Problem Set 3

Physics 180

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8.1 Assume that in the shell model the nucleon energy levels are ordered as shown in Figure 8.4. Write down the shell model configuration of the nucleus $^{7}_{3}$Li and hence find its spin, parity, and magnetic moment (in nuclear magnetons). Give the two most likely configurations for the first excited state, assuming that only protons are excited.

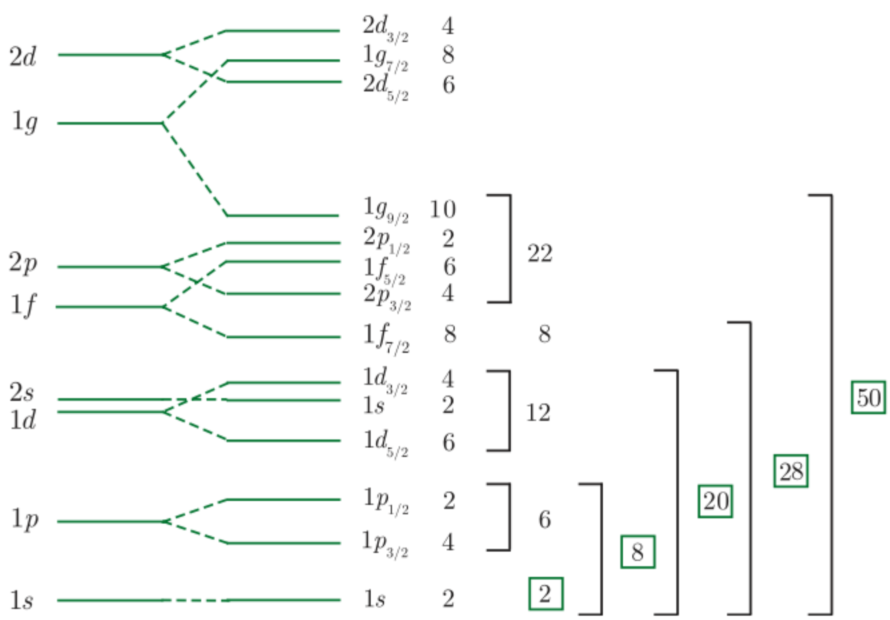

Figure 8.4 Low-lying energy levels in a single-particle shell model using a Woods–Saxon potential plus spin–orbit term. The integers in boxes correspond to nuclear magic numbers.

$^{7}_{3}$Li has $Z = 3$ protons and $N = A - Z = 4$ neutrons (even-odd nuclei). From this, based on Figure 8.4, then we have the configurations:

$$(1s_{1/2})^2(1p_{3/2})^1 \quad \text{for protons}$$
$$(1s_{1/2})^2(1p_{3/2})^2 \quad \text{for neutrons}$$

with all neutrons being paired, we only check the protons for contributions on spin, parity, and magnetic moment. We note that the protons have the following:

$$j = 3/2 \quad l = 1$$

with no net spin contribution from then neutrons, then the total spin is:

$$\boxed{j = 3/2} \tag{1}$$

its parity P is then given by Equation 8.27:

$$P = (-1)^l = -1^1$$

$$\boxed{P = -1} \tag{2}$$

Its nuclear magnetic moment μ in nuclear magnetons is then given by Equation 8.28 and Equation 8.31 :

$$\begin{aligned} \mu &= g_j j = j + 2.3 \\ &= \frac{3}{2} + 2.3 \end{aligned}$$

$$\boxed{\mu = 3.8 \text{ nuclear magnetons}} \tag{3}$$

Assuming that only the protons are excited, then either the proton at $p_{3/2}$ gets excited to $p_{1/2}$, or a proton from $s_{1/2}$ gets excited to $p_{3/2}$, giving the following possible configurations:

$$\boxed{(1s_{1/2})^2(1p_{1/2})^1 \text{ for protons } \ (1s_{1/2})^2(1p_{3/2})^2 \text{ for neutrons}} \tag{4}$$

$$\boxed{(1s_{1/2})^1(1p_{3/2})^2 \text{ for protons } \ (1s_{1/2})^2(1p_{3/2})^2 \text{ for neutrons}} \tag{5}$$

8.5 Show explicitly that a uniformly charged ellipsoid at rest with total charge Ze and semi-axes defined in Figure 2.17 has a quadrupole moment $Q = 2Z(a^2 - b^2)/5$.

