Final Oral Exam

Relational Learning and Fairness

Guy W. Cole

November 26, 2019

Outline

Topic Blockmodel: Introduction

- *Problem*: Lots of communications observed over network connections, how do we improve inference about the people based on the communications?

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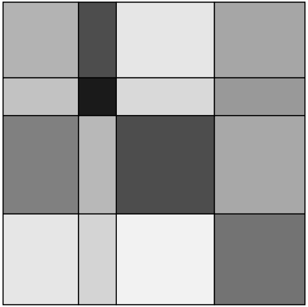
- *Problem*: Lots of communications observed over network connections, how do we improve inference about the people based on the communications?
- *Intuition*: Individuals in one homogeneous group should behave similarly when interacting with individuals of another homogeneous group.
- *Model*: Use a stochastic blockmodel to assign individuals to communities, and a topic model for the communications from each community to each other.

Background: Stochastic Blockmodels

- Boolean Stochastic Blockmodel:[a]
  - Each of n individuals are assigned to one of K communities
  - Each pair of communities (i, j) has an edge probability P_{ij}, probability that node in i has edge with any node in j.
- Can move from boolean to other distributions...

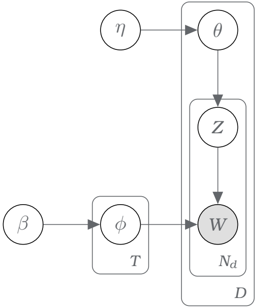


[a] Y. Wang and Wong 1987; Snijders and Nowicki 1997.

Background: Latent Dirichlet Allocation

Topic Modeling:[a] Each document has a topic distribution θ from which each topic z is drawn, and each word w is drawn from the word distribution ϕ for its corresponding topic.

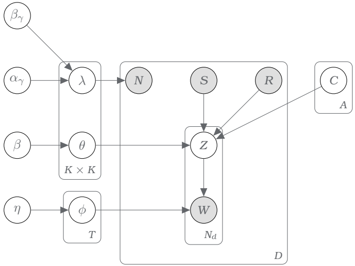
$$
\begin{aligned}
\theta_d &\sim \text{Dirichlet}(\eta) \\
\phi_t &\sim \text{Dirichlet}(\beta) \\
z_{dn} &\sim \text{Multinomial}(\theta_d) \\
w_{dn} &\sim \text{Multinomial}(\phi_{(z_d n)})
\end{aligned}
$$



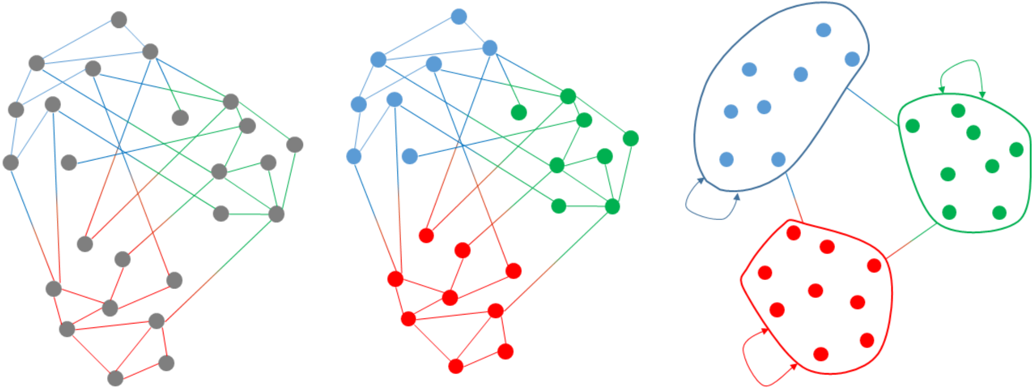
[a] Blei, Ng, and Jordan 2003.

Topic Blockmodel

- Draw community memberships: $Z \sim \text{CRP}(\alpha)$
- Draw expected of word counts $\lambda_{ij} \sim \text{Gamma}(\alpha_\gamma, \beta_\gamma)$
- Draw topic distributions: $\theta_{ij} \sim \text{Dirichlet}(\eta)$
- Draw word distributions: $\phi_t \sim \text{Dirichlet}(\beta)$
- For each sender s to receiver r,
  - Draw a number of words $n_{sr} \sim \text{Poisson}(\lambda_{(C_s)(C_r)})$
  - For each of n_{ij} words:
    - Draw a topic: $Z_{wsr} \sim \text{Multinomial}(\theta_{(C_s)(C_r)})$
    - Draw a word: $W_{wsr} \sim \text{Multinomial}(\phi_{(Z_{wsr})})$



Topic Blockmodel



Count Modeling

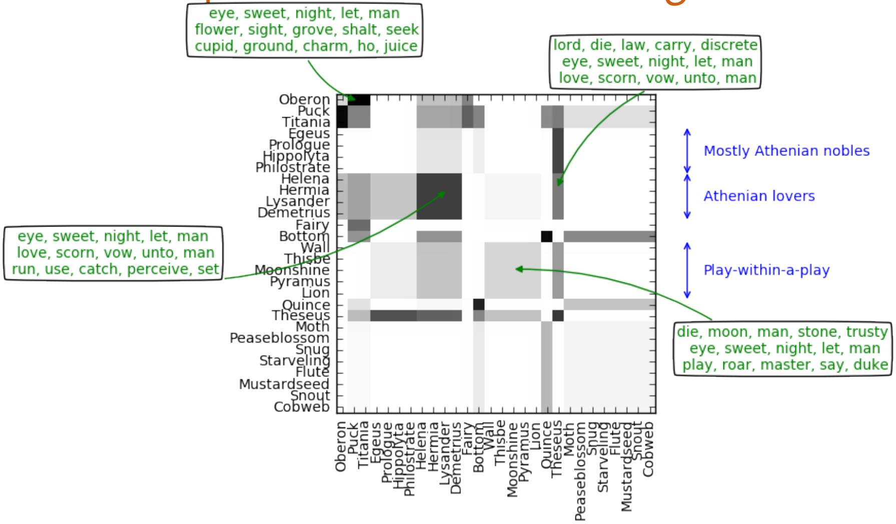
We model not just the *content* but also the *frequency*

