Financial Econometrics

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Contents

Lecture 1: Introduction Lecture

Wed 31 Jan 11:19

1 Financial Time Series and their Characteristics

1.1 Asset Returns

Financial studies involve returns, instead of prices of assets.

Returns :

- is a complete and scale free summary of the investment opportunity
- are easier to handle than price series

p_t is the price of an asset at time index t. And assuming an asset pays no dividends.

Continuous Compounding

One period Simple Returns

Holding the asset for one period from date $t-1$ to date t would result in a simple gross return :

$$1 + R_t = \frac{P_t}{P_{t-1}} \text{ or } P_t = P_{t-1}(1 + R_t).$$

The corresponding one period simple net return or simple return is :

$$R_t = \frac{P_t}{P_{t-1}} - 1 = \frac{P_t - P_{t-1}}{P_{t-1}}.$$

Multi period Simple Returns

Holding the asset for k periods between dates t-k and t gives a k-period simple gross return :

$$1 + R_t[k] = \frac{P_t}{P_{t-k}} = \frac{P_t}{P_{t-1}} \times \frac{P_{t-1}}{P_{t-2}} \times \ldots \times \frac{P_{t-k+1}}{P_{t-k}}.$$
$$= (1 + R_t)(1 + R_{t-1}) \ldots (1 + R_{t-k+1}).$$
$$= \prod_{j=0}^{k-1} (1 + R_{t-j}).$$

That is, the k-period simple gross return is just the product of the k one period simple gross returns involved. A compound return.

The actual time interval is important in discussing and comparing returns, if not given, it is implicitly assumed to be one year.

If an asset is gold for k years, then the annualized average return is defined as

$$\text{Annualized} R_t[k] = (\prod_{j=0}^{(k-1)} (1 + R_{t-j}))^{(\frac{1}{k})} - 1.$$

Which is a geometric mean of the k one period simple gross returns involved and can be computed by

$$= \exp(\frac{1}{k} \sum_{j=0}^{(k-1)} \ln(1 + r_{t-j})) - 1$$

Where it is easier to compute the arithmetic average than the geometric mean and the one-period returns tend so be small, one can use a first order Taylor expansion to approximate the annualized return and obtain

$$\approx \frac{1}{k} \sum_{j=0}^{(k-1)} R_{t-j}.$$

Continuous Compounding

Assume the interest rate of a bank deposit is 10% per annul, and the initial deposit is \$1

If the bank pays interest once a year, then the net value of the deposit becomes 1.1\$. If the bank pays interest semi-annually, the 6-month interest rate is 5% and the net value is $1(1 + \frac{0.1}{2})^2 = \$1.1025$ after the first year.

In general if the bank pays interest m times a year, then the interest rate for each payment is $10\%/m$ and the net value of the deposit becomes $1(1 + \frac{0.1}{m})^{(m)}$ one year later.

Continuously Compounded Returns

The natural logarithm of the simple gross return of an asset is called the continuously compounded return or log return :

$$r_t = \ln(1 + R_t) = \ln(P_t/P_{t-1}) = p_t - p_{t-1} \tag{1}$$

where $p_t = \ln(P_T)$. Continuously compounded returns are advantageous since they are the sum of continuously compounded multi period return.

Portfolio Return

Simple net return of a portfolio consisting of N assets is a weighted average of the simple net returns of the assets involved, where the weight on each asset is the percentage of the portfolio's value invested in that asset. Where p is a portfolio that places weight w_i on asset i. Then the simple return of p at time t is

$$R_{p,t} = \sum_{i=1}^{N} w_i R_{it}.$$

where R_{it} is the simple return of asset i.

The continuously compounded returns of a portfolio, do not have this property. Instead,

$$r_{p,t} sim \sum_{i=1}^{N} w_i r_{it}.$$

Where $r_{p,t}$ is the continuously compounded return of the portfolio at time t

Dividend Payment

If an asset pays periodically. Let D_t be the dividend payment of an asset between dates $t-1$ and P_t be the price of the asset at the end of period t. The dividend is this not included in P_t The simple net return and continuously compounded return at time t become

$$R_t = \frac{P_t + D_t}{P_{t-1}} - 1 \quad , \quad r_t = \ln(P_t + D_t) - \ln(P_{t-1}).$$

Excess Return

The difference between the asset's return and return on some reference asset, often taken to be rissoles such as short term US treasury bill. Simple excess return and log excess return of an asset are then defined as

$$Z_t = R_t - R_{0t} \quad , \quad z_t = r_t - r_{0t}.$$

Where R_{0t} and r_{0t} are the simple and log returns of the reference asset (resp)

Distributional Properties of Returns

Review of statistical distributions and their moments

Joint Distribution

$$F_{X,Y}(x, y : \theta) = P(X \leq x, Y \leq y : \theta).$$

where $x \in R^{(p)}$, $y \in R^{(q)}$ and the inequality $\leq$ is a joint distribution function of X and Y with parameter θ. The behavior of X and Y is characterized by $F_{X,Y}(x, y : \theta$

