Introduction to PDEs, Fall 2024

$\mathbf{Mid\text{-}term}\ \mathbf{1}\ \mathrm{Make\text{-}up}$

2 Hours and 30 Minutes

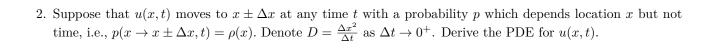
Name(Print):		
Student No:		
Signature:		
There are 10 problems, 10 points each, 100 points in total.		
Show details to get full credits. Make your justifications clear and direct.		

Leave the following table blank

Score Table			
Problem	Points	Score	
1	10		
2	10		
3	10		
4	10		
5	10		
6	10		
7	10		
8	10		
9	10		
10	10		
	Total score		

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1. Suppose that u(x,t) moves to $x \pm \Delta x$ at any time t with a probability $\frac{1}{2}$, i.e., $p(x \to x \pm \Delta x, t) = \frac{1}{2}$. Denote $D = \frac{\Delta x^2}{\Delta t}$ as $\Delta t \to 0^+$. Derive the PDE for u(x,t).



3. Let us consider a similar scenario in a 2D lattice with mesh size $\Delta x = \Delta y$. Let u(x,y,t) be the number of particles at location $(x,y) \in \mathbb{R}^2$ and time t>0. Suppose that each particle, at the next time $t+\Delta t$, moves northwards, southwards, westwards or eastwards with probability $\frac{1}{4}$, i.e., $p((x,y) \to (x \pm \Delta x,y),t) = p((x,y) \to (x,y \pm \Delta y),t) = \frac{1}{4}$. Assume that $D = \frac{\Delta x^2}{\Delta t}$ as $\Delta t \to 0^+ > 0$. Derive the PDE for u(x,y,t).

- 4. 1. Let $f(\mathbf{x})$ be a differentiable function defined over $\Omega \subset \mathbb{R}^n$, $n \geq 1$. Write down the definition of
 - 1). the gradient operator ∇f ;
 - 2). the Laplace operator Δf ;
 - 2. Consider the following reaction-diffusion equation for $u(\mathbf{x},t)$

$$u_t = D\Delta u + f(u, \mathbf{x}, t), \quad x \in \Omega, t > 0.$$

- 1). Suppose that u denotes the temperature at location-time (\mathbf{x}, t) . What does it mean or model if t > 0?
- 2). Suppose that u denotes the population at location-time (\mathbf{x},t) . What does it mean or model if t>0?

- 5. Let Ω be a bounded domain in \mathbb{R}^N , $N \geq 1$.
 - (i) Write down the definitions of Dirichlet, Neumann, Robin boundary conditions.
 - (ii) What are the physical interpretations of DBC and NBC if u(x,t) represents the temperature?
 - (iii) What are the biological interpretations of DBC and NBC if u(x,t) represents the population density?

6. Consider the following initial boundary value problem

$$\begin{cases} u_t = D\Delta u + f(x,t), & x \in \Omega, t > 0, \\ u(x,0) = \phi(x), & x \in \Omega, \\ u(x,t) = \psi(x), & x \in \partial\Omega, t > 0. \end{cases}$$
 (1)

Use the energy method to prove the uniqueness of (1).

7. Let us reconsider the following reaction-diffusion system with mixed boundary condition

$$\begin{cases}
 u_t = D\Delta u + f(x,t), & x \in \Omega, t > 0, \\
 u(x,0) = \phi(x), & x \in \Omega, \\
 u(x,t) = \psi_1(x,t), & x \in \partial\Omega_1, t > 0, \\
 \partial_{\mathbf{n}} u(x,t) = \psi_2(x,t), & x \in \partial\Omega_2, t > 0,
\end{cases} \tag{2}$$

where $\partial\Omega_1$ denotes part of the boundary Ω and $\partial\Omega_2$ denotes the rest part. Prove the uniqueness of (2).

8. Denote $u_n(x,t) := e^{-D(\frac{n\pi}{L})^2 t} \sin \frac{n\pi x}{L}$.

i) show that, by formal calculation, for each $N < \infty$, the series $\sum_{n=1}^{N} c_n u_n(x,t)$ is a solution of the following baby heat equation

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2},$$

where c_n are constants.

ii) Fixed L and D, for example, by assuming that $L=\pi$ and D=0.05. Sketch $u_1(x,t)$ over $x\in(0,\pi)$ with $t=0,\,t=0.5,\,t=2$ in the same coordinate.

- 9. 1) Write down the definition of $L^2(\Omega)$ and L^2 -norm;
 - 2) Write down the definition that functions f and g are orthogonal in $L^2(\Omega)$;
 - 3) Perform straightforward calculations to verify that $\sin \frac{m\pi x}{L}$ and $\sin \frac{n\pi x}{L}$ are orthogonal in $L^2(0,L)$ as

$$\int_0^L \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L} = \frac{L}{2} \delta_{mn} = \begin{cases} \frac{L}{2}, & \text{if } m = n, \\ 0, & \text{if } m \neq n; \end{cases}$$

here δ is the so-called Kronecker delta function.

- 10. 1). Show that $f(x) = \frac{1}{\sqrt{x}} \in L^1(0,1)$, but not in $L^2(0,1)$.
 - 2). What would be the general conditions for a function of the form $f(x) = x^{\alpha}$ to belong to $L^{p}(0,1)$, but not $L^{q}(0,1)$, assuming $p,q \in (1,\infty)$ for simplicity?