

# Micro Econometrics

Sol Yates

April 25, 2024

## Contents

<b>1</b>	<b>Regression</b>	<b>2</b>
1.1	Classical Linear Model . . . . .	3
1.2	Multiple Regression . . . . .	4
1.3	Gauss Markov Assumptions . . . . .	4
1.4	Small Sample Properties . . . . .	5
1.5	Asymptotic Properties . . . . .	8
1.6	Interpretation and Modelling . . . . .	10
1.7	Timeout: Testing Equal Means . . . . .	12
<b>2</b>	<b>Standard Errors</b>	<b>15</b>
2.1	Introduction . . . . .	15
2.2	Heteroskedacity - Robust Standard Errors . . . . .	15
2.3	Heteroskedacity Problems . . . . .	15
2.4	Breusch-Pagan Test for Heteroskedacity . . . . .	18
2.5	Clustered Standard Errors . . . . .	22
2.6	Basic Intro to Bootstrap . . . . .	28
2.7	Non-parametric Bootstrap . . . . .	28
<b>3</b>	<b>Instrumental Variables</b>	<b>33</b>
3.1	Forms of Endogeneity . . . . .	33
3.2	IV Estimator . . . . .	36
3.3	Case : $\text{length}(z) = \text{length}(x)$ . . . . .	38
3.4	2SLS/GIVE . . . . .	39
3.5	Properties . . . . .	40
3.6	Group Mean Estimator . . . . .	40
<b>4</b>	<b>Randomised Experiments</b>	<b>44</b>
4.1	RCTs - imperfect compliance . . . . .	45
4.2	Late and Bloom Result . . . . .	46
<b>5</b>	<b>Panel Data Methods</b>	<b>46</b>
<b>6</b>	<b>Differences-in-Differences</b>	<b>48</b>

## Lecture 1: First Lecture

Wed 31 Jan 11:21

## 1 Regression

## [L1 Regression]

## Regression Fundamentals

As an empiricist, differences in economic fortune are hard to explain, as applied econometricians, we believe we can summarise and interpret 'randomness' in a useful way.

An example of systematic randomness is the connection between education and earnings. Even without considering the causality, it is clear that education predicts earnings in a narrow statistical sense.

This can be compellingly summarised by the conditional expectation function (CEF).

The CEF for a dependent variable  $y_i$  is given a  $k \times 1$  vector of covariates  $X_i$  (elements  $x_{ki}$ ) is the *expectation* or population average of  $y_i$ , with  $X_i$  held fixed.

The population average can be thought of the mean in an infinitely large sample, or the average in a completely enumerated finite population.

Written  $E[y_i|X_i]$  and is a function of  $X_i$ . Then, because  $X_i$  is random the CEF is random.

But we are often most interested in CEFs that are functions of many variables, conventionally in the vector  $X_i$ , then for a specific value of  $X_i$  we write  $E[y_i|X_i = x]$

For a continuous  $Y_i$  with conditional density  $\{y(t|X_i = x)$  at  $Y_i = t$  the CEF is

$$E[y_i|X_i = x] = \int t\{y(t|X_i = x)dt$$

Then, if  $Y_i$  is discrete,  $E[y_i|X_i = x]$  equals the sum  $\sum_i tP(Y_i = t|X_i = x)$  where  $P(Y_i = t|X_i = x)$  is the *conditional probability mass function* for  $Y_i$  given  $X_i = x_i$

Whilst expectation is a population concept, in practice we have samples which rarely consist of the entire population. We use these to make inferences about the population, so the sample CEF is used to learn about the population CEF.

**Law of Iterated Expectations** An unconditional expectation can be written as the unconditional average of the CEF. Or,

$$E[Y_i] = E\{E[Y_i|X_i]\}$$

Where the outer expectation uses the distribution of  $X_i$

**Theorem 1 :**

## 3.1.1 CEF Decomposition property

$$Y_i = E[Y_i|X_i] + \varepsilon_i$$

Where  $\varepsilon_i$  is mean independent of  $X_i$ , that is  $E[\varepsilon_i|X_i] = 0$  and therefore  $\varepsilon_i$  is uncorrelated with any function of  $X_i$

## Regression

Studying the relationship between one (dependent) variable  $y$  and  $k$  other independent variables  $x_j$ , ( $j = 1, \dots, k$ )

1. Does the Covid vaccine work
2. What are the returns to schooling
3. What is the effect of having internet at home on student's grades
4. Does a job training program decrease the time of getting out of unemployment

We are often interested in a single variable, including other regressors as controls

### 1.1 Classical Linear Model

#### Linear regression

- Relies on 5 main Gauss-Markov assumptions
- In small samples is unbiased and BLUE
- In large samples is consistent and asymptotically normal. There is no need for the normality assumption to establish asymptotic distribution

The key assumption for OLS to consistently estimate  $\beta$  is the **population orthogonality condition**

#### Assumption 1 ((OLS.1))

$$E[x'u] = 0$$

Because  $x$  contains a constant, this assumption is equivalent to saying that  $u$  has mean zero and is uncorrelated with each regressor

The zero conditional mean assumption is sufficient for this

The vector  $(x, u)$  represents a population and this assumption is a restriction on the joint distribution in this population.

If  $x$  contains years of schooling and experience, and the main component of  $u$  is innate ability then this assumption implies that ability is uncorrelated with education and experience in the population.

#### Assumption 2 (OLS.2)

$$\text{rank} E[x'x] = K$$

Since  $E[x'x]$  is a symmetric  $K \times K$  matrix, this assumption is equivalent to assuming that  $E[x'x]$  is positive definite. Since  $x_1 = 1$  this is also equivalent to saying the (population) variance matrix is non-singular. This fails iff at least one of the regressors can be written as a linear function of the other regressors.

#### Assumption 3 ((OLS.3))

$$E[u^2 x'x] = s^2 E[x'x] \quad \text{where} \quad s^2 \equiv E[u^2]$$

Then because  $E[u] = 0$ ,  $s^2$  is also equal to  $\text{Var}(u)$ . This is essentially the weakest form of the homoskedasticity assumption since this assumption is equivalent to assuming that  $u^2$  is uncorrelated with each  $x_j$ ,  $x_j^2$  and all cross products of form  $x_j, x_k$ .

By the LIE, sufficient for this assumption is  $E[u^2|x] = \sigma^2$  which is the same as  $\text{Var}(u|x) = \sigma^2$  when  $E[u|x] = 0$ .

## 1.2 Multiple Regression

Classical linear model

$$\begin{aligned} Y_i &= \beta_0 + \beta_1 x_{i,1} + \beta_2 x_{i,2} + \dots + \beta_k x_{i,k} + u_i, i = 1, \dots, n \\ &= x_i \beta + u_i \quad \text{vector notation} \\ Y &= X\beta + U \quad \text{matrix notation.} \end{aligned}$$

1. Where  $\beta_0$  is the intercept,  $\beta_j$  is the parameter (slope) associated with  $x_j$
2.  $u_i$  is the unobserved error term : containing factors other than  $x_j$  's explaining  $y$
3.  $N$  is the number of observations

Least Squares Estimator

**Objective** : to estimate the effect of  $x_j$  on  $y$ , we need to estimate the population parameters  $\beta_0, \dots, \beta_k$

**Ordinary Least squares estimates**  $\beta$  by minimising the sum of squared residuals :

$$\hat{\beta} = \underset{\beta}{\operatorname{argmin}} \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_{i,1} - \dots - \beta_k x_{i,k})^2 = \|Y - X\beta\|^2$$

Taking first order conditions

$$\hat{\beta} = (X'X)^{-1}X'Y$$

It can be shown that

1. Residuals :  $\hat{u}_i = y_i - x_i \hat{\beta}$  with  $\sum_{i=1}^n \hat{u}_i = 0$
2. Fitted values :  $\hat{y}_i = x_i \hat{\beta}$

In the single regressor model

Model  $y = \beta_0 + \beta_1 x_1 + u$

$$\begin{aligned} \hat{\beta}_1 &= \frac{\sum_{i=1}^n (x_{i,1} - \bar{x}_1)(y_i - \bar{y})}{\sum_{i=1}^n (x_{i,1} - \bar{x}_1)^2} \\ &= \operatorname{cov} \frac{x_1, y}{\hat{v}(x_1)} \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}_1 \end{aligned}$$

## 1.3 Gauss Markov Assumptions

**Assumption 4** (Linear in parameters (*MLR.1*)) The model in the population can be written as

$$Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_k x_k + u$$

Where  $\beta_0, \beta_1, \dots, \beta_k$  are the unknown parameters (constants) of interest and  $u$  is an unobservable random error or disturbance term

**Assumption 5** ( Random Sampling (*MLR.2*)) We have a random sample of  $n$  observations, following the population model assumption in *MLR.1*.