$$\lambda_{ij} \sim \text{Gamma}(\alpha_\gamma, \beta_\gamma)$$

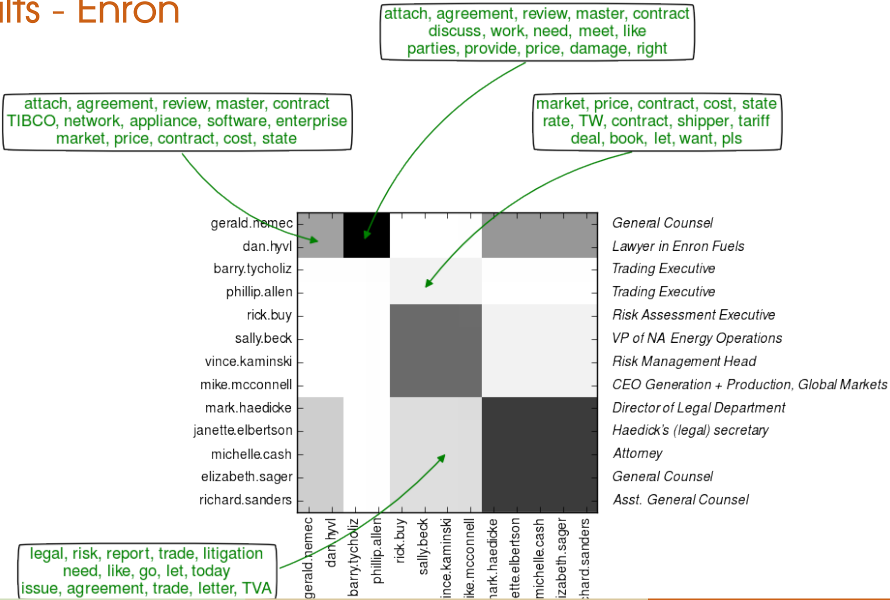
$$N_{sr} \sim \text{Poisson}(\lambda_{(C_s)(C_r)})$$

Captures information from communication frequency / intensity, not just what they discuss

Results - Shakespeare's *A Midsummer Night's Dream*



Results - Enron



Results - Quantitative Tasks

Table: Log predictive likelihood ($\pm$ one standard error) of document text, conditioned on sender and recipient where applicable.

Model	ENRON	Shakespeare
LDA	-410,110.2 $\pm$ 50.8	--48,716.2 $\pm$ 4.6
ART	-365,600.5 $\pm$ 47.7	-47,495.5 $\pm$ 4.8
CNT	-368,983.5 $\pm$ 89.2	**-46,076.6** $\pm$ **3.9**
Topic Blockmodel	**-345.632.5** $\pm$ **4.1**	-46,275.9 $\pm$ 4.0

Results - Quantitative Tasks

Table: Log predictive likelihood (± one standard error) of document recipient, conditioned on document content and sender where applicable.

Model	ENRON	Shakespeare
ART	-204,585.3 ± 6.4	-19,809.7 ± 1.1
CNT	-216,278.9 ± <0.1	-19,703.3 ± <0.1
Poisson-SBM	-160,984.7 ± 148.6	-14,587.2 ± 35.9
Topic Blockmodel	**-137,199.8** ± **53.2**	**-12,997.8** ± **20.6**

Results - Quantitative Tasks

Table: Log predictive likelihood ($\pm$ one standard error) of document sender and recipient, conditioned on document content where applicable.

Model	ENRON	Shakespeare
ART	-416,588.6 $\pm$ 6.8	-39,580.0 $\pm$ 1.0
CNT	-432,557.7 $\pm$ $<$0.1	-39,406.7 $\pm$ $<$0.1
Poisson-SBM	-347,479.6 $\pm$ 148.6	-31,400.3 $\pm$ 35.9
Topic Blockmodel	**-321,127.8 $\pm$ 53.3**	**-29,614.0 $\pm$ 20.6**

Results - Quantitative Tasks

Table: Log predictive likelihood ($\pm$ one standard error) of sender and recipient counts.

Model	ENRON	Shakespeare
Poisson-SBM	-92,851.2 $\pm$ 12.1	-103,411.4 $\pm$ 0.6
Topic Blockmodel	**-88,730.4 $\pm$ 3.1**	**-102,549.8 $\pm$ 0.2**

Conclusion

Questions?

Monotonic Fairness: Introduction

- *Problem*: Although we can create fair(er) prediction and classification systems, they tend to create resentment which undermines their support.

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Monotonic Fairness: Introduction

- *Problem*: Although we can create fair(er) prediction and classification systems, they tend to create resentment which undermines their support.
- *Intuition*: If we define resentment as seeing someone ``worse''get a ``better'' outcome, we can form models that avoid that outcome.
- *Model*: Modify existing fair neural network models with a monotonic neural network to guarantee that resentment doesn't occur

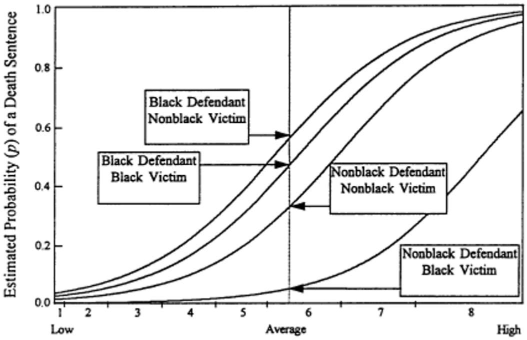
Fairness in machine learning

Machine learning models are powerful, but imperfect

- Tend to reproduce historic biases
- Can overfit, producing functions that are locally nonsensical

