HS 16
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Numerical Methods
Exercise Sheet 5

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http://www.physik.uzh.ch/en/teaching/PHY233/HS2016.html

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In this exercise sheet you will be asked to implement different algorithms to exactly solve the linear system $A \cdot \vec{x} = \vec{b}$, with:

$$A = \begin{bmatrix} 6 & -8 & -4 & 8 & -5 & -9 & -9 & -8 & -8 & 4 \\ -1 & -4 & -5 & 5 & -5 & 9 & 5 & -4 & 4 & 1 \\ 4 & 1 & -3 & -6 & -9 & -6 & 7 & 7 & -3 & 0 \\ -1 & -4 & -3 & -3 & 7 & -7 & -6 & -7 & 7 & 8 \\ -6 & -8 & 4 & -6 & -3 & -2 & 7 & -7 & -3 & -6 \\ -6 & -3 & -4 & -2 & 9 & 3 & -7 & 4 & 1 & 5 \\ 3 & -7 & 5 & -7 & 1 & 6 & 0 & -4 & -3 & -5 \\ 3 & -3 & -4 & -6 & -9 & 4 & 2 & -4 & 8 & -3 \\ 3 & -8 & 9 & -4 & 1 & -7 & 2 & 2 & -7 & -8 \\ 5 & 4 & -6 & 3 & -6 & -2 & 6 & -4 & 8 & -2 \end{bmatrix} \quad ; \quad \vec{b} = \begin{bmatrix} 44 \\ -38 \\ 31 \\ -23 \\ 40 \\ -13 \\ 28 \\ -15 \\ 79 \\ -37 \end{bmatrix} .$$

You can copy a .txt file with the above matrices at: `https://git.physik.uzh.ch/gitbucket/pattori/num-meth/blob/master/linear_system_class4.txt`.

Exercise 1: Cramer's method (40 Pts.)

a) Implement a function that, given A and $\vec{b}$ (and i), returns the matrix A_i, defined as the matrix where the i-th column of A has been substituted with $\vec{b}$. Explicitly:

$$A_i = \begin{bmatrix} a_{1,1} & \dots & a_{1,i-1} & b_1 & a_{1,i+1} & \dots \\ a_{2,1} & \dots & a_{2,i-1} & b_2 & a_{2,i+1} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n,1} & \dots & a_{n,i-1} & b_n & a_{n,i+1} & \dots \end{bmatrix} .$$

(10 points)

b) Explicitly implement a function that compute the determinant of a matrix A. This can be done recursively through the Laplace's formula. If A is a $n \times n$ matrix, then:

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{i,j} \det(M_{i,j})$$

where $a_{i,j}$ is the A matrix element at position $\{i,j\}$, while $M_{i,j}$ is the $(n-1) \times (n-1)$ matrix obtained by removing the i-th row and the j-th column from A. The formula work for arbitrary i. (20 points)

c) Find the solution $\vec{x}$ to the linear system $A \cdot \vec{x} = \vec{b}$ through the Cramer method, that is:

$$x_i = \frac{\det A_i}{\det A} \ .$$

(10 points)

Exercise 2: Gauss method, modified Gauss method and Jordan method (60 Pts.)

a) Taken $A^{[0]} \equiv A$ and $\vec{b}^{[0]} \equiv \vec{b}$, implement the 10×11 matrix obtained by adding $\vec{b}$ as an extra column to A: $M^{[0]} = \left[A^{[0]} \,|\, \vec{b}^{[0]} \right]$. (5 points)

b) Implement the iterative step $M^{[i-1]} \to M^{[i]}$. At each step, one first sets $M^{[i]} = M^{[i-1]}$ and then substitutes:

$$M^{[i]}_{k,l} \leftarrow \left(M^{[i]}_{k,l} - M^{[i]}_{i,l} * \frac{M^{[i]}_{k,i}}{M^{[i]}_{i,i}} \right) \quad ; \quad j \in [1, 11] \ ; k \in [i+1, 10] \ .$$

(15 points)

c) If the iterative step of task (b) is correctly implemented, $M^{[9]}$ should have the form:

$$M^{[9]} = \left[A^{[9]} \,|\, \vec{b}^{[9]} \right] \left[\begin{array}{cccccc|c} a^{[9]}_{1,1} & a^{[9]}_{1,2} & \dots & \dots & a^{[9]}_{1,10} & b^{[9]}_1 \\ 0 & a^{[9]}_{2,2} & \dots & \dots & a^{[9]}_{2,10} & b^{[9]}_2 \\ 0 & 0 & a^{[9]}_{3,3} & \dots & a^{[9]}_{3,10} & b^{[9]}_3 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & a^{[9]}_{10,10} & b^{[9]}_{10} \end{array} \right]$$

