Calculations in perturbation theory

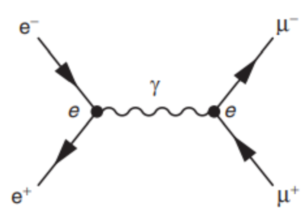
Spin sums

For a given process, the total amplitude $\mathcal{M}_{fi}$ is the sum of all individual amplitudes (possible Feynman diagrams), but terms aside fromt the matrix element of the lowest-order term are negligible. Including the spin, then the **spin-averaged matrix element** is given by:

$$\begin{aligned}
\langle|\mathcal{M}_{fi}|^2\rangle &= \frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 && (1)\\
&= \frac{1}{4}(|\mathcal{M}_{\text{RR}}|^2+|\mathcal{M}_{\text{RL}}|^2+|\mathcal{M}_{\text{LR}}|^2+|\mathcal{M}_{\text{LL}}|^2) && (2)\\
&= \frac{1}{4}(|\mathcal{M}_{\text{RR}\to\text{RR}}|^2+|\mathcal{M}_{\text{RR}\to\text{RL}}|^2+\cdots+|\mathcal{M}_{\text{RL}\to\text{RR}}|^2+\dots) && (3)\\
& && (4)
\end{aligned}$$

Helicity amplitudes

We consider an electron-positron annihilation:



For a helicity combination, the matrix element is given by:

$$\mathcal{M} = -\frac{e^2}{s}j_e\cdot j_\mu \tag{5}$$

with the currents given by:

$$\begin{aligned}
j_\mu^0 &= \bar{u}_\uparrow(p_3)\gamma^0 v_\downarrow(p_4)\\
j_\mu^1 &= \bar{u}_\uparrow(p_3)\gamma^1 v_\downarrow(p_4)\\
j_\mu^2 &= \bar{u}_\uparrow(p_3)\gamma^2 v_\downarrow(p_4)\\
j_\mu^3 &= \bar{u}_\uparrow(p_3)\gamma^3 v_\downarrow(p_4)
\end{aligned}$$

and each component may be evaluated using:

$$\bar{\psi}\gamma^0\phi \equiv \psi^\dagger\gamma^0\gamma^0\phi = \psi_1^*\phi_1 + \psi_2^*\phi_2 + \psi_3^*\phi_3 + \psi_4^*\phi_4$$
$$\bar{\psi}\gamma^1\phi \equiv \psi^\dagger\gamma^0\gamma^1\phi = \psi_1^*\phi_4 + \psi_2^*\phi_3 + \psi_3^*\phi_2 + \psi_4^*\phi_1$$
$$\bar{\psi}\gamma^2\phi \equiv \psi^\dagger\gamma^0\gamma^2\phi = -i(\psi_1^*\phi_4 - \psi_2^*\phi_3 + \psi_3^*\phi_2 - \psi_4^*\phi_1)$$
$$\bar{\psi}\gamma^3\phi \equiv \psi^\dagger\gamma^0\gamma^3\phi = \psi_1^*\phi_3 + \psi_2^*\phi_4 + \psi_3^*\phi_1 + \psi_4^*\phi_2$$

Trace techniques

In terms of trace, the sum of matrix element square for all matrices is given by:

$$\sum_{\text{spins}} |\mathcal{M}_{fi}|^2 = \frac{e^4}{q^4}\mathcal{L}_{(e)}^{\mu\nu}\mathcal{L}^{(\mu)}_{\mu\nu} \tag{6}$$

$$= \frac{e^4}{q^4}\,\mathrm{Tr}([\not{p}_2 - m]\gamma^\mu[\not{p}_1 + m]\gamma^\nu) \times \mathrm{Tr}([\not{p}_3 + M]\gamma_\mu[\not{p}_4 - M]\gamma_\nu) \tag{7}$$

to evaluate the traces, the following theorems are applied:

(a) $\mathrm{Tr}\,(I) = 4$;

(b) the trace of any odd number of γ-matrices is zero;

(c) $\mathrm{Tr}\,(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$;

(d) $\mathrm{Tr}\,(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4g^{\mu\nu}g^{\rho\sigma} - 4g^{\mu\rho}g^{\nu\sigma} + 4g^{\mu\sigma}g^{\nu\rho}$;

(e) the trace of γ^5 multiplied by an odd number of γ-matrices is zero;

(f) $\mathrm{Tr}\left(\gamma^5\right) = 0$;

(g) $\mathrm{Tr}\left(\gamma^5\gamma^\mu\gamma^\nu\right) = 0$; and

(h) $\mathrm{Tr}\left(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) = 4i\varepsilon^{\mu\nu\rho\sigma}$, where $\varepsilon^{\mu\nu\rho\sigma}$ is antisymmetric under the interchange of any two indices.