Introduction to PDEs, Fall 2024

Homework 9 solutions

Name:________________

1. Consider the following problem over the half-plane

$$\begin{cases} u_t = Du_{xx}, & x \in (0,\infty), t > 0, \\ u(x,0) = 0, & x \in (0,\infty), \\ u(0,t) = N_0, & t > 0, \end{cases} \tag{0.1}$$

where N_0 is a point heating source at the endpoint.

(1) Solve for $u(x,t)$ in terms of an integral. *Suggested solution*:

$$u(x,t) = \frac{N_0}{\sqrt{4\pi Dt}} \int_0^\infty e^{-\frac{|x+\xi|^2}{4Dt}} d\xi.$$

(2) Set $D = N_0 = 1$. Plot your solution for $t = 0.001$, 0.01, 0.1 and 1. The graphs should match the physical description of the problem.

Solution 1. *Let us first consider its counterpart over $\mathbb{R}$ in the following form*

$$\begin{cases} u_t = Du_{xx}, & x \in (-\infty,\infty), t > 0, \\ u(x,0) = 0, & x \in (-\infty,\infty), \\ u(0,t) = \phi(x), & t > 0, \end{cases} \tag{0.2}$$

with

$$\phi(x) = \begin{cases} 0, & x > 0, \\ \Phi(x), & x < 0, \end{cases}$$

$\Phi(x)$ to be chosen. We observe that the solution to (0.2) solves (0.4) except the boundary condition. Therefore we shall choose $\Phi(x)$ to this end. Note that the solution to (0.2) is

$$u(x,t) = \int_{-\infty}^{\infty} \phi(\xi) G(x,t;\xi) d\xi = \int_{-\infty}^{0} \Phi(\xi) G(x,t;\xi) d\xi.$$

Coping it with the initial condition $u(x,0) = N_0$ gives us

$$N_0 = \int_{-\infty}^{0} \Phi(\xi) G(0,t;\xi) d\xi.$$

There are various choices of $\Phi(x)$ through which one can achieve this identity and the simplest one is a constant $\Phi \equiv K$. In this case, using the fact that

$$\int_{-\infty}^{0} G(0,t,\xi) d\xi = \frac{1}{2}$$

gives us that $\Phi(x) \equiv K = 2N_0$. Therefore we have that

$$u(x,t) = 2N_0 \int_{-\infty}^{0} G(x,t;\xi) d\xi$$

or an equivalent form

$$u(x,t) = 2N_0 \int_0^\infty G(x,t;-\xi) d\xi = \frac{N_0}{\sqrt{4\pi Dt}} \int_0^\infty e^{-\frac{|x+\xi|^2}{4Dt}} d\xi.$$

Again, I would like to remark that here the choice of such Φ is apparently not unique, however different Φ gives rise to different integral, while in the end, $u(x,t)$ end up the same since it is unique (though we have not proved the uniqueness for the Cauchy's problem). This is very similar to what we convert inhomogeneous boundary conditions into homogeneous ones, one has infinitely many choices of $w(x,t)$, and different ones give rise to different problems, while eventually ending up with the same solution for the original problem.

Even though the approach above may seem awkward or not natural to some of you, it is wrong to solve this problem by the kernel of half-plane

$$G^*(x,t;\xi) = G(x,t;\xi) - G(x,t;-\xi)$$

since the boundary condition is inhomogeneous. Some of your peer students have redone the whole problem from the beginning, that being said, first consider the problem over $(0,L)$

$$\begin{cases} u_t = Du_{xx}, & x \in (0,L), t>0, \\ u(x,0) = 0, & x \in (0,L), \\ u(0,t) = N_0, & t>0, \end{cases} \tag{0.3}$$

work on it by converting the inhomogeneous boundary condition into a homogeneous one, and then send L to infinity, which I have no problem with. In this spirit, one can simply denote $v(x,t) := u(x,t) - N_0$, then $v(x,t)$ satisfies

$$\begin{cases} v_t = Dv_{xx}, & x \in (0,\infty), t>0, \\ v(x,0) = -N_0, & x \in (0,\infty), \\ v(0,t) = 0, & t>0, \end{cases} \tag{0.4}$$

therefore by the formula for the half-line problem, we have that

$$v(x,t) = -N_0 \int_0^\infty G(x,t;\xi) - G(x,t;-\xi) d\xi$$

hence

$$u(x,t) = N_0 - N_0 \int_0^\infty G(x,t;\xi) - G(x,t;-\xi) d\xi,$$

which, in light of the identity

$$1 - \int_0^\infty G(x,t;\xi)d\xi = 1 - \int_{-\infty}^0 G(x,t;-\xi)d\xi = \int_0^\infty G(x,t;-\xi)d\xi,$$

also gives rise to the desired expression of $u(x,t)$ as above.

