Problem Set 7

5.1 Draw the two time-ordered diagrams for the *s*-channel process shown in Figure 5.5. By repeating the steps of Section 5.1.1, show that the propagator has the same form as obtained for the *t*-channel process.

Hint: one of the time-ordered diagrams is non-intuitive, remember that in second-order perturbation theory the intermediate state does not conserve energy.

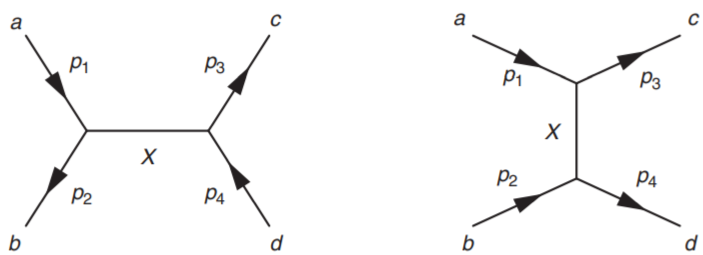
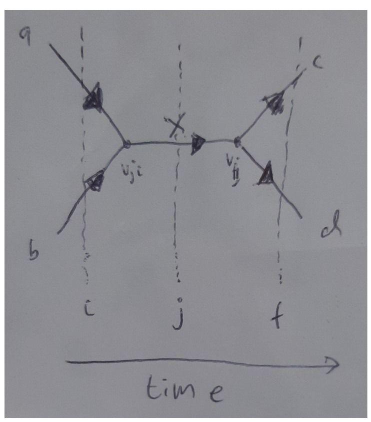
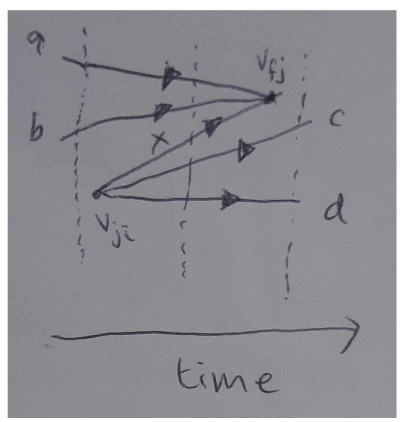


Fig. 5.5 Feynman diagrams for illustrative *s*-channel annihilation and *t*-channel scattering processes.

For the given s-channel, the processes can have different sequences on which they happened, as all we know is that there are incoming a and b and outgoing c and d. For one, an incoming a splits into an outgoing c and X, and this X then interacts with an incoming b to produce d. In another, c, d, and X are produced first, then incoming a and b interact with X to disappear. These time orderings may be represented as:





For the first time ordering, we may obtain the transition matrix element T_{fi}^{ab} as given by Equation 5.1 and by noting the different regions (c and d in the f region, X in the j region, and a and b in the i region):

$$T_{fi}^{ab} = \frac{\langle f| V |j\rangle \langle j| V |i\rangle}{E_i - E_j} \tag{1}$$

$$T_{fi}^{ab} = \frac{\langle c+d| V |X\rangle \langle X| V |a+b\rangle}{(E_a + E_b) - E_X} \tag{2}$$

$$\tag{3}$$

The non-invariant matrix elements $\langle f|\, V\, |j\rangle$ and $\langle j|\, V\, |i\rangle$ are given by:

$$\langle f|\, V\, |j\rangle = \langle c+d|\, V\, |X\rangle = \mathcal{M}_{fj} \prod_k (2E_k)^{-1/2} \tag{4}$$
$$\langle j|\, V\, |i\rangle = \langle X|\, V\, |a+b\rangle = \mathcal{M}_{ji} \prod_k (2E_k)^{-1/2} \tag{5}$$
$$\tag{6}$$

we can also rewrite the matrix as the coupling constants:

$$\mathcal{M}_{fj}\mathcal{M}_{ji} = g_a g_b = g^2 \tag{7}$$

thus we have:

$$T_{fi}^{ab} = \frac{1}{2E_X \cdot 2(2E_aE_bE_cE_d)^{1/2}} \frac{g^2}{(E_a+E_b)-E_X} \tag{8}$$
$$\tag{9}$$

we can get the overall Lorentz-invariant matrix in the form of:

$$\mathcal{M}_{fi}^{ab} = 2(2E_aE_bE_cE_d)^{1/2} T_{fi}^{ab} \tag{10}$$
$$\mathcal{M}_{fi}^{ab} = \frac{1}{2E_X} \frac{g^2}{(E_a+E_b)-E_X} \tag{11}$$

we repeat the same steps for the second time ordering:

$$T_{fi}^{cd} = \frac{\langle f|\, V\, |j\rangle \langle j|\, V\, |i\rangle}{E_i - E_j} \tag{12}$$
$$T_{fi}^{cd} = \frac{\langle c+d+X|\, V\, |0\rangle \langle 0|\, V\, |a+b+X\rangle}{(E_a+E_b)-(E_a+E_b+E_c+E_d+E_X)} \tag{13}$$
$$\tag{14}$$

with $E_c + E_d = E_a + E_b$, we have:

$$T_{fi}^{cd} = \frac{1}{2E_X \cdot 2(2E_aE_bE_cE_d)^{1/2}} \frac{g^2}{-E_a-E_b-E_X} \tag{15}$$
$$\tag{16}$$

we proceed with obtaining the matrix element:

$$\mathcal{M}_{fi}^{cd} = 2(2E_aE_bE_cE_d)^{1/2}T_{fi}^{cd} \tag{17}$$
$$\mathcal{M}_{fi}^{cd} = -\frac{1}{2E_X}\frac{g^2}{(E_a+E_b)+E_X} \tag{18}$$

to get the combined Lorentz-invariant matrix element of both time orderings, we get their sum:

$$\mathcal{M} = \mathcal{M}_{fi}^{ab} + \mathcal{M}_{fi}^{cd} \tag{19}$$
$$\mathcal{M} = \left[\frac{1}{2E_X}\frac{g^2}{(E_a+E_b)-E_X}\right] + \left[-\frac{1}{2E_X}\frac{g^2}{(E_a+E_b)+E_X}\right] \tag{20}$$
$$\mathcal{M} = \frac{g^2}{2E_X}\left[\frac{1}{(E_a+E_b)-E_X} - \frac{1}{(E_a+E_b)+E_X}\right] \tag{21}$$
$$\mathcal{M} = \frac{g^2}{2E_X}\left[\frac{2E_X}{(E_a+E_b)^2-E_X^2}\right] \tag{22}$$
$$\tag{23}$$

we can rewrite E_X^2 as:

$$E_X^2 = \mathbf{p}_X^2 + m_X^2 \tag{24}$$
$$E_X^2 = (\mathbf{p}_a+\mathbf{p}_b)^2 + m_X^2 \tag{25}$$

thus the matrix element becomes:

$$\mathcal{M} = \frac{g^2}{(E_a+E_b)^2 - (\mathbf{p}_a+\mathbf{p}_b)^2 - m_X^2} \tag{26}$$
$$\mathcal{M} = \frac{g^2}{(p_a+p_b)^2 - m_X^2} \tag{27}$$

