

Problem Set 7

- 5.1** Draw the two time-ordered diagrams for the *s*-channel process shown in Figure 5.5. By repeating the steps of Section 5.1.1, show that the propagator has the same form as obtained for the *t*-channel process.

Hint: one of the time-ordered diagrams is non-intuitive, remember that in second-order perturbation theory the intermediate state does not conserve energy.

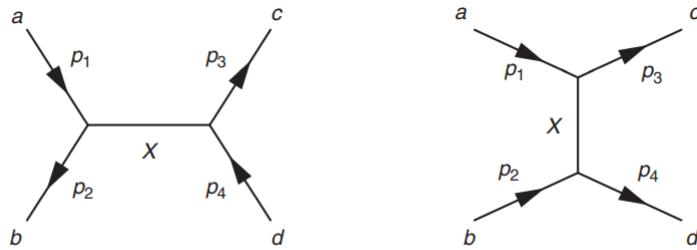
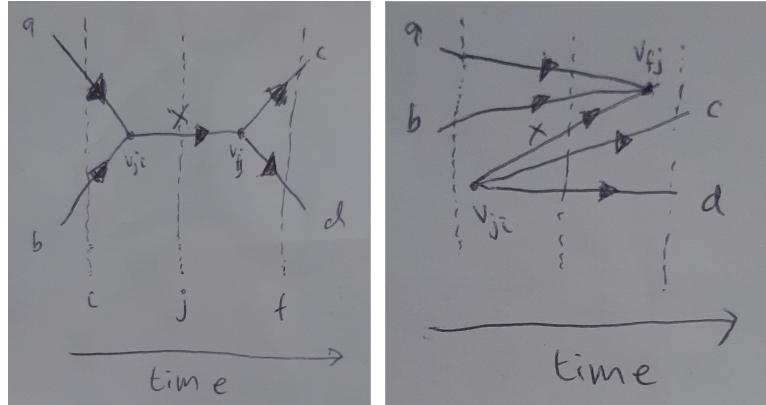


Fig.5.5 Feynman diagrams for illustrative *s*-channel annihilation and *t*-channel scattering processes.

For the given *s*-channel, the processes can have different sequences on which they happened, as all we know is that there are incoming *a* and *b* and outgoing *c* and *d*. For one, an incoming *a* splits into an outgoing *c* and *X*, and this *X* then interacts with an incoming *b* to produce *d*. In another, *c*, *d*, and *X* are produced first, then incoming *a* and *b* interact with *X* to disappear. These time orderings may be represented as:



For the first time ordering, we may obtain the transition matrix element T_{fi}^{ab} as given by Equation 5.1 and by noting the different regions (*c* and *d* in the *f* region, *X* in the *j* region, and *a* and *b* in the *i* region):

$$T_{fi}^{ab} = \frac{\langle f | V | j \rangle \langle j | V | i \rangle}{E_i - E_j} \quad (1)$$

$$T_{fi}^{ab} = \frac{\langle c + d | V | X \rangle \langle X | V | a + b \rangle}{(E_a + E_b) - E_X} \quad (2)$$

$$(3)$$

The non-invariant matrix elements $\langle f | V | j \rangle$ and $\langle j | V | i \rangle$ are given by:

$$\langle f | V | j \rangle = \langle c + d | V | X \rangle = \mathcal{M}_{fj} \prod_k (2E_k)^{-1/2} \quad (4)$$

$$\langle j | V | i \rangle = \langle X | V | a + b \rangle = \mathcal{M}_{ji} \prod_k (2E_k)^{-1/2} \quad (5)$$

$$(6)$$

we can also rewrite the matrix as the coupling constants:

$$\mathcal{M}_{fj} \mathcal{M}_{ji} = g_a g_b = g^2 \quad (7)$$

thus we have:

$$T_{fi}^{ab} = \frac{1}{2E_X \cdot 2(2E_a E_b E_c E_d)^{1/2}} \frac{g^2}{(E_a + E_b) - E_X} \quad (8)$$

$$(9)$$

we can get the overall Lorentz-invariant matrix in the form of:

$$\mathcal{M}_{fi}^{ab} = 2(2E_a E_b E_c E_d)^{1/2} T_{fi}^{ab} \quad (10)$$

$$\mathcal{M}_{fi}^{ab} = \frac{1}{2E_X} \frac{g^2}{(E_a + E_b) - E_X} \quad (11)$$

we repeat the same steps for the second time ordering:

$$T_{fi}^{cd} = \frac{\langle f | V | j \rangle \langle j | V | i \rangle}{E_i - E_j} \quad (12)$$

$$T_{fi}^{cd} = \frac{\langle c + d + X | V | 0 \rangle \langle 0 | V | a + b + X \rangle}{(E_a + E_b) - (E_a + E_b + E_c + E_d + E_X)} \quad (13)$$

$$(14)$$

with $E_c + E_d = E_a + E_b$, we have:

$$T_{fi}^{cd} = \frac{1}{2E_X \cdot 2(2E_a E_b E_c E_d)^{1/2}} \frac{g^2}{-E_a - E_b - E_X} \quad (15)$$

$$(16)$$

we proceed with obtaining the matrix element:

$$\mathcal{M}_{fi}^{cd} = 2(2E_a E_b E_c E_d)^{1/2} T_{fi}^{cd} \quad (17)$$

$$\mathcal{M}_{fi}^{cd} = -\frac{1}{2E_X} \frac{g^2}{(E_a + E_b) + E_X} \quad (18)$$

to get the combined Lorentz-invariant matrix element of both time orderings, we get their sum:

$$\mathcal{M} = \mathcal{M}_{fi}^{ab} + \mathcal{M}_{fi}^{cd} \quad (19)$$

$$\mathcal{M} = \left[\frac{1}{2E_X} \frac{g^2}{(E_a + E_b) - E_X} \right] + \left[-\frac{1}{2E_X} \frac{g^2}{(E_a + E_b) + E_X} \right] \quad (20)$$

$$\mathcal{M} = \frac{g^2}{2E_X} \left[\frac{1}{(E_a + E_b) - E_X} - \frac{1}{(E_a + E_b) + E_X} \right] \quad (21)$$

$$\mathcal{M} = \frac{g^2}{2E_X} \left[\frac{2E_X}{(E_a + E_b)^2 - E_X^2} \right] \quad (22)$$

(23)

we can rewrite E_X^2 as:

$$E_X^2 = \mathbf{p}_X^2 + m_X^2 \quad (24)$$

$$E_X^2 = (\mathbf{p}_a + \mathbf{p}_b)^2 + m_X^2 \quad (25)$$

thus the matrix element becomes:

$$\mathcal{M} = \frac{g^2}{(E_a + E_b)^2 - (\mathbf{p}_a + \mathbf{p}_b)^2 - m_X^2} \quad (26)$$

$$\mathcal{M} = \frac{g^2}{(p_a + p_b)^2 - m_X^2} \quad (27)$$

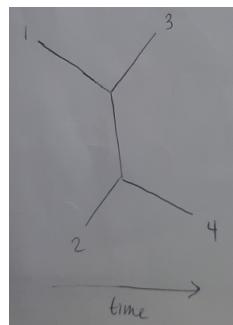
rewriting this in terms of the Mandelstam s variable $q = (p_a + p_b)^2$, we get:

$$\mathcal{M} = \frac{g^2}{q^2 - m_X^2} \quad (28)$$

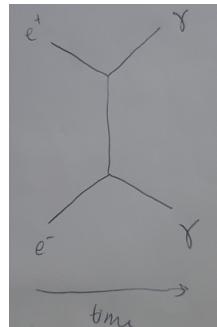
This form recovers the $\frac{1}{q^2 - m_X^2}$ propagator.

- 5.3** Draw the lowest-order *t*-channel and *u*-channel Feynman diagrams for $e^+e^- \rightarrow \gamma\gamma$ and use the Feynman rules for QED to write down the corresponding matrix elements.