- Often referred to as IID assumption
- Ensures that our sample's representative for the population
- Would fail if we observed only part of the population in our sample

**Assumption 6** (No perfect colinearity *MLR.3*) In the sample (thus pop too), none of the independent variables are constant, and there are no exact linear relationships between the independent variables

- Often referred to as full rank assumption
- Dummy variable trap - not to include a binary for both male and female
- Not to confuse with highly but not perfectly correlated variables (multicollinearity)

**Assumption 7** (zero conditional mean *MLR.4*) The error  $u$  has an expected value of zero given any value of the explanatory variable,

$$E[u|x_1, x_2, \dots, x_k] = 0$$

- Key to deriving unbiasedness
- If it holds for variable  $x_j$ , the variable is exogenous
- Requires at a minimum : all factors in the observed error term must be uncorrelated with the explanatory variables
- Any problem that causes  $u$  to be correlated with any of the  $x_j$  's causes this assumption to fail and OLS to be biased !
- Examples for endogeneity : misspecified functional form, omitting important variables, measurement error and any  $x_j$  being jointly determined with  $y$

**Assumption 8** (Homoskedasticity *MLR.5*) The error  $u$  has the same variance given any value of explanatory variables, in other words

$$V[u|x_1, \dots, x_k] = \sigma^2$$

- The variance of the unobserved error  $u$  conditional on the explanatory variables is the same for all combinations of the outcomes of the explanatory variables
- If this assumption fails, we speak of heteroskedastic errors
- This assumption is not needed for unbiased/ consistency but for efficiency of OLS
- This also means that  $V[y|x] = \sigma^2$

## 1.4 Small Sample Properties

### Unbiasedness of OLS

Under assumption 1-4, the OLS estimator is unbiased

$$E[\hat{\beta}_j] = \beta_j \text{ for } j = 0, \dots, k$$

For any values of the population parameter  $\beta_j$ .

The OLS estimators are unbiased estimators of the population parameters

- Might not exactly be the population value
- Deviations from the population value are not systematic

- If we were to repeat the estimation on several random samples the deviations should average out to zero

### Variance

Sampling variance of the OLS slope estimators

Under assumptions 1-5, conditional on the sample values of the independent variables the variance is

$$V[\hat{\beta}_j] = \frac{\sigma^2}{SST_j(1 - R_j^2)} \text{ for } j = 0, \dots, k$$

Where  $SST_j = \sum_{i=1}^n (x_{ij} - \bar{x}_j)^2$  is the sum of total sample variation in  $x_j$  and  $R_j^2$  is the R-squared from regressing  $x_j$  on all other independent variables (including an intercept)

- The standard error formulas make it apparent that we need variation in the regressors to increase precision
- The  $R_j^2$  representation makes it also apparent that a high multicollinearity increases the variance of the estimator

**Matrix Representation** General formula in matrix form (including the intercept)

$$V[\hat{B}_j] = \sigma^2 (X'X)^{-1}$$

The variance of the j-th parameter estimate

$$\sigma^2 (X'X)^{-1}_{[j+1, j+1]}$$

### Gauss Markov Theorem

#### Theorem 2 : Gauss Markov Theorem

Under assumptions 1-5  $\hat{\beta}_0, \hat{\beta}_1, \dots, \hat{\beta}_k$  are the *best linear unbiased estimators* (BLUE)s of

$$\beta_0, \beta_1, \dots, \beta_k$$

A more general (and asymptotic) version of the GM theorem ([JW-CS- 14]). While stating that OLS has the smallest variance in the class of linear, unbiased estimators, it does not allow us to compare OLS to unbiased estimators that are not linear in the vector of observations on the dependent variable

#### Theorem 3 : (Efficiency in a Class of Estimators)

Let  $\hat{\theta}_\tau : \tau \in \mathcal{F}$  be a class of  $\sqrt{N}$ -asymptotically normal estimators with variances matrix's of the form  $V = A^{-1} E[s(w)s(w)'] (A')^{-1}$

If for some  $\tau^* \in \mathcal{F}$  and  $\rho > 0$

$$E[s_\tau(w)s_{\tau^*}(w)'] = \rho A_{\tau^*} \quad \text{all } \tau \in \mathcal{F} \quad (1)$$

Then  $\hat{\theta}_{\tau^*}$  is asymptotically relatively efficient in the class  $\{\hat{\theta}_\tau : \tau \in \mathcal{F}\}$

That is, if we specify a class of estimators by defining the index set  $\mathcal{F}$  then the estimator  $\hat{\theta}_{\tau^*}$  is more efficient than all other estimators in the class if we can show this

- If the assumptions hold, we do not need to look for another unbiased estimator since this see the best
- Best meaning most efficient with smallest variance

### Small Sample Inference

We are interested in performing inference, we need : variance (standard error) and distribution of parameter estimator

Firstly Estimation of the error variance :  $\sigma^2 = \frac{1}{n-k-1} \hat{u}_i^2 = \frac{SSR}{n-k-1}$

We can show this estimator is unbiased under

#### Theorem 4 : unbiased estimation of $\sigma^2$

Under the GM assumptions (1-5),

$$E[\hat{\sigma}^2] = \sigma^2$$

**Standard Errors** Is  $\sqrt{\text{of variance}}$

$$Sd(\hat{\beta}_j) = \sqrt{\frac{\sigma^2}{SST_j(1 - r_j^2)}}$$

$$Se(\hat{\beta}_j) = \sqrt{\frac{\sigma^2}{SST_j(1 - R_j^2)}} = \frac{\hat{\sigma}}{\sqrt{n}sd(x_j)\sqrt{1 - R_j^2}} \quad \text{where } sd(x_j) = \sqrt{n^{-1} \sum_i (x_{ij} - \bar{x}_j)^2}$$

Standard errors shrink to zero at the rate  $\frac{1}{\sqrt{n}}$  (since in denominator)

**Assumption 9** (Normality MLR.6) The population error  $u$  is independent of the explanatory variables  $x_1, x_2, \dots, x_k$  and is normally distributed with zero mean and variance  $\sigma^2$  :  $u \sim \mathcal{N}(0, \sigma^2)$

This is a stronger assumption than 1-5 and means we are necessarily assuming zero conditional mean (4) and homoskedasticity (5).

#### Theorem 5 : Normal Sampling Distributions

Under assumptions 1-6, conditional on the sample values of the independent variables

$$\hat{\beta}_j \sim \mathcal{N}(\beta_j, V(\hat{\beta}_j))$$

(variance expression)

Therefore,

$$(\hat{\beta}_j - \beta_j)/sd(\hat{\beta}_j) \sim \mathcal{N}(0, 1)$$

Or,  $\hat{\beta}|X \sim MVN(\beta, \sigma^2(X'X)^{-1})$  (matrix notation)

## 1.5 Asymptotic Properties

### Consistency

**Assumption 10** (Zero Mean and Zero correlation (MLR.4'))

$$E[u] = 0 \text{ and } Cov[x_j, u] = 0, \text{ for } j = 1, 2, \dots, k.$$

- If we are only interested in consistency : this replace zero conditional mean (MLR.4)
- However, zero conditional mean important for finite sample and to ensure that we have properly modelled the population regression function  $E[y|x] = \beta_0 + \beta_1 x_1 + \dots + \beta_k x_k$
- This gives us the average or partial effects of  $x_j$  on the expected value of  $y$ .

#### Theorem 6 : Consistency of OLS

Under assumptions MLR.1 - MLR.4, (or replacing 4 with 7), the OLS estimator  $\hat{\beta}_j$  is consistent for  $\beta_j$  for all  $j = 1, 2, \dots, k$

Consistency means that when  $n$  goes to  $\infty$ , the estimator will recover the population value in probability :

$$\text{plim}_{n \rightarrow \infty} \hat{\beta}_j = \beta_j$$

Essentially, the asymptotic bias shrinks to 0.

