De Zolt's Postulate in Three-Dimensions: The Geometrical and the Abstract Paths DRAFT

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1 Introduction

In [3], we presented an abstract approach to Zolt's postulate proof, in its orginal form, that is, for polygons. We proved Zolt's postulate under the following utterance:

Postulate (De Zolt)**.** *Given a polygon $\mathcal{P}$ and a decomposition $T = \{t_1, \ldots, t_k\}$ of $\mathcal{P}$ into k polygons. Let $t_i \in T$, then $T - \{t_i\}$ is not equivalent to T in the theory of equivalence of plane polygons.*

Decompositions are taken as special list of polygons, which are in their turns lists of segments. The abstract approach then compare the original decomposition with the reduced one using operations for constructing and deconstructiing in an algebraic setting, see previous article.

A natural question that arises is to which extent we can adapt the abstract approach to producing a proof of Zolt's postulate for three-dimensions. There are some issues in extending Zolt postulate to higher dimensions. The well-known ball paradoxes, namely, Hausdorf and the two versions of Banach-Tarski paradoxes, teach us that Zolt postulate cannot be proved inside **ZFC**. We explain this in the sequel.

2 The Sphere Paradox and De Zolt's Postulate

Theorem 1 is the general version of the ball[1] paradox, see [5] and [1]. Given A and B subsets of $\mathbb{R}^3$, we say that A and B, are finitely equidecomposable, i.e., $A \sim B$, iff, there are two families of parwise disjoint sets $A_0, \ldots, A_k$, and, $B_0, \ldots, B_k$, and a family $g_0, \ldots, g_k$ of translations and rotations, such that, for all $i = 0, \ldots, k$, $g_i(A_i) = B_i$, $A = \bigcup_{i=0,\ldots,k} A_i$ and $B = \bigcup_{i=0,\ldots,k} B_i$. An easy corolary of Theorem 1 is corollary 1, stated integrally inside the context of the Ball paradoxes. The (apparent) paradox derived from this corollary is called the pea and earth paradox, that states that we can take a ball of the size of a pea, cut it into finitely many pieces and by rotations and translations of each of them, we form a ball of the size of the earth. In fact this works for any pair of balls in the three-dimensional space.

Theorem 1 (Banach-Tarski)**.** *Let A and B be bounded subsets of $\mathbb{R}^3$ with non-empty interior. We have that $A \sim B$.*

Corollary 1 (Banach-Tarski)**.** *Any two balls in $\mathbb{R}^3$ are equiodecomposable.*

Using Corollary 1 we show that Zolt postulate cannot be valid in the same theory that supports this corollary. Consider an octahedron circumscribed in the unit ball, as in figure 1 and a bigger one circumscribed in the ball with diameter 5, as in figure 2. The decompostions, in the sense of Zolt's postulate, of the two octahebra are respectively, $\{w, z\}$ and $\{w, x, y, z\}$. By Corollary 1, the two balls are equidecomposable. The family of isometries that witnesses this equidecomposition carry the Zolt's decomposition of the bigger ball into the first one. On the other hand, if Zolt's postulate holds in $\mathbb{R}^3$ then such a equidecomposability is impossible, since $\{w, z\} \precneqq \{w, x, z\} \precneqq \{w, x, y, z\}$ is a consequence of Zolt's postulate.

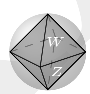

Figure 1: Octahedron circumscribed in the unit sphere

Consider the following Theorem 2, apparently a weaker version of the ball paradox, also known as Hausdorff paradox[2]. It mentions the surface, S^2, of the 3D sphere, instead of solid ball itself as Corollary 1 does. The definition of G-paradoxical seems to be the origin, in chronological sense, of the ball paradoxes mentioned here. It is a definition relative to a group G of geometrical rotations in $\mathbb{R}^3$, for example the set of rotations on the axes of S^2 with the composition of rotations as the group operation. We say that a bounded set A of $\mathbb{R}^3$ is G-paradoxical, if there are two disjoint subsets of A, we can partition them into a finite number of pieces and then by actions from the group on these pieces, we can create two copies of the original set.

