An EM Algorithm for Learning in Controlled Linear Dynamical Systems

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Abstract

In this report, we derive an expectation-maximization (EM) algorithm for learning in controlled dynamical systems. This report should be thought of as a simple extension of the algorithm presented in [1]. But in our case, we assume that the physical models of the objects and observation models are known, however the 'policies' of objects are not known. Starting from this point, we derive an EM algorithm for learning policies from data.

1 Introduction

In [1], the authors derive an EM algorithm to learn the parameters of a linear dynamical system. Simply, they consider the following dynamical model,

$$x_t = Ax_{t-1} + w_{t-1}, \tag{1}$$
$$y_t = Cx_t + n_t, \tag{2}$$

where $(x_t)_{t\geq 0}$ is a hidden stochastic process and $(y_t)_{t\geq 0}$ is the observation process and w_t and n_t are white noise sequences. They derive update rules (for EM) for A, C, Q, R, where Q and R noise covariances.

In this report, we proceed in a similar fashion (as an exercise). However, we assume our dynamical system is *controlled* [2]. A controlled linear dynamical system (CLDS) can be written as,

$$x_t = Ax_{t-1} + BLx_{t-1} + w_{t-1}, \tag{3}$$
$$y_t = Cx_t + n_t. \tag{4}$$

In general, control inputs are denoted by u_t. In our CLDS, we denote them with Lx_{t-1}. In theory, the matrix L is determined by minimizing a quadratic cost function[1]

$$\mathbb{E}\left\{x_N^T Q_N x_N + \sum_{k=0}^{N-1}(x_k^T Q_k x_k + u_k^T R_k u_k).\right\} \tag{5}$$

For the derivation of L by minimizing this cost function, see [2].

The idea of learning this cost function from observations is called as inverse optimal control or inverse reinforcement learning. However, from an EM viewpoint, deriving the EM algorithm to learn cost this function is pretty hard due to the recursive structure of the Algebraic Riccati Equation [2]. Rather, by focusing on only to L matrix, an EM algorithm can be devised. In this repor, we assume A, B, C is known. However, if they are unknown, one can also derive the EM algorithm following the work [1] and our derivations. The assumption about A, B, C makes sense: In general, physical models of objects are approximately known. For example, if one wants to track a vehicle, one usually knows how its physical location evolves. However, we would not know if the driver accelerating or decelerating the vehicle. This is what we call as a policy.

In this work, we try to estimate the policy matrix L. For a detailed discussion and derivation of policy matrix, see [2]. As we mentioned, we assume A, B, C is known, and further assume everything else is known including initial conditions to simplify derivation – since the update rules for those quantities are known. Note that, in order to assume that the dynamical system is controlled, we need the pair (A, B) should be controllable and the pair (A, C) should be observable [2]. We assume these assumptions hold.

2 The EM algorithm

The EM algorithm aims to maximize the log likelihood of complete data given the observed data. For an excellent treatment of EM algorithm, see [3]. First of all, we write the likelihood

$$p(x_{1:T}, y_{1:T}|L) = p(x_1)\prod_{t=2}^{T} p(x_t|x_{t-1}, L)\prod_{t=1}^{T} p(y_t|x_t), \tag{6}$$

[1]Note that the matrices Q and R in the cost function are *not* noise covariances. They characterise the interaction over state variables.

by using the conditional independence assumptions implicit in our model. This model is an instance of a hidden Markov model and these models factorize over observation and transition distributions [4]. By departing this point, we write the log-likelihood

$$\log p(x_{1:T}, y_{1:T}|L) = \log p(x_1) + \sum_{t=2}^{T} \log p(x_t|x_{t-1}, L) + \sum_{t=1}^{T} \log p(y_t|x_t). \tag{7}$$

In a typical expectation maximization setup, we aim to perform two step: (1) E-step, (2) M-step. In E-step, we aim at computing the following expectation,

$$\langle \log p(x_{1:T}, y_{1:T}|L) \rangle_{p(x_{1:T}|y_{1:T}, L^\star)}, \tag{8}$$

where $L^\star$ is a fixed parameter. Then, in the maximization step, we aim at updating this $L^\star$ by setting derivative of this expectation with respect to L to zero. We start with the maximization step (M-step) and write expectations in the equations explicitly. Then, we show the E-step – how to compute these expectations.