For the simple model with one regressor :  $y_i = \beta_0 + \beta_1 x_{i,1} + u_i$ , to show consistency :

1. Write down the formula for  $\hat{\beta}_1$  and plug in  $y_i$  :

$$\begin{aligned} \hat{\beta}_1 &= \left( \sum_{i=1}^n (x_{i,1} - \bar{x})(y_i - \bar{y}) \right) / \left( \sum_{i=1}^n (x_{i,1} - \bar{x})^2 \right) \\ &= \beta_1 + \left( \frac{1}{n} \sum_{i=1}^n (x_{i,1} - \bar{x})(u_i - \bar{u}) \right) / \left( \frac{1}{n} \sum_{i=1}^n (x_{i,1} - \bar{x})^2 \right) \end{aligned}$$

2. Apply the LLN :

$$\text{plim}_{n \rightarrow \infty} \hat{\beta}_j = \beta_1 + \frac{Cov[u, x_1]}{V(x_1)} = \beta_1$$

Since  $Cov[u, x_1] = 0$  (previous assumption)

However, we need to assume finite moments for the LLN to hold. Since it assumes iid observations

LLN :  $\bar{X}_n \xrightarrow{P} X$  as  $n \rightarrow \infty$

### Remarks

- In a single regressor model :  $\beta_1 = \frac{Cov(y, x_1)}{V(x_1)}$
- Including more regressor changes this expression for the population estimate  $\beta_j$  but since the effect of the other covariates is partialled out, we still recover  $\beta_j$
- Multicollinearity only affects the variance of the estimator but not consistency



**Theorem 7 : Asymptotic Normality of OLS**

Under the Gauss-Markov assumptions (1-5),

1.  $\sqrt{n}(\hat{\beta}_j - \beta_j) \sim \mathcal{N}(\beta_j, \frac{\sigma^2}{a_j^2})$  where  $(\frac{\sigma^2}{a_j^2} > 0)$  is the asymptotic variance of  $\sqrt{n}(\hat{\beta}_j - \beta_j)$  : for the slope coefficients,  $a_j^2 = \text{plim} \frac{1}{n} \sum_{i=1}^n \hat{r}_{ij}^2$  where the  $\hat{r}_{ij}^2$  are the residuals from regressing  $x_{ji}$  on the other independent variables. And we can say that  $\hat{\beta}_j$  is asymptotically normally distributed
2.  $\text{hat}\sigma^2$  is a consistent estimator of  $\sigma^2 = V(u)$
3. For each j,  $(\frac{\hat{\beta}_j - \beta_j}{\text{sd}(\hat{\beta}_j)} \sim \mathcal{N}(0, 1)$  (where sd unobserved)
4. For each j,  $(\frac{\hat{\beta}_j - \beta_j}{\text{se}(\hat{\beta}_j)} \sim \mathcal{N}(0, 1)$  (where se estimated)

Where  $\text{se}(\hat{\beta}_j)$  is the usual OLS estimator

**Matrix Form**

$$\sqrt{n}(\hat{\beta} - \beta) \sim \mathcal{N}[0, \sigma^2(\text{plim} \frac{X'X}{n})^{-1}]$$

With  $\text{plim} \frac{X'X}{n} = E[x'x]$

- For convergence, one needs the asymptotic normalisation  $\sqrt{n}$
- However we are interested in the variance of  $\hat{\beta}$ . For estimation, we use the sample analog of the variance covariance and remove the asymptotic normalisation again by dividing by n
- We obtain the asymptotic variance :  $\hat{V} = \hat{\sigma}^2(X'X)^{-1}$

This is a very important result for inference. The normality assumption is not needed in large sample. Therefore, regardless of the error distribution, if properly standardised, we have approximate normal standard distributions. We can use the (unobserved)  $\text{sd}(\hat{\beta}_j)$  or the observed  $\text{se}(\hat{\beta}_j)$  to achieve this result, where we can estimate the latter since it depends on  $\hat{\sigma}^2$

Then because the t distribution approaches the normal distribution for large df, we can also say that  $(\hat{\beta}_j - \beta_j)/\text{se}(\hat{\beta}_j) \sim t_{(n-k-1)}$ . But we still need homoskedacity, and with large sample, all the testing issues still apply.

**Partialling Out**

Intuitively,  $\hat{\beta}_1$  measures the sample relationship between y and  $x_1$  after the other regressors have been partialled out

1. Model :

$$Y = \beta_0 + \beta_1 x_1 + \dots + \beta_k x_k + u, \quad E[u, x_1, \dots, x_k] = 0$$

2. Regress  $x_1 \sim x_1 + x_2 + \dots + x_k$  and compute the residual  $\hat{r}_{i1}$
3. Regress  $y \sim \hat{r}_{i1}$  which yields the OLS estimate  $\hat{\beta}_1$
4. One can show that the resulting OLS estimator from the regression in 3, equals the OLS estimator for  $\beta_1$  from a regression based on the model in 1

In general form, this is called the **Frisch-Waugh Theorem**

Also giving us the regression anatomy formula :

$$\beta_j = \frac{Cov(y_i, \hat{r}_{i,j})}{V(\hat{r}_{i,j})}$$

## 1.6 Interpretation and Modelling

1.  $y = \beta_0 + \beta_1 x_1 + u$
2.  $y = \beta_0 + \beta_1 \log(x) + u$
3.  $\log(y) = \beta_0 + \beta_1 x + u$
4.  $\log(y) = \beta_0 + \beta_1 \log(x) + u$

Model	Dep. Variable	Ind. Variable	Interpretation of $\beta_1$
Level-level	y	x	$\Delta y = \beta_1 \Delta x$
Level-log	y	$\log(x)$	$\Delta y = (\beta_1/100)\% \Delta x$
Log-level	$\log(y)$	x	$\% \Delta y = (100\beta_1) \Delta x$
Log-log	$\log(y)$	$\log(x)$	$\% \Delta y = (\beta_1)\% \Delta x$

Source: Wooldridge (2009)

Figure 1: Interpretation And Log Transformation Table

### Linear Probability model

Model  $y = \beta_0 + \beta_1 x + u$  such that  $y = \{0, 1\}$  is a binary dependent variable

- Since y is binary

$$E[y|x] = P(y = 1|x) = \beta_0 + \beta_1 x$$

$$1 - E[y|x] = P(y = 0|x) = 1 - \beta_0 - \beta_1 x$$

With marginal effects

$$\frac{\partial E[y|x]}{\partial x} = \beta_1$$

- The predicted values are probabilities of the outcome being equal to 1
- Interpretation :  $\beta_1$  is the change in the probability that  $y = 1$  for a 1 unit increase in  $x_1$  (percentage points)

### Aside

- Pros are the estimation and interpretation is straight forward
- Cons are for prediction, the predicted probabilities can be outside the interval  $[0, 1]$
- Another con is the errors are Heteroskedacity and hence violate the gauss Markov assumption

$$V(y|x) = P(y = 1|x)(1 - P(y = 1|x))$$

Note other binary dependent variable models are probit and logit

## Binary Regressors

Model

$$Y = \beta_0 + \beta_1 x + u \quad E[u|x] = 0$$

Where

- $x \in \{0, 1\}$  is binary variable
- Often also referred to as 'dummy' variable
- Example : effect of gender on hourly wage. Let  $x = 1$  if the individual is a woman ( $x = 0$  man)
- Often used to evaluate a treatment effect such as the effect of an intervention, a policy, a program
- OLS results in comparing group averages

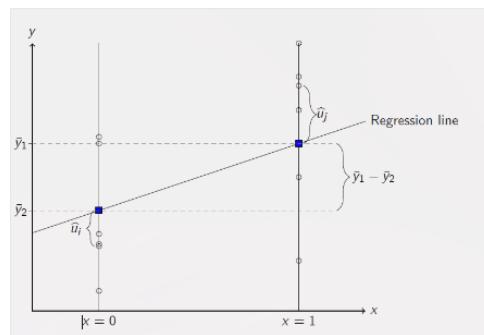


Figure 2: Relationship between  $y$  and  $x$  when  $x$  is binary

The white circles represent typical datapoints and the blue rectangles represent sample averages.

The average effect on  $y$  is the difference between the averages of both groups

## Population Raw Differential

The CEF for  $x = 1$  and  $x = 0$  :

$$\mu_1 \equiv E[y|x = 1] = \beta_0 + \beta_1 + E[u|x = 1]$$

$$\mu_0 \equiv E[y|x = 0] = \beta_0 + E[u|x = 0]$$

under zero conditional mean:

$$E[y|x = 1] = \beta_0 + \beta_1$$

$$E[y|x = 0] = \beta_0$$

hence

$$\beta_1 E[y|x = 1] - E[y|x = 0]$$

Where the parameter to be estimated by OLS is the **population raw differential**

**Sample Raw Differential** Now we can replace the conditional expectations by their sample analogue, that the conditional expectation of  $y$  for women is the mean outcome of women and the conditional expectation of  $y$  for men is the mean outcome of men.

