# Problem Set 1

Physics 180
Martin 3rd Ed

**2.3** Show explicitly that (2.39) follows from (2.37).

Equation 2.37 gives an expression for the form factor:

$$F(\mathbf{q}^2) = \frac{1}{Ze} \int f(\mathbf{r}) \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{i|\mathbf{q}| r \cos\theta}{\hbar} \right)^n d^3\mathbf{r} \tag{2.37}$$

from here on, we can assume spherical symmetry to expand $d^3\mathbf{r} = r^2 \sin\theta dr d\theta d\phi$ where it appears, and have $f(\mathbf{r}) \to f(r)$:

$$F(\mathbf{q}^2) = \frac{1}{Ze} \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=\infty} f(r) \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{i|\mathbf{q}| r \cos\theta}{\hbar} \right)^n r^2 \sin\theta dr d\theta \int_{\phi=0}^{\phi=2\pi} d\phi \tag{2.37.1}$$

$$F(\mathbf{q}^2) = \frac{2\pi}{Ze} \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=\infty} f(r) \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{i|\mathbf{q}| r \cos\theta}{\hbar} \right)^n r^2 \sin\theta dr d\theta \tag{2.37.2}$$

we expand to show the first few terms of the summation:

$$\begin{aligned} F(\mathbf{q}^2) = \frac{2\pi}{Ze} \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=\infty} f(r) \Big[ & r^2 \sin\theta dr d\theta + \frac{i|\mathbf{q}|}{\hbar} r^3 \cos\theta \, dr \, \sin\theta \, d\theta \\ & - \frac{|\mathbf{q}|^2}{2\hbar^2} r^4 \, dr \cos^2\theta \sin\theta \, d\theta + \dots \Big] \end{aligned} \tag{2.37.3}$$

$$F(\mathbf{q}^2) = \frac{2\pi}{Ze} \int_{r=0}^{r=\infty} f(r) \left[ 2r^2 dr - \frac{|\mathbf{q}|^2}{3\hbar^2} r^4 \, dr + \dots \right] \tag{2.37.4}$$

$$F(\mathbf{q}^2) = \frac{4\pi}{Ze} \int_{r=0}^{r=\infty} f(r) r^2 dr - \frac{2\pi |\mathbf{q}|^2}{3Ze\hbar^2} \int_{r=0}^{r=\infty} f(r) r^4 \, dr + \dots \tag{2.37.5}$$

from Equation 2.23 we have an expression for $Ze$ as a static charge distribution (a.k.a. the normalization condition):

$$\int f(\mathbf{r}) d^3\mathbf{r} = Ze \tag{2.23}$$

expanding in spherical coordinates once more, we have:

$$\int_{r=0}^{r=\infty} f(r)r^2\ dr\ \int_{\theta=0}^{\theta=\pi} \sin\theta d\theta\ \int_{\phi=0}^{\phi=2\pi} d\phi = Ze \tag{2.23.1}$$

$$4\pi\int_{r=0}^{r=\infty} f(r)r^2\ dr = Ze \tag{2.23.1}$$

the left hand side is the same as the first term of Equation 2.37.5, thus we get:

$$F(\mathbf{q}^2) = \cancelto{1}{4\pi\int_{r=0}^{r=\infty} f(r)r^2 dr\left(4\pi\int_{r=0}^{r=\infty} f(r)r^2\ dr\right)^{-1}} - \frac{2\pi|\mathbf{q}|^2}{3Ze\hbar^2}\int_{r=0}^{r=\infty} f(r)r^4\ dr + \ldots \tag{2.37.6}$$

$$F(\mathbf{q}^2) = 1 - \frac{2\pi|\mathbf{q}|^2}{3Ze\hbar^2}\int_{r=0}^{r=\infty} f(r)r^4\ dr + \ldots \tag{2.37.7}$$

from Equation 2.36 we have an expression for mean square charge radius:

$$\langle r^2\rangle = \frac{1}{Ze}\int r^2 f(\mathbf{r})d^3\mathbf{r} \tag{2.36}$$

expanding again in spherical coordinates, this becomes:

$$\langle r^2\rangle = \frac{1}{Ze}\int_{r=0}^{r=\infty} r^4 f(r)\ dr\ \int_{\theta=0}^{\theta=\pi} \sin\theta d\theta\ \int_{\phi=0}^{\phi=2\pi} d\phi \tag{2.36.1}$$

$$\langle r^2\rangle = \frac{4\pi}{Ze}\int_{r=0}^{r=\infty} r^4 f(r)\ dr \tag{2.36.2}$$

we can insert this into the second term of Equation 2.37.7 and get:

$$F(\mathbf{q}^2) = 1 - \left(\frac{4\pi}{Ze}\int_{r=0}^{r=\infty} f(r)r^4\ dr\right)\frac{|\mathbf{q}|^2}{6\hbar^2} + \ldots \tag{2.38.1}$$

$$F(\mathbf{q}^2) = 1 - \frac{|\mathbf{q}|^2}{6\hbar^2} \left\langle r^2 \right\rangle + \dots \tag{2.38.2}$$

we can drop the absolute value sign and thus arrive at Equation 2.39:

$$\boxed{F(\mathbf{q}^2) = 1 - \frac{\mathbf{q}^2}{6\hbar^2} \langle r \rangle + \dots} \tag{2.39}$$