2.1 M-step

The important term for us is $p(x_t|x_{t-1}, L)$. Let us write it explicitly:

$$p(x_t|x_{t-1}, L) = \mathcal{N}(x_t; (A + BL)x_{t-1}, Q). \tag{9}$$

Suppose A, B, Q is known. More explicitly

$$p(x_t|x_{t-1}) = \exp\left\{-\frac{1}{2}\left(x_t - (A+BL)x_{t-1}\right)^\top Q^{-1}\left(x_t - (A+BL)x_{t-1}\right)\right\}(2\pi)^{-k/2}|Q|^{-1/2}. \tag{10}$$

If we would like to devise a formula to update L, we have to minimize following cost function[2] [3]

$$\mathcal{Q}(\theta, \theta^{\text{old}}) = -\frac{T-1}{2}\log|Q| - \mathbb{E}_{x_{1:T}|\theta^{\text{old}}}\left[\frac{1}{2}\sum_{t=2}^{T}(x_t - (A+BL)x_{t-1})^\top Q^{-1}(x_t - (A+BL)x_{t-1})\right]. \tag{11}$$

First, consider the term $(x_t - (A+BL)x_{t-1})^\top Q^{-1}(x_t - (A+BL)x_{t-1})$. We can expand this term as

$$\begin{aligned} x_t^\top Q^{-1} x_t - x_t^\top Q^{-1}(A+BL)x_{t-1} &- ((A+BL)x_{t-1})^\top Q^{-1} x_t \\ &+ ((A+BL)x_{t-1})^\top Q^{-1}((A+BL)x_{t-1}). \end{aligned}$$

[2] $(\cdot)^\top$ denotes transpose.

By playing with this expression, we arrive at

$$\begin{aligned} x_t^\top Q^{-1} x_t - x_t^\top Q^{-1}(A+BL)x_{t-1} - x_{t-1}^\top (A+BL)^\top Q^{-1} x_t \\ + x_{t-1}^\top (A+BL)^\top Q^{-1}(A+BL)x_{t-1}, \end{aligned}$$

which, by expanding the last term, yields

$$\begin{aligned} x_t^\top Q^{-1} x_t - x_t^\top Q^{-1}(A+BL)x_{t-1} - x_{t-1}^\top (A+BL)^\top Q^{-1} x_t \\ + x_{t-1}^\top A^\top Q^{-1}(A+BL)x_{t-1} + x_{t-1}^\top (BL)^\top Q^{-1}(A+BL)x_{t-1}. \end{aligned}$$

We can in turn write this as

$$\begin{aligned} x_t^\top Q^{-1} x_t - x_t^\top Q^{-1}(A+BL)x_{t-1} - x_{t-1}^\top (A+BL)^\top Q^{-1} x_t + x_{t-1}^\top A^\top Q^{-1} A x_{t-1} \\ + x_{t-1}^\top A^\top Q^{-1} BL x_{t-1} + x_{t-1}^\top (BL)^\top Q^{-1} A x_{t-1} + x_{t-1}^\top (BL)^\top Q^{-1} BL x_{t-1}. \end{aligned}$$

Hence, by expanding all terms, we can write this expression as

$$\begin{aligned} x_t^\top Q^{-1} x_t - x_t^\top Q^{-1} A x_{t-1} - x_t^\top Q^{-1} BL x_{t-1} - x_{t-1}^\top A^\top Q^{-1} x_t - x_{t-1}^\top (BL)^\top Q^{-1} x_t + \\ x_{t-1}^\top A^\top Q^{-1} A x_{t-1} + x_{t-1}^\top A^\top Q^{-1} BL x_{t-1} + x_{t-1}^\top (BL)^\top Q^{-1} A x_{t-1} + x_{t-1}^\top (BL)^\top Q^{-1} BL x_{t-1}. \end{aligned}$$