Theorem 2 (Hausdorff)**.** *There is countable set $D \subset S^2$ such that, $S^2 - D$ is rotation-paradoxical.*

[1] Also known as General Banach-Tarski paradox

[2] The Hausdorff paradox is used in the proof of the Banach-Tarski paradox

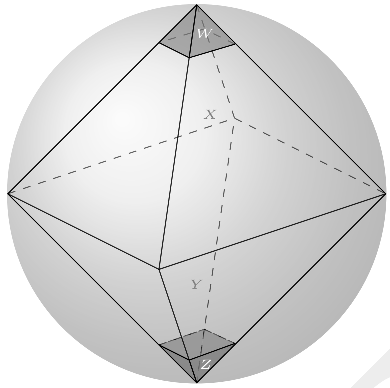

Figure 2: Octahedron circumscribed in the sphere of diameter 5

We have to observe that in well-known proof of the this theorem we can see that the cardinality of D is infinite, see [5] for example.

The above Theorem 2 can be used to provide one more argument against Zolt's postulate validity. Take the octahedron circumsbribed in the unit sphere as in figure 1. There are six points, the vertexes of the octahedron, that belong to S^2. If we apply Theorem 2 then there is a set $D \subset S^2$, such that S^2
D is rotation paradoxical. Since $D$is infinite, we can consider, without loss of generality, that the the six vertexes of the octahedron do not belong to D. Thus, there are two copies of this octahedron that are isometric to the original octahedron. Considering the decomposition of the $\{w, z\}$ decomposition of the octahedron, and, if Zolt's postulate holds then $\{x, y\}$ is equivalent to $\{x, y, x', y'\}$, where x' and y' are the copy of the decomposition of the second copy of the octahedron. However, this is an absurd, for the first decomposition has 2 elements and the second has 4 elements.

De Zolt's postulate is not the only proposition that is is provable in the plane geometry and that does not hold in the space (3D) geometry. The Bolyai-Gerwain theorem[3], that is valid in the plane geometry cannot be provable in 3D geometry, due to its falsity in this case. We say that two polygons are congruent by dissection, if and only if, one of them can be cut up into polygonal pieces and reassembled in to form the other polygon. This is a classical problem in geometry. Bolyai-Gerwain theorem states that two polygons P_1 and P_2 are congruent by dissection if and only if they have the same area. Max Dehn proved, in [2], that the regular tetrahedron and the cube are not congruent by dissection[4]

Since Zermelo-Fraenkel is consistent with the Axiom of Choice (AC) and, the Banach-Tarski theorems (paradoxes) are obtained in ZFC (Zermelo-Fraenkel + AC) we have to conclude that De Zolt's in higher dimension might not be a theorem of ZF,

[3]Proved independently by F.Bolyai (1832) and P.Gerwain (1833)

[4]Knowning if every pair of polyhedra with a same volume are, or not, congruent by dissection is the third Hilbert problem in the list posed to the mathematical community in Paris, 1900. Max Dehn, in his habilitationshirift defined a notion of invariance, nowadays known as Dehn-invariance, and showed that for any pair of polyhedra with the same volume, they are invariant if and only if they are congruent by dissection.

but in a weaker theory. The failure of De Zolt's in ZFC does not seem to be a direct consequence of AC, indeed. This article could have a twofold goal, namely, to discuss at what extend the known proofs of De Zolt's can be extended to the three-dimension or higher-dimension case and, to provide an alternative and quite basic proof of De Zolt for higher-dimension that is conveyed in a quite weak sub-theory of ZF. We take this last pathway in this article, however. In Section 3 we motivate an abstract approach to prove a Zolt's in a higher-dimension setting. In section 4 we propose our approach to prove Zolt's postulate in higher-dimension.

3 An Abstract Proof of De Zolt's

In this section we show by means of an example, how an abstract, and somehow naive, constructive approach to De Zolt's postulate can be effective in proving it for three-dimension. Consider a polyhedron and one of its decompostions as shown in figure 3

