SOLUTIONS TO ALL PROBLEMS (FOR INSTRUCTORS)

Note: Accurate numerical values of the proton and neutron masses and many other quantities are given in the Table of Physical Constants on the inside of the rear cover of this book. Values of atomic masses are available online, for example at https://www.webelements.com.

PROBLEMS 1

1.1 (a) The nucleus contains Z protons and $(A-Z)$ neutrons and there are Z orbital electrons. Hence the neutral atom is a bound state of a total of $(A+Z)$ fermions. It therefore has integer spin, and hence is a boson, if $(A+Z)$ is even; and non-integer spin and is a fermion if $(A+Z)$ is odd.

(b) In the proton-electron model of the nucleus, it must contain A protons to give the correct mass number and $(A-Z)$ electrons to give the correct nuclear charge $+Ze$. Together with the Z orbital electrons, this gives a total of $A+(A-Z)+Z=2A$ fermions, so that all neutral atoms would be bosons, in contradiction to experiment.

1.2 Using the relations (1.11), gives

$$\hat{P}Y_1^1 = \sqrt{\frac{3}{8\pi}}\sin(\pi-\theta)e^{i(\pi+\phi)} = -\sqrt{\frac{3}{8\pi}}\sin(\theta)e^{i\phi} = -Y_1^1,$$

and hence Y_1^1 is an eigenfunction of parity with eigenvalue -1.

1.3 Because the initial state is at rest, it has $L=0$ and thus its parity is $P_i = P_p P_{\bar{p}}(-1)^L = -1$, where we have used the fact that the fermion-antifermion pair has overall negative intrinsic parity. In the final state, the neutral pions are identical bosons and so their wavefunction must be totally symmetric under their interchange. This implies even orbital angular momentum L' between them and hence $P_f = P_\pi^2(-1)^{L'} = 1 \neq P_i$. The reaction does not conserve parity and is thus forbidden as a strong interaction.

1.4 Since $\hat{C}^2 = 1$, we must have

$$\hat{C}^2\left|b,\psi_b\right\rangle = C_b\hat{C}\left|\bar{b},\psi_{\bar{b}}\right\rangle = \left|b,\psi_b\right\rangle,$$

implying that

$$\hat{C}\left|\bar{b},\psi_{\bar{b}}\right\rangle = C_{\bar{b}}\left|b,\psi_b\right\rangle$$

with $C_b C_{\bar{b}} = 1$ independent of C_b. The result follows because an eigenstate of $\hat{C}$ must contain only particle-antiparticle pairs $b\bar{b}$, leading to the intrinsic parity factor $C_b C_{\bar{b}} = 1$, independent of C_b.

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Companion website: www.wiley.com/go/martin/nuclear3

1.5 (a) Because electric charges change sign, we have $\hat{C}\,\mathbf{E}(\mathbf{r},t) = -\mathbf{E}(\mathbf{r},t)$, Thus, if $\mathbf{E} = -\partial\mathbf{A}/\partial t$ is invariant under charge conjugation, it follows directly that $C_\gamma = -1$.

(b) From the invariance of Maxwell's equations,

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{1}{\varepsilon_0}\rho, \quad \boldsymbol{\nabla}\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t},$$

$\mathbf{B}\to\mathbf{B}$ and $\mathbf{B}\to-\mathbf{B}$ under P and T, respectively.

1.6 The interaction of an electric dipole moment **d** with an electric field **E** is of the form $-\mathbf{d}\cdot\mathbf{E}$. But the only vector available to define a direction is **J**, and so **d** must be collinear with **J**, and we can write the Hamiltonian as

$$H = -\mathbf{d}\cdot\mathbf{E} = -\frac{d}{J}\mathbf{J}\cdot\mathbf{E}\;,$$

where d and J are the moduli of their respective vector quantities. However, **E** is even under time reversal, whereas **J** is odd, so H is also odd and therefore a non-zero value of d can only exist if time reversal invariance is violated.

1.7 From (1.73) and (1.74), we have

$$R = \frac{\mathrm{d}\sigma(pp\to\pi^+d)/\mathrm{d}\Omega}{\mathrm{d}\sigma(\pi^+d\to pp)/\mathrm{d}\Omega} = \frac{(2s_\pi+1)(2s_d+1)}{(2s_p+1)^2}\,\frac{p_\pi^2}{p_p^2}\,\frac{\overline{\left|\mathcal{M}_{if}\right|^2}}{\overline{\left|\mathcal{M}_{fi}\right|^2}}\,.$$

By detailed balance (1.27), the two spin-averaged squared matrix elements are equal, and so, using $S_p = 1/2$ and $S_d = 1$, gives the result.

1.8 The parity of the deuteron is $P_d = P_pP_n(-1)^{L_{pn}}$. Since the deuteron is an S-wave bound state, $L_{pn} = 0$ and so, using $P_p = P_n = 1$, gives $P_d = 1$. The parity of the initial state is therefore

$$P_i = P_{\pi^-}P_d(-1)^{L_{\pi d}} = P_{\pi^-},$$

because the pion is at rest, and so $L_{\pi d} = 0$. The parity of the final state is

$$P_f = P_nP_n(-1)^{L_{nn}} = (-1)^{L_{nn}}$$

and therefore $P_{\pi^-} = (-1)^{L_{nn}}$. The value of L_{nn} may be found by imposing the Pauli principle condition that $\psi_{nn} = \psi_{\text{space}}\psi_{\text{spin}}$ must be antisymmetric, together with angular momentum conservation. Examining the spin wavefunctions (1.17) shows that there are two possibilities for ψ_{spin}: either the symmetric $S=1$ state or the $S=0$

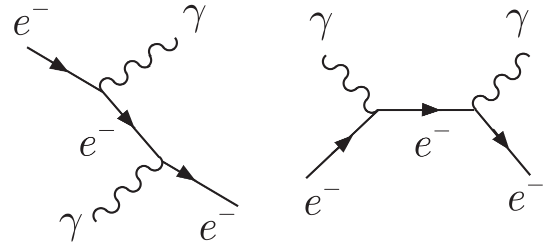
antisymmetric state. If $S = 0$, then ψ_{space} would have to be symmetric, implying L_{nn}, and hence the total angular momentum in the final state would be even. But the total angular momentum would not then be conserved, since in the initial state the deuteron has spin-1 and there is no orbital angular momentum. Thus $S = 1$ is implied and ψ_{space} is antisymmetric, i.e. $L_{nn} = 1, 3, \cdots$ The only way to combine L_{nn} and S to give $J = 1$ is with $L_{nn} = 1$ and hence $P_{\pi^-} = -1$.

1.9 (a) $\bar{\nu}_e + e^+ \to \bar{\nu}_e + e^+$;

(b) $p + p \to p + p + \pi^0 + \pi^0$;

(c) $\bar{p} + n \to \pi^- + \pi^0 + \pi^0, \quad \pi^- + \pi^+ + \pi^-$.

1.10 (a)



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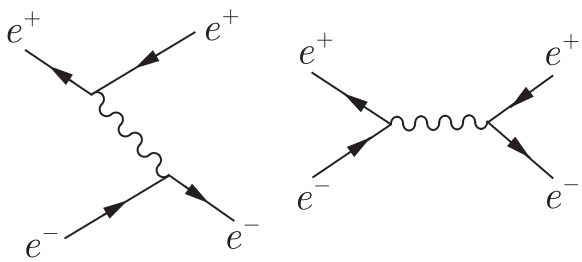

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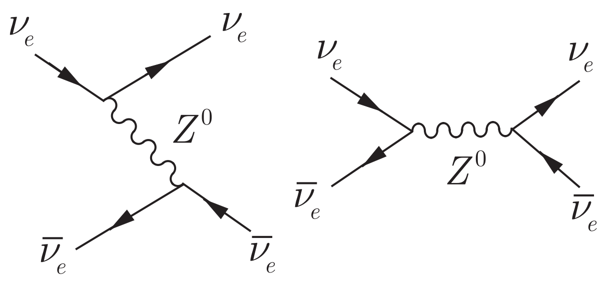
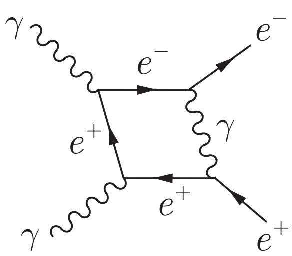


1.11 (a)



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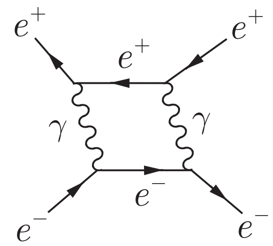


1.12 For a spherically symmetric static solution we can set $\Psi(r,t) = \phi(r) = \phi$, where $r = |\mathbf{r}|$, and use

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r},$$

giving

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r} = \left(\frac{mc}{\hbar}\right)^2 \phi .$$

Substituting $\phi = u(r)/r$ gives

$$\frac{\mathrm{d}^2 u(r)}{\mathrm{d}r^2} = \left(\frac{mc}{\hbar}\right)^2 u(r)$$

and the result follows by solving for u, and imposing $\phi \to 0$ as $r \to \infty$.

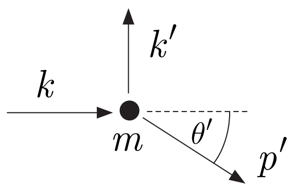
1.13 Using spherical polar co-ordinates, we have

$$\mathbf{q}\cdot\mathbf{r} = qr\cos\theta \quad \text{and} \quad \mathrm{d}^3\mathbf{r} = r^2 \mathrm{d}r\,\mathrm{d}\cos\theta\,\mathrm{d}\phi,$$

where $q = |\mathbf{q}|$. Thus, from (1.47),

$$\mathcal{M}(q^2) = \frac{-g^2}{4\pi}\int_0^{2\pi}\mathrm{d}\phi\int_0^{\infty} r^2\frac{e^{-r/R}}{r}\mathrm{d}r\int_{-1}^{+1}\exp(iqr\cos\theta/\hbar)\,\mathrm{d}\cos\theta = \frac{-g^2\hbar^2}{q^2 + m^2c^2},$$

1.14 The kinematics of the scattering process are



where k' and p' are the magnitudes of the momenta of the final-state electron and proton, respectively. By momentum conservation,

$$k = p'\cos\theta' \tag{1}$$

and

$$k' = p' \sin\theta' \tag{2}$$

implying

$$p'^2 = k^2 + k'^2 \tag{3}$$

and

$$k' = k \tan\theta' \tag{4}$$

By energy conservation,

$$mc^2 + kc = k'c + (p'^2c^2 + m^2c^4)^{1/2} \quad ,$$

where we have neglected electron masses, so that

$$\begin{aligned} p'^2c^2 + m^2c^4 &= (mc^2 + kc - k'c)^2 \\ &= m^2c^4 + k^2c^2 + k'^2c^2 + 2mc^2kc - 2mc^2k'c - 2kk'c, \end{aligned}$$

which using (3) gives

$$0 = 2mc^2kc - 2mc^2k'c - 2kk'c^2 .$$

Hence

$$k' = \left(\frac{mc}{mc + k}\right)k$$

and by (4)

$$\theta' = \text{atan}\left(\frac{k'}{k}\right) = \text{atan}\left(\frac{mc}{mc + k}\right).$$

1.15 From (1.57c), $\sigma = WM_A \big/ I(\rho t)N_A$. Since the scattering is isotropic, the total number of protons emitted from the target is

$$W = 20 \times (4\pi \big/ 2 \times 10^{-3}) = 1.25 \times 10^5 \ \text{s}^{-1} .$$

I can be calculated from the current, noting that the alpha particles carry two units of charge, and is $I = 3.13 \times 10^{10}\ \text{s}^{-1}$. The density of the target is $\rho t = 10^{-32}\ \text{kg fm}^{-2}$. Putting everything together gives $\sigma = 161\ \text{mb}$.

1.16 Write

$$\sigma = \frac{8\pi\alpha^2}{3m_e^2}\hbar^a c^b$$

and impose the dimensional condition $[\sigma] = [L]^2$. This gives $a = -b = 2$ and hence

$$\sigma = \frac{8\pi\alpha^2(\hbar c)^2}{3(m_e c^2)^2} .$$

Evaluating this gives $\sigma = 6.65 \times 10^{-29}\ \text{m}^2 = 665\ \text{mb}$.

PROBLEMS 2

2.1 From (2.22), the precision P is proportional to $(t\sqrt{N})^{-1}$, where $t = T_{\mathrm{obs}}$. For radioisotopes, $N \propto \exp(-\lambda t)$, where $\lambda = \ln 2 / t_{1/2}$, and $t_{1/2}$ is the half-life. Thus,

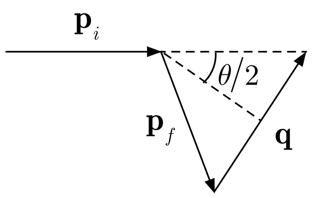
$$P \propto \frac{1}{t}\exp\left(\frac{t\ln 2}{2t_{1/2}}\right),$$

which has a maximum at $t/t_{1/2} \approx 2.9$.

2.2 From (2.32), the form factor is

$$F(\mathbf{q}^2) = 3\left[\sin b(a) - b(a)\cos b(a)\right]b^{-3},$$

where $b = qa/\hbar$. To evaluate this we need to find a and q. We can find the latter from the figure below.



Using $p = |\mathbf{p}_i| = |\mathbf{p}_f|$ and $q = |\mathbf{q}|$, gives

$$q = 2p\sin(\theta/2) = 57.5\,\mathrm{MeV}/c.$$

Also, we know that $a = 1.21A^{1/3}\,\mathrm{fm}$ and so for $A = 56$, $a = 4.63\ \mathrm{fm}$, and $qa/\hbar = 1.35$ radians. Finally, using this in the expression for F, gives $F = 0.829$ and hence the reduction is $F^2 = 0.69$.

2.3 Setting $q = |\mathbf{q}|$ in (2.37), we have

$$F(\mathbf{q}^2) = \frac{1}{Ze}\int f(\mathbf{r})\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{iqr\cos\theta}{\hbar}\right)^n \mathrm{d}^3\mathbf{r}.$$

Using $\mathrm{d}^3\mathbf{r} = r^2\mathrm{d}\cos\theta\,\mathrm{d}\phi\,\mathrm{d}r$ and doing the ϕ integral, gives

$$\begin{aligned} F(\mathbf{q}^2) &= \frac{2\pi}{Ze}\iint f(r)r^2\left[1 + \frac{iqr\cos\theta}{\hbar} - \frac{q^2r^2\cos^2\theta}{2\hbar^2} + \ldots\right]\mathrm{d}r\,\mathrm{d}\cos\theta \\ &= \frac{4\pi}{Ze}\int_0^{\infty} f(r)r^2\mathrm{d}r - \frac{4\pi q^2}{6Ze\hbar^2}\int_0^{\infty} f(r)r^4\mathrm{d}r + \ldots \end{aligned}$$

But from (2.23),

$$Ze = 4\pi\int_0^\infty f(r)r^2\mathrm{d}r$$

and from (2.36)

$$Ze\left\langle r^2\right\rangle = 4\pi\int_0^\infty f(r)r^4\mathrm{d}r\,,$$

so

$$F(\mathbf{q}^2) = 1 - \frac{\mathbf{q}^2}{6\hbar^2}\left\langle r^2\right\rangle + \cdots$$

2.4 From (2.39),

$$\left\langle r^2\right\rangle = 6\hbar^2\left[1 - F(q^2)\right]\Big/q^2\,,$$

where $q = 2(E/c)\sin(\theta/2) = 43.6\,\mathrm{MeV}/c^2$. Also, $F^2 = 0.65$ and so

$$\sqrt{\left\langle r^2\right\rangle} = 4.9\ \mathrm{fm}\,.$$

2.5 The charge distribution is spherically symmetric, so the angular integrations in the general result (2.27) may be done, giving (2.28), and

$$F(\mathbf{q}^2) \propto \frac{1}{q}\int_0^\infty r\rho(r)\sin\left(qr/\hbar\right)\mathrm{d}r\,,$$

which using

$$\rho(r) = \rho_0\exp(-r/a)/r$$

gives

$$F(\mathbf{q}^2) \propto \frac{\rho_0}{q}\int_0^\infty \exp(-r/a)\sin(qr/\hbar)\mathrm{d}r\,.$$

The integral may be evaluated by twice integrating by parts, giving

$$F(\mathbf{q}^2) \propto \frac{1}{1 + a^2q^2/\hbar^2}\,.$$

2.6 In 1g of the isotope there are initially

$$N_0 = (1\mathrm{g}/208)\times N_A = 2.9\times10^{21}\ \text{atoms},$$

where $N_A = 6.02\times10^{23}$ is Avogadro's number. At time t there are $N(t) = N_0e^{-t/\tau}$ atoms, where τ is the mean life of the isotope. Provided $t \ll \tau$, the average number of counts per hour, given that the detector is only 10% efficient, is

$$\frac{N_0 - N(t)}{t} \approx \frac{N_0}{\tau} = \frac{75}{0.1\times24}\mathrm{hr}^{-1}\,.$$

Thus, $\tau = 2.4N_0/75\,\text{hr} \approx 1.06\times10^{16}\,\text{yr}$.

2.7 If the transition rate for ${}^{212}_{86}\text{Rn}$ decay is λ_1 and that for ${}^{208}_{84}\text{Po}$ is λ_2 and if the numbers of each of these atoms at time t is $N_1(t)$ and $N_2(t)$, respectively, then the decays are governed by (2.67), i.e.

$$N_2(t) = \lambda_1 N_1(0)\left[\exp(-\lambda_1 t) - \exp(-\lambda_2 t)\right]\left(\lambda_2 - \lambda_1\right)^{-1}.$$

This is a maximum when $\mathrm{d}N_2(t)/\mathrm{d}t = 0$, i.e. when

$$\lambda_2 \exp(-\lambda_2 t) = \lambda_1 \exp(-\lambda_1 t),$$

with

$$t_{\text{max}} = \ln\left(\lambda_1/\lambda_2\right)\left(\lambda_1 - \lambda_2\right)^{-1}.$$

Using

$$\lambda_1 = 4.18\times10^{-2}\,\text{min}^{-1} \quad \text{and} \quad \lambda_2 = 6.56\times10^{-7}\,\text{min}^{-1},$$

gives $t_{\text{max}} = 265\,\text{min}$.

2.8 The total decay rate of both modes of ${}^{138}_{57}\text{La}$ is

$$(1+0.5)\times(7.8\times10^2)\,\text{kg}^{-1}\text{s}^{-1} = 1.17\times10^3\,\text{kg}^{-1}\text{s}^{-1}.$$

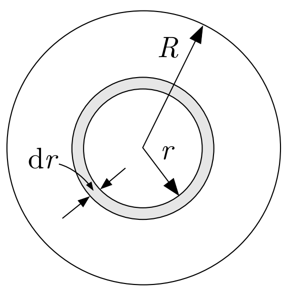
Also, since this isotope is only 0.09% of natural lanthanum, the number of ${}^{138}_{57}\text{La}$ atoms/kg is

$$N = (9\times10^{-4})\times\left(1000/138.91\right)\times(6.022\times10^{23}) = 3.90\times10^{21}\,\text{kg}^{-1}.$$

The rate of decays is $-\mathrm{d}N/\mathrm{d}t = \lambda N$, where λ is the transition rate and in terms of this, the mean lifetime $\tau = 1/\lambda$. Thus,

$$\tau = \frac{N}{-\mathrm{d}N/\mathrm{d}t} = 3.33\times10^{18}\,\text{s} = 1.06\times10^{11}\,\text{yr}.$$

2.9 Consider a sphere of radius R with uniform charge density $\rho = (3Ze/4\pi R^3)$. Then, referring to the figure below, the charge



enclosed inside the inner sphere of radius r is Zer^3/R^3, and the charge contained in the shell of thickness dr is $4\pi r\rho \mathrm{d}r$. The contribution to the self-energy is then

$$\frac{Zer^3}{R^3}\frac{1}{4\pi\varepsilon_0 r}4\pi r^2\rho \mathrm{d}r = \frac{3Z^2e^2r^4}{4\pi\varepsilon_0 R^6}\mathrm{d}r$$

and integrating from 0 to R gives

$$E = \frac{3}{5}\frac{Z^2e^2}{4\pi\varepsilon_0 R} = \frac{1}{2}\frac{Z^2}{A^{1/3}}\alpha\hbar c = \frac{aZ^2}{A^{1/3}},$$

where

$$a = (\alpha\hbar c)/(2\,\mathrm{fm}) = 0.720\,\mathrm{MeV}.$$

The corresponding contribution to the mass is $0.720\,\mathrm{MeV}/c^2$, compared to the empirical value $a_c = 0.697\,\mathrm{MeV}/c^2$ given in (2.57) obtained from a fit of the semi-empirical mass formula to a range of real nuclei with $A>20$.

2.10 The energy released is the increase in binding energy. Now from the SEMF,

$$BE(35,87) = a_v(87) - a_s(87)^{2/3} - a_c\frac{(35)^2}{(87)^{1/3}} - a_a\frac{(87-70)^2}{348},$$

$$BE(57,145) = a_v(145) - a_s(145)^{2/3} - a_c\frac{(57)^2}{(145)^{1/3}} - a_a\frac{(145-114)^2}{580},$$

$$BE(92,235) = a_v(235) - a_s(235)^{2/3} - a_c\frac{(92)^2}{(235)^{1/3}} - a_a\frac{(235-184)^2}{940}.$$

The energy released is thus

$$\begin{aligned} E &= BE(35,87) + BE(57,145) - BE(92,235) \\ &= -3a_v - 9.153a_s + 476.7a_c + 0.280\,a_a \end{aligned}$$

which, using the values given in (2.57), gives $E = 154\,\mathrm{MeV}$.

2.11 The most stable nucleus for fixed A has a Z-value given by i.e. $Z = \beta/2\gamma$, where from (2.71),

$$\beta = a_a + (M_n - M_p - m_e) \quad \text{and} \quad \gamma = a_a/A + a_c/A^{1/3}.$$

Changing α would not change a_a, but would affect the Coulomb coefficient because a_c is proportional to α. For $A = 111$, using the value of a_a from (2.57) gives

$$\beta = 93.93 \ \text{MeV/c}^2 \quad \text{and} \quad \gamma = 0.839 + 0.208\,a_c \ \text{MeV/c}^2 \, .$$

For $Z = 47$, $a_c = 0.770 \ \text{MeV/c}^2$. This is a change of about 10% from the value given in (2.57) and so α would have to change by the same percentage.