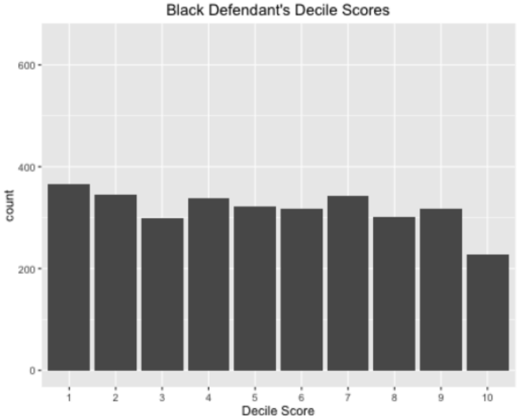
Old Systemic Bias

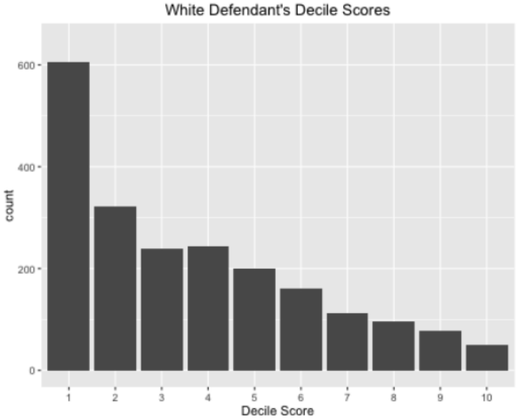
Estimated Race of Defendant and Race of Victim Effects
in Jury Death Sentencing Decisions Among All Death Eligible Cases
Philadelphia 1983-93



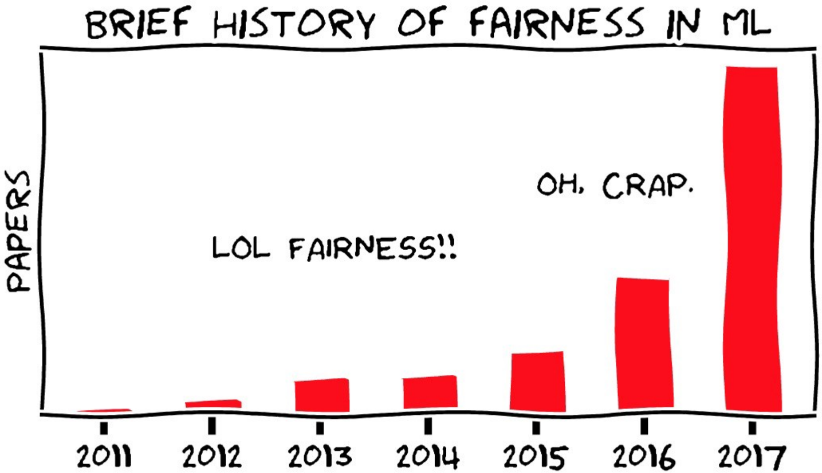
Defendant Culpability Index and Scale Score
Estimated in a Logistic Regression Analysis

New Systemic Bias





Growing Awareness



Fairness in machine learning

Fairness in ML generally has 3 steps:

- Conceptualize

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Fairness in machine learning

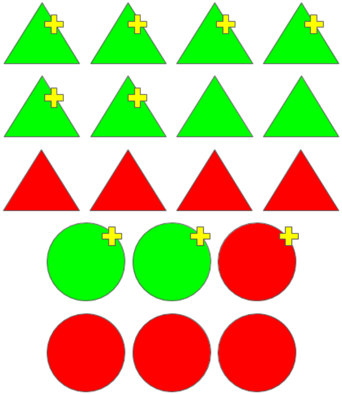
Fairness in ML generally has 3 steps:

- Conceptualize
- Measure
- Prevent
  - Cheaply, hopefully

Concepts of fairness: Equality of Outcome

Equality of Outcome: ``Each group should have the same outcome on average''

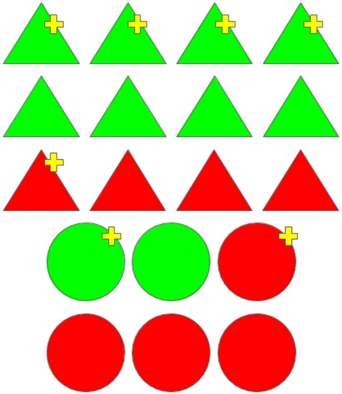
$$\mathbb{E}[\hat{Y}|A = a] = \mathbb{E}[\hat{Y}|A = a']$$



Concepts of fairness: Equality of Odds

Equality of Odds: ``The average prediction should be independent of protected class for people with the same outcome.''

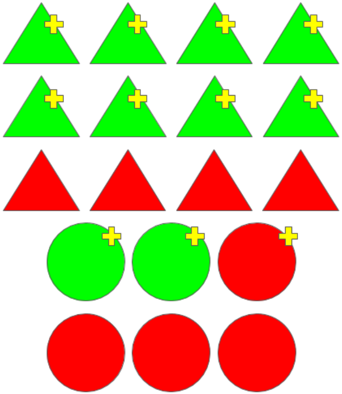
$$\mathbb{E}[\hat{Y}|A = a, Y = y] = \mathbb{E}[\hat{Y}|A = a', Y = y]$$



Concepts of fairness: Equality of Opportunity

Equality of Opportunity: ``For people who deserve the favorable outcome, the probability of receiving the favorable prediction should be independent of class

$$\Pr[\hat{Y} = 1 | A = a, Y = 1] = \Pr[\hat{Y} = 1 | A = a$$



Concepts of fairness: Individual Fairness

Individual fairness: similar individuals should be treated similarly

- $d\left(\hat{f}(X_i), \hat{f}(X_j)\right) \leq D(X_i, X_j)$
- Essentially a requirement of Lipschitz continuity with bounded smoothness.

