Problem Set 11

8.3 In fixed-target electron–proton inelastic scattering:

(a) show that the laboratory frame differential cross section for deep-inelastic scattering is related to the Lorentz-invariant differential cross section of Equation (8.11) by

$$\frac{d^2\sigma}{dE_3\, d\Omega} = \frac{E_1 E_3}{\pi}\frac{d^2\sigma}{dE_3\, dQ^2} = \frac{E_1 E_3}{\pi}\frac{2m_p x^2}{Q^2}\frac{d^2\sigma}{dx\, dQ^2},$$

where E_1 and E_3 are the energies of the incoming and outgoing electron.

(b) Show that

$$\frac{2m_p x^2}{Q^2}\cdot\frac{y^2}{2} = \frac{1}{m_p}\frac{E_3}{E_1}\sin^2\frac{\theta}{2} \quad \text{and} \quad 1 - y - \frac{m_p^2 x^2 y^2}{Q^2} = \frac{E_3}{E_1}\cos^2\frac{\theta}{2}.$$

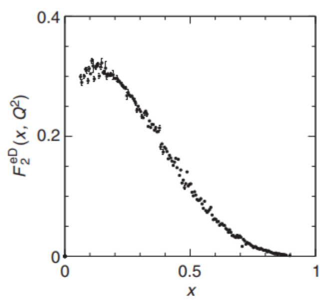

Fig. 8.18 SLAC measurements of $F_2^{eD}(x, Q^2)$ in for $2 < Q^2/\text{GeV}^2 < 30$. Data from Whitlow *et al.* (1992).

(c) Hence, show that the Lorentz-invariant cross section of Equation (8.11) becomes

$$\frac{d^2\sigma}{dE_3\, d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4\theta/2}\left[\frac{F_2}{\nu}\cos^2\frac{\theta}{2} + \frac{2F_1}{m_p}\sin^2\frac{\theta}{2}\right].$$

(d) A fixed-target ep scattering experiment consists of an electron beam of maximum energy 20 GeV and a variable angle spectrometer that can detect scattered electrons with energies greater than 2 GeV. Find the range of values of θ over which deep inelastic scattering events can be studied at $x = 0.2$ and $Q^2 = 2\,\text{GeV}^2$.

a)

Equation (8.11) gives:

$$\frac{d^2\sigma}{dxdQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[\left(1 - y - \frac{m_p^2 y^2}{Q^2}\right)\frac{F_2(x,Q^2)}{x} + y^2 F_1(xQ^2)\right] \tag{1}$$

for a differential cross-section, we may express $d\Omega$ as:

$$d\Omega = 2\pi d\cos\theta \tag{2}$$

in order to express this in terms of Q^2, we use the approximation of Q^2:

$$Q^2 \approx 2E_1E_3(1-\cos\theta) \tag{3}$$
$$\left|\frac{dQ^2}{d\cos\theta}\right| = 2E_1E_3 \tag{4}$$

thus we have:

$$\frac{d^2\sigma}{dE_3 d\Omega} = \frac{d^2\sigma}{dE_3(2\pi d\cos\theta)} \tag{5}$$
$$= \frac{1}{2\pi}\frac{d^2\sigma}{dE_3\left(\frac{dQ^2}{2E_1E_3}\right)} \tag{6}$$

$$\boxed{\frac{d^2\sigma}{dE_3 d\Omega} = \frac{E_1E_3}{\pi}\frac{d^2\sigma}{dE_3 dQ^2}} \tag{7}$$

we may rewrite dE_3 in terms of x by noting:

$$x = \frac{Q^2}{2m_p v} \tag{8}$$
$$v = \frac{Q^2}{2m_p x} \tag{9}$$
$$v = E_1 - E_3 \tag{10}$$
$$\frac{dv}{dE_3} = -1 \tag{11}$$
$$\frac{dv}{dx} = -\frac{Q^2}{2m_p x^2} \tag{12}$$
$$\tag{13}$$

we rewrite the RHS of the equation obtained earlier as:

$$\frac{E_1E_3}{\pi}\frac{d^2\sigma}{dE_3 dQ^2} = -\frac{E_1E_3}{\pi}\frac{d^2\sigma}{dv dQ^2} \tag{14}$$
$$= \frac{E_1E_3}{\pi}\frac{2m_p x^2}{Q^2}\frac{d^2\sigma}{dx dQ^2} \tag{15}$$

equating this to the LHS, we get:

$$\boxed{\frac{d^2\sigma}{dE_3 d\Omega} = \frac{E_1E_3}{\pi}\frac{2m_p x^2}{Q^2}\frac{d^2\sigma}{dx dQ^2}} \tag{16}$$

b)

We first aim to show that:

$$\frac{2m_p x^2}{Q^2} \cdot \frac{y^2}{2} = \frac{1}{m_p} \frac{E_3}{E_1} \sin^2 \frac{\theta}{2} \tag{17}$$

We note the approximation for Q^2 given negligible electron mass:

$$Q^2 \approx 2E_1E_3(1 - \cos\theta) = 4E_1E_3 \sin^2 \frac{\theta}{2} \tag{18}$$
$$\tag{19}$$