2.12 In the rest frame of the ${}^{269}_{108}\text{Hs}$ nucleus, $m_\alpha \upsilon_\alpha = m_{\text{Sg}} \upsilon_{\text{Sg}}$, where υ is the appropriate velocity. The ratio of the kinetic energies in the centre-of-mass is therefore $E_{\text{Sg}} / E_\alpha = m_\alpha / m_{\text{Sg}}$ and the total kinetic energy is

$$E_\alpha \left(1 + m_\alpha / m_{\text{Sg}}\right) = 9.369 \ \text{MeV} \, .$$

Thus,

$$m_{\text{Hs}} c^2 = [(m_{\text{Sg}} + m_\alpha)c^2 + 9.369] \ \text{MeV} = 269.13 \ \text{u} \, .$$

2.13 If there are N_0 atoms of ${}^{238}_{94}\text{Pu}$ at launch, then after t years the activity of the source will be $\mathcal{A}(t) = N_0 \exp(-t/\tau)/\tau$, where τ is the lifetime. The instantaneous power is then

$$P(t) = \mathcal{A}(t) \times \text{efficiency} \times \text{energy released}$$

which, converting to Watts, is

$$P(t) = \mathcal{A}(t) \times 0.05 \times (5.49 \times 10^6) \times 1.602 \times 10^{-19} \ \text{W} > 200 \ W$$

and hence $\mathcal{A}(t) > 4.55 \times 10^{15} \text{s}^{-1}$. Substituting the value given for τ , gives $N_0 = 1.88 \times 10^{25}$ and hence the weight of ${}^{238}_{94}\text{Pu}$ at launch would have to be at least

$$\left(\frac{1.88 \times 10^{25}}{6.02 \times 10^{23}}\right)\left(\frac{238}{1000}\right) \text{kg} = 7.43 \ \text{kg} \, .$$

2.14 We first calculate the mass difference between $\left(p + {}^{46}_{21}\text{Sc}\right)$ and $\left(n + {}^{46}_{22}\text{Ti}\right)$. Using the information given, we have

$$M(21,46) - [M(22,46) + m_e] = 2.37 \ \text{MeV/c}^2 \ \text{ and } \ M_n - (M_p + m_e) = 0.78 \ \text{MeV/c}^2$$

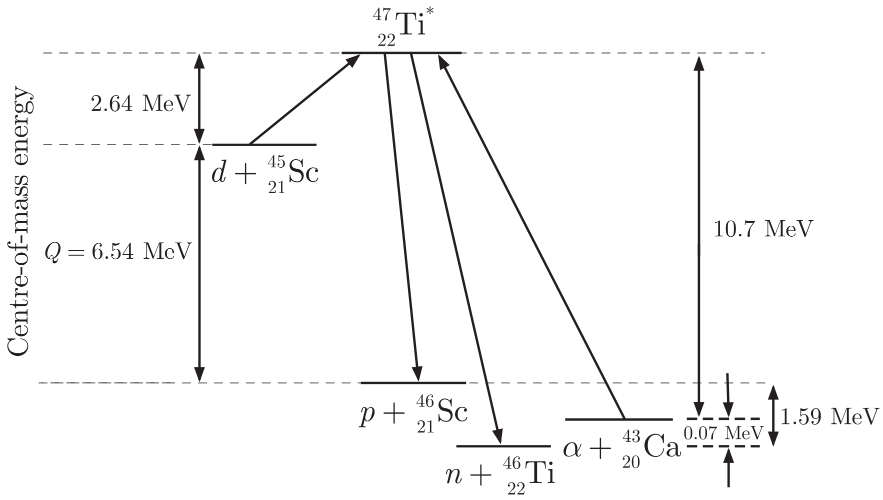
and hence

$$[M_p + M(21,46)] - [M_n + M(22,46)] = 1.59 \ \text{MeV/c}^2 \, .$$

We also need the mass differences

$$[M_\alpha + M(20,43)] - [M_n + M(22,46)] = 0.07 \ \text{MeV/c}^2 \, .$$

We can now draw the energy level diagram where the centre-of-mass energy of the resonance is (see (2.8)) $2.76 \times (45/47) = 2.64$ MeV.



Thus the resonance could be excited in the ${}^{43}_{20}\mathrm{Ca}(\alpha, n){}^{46}_{22}\mathrm{Ti}$ reaction at an α particle laboratory energy of $10.7 \times (47/43) = 11.7$ MeV.

2.15 We have $\mathrm{d}N(t)/\mathrm{d}t = P - \lambda N$, from which

$$Pe^{\lambda t} = e^{\lambda t}\left(\lambda N + \frac{\mathrm{d}N(t)}{\mathrm{d}t}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(Ne^{\lambda t}\right)$$

Integrating, and using the fact that $N = 0$ at $t = 0$ to determine the constant of integration, gives the required result.

2.16 The number of ${}^{35}\mathrm{Cl}$ atoms in 1 g of the natural chloride is

$$N = 2 \times 0.758 \times N_A / \text{molecular weight} = 7.04 \times 10^{21}.$$

From Problem 2.15, the activity is

$$\mathcal{A}(t) = \lambda N = P\left(1 - e^{-\lambda t}\right) \approx P\lambda t, \text{ since } \lambda t \ll 1.$$

So

$$t = \frac{\mathcal{A}(t)}{P\lambda} = \frac{\mathcal{A}(t) t_{1/2}}{\ln 2 \times \sigma \times F \times N}.$$

Substituting $\mathcal{A}(t) = 3 \times 10^5$ Bq and using the other constants given, yields $t = 1.55$ days.

2.17 At very low energies we may assume the scattering has $l = 0$ and so in (1.84) we have $j = 1/2$, $s_n = 1/2$ and $s_u = 0$. Thus,

$$\sigma_{\max} = \frac{\pi\hbar^2}{q_n^2}\frac{(\Gamma_n\Gamma_n + \Gamma_n\Gamma_\gamma)}{\Gamma^2/4} = \frac{4\pi\hbar^2\Gamma_n}{q_n^2\Gamma}\text{, since } \Gamma = \Gamma_n + \Gamma_\gamma .$$

Therefore,

$$\Gamma_n = q_n^2\Gamma\sigma_{\max}\big/4\pi\hbar^2 = 0.35\times10^{-3}\ \text{eV}$$

and

$$\Gamma_\gamma = \Gamma - \Gamma_n = 9.65\times10^{-3}\ \text{eV}.$$

PROBLEMS 3

3.1 (a) Forbidden: violates L_μ conservation, because $L_\mu(\nu_\mu) = 1$, but $L_\mu(\mu^+) = -1$.

(b) Forbidden: violates electric charge conservation, because Q (left-hand side) = 1, but Q (right-hand side) = 0

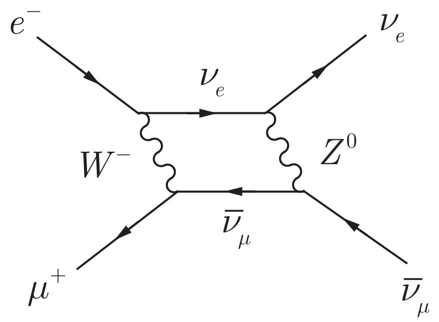
(c) Forbidden: violates baryon number conservation because B (left-hand side) = 1, but B (right-hand side) = 0

(d) Allowed as a weak interaction: conserves L_μ, B, Q etc., violates S. (It also has $\Delta S = \Delta Q$ for the hadrons – this is discussed in Section 6.2.4.)

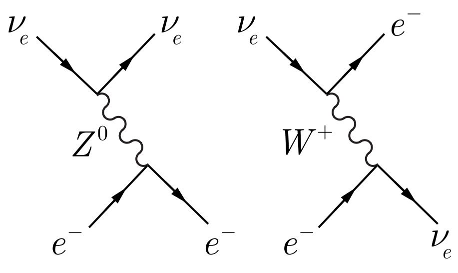
(e) Allowed as a weak interaction: conserves L_e, B, Q etc.

(f) Forbidden: violates L_μ and L_τ

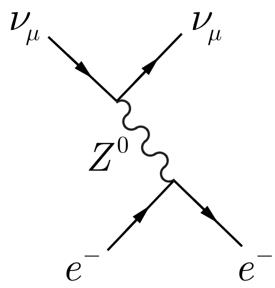
3.2 An example of a fourth-order Feynman diagram is:



3.3 The electron neutrino may interact with electrons via both Z^0 and W^- exchange:



whereas, because of lepton number conservation, the muon neutrino can only interact via Z^0 exchange:



3.4 From (3.31b) we have $L_0 = 4E(\hbar c)\big/\Delta m_{ij}^2 c^4$. Then if L_0 is expressed in km, E in GeV and Δm_{ij}^2 in $(\mathrm{eV/c^2})^2$, we have

$$L_0 = \frac{4E \times (1.97 \times 10^{-13}) \times 10^{18}}{\Delta m_{ij}^2} = \frac{E}{1.27 \Delta m_{ij}^2} \text{ km}.$$

3.5 From (3.31a), we have

$$P(\bar{\nu}_e \to \bar{\nu}_x) = \sin^2(2\theta)\sin^2[\Delta(m^2c^4)L\big/(4\hbar cE)],$$

which for maximal mixing ($\theta = \pi/4$) gives

$$P(\bar{\nu}_e \to \bar{\nu}_x) = \sin^2[1.27\,\Delta(m^2c^4)L\big/E],$$

where L is measured in metres, E in MeV, $\Delta(m^2c^4)$ in $(\mathrm{eV/c^2})^2$ and

$$\Delta m^2 \equiv m^2(\bar{v}_e) - m^2(\bar{v}_x).$$

If $P(\bar{\nu}_e \to \bar{\nu}_e) = 0.9 \pm 0.1$, then $0.3 \geq P(\bar{\nu}_e \to \bar{\nu}_x) \geq 0$ at 95% confidence level, and hence

$$0 \leq \Delta(m^2c^4) \leq 6.9 \times 10^{-3}\ (\mathrm{eV/c^2})^2.$$

3.6 From the data given, the total number of nucleons is $N = 1.2 \times 10^{57}$ and hence $n = 8.3 \times 10^{38}\ \mathrm{km}^{-3}$. Also the mean energy of the neutrinos from reaction (3.38) is 0.26 MeV, so the cross-section is $\sigma = 1.8 \times 10^{-46}\ \mathrm{m}^2$. Thus 7×10^{12} km, i.e. about 10^7 times the solar radius.

3.7 The doublet of $S = +1$ mesons (K^+, K^0) has isospin $I = 1/2$, with $I_3(K^+) = 1/2$ and $I_3(K^0) = -1/2$. The triplet of $S = -1$ baryons $(\Sigma^+, \Sigma^0, \Sigma^-)$ has $I = 1$, with $I_3 = 1, 0, -1$ for Σ^+, Σ^0 and Σ^- , respectively. Thus (K^+, K^0) is analogous to the (p, n) isospin doublet and $(\Sigma^+, \Sigma^0, \Sigma^-)$ is analogous to the (π^+, π^0, π^-) isospin triplet. Hence, by analogy with (3.61a,b),

$$\mathcal{M}(\pi^- p \to \Sigma^- K^+) = \tfrac{1}{3}\mathcal{M}_{3/2} + \tfrac{2}{3}\mathcal{M}_{1/2}\,,$$

$$\mathcal{M}(\pi^- p \to \Sigma^0 K^0) = \tfrac{\sqrt{2}}{3}\mathcal{M}_{3/2} - \tfrac{\sqrt{2}}{3}\mathcal{M}_{1/2},$$

and

$$\mathcal{M}(\pi^+ p \to \Sigma^+ K^+) = \mathcal{M}_{3/2}\,,$$

where $\mathcal{M}_{1/2,3/2}$ are the amplitudes for scattering in a pure isospin state $I = 1/2, 3/2$, respectively. Thus,

$$\begin{aligned}&\sigma(\pi^+ p \to \Sigma^+ K^+) : \sigma(\pi^- p \to \Sigma^- K^+) : \sigma(\pi^- p \to \Sigma^0 K^o)\\ &\qquad = \left|\mathcal{M}_{3/2}\right|^2 : \tfrac{1}{9}\left|\mathcal{M}_{3/2} + 2\mathcal{M}_{1/2}\right|^2 : \tfrac{2}{9}\left|\mathcal{M}_{3/2} - \mathcal{M}_{1/2}\right|^2\end{aligned}$$

3.8 Under charge symmetry, $n(udd) \rightleftharpoons p(duu)$ and $\pi^+(u\bar{d}) \rightleftharpoons \pi^-(d\bar{u})$, and since the strong interaction is approximately charge symmetric, we would expect $\sigma(\pi^+ n) \approx \sigma(\pi^- p)$ at the same energy, with small violations due to electromagnetic effects and quark mass differences. However, $K^+(u\bar{s})$ and $K^-(s\bar{u})$ are not charge symmetric and so there is no reason why $\sigma(K^+ n)$ and $\sigma(K^- p)$ should be equal.

3.9 'Lowest-lying' implies that the internal orbital angular momentum between the quarks is zero. Hence the parity is $P = +$ and ψ_{space} is symmetric. Since the Pauli principle requires the overall wavefunction to be antisymmetric under the interchange of any pair of like quarks, it follows that ψ_{spin} is antisymmetric. Thus, any pair of like quarks must have antiparallel spins, i.e. be in a spin-0 state. Consider all possible baryon states qqq, where $q = u$, d, s. There are six combinations with a single like pair: uud, uus, ddu, dds, ssu, ssd, with the spin of uu etc equal to zero. Adding the spin of the third quark leads to six states with $J^P = 1/2^+$. In principle there could be six combinations with all three quarks the same: uuu, ddd, sss, but in practice these do not occur because it is impossible to arrange all three spins in an antisymmetric way. Finally, there is one combination where all three quarks are different: uds. Here there are no restrictions from the Pauli principle, so the ud pair could have spin 0 or spin 1. Adding the spin of the s quark leads to two states with $J^P = 1/2^+$ and one with $J^P = 3/2^+$. Collecting the results, gives an octet of $J^P = 1/2^+$ states and a singlet $J^P = 3/2^+$ state. This is *not* what is observed in nature. In Chapter 5 we discuss what additional assumptions have to be made to reproduce the observed spectrum.

3.10 The quantum number combination (2, 1, 0, 1, 0) corresponds to a baryon qqq. It has $S = 0$, $C = 1$ and $\widetilde{B} = 0$, so must be of the form cxy, where x and y are u or d quarks. The charge $Q = 2$ requires both x and y to be u quarks, i.e. cuu. The others are established by similar arguments and so the full set is:

$$(2,1,0,1,0) = cuu, \qquad (0,1,-2,1,0) = css, \qquad (0,0,1,0,-1) = b\bar{s},$$
$$(0,-1,1,0,0) = \bar{s}\bar{d}\bar{u}, \qquad (0,1,-1,1,0) = csd, \qquad (-1,1,-3,0,0) = sss.$$

They are called Σ_c^{++}, Ω_c^0, $\bar{B}_s^0$, $\bar{\Lambda}$, Σ_c^0 and Ω^-, respectively.

3.11 The ground state mesons all have $L=0$ and $S=0$. Therefore they all have $P=-1$. Only in the case of the neutral pion is their constituent quark and antiquark also particle and antiparticle. Thus C is only defined for the π^0 and is $C=1$ by (1.18). For the excited states, $L=0$ still and thus $P=-1$ as for the ground states. However, the total spin of the constituent quarks is $S=1$ and so for the ρ^0, the only state for which C is defined, $C=-1$.

For the excited states, by definition there is a lower mass configuration with the same quark flavours. As the mass differences between the excited states and their ground states is greater than the mass of a pion, they can all decay by the strong interaction. In the case of the charged pions, there are no lower mass hadron states with the same flavour, and so the only possibility is to decay via the weak interaction, with much longer lifetimes.

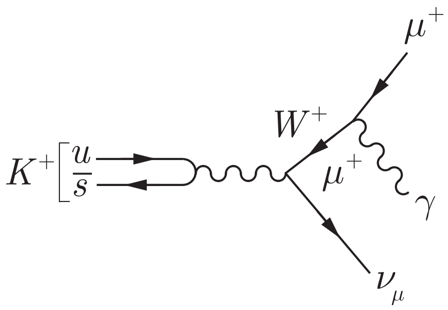
In the case of ρ^0 decay, the initial state has a total angular momentum of 1 and since the pions have zero spin, the $\pi\pi$ final state must have $L=1$. While this is possible for $\pi^+\pi^-$, for the case of $\pi^0\pi^0$ it violates the Pauli Principle and so is forbidden.

3.12 Reactions (a), (d) and (f) conserve all quark numbers individually and hence are strong interactions. Reaction (e) violates strangeness and is a weak interaction. Reaction (c) conserves strangeness and involves photons and hence is an electromagnetic interaction. Reaction (b) violates both baryon number and electron lepton number and is therefore forbidden.

3.13 In the initial state, $S=-1$ and $B=1$. To balance strangeness (conserved in strong interactions) in the final state $S(Y^-)=-2$ and to balance baryon number, $B(Y^-)=1$. As charm and bottom for the initial state are both zero, these quantum numbers are zero for the *Y*. The quark content is therefore *dss.* In the decay, the strangeness of the Λ is $S(\Lambda)=-1$ and so strangeness is not conserved. This is therefore a weak interaction and its lifetime will be in the range 10^{-7} to $10^{-13}\,\text{s}$.

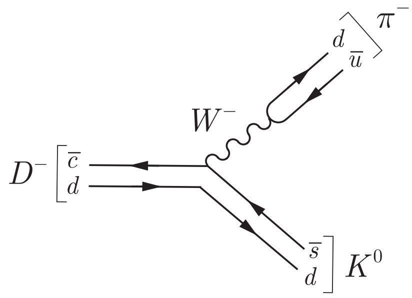
3.14 The initial reaction is strong because it conserves all individual quark numbers. The Ω^- decay is weak because strangeness changes by one unit and the same is true for the decays of the Ξ^0, K^+ and K^0. The decay of the π^+ is also weak because it involves neutrinos and finally the decay of the π^0 is electromagnetic because only photons are involved.

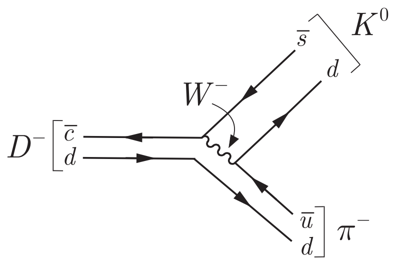
3.15 The Feynman diagram is:



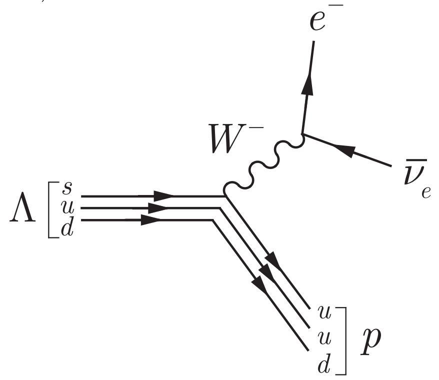
The two vertices where the W boson couples are weak interactions and have strengths $\sqrt{\alpha_W}$. The remaining vertex is electromagnetic and has strength $\sqrt{\alpha_{EM}}$. So the overall strength of the diagram is $\alpha_W\sqrt{\alpha_{EM}}$.

3.16 (a) The quark compositions are: $D^- = d\bar{c}$; $K^0 = d\bar{s}$; $\pi^- = d\bar{u}$ and since the dominant decay of a c-quark is $c \to s$, we have





(b) The quark compositions are: $\Lambda = sud$; $p = uud$ and since the dominant decay of an s-quark is $s \to u$, we have



3.17 In the $\Sigma^0(1193) = uds$, the ud pair is in a spin-1 state, as discussed in Section 3.2.2. This pair has a magnetic moment $\mu_u + \mu_d$, since the u and d spins are parallel. An argument similar to that given for the magnetic moments of baryons $B = aab$ in Section 3.3.4(a) then gives

$$\mu_{\Sigma^0} = \tfrac{2}{3}(\mu_u + \mu_d) - \tfrac{1}{3}\mu_s = 0.83\mu_N$$

where μ_N is the nuclear magneton and we have used the quark masses given in (3.81). (Unfortunately, this prediction cannot easily be checked experimentally because the Σ^0 is too short-lived.)

In the $\Omega^-(1672)$, the three s quarks must have parallel spins to give $J = 3/2$, so that the magnetic moment $\mu_{\Omega^-} = 3\mu_s = -1.83\mu_N$. The measured value is $-2.02 \pm 0.05\mu_N$.

3.18 The quark composition is $\Sigma = uds$. Now

$$(\mathbf{S}_u + \mathbf{S}_d)^2 = \mathbf{S}_u^2 + \mathbf{S}_d^2 + 2\mathbf{S}_u \cdot \mathbf{S}_d = 2\hbar^2$$

and hence

$$\mathbf{S}_u \cdot \mathbf{S}_d = \hbar^2/4.$$

Then, from the general formula (3.91), setting $m_u = m_d = m$, we have

$$\begin{aligned} M_\Sigma &= 2m + m_s + \frac{b}{\hbar^2}\left[\frac{\mathbf{S}_u \cdot \mathbf{S}_d}{m^2} + \frac{\mathbf{S}_d \cdot \mathbf{S}_s + \mathbf{S}_u \cdot \mathbf{S}_s}{mm_s}\right] \\ &= 2m + m_s + \frac{b}{\hbar^2}\left[\frac{\mathbf{S}_u \cdot \mathbf{S}_d}{m^2} + \frac{\mathbf{S}_1 \cdot \mathbf{S}_2 + \mathbf{S}_1 \cdot \mathbf{S}_3 + \mathbf{S}_2 \cdot \mathbf{S}_3 - \mathbf{S}_u \cdot \mathbf{S}_d}{mm_s}\right], \end{aligned}$$

which using

$$\mathbf{S}_1 \cdot \mathbf{S}_2 + \mathbf{S}_1 \cdot \mathbf{S}_3 + \mathbf{S}_2 \cdot \mathbf{S}_3 = -3\hbar^2/4$$

from (3.92), gives

$$M_\Sigma = 2m + m_s + \frac{b}{4}\left[\frac{1}{m^2} - \frac{4}{mm_s}\right].$$

3.19 The first two quantum number combinations are compatible with the assignments

$$(1,0,0,1,1) = c\bar{b}, \quad (-1,1,-2,0,-1) = ssb$$

and can exist within the simple quark model. There are no combinations $q\bar{q}$ or qqq which are compatible with the second two combinations, so these cannot exist within the simple quark model. The combination $(0,0,1,0,1)$ must be a meson $q\bar{q}$ because $B = 0$, but must contain both an $\bar{s}$ antiquark and a $\bar{b}$ antiquark since $S = \tilde{B} = 1$. These are incompatible requirements. The combination $(-1,1,0,1,-1)$ must be a baryon qqq of the form xcb (where $x = u$ or d) since $B = 1$, $S = 0$, $C = 1$ and $\tilde{B} = -1$. The possible charges are $Q = 1$ and 0, corresponding to ucd and dcb, respectively, which are incompatible with the requirement $Q = -1$.

3.20 The cross-section for a two-body collision $i \to f$ in the vicinity of a resonance is given by the Bret-Wigner formula (1.83b) which is

$$\sigma_{if}(E)=\frac{\pi\hbar^2}{q_i^2}\frac{2J+1}{(2S_1+1)(2S_2+1)}\frac{\Gamma_i\Gamma_f}{[(E-Mc^2)^2+\Gamma^2/4]},$$

where J is the spin of the resonance, S_1 and S_2 are the spins of the initial-state particles, and q_i is their centre-of-mass momentum. For e^+e^- collisions at high energy, the electron mass may be neglected and so in the vicinity of the resonance

$$q_i^2c^2=E_i^2-m_e^2c^4\approx E_i^2\approx M^2c^4/4\,.$$

Then the Breit-Wigner formula for the production of a spin-1 resonance may be written

$$\sigma_f\equiv\sigma_{ef}=12\pi(\hbar c)^2\frac{\Gamma_e\Gamma_f}{\Gamma^2M^2c^4}\frac{1}{[4(E-Mc^2)/\Gamma]^2+1}\,.$$

Integrating both sides of the above expression using the substitution $\tan\theta=4(E-Mc^2)/\Gamma$ gives

$$\int\sigma_f(E)\mathrm{d}E=\frac{6\pi^2(\hbar c)^2\Gamma_e\Gamma_f}{\Gamma M^2c^4},$$

where, since the only significant contribution comes from the immediate vicinity of the peak, the integral has been taken from $E=-\infty$ to $E=+\infty$.
For the total cross-section, $\Gamma_f=\Gamma$, so that

$$\int\sigma_f(E)\mathrm{d}E=\frac{6\pi^2\Gamma_e(\hbar c)^2}{(Mc^2)^2}\qquad(1)$$

Finally, we can evaluate (1) using the numerical values given for the integral. The result is $\Gamma_e=1.05\,\mathrm{keV}$, and by universality $\Gamma_\mu=\Gamma_e$. The value of Γ follows from the ratio

$$\Gamma=\Gamma_e\left(\int\sigma_t\,\mathrm{d}E\Big/\int\sigma_e\,\mathrm{d}E\right)=30\,,$$

giving $\Gamma=31.5\,\mathrm{keV}$.

