Problem Set 8

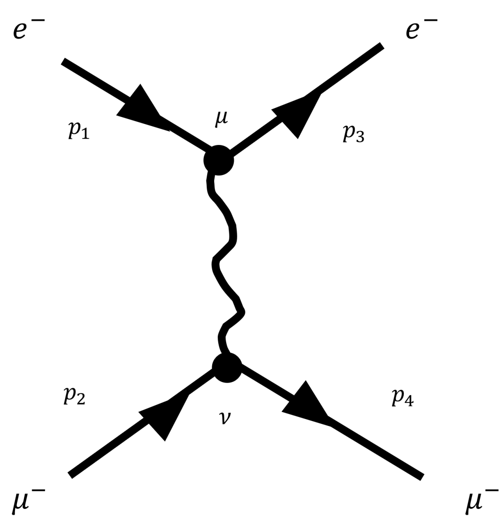
6.7 Using helicity amplitudes, calculate the differential cross section for $e^-\mu^- \to e^-\mu^-$ scattering in the following steps:

(a) From the Feynman rules for QED, show that the lowest-order QED matrix element for $e^-\mu^- \to e^-\mu^-$ is

$$\mathcal{M}_{fi} = -\frac{e^2}{(p_1 - p_3)^2} g_{\mu\nu} \left[\bar{u}(p_3)\gamma^\mu u(p_1)\right] \left[\bar{u}(p_4)\gamma^\nu u(p_2)\right],$$

where p_1 and p_3 are the four-momenta of the initial- and final-state e^-, and p_2 and p_4 are the four-momenta of the initial- and final-state μ^-.

For the $e^-\mu^- \to e^-\mu^-$ scattering, the Feynmann diagram will follow the t configuration as such:



we recall the following from QED:

$$\begin{aligned}
\text{initial state particle } (e^-) &: \; u(p) \\
\text{final state particle } (e^-) &: \; \bar{u}(p) \\
\text{photon propagator} &: \; -\frac{ig_{\mu\nu}}{q^2} \\
\text{interaction vertex factor} &: \; ie\gamma^\mu
\end{aligned}$$

getting each contribution, we have the contribution from the initial state electron (particle 1) to the final state electron (particle 3):

$$[\bar{u}(p)]\,[ie\gamma^{\mu}]\,[u(p)] \tag{1}$$
$$[\bar{u}(p_3)]\,[ie\gamma^{\mu}]\,[u(p_1)] \tag{2}$$
$$\tag{3}$$

we also have sismilar contribution for the initial state muon (particle 2) and the final state muon (particle 4):

$$[\bar{u}(p)]\,[ie\gamma^{\mu}]\,[u(p)] \tag{4}$$
$$[\bar{u}(p_4)]\,[ie\gamma^{\mu}]\,[u(p_2)] \tag{5}$$
$$\tag{6}$$

combining these with the photon propagator, we then have:

$$-i\mathcal{M} = [\bar{u}(p_3)]\,[ie\gamma^{\mu}]\,[u(p_1)]\left[-\frac{ig_{\mu\nu}}{q^2}\right][\bar{u}(p_4)]\,[ie\gamma^{\mu}]\,[u(p_2)] \tag{7}$$
$$\mathcal{M} = -\frac{e^2}{q^2}g_{\mu\nu}[\bar{u}(p_3)]\,[\gamma^{\mu}]\,[u(p_1)][\bar{u}(p_4)]\,[\gamma^{\mu}]\,[u(p_2)] \tag{8}$$
$$\tag{9}$$

for the t channel configuration, $q = p_1 - p_3 = p_2 - p_4$, thus we can rewrite this as:

$$\boxed{\mathcal{M} = -\frac{e^2}{(p_1 - p_3)^2}g_{\mu\nu}[\bar{u}(p_3)\gamma^{\mu}u(p_1)][\bar{u}(p_4)\gamma^{\mu}u(p_2)]} \tag{10}$$

(b) Working in the centre-of-mass frame, and writing the four-momenta of the initial- and final-state e^- as $p_1^\mu = (E_1, 0, 0, p)$ and $p_3^\mu = (E_1, p\sin\theta, 0, p\cos\theta)$ respectively, show that the electron currents for the four possible helicity combinations are

$$\bar{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) = 2(E_1 c, ps, -ips, pc),$$
$$\bar{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1) = 2(ms, 0, 0, 0),$$
$$\bar{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) = 2(E_1 c, ps, ips, pc),$$
$$\bar{u}_\downarrow(p_3)\gamma^\mu u_\uparrow(p_1) = -2(ms, 0, 0, 0),$$

where m is the electron mass, $s = \sin(\theta/2)$ and $c = \cos(\theta/2)$.