Individual Resentment

An individual may still experience *resentment* when:

- They receive a less favorable outcome,
- Another person receives a more favorable outcome,
- And either:
  - The other person is identical except for a protected attribute
  - The other person has "worse" non-protected attributes

Individual Resentment

Definition

Protected Attribute Resentment (Class Resentment): Individual u experiences *class resentment* under function f if $\exists\, A' \in \mathcal{A}$ s.t. $f(X_u, A') > f(X_u, A_u)$.

Definition

Non-Protected Attribute Resentment (Score Resentment): Individual u experiences *score resentment* under function f if $\exists\, (X', A') \in (\mathcal{X}, \mathcal{A})$ such that X_u is objectively ``better'' than X' but $f(X', A_u) > f(X_u, A_u)$. (A' may be A_u.)

COMPAS Resentment

Defendant A

Caucasian

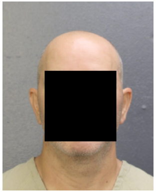
25 y.o

4 priors

No juvenile charges

Felony, Violent charge

Robbery, no weapon



Defendant B

African-American

25 y.o

3 priors

No juvenile charges

Felony, Non-Violent charge

Grand Theft, 3rd Deg.

COMPAS Resentment

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No juvenile charges

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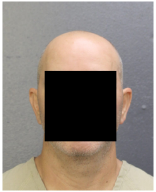


Decile Score: 4

Defendant B

African-American

25 y.o

3 priors

No juvenile charges

Felony, Non-Violent charge

Grand Theft, 3rd Deg.

Decile Score: 10

Preventing Resentment

In order to prevent resentment, we propose a system which:

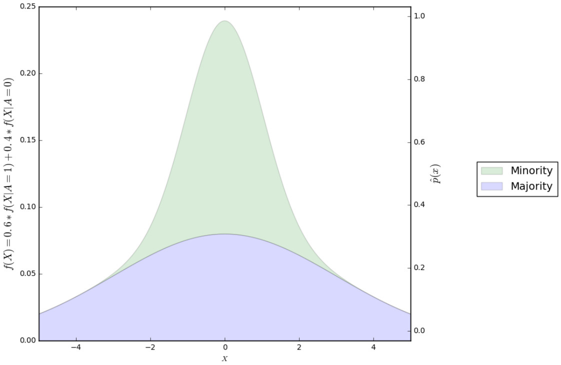
- Takes only non-protected attributes X (not protected attributes A) as input for prediction.
- Use a neural network which has an output function that is monotonic w.r.t. those dimensions X that are user specified

Preventing Resentment

We then train that function to minimize a weighted sum of prediction loss and group fairness loss.

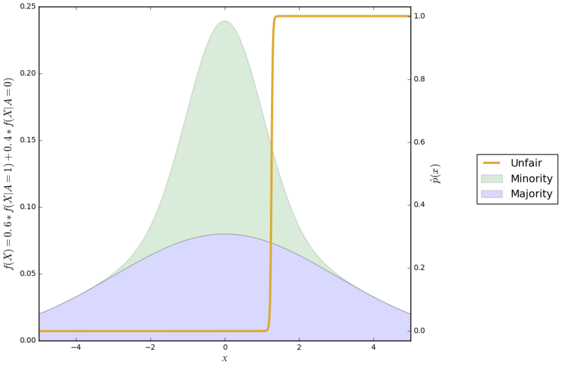
$$\text{Loss} = (1 - \alpha)\ \text{Loss}_{Acc} + \alpha\ \text{Loss}_{Fair}$$

Resentment Graphically



Resentment Graphically

Learn $f : \mathbb{R} \rightarrow [0, 1]$
Maximize $\sum f(X_i)X_i / \sum f(X_i)$
s.t. $\sum f(X_i)/n = 0.25$



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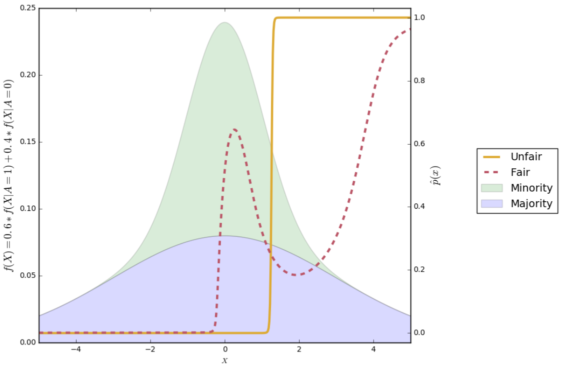
Add fairness:
$\mathbb{E}[f|A = 0] = \mathbb{E}[f|A = 1]$



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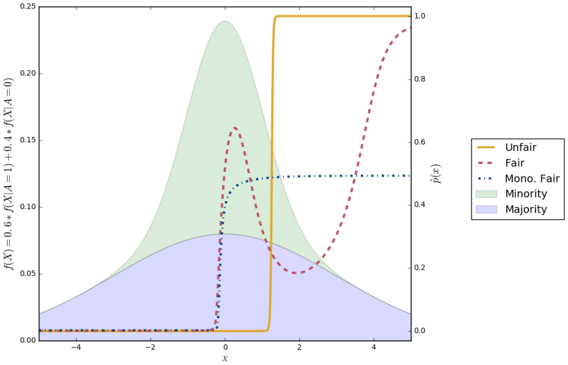
...and no resentment