3.21 An argument similar to that given in Section 3.3.6 gives the following allowed combinations:

Baryons		*Mesons*	
C	Q	C	Q
3	2	1	1,0
2	2,1	0	1,0,−1
1	2,1,0	−1	0,−1
0	2,1,0,−1		

PROBLEMS 4

4.1 In an obvious notation,

$$\begin{aligned}E_{\mathrm{CM}}^2 &= (E_e + E_p)^2 - (\mathbf{p}_e c + \mathbf{p}_p c)^2 \\ &= (E_e^2 - \mathbf{p}_e^2 c^2) + (E_p^2 - \mathbf{p}_p^2 c^2) + 2E_e E_p - 2\mathbf{p}_e \cdot \mathbf{p}_p c^2 \\ &= m_e^2 c^4 + m_p^2 c^4 + 2E_e E_p - 2\mathbf{p}_e \cdot \mathbf{p}_p c^2.\end{aligned}$$

At the energies of the beams, masses may be neglected and so with $p = |\mathbf{p}|$,

$$E_{\mathrm{CM}}^2 = 2E_e E_p - 2p_e p_p c^2 \cos(\pi - \theta) = 2E_e E_p \left[1 - \cos(\pi - \theta)\right],$$

where θ is the crossing angle. Using the values given, gives $E_{\mathrm{CM}} = 154\ \mathrm{GeV}$. In a fixed-target experiment, and again neglecting masses,

$$E_{\mathrm{CM}}^2 = 2E_e E_p - 2\mathbf{p}_\mathbf{e} \cdot \mathbf{p}_\mathbf{p} c^2 ,$$

where

$$E_e = E_L\ ,\ \ E_p = m_p c^2\ ,\ \ \mathbf{p}_p = \mathbf{0}.$$

Thus, $E_{\mathrm{CM}} = \left(2m_p c^2 E_L\right)^{1/2}$ and for $E_{\mathrm{CM}} = 154\ \mathrm{GeV}$, this gives $E_L = 1.26 \times 10^4\ \mathrm{GeV}$.

4.2 For constant acceleration, the ions must travel the length of the drift tube in half a cycle of the rf field. Thus, $L = v/2f$, where v is the velocity of the ion. Since the energy is far less than the rest mass of the ion, we can use nonrelativistic kinematics to find v, i.e. $v = 4.01 \times 10^7\ \mathrm{m\ s^{-1}}$ and finally $L = 1\ \mathrm{m}$.

4.3 A particle with mass m, charge q and speed v moving in a plane perpendicular to a constant magnetic field of magnitude B will traverse a circular path with radius of curvature $r = mv/qB$ and hence the cyclotron frequency is $f = v/2\pi r = qB/2\pi m$. At each traversal the particle will receive energy from the rf field, so if f is kept fixed, r will increase (i.e. the trajectory will be a spiral). Thus if the final energy is E, the extraction radius will be $R = \sqrt{2mE}/qB$. To evaluate these expressions we use $q = 2e = 3.2 \times 10^{-19}\mathrm{C}$, together with

$$B = 0.8\ \mathrm{T} = 0.45 \times 10^{30}\ (\mathrm{MeV/c^2})\ \mathrm{s^{-1}\ C^{-1}}$$

and thus $f = 6.15\ \mathrm{MHz}$ and $R = 62.3\ \mathrm{cm}$.

4.4 The equations of motion are $\mathrm{d}\mathbf{p}/\mathrm{d}t = e\mathbf{v} \times \mathbf{B}$ and $\mathrm{d}E/\mathrm{d}t = 0$, where the second follows from the fact that the Lorentz force is perpendicular to the direction of motion and so does no work on the particle. The magnitude of the velocity $v = |\mathbf{v}|$ and $\gamma(v)$ are therefore constant, so that the first equation becomes

$$m\gamma \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = e\mathbf{v} \times \mathbf{B},$$

which reduced to $m\gamma v^2/\rho = evB$ for a circular orbit. Hence $p = m\gamma v = e\rho B$, and in S.I.units, the result follows from

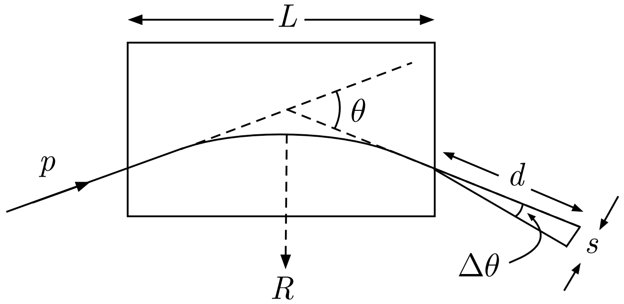
$$pc = ec\rho B = 4.8 \times 10^{-11} \rho B \text{ joules.}$$

Since $1 \text{ joule} = 0.62 \times 10^{19} \text{ eV}$ this gives $p = 0.3B\rho \text{ GeV/c}$, as required.

4.5 A particle with unit charge e and momentum p in the uniform magnetic field B of the bending magnet will traverse a circular trajectory of radius R, given by $p = BR$. If B is in Teslas, R in metres and p in GeV/c, then $p = 0.3BR$ (see Problem 4.3). Referring to the figure below, we have

$$\theta \approx L/R = 0.3LB/p \quad \text{and} \quad \Delta\theta = s/d = 0.3BL\Delta p/p^2 .$$

Solving for d using the data given, gives $d = 4.63$ m.



4.6 Luminosity may be calculated from (4.7a) for colliders, $L = N_1 N_2 f/A$. Here $N_{1,2}$ are the numbers of particles in each bunch, A is the cross-sectional area of the beam, and the frequency between collisions $f = n\omega_c$, where n is the number of bunches in each beam and ω_c is the cyclotron frequency. The numerical values are:

$$n = 12,\ N_1 = N_2 = 3 \times 10^{11},\ A = (2 \times 10^{-4})\text{cm}^2,\ \omega_c = (3 \times 10^{10}/8\pi \times 10^5)\text{s}^{-1},$$

so finally $L = 6.43 \times 10^{31} \text{cm}^{-2}\text{s}^{-1}$.

4.7 The average distance between collisions of a neutrino and an iron nucleus is the mean free path $\lambda = 1/n\sigma_\nu$, where $n \approx \rho/m_p$ is the number of nucleons per cm^3. Using the data given, $n \approx 4.7 \times 10^{24} \text{ cm}^{-3}$ and $\sigma_\nu \approx 3 \times 10^{-36} \text{ cm}^2$, so that $\lambda \approx 7.1 \times 10^{10}$ cm. Thus if 1 in 10^9 neutrinos is to interact, the thickness of iron required is 71 cm.

4.8 Radiation energy losses are given by $-\mathrm{d}E/\mathrm{d}x = E/L_R$, where L_R is the radiation length. This implies that $E = E_0 \exp(-x/L_R)$, where E_0 is the initial energy. Using the data given, gives $E = 1.51\ \text{GeV}$. Radiation losses at fixed E are proportional to m^{-2}, where m is the mass of the projectile. Thus for muons, they are negligible at this energy.

4.9 The total cross-section is $\sigma_{\text{tot}} = \sigma_{\text{el}} + \sigma_{\text{cap}} + \sigma_{\text{fission}} = 4\times 10^2\ \text{b}$ and the attenuation is $\exp(-nx\sigma_{\text{tot}})$ where $nx = 10^{-1}N_A/A = 2.56\times 10^{23}\ \text{m}^{-2}$. Thus $\exp(-nx\sigma_{\text{tot}}) = 0.9898$, i.e 1.02% of the incident particles interact and of these the fraction that elastically scatter is given by the ratio of the cross-sections, i.e. 0.75×10^{-4}. Thus the intensity of elastically scattered neutrons is $0.765\ \text{s}^{-1}$ and finally the flux at 5m is $2.44\times 10^{-3}\ \text{m}^{-2}\text{s}^{-1}$.

4.10 The total centre-of-mass energy is given by

$$E_{\text{CM}} \approx (2mc^2E_L)^{1/2} = 0.226\ \text{GeV},$$

and so the cross-section is $\sigma = 1.70\times 10^{-34}\,\text{m}^2$. The interaction length is $l = 1/n\sigma$, where n is the number density of electrons in the target. This is given by $n = \rho N_A Z/m_{\text{Pb}}$, where N_A is Avogadro's number and for lead, $Z = 82$ and m_{Pb}, the atomic mass of lead is $208\,\text{g}$. Thus $n = 2.71\times 10^{33}\ \text{m}^{-3}$ and $l = 2.17\,\text{m}$.

4.11 The target contains $n = 5.30\times 10^{24}$ protons and so the total number of interactions per second is

$$N = n\times\text{flux}\times\sigma_{\text{tot}} = (5.30\times 10^{24})\times(2\times 10^{7})\times(40\times 10^{-31}) = 424\ \text{s}^{-1}.$$

Since each neutral pion decays to two photons, the rate of production of photons from the target is thus 848 photons.

4.12 For small υ, the Bethe-Bloch formula may be written

$$S \equiv -\frac{\mathrm{d}E}{\mathrm{d}x} \propto \frac{1}{\upsilon^2}\ln\left(\frac{2m_e\upsilon^2}{I}\right) \quad\text{with}\quad \frac{\mathrm{d}S}{\mathrm{d}\upsilon} \propto \frac{2}{\upsilon^3}\left[1-\ln\left(\frac{2m_e\upsilon^2}{I}\right)\right].$$

the former has a maximum at $\upsilon^2 = eI/2m_e$. For a proton in iron we can use $I = 10Z\ \text{eV} = 260\ \text{eV}$, so that

$$E_p = m_p\upsilon^2/2 = m_pIe/4m_e = 324\ \text{keV}.$$

4.13 Using $E = \gamma Mc^2$ and $\gamma = (1-\beta^2)^{-1/2}$, where M is the mass of the projectile, we have

$$\frac{\mathrm{d}E}{\mathrm{d}\beta} = Mc^2 \frac{\beta}{(1-\beta^2)^{3/2}}\,.$$

So, using this and (4.14) in (4.15), gives

$$R = \frac{M}{q^2 n_e} \int_0^{\beta_{\text{initial}}} f(\beta)\,\mathrm{d}\beta\,,$$

where $f(\beta)$ is a function of β. The result follows directly.

$$R = \int_0^R \mathrm{d}x = \int_0^{\beta_{\text{initial}}} \left(-\frac{\mathrm{d}E}{\mathrm{d}x}\right)^{-1} \left[\frac{\mathrm{d}E}{\mathrm{d}\beta}\,\mathrm{d}\beta\right] = \frac{M}{q^2 n_e} F(\beta_{\text{initial}})\,.$$

4.14 From (4.24), $E(r) = V/r\ln(r_c/r_a)$ and at the surface of the anode this is $4023\ \text{kV m}^{-1}$. Also, if $E_{\text{threshold}}(r) = 750\ \text{kV m}^{-1}$, then from (4.24) $r = 0.107\ \text{mm}$ and so the distance to the anode is $0.087\ \text{mm}$. This contains 21.7 mean free paths and so assuming each collision produces an ion pair, the multiplication factor is $2^{21.7} = 3.41\times 10^6 \approx 10^{6.53}$.

4.15 A particle with velocity v will take time $t = L/v$ to pass between the scintillation counters. Relativistically, $p = m\gamma v$, which gives $v = c(1+m^2c^2/p^2)^{-1/2}$ on solving for v. Thus, the difference in flight times is (taking $m_1 > m_2$)

$$\Delta t = \frac{L}{c}\left[\left(1+\frac{m_1^2c^2}{p^2}\right)^{1/2} - \left(1+\frac{m_2^2c^2}{p^2}\right)^{1/2}\right].$$

For large momenta such that $p^2 \gg m_1^2c^2 > m_2^2c^2$, we can expand the brackets to give

$$\Delta t \approx \frac{L(m_1^2c^2 - m_2^2c^2)}{2p^2c}\,,$$

which decreases like p^{-2}. For pions and kaons with momentum 3 GeV/c, we can use the approximate formula and the minimum value of the flight path is $L \approx 4.8\ \text{m}$.

4.16 The Čerenkov condition is $\beta n \geq 1$. For the pion to give a signal, but not the kaon, we have $\beta_\pi n \geq 1 \geq \beta_K n$. The momentum is given by $p = mv\gamma$, so eliminating γ gives

$$\beta = v/c = (1+m^2c^2/p^2)^{-1/2}\,.$$

For $p = 20\ \mathrm{GeV/c}$, $\beta_\pi = 0.99997$, $\beta_K = 0.99970$, so the condition on the refractive index is

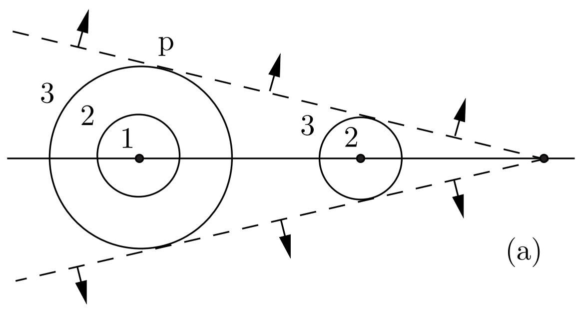
$$3\times 10^{-4} \geq (n-1)/n \geq 3\times 10^{-5}.$$

Using the largest value of $n = 1.0003$, we have

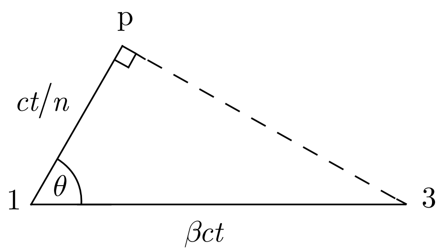
$$N = 2\pi\alpha\left(1 - \frac{1}{\beta_\pi^2 n^2}\right)\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)$$

as the number of photons radiated per metre, where $\lambda_1 = 400\,\mathrm{nm}$ and $\lambda_2 = 700\,\mathrm{nm}$. Numerically, $N = 26.5$ photons/m, and hence to obtain 200 photons requires a detector of length 7.5 m.

4.17 Referring to the figure below, the distance between two positions of the particle Δt apart in time is $v\Delta t$. The wave fronts from these two positions have a difference in their distance travelled of $c\Delta t/n$.



(a)



(b)

They constructively interfere at an angle θ, where

$$\cos\theta = \frac{c\Delta t/n}{v\Delta t} = \frac{1}{\beta n}.$$

The maximum value of θ corresponds to the minimum of $\cos\theta$ and hence the maximum of β. This occurs as $\beta \to 1$, when $\theta_{\max} = \cos^{-1}(1/n)$ and corresponds to the ultra-relativistic or massless limit.

The quantity β may be expressed as

$$\beta = pc/E = pc(p^2c^2 + m^2c^4)^{-1/2} .$$

Hence,

$$\cos\theta = \frac{1}{n}\frac{\sqrt{p^2c^2 + m^2c^4}}{pc} ,$$

which, after rearranging, gives

$$x = (mc^2)^2 = p^2c^2(n^2\cos^2\theta - 1) .$$

Differentiating this formula gives

$$\mathrm{d}x/\mathrm{d}\theta = -2p^2c^2n^2\cos\theta\sin\theta$$

and the error on x is then given by $\sigma_x = \left|\mathrm{d}x/\mathrm{d}\theta\right|\sigma_\theta$. For very relativistic particles, the derivative can be approximated by using θ_{max} , for which

$$\cos\theta_{\mathrm{max}} = 1/n, \quad \sin\theta_{\mathrm{max}} = \sqrt{n^2 - 1}/n .$$

Hence

$$\sigma_x \approx 2p^2c^2n^2\frac{1}{n}\frac{\sqrt{n^2-1}}{n}\sigma_\theta = 2p^2c^2\sqrt{n^2-1}\,\sigma_\theta .$$

4.18 From (4.26a) we have

$$\hbar\omega_p = \frac{1}{\alpha}(4\pi N_e r_e^3)^{1/2} m_3 c^2$$

Consider the factor in brackets. Using the formula for the classical radius given following (4.26a), we have $r_e = 2.818\times10^{-15}\,\mathrm{m}$. In addition, $N_e = \rho Z/AM_u$, where $M_u = 1.660\times10^{-27}\,\mathrm{kg}$ is the atomic mass unit. Hence

$$4\pi N_e r_e^3 = 169.4\times10^{-15}\,\rho Z/A ,$$

with ρ in $\mathrm{g\,cm^{-3}}$. Substituting in (4.26a) with $m_ec^2 = 0.511\times10^6\,\mathrm{eV}$ and $\alpha = 1/137.0$, gives finally

$$\hbar\omega_p = 28.81\left(\frac{\rho Z}{A}\right)^{1/2}\mathrm{eV} ,$$

in agreement with (4.26b).

4.19 To be detected, the event must have $150^0 > \theta > 30^0$, i.e. $\left|\cos\theta\right| < 0.866$. Setting $x = \cos\theta$, the fraction of events in this range is

$$f = \int_{-0.866}^{+0.866}\frac{\mathrm{d}\sigma}{\mathrm{d}x}dx \Bigg/ \int_{-1.0}^{+1.0}\frac{\mathrm{d}\sigma}{\mathrm{d}x}dx = 0.812 .$$

The total cross-section is given by

$$\sigma = \int \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} d\Omega = \int_0^{2\pi} \mathrm{d}\phi \int_{-1}^{+1} \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \mathrm{d}\cos\theta = 2\pi \frac{\alpha^2\hbar^2c^2}{4E_{\mathrm{CM}}^2} \int_{-1}^{+1} \left(1+\cos^2\theta\right) \mathrm{d}\cos\theta \,.$$

Using $E_{\mathrm{CM}} = 10\,\mathrm{GeV}$, gives

$$\sigma = 4\pi\alpha^2\hbar^2c^2 \big/ 3E_{\mathrm{CM}}^2 = 0.868\,\mathrm{nb}\,.$$

The rate of production of events is given by $L\sigma$ and since L is a constant, the total number of events produced will be $L\sigma t = 8.68 \times 10^4$.

The $\tau^\pm$ decay too quickly to leave a visible track in the drift chamber. The e^+ and the μ^- will leave tracks in the drift chamber and the e^+ will produce a shower in the electromagnetic calorimeter. If it has enough energy, the μ^- will pass through the calorimeters and leave a signal in the muon chamber. There will no signal in the hadronic calorimeter.

PROBLEMS 5

5.1 We have

$$m = \alpha + \beta + \gamma \geq n = \bar{\alpha} + \bar{\beta} + \bar{\gamma}\,,$$

where the inequality is because baryon number $B > 0$. Using the values of the colour charges I_3^C and Y^C from Table 5.1, the colour charges for the state are:

$$I_3^C = (\alpha - \bar{\alpha})/2 - (\beta - \bar{\beta})/2$$

and

$$Y^C = (\alpha - \bar{\alpha})/3 + (\beta - \bar{\beta})/3 - 2(\gamma - \bar{\gamma})/3\,.$$

By colour confinement, both these colour charges must be zero for observable hadrons, which implies

$$\alpha - \bar{\alpha} = \beta - \bar{\beta} = \gamma - \bar{\gamma} \equiv p \text{ and hence } m - n = 3p\,,$$

where p is a non-negative integer. Thus the only combinations allowed by colour confinement are of the form

$$(3q)^p (q\bar{q})^n \qquad (p, n \geq 0)\,.$$

It follows that a state with the structure qq is not allowed, as no suitable values of p and n can be found.

5.2 The most general baryon colour wavefunction is

$$\chi_B^C = \alpha_1 r_1 g_2 b_3 + \alpha_2 g_1 r_2 b_3 + \alpha_3 b_1 r_2 g_3 + \alpha_4 b_1 g_2 r_3 + \alpha_5 g_1 b_2 r_3 + \alpha_6 r_1 b_2 g_3$$

where the α_i $(i = 1, 2, \ldots, 6)$ are constants. If we apply the operator $\hat{F}_1$ to the first term and use the relations

$$\hat{F}_1 r = \tfrac{1}{2} g, \quad \hat{F}_1 g = \tfrac{1}{2} r, \quad \hat{F}_1 b = 0 \ ,$$

we have

$$\alpha_1 \hat{F}_1 (r_1 g_2 b_3) = \alpha_1 (\hat{F}_1 r_1) g_2 b_3 + \alpha_1 r_1 (\hat{F}_1 g_2) b_3 + \alpha_1 r_1 g_2 (\hat{F}_1 b_3) = \frac{\alpha_1}{2}\left(g_1 g_2 b_3 + r_1 r_2 b_3\right).$$

Similar contributions are obtained by acting with $\hat{F}_1$ on the other terms in χ_B^C and collecting these together we obtain

$$\hat{F}_1 \chi_B^C = \frac{(\alpha_1 + \alpha_2)}{2}\left(g_1 g_2 b_3 + r_1 r_2 b_3\right) + \frac{(\alpha_3 + \alpha_4)}{2}\left(b_1 g_2 g_3 + b_1 r_2 r_3\right) + \frac{(\alpha_5 + \alpha_6)}{2}\left(g_1 b_2 g_3 + r_1 b_2 r_3\right).$$

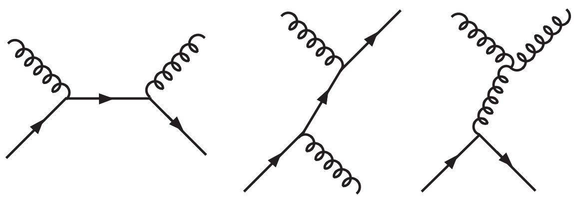
This is only compatible with the confinement condition $\hat{F}_i \chi_B^C = 0$ for $i = 1$ if

$$\alpha_1 = -\alpha_2 \qquad \alpha_3 = -\alpha_4 \qquad \alpha_5 = -\alpha_6$$

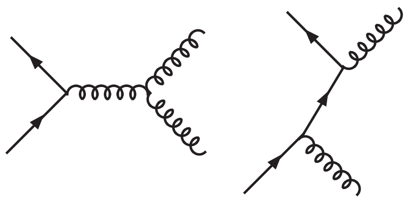
The full set of such conditions leads to the antisymmetric form (5.2).

5.3

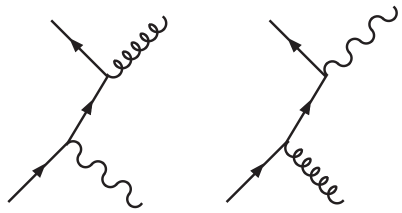
(a)



(b)



(c)



5.4 From (5.6) the magnitude of the typical momentum transfer is of order $|\mathbf{q}| = \hbar/r$ which gives $|\mathbf{q}| = 2\,\mathrm{GeV}/c$ and $|\mathbf{q}| = 200\,\mathrm{GeV}/c$ at $r = 0.1\,\mathrm{fm}$ and $r = 0.001\,\mathrm{fm}$, respectively. From (5.5a), the potential is dominated by one-gluon exchange and since there is no energy transferred in scattering by a static potential, the appropriate scale (5.8) for evaluating α_s is $\mu = |\mathbf{q}|c$. This gives $\alpha_s(2\,\mathrm{GeV}) \simeq 0.3$ and $\alpha_s(200\,\mathrm{GeV}) \simeq 0.1$ from Figure 5.3, so that $V(0.001\,\mathrm{fm})/V(0.1\,\mathrm{fm})$ is of order $1/3$.

5.5 By analogy with the QED formula, we have

$$\Gamma(3g) = 2(\pi^2 - 9)\alpha_s^6 m_c c^2 / 9\pi ,$$

where $m_c \approx 1.5\ \mathrm{GeV/c^2}$ is the constituent mass of the c–quark. Evaluating this gives $\alpha_s = 0.31$. In the case of the radiative decay,

$$\Gamma(gg\gamma) = 2(\pi^2 - 9)\alpha_s^4 \alpha^2 m_b c^2 / 9\pi ,$$

where $m_b \approx 4.5\ \mathrm{GeV/c^2}$ is the constituent mass of the b–quark. Evaluating this gives $\alpha_s = 0.32$. (These values are a little too large because in practice α in the QED formulas should be replaced by $4\alpha_s/3$, as in (5.5a)).

