Alastair Hall

ECON61001: Econometric Methods

Semester 1

Study Group Assignment # 5

Our large sample justification of our OLS-based inference methods has rested on applications of the Weak Law of Large Numbers and Central Limit Theorem to appropriate functions of the data. In lectures, we assume that our data are realizations of stationary, weakly dependent time series but these conditions are sufficient and not necessary in order to invoke these limit theorems. In these questions, we explore the properties of two simple non-stationary autoregressive time series models. In the Tutorial session, we consider an AR(1) process that converges to a stationary AR(1) process sufficiently quickly that we obtain the same large sample results for our OLS statistics as would be obtained if the series is in fact a stationary AR(1) process. In Questions 1-3 below, we consider a non-stationary AR(1) process called a *unit root process*. If the data are generated via a unit root process then we cannot invoke our limit theorems and our OLS-based inference methods are invalid even in large samples. Many economic time series exhibit the traits of unit root processes and so it is standard practice to test whether our time series follow a unit root process prior to our regression analysis using a procedure known as the (Augmented) Dickey-Fuller test. In Question 4, you examine the regression model on which this test is based and then you implement versions of this test in the computer exercises.

Please enter the details of the group:

Group name	Group Matrix
Student ID 1	11335127
Student ID 2	10710007
Student ID 3	10704589
Student ID 4	11465531
Student ID 5	

1. Consider the process

$$v_t = \theta v_{t-1} + u_t, \qquad t = 1, 2, \ldots, T,$$

where $v_0 = 0$ and $\{u_t\}_{t=1}^T$ is a white noise sequence with variance σ^2. Suppose that $\theta = 1$. Show that $Var[v_t] = t\sigma^2$ and hence that $Var[v_t] \to \infty$ and $t \to \infty$.

We use : $Var[v_t] = E[(v_t - E[v_t])(v_t - E[v_t])']$

$\Rightarrow$ To calculate $Var[v_t]$, we need to know $E[v_t]$.

From $v_t = \theta v_{t-1} + u_t$, it follows that :

$v_1 = \theta v_0 + u_1 = u_1$ (since $v_0 = 0$)

$v_2 = \theta v_1 + u_2 = \theta u_1 + u_2$

$v_3 = \theta v_2 + u_3 = \theta(\theta u_1 + u_2) + u_3$

$= \theta^2 u_1 + \theta u_2 + u_3$

$v_4 = \theta v_3 + u_4 = \theta[\theta(\theta u_1 + v_2) + u_3] + u_4.$

$= \theta^3 u_1 + \theta^2 u_2 + \theta u_3 + u_4.$

$\vdots$

$\Rightarrow v_t = \theta^{t-1} u_1 + \theta^{t-2} u_2 + \ldots + \theta^1 u_{t-1} + u_t.$

$= \sum_{i=0}^{t-1} \theta^i u_{t-i}$

$\Rightarrow E[v_t] = E[\sum_{i=0}^{t-1} \theta^i u_{t-i}] = \sum_{i=0}^{t-1} \theta^i E[u_{t-i}] = 0$ (since $E[u_t] = 0$).

Therefore, it follows that :

$Var[v_t] = E[(v_t - E[v_t])(v_t - E[v_t])']$

$= E[v_t^2]$

$= E[(\sum_{i=0}^{t-1} \theta^i u_{t-i})^2].$

$= E[\sum_{i=0}^{t-1} \sum_{j=0}^{t-1} \theta^i u_{t-i} \theta^j u_{t-j}]$

$\Rightarrow$ If i and j coincide, $Var[v_t] = E[\sum_{i=0}^{t-1} \theta^{2i} u_{t-i}^2] = \sigma^2 \sum_{i=0}^{t-1} \theta^{2i}$

$\Rightarrow$ If i and j do not coincide, the off-diagonal elements of $Var[v_t]$:

$E[\sum_{i=0}^{t-1} \sum_{j=0}^{t-1} \theta^{i+j} u_{t-i} u_{t-j}] = 0$ since $cov[u_{t-i}, u_{t-j}] = 0$

$\Rightarrow Var[v_t] = \sigma^2 \sum_{i=0}^{t-1} \theta^{2i} = t.\sigma^2$ given $\theta = 1$.

Examine $\lim_{t\to\infty} Var[v_t] = \lim_{t\to\infty} t.\sigma^2$

It's easy to see that when $t \to \infty$, $var[v_t] \to \infty$.

Now consider the process

$$v_t \;=\; \alpha \,+\, \theta v_{t-1} \,+\, u_t, \qquad t = 1, 2, \ldots, T,$$

where $v_0 = 0$ and $\{u_t\}_{t=1}^{T}$ is a white noise sequence.