If the joint probability density function $f_{x,y}(x, y : \theta)$ exists then

$$F_{X,Y}(x, y : \theta) = \int_{-\infty}^{x} \int_{-\infty}^{Y} f_{x,y}(w; z; \theta) dz dw.$$

Where X and Y are continuous random vectors

Marginal Distribution

Given by

$$F_X(X; \theta) = F_{X,Y}(x, \infty, \ldots, \infty, \theta).$$

Thus, the marginal distribution of X is obtained by integrating out Y. A similar definition applies to the marginal distribution of Y if $k = 1$ X is a scalar random variable and the distribution function becomes

$$F_X(x) = P(X \leq x; \theta).$$

which is the CDF of X. The CDF of a random variable is nondecreasing and satisfies $F_X(-\infty) = 0$and $f_X(\infty) = 1$ For a given probability p, the smallest real number x_p such that $p \leq F_X(x_p)$ is called the 100 p th quantile of the random variable X

Conditional Distribution
The conditional distribution of X given $y \leq y$ is given by

$$F_{X|Y \leq y}(x; \theta) = \frac{P(X \leq X, Y \leq Y : \theta}{P(Y \leq Y : \theta)}.$$

Moments of a Random Variable
The l-th moment of a continuous random variable X is defined as

$$m_l' = E[X^l] = \int_{-\infty}^{\infty} x^l f(x) dx$$

Where E stands for expectation and $f(x)$ is the probability density function of x. The first moment is called the mean or expectation, measuring the central location of the distribution.

The l-th central moment of X is defined as

$$m_l = E[(X - \mu_x)^l] = \int_{-\infty}^{\infty} (x - \mu_x)^l f(x) dx$$

The second central moment, denoted σ_x^2 measures the variability of X and is called the variance of X. The positive square root σ_x of variance is the *standard deviation* of X.

The first two moments of a random variable uniquely determine a normal distribution.

The Third Central moment measures the symmetry of X with respect to its mean, whereas the *fourth central moment* measures the tail behaviour of X.

Skewness and *kurtosis* are normalised third and fourth central moments of X, are often used to summarise the extent of asymmetry and tail thickness

1.2 Descriptive Statistics

Let Y_t be a time-series of random variables with a history of realisations y_t with $t = 1, \ldots, T$

Mean

$$E[Y_t] = \mu \quad , \quad \hat{mu} = \frac{1}{T} \sum_{t=1}^{T} y_t$$

Variance

$$V[Y_t] = E[(Y_t - \mu)^2] \quad , \quad \hat{\sigma}^2 = \frac{1}{T} \sum_{t=1}^{T} (y_t - \hat{mu})^2$$

Skewness

$$S[Y_t] = E[\frac{(Y_t - \mu)^3}{\sigma^3}] \quad , \quad \hat{S} = \frac{1}{T} \sum_{t=1}^{T} [\frac{(Y_t - \mu)^3}{\sigma^3}][]$$

Kurtosis

$$K[Y_t] = E[\frac{(Y_t - \mu)^4}{\sigma^4}] \quad , \quad \hat{S} = \frac{1}{T} \sum_{t=1}^{T} [\frac{(Y_t - \mu)^4}{\sigma^4}][]$$

Jargue-Bera test, tests H_0 of normality of the series :

$$JB = \frac{T}{6}(\hat{S}^2 + \frac{(\hat{K} - 3)^2}{4})$$

Where k is the number of estimated parameters. This test statistic has a χ^2 distribution with 2 degrees of freedom (always). Tests 2 parameters jointly. Rejection when skewness is not 0 or kurtosis is not 3. Skewed or heavy tailed. Then use individual tests against 0 or 3 using WLLN and CLT. T test, standardizing appropriately.

Quantile-Quantile plots : plot theoretical quantiles against the empirical ones

Stylized Facts

- Return series do not follow a normal distribution
- the normal distribution does not explain the occurrence probability of extreme events
- better assumptions are student-t or stable distributions
- on higher frequencies (intraday) the deviation from normality is more pronounced than on lower frequencies
- aggregated return series, do however, tend to normality
- Return series posses fat tails
- Return series are leptokurtic or posses an overkurtosis (kurtosis >3)
- large returns occur more often than expected
- large returns are more often negative than positive which yields left skewed returns (skewness <0)
- intraday returns are subject to typical trading session effects (seasonality, opening and closing issues)
- returns are subject to volatility clustering, which is again more pronounced on higher frequencies
- volatility is time varying
- financial time series are correlated
- correlations are also time varying

Standardized Return

$$(\frac{r_t - \mu}{\hat{\sigma}}).$$

Kurtosis is probably the most important, telling you about the number of extreme events. Say coca-cola vs tesla (kurtosis of 50). Can be seen as number of outliers around mean

Plotting histogram, kurtosis is heavy tails, extreme distribution lands exactly to the tails.