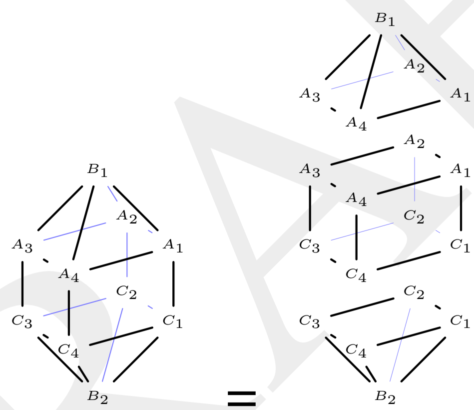

Figure 3: A polyhedron and one of its decompostions

To facilitate the denotation of the decomposition shown in figure 3 and its parts, we denote by P_1 the polyhedron formed by B_1, A_1, A_2, A_3, A_4, by P_2 the polyhedron formed by $A_1, A_2, A_3, A_4, C_1, C_2, C_3, C_4$ and, finally, by P_3 the polyhedron formed by B_2, C_1, C_2, C_3, C_4. By identifying the common points A_1, A_2, A_3 and A_4 existing in P_1 and P_2 we have the upper polyhedron, which we denote by $P_1; P_2$. Finally, we denote by $(P_1; P_2); P_3$ the whole decompostion, by a similar identification of points common in $P_1; P_2$ and P_3. The expression $P_1; (P_2; P_3)$ represents a different and equivalent way of denoting this decomposition. The difference is the order of identification of the common points. This is of course a naive way of denoting polyhedra decompositions, but it helps to provide a justification that the polyhedron obtained by removing P_1, i.e., $P_2; P_3$, from the decompositon is stricly smaller then the decomposition itself, that is, DeZolt statement holds for this decomposition reduction represented by figure 4.

We use basic mereological judgements and the concept of trivial and proper magnitudes, as it was introduced in [3] to deal with the two-dimensional, to provide a formal rule based justification for drawing the conclusion that the polyhedron at the left side of figure 4 is stricly smaller than the right-side one, i.e, $(P_2; P_3) \prec P_1; (P_2; P_3)$. This is shown by the deduction below.

$$\frac{P_2; P_3 \preceq \epsilon; (P_2; P_3) \qquad \dfrac{\epsilon \prec P_1 \qquad \dfrac{P_2 \preceq P_2 \qquad P_3 \preceq P_3}{P_2; P_3 \preceq P_2; P_3}}{\epsilon; (P_2; P_3) \prec P_1; (P_2; P_3)}}{P_2; P_3 \prec P_1; (P_2; P_3)}$$

In the following section we better formalize the above rules and the system, through a typed language and rules, i.e., a typing system. It is important to note that the use of types in different of the use of sorts. The system used to prove Zolt 3D is a very weak typing system. It is not even capable of define the natural numbers. It has only three types $\mathfrak{p}$ for points, $\mathfrak{s}$ for segments or segment-like objects, $\mathfrak{f}$ for faces or facetal surfaces, and, $\mathfrak{v}$ for volumes or polyhedral objects. We can observe that it is not a sorted language, although, we could use a sorted approach, $\mathfrak{p}, \mathfrak{s}, \mathfrak{f}$ and $\mathfrak{v}$ are not sorts. In fact, there is a dependency between these types. For example, terms of type $\mathfrak{f}$ (polyhedral surfaces or facetal surfaces) depend on the hypothesis on the polygonal lines (of type $\mathfrak{s}$) that depend on points. This dependency is not explicitly annotated, for the sake of a cleaner representation of the decompositions. We do not want to have the Product of families type here to bind these dependencies. The proof inference itself takes care of these dependencies. One of the main reasons to have these dependencies is that we have the concepts of $Jordan(\_, \ldots, \_)$, $Colinear(\_, \ldots, \_)$, and $Closed(\_, \ldots, \_)$ in order to assign the types $\mathfrak{f}$, $\mathfrak{s}$, and $\mathfrak{p}$ to the corresponding terms. These notions are not needed to be formalized in a particular proof, but they are taken as holding from the hypothesis of polyhedron decompositions as used in the proof of Zolt theorem. Another reason is that if the system were a sorted one, it would be equivalent fo first-order logic, and hence, unable to formalize $Jordan(\_, \ldots, \_)$, $Closed(\_, \ldots, \_)$ and $Colinear(\_, \ldots, \_)$, unless we introduce the killing[5] Real numbers concept.

We can note that the main reason to have the impression of a sorted system is the fact that the system has not constructors (like Product types, co-product and so). These constructors have a global effect. Our approach is a local one. The typing system works in a more or less analytical way.¨

In the next section we present the typing system Z_p. We also show deductions that formalize in Z_p the above example.

4 A Type Theoretical System for Geometric Mereology: Z_p

We consider that points are primitive objects. Thus, all we need to denote a point is a declaration. With an statement of the form $x : \mathfrak{p}$ we can declare that x is of type point.