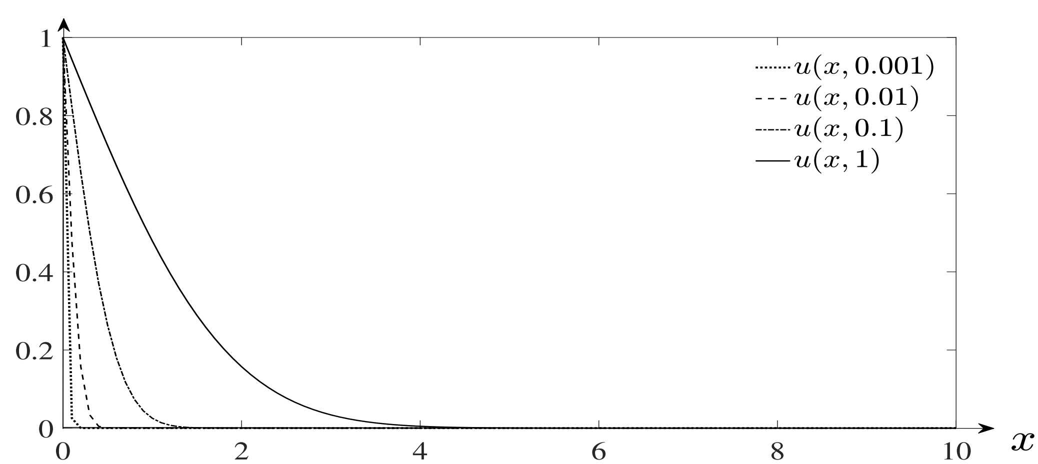

Figure 1: Solution $u(x,t)$ at different times. One observes that the DBC acts as a source that keeps pumping the heat into the region, hence for each location the temperature is increasing with respect to time as one can observe.

2. We know that the solution $u(x,t)$ to the following Cauchy problem

$$\begin{cases} u_t = Du_{xx}, & x \in (-\infty,\infty), t>0, \\ u(x,0)=\phi(x), & x\in(-\infty,\infty), \end{cases} \tag{0.5}$$

is given by

$$u(x,t)=\int_{\mathbb{R}} \phi(\xi)G(x,t;\xi)d\xi,$$

with

$$G(x,t;\xi)=\frac{1}{\sqrt{4\pi Dt}}e^{-\frac{(x-\xi)^2}{4Dt}}{}^{*}.$$

Suppose that $\phi(x)\in L^\infty(\mathbb{R})\cap C^0(\mathbb{R})$ (i.e., continuous and bounded).

(a) Prove that $u(x,t)\in C_x(\mathbb{R})$, i.e., continuous with respect to x. Hint: you can either use $\epsilon-\delta$ language or show that $u(x_n,t)\to u(x,t)$ as $x_n\to x$ for each fixed (x,t). Similarly, you can continue to prove that $u\in C^k_x(\mathbb{R})$ for any $k\in\mathbb{N}^+$, hence C^∞_x, while you can skip this part;

(b) Indeed, as you may have seen from your proof, the continuity condition on initial data $\phi(x)$ above is not required, i.e., $u(x,t)\in C^\infty(\mathbb{R}\times[\epsilon,\infty))$ for any $\epsilon>0$, if $\phi(x)\in L^\infty(\mathbb{R})$ (even if it has jump or discontinuity). To illustrate this, let us assume $\phi(x)=1$ for $x\in(-1,1)$ and $\phi(x)=0$ elsewhere, therefore it has jumps at $x=\pm1$. Choose $D=1$, then use MATLAB to plot the integral solution $u(x,t)$ above for time $t=0.001$, $t=0.01$, $t=0.1$ and $t=1$ in the same coordinate–choose the integration limit to be $(-M,M)$ for some M large enough so it approximates the exact solution. One shall see that $u(x,t)$ becomes smooth at $x=\pm1$ for any small time $t>0$, though two jumps are present in the initial data; this is the so-called smoothing or regularizing effect of diffusion.

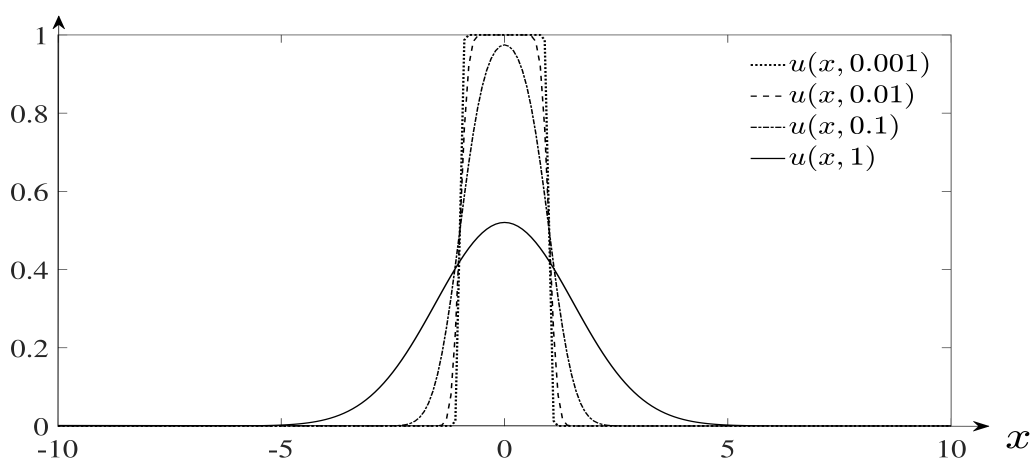

Figure 2: Smoothing effect of diffusion with diffusion rate chosen $D=1$. The initial condition is a characteristic function supported on $[-1,1]$. We see that the singularities at $x\pm1$ are smeared out immediately after even a tiny time $t=0.001$. Indeed, $u(x,t)$ is C^∞ smooth in x for each $t>0$ as one can prove.