Monotonic Neural Networks

As proposed by Sill 1998 , redefine hidden nodes as:

$$h_{j,l} = \sigma\left(\sum_{i \in 1 \ldots |H_{l-1}|} \tau(w_{i,j}^{l}) h_{j,l-1} + b_j^l\right)$$

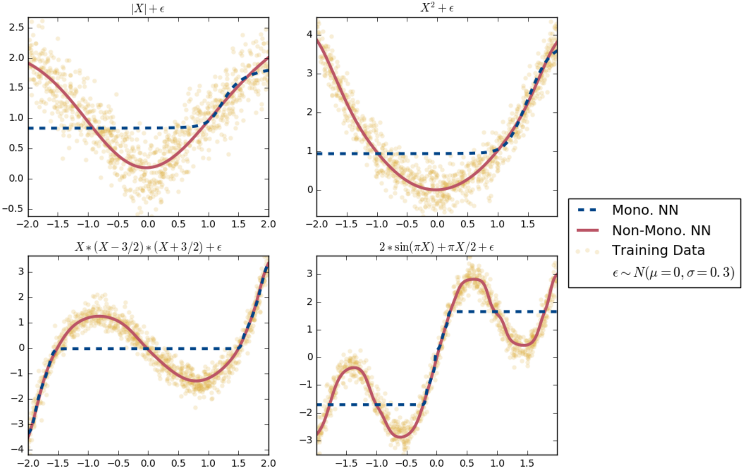
where $\tau : \mathbb{R} \to \mathbb{R}_+$ and σ is monotonically non-decreasing

Mixed Monotonicity

If we *don't apply* τ to the weights in the first layer corresponding to dimension of X_j, then the function *will not be monotonic* w.r.t. X_j, even if we transform the weights in subsequent layers.

If we replace $\tau(w^1_{i,j})$ with $-\tau(w^1_{i,j})$ for some input dimension j, then the first layer (and all subsequent layers) will have a monotonic non-increasing relationship with that dimension.

Montonicity Demo



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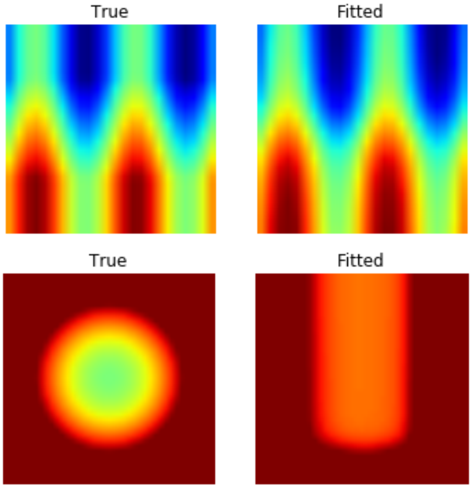

Datasets

We evaluate our model on three fairness-related data sets:

- *Law school* : 17,400 law school applicants, trying to predict law school grade as function of LSAT and undergrad GPA, protecting gender.

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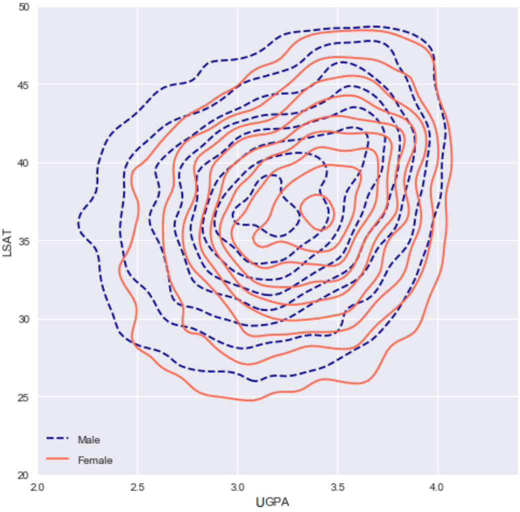
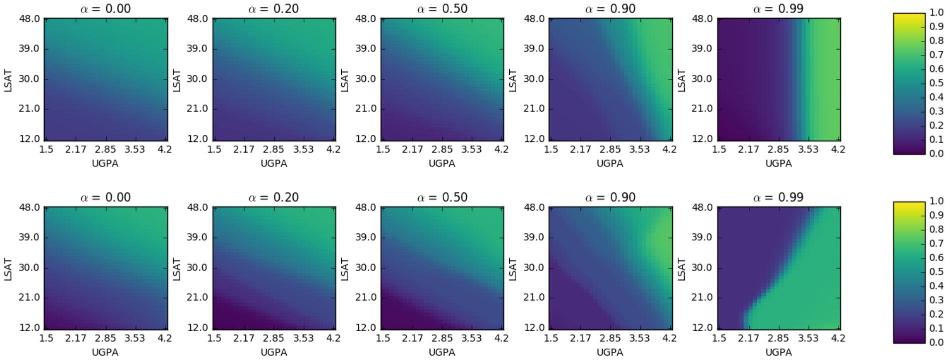
- *German Credit Data* : 1,000 West German credit applications. Predict loan repayment based on employment, financial, and residency information while protecting (binary) age. 58 attributes with mixture of monotonicity.