[5] Admiting the Real Numbers, there can be enough expression power to get back the Ball paradoxes

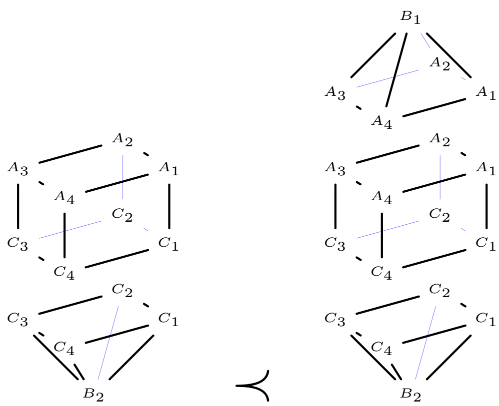

Figure 4: Comparing a polyhedron with one of its reduced decompositions

The types in Z_p, as stated in a previous section, are points $\mathfrak{p}$, segments or segment-like objects $\mathfrak{s}$, facetal surfaces $\mathfrak{f}$ and volumes or polyhedra $\mathfrak{v}$.

The premisses that mention $Jordan$, $Closed$ and $Colinear$ are basically proof obligations or assumptions provided as free. $Jordan(t_1, t_2)$ holds whenever t_1 and t_2 are two coplanar segment-like objects that form a Jordan polygonal region in the plane. $Colinear(t_1, t_2)$ holds whenever t_1 and t_2 are not colinear, in the case that there is a unique plane that contains both segments, or segment-like object.

Geometric figures are represented by (lists (of lists (of lists))...). We alternate the many formats of bracket delmiters such that we gain extra readability. For example, we use $[([\ldots])]$ or even $[(\langle\ldots\rangle)]$ instead of $(((\ldots)))$ to get more readable terms. Figure 5 presents the set of rules of system Z_p.

The last four rules, cmp_i, $i = 0, 1, 2, 3$, allow the derivation of a compatibility condition of the form $p\ cmp\ q$, i.e., the rule concludes that the two terms represent geometrical objects, such that, p and q take part in an uppermost term formed just by both of them.. All of these rules are abstraction rules, for their premisses have more informational content than the conclusion. The rules ϵ_i, $i = 0, 1, 2$, deal with the trivial polyhedron and its relationship with composition of parts of polyhedra. The merealogica rules, $\preceq_i$ and $\prec_j$ relate the magnitudes of composition with those of its components. The composition rules, $\mathfrak{p}$, $\mathfrak{s}_i$, $\mathfrak{f}_i$ and $\mathfrak{v}_i$, $i = 1, 2$, are typing rules that state how a composition is structured from the (types of) its components. In the sequel, we show how the example presented in the previous section is formalized using Z_p.

In the sequel we show a series of deductions that are used to derive a typed term of type $\mathfrak{v}$ that represents the polyhedron in figure 6. The proof in Z_p that derives the respective term of type $\mathfrak{v}$ is in figure 13.

Types $\mathfrak{T}$: $\mathfrak{p}$, $\mathfrak{s}$, $\mathfrak{f}$, $\mathfrak{v}$

Rules for typing inference

$$\mathfrak{s}_1\ \frac{n:\mathfrak{p}\quad m:\mathfrak{p}\quad n\neq m}{\langle n,m\rangle:\mathfrak{s}}\qquad \mathfrak{s}_2\ \frac{p:\mathfrak{s}\quad q:\mathfrak{s}\quad \neg Colinear(p,q)\quad p\cap q:\mathfrak{p}}{\langle p:\mathfrak{s},q:\mathfrak{s}\rangle:\mathfrak{s}}$$

$$\mathfrak{f}_1\ \frac{p:\mathfrak{s}\quad q:\mathfrak{s}\quad Jordan(p\cup q)}{p\cup q:\mathfrak{f}}\qquad \mathfrak{f}_2\ \frac{p:\mathfrak{f}\quad q:\mathfrak{f}\quad p\cap q:\mathfrak{s}}{p\cup q:\mathfrak{f}}$$

$$\mathfrak{v}_1\ \frac{p:\mathfrak{f}\quad q:\mathfrak{f}\quad Closed(p\cup q)}{p\cup q:\mathfrak{v}}\qquad \mathfrak{v}_2\ \frac{p:\mathfrak{v}\quad q:\mathfrak{v}\quad p\cap q:\mathfrak{f}}{p\cup q:\mathfrak{v}}$$