Solution 2. *(a). I shall take the second approach here, i.e., by showing that $|u_n(x)-u(x)|\to0$ if $|x_n-x|\to0$ $(n\to\infty)$ with (x,t) fixed. To this end, we observe that for each fixed pair $(x,t)\in\mathbb{R}\times\mathbb{R}^+$*

$$\begin{aligned}|u(x_n,t)-u(x,t)|=&\Big|\int_{\mathbb{R}^n}\phi(\xi)\Big(G(x_n,t,\xi)-G(x,t,\xi)\Big)d\xi\Big|\\ \le&\int_{\mathbb{R}^n}|\phi(\xi)|\Big|G(x_n,t,\xi)-G(x,t,\xi)\Big|d\xi\\ \le&\|\phi(\xi)\|_{L^\infty(\mathbb{R}^n)}\int_{\mathbb{R}^n}\Big|G(x_n,t,\xi)-G(x,t,\xi)\Big|d\xi.\end{aligned}$$

*I would like to point out that there should be no confusion on the notation here, where $G(x-\xi;t)$ was adopted in class. It is a matter of the integration variable and the parameter.

It is easy to see that, in the last integral, the integrand $|G(x_n,t,\xi)-G(x,t,\xi)|$ converges to zero point wisely, therefore to obtain the convergence of the integral to zero through Lebesgue's dominated convergence theorem, one only needs to that the integrand is bounded by an integrable function or a constant. The latter case is impossible since it has singularities at both x_n and x. To show the former, one can apply the Mean Value Theorem to have that

$$|G(x_n,t,\xi)-G(x,t,\xi)| = G_x(\tilde{x},t,\xi)(x_n-x),$$

for some $\tilde{x}\in(x-1,x+1)$, for n large enough, show that G_x is absolutely integrable. I leave this to the student to verify. An alternative way is to follow the approach in the class by breaking $\mathbb{R}^n$ into two regions (three indeed) as follows: it is easy to know that, for any $\epsilon>0$, one can choose M large enough such that

$$\int_{I_M}\Big|G(x_n,t,\xi)-G(x,t,\xi)\Big|d\xi < \frac{\epsilon}{2},$$

where I denote for notational simplicity the outer region to be

$$I_M := \{|\xi|\geq M\}.$$

Note that I_M may depend on x and t, while it can be uniform in x_n as long as n is large enough; on the other hand, in this inner region, we have that, for n sufficiently large $G(x_n,t,\xi)<G(x,t,\xi)+1$ hence

$$|G(x_n,t,\xi)-G(x,t,\xi)| < 2G(x,t,\xi)+1,$$

which is, though not bounded in $\mathbb{R}^n\backslash I_M$, absolutely integrable over the inner region. Therefore, since the integrand converges to zero point wisely, one can apply the dominated convergence theorem to obtain that for n sufficiently large

$$\int_{\mathbb{R}^n\backslash I_M}\Big|G(x_n,t,\xi)-G(x,t,\xi)\Big|d\xi < \frac{\epsilon}{2}.$$

I would like to remark that this is why sometimes it is necessary or friendly to students/beginners to split $\mathbb{R}^n$ in this form.

(b). See figure 2.

3. This problem introduces how PDEs are connected to and applied in finance. Specifically, we will derive the price of a European option explicitly by solving the classical 1D heat equation, ultimately leading to the seminal Black-Scholes formula.

   In mathematical finance, the Black-Scholes or Black-Scholes-Merton model is a PDE that describes the price evolution of a European call or put option under the Black-Scholes framework. Let S_t denote the price of the underlying risky asset at time t, typically a stock, which is treated as an independent variable that investors adjust to optimize their investment strategies. The model assumes that the price of S_t follows a geometric Brownian motion

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t{}^{\dagger},$$

   Where W_t is Brownian motion (a continuous stochastic process with Gaussian increments, i.e., $W_{t+s}-W_s\sim N(0,s)$ for $t,s>0$), it can be intuitively treated as a process that randomly increases or decreases over time. The parameter $\sigma>0$ represents the magnitude of this randomness, quantifying the risk associated with investing in the stock. In finance, σ is known as *volatility* and can be thought of as analogous to the variance of a random variable in statistics. It follows that S_t, being influenced by W_t, becomes a stochastic process, meaning that at each fixed time t, S_t is a random variable dependent on a hidden random event ω. While financial time-series data suggest that volatility σ often depends on the stock price S_t and other factors—such as time delay, term structure, component structure, or phenomena like the volatility smile—for the sake of mathematical simplicity (as all models are approximations of reality), the Black-Scholes model assumes σ to be constant. Admittedly, this assumption limits the model's practical utility.

