda w(x)y,$$

where p, w are assumed to be of the same sign (typically assumed positive), p, p', q and w are continuous functions of x. If this equation is endowed with the following boundary conditions

$$\begin{aligned} \alpha_1 y(a) + \alpha_2 y'(a) = 0 \quad \alpha_1^2 + \alpha_2^2 > 0, \\ \beta_1 y(b) + \beta_2 y'(b) = 0 \quad \beta_1^2 + \beta_2^2 > 0, \end{aligned}$$

then the statements in our class hold. Let (y_i, λ_i), $i = 1, 2$, be two eigen-pairs of the problem with $\lambda_1 \neq \lambda_2$. Prove that y_i are orthogonal in (weighted) L^2 as

$$\langle y_1, y_2 \rangle := \int_a^b y_1(x) y_2(x) w(x) dx = 0.$$

Remark: Indeed, one can show that

$$\langle y_n, y_m \rangle = \int_a^b y_n(x) y_m(x) w(x) dx = \delta_{mn},$$

where δ_{mn} is the Kronecker delta function.

2. Some of you may be wondering why we care about and look for a solution (only) in L^2. This is a very good question, and here are some of the main reasons. First of all, it is obvious that a solution in L^2 is not enough, and ultimately we are interested in finding smooth solutions. However, you will find that you lack the tools to do this at this stage. Therefore, one can first find some solutions in L^2, and then prove (if we can and if they are) that they are actually in L^4, L^6,...L^∞ or other spaces (Sobolev spaces), and then finally smooth (or not) by techniques called *a prior estimate* or *regularity estimate*. Of course, other advanced mathematics must come into play, and they are beyond the scope of this course. For example, one of the seven Millennium Prize Problems in Mathematics proposed by the Clay Mathematics Institute in May 2000 is to prove the existence or nonexistence of a smooth and globally well-defined solution to the Navier-Stokes equations in 3D, but it remains open to this day. Second, some equations (e.g., wave equations) do not have smooth solutions and can develop a shock or singularity in finite time. Many reaction-diffusion equations/systems also admit solutions with their L^∞ going to infinity in finite time, and this is called blow–up in finite time. Therefore, finding a smooth solution for all time $t \in \mathbb{R}^+$, is impossible for many problems. I also want to mention that saying that L^2 is not enough does not mean that it is not interesting, people nowadays are still very interested in finding solutions in L^p spaces, or the so called weak solutions, but the method of separation of variables does not work very well, not only because of the PDE, but also because of the properties of the space itself, especially if the function (solution) has a jump discontinuity. The eigen–expansion or Fourier expansion of $f(x)$ is the limit of the following sum as $N \to \infty$

$$f^N(x) := \sum_{n=0}^{\infty} C_n X_n(x),$$

where $\{X_n(x)\}$ is the orthogonal basis of $L^2(0, L)$ and

$$C_n = \frac{\int_0^L f(x)X_n(x)dx}{\int_0^L X_n^2(x)dx}.$$

then the general theory states that $f^N \to f$ pointwise in $(0, L)$ if $f(x)$ is continuous, uniformly if $f(x)$ is differentiable, while in L^2 if f is merely square integrable. In the last case, in particular, when $f(x)$ has jump discontinuity, one may see that the approximation can go wild, and this is usually referred to as the so-called *Gibbs phenomenon*. To see this yourself, let us consider the following example: let $f(x)$ be a function with a jump at $x = 0$ defined over $(-\pi, \pi)$ as: $f(x) = 1$ for $x \in [0, \pi)$ and $f(x) = -1$ for $x \in (-\pi, 0)$. First of all, write $f(x)$ into its series as

$$f(x) = \sum_{n=1}^{\infty} C_n \sin nx. \tag{0.1}$$

Find C_n. Now let us approximate it by the sum of the first N terms as before

$$f_N(x) := \sum_{n=1}^{N} C_n \sin nx \tag{0.2}$$

for some large N. Plot $f_N(x)$ over $(-\pi, \pi)$ for $N = 2, 4, 8, 16$ on the same graph. Try $N = 16, 32$ and 64 again. What are your observations? You can try with even larger N.