2. Suppose that $|\theta| < 1$. Show that

$$E[v_t] \;=\; \alpha \sum_{i=0}^{t-1} \theta^i,$$

and hence that

$$lim_{t\to\infty} E[v_t] \;=\; \frac{\alpha}{1-\theta}.$$

From the given process $v_t = \alpha + \theta v_{t-1} + u_t$, we have:

$$v_1 = \alpha + \theta v_0 + u_1 = \alpha + u_1 \quad (\text{as } v_0 = 0)$$

$$v_2 = \alpha + \theta v_1 + u_2 = \alpha + \theta(\alpha + u_1) + u_2$$

$$= \alpha(1+\theta) + \theta u_1 + u_2$$

$$v_3 = \alpha + \theta v_2 + u_3 = \alpha + \theta[\alpha(1+\theta) + \theta u_1 + u_2] + u_3$$

$$= (\theta^2 + \theta)\alpha + \theta^2 u_1 + \theta u_2 + u_3$$

$$\vdots$$

$$v_t = (\theta^{t-1} + \theta^{t-2} + \ldots + \theta^1 + \theta^0)\alpha + \theta^{t-1} u_1 + \theta^{t-2} u_2 + \ldots + \theta^1 u_{t-1} + u_t$$

$$\Rightarrow \quad v_t = \alpha \sum_{i=0}^{t-1} \theta^i + \sum_{i=0}^{t-1} \theta^i u_{t-i}$$

$$\Rightarrow \quad E[v_t] = E\left[\alpha \sum_{i=0}^{t-1} \theta^i + \sum_{i=0}^{t-1} \theta^i u_{t-i}\right]$$

$$= E\left[\alpha \sum_{i=0}^{t-1} \theta^i\right] + E\left[\sum_{i=0}^{t-1} \theta^i u_{t-i}\right]$$

$$= \alpha \sum_{i=0}^{t-1} \theta^i \quad (\text{as } \alpha, \theta \text{ are fixed and } E[u_t] = 0).$$

Examine $\lim_{t\to\infty} E[v_t] = \lim_{t\to\infty} \alpha \sum_{i=0}^{t-1} \theta^i = \alpha \cdot \lim_{t\to\infty} \sum_{i=0}^{t-1} \theta^i$

Notice that $\sum_{i=0}^{t-1} \theta^i$ is the t first terms of a geometric sequence

$$\Rightarrow \quad \sum_{i=0}^{t-1} \theta^i = \frac{1-\theta^t}{1-\theta}$$

$$\Rightarrow \quad \lim_{t\to\infty} \sum_{i=0}^{t-1} \theta^i = \lim_{t\to\infty} \frac{1-\theta^t}{1-\theta} = \frac{1}{1-\theta} \quad (\text{as } |\theta| < 1)$$

$$\Rightarrow \quad \lim_{t\to\infty} E[v_t] = \frac{\alpha}{1-\theta}$$

3. Suppose that $\theta = 1$. Show that $E[v_t] = t\alpha$. Characterize the limiting behaviour of $E[v_t]$ as $t \to \infty$.

From Q2, we have that: $v_t = \alpha \sum_{i=0}^{t-1} \theta^i + \sum_{i=0}^{t-1} \theta^i u_{t-i}$
and $E[v_t] = \alpha \sum_{i=0}^{t-1} \theta^i$

If $\theta = 1 \Rightarrow \sum_{i=0}^{t-1} \theta^i = t \Rightarrow E[v_t] = \alpha t.$

Examine $\lim_{t \to \infty} E[v_t] = \lim_{t \to \infty} \alpha t = \infty$

$\Rightarrow$ As $t \to \infty$, $E[v_t] \to \infty$ if $\theta = 1$.

From our discussion in the Tutorial session and Questions 1-3 above it can be seen that the time series behaves very differently if $|\theta| < 1$ or $\theta = 1$. A popular test for the null hypothesis that $\theta = 1$ is the (Augmented) Dickey-Fuller test. In the next question you consider the regression model upon which the Dickey-Fuller test is based.

4. Suppose that v_t is generated by

$$v_t \ = \ \theta v_{t-1} \ + \ u_t$$

and consider the regression model

$$\Delta v_t \ = \ \gamma v_{t-1} \ + \ u_t.$$

Show that if $\theta = 1$ then $\gamma = 0$ and that if $|\theta| < 1$ then $\gamma < 0$.

Consider that $\Delta v_t = v_t - v_{t-1}$.

With $v_t = \theta v_{t-1} + u_t$

$\Rightarrow \Delta v_t = \theta v_{t-1} + u_t - v_{t-1}$

$= (\theta - 1) v_{t-1} + u_t$. (1)

Also, from given regression model, we have:

$\Delta v_t = \gamma v_{t-1} + u_t$ (2)

(1)(2) $\Rightarrow \theta - 1 = \gamma$

$\Rightarrow \begin{cases} \text{If } \theta = 1 \rightarrow \gamma = 0 \\ \text{If } |\theta| < 1 \Rightarrow \gamma < 0. \end{cases}$