Introduction to PDEs, Fall 2024

Homework 9 due Dec 12

Name:______________

1. Consider the following problem over the half-plane

$$\begin{cases} u_t = Du_{xx}, & x \in (0,\infty), t > 0, \\ u(x,0) = 0, & x \in (0,\infty), \\ u(0,t) = N_0, & t > 0, \end{cases} \tag{0.1}$$

where N_0 is a point heating source at the endpoint.

(1) Solve for $u(x,t)$ in terms of an integral. *Suggested solution*:

$$u(x,t) = \frac{N_0}{\sqrt{4\pi Dt}} \int_0^\infty e^{-\frac{|x+\xi|^2}{4Dt}} d\xi.$$

(2) Set $D = N_0 = 1$. Plot your solve for $t = 0.001$, 0.01, 0.1 and 1. The graphs should match the physical description of the problem.

2. We know that the solution $u(x,t)$ to the following Cauchy problem

$$\begin{cases} u_t = Du_{xx}, & x \in (-\infty,\infty), t > 0, \\ u(x,0) = \phi(x), & x \in (-\infty,\infty), \end{cases} \tag{0.2}$$

is given by

$$u(x,t) = \int_{\mathbb{R}} \phi(\xi) G(x,t;\xi) d\xi,$$

with

$$G(x,t;\xi) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{(x-\xi)^2}{4Dt}}{}^{*}.$$

Suppose that $\phi(x) \in L^\infty(\mathbb{R}) \cap C^0(\mathbb{R})$ (i.e., continuous and bounded).

(a) Prove that $u(x,t) \in C_x(\mathbb{R})$, i.e., continuous with respect to x. Hint: you can either use $\epsilon - \delta$ language or show that $u(x_n,t) \to u(x,t)$ as $x_n \to x$ for each fixed (x,t). Similarly, you can continue to prove that $u \in C_x^k(\mathbb{R})$ for any $k \in \mathbb{N}^+$, hence C_x^∞, while you can skip this part;

(b) Indeed, as you may have seen from your proof, the continuity condition on initial data $\phi(x)$ above is not required, i.e., $u(x,t) \in C^\infty(\mathbb{R} \times [\epsilon,\infty))$ for any $\epsilon > 0$, if $\phi(x) \in L^\infty(\mathbb{R})$ (even if it has jump or discontinuity). To illustrate this, let us assume $\phi(x) = 1$ for $x \in (-1,1)$ and $\phi(x) = 0$ elsewhere, therefore it has jumps at $x = \pm 1$. Choose $D = 1$, then use MATLAB to plot the integral solution $u(x,t)$ above for time $t = 0.001$, $t = 0.01$, $t = 0.1$ and $t = 1$ in the same coordinate–choose the integration limit to be $(-M, M)$ for some M large enough so it approximates the exact solution. One shall see that $u(x,t)$ becomes smooth at $x = \pm 1$ for any small time $t > 0$, though two jumps are present in the initial data; this is the so-called smoothing or regularizing effect of diffusion.

3. This problem introduces how PDEs are connected to and applied in finance. Specifically, we will derive the price of a European option explicitly by solving the classical 1D heat equation, ultimately leading to the seminal Black-Scholes formula.

[*] I would like to point out that there should be no confusion on the notation here, where $G(x-\xi;t)$ was adopted in class. It is a matter of the integration variable and the parameter.

In mathematical finance, the Black-Scholes or Black-Scholes-Merton model is a PDE that describes the price evolution of a European call or put option under the Black-Scholes framework. Let S_t denote the price of the underlying risky asset at time t, typically a stock, which is treated as an independent variable that investors adjust to optimize their investment strategies. The model assumes that the price of S_t follows a geometric Brownian motion

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t{}^{\dagger},$$

Where W_t is Brownian motion (a continuous stochastic process with Gaussian increments, i.e., $W_{t+s} - W_s \sim N(0, s)$ for $t, s > 0$), it can be intuitively treated as a process that randomly increases or decreases over time. The parameter $\sigma > 0$ represents the magnitude of this randomness, quantifying the risk associated with investing in the stock. In finance, σ is known as *volatility* and can be thought of as analogous to the variance of a random variable in statistics. It follows that S_t, being influenced by W_t, becomes a stochastic process, meaning that at each fixed time t, S_t is a random variable dependent on a hidden random event ω. While financial time-series data suggest that volatility σ often depends on the stock price S_t and other factors—such as time delay, term structure, component structure, or phenomena like the volatility smile—for the sake of mathematical simplicity (as all models are approximations of reality), the Black-Scholes model assumes σ to be constant. Admittedly, this assumption limits the model's practical utility.

Under this framework, the dynamics described above lead to the following PDE:

$$\begin{cases} \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, & S > 0, 0 < t < T, \\ V(S, T) = \phi(S), & S > 0, \end{cases} \tag{0.3}$$

where $T > 0$ represents a pre-specified time, known as the *strike time* or maturity. The function V denotes the option value (the dependent variable). In finance, the term *option* refers to the right to buy or sell an asset at a predetermined price—called the *strike price* K—before or at the strike time T. Both K and T are specified in the option contract. To derive this PDE, you need a foundational understanding of stochastic calculus or Itô calculus. However, since this topic lies outside the scope of this homework or course, we will not delve into the derivation. For now, let us focus on formulating this PDE. It is important to note that, unlike many PDEs with initial conditions, the Black-Scholes PDE is paired with a *terminal condition.* This is because when reformulated as a heat equation, the diffusion coefficient becomes $-\frac{1}{2}\sigma^2 S^2$, which might seem unusual. However, this does not contradict the principle that the diffusion rate cannot be negative, as the transformation accounts for the sign change appropriately.