   Under this framework, the dynamics described above lead to the following PDE:

$$\begin{cases} \frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0, & S>0, 0<t<T,\\ V(S,T)=\phi(S), & S>0, \end{cases} \tag{0.6}$$

[†] Here the subindex t, like it or not, merely means a notation, not the partial derivative.

where $T > 0$ represents a pre-specified time, known as the *strike time* or maturity. The function V denotes the option value (the dependent variable). In finance, the term *option* refers to the right to buy or sell an asset at a predetermined price—called the *strike price* K—before or at the strike time T. Both K and T are specified in the option contract. To derive this PDE, you need a foundational understanding of stochastic calculus or Itô calculus. However, since this topic lies outside the scope of this homework or course, we will not delve into the derivation. For now, let us focus on formulating this PDE. It is important to note that, unlike many PDEs with initial conditions, the Black-Scholes PDE is paired with a *terminal condition.* This is because when reformulated as a heat equation, the diffusion coefficient becomes $-\frac{1}{2}\sigma^2 S^2$, which might seem unusual. However, this does not contradict the principle that the diffusion rate cannot be negative, as the transformation accounts for the sign change appropriately.

The goal of this homework is to demonstrate that equation (0.6) can be transformed into the Cauchy problem for the heat equation and subsequently solved explicitly. To achieve this, complete the following steps:

(a). Introduce the new variables

$$S = Ke^x, t = T - \frac{\tau}{\sigma^2/2},$$

where the constant K is the strike price. Let $v(x,\tau) = V(S,t)$. Show that

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + \Big(\frac{2r}{\sigma^2} - 1\Big)\frac{\partial v}{\partial x} - \frac{2r}{\sigma^2}v.$$

(b). Introduce

$$u(x,\tau) = e^{ax+b\tau}v(x,\tau).$$

Choose constants a and b such that $u(x,\tau)$ satisfies

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}. \tag{0.7}$$

(c). A European option can only be exercised at its expiration date, T. Specifically, the call option for a European option is defined mathematically as:

$$\phi(S) = \max\{S - K, 0\}.$$

Solve the classical heat equation (0.7) subject to this terminal condition. Express your solution $u(x,\tau)$ in terms of integrals.

(d). In terms of the solutions in (c) and the transformations. Show that the solution $V(S,t)$ to (0.6) is

$$V = SN(d_1) - Ke^{-r(T-t)}N(d_2),$$

where

$$d_1 = \frac{\ln S/K + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = \frac{\ln S/K + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$

and N is the cumulative normal distribution that

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}\,dy.$$

(e) Suppose $S = 200$, $K = 210$, $r = 2\%$ (the annual interest rate), $\sigma = 0.58$, and the expiration date is in two months. Find the call value V. To use this formula, ensure that all parameters are converted to the same scale.

Solution 3. *(a). Since $S = Ke^x, t = T - \frac{\tau}{\sigma^2/2}$, we have that $\tau = (T-t)\frac{\sigma^2}{2}, x = \ln\frac{S}{K}$ and therefore*

$$\frac{\partial \tau}{\partial t} = -\frac{\sigma^2}{2}, \frac{\partial x}{\partial S} = \frac{1}{S},$$

$$\frac{\partial V}{\partial t}=\frac{\partial \tau}{\partial t}\frac{\partial v}{\partial \tau}=-\frac{\sigma^2}{2}\frac{\partial v}{\partial \tau},\frac{\partial V}{\partial S}=\frac{\partial x}{\partial S}\frac{\partial v}{\partial x}=\frac{1}{S}\frac{\partial v}{\partial x}$$

and

$$\frac{\partial^2 V}{\partial S^2}=\frac{\partial}{\partial S}(\frac{\partial V}{\partial S})=(-\frac{1}{S^2})\frac{\partial v}{\partial x}+\frac{1}{S}\frac{\partial x}{\partial S}\frac{\partial}{\partial x}(\frac{\partial v}{\partial x})=(-\frac{1}{S^2})\frac{\partial v}{\partial x}+\frac{1}{S^2}\frac{\partial^2 v}{\partial x^2}.$$

Substituting these identities into the PDE leads us to

$$-\frac{\sigma^2}{2}\frac{\partial v}{\partial \tau}+\frac{\sigma^2}{2}S^2(-\frac{1}{S^2}\frac{\partial v}{\partial x}+\frac{1}{S^2}\frac{\partial^2 v}{\partial x^2})+rS\frac{1}{S}\frac{\partial v}{\partial x}-rv=0,$$

which implies

$$\frac{\partial v}{\partial \tau}=\frac{\partial^2 v}{\partial x^2}+\Big(\frac{2r}{\sigma^2}-1\Big)\frac{\partial v}{\partial x}-\frac{2r}{\sigma^2}v.$$