1.3 Distribution of Returns

The most general model for the log returns is its joint distribution function $F_r(r_{11}, \ldots, r_N : r_{12}, \ldots, r_{N2} : \ldots r_{IT} \ldots r_{NT} : Y; \theta)$

Where Y is a state vector consisting of variables that summarise the environment in which asset returns are determined and θ is a vector of parameters that uniquely determines the distribution funcion $F_r(\cdot)$, which governs the stochastic behaviour the returns r_{it} and $\mathbf{Y}$.

Often the state vector Y is treated as given and the main concern is the conditional distribution of $\{r)it\}$ given Y.

Some financial theories (CAPM) focus on the joint distirbution of N returns at a single tome index t. Whilst others look at the dynamic structure of individual asset returns

Since our main concern is the joint distribution of $\{r_{it}\}_{t=1}^{T}$ for asset i, it is useful to partition the joint distribution as :

$$F(r_{i_1}, \ldots, r_{iT} : \theta) = F(r_{i1})F(r_{i2}|r_{i1}) \ldots F(r_{iT}|r_{iT-1}, \ldots, r_{i1})$$
$$= F(r_{i1}) \prod_{t=2}^{T} F(r_{iT}|r_{it-1}, \ldots, r_{i1})$$

Where the parameter θ is omitted for brevity.

This partition the temporal dependencies of the log return r_{it}. With the main issue the specification of the conditional distribution $F(r_{it}|r_{i\ t-1})$ since different distributional specification lead to different theories in finance.

For instance the *random walk hypothesis* in which one version entails the conditional distribution $F(r_{it}|r_{i\ \ t-1}, \ldots r_{i1})$ is equal to the marginal distribution $F(r_{it})$ meaning returns are temporally independent and thus not predictable.

Normal Distribution

A traditional assumption is that the simple returns $\{R_{it}|t = 1, \ldots, T\}$ are **independently and identically distributed** as normal with fixed mean and variance.

However, this assumption encounters difficulties empirically,

- the lower bound of a simple return is -1, but the normal distribution may assume any value in the real line and hence has no lower bound
- if R_{it} is normally distributed then the multi period simple return $R_{it}[k]$ is not normally distributed because it is a product of one period returns
- the normality assumption is not supported by many empirical asset returns

Log normal Distribution

Another commonly used assumption is that he long returns r_t of an asset are independent and identically distributed (iid) as normal with mean μ and variance σ^2. The simple returns are then iid lognormal random variables with mean and variance given by

$$E[R_t] = \exp(\mu + \frac{\sigma^2}{2}) - 1$$

and

$$Var[R_t] = \exp(2\mu + \sigma^2)[\exp(\sigma^2) - 1]$$

Stable Distribution

The stable distribution are a natural generalisation of normal in that they are stable under addition, meeting the need of contionusly compounded returns r_t. Furthermore, stable disributinos are capabale of capturing excess kurtosis, shown by histroical stock returns

Hypothesis Test

null $H_0 : s = 0$ vs $H_1 : S \neq 0$

$$\hat{t} + CLT \to^{(d)} N(0, 1).$$

Tells you distribution under the null, then 95% of probability mass is between critical values, then outside of this, either suff evidence against the null or a type I error (5%) (at tails). Fundamentally, we cannot trust the null hypothesis.

Whatever we want to test, we put into the alternative. NO conclusion can be made if we fail to reject the null. If we collect evidence against the null then this is fundamentally different.

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Lecture 2: Second Lecture - Review TS

Fri 02 Feb 15:45

2 Time Series Basics

Stochastic Processes

- chronologically ordered equidistant observations
- generated by stochastic process
- stochastic process - collection of RV (each Y_i is generated by different member of stochastic processes)
- **assumption** time series data has been generated by *stochastic process*

Definition 1. **stochastic process** is a family of random variables defined on a probability space

Definition 2. **time series** is a realisation of a stochastic processs

Definition 3. **time series analysis** - only one history $Y_t(w)$, one state of the world $w \in \omega$ is available, but the goal is to derive the properties of $Y_t(\cdot)$ for a given t for different states of the world

idea - how can we understand what is driving omegas? Different states of the world, since we observe y_t. So place some structure on y_t

Should be able to recognise :

- non-stationary time series
- autoregressive time series
- kurtosis time series

Definition 4. auto covariance
Definition

time series often show correlation between successive observations, this feature is called serial correlating or **autocorrelation**

dependencies over time are described by auto covariance and autocorrelation functions

the j-th autocovariance of Y_t is given by

$$Cov[Y_t, y_{t-j}] = \gamma_{t,t-j} = E[Y_t - E[Y_t][Y_{t-j} - E[Y_{t-j}]$$

correspondingly the variance of Y_t is defined as :

$$V[Y_T] = \gamma_{t,t} = E[Y_t - E[Y_t])^2]$$

Definition 5. autocorrelation

the j-th autocorrelation of Y_t is given by :

$$\rho_{t,t-j} = \frac{Cov[Y_t, Y_{t-j}]}{V[Y_t]^{(\frac{1}{2})} V[Y_{t-j}^{(\frac{1}{2})}}$$