Finally, we can write it as (by using the properties of trace [5])[3],

$$\begin{aligned} F_t = \mathrm{Tr}\Bigg[& Q^{-1} x_t x_t^\top - A^\top Q^{-1} x_t x_{t-1}^\top - (BL)^\top Q^{-1} x_t x_{t-1}^\top - A^\top Q^{-1} x_t x_{t-1}^\top - (BL)^\top Q^{-1} x_t x_{t-1}^\top \\ & + A^\top Q^{-1} A x_{t-1} x_{t-1}^\top + A^\top Q^{-1} BL x_{t-1} x_{t-1}^\top + (BL)^\top Q^{-1} A x_{t-1} x_{t-1}^\top + (BL)^\top Q^{-1} BL x_{t-1} x_{t-1}^\top \Bigg]. \end{aligned}$$

We call this quantity as F_t to use in below equations. Now, if we compute,

$$\frac{\partial \mathcal{Q}(\theta, \theta^{\text{old}})}{\partial L} = -\frac{1}{2} \mathbb{E}_{x_{1:T}|\theta^{\text{old}}} \left[\sum_{t=2}^{T} \frac{\partial (x_t - (A+BL)x_{t-1})^\top Q^{-1} (x_t - (A+BL)x_{t-1})}{\partial L} \right] \tag{12}$$

If we use the properties derived in above equations, we obtain

$$\frac{\partial \mathcal{Q}(\theta, \theta^{\text{old}})}{\partial L} = -\frac{1}{2} \mathbb{E}_{x_{1:T}|\theta^{\text{old}}} \left[\sum_{t=2}^{T} \frac{\partial F_t}{\partial L} \right]. \tag{13}$$

[3]Since $Q^{-1} = {Q^{-1}}^\top$

We can expand the derivatives

$$\frac{\partial \mathcal{Q}(\theta,\theta^{\text{old}})}{\partial L} = -\frac{1}{2}\mathbb{E}_{x_{1:T}|\theta^{\text{old}}}\left[\sum_{t=2}^{T}\left(-\frac{\partial \text{Tr}\left[(BL)^\top Q^{-1}x_t x_{t-1}^\top\right]}{\partial L} - \frac{\partial \text{Tr}\left[(BL)^\top Q^{-1}x_t x_{t-1}^\top\right]}{\partial L}\right.\right.$$
$$\left.\left.+\frac{\partial \text{Tr}\left[A^\top Q^{-1}BLx_{t-1}x_{t-1}^\top\right]}{\partial L} + \frac{\partial \text{Tr}\left[(BL)^\top Q^{-1}Ax_{t-1}x_{t-1}^\top\right]}{\partial L} + \frac{\partial \text{Tr}\left[(BL)^\top Q^{-1}BLx_{t-1}x_{t-1}^\top\right]}{\partial L}\right)\right]. \tag{14}$$

With a bit more painful algebra (put $L^\top B^\top$ instead of $(BL)^\top$,

$$\frac{\partial \mathcal{Q}(\theta,\theta^{\text{old}})}{\partial L} = -\frac{1}{2}\mathbb{E}_{x_{1:T}|\theta^{\text{old}}}\left[\sum_{t=2}^{T}\left(-\frac{\partial \text{Tr}\left[L^\top B^\top Q^{-1}x_t x_{t-1}^\top\right]}{\partial L} - \frac{\partial \text{Tr}\left[L^\top B^\top Q^{-1}x_t x_{t-1}^\top\right]}{\partial L}\right.\right.$$
$$\left.\left.+\frac{\partial \text{Tr}\left[A^\top Q^{-1}BLx_{t-1}x_{t-1}^\top\right]}{\partial L} + \frac{\partial \text{Tr}\left[L^\top B^\top Q^{-1}Ax_{t-1}x_{t-1}^\top\right]}{\partial L} + \frac{\partial \text{Tr}\left[L^\top B^\top Q^{-1}BLx_{t-1}x_{t-1}^\top\right]}{\partial L}\right)\right]. \tag{15}$$