Law School Gender Density



Law School Acceptance Functions



(α is fraction of loss from fairness)

Evaluation metrics

We follow the evaluation metrics of Zemel 2013 *Learning Fair Representations*:

- Discrimination: disparate impact i.e. absolute difference in expectation:

$$\left| \frac{\sum_{n:s_n=1} \hat{y}_n}{\sum_{n:s_n=1} 1} - \frac{\sum_{n:s_n=0} \hat{y}_n}{\sum_{n:s_n=0} 1} \right|$$

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- Accuracy: inverse of mean absolute error

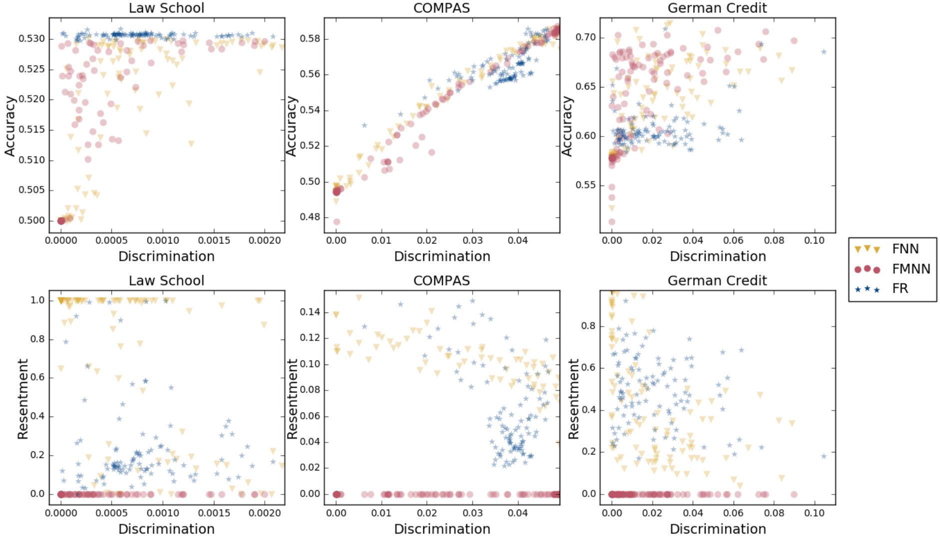
$$1 - \frac{1}{N} \sum_{n=1}^{N} |y_n - \hat{y}_n|$$

Evaluation metrics

We add Resentment: fraction of people that find example in sample they resent

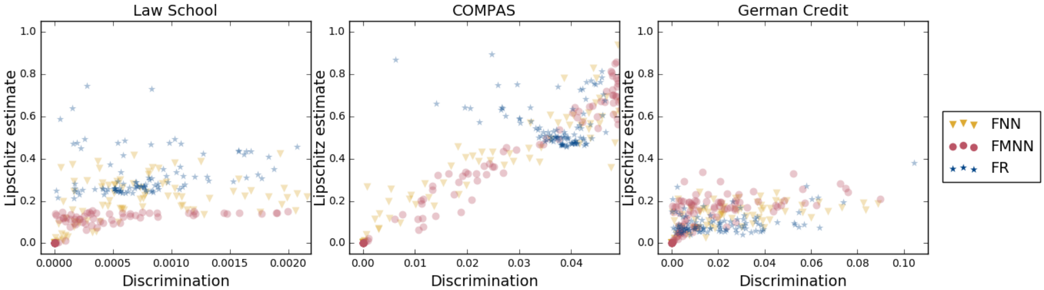
$$\frac{1}{N}\sum_{i=1}^{N}\max_{j\in\mathcal{N}_i}\left(1_{\hat{y}_i<\hat{y}_j}\right)$$

Results



Law School
COMPAS
German Credit
Accuracy
Resentment
Discrimination
0.530
0.525
0.520
0.515
0.510
0.505
0.500
0.0000
0.0005
0.0010
0.0015
0.0020
0.58
0.56
0.54
0.52
0.50
0.48
0.00
0.01
0.02
0.03
0.04
0.70
0.65
0.60
0.55
0.02
0.04
0.06
0.08
0.10
1.0
0.8
0.6
0.4
0.2
0.0
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0.00
FNN
FMNN
FR

Lipschitz Smoothness



Eliciting Montonic Fairness: Introduction

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- *Problem*: Axis-based monotonicity is too strict for fairness, and oracle info about which individuals are ``better'' is hard to systematize (even if we had an oracle).
- *Intuition*: Lay people experience resentment, so we should be able to survey them and use that data to figure out what is a ``better'' set of attributes.
- *Model*: Collect non-expert arbiter ratings on what the relative treatment of pairs of individuals should be. Combine this with historical data as input to a conditional neural network that can post hoc adjust between accuracy and resentment prevention.

Introduction

In practice we often which to capture more complex definitions of ``better" attribute sets that consider multiple attributes at once.

Imagine you're evaluating whether two defendants should get bail:

- Defendant A: 0 prior felonies, 1 prior misdemeanors
- Defendant B: 10 prior felonies, 0 prior misdemeanors

Monotonicity as previously discussed would say A and B are incomparable, but most people would agree that B should be less likely to get bail.

Preference Learning

Ultimately we have a problem of *preference learning*. We have two options, X_i and X_j, and want to learn a preference function between them.

Many approaches in the literature aim to learn personalized preference functions to recommend the best product *for an individual*. We wish to learn a population-wide preference function which will be applied universally.

Model Structure

Re-encode Y_i into pairwise data Z_{ij}:

$$Z_{ij}^{obs} = \begin{cases} 1 & \text{if} \quad Y_i^{obs} = 1 \text{ and } Y_j^{obs} = 0 \\ 2 & \text{if} \quad Y_i^{obs} = 0 \text{ and } Y_j^{obs} = 1 \\ 3 & \text{if} \quad Y_i^{obs} = Y_j^{obs} \end{cases} .$$

Will use the same encodings for survey data for ``more likely'', ``less likely'', and ``similarly likely''