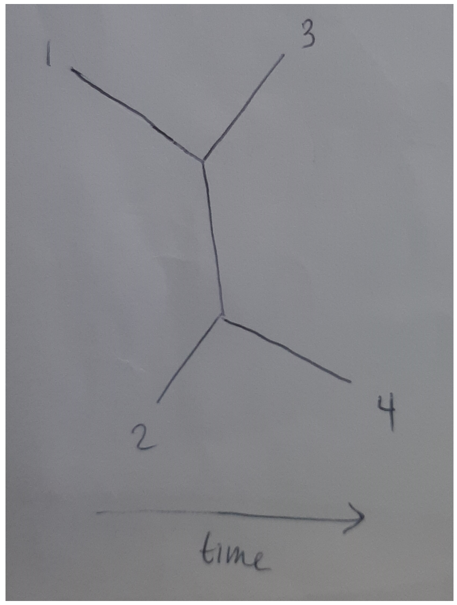
rewriting this in terms of the Mandelstam s variable $q = (p_a+p_b)^2$, we get:

$$\mathcal{M} = \frac{g^2}{q^2 - m_X^2} \tag{28}$$

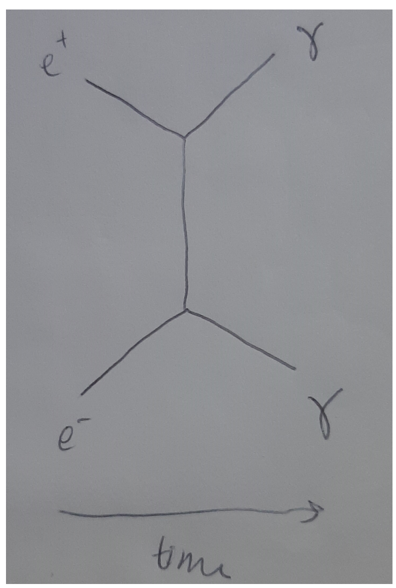
This form recovers the $\frac{1}{q^2 - m_X^2}$ propagator.

5.3 Draw the lowest-order t-channel and u-channel Feynman diagrams for $e^+e^- \to \gamma\gamma$ and use the Feynman rules for QED to write down the corresponding matrix elements.

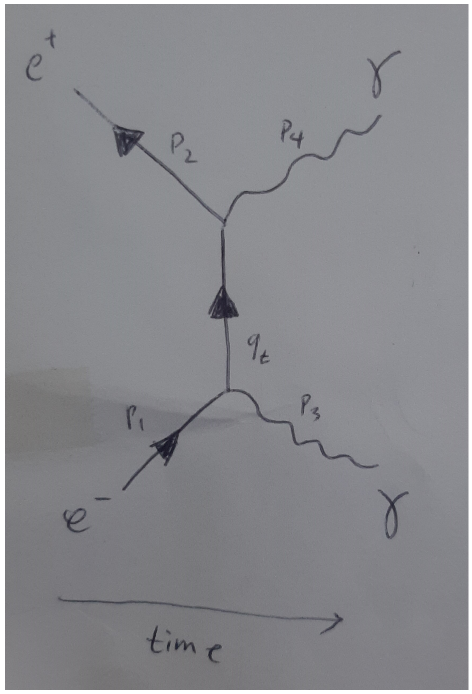
In the case of $e^+e^- \to \gamma\gamma$, two identical particles are produced. Hence, we use either t-channel or u-channel diagrams. The t-channel diagram generally looks like:



we can place the particles given into the above general configuration:



reconfiguring this so that particle arrows go with the time arrow, antiparticle arrows go against the time arrow, each vertex has one γ photon as allowed in vertex rules, and having the fermion propagator as the virtual particle, we have:



we note the following contributions to the matrix element $-i\mathcal{M}_t$:

$$
\begin{aligned}
\text{initial state particle } (e^-) &: u(p) \\
\text{initial state antiparticle } (e^+) &: \bar{v}(p) \\
\text{final state photons } (\gamma) &: \varepsilon^*_\mu(p) \\
\text{fermion propagator } (q) &: -\frac{i(\gamma^\mu q_\mu + m)}{q^2 - m^2} \\
\text{interaction vertex factor } &: ie\gamma^\mu
\end{aligned}
$$

the product of the contributions from the initial state electron at the 1 position and the final state photon at position 3 along with the corresponding interaction vertex factor is given by:

$$\varepsilon^*_\mu(p)ie\gamma^\mu u(p) \tag{29}$$

$$\varepsilon^*_\mu(p_3)ie\gamma^\mu u(p_1) \tag{30}$$

(31)

the product of the contributions from the initial state positron at the 2 position and the final state photon at position 4 along with the corresponding interaction vertex factor is given by:

$$\varepsilon_\nu^*(p)ie\gamma^\nu\bar{v}(p) \tag{32}$$

$$\varepsilon_\nu^*(p_4)ie\gamma^\nu\bar{v}(p_2) \tag{33}$$

$$\tag{34}$$

lastly, we have the fermion propagator in the middle:

$$-\frac{i(\gamma^\mu q_\mu + m)}{q^2 - m^2} \tag{35}$$

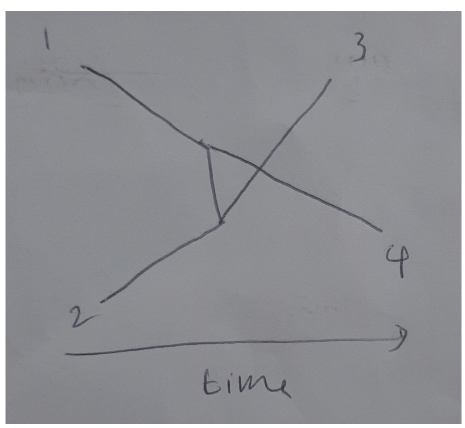
$$\tag{36}$$

thus for the matrix element of the t-channel diagram, rearranging the terms such that the initial state electron and positron terms are nearer to the fermion propagator, we have:

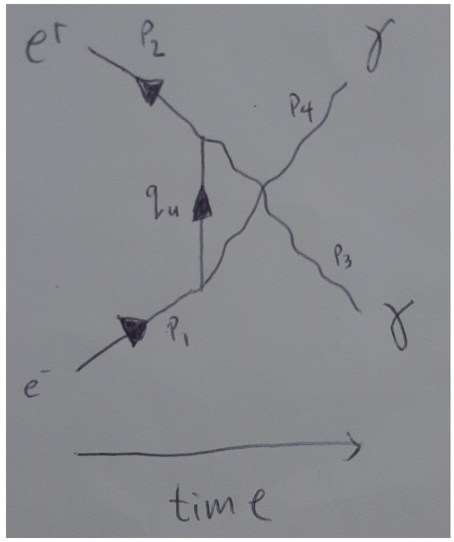
$$-i\mathcal{M}_t = \left[\varepsilon_\mu^*(p_3)ie\gamma^\mu u(p_1)\right]\left[-\frac{i(\gamma^\mu q_\mu + m)}{q^2 - m^2}\right]\left[\bar{v}(p_2)ie\gamma^\nu\varepsilon_\nu^*(p_4)\right] \tag{37}$$

$$\boxed{\mathcal{M}_t = \left[\varepsilon_\mu^*(p_3)e\gamma^\mu u(p_1)\right]\left[-\frac{(\gamma^\mu q_\mu + m)}{q^2 - m^2}\right]\left[\bar{v}(p_2)e\gamma^\nu\varepsilon_\nu^*(p_4)\right]} \tag{38}$$

for the u-channel diagram, we have the general configuration:



particles inserted along with the proper corrections, we have:



we notice from the general diagram that particle 4 and 3 just switched, and thus for its matrix elements $\mathcal{M}_u$ we can interchange the terms referring to those particles from the earlier $\mathcal{M}_t$:

$$\boxed{\mathcal{M}_u = \left[\varepsilon_\mu^*(p_4)e\gamma^\mu u(p_1)\right]\left[-\frac{(\gamma^\mu q_\mu + m)}{q^2 - m^2}\right]\left[\bar{v}(p_2)e\gamma^\nu \varepsilon_\nu^*(p_3)\right]} \tag{39}$$