Problem Set 10

7.4 For a spherically symmetric charge distribution $\rho(r)$, where

$$\int \rho(r)\, d^3\mathbf{r} = 1,$$

show that the form factor can be expressed as

$$F(\mathbf{q}^2) = \frac{4\pi}{q}\int_0^\infty r\sin(qr)\rho(r)\,dr,$$
$$\simeq 1 - \frac{1}{6}q^2\langle R^2\rangle + \cdots,$$

where $\langle R^2\rangle$ is the mean square charge radius. Hence show that

$$\langle R^2\rangle = -6\left[\frac{dF(\mathbf{q}^2)}{dq^2}\right]_{q^2=0}.$$

We begin with the definition of the form factor as:

$$F(\mathbf{q^2}) = \int \rho(\mathbf{r})e^{i\mathbf{q}\cdot\mathbf{r}}\; d^3\mathbf{r} \tag{1}$$

evalutating in spherical coordinates, noting that the dot product $\mathbf{q}\cdot\mathbf{r} = qr\cos\theta$, this becomes:

$$\begin{aligned}
F(\mathbf{q^2}) &= \int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi}\int_{r=0}^{r=\infty} \rho(\mathbf{r})e^{i\mathbf{q}\cdot\mathbf{r}}\; r^2\sin\theta\; dr\; d\phi\; d\theta && (2)\\
&= \int_{\phi=0}^{\phi=2\pi} d\phi \int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=\infty} \rho(\mathbf{r})e^{iqr\cos\theta}\; r^2\; dr\;\; \sin\theta\; d\theta && (3)\\
&= 2\pi\int_{r=0}^{r=\infty}\left[\int_{\theta=0}^{\theta=\pi} e^{iqr\cos\theta}\sin\theta\; d\theta\right] r^2\rho(\mathbf{r})\; dr && (4)
\end{aligned}$$

rewriting $\sin\theta\; d\theta = d(\cos\theta)$, we get:

$$\begin{aligned}
F(\mathbf{q^2}) &= 2\pi\int_{r=0}^{r=\infty}\left[\int_{\theta=0}^{\theta=\pi} e^{iqr\cos\theta} d(\cos\theta)\right] r^2\rho(\mathbf{r})\; dr && (5)\\
&= 2\pi\int_{r=0}^{r=\infty}\left[\int_{\cos\theta=-1}^{\cos\theta=1} e^{iqr\cos\theta} d(\cos\theta)\right] r^2\rho(\mathbf{r})\; dr && (6)\\
&= 2\pi\int_{r=0}^{r=\infty}\left[\frac{-i\left(e^{iqr}-e^{-iqr}\right)}{qr}\right] r^2\rho(\mathbf{r})\; dr && (7)\\
&= 2\pi\int_{r=0}^{r=\infty}\left[\frac{\left(e^{iqr}-e^{-iqr}\right)}{iq}\right] r\rho(r)\; dr && (8)\\
& && (9)
\end{aligned}$$

noting that $\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$, we may rewrite this as:

$$F(\mathbf{q^2}) = 2\pi \int_{r=0}^{r=\infty} \left[\frac{\left(e^{iqr} - e^{-iqr}\right)}{iq} \right] \left(\frac{2}{2}\right) r\rho(r)\ dr \tag{10}$$

$$\tag{11}$$

$$\boxed{F(\mathbf{q^2}) = \frac{4\pi}{q} \int_{r=0}^{r=\infty} r\sin(qr)\rho(r)\ dr} \tag{12}$$

expanding the $\sin(qr)$ term as its Taylor expansion $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$ we have:

$$\begin{aligned}
F(\mathbf{q^2}) &= \frac{4\pi}{q} \int_{r=0}^{r=\infty} \sin(qr) r\rho(r)\ dr && (13) \\
&= \frac{4\pi}{q} \int_{r=0}^{r=\infty} r\rho(r) \sin(qr)\ dr && (14) \\
&= \frac{4\pi}{q} \int_{r=0}^{r=\infty} r\rho(r) \left[qr - \frac{(qr)^3}{3!} + \ldots \right]\ dr && (15) \\
&= \frac{4\pi}{q} \int_{r=0}^{r=\infty} r\rho(r) \left[qr - \frac{(qr)^3}{6} + \ldots \right]\ dr && (16)
\end{aligned}$$

we note the given condition for a given spherically symmetric charge distribution $\rho(r)$ such that:

$$\int \rho(r) d^3\mathbf{r} = 1 \tag{17}$$

performing the integral in spherical coordinates gives:

$$\int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=\infty} \rho(r) r^2 \sin\theta\ dr\ d\phi\ d\theta = 1 \tag{18}$$

$$4\pi \int_{r=0}^{r=\infty} r^2\ \rho(r) dr = 1 \tag{19}$$

thus the form factor becomes:

$$F(\mathbf{q^2}) = 4\pi \int_{r=0}^{r=\infty} r^2 \rho(r)\; dr - \frac{1}{6} q^2 \left(4\pi \int_{r=0}^{r=\infty} r^4 \; dr \right) + \dots \tag{20}$$