Since we can change the order in the trace operator, we can make this expression more differentiation-friendly:

$$\frac{\partial \mathcal{Q}(\theta,\theta^{\text{old}})}{\partial L} = -\frac{1}{2}\mathbb{E}_{x_{1:T}|\theta^{\text{old}}}\left[\sum_{t=2}^{T}\left(-\frac{\partial \text{Tr}\left[B^\top Q^{-1}x_t x_{t-1}^\top L^\top\right]}{\partial L} - \frac{\partial \text{Tr}\left[B^\top Q^{-1}x_t x_{t-1}^\top L^\top\right]}{\partial L}\right.\right.$$
$$\left.\left.+\frac{\partial \text{Tr}\left[x_{t-1}x_{t-1}^\top A^\top Q^{-1}BL\right]}{\partial L} + \frac{\partial \text{Tr}\left[B^\top Q^{-1}Ax_{t-1}x_{t-1}^\top L^\top\right]}{\partial L} + \frac{\partial \text{Tr}\left[B^\top Q^{-1}BLx_{t-1}x_{t-1}^\top L^\top\right]}{\partial L}\right)\right]. \tag{16}$$

Now we can take derivatives[4],

$$\begin{aligned}\frac{\partial \mathcal{Q}(\theta,\theta^{\text{old}})}{\partial L} = -\frac{1}{2}\mathbb{E}_{x_{1:T}|\theta^{\text{old}}}\Bigg[\sum_{t=2}^{T}\big(&-\left[B^\top Q^{-1}x_t x_{t-1}^\top\right] - \left[B^\top Q^{-1}x_t x_{t-1}^\top\right] \\ &+\left[B^\top Q^{-1}Ax_{t-1}x_{t-1}^\top\right] + \left[B^\top Q^{-1}Ax_{t-1}x_{t-1}^\top\right] \\ &+\left[B^\top Q^{-1}BLx_{t-1}x_{t-1}^\top + B^\top Q^{-1}BLx_{t-1}x_{t-1}^\top\right]\big)\Bigg]\end{aligned} \tag{17}$$

Our aim is to leave L alone by setting

$$\frac{\partial \mathcal{Q}(\theta,\theta^{\text{old}})}{\partial L} = 0$$

[4]From [5], we have $\frac{\partial Tr[MXNX^\top]}{\partial X} = M^\top X N^\top + MXN$. Here we have $M = B^\top Q^{-1}B$ and $N = x_{t-1}x_{t-1}^T$. Luckily, $M = M^T$ and $N = N^T$.

For the sake of simplicity, let us write

$$E_{t|t-1} = \mathbb{E}_{x_{1:T}|\theta^{\text{old}}}\left[x_t x_{t-1}^\top\right] \tag{18}$$

$$E_{t-1|t-1} = \mathbb{E}_{x_{1:T}|\theta^{\text{old}}}\left[x_{t-1} x_{t-1}^\top\right], \tag{19}$$

and

$$E^s_{t|t-1} = \sum_{t=2}^{T} E_{t|t-1}, \tag{20}$$

$$E^s_{t-1|t-1} = \sum_{t=2}^{T} E_{t-1|t-1}, \tag{21}$$

$$\tag{22}$$

where s stands for summation from 2 to T. Then, if we set the derivative to zero, we have,

$$\begin{aligned} &B^\top Q^{-1} E^s_{t|t-1} + B^\top Q^{-1} E^s_{t|t-1} - B^\top Q^{-1} A E^s_{t-1|t-1} - B^\top Q^{-1} A E^s_{t-1|t-1} \\ &= \left(B^\top Q^{-1} B L E^s_{t-1|t-1} + B^\top Q^{-1} B L E^s_{t-1|t-1}\right) \end{aligned} \tag{23}$$

Let us put $M = B^\top Q^{-1} B$ and $N = E^s_{t-1|t-1}$. Denote $Z = B^\top Q^{-1} E^s_{t|t-1} - B^\top Q^{-1} A E^s_{t-1|t-1}$. Hence our problem reduces to finding an expression for L by using the following equation

$$Z = MLN. \tag{24}$$

Then, we compactly obtain

$$L = M^{-1} Z N^{-1}. \tag{25}$$

2.2 E-step

For a given estimate of L, one can use the exact same E-step given in [1]. One has to perform a forward-filtering, backward-smoothing algorithm for linear dynamical systems, i.e., the Kalman filtering followed by the RTS smoothing to estimate the expectations given in the M-step. One can consult to [1] to employ this algorithm, only by modifying the A matrix with $A + B\hat{L}$ where $\hat{L}$ is the current estimate of the matrix L.

References

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