We begin with the equation for the quadrupole Q given by Equation 8.32:

$$eQ \equiv \int \rho(\mathbf{r})(3z^2 - r^2)d^3\mathbf{r} \tag{8.32}$$

$$Q = \frac{1}{e}\int \rho(\mathbf{r})(3z^2 - r^2)d^3\mathbf{r} \tag{8.32.1}$$

we then consider the given transformation:

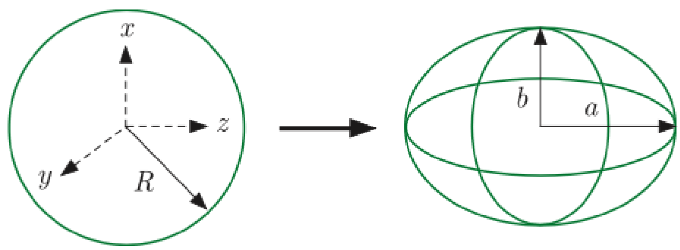

Figure 2.17 Deformation of a heavy nucleus.

from the spherical model, we can express r as:

$$r^2 = x^2 + y^2 + z^2$$

after the deformation to a prolate ellipsoid, $x,\ y,\ z$ become:

$$x \to bx', \quad y \to by', \quad z \to az' \implies r^2 = b^2x'^2 + b^2y'^2 + a^2z'^2$$

thus our quadrupole moment becomes:

$$Q = \frac{1}{e}\int\int\int \rho(\mathbf{r})(2a^2z'^2 - b^2x'^2 - b^2y'^2)\ dx'\ dy'\ dz' \tag{8.32.2}$$

we can rewrite the charge density $\rho(\mathbf{r})$ as the total charge over the volume, and with the given total charge Ze we have:

$$\begin{aligned}
\rho(\mathbf{r}) &= \frac{Ze}{V} \\
&= \frac{Ze}{\frac{4}{3}\pi ab^2} \\
&= \frac{3Ze}{4\pi ab^2}
\end{aligned}$$

and for the quadrupole moment we get:

$$Q = \frac{3Z}{4\pi ab^2} \int\int\int (2a^2 z'^2 - b^2 x'^2 - b^2 y'^2)\ dx'\ dy'\ dz' \tag{8.32.3}$$

we proceed to evaluate each integral term in spherical coordinates, noting the following change of variables:

$$\begin{aligned}
x' &\to s\cos\varphi\sin\theta \qquad 0 \le s \le 1 \\
y' &\to s\sin\varphi\sin\theta \qquad 0 \le \varphi \le \pi/2 \\
z' &\to s\cos\theta \qquad\qquad 0 \le \theta \le \pi/2 \\
dx'dy'dz' &\to 8s^2ab^2\sin\theta ds d\varphi d\theta
\end{aligned}$$

evaluating each integral, we have:

$$\begin{aligned}
\int\int\int 2a^2 z'^2\ dx'\ dy'\ dz' &= 2a^2 \int\int\int s^2\cos^2\theta\ s^2ab^2\sin\theta\ 8ds d\varphi d\theta \\
&= 16a^3b^2 \int_0^1 s^4\ ds \int_0^{\pi/2} \sin\theta\cos^2\theta\ d\theta \int_0^{\pi/2} d\varphi \\
&= 16a^3b^2 \left(\frac{1}{5}\right)\left(\frac{1}{3}\right)\left(\frac{\pi}{2}\right) \\
&= 8a^3b^2\frac{\pi}{15}
\end{aligned}$$

$$\begin{aligned}
\int\int\int b^2 x'^2\ dx'\ dy'\ dz' &= b^2 \int\int\int s^2\cos^2\varphi\sin^2\theta\ s^2ab^2\sin\theta\ 8ds d\varphi d\theta \\
&= 8ab^4 \int_0^1 s^4\ ds \int_0^{\pi/2} \cos^2\varphi\ d\varphi \int_0^{\pi/2} \sin^3\theta\ d\theta \\
&= 8ab^4 \left(\frac{1}{5}\right)\left(\frac{\pi}{4}\right)\left(\frac{2}{3}\right) \\
&= 8ab^4\frac{\pi}{30}
\end{aligned}$$

$$
\begin{aligned}
\int\int\int b^2 y'^2 \, dx' \, dy' \, dz' &= b^2 \int\int\int s^2 \sin^2\varphi \sin^2\theta \; s^2 a b^2 \sin\theta \; 8 ds d\varphi d\theta \\
&= 8ab^4 \int_0^1 s^4 \, ds \int_0^{\pi/2} \sin^2\varphi \, d\varphi \int_0^{\pi/2} \sin^3\theta \, d\theta \\
&= 8ab^4 \left(\frac{1}{5}\right)\left(\frac{\pi}{4}\right)\left(\frac{2}{3}\right) \\
&= 8ab^4 \frac{\pi}{30}
\end{aligned}
$$

thus we have:

$$
Q = \frac{3Z}{4\pi a b^2}\left(8a^3 b^2 \frac{\pi}{15} - 8ab^4 \frac{\pi}{15}\right)
$$

$$
\boxed{Q = \frac{2Z(a^2 - b^2)}{5}} \tag{1}
$$