Problem Set 9

6.10* Use the trace formalism to calculate the QED spin-averaged matrix element squared for $e^+e^- \to f\bar{f}$ including the electron mass term.

$e^+e^- \to f\bar{f}$ gives the following LO Feynmann diagram as given in Thomson:

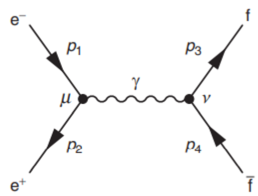

Fig. 6.11 The lowest-order QED Feynman diagram for $e^+e^- \to f\bar{f}$.

With this, we may use QED rules to get the matrix elements, noting the additional charge of the fermion Q_f:

$$-i\mathcal{M}_{fi} = Q_f[\bar{v}(p_2)ie\gamma^\mu u(p_1)]\left[\frac{-ig_{\mu\nu}}{q^2}\right][\bar{u}(p_3)ie\gamma^\nu u(p_4)] \tag{1}$$

$$\mathcal{M}_{fi} = -Q_f[\bar{v}(p_2)e\gamma^\mu u(p_1)]\left[\frac{g_{\mu\nu}}{q^2}\right][\bar{u}(p_3)e\gamma^\nu u(p_4)] \tag{2}$$

$$\mathcal{M}_{fi} = \frac{-Q_f e^2}{q^2}[\bar{v}(p_2)\gamma^\mu u(p_1)]g_{\mu\nu}[\bar{u}(p_3)\gamma^\nu u(p_4)] \tag{3}$$

$$\mathcal{M}_{fi} = \frac{-Q_f e^2}{q^2}[\bar{v}(p_2)\gamma^\mu u(p_1)][\bar{u}(p_3)\gamma_\nu u(p_4)] \tag{4}$$

$$\tag{5}$$

we may rearrange the elements in trace formalism as in Equation (6.51) to get the spin-summed matrix element squared $\sum_{\text{spins}}|\mathcal{M}_{fi}|^2$:

$$\sum_{\text{spins}}|\mathcal{M}_{fi}|^2 = \frac{Q_f^2 e^4}{q^4}\text{Tr}([\not{p}_2 - m_e]\gamma^\mu[\not{p}_1 + m_e]\gamma^\nu) \times \text{Tr}([\not{p}_3 + m_f]\gamma_\mu[\not{p}_4 - m_f]\gamma_\nu) \tag{6}$$

we proceed with evaluating the traces:

$$\text{Tr}([\not{p}_2 - m_e]\gamma^\mu[\not{p}_1 + m_e]\gamma^\nu) = \text{Tr}(\not{p}_2\gamma^\mu\not{p}_1\gamma^\nu + \not{p}_2\gamma^\mu m_e\gamma^\nu - \not{p}_1\gamma^\nu m_e\gamma^\mu - m_e^2\gamma^\mu\gamma^\nu) \tag{7}$$

$$\tag{8}$$

we may rewrite $\not{p}_1 = \gamma^\sigma p_{1\sigma}$ and $\not{p}_2 = \gamma^\rho p_{2\rho}$, thus we have:

$$\begin{aligned}\mathrm{Tr}([\gamma^\rho p_{2\rho} - m_e]\gamma^\mu[\gamma^\sigma p_{1\sigma} + m_e]\gamma^\nu) &= \mathrm{Tr}(\gamma^\rho p_{2\rho}\gamma^\mu\gamma^\sigma p_{1\sigma}\gamma^\nu + \gamma^\rho p_{2\rho}\gamma^\mu m_e\gamma^\nu && (9)\\ &- \gamma^\sigma p_{1\sigma}\gamma^\nu m_e\gamma^\mu - m_e^2\gamma^\mu\gamma^\nu) && (10)\end{aligned}$$

we note the following trace theorems:

(b) the trace of odd-numbered γ-matrices zero out
(c) $\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$
(d) $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4g^{\mu\nu}g^{\rho\sigma} - 4g^{\mu\rho}g^{\nu\sigma} + 4g^{\mu\sigma}g^{\nu\rho}$

with these, our trace becomes:

$$\begin{aligned}\mathrm{Tr}([\not{p}_2 - m_e]\gamma^\mu[\not{p}_1 + m_e]\gamma^\nu) &= \mathrm{Tr}(\gamma^\rho p_{2\rho}\gamma^\mu\gamma^\sigma p_{1\sigma}\gamma^\nu) - m_e^2\,\mathrm{Tr}(\gamma^\mu\gamma^\nu) && (11)\\ &= p_{2\rho}p_{1\sigma}\,\mathrm{Tr}(\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu) - m_e^2\,\mathrm{Tr}(\gamma^\mu\gamma^\nu) && (12)\\ &= 4p_{2\rho}p_{1\sigma}(g^{\rho\mu}g^{\sigma\nu} - g^{\rho\sigma}g^{\mu\nu} + g^{\rho\nu}g^{\mu\sigma}) - 4m_e^2 g^{\mu\nu} && (13)\\ &= 4p_2^\mu p_1^\nu - 4g^{\mu\nu}(p_1\cdot p_2) + 4p_2^\nu p_1^\mu - 4m_e^2 g^{\mu\nu} && (14)\\ & && (15)\end{aligned}$$

we do the same steps for the other trace:

$$\begin{aligned}\mathrm{Tr}([\not{p}_3 + m_f]\gamma_\mu[\not{p}_4 - m_f]\gamma_\nu) &= \mathrm{Tr}(\not{p}_3\gamma_\mu\not{p}_4\gamma_\nu - \not{p}_3\gamma_\mu m_f\gamma_\nu + m_f\gamma_\mu\not{p}_4\gamma_\nu - m_f^2\gamma_\mu\gamma_\nu) && (16)\\ &= \mathrm{Tr}(\gamma^\rho p_{3\rho}\gamma_\mu\gamma^\sigma p_{4\sigma}\gamma_\nu - \gamma^\rho p_{3\rho}\gamma_\mu m_f\gamma_\nu && (17)\\ &\quad + m_f\gamma_\mu\gamma^\sigma p_{4\sigma}\gamma_\nu - m_f^2\gamma_\mu\gamma_\nu) && (18)\\ &= p_{3\rho}p_{4\sigma}\,\mathrm{Tr}(\gamma^\rho\gamma_\mu\gamma^\sigma\gamma_\nu) - m_f^2\,\mathrm{Tr}(\gamma_\mu\gamma_\nu) && (19)\\ &= 4p_{3\mu}p_{4\nu} - 4g_{\mu\nu}(p_3\cdot p_4) + 4p_{3\nu}p_{4\mu} - 4m_f^2 g_{\mu\nu} && (20)\end{aligned}$$

thus the spin-summed matrix element squared becomes:

$$\sum_{\text{spins}} |\mathcal{M}_{fi}|^2 = \frac{Q_f^2 e^4}{q^4}\left(4p_2^\mu p_1^\nu - 4g^{\mu\nu}(p_1 \cdot p_2) + 4p_2^\nu p_1^\mu - 4m_e^2 g^{\mu\nu}\right) \times \tag{21}$$

$$\left(4p_{3\mu}p_{4\nu} - 4g_{\mu\nu}(p_3 \cdot p_4) + 4p_{3\nu}p_{4\mu} - 4m_f^2 g_{\mu\nu}\right) \tag{22}$$

$$= \frac{16Q_f^2 e^4}{q^4}\left(p_2^\mu p_1^\nu - g^{\mu\nu}(p_1 \cdot p_2) + p_2^\nu p_1^\mu - m_e^2 g^{\mu\nu}\right) \times \tag{23}$$

$$\left(p_{3\mu}p_{4\nu} - g_{\mu\nu}(p_3 \cdot p_4) + p_{3\nu}p_{4\mu} - m_f^2 g_{\mu\nu}\right) \tag{24}$$

$$\tag{25}$$

we may contract the indices using the following:

$$g^{\mu\nu}g_{\mu\nu} = 4 \quad p_2^\mu p_1^\nu g_{\mu\nu} = (p_1 \cdot p_2) \quad p_2^\mu p_1^\nu p_{3\mu} p_{4\nu} = (p_2 \cdot p_3)(p_1 \cdot p_4)$$

thus we have:

$$\sum_{\text{spins}} |\mathcal{M}_{fi}|^2 = \frac{16Q_f^2 e^4}{q^4}\Bigg((p_1 \cdot p_4)(p_2 \cdot p_3) - (p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_3)(p_2 \cdot p_4) - m_f^2 (p_1 \cdot p_2) \tag{26}$$

$$- (p_1 \cdot p_2)(p_3 \cdot p_4) + 4(p_1 \cdot p_2)(p_3 \cdot p_4) - (p_1 \cdot p_2)(p_3 \cdot p_4) + 4m_f^2 (p_1 \cdot p_2) \tag{27}$$

$$+ (p_1 \cdot p_3)(p_2 \cdot p_4) - (p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) - m_f^2 (p_1 \cdot p_2) \tag{28}$$

$$- m_e^2 (p_3 \cdot p_4) + 4m_e^2 (p_3 \cdot p_4) - m_e^2 (p_3 \cdot p_4) + 4m_e^2 m_f^2\Bigg) \tag{29}$$

$$\sum_{\text{spins}} |\mathcal{M}_{fi}|^2 = \frac{16Q_f^2 e^4}{q^4}\left(2(p_1 \cdot p_4)(p_2 \cdot p_3) + 2(p_1 \cdot p_3)(p_2 \cdot p_4) + 2m_f^2 (p_1 \cdot p_2) + 2m_e^2 (p_3 \cdot p_4) + 4m_e^2 m_f^2\right) \tag{30}$$

$$= \frac{32Q_f^2 e^4}{q^4}\left((p_1 \cdot p_4)(p_2 \cdot p_3) + (p_1 \cdot p_3)(p_2 \cdot p_4) + m_f^2 (p_1 \cdot p_2) + m_e^2 (p_3 \cdot p_4) + 2m_e^2 m_f^2\right) \tag{31}$$

$$\tag{32}$$

to get the spin-averaged matrix element squared, we evaluate:

$$\left\langle |\mathcal{M}_{fi}|^2 \right\rangle = \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_{fi}|^2 \tag{33}$$

$$\tag{34}$$

from the spin-summed matrix element squared, then we have:

$$\left\langle |\mathcal{M}_{fi}|^2 \right\rangle = \frac{8Q_f^2 e^4}{q^4} \left[(p_1 \cdot p_4)(p_2 \cdot p_3) + (p_1 \cdot p_3)(p_2 \cdot p_4) + m_f^2 (p_1 \cdot p_2) + m_e^2 (p_3 \cdot p_4) + 2m_e^2 m_f^2 \right] \tag{35}$$

we may rewrite the four-momentum squared of the virtual phton as $q^2 = (p_1 + p_2)^2$, thus getting:

$$\boxed{\left\langle |\mathcal{M}_{fi}|^2 \right\rangle = \frac{8Q_f^2 e^4}{(p_1 + p_2)^4} \left[(p_1 \cdot p_4)(p_2 \cdot p_3) + (p_1 \cdot p_3)(p_2 \cdot p_4) + m_f^2 (p_1 \cdot p_2) + m_e^2 (p_3 \cdot p_4) + 2m_e^2 m_f^2 \right]} \tag{36}$$