Introduction to PDEs, Fall 2024

Homework 2 solutions

Name:_______________

1. Suppose that $u(x,t)$ moves to $x \pm \Delta x$ at any time t with a probability p which depends location x but not time, i.e.,
$$p(x \to x \pm \Delta x, t) = \rho(x)$$
Put $D = \frac{\Delta x^2}{\Delta t}$ as $\Delta t \to 0^+$. Derive the PDE for $u(x,t)$. What if the probability also depends on time, i.e.,
$$p(x \to x \pm \Delta x, t) = \rho(x,t)$$
or
$$p(x \to x \pm \Delta x, t) = \rho(x, t + \Delta t).$$

Solution 1. *First, we begin with the scenario where $p(x \to x \pm \Delta x, t) = \rho(x)$. Consequently, we have $p(x \pm \Delta x \to x, t) = \rho(x \pm \Delta x)$ and $p(x \to x, t) = 1 - 2\rho(x)$.*

Following similar reasoning as above, we can express the evolution of $u(x,t)$ as follows:

$$\begin{aligned} u(x,t+\Delta t) =& u(x+\Delta x,t)p(x+\Delta x \to x,t) + u(x-\Delta x,t)p(x-\Delta x \to x,t) + u(x,t)p(x \to x,t) \\ =& u(x+\Delta x,t)\rho(x+\Delta x) + u(x-\Delta x,t)\rho(x-\Delta x) + u(x,t)(1-2\rho(x)). \end{aligned} \tag{0.1}$$

To simplify equation (0.1), we can expand both $u(x \pm \Delta x, t)$ and $\rho(x \pm \Delta x)$. However, it may be more straightforward to define $w(x \pm \Delta x, t) := u(x \pm \Delta x, t)\rho(x \pm \Delta x)$. This allows the right-hand side of (0.1) to be rewritten as:

$$w(x+\Delta x,t) + w(x-\Delta x,t) + u(x,t)(1-2\rho(x)) = \frac{\partial^2 w}{\partial x^2}\Delta x^2 + u(x,t) + O(\Delta x^4).$$

By equating this expression with the left-hand side and taking the limit as $\frac{\Delta x^2}{\Delta t} \to 0^+$, we arrive at the partial differential equation (PDE):

$$u_t = D(\rho(x)u(x,t))_{xx}.$$

Secondly, if the probability also depends on time, we have $p(x \to x \pm \Delta x, t) = \rho(x,t)$. The right-hand side of (0.1) remains unchanged, as we do not need to expand it with respect to time t. Thus, we retain the same partial differential equation as above:

$$u_t = D(\rho(x)u(x,t))_{xx}.$$

However, if the transition probability depends on time as $p(x \to x \pm \Delta x, t) = \rho(x, t+\Delta t)$, then (0.1) should be expressed as:

$$\begin{aligned} u(x,t+\Delta t) =& u(x+\Delta x,t)p(x+\Delta x \to x,t) + u(x-\Delta x,t)p(x-\Delta x \to x,t) + u(x,t)p(x \to x,t) \\ =& u(x+\Delta x,t)\rho(x+\Delta x,t+\Delta t) + u(x-\Delta x,t)\rho(x-\Delta x,t+\Delta t) \\ &+ u(x,t)(1-2\rho(x,t+\Delta t)). \end{aligned} \tag{0.2}$$

We will need to analyze (0.2) in more detail. Although we could expand each term and multiply them out, we can instead define $w(x \pm \Delta x, t) := u(x \pm \Delta x, t)\rho(x \pm \Delta x, t + \Delta t)$ as stated above, allowing us to rewrite (0.2) as follows:

$$\begin{aligned} u(x, t+\Delta t) =& u(x+\Delta x, t)\rho(x+\Delta x, t+\Delta t) + u(x-\Delta x, t)\rho(x-\Delta x, t+\Delta t) \\ &+ u(x,t)(1-2\rho(x, t+\Delta t)) \\ =& w(x+\Delta x, t) + w(x-\Delta x, t) + u(x,t)(1-2\rho(x,t+\Delta t)) \\ &+ u(x+\Delta x, t)\Big(\rho(x+\Delta x, t+\Delta t) - \rho(x+\Delta x, t)\Big) \\ &+ u(x-\Delta x, t)\Big(\rho(x-\Delta x, t+\Delta t) - \rho(x-\Delta x, t)\Big) \\ =& 2w(x,t) + w_{xx}(x,t)\Delta x^2 + O(\Delta x^4) + u(x,t)(1-2\rho(x,t+\Delta t)) \\ &+ u(x+\Delta x, t)\Big(\rho_t(x+\Delta x, t)\Delta t + O(\Delta t^2)\Big) \\ &+ u(x-\Delta x, t)\Big(\rho_t(x-\Delta x, t)\Delta t + O(\Delta t^2)\Big). \end{aligned} \tag{0.3}$$