$$\epsilon_0\ \frac{p:\mathfrak{T}}{p\preceq p}\quad \epsilon_1\ \frac{p:\mathfrak{T}\quad q:\mathfrak{T}\quad p\ cmp\ q\quad q\neq\epsilon}{p\prec p;q}\quad \epsilon_2\ \frac{p:\mathfrak{T}\quad q:\mathfrak{T}\quad p\ cmp\ q\quad p\neq\epsilon}{q\prec p;q}$$

$$\preceq_1\ \frac{p_i:\mathfrak{T}\quad q_i:\mathfrak{T}\quad p_i\preceq q_i\quad i=1,2}{p_1;p_2\preceq q_1;q_2}\quad \prec_1\ \frac{p_i:\mathfrak{T}\quad q_i:\mathfrak{T}\quad p_1\prec q_1\quad p_2\preceq q_2}{p_1;p_2\prec q_1;q_2}$$

$$\epsilon\ \frac{}{\epsilon:\mathfrak{T}}\qquad \prec_2\ \frac{p_i:\mathfrak{T}\qquad q_i:\mathfrak{T}\qquad p_1\preceq q_1\qquad p_2\prec q_2}{p_1;p_2\prec q_1;q_2}$$

$$cmp_0\ \frac{\langle n,m\rangle:\mathfrak{s}}{n\ cmp\ m}\qquad cmp_1\ \frac{\langle p:\mathfrak{s},q:\mathfrak{s}\rangle:\mathfrak{s}}{p\ cmp\ q}\qquad cmp_2\ \frac{p\cup q:\mathfrak{f}}{p\ cmp\ q}\qquad cmp_3\ \frac{p\cup q:\mathfrak{v}}{p\ cmp\ q}$$

Figure 5: The type system Z_p for polyhedral mereology

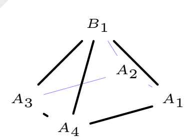

Figure 6: Polyhedron used to illustrate how Z_p works.

We start by showing, in figure 8 the derivation that represents that the face in figure 7 is of type $\mathfrak{f}$.

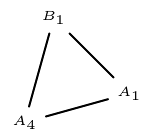

Figure 7: Face represented by the term of type $\mathfrak{f}$ in the derivation in figure 8

$$
\mathfrak{f}_1\,\dfrac{\mathfrak{s}_2\,\dfrac{\mathfrak{s}_1\,\dfrac{\begin{array}{c}A_4 \neq B_1\\ A_4 : \mathfrak{p} \quad B_1 : \mathfrak{p}\end{array}}{\langle A_4, B_1\rangle : \mathfrak{s}} \quad \mathfrak{s}_1\,\dfrac{\begin{array}{c}A_1 \neq B_1\\ B_1 : \mathfrak{p} \quad A_1 : \mathfrak{p}\end{array}}{\langle B_1, A_1\rangle : \mathfrak{s}} \quad \langle A_4, B_1\rangle \cap \langle B_1, A_1\rangle : \mathfrak{p} \quad \neg Colinear(\langle A_4, B_1\rangle, \langle B_1, A_1\rangle)}{[\langle A_4, B_1\rangle, \langle B_1, A_1\rangle] : \mathfrak{s}} \quad \mathfrak{s}_1\,\dfrac{\begin{array}{c}A_1 \neq A_4\\ A_1 : \mathfrak{p} \quad A_4 : \mathfrak{p}\end{array}}{\langle A_1, A_4\rangle : \mathfrak{s}} \quad Jordan(A_4, B_1, A_1)}{\langle [\langle A_4, B_1\rangle, \langle B_1, A_1\rangle], \langle A_1, A_4\rangle\rangle : \mathfrak{f}}
$$

Figure 8: The derivation of the term representing the face in figure 7

The derivation in figure 8 is π_1. This derivation depends on $Jordan(A_4, B_1, A_1)$[6]. We do not need to prove this assertion, unless there is need to provide a closed derivation, i.e., a proof.

There is an analogous derivation π_2 that provides support for stating that the face $A_4B_1A_3$ is of type $\mathfrak{f}$. The derivation in figure 10 shows that the facetal surface, in figure 9, formed by gluing faces $A_4B_1A_1$ and $A_4B_1A_3$ among their intersection segment A_4B_1, of type $\mathfrak{s}$, is of type $\mathfrak{f}$.