5.6 The chi states would be expected to have broader widths, because they have $C = +1$ rather than $C = -1$ for $J/\psi(3097)$ and $\psi(3686)$, and can they decay via two rather than the three gluons shown in Figure 5.4, and so are less heavily suppressed In addition, the widths of the $J/\psi(3097)$ and $\psi(3686)$ are further suppressed by an angular momentum barrier because they are P-wave states. Experimentally, the three chi states indeed much broader than the J/ψ and $\psi(3686)$ states.

5.7 In general, a neutral meson with spin J, which is an eigenstate of C parity, can have $C = (-1)^J$ or $C = (-1)^{J+1}$, and $P = (-1)^J$ or $P = (-1)^{J+1}$, giving four possible combinations of C and P. Since the parity of a fermions and its antifermion are opposite, we have $P = (-1)^{L+1}$ for a bound state of a quark with its own antiquark, while by (1.18) its C-parity is given by $C = (-1)^{L+S}$. Hence for $J = 0$ the combinations 0^{--} and 0^{+-} are forbidden in the simple quark model, as shown in the text. For arbitrary $J > 0$, the possibilities are: $S = 0, J = L$, leading to $P = (-1)^{J+1}, C = (-1)^J$; $S = 1, J = L \pm 1$, leading to $P = (-1)^J, C = (-1)^J$; and $S = 1, J = L$, leading to $P = (-1)^{J+1}, C = (-1)^{J+1}$. This exhausts the possibilities, so that the combination $P = (-1)^J, C = (-1)^{J+1}$ does not occur for $J > 0$, and mesons with $J^{PC} = 0^{-+}$ and $J^{PC} = 0^{+-}, 1^{-+}, 2^{+-}, 3^{-+}, \ldots$ are forbidden in the simple quark model as stated.

5.8 We need to show that the $I = 1, I_3 = 0$ state of two pions π_1 and π_2 has no $\pi^0\pi^0$ component. Using the notation $\left|\pi\pi, I, I_3\right\rangle$ for the $\pi\pi$ states and $\left|I, I_3\right\rangle_i$ for the $\pi_1\pi_2$ states, we have

$$\left|\pi\pi, 2, 2\right\rangle = \left|1,1\right\rangle_1 \left|1,1\right\rangle_2 ,$$

where we ignore any overall phase factors throughout. Using (A.52), then gives

$$\left|\pi\pi,2,1\right\rangle = \frac{1}{\sqrt{2}}\left[\left|1,1\right\rangle_1\left|1,0\right\rangle_2 + \left|1,0\right\rangle_1\left|1,1\right\rangle_2\right] ,$$

so that by orthogonality

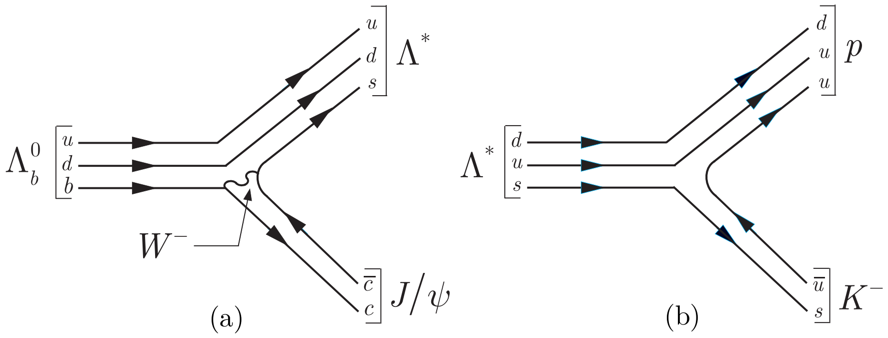
$$\left|\pi\pi,1,1\right\rangle = \frac{1}{\sqrt{2}}\left[\left|1,1\right\rangle_1\left|1,0\right\rangle_2 - \left|1,0\right\rangle_1\left|1,1\right\rangle_2\right].$$

Finally, using (A.52) again gives

$$\begin{aligned}\left|\pi\pi,1,0\right\rangle &= \frac{1}{\sqrt{2}}\left[\left|1,1\right\rangle_1\left|1,-1\right\rangle_2 - \left|1,-1\right\rangle_1\left|1,1\right\rangle_2\right] \\ &= -\frac{1}{\sqrt{2}}\left[\left|\pi^+\right\rangle_1\left|\pi^-\right\rangle_2 - \left|\pi^-\right\rangle_1\left|\pi^+\right\rangle_2\right],\end{aligned}$$

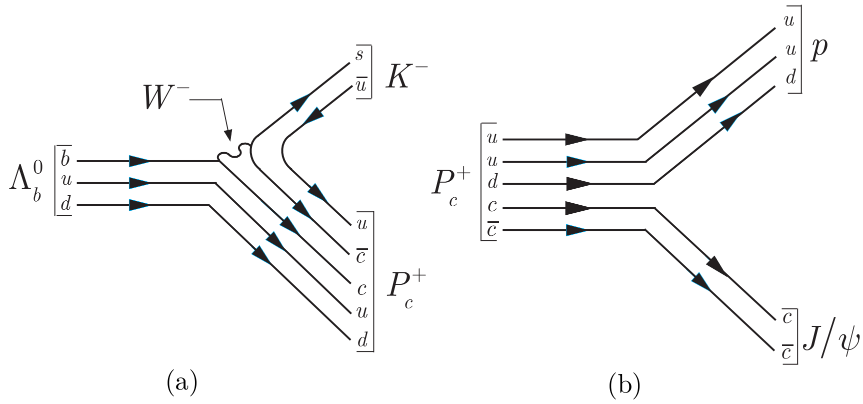
using (A.62). Thus there is no $\pi^0\pi^0$ component.

5.9 The decay $\Lambda_b^0 \to \Lambda^* + J/\Psi$ corresponds to the transition $udb \to uds + c\bar{c}$ in which $b \to c$ and an $s\bar{c}$ pair is produced. Since $\widetilde{B}, C$ and S are violated, it is a weak interaction, given in lowest order by the diagram (a) below.



The decay $\Lambda^* \to K^- + p$ corresponds to $uds \to s\bar{u} + uud$ at quark level and is a strong interaction in which a $u\bar{u}$ is created. The required diagram is therefore (b) above.

5.10 The decay $\Lambda_b^0 \to K^- + P_c^+$ corresponds to the weak interaction $udb \to s\bar{u} + uudc\bar{c}$ at quark level, in which $b \to c$ and both $s\bar{c}$ and $u\bar{u}$ pairs are created. The diagram is (a) below.



The decay $P_c^+ \to J/\psi + p$ is the strong reaction $uudc\bar{c} \to c\bar{c} + uud$, so no quarks are created. The quark diagram is (b) above.

5.11 Substituting (5.22) into (5.23) and setting $N_C = 3$, gives

$$R = 3(1+\alpha_s/\pi)\sum e_q^2 ,$$

where α_s is given by (5.9) and (5.10), evaluated at $\mu^2 = E_{\rm CM}^2$ and the sum is over those quarks that can be produced in pairs at the energy considered. At 2.8 GeV the u, d and s quarks can contribute and at 15 GeV the u, d, s, c and b quarks can contribute. Evaluating R then gives $R \approx 2.17$ at $E_{\rm CM} = 2.8$ GeV and $R \approx 3.89$ at $E_{\rm CM} = 15$ GeV.

5.12 Energy-momentum conservation gives,

$$W^2c^4 = \left[(E-E') + E_P\right]^2 - \left[(\mathbf{p}-\mathbf{p}) + \mathbf{P}\right]^2 c^2 = \text{invariant mass squared of } X ,$$

so that, in the rest frame of the proton,

$$W^2c^4 = [(E-E') + Mc^2]^2 - (\mathbf{p}-\mathbf{p}')c^2 = 2Mc^2(E-E') + Mc^4 - Q^2 \ ,$$

where M is the proton mass and we have used the definition of Q^2. Hence

$$2M\nu c^2 = 2Mc^2(E'-E)$$

by (5.24) and the result follows. Since *some* energy must be transferred to the outgoing electron, it follows that $E \geq E'$, i.e. $\nu \geq 0$. Also, since the lightest state X is the proton, $W^2 \geq M^2$. Thus,

$$2M\nu = Q^2 + (W^2 - M^2)c^2 \geq Q^2 .$$

From the definition of x, it follows that $x \le 1$. Finally, $x > 0$ because both Q^2 and $2M\nu$ are positive.

5.13 For elastic scattering, $W^2 = M^2$ and so from the definition (5.24)

$$2Mc^2\nu = Q^2$$

and hence

$$x \equiv Q^2 / 2Mc^2\nu = 1 .$$

From the definition

$$Q^2 \equiv (\mathbf{p} - \mathbf{p}')^2 c^2 - (E - E')^2 ,$$

we have

$$Q^2 = \mathbf{p}^2 c^2 - 2\mathbf{p}\cdot\mathbf{p}' c^2 + \mathbf{p}'^2 c^2 - E^2 + +2EE' - E'^2 .$$

If we ignore the lepton masses, then $p \equiv |\mathbf{p}| = E/c$ and $p' \equiv |\mathbf{p}'| = E'/c$, so that

$$Q^2 = 2EE'(1 - \cos\theta) .$$

Also, in the rest frame of the proton, $\nu = E - E'$, so substituting this and the above expression for Q^2 into the relation $2Mc^2\nu = Q^2$ gives the result.

5.14 From (5.33), the F_2 structure function for charged lepton scattering is given by

$$F_2\left(x, Q^2\right) = \sum_{a,\bar{a}} \left[e_a^2\, x\, f_a(x) + e_a^2\, x\, f_{\bar{a}}(x) \right],$$

where $a = u, d, s, \ldots$ and $f_a, f_{\bar{a}}$ are the corresponding quark and antiquark distribution functions $u(x), d(x), s(x), \ldots,\ \bar{u}(x), \bar{d}(x), \bar{s}(x), \ldots$. The distribution functions for protons and neutrons are related by isospin symmetry, i.e. interchanging u and d quarks changes neutron to proton, so

$$u^p = d^n \equiv u, \quad d^p = u^n \equiv d, \quad s^p = s^n \equiv s, \quad c^p = c^n \equiv c, \quad b^p = b^n \equiv b,$$

with similar relations for the antiquarks. Then the F_2 structure function for protons and neutrons may be written

$$F_2^p(x) = x\left[\tfrac{1}{9}(d + \bar{d}) + \tfrac{4}{9}(u + \bar{u}) + \tfrac{1}{9}(s + \bar{s}) + \tfrac{4}{9}(c + \bar{c}) + \tfrac{1}{9}(b + \bar{b}) \right]$$

and

$$F_2^n(x) = x\left[\tfrac{4}{9}(d + \bar{d}) + \tfrac{1}{9}(u + \bar{u}) + \tfrac{1}{9}(s + \bar{s}) + \tfrac{4}{9}(c + \bar{c}) + \tfrac{1}{9}(b + \bar{b}) \right],$$

so that

$$\int_0^1 \left[F_2^p(x) - F_2^n(x) \right] \frac{\mathrm{d}x}{x} = \frac{1}{3}\int_0^1 [u(x) + \bar{u}(x)]\,\mathrm{d}x - \frac{1}{3}\int_0^1 [d(x) + \bar{d}(x)]\,\mathrm{d}x .$$

But summing over all contributions we must recover the quantum numbers of the proton, i.e,

$$\int_0^1 [u(x) - \overline{u}(x)]\mathrm{d}x = 2\ ; \qquad \int_0^1 [d(x) - \overline{d}(x)]\mathrm{d}x = 1\,.$$

Finally, eliminating the integrals over u and d gives the Gottfried sum rule:

$$\int_0^1 \left[F_2^p(x) - F_2^n(x)\right]\frac{\mathrm{d}x}{x} = \frac{1}{3} + \frac{2}{3}\int_0^1 \left[\overline{u}(x) - \overline{d}(x)\right]\mathrm{d}x\,,$$

The experimental value of the left hand side is about 0.24, so that the difference between the antiquark distributions on the right hand side makes a significant contribution.

5.15 A proton has the valence quark content $p = uud$. Hence

$$\int_0^1 u(x)\mathrm{d}x = 2 \quad \text{and} \quad \int_0^1 \mathrm{d}(x)\mathrm{d}x = 1,$$

Substituting the given expressions for $u(x)$ and $d(x)$ and evaluating the integral gives $a = 2b\ = 2.19$ and hence

$$xV(x) = xu(x) + xd(x) = 3.28x^{1/2}(1-x)^3\,.$$

This has a maximum value of 0.78 at $x = 0.143$, while integrating $xV(x)$ from 0 to 1 gives 0.33 as the fraction of the proton's momentum carried by the valence quarks. While not precise, these values are in qualitative agreement with the behaviour shown in Figure 5.23.

5.16 (a) We need to show that $y_1 - y_2 \to y_1' - y_2' = y_1 - y_2$ for any y_1, y_2, i.e. that for any y, y' can be written in the form $y' = y + a$, where a is independent of y. Under a Lorentz transformation

$$y \to y' = \frac{1}{2}\ln\left(\frac{E' + p_z'c}{E' - p_z'c}\right) = \frac{1}{2}\ln\left[\frac{\gamma(E - \beta p_z c) + \gamma(p_z c - \beta E)}{\gamma(E - \beta p_z c) - \gamma(p_z c - \beta E)}\right],$$

where $\beta = v/c$ and $\gamma = (1-\beta^2)^{1/2}$ in natural units. Cancelling γ and factorising the numerator and denominator gives

$$y' = \frac{1}{2}\ln\left[\frac{(E + p_z c)(1-\beta)}{(E - p_z c)(1+\beta)}\right] = y + \frac{1}{2}\ln\left(\frac{1-\beta}{1+\beta}\right),$$

implying $y_1' - y_2' = y_1 - y_2$ as required.

(b) If the mass is negligible, $p_z c = E\cos\theta$, so that the rapidity becomes

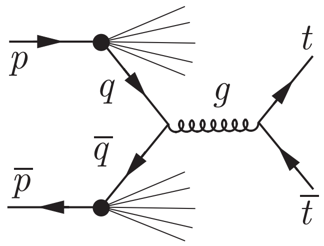
$$y = \frac{1}{2}\ln\left(\frac{E + p_z c}{E - p_z c}\right) = \frac{1}{2}\ln\left(\frac{1+\cos\theta}{1-\cos\theta}\right),$$

which reduces to the pseudorapidity $\eta = -\ln(\tan\theta/2)$ on substituting

$$\cos\theta = 2\cos^2(\theta/2) - 1 \quad \text{and} \quad \cos\theta = 1 - 2\sin^2(\theta/2)$$

into the numerator and denominator of the term in brackets.

5.17 The mechanism for producing $t\bar{t}$ pairs in $p\bar{p}$ collisions is shown in below.



Thus to produce significant numbers of $t\bar{t}$ pairs, we need significant numbers of $q\bar{q}$ pairs with $E_{\text{CM}} > 2m_t$. From Figure 5.23, we see that the quark distribution in the proton, and hence the antiquark distribution in the antiproton, both peak around $x = 0.15$, which corresponds to a quark enegy $0.15E$, where E is the beam energy and we have neglected the proton mass. If they collide head-on, $E_{\text{CM}} = 0.3E$, which equals the threshold energy $2m_t c^2$ for pair production at $E \approx 1\,\text{TeV}$. From Figure 5.23, we also see that there are many quarks and antiquarks with higher energies then $0.15E$, so that $t\bar{t}$ pairs should readily be produced at $E \approx 1\,\text{TeV}$. (The t-quark was actually discovered at the FNAL Tevatron, with a beam energy of 0.9 TeV.)

5.18 (a) The peak value of the cross-section is where $E = Mc^2$, i.e.

$$\sigma_{\max} = \frac{12\pi(\hbar c)^2}{M^2c^4}\frac{\Gamma_{u\bar{d}}}{\Gamma} = \frac{12\pi(\hbar c)^2}{M^2}\text{BR}(W^+ \to u\bar{d}),$$

which is 760 nb.

(b) The cross section $\sigma(p\bar{p} \to W^+ + \cdots)$ is given by the integral

$$\sigma_{p\bar{p}}(s) = \int_0^1\int_0^1 \sigma_{u\bar{d}}(E)\,u(x_u)\,d(x_d)\,\mathrm{d}x_u\,\mathrm{d}x_d$$

where u and d are the valence quark functions for protons, and we have used charge conjugation invariance to replace the $\bar{d}$ distribution in the antiproton by the d

distribution in the proton. In the narrow width approximation for $\sigma_{u\bar{d}}$, this integral may be written

$$\sigma_{p\bar{p}} = \frac{\pi\Gamma}{Mc^2}\sigma_{\max}\int_0^1\int_0^1 u(x_u)d(x_d)\delta\left(1-\frac{E^2}{M^2c^4}\right)\mathrm{d}x_u\mathrm{d}x_d,$$

which using the relation $E^2 = x_u x_d s$ becomes

$$\sigma_{p\bar{p}} = \frac{\pi\Gamma}{Mc^2}\sigma_{\max}\int_0^1 u(x_u)\,\mathrm{d}x_u\int_0^1 d(x_d)\delta\left(1-\frac{x_u x_d s}{M^2c^4}\right)\mathrm{d}x_d.$$

The integral over x_d may be done using the properties of the delta function. (See, for example, Section B.5.2 of Martin and Shaw (2016). This gives

$$\sigma_{p\bar{p}} = \frac{\pi\Gamma}{Mc^2}\sigma_{\max}\int_0^1 u(x_u)\,\mathrm{d}x_u\, d\left(\frac{M^2c^4}{x_u s}\right)\int_0^1 \delta\left(1-\frac{x_u x_d s}{M^2c^4}\right)\mathrm{d}x_d,$$

and again using the properties of the delta function, the second integral is $M^2c^4/(sx_u)$. So finally,

$$\sigma_{p\bar{p}} = \frac{\pi Mc^2\Gamma}{s}\sigma_{\max}\int_0^1 \frac{u(x_u)d(M^2c^4/x_u s)}{x_u}\mathrm{d}x_u,$$

which is the required integral, and may be evaluated for given forms of the structure functions.

5.19 The formula (5.56a) remains unchanged, but for $\pi^- p \to \mu^+\mu^- X$ (5.56b) becomes

$$F(x_1,x_2) = \sum_a e_a^2\left[F_a^-(x_1)f_{\bar{a}}(x_2) + F_{\bar{a}}^-(x_1)f_a(x_2)\right], \tag{A}$$

where $F_i^-(x)$ is the probability of finding a parton of type i with fractional momentum x in the incoming π^-. If we assume the cross-sections are just the sums of the cross-sections on the constituent nucleons, then since C has six protons and six neutrons, the ratio R is given by

$$R = \frac{\mathrm{d}^2\sigma(\pi^- p\to\mu^+\mu^- X) + \mathrm{d}^2\sigma(\pi^- n\to\mu^+\mu^- X)}{\mathrm{d}^2\sigma(\pi^+ p\to\mu^+\mu^- X) + \mathrm{d}^2\sigma(\pi^+ n\to\mu^+\mu^- X)}.$$

In addition, $M^2c^4 \to s$ implies $x_1x_2 \to 1$ by (5.52), and since $x_i < 1$, this implies both $x_1 \to 1$ and $x_2 \to 1$. Assuming that valence quarks dominate in the pion as well as the proton at large x, then (A) reduces to

$$F(x_1, x_2) = \tfrac{4}{9} F_{\overline{u}}^{-}(x_1) u(x_2), \quad (\pi^- p)$$

since π^- has the valence quark composition $d\overline{u}$. The corresponding result for neutrons is

$$F(x_1, x_2) = \tfrac{4}{9} F_{\overline{u}}^{-}(x_1) u_n(x_2) = \tfrac{4}{9} F_{\overline{u}}^{-}(x_1) d(x_2), \quad (\pi^- n)$$

since by isospin symmetry, the distribution of a u quark in a neutron is the same as the distribution of a d quark in the proton, as discussed in Section 5.5.4. For an incident π^+ , these results become

$$F(x_1, x_2) = \tfrac{1}{9} F_{\overline{d}}^{+}(x_1) d(x_2), \quad (\pi^+ p)$$

and

$$F(x_1, x_2) = \tfrac{1}{9} F_{\overline{d}}^{+}(x_1) u(x_2), \quad (\pi^+ n) \,,$$

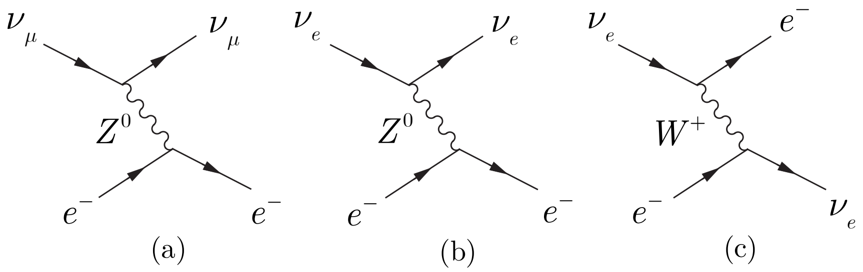
where $F_{\overline{d}}^{+}(x_1)$ is the probability of finding a $\overline{d}$ quark with fractional momentum x_1 in the π^+ . Since the cross-sections are proportional to $F(x_1, x_2)$ in each case, this implies

$$R = \frac{4F_{\overline{u}}^{-}(x_1)[u(x_2) + d(x_2)]}{F_{\overline{d}}^{+}(x_1)[u(x_2) + d(x_2)]} = 4$$

because, by isospin invariance, the probability of finding a $\overline{u}$ quark with fractional momentum x_1 in the π^- is the same as the probability of finding a $\overline{d}$ quark with fractional momentum x_1 in the π^+ , since the two differ by the substitutions $u \to d$ and $\overline{d} \to \overline{u}$, with the wavefunctions unchanged.

PROBLEMS 6

6.1 A charged current weak interaction is one mediated by the exchange of a charged $W^{\pm}$ boson. An example is $n \to p + e^- + \bar{\nu}_e$. A neutral current weak is one mediated by a neutral Z^0 boson. An example is $\nu_\mu + p \to \nu_\mu + p$. Charged current weak interactions do not conserve the strangeness quantum number, whereas neutral current weak interaction do. For $\nu_\mu + e^- \to \nu_\mu + e^-$, the only Feynman diagram that conserves both L_e and L_μ is (a) below, which is a weak neutral current. However, for $\nu_e + e^- \to \nu_e + e^-$, there are two diagrams (b) and (c) below.



Thus the reaction has both neutral and charged current components and is not unambiguous evidence for weak neutral currents.

6.2 If an exchanged particle approaches to within a distance $d\,\mathrm{fm}$, this is equivalent to a three-momentum transfer of order $|\mathbf{q}| = \hbar/d = (0.2/d)\mathrm{GeV}/c$, so that $|\mathbf{q}| = 0.2\mathrm{GeV}/c$ for $d = 1\mathrm{fm}$ and $|\mathbf{q}| = 200\mathrm{GeV}/c$ for $d = 10^{-3}\,\mathrm{fm}$; and since for elastic scattering of identical particles there is no energy transfer in the centre-of-mass, typical q^2 values are of order $-4\times 10^{-2}(\mathrm{GeV}/c)^2$ and $-4\times 10^4(\mathrm{GeV}/c)^2$, respectively. The scattering amplitude is given by (1.51),

$$\mathcal{M}(q^2) = g^2\hbar^2(q^2 - m^2c^2)^{-1}$$

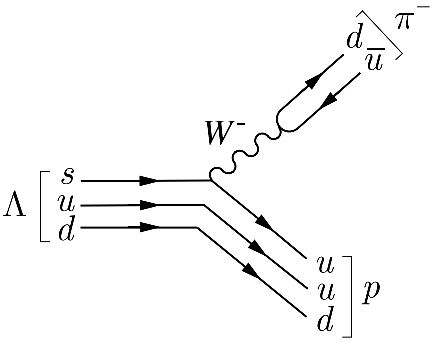
where m is the mass of the exchanged particle. Thus,

$$\frac{M_{\mathrm{EM}}(q^2)}{M_{\mathrm{Weak}}(q^2)} = \frac{q^2c^2 - m_Z^2c^4}{q^2c^2 - m_\gamma^2c^4}$$

for $g_{\mathrm{EM}} \simeq g_{\mathrm{Weak}}$. Using $m_\gamma = 0$ and $m_Z = 91\,\mathrm{GeV}/c^2$ gives ratios of order 2×10^5 and unity for distances of closest approach of 1fm and $10^{-3}\,\mathrm{fm}$, respectively.