- Replacing the population means by their sample averages, we obtain the OLS estimators
- $\hat{\beta}_1$  is the **sample raw differential**

$$\hat{\beta}_1 = \frac{1}{n_1} \sum_{i:x_i=1} y_i - \frac{1}{n_0} \sum_{i:x_i=0} y_i = \bar{y}_1 - \bar{y}_0, \quad \hat{\beta}_0 = \bar{y}_0$$

Where  $n_1$  ( $n_0$ ) is the number of observations with  $x = 1$  ( $x = 0$ )

Let  $\hat{\beta}_0 = 7$  and  $\hat{\beta}_1 = -2.5$  then men earn on average 7 GBP per hour, and women earn on average 2.5 GBP per hour less than the *average* man

## 1.7 Timeout: Testing Equal Means

To test whether the population means for 2 sub-samples are the same

$$H_0 = E[y|x=1] = E[y|x=0] \equiv \beta_1 = 0$$

With Test stat :

$$\hat{\beta}_1 / se(\hat{\beta}_1) = \frac{\bar{y}_1 - \bar{y}_0}{\sqrt{\hat{\sigma}_1^2/n_1 + \hat{\sigma}_0^2/n_0}}$$

Where  $\hat{\sigma}^2$  are the estimated group specific error variances

### (y) and Binary Regressors

Model :  $\log(y) = \beta_0 + \beta_1 x$  for  $x \in \{0, 1\}$

$$\log(y) = \begin{cases} \log(y_1) = \beta_0 + \beta_1 + u & \text{if } x = 1 \\ \log(y_0) = \beta_0 + u & \text{if } x = 0 \end{cases}$$

**Log Point Interpretation** If the resulting  $\% \Delta y / 100$  is small :

$$\begin{aligned} \beta_1 &= \log(y_1) - \log(y_0) = \log\left(\frac{y_1}{y_0}\right) = \log\left(1 + \frac{y_1 - y_0}{y_0}\right) \\ &= \log\left(1 + \frac{\% \Delta y}{100}\right) \approx \% \Delta y / 100 \\ 100\beta_1 &\approx \% \Delta y \end{aligned}$$

If  $\% \Delta y / 100$  is small, one can interpret  $\beta_1$  as the *raw differential*

**Exact Interpretation - Percentage Change**

$$\frac{\Delta y}{y_0} = \frac{y_1 - y_0}{y_0} = \frac{y_1}{y_0} - 1 = \frac{\exp(\beta_0 + \beta_1 + u)}{\exp(\beta_0 + u)} - 1 = \exp(\beta_1) - 1$$

Then, plugging the estimate into the equation

$$\% \Delta y = 100[\exp(\hat{\beta}_1) - 1]$$

**interpretation** -  $\frac{\Delta y}{y_0} = -0.26$  means that a woman's wage is 26% below that of a comparable man's wage

**Categorical Regressors**

- Some characteristics such as regions are originally categorical
- We can render categorical variables based on an originally continuous variable, say bins based on firm size
- The solution is to create multi-category dummies  $d_k$  to account for differential effects
- Say the effect of law school ranking on median starting salaries,  $\ln(y_i)$  where they found better ranked schools result in higher wages

In order to estimate whether there is a differential effect for the different ranks include the categories as dummies, say

$$D_{i,1} = \begin{cases} 1, & \text{if } 1 \leq \text{rank} \leq 10 \\ 0, & \text{otherwise} \end{cases} \quad ; D_{i,6} = \begin{cases} 1, & \text{if } \text{rank} > 100 \\ 0, & \text{otherwise} \end{cases}$$

The model

$$\ln(y_i) = \beta_0 + \beta_1 d_{i,1} + \dots + \beta_5 d_{i,5} + x_i \gamma + u_i$$

Where we exclude one dummy  $d_{i,6}$  to avoid perfect-collinearity (dummy var trap)

Interpretation

- Usual interpretation for a binary regressor wrt base category
- $\beta_j = E[\ln(y)|d_j = 1] - E[\ln(y)|d_6 = 1]$  for  $j = 1, \dots, 5$
- $\beta_0$  : log median starting salary for the omitted (base) category, the largest rank category
- Estimated percentage change for the first rank bin compared to the largest bin 101.29%

**Interaction Terms**

Model :  $y = \beta_0 + \beta_1 x_1 + \beta_2 d + \beta_3 x_1 \times d + u$

- Where  $x_1$  is continuous
- Then the effect of  $x_1$  is different for each group :  $\frac{\partial E[y|x_1, d]}{\partial x_1} = \beta_1 + \beta_3 \times d$
- If  $d = 1$ , then  $\frac{\partial E[y|x_1, d]}{\partial x_1} = \beta_1 + \beta_3$
- If  $d = 0$ , then  $\frac{\partial E[y|x_1, d]}{\partial x_1} = \beta_1$

- If  $d=1$  for women, then a unit increase in  $x_1$  leads to a  $\beta_1 + \beta_3$  increase for women and an increase for men of  $\beta_1$ . That is, the returns to  $x_1$  are for women  $\beta_3$  higher.
- If the independent variable is in log, we can interpret the coefficient as  $100 \times [\text{parameter}] \%$

Slightly different model, where both regressors are binary,  $y = \beta_0 + \beta_1 d_1 + \beta_2 d_2 + \beta_3 d_1 \times d_2 + u$

- We can compute expectation for each case:

$$\begin{aligned} E[y|d_1 = 0, d_2 = 0] &= \beta_0 \\ E[y|d_1 = 0, d_2 = 1] &= \beta_0 + \beta_2 \\ E[y|d_1 = 1, d_2 = 0] &= \beta_0 + \beta_1 \\ E[y|d_1 = 1, d_2 = 1] &= \beta_0 + \beta_1 + \beta_2 + \beta_3 \end{aligned}$$

- We can now interpret the obtained regression coefficients according to these differentials
- For example, if  $d_1 = 1$  for female and  $d_2 = 1$  for being married, then the outcome is on average for married women by  $\beta_1 + \beta_2 + \beta_3$  higher than for single men

## Polynomials

We often use to model non-linear relationships such as the diminishing returns to experience

- Model :  $y = \beta_0 + \beta_1 x + \beta_2 x^2$
- Interpretation :  $\frac{\partial E[y|x]}{\partial x} = \beta_1 + 2 \times \beta_2 x$
- We can compute the average change in  $y$  by a one unit change in  $x$  for a specific point in time (say for 0, 1, 2 years of experience)

## Econometrics Techniques

- Nonlinear relationships : EG modelling, non-parametric regression
- Standard errors: robust, clustered and bootstrap standard errors
- Addressing homogeneity :

Estimator	Min Data Requirement	Notes
Regression, Matching	Single cross-section	Observables
Instrumental Variables	Single cross-section	Valid instrument
Randomized Controlled Trials	Single cross-section	Program manipulation
Fixed Effects	Panel Data	Only FEs omitted
Random Effects	Panel Data	FEs uncorrelated
Difference-in-Differences	Repeated cross-sections	Common trends
Regression Discontinuity	Single cross-section	Running variable

Figure 3

## Lecture 2: Standard Errors

Sun 04 Feb 17:54

[L2]

## 2 Standard Errors

### 2.1 Introduction

- After a point estimate, we want to know the statistical significance
- Requiring the standard error and the distribution
- If the standard errors are wrong we cannot use the usual t / F statistics for drawing inference
- We either have too large SEs,
  - Zero might be included in the CI when it should not be
  - There is a risk of not detecting an effect even there was
- Or, too small SEs
  - Zero might not be in the CI when it should be
  - We may claim the existence of an effect when in reality there is none
  - Of course, this is worse - a **wrong** SE can lead to a **wrong** conclusion!
- Robust SE
  - Traditionally, inference assumes homoskedacity
  - But the variance of error terms might be different for different observations depending on their characteristics
  - Heteroskedacity robust SE accounts for this
- SE
  - Traditional estimation relied on random sampling
  - In the case of data with a group structure, the error terms might be correlated
  - To account we use clustered SE
- Bootstrap
  - Bootstrap is a re sampling method that offers an alternative to inference based on asymptotic formulas convenient in cases where the sampling distribution is unknown

- Note.**
1. If we can estimate a model parameter consistently, why do we care about inference?
  2. Do heteroskedatic errors or clustering affect the OLS point estimate for model parameters
  3. An example where heteroskedacity / clustering occurs
  4. When would bootstrap be useful?