3. We recall from baby calculus that a number sequence is called a Cauchy sequence if $|a_{n+1} - a_n|$ is arbitrarily small if n is arbitrarily large; for example $a_n = \frac{1}{n}$ is Cauchy because $|a_{n+1} - a_n| = \frac{1}{n(n+1)} \to 0$ as $n \to \infty$. And a region is called **complete** if every Cauchy converges in it. For example, $(0, 1)$ is not closed since $a_n = \frac{1}{n}$ is Cauchy but $a_n \to 0 \notin (0, 1)$, while $[0, 1]$ is closed. There are more or less rigorous definitions of a closed region. We have a cousin of closedness when it comes to function space, i.e., completeness. For a function space $\mathcal{X}$ endowed with a norm (called a normed space or a metric space) denoted by $\|\cdot\|_{\mathcal{X}}$, we say that a function sequence is Cauchy if $\|f_{n+1}(x) - f_n(x)\|_{\mathcal{X}} \to 0$ as $n \to \infty$. Here the only difference/generalization is that the absolute value distance is replaced by the so-called "norm"; then we say that space $\mathcal{X}$ is **complete** if every Cauchy sequence converges (in the space or that norm). For instance, one can prove that $L^p(\Omega)$, $p \in (1, \infty)$ is complete (well, you should have learned in your Analysis course or learned it by yourself otherwise); moreover, in Euclidean space complete is equivalent as closed and bounded.

   Going back to the completeness of L^p, if f_n is Cauchy in L^p its limit must be in L^p. Then one usually needs to verify the Cauchyness of the sequences.

   i) prove that the sequence f_N given by (0.2) is Cauchy in L^2. Then according to i), its limit (0.1) belongs to L^2 and this gives you proof of the convergence fashion we had in class;

   ii) plot $\|f_{n+1} - f_n\|_{L^2((-\pi,\pi))}$ for $n = 1, 2, ...$ and you should observe the convergence; indeed, to better illustrating the convergence it makes sense to plot the log error $\mathcal{E}_n := \log(\|f_{n+1} - f_n\|_{L^2})$ for n large, for instance starting from 10. Remark: MATLAB can evaluate the integrals hence you do not have to do it by brutal force;

   iii) try different L^p norms with $p = 5, 10, 20, 30, ...$ and do the same as in ii); plot them in the same graph; what are your observations?

   iv) now find the max-norm and do the same as in ii).

4. This problem is to warm up and prepare you for the fashions of convergence of sequences of functions. The formal way (using ϵ–δ language) to define **pointwise convergence** of $f_n(x)$ over set E is, $\forall x \in E$ and $\forall \epsilon > 0$, there exists $N_0 \in \mathbb{N}^+$ such that for all $n \geq N_0$ we have that $|f_n(x) - f(x)| \leq \epsilon$. **Uniform convergence** of $f_n \to f$ is that, $\forall \epsilon > 0$, there exists $N_1 \in \mathbb{N}^+$ such that for all $n \geq N_1$ we have that $|f_n(x) - f(x)| \leq \epsilon$ $\forall x$ in E. The difference is the order of $\forall x$ and N, therefore N_0 depends on both ϵ and x, while N_1 does not depend on x hence by saying uniform convergence we meant that it is uniform with respect to x, or the convergence speed does not depend on x.