6.3 If a corresponds to an electron neutrino line directed into the vertex, the conservation laws restrict b to be an electron line directed out of the vertex, corresponding to Figure 6.4. Similarly, when the conservation laws are taken into account, all the other possibilities for a lead to one or other of the vertices of Figure 6.4 and 6.5.

6.4 The Feynman diagram is:



The amplitude has two factors of the weak coupling g_W and one W propagator carrying a momentum q, i.e.

$$\text{amplitude} \propto \frac{g_W^2}{q^2c^2 - M_W^2c^4} \propto \frac{g_W^2}{M_W^2} \, ,$$

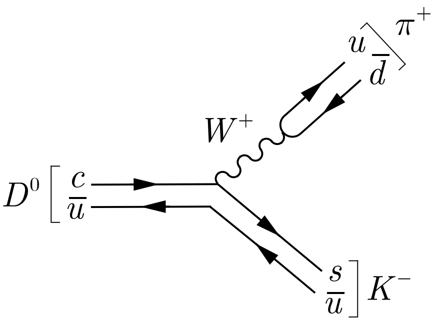
because $qc \approx M_\Lambda c^2 \ll M_W c^2$. Now,

$$\Gamma(\Lambda \to p\pi^-) \propto (\text{amplitude})^2 \propto g_W^4 \big/ M_W^4$$

and so doubling g_W and reducing M_W by a factor of four will increase the rate by a factor

$$\left[2^4\right] \Big/ \left[\left(1/4\right)^4\right] = 4096 \, .$$

6.5 The quark compositions are: $D^0 = c\overline{u}$; $K^- = s\overline{u}$; $\pi^+ = u\overline{d}$. Since preferentially $c \to s$, we have:

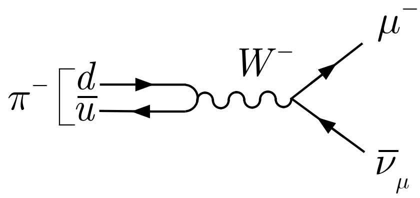


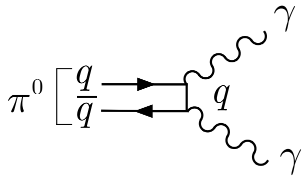
i.e. a lowest-order charge current weak interaction. However, for $D^+ \to K^0 + \pi^+$, we have

$$D^+ = c\overline{d} \; ; \; K^0 = d\overline{s} \; ; \; \pi^+ = u\overline{d} \; .$$

Thus we could arrange $c \to d$ via W emission and the W^+ could then decay to $u\bar{d}$, i.e. π^+. However, this would leave the $\bar{d}$ quark in the D^+ to decay to an $\bar{s}$ quark in the K^0 which is not possible as they both have the same charge.

6.6 The relevant Feynman diagrams are:





In the case of the charged pion, there are two vertices of strength $\sqrt{\alpha_W}$, and there will be a propagator

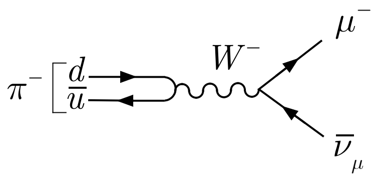
$$\frac{1}{Q^2 + M_W^2 c^2} \approx \frac{1}{M_W^2 c^2},$$

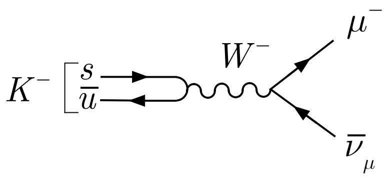
because the momentum transfer (squared) Q^2 carried by the W is very small compared to the mass of the W. Thus the decay rate will be proportional to

$$\left(\frac{\sqrt{\alpha_W}\sqrt{\alpha_W}}{M_W^2} \right)^2 = \frac{\alpha_W^2}{M_W^4}.$$

In the case of the neutral pion, there are two vertices of strength $\sqrt{\alpha_{\text{em}}}$, but no propagator with a W-mass factor. (The effective propagator will depend on the mass of the quark.) Thus the decay rate will be proportional to α_{em}^2 and since $\alpha_{\text{em}} \approx \alpha_W$, the decay rate for the charged pion will be much smaller than that for the neutral decay, i.e. the lifetime of the π^0 will be much shorter.

6.7 The two Feynman diagrams are:





Using lepton-quark symmetry and the Cabibbo hypothesis, the two hadron vertices are given by

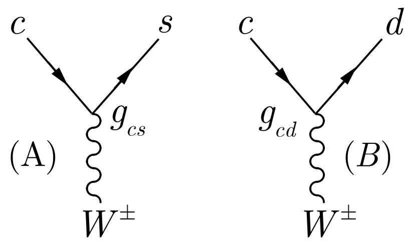
$$g_{udW} = g_W \cos\theta_C \quad \text{and} \quad g_{usW} = g_W \sin\theta_C.$$

So, if we ignore kinematic differences and spin effects, and use $\theta_C = 13^0$, we would expect the ratio of decay rates is given by

$$R = \frac{\text{Rate}(K^- \to \mu^- + \bar{\nu}_\mu)}{\text{Rate}(\pi^- \to \mu^- + \bar{\nu}_\mu)} \propto \frac{g_{usW}^2}{g_{udW}^2} = \tan^2\theta_C \approx 0.05$$

The measured ratio is actually about 1.3, which shows the importance of the neglected effects. For example, the Q–value for the kaon decay is about 10 times that for pion decay.

6.8 Cabibbo-allowed decays involve the csW vertex of Figure (A) below, giving rise to the selection rule $\Delta C = \Delta S = \Delta Q = \pm 1$. Cabibbo-suppressed decays involve the cdW vertex of Figure (B), giving rise to the selection rules $\Delta C = \Delta Q = \pm 1,\ \Delta S = 0$. Using these rules one sees that the decays are: (a) Cabibbo-allowed, (b) forbidden, (c) forbidden, and (d) Cabibbo-suppressed.



6.9 The decays (a) and (c) are forbidden by the $\Delta S = \Delta Q$ rule (6.31) for semi-leptonic decays and (e) is forbidden by the $\Delta S = 0, \pm 1$ rule (6.33) for purely hadronic decays. Decays (b), (d) and (f) are allowed decays and have all been observed experimentally.

6.10(a) In addition to the decay $b \to c + e^- + \bar{\nu}_e$, there are two other leptonic decays $(\ell = \mu^-, \tau^-)$ and by lepton universality they will all have equal decay rates. There are also hadronic decays of the form $b \to c + X$ where $Q(X) = -1$. Examining the allowed $Wq\bar{q}$ vertices using lepton-quark symmetry shows that the only forms that X can have, if we ignore Cabibbo-suppressed modes, are $d\bar{u}$ and $s\bar{c}$. Each of these hadronic decays has a probability three times that of a leptonic decay because the quarks exist in three colour states. Thus, there are effectively 6 hadronic channels and 3 leptonic ones and if we neglect the masses of the final-state quarks and leptons and set $\cos^2\theta_C$ to unity instead of 0.95, they all have the same rate. So we have approximately,

$$BR(b \to c + e^- + \bar{\nu}_e) = 1/9\,.$$

(b) The argument is similar to that of (a) above. Thus, in addition to the decay $\tau^- \to e^- + \bar{\nu}_e + \nu_\tau$, there is also the leptonic decay $\tau^- \to \mu^- + \bar{\nu}_\mu + \nu_\tau$ with equal probability and the hadronic decays $\tau^- \to \nu_\tau + X$. In principle, $X = d\bar{u}$ and $s\bar{c}$, but the latter is not allowed because $m_s + m_c > m_\tau$. So the only allowed hadronic decay is $\tau^- \to d + \bar{u} + \nu_\tau$ with a relative probability of 3 because of colour. So finally, we have aproximately

$$BR(\tau^- \to e^- + \bar{\nu}_e + \nu_\tau) = 1/5\,.$$

(The measured branching ratio is 0.18.)

6.11 An arbitrary complex $n \times n$ matrix $\mathbf{U}$ has $2n^2$ real parameters. The matrix $\mathbf{F} \equiv \mathbf{U}^\dagger\mathbf{U}$ is Hermitian by construction, so $F_{ij} = F_{ji}^*$ and it has n^2 real parameters. Hence, the condition $\mathbf{U}^\dagger\mathbf{U} = \mathbf{1}$ imposes n^2 conditions on $\mathbf{U}$, leaving n^2 real parameters undetermined. Since (A) has $n^2 = 4$ real parameters and satisfies $\mathbf{U}^\dagger\mathbf{U} = \mathbf{1}$, it is the most general 2×2 unitary matrix.

Substituting (A) into (B) gives

$$\begin{aligned} d' &= e^{-i\alpha}(e^{i\beta} d\cos\theta_C + e^{i\gamma} s\sin\theta_C) \\ s' &= e^{-i\alpha}(-e^{-i\gamma} d\sin\theta_C + e^{-i\beta} s\cos\theta_C) \end{aligned},$$

which can be written

$$\begin{aligned} e^{i(\alpha-\beta)}d' &= d\cos\theta_C + e^{i(\gamma-\beta)} s\sin\theta_C \\ e^{i(\alpha+\gamma)}s' &= -d\sin\theta_C + e^{i(\gamma-\beta)} s\cos\theta_C \end{aligned}.$$

Redefining the phases of the quark states by

$$e^{i(\alpha-\beta)}d' \to d', \qquad e^{i(\alpha+\gamma)}s' \to s', \qquad e^{i(\gamma-\beta)}s \to s$$

gives the required result.

6.12 The reactions are assumed to be

$$\text{(a) } \Xi^- + S^0 \to \Lambda + \pi^- \quad \text{and} \quad \text{(b) } \Xi^0 + S^0 \to \Lambda + \pi^0,$$

where S^0 has $I = 1/2$ and isospin is conserved. Conservation of I_3 then requires $I_3(S^0) = -1/2$, since Ξ^0 and Ξ^- have $I = 1/2$ and $I_3 = 1/2, -1/2$, respectively. In reaction (a), the initial and final states have $I_3 = -1$ and hence can only be pure $I = 1$, with a rate $\left|M_1\right|^2$, say. In reaction (b), the final state has $I = 1$, and we are given that the initial state has only a 50% probability of being in an $I = 1$ state, so the rate is $\frac{1}{2}\left|M_1\right|^2$. Hence the ratio is predicted to be 2. (The measured value is about 1.8. That the state $\left|\Xi^0, S^0\right\rangle$ is an equal mixture of states with $I = 0$ and $I = 1$, as given, can be derived using the methods of Appendix A.4.)

6.13 In the simple Bohr model, the strong attraction between the two t quarks is balanced by the centripetal force, i.e.

$$\frac{\mu v^2}{r} = \frac{4}{3}\frac{\hbar c \alpha_s}{r^2},$$

where r is the radius of the orbit, v is the quark velocity and μ the reduced mass $(\mu = m_t/2)$. In addition, the angular momentum is quantised, i.e. $\mu v r = n\hbar$. From these two equations, the radius of the ground state $(n = 1)$ is given by

$$r = \frac{3}{4}\frac{\hbar}{c\mu\alpha_s}.$$

Finally, using $m_t = 173\ \mathrm{GeV}/\mathrm{c}^2$, gives

$$r = \frac{3(\hbar c)}{4(\mu c^2)\alpha_s} = 1.7\times 10^{-2}\ \mathrm{fm}.$$

The time taken to traverse a single orbit of the ground state is

$$t = 2\pi r/v = 2\pi\mu r^2/\hbar = 2.7\times 10^{-24}\,\mathrm{s}.$$

This is much longer than the expected lifetime of the top quark, which is about 4×10^{-25} s.

6.14 The decay (a) is a charged current weak interaction satisfying the $\Delta S = \Delta Q$ rule (6.31) for semi-leptonic decays and so is allowed by single W-exchange while (c) is an allowed electromagnetic process. Decays (b) and (d) are both forbidden as electromagnetic interactions because $\Delta S \neq 0$, and are also forbidden as weak interactions because there are no strangeness-changing weak neutral currents

6.15 Integrating the differential cross-sections over y (from 0 to 1) gives for a target with a specific quark distribution

$$\frac{\sigma^{NC}(\nu)}{\sigma^{CC}(\nu)} = \left[\int_0^1 [g_L^2 + g_R^2(1-y)^2]\,\mathrm{d}y\right]\left[\int_0^1 \mathrm{d}y\right]^{-1} = g_L^2 + \tfrac{1}{3}g_R^2$$

and

$$\frac{\sigma^{NC}(\bar{\nu})}{\sigma^{CC}(\bar{\nu})} = \left[\int_0^1 [g_L^2(1-y)^2 + g_R^2]\,\mathrm{d}y\right]\left[\int_0^1 (1-y)^2\,\mathrm{d}y\right]^{-1} = g_L^2 + 3g_R^2.$$

For an isoscalar target, we must add the contributions for u and d quarks in equal amounts, i.e.

$$\frac{\sigma^{NC}(\nu)}{\sigma^{CC}(\nu)}(\text{isoscalar}) = g_L^2(u) + \tfrac{1}{3}g_R^2(u) + g_L^2(d) + \tfrac{1}{3}g_R^2(d)$$

and

$$\frac{\sigma^{NC}(\bar{\nu})}{\sigma^{CC}(\bar{\nu})}(\text{isoscalar}) = g_L^2(u) + 3g_R^2(u) + g_L^2(d) + 3g_R^2(d)$$

Substituting for the couplings finally gives for an isoscalar target

$$\frac{\sigma^{NC}(\nu)}{\sigma^{CC}(\nu)} = \frac{1}{2} - \sin^2\theta_W + \frac{20}{27}\sin^4\theta_W, \quad \frac{\sigma^{NC}(\overline{\nu})}{\sigma^{CC}(\overline{\nu})} = \frac{1}{2} - \sin^2\theta_W + \frac{20}{9}\sin^4\theta_W$$

6.16 The required number of events produced must be 20000, taking account of the detection efficiency. If the cross-section is $60\,\text{fb} = 6\times10^{-38}\text{cm}^2$, then the integrated luminosity required is

$$2\times10^4/6\times10^{-38} = (1/3)\times10^{42}\ \text{cm}^{-2}$$

and hence the instantaneous luminosity must be $3.3\times10^{34}\text{cm}^{-2}\text{s}^{-1}$.
The branching ratio for $Z^0 \to b\overline{b}$ is found from the partial widths to be 15%. Thus, if b quarks are detected, the much greater branching ratio for $H \to b\overline{b}$ will help distinguish this decay from the background of $Z^0 \to b\overline{b}$.

6.17 From (6.65a) at $E_{\text{CM}} = E = M_H c^2$, we have

$$\sigma_\gamma(e^+e^- \to \mu^+\mu^-) \approx \frac{\alpha^2(\hbar c)^2}{(M_H c^2)^2} \approx \frac{2\times10^{-6}\text{GeV}^2\text{fm}^2}{(M_H c^2)^2}.$$

for the contribution of Figure 6.25a alone. Because the Higgs boson has spin zero, the Breit Wigner formula (1.84) becomes

$$\sigma(e^+ + e^- \to X) = \frac{\pi\hbar^2}{q_e^2}\left[\frac{\Gamma(H^0 \to e^+ + e^-)\Gamma(H^0 \to X)}{(E_{\text{CM}} - M_H c^2)^2 + \Gamma_H^2/4}\right],$$

where q_e is the centre-of mass momentum of the initial particles. The value of the cross section at the peak is then

$$\sigma_{H^0}(e^+e^- \to \mu^+\mu^-) = \frac{4\pi(\hbar c)^2}{(M_H c^2)^2} B(H^0 \to e^+e^-)B(H^0 \to \mu^+\mu^-)\ .$$

Since the branching ratio for $H^0 \to \tau^+\tau^-$ is about 6×10^{-2}, from Table 6.1, and by (6.74) the decay widths are proportional to the squared lepton masses, this gives

$$\sigma_{H^0}(e^+e^- \to \mu^+\mu^-) \approx \frac{5\times10^{-14}\text{GeV}^2\text{fm}^2}{(M_H c^2)^2}.$$

The amplitudes are proportional to the square roots of the above hypothetical cross-sections, but the Higgs contribution is still negligible.
At the Z^0 peak, we have

$$\sigma_\gamma(e^+e^- \to \mu^+\mu^-) \approx \frac{\alpha^2(\hbar c)^2}{(M_Z c^2)^2} \approx \frac{2\times10^{-6}\text{GeV}^2\text{fm}^2}{(M_Z c^2)^2}$$

for the contribution from Figure 6.25(a) alone, whereas for the contribution of Figure 6.25b the Breit–Wigner formula gives

$$\sigma_{Z^0}(e^+e^- \to \mu^+\mu^-) = \frac{12\pi(\hbar c)^2}{(M_Z c^2)^2} B(Z^0 \to e^+e^-)B(Z^0 \to \mu^+\mu^-) \approx \frac{2\times 10^{-2}\,\mathrm{GeV}^2\mathrm{fm}^2}{(M_Z c^2)^2}$$

since the branching ratios are both 3.4%. Hence, in contrast to the Higgs case, in this case the resonant peak is the dominant contribution.

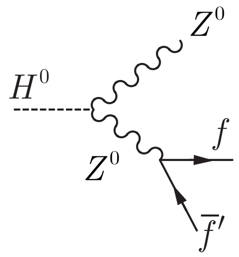
6.18 The decay of the Higgs boson to four charged leptons, denoted $H^0 \to 4\ell$, is dominated by the mechanism shown in Figure 6.40, where one of the Z^0 bosons is 'on shell', i.e. real, so that the decay is

$$H^0 \to Z^0 + \ell^+ + \ell^- \quad \text{with} \quad Z^0 \to \ell'^+ + \ell'^- \;, \tag{A}$$

where we have to sum over the possibilities $\ell,\ \ell' = e,\ \mu$. This gives

$$\begin{aligned}\Gamma(H^0 \to 4\ell) &\approx \Gamma(H^0 \to Z^0\ell^+\ell^-)B(Z^0 \to \ell^+\ell^-) \\ &= \Gamma(H^0 \to Z^0 f\bar{f})B(Z^0 \to \ell^+\ell^-)\frac{\Gamma(H^0 \to Z^0\ell^+\ell^-)}{\Gamma(H^0 \to Z^0 f\bar{f})},\end{aligned} \tag{B}$$

where $f\bar{f}$ is any fermion-antifermion pair. The obvious mechanism for this decay is shown in the figure below, and corresponds to the first step in the sequence (A) for the case of charged leptons.



Since the fermion masses are much less than the mass $(M_H - M_Z)$ released in the decay, they can be neglected to a good approximation. Then the relative rates for different fermions is controlled by the relative $Z^0 f\bar{f}$ couplings, so that they are the same as those observed in Z^0 decays, i.e.

$$\frac{\Gamma(H^0 \to Z^0\ell^+\ell^-)}{\Gamma(H^0 \to Z^0 f\bar{f})} = B(Z^0 \to \ell^+\ell^-).$$

Together with (B), this implies

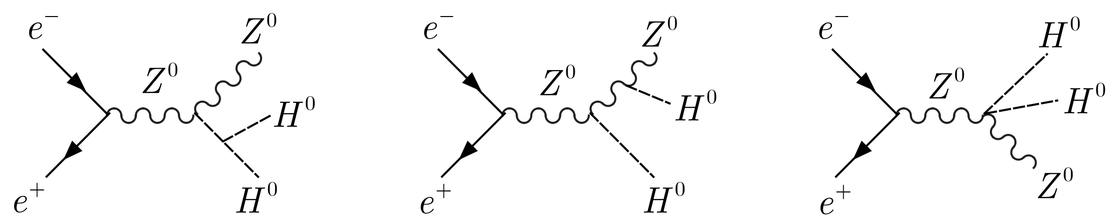
$$B(H^0 \to 4\ell) \approx B(H^0 \to Z^0 f\bar{f})[B(Z^0 \to \ell^+\ell^-)]^2 \approx 1.2\times 10^{-4} \;, \tag{C}$$

where we have used the result for $B(H^0 \to Z^0\ell^+\ell^-)$ given in Table 6.1 together with the value $B(Z^0 \to \ell^+\ell^-) = 0.068$ when summed over $\ell = e, \mu$. The corresponding result for (6.86), denoted $H^0 \to 2\ell 2\nu$, is

$$B(H^0 \to 2\ell 2\nu) \approx B(H^0 \to Wf\bar{f})[B(W \to \ell\nu)]^2 \approx 9.8 \times 10^{-3} \ ,$$

using $B(H^0 \to Wf\bar{f})$ from Table 6.1 and $B(W \to \ell\nu) = 0.21$, again when summed over $\ell = e, \mu$. (The predictions from a full calculation are 1.25×10^{-4} and 1.06×10^{-2} , respectively.) Substituting the numerical value (C), together with the given values of the integrated luminosity and cross-section, into (6.83), gives $N = 60$ decays with a statistical error of order $\sqrt{N} \approx 8$ and an error of about 8 from the uncertainty in the cross-section, giving $N = 60 \pm 11$, ignoring errors in the width. The number observed in Figure 6.42 is smaller, about 15, presumably because the cuts used to eliminate background also eliminate genuine events. In particular, any of the latter in which just one of the four leptons had a transverse momentum of less than 5 GeV would be eliminated from the sample.

6.19 We need only consider diagrams in which Higgs bosons couple to heavy particles. The lowest-order diagrams are:



This reaction is therefore ideally suited to measure the triple H^0 vertex, which does not contribute significantly to Higgs decays.

PROBLEMS 7

7.1 The initial total spin $J = 5$ of the ^{60}Co nuclei can only be conserved if the spins of all the final-state particles point in the same direction. For $\theta = 0$, the electrons are emitted in this direction so that they must have positive helicity; i.e. they must be in right-handed states e_R . For relativistic electrons, when $v \to c$, this is forbidden by the $V - A$ interaction, as discussed in Section 7.2.2. Hence $I(v = c, \theta = 0) = 0$, implying the $\alpha = -1$, which is the observed value.

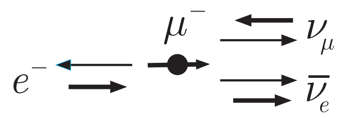
7.2 The most probable energy is given by

$$\frac{\mathrm{d}}{\mathrm{d}E_e}\left(\frac{\mathrm{d}\Gamma}{\mathrm{d}E_e}\right) = 0,$$

which gives

$$\frac{2G_F^2(m_\mu c^2)^2}{(2\pi)^3(\hbar c)^6}\left(2E_e - \frac{4E_e^2}{m_\mu c^2}\right) = 0, \quad \text{i.e. } E_e = m_\mu c^2 \big/ 2 .$$

When $E_e \approx m_\mu c^2 \big/ 2$, the electron has its maximum energy and the two neutrinos must be recoiling in the opposite direction. If the masses of all final-state particles are neglected, only left-handed particles and right-handed antiparticles can be emitted. Hence, since angular momentum must be conserved along the axis of motion, the orientations of the momenta and spins are



where the thick (thin) arrows denoted the particle's spin (momentum). Integrating the energy spectrum gives a total decay width

$$\Gamma = \frac{2G_F^2(m_\mu c^2)^2}{(2\pi)^3(\hbar c)^6}\int_0^{m_\mu c^2/2}\left[E_e^{\,2} - \frac{4E_e^3}{3m_\mu c^2}\right]\mathrm{d}E_e = \frac{G_F^2(m_\mu c^2)^5}{192\pi^3(\hbar c)^6} .$$

Numerically, $\Gamma = 3.01\times 10^{-19}\,\mathrm{GeV}$, which gives a lifetime $\tau = \hbar/\Gamma = 2.19\times 10^{-6}\,\mathrm{s}$.