**2.10** Use the SEMF to estimate the energy released in the spontaneous fission reaction

$$^{235}_{92}\mathrm{U} \to ^{87}_{35}\mathrm{Br} + ^{145}_{57}\mathrm{La} + 3n.$$

To obtain the released energy, we get difference between the binding energies B of the products and the reactant:

$$E = (\mathrm{B_{Br}} + \mathrm{B_{La}}) - \mathrm{B_U} \tag{2.48.1}$$
$$E = [\mathrm{B}(35, 87) + \mathrm{B}(57, 145)] - \mathrm{B}(92, 235) \tag{2.48.2}$$

we proceed to approximate each binding energy B by getting their respective atomic masses with SEMF as given by Equation 2.49:

$$\mathrm{M(Z,A)} = \sum_{i=0}^{5} f_i(\mathrm{Z,A}) \tag{2.49}$$

where each $f_i(\mathrm{Z,A})$ is:

$$f_0(\mathrm{Z,A}) = \mathrm{Z}(M_p + m_e) + (\mathrm{A} - \mathrm{Z})M_n \tag{2.50}$$
$$f_1(\mathrm{Z,A}) = -a_v A \tag{2.51}$$
$$f_2(\mathrm{Z,A}) = a_s A^{2/3} \tag{2.52}$$
$$f_3(\mathrm{Z,A}) = a_c \frac{Z^2}{A^{1/3}} \tag{2.53}$$
$$f_4(\mathrm{Z,A}) = a_a \frac{(Z - A/2)^2}{A} \tag{2.54}$$
$$\begin{aligned} f_5(\mathrm{Z,A}) &= -f(A) \quad \text{if both } Z, N \text{ are even} \\ &= +f(A) \quad \text{if both } Z, N \text{ are odd} \\ &= 0 \quad \text{if either one of } Z, N \text{ is odd and the other is even} \end{aligned} \tag{2.55}$$

where $N = A - Z$ and $f(A) = a_p A^{-1/2}$ is usually used for Equation 2.55. Getting each atomic mass we have:

$$\begin{aligned} &\mathrm{M}(92, 235) \\ &= 92(M_p + m_e) + 143M_n - 235a_v + 235^{2/3}a_s + \frac{92^2}{235^{1/3}}a_c + \frac{(92 - \frac{235}{2})^2}{235}a_a + 0 \end{aligned} \tag{2.56.1}$$

$$
\begin{aligned}
&\text{M}(35, 87)\\
&= 35(M_p + m_e) + 52M_n - 87a_v + 87^{2/3}a_s + \frac{35^2}{87^{1/3}}a_c + \frac{(35 - \frac{87}{2})^2}{87}a_a + 0
\end{aligned} \tag{2.56.2}
$$

$$
\begin{aligned}
&\text{M}(57, 145)\\
&= 57(M_p + m_e) + 88M_n - 145a_v + 145^{2/3}a_s + \frac{57^2}{145^{1/3}}a_c + \frac{(57 - \frac{145}{2})^2}{145}a_a + 0
\end{aligned} \tag{2.56.3}
$$

using these, we can get the binding energies using Equation 2.48:

$$
\text{B}(\text{Z}, \text{A}) = -\frac{1}{c^2}[\text{M}(\text{Z}, \text{A}) - \text{Z}(M_p + m_e) - \text{N}M_n] \tag{2.48}
$$

for convenience we absorb the $1/c^2$ into the $a_i$ constants, thus we have:

$$
\text{B}(92, 235) = 235a_v - 235^{2/3}a_s - \frac{92^2}{235^{1/3}}a_c - \frac{(92 - \frac{235}{2})^2}{235}a_a - 0 \tag{2.57.1}
$$

$$
\text{B}(35, 87) = 87a_v - 87^{2/3}a_s - \frac{35^2}{87^{1/3}}a_c - \frac{(35 - \frac{87}{2})^2}{87}a_a - 0 \tag{2.57.2}
$$

$$
\text{B}(57, 145) = 145a_v - 145^{2/3}a_s - \frac{57^2}{145^{1/3}}a_c - \frac{(57 - \frac{145}{2})^2}{145}a_a - 0 \tag{2.57.3}
$$

getting energy released E per Equation 2.48.2:

$$
\begin{aligned}
E &= [87 + 145 - 235]a_v\\
&\quad + [-87^{2/3} - 145^{2/3} + 235^{2/3}]a_s\\
&\quad + \left[-\frac{35^2}{87^{1/3}} - \frac{57^2}{145^{1/3}} + \frac{92^2}{235^{1/3}}\right]a_c\\
&\quad + \left[-\frac{(35 - \frac{87}{2})^2}{87} - \frac{(57 - \frac{145}{2})^2}{145} + \frac{(92 - \frac{235}{2})^2}{235}\right]a_a
\end{aligned} \tag{2.57.4}
$$

$$E = -3a_v - 9.15a_s + 476.68a_c + 0.28a_a \tag{2.57.5}$$

we use the following values for the $a_i$ terms (in $MeV/c^2$ units):

$$a_v = 15.56, \quad a_s = 17.23, \quad a_c = 0.697, \quad a_a = 93.14 \tag{2.58}$$

thus for released energy we have:

$$\boxed{E = 154.00\ MeV}$$