(b). Let $u(x,t)=e^{ax+b\tau}v(x,t)$ for some constants a and b to be determined. Then

$$\frac{\partial u}{\partial \tau}=(bv+\frac{\partial v}{\partial \tau})e^{ax+b\tau}$$

and

$$\frac{\partial u}{\partial x}=(av+\frac{\partial v}{\partial x})e^{ax+b\tau},\frac{\partial^2 u}{\partial x^2}=\Big(\frac{\partial^2 v}{\partial x^2}+2a\frac{\partial v}{\partial x}+a^2v\Big)e^{ax+b\tau}.$$

Therefore, we have that

$$\begin{aligned}\frac{\partial u}{\partial \tau}-\frac{\partial^2 u}{\partial x^2}=&\Big(bv+\frac{\partial v}{\partial \tau}-\frac{\partial^2 v}{\partial x^2}-2a\frac{\partial v}{\partial x}-a^2v\Big)e^{ax+b\tau}\\=&\Big(\frac{\partial v}{\partial \tau}-\frac{\partial^2 v}{\partial x^2}-2a\frac{\partial v}{\partial x}+(b-a^2)v\Big)e^{ax+b\tau}\end{aligned}$$

Now we choose a and b such that

$$\begin{cases}2a=k-1\\ b-a^2=k\end{cases}$$

with $k=\frac{2r}{\sigma^2}$, i.e.,

$$a=\frac{k-1}{2}=\frac{2r-\sigma^2}{2\sigma^2},b=\frac{(k+1)^2}{4}=\frac{(2r+\sigma^2)^2}{4\sigma^4}.$$

Then we readily see that $u(x,\tau)$ satisfies

$$\frac{\partial u}{\partial \tau}=\frac{\partial^2 u}{\partial x^2}.$$

(c). $\tau=0$ for $t=T$, then the terminal condition becomes

$$V(S,T)=v(x,0)=\phi(S)=\phi(Ke^x)=\max\{Ke^x-K,0\}=K(e^x-1)^+$$

Thus we can obtain $u(x,0)$ is

$$u(x,0)=e^{ax}v(x,0)=e^{\frac{k-1}{2}x}v(x,0)=K(e^{\frac{(k+1)x}{2}}-e^{\frac{(k-1)x}{2}})^+:=\Phi(x),$$

where $k=\frac{2r}{\sigma^2}$ as given above.
On the other hand, we know that the solution to the heat equation takes the integral form

$$u(x,\tau)=G*\Phi(x,\tau)=\int_{\mathbb{R}}G(x-\xi,\tau)\Phi(\xi)d\xi,$$

where $G(x-\xi,\tau)=\frac{1}{\sqrt{4\pi\tau}}e^{-\frac{(x-\xi)^2}{4\tau}}$ *and* $\Phi(\xi)=K(e^{\frac{(k+1)\xi}{2}}-e^{\frac{(k-1)\xi}{2}})^+$. *To be precise, we can rewrite* $u(x,\tau)$ *as*

$$\begin{aligned} u(x,\tau)=&\frac{K}{\sqrt{4\pi\tau}}\int_{\mathbb{R}}(e^{\frac{(k+1)\xi}{2}}-e^{\frac{(k-1)\xi}{2}})^+e^{-\frac{(x-\xi)^2}{4\tau}}d\xi\\ =&\frac{K}{\sqrt{4\pi\tau}}\int_0^\infty(e^{\frac{(k+1)\xi}{2}}-e^{\frac{(k-1)\xi}{2}})e^{-\frac{(x-\xi)^2}{4\tau}}d\xi, \end{aligned}$$

because

$$\Phi(\xi)=\begin{cases} K(e^{\frac{(k+1)\xi}{2}}-e^{\frac{(k-1)\xi}{2}}), & \xi>0,\\ 0, & \textit{otherwise.}\end{cases}$$

(d). Now we change the variable ξ *as* $\xi=x+\sqrt{2\tau}\eta$ *and obtain*

$$\begin{aligned} u(x,\tau)=&\frac{K}{\sqrt{4\pi\tau}}\int_0^\infty(e^{\frac{(k+1)\xi}{2}}-e^{\frac{(k-1)\xi}{2}})e^{-\frac{(x-\xi)^2}{4\tau}}d\xi\\ =&\frac{K}{\sqrt{4\pi\tau}}\int_{-\frac{x}{\sqrt{2\tau}}}^\infty(e^{\frac{(k+1)(x+\sqrt{2\tau}\eta)}{2}}-e^{\frac{(k-1)(x+\sqrt{2\tau}\eta)}{2}})e^{-\frac{\eta^2}{2}}\sqrt{2\tau}d\eta\\ =&\frac{K}{\sqrt{2\pi}}e^{\frac{(k+1)x}{2}+\frac{1}{4}\tau(k+1)^2}\int_{-\frac{x}{\sqrt{2\tau}}}^\infty e^{-\frac{(\eta-\frac{1}{2}\sqrt{2\tau}(k+1))^2}{2}}d\eta\\ &-\frac{K}{\sqrt{2\pi}}e^{\frac{(k-1)x}{2}+\frac{1}{4}\tau(k-1)^2}\int_{-\frac{x}{\sqrt{2\tau}}}^\infty e^{-\frac{(\eta-\frac{1}{2}\sqrt{2\tau}(k-1))^2}{2}}d\eta. \end{aligned}$$