7.3 In the centre-of-mass frame, the electron energy E remains unchanged on scattering, so that the squared four-momentum transfer is

$$q^2 = -(\mathbf{p}-\mathbf{p}')^2 = -2p^2(1-\cos\theta_{\mathrm{CM}}) \ ,$$

where p and $\mathbf{p}'$ are the initial and final momenta, $p = |\mathbf{p}| = |\mathbf{p}'|$ and θ_{CM} is the scattering angle. Now

$$4p^2 = \frac{4E^2 - 4m^2c^4}{c^2} = \frac{W^2c^4 - 4m^2c^4}{c^2} \ ,$$

where W is the invariant mass (cf B.18), given by $Wc^2 = 2E$ in the centre-of-mass frame (see B.19b). In the laboratory frame, the invariant mass is given by (cf. B19a)

$$W^2c^4 = 2m^2c^4 + 2mc^2E_L$$

and combining these equations gives

$$q^2 = (2m^2c^2 - 2mE_L)\sin^2(\theta_{\mathrm{CM}}/2)$$

as required. The maximum value of $|q^2|$ corresponds to $\theta_{\rm CM} = \pi/2$ and is $0.051(\mathrm{GeV}/c)^2$ for $E_L = 50\,\mathrm{GeV}$.
As noted in Section 7.1.2, the asymmetry parameter is proportional to

$$0.25 - \sin^2\theta_W \approx 0.018$$

since $\sin^2\theta_W \approx 0.232$. So as a 10% error on the asymmetry parameter gives an error of about 0.002 in $\sin^2\theta_W$, which is about 1%.

7.4 Integrating over the azimuthal angle ϕ gives

$$\begin{aligned}\sigma_F &= \frac{\pi\alpha^2(\hbar c)^2}{2s}\int_0^1\left[F(s)(1+\cos^2\theta)+G(s)\cos\theta\right]\mathrm{d}\cos\theta\\ &= \frac{\pi\alpha^2(\hbar c)^2}{2s}\left[\frac{4}{3}F(s)+\frac{1}{3}G(s)\right]\end{aligned}$$

and

$$\begin{aligned}\sigma_B &= \frac{\pi\alpha^2(\hbar c)^2}{2s}\int_{-1}^0\left[F(s)(1+\cos^2\theta)+G(s)\cos\theta\right]\mathrm{d}\cos\theta\\ &= \frac{\pi\alpha^2(\hbar c)^2}{2s}\left[\frac{4}{3}F(s)-\frac{1}{3}G(s)\right],\end{aligned}$$

so that $A_{FB} = 3G(s)/8F(s)$.
For unpolarised particles we have

$$e^+(\mathbf{p}) + e^-(-\mathbf{p}) \to \mu^+(\mathbf{p}') + \mu^-(-\mathbf{p}')\ , \tag{A}$$

which becomes

$$e^+(-\mathbf{p}) + e^-(\mathbf{p}) \to \mu^+(-\mathbf{p}') + \mu^-(\mathbf{p}')$$

under a parity transformation. A rotation through π about an axis lying in the scattering plane and perpendicular to the beam direction (cf. Figure 7.4 for Möller scattering) then gives

$$e^+(\mathbf{p}) + e^-(-\mathbf{p}) \to \mu^+(\mathbf{p}') + \mu^-(-\mathbf{p}')$$

which is identical to (A). Hence parity does not lead to a restriction on the angular distribution and a non-zero value of the forward-backward asymmetry does not violate parity conservation.

7.5 For neutrinos, $g_R(\nu) = 0$ and $g_L(\nu) = 1/2$. So,

$$\Gamma_{\nu_e} = \Gamma_{\nu_\mu} = \Gamma_{\nu_\tau} = \Gamma_0/4\,,$$

where

$$\Gamma_0 = \frac{G_F M_Z^3 c^6}{3\pi\sqrt{2}(\hbar c)^3} = 668\,\mathrm{MeV}\,.$$

Thus the partial width for decay to three neutrino pairs is predicted to be $\Gamma_\nu = 501\,\mathrm{MeV}$. For quarks,

$$g_R(u,\,c,\,t) = -1/6 \quad \text{and} \quad g_L(u,\,c,\,t) = 1/3\,.$$

Thus, $\Gamma_u = \Gamma_c = 10\Gamma_0/72$. Also,

$$g_R(d,\,s,\,b) = 1/12 \;\text{ and }\; g_L(b,\,s,\,d) = -5/12\,.$$

Thus, $\Gamma_d = \Gamma_s = \Gamma_b = 13\Gamma_0/72$. Finally, $\Gamma_q = \sum_i \Gamma_i$, where $i = u,\,c,\,d,\,s,\,b$ (no top quark because $2M_t > M_Z$). So

$$\Gamma_q = \left(\frac{3\times 13}{72} + \frac{2\times 10}{72}\right)\Gamma_0 = \frac{59}{72}\Gamma_0 = 547\,\mathrm{MeV}\,.$$

Hadron production is assumed to be equivalent to the production of $q\bar{q}$ pairs followed by fragmentation with probability unity. Thus $\Gamma_{\mathrm{hadron}} = 3\Gamma_q$, where the factor of three is because each quark exists in one of three colour states. Thus the prediction is $\Gamma_{\mathrm{hadron}} = 1641\ \mathrm{MeV}$.

If there are N_ν generations of neutrinos with $M_\nu < M_Z/2$, so that $Z^0 \to \nu\bar{\nu}$ is allowed, then

$$\Gamma_{\mathrm{tot}} = \Gamma_{\mathrm{hadron}} + \Gamma_{\mathrm{lepton}} + N_\nu \Gamma_{\nu\bar{\nu}}$$

where $\Gamma_{\nu\bar{\nu}}$ is the width to a specific $\nu\bar{\nu}$ pair. Thus, using the experimental data and the predicted width to neutrino pairs,

$$\begin{aligned} N_\nu &= \frac{\Gamma_{\mathrm{tot}} - \Gamma_{\mathrm{hadron}} - \Gamma_{\mathrm{lepton}}}{\Gamma_{\nu\bar{\nu}}} \\ &= \frac{(2495\pm 2) - (1744\pm 2) - (251.9\pm 0.3)}{167} = 2.99 \pm 0.02, \end{aligned}$$

which rules out values of N_ν greater than 3.

7.6 The argument used in Section 7.3.1 to deduce the parities of $\pi^0\pi^0$ and $\pi^0\pi^0\pi^0$ states shows that the $\pi^+\pi^0$ state has $P = 1$ and that the $\pi^+\pi^+\pi^-$ state has $P = -1$. Hence parity must be violated in one of the reactions irrespective of which spin value is assigned to the kaon.

7.7 Consider the decay of a K^0 at rest, i.e.

$$K^0(\mathbf{p}=\mathbf{0}) \to \pi^-(\mathbf{p}_1) + e^+(\mathbf{p}_2, s_2) + \nu_e(\mathbf{p}_3, s_3)$$

where the momenta satisfy $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = \mathbf{0}$ and s_2 and s_3 are the electron and neutrino spins. *CP* changes particles into antiparticles, reverses the signs of the momenta, and leaves the spins unchanged. Hence, by *CP* invariance the rates for the above decay and for

$$\bar{K}^0(\mathbf{p}=\mathbf{0}) \to \pi^+(-\mathbf{p}_1) + e^-(-\mathbf{p}_2, s_2) + \bar{\nu}_e(-\mathbf{p}_3, s_3)$$

are equal. The desired result follows on summing over all possible final state momenta and spins.

7.8 On substituting (7.25) into (7.33) and (7.34), one obtains

$$\left|K_L^0\right\rangle = N\left[(1+\varepsilon)\left|K^0\right\rangle - (1-\varepsilon)\left|\bar{K}^0\right\rangle\right]$$

and

$$\left|K_S^0\right\rangle = N\left[(1+\varepsilon)\left|K^0\right\rangle + (1-\varepsilon)\left|\bar{K}^0\right\rangle\right]$$

where $N = [2(1+\left|\varepsilon\right|^2]^{-1/2}$. Hence the ratio η_{00} defined in (7.39) may be written

$$\mathcal{M}_S^I(K_S^0 \to \pi\pi) = N[(1+\varepsilon)A_I + (1-\varepsilon)A_I^*],$$

where I is the isospin of the $\pi\pi$ final state and A_I and A_I^* are defined in (B) and (C) in the question. Thus, since A_0 is real,

$$\mathcal{M}_S^0 = 2NA_0 \quad \text{and} \quad \mathcal{M}_S^2 = 2N(\mathrm{Re}\,A_2 + i\varepsilon\,\mathrm{Im}\,A_2)$$

Substituting into (A) and again using A_0 as real, gives

$$\mathcal{M}(K_S^0 \to \pi^0\pi^0) = 2N\sqrt{\tfrac{2}{3}}e^{i\delta_2}[\mathrm{Re}\,A_2 + i\varepsilon\,\mathrm{Im}\,A_2] - 2N\sqrt{\tfrac{1}{3}}A_0 e^{i\delta_0} \approx \frac{2N}{\sqrt{3}}\left[\sqrt{2}e^{i\delta_2}\,\mathrm{Re}\,A_2 - e^{i\delta_0}A_0\right],$$

where we have also used $A_2 \ll A_0$ and $\left|\varepsilon\right| \ll 1$ to drop terms that are second order in these quantities. In an analogous way, using (A), we find that

$$\mathcal{M}(K_L^0 \to \pi^0\pi^0) = \frac{2N}{\sqrt{3}}\left[\sqrt{2}e^{i\delta_2}\left(\varepsilon\,\mathrm{Re}\,A_2 + i\,\mathrm{Im}\,A_2\right) - \varepsilon e^{i\delta_0}A_0\right] \approx \frac{2N}{\sqrt{3}}\left[i\sqrt{2}e^{i\delta_2}\,\mathrm{Im}\,A_2 - \varepsilon e^{i\delta_0}A_0\right].$$

Finally, substituting into (7.39) gives

$$\eta_{00} = \frac{\varepsilon - i\sqrt{2}e^{i\Delta}\,\mathrm{Im}\,A_2/A_0}{1-\sqrt{2}e^{i\Delta}\,\mathrm{Re}\,A_2/A_0} \approx \varepsilon - i\sqrt{2}\exp(i\Delta)\frac{\mathrm{Im}\,A_2}{A_0},$$

where $\Delta \equiv (\delta_2 - \delta_0)$ and again small terms have been neglected.

7.9 Each $B^{\pm}$ particle will have an energy 5.5 GeV, giving

$$\gamma = E/m = 1.04,\quad \beta = \upsilon/c = (1-1/\gamma^2)^{1/2} = 0.27\,,$$

so that the distance travelled is $d = \gamma\tau_0\upsilon = 1.4\times 10^{-2}\mathrm{cm}$.

7.10 (a) The semi-leptonic decay $\bar{D}^0\pi^-\mu^+\nu_\mu$ can only proceed in lowest-order interaction via $\bar{b} \to \bar{c} + W^+$ with $W^+ \to \ell^+ + \nu_\ell$, together with $q\bar{q}$ pair production. Thus it can only give a positive charged lepton, whereas a $\bar{B}^0$ semi-leptonic decay gives a negative charged lepton. Therefore (a) is a B^0 decay.

(b) The decay $\bar{B}^0 \to \rho^+ + K^-$ at quark level is $b\bar{d} \to u\bar{d} + s\bar{u}$, and so involves the transition $b \to us\bar{u}$. This can occur via $b \to u + W^-$ with $W^- \to s + \bar{u}$. In contrast, $B^0 \to \rho^+ + K^-$ is $\bar{b}d \to u\bar{d} + s\bar{u}$ at the quark level. However, while $\bar{b} \to \bar{u} + W^+$, $W^+ d \not\to u\bar{d}s$ in lowest order. Hence (b) is a B^0 decay.

(c) The decay $B^0 \to \rho^+ + \pi^-$ at quark level is $d\bar{b} \to u\bar{d} + d\bar{u}$, and so involves the transition $\bar{b} \to u\bar{d}u$. This is allowed via $\bar{b} \to \bar{u}W^+$ with $W^+ \to d\bar{u}$. $\bar{B}^0 \to \rho^-\pi^+$ at the quark level is $b\bar{d} \to u\bar{d} + d\bar{u}$, and so involves the transition $b \to ud\bar{u}$. This can occur via $b \to uW^-$ with $W^- \to d\bar{u}$. Hence (c) could be either B^0 or $\bar{B}^0$.

(d) The decay $B^0 \to D^- D_s^+$ at quark level is $\bar{b}d \to d\bar{c} + c\bar{s}$, i.e. it involves the transition $\bar{b} \to \bar{c}c\bar{s}$, which can proceed via $\bar{b} \to \bar{c}W^+$ with $W^+ \to c\bar{s}$. $\bar{B}^0 \to D^- D_s^+$ at quark level is $b\bar{d} \to d\bar{c} + c\bar{s}$. The b quark can decay via $b \to cW^-$, but $W^-\bar{d} \to d\bar{c}\,\bar{s}$ is not possible in lowest order. Hence (d) is a B^0 meson.

7.11 Substituting (7.76a,b) into (7.75) gives

$$I\left[B^0(t)\right] \to f] = \left|\mathcal{M}_f\right|^2 \left[\left|A(t)\right|^2 + \left|\lambda\right|^2\left|\bar{A}(t)\right|^2 + \lambda^* A(t)\bar{A}^*(t) + \lambda A^*(t)\bar{A}(t)\right]$$

and

$$I\left[\bar{B}^0(t)\right] \to f] = \left|\mathcal{M}_f\right|^2 \left[\left|\lambda\right|^2\left|A(t)\right|^2 + \left|\bar{A}(t)\right|^2 + \lambda A(t)\bar{A}^*(t) + \lambda^* A^*(t)\bar{A}(t)\right].$$

The terms on the right-hand side of this equation can be found from (7.68a,b). They are

$$\left|A(t)\right|^2 = e^{-\Gamma t}\cos^2(\Delta mt/2) \quad \text{and} \quad \left|\bar{A}(t)\right|^2 = e^{-\Gamma t}\sin^2(\Delta mt/2)\,,$$

and, after some manipulation,

$$\lambda^* A(t)\bar{A}^*(t) + \lambda A^*(t)\bar{A}(t) = -e^{-\Gamma t}\,\mathrm{Im}\,\lambda\sin(\Delta mt)\,.$$

and

$$\lambda A(t)\bar{A}^*(t) + \lambda^* A^*(t)\bar{A}(t) = e^{-\Gamma t}\,\mathrm{Im}\,\lambda\sin(\Delta mt)\,,$$

so that

$$I[B^0(t)\to f] = \left|\mathcal{M}_f\right|^2 e^{-\Gamma t}\left[\cos^2(\Delta mt/2) + \left|\lambda\right|^2\sin^2(\Delta mt/2) - \mathrm{Im}\,\lambda\sin(\Delta mt)\right]$$

and

$$I[\bar{B}^0(t)\to f] = \left|\mathcal{M}_f\right|^2 e^{-\Gamma t}\left[\left|\lambda\right|^2\cos^2(\Delta mt/2) + \sin^2(\Delta mt/2) + \mathrm{Im}\,\lambda\sin(\Delta mt)\right].$$

Finally, on substituting

$$\cos^2(\Delta mt/2) = \tfrac{1}{2}[1+\cos(\Delta mt)] \;\text{ and }\; \sin^2(\Delta mt/2) = \tfrac{1}{2}[1-\cos(\Delta mt)]$$

we get

$$I[B^0(t)\to f] = \left|\mathcal{M}_f\right|^2 e^{-\Gamma t}\left[\frac{1}{2}(1+\left|\lambda\right|^2) + \tfrac{1}{2}(1-\left|\lambda\right|^2)\cos(\Delta mt) - \mathrm{Im}\,\lambda\sin(\Delta mt)\right]$$

and

$$I[\bar{B}^0(t)\to f] = \left|\mathcal{M}_f\right|^2 e^{-\Gamma t}\left[\frac{1}{2}(1+\left|\lambda\right|^2) - \tfrac{1}{2}(1-\left|\lambda\right|^2)\cos(\Delta mt) + \mathrm{Im}\,\lambda\sin(\Delta mt)\right],$$

as required.

7.12 Substituting (7.25) into (7.33 and 7.34) gives

$$\left|K_S^0\right\rangle = \alpha\left[(1+\varepsilon)\left|K^0\right\rangle + (1-\varepsilon)\left|\bar{K}^0\right\rangle\right] \;\text{ and }\; \left|K_L^0\right\rangle = \alpha\left[(1+\varepsilon)\left|K^0\right\rangle - (1-\varepsilon)\left|\bar{K}^0\right\rangle\right],$$

and so by analogy

$$\left|M_a^0\right\rangle = \alpha\left[(1+\varepsilon)\left|M^0\right\rangle + (1-\varepsilon)\left|\bar{M}^0\right\rangle\right] \;\text{ and }\; \left|M_b^0\right\rangle = \alpha\left[(1+\varepsilon)\left|M^0\right\rangle - (1-\varepsilon)\left|\bar{M}^0\right\rangle\right],$$

where $\alpha = [2(1+\left|\varepsilon\right|^2)]^{-1/2}$. On comparing with (7.62), this gives

$$p = \alpha(1+\varepsilon),\;\; q = \alpha(1-\varepsilon)$$

so that (7.64c) is

$$\xi = \frac{q}{p} = \frac{(1-\varepsilon)}{(1+\varepsilon)}\;,$$

and hence $\varepsilon = (1-\xi)/(1+\xi)$. Then substituting $\xi = \exp(-2i\beta)$ gives

$$\varepsilon = \frac{1-e^{-2i\beta}}{1+e^{-2i\beta}} = \frac{e^{i\beta}-e^{-i\beta}}{e^{i\beta}+e^{-i\beta}} = i\tan\beta.$$

7.13 Using the Wolfenstein parameterisation (7.84), we see that the tree diagram Figure (7.14a) involves the vertices

$$V_{ub}V_{us} \approx A\lambda^4(\rho - i\eta)\,,$$

whereas the three penguin diagrams Figure (7.14b) involve the vertices

$$V_{qb}V_{qs},\quad q = u,\, c,\, t\ .$$

The largest contributions from penguin diagrams come from $q = c,\, t$, and taking into account the suppression factor $L = 0.2 - 0.3$ associated with loops in penguin diagrams, they are of order

$$L\,V_{qb}V_{qs} \approx A\lambda^2 L.$$

Using the parameter values given following (7.85), we see that the largest contribution comes from the penguin diagrams Figure (7.14b), and that the tree diagram is suppressed relative to it by a factor of order

$$\left|\frac{\lambda^2(\rho - i\eta)}{L}\right| \approx \frac{1}{10}.$$

7.14 At the quark level, the reaction $\bar{B}^0 \to \pi^+\pi^-$ is $b\bar{d} \to u\bar{d} + d\bar{u}$, i.e. it involves the transition $b \to ud\bar{u}$, corresponding to the diagrams of Figure (7.19a,b) with $q = u$. Hence the tree diagram is proportional to

$$V_{ub}V_{ud} \approx A\lambda^3(\rho - i\eta)\,,$$

while the leading penguin diagram is for $q' = t$, and is proportional to

$$L\,V_{tb}V_{td} \approx LA\lambda^3\,,$$

where L is the loop suppression factor of order 0.2–0.3. Clearly they are of comparable importance, with strong direct CP violation in the diagram.
The decay $\bar{B}^0 \to D^+D^-$ at the quark level is $b\bar{d} \to c\bar{d} + d\bar{c}$, which is again described by Figure (7.19) but with $q = c$. Thus the tree diagram is proportional to

$$V_{cb}V_{cd} \approx A\lambda^3\,,$$

with a negligible CP–violating contribution, while the leading penguin diagram is for $q' = t$ and is proportional to

$$LV_{tb}V_{td} \approx LA\lambda^3(1-\rho+i\eta)$$

Thus the penguin diagram is quite strongly CP violating and is only suppressed by a loop factor, so that direct CP violation again cannot be neglected.

PROBLEMS 8

8.1 For the ${}^{7}_{3}\mathrm{Li}$ nucleus, $Z=3$ and $N=4$. Hence, from Figure 8.4, the configuration is:

$$\text{protons:} \quad \left(1s_{1/2}\right)^2\left(1p_{3/2}\right)^1; \qquad \text{neutrons:} \quad \left(1s_{1/2}\right)^2\left(1p_{3/2}\right)^2$$

By the pairing hypothesis, the two neutrons in the $1p_{3/2}$ sub-shell will have a total orbital angular momentum and spin $\mathbf{L}=\mathbf{S}=\mathbf{0}$ and hence $\mathbf{J}=\mathbf{0}$. Therefore they will not contribute to the overall nuclear spin, parity or magnetic moment. These will be determined by the quantum numbers of the unpaired proton in the $1p_{3/2}$ subshell. This has $J=3/2$ and $l=1$, hence for the spin-parity we have $J^P = 3/2^-$. From (8.31), the magnetic moment is given by

$$m = j\,g_{\text{proton}} = j + 2.3 \text{ (since } j = l + 1/2) = 1.5 + 2.3 = 3.8 \text{ nuclear magnetons}$$

If only protons are excited, the two most likely excited states are:

$$\text{protons:} \quad \left(1s_{1/2}\right)^2\left(1p_{1/2}\right)^1; \qquad \text{neutrons:} \quad \left(1s_{1/2}\right)^2\left(1p_{3/2}\right)^2,$$

which corresponds to exciting a proton from the $p_{3/2}$ subshell to the $p_{1/2}$ subshell, and

$$\text{protons:} \quad \left(1s_{1/2}\right)^{-1}\left(1p_{3/2}\right)^2; \qquad \text{neutrons:} \quad \left(1s_{1/2}\right)^2\left(1p_{3/2}\right)^2,$$

which corresponds to exciting a proton from the $s_{1/2}$ subshell to the $p_{3/2}$ subshell.

8.2 A state with quantum number $j = l \pm 1/2$ can contain a maximum number $N_j = 2(2j+1)$ nucleons. Therefore, if $N_j = 16$ if follows that $j = 7/2$ and $l = 3$ or 4. But we know that the parity is odd, and since $P = (-1)^l$, it follows that $l = 3$.

8.3 The configuration of the ground state is:

$$\text{protons: } (1s_{1/2})^2(1p_{3/2})^4(1p_{1/2})^2(1d_{5/2})\ ; \qquad \text{neutrons: } (1s_{1/2})^2(1p_{3/2})^4(1p_{1/2})^2$$

To get $j^P = 1/2^-$, one could promote a $p_{1/2}$ proton to the $d_{5/2}$ shell, giving:

$$\text{protons: } (1s_{1/2})^2(1p_{3/2})^4(1p_{1/2})^{-1}(1d_{5/2})^2.$$

Then by the pairing hypothesis, the two $d_{5/2}$ protons could combine to give $j^P = 0^+$, so that the total spin-parity would be determined by the unpaired $p_{1/2}$ proton, i.e. $j^P = 1/2^-$. Alternatively, one of the $p_{3/2}$ neutrons could be promoted to the $d_{5/2}$ shell, giving

$$\text{neutrons: } (1s_{1/2})^2(1p_{3/2})^{-1}(1p_{1/2})^2(1d_{5/2})$$

and the $d_{5/2}$ proton and $d_{5/2}$ neutron and could combine to give $j^P = 2^+$, so that when this combines with the single unpaired $j^P = 3/2^-$ neutron the overall spin-parity is $j^P = 1/2^-$ (the other two $j^P = 3/2^-$ neutrons would pair to give $j^P = 0^+$), There are other possibilities.