Since we will send both Δx and Δt to zero while maintaining the ratio $\frac{\Delta x^2}{\Delta t} = D > 0$, we only need to keep the residual terms (or the so-called remainders) up to order Δx^2 and Δt. By continuing the expansion in (0.3), we can collect the relevant terms:

$$\begin{aligned} u(x, t+\Delta t) =& 2w(x,t) + w_{xx}(x,t)\Delta x^2 + O(\Delta x^4) + u(x,t)(1 - 2\rho(x,t) - 2\rho_t(x,t)\Delta t + O(\Delta t^2)) \\ &+ u(x+\Delta x, t)\rho_t(x+\Delta x, t)\Delta t + u(x-\Delta x, t)\rho_t(x-\Delta x, t)\Delta t + O(\Delta t^2) \\ =& u(x,t) + w_{xx}(x,t)\Delta x^2 - 2u(x,t)\rho_t(x,t)\Delta t + O(\Delta x^4) + O(\Delta t^2), \end{aligned} \tag{0.4}$$

where we used the notation $w := u\rho$ to derive the last identity. Finally, we divide both sides of (0.4) by Δt and take the limit as Δt approaches zero, while maintaining $D = \frac{\Delta x^2}{\Delta t} > 0$. This allows us to collect the partial differential equation:

$$u_t + 2u\rho_t = D(u\rho)_{xx}.$$

For instance, if ρ is independent of time t, then $\rho_t = 0$, and this PDE reduces to the previous form.

2. Now change the probability above as arrival–dependent, i.e.,

$$p(x \to x \pm \Delta x, t) = \rho(x \pm \Delta x).$$

Derive the PDE for $u(x,t)$. Is the PDE the same as the one above? Compare them and state your observations.

Solution 2. *When the probability is arrival-dependent, we have $p(x \pm \Delta x \to x, t) = \rho(x)$ and $p(x \to x, t) = 1 - \rho(x+\Delta x) - \rho(x - \Delta x)$. Therefore, similarly to the previous derivation, we can establish the following difference equation:*

$$\begin{aligned} u(x, t+\Delta t) =& u(x+\Delta x, t)p(x+\Delta x \to x, t) + u(x-\Delta x, t)p(x-\Delta x \to x, t) + u(x,t)p(x \to x, t) \\ =& u(x+\Delta x, t)\rho(x) + u(x-\Delta x, t)\rho(x) + u(x,t)\left(1 - \rho(x+\Delta x) - \rho(x-\Delta x)\right). \end{aligned} \tag{0.5}$$

It is not difficult to simplify (0.5), leading us to the partial differential equation:

$$u_t = D(\rho u_{xx} - u\rho_{xx}).$$

3. Let us now consider heat diffusion in an inhomogeneous material in $\mathbb{R}^3$. That is, the density $\rho(x)$ (*gram/cm.*^{3}), the heat capacity $c(x)$ (*calorie/gram.degree*), and the thermal conductivity $\kappa(x)$ (*calorie/cm.degree.second*) are all functions of x. Let $u(x,t)$ be the temperature at location x and time t. Show that the equation for such a heat flow is

$$c(x)\rho(x)\frac{\partial u}{\partial t} = \nabla \cdot \big(\kappa(x)\nabla u\big), \tag{0.6}$$

You must present your justifications in a logically well-ordered way.

Solution 3. *I would like to mention that you can work it out similar to what we did in the lecture: consider any domain in $\mathbb{R}^n$. Then it is easy to know that the total thermal energy within this region is*

$$E(t) = \int_\Omega c(x)\rho(x)u(x,t)dx$$

and the rate of change of the energy is

$$\frac{dE(t)}{dt} = \frac{d}{dt}\int_\Omega c(x)\rho(x)u(x,t)dx = \int_\Omega c(x)\rho(x)\frac{\partial u(x,t)}{\partial t}dx.$$

On the other hand, by the conservation of thermal energy, we know that the rate of change must equals those across the boundary, which takes the form

$$\int_{\partial\Omega} \kappa(x)\partial_{\boldsymbol{n}} u(x,t)dS = \int_\Omega \nabla\cdot\big(\kappa(x)\nabla u(x,t)\big)dx,$$

where the last identity follows from the divergence theorem. Equating the equations above, we have

$$\int_\Omega c(x)\rho(x)\frac{\partial u(x,t)}{\partial t}dx = \int_\Omega \nabla\cdot\big(\kappa(x)\nabla u(x,t)\big)dx$$

hold for any Ω. Therefore, we can show that, through the same contradiction argument, $u(x,t)$ satisfies PDE (0.6) as

$$c(x)\rho(x)\frac{\partial u(x,t)}{\partial t} = \nabla\cdot\big(\kappa(x)\nabla u(x,t)\big), \quad x\in\Omega, t>0.$$