### 2.2 Heteroskedacity - Robust Standard Errors

### 2.3 Heteroskedacity Problems

- Traditional inference assumes homoskedatic errors  $V(u|x) = \sigma^2$

- This implies that the variance of the unobserved error  $u$ , is constant for all possible values of all the regressor  $x$ 's
- Since the proofs for unbiasedness and consistency do not depend on this assumption we still obtain unbiased and consistent OLS estimates
- However, if this is not true ( $\sigma_i^2$ ) then the errors are called **heteroskedastic** and traditional variance estimators are biased
- Heteroskedasticity robust SE specifically in the CS case
- If the degree of heteroskedasticity is low, the traditional variance estimator might be less biased

**Example (Returns to education).** Regressing  $wage \sim educ$

It is reasonable to believe that the variance is unobserved factors hidden in the error term differs by educational attainment

Individuals with higher education : potentially more diverse interests and more job opportunities affecting their wage

Individuals with very low education : fewer opportunities and often must work at the minimum wage, the error variance is typically lower

**Variance estimation with heteroskedasticity**

Simple regression :  $y = \beta_0 + \beta_1 x + u$

We know  $\hat{\beta}_1 = \beta_1 + \frac{\sum_{i=1}^n (x_i - \bar{x}) u_i}{\sum_{i=1}^n (x_i - \bar{x})^2}$

Which is a function of the error terms

Therefore :

$$V(\hat{\beta}_1) = \frac{\sum_{i=1}^n (x_i - \bar{x})^2 \sigma_i^2}{SST_x^2}$$

Where  $SST_x = \sum_{i=1}^n (x_i - \bar{x})^2$

- Where  $\sigma_i^2$  is conditional variance of error term (depending on each individual)
- If  $\sigma_i^2 = \sigma^2$  the formula reduces to the traditional (OLS variance) formula :  $V(\hat{\beta}_1) = \frac{\sigma^2}{SST_x}$
- We have to estimate the conditional variance of the error, we do this by taking the residuals of OLS, squaring them and replacing them in the following formula for the error variance
- This leads to the following heteroskedasticity robust estimator (simple regression model) :

$$\hat{V}(\hat{\beta}_1) = \frac{\sum_{i=1}^n (x_i - \bar{x})^2 \hat{u}_i^2}{SST_x^2}$$

Where  $\hat{u}_i^2$  are the OLS residuals



## Generalisation

The formula generalises to

$$\hat{V}(\hat{\beta}_j) = \frac{\sum_{i=1}^n \hat{r}_{ij}^2 \hat{u}_i^2}{SSR_j^2}$$

Where the  $\sigma_i^2$  are replaced by residuals sourced from OG regression and the  $\hat{r}_{ij}$  are the residuals from regressing  $x_j$  on all other independent variables.

Where  $\hat{r}_{ij}$  is the  $i$ -th residual from regressing  $x_j$  on all other independent variables and  $SSR_j$  the sum of squared residuals from this regression

- Robust to heteroskedacity **of any form** (inc homoskedacity)
- Often also called white, huber, eicker SE
- Sometimes degrees of freedom adjustment by multiplying  $\frac{n}{n-k-1}$
- But with **drawback** that it only has asymptotic justification (need large sample for it to be valid)

## Matrix Representation - Asymptotic Variance

Model :  $y = X\beta + U$

We know  $\sqrt{n}(\hat{\beta} - \beta) \xrightarrow{d} \mathcal{N}(0, V)$  Where

$$\begin{aligned} V &= E[E[X'X]]^{-1} [E[X'Xu^2]] [E[X'X]]^{-1} \\ &\text{with fixed regressors( replace with sample analog )} \\ &= \left[ \frac{1}{n} X'X \right]^{-1} \left[ \frac{1}{n} X' \psi X \right] \left[ \frac{1}{n} X'X \right]^{-1} \end{aligned}$$

And the variance-covariance matrix  $\psi$

$$\psi = \begin{bmatrix} V(u_1|x) & 0 & \dots & 0 \\ 0 & V[u_2|x] & \dots & 0 \\ \dots & \dots & \ddots & \vdots \\ 0 & 0 & \dots & V[u_n|X] \end{bmatrix}$$

Eventually,  $\frac{V}{n} = AV(\hat{\beta})$

## Matrix Representation - Estimation

- We can then find an estimate the middle term by :  $\frac{1}{n} \sum_{i=1}^n u_i^2 \hat{x}_i' x_i = \frac{1}{n} X' \hat{\psi} X$
- Where  $\hat{\psi} = \text{diag}[\hat{u}_1^2, \dots, \hat{u}_n^2]$

$$\hat{V} = \left[ \frac{1}{n} X'X \right]^{-1} \frac{1}{n X' \hat{\psi} X} X \left[ \frac{1}{n} X'X \right]^{-1}$$

- In order to estimate the Asymptotic Variance (AV)  $\hat{\beta}_j$ , we need to remove the asymptotic normalisation by dividing by  $n$
- Resulting Estimator :

$$\hat{AV} = n[X'X]^{-1} \frac{\sum_{i=1}^n \hat{u}_i^2 x_i' x_i}{n} [X'X]^{-1}$$

- Sometimes corrected by the degrees of freedom  $n/n - k - 1$  to improve finite sample properties

- SEs : square root of the diagonal elements
- Recall that under homoskedasticity, we obtain  $\sigma^2(X'X)^{-1}$

**Example.** Returns to Education `Reg1 = lm(wage ~ educ, data = wage1)`

```
Residuals:
    Min       1Q   Median       3Q      Max
-2.21158 -0.36393 -0.07263  0.29712  1.52339

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 0.583773   0.097336   5.998 3.74e-09 ***
educ         0.082744   0.007567  10.935 < 2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.4801 on 524 degrees of freedom
Multiple R-squared:  0.1858,    Adjusted R-squared:  0.1843
F-statistic: 119.6 on 1 and 524 DF,  p-value: < 2.2e-16

> coeftest(reg1, vcov = vcovHC(reg1, "HC1"))

t test of coefficients:

            Estimate Std. Error t value Pr(>|t|)
(Intercept) 0.5837727   0.0982339   5.9427 5.118e-09 ***
educ         0.0827444   0.0077389  10.6920 < 2.2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Figure 4: R regression output

In which we have used `coeftest` and `vcovHC` HC1 variance covariance matrix for one form of the robust one. We have obtained the estimates in both cases.

Comparing, we have the **same estimate**, however the **SE** in the robust case are slightly bigger. This isn't a great example since it doesn't change significance however it shows both estimators can give different SE, but the estimate from OLS remains the same.

## 2.4 Breusch-Pagan Test for Heteroskedasticity

- Testing hypothesis

$$H_0 : V(u|x_1, \dots, x_k) = E(u^2|x_1, \dots, x_k) = \sigma^2$$

Where  $V[u|x] = E[u^2|x] - \underbrace{0}_{E[u|x]}$

- Assume a linear relationship :

$$U^2 = \delta_0 + \delta_1 x_1 + \dots + \delta_k x_k + v, E[v|x_1, \dots, x_k] = 0$$

- Since we cannot observe the errors ( $u^2$ ), we replace them with the residuals and estimate the regression

1. Estimate

$$\hat{u}^2 = \delta_0 + \delta_1 x_1 + \dots + error$$

Recover the  $R_u^2$

2. Hypotheses :  $\delta_1 = \dots = \delta_k = 0$
3. Test stat :  $F = \frac{R_u^2/k}{(1-R_u^2)/(n-k-1)} \sim^{H_0} \mathcal{F}_{k, n-k-1}$  Or  $LM = nR_u^2 \sim^{H_0} \chi_k^2$  where k dof, F follows fisher dist, LM follows chi squared dist.
4. Decision : if the p-value is small enough (typically  $< 0.05$ ), we **reject** the null of homoskedacity

**Exercise 1 (Heteroskedacity with 2 Categories).** Model  $y_i = \beta_0 + \beta_1 d_i + u_i$ ,  $i = 1, \dots, n$  where  $d_i$  is a binary variable

Let  $n_1 = \sum_i d_i$ ,  $n_0 = \sum_i (1 - d_i)$ ,  $n = n_1 + n_0$  and  $p = \frac{n_1}{n}$  (probability of being trated, share of treated ind in samp / n)

We have seen that  $\hat{\beta}_1 = \bar{y}_1 - \bar{y}_0$  and  $\hat{\beta}_0 = \bar{y}_0$  (differences in group mean outcomes) ( $\hat{\beta}_0$  is intercept, mean of untreated)

Under homoskedacity in small sample conventional t statistic has a t-distribution

Heteroskedacity here means that the variances in the  $d_i = 1$  and  $d_i = 0$  population are different: the exact small sample distribution for this problem is unknown

Differences in the standard error formulae depend on how the variance in  $d_i$  is modelled (residual as difference between outcome and group mean outcome )

- Note  $\hat{u}_i = y_i - \bar{y}_I$  for  $d_i = I$ ,  $I \in \{0, 1\}$
- Define  $s_I^2 = \sum_{i:d=I} (y_i - \bar{y}_I)^2$  (which is the estimated sum of squared residuals in each group)
- Under conventional SEs:  $\hat{\sigma}^2(X'X)^{-1}$  with estimate of  $\hat{\sigma}^2 = \frac{1}{n-k-1} \sum_{i=1}^n \hat{u}_i^2$
- Where  $\sum_{i=1}^n \hat{u}_i^2 = \sum_{i:d=1} \hat{u}_i^2 + \sum_{i:d=0} \hat{u}_i^2 = s_1^2 + s_0^2$  (sum od squared resid = sum of residuals squared for treated and untreated ind)
- Hence  $\hat{\sigma}^2 = \frac{s_1^2 + s_0^2}{n-2}$  (is equal to n-2 since have single regressor and intercept)
- Now,  $(X'X)^{-1}_{[2,2]} = \frac{n}{nn_1 - n_1^2}$  (if interested in slope, take X and 2,2 element equal to this expression, using this we can take estimator for variance )
- Hence  $\hat{V}(\hat{\beta}_1)_c = \frac{n}{n_1 n_0} \frac{s_1^2 + s_0^2}{n-2}$  (conventional variance estimator if replace elements by percentage shares)
- It can be shown that  $\hat{V}(\hat{\beta}_1)_c = \frac{1}{np(1-p)} \frac{s_1^2 + s_0^2}{n-2}$
- For robust SEs :  $\hat{\sigma}^2(X'X)^{-1}(X\hat{\psi}Z)(X'X)^{-1} \rightarrow \hat{V}(\hat{\beta}_1)_r = \frac{s_1^2}{n_1^2} + \frac{s_0^2}{n_0^2}$
- When  $\frac{s_1^2}{n_1} = \frac{s_0^2}{n_0}$ , both estimates coincide (for large n)
- When  $n_1 = n_0 = \frac{n}{2}$  they also coincide, when the data are balanced, the robust SE won't differ much from the traditional one under heteroskedacity
- If both groups variances are the same, then both estimates coincide, because then we have homoskedacity

- Also if we have the same individuals for treated and untreated groups, then they also coincide, so if we have very balanced data (2 cat) the robust SE won't differ much from the traditional one

### BP test

- Interpretation of the BP test
- Recall the regression  $\hat{u}_i^2 = \delta_0 + \delta_1 d_i + v$

$$\hat{\delta}_0 = \frac{\sum_{i:d=0} \hat{u}_i^2}{n_0} = \frac{s_0^2}{n_0} \hat{\delta}_1 = \frac{\sum_{i:d=1} \hat{u}_i^2}{n_1} - \frac{\sum_{i:d=0} \hat{u}_i^2}{n_0} = \frac{s_1^2}{n_1} - \frac{s_0^2}{n_0}$$

- Testing  $H_0 : \delta_1 = 0$  is equivalent to testing  $\sigma_1^2 = \sigma_0^2$

**Example (Housing Price Equation).** Log is sometimes used to get rid of heteroskedacity

Model :  $\text{price} = \beta_0 + \beta_1 \text{lotsize} + \beta_2 \text{sqrft} + \beta_3 \text{bdrms} + u$

Where price is the housing price, lotsize the size of the lot, size size of house in sq ft

We want to estimate the above regression and test for heteroskedacity and see whether using logs in the dependent variable changes our conclusion