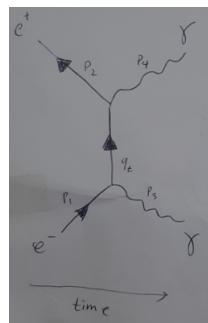
In the case of $e^+e^- \rightarrow \gamma\gamma$, two identical particles are produced. Hence, we use either *t*-channel or *u*-channel diagrams. The *t*-channel diagram generally looks like:



we can place the particles given into the above general configuration:



reconfiguring this so that particle arrows go with the time arrow, antiparticle arrows go against the time arrow, each vertex has one γ photon as allowed in vertex rules, and having the fermion propagator as the virtual particle, we have:



we note the following contributions to the matrix element $-i\mathcal{M}_t$:

$$\begin{aligned}
\text{initial state particle } (e^-) &: u(p) \\
\text{initial state antiparticle } (e^+) &: \bar{v}(p) \\
\text{final state photons } (\gamma) &: \varepsilon_\mu^*(p) \\
\text{fermion propagator } (q) &: -\frac{i(\gamma^\mu q_\mu + m)}{q^2 - m^2} \\
\text{interaction vertex factor} &: ie\gamma^\mu
\end{aligned}$$

the product of the contributions from the initial state electron at the 1 position and the final state photon at position 3 along with the corresponding interaction vertex factor is given by:

$$\varepsilon_\mu^*(p)ie\gamma^\mu u(p) \quad (29)$$

$$\varepsilon_\mu^*(p_3)ie\gamma^\mu u(p_1) \quad (30)$$

$$(31)$$

the product of the contributions from the initial state positron at the 2 position and the final state photon at position 4 along with the corresponding interaction vertex factor is given by:

$$\varepsilon_\nu^*(p)ie\gamma^\nu \bar{v}(p) \quad (32)$$

$$\varepsilon_\nu^*(p_4)ie\gamma^\nu \bar{v}(p_2) \quad (33)$$

$$(34)$$

lastly, we have the fermion propagator in the middle:

$$-\frac{i(\gamma^\mu q_\mu + m)}{q^2 - m^2} \quad (35)$$

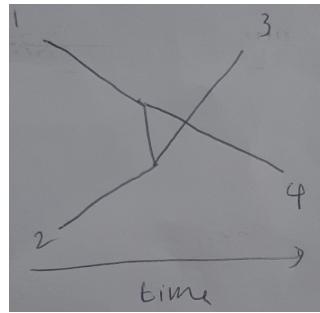
$$(36)$$

thus for the matrix element of the t-channel diagram, rearranging the terms such that the initial state electron and positron terms are nearer to the fermion propagator, we have:

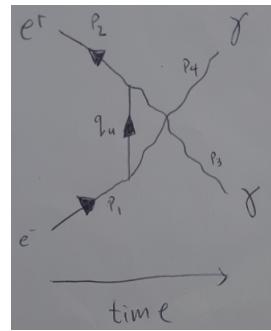
$$-i\mathcal{M}_t = [\varepsilon_\mu^*(p_3)ie\gamma^\mu u(p_1)] \left[-\frac{i(\gamma^\mu q_\mu + m)}{q^2 - m^2} \right] [\bar{v}(p_2)ie\gamma^\nu \varepsilon_\nu^*(p_4)] \quad (37)$$

$$\mathcal{M}_t = [\varepsilon_\mu^*(p_3)ie\gamma^\mu u(p_1)] \left[-\frac{(\gamma^\mu q_\mu + m)}{q^2 - m^2} \right] [\bar{v}(p_2)ie\gamma^\nu \varepsilon_\nu^*(p_4)]$$

for the u-channel diagram, we have the general configuration:



particles inserted along with the proper corrections, we have:



we notice from the general diagram that particle 4 and 3 just switched, and thus for its matrix elements \mathcal{M}_u we can interchange the terms referring to those particles from the earlier \mathcal{M}_t :

$$\mathcal{M}_u = [\varepsilon_\mu^*(p_4) e \gamma^\mu u(p_1)] \left[-\frac{(\gamma^\mu q_\mu + m)}{q^2 - m^2} \right] [\bar{v}(p_2) e \gamma^\nu \varepsilon_\nu^*(p_3)] \quad (39)$$