HS 16
M. Chrzaszcz
D. van Dyk

Numerical Methods
Exercise Sheet 10

I. Bezshyiko, A. Pattori
http://www.physik.uzh.ch/en/teaching/PHY233/HS2016.html

Issued: 23.11.2016
Due: 28.11.2016

Save the date! The class scheduled for Wednesday 30 November is moved to: **Monday 28 November**

Exercise 1: ODE solution with finite difference method (120 Pts.)

In this exercise you will be asked to numerically solve an ordinary, autonomous, second order differential equation with Dirichlet boundary conditions (BCs):

$$\begin{cases} y''(x) + y'(x) + 25\,y(x) = 0 \\ y(0) = 1.3 \\ y(2) = -0.4 \end{cases}, \tag{1}$$

using the so called "finite difference method".

In a finite difference method, one converts the ODE into a linear system of equations. To achieve this, the integration domain $x \in [0, 2]$ is discretized into $N+1$ equally distanced points x_k $(k = 0, \dots, N)$. Once evaluated the ODE (1) at the points x_k and translated $y'(x_k), y''(x_k)$ into finite differences, we are left with $N-1$ linear equations in the variables $y_k = y(x_k)$ $(k = 1, \dots, N-1)$. Such linear system of $N-1$ equations can then be solved to find $\{y_k\}$.

<u>In this exercise, use $N = 1000$</u>.

a) Implement a finite difference method using forward differences. This means that, to "build" the linear equations, the following substitutions should be performed:

$$y'(x_k) = \frac{1}{h}(y_{k+1} - y_k)$$
$$y''(x_k) = \frac{1}{h^2}(y_{k+2} - 2\,y_{k+1} + y_k)$$

- Find the correct $N-1$ linear equations. (10 points)
- Implement correctly the linear system to be solved. Make sure to use the BCs for y_0, y_N properly. (20 points)
- Solve the linear system and find the approximate solution $\{y_k\}$. (20 points)

b) Implement a finite difference method using central differences. This means that, to "build" the linear equations, the following substitutions should be performed:

$$y'(x_k) = \frac{1}{2h}(y_{k+1} - y_{k-1})$$
$$y''(x_k) = \frac{1}{4h^2}(y_{k+2} - 2\,y_k + y_{k-2})$$

- Find the correct $N-1$ linear equations.

  Hint: for $k=1$ and $k=N-1$, implementing central differences would introduce y_{-1} and y_{N+1}, which is bad. To overcome this difficulty, for $k=1$ one should use forward differences and for $k=N-1$ backward differences. (10 points)

- Implement correctly the linear system to be solved. Make sure to use the BCs for y_0, y_N properly. (20 points)

- Solve the linear system and find the approximate solution $\{y_k\}$. (20 points)

c) Optional: The exact solution to ODE (1) is:

$$y(x) = e^{-\xi\,\omega_0\,x}\left(c_1\cos(\sqrt{1-\xi^2}\omega_0\,x) + c_2\sin(\sqrt{1-\xi^2}\omega_0\,x)\right)\;, \tag{2}$$

with $\xi = 0.1$, $\omega_0 = 5$, $c_1 = 1.3$, $c_2 \approx -0.0748969$.

Use the exact formula (2) to extract and compare the absolute errors of the solutions of tasks (a) and (b). (20 point)

Save the date! The class scheduled for Wednesday 30 November is moved to:
Monday 28 November

Maximum number of points for mandatory tasks on 28.11 : 100

Maximum possible number of points for tasks on 28.11 : 120