Nuclear phenomenology

Nuclides

Nuclides are typically written as:

$$^{\mathrm{A}}_{\mathrm{Z}}\mathrm{Y}$$

where we note that:

A is the **mass/nucleon number** (# of nucleons)
Z is the **proton/atomic number** (# of protons)
N is the **neutron number** (# of neutrons)

with $\mathbf{A} = \mathbf{Z} + \mathbf{N}$. We also note that isotopes with: same A = isobars; same Z = isotopes; same N = isotones. Some elements have multiple isotopes, with different stabilty and abundance.

Nuclear shapes and sizes

Nuclei may be treated as static charge distributions with normalization:

$$\int f(\mathbf{r})\mathrm{d}^3\mathbf{r} = Ze \tag{1}$$

where e is the electron charge. Under the Born approximation, the cross-section $\frac{d\sigma}{d\Omega}$ is given by:

$$\left(\frac{d\sigma}{d\Omega}\right)_0 = \frac{Z^2\alpha^2(\hbar c)^2}{4\beta^4 E^2 \sin^4(\theta/2)} \tag{2}$$

including the electron spin, the Mott cross-section is given by:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} = \left(\frac{d\sigma}{d\Omega}\right)_0 [1 - \beta^2 \sin^2(\theta/2)] \tag{3}$$

In the nonrelativistic limit with no spin dependence, the Rutherford cross-section is given by:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Rutherford}} = \frac{(\hbar c)^2(\alpha Z)^2}{4m^2v^4 \sin^4(\theta/2)} \tag{4}$$

Given $\mathbf{q} = \mathbf{p} - \mathbf{p}'$ where $\mathbf{p}$ and $\mathbf{p}'$ are initial and final electron momenta, the form factor $F(\mathbf{q}^2)$ (Fourier transform of charge distribution) is given by:

$$F(\mathbf{q}^2) = \frac{1}{Ze}\int e^{i\mathbf{q}\cdot\mathbf{r}/\hbar} f(\mathbf{r})\, \mathrm{d}^3\mathbf{r} \tag{5}$$

$$F(\mathbf{q}^2) = \frac{4\pi\hbar}{Zeq}\int_0^\infty r\rho(r)\sin\left(\frac{qr}{\hbar}\right)\, \mathrm{d}r \tag{6}$$

$$F(\mathbf{q}^2) = \frac{4\pi}{Ze}\int_0^\infty f(r)r^2\, \mathrm{d}r - \frac{4\pi\mathbf{q}^2}{6Ze\hbar^2}\int_0^\infty f(r)r^4\, \mathrm{d}r + \dots \tag{7}$$

$$\tag{8}$$

Experimental cross-section may be approximately related to the Mott cross-section by:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{expt}} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} |F(\mathbf{q}^2)| \tag{9}$$

Representing the nuclear charge distribution as a hard sphere where $\rho(r)$ is constant for $r \leq a$ and 0 otherwise, then the form factor simplifies to $F(\mathbf{q}^2) = 3[\sin(b) - b\cos(b)]b^{-3}$.

The charge distribution may be obtained from the form factor using:

$$f(\mathbf{r}) = \frac{Ze}{(2\pi)^3}\int F(\mathbf{q}^2)e^{-i\mathbf{q}\cdot\mathbf{r}/\hbar}\mathrm{d}^3\mathbf{q} \tag{10}$$

which can be simplified into:

$$f(r) = \rho_{\text{ch}}(r) = \frac{\rho^0_{\text{ch}}}{1 + e^{(r-a)/b}} \tag{11}$$

where $a \approx 1.07A^{1/3}$ fm and $b \approx 0.54$ fm.

Another important quantity would be the mean square charge radius:

$$\langle r^2 \rangle = \frac{1}{Ze}\int r^2 f(\mathbf{r})\, \mathrm{d}^3\mathbf{r} \tag{12}$$

$$\langle r^2 \rangle = -6\hbar^2 \frac{\mathrm{d}F(\mathbf{q}^2)}{\mathrm{d}\mathbf{q}^2}\bigg|_{\mathbf{q}^2=0} \tag{13}$$

for very small values of $\mathbf{q}^2$, for medium and heavy nuclei, $\langle r^2 \rangle^{1/2} = 0.94A^{1/3}$ fm.

Semi-empirical mass formula (SEMF)

The forces that bind nucleons also contribute to the total mass $M(Z,A)$ of an atom aside from the contributions of the proton (with mass M_p), neutron (with mass M_n), and electron (with mass m_e). This mass deficit is given by:

$$\Delta M(Z,A) = M(Z,A) - Z(M_p + m_e) - NM_n \tag{14}$$

multiplying each side of the equation by $-c^2$ gives the **binding energy** B.

The atomic mass $M(Z,A)$ is given by a sum of terms:

$$M(Z,A) = \sum_{i=0}^{5} f_i(Z,A) \tag{15}$$

where each term is given by:

$$\begin{aligned}
\text{(mass of nucleons \& electrons)} \quad f_0(Z,A) &= Z(M_p + m_e) + (A-Z)M_n \\
\text{(volume term)} \quad f_1(Z,A) &= -a_1 A \\
\text{(surface term)} \quad f_2(Z,A) &= a_2 A^{2/3} \\
\text{(Coulomb term)} \quad f_3(Z,A) &= a_3 \frac{Z(Z-1)}{A^{1/3}} \approx a_3 \frac{Z^2}{A^{1/3}} \\
\text{(asymmetry term)} \quad f_4(Z,A) &= a_4 \frac{(Z-A/2)^2}{A} \\
\text{(pairing term)} \quad f_5(Z,A) &= -a_5 A^{-1/2} \text{ if } Z \text{ and } N \text{ are even} \\
&= a_5 A^{-1/2} \text{ if } Z \text{ and } N \text{ are odd} \\
&= 0 \text{ otherwise}
\end{aligned}$$

the constants of each term have the respective values:

$$\begin{aligned}
a_1 &= a_v = 15.56\ \mathrm{MeV/c^2} \\
a_2 &= a_s = 17.23\ \mathrm{MeV/c^2} \\
a_3 &= a_c = 0.697\ \mathrm{MeV/c^2} \\
a_4 &= a_a = 93.14\ \mathrm{MeV/c^2} \\
a_5 &= a_p = 12\ \mathrm{MeV/c^2}
\end{aligned}$$