For the first term, we denote $y=\eta-\frac{1}{2}\sqrt{2\tau}(k+1)$ *and have*

$$\begin{aligned} \frac{1}{2\pi}\int_{-\frac{x}{\sqrt{2\tau}}}^\infty e^{-\frac{(\eta-\frac{1}{2}\sqrt{2\tau}(k+1))^2}{2}}d\eta=&\frac{1}{2\pi}\int_{-\frac{x}{\sqrt{2\tau}}-\frac{1}{2}\sqrt{2\tau}(k+1)}^\infty e^{-\frac{y^2}{2}}dy=\frac{1}{2\pi}\int_{-\infty}^{\frac{x}{\sqrt{2\tau}}+\frac{1}{2}\sqrt{2\tau}(k+1)}e^{-\frac{y^2}{2}}dy\\ =&N\Big(\frac{x}{\sqrt{2\tau}}+\frac{1}{2}\sqrt{2\tau}(k+1)\Big)=N(d_1), \end{aligned}$$

where

$$d_1=\frac{x}{\sqrt{2\tau}}+\frac{1}{2}\sqrt{2\tau}(k+1)=\frac{\ln\frac{S}{K}+(\frac{2r}{\sigma^2}+1)\frac{\sigma^2}{2}(T-t)}{\sqrt{2\frac{\sigma^2}{2}(T-t)}}=\frac{\ln S/K+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.$$

Similarly, we let $y=\eta-\frac{1}{2}\sqrt{2\tau}(k-1)$ *and have that*

$$\begin{aligned} \frac{1}{2\pi}\int_{-\frac{x}{\sqrt{2\tau}}}^\infty e^{-\frac{(\eta-\frac{1}{2}\sqrt{2\tau}(k-1))^2}{2}}d\eta=&\frac{1}{2\pi}\int_{-\infty}^{\frac{x}{\sqrt{2\tau}}+\frac{1}{2}\sqrt{2\tau}(k-1)}e^{-\frac{y^2}{2}}dy\\ =&N\Big(\frac{x}{\sqrt{2\tau}}+\frac{1}{2}\sqrt{2\tau}(k-1)\Big)=N(d_2), \end{aligned}$$

where

$$d_2=\frac{x}{\sqrt{2\tau}}+\frac{1}{2}\sqrt{2\tau}(k-1)=\frac{\ln\frac{S}{K}+(\frac{2r}{\sigma^2}-1)\frac{\sigma^2}{2}(T-t)}{\sqrt{2\frac{\sigma^2}{2}(T-t)}}=\frac{\ln S/K+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

Finally, using $u=e^{ax+b\tau}v$ *leads us to the desired*

$$\begin{aligned} V=&e^{-ax-b\tau}u=e^{-\frac{k-1}{2}x-\frac{(k+1)^2}{4}\tau}\Big(Ke^{\frac{(k+1)x}{2}+\frac{1}{4}\tau(k+1)^2}N(d_1)-Ke^{\frac{(k-1)x}{2}+\frac{1}{4}\tau(k-1)^2}N(d_2)\Big)\\ =&Ke^xN(d_1)-Ke^{k\tau}N(d_2)\\ =&SN(d_1)-Ke^{r(T-t)}N(d_2), \end{aligned}$$

where

$$d_1 = \frac{\ln S/K + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = \frac{\ln S/K + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$

and N is the cumulative normal distribution that

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy.$$

(e). Choosing $S = 200, K = 210, r = 0.02, \sigma = 0.58, T = \frac{2}{12}$ and $t = 0$, we can calculate

$$\begin{aligned} d_1 &= \frac{\ln S/K + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \approx -0.0736, \\ d_2 &= \frac{\ln S/K + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \approx -0.3104, \\ N(d_1) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_1} e^{-\frac{y^2}{2}} dy \approx 0.470671, \\ N(d_2) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_2} e^{-\frac{y^2}{2}} dy \approx 0.378141, \end{aligned}$$

and eventually evaluate that $V = SN(d_1) - Ke^{-r(T-t)}N(d_2) \approx 14.988848$.

4. Show the following facts for Dirac delta function $\delta(x)$:

(1). $\delta(x) = \delta(-x)$, i.e., show that they hold in the distribution sense for any $\phi \in C_c(\mathbb{R})$:

$$\int_{\mathbb{R}} \delta(x)\phi(x)dx = \int_{\mathbb{R}} \delta(-x)\phi(x)dx.$$

This also applies to the rest.