The non-relativistic helicity spinors are given by Equation (4.65) of Thomson:

$$u_\uparrow = \sqrt{E+m}\begin{pmatrix} c \\ se^{i\phi} \\ \frac{p}{E+m}c \\ \frac{p}{E+m}se^{i\phi} \end{pmatrix} \quad u_\downarrow = \sqrt{E+m}\begin{pmatrix} -s \\ ce^{i\phi} \\ \frac{p}{E+m}s \\ -\frac{p}{E+m}ce^{i\phi} \end{pmatrix} \tag{11}$$
$$\tag{12}$$

with the given $p_1 = (E_1, 0, 0, p)$ for the incoming electron, noting that $\theta = 0$ and $\phi = 0$ then we have:

$$s = \sin(\theta/2) \to \sin(0) = 0 \quad c = \cos(\theta/2) \to \cos(0) = 1 \quad e^{i\phi} \to e^0 = 1 \tag{13}$$

$$u_\uparrow(p_1) = \sqrt{E_1+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p}{E_1+m} \\ 0 \end{pmatrix} \quad u_\downarrow(p_1) = \sqrt{E_1+m}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -\frac{p}{E_1+m} \end{pmatrix} \tag{14}$$
$$\tag{15}$$

for the outgoing electron with $p_3 = (E_1, p\sin\theta, 0, p\cos\theta)$, noting that $\theta = \theta$ and $\phi = 0$ we have:

$$s = \sin(\theta/2) \quad c = \cos(\theta/2) \quad e^{i\phi} \to e^0 = 1 \tag{16}$$

$$u_\uparrow(p_3) = \sqrt{E_1+m}\begin{pmatrix} c \\ s \\ \frac{p}{E_1+m}c \\ \frac{p}{E_1+m}s \end{pmatrix} \quad u_\downarrow(p_3) = \sqrt{E_1+m}\begin{pmatrix} -s \\ c \\ \frac{p}{E_1+m}s \\ -\frac{p}{E_1+m}c \end{pmatrix} \tag{17}$$
$$\tag{18}$$

we note Equations (6.12) to (6.15) of how the components of $\bar{\psi}\gamma^\mu\phi \equiv \psi^\dagger\gamma^0\gamma^\mu\phi$ may be evaluated:

$$\bar{\psi}\gamma^0\phi \equiv \psi^\dagger\gamma^0\gamma^0\phi = \psi_1^*\phi_1 + \psi_2^*\phi_2 + \psi_3^*\phi_3 + \psi_4^*\phi_4 \tag{19}$$
$$\bar{\psi}\gamma^1\phi \equiv \psi^\dagger\gamma^0\gamma^1\phi = \psi_1^*\phi_4 + \psi_2^*\phi_3 + \psi_3^*\phi_2 + \psi_4^*\phi_1 \tag{20}$$
$$\bar{\psi}\gamma^2\phi \equiv \psi^\dagger\gamma^0\gamma^2\phi = -i(\psi_1^*\phi_4 - \psi_2^*\phi_3 + \psi_3^*\phi_2 - \psi_4^*\phi_1) \tag{21}$$
$$\bar{\psi}\gamma^3\phi \equiv \psi^\dagger\gamma^0\gamma^3\phi = \psi_1^*\phi_3 + \psi_2^*\phi_4 + \psi_3^*\phi_1 + \psi_4^*\phi_2 \tag{22}$$
$$\tag{23}$$

we first evaluate $\bar{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1)$ using the equations for $\bar{\psi}\gamma^\mu\phi \equiv \psi^\dagger\gamma^0\gamma^\mu\phi$:

$$\bar{u}_\downarrow(p_3)\gamma^0 u_\downarrow(p_1) = u_\downarrow(p_3)_1^* u_\downarrow(p_1)_1 + u_\downarrow(p_3)_2^* u_\downarrow(p_1)_2 + u_\downarrow(p_3)_3^* u_\downarrow(p_1)_3 + u_\downarrow(p_3)_4^* u_\downarrow(p_1)_4 \tag{24}$$
$$= (E_1+m)\left(0 + c + \frac{cp^2}{(E_1+m)^2}\right) \tag{25}$$
$$\approx (2E_1 - p^2)c + cp^2 \tag{26}$$
$$= 2E_1 c \tag{27}$$
$$\bar{u}_\downarrow(p_3)\gamma^1 u_\downarrow(p_1) = u_\downarrow(p_3)_1^* u_\downarrow(p_1)_4 + u_\downarrow(p_3)_2^* u_\downarrow(p_1)_3 + u_\downarrow(p_3)_3^* u_\downarrow(p_1)_2 + u_\downarrow(p_3)_4^* u_\downarrow(p_1)_1 \tag{28}$$
$$= (E_1+m)\left(\frac{sp}{E_1+m} + 0 + \frac{p}{E_1+m}s + 0\right) \tag{29}$$
$$= 2ps \tag{30}$$
$$\bar{u}_\downarrow(p_3)\gamma^2 u_\downarrow(p_1) = -i(u_\downarrow(p_3)_1^* u_\downarrow(p_1)_4 - u_\downarrow(p_3)_2^* u_\downarrow(p_1)_3 + u_\downarrow(p_3)_3^* u_\downarrow(p_1)_2 - u_\downarrow(p_3)_4^* u_\downarrow(p_1)_1) \tag{31}$$
$$= -i(E_1+m)\left(\frac{sp}{E_1+m} - 0 + \frac{p}{E_1+m}s - 0\right) \tag{32}$$
$$= -2ips \tag{33}$$
$$\bar{u}_\downarrow(p_3)\gamma^3 u_\downarrow(p_1) = u_\downarrow(p_3)_1^* u_\downarrow(p_1)_3 + u_\downarrow(p_3)_2^* u_\downarrow(p_1)_4 + u_\downarrow(p_3)_3^* u_\downarrow(p_1)_1 + u_\downarrow(p_3)_4^* u_\downarrow(p_1)_2 \tag{34}$$
$$= (E_1+m)\left(0 + \frac{-cp}{E_1+m} + 0 + \frac{-p}{E_1+m}c\right) \tag{35}$$
$$= 2pc \tag{36}$$

for the first component we used the approximation:

$$E^2 \approx p^2 + m \quad \text{(in natural units } c = 1\text{, and } m^2 \approx m \text{ since } m \text{ is very small)} \tag{37}$$
$$\tag{38}$$

combining all the components, we have:

$$\boxed{\bar{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) = 2(E_1 c, ps, -ips, pc)} \tag{39}$$

we next evaluate the components of $\bar{u}_{\uparrow}(p_3)\gamma^\mu u_{\downarrow}(p_1)$:

$$\begin{aligned}
\bar{u}_{\uparrow}(p_3)\gamma^0 u_{\downarrow}(p_1) &= u_{\uparrow}(p_3)_1^* u_{\downarrow}(p_1)_1 + u_{\uparrow}(p_3)_2^* u_{\downarrow}(p_1)_2 + u_{\uparrow}(p_3)_3^* u_{\downarrow}(p_1)_3 + u_{\uparrow}(p_3)_4^* u_{\downarrow}(p_1)_4 && (40)\\
&= (E_1+m)\left(0 + s + 0 - \frac{p^2}{(E_1+m)^2}s\right) && (41)\\
&\approx (2m+p^2)s - p^2 s && (42)\\
&= 2ms && (43)\\
\bar{u}_{\uparrow}(p_3)\gamma^1 u_{\downarrow}(p_1) &= u_{\uparrow}(p_3)_1^* u_{\downarrow}(p_1)_4 + u_{\uparrow}(p_3)_2^* u_{\downarrow}(p_1)_3 + u_{\uparrow}(p_3)_3^* u_{\downarrow}(p_1)_2 + u_{\uparrow}(p_3)_4^* u_{\downarrow}(p_1)_1 && (44)\\
&= (E_1+m)\left(-\frac{p}{E_1+m}c + 0 + \frac{p}{E_1+m}c + 0\right) && (45)\\
&= 0 && (46)\\
\bar{u}_{\uparrow}(p_3)\gamma^2 u_{\downarrow}(p_1) &= -i(u_{\uparrow}(p_3)_1^* u_{\downarrow}(p_1)_4 - u_{\uparrow}(p_3)_2^* u_{\downarrow}(p_1)_3 + u_{\uparrow}(p_3)_3^* u_{\downarrow}(p_1)_2 - u_{\uparrow}(p_3)_4^* u_{\downarrow}(p_1)_1) && (47)\\
&= -i(E_1+m)\left(-\frac{p}{E_1+m}c - 0 + \frac{p}{E_1+m}c - 0\right) && (48)\\
&= 0 && (49)\\
\bar{u}_{\uparrow}(p_3)\gamma^3 u_{\downarrow}(p_1) &= u_{\uparrow}(p_3)_1^* u_{\downarrow}(p_1)_3 + u_{\uparrow}(p_3)_2^* u_{\downarrow}(p_1)_4 + u_{\uparrow}(p_3)_3^* u_{\downarrow}(p_1)_1 + u_{\uparrow}(p_3)_4^* u_{\downarrow}(p_1)_2 && (50)\\
&= (E_1+m)\left(0 - \frac{sp}{E_1+m} + 0 + \frac{p}{E_1+m}s\right) && (51)\\
&= 0 && (52)
\end{aligned}$$

combining all the components, we have:

$$\boxed{\bar{u}_{\uparrow}(p_3)\gamma^\mu u_{\downarrow}(p_1) = 2(ms, 0, 0, 0)} \tag{53}$$

we then evaluate the components of $\bar{u}_{\uparrow}(p_3)\gamma^\mu u_{\uparrow}(p_1)$:

$$\begin{aligned}
\bar{u}_\uparrow(p_3)\gamma^0 u_\uparrow(p_1) &= u_\uparrow(p_3)_1^* u_\uparrow(p_1)_1 + u_\uparrow(p_3)_2^* u_\uparrow(p_1)_2 + u_\uparrow(p_3)_3^* u_\uparrow(p_1)_3 + u_\uparrow(p_3)_4^* u_\uparrow(p_1)_4 && (54)\\
&= (E_1+m)\left(c + 0 + \frac{p^2}{E_1+m}c + 0\right) && (55)\\
&\approx (2E_1 - p^2)c + p^2 c && (56)\\
&= 2E_1 c && (57)\\
\bar{u}_\uparrow(p_3)\gamma^1 u_\uparrow(p_1) &= u_\uparrow(p_3)_1^* u_\uparrow(p_1)_4 + u_\uparrow(p_3)_2^* u_\uparrow(p_1)_3 + u_\uparrow(p_3)_3^* u_\uparrow(p_1)_2 + u_\uparrow(p_3)_4^* u_\uparrow(p_1)_1 && (58)\\
&= (E_1+m)\left(0 + \frac{sp}{E_1+m} + 0 + \frac{p}{E_1+m}s\right) && (59)\\
&= 2ps && (60)\\
\bar{u}_\uparrow(p_3)\gamma^2 u_\uparrow(p_1) &= -i(u_\uparrow(p_3)_1^* u_\uparrow(p_1)_4 - u_\uparrow(p_3)_2^* u_\uparrow(p_1)_3 + u_\uparrow(p_3)_3^* u_\uparrow(p_1)_2 - u_\uparrow(p_3)_4^* u_\uparrow(p_1)_1) && (61)\\
&= -i(E_1+m)\left(0 - \frac{sp}{E_1+m} + 0 - \frac{p}{E_1+m}s\right) && (62)\\
&= 2ips && (63)\\
\bar{u}_\uparrow(p_3)\gamma^3 u_\uparrow(p_1) &= u_\uparrow(p_3)_1^* u_\uparrow(p_1)_3 + u_\uparrow(p_3)_2^* u_\uparrow(p_1)_4 + u_\uparrow(p_3)_3^* u_\uparrow(p_1)_1 + u_\uparrow(p_3)_4^* u_\uparrow(p_1)_2 && (64)\\
&= (E_1+m)\left(\frac{cp}{E_1+m} + 0 + \frac{p}{E_1+m}c + 0\right) && (65)\\
&= 2pc && (66)
\end{aligned}$$