Definition 6. Covariance stationary A time series $\{Y_t\}_{t=-\infty}^{(\infty)}$ is called covariance stationary, or weakly stationary, if :

$$E[Y_t] = \mu_Y$$
$$V[Y_t] = \gamma_{t,t} = \gamma_0 = \sigma_Y^2 < \infty$$
$$Cov[Y_t, Y_{t-j}] = \gamma_{t,t-j} = \gamma_j < \infty$$

for a covaraince stationary process the j-th autocorrelation is given by :

Definition 7. white noise a TS is called this if it satisfies the following

$$E[Y_t] = 0 V[Y_t] = \sigma_Y^2 COv[Y_t, Y_s] = E[Y_t, Y_s] = 0$$

white noise is a weakly stationary process

Definition 8. Autocorrelation function of a covaraince stationary process $\{Y_t\}_{t=-\infty}^{(\infty)}$ is the sequence of autocorrelations ρ_j for all $j = 0, 1, 2, \ldots$

Definition 9. the empirical (or sample) autocorrelatino function of a time series Y_t is the sequence of sample autocorrelation coefficients $\hat{\rho}_j$ for all $j = 0, 1, 2, \ldots$:

$$\hat{\rho}_j = \frac{\hat{\gamma}_j}{\hat{\gamma}_0} = \frac{\sum_{t=j+1}^{T} (Y_t - \overline{Y}(Y_{t-j} - \overline{Y})}{\sum_{t=1}^{T} (Y_t - \overline{Y}^2}$$

and

$$\hat{\gamma}_j = \frac{1}{T} \sum_{t=j+1}^{T} (Y_t - \overline{Y})(Y_{t-j} - \overline{Y}) \qquad \overline{Y} = \frac{1}{T} \sum_{t=1}^{T} Y_t$$

the graphical depictions of the empirical autocorrelation function is called an autocorrelogram

Definition 10. partial autocorrelation function

partial autocorrelation between Y_t and Y_{t-j} is the conditional correlation between Y_t and Y_{t-j} given (holding fixed) $Y_{t-1}, \ldots, Y_{t-j+1}$

$$a_j = Cor[Y_t, Y_{t-j} | Y_{t-1}, \ldots, Y_{t-j+1}]$$

effects of in-between values are controlled for corresponding sample quantity $\hat{a}_j$ is called sample partial autocorrelation and is obtained as the OLS estimator of the coefficient a_j in model

$$Y_t = a_0 + a_1 Y_{t-1} + \ldots + a_j Y_{t-j} + \mu_t$$

Definition 11. sample autocorrelation function

if data generating process is a white noise process, then for large T:

$$\hat{\rho}_j \approx N(0, \frac{1}{T}), j = 1, 2, \ldots$$

Means : $H_0 : \hat{\rho}_j = 0$ is rejected, if zero does not fall within the approximate 95% confidence interval

$$[r\hat{h}o_j - \frac{2}{\sqrt{T}}, r\hat{h}o_j + \frac{2}{\sqrt{T}}]$$

Equivalently, autocorrelations are not significant when $\hat{\rho}_j$ is within the approximate two standard error bound $\pm 2/\sqrt{T}$

3 Arma Processes

Definition 12. a time series is called an autoregressive process of order p if it satisfies a relationship of the type :

$$Y_t = c + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \rho_p Y_{t-p} + \varepsilon_t$$

where ε_t is a white noise error term

A(1) process : the simplest form of an A(p) process is obtained for $p = 1$ as

$$Y_t = c + \phi_1 Y_{t-1} + \varepsilon_t$$

Definition 13. MA(q)-Process a time series is called a **moving average process of order** q if it satisfies a relatinoship of the type

$$Y_t = \mu = \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \ldots + \theta_q \varepsilon_{t-q}$$

where ε_t is a white noise error term

MA(1) Process: the simplest form of an MA(q) process os obtained for $q = 1$ as

$$Y_t = \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1}$$

Example. AR(1) Process

$$\sum_{i=0}^{\infty} \rho^{(i)} u_{t-i} =^{(wald)} MA(\infty)$$

3.1 ARMA

Lag operator let $\{Y_t\}_{t=-\infty}^{(\infty)}$ be a time series, then the lag operator $\mathcal{L}$ is defined by the relation

$$L^{(J)} \equiv Y_{t-j}$$

If $\{Y_t = c\}_{t=-\infty}^{(\infty)}$ where $c \in \mathbb{R}$, then $\mathcal{L}^{(j)}Y_t = L^{(j)}c = c$

ARMA(p,q) is a time series $\{Y_t\}_{t=-\infty}^{(\infty)}$ of the following form

$$\phi_p(L)Y_t = c + \Theta(L)\varepsilon_t where$$
$$\phi_p(L) = 1 - \phi_1 L - \phi_2 L^{(2)} - \ldots . - \phi_p L^{(p)}$$
$$\Theta(L) = 1 + \theta_1 L + \theta_2 L^{(2)} + \ldots + \theta_q L^{(q)}$$

with ε_t being a white noise and ϕ_p and Θ_q are called lag polynomials

3.2 ARMA estimation

ARMA(p, q) process:

$$Y_t = c + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \phi_p Y_{t-p} + \varepsilon_t + \theta_1 e_{t-1} + \theta_2 \varepsilon_{t-2} + \ldots + \theta_q \varepsilon_{t-q}$$