```
Call:
lm(formula = price ~ lotsize + sqrft + bdrms, data = data)

Residuals:
    Min       1Q   Median       3Q      Max
-120.026  -38.530   -6.555    32.323   209.376

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) -2.177e+01  2.948e+01  -0.739  0.46221
lotsize      2.068e-03  6.421e-04   3.220  0.00182 **
sqrft       1.228e-01  1.324e-02   9.275 1.66e-14 ***
bdrms       1.385e+01  9.010e+00   1.537  0.12795
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 59.83 on 84 degrees of freedom
Multiple R-squared:  0.6724,    Adjusted R-squared:  0.6607
F-statistic: 57.46 on 3 and 84 DF,  p-value: < 2.2e-16
```

Figure 5: Housing Price Equation Output 1

We cannot really learn much about heteroskedacity, although lot and size is statistically significant. Testing for heteroskedacity using BP test, predicting residuals from previous regression and squared them, then we take the squared residuals and regress on independent variables

```

Call:
lm(formula = u.hat2 ~ lotsize + sqrft + bdrms, data = data)

Residuals:
    Min       1Q   Median       3Q      Max
-9044   -2212   -1256    -97   42562

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) -5.523e+03  3.259e+03  -1.694  0.09390 .
lotsize      2.015e-01  7.101e-02   2.838  0.00569 **
sqrft        1.691e+00  1.464e+00   1.155  0.25128
bdrms        1.042e+03  9.964e+02   1.046  0.29877
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 6617 on 84 degrees of freedom
Multiple R-squared:  0.1601,    Adjusted R-squared:  0.1301
F-statistic: 5.339 on 3 and 84 DF,  p-value: 0.002048

>
> # Compute F-stat by hand: recover the R2:
> R_u2 = summary(reg.res)$r.squared
> df = reg.res$df # n-k-1
> k = 3
>
> # F-stat:
> F = (R_u2/k) / ((1-R_u2)/(df))
> F
[1] 5.338919

```

Figure 6: Housing Price equation output 2

We obtain the f stat, testing for joint normality of parameter estimate, 5.3 with p value  $< 0.05$ , testing for heteroskedacity using BP test leads us to reject the null of homoskedacity

```

Call:
lm(formula = log(price) ~ log(lotsize) + log(sqrft) + bdrms,
    data = data)

Residuals:
    Min       1Q   Median       3Q      Max
-0.68422 -0.09178 -0.01584  0.11213  0.66899

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) -1.29704    0.65128  -1.992  0.0497 *
log(lotsize)  0.16797    0.03828   4.388 3.31e-05 ***
log(sqrft)    0.70023    0.09287   7.540 5.01e-11 ***
bdrms         0.03696    0.02753   1.342  0.1831
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.1846 on 84 degrees of freedom
Multiple R-squared:  0.643,    Adjusted R-squared:  0.6302
F-statistic: 50.42 on 3 and 84 DF,  p-value: < 2.2e-16

```

Figure 7

Does our question change if we use logs? Running the regression we obtain the above, not telling us much again, but helps us to predict residuals based on this regression, then we can test for heteroskedacity

```

Call:
lm(formula = u.hat.ln2 ~ log(lotsize) + log(sqft) + bdrms, data = data)

Residuals:
    Min       1Q   Median       3Q      Max
-0.05601 -0.03011 -0.01687  0.00523  0.40978