combining all the components, we have:

$$\boxed{\bar{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) = 2(E_1 c, ps, ips, pc)} \tag{67}$$

lastly evaluate the components of $\bar{u}_\downarrow(p_3)\gamma^\mu u_\uparrow(p_1)$:

$$\begin{aligned}
\bar{u}_\downarrow(p_3)\gamma^0 u_\uparrow(p_1) &= u_\downarrow(p_3)_1^* u_\uparrow(p_1)_1 + u_\downarrow(p_3)_2^* u_\uparrow(p_1)_2 + u_\downarrow(p_3)_3^* u_\uparrow(p_1)_3 + u_\downarrow(p_3)_4^* u_\uparrow(p_1)_4 && (68)\\
&= (E_1+m)\left(0 + s + 0 - \frac{p^2}{E_1+m}s\right) && (69)\\
&\approx (2m+p^2)s - p^2 s && (70)\\
&= -2ms && (71)\\
\bar{u}_\downarrow(p_3)\gamma^1 u_\uparrow(p_1) &= u_\downarrow(p_3)_1^* u_\uparrow(p_1)_4 + u_\downarrow(p_3)_2^* u_\uparrow(p_1)_3 + u_\downarrow(p_3)_3^* u_\uparrow(p_1)_2 + u_\downarrow(p_3)_4^* u_\uparrow(p_1)_1 && (72)\\
&= (E_1+m)\left(-\frac{cp}{E_1+m} + 0 + \frac{p}{E_1+m}c + 0\right) && (73)\\
&= 0 && (74)\\
\bar{u}_\downarrow(p_3)\gamma^2 u_\uparrow(p_1) &= -i(u_\downarrow(p_3)_1^* u_\uparrow(p_1)_4 - u_\downarrow(p_3)_2^* u_\uparrow(p_1)_3 + u_\downarrow(p_3)_3^* u_\uparrow(p_1)_2 - u_\downarrow(p_3)_4^* u_\uparrow(p_1)_1) && (75)\\
&= -i(E_1+m)\left(-\frac{cp}{E_1+m} - 0 + \frac{p}{E_1+m}c - 0\right) && (76)\\
&= 0 && (77)\\
\bar{u}_\downarrow(p_3)\gamma^3 u_\uparrow(p_1) &= u_\downarrow(p_3)_1^* u_\uparrow(p_1)_3 + u_\downarrow(p_3)_2^* u_\uparrow(p_1)_4 + u_\downarrow(p_3)_3^* u_\uparrow(p_1)_1 + u_\downarrow(p_3)_4^* u_\uparrow(p_1)_2 && (78)\\
&= (E_1+m)\left(0 - \frac{sp}{E_1+m} + 0 - \frac{p}{E_1+m}s\right) && (79)\\
&= 0 && (80)
\end{aligned}$$

combining all the components, we have:

$$\boxed{\bar{u}_\downarrow(p_3)\gamma^\mu u_\uparrow(p_1) = -2(ms, 0, 0, 0)} \tag{81}$$

thus we have the four possible helicity combinations:

$$\boxed{\begin{aligned}
\bar{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) &= 2(E_1 c, ps, -ips, pc)\\
\bar{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1) &= 2(ms, 0, 0, 0)\\
\bar{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) &= 2(E_1 c, ps, ips, pc)\\
\bar{u}_\downarrow(p_3)\gamma^\mu u_\uparrow(p_1) &= -2(ms, 0, 0, 0)
\end{aligned}}$$

(c) Explain why the effect of the parity operator $\hat{P} = \gamma^0$ is

$$\hat{P}u_\uparrow(\mathrm{p}, \theta, \phi) = \hat{P}u_\downarrow(\mathrm{p}, \pi - \theta, \pi + \phi).$$

Hence, or otherwise, show that the muon currents for the four helicity combinations are

$$\begin{aligned}
\bar{u}_\downarrow(p_4)\gamma^\mu u_\downarrow(p_2) &= 2(E_2c, -\mathrm{p}s, -i\mathrm{p}s, -\mathrm{p}c),\\
\bar{u}_\uparrow(p_4)\gamma^\mu u_\downarrow(p_2) &= 2(Ms, 0, 0, 0),\\
\bar{u}_\uparrow(p_4)\gamma^\mu u_\uparrow(p_2) &= 2(E_2c, -\mathrm{p}s, i\mathrm{p}s, -\mathrm{p}c),\\
\bar{u}_\downarrow(p_4)\gamma^\mu u_\uparrow(p_2) &= -2(Ms, 0, 0, 0),
\end{aligned}$$

where M is the muon mass.

