Problem Set 2

Physics 180

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2.13 The isotope ${}^{238}_{94}$Pu decays via α emission to the essentially stable isotope ${}^{234}_{92}$U with a mean lifetime of 126.7 yr and a release of 5.49 MeV of kinetic energy. This energy is converted to electrical power in a space probe designed to reach planet X in a journey planned to last four years. If the efficiency of power conversion is 5% and on reaching planet X the probe requires at least 200 W of power to perform its landing tasks, how much ${}^{238}_{94}$Pu would be needed at launch?

Looking at the required power and the power conversion efficiency, then we can find the total power P_{req} we need to be released in the α emission:

$$P_{\text{req}} = \frac{200\text{ W}}{5\%} \tag{1}$$

$$P_{\text{req}} = 4000\text{ W} \tag{2}$$

With the given released kinetic energy, we can then get the power P_α given by the α emission:

$$P_\alpha = \mathcal{A}(t) \times 5.49\text{MeV} \times \frac{1.602 \times 10^{-13}\text{Ws}}{1\text{MeV}} \tag{3}$$

To find the power released in the α emission, we check its activity given by $\mathcal{A}(t) = \lambda N_0 \exp(-\lambda t)$. With the given mean lifetime τ, then we can say that the decay constant λ can be expressed as:

$$\lambda = \frac{1}{\tau} \tag{4}$$

thus using this, we get an expression for the activity:

$$\mathcal{A}(t) = \frac{1}{\tau} N_0 \exp(-t/\tau) \tag{5}$$

if we equate the power given by the reaction P_α and the power needed P_{req}, then we can arrive at a value for the initial number of ${}^{238}_{94}$Pu atoms required (N_0):

$$4000\text{ W} = \frac{1}{\tau} N_0 \exp(-t/\tau) \times 5.49\text{MeV} \times \frac{1.602 \times 10^{-13}\text{Ws}}{1\text{MeV}} \tag{6}$$

using the given $t = 4$ yrs and $\tau = 126.7$ yrs and converting them to seconds, we get:

$$4000\text{ W} \times \frac{1\text{MeV}}{1.602 \times 10^{-13}\text{Ws}} \times \frac{1}{5.49\text{MeV}} \times \frac{126.7\text{ yrs}}{\exp(-4/126.7)} \times \frac{365\text{ days}}{1\text{ yr}} \times \frac{24\text{ hrs}}{1\text{ day}} \times \frac{60\text{ min}}{1\text{ hr}} \times \frac{60\text{ s}}{1\text{ min}} = N_0 \tag{7}$$

$$N_0 = 1.88 \times 10^{25}\ \ {}^{238}_{94}\text{Pu atoms} \tag{8}$$

to get the needed weight W, then we use its molar mass:

$$W = (1.88 \times 10^{25}\text{ atoms}) \times \frac{1\text{ mol}}{6.02 \times 10^{23}\text{ atoms}} \times \frac{238\text{ g}}{1\text{ mol}} \times \frac{1\text{ kg}}{1000\text{ g}} \tag{9}$$

$$\boxed{W = 7.43\text{ kg}} \tag{10}$$

2.15 A radioisotope with decay constant λ is produced at a constant rate P starting at time $t = 0$. Show that the number of atoms at time $t > 0$ is $N(t) = P[1 - \exp(-\lambda t)]/\lambda$.

From the given decay constant λ and rate P starting at $t = 0$, then we can express $dN(t)/dt$ as:

$$\frac{dN(t)}{dt} = P - \lambda N(t) \tag{1}$$

$$P = \frac{dN(t)}{dt} + \lambda N(t) \tag{2}$$

to get the integrating factor μ to be multiplied to both sides for the differential equation $q(t) = y' + p(t)y$, we have:

$$\mu = e^{\int p(t)\,dt} \tag{3}$$

$$\mu = e^{\int \lambda\,dt} \tag{4}$$

$$\mu = e^{\lambda t} \tag{5}$$

multiplying each side by $e^{\lambda t}$, we get:

$$Pe^{\lambda t} = \frac{dN(t)}{dt}e^{\lambda t} + \lambda N(t)e^{\lambda t} \tag{6}$$

$$\tag{7}$$

we note that the derivative of $f(t)e^{\lambda t}$ with respect to t is:

$$\frac{d}{dt}[f(t)e^{\lambda t}] = f(t)\lambda e^{\lambda t} + e^{\lambda t}\frac{df(t)}{dt} \tag{8}$$

noting that the right hand side looks similar to the derivative above, then we can rewrite it as:

$$Pe^{\lambda t} = \frac{dN(t)e^{\lambda t}}{dt} \tag{9}$$

integrating both sides with respect to t, we get:

$$\int Pe^{\lambda t}dt = \int \frac{dN(t)e^{\lambda t}}{dt} \tag{10}$$

$$\frac{P}{\lambda}e^{\lambda t} + C = N(t)e^{\lambda t} \tag{11}$$

to evaluate for the integration constant C, we note the condition $N(0) = 0$. Thus at $t = 0$ we get:

$$\frac{P}{\lambda} + C = 0 \tag{12}$$

$$C = -\frac{P}{\lambda} \tag{13}$$

using this integration constant, then our original equation becomes:

$$\frac{P}{\lambda}e^{\lambda t} - \frac{P}{\lambda} = N(t)e^{\lambda t} \tag{14}$$

$$\boxed{N(t) = \frac{P\left(1 - e^{-\lambda t}\right)}{\lambda}} \tag{15}$$