Rearranging terms, we have:

$$E_3 \sin^2 \frac{\theta}{2} = \frac{Q^2}{4E_1} \tag{20}$$
$$\tag{21}$$

We may divide both sides by E_1 to have:

$$\frac{E_3}{E_1} \sin^2 \frac{\theta}{2} = \frac{Q^2}{4E_1^2} \tag{22}$$
$$\tag{23}$$

We next note that in the frame where the proton is at rest (meaning $p_2 = (m_p, 0, 0, 0)$), the inelasticity y may be written as:

$$y = \frac{m_p(E_1 - E_3)}{m_p E_1} = 1 - \frac{E_3}{E_1} \tag{24}$$
$$1 - y = \frac{E_3}{E_1} \tag{25}$$

and v may be written as:

$$v = \frac{p_2 \cdot q}{m_p} = E_1 - E_3 \tag{26}$$

the relationship between y and v may then be described by:

$$y = \frac{v}{E_1} \tag{27}$$

With the relations between Q^2 and x given in Equation 8.6 of Thomson, we have:

$$x = \frac{Q^2}{2m_p v} \tag{28}$$

$$\tag{29}$$

using these on the LHS of Equation 17, we get:

$$\begin{aligned}
\frac{2m_p x^2}{Q^2} \cdot \frac{y^2}{2} &= \frac{2m_p}{Q^2}\left(\frac{Q^2}{2m_p v}\right)^2 \cdot \frac{y^2}{2} && (30)\\
&= \frac{Q^2}{2m_p v^2} \cdot \frac{y^2}{2} && (31)\\
&= \frac{Q^2}{4m_p}\left(\frac{1}{E_1^2}\right) && (32)\\
&= \frac{1}{4m_p}\left(4E_1E_3\sin^2\frac{\theta}{2}\right)\left(\frac{1}{E_1^2}\right) && (33)\\
& && (34)
\end{aligned}$$

Finally, we get:

$$\boxed{\frac{2m_p x^2}{Q^2} \cdot \frac{y^2}{2} = \frac{1}{m_p}\frac{E_3}{E_1}\sin^2\frac{\theta}{2}} \tag{35}$$

The next equation we prove is:

$$1 - y - \frac{m_p^2 x^2 y^2}{Q^2} = \frac{E_3}{E_1}\cos^2\frac{\theta}{2} \tag{36}$$

expanding the RHS, we get:

$$\begin{aligned}
\frac{E_3}{E_1}\cos^2\frac{\theta}{2} &= \frac{E_3}{E_1}\left[1-\sin^2\frac{\theta}{2}\right] && (37)\\
&= \left[\frac{E_3}{E_1}-\frac{E_3}{E_1}\sin^2\frac{\theta}{2}\right]\frac{m_p}{m_p} && (38)\\
&= \frac{m_pE_3}{m_pE_1}-\frac{m_p}{m_p}\frac{E_3}{E_1}\sin^2\frac{\theta}{2} && (39)\\
&= \frac{E_3}{E_1}-\left(\frac{m_p^2x^2y^2}{Q^2}\right) && (40)\\
& && (41)
\end{aligned}$$

where in the last step we used the recently obtained Equation 35. This leads us to:

$$\boxed{1-y-\frac{m_p^2x^2y^2}{Q^2}=\frac{E_3}{E_1}\cos^2\frac{\theta}{2}} \tag{42}$$

c)

Equation (8.11) gives:

$$\frac{d^2\sigma}{dxdQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[\left(1 - y - \frac{m_p^2 y^2}{Q^2}\right)\frac{F_2(x,Q^2)}{x} + y^2 F_1(x,Q^2)\right] \tag{43}$$

Recalling Equation 16, we get:

$$\frac{d^2\sigma}{dE_3 d\Omega} = \frac{E_1 E_3}{\pi}\frac{2m_p x^2}{Q^2}\frac{4\pi\alpha^2}{Q^4}\left[\left(1 - y - \frac{m_p^2 y^2}{Q^2}\right)\frac{F_2(x,Q^2)}{x} + y^2 F_1(x,Q^2)\right] \tag{44}$$

recalling Equation 42, we get:

$$\frac{d^2\sigma}{dE_3 d\Omega} = \frac{E_1 E_3}{\pi}\frac{2m_p x^2}{Q^2}\frac{4\pi\alpha^2}{Q^4}\left[\left(\frac{E_3}{E_1}\cos^2\frac{\theta}{2}\right)\frac{F_2(x,Q^2)}{x} + y^2 F_1(x,Q^2)\right] \tag{45}$$

distributing $\frac{x^2}{Q^2}$ in the bracket terms (and shortening $F(x,Q^2) \to F$), this becomes:

$$\frac{d^2\sigma}{dE_3 d\Omega} = \frac{8m_p E_1 E_3 \alpha^2}{Q^4}\left[\left(\frac{E_3}{E_1}\cos^2\frac{\theta}{2}\right)\frac{xF_2}{Q^2} + y^2\frac{x^2 F_1}{Q^2}\right] \tag{46}$$