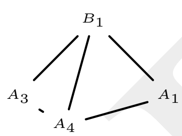

Figure 9: The facetal surface of type $\mathfrak{f}$. Vertex A_2 does not belong to it.

$$
\mathfrak{f}_2\,\dfrac{\mathfrak{f}_1\,\dfrac{\pi_1}{\langle [\langle A_4, B_1\rangle, \langle B_1, A_1\rangle], \langle A_1, A_4\rangle\rangle : \mathfrak{f}} \quad \mathfrak{f}_1\,\dfrac{\pi_2}{\langle [\langle A_4, B_1\rangle, \langle B_1, A_3\rangle], \langle A_3, A_4\rangle\rangle : \mathfrak{f}} \quad \mathfrak{f}_2\,\dfrac{\pi_3}{\langle [\langle A_4, B_1\rangle, \langle B_1, A_1\rangle], \langle A_1, A_4\rangle\rangle \cap \langle [\langle A_4, B_1\rangle, \langle B_1, A_3\rangle], \langle A_3, A_4\rangle\rangle : \mathfrak{s}}}{\langle [\langle A_4, B_1\rangle, \langle B_1, A_1\rangle], \langle A_1, A_4\rangle\rangle \cup \langle [\langle A_4, B_1\rangle, \langle B_1, A_3\rangle], \langle A_3, A_4\rangle\rangle : \mathfrak{f}}
$$

Figure 10: The derivation $\pi_\cap$ of the term representing the facetal surface in figure 9

Similarly, there is the derivation π_4 stating that $A_3B_1A_2$ is represented by a term of type $\mathfrak{f}$. There is also a derivation, which in figure 12 is π_5, showing that the facetal surface $B_1A_1A_4A_3$ has a common segment with $A_3B_1A_2$. Thus, we can glue these two facetal surfaces along this common segment, deriving that the facetal surface in figure 14 is of type $\mathfrak{f}$, as shown in figure 12.

[6]We soemtimes use a (informal) list of objects as an abreviation of the corresponding set of objects. Thus, the set $\{A_4, B_1, A_1\}$ is also represented by A_4, B_1, A_1 simply

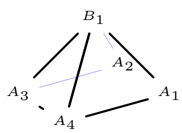

Figure 11: Facetal surface $B_1A_1A_4A_3$

$$\dfrac{\dfrac{\pi_\cap}{\langle [A_4,B_1,A_1],[A_4,B_1,A_3]\rangle : \mathfrak{f}} \quad \dfrac{\pi_4}{\langle [A_3,B_1,A_2]\rangle : \mathfrak{f}} \quad \dfrac{\pi_5}{\langle [A_4,B_1,A_1],[A_4,B_1,A_3]\rangle \cap \langle [A_3,B_1,A_2]\rangle : \mathfrak{s}}}{\langle [A_4,B_1,A_1],[A_4,B_1,A_3]\rangle \cup \langle [A_3,B_1,A_2]\rangle : \mathfrak{f}}$$

Figure 12: The derivation $\pi^3_\cap$ of the term representing the facetal surface in figure 14

Finally, assuming that by adding the face $A_2B_1A_1$, of type $\mathfrak{f}$ according to derivation π_6, to the facetal surface represented by the conclusion of the above derivation $\pi_\cap$ we have a closed surface, hence a polyhedron. We reach to a term of type $\mathfrak{v}$, as shows the derivation in figure 13.

$$\dfrac{\dfrac{\pi^3_\cap}{\langle [A_4,B_1,A_1],[A_4,B_1,A_3]\rangle \cup \langle [A_3,B_1,A_2]\rangle : \mathfrak{f}} \quad \dfrac{\pi_6}{\langle [A_2,B_1,A_1]\rangle : \mathfrak{f}} \quad Closed(\langle [A_4,B_1,A_1],[A_4,B_1,A_3],[A_3,B_1,A_2]\rangle, \langle [A_2,B_1,A_1]\rangle}{\langle [A_4,B_1,A_1],[A_4,B_1,A_3],[A_3,B_1,A_2]\rangle \cup \langle [A_2,B_1,A_1]\rangle : \mathfrak{v})}$$

Figure 13: The Z_p proof deriving the term that represents the polyhedron in figure 6

Analogous derivations for each of the polyhedra that take part in the decomposition of our polyhedron in the previous section allows us to derive that it is of type $\mathfrak{v}$.