$$= 1 - \frac{1}{6} q^2 \left(4\pi \int_{r=0}^{r=\infty} r^4 \; dr \right) + \dots \tag{21}$$

$$\tag{22}$$

with the mean square charge radius $\langle R^2 \rangle = 4\pi \int_{r=0}^{r=\infty} r^4 \; dr$, then this becomes:

$$\boxed{F(\mathbf{q^2}) \approx 1 - \frac{1}{6} q^2 \langle R^2 \rangle} \tag{23}$$

differentiating this with respect to q^2, we have:

$$\frac{dF(\mathbf{q}^2)}{dq^2} = \frac{d}{dq^2} \left[1 - \frac{1}{6} q^2 \langle R^2 \rangle \right] \tag{24}$$

$$\frac{dF(\mathbf{q}^2)}{dq^2} = -\frac{1}{6} \langle R^2 \rangle \tag{25}$$

$$\langle R^2 \rangle = -6 \left[\frac{dF(\mathbf{q}^2)}{dq^2} \right] \tag{26}$$

since for higher values of $\mathbf{q}^2$, the elastic scattering cross section tends to zero ($F(\mathbf{q}^2 \to \infty) = 0$), then:

$$\boxed{\langle R^2 \rangle = -6 \left[\frac{dF(\mathbf{q}^2)}{dq^2} \right]_{q^2=0}} \tag{27}$$

7.8 The experimental data of Figure 7.8 can be described by the form factor

$$G(Q^2) = \frac{G(0)}{(1 + Q^2/Q_0^2)^2},$$

with $Q_0 = 0.71$ GeV. Taking $Q^2 \approx \mathbf{q}^2$, show that this implies that proton has an exponential charge distribution of the form

$$\rho(\mathbf{r}) = \rho_0 e^{-r/a},$$

and find the value of a.

The form factor in spherical coordinates is given by:

$$G(q^2) = \int \rho(\mathbf{r}) e^{i\mathbf{q}\cdot\mathbf{r}}\ d^3\mathbf{r} \tag{28}$$

$$= \int_{\theta=0}^{\theta=\pi} \int_{\phi=0}^{\phi=2\pi} \int_{r=0}^{r=\infty} \rho(\mathbf{r}) e^{i\mathbf{q}\cdot\mathbf{r}}\ r^2 \sin\theta\ dr\ d\phi\ d\theta \tag{29}$$

$$\tag{30}$$

this is similar to the earlier problem, where we get:

$$G(q^2) = \frac{4\pi}{q} \int_{r=0}^{r=\infty} r \sin(qr) \rho(r)\ dr \tag{31}$$

with the given charge distribution $\rho(\mathbf{r}) = \rho_0 e^{-r/a}$, this becomes:

$$G(q^2) = \frac{4\pi\rho_0}{q} \int_{r=0}^{r=\infty} r \sin(qr) e^{-r/a}\ dr \tag{32}$$

rewriting the sine term in its exponential form, we have:

$$G(q^2) = \frac{4\pi\rho_0}{q} \int_{r=0}^{r=\infty} r \left[\frac{e^{iqr} - e^{-iqr}}{2i}\right] e^{-r/a}\ dr \tag{33}$$

$$= \frac{2\pi\rho_0}{iq} \int_{r=0}^{r=\infty} r e^{-r/a+iqr}\ dr - \frac{2\pi\rho_0}{iq} \int_{r=0}^{r=\infty} r e^{-r/a-iqr}\ dr \tag{34}$$

$$\tag{35}$$

with $u = r$ and v as the exponential term, integration by parts $\int udv = uv - \int vdu$ gives:

$$G(q^2) = \frac{2\pi\rho_0}{iq}\left(\frac{1}{[(1/a)-iq]^2} - \frac{1}{[(1/a)+iq]^2}\right) \tag{36}$$

$$G(q^2) = \frac{2\pi\rho_0}{iq}\left(i\frac{4(q/a)}{(1/a)^4+q^4+2(1/a)^2q^2}\right) \tag{37}$$

$$G(q^2) = 8\pi\rho_0 a^3\left(\frac{1}{(1+a^2q^2)^2}\right) \tag{38}$$

we note that at $q^2 = 0$, we have:

$$G(0) = 8\pi\rho_0 a^3 \tag{39}$$

we may rewrite the form factor as:

$$G(q^2) = \frac{G(0)}{(1+a^2q^2)^2} \tag{40}$$

comparing this to the given form factor, we see that:

$$G(q^2) = \frac{G(0)}{(1+a^2q^2)^2} \qquad G(Q^2) = \frac{G(0)}{(1+Q^2/Q_0^2)^2} \tag{41}$$

$$\implies q^2 = Q^2, \quad a^2 = 1/Q_0^2 \tag{42}$$

from this we may find the value for a with the given $Q_0 = 0.71$ GeV:

$$a = \sqrt{\frac{1}{(0.71 \text{ GeV})^2}} \tag{43}$$

$$\boxed{a = 1.41 \text{ GeV}} \tag{44}$$