The goal of this homework is to demonstrate that equation (0.3) can be transformed into the Cauchy problem for the heat equation and subsequently solved explicitly. To achieve this, complete the following steps:

(a). Introduce the new variables

$$S = Ke^x, t = T - \frac{\tau}{\sigma^2/2},$$

where the constant K is the strike price. Let $v(x, \tau) = V(S, t)$. Show that

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + \Big(\frac{2r}{\sigma^2} - 1\Big)\frac{\partial v}{\partial x} - \frac{2r}{\sigma^2} v.$$

(b). Introduce

$$u(x, \tau) = e^{ax+b\tau} v(x, \tau).$$

Choose constants a and b such that $u(x, \tau)$ satisfies

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}. \tag{0.4}$$

(c). A European option can only be exercised at its expiration date, T. Specifically, the call option for a European option is defined mathematically as:

$$\phi(S) = \max\{S - K, 0\}.$$

†Here the subindex t, like it or not, merely means a notation, not the partial derivative.

Solve the classical heat equation (0.4) subject to this terminal condition. Express your solution $u(x,\tau)$ in terms of integrals.

(d). In terms of the solutions in (c) and the transformations. Show that the solution $V(S,t)$ to (0.3) is

$$V = SN(d_1) - Ke^{-r(T-t)}N(d_2),$$

where

$$d_1 = \frac{\ln S/K + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$
$$d_2 = \frac{\ln S/K + (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$

and N is the cumulative normal distribution that

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}dy.$$

(e) Suppose $S = 200$, $K = 210$, $r = 2\%$ (the annual interest rate), $\sigma = 0.58$, and the expiration date is in two months. Find the call value V. To use this formula, ensure that all parameters are converted to the same scale.

4. Show the following facts for Dirac delta function $\delta(x)$:

(1). $\delta(x) = \delta(-x)$, i.e., show that they hold in the distribution sense for any $\phi \in C_c(\mathbb{R})$:

$$\int_{\mathbb{R}} \delta(x)\phi(x)dx = \int_{\mathbb{R}} \delta(-x)\phi(x)dx.$$

This also applies to the rest.

(2). $\delta(kx) = \frac{\delta(x)}{|k|}$, where k is a non-zero constant;

(3). $\int_{\mathbb{R}} f(x)\delta(x-x_0)dx = f(x_0)$; $\delta(x-x_0)$ is occasionally written as $\delta_{x_0}(x)$;

(4). Let $f(x)$ be continuous except for a jump–discontinuity at 0. Show that

$$\frac{f(0^-)+f(0^+)}{2} = \int_{-\infty}^{\infty} f(x)\delta(x)dx$$

5. We know that the Heaviside step function

$$H(x) = \begin{cases} 1 & x > 0, \\ 0, & x < 0 \end{cases} \tag{0.5}$$

has the Dirac delta function $\delta(x)$ as its weak derivative. As I mentioned in class, you might encounter textbooks where the Heaviside function is defined differently, such as:

$$H(x) = \begin{cases} 1 & x > 0, \\ \frac{1}{2} \text{ (or any other number)} & x = 0, \\ 0, & x < 0. \end{cases} \tag{0.6}$$

According to Lebesgue's theory, the value of a function at a single point (or on a set of measure zero) does not affect its general properties. Two functions that are equal almost everywhere are considered identical. Accordingly, the two forms of $H(x)$ are regarded as equivalent, although we shall adopt the former in this course. Similarly, the weak derivative of a function is unique up to a set of measure zero. In other words, if $f(x)$ is a weak derivative of $F(x)$, then $g(x)$ is also a weak derivative if $f(x)$ and $g(x)$ differ only on a set of measure zero. This principle extends further. A function known as a bump function is defined as $B(x) = xH(x)$. Using the definition of a weak derivative, show that the weak derivative of $B(x)$ is $H(x)$.

6. Find the weak derivative of $F(x)$, denoted by $f(x)$

$$F(x) = \begin{cases} x, & 0 < x \le 1, \\ 1, & 1 \le x < 2. \end{cases} \tag{0.7}$$

7. One can easily generalize the second–order operator to higher dimension, the Laplace operator Δ over $\Omega \subset \mathbb{R}^n$, $n \geq 1$, with $x = (x_1, x_2, ..., x_n)$.

   (a) We say that f is radially symmetric if $f(x) = f(r)$, $r = |x| := \sqrt{\sum_{i=1}^n x_i^2}$. Prove that

   $$\Delta f(r) = f''(r) + \frac{n-1}{r} f'(r),$$

   where the prime denotes a derivative taken with respect to r.

   (b) Denote that $G(r) := \frac{1}{2\pi} \ln r$ for $n = 2$. We shall show that $\Delta G = \delta(r)$. For this moment, let us consider its regularization over $2D$ of the form

   $$G_\epsilon(r) = \frac{1}{2\pi} \ln(r + \epsilon), \epsilon > 0.$$

   Show that $\Delta G_\epsilon(r)$ converges to $\delta(x)$ in distribution as $\epsilon \to 0^+$. Hint: you can either apply Lebesgue's dominated convergence theorem, or use ϵ–δ language. Make sure you have checked all the conditions when applying the former one.

   (c) Denote $G(r) := -\frac{1}{4\pi r}$ for $n = 3$. Mimic (b) by finding an approximation G_ϵ and then show that this approximation ΔG_ϵ convergence to $\delta(x)$ in distribution.