8.4 For ${}^{93}_{41}\text{Nb}$, $Z = 41$ and $N = 52$. From the filling diagram Figure 8.4, the configuration is predicted to be:

$$\text{proton}: \ \ldots(2p_{3/2})^4(1f_{5/2})^6(2p_{1/2})^2(1g_{9/2})^1\ ; \qquad \text{neutron}: \ \ldots(2d_{5/2})^2.$$

So $l = 4,\ j = 9/2 \Rightarrow j^P = 9/2^+$ (which agrees with experiment). The magnetic dipole moment follows from the expression for j_{proton} in (8.31) with $j = l + 1/2$, i.e. $\mu = (j + 2.3)\mu_N = 6.8\mu_N$. (The measured value is $6.17\mu_N$.) For ${}^{33}_{16}\text{S}$, $Z = 16$ and $N = 17$. From the filling diagram Figure 8.4, the configuration is predicted to be:

$$\text{proton: } \cdots(1d_{5/2})^6(2s_{1/2})^2\ ; \qquad \text{neutron}: \ \cdots(1d_{5/2})^7(2s_{1/2})^2(1d_{3/2})^1.$$

So $l = 2,\ j = 3/2 \Rightarrow j^P = 3/2^+$ (which agrees with experiment). The magnetic dipole moment follows from the expression for j_{neutron} in (8.31) with $j = l - 1/2$, i.e. $\mu = [(1.9j)/(j+1)]\mu_N = 1.14\mu_N$. (The measured value is $0.64\mu_N$.)

8.5 From (8.32),

$$eQ = \int \rho(2z^2 - x^2 - y^2)\mathrm{d}\tau$$

with $\rho = Ze\big/(\frac{4}{3}\pi b^2 a)$ and the integral is through the volume of the spheroid $(x^2+y^2)\big/b^2 + z^2\big/a^2 \le 1$. The integral can be transformed to one over the volume of a sphere by the transformations $x = bx'$, $y = by'$ and $z = az'$. Then

$$Q = \frac{3Z}{4\pi}\iiint \mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z'\,(2a^2z'^2 - b^2x'^2 - b^2y'^2)\,.$$

But

$$\iiint x'^2\,\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z' = \frac{1}{3}\int_0^1 r'^2\,4\pi r'^2\mathrm{d}r' = \frac{4\pi}{15}\,,$$

and similarly for the other integrals. Thus, by direct substitution, $Q = 2Z(a^2-b^2)\big/5$.

8.6 From Question 8.5 we have $Q = 2Z(a^2-b^2)\big/5$ and using $Z = 67$ this gives $a^2 - b^2 = 13.1\ \mathrm{fm}^2$. Also, from (2.32) we have $A = 4\pi ab^2\rho/3$, where $\rho = 0.17\ \mathrm{fm}^{-3}$ is the nuclear density. Thus, $ab^2 = 231.7\ \mathrm{fm}^3$. The solution of these two equations gives $a \approx 6.85$ fm and $b \approx 5.82$ fm.

8.7 From (8.54) and (8.56),

$$t_{1/2} = \ln 2\big/\lambda = aR\ln 2\,\exp(G)\,,$$

where a is a constant formed from the frequency and the probability of forming alpha particles in the nucleus. Thus

$$t_{1/2}({}^{228}_{90}\mathrm{Th}) = t_{1/2}({}^{226}_{90}\mathrm{Th})R({}^{228}_{90}\mathrm{Th})\exp[G({}^{228}_{90}\mathrm{Th}) - G({}^{226}_{90}\mathrm{Th})]\big/R({}^{226}_{90}\mathrm{Th})\,.$$

The Gamow factors may be calculated from the data given by first finding the values of r_C from (8.49), recalling that (Z, A) refers to the daughter nucleus. This gives $r_C({}^{228}_{90}\mathrm{Th}) = 45.90\,\mathrm{fm}$, and $r_C({}^{226}_{90}\mathrm{Th}) = 39.28\,\mathrm{fm}$. Then the values of R are found using $R = 1.21(A^{1/3} + 4^{1/3})$, which gives $R({}^{228}_{90}\mathrm{Th}) = 9.269$ fm and $R({}^{226}_{90}\mathrm{Th}) = 9.247$ fm. Substituting these values in (8.52) then gives $G({}^{228}_{90}\mathrm{Th}) = 65.79$ and $G({}^{226}_{90}\mathrm{Th}) = 55.50$ and finally $t_{1/2}({}^{228}_{90}\mathrm{Th}) = 1.72\,\mathrm{yr}$. (The measured value is 1.9 yr.)

8.8(a) To balance the atomic and mass numbers, the decay is

$${}^{238}_{92}\mathrm{U} \to {}^{234}_{90}\mathrm{Th} + {}^{4}_{2}\mathrm{He}$$

and referring to Figure 8.9,

$$V_C(R) = 2Z\alpha\hbar c\big/R\,,$$

where Z refers to the daughter nucleus, and from (2.42), $R = 1.21A^{1/3}\mathrm{fm}$, where again A refers to the daughter nucleus. Thus

$$V_C(R) = \frac{2\times 90\times 197.33}{137\times 1.21\times (234)^{1/3}} \approx 35\,\text{MeV}\,.$$

In practice R is the effective radius of the combined thorium and alpha particle system and is closer to $R = 1.44A^{1/3}\text{fm}$, which would give $V_C(R) \approx 29\,\text{MeV}$.

(b) The kinetic energy released in the decay is given by

$$E_\alpha = M_{\text{U}}c^2 - (M_{\text{Th}} + M_\alpha)c^2 = 4.248\,\text{MeV}\,.$$

Then from Equation (8.49), $r_C = 2Z(\hbar c)\alpha / E_\alpha$, where α is the fine structure constant. Evaluating this gives

$$r_C = \frac{(2\times 90)\times(197.33\,\text{MeV fm})}{137(4.248\,\text{MeV})} = 61\,\text{fm}.$$

8.9 From the Geiger-Nuttall relation (8.44),

$$\ln_{10}\lambda = \log\left(\frac{\ln 2}{t_{1/2}}\right) = \log\left(\frac{0.693}{t_{1/2}}\right) = c + d\log r\,.$$

For nuclei A,

$$c + d\log(3) = \log\left(\frac{0.693}{3.65\times 10^5}\right),$$

and for nuclei B,

$$c + d\log(4) = \log\left(\frac{0.693}{10^2}\right).$$

where we have used days for units of lifetime and centimetres for units of range. Solving for c and d gives $c = -19.32$ and $d = 28.50$. Finally, using these values in the Geiger-Nuttall relation for nuclei C gives $t_{1/2} \approx 9.63\times 10^{-4}$ days, i.e. 1.39 mins.

8.10 In the centre-of-mass system, the threshold for ${}^{34}\text{S} + p \to n + {}^{34}\text{Cl}$ is $6.45\times(34/35) = 6.27$ MeV. Correcting for the neutron-proton mass difference gives the Cl–S mass difference as 5.49 MeV, and since in the positron decay mode ${}^{34}\text{Cl} \to {}^{34}\text{S} + e^+ + \nu_e$ the energy released is

$$Q = M(A,Z) - M(A,Z-1) - 2m_e\,,$$

the maximum positron energy is 4.47 MeV.

8.11 We need to calculate the fraction

$$F \equiv \left[\int_{T_0-\Delta}^{T_0} I(T)\,\mathrm{d}T\right]\left[\int_0^{T_0} I(T)\,\mathrm{d}T\right]^{-1},$$

where $\Delta = 5$ eV and $I(T) = T^{1/2}(T_0 - T)^2$ with $T_0 = 18.6$ keV . Evaluating the denominator gives $16T_0^{7/2}/105$ and using the integral given, the numerator is $T_0^{1/2}\Delta^3/3$. Thus

$$F = \frac{T_0^{1/2}\Delta^3}{3} \cdot \frac{105}{16T_0^{7/2}} = 2.19\left(\frac{\Delta}{T_0}\right)^3 = 4.25 \times 10^{-11}.$$

8.12 The mean energy $\bar{T}$ is defined by

$$\bar{T} \equiv \left[\int_0^{T_0} T\,\mathrm{d}\omega(T)\right]\left[\int_0^{T_0} \mathrm{d}\omega(T)\right]^{-1}.$$

The integrals are:

$$\int T^{3/2}(T_0 - T)^2 \mathrm{d}T = \tfrac{2}{315} T^{5/2}\left[63T_0^2 - 90T_0T + 35T^2\right]$$

and

$$\int T^{1/2}(T_0 - T)^2 \mathrm{d}T = \tfrac{2}{105} T^{3/2}\left[35T_0^2 - 42T_0T + 15T^2\right].$$

Substituting the limits gives $\bar{T} = T_0/3$.

8.13 The possible transitions are as follows:

Initial	Final	L	ΔP	Multipoles
$3/2^-$	$5/2^-$	1, 2, 3, 4	No	M1, E2, M3, M4
$3/2^-$	$1/2^-$	1, 2	No	M1, E2
$5/2^-$	$1/2^-$	2, 3	No	E2, M3

From Figure 8.14, the dominant multipole for a fixed transition energy will be M1 for the $3/2^- \to 5/2^-$ and $3/2^- \to 1/2^-$ transitions and E2 for the $5/2^- \to 1/2^-$ transition. Thus we need to calculate the rate for a M1 transition with $E_\gamma = 178$ keV . This can be done using equation (8.89) together with the relation $\tau_{1/2} = \hbar \ln 2/\Gamma$ and gives $\tau_{1/2} \approx 3.9 \times 10^{-12}$ s . The measured value is 3.5×10^{-10} s , which confirms that the Weisskopf approximation is not very accurate.

8.14 The J^P values of the Σ^0 and the Λ are both $1/2^+$, so the photon has $L=1$ and as there is no change of parity the decay proceeds via an M1 transition. The Δ^0 has $J^P = 3/2^+$ and again there is no parity change. Therefore both M1 and E2 multipoles could be involved, with M1 dominant (see Section 8.8.2). If we assume that the reduced transition probabilities are equal in the two cases, then from (8.89), in an obvious notation,

$$\tau(\Sigma^0) = \left[\frac{E_\gamma(\Delta^0)}{E_\gamma(\Sigma^0)}\right]^3 \frac{\tau(\Delta^0)}{B},$$

where B is the branching ration for the decay $\Delta^0(1232) \to n + \gamma$.

$$\tau(\Sigma^0) = (292/77)^3 \times (0.6\times 10^{-23})/0.0056 = 5.8\times 10^{-20}\ \text{s}.$$

(The measured value is $(7.4 \pm 0.7)\times 10^{-20}\text{s}$)

8.15 Set $L=3$ in (8.88a), substitute the result into (8.86) to give

$$\Gamma = \hbar T^E = \alpha\hbar c\left(E_\gamma/\hbar c\right)^7 R^6 N(3),$$

where $N(3) = 2/[3(105)^2]$. Then using $R = 1.21A^{1/3}\text{fm}$ gives

$$\Gamma = (2.35\times 10^{-14})E_\gamma^7 A^2\,\text{eV},$$

where E_γ is measured in MeV.

PROBLEMS 9

9.1 To balance the number of protons and neutrons, the fission reaction must be

$$n + {}^{235}_{92}\text{U} \to {}^{92}_{37}\text{Rb} + {}^{140}_{55}\text{Cs} + 4n$$

The energy released is the differences in binding energies of the various nuclei, because the mass terms in the SEMF (2.49) cancel out. We have, in an obvious notation,

$$\Delta(A) = 3\,;\ \ \Delta(A^{2/3}) = -9.26\,;\ \ \Delta\left[\frac{(Z-N)^2}{4A}\right] = 0.28\,;\ \ \Delta\left[\frac{Z^2}{A^{1/3}}\right] = 485.9\,.$$

The contribution from the pairing term is negligible (about 1MeV). Using the numerical values for the coefficients in the SEMF given in (2.57), the energy released per fission is $E_F = 158.5$ MeV and the power of the nuclear reactor is

$$P = nE_F = 100\,\mathrm{MW} = 6.24\times 10^{20}\,\mathrm{MeV s^{-1}}\,,$$

where n is the number of fissions per second. Since one neutron escapes per fission and contributes to the flux, the flux F is equal to the number of fissions per unit area per second, i.e.

$$F = \frac{n}{4\pi r^2} = \frac{P}{4\pi r^2 E_F} = 3.13\times 10^{17}\,\mathrm{s^{-1}\,m^{-2}}\,.$$

The interaction rate R_I is given by

$$R_I = \sigma \times F \times n_T\,,$$

where n_T is the number of target particles, given by $n_T = n \times N_A$, N_A is Avogadro's number, and n is found from the ideal gas law to be $n = PV/RT$, where R is the ideal gas constant. Using $T = 298K$, $P = 1\times 10^5\,\mathrm{Pa}$ and $R = 8.31\,\mathrm{Pa\ m^3\,mol^{-1}\,K^{-1}}$, gives $n = 52.5$ mol and hence $n_T = 3.16\times 10^{25}$ Finally, using the cross-section $\sigma = 10^{-31}\,\mathrm{m}^2$, the interaction rate is $R_I = 0.989\times 10^{12}\,\mathrm{s^{-1}}$.

9.2 The neutron speed in the CM system is

$$\upsilon - m\upsilon/(M+m) = M\upsilon/(M+m)$$

and if the scattering angle in the CM system is θ, then after the collision the neutron will have a speed

$$\upsilon(m + M\cos\theta)/(M+m)$$

in the original direction and

$$M\upsilon\sin\theta/(M+m)$$

perpendicular to this direction. Thus the kinetic energy is

$$E(\cos\theta) = \frac{m\upsilon^2(M^2 + 2mM\cos\theta + m^2)}{2(M+m)^2}$$

and the average value is

$$E_{\text{final}} = \overline{E} \equiv \left[\int_{-1}^{1} E(\cos\theta)\,\mathrm{d}\cos\theta\right]\left[\int_{-1}^{1} \mathrm{d}\cos\theta\right]^{-1} = RE_{\text{initial}}\,,$$

where the reduction factor is $R = (M^2 + m^2)/(M+m)^2$.

For neutron scattering from graphite, $R \approx 0.86$ and after N collisions the energy will be reduced to $E_{\text{final}} = R^N E_{\text{initial}}$. As stated in Section 9.1.1(a), the average initial energy of fission neutrons from ^{235}U is 2 MeV and to thermalise them their energy would have to be reduced to about 0.025 eV, which implies

$$N \approx \ln(E_{\text{final}}/E_{\text{initial}})/\ln(0.86) \approx 121 .$$

9.3 From (1.57a), for the fission of ^{235}U, $W_f = JN(235)\sigma_f$ and the total power output is $P = W_f E_f$, where E_f is the energy released per fission. For the capture by ^{238}U, $W_c = JN(238)\sigma_c$. Eliminating the flux J, gives

$$W_c = \frac{N(238)\sigma_c}{N(235)\sigma_f}\left(\frac{P}{E_f}\right).$$

Using the data supplied, gives $W_c = 1.08 \times 10^{19}$ atoms $\text{s}^{-1} \approx 135$ kg y^{-1}.

9.4 From the definition of the multiplication factor, $F = k/N = 0.38$, and hence approximately 11 neutrons go on to induce fission. Each of these will produce k neutrons in the first cycle, k^2 in the second cycle and so on. Thus in n cycles, there will be $11(1 + k + k^2 \cdots k^n)$ neutrons for each initial 1 GeV proton, and for large n this is approximately $11/(1-k)$. Finally,

$$\text{energy gain} = \frac{11}{(1-k)} \times \frac{\text{energy per fission}}{\text{energy per proton}} = 44.$$

9.5 The PPI chain overall is:

$$4(^1\text{H}) \to {}^4\text{He} + 2e^+ + 2\nu_e + 2\gamma + 24.68\ \text{MeV}.$$

Two corrections have to be made to this. Firstly, the positrons will annihilate with electrons in the plasma releasing a further $2m_e c^2 = 1.02\,\text{MeV}$ per positron. Secondly, each neutrino carries off 0.26 MeV of energy into space that will not be detected. So, making these corrections, the total output per hydrogen atom is 6.55 MeV. The total energy produced to date is 5.60×10^{43} Joules $= 3.50 \times 10^{56}$ MeV. Thus, the total number of hydrogen atoms consumed is 5.34×10^{55} and so the fraction of the Sun's hydrogen used is 5.9% and as this corresponds to 4.6 billion years, the Sun has another 73 billion years to burn before its supply of hydrogen is exhausted.

9.6 A solar constant of 8.4 J cm^{-2} s^{-1} is equivalent to 5.25×10^{13} MeV cm^{-2} s^{-1} of energy deposited. If this is due to the PPI reaction

$$4(^1\text{H}) \to {}^4\text{He} + 2e^+ + 2\nu_e + 2\gamma,$$

then this rate of energy deposition corresponds to a flux of $\left(5.25\times10^{13}/2\times6.55\right)\approx 4\times10^{12}$ neutrinos per cm^2 per second.

9.7 For the Lawson criterion to be just satisfied, from (9.52),

$$L=\frac{n_d\left\langle\sigma_{dt}v\right\rangle t_c Q}{6kT}=1\,.$$

We have $kT=10$ keV and from Figure 9.8 we can estimate $\left\langle\sigma_{dt}v\right\rangle\approx10^{-22}\ \text{m}^3\ \text{s}^{-1}$. Also, from (9.51), $Q=17.6$ MeV. So, finally, $n_d=6.8\times10^{18}\ \text{m}^{-3}$.

9.8 The mass of a d-t pair is $5.03\ \text{u}=8.36\times10^{-24}\,\text{g}$. The number of d-t pairs in a 1 mg pellet is therefore 1.2×10^{20}. From (9.51), each d-t pair releases 17.6 MeV of energy. Thus, allowing for the efficiency of conversion, each pellet releases $5.3\times10^{26}\,\text{eV}$. The output power is $750\ \text{MW}=4.7\times10^{27}\ \text{eV/s}$. Thus the rate of pellets required is $8.9\approx9$ per sec.

9.9 Assume a typical body mass of 70 kg. Since approximately 70% of the body consists of water, roughly half of the mass is protons. This corresponds to 2.1×10^{28} protons and after 1 yr the number that will have decayed is $2.1\times10^{28}[1-\exp(-1/\tau)]$, where τ is the lifetime of the proton in years. Each proton will eventually deposit almost all of its rest energy, i.e. approximately 0.938 GeV, in the body. Thus in 1 yr the total energy in Joules deposited per kg of body mass would be $4.5\times10^{16}[1-\exp(-1/\tau)]$ and this amount will be lethal if greater than 5Gy. Expanding the exponential gives the result that the existence of humans implies $\tau\gtrsim10^{16}\,\text{yr}$.

9.10 The approximate rate of whole-body radiation absorbed is given by Equation (9.58a). Substituting the data given, we have

$$\frac{\mathrm{d}D}{\mathrm{d}t}(\mu\text{Sv h}^{-1})=\frac{A(\text{MBq})\times E_\gamma(\text{MeV})}{6r^2(\text{m}^2)}=1.67\times10^{-2}\mu\text{Sv h}^{-1}$$

and so in 18h, the total absorbed dose is $0.30\ \mu\text{Sv}$.

9.11 If the initial intensity is I_0, then from (4.21), the intensities after passing through the bone (surrounded by tissue), I_b, and the same thickness of tissue only I_t, are

$$I_b\approx I_0\exp[-(b/\lambda_b)+2(t/\lambda_t)]\ \text{ and }\ I_t\approx I_0\exp[-(b+2t)/\lambda_t]\,.$$

Thus

$$R=\exp[-b(\mu_b-\mu_t)]=0.7$$

and hence

$$b = -\ln(0.7)/(\mu_b - \mu_t) = 2.5 \text{ cm}.$$

9.12 From Figure 4.10, the rate of ionisation energy losses is only slowly varying for momenta above about 1 GeV/c and given that living matter is mainly water and hydrocarbons a reasonable estimate is $3\,\text{MeV g}^{-1}\text{ cm}^2$. Thus the energy deposited in one year is 2.37×10^9 MeV kg^{-1}, which is 3.8×10^{-4} Gy.

9.13 The nuclear magnetic resonance frequency is given in Section 9.4.4 as $f = |\mu| B/jh$. The spin of the nucleus may be found from the discussion in Section 8.3.3. There it was shown from the pairing hypothesis that the spin of an odd-A nucleus is determined by the sole unpaired nucleon and may be found using a filling diagram such as Figure 8.4. This gives $j = 5/2$ for $^{55}_{25}\text{Mn}$. The other numerical data we use are: $B = 2$ T, $\mu = 3.46\ \mu_N$. Substituting then gives $f = 21.1$ MHz.

9.14 The count rate is proportional to the number of ^{14}C atoms present in the sample. If we assume that the abundance of ^{14}C has not changed with time, and that the artefact was made from living material and is predominantly carbon, then at the time it was made $(t = 0)$, 1g would have contained 5×10^{22} carbon atoms of which $N_0 = 6\times 10^{10}$ would have been ^{14}C. Thus the average count rate would have been $N_0/\tau = 13.8$ per minute. At time t, the number of ^{14}C atoms would be $N(t) = N_0 \exp(-t/\tau)$ and

$$N(t)/N_0 = e^{-t/\tau} = 2.1/13.8,$$

from which $t = \tau \ln 6.57 = 1.56\times 10^4\,\text{yr}$. The artifact is therefore approximately 15600 years old.

9.15 If there were N_0 atoms of each isotope at the formation of the planet $(t = 0)$, then after time t the number of atoms of ^{205}Pb is

$$N_{205}(t) = N_0 \exp(-t/\tau_{205}), \quad \text{with} \quad N_{204}(t) = N_0,$$

so that

$$\frac{N_{205}(t)}{N_{204}(t)} = \exp\left(-\frac{t}{\tau_{205}}\right) = \frac{n_{205}}{n_{204}} = 2\times 10^{-7}$$

and $t = -\tau_{205} \ln(2\times 10^{-7}) = 2.4\times 10^8\,\text{y}$.

PROBLEMS A

A.1 Using the definition (A.42) we have

$$[\hat{I}_+,\hat{I}_-]=[(\hat{I}_x+i\hat{I}_y),(\hat{I}_x-i\hat{I}_y)]$$
$$=[\hat{I}_x,\hat{I}_x]+i[\hat{I}_y,\hat{I}_x]-i[\hat{I}_x,\hat{I}_y]+[\hat{I}_y,\hat{I}_y].$$

Then evaluating the brackets using (A.40) gives

$$[\hat{I}_+,\hat{I}_-]=0+\hat{I}_z+\hat{I}_z+0=2\hat{I}_z,$$

as required. Similarly,

$$[\hat{I}_z,\hat{I}_\pm]=[\hat{I}_z,(\hat{I}_x\pm i\hat{I}_y)]=[\hat{I}_z,\hat{I}_x]\pm i[\hat{I}_z,\hat{I}_y]$$
$$=i\hat{I}_y\pm\hat{I}_x=\pm i\hat{I}_\pm.$$

A.2 Suppose $\left|I,I_3\right\rangle$ and $\left|I,I_3\pm1\right\rangle$ are two states in the same multiplet with energies E and E', respectively. Since both exist, they must be related by (A.52a), i.e.

$$\hat{I}_\pm\left|I,I_3\right\rangle=C_\pm\left|I,I_3\pm1\right\rangle \qquad \text{(A)}$$

where $C_\pm\equiv C_\pm(I,I_3)\neq0$. Then since $I_\pm$ and H commute, because isospin is conserved, acting on the left-hand side of (A) gives

$$H\hat{I}_\pm\left|I,I_3\right\rangle=\hat{I}_\pm H\left|I,I_3\right\rangle=E\hat{I}_\pm\left|I,I_3\right\rangle=EC_\pm\hat{I}_\pm\left|I,I_3\right\rangle,$$

while acting on the right-hand side gives

$$C_\pm H\left|I,I_3\pm1\right\rangle=E'C_\pm\left|I,I_3\pm1\right\rangle$$

and hence, since $C_\pm\neq0$, $E=E'$, as required.