(2). $\delta(kx) = \frac{\delta(x)}{|k|}$, where k is a non-zero constant;

(3). $\int_{\mathbb{R}} f(x)\delta(x - x_0)dx = f(x_0)$; $\delta(x - x_0)$ is occasionally written as $\delta_{x_0}(x)$;

(4). Let $f(x)$ be continuous except for a jump–discontinuity at 0. Show that

$$\frac{f(0^-) + f(0^+)}{2} = \int_{-\infty}^{\infty} f(x)\delta(x)dx$$

Solution 4. *(1). I would like to mention that, whenever checking identities involving $\delta(x)$, it is necessary to check that it holds both points wisely and in the distribution sense. First of all, it is easy to check that $\delta(x) = \delta(-x)$ point wisely (which is only a formal identity), hence I skip it here. To show that they are the same in the distribution sense, we choose an arbitrary continuous function $\phi(x)$ and have that*

$$\int_{-\infty}^{\infty} \phi(x)\delta(x)dx = \phi(0) = \phi(-0) = \int_{-\infty}^{\infty} \phi(-x)\delta(x)dx = \int_{-\infty}^{\infty} \phi(\xi)\delta(-\xi)d\xi,$$

where we denote $\xi := -x$ in the last identity. Therefore we have the desired identity. The verification of (2) and (3) follows the same approach and I skip it here.

(4). There are several ways that one can obtain this result and here is one of them. Let us introduce

$$g(x) = \frac{f(x) + f(-x)}{2}, \forall x \in \mathbb{R};$$

if we further define

$$g(0) := \frac{f(0^+) + f(0^-)}{2},$$

then it is easy to see that $g(x)$ is continuous over $\mathbb{R}^n$. Therefore, by the definition of a Dirac–delta function, we have that

$$\int_{\mathbb{R}} g(x)\delta(x)dx = g(0) = \frac{f(0^+) + f(0^-)}{2};$$

on the other hand, we have that

$$\begin{aligned} &\int_{\mathbb{R}} g(x)\delta(x)dx = \frac{1}{2}\int_{\mathbb{R}} (f(x) + f(-x))\delta(x)dx \\ =&\frac{1}{2}\int_{\mathbb{R}} f(x)\delta(x)dx + \frac{1}{2}\int_{\mathbb{R}} f(-x)\delta(x)dx \\ =&\int_{\mathbb{R}} f(x)\delta(x)dx, \end{aligned}$$

where the last identity follows from a change of variable as above. Another approach of the same spirit is to introduce $F(x) = 2g(x)$ and define $F(0)$ as it should be. This should give a method that is intuitively not straightforward, but rigorous.

5. We know that the Heaviside step function

$$H(x) = \begin{cases} 1 & x > 0, \\ 0, & x < 0 \end{cases} \tag{0.8}$$

has the Dirac delta function $\delta(x)$ as its weak derivative. As I mentioned in class, you might encounter textbooks where the Heaviside function is defined differently, such as:

$$H(x) = \begin{cases} 1 & x > 0, \\ \frac{1}{2} \text{ (or any other number)} & x = 0, \\ 0, & x < 0. \end{cases} \tag{0.9}$$

According to Lebesgue's theory, the value of a function at a single point (or on a set of measure zero) does not affect its general properties. Two functions that are equal almost everywhere are considered identical. Accordingly, the two forms of $H(x)$ are regarded as equivalent, although we shall adopt the former in this course. Similarly, the weak derivative of a function is unique up to a set of measure zero. In other words, if $f(x)$ is a weak derivative of $F(x)$, then $g(x)$ is also a weak derivative if $f(x)$ and $g(x)$ differ only on a set of measure zero. This principle extends further. A function known as a bump function is defined as $B(x) = xH(x)$. Using the definition of a weak derivative, show that the weak derivative of $B(x)$ is $H(x)$.

Solution 5. *Let $v(x) \in L^1_{loc}(\mathbb{R}^1)$ be a weak derivative of $R(x)$, then we have from the definition of the weak derivative that, for M being large (or, you can just work on $(-\infty, \infty)$)*

$$\int_{-M}^{M} R(x)\phi'(x)dx = -\int_{-M}^{M} v(x)\phi(x)dx$$

for all $\phi(x) \in C^1_0(-M, M)$, i.e., $\phi(x) \in C^1(-M, M)$ and $\phi(x) = 0$ for $|x| > M$. Note that we usually send $M = \infty$, and the test function in C^∞_0, however, this alternative just gives you the impression that they are equivalent in the definition. Then we have from integration by parts that

$$\int_{-M}^{M} R(x)\phi'(x)dx = \int_0^M x d\phi(x) = -\int_0^M \phi(x)dx = -\int_{-M}^{M} H(x)\phi(x)dx,$$

therefore $H(x)$ is a weak derivative of $R(x)$. Finally, I want to remark that the weak derivative is unique in the sense of measure zero, i.e., out of a region of zero measure.

6. Find the weak derivative of $F(x)$, denoted by $f(x)$

$$F(x) = \begin{cases} x, & 0 < x \le 1, \\ 1, & 1 \le x < 2. \end{cases} \tag{0.10}$$

$$F(x) = \begin{cases} x, & 0 < x \le 1, \\ 1, & 1 \le x < 2. \end{cases} \tag{0.11}$$

Solution 6. *Formally we see that the weak derivative of $F(x)$, denoted by $f(x)$, is*

$$f(x)=\begin{cases}1, & 0<x\le 1,\\ 0, & 1\le x<2.\end{cases} \tag{0.12}$$

To prove this by definition, we choose any $\phi\in C_c^\infty(0,2)$ and can easily find that

$$\int_0^2 F\phi' dx=\int_0^1 F\phi' dx+\int_1^2 F\phi' dx=-\int_0^1 \phi dx=\int_0^2 f\phi dx,$$

with f given above. This is done. Note that function $F(x)$ is define, in this example, over $(0,2)$, therefore the test function must be compactly supported over $(0,2)$, not $(-\infty,\infty)$ any more. I wish this logic is not too difficult to follow.

