Alastair Hall

ECON61001: Econometric Methods

Semester 1

Study Group Questions # 3

In these questions you consider the relationship between correlation, partial correlations and the multiple correlation coefficient. In Question 4, you consider the partitioned regression model

$$y = X_1\beta_1 + X_2\beta_2 + u \tag{1}$$

where X_1 is a $T \times k_1$ matrix, X_2 is a $T \times k_2$ matrix and y and u are $T \times 1$ vectors. Let $\hat{\beta} = (\hat{\beta}_1', \hat{\beta}_2')'$ denote the OLS estimator of $\beta = (\beta_1', \beta_2')'$, $\tilde{\beta}_1$ denote the OLS coefficient estimator from the regression of y on X_1, and $\tilde{y}$ is the prediction of y based on the OLS regression of y on X_1.

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1. Show that the residual vector can be written as:

$$e = \{ I_T - P_1 - \bar{P}_2 \} y,$$

where $P_1 = X_1(X_1'X_1)^{-1}X_1'$ and $\bar{P}_2 = \bar{X}_2(\bar{X}_2'\bar{X}_2)^{-1}\bar{X}_2'$ for $\bar{X}_2 = M_1 X_2$, and hence that the residual sum of squares, RSS, is given by

$$RSS = y' \{ I_T - P_1 - \bar{P}_2 \} y = RSS_1 - y'\bar{P}_2 y,$$

where RSS_1 is the residual sum of squares from the OLS regression of y on X_1.

Consider the partitioned regression model:

$$y = X_1\beta_1 + X_2\beta_2 + u$$

Let $X = [X_1 \;\; X_2] \Rightarrow X' = \begin{bmatrix} X_1' \\ X_2' \end{bmatrix}$

$$\hat{\beta} = \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix}$$

From $\hat{\beta}$ being OLS estimator, we have:

$$\hat{\beta} = (X'X)^{-1}X'y$$

Times both sides with $(X'X)$, we have:

$$(X'X)\hat{\beta} = X'y$$

This equals:

$$\begin{bmatrix} X_1' \\ X_2' \end{bmatrix} [X_1 \;\; X_2] \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} X_1' \\ X_2' \end{bmatrix} y$$

$$\Leftrightarrow \begin{bmatrix} X_1'X_1 & X_1'X_2 \\ X_2'X_1 & X_2'X_2 \end{bmatrix} \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} X_1'y \\ X_2'y \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} X_1'X_1\hat{\beta}_1 + X_1'X_2\hat{\beta}_2 \\ X_2'X_1\hat{\beta}_1 + X_2'X_2\hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} X_1'y \\ X_2'y \end{bmatrix}$$

Then we have the system:

$$\begin{cases} X_1'X_1\hat{\beta}_1 + X_1'X_2\hat{\beta}_2 = X_1'y & (1) \\ X_2'X_1\hat{\beta}_1 + X_2'X_2\hat{\beta}_2 = X_2'y. & (2) \end{cases}$$

From (1):

$$\hat{\beta}_1 = (X_1'X_1)^{-1}X_1'y - (X_1'X_1)^{-1}X_1'X_2\hat{\beta}_2 \qquad (1')$$

Substitute $\hat{\beta}_1$ in (2), we have:

$$X_2'X_1(X_1'X_1)^{-1}X_1'y - X_2'X_1(X_1'X_1)^{-1}X_1'X_2\hat{\beta}_2 + X_2'X_2\hat{\beta}_2 = X_2'y. \qquad (3)$$

Let $\begin{cases} P_1 = X_1(X_1'X_1)^{-1}X_1' \\ M_1 = I_T - P_1 \end{cases}$

We can rewrite (3) as:

$$X_2'P_1y - X_2'P_1X_2\hat{\beta}_2 + X_2'I_TX_2\hat{\beta}_2 = X_2'I_Ty$$

$$\Leftrightarrow (X_2'I_TX_2 - X_2'P_1X_2)\hat{\beta}_2 = X_2'I_Ty - X_2'P_1y$$

$$\Leftrightarrow X_2'M_1X_2\hat{\beta}_2 = X_2'M_1y$$

$$\Leftrightarrow \hat{\beta}_2 = (X_2'M_1X_2)^{-1}X_2'M_1y.$$

Then, combine with (1'), we have $\hat{\beta}_1$:

$$\hat{\beta}_1 = (X_1'X_1)^{-1}X_1'y - (X_1'X_1)^{-1}X_1'X_2(X_2'M_1X_2)^{-1}X_2'M_1y.$$