- estimation via conditional max likelihood
- **conditional** : derive the likelihood function under the assumption that the initial values of Y_t and ε_t are available
- **assume** : $\varepsilon_t \sim^{(iid)} N(0, \sigma^2)$
- ML parameter estimators are derided under the assumption of normality are quasi ML estimators
- our goal is to estimate the vector $\theta = (c, \phi_1, \phi_2, \ldots, \theta_1, \theta_2, \ldots, \theta_q, \sigma^2)'$

ARMA estimation
Conditional log likelihood
estimation is done under asssumption that error term is normal.
LBQ test
whether p is sufficiently long, if model specified correctly, then residuals shouldn't be correlated with each other.
Tells whether white noise property is plausible assumption
critical values is from chi-squared dist, we test for absence of autocorrelation upto chosen lag order, leading into next weeks lecture of conditional heteroskedacity.
ARCH-LM test
tests for conditional heteroskecadity in regression residuals
pick ARMA based on this, if modelled succesfully then null of LBQ test shouldnt be rejected and there shouldnt be any conditional heterskedacity

Lecture 3: ARCH Models

Mon 12 Feb 09:01

Review

Week 1

Leptokurtic Property - How to measure a lot of outliers? Kurtosis. The kurtosis of our distribution is larger than 3 (4th moment of distribution). Since $K[r_t] > 3$ where $e \sim \mathcal{N}(\mu, \sigma^2)$

Left-Skewness - more negative returns than positive ones. $S[r_t] < 0$

Volatility clustering - periods of high volatility are followed by periods of high volatility. The volatile periods on the markets (across S of return distribution) they *cluster*. Market volatility is persistent.

Shape of daily returns - Compared to say a normal bell curve, is this a good distribution? Weekly more normal then daily, monthly more normal than weekly. Thus *aggregate returns tend to normality*

- Should know these by heart
- And be able to apply them and tell graphically

Week 2

Time series analysis

ARMA Models ($ARMA(1,1)$ $y_t = c + \rho_1 + y_{t-1} + \varepsilon_t + \theta\varepsilon_{t-1}$

Stationarity ADF test

Model selection ACF /DACF

Information criteria - Bayesian information criteria helps to choose whether ARMA(1, 2) or MA(1) is better for data.

At the end we estimate by quasi-likelihood since $\varepsilon_t \sim \mathcal{N}(0, \sigma^2)$, which is key to today's material.

It is important to realise this assumption is quite strong, the shortcut for this type of estimator is quasi-likelihood.

$\Theta = (c, \rho_1, \theta_1, \sigma^2)$, in empirical terms the maximum likelihood estimators minimise the negative log-likelihood, we can only find the minimum using gradient descent, hence minimising the negative.

$\hat{\theta}_{QML} = \underset{\Theta}{argmin}$

Autoregressive process order 1

Log likelihood - maximises function to find combination of parameters of model such that our ε_t's are normal

For Financial Econometrics, once plot ACF and PCF, once looking at squared residuals, we have seen a lot of significant lags in the squared residuals. LBQ test and ARCH-LM test whether there is remaining autocorrelation within the squares residuals.

These tests tell us that $\hat{\sigma^2}$ tell us there is autocorrelation across time within the residuals, only problem of model misspecification comes from squared residuals, variance of error term.

3.3 Conditional Heteroskedacity

In any ARMA model there is some expectation

$$y_t = E[y_t|F_{t-1}] + \varepsilon_t$$

$c + \rho y_{t-1} + \theta\varepsilon_{t-1}$. F is filtration, past information and ε_t is new information/shock today.

White noise (DSA) :

$$\begin{aligned} \varepsilon_t &\sim WN \\ E[\varepsilon_t] &= 0 \\ V[\varepsilon_t] &= \sigma^2 \\ cov(\varepsilon_t, \varepsilon_t) &= 0 \end{aligned}$$

What is the difference between conditional and unconditional moments?
Conditional : $V[\varepsilon_t]$ and Unconditional $V[\varepsilon_t|F_{t-1}]$

$$
\begin{aligned}
y_t &= c + \rho y_{t-1} + \varepsilon_t \\
E[y_t] &= E[c + \rho y_{t-1} + \varepsilon_t] \\
&= c + \rho E[y_{t-1}] \\
E[y_t] &= E[y_{t-1}] \\
E[y_t] &= \frac{c}{1-\rho}
\end{aligned}
$$

That is,

$$\frac{c}{1-\rho} \quad vs \quad c + \phi y_{t-1} \qquad (*)$$

$$
\begin{aligned}
&E[y_t|F_{t-1}] \\
&E[c + \rho y_{t-1} + \varepsilon_t|F_{t-1}] \\
c + \rho E[y_{t-1}&|F_{t-1}] + E[\varepsilon_t|F_{t-1}] \\
&c + \rho y_{t-1} + 0
\end{aligned}
$$