   To elaborate, we recall that a function $f(x)$ is **continuous** at x_0 if $\forall \epsilon > 0$, we can find $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \epsilon$. Here δ depends on ϵ and also x_0. If δ only depends on ϵ and is independent of x_0, i.e., $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$ for any $x, y \in E$, then f is **uniformly**

continuous. There is another type of continuity called **equicontinuity** for a sequence of functions $\{f_n(x)\}$, which states that $|x - x_0| < \delta$ implies $|f_n(x) - f_n(x_0)| < \epsilon$ for all n, while δ may depend on x_0 but does not depend on n. Similarly, if δ only depends on ϵ, we say that $\{f_n\}$ is **uniform equicontinuous**. An important property of equicontinuous function sequence according to **Arzela-Ascoli** Theorem is that any uniformly bounded and equicontinuous sequence has a subsequence that converges uniformly. This theorem is one of two cornerstones in modern mathematical analysis (the other being semi-continuity), and you are not required to understand equicontinuity for this course.

We already know that uniform convergence implies pointwise convergence but not the other way. However, if the limit is also continuous, then pointwise convergence implies uniform convergence (You might be asked to prove this in an advanced analysis course).

(a). Prove that if a sequence of continuous functions f_n converges to f uniformly in (a, b), then f is continuous. (Hint: use the $\epsilon - \delta$ language; for each x_0 in E, choose $f_n(x_0)$ that converges to $f(x_0)$; do the same for $f_n(x)$ with x being in the neighbourhood of x_0.)

(b). Another important application of uniform convergence is the switch of the order of integral and limit. Consider the following so-called tent–function

$$f_n(x) = \begin{cases} n^2 x, & x \in [0, \frac{1}{n}], \\ 2n - n^2 x, & x \in (\frac{1}{n}, \frac{2}{n}], \\ 0, & x \in (\frac{2}{n}, 1], \end{cases} \tag{0.3}$$

Find the pointwise limit $f(x)$ of $f_n(x)$; evaluate the limits

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx \text{ and } \int_0^1 \lim_{n\to\infty} f_n(x)dx.$$

Are they equal?

(c). What is Lebesgue's Dominated Convergence Theorem? Why does it fail in the example above?

(d). Let $f_n(x)$ be a sequence that converges to $f(x)$ uniformly in interval (a, b). Prove that

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx = \int_a^b f(x)dx.$$

5. It is known that strong convergence implies weak convergence, while not the converse. One counter-example we mentioned in class is $f_n(x) := \sin nx$ over $(0, 2\pi)$ (or $(0, \pi)$).

   (i) Prove that $\sin nx \nrightarrow 0$ in $L^2((0, 2\pi))$.

   (ii) Prove that $\sin nx \rightharpoonup 0$ weakly by showing

   $$\int_0^{2\pi} g(x) \sin nx dx \to 0 = \Big(\int_0^{2\pi} g(x) 0 dx\Big) \quad \forall g \in L^2((0, 2\pi)).$$

   If suffices even if $g \in L^1$. Hint: Riemann–Lebesgue lemma.

   (iii). Find another counter-example in textbook or online.

6. (**only for motivated students**) We recall that $f_n(x) \rightharpoonup f(x)$ weakly in L^p (resp. convergence in distribution) if for any $\phi \in L^q$ (resp. continuous and bounded), which is its conjugate space with $\frac{1}{p} + \frac{1}{q} = 1$, we have that

   $$\int_\Omega f_n \phi dx \to \int_\Omega f \phi dx.$$

   Here we see that for any g in L^q

   $$< \cdot, g >= \int_\Omega \cdot g$$

   defines a bounded linear functional for L^p. Then we also call L^q *the dual space* of L^p since any element in L^q defines a functional for L^q.

   (i) Another type of convergence that you may see sometimes is $\|f_n\|_p \to \|f\|_p$, which merely states the convergence of a sequence of real numbers. Prove that if $f_n \to f$ in L^p, then $\|f_n\|_p \to \|f\|_p$ (Use Minkowski triangle inequality); however the opposite statement is not necessarily true. Give a counter-example and show it;

   (ii) Prove that, if $f_n \rightharpoonup f$ weakly and $\|f_n\|_p \to \|f\|_p$, then $f_n \to f$ strongly.