7. One can easily generalize the second–order operator to higher dimension, the Laplace operator Δ over $\Omega\subset\mathbb{R}^n$, $n\ge 1$, with $x=(x_1,x_2,...,x_n)$.
   (a) We say that f is radially symmetric if $f(x)=f(r)$, $r=|x|:=\sqrt{\sum_{i=1}^n x_i^2}$. Prove that
   $$\Delta f(r)=f''(r)+\frac{n-1}{r}f'(r),$$
   where the prime denotes a derivative taken with respect to r.
   (b) Denote that $G(r):=\frac{1}{2\pi}\ln r$ for $n=2$. We shall show that $\Delta G=\delta(r)$. For this moment, let us consider its regularization over $2D$ of the form
   $$G_\epsilon(r)=\frac{1}{2\pi}\ln(r+\epsilon),\epsilon>0.$$
   Show that $\Delta G_\epsilon(r)$ converges to $\delta(x)$ in distribution as $\epsilon\to 0^+$. Hint: you can either apply Lebesgue's dominated convergence theorem, or use ϵ–δ language. Make sure you have checked all the conditions when applying the former one.
   (c) Denote $G(r):=-\frac{1}{4\pi r}$ for $n=3$. Mimic (b) by finding an approximation G_ϵ and then show that this approximation ΔG_ϵ convergence to $\delta(x)$ in distribution.

Solution 7. *(a) We show this by straightforward calculations. First of all, we have that $\frac{\partial r}{\partial x_i}=\frac{x_i}{r}$. Then by chain rule, we have*

$$\frac{\partial f(r)}{\partial x_i}=\frac{\partial f(r)}{\partial r}\frac{\partial r}{\partial x_i}=\frac{\partial f(r)}{\partial r}\frac{x_i}{r};$$

moreover, we have

$$\frac{\partial^2 f}{\partial x_i^2}=\frac{\partial}{\partial x_i}\Big(\frac{\partial f}{\partial r}\frac{x_i}{r}\Big)=\frac{\partial^2 f}{\partial r^2}\big(\frac{x_i}{r}\big)^2+\frac{\partial f}{\partial r}\frac{\partial}{\partial x_i}\big(\frac{x_i}{r}\big);$$

finally, using the fact that

$$\frac{\partial}{\partial x_i}\big(\frac{x_i}{r}\big)=\frac{1}{r}-\frac{x_i^2}{r^3}$$

leads us to the expected identity.

(b) By the identity in (a), we have that

$$\Delta G_\epsilon(r)=G_\epsilon''(r)+\frac{1}{r}G_\epsilon'(r)=\frac{1}{2\pi}\Big(\frac{1}{r+\epsilon}+\frac{1}{r}\cdot\frac{-1}{(r+\epsilon)^2}\Big)=\delta_\epsilon(x):=\frac{\epsilon}{2\pi r(r+\epsilon)^2},$$

and we shall show $\delta_\epsilon(x)$ converges to $\delta(x)$ is distribution. Note that here its distribution limit $\delta(x)$ satisfies all the properties except that it is multi-dimensional.

To this end, we first see that formally $\delta_\epsilon(x)\to\infty$ if $x=0$, $\to 0$ if $x\ne 0$. Next, we have that

$$\int_{\mathbb{R}^2}\delta_\epsilon(x)dx=\int_{\mathbb{R}^2}\frac{\epsilon}{2\pi r(r+\epsilon)^2}dx=\int_0^\infty\frac{\epsilon}{(r+\epsilon)^2}dr=-\frac{\epsilon}{r+\epsilon}\Big|_0^\infty=1.$$

Now, we only need to show that for any text function $\phi(x) \in C(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$

$$\int_{\mathbb{R}^2} \delta_\epsilon(x)\phi(x)dx \to \phi(0),$$

or equivalently

$$\int_{\mathbb{R}^2} \frac{\epsilon}{2\pi(r+\epsilon)^2}\phi(x)dx \to \phi(0).$$

To show this, we observe that for any δ small, one can choose R large enough such that

$$\int_{\mathbb{R}^2 \setminus B_0(M)} \frac{\epsilon}{2\pi(r+\epsilon)^2}\phi(x)dx = \delta;$$

on the other hand, one has from the dominated convergence theorem that as $\epsilon \to 0$

$$\int_{B_0(M)} \frac{\epsilon}{2\pi(r+\epsilon)^2}\phi(x)dx = \phi(x_\epsilon)\int_{B_0(M)} \frac{\epsilon}{2\pi(r+\epsilon)^2}dx \le \phi(x_\epsilon)(1-\delta) \to \phi(0),$$

since δ is arbitrary. I would like to mention that one can also apply the standard ϵ-δ to prove this.

(c) I skip the proof here. It follows from straightforward calculations as above.