We note that the parity operator $\hat{P} = \gamma^0$ reverses the momentum but retains spin state, such that $\hat{P}u_1(E, \mathbf{p}) = u_1(E, -\mathbf{p})$. For a given $u_\uparrow(p, \theta, \phi)$, having the parity operator act on it gives:

$$\hat{P}u_\uparrow(p, \theta, \phi) = \gamma^0 u_\uparrow(p, \theta, \phi) \tag{82}$$

we recall their Dirac-Pauli representations:

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad u_\uparrow(p, \theta, \phi) = \sqrt{E+m}\begin{pmatrix} c \\ se^{i\phi} \\ \frac{p}{E+m}c \\ \frac{p}{E+m}se^{i\phi} \end{pmatrix} \tag{83}$$

we then have:

$$\hat{P}u_\uparrow(p, \theta, \phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\sqrt{E+m}\begin{pmatrix} c \\ se^{i\phi} \\ \frac{p}{E+m}c \\ \frac{p}{E+m}se^{i\phi} \end{pmatrix} \tag{84}$$

$$\hat{P}u_\uparrow(p, \theta, \phi) = \sqrt{E+m}\begin{pmatrix} c \\ se^{i\phi} \\ -\frac{p}{E+m}c \\ -\frac{p}{E+m}se^{i\phi} \end{pmatrix} \tag{85}$$

$$\tag{86}$$

if we take a look at $u_\downarrow(p, \pi - \theta, \pi + \phi)$, we have:

$$u_{\downarrow}(p,\pi-\theta,\pi+\phi) = \sqrt{E+m}\begin{pmatrix} -\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right) \\ \cos\left(\frac{\pi}{2}-\frac{\theta}{2}\right)e^{i(\phi+\pi)} \\ \frac{p}{E+m}\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right) \\ -\frac{p}{E+m}\cos\left(\frac{\pi}{2}-\frac{\theta}{2}\right)e^{i(\phi+\pi)} \end{pmatrix} \tag{87}$$

$$u_{\downarrow}(p,\pi-\theta,\pi+\phi) = \sqrt{E+m}\begin{pmatrix} -c \\ -se^{i\phi} \\ \frac{p}{E+m}c \\ \frac{p}{E+m}se^{i\phi} \end{pmatrix} \tag{88}$$

then we have the relation:

$$\boxed{\hat{P}u_{\uparrow}(p,\theta,\phi) = u_{\downarrow}(p,\pi-\theta,\pi+\phi)} \tag{89}$$

Thus we can claim that we can get the muon currents $\bar{u}(p_4)\gamma^\mu u(p_2)$ from the electron currents $\bar{u}(p_3)\gamma^\mu u(p_1)$ such that:

$$\bar{u}_{\downarrow}(p_4)\gamma^\mu u_{\downarrow}(p_2) = \overline{\hat{P}u_{\uparrow}(p_3)}\gamma^\mu \hat{P}u_{\uparrow}(p_1) \tag{90}$$

$$\bar{u}_{\downarrow}(p_4)\gamma^\mu u_{\downarrow}(p_2) = [\hat{P}u_{\uparrow}(p_3)]^\dagger\gamma^\mu \hat{P}u_{\uparrow}(p_1) \tag{91}$$

$$\tag{92}$$

using the parity operator $\hat{P} = \gamma^0$, we have:

$$\bar{u}_{\downarrow}(p_4)\gamma^\mu u_{\downarrow}(p_2) = [\gamma^0 u_{\uparrow}(p_3)]^\dagger\gamma^\mu\gamma^0 u_{\uparrow}(p_1) \tag{93}$$