Model Structure

Define a pairwise loss function using Z:

$$\mathcal{L}_Z(Z, \hat{p}, \mathcal{Z}) = -\frac{1}{|\mathcal{Z}|} \sum_{(i,j) \in \mathcal{Z}} \left(\begin{array}{l} \mathbf{1}_{Z_{ij}=1} \ \log\left(\hat{p}_i(1-\hat{p}_j)\right) \ + \\ \mathbf{1}_{Z_{ij}=2} \ \log\left((1-\hat{p}_i)\hat{p}_j\right) \ + \\ \mathbf{1}_{Z_{ij}=3} \ \log\left(\hat{p}_i\hat{p}_j + (1-\hat{p}_i)(1-\hat{p}_j)\right) \end{array} \right)$$

Model Loss

Define our neural network:

$$\text{logit}\left(\hat{p}_i\right) = f_\theta(X_i, c)$$

Define the loss the minimize:

$$\mathcal{L} = \underbrace{\mathcal{L}_Z\left(Z_{ij}^{obs}, \hat{p}_i = f(X_i, c = 0), \mathcal{O}\right)}_{\mathcal{L}_Z^{obs}} + \underbrace{\mathcal{L}_Z\left(Z_{ij}^{arb}, \hat{p}_i = f(X_i, c = 1), \mathcal{A}\right)}_{\mathcal{L}_Z^{arb}} + g(\theta)$$

Optimize $\hat{\theta} = \arg\min_\theta \mathcal{L}$.

Synthetic Experiment Setup

$$X_i \sim N(0,1)^2$$

$$\beta_{obs} = [0.9, 1.1]$$

$$p_i^{obs} = \frac{1}{1 + e^{-(X_i \beta^{obs} - 1)}}$$

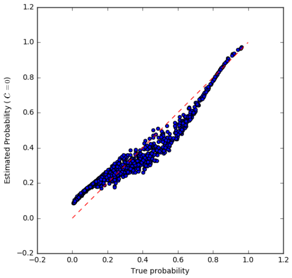
$$Y_i \sim \text{Bernoulli}(p_i^{obs})$$

$$\beta_{arb} = [1.1, 0.9]$$

$$Z_{ij}^{arb} = \begin{cases} 1 & \text{if} \quad X_i \beta^{arb} > X_j \beta^{arb} + 0.25 \\ 2 & \text{if} \quad X_j \beta^{arb} > X_i \beta^{arb} + 0.25 \\ 3 & \text{if} \quad |X_i \beta^{arb} - X_j \beta^{arb}| < 0.25 \end{cases} .$$

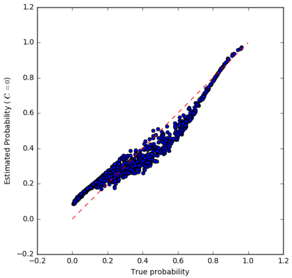
Synthetic Experiment Results

Recover $\hat{p}$

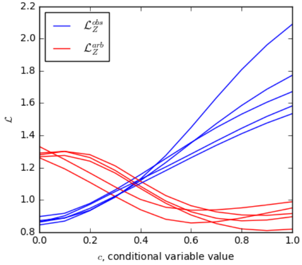


Synthetic Experiment Results

Recover $\hat{p}$

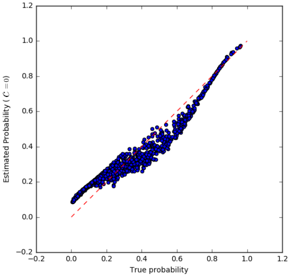


Losses respond to c

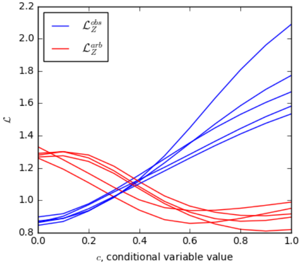


Synthetic Experiment Results

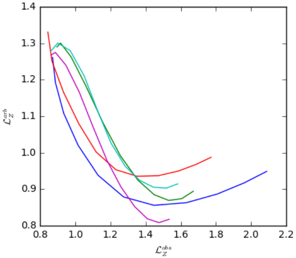
Recover $\hat{p}$



Losses respond to c



Tradeoff



COMPAS Experiment Setup

Five arbiters each shown 100 pairs on individuals' attributes: age, (adult) priors count, juv. felony count, juv. misdemeanor count, juv. other counts, charge degree (fel. or mis.), and violent charge (T/F). Asked to rate as:

- ``A is at least as likely to (re)offend" ($Z = 1$)
- ``B is at least as likely to (re)offend" ($Z = 2$)
- ``A and B are similarly likely to (re)offend" ($Z = 3$)
- ``No preference / any of the others are fair" (Ignored)

COMPAS Arbiter Results

- 298 dissimilar ($Z \in \{1, 2\}$) ratings, 185 similar ($Z = 3$) responses, 18 ratings ignored
- Surprisingly accurate: 78% of dissimilar ratings correct, similar to COMPAS decile score difference of 3 ($\sim 54\%$ of pairs have decile score difference ≥ 3)
- Disparate impact: when comparing African-America to Caucasian, rate former more likely to re-offend 65% of the time

COMPAS Experiment Loss

Add Equality of Odds loss:

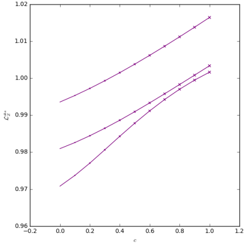
$$\mathcal{L}_F = \sum_y \sum_a \left(\bar{\hat{y}}_{ay} - \bar{\hat{y}}_{\cdot y}\right)^2$$

$$\mathcal{L} = \mathcal{L}_Z^{obs} + \mathcal{L}_Z^{arb} + \lambda_F \mathcal{L}_F + g(\theta)$$

Set $\lambda_F = 0.001$ for experiments.