Note that both hand sides have the unit

$$\frac{calorie}{gram\times degree}\times\frac{gram}{cm^3}\times cm^2\times\frac{degree}{second} = \frac{calorie}{cm\times second}$$

4. Perform straightforward calculations to verify that

$$\int_0^L \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L} = \int_0^L \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L} = \frac{L}{2}\delta_{mn} = \begin{cases} \frac{L}{2}, & \text{if } m=n,\\ 0, & \text{if } m\neq n;\end{cases}$$

here δ is the so-called Kronecker delta function.

Solution 4. *(i). For the sine functions: if $m\neq n$, we can compute that*

$$\begin{aligned}\int_0^L \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx &= \int_0^L -\frac{\cos\frac{(m+n)\pi}{L} - \cos\frac{(m-n)\pi}{L}}{2}dx\\ &= (-\frac{1}{2})\Big[\frac{L}{(m+n)\pi}\sin\frac{(m+n)\pi x}{L}\Big|_0^L - \frac{L}{(m-n)\pi}\sin\frac{(m-n)\pi x}{L}\Big|_0^L\Big] = 0;\end{aligned}$$

if $m=n$, we have that

$$\int_0^L \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx = \int_0^L\frac{1-\cos\frac{2m\pi x}{L}}{2}dx = \frac{L}{2} - \frac{L}{4m\pi}\sin\frac{2m\pi x}{L}\Big|_0^L = \frac{L}{2}.$$

(ii). For the cosine function: if $m\neq n$, we can compute that

$$\begin{aligned}\int_0^L \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx &= \int_0^L \frac{\cos\frac{(m+n)\pi}{L} + \cos\frac{(m-n)\pi}{L}}{2}dx\\ &= \frac{1}{2}\Big[\frac{L}{(m+n)\pi}\sin\frac{(m+n)\pi x}{L}\Big|_0^L + \frac{L}{(m-n)\pi}\sin\frac{(m-n)\pi x}{L}\Big|_0^L\Big] = 0;\end{aligned}$$

if $m=n$, we have that

$$\int_0^L \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}dx = \int_0^L\frac{1+\cos\frac{2m\pi x}{L}}{2}dx = \frac{L}{2} + \frac{L}{4m\pi}\sin\frac{2m\pi x}{L}\Big|_0^L = \frac{L}{2}.$$

Therefore, we can conclude that

$$\int_0^L \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L} = \int_0^L \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L} = \frac{L}{2}\delta_{mn} = \begin{cases} \frac{L}{2}, & \textit{if } m = n \\ 0, & \textit{if } m \neq n \end{cases}$$

I'd like to point out that later in this course we will encounter another important delta function, known as the Dirac delta function. In general, when people talk about the delta function in the context of partial differential equations, they mean the Dirac delta function, not the Kronecker delta.

5. Consider the following generalized heat equation for $m \geq 1$.

$$u_t = \Delta(u^m), \mathbf{x} \in \mathbb{R}^N, t \in \mathbb{R}^+, \tag{0.7}$$

which reduces to the classical heat equation when $m = 1$, and to Boussinesq's equation when $m = 2$. Note that one can rewrite $u_t = \nabla\cdot(mu^{m-1}\nabla u)$, thus recognizing the diffusion rate as mu^{m-1}. This equation was proposed in the study of ideal gas flowing isentropically in a homogeneous medium, or the flow of fluid through porous media (such as oil through the soil), where instead of the Fourier law of constant diffusivity, the law $\mathbf{J} = -u^{m-1}\nabla u$ is usually observed. It is not necessary to know why this particular form was chosen, but it is not surprising to imagine that the flow of dye in water behaves differently from that of oil in soil.

A fundamental solution to the problem in 1D was obtained in the 1950s by the Russian mathematician Barenblatt, where in $\mathbb{R}^N$, $N \geq 1$, one has

$$u(\mathbf{x}, t) = t^{-\alpha}\left(C - \kappa|\mathbf{x}|^2 t^{-2\beta}\right)_+^{\frac{1}{m-1}}, \tag{0.8}$$

where

$$(f)_+ := \max\{f, 0\}, \alpha := \frac{N}{(m-1)N+2}, \beta := \frac{\alpha}{N}, \kappa = \frac{\alpha(m-1)}{2mN}$$

and C is an arbitrary positive constant.