$$\bar{u}_{\downarrow}(p_4)\gamma^\mu u_{\downarrow}(p_2) = u_{\uparrow}^\dagger(p_3)\gamma^{0\dagger}\gamma^\mu\gamma^0 u_{\uparrow}(p_1) \tag{94}$$

$$\bar{u}_{\downarrow}(p_4)\gamma^\mu u_{\downarrow}(p_2) = \bar{u}_{\uparrow}(p_3)\gamma^{0\dagger}\gamma^\mu\gamma^0 u_{\uparrow}(p_1) \tag{95}$$

$$\tag{96}$$

since we know that $\gamma^{0\dagger} = \gamma^0$, we then have:

$$\bar{u}_{\downarrow}(p_4)\gamma^\mu u_{\downarrow}(p_2) = \bar{u}_{\uparrow}(p_3)\gamma^0\gamma^\mu\gamma^0 u_{\uparrow}(p_1) \tag{97}$$

$$\tag{98}$$

in Problem Set 5, we know that:

$$\gamma^0\gamma^\mu\gamma^0 = \begin{cases} \gamma^\mu & \mu = 0 \\ -\gamma^\mu & \mu = 1,2,3 \end{cases} \tag{99}$$

then we have:

$$\bar{u}_\downarrow(p_4)\gamma^\mu u_\downarrow(p_2) = \begin{cases} \bar{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) & \mu = 0 \\ -\bar{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) & \mu = 1,2,3 \end{cases} \tag{100}$$

meaning $\bar{u}_\downarrow(p_4)\gamma^\mu u_\downarrow(p_2)$ has similar 4 components with $\bar{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1)$ (the arrows reversed), but with the γ^1,γ^2, and γ^3 components multiplied by -1. All in all, we have:

$$\bar{u}_\downarrow(p_4)\gamma^\mu u_\downarrow(p_2) = \kappa(\mu)\bar{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) \tag{101}$$
$$\bar{u}_\uparrow(p_4)\gamma^\mu u_\downarrow(p_2) = \kappa(\mu)\bar{u}_\downarrow(p_3)\gamma^\mu u_\uparrow(p_1) \tag{102}$$
$$\bar{u}_\uparrow(p_4)\gamma^\mu u_\uparrow(p_2) = \kappa(\mu)\bar{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) \tag{103}$$
$$\bar{u}_\downarrow(p_4)\gamma^\mu u_\uparrow(p_2) = \kappa(\mu)\bar{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1) \tag{104}$$
$$\tag{105}$$

where:

$$\kappa(\mu) = \begin{cases} 1 & \mu = 0 \\ -1 & \mu = 1,2,3 \end{cases} \tag{106}$$

applying this to what we obtained earlier, then we get the expected helicity combinations (replacing m with M and E_1 with E_2) :

$$\boxed{\begin{aligned} \bar{u}_\downarrow(p_4)\gamma^\mu u_\downarrow(p_2) &= 2(E_2c, -ps - ips, -pc) \\ \bar{u}_\uparrow(p_4)\gamma^\mu u_\downarrow(p_2) &= 2(Ms, 0, 0, 0) \\ \bar{u}_\uparrow(p_4)\gamma^\mu u_\uparrow(p_2) &= 2(E_2c, -ps, ips, -pc) \\ \bar{u}_\downarrow(p_4)\gamma^\mu u_\uparrow(p_2) &= -2(Ms, 0, 0, 0) \end{aligned}}$$

(d) For the relativistic limit where $E \gg M$, show that the matrix element squared for the case where the incoming e^- and incoming μ^- are both left-handed is given by

$$|\mathcal{M}_{LL}|^2 = \frac{4e^4 s^2}{(p_1 - p_3)^4},$$

where $s = (p_1 + p_2)^2$. Find the corresponding expressions for $|\mathcal{M}_{RL}|^2$, $|\mathcal{M}_{RR}|^2$ and $|\mathcal{M}_{LR}|^2$.

Ifn the relativistic limit (M and m are negligible, $p = E$), then from what we have obtained, with $E_1 = E_2 = E$, we have for the electron current j_e:

$$j_{e,LL} = \bar{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) = 2E(c, s, -is, c) \tag{107}$$
$$j_{e,RR} = \bar{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) = 2E(c, s, is, c) \tag{108}$$

for the muon current j_μ we have:

$$j_{\mu,LL} = \bar{u}_\downarrow(p_4)\gamma^\mu u_\downarrow(p_2) = 2E(c, -s, -is, -c) \tag{109}$$
$$j_{\mu,RR} = \bar{u}_\uparrow(p_4)\gamma^\mu u_\uparrow(p_2) = 2E(c, -s, is, -c) \tag{110}$$
$$\tag{111}$$

using these currents we may obtain the matrix element squared. We may first get $|\mathcal{M}_{LL}|^2$:

$$|\mathcal{M}_{LL}|^2 = \left[-\frac{e^2}{t^2} j_{e,LL} \cdot j_{\mu,LL}\right]^2 \tag{112}$$
$$= \frac{e^4}{(p_1 - p_3)^4}\left[2E(c, s, -is, c) \cdot 2E(c, -s, -is, -c)\right]^2 \tag{113}$$
$$= \frac{e^4}{(p_1 - p_3)^4}\left[4E^2(c^2 + s^2 + s^2 + c^2)\right]^2 \tag{114}$$

with $s = 4E^2$ (taken from Chapter 6.2.4), this becomes:

$$|\mathcal{M}_{LL}|^2 = \frac{e^4}{(p_1 - p_3)^4}\left[s(2)\right]^2 \tag{115}$$

$$\boxed{|\mathcal{M}_{LL}|^2 = \frac{4e^4 s^2}{(p_1 - p_3)^4}} \tag{116}$$

next we try for $|\mathcal{M}_{RR}|^2$

$$
\begin{aligned}
|\mathcal{M}_{RR}|^2 &= \left[-\frac{e^2}{t^2} j_{e,RR} \cdot j_{\mu,RR}\right]^2 && (117)\\
&= \frac{e^4}{(p_1-p_3)^4}\left[2E(c,s,is,c)\cdot 2E(c,-s,is,-c)\right]^2 && (118)\\
&= \frac{e^4}{(p_1-p_3)^4}\left[4E^2(c^2+s^2+s^2+c^2)\right]^2 && (119)\\
&= \frac{e^4}{(p_1-p_3)^4}\left[s(2)\right]^2 && (120)
\end{aligned}
$$

$$
\boxed{|\mathcal{M}_{RR}|^2 = \frac{4e^4s^2}{(p_1-p_3)^4}} \qquad (121)
$$

we proceed to the other matrix element squared, with $|\mathcal{M}_{LR}|^2$ and $|\mathcal{M}_{RL}|^2$:

$$
\begin{aligned}
|\mathcal{M}_{LR}|^2 &= \left[-\frac{e^2}{t^2} j_{e,LL} \cdot j_{\mu,RR}\right]^2 && (122)\\
&= \frac{e^4}{(p_1-p_3)^4}\left[2E(c,s,-is,c)\cdot 2E(c,-s,is,-c)\right]^2 && (123)\\
&= \frac{e^4}{(p_1-p_3)^4}\left[4E^2(c^2+s^2-s^2+c^2)\right]^2 && (124)\\
&= \frac{e^4}{(p_1-p_3)^4}\left[s(2c^2)\right]^2 && (125)\\
& && (126)
\end{aligned}
$$

$$
\boxed{|\mathcal{M}_{LR}|^2 = \frac{e^4s^2}{(p_1-p_3)^4}(1+\cos\theta)} \qquad (127)
$$

$$
\begin{aligned}
|\mathcal{M}_{RL}|^2 &= \left[-\frac{e^2}{t^2} j_{e,RR} \cdot j_{\mu,LL}\right]^2 && (128)\\
&= \frac{e^4}{(p_1-p_3)^4}\left[2E(c,s,is,c)\cdot 2E(c,-s,-is,-c)\right]^2 && (129)\\
&= \frac{e^4}{(p_1-p_3)^4}\left[4E^2(c^2+s^2-s^2+c^2)\right]^2 && (130)\\
&= \frac{e^4}{(p_1-p_3)^4}\left[s(2c^2)\right]^2 && (131)\\
& && (132)
\end{aligned}
$$

$$\boxed{|\mathcal{M}_{RL}|^2 = \frac{e^4 s^2}{(p_1 - p_3)^4}(1 + \cos\theta)} \tag{133}$$