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  0.509994   0.257857   1.978  0.0512 .
log(lotsize) -0.007016   0.015156  -0.463  0.6446
log(sqft)    -0.062737   0.036767  -1.706  0.0916 .
bdrms        0.016841   0.010900   1.545  0.1261
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.07309 on 84 degrees of freedom
Multiple R-squared:  0.04799,    Adjusted R-squared:  0.01399
F-statistic: 1.421 on 3 and 84 DF,  p-value: 0.2451

```

Figure 8

Doing the same as before (without logs) we take our residuals, square the, then regress on individual variables. Giving us f stat of 1.4, which given the p val of 0.2 leads us to *failing to reject the null of homoskedasticity*. Thus our initial SE weren't very useful, but using the logs we can assume homoskedasticity.

### Heteroskedasticity Conclusion

- Use robust SE when heteroskedastic errors
- But there is a danger of small sample bias from robust SE (arising from asymptotic justification)
- Under homoskedasticity or little heteroskedasticity, it might be preferable to use the traditional OLS variance estimator
- It is recommended to report both the robust and conventional standard error and suggest to take the maximum of both for inference
- White test for heteroskedasticity includes the squares and cross-products of the independent variables
- LPM : built in heteroskedasticity → need to compute robust SEs
- Using logs in the dependent variables has been seen to improve in terms of heteroskedasticity in many applications

## 2.5 Clustered Standard Errors

### Illustration of Moulton Problem

The Moulton Problem - biased standard errors where observations are not independent within groups, but the regression model incorrectly specifies that they are.

The clustering of data can lead to an underestimation of standard errors → statistical tests likely overly optimistic about the significance.

Closely related to correlation over time in DiD, state average employment rates are correlated over time [ME-AP]

- Pillar assumption is random sampling
- There is potential dependence of data within a group structure
  - Exam grades of children from same class or school : grades are correlated because of the same school, teacher and background / class environment

- Health outcomes in the same village, Errors are correlated because of the same medical and food supply and similar cultural background
- Earning in the same region might be correlated because of the same industrial structure
- Analysing workers in firms (earnings, tenure, promotion) will suffer from common firm effects

### The Moulton Problem

- Illustration using a simple model with a group structure
- Intuitively, effect of a macro variable on an individual level outcomes
  - Effect of school-type on exam-grades
  - Effect of regional unemployment on individuals' wages
- Model

$$Y_{ig} = \beta_0 + \beta_1 x_g + e_{ig}$$

- With  $g = 1, \dots, G$  and  $i = 1, \dots, n$
- $y_{ig}$  is the outcome for individual  $i$  in group  $g$
- Here  $x_g$  varies only at the group level

**Note.** Lecture : if we estimate a model parameter consistently, why do we care about inference?

- We would like to investigate a problem
- Policymaker would like to know whether to implement school building program
- But what is decision rule? Typically, think about Statistical significance and sufficient magnitude then the policymaker wants to adopt the program, if not then not adoptable.
- We need CI or at least a statistical test. For this we need SE and distribution
- Need to estimate SE correctly to get correct CI, if we have too large CI (SE wrong), the implication/ error is that we risk not detecting an effect, when there is
- But the other way around too small SE (forgot to cluster), might think building schools help and invest a lot of money, but the effect is 0
- This is a danger and the problem is *incorrect standard errors lead to incorrect confidence intervals*

**Note.** Lecture : Once we have accounted for clustering using the Moulton approach compared to the standard errors, is it more likely that the clustered standard errors are larger or smaller than the OLS

**Note.** Larger

**Note.** Lecture : what solutions exist to account for clustering

- Group averages (only valid for regressors that don't vary within each individual within a group)

- Parametric - estimate the Moulton factor
- Clustering SE
- Block bootstrap

### Data Structure

- Recall  $E[e_{ig}] = 0$  &  $v(E_{ig}) = \sigma_e^2$
- Recall correlation:  $\rho_e = \frac{Cov(e_{ig}, e_{jg})}{sd(e_{ig})sd(e_{jg})}$
- Likely : for individual I and j from the same group g :

$$Cov[e_{ig}, e_{jg}] = \rho_e \sigma_e^2 > 0$$

### Additive Random Effects

- Group correlation often modelled using additive random effects, assume  $e_{ig} = v$
- $v_g$  : group specific error term which captures all the within-group correlation with  $E[v_g] = 0$  &  $V(v_g) = \sigma_b^2$
- $n_{ig}$  : individual level specific error term with  $E[n_{ig}] = 0$  &  $V(n_{ig}) = \sigma_n^2$
- Assuming  $v_g$  and  $n_{ig}$  are uncorrelated
- We note that  $n_{ig}$  and  $n_{jg}$  are uncorrelated

### Data Structure

- Recall  $E[e_{ig}] = 0$  &  $v(E_{ig}) = \sigma_e^2$
- Recall correlation:  $\rho_e = \frac{Cov(e_{ig}, e_{jg})}{sd(e_{ig})sd(e_{jg})}$
- Likely : for individual I and j from the same group g :

$$Cov[e_{ig}, e_{jg}] = \rho_e \sigma_e^2 > 0$$

### Additive Random Effects

- Group correlation often modelled using additive random effects, assume  $e_{ig} = v$
- $v_g$  : group specific error term which captures all the within-group correlation with  $E[v_g] = 0$  &  $V(v_g) = \sigma_b^2$
- $n_{ig}$  : individual level specific error term with  $E[n_{ig}] = 0$  &  $V(n_{ig}) = \sigma_n^2$
- Assuming  $v_g$  and  $n_{ig}$  are uncorrelated
- We note that  $n_{ig}$  and  $n_{jg}$  are uncorrelated

$$\begin{aligned} Cov(e_{ig}, e_{jg}) &= E[(v_g, n_{ig})(v_g + n_{jg})] = E[v_g^2] = \sigma_v^2 \\ V[e_{ig}] &= E[(v_g + n_{ig})^2] = E(v_g^2 + n_{ig}^2) = \sigma_v^2 + \sigma_n^2 \end{aligned}$$

### Intraclass Correlation Coefficient



- The intraclass correlation coefficient as the proportion of variation in  $(v + n)$  due to  $v$  :

$$\rho_e = \frac{\text{Cov}(e_{ig}, e_{jg})}{\text{sd}(e_{ig})\text{sd}(e_{jg})} = \frac{\sigma_v^2}{\sigma_v^2 + \sigma_n^2}$$

- When the regressor of interest varies only at group level, then this error structure can increase standard errors sharply
- By how much is the conventional variance of the OLS estimate inflated?
- Let  $V_c(\hat{\beta}_1)$  denote the conventional OLS variance expression and  $V(\hat{\beta}_1)$  be the correct sampling variance with this error structure
- Depending on the data structure. There are various versions to quantify  $\frac{V(\hat{\beta}_1)}{V_c[\hat{\beta}_{s1:1}]}$

For the following data structure :

- Nonstochastic regressors fixed at the group level (that is, all regressors are the same for each individual in a group)
- Equal group sizes  $N = n_1 = \dots = n_G$  with total sample size  $n = G * N$

$$\frac{V[\hat{\beta}_{s1:1}]}{V_c[\hat{\beta}_{s1:1}]} = 1 + (N - 1)\rho_e$$

Moulton Factor  $\sqrt{1 + (N - 1)\rho_e}$

Which quantifies how much we over estimate precision by ignoring intraclass correlation

**Remark.** • Conventional standard errors become increasingly misleading as group size  $N$  and / or  $\rho_e$  increase

- If there is no error correlation ( $\rho_e = 0$ ) , there is no overestimation
- If  $\rho_e = 1$  (or  $n_{ig} = 0$ ), then within a group, all 's are the same : the conventional variance is scaled up by  $(N-1)$  since we copy each information  $N$  times without generating new information
- With the total sample size fixed, increasing the group sizes  $N$  just decreases the number of clusters which leads to less independent information
- The Moulton factor can be very big even with a small correlation. Assume 100 observations per group and a  $\rho_e = 0.1$  leads to a Moulton factor of 3.3. The conventional standard errors are only roughly  $\frac{1}{3}$  of what they should be

## Generalisations

The most general form where  $x$  varies by  $g$  and  $I$  with variations in  $g$  :

$$\frac{V[\hat{\beta}_{s1:1}]}{V_c[\hat{\beta}_{s1:1}]} = 1 + \left[ \frac{V(N_g)}{\bar{n}_g} + \bar{n}_g - 1 \right] \rho_r \rho_x$$

where  $\rho_{ho_x}$  is the within cluster correlation coefficient for  $x$ :

$$\rho_x = \frac{\sum_g \sum_j \sum_{i \neq j} (x_{ig} - \bar{x})(x_{jg} - \bar{x})}{V[x_g] \sum_g n_g(n_g - 1)}$$

- $\rho_x$  is a generic measure of the correlation of the regressors within the group. If this correlation is zero, the Moulton effect disappears
- Clustering has a bigger impact on standard errors with variable group sizes and when  $\rho_x$  is large
- If the group size is fixed but  $x$  varies by  $g$  and  $I$ , the Moulton factor becomes the square root of  $1 + (N - 1)\rho_E\rho_x$

### Moulton Problem - Solutions

Model  $y = \beta_0 + \beta_1 x_{ig} + e_{ig}$  with  $g = 1, \dots, G$

1. Parametric approach
  - Fix the conventional standard errors using the general formula for the Moulton factor by estimating the intraclass correlations  $\rho_e$  and  $\rho_x$
2. Cluster standard errors
  - (a) Generalisation of white's robust covariance matrix

$$\hat{AV}(\hat{\beta}_{s1:1}) = (X'X)^{-1} \left( a \sum_{g=1}^G X_g' \hat{e}_g \hat{e}_g' X_g \right) (X'X)^{-1}$$

- (b) Where  $\hat{e}_g$  is a  $n_g \times 1$  vector of residuals for observations in the  $g$ -th cluster and  $X_g$  is a  $n_g \times k$  matrix of regressors for observations in the  $g$ -th cluster
  - (c) Typically, there is a degrees of freedom adjustment  $a = \frac{G(n-1)}{(G-1)(n-k)}$
  - (d) Consistent if number of clusters is large but not consistent with fixed number of groups (even when group sizes tend to  $\infty$ )
  - (e) No assumptions about within-group correlation structure (not just parametric such as in the additive error structure)
  - (f) If each individual is his own group ( $I = g$  and  $G = n$ ) then the formula collapses back to the robust estimator
3. Use group averages instead of microdata
    - (a) Model :  $y_{ig} = \beta_0 + \beta_1 x_g + e_{ig}$ ,  $g = 1, \dots, G$
    - (b) We estimate  $\bar{y}_g = \beta_0 + \beta_1 \bar{x}_g + \bar{e}_g$  by weighted least squares using  $n_g$  as weights
    - (c) However, neglecting heteroskedasticity unless the group sizes are equal
    - (d) Relying on asymptotics for group number, not group sizes
    - (e) With modest group sizes, it is expected to have good finite sample properties of regressions with normal errors
    - (f) And is likely to be more reliable than clustered standard errors with few clusters
    - (g) But does not work if  $x$  varies within groups and ignores any other micro-level covariates
    - (h) But there exists a 2 step approach to include micro level covariates (A & P)
  4. Block bootstrap

(a) To be discussed

## 5. GLS or Max Likelihood approaches

GLS :

- (a) In some cases is possible to estimate GLS or maximum likelihood model
- (b) Requires a model for error structure

**Example (Star Experiment).** Krueger (1999) uses IV to estimate the effect of class size on students' achievements  $y_{ig}$  is the test score of student  $I$  in class  $g$  and class size  $x_g$

Students were randomly assigned to each class but data are unlikely to be independent across observations.

Test scores in the same classes are correlated because students in the same class share background characteristics and are exposed to the same teacher and classroom environment

It is likely for students  $I$  and  $j$  from the same class  $g$  :

$$E[e_{ig}, e_{jg}] = \rho_r \sigma_e^2 > 0$$

The estimation strategy is for now not in our focus, though we can compare the different standard error estimates

Standard errors for class size effects in the STAR data (318 clusters)	
Variance Estimator	Std. Err.
Robust ( $HC_1$ )	.090
Parametric Moulton correction (using Moulton intraclass correlation)	.222
Parametric Moulton correction (using Stata intraclass correlation)	.230
Clustered	.232
Block bootstrap	.231
Estimation using group means (weighted by class size)	.226
<i>Notes:</i> The table reports standard errors for the estimates from a regression of kindergartners' average percentile scores on class size using the public use data set from Project STAR. The coefficient on class size is $-.62$ . The group level for clustering is the classroom. The number of observations is 5,743. The bootstrap estimate uses 1,000 replications.	

Figure 9: Robust standard errors after correcting for clustering

[Moulton Derivation]

## Lecture 3: Bootstrap and IV

Tue 13 Feb 15:20

[L3]

### 2.6 Basic Intro to Bootstrap

Based on the data we have we simulate and pretend we many more 'made-up' datasets we didn't have previously. Runs into issues when estimating min or max, rather than mean and under non-Gaussian distribution. When asymptotically normal or ..., bootstrap good choice.

OvB only creates bias if correlation with regressors, if going to argue variable is non-correlated, it is fine. But including too many regressors may be problematic too, end up including too many variables correlated with regressor, on top of fact it creates noise.

Attenuation Bias - if measurement error, nothing can do about it. As long as variance and this measurement error, it exists. But if less variance in measurement error, then it disappears. Of course, provided error isn't systematic.

- Another method for estimating variance, CI and dist on statistic
- Often used when exact distribution is unknown
- Different versions but non parametric most common

### 2.7 Non-parametric Bootstrap

- $X$  is distributed according to some distribution  $F$ :  $X \sim F$
- $x = (x_1, \dots, x_n)$  represents an iid sample from this variable
- Suppose we want to estimate the variance and the distribution of a statistic  $T_n = g(x_1, \dots, x_n)$
- Ultimately interested in variance of distribution of this statistic  $T_n = g(x_1, \dots, x_n)$

#### NP Bootstrap - Variance

- Let  $V_F$  denote the variance of  $T_n$  where the subscript  $F$  indicates that the variance is a function of  $F$
- If we knew  $F$ , we could compute the variance
- For example for  $T_n = \frac{1}{n} \sum_{i=1}^n x_i$ ,

$$V_F(T_n) = \frac{V(x)}{n} = \frac{\int x^2 dF(x) - (\int x dF(x))^2}{n}$$

Where  $dF(x)$  is the pdf in integral form (2nd term)

- Which is a function of  $F$
- Idea is to estimate  $V_F(T_n)$  with  $V_{\hat{F}}(T_n)$
- Or, *use a plug in estimator* of the variance
- Since  $V_{\hat{F}}(T_n)$  may be difficult to compute, we approximate it with a simulation estimate denoted by  $v_{boot}$

**Key Idea**

Put the initial sample  $x = (x_1, \dots, x_n)$  into an urn

1. Draw  $n$  observations from  $x$  with replacement
  - Each observation has the probability of  $\frac{1}{n}$  of being drawn
  - Gives each bootstrap sample  $x_1^* = (x_{11}^*, \dots, x_{n1}^*)$
2. Based on the single bootstrap sample, we estimate (compute bootstrap statistic)

$$T_{n1}^* = g(x_{11}^*, \dots, x_{n1}^*)$$

3. Repeat steps 1 and 2  $B$  times to get  $T_{n1}^*, \dots, T_{nB}^*$  where :

$$T_{nb}^* = g(x_{1b}^*, \dots, x_{nb}^*) \text{ for } b = 1, \dots, B$$

Where  $B$  is the number of bootstrap replications

$$V_{boot} = \frac{1}{B} \sum_{b=1}^B (T_{nb}^* - \frac{1}{B} \sum_{r=1}^B T_{nr}^*)$$

(then take sample analog of variance)

Then by the law of large numbers  $v_{boot} \xrightarrow{a} V_{\hat{F}(T_n)}$  as  $B \rightarrow \infty$  (bootstrap variance tends to variance of stat we were after)

Need to reiterate quite often, in real world we have initial sample from true distribution  $F$  which gives us stat  $T_n$  in bootstrap world we have bootstrap sample which comes from resampling our initial sample which gives us our bootstrap stat :  $T_n^*$

Imagine in real world, initial sample with 4 obs, giving us stat which is a function of these 4 obs (say the average over these 4 obs). In order to get into bootstrap world, we place sample in urn, we draw  $b$  times 4 observations each time with replacement

1 <sup>st</sup> draw:	$x_1^* = \{1, 3, 1, 2\}$	$\rightarrow g(1, 3, 1, 2) = T_{n1}^*$
2 <sup>nd</sup> draw:	$x_2^* = \{1, 4, 4, 4\}$	$\rightarrow g(1, 4, 4, 4) = T_{n2}^*$
...		
$b^{th}$ draw:	$x_b^* = \{2, 4, 1, 1\}$	$\rightarrow g(2, 4, 1, 1) = T_{nb}^*$
...		
$B^{th}$ draw:	$x_B^* = \{1, 3, 2, 4\}$	$\rightarrow g(1, 3, 2, 4) = T_{nB}^*$

Figure 10: Bootstrap World

When we do the 1-st draw we get 1, then since draw with replacement, it could happen we draw this again, second is 3, we also put this back, we do this even further then we got again observation with 1.

Then eventually we get the observation with 2 then we can compute the bootstrap statistic  $b$  times to obtain bootstrap samples

**Use of the Bootstrap**

- The empirical distribution of the B bootstrap samples gives us the approximated distribution / moments of  $T_n$
- EEG standard Errors :  $\hat{se} = \sqrt{v_{boot}}$
- Approximate the CDF of  $T_n$ . Let  $G_n(t) = (T_n < t)$  be the CDF of  $T_n$
- The bootstrap appropriate to  $G_n$  is

$$\hat{G}_n^*(t) = \frac{1}{B} \sum_{b=1}^B 1_{\{T_{nb}^* \leq t\}}$$

Where the binary variable obtains probability

- Confidence intervals based on SE or quantiles
- Normal interval :

$$T_n \pm z_{\frac{\alpha}{2}} \hat{se}_{boot}$$

- Where  $\hat{se}_{boot}$  is the bootstrap estimate for the SE
- Where  $z_{\frac{\alpha}{2}}$  is the  $\frac{\alpha}{2}$  quantile of the standard normal distribution
- The interval is not accurate unless the distribution of  $T_n$  is close to normal