White Noise

- $E[\varepsilon_t] = 0$
- $V[\varepsilon_t] = \sigma^2$
- $cov[\varepsilon_t, \varepsilon_t] = 0$

The unconditional moment in (*) is more important.
White noise assumption, assumes both conditional and unconditional are constant over time, that is

$$V[\varepsilon_t] = V[\varepsilon_t, |F_{t-1}] = \sigma^2$$

$V[\varepsilon_t] = \sigma^2$ but $V[\varepsilon_t, F_{t-1}]$ is time varying (conditional second moment).
We start with $\varepsilon_t = \mathcal{L}_t \cdot \sigma_t$ where $\mathcal{L}_t \sim \mathcal{N}(0,1)$ and ARCH (1) : $\sigma_t^2 = w + \alpha\varepsilon_{t-1}$
As we have just done with AR1, now look at conditional and unconditional second moment of ARCH(1).
$V[\varepsilon_t]$ and

$$
\begin{aligned}
E[\varepsilon_t] &= E[\mathcal{L}_t\sigma_t] = \\
&E[\mathcal{L}_t]E[\sigma_t] \\
&0 \cdot E[\sigma_t] = 0 \\
V[\varepsilon_t] &= E[\varepsilon_t]^2 E[\mathcal{L}_t^2 \cdot \sigma_t^2] = \\
&E[\mathcal{L}_t^2] \cdot E[\sigma_t^2] = E[\sigma_t^2]
\end{aligned}
$$

$V[\varepsilon_t|F_{t-1}]$ (not right yet)

$$\begin{aligned} E[\varepsilon_t|F_{t-1}] = \\ E[\mathcal{L}_t]E[\sigma_t] \\ 0 \cdot E[\sigma_t] = 0 \\ V[\varepsilon_t] = E[\varepsilon_t]^2 E[\mathcal{L}_t^2 \cdot \sigma_t^2] = \\ E[\mathcal{L}_t^2] \cdot E[\sigma_t^2] = E[\sigma_t^2] \end{aligned}$$

General Settings

So far we have focused on the estimation of the conditional mean function $E[Y_t|F_{t-1}]$:

$$Y_t = E[Y_t|F_{t-1}] + \varepsilon_t$$

where ε_t is a weak white noise, that is, ε_t is serially uncorrelated : $Cov[\varepsilon_t, \varepsilon_{t-j}] = 0 \ \ \forall j \neq 1$

ARCH(1) Processes

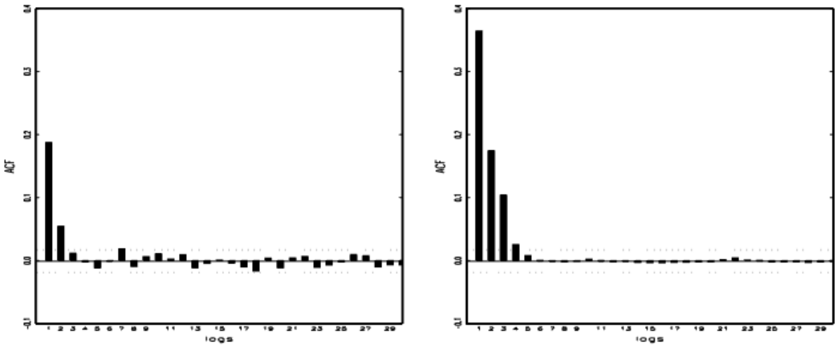
A process σ_t^2 is called an ARCH(1) process if

$$\sigma_t^2 + w + \alpha\varepsilon_{t-1}^2$$

with w>0 and $\alpha \geq 0$

Properties of Arch(1)

- Arch (!) Conditional variance σ_t^2 is strictly positive if $w > 0$ and $a \geq 0$
- Opposite to the historical volatility estimator, the arch 1 volatility is a weighted average of past information that gives more weight to the recent information than to the distant one
- Arch 1 process can be written as an A(1) process in ε_t^2
- Consequently ε_t^2 is stationary if $|\alpha| < 1$
- Given that both process ε_t and ε_t^2 and $E[\varepsilon_t] = 0$ then the unconditional variance of ε_t, $E[\varepsilon_t]$ is given by
$$\sigma_\varepsilon^2 = V[\varepsilon_t] = E[\varepsilon_t^2] = \frac{w}{1-\alpha}$$
- ARCH(1) captures the clustering effect : when volatility is high, it more probably stays high
- The kurtosis is always large t



utocorrelation functions of squared time series with ARCH(1) conditional variance with $\alpha = 0.2$ (left panel) and $\alpha = 0.7$ (right panel)

Figure 1

Conditional variance moment, we observe a high persistence in daily log returns in order to cauterises this lag persistence, this lag has to be large too. But the estimation of this A(50) model becomes very cumbersome, likelihoods optimise numerically, once you start imposing Stationarity conditions this it rot ensure generating something with a stationary second moments, these are some solutions to polynomial equations so we run into large p issues.

In tutorial we look at arch's in simulation study

Lecture 4: GARCH

Mon 19 Feb 09:00

Recap

ARMA

1. $E[\varepsilon_t] = 0$
2. $V\varepsilon_t = \sigma^2$
3. $\text{Cov}(\varepsilon_t, \varepsilon_s) = 0$ that is no serial correlation

Tutorial 2 : S&P 500 Daily log returns $\rightarrow$ ARMA(p,q) $\rightarrow$ BIC then use residual diagnostics

$$@_t = y_t - \hat{E}[y_t|F_{t-1}] \rightarrow MA(\mathcal{L})$$

Week 3
NP / Rob Engel 2003

$$\varepsilon_t = \sigma_t \mathcal{L}_t$$
$$\mathcal{L}_t \sim \mathcal{N}(0,1)$$
$$\sigma_t^2 = w + \alpha\varepsilon_{t-1}^2 < -$$

1. $a \geq 0$ and $\omega > 0$ - to ensure positivity of conditional variance
2. $|\alpha| < 1$ Stationarity of conditional variance

ARCH(1)

$$\begin{cases} \sigma = w + \alpha\varepsilon_{t-1} < - \\ \varepsilon_t + \mathcal{L}_t \sigma_t \\ \quad \text{rewrite } \sigma_t^2 = w + \alpha\varepsilon_{t-1}^2 + \varepsilon_t^2 - \varepsilon_t^2 \\ AR(1)\ in\ \varepsilon_t^2 \rightarrow \varepsilon_t^2 = w + \alpha\varepsilon_{t-1}^2 + \left(\varepsilon_t^2 - \sigma_t^2\right) \end{cases} \qquad Video \begin{cases} E[V_t] = 0 \\ V[v_t] < \infty v_t = \sigma^2 \\ cov(v_t, v_{t-s}) = 0 \end{cases}$$

Pros

- Volatility clustering (video)
- rise persistence at the cost of ARCH (p)
- Leptokurtic property $\alpha^2 \in (0, \frac{1}{3})$

Cons

- leverage effect : $E[\mathcal{L}_t^3] = 0$
- Long memory (ACF)

What can we do with our Garch models to capture all remaining things in ACF?

3.4 GARCH

A process σ_t^2 is called an GARCH(1, 1) process if

$$\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2$$

with $\omega > 0$, $\alpha \geq 0$ and $\beta \geq 0$

Properties

- ε_t^2 is stationary if $\alpha + \beta < 1$
- both processes ε_t and ε_t^2 are stationary and $E[\varepsilon_t] = 0$ then the unconditional variance of ε_t $V[\varepsilon_t]$ which is equal to the unconditional mean of ε_t^2
- no leverage effects as in the ARCH
-

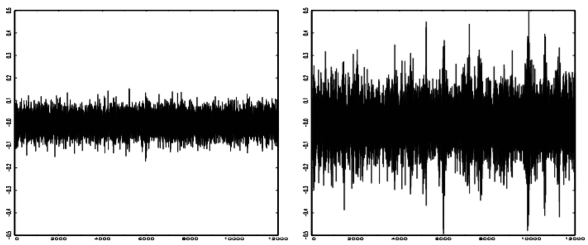


Figure 2: Simulated GARCH Models

left $\alpha = 0.01$ and $\beta = 0.8$. Right $\alpha = 0.08$ and $\beta = 0.9$ If allow close to 1 then can generate longer persistence, usually the memory of the daily log returns is us more persistent. Most have very low memory, thus people came up with GARCH(p, q)

M1 GARCH(1, 1)

- it takes into account / able to model more persistent conditional volatility processes
- mitigating the tradeoff between generating a leptokurtic distribution of ε_t and the persistence iof the ACF if ε_t^2 as compared to ARCH(1)

M2 ARCH(1)

GARCH captures over kurtosis, even if we could like sum of $\alpha + \beta$ to 1, we still have an opportunity to generate a over kurtosis (>3)

We can also show GARCH reveals larger excess kurtosis than the arch model, we can compare which is larger than the other, $\frac{6\alpha^2}{1-2a^2-(\alpha+\beta)^2)}$

can show A(1) is equal to $MA(\infty)$, same applies for GARCH for $ARCH(\infty)$

$\alpha + \beta$ providers the necessary information on the degree of volatility clustering

GARCH(p, q)

Just extension of GARCH(1, 1), key notation is polynomial for lag operator, lags shift an observation 1 period ahead (power 2 = 2 period ahead). But except for notation, nothing fundamental changes.

To lie outside of the root circle, in practice to estimate such a model, ensure positivity constraints, then also have to ensure process modelling is stationary - the constraints on stationary on highly non linear. This very quickly becomes a complicated non linear constraint, thus a numerical issue driven by Stationarity constraint (non linear) imposed by IRMA (p, q), but if allow for more p and q lags, then model is able to generate over kurtosis then the persistence of the series, the properties become better but at the cost of optimising over something with highly non-linear constraint.

Further Types of GARCH models

ARCH providers an exponential decay, have to know GARCHS for risk modelling.

Integrated GARCH(1, 1)

- specific to high frequency time series
- describes a very large persistence in the conditional variance
- is strictly stationary
- propose α and β sum upto 1, GARCH STRUCTURE there to ensure non stationary process
- risk metrics assumes that daily log returns follows process with infinite variance, that is we are not dealing with well defined statistical processes in real life, as seen by lack of first 2 moments

RiskMetrics^{TM} A special case of the IGARCH(1, 1) process

- From estimating the
- Gives forecast
- λ calibrates on loads of different stocks in the 90s
- Fix the β with λ

Exponential GARCH aimed at capturing asymmetric shocks, now modelling h_{t-1} log transformation of σ_t^2, assuming it follows GARCH looking process, and modify the ARCH part

- modelling logs of variance because we want to get rid of parameter constraints, if modelling logs can be positive, negative, get rid of these issues by modelling logs
-

Threshold GARCH TGARCH (1, 1) with indicator function, if shock was negative, bit easier to look at, if γ is positive, then ...

Tgarch, E garch if model left skewed

Tgarch(1,1) GJR-Garch

usual garch(1, 1) : $\sigma_t^2 = w + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2$

tgarch(1, 1) : $\sigma_t^2 = w + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2$

News impact curve : $NIC(\varepsilon_t|\sigma_{t-1}^2 = \sigma_{t-2}^2, \ldots, = \sigma_t^2)$

GARCH(1, 1) : $w + \beta\sigma_t^2 + \alpha\varepsilon_{t-1}^2$ TGARCH(1, 1) =

$$\begin{cases} w + \beta\sigma_t^2 + \alpha\varepsilon_{t-1}^2 \varepsilon_{t-1} < 0 \\ w + \beta\sigma_t^2 + \alpha + \delta\varepsilon_{t-1}^2, \varepsilon_{t-1} < 0 \end{cases}$$

NIC : Egarch(1,1)

$$h_t = \ln(\sigma_t^2) = w + \alpha\mathcal{L}_{t-1} + \gamma(|z_{t-1}| - \sqrt{\frac{2}{a}}) \exp(h_t) = \sigma_t^2 = \exp^w \cdot \exp^{\alpha z_{t-1}} \cdot \exp^{\gamma(|z_{t-1}| - \sqrt{\frac{2}{a}})} \sigma_t^2 = \exp^w \cdot \sigma^2$$

$$\varepsilon_t > 0$$
$$\varepsilon_t < 0$$

If shock positive then $\exp^{\alpha+\gamma} \cdot \varepsilon_t/\sigma_t$

NIC: once you write down NIC, then it becomes more evident what model parameters give you which response, EGarch $\alpha < 0$, z_t between 0 and 1

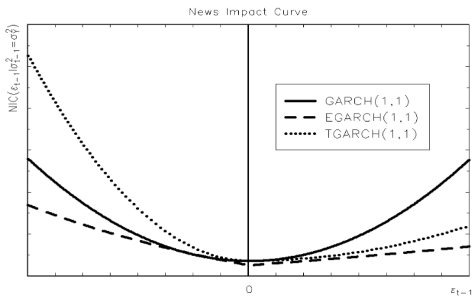


Figure 3: News Impact Curve

Model quality based on one picture, isn't exactly true, in order to plot NIC. plug in σ, γ, β (ω) , plot is based on one set of parameters, can easily be reversed.

So has something to do with data rather than overall quality of model,

Recap ARCH, GARCH, IGARCH, EGARCH, TGARCH. Financial econometrics model conditional second moment, but what about first moment?

- conditional mean? (1st moment), why are we interested in the second moment?
- We are risk averse etc, but
- in week 2 we have talked about how to model, ARMA - expected value of y_t then T2 we estimated conditional mean models, but the returns are on average 0, there is very slight autoregressive coefficients, but overall there is **no time series structure** in the conditional mean :

$$E[r_t|F_{t-1}] = 0$$

- WE have compared the ACF for daily log returns r_t, but in the actual return series, the history of returns is completely uninformative of the future
- in autocorrelation function few squared return we see a lot going on, and it doesn't die out, squared return is a proxy of conditional variance

Why do we model conditional second moment?

There is no time series structure to first moment, but there is in conditional second moment. Then we think how can we model our conditional variance of return process?

Nobel prize given for ARMA framework where ε_t can be white noise process. Then, even GARCH is not enough.

Then RiskMetrics comes and assumes infinite variance of daily returns, albeit a popular way of thinking. How much does turbulence persevere in market, how long after do we have to be conservative in our risk approaches

EGARCH, TGARCH. TGARCH more intuitive, EGARCH model the log variances and so can relax the positivity constraints, we don't care whether shocks are negative. Essentially a philosophical introduction to risk-modelling

For further references see Something Linky and [**PS2-part2-Q**] or local file normally : File.txt

[**On Optimal Set Estimation**] yes no abcd