From the model, we have:

$$e = y - \hat{\beta}_1X_1 - \hat{\beta}_2X_2$$

Plug in $\hat{\beta}_1$ and $\hat{\beta}_2$, we have:

$$e = y - [X_1(X_1'X_1)^{-1}X_1'y - X_1(X_1'X_1)^{-1}X_1'X_2(X_2'M_1X_2)^{-1}X_2'M_1y] - X_2(X_2'M_1X_2)^{-1}X_2'M_1y$$

$$\Leftrightarrow e = [I_T - P_1 - (I_T - P_1)X_2(X_2'M_1X_2)^{-1}X_2'M_1]y$$

$$= \{I_T - P_1 - M_1X_2(X_2'M_1'M_1X_2)^{-1}X_2'M_1'\}y \qquad (4)$$

because M_1 is an orthogonal projection matrix.

Let $\begin{cases} \bar{P}_2 = \bar{X}_2(\bar{X}_2'\bar{X}_2)^{-1}\bar{X}_2' \\ \bar{X}_2 = M_1X_2 \end{cases}$

We can rewrite (4) as:

$$e = \{I_T - P_1 - \bar{X}_2(\bar{X}_2'\bar{X}_2)^{-1}\bar{X}_2'\}y$$

$$= [I_T - P_1 - \bar{P}_2]y. \qquad \text{(Q.E.D)}$$

We have $RSS = \sum_{i=1}^{T} e_i^2 \quad (i = 1, 2, 3, \ldots, T)$

$$\Rightarrow RSS = \{[I_T - P_1 - \bar{P}_2]y\}^2$$

$$= y'(I_T - P_1 - \bar{P}_2)'(I_T - P_1 - \bar{P}_2)y$$

$$= y'(I_T - P_1 - \bar{P}_2)y \qquad (5)$$

because $(I_T - P_1 - \bar{P}_2)$ is an orthogonal projection matrix

Consider the model $y = \beta_1X_1 + u$

we can derive $\hat{\beta}_1 = (X_1'X_1)^{-1}X_1'y$

$$\Rightarrow u = y - X_1(X_1'X_1)^{-1}X_1'y$$

$$= I_Ty - P_1y$$

$$= M_1y.$$

$$\Rightarrow RSS_1 = y'M_1y.$$

With that, (5) can be rewritten as:

$$RSS = y'(M_1 - \bar{P}_2)y$$

$$= y'M_1y - y'\bar{P}_2y$$

$$= RSS_1 - y'\bar{P}_2y \qquad \text{(Q.E.D)}$$

Now consider the version of (1) in which $k_1 = 2$, $k_2 = 1$ and $X_1 = [\iota_T, x_2]$, $X_2 = [x_3]$ that is, the model can be written as

$$y_t \;=\; \delta_1 \,+\, \delta_2 x_{2,t} \,+\, \delta_3 x_{3,t} \,+\, u_t, \qquad t = 1, 2, \ldots, T, \tag{2}$$

where $\beta_1 = (\delta_1, \delta_2)'$ and $\beta_2 = \delta_3$. Re-define R^2 to be the multiple correlation coefficient from the OLS estimation of (2). Define r_2 to be the correlation between y and x_2, and $r_{y,3|2}$ to be the partial correlation between y and x_3 given (the intercept and) x_2.

2. Show that:

$$1 - R^2 \;=\; (1 - r_2^2)(1 - r_{y,3|2}^2).$$

Hint: Use the result in Question 1.