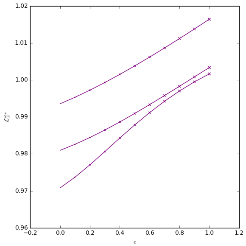
COMPAS Experiment Results

$\mathcal{L}_Z^{obs}$ vs c

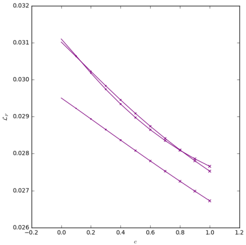


COMPAS Experiment Results

$\mathcal{L}_Z^{obs}$ vs c

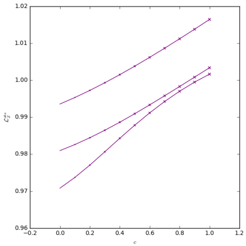


$\mathcal{L}_F$ vs c

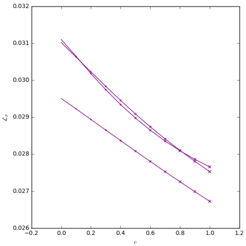


COMPAS Experiment Results

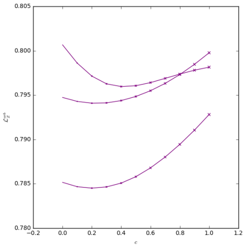
$\mathcal{L}_Z^{obs}$ vs c



$\mathcal{L}_F$ vs c

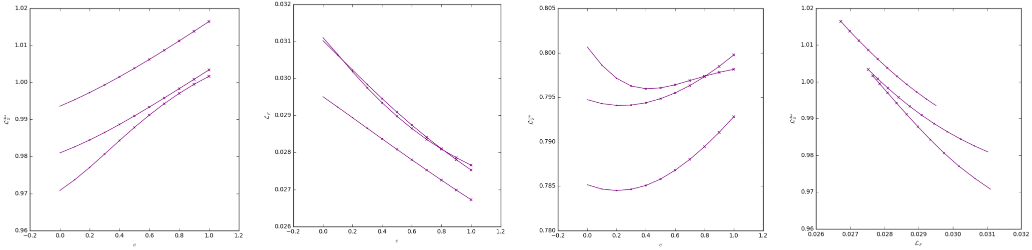


$\mathcal{L}_Z^{arb}$ vs c



COMPAS Experiment Results

$\mathcal{L}_Z^{obs}$ vs c | $\mathcal{L}_F$ vs c | $\mathcal{L}_Z^{arb}$ vs c | $\mathcal{L}_Z^{obs}$ vs $\mathcal{L}_F$



Conclusion

Questions?

Selected Bibliography I

- Blei, D.M., A.Y. Ng, and M.I. Jordan (2003). ``Latent Dirichlet allocation''. In: *The Journal of Machine Learning Research* 3, pp. 993–1022.
- Larson, Jeff et al. (2016). ``How we analyzed the COMPAS recidivism algorithm''. In: *ProPublica (5 2016)* 9.
- Lichman, M (2013). *UCI machine learning repository*. URL: https://archive.ics.uci.edu/ml/datasets/statlog+(german+credit+data).
- McCallum, A., A. Corrada-Emmanuel, and X. Wang (2005). ``The author-recipient-topic model for topic and role discovery in social networks, with application to Enron and academic email''. In: *Workshop on Link Analysis, Counterterrorism and Security*.
- Sill, Joseph (1998). ``Monotonic networks''. In: *Advances in Neural Information Processing Systems*, pp. 661–667.

Selected Bibliography II

- Snijders, T.A.B. and T. Nowicki (1997). ``Estimation and prediction for stochastic blockmodels for graphs with latent block structure''. In: *Journal of Classification* 14.1, pp. 75–100.
- Tu, Y. et al. (2010). ``Citation author topic model in expert search''. In: *ACL International Conference on Computational Linguistics*.
- Wang, Y.J. and G.Y. Wong (1987). ``Stochastic blockmodels for directed graphs''. In: *Journal of the American Statistical Association* 82.397, pp. 8–19.
- Wightman, Linda F and Henry Ramsey (1998). *LSAC national longitudinal bar passage study*. Law School Admission Council.
- Zemel, Rich et al. (2013). ``Learning fair representations''. In: *International Conference on Machine Learning*, pp. 325–333.

Proving Monotonicity

Define a *monotonic non-decreasing (MND) function* $f : \mathbb{R} \to \mathbb{R}$
s.t. $f(x + dx) - f(x) = df \geq 0 \;\forall\; dx \geq 0$.
Assume f, g are MND, h is monotone non-increasing (MNI), then:

Recursion:	$f \circ g$ is MND
Negation:	$f \circ h$ and $h \circ f$ are MNI
Linearity:	if $a > 0$, $af(x) + b$ is MND
Addition:	$f + g$ is MND

Proving Monotonicity

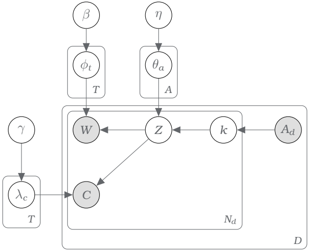
We can then prove the properties of our network:

- By Linearity, if $\tau(w^l_{i,j}) > 0$ then $\tau(w^l_{i,j})h_{j,l-1} + b^l_j$ is MND w.r.t. $h_{j,l-1}$
- By Addition, $\sum_{i \in 1 \ldots |H_{l-1}|} \tau(w^l_{i,j})h_{j,l-1} + b^l_j$ is MND w.r.t. each of $h_{j,l-1}$
- By Recursion, if σ is MND then $h^l_{i,j}\sigma\left(\sum_{i \in 1 \ldots |H_{l-1}|} \tau(w^l_{i,j})h_{j,l-1} + b^l_j\right)$ is MND w.r.t. each of $h_{j,l-1}$

(

noframenumbering)Related Models

Citation-Author Topic Model Tu et al. 2010



Author-Recipient Topic Model McCallum, Corrada-Emmanuel, and X. Wang 2005