A.3 Consider the antiquark assignments of (A.54), i.e.

$$\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle=\bar{d},\quad \left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle=-\bar{u}\ .$$

Then using (A.45b) for the $\bar{u}$ state, for example, gives

$$\hat{I}_3\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle=-\hat{I}_3\bar{u}=\tfrac{1}{2}\bar{u}=-\tfrac{1}{2}\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle,$$
$$\hat{I}_-\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle=-\hat{I}_-\bar{u}=0,$$
$$\hat{I}_+\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle=-\hat{I}_+\bar{u}=\bar{d}=\left|\tfrac{1}{2},\tfrac{1}{2}\right\rangle,$$

which coincide precisely with the results obtained by applying (A.52a) and (A.52b) to $\left|I,I_3\right\rangle=\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle$. To deduce the result for I^2, we use (A.49a)) together with (A.45b)) to obtain

$$\hat{I}^2\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle = -\hat{I}^2\overline{u} = -\tfrac{1}{2}[\hat{I}_+\hat{I}_- + \hat{I}_-\hat{I}_+]\overline{u} - \hat{I}_3\overline{u}$$
$$= -\tfrac{3}{4}\,\overline{u} = -\tfrac{3}{4}\left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle,$$

which coincides precisely with the result obtained from (A.50a)), as required. The results for the other $\left|I,I_3\right\rangle$ states follow in a similar way.

A.4 We need to identify the isospin of the final state. This is either $I = 3/2,\, I_3 = -1/2$, or $I = 1/2,\, I_3 = -1/2$. The former may be obtained from the unique state

$$\left|N\pi;\tfrac{3}{2},\tfrac{3}{2}\right\rangle = \left|N;\tfrac{1}{2},\tfrac{1}{2}\right\rangle\left|\pi;1,1\right\rangle$$

by using the lowering operator $\hat{I}_-$. From the left-hand side,

$$\hat{I}_-\left|N\pi;\tfrac{3}{2},\tfrac{3}{2}\right\rangle = \sqrt{3}\left|N\pi;\tfrac{3}{2},\tfrac{1}{2}\right\rangle,$$

and for the right-hand side

$$\hat{I}_-\left|N\pi;\tfrac{3}{2},\tfrac{1}{2}\right\rangle = \left|N;\tfrac{1}{2},-\tfrac{1}{2}\right\rangle\left|\pi;1,1\right\rangle + \sqrt{2}\left|N;\tfrac{1}{2},\tfrac{1}{2}\right\rangle\left|\pi;1,0\right\rangle$$

Comparing these, we obtain the desired results

$$\left|N\pi;\tfrac{3}{2},\tfrac{1}{2}\right\rangle = \tfrac{1}{\sqrt{3}}\left|N;\tfrac{1}{2},-\tfrac{1}{2}\right\rangle\left|\pi;1,1\right\rangle + \sqrt{\tfrac{2}{3}}\left|N;\tfrac{1}{2},\tfrac{1}{2}\right\rangle\left|\pi;1,0\right\rangle$$
$$= -\tfrac{1}{\sqrt{3}}\left|n\pi^+\right\rangle + \sqrt{\tfrac{2}{3}}\left|p\pi^0\right\rangle,$$

where in the second equation we have identified the pion and nucleon charge states. The state with $I = 1/2,\, I_3 = -1/2$ must be orthogonal to this and is therefore

$$\left|N\pi;\,\tfrac{1}{2},\,-\tfrac{1}{2}\right\rangle = \sqrt{\tfrac{1}{3}}\left|n\pi^0\right\rangle - \sqrt{\tfrac{2}{3}}\left|p\pi^-\right\rangle,$$

up to an irrelevant overall phase factor. The branching fractions for $X^0 \to n\pi^0$ and $X^0 \to p\pi^-$ are therefore in the ratio $2:1$ for $I = 3/2$ and $1:2$ for $I = 1/2$, since isospin is conserved in the strong interaction. From the data given, $I = 1/2$. (The X^0 is actually the $N^0(1520)$ and is an excited state of the neutron.)

A.5 We need to identify the $\Sigma\pi$ state with $I = I_3 = 0$. Since $I_3 = 0$, this must be of the form

$$\left|\Sigma\pi;\,0,\,0\right\rangle = \alpha\left|\Sigma;\,1,\,1\right\rangle\left|\pi;\,1,-1\right\rangle + \beta\left|\Sigma;\,1,\,0\right\rangle\left|\pi;\,1,0\right\rangle$$
$$+\,\gamma\left|\Sigma;\,1,\,-1\right\rangle\left|\pi;\,1,1\right\rangle,$$

and because it has $I = 0$, it must satisfy

$$I_+ \left| \Sigma\pi;\, 0,\, 0 \right\rangle = I_- \left| \Sigma\pi;\, 0,\, 0 \right\rangle = 0 \,.$$

Either of these conditions is sufficient to determine b and γ in terms of a, and the resulting normalised state is

$$\left| \Sigma\pi;\, 0,\, 0 \right\rangle = \sqrt{\tfrac{1}{3}} \left\{ \left| \Sigma^+\pi^- \right\rangle - \left| \Sigma^0\pi^0 \right\rangle - \left| \Sigma^-\pi^+ \right\rangle \right\}$$

up to an irrelevant overall phase factor. Thus the branching ratios to each charged state are equal; i.e. they must each be 14% to give a total $\Sigma\pi$ branching ratio of 42% as stated.

PROBLEMS B

B.1 (a) From the definitions of s, t and u, we have

$$\begin{aligned}(s+t+u)c^2 = {} & (p_A^2 + 2p_A p_B + p_B^2) \\ & + (p_A^2 - 2p_A p_C + p_C^2) + (p_A^2 - 2p_A p_D + p_D^2)\end{aligned},$$

which using $p_A^2 = m_A^2 c^2$ etc, becomes

$$(s+t+u)c^2 = 3m_A^2c^2 + m_B^2c^2 + m_C^2c^2 + m_D^2c^2 + 2p_A(p_B - p_C - p_D)\,.$$

But from 4-momentum conservation, $p_A + p_B = p_C + p_D$, so that

$$(s+t+u)c^2 = 3m_A^2c^2 + m_B^2c^2 + m_C^2c^2 + m_D^2c^2 - 2p_A^2$$

and hence

$$(s+t+u) = \sum_{j=A,B,C,D} m_j^2 \,.$$

(b) From the definition of t,

$$c^2 t = p_A^2 + p_C^2 - 2p_A p_C = m_A^2c^2 + m_C^2c^2 - 2\left(\frac{E_A E_C}{c^2} - \mathbf{p}_A \cdot \mathbf{p}_C \right).$$

For elastic scattering, $A \equiv C$. Thus $E_A = E_C$ and $\left|\mathbf{p}_A\right| = \left|\mathbf{p}_C\right| = p$, so that $\mathbf{p}_A \cdot \mathbf{p}_C = p^2 \cos\theta$. Then

$$c^2 t = 2m_A^2c^2 - 2\left(E_A^2 / c^2 - p^2 \cos\theta \right)$$

and using $E_A^2 = p^2c^2 + m_A^2c^4$, gives $t = -2p^2(1-\cos\theta)/c^2$.

B.2 Energy conservation gives $E_\pi = E_\mu + E_\nu$, where

$$E_\pi = \gamma m_\pi c^2, \qquad E_\mu = c(m_\mu^2 c^2 + p_\mu^2)^{1/2}, \qquad E_\nu = p_\nu c$$

and hence

$$\left(\gamma m_\pi c^2 - p_\nu c\right)^2 = c^2\left(m_\mu^2 c^2 + p_\mu^2\right) \tag{1}$$

But 3-momentum conservation gives

$$p_\mu \cos\theta = p_\pi = \gamma m_\pi v, \qquad p_\mu \sin\theta = p_v = E_\nu / c \tag{2}$$

Eliminating p_μ and p_ν between (1) and (2) and simplifying, gives

$$\tan\theta = \frac{(m_\pi^2 - m_\mu^2)}{2\beta\gamma^2 m_\pi^2}.$$

B.3 Conservation of 4-momentum is $p_\mu = p_\pi - p_\nu$, from which $p_\mu^2 = p_\pi^2 + p_\nu^2 - 2p_\pi p_\nu$. Now $p_j^2 = m_j^2 c^2$ for $j = \pi,\ \mu$ and ν, and

$$p_\pi p_v = \frac{E_\pi E_\nu}{c^2} - \mathbf{p}_\pi \cdot \mathbf{p}_\nu = m_\pi E_\nu = m_\pi \left|\mathbf{p}_\nu\right| c\,,$$

because $\mathbf{p}_\pi = \mathbf{0}$ and $E_\pi = m_\pi c^2$ in the rest frame of the pion. But $\left|\mathbf{p}_\nu\right| = \left|\mathbf{p}_\mu\right| \equiv p$ because the muon and neutrino emerge back-to-back. Thus, $p = (m_\pi^2 - m_\mu^2)c / 2m_\pi$. But $p = \gamma m_\mu v$, from which $v = pc(p^2 + m_\mu^2 c^2)^{-1/2}$. Finally, substituting for p gives

$$v = \left(\frac{m_\pi^2 - m_\mu^2}{m_\pi^2 + m_\mu^2}\right) c\,.$$

B.4 By momentum conservation, the momentum components of X^0 are:

$$p_x = -0.743 \text{ (GeV/c)}, \quad p_y = -0.068 \text{ (GeV/c)}, \quad p_z = 2.595 \text{ (GeV/c)}$$

and hence $p_X^2 = 7.291$. Also,

$$p_A^2 = 4.686 \text{ (GeV/c)}^2 \text{ and } p_B^2 = 0.304 \text{ (GeV/c)}^2\,.$$

Under hypothesis (a):

$$E_A = (m_\pi^2 c^4 + p_A^2 c^2)^{1/2} = 2.169 \text{ GeV}$$

and

$$E_B = (m_K^2 c^4 + p_B^2 c^2)^{1/2} = 0.740 \text{ GeV}.$$

Thus $E_X = 2.909$ GeV and

$$M_X = (E_X^2 - p_X^2 c^2)^{1/2} c^{-2} = 1.082 \text{ GeV/c}^2.$$

Under hypothesis (b):

$$E_A = (m_p^2 c^4 + p_A^2 c^2)^{1/2} = 2.359 \text{ GeV}$$

and

$$E_B = (m_\pi^2 c^4 + p_B^2 c^2)^{1/2} = 0.569 \text{ GeV}.$$

Thus $E_X = 2.928$ GeV and $M_X = (E_X^2 - p_X^2 c^2)^{1/2} c^{-2} = 1.132 \text{ GeV/c}^2$. Since $M_D = 1.86 \text{ GeV/c}^2$ and $M_\Lambda = 1.12 \text{ GeV/c}^2$, the decay is $\Lambda \to p + \pi^-$.

B.5 The total 4-momentum of the initial state is

$$p_{tot} = \left[\left(E + m_p c^2\right)/c, \mathbf{p}_L\right].$$

Hence the invariant mass W is given by

$$(Wc^2)^2 = (E_L + m_p c^2)^2 - p_L^2 c^2,$$

where $p_L \equiv |\mathbf{p}_L|$. The invariant mass squared in the final state evaluated in the centre-of-mass frame has a minimum value $(4m_p c)^2$ when all four particles are stationary. Thus, E_{min} is given by

$$(E_{min} + m_p c^2)^2 - p_L^2 c^2 = (4m_p c^2)^2$$

which expanding and using

$$E_{\min}^2 - p_L^2 c^2 = m_p^2 c^4,$$

gives $E_{\min} = 7m_p c^2 = 6.6$ GeV.

For a bound proton, the initial 4-momentum of the projectile is $(E'_L/c, \mathbf{p}'_L)$ and that of the target is $(E/c, -\mathbf{p})$, where p is the internal momentum of the nucleons, which we have taken to be in the opposite direction of the beam because this gives the maximum invariant mass for a given E'_L. The invariant mass W' is now given by

$$(W'c^2)^2 = (E'_L + E)^2 - (p'_L - p)^2 c^2 = 2m_p^2 c^4 + 2EE'_L + 2pp'_L c^2.$$

Since the thresholds E_{min} and E'_{min} correspond to the same invariant mass $4m_p$, we have

$$2m_p c^2 E_{min} = 2EE'_{min} + 2pp'_{min}c^2 .$$

Finally, since the internal momentum of the nucleons is $\sim 250\,\mathrm{MeV}/\mathrm{c}$ (see Chapter 7), $E \approx m_p c^2$, while for the relativistic incident protons $p'_{min} \approx E'_{min}/c$, so using these gives

$$E'_{min} \approx \left(1 - p/m_p c\right) E_{min} = 4.8\,\mathrm{GeV} .$$

B.6 The initial total energy is $E_i = E_A = m_A c^2$ and the final total energy is $E_f = E_B + E_C$, where

$$E_B = (m_B^2 c^4 + p_B^2 c^2)^{1/2}, \text{ and } E_C = (m_C^2 c^4 + p_C^2 c^2)^{1/2} ,$$

with $p_B = \left|\mathbf{p}_B\right|$ and $p_C = \left|\mathbf{p}_C\right|$. But by momentum conservation, $\mathbf{p}_B = -\mathbf{p}_C \equiv \mathbf{p}$ and so

$$\left[m_A c^2 - (m_B^2 c^4 + p^2 c^2)^{1/2}\right]^2 = \left(m_C^2 c^4 + p^2 c^2\right),$$

which on expanding gives $E_B = \left(m_A^2 + m_B^2 - m_C^2\right) c^2 / 2m_A$.

B.7 If the 4-momenta of the photons are $p_i = (E_i/c, \mathbf{p}_i)$ $(i = 1,2)$, then the invariant mass of M is given by

$$M^2 c^4 = (E_1 + E_2)^2 - (\mathbf{p}_1 + \mathbf{p}_2)c^2 = 2E_1 E_2 (1 - \cos\theta) ,$$

since $\mathbf{p}_1 \cdot \mathbf{p}_2 = E_1 E_2 \cos\theta / c^2$ for zero-mass photons. Thus,

$$\cos\theta = 1 - M^2 c^4 / 2E_1 E_2 .$$

B.8 In an obvious notation, the kinematics in the lab frame are:

$$\gamma(E_\gamma, \mathbf{p}_\gamma) + e^-(mc^2, 0) \to \gamma(E'_\gamma, \mathbf{p}'_\gamma) + e^-(E, \mathbf{p}) .$$

Energy conservation gives $E_\gamma + mc^2 = E'_\gamma + E_e$ and momentum conservation gives $\mathbf{p}_\gamma = \mathbf{p}'_\gamma + \mathbf{p}_e$. From the latter we have

$$E_e^2 - m^2 c^4 = c^2 (\mathbf{p}_\gamma{}^2 + \mathbf{p}'^2_\gamma - 2\mathbf{p}_\gamma \cdot \mathbf{p}'_\gamma) .$$

But $p_\gamma c = E_\gamma$, $p'_\gamma c = E'_\gamma$ and the scattering angle is θ, so we have

$$E_e^2 - m^2c^4 = E_\gamma^2 + E_\gamma'^2 - 2E_\gamma E'_\gamma \cos\theta \ .$$

Eliminating E_e between this equation and the equation for energy conservation gives

$$E'_\gamma = E_\gamma[1 + E_\gamma(1-\cos\theta)/mc^2]^{-1} \ .$$

Finally, using $E'_\gamma = E_\gamma/2$ and $\theta = 60^0$, gives $E_\gamma = 2mc^2 = 1.02\,\mathbf{MeV}$.

PROBLEMS C

C.1 The assumptions are: ignore recoil of target nucleus because its mass is much greater than the total energy of the projectile α particle; use nonrelativistic kinematics because the kinetic energy of the α particle is very much less than its rest mass; assume Rutherford formula (i.e. the Born approximation) is valid for small-angle scattering. The relevant formula is then (C.13) and it may be evaluated using $z = 2, Z = 83$, $E_{kin} = 20$ MeV and $\theta = 20^0$. The result is $\mathrm{d}\sigma/\mathrm{d}\Omega = 98.3$ b/sr.

C.2 From Figure C.2, the distance of closest approach d is when $x = 0$. For $x < 0$, the sum of the kinetic and potential energies is $E_{ke} = \frac{1}{2}mv^2$ and the angular momentum is mvb. At $x = 0$, the total mechanical energy is

$$\frac{mu^2}{2} + \frac{Zze^2}{4\pi\varepsilon_0 d}$$

and the angular momentum is mud, where u is the instantaneous velocity. From angular momentum conservation, $u = vb/d$ and using this in the conservation of total mechanical energy gives $d^2 - Kd - b^2 = 0$, where, using (C.9), $K \equiv 2b/\cot(\theta/2)$. The solution for $d \ge 0$ is

$$d = b\big[1 + \mathrm{cosec}(\theta/2)\big]\big/\cot(\theta/2).$$

C.3 The result for small-angle scattering follows directly from (C.9) in the limit $\theta \to 0$. Evaluating b, we have, using the data given,

$$b = \frac{zZe^2}{2\pi\varepsilon_0 mv^2\theta} = 2zZ\left(\frac{e^2}{4\pi\varepsilon_0\hbar c}\right)\frac{\hbar c}{mc^2}\frac{1}{(v/c)^2\theta} = 1.55\times10^{-13}\,\mathrm{m} \ .$$

The cross-section for scattering through an angle greater than 5^0 is thus $\sigma = \pi b^2 = 7.55\times10^{-26}\mathrm{m}^2$ and the probability that the proton scatters through an angle greater than 5^0 is $P = 1 - \exp(-n\sigma t)$, where n is the number density of the target. Using $n = 6.658\times10^{28}\mathrm{m}^{-3}$, gives $P = 4.91\times10^{-2}$. Since P is very small but the number of scattering centres is very large, the scattering is governed by the Poisson distribution and the probability for a single scatter is

$$P_1(m) = me^{-m} = 4.91\times10^{-2}\text{, giving } m \approx 0.052\,.$$

Finally, the probability for two scattering is

$$P_2 = m^2 \exp(-m)/2! \approx 1.3\times10^{-3}\ .$$

PROBLEMS D

D.1 From (D.4) we obtain (recall that $\mathbf{r} = x_1,\, x_2,\, x_3$)

$$\dot{x}_i = \frac{\partial H}{\partial p_i} = \frac{(p_i - qA_i)}{m} \tag{1}$$

so that

$$p_i = m\dot{x}_i + qA_i(\mathbf{r},t); \tag{2}$$

and

$$\begin{aligned}\dot{p}_i &= -\frac{\partial H}{\partial x_i} = -q\frac{\partial \phi}{\partial x_i} + \frac{q}{m}\sum_j (p_j - qA_j)\frac{\partial A_j}{\partial x_i} \\ &= -q\frac{\partial \phi}{\partial x_i} + q\sum_j \dot{x}_j\frac{\partial A_j}{\partial x_i}.\end{aligned} \tag{3}$$

But from (2),

$$\dot{p}_i = m\ddot{x}_i + q\frac{\partial A_i}{\partial t} + q\sum_j \frac{\partial A_i}{\partial x_j}\dot{x}_j\,,$$

which combined with (3) and (D.1) gives

$$m\ddot{x}_i = qE_i + q\sum_j \dot{x}_j\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right).$$

This is identical with the equation of motion given in the question as required if

$$(\mathbf{v}\times\mathbf{B})_i = (\dot{\mathbf{x}}\times\mathbf{B})_i = \sum_j \dot{x}_j\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right) \tag{4}$$

This is easily checked by inspecting components; for example, in the x direction, using B = curl A, we have

$$(\dot{\mathbf{x}}\times\mathbf{B})_1 = \dot{x}_2B_3 - \dot{x}_3B_2 = \dot{x}_2\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right) - \dot{x}_3\left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right)$$

which is identical with the right-hand side of (4) for $i = 1$.

D.2 The first two Maxwell equations are automatically satisfied on substituting (D.1) since $\operatorname{div}\operatorname{curl}\mathbf{F} = 0$ (any F) and $\operatorname{curl}\operatorname{grad} f = 0$ (any *f*). On substituting (D.1), div E = 0 becomes

$$-\nabla^2\phi - \boldsymbol{\nabla}\cdot\frac{\partial\mathbf{A}}{\partial t} = 0\ ,$$

which becomes

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\phi - \frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{A}\right) = 0\ , \tag{D.6a}$$

on adding and subtracting $\partial^2\phi/\partial t^2$. The remaining equation $\operatorname{curl}\mathbf{B} = \partial\mathbf{E}/\partial t$, reduces to (D.6b) using the identity

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{A} = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \nabla^2\mathbf{A}\ .$$

D.3 Suppose

$$\frac{\partial\tilde{\phi}}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{A} = -\alpha(\mathbf{r},t) \neq 0\ ,$$

where $\alpha(\mathbf{r},t) \to 0$ as $|\mathbf{r}| \to \infty$ if the potentials vanish at infinity. Then

$$\frac{\partial\phi}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{A} = \frac{\partial\tilde{\phi}}{\partial t} + \boldsymbol{\nabla}\cdot\tilde{\mathbf{A}} + \left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)f = 0\ ,$$

provided

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)f(\mathbf{r},t) = \alpha(\mathbf{r},t)\ .$$

This is just a wave equation with a source $\alpha(\mathbf{r},t)$, and can be solved by standard methods. [1] In addition, if f_0 is a solution, then $f_0 + \delta f$ is also a solution, provided that

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\delta f(\mathbf{r},t) = 0$$

which is satisfied by any

$$\delta f = \int \mathrm{d}^3\mathbf{k}\, a(\mathbf{k}) \exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$$

[1] See, for example, Section 6.6 of Jackson (1975)

where $\omega = |\mathbf{k}|$. In other words, there are an infinite number of gauge choices that satisfy the Lorentz condition.

D.4 Under a gauge transformation (D.2) and (D.5), $(\phi, \mathbf{A}, \Psi) \to (\phi', \mathbf{A}', \Psi')$ and the terms in (D.12) become

$$\beta m \Psi' = e^{-iqf}(\beta m \Psi),$$

$$i\left(\frac{\partial}{\partial t} + iq\phi'\right)\Psi' = i\left(\frac{\partial}{\partial t} + iq\phi + iq\frac{\partial f}{\partial t}\right)e^{-iqf}\Psi = e^{-iqf}\left[i\left(\frac{\partial}{\partial t} + iq\phi\right)\Psi\right]$$

and

$$-i\boldsymbol{\alpha}\cdot(\boldsymbol{\nabla} - iq\mathbf{A}')\Psi' = -i\boldsymbol{\alpha}\cdot(\boldsymbol{\nabla} - iq\mathbf{A} + iq\boldsymbol{\nabla} f)e^{-iqf}\Psi = e^{-iqf}\left[-i\boldsymbol{\alpha}\cdot(\boldsymbol{\nabla} - iq\mathbf{A})\Psi\right].$$

Hence, if the Dirac equation (D.12) holds for $(\phi, \mathbf{A}, \Psi)$, it also holds for $(\phi', \mathbf{A}', \Psi')$.

D.5 The transformation (D.5) can be rewritten using (D.16) as

$$\Psi \to \Psi' = e^{-iqf}\Psi = e^{-ieQf}\Psi = \exp(-ieI_3^W f)\exp(-ieY^W f)\Psi = \exp(-ieI_3^W f)\Psi''$$

where

$$\Psi'' = \exp(-ieY^W f)\Psi.$$

In other words, it can be written as a gauge transformation $\Psi \to \Psi''$ of the form (D.17) with $g'\omega(\mathbf{r},t) = ef(\mathbf{r},t)$, followed by a transformation $\Psi'' \to \Psi'$ of the form (D.15) with $gf_3 = ef$, and $f_i = 0\ (i \neq 3)$.