The model can be rewritten as $y = X_1\beta_1 + X_2\beta_2 + u$ with:

$X_1 = [\iota_T \;\; x_2]$ and $X_2 = [x_3]$

From Q_1, we have $1 - R^2 = RSS_1 - y'\bar{P}_2 y$ with RSS_1 is from the model $y = \alpha_1 \iota_T + \alpha_2 x_2 + u_1$

$RSS_1 = \Sigma u_1^2 = \Sigma (y - \alpha_1 \iota_T - \alpha_2 x_2)^2 = S_{yy} - \alpha_2^2 S_{22}$

$\Rightarrow R_1^2 = r_2^2 = \dfrac{S_{y2}^2}{S_{yy} S_{22}} \Rightarrow RSS_1 = S_{yy}(1 - r_2^2)$

Then we can rewrite: $1 - R^2 = \dfrac{RSS}{S_{yy}} = \dfrac{S_{yy}(1 - r_2^2) - y'\bar{P}_2 y}{S_{yy}}$

Which means, to prove $1 - R^2 = (1 - r_2^2)(1 - r_{y,3|2}^2)$, we need to prove:

$$\frac{y'\bar{P}_2 y}{S_{yy}} = (1 - r_2^2)\, r_{y,3|2}^2 \quad (1)$$

Start with the RHS. We have $r^2_{y,3|2} = \dfrac{(r_{y3} - r_2 r_{32})^2}{(1-r_2^2)(1-r_{32}^2)}$

$$\Rightarrow \text{RHS}(1) = \frac{r_{y3}^2 - r_2^2 r_{32}^2 + 2 r_{y3} r_2 r_{32}}{1 - r_{32}^2}$$

$$= \frac{\dfrac{S_{y3}^2}{S_{yy} S_{33}} - \dfrac{S_{y2}^2 S_{32}^2}{S_{yy} S_{22}^2 S_{33}} + 2\dfrac{S_{y3} S_{y2} S_{32}}{S_{yy} S_{22} S_{32}}}{\dfrac{S_{33} S_{32} - S_{32}^2}{S_{33} S_{32}}}$$

$$= \frac{S_{y3}^2 S_{22}^2 + S_{y2}^2 S_{32}^2 - 2 S_{y3} S_{32} S_{y2} S_{22}}{S_{yy} S_{22} (S_{33} S_{22} - S_{32}^2)}$$

Turn to $y'\bar{P}_2 y$ in LHS(1): $y'\bar{P}_2 y = y'[I_T - P_1]x_3[x_3'(I_T - P_1)x_3]^{-1}x_3'[I_T - P_1]y$

with $P_1 = X_1(X_1'X_1)^{-1}X_1' = [\iota_T \ \ x_2]\left(\begin{bmatrix} \iota_T' \\ x_2' \end{bmatrix}[\iota_T \ \ x_2]\right)^{-1}\begin{bmatrix} \iota_T' \\ x_2' \end{bmatrix}$

We have $(X_1'X_1)^{-1} = \begin{bmatrix} \iota_T'\iota_T & \iota_T' x_2 \\ x_2'\iota_T & x_2' x_2 \end{bmatrix}^{-1} = \begin{bmatrix} T & \Sigma x_2 \\ \Sigma x_2 & \Sigma x_2^2 \end{bmatrix}^{-1} = \dfrac{1}{T.S_{22}}\begin{bmatrix} \Sigma x_2^2 & -\Sigma x_2 \\ -\Sigma x_2 & T \end{bmatrix}$

because $\det(X_1'X) = T.\Sigma x_2^2 - (\Sigma x_2)^2 = T^2\left[\frac{1}{T}\Sigma x_2^2 - \left(\frac{1}{T}\Sigma x_2\right)^2\right]$

$$= T(\Sigma x_2^2 - T\bar{x}_2^2) = T.S_{22}.$$

$\Rightarrow (X_1'X_1)^{-1}$ can also be written as $(X_1'X_1)^{-1} = \dfrac{1}{T.S_{22}}\begin{bmatrix} x_2'x_2 & -\iota_T' x_2 \\ -x_2'\iota_T & \iota_T'\iota_T \end{bmatrix}$

$$\Rightarrow P_1 = \frac{1}{T.S_{22}}[\iota_T \ \ x_2]\begin{bmatrix} x_2'x_2 & -\iota_T' x_2 \\ -x_2'\iota_T & \iota_T'\iota_T \end{bmatrix}\begin{bmatrix} \iota_T' \\ x_2' \end{bmatrix}$$

$$= \frac{1}{T.S_{22}}[\iota_T x_2' x_2 \iota_T' - x_2 \iota_T' x_2 \iota_T' - \iota_T \iota_T' x_2 x_2' + T x_2 x_2']$$