Now the linear system $A^{[9]} \cdot \vec{x} = \vec{b}^{[9]}$ is easily solvable, since $A^{[9]}$ has a triangular form. The solution reads:

$$x_{10} = \frac{b^{[9]}_{10}}{A^{[9]}_{10,10}} \ , \tag{1}$$

$$x_j = \frac{1}{A^{[9]}_{j,j}} \left(b^{[9]}_j - \sum_{k=j+1}^{10} A^{[9]}_{j,k} \cdot x_k \right) \quad ; \quad j = 9, 8, \dots, 1 \ . \tag{2}$$

Implement this formula and find $\vec{x}$. (15 points)

d) Implement the modified Gauss method. This means that, before the iterative step $M^{[i-1]} \to M^{[i]}$ is taken, one should reshuffle the rows of $M^{[i-1]}$ in the following way:

- Among the rows k with $k \geq i$, find the one corresponding to the biggest $M^{[i-1]}_{k,i}$ value.
- In $M^{[i-1]}$, permute the i-th and k-th row. Please notice that this permutation should be applied to the variable vector $\vec{x} = \{x_1, x_2, \dots, x_{10}\}$ as well, otherwise one would find the right solutions, but in the wrong order.
- Now apply the $M^{[i-1]} \to M^{[i]}$ iterative step implemented in (b).

(15 points)

e) Implement the Jordan elimination method. One just need to slightly modify the implementation made in (b). Now after having set $M^{[i]} = M^{[i-1]}$, one substitutes:

$$M^{[i]}_{k,l} \leftarrow \left(M^{[i]}_{k,l} - M^{[i]}_{i,l} * \frac{M^{[i]}_{k,i}}{M^{[i]}_{i,i}} \right) \quad ; \quad j \in [1,11] \; ; k \in [1, i-1] \cup [i+1, 10] \; .$$

Please notice the different range for k. (10 points)

Exercise 3: Optional: LU method (40 Pts.)

a) Implement a function that, given $A^{[i]}$, returns the matrix $L^{[i]}$, defined as:

$$L^{[i]}_{j,k} = \begin{cases} 1 & \text{for } j = k \\ -\frac{A^{[i]}_{j,k}}{A^{[i]}_{k,k}} & \text{for } i = k \text{ and } k > j \\ 0 & \text{elsewhere} \end{cases} \quad .$$

(10 points)

b) Using the iterative step $A^{[i+1]} = L^{[i]} \cdot A^{[i]}$, find the following matrices:

$$A^{[10]}, L^{[1]}, L^{[2]}, \ldots, L^{[9]} \; .$$

(10 points)

c) Set $U = A^{[10]}$. Then finds $L = \left(L^{[1]}\right)^{-1} \cdot \left(L^{[2]}\right)^{-1} \ldots \left(L^{[9]}\right)^{-1}$. The inverse of $L^{[i]}$ is simply:

$$\left(L^{[1]}\right)^{-1} = -L^{[i]} + 2 \cdot \mathbb{1}_{10} \; ,$$

where $\mathbb{1}_{10}$ is the 10×10 identity matrix. (10 points)

d) Solve the two linear systems (in triangular form):

$$\begin{cases} L \cdot \vec{y} &= \vec{b} \\ U \cdot \vec{x} &= \vec{y} \end{cases} \; .$$

One can use Eqs. (1), (2) to solve $U \cdot \vec{x} = \vec{y}$, and

$$y_1 = \frac{b_1}{L_{1,1}} \; ,$$

$$y_j = \frac{1}{L_{j,j}} \left(b_j - \sum_{k=1}^{j-1} L_{j,k} \cdot y_k \right) \quad ; \quad j = 1, 2, \ldots, 9 \; .$$

to solve $L \cdot \vec{y} = \vec{b}$. (10 points)

Exercise 4: Optional: Efficiency (10 Pts.)

a) Compare the speed of the above algorithms. (10 points)

Maximum number of points for mandatory tasks on 26.10 (or on 02.11) : 100

Maximum possible number of points for tasks on 26.10 (or on 02.11) : 150