The first thing to observe, concerning the system Z_p is that it can formalize any decomposition of any polyhedron, type $\mathfrak{v}$, as well facetal surfaces, type $\mathfrak{f}$, and polygonal lines, type $\mathfrak{s}$. In particular, the kind of decomposition for polyhedra called **tetrahedralization** is formed only by tetrahedra that share faces, i.e., the intersection of any two tetrahedra is either a face, or it is empty, and the union of all the tetrahedra in this tetrahedralization is the whole polyhedra. We advice that not all polyhedra has a triangulation or tetralization, but all convex polyhedra have. This is reported as an easy result and seems to take part of the mathematical folklore theorems. However, [4] is one of the first articles reporting possible extentions of the tetrahedralization to more than convex polyhedra. There are many open problems currently under investigations. Obviously our work is out of the scope of this extentions. We can state that convex polyhedra is perhaps a bit far from a traditional view of polyhedra. The following lemma 1 shows that the system Z_p can express any triangulation or tetralization easily. So, as expected we can include 3D convex polyhedra in the scope of 3D Zolt's postulate.

We observe that any tetrahedron $ABCD$, as depicted in figure 14, can be expressed as the union of the faces ABC, ADC, BDC and BAD. As we already noticed in the last section, these triangles have terms $t_{ABC}, t_{ADC}, t_{BDC}$ and t_{BAD} all of type $\mathfrak{f}$ in

Z_p. For example t_{ABC} can be $\langle[\langle B, A\rangle, \langle A, C\rangle], \langle C, B\rangle\rangle$ with a derivation analogous to the one in figure 8. Finally, using rules $\mathfrak{f}_1$ and $\mathfrak{v}_1$ we can have a derivation that draws that the there is a term of type $\mathfrak{v}$ that corresponds to the triangulation of $ABCD$ as considered above. Given a triangulation Δ, the Z_p term obtained from it, as done above for a sole tetrahedron, is called $r(\Delta)$.

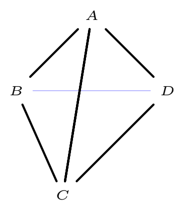

Figure 14: Facetal surface and tetrahedron $ABCD$

Lemma 1. *Let Δ be a triangulation of a polyhedron. Then there is a term $r(\Delta)$ and a proof π, in Z_p, that $r(\Delta)$ is of type $\mathfrak{v}$ and a derivation π of $r(\Delta) : \mathfrak{v}$ in Z_p.*

Proof: We prove it by induction on the number of tetrahedra in Δ. If $card(\Delta) = 1$, then we already have the base step discussed in the paragraph above the lemma statement. Let $\Delta = \Delta_1 \cup \{\theta\}$, where θ is a tetrahedron sharing a face with the remaining triangulation. BY inductive hypothesis there is a derivation π_1 of $t_1 : \mathfrak{v}$, with $r(\Delta_1) = t_1$. Moreover, we have derivations of $t_\theta : \mathfrak{v}$ and $t_1 \cap t_\theta : \mathfrak{f}$, so by applying rule $\mathfrak{v}_2$ we can derive $t_1 \cup t_\theta : \mathfrak{v}$. Thus, $r(\Delta) = t_1 \cup t_\theta$.

□

An interesting point to observe is that Z_p can express decompositions that more general than triangulations or tetrahebralizations. For example, the decomposition discussed in section 5 is not a tetrahedralization and, $Z_p$can express it.

Theorem 3 (Completeness)**.** *Let Δ_p be a polyhedron decomposition of P and Δ_q a truncation of Δ_p then $r(\Delta_q) \prec r(\Delta_p)$ is provable in Z_p.*

Proof: Sketch: From Δ_p and Δ_q, by lemma 1, we have π_1 and π_2, derivations of $r(\Delta_p) : \mathfrak{v}$ and $r(\Delta_q) : \mathfrak{v}$. Combine π_1 and π_2 to obtain a proof of $r(\Delta_q) \prec r(\Delta_p)$.

Obs: Removing one of the polyhedra that takes part in the decomposition is equivalent ot have ϵ in its place. We compare part by part of the truncated polyhedron with the integral one, using the mereological rules that deal with $\prec$ and $\preceq$, to obtain the desired conclusion. The proof proceeds by induction on the size of π_1.

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