# Problem Set 5

**4.7** By operating on the Dirac equation,

$$(i\gamma^\mu \partial_\mu - m)\psi = 0,$$

with $\gamma^\nu \partial_\nu$, prove that the components of $\psi$ satisfy the Klein–Gordon equation,

$$(\partial^\mu \partial_\mu + m^2)\psi = 0.$$

We start with the given equation:

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

operating on it with $\gamma^\nu \partial_\nu$, we get:

$$\begin{aligned} \gamma^\nu \partial_\nu \left[(i\gamma^\mu \partial_\mu - m)\psi = 0\right] \\ (i\gamma^\nu \gamma^\mu \partial_\nu \partial_\mu - m\gamma^\nu \partial_\nu)\psi = 0 \end{aligned}$$

we can operate on this with $-i$ to get:

$$\begin{aligned} -i\left[(i\gamma^\nu \gamma^\mu \partial_\nu \partial_\mu - m\gamma^\nu \partial_\nu)\psi = 0\right] \\ (\gamma^\nu \gamma^\mu \partial_\nu \partial_\mu + im\gamma^\nu \partial_\nu)\psi = 0 \end{aligned}$$

we note that the wavefunction satisfies the Dirac equation (Eqn. 4.39) $(i\gamma^\mu \partial_\mu - m)\psi = 0$:

$$i\gamma^\nu \partial_\nu \psi = m\psi$$

we can use this with our original equation:

$$\begin{aligned} (\gamma^\nu \gamma^\mu \partial_\nu \partial_\mu + m[i\gamma^\nu \partial_\nu])\psi = 0 \\ (\gamma^\nu \gamma^\mu \partial_\nu \partial_\mu + m^2)\psi = 0 \end{aligned}$$

we note that this holds true even if we switch indices:

$$(\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu + m^2)\psi = 0$$

adding these two, we get:

$$\begin{aligned}(\gamma^\nu\gamma^\mu\partial_\nu\partial_\mu + m^2)\psi + (\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + m^2)\psi &= 0\\ ([\gamma^\nu\gamma^\mu\partial_\nu\partial_\mu + \gamma^\mu\gamma^\nu\partial_\mu\partial_\nu] + 2m^2)\psi &= 0\\ \left(\frac{1}{2}\left[\gamma^\nu\gamma^\mu\partial_\nu\partial_\mu + \gamma^\mu\gamma^\nu\partial_\mu\partial_\nu\right] + m^2\right)\psi &= 0\end{aligned}$$

we note that we can switch up the order of differentiation in the second term in brackets:

$$\begin{aligned}\frac{1}{2}\left[\gamma^\nu\gamma^\mu\partial_\nu\partial_\mu + \gamma^\mu\gamma^\nu\partial_\mu\partial_\nu\right] &= \frac{1}{2}\left[\gamma^\nu\gamma^\mu\partial_\nu\partial_\mu + \gamma^\mu\gamma^\nu\partial_\nu\partial_\mu\right]\\ \frac{1}{2}\left[\gamma^\nu\gamma^\mu\partial_\nu\partial_\mu + \gamma^\mu\gamma^\nu\partial_\mu\partial_\nu\right] &= \frac{1}{2}\left[\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu\right]\partial_\nu\partial_\mu\end{aligned}$$

our original equation becomes:

$$\left(\frac{1}{2}\left[\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu\right]\partial_\nu\partial_\mu + m^2\right)\psi = 0$$

we note of the anticommutation relation (Eqn. 4.33) $\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu = 2g^{\mu\nu}$:

$$\frac{1}{2}\left[\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu\right]\partial_\nu\partial_\mu = g^{\mu\nu}\partial_\nu\partial_\mu$$

then we get:

$$\left(g^{\mu\nu}\partial_\nu\partial_\mu + m^2\right)\psi = 0$$

we note that we can rewrite $g^{\mu\nu}\partial_\nu\partial_\mu$ as:

$$g^{\mu\nu}\partial_\nu\partial_\mu = \partial^\mu\partial_\mu$$

thus our original equation becomes the Klein-Gordon equation:

$$\boxed{\left(\partial^\mu\partial_\mu + m^2\right)\psi = 0} \tag{1}$$

**4.8** Show that

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0.$$

To show that $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$, we show that this holds for $\mu = 0$ and $\mu = k \neq 0$.

$\underline{\mu = 0}$

For $\mu = 0$, the LHS of our equation becomes:

$$\text{LHS: } (\gamma^0)^\dagger$$

As $\gamma^0$ is Hermitian (Eqn. 4.34), then this becomes:

$$\text{LHS: } (\gamma^0)^\dagger = \gamma^0$$

The original equation becomes:

$$\gamma^0 = \gamma^0\gamma^0\gamma^0$$

with the $\gamma^0$ defined as $\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$, we evaluate the RHS of the equation:

$$\text{RHS: } \gamma^0\gamma^0\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\text{RHS: } \gamma^0\gamma^0\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\text{RHS: } \gamma^0\gamma^0\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \gamma^0$$

the RHS = LHS, and thus we now have shown that $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$ for $\mu = 0$. We proceed with showing this is also true for $\mu = k \neq 0$.

$\underline{\mu = k \neq 0}$

For $\mu = k \neq 0$, the LHS of our equation becomes:

$$\text{LHS: } (\gamma^k)^\dagger$$

As $\gamma^k$ is anti-Hermitian (Eqn. 4.34), then this becomes:

$$\text{LHS: } (\gamma^k)^\dagger = -\gamma^k$$

The original equation becomes:

$$-\gamma^k = \gamma^0\gamma^k\gamma^0$$

we note that we have the given Dirac-Pauli representation of the $\gamma$-matrices for $k = 1, 2, 3$:

$$\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$
$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$
$$\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

noting the zeros common with them then in general, we can then express the matrices as:

$$\gamma^k = \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & c & d \\ f & g & 0 & 0 \\ h & j & 0 & 0 \end{pmatrix}$$

using this, then when we evaluate the RHS of the equation, we get:

$$\text{RHS:}\ \ \gamma^0\gamma^k\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & c & d \\ f & g & 0 & 0 \\ h & j & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\text{RHS:}\ \ \gamma^0\gamma^k\gamma^0 = \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & c & d \\ -f & -g & 0 & 0 \\ -h & -j & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\text{RHS:}\ \ \gamma^0\gamma^k\gamma^0 = \begin{pmatrix} 0 & 0 & -a & -b \\ 0 & 0 & -c & -d \\ -f & -g & 0 & 0 \\ -h & -j & 0 & 0 \end{pmatrix} = -\gamma^k$$

the RHS = LHS, and thus we now have shown that $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$ for $\mu = k \neq 0$. Since the conditions have been met for $\mu = 0$ and $\mu \neq 0$, then we have shown that in general:

$$\boxed{(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0}$$