Rewrite $x_3'(I_T - P_1)y = x_3'y - x_3'P_1 y$

we have $x_3'P_1 y = \dfrac{1}{T.S_{22}} x_3'[\iota_T x_2' x_2 \iota_T' - x_2 \iota_T' x_2 \iota_T' - \iota_T \iota_T' x_2 x_2' + T x_2 x_2']y$

$$= \frac{1}{T.S_{22}}[\Sigma x_3 \Sigma y\, x_2' x_2 - \Sigma x_2 \Sigma y\, x_3' x_2 - \Sigma x_3 \Sigma x_2\, x_2' y + T x_3' x_2 x_2' y]$$

$$= \frac{1}{S_{22}}[T\bar{x}_3 \bar{y}\, x_2' x_2 - T\bar{x}_2 \bar{y}\, x_3' x_2 - T\bar{x}_3 \bar{x}_2\, x_2' y + x_3' x_2 x_2' y]$$

$$= \frac{1}{S_{22}}\Big[(T\bar{x}_3 \bar{y}\, x_2' x_2 - T^2 \bar{x}_3 \bar{y}\, \bar{x}_2^2) + (T^2 \bar{x}_3 \bar{y}\, \bar{x}_2^2 - T\bar{x}_2 \bar{y}\, x_3' x_2)$$
$$+ (x_3' x_2\, x_2' y - T\bar{x}_3 \bar{x}_2\, x_2' y)\Big]$$

$$= \frac{1}{S_{22}}(T\bar{x}_3 \bar{y}\, S_{22} - T\bar{x}_2 \bar{y}\, S_{32} + S_{32}\, x_2' y)$$

$$\Rightarrow x_3'[I_T - P_1]y = x_3'y - T\bar{x}_3\bar{y} - \frac{S_{32}S_{y2}}{S_{22}} = S_{y3} - \frac{S_{32}S_{y2}}{S_{22}}$$

With the same calculation, we can get $\begin{cases} y'[I_T - P_1]x_3 = S_{y3} - \dfrac{S_{32}S_{y2}}{S_{22}} \\ x_3'(I_T - P_1)x_3 = S_{33} - \dfrac{S_{32}^2}{S_{22}} \end{cases}$

$\Rightarrow$ We can rewrite $y'\bar{P}_2 y = \dfrac{S_{22}S_{y3} - S_{32}S_{y2}}{S_{22}} \cdot \dfrac{S_{22}}{S_{33}S_{22} - S_{32}^2} \cdot \dfrac{S_{y3}S_{22} - S_{32}S_{y2}}{S_{22}}$

$$= \frac{S_{y3}^2 S_{22}^2 + S_{32}^2 S_{y2}^2 - 2 S_{22} S_{y3} S_{32} S_{y2}}{S_{22}(S_{33}S_{22} - S_{32}^2)}$$

$\Rightarrow$ LHS(1) $= \dfrac{y'\bar{P}_2 y}{S_{yy}} =$ RHS(1) (Q.E.D).

For your information: The result in Question 2 holds if we reverse the roles of x_2 and x_3 so that,

$$1 - R^2 = (1 - r_3^2)(1 - r_{y,2|3}^2),$$

where r_3 is the correlation between y and x_3 and $r_{y,2|3}$ is the partial correlation between y and x_2 given (the intercept and) x_3. This relationship between the correlations also extends to larger models. For example, if we consider the model

$$y_t = \delta_1 + \delta_2 x_{2,t} + \delta_3 x_{3,t} + \delta_4 x_{4,t} + u_t, \qquad t = 1, 2, \ldots, T, \tag{3}$$

then the R^2 from this regression satisfies:

$$1 - R^2 = (1 - r_2^2)(1 - r_{y,3|2}^2)(1 - r_{y,4|3,2}^2)$$

where $r_{y,4|3,2}$ is the partial correlation between y and x_4 given (ι_T, x_2, x_3) and $r_{y,3|2}$ is the partial correlation between y and x_3 given (ι_T, x_2). As in the model in Question 2, the subscripts can be interchanged in any order so that, for example, we also have

$$1 - R^2 = (1 - r_3^2)(1 - r_{y,4|3}^2)(1 - r_{y,2|3,4}^2),$$

in the obvious notation for the model in (3).