(1) In 1D, prove that (0.8) satisfies the PME (0.7). You can, but are not required to, prove this in any dimension;

(2) In 1D ($n = 1$), choose $m = 2$ and $C = 10$, and then plot $u(x, t)$ for $t = 1$, 2, 5, and 10.You can also plot this in 2D or higher dimensions.

Solution 5. *(1) can be verified by straightforward calculations;*

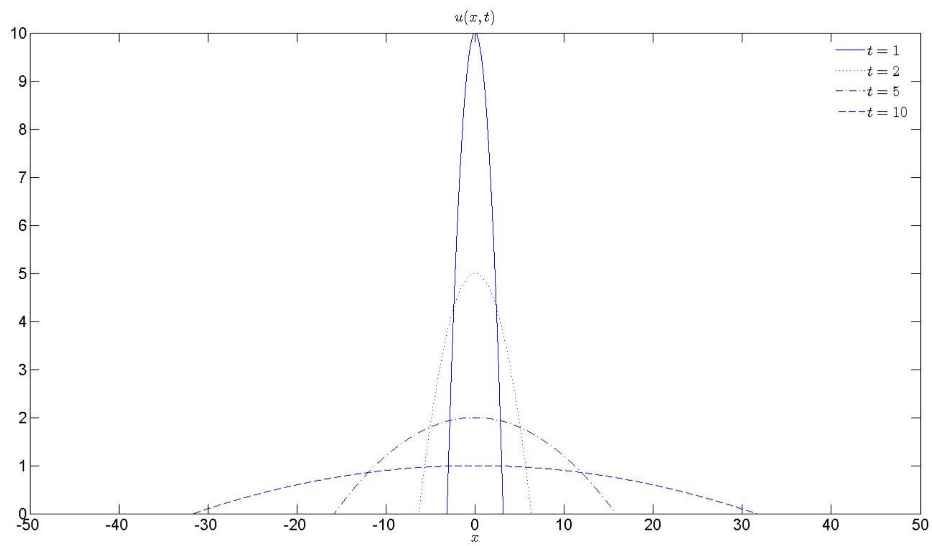

Figure 1: The solutions at the given times with the given coefficients. I have chosen my own values for α, β, κ in the problem, but you are free to choose your own set of parameters. You can see from these plots that the solution *travels* a finite distance in a finite time, and this is called the finite rate of propagation. We will see later in this course that the classical heat equation, not the porous media equation introduced here, has an infinite rate of propagation.

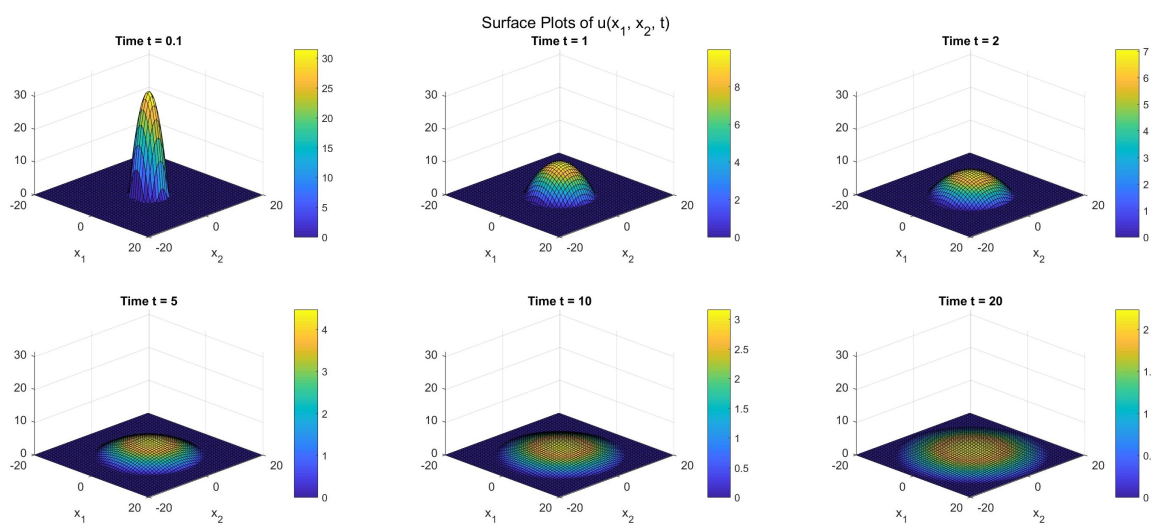

Figure 2: The 2D solutions at the specified times with the same coefficients as above.