```
> set.seed(1000)
> n = 100
> mu = 5
> sigma2 = 2
> X = rnorm(n, mu, sd = sqrt(sigma2))
> #Tn = mean(X)
>
> B = 500
> TnStar = c()
> for (i in 1:B){
+   XStar = sample(X, size = n, replace = TRUE)
+   TnStar[i] = mean(XStar)
+ }
> # Bootstrap Mean:
> mean(TnStar)
[1] 5.031754
> # True mean: 5
>
> # Bootstrap Variance:
> var(TnStar)
[1] 0.02003969
> #True Variance:
> V = sigma2/n
> V
[1] 0.02
```

Figure 11: Bootstrap Code

Set seed to ensure RV is same on diff computers, then 100 obs, mean = 5, variance = 2

We want 500 bootstrap replications, then we initiate an empty vector  $t_n^*$  to collect bootstrap replications, then iterate over 500 i's. For each I in 1:500 we sample from our initial vector, with replacement 100 observations, giving us bootstrap sample, then we take mean to obtain bootstrap mean

$T_n^*$  has 500 bootstrap means, then we take mean over these 500 and compare to true expectation. 5.03 is very close to the true mean,

We proceed the same to estimate variance based on bootstrap replications, we are also close to variance also.

Practically, it depends on the situation to normalise test stat (demean or standardise in order to ensure normal distribution)

### Regression Estimates

Procedure quite similar, but with at least 2 characteristics for each individual parameters

Instead of drawing directly from RV, we Draw pairs of  $\{y_i, x_i\}$  to

- Sometimes called the *pairs bootstrap*
- Instead of drawing directly from the random variable, you would sample the indices of the observations

Empirical, non parametric, standard, pairs.

### Wild Bootstrap

relies on assumption that error term at disposal

- Model  $y_i = \beta_0 + \beta_1 x_i + u_i$  (one regressor)
- Preserves heteroskedastic behaviour since don't destroy link between x's and error terms
- Initial sample  $z