recalling Equation 35, we get:

$$\frac{d^2\sigma}{dE_3 d\Omega} = \frac{8m_p E_1 E_3 \alpha^2}{Q^4}\left[\left(\frac{E_3}{E_1}\cos^2\frac{\theta}{2}\right)\frac{xF_2}{Q^2} + \left(\frac{Q^2 E_3}{m_p^2 E_1 x^2}\sin^2\frac{\theta}{2}\right)\frac{x^2 F_1}{Q^2}\right] \tag{47}$$

$$= \frac{8m_p E_3^2 \alpha^2}{Q^4}\left[\left(\cos^2\frac{\theta}{2}\right)\frac{xF_2}{Q^2} + \left(\frac{1}{m_p^2}\sin^2\frac{\theta}{2}\right)F_1\right] \tag{48}$$

$$\tag{49}$$

from the definition of x we note that:

$$x = \frac{Q^2}{2m_p v} \tag{50}$$

$$\frac{x}{Q^2} = \frac{1}{2m_p v} \tag{51}$$

plugging this, we get:

$$\frac{d^2\sigma}{dE_3 d\Omega} = \frac{8m_p E_3^2 \alpha^2}{Q^4}\left[\left(\frac{1}{2m_p v}\cos^2\frac{\theta}{2}\right)F_2 + \left(\frac{1}{m_p^2}\sin^2\frac{\theta}{2}\right)F_1\right] \quad (52)$$

$$= \frac{4E_3^2\alpha^2}{Q^4}\left[\left(\frac{1}{v}\cos^2\frac{\theta}{2}\right)F_2 + \left(\frac{2}{m_p}\sin^2\frac{\theta}{2}\right)F_1\right] \quad (53)$$

$$(54)$$

rewriting $Q^2 = 4E_1E_3\sin^2\frac{\theta}{2}$ this becomes:

$$\frac{d^2\sigma}{dE_3 d\Omega} = \frac{4E_3^2\alpha^2}{16E_1^2E_3^2\sin^4\theta/2}\left[\left(\frac{1}{v}\cos^2\frac{\theta}{2}\right)F_2 + \left(\frac{2}{m_p}\sin^2\frac{\theta}{2}\right)F_1\right] \quad (55)$$

$$(56)$$

$$\boxed{\frac{d^2\sigma}{dE_3 d\Omega} = \frac{\alpha^2}{4E_1^2\sin^4\theta/2}\left[\frac{F_2}{v}\cos^2\frac{\theta}{2} + \frac{2F_1}{m_p}\sin^2\frac{\theta}{2}\right]} \quad (57)$$

d)

Using the given $x = 0.2$ and $Q^2 = 2$ GeV^2, and with proton mass $m_p = 0.938$ GeV$/c^2 = 0.938$ GeV, we may get v:

$$v = \frac{Q^2}{2m_p x} \tag{58}$$

$$= \frac{2 \text{ GeV}^2}{2(0.938 \text{ GeV})(0.2)} = 5.33 \text{ GeV} \tag{59}$$

With v being the energy lost by the electron in the frame where the initial-state proton is at rest, then with $v = E_1 - E_3$ we have:

$$E_1 - E_3 = 5.33 \text{ GeV} \tag{60}$$

in order to get possible values for θ, we recall:

$$Q^2 = 4E_1E_3 \sin^2 \frac{\theta}{2} \tag{61}$$

$$\sin^2 \frac{\theta}{2} = \frac{Q^2}{4E_1E_3} \tag{62}$$

we note that since v must be positive, then $E_1 > E_3$, and that E_1 would correspond to maximum beam energy and E_3 would correspond to minimum beam energy. Starting with the given $E_1 =$ 20 GeV, we find one extremum value of θ, noting Equation 60 to get the value of E_3:

$$\sin^2 \frac{\theta}{2} = \frac{Q^2}{4E_1E_3} \tag{63}$$

$$= \frac{2 \text{ GeV}}{4(20 \text{ GeV})(14.67 \text{ GeV})} = 1.70 \times 10^{-3} \tag{64}$$

$$\tag{65}$$

getting θ, we have:

$$\theta_{E_1} = 2 \arcsin\left(\sqrt{1.70 \times 10^{-3}}\right) \tag{66}$$

$$\theta_{E_1} = 4.73^o \tag{67}$$

doing the same for the minimum detectable value $E_3 = 2$ GeV we have:

$$\sin^2\frac{\theta}{2} = \frac{Q^2}{4E_1E_3} \tag{68}$$
$$= \frac{2\text{ GeV}}{4(7.33\text{ GeV})(2\text{ GeV})} = 3.41\times10^{-2} \tag{69}$$
$$\tag{70}$$

$$\theta_{E_3} = 2\arcsin\left(\sqrt{3.41\times10^{-2}}\right) \tag{71}$$
$$\theta_{E_3} = 21.28^o \tag{72}$$

we may then treat θ_{E_1} and θ_{E_3} as the minimum and maximum values of θ, so we get the possible θ values of:

$$\boxed{4.73^o \leq \theta \leq 21.28^o} \tag{73}$$