Alastair Hall

ECON61001:
Econometric Methods

Semester 1

Study Group Questions # 3

In these questions you consider the relationship between correlation, partial correlations and the multiple correlation coefficient. In Question 4, you consider the partitioned regression model

$$y \;=\; X_1\beta_1 \,+\, X_2\beta_2 \,+\, u \tag{1}$$

where X_1 is a $T \times k_1$ matrix, X_2 is a $T \times k_2$ matrix and y and u are $T \times 1$ vectors. Let $\hat{\beta} = (\hat{\beta}_1', \hat{\beta}_2')'$ denote the OLS estimator of $\beta = (\beta_1', \beta_2')'$, $\tilde{\beta}_1$ denote the OLS coefficient estimator from the regression of y on X_1, and $\tilde{y}$ is the prediction of y based on the OLS regression of y on X_1.

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1. Show that the residual vector can be written as:

$$e \;=\; \{\, I_T \;-\; P_1 \;-\; \bar{P}_2 \,\}\, y,$$

where $P_1 = X_1(X_1'X_1)^{-1}X_1'$ and $\bar{P}_2 = \bar{X}_2(\bar{X}_2'\bar{X}_2)^{-1}\bar{X}_2'$ for $\bar{X}_2 = M_1X_2$, and hence that the residual sum of squares, RSS, is given by

$$RSS \;=\; y'\,\{\, I_T \;-\; P_1 \;-\; \bar{P}_2 \,\}\, y \;=\; RSS_1 \;-\; y'\bar{P}_2 y,$$

where RSS_1 is the residual sum of squares from the OLS regression of y on X_1.

Consider the partitioned regression model:

$$y = X_1\beta_1 + X_2\beta_2 + u$$

Let $X = [X_1 \;\; X_2] \Rightarrow X' = \begin{bmatrix} X_1' \\ X_2' \end{bmatrix}$

$$\hat{\beta} = \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix}$$

From $\hat{\beta}$ being OLS estimator, we have:

$$\hat{\beta} = (X'X)^{-1}X'y$$

Times both sides with $(X'X)$, we have:

$$(X'X)\hat{\beta} = X'y$$

This equals:

$$\begin{bmatrix} X_1' \\ X_2' \end{bmatrix} [X_1 \;\; X_2] \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} X_1' \\ X_2' \end{bmatrix} y$$

$$\Leftrightarrow \begin{bmatrix} X_1'X_1 & X_1'X_2 \\ X_2'X_1 & X_2'X_2 \end{bmatrix} \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} X_1'y \\ X_2'y \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} X_1'X_1\hat{\beta}_1 + X_1'X_2\hat{\beta}_2 \\ X_2'X_1\hat{\beta}_1 + X_2'X_2\hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} X_1'y \\ X_2'y \end{bmatrix}$$

Then we have the system:

$$\begin{cases} X_1'X_1\hat{\beta}_1 + X_1'X_2\hat{\beta}_2 = X_1'y & (1) \\ X_2'X_1\hat{\beta}_1 + X_2'X_2\hat{\beta}_2 = X_2'y. & (2) \end{cases}$$

From (1):

$$\hat{\beta}_1 = (X_1'X_1)^{-1}X_1'y - (X_1'X_1)^{-1}X_1'X_2\hat{\beta}_2 \qquad (1')$$

Substitute $\hat{\beta}_1$ in (2), we have:

$$X_2'X_1(X_1'X_1)^{-1}X_1'y - X_2'X_1(X_1'X_1)^{-1}X_1'X_2\hat{\beta}_2 + X_2'X_2\hat{\beta}_2 = X_2'y. \qquad (3)$$

Let $\begin{cases} P_1 = X_1(X_1'X_1)^{-1}X_1' \\ M_1 = I_T - P_1 \end{cases}$

We can rewrite (3) as:

$$X_2'P_1y - X_2'P_1X_2\hat{\beta}_2 + X_2'I_TX_2\hat{\beta}_2 = X_2'I_Ty$$

$$\Leftrightarrow (X_2'I_TX_2 - X_2'P_1X_2)\hat{\beta}_2 = X_2'I_Ty - X_2'P_1y$$

$$\Leftrightarrow X_2'M_1X_2\hat{\beta}_2 = X_2'M_1y$$

$$\Leftrightarrow \hat{\beta}_2 = (X_2'M_1X_2)^{-1}X_2'M_1y.$$

Then, combine with (1'), we have $\hat{\beta}_1$:

$$\hat{\beta}_1 = (X_1'X_1)^{-1}X_1'y - (X_1'X_1)^{-1}X_1'X_2(X_2'M_1X_2)^{-1}X_2'M_1y.$$

From the model, we have:

$$e = y - \hat{\beta}_1X_1 - \hat{\beta}_2X_2$$

Plug in $\hat{\beta}_1$ and $\hat{\beta}_2$, we have:

$$e = y - [X_1(X_1'X_1)^{-1}X_1'y - X_1(X_1'X_1)^{-1}X_1'X_2(X_2'M_1X_2)^{-1}X_2'M_1y] - X_2(X_2'M_1X_2)^{-1}X_2'M_1y$$

$$\Leftrightarrow e = [I_T - P_1 - (I_T - P_1)X_2(X_2'M_1X_2)^{-1}X_2'M_1]y$$

$$= \{I_T - P_1 - M_1X_2(X_2'M_1'M_1X_2)^{-1}X_2'M_1'\}y \qquad (4)$$

because M_1 is an orthogonal projection matrix.

Let $\begin{cases} \bar{P}_2 = \bar{X}_2(\bar{X}_2'\bar{X}_2)^{-1}\bar{X}_2' \\ \bar{X}_2 = M_1X_2 \end{cases}$

We can rewrite (4) as:

$$e = \{I_T - P_1 - \bar{X}_2(\bar{X}_2'\bar{X}_2)^{-1}\bar{X}_2'\}y$$

$$= [I_T - P_1 - \bar{P}_2]y. \qquad \text{(Q.E.D)}$$

We have $RSS = \sum_{i=1}^{T} e_i^2 \quad (i = 1, 2, 3, \ldots, T)$

$$\Rightarrow RSS = \{[I_T - P_1 - \bar{P}_2]y\}^2$$

$$= y'(I_T - P_1 - \bar{P}_2)'(I_T - P_1 - \bar{P}_2)y$$

$$= y'(I_T - P_1 - \bar{P}_2)y \qquad (5)$$

because $(I_T - P_1 - \bar{P}_2)$ is an orthogonal projection matrix

Consider the model $y = \beta_1X_1 + u$

we can derive $\hat{\beta}_1 = (X_1'X_1)^{-1}X_1'y$

$$\Rightarrow u = y - X_1(X_1'X_1)^{-1}X_1'y$$

$$= I_Ty - P_1y$$

$$= M_1y.$$

$$\Rightarrow RSS_1 = y'M_1y.$$

With that, (5) can be rewritten as:

$$RSS = y'(M_1 - \bar{P}_2)y$$

$$= y'M_1y - y'\bar{P}_2y$$

$$= RSS_1 - y'\bar{P}_2y \qquad \text{(Q.E.D)}$$

Now consider the version of (1) in which $k_1 = 2$, $k_2 = 1$ and $X_1 = [\iota_T, x_2]$, $X_2 = [x_3]$ that is, the model can be written as

$$y_t \;=\; \delta_1 \,+\, \delta_2 x_{2,t} \,+\, \delta_3 x_{3,t} \,+\, u_t, \qquad t = 1, 2, \ldots, T, \tag{2}$$

where $\beta_1 = (\delta_1, \delta_2)'$ and $\beta_2 = \delta_3$. Re-define R^2 to be the multiple correlation coefficient from the OLS estimation of (2). Define r_2 to be the correlation between y and x_2, and $r_{y,3|2}$ to be the partial correlation between y and x_3 given (the intercept and) x_2.

2. Show that:

$$1 \,-\, R^2 \;=\; (1 \,-\, r_2^2)(1 \,-\, r_{y,3|2}^2).$$

Hint: Use the result in Question 1.

For your information: The result in Question 2 holds if we reverse the roles of x_2 and x_3 so that,

$$1 - R^2 = (1 - r_3^2)(1 - r_{y,2|3}^2),$$

where r_3 is the correlation between y and x_3 and $r_{y,2|3}$ is the partial correlation between y and x_2 given (the intercept and) x_3. This relationship between the correlations also extends to larger models. For example, if we consider the model

$$y_t = \delta_1 + \delta_2 x_{2,t} + \delta_3 x_{3,t} + \delta_4 x_{4,t} + u_t, \qquad t = 1, 2, \ldots, T, \tag{3}$$

then the R^2 from this regression satisfies:

$$1 - R^2 = (1 - r_2^2)(1 - r_{y,3|2}^2)(1 - r_{y,4|3,2}^2)$$

where $r_{y,4|3,2}$ is the partial correlation between y and x_4 given (ι_T, x_2, x_3) and $r_{y,3|2}$ is the partial correlation between y and x_3 given (ι_T, x_2). As in the model in Question 2, the subscripts can be interchanged in any order so that, for example, we also have

$$1 - R^2 = (1 - r_3^2)(1 - r_{y,4|3}^2)(1 - r_{y,2|3,4}^2),$$

in the obvious notation for the model in (3).