Dirac Equation

Klein-Gordon Equation

The Einstein energy-momentum relationship when expressed in terms of operators becomes:

$$E^2 = \mathbf{p}^2 + m^2$$
$$\hat{E}^2\psi(\mathbf{x},t) = \hat{\mathbf{p}}^2\psi(\mathbf{x},t) + m^2\psi(\mathbf{x},t)$$

with the operators rewritten as $\hat{\mathbf{p}} = -i\nabla$ and $\hat{E} = i\frac{\partial}{\partial t}$, the **Klein-Gordon wave equation** is given by:

$$\frac{\partial^2\psi}{\partial t^2} = \nabla^2\psi - m^2\psi \tag{1}$$

or in Lorentz-invariant form,

$$(\partial^\mu\partial_\mu + m^2)\psi = 0 \tag{2}$$

where

$$\partial^\mu\partial_\mu = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$$

This has plane wave solutions in the form of:

$$\psi(\mathbf{x},t) = Ne^{i(\mathbf{p}\cdot\mathbf{x} - Et)} \tag{3}$$

The Dirac Equation

Dirac sought to find a wave equation that was first order in space and time derivatives (KGE had seecond order):

$$\hat{E}\psi = (\alpha\cdot\hat{\mathbf{p}} + \beta m)\psi \tag{4}$$
$$i\frac{\partial}{\partial t}\psi = \left(-i\alpha_x\frac{\partial}{\partial x} - i\alpha_y\frac{\partial}{\partial y} - i\alpha_z\frac{\partial}{\partial z} + \beta m\right)\psi \tag{5}$$

in order to satisfy the KGE, α and β must satisfy:

$$\begin{aligned}\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 = I \\ \alpha_j\beta + \beta\alpha_j = 0 \\ \alpha_j\alpha_k + \alpha_k\alpha_j = 0 \quad (j \neq k)\end{aligned}$$

α components and β are usually explicitly written using the **Dirac-Pauli representation**:

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$
$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Probability density and probability current

Starting with the KGE and after exploiting the Hermitian property of the α and β matrices, a continuity equation may be obtained:

$$\nabla \cdot (\psi^\dagger \alpha \psi) + \frac{\partial(\psi^\dagger \psi)}{\partial t} = 0 \tag{6}$$

with $\psi^\dagger = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*)$. The probability density and probability current then take the forms of:

$$\begin{aligned}&\rho = \psi^\dagger \psi = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2 \\ &\mathbf{j} = \psi^\dagger \alpha \psi\end{aligned}$$

Covariant form of the Dirac equation

The covariant form of the Dirac equation is given by:

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \tag{7}$$

the gamma matrices are as follow:

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$
$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \quad \gamma^4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

the matrices satisfy conditions such as:

$$\begin{aligned} (\gamma^0)^2 &= I \\ (\gamma^k)^2 &= -I \\ \gamma^\mu \gamma^\nu &= -\gamma^\nu \gamma^\mu \quad (\mu \neq \nu) \\ \{\gamma^\mu, \gamma^\nu\} &\equiv \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \\ \gamma^{0\dagger} &= \gamma^0 \\ \gamma^{k\dagger} &= -\gamma^k \end{aligned}$$

The adjoint spinor and covariant current

The adjoint spinor may be defined as:

$$\bar{\psi} = \psi^\dagger \gamma^0 = (\psi_1^*, \psi_2^*, -\psi_3^*, -\psi_4^*) \tag{8}$$

the four-vector current may then be written as:

$$j^\mu = \bar{\psi} \gamma^\mu \psi \tag{9}$$

Solutions to Dirac equation

Solutions may take the form of:

$$\psi(\mathbf{x}, t) = u(E, \mathbf{p}) e^{i(\mathbf{p}\cdot\mathbf{x} - Et)} \tag{10}$$

satisfying the Dirac equation then the Dirac spinor u satisfies:

$$(\gamma^\mu p_\mu - m)u = 0 \tag{11}$$

Particles at rest

Particles at rest have $\mathbf{p} = 0$, thus we have:

$$\psi = u(E, 0)e^{-iEt} \tag{12}$$
$$E\gamma^0 u = mu \tag{13}$$

this gives solutions of:

$$\psi_1 = N \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-imt} \quad \psi_2 = N \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{-imt}$$
$$\psi_3 = N \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{+imt} \quad \psi_1 = N \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{+imt}$$

General free-particle solutions

General free-particle solutions take the form of

$$\psi_i = u(E, \mathbf{p})e^{i(\mathbf{p}\cdot\mathbf{x} - Et)} \tag{14}$$
$$\tag{15}$$

where u_i take form of:

$$u_1 = N_1 \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix} \quad u_2 = N_2 \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$
$$u_3 = N_3 \begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x+ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix} \quad u_4 = N_4 \begin{pmatrix} \frac{p_x-ip_y}{E-m} \\ \frac{-p_z}{E-m}0 \\ 1 \end{pmatrix}$$

Antiparticle solutions

Antiparticle solutions take the form of

$$\psi_i = v(E, \mathbf{p})e^{i(\mathbf{p}\cdot\mathbf{x}-Et)} \tag{16}$$
$$\tag{17}$$

where v_i take form of:

$$v_1 = N\begin{pmatrix} \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} \quad v_2 = N\begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}$$
$$v_3 = N\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E-m} \\ \frac{p_x + ip_y}{E-m} \end{pmatrix} \quad u_4 = N_4\begin{pmatrix} 0 \\ 1 \\ \frac{p_x - ip_y}{E-m} \\ \frac{-p_z}{E-m} \end{pmatrix}$$

Usually, it is more natural to work with positive energy solutions for particles and antiparticles ($\{u_1, u_2, v_1, v_2\}$)

Helicity

Particle helicity spinors take the form of:

$$u_\uparrow = \sqrt{E+m}\begin{pmatrix} c \\ se^{i\phi} \\ \frac{p}{E+m}c \\ \frac{p}{E+m}se^{i\phi} \end{pmatrix} \quad u_\downarrow = \sqrt{E+m}\begin{pmatrix} -s \\ ce^{i\phi} \\ \frac{p}{E+m}s \\ -\frac{p}{E+m}ce^{i\phi} \end{pmatrix}$$
$$u_\uparrow \approx \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix} \quad u_\downarrow \approx \sqrt{E}\begin{pmatrix} -s \\ ce^{i\phi} \\ s \\ -ce^{i\phi} \end{pmatrix} \quad (E \gg m)$$

Antiparticle helicity spinors take the form of:

$$v_\uparrow = \sqrt{E+m}\begin{pmatrix} \frac{p}{E+m}s \\ -\frac{p}{E+m}ce^{i\phi} \\ -s \\ ce^{i\phi} \end{pmatrix} \quad v_\downarrow = \sqrt{E+m}\begin{pmatrix} \frac{p}{E+m}c \\ \frac{p}{E+m}se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}$$
$$v_\uparrow = \sqrt{E}\begin{pmatrix} s \\ -ce^{i\phi} \\ -s \\ ce^{i\phi} \end{pmatrix} \quad v_\downarrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix} \quad (E \gg m)$$