Problem Set 6

4.3 Verify the statement that the Einstein energy–momentum relationship is recovered if any of the four Dirac spinors of (4.48) are substituted into the Dirac equation written in terms of momentum, $(\gamma^\mu p_\mu - m)u = 0$.

The Dirac equation reads:

$$(\gamma^\mu p_\mu - m)u = 0 \tag{1}$$

working back and substituting corresponding μ for the spinors (from Equations 4.40-4.41 of Thomson):

$$(\gamma^0 E - \gamma^1 p_x - \gamma^2 p_y - \gamma^3 p_z - m)u(E, \mathbf{p})e^{i(\mathbf{p}\cdot\mathbf{x} - Et)} = 0 \tag{2}$$

we have the following matrix expressions for γ (from Equation 4.35 of Thomson):

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$
$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

thus plugging these matrices into the Dirac equation in Equation 2 (turning m to matrix form as mI), we have:

$$\left[\begin{pmatrix} E & 0 & 0 & 0 \\ 0 & E & 0 & 0 \\ 0 & 0 & -E & 0 \\ 0 & 0 & 0 & -E \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & p_x \\ 0 & 0 & p_x & 0 \\ 0 & -p_x & 0 & 0 \\ -p_x & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & -ip_y \\ 0 & 0 & ip_y & 0 \\ 0 & ip_y & 0 & 0 \\ -ip_y & 0 & 0 & 0 \end{pmatrix} \right.$$
$$\left. - \begin{pmatrix} 0 & 0 & p_z & 0 \\ 0 & 0 & 0 & -p_z \\ -p_z & 0 & 0 & 0 \\ 0 & p_z & 0 & 0 \end{pmatrix} - \begin{pmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & m \end{pmatrix}\right] u(E, \mathbf{p})e^{i(\mathbf{p}\cdot\mathbf{x} - Et)} = 0$$

$$\left[\begin{pmatrix} E-m & 0 & -p_z & -p_x+ip_y \\ 0 & E-m & -p_x-ip_y & p_z \\ p_z & p_x-ip_y & -E-m & 0 \\ p_x+ip_y & -p_z & 0 & -E-m \end{pmatrix}\right] u(E,\mathbf{p})e^{i(\mathbf{p}\cdot\mathbf{x}-Et)} = 0 \tag{3}$$

this equation needs to be satisfied. Since the exponential term does not zero out, then we have the equation:

$$\left[\begin{pmatrix} E-m & 0 & -p_z & -p_x+ip_y \\ 0 & E-m & -p_x-ip_y & p_z \\ p_z & p_x-ip_y & -E-m & 0 \\ p_x+ip_y & -p_z & 0 & -E-m \end{pmatrix}\right] u(E,\mathbf{p}) = 0 \tag{4}$$

for the matrix form of $u(E,\mathbf{p})$, we have the following plane wave solutions (from Equation 4.48 of Thomson):

$$u_1 = N_1 \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix} \quad u_2 = N_2 \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$
$$u_3 = N_3 \begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x+ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix} \quad u_4 = N_4 \begin{pmatrix} \frac{p_x-ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix}$$

we check each form of $u(E,\mathbf{p})$ on what conditions must be met for Equation 4 to hold true. Starting with u_1 we have:

$$\begin{pmatrix} E-m & 0 & -p_z & -p_x+ip_y \\ 0 & E-m & -p_x-ip_y & p_z \\ p_z & p_x-ip_y & -E-m & 0 \\ p_x+ip_y & -p_z & 0 & -E-m \end{pmatrix} N_1 \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix} = 0 \tag{5}$$

since N_1 is nonzero too (as well as the other N), we can ignore it, and our equation becomes:

$$\begin{pmatrix} (E-m)+0-\frac{p_z^2}{E+m}+\frac{-p_x^2-p_y^2}{E+m} \\ 0+0+\frac{-p_xp_z-ip_yp_z}{E+m}+\frac{p_xp_z+ip_yp_z}{E+m} \\ p_z+0-p_z+0 \\ (p_x+ip_y)+0+0+-(px+ip_y) \end{pmatrix} = 0 \tag{6}$$

$$\begin{pmatrix} E-m-\frac{p_z^2}{E+m}+\frac{-p_x^2-p_y^2}{E+m} \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0 \tag{7}$$

for this condition to hold true, then the first element of the matrix must zero out:

$$E-m-\frac{p_z^2}{E+m}+\frac{-p_x^2-p_y^2}{E+m} = 0 \tag{8}$$

$$\frac{1}{E+m}\left(E^2-m^2-p_z^2-p_x^2-p_y^2\right) = 0 \tag{9}$$

$$\boxed{E^2 = \mathbf{p}^2+m^2} \tag{10}$$

we have arrived at the Einstein energy-momentum relationship. We can proceed to check for other spinors:

u_2 spinor

$$\begin{pmatrix} E-m & 0 & -p_z & -p_x+ip_y \\ 0 & E-m & -p_x-ip_y & p_z \\ p_z & p_x-ip_y & -E-m & 0 \\ p_x+ip_y & -p_z & 0 & -E-m \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix} = 0 \tag{11}$$

$$\begin{pmatrix} 0+0+\frac{-p_xp_z+ip_yp_z}{E+m}+\frac{p_xp_z-ip_yp_z}{E+m} \\ 0+(E-m)+\frac{-p_x^2-p_y^2}{E+m}+\frac{-p_z^2}{E+m} \\ 0+(p_x-ip_y)+-(p_x-ip_y)+0 \\ 0+-p_z+0+-(-p_z) \end{pmatrix} = 0 \tag{12}$$

$$\begin{pmatrix} 0 \\ E-m-\frac{p_z^2}{E+m}+\frac{-p_x^2-p_y^2}{E+m} \\ 0 \\ 0 \end{pmatrix} = 0 \tag{13}$$

we find a matrix similar to the one for u_1, but the nonzero term is now the second element instead of being the first. Again, this would mean that the Einstein energy-momentum relationship must hold.

<u>u_3 spinor</u>

$$\begin{pmatrix} E-m & 0 & -p_z & -p_x+ip_y \\ 0 & E-m & -p_x-ip_y & p_z \\ p_z & p_x-ip_y & -E-m & 0 \\ p_x+ip_y & -p_z & 0 & -E-m \end{pmatrix} \begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x+ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix} = 0 \tag{14}$$

$$\begin{pmatrix} p_z + 0 - p_z + 0 \\ 0 + (p_x+ip_y) - p_x - ip_y + 0 \\ \frac{p_z^2}{E-m} + \frac{p_x^2+p_y^2}{E-m} + (-E-m) + 0 \\ \frac{p_xp_z+ip_yp_z}{E-m} - \frac{p_xp_z+ip_yp_z}{E-m} + 0 + 0 \end{pmatrix} = 0 \tag{15}$$

$$\begin{pmatrix} 0 \\ 0 \\ \frac{p_z^2}{E-m} + \frac{p_x^2+p_y^2}{E-m} + (-E-m) \\ 0 \end{pmatrix} = 0 \tag{16}$$

$$\tag{17}$$

for this to hold true, then the third element must zero out:

$$\frac{p_z^2}{E-m} + \frac{p_x^2+p_y^2}{E-m} + (-E-m) = 0 \tag{18}$$

$$\frac{p_z^2}{E-m} + \frac{p_x^2+p_y^2}{E-m} + -(E+m) = 0 \tag{19}$$

$$\frac{1}{E-m}\left(p_z^2 + p_x^2 + p_y^2 - (E^2 - m^2)\right) = 0 \tag{20}$$

$$\boxed{E^2 = \mathbf{p}^2 + m^2} \tag{21}$$

the Einstein energy-momentum relationship still pops up.

<u>u_4 spinor</u>

$$\begin{pmatrix} E-m & 0 & -p_z & -p_x+ip_y \\ 0 & E-m & -p_x-ip_y & p_z \\ p_z & p_x-ip_y & -E-m & 0 \\ p_x+ip_y & -p_z & 0 & -E-m \end{pmatrix} \begin{pmatrix} \frac{p_x-ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix} = 0 \tag{22}$$

$$\begin{pmatrix} (p_x-ip_y)+0+0+(-p_x+ip_y) \\ 0+-p_z+0+p_z \\ \frac{p_xp_z+ip_yp_z}{E-m}+\frac{-p_xp_z-ip_yp_z}{E-m}+0+0 \\ \frac{(p_x^2+p_y^2)}{E-m}+\frac{p_z^2}{E-m}+0+(-E-m) \end{pmatrix} = 0 \tag{23}$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{(p_x^2+p_y^2)}{E-m}+\frac{p_z^2}{E-m}+(-E-m) \end{pmatrix} = 0 \tag{24}$$

$$\tag{25}$$

we find a matrix similar to the one for u_3, but the nonzero term is now the fourth element instead of being the third. Again, this would mean that the Einstein energy-momentum relationship must hold. By now, we have shown that plugging in any of the four Dirac spinors recovers the Einstein energy-momentum relationship.

4.9 Starting from

$$(\gamma^\mu p_\mu - m)u = 0,$$

show that the corresponding equation for the adjoint spinor is

$$\bar{u}(\gamma^\mu p_\mu - m) = 0.$$

We start with:

$$(\gamma^\mu p_\mu - m)u = 0 \tag{26}$$

taking the Hermitian conjugate of each side, we get:

$$u^\dagger(\gamma^{\mu\dagger} p_\mu - m) = 0 \tag{27}$$

we note we have shown before that:

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0 \tag{28}$$

thus our equation becomes:

$$u^\dagger(\gamma^0\gamma^\mu\gamma^0 p_\mu - m) = 0 \tag{29}$$

with $\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$, we can show that:

$$\gamma^0\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{30}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I \tag{31}$$

thus we can manipulate our original equation as:

$$u^\dagger(\gamma^0\gamma^\mu\gamma^0\gamma^0 p_\mu - m\gamma^0) = 0 \tag{32}$$
$$u^\dagger(\gamma^0\gamma^\mu p_\mu - m\gamma^0) = 0 \tag{33}$$
$$\tag{34}$$

by definition (Equation 4.36-4.37 of Thomson), $\bar{u} = u^\dagger\gamma^0$ so we can rewrite our equation as:

$$\boxed{\bar{u}(\gamma^\mu p_\mu - m) = 0} \tag{35}$$

Hence, without using the explicit form for the u spinors, show that the normalisation condition $u^\dagger u = 2E$ leads to

$$\bar{u}u = 2m,$$

and that

$$\bar{u}\gamma^\mu u = 2p^\mu.$$

We start with the original Dirac equation and the corresponding equation for the adjoint spinor:

$$(\gamma^\mu p_\mu - m)u = 0 \tag{36}$$
$$\bar{u}(\gamma^\mu p_\mu - m) = 0 \tag{37}$$

since both equations zero out, we can manipulate each of them; by multiplying $\bar{u}\gamma^\nu$ by the first equation and multiplying the second equation by $\gamma^\nu u$:

$$\bar{u}\gamma^\nu[(\gamma^\mu p_\mu - m)u = 0] \tag{38}$$
$$[\bar{u}(\gamma^\mu p_\mu - m) = 0]\gamma^\nu u \tag{39}$$

$$\bar{u}\gamma^\nu(\gamma^\mu p_\mu - m)u = 0 \tag{40}$$
$$\bar{u}(\gamma^\mu p_\mu - m)\gamma^\nu u = 0 \tag{41}$$

adding these, we can get:

$$\bar{u}\gamma^\nu\gamma^\mu p_\mu u - \bar{u}\gamma^\nu u m + \bar{u}\gamma^\mu\gamma^\nu p_\mu u - \bar{u}\gamma^\nu u m = 0 \tag{42}$$
$$\bar{u}(\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu)p_\mu u - 2\bar{u}\gamma^\nu u m = 0 \tag{43}$$

by Equation 4.33 of Thomson, we note of the anticommutation relation $\gamma^\nu\gamma^\mu + \gamma^\mu\gamma^\nu = 2g^{\mu\nu}$ and rewrite this as:

$$\bar{u}2g^{\mu\nu}p_\mu u - 2\bar{u}\gamma^\nu um = 0 \tag{44}$$

$$\bar{u}g^{\mu\nu}p_\mu u - \bar{u}\gamma^\nu um = 0 \tag{45}$$

having the contravariant metric tensor $g^{\mu\nu}$ act on p_μ switches and raises its index, thus we have:

$$\bar{u}p^\nu u - \bar{u}\gamma^\nu um = 0 \tag{46}$$

now for the case of Equation 46 of $\nu = 0$, we have:

$$\bar{u}p^0 u - \bar{u}\gamma^0 um = 0 \tag{47}$$

again using Equation 4.36-4.37 of Thomson, $\bar{u} = u^\dagger\gamma^0$, and noting that $p^0 = E$, then we have:

$$\bar{u}Eu - u^\dagger\gamma^0\gamma^0 um = 0 \tag{48}$$

with $\gamma^0\gamma^0 = I$ and the given normalisation condition $u^\dagger u = 2E$, then we get:

$$\bar{u}Eu - u^\dagger um = 0 \tag{49}$$

$$\bar{u}Eu = 2Em \tag{50}$$

we finally arrive at:

$$\boxed{\bar{u}u = 2m} \tag{51}$$

we can use Equation 51 for the case of Equation 46 where $\nu \neq 0$:

$$\bar{u}p^\nu u - \bar{u}\gamma^\nu um = 0 \tag{52}$$

$$2mp^\nu = \bar{u}\gamma^\nu um \tag{53}$$

changing indices from ν to μ we then arrive at:

$$\boxed{\bar{u}\gamma^\mu u = 2p^\mu} \tag{54}$$