Introduction to PDEs, Fall 2024

Homework 11 due Dec 26

Name:________________

1. Find a fundamental solution of Laplacian Δ in 3D. Hint: $G(r) = -\frac{1}{4\pi}\frac{1}{|\mathbf{x}|} = -\frac{1}{4\pi}\frac{1}{r}$.

2. Find the Green's function over $\mathbb{R}^3_+ := \{(x, y, z) \in (-\infty, \infty) \times (-\infty, \infty) \times (0, \infty)\}$, and then solve

$$\begin{cases} \Delta u = 0, & x \in \mathbb{R}^3_+, \\ \frac{\partial u}{\partial \mathbf{n}} = g, & x \in \partial\mathbb{R}^3_+. \end{cases} \tag{0.1}$$

3. Verify that the Laplacian of $u(x, y)$ in the polar coordinates $x = r\cos\theta, y = r\sin\theta$ is

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

4. Solve the following Poisson's equation for $u(r, \theta)$

$$\begin{cases} \Delta u = \cos\theta, & r \in (1, 2)\ \theta \in [0, 2\pi), \\ u|_{r=1} = 0,\ u|_{r=2} = 2. \end{cases} \tag{0.2}$$

   Hint: There are two methods you can tackle this problem. The first is to use the method of separation of variables. For each fixed $r \in (1, 2)$, $u(r, \theta)$ is a 2π–periodic function of θ. Show that it can expands into

$$u(r, \theta) = A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta + B_n \sin n\theta,$$

   where A_n and B_n are functions of r. Then substitute this into the polar coordinate of the PDE and collect the ODEs for A_n and B_n. You may need to solve some Euler–type ODE. Then find the coefficients by the boundary conditions. The second method is to find $u = v + w$ such that $\Delta v = 0$ and $\Delta w = \cos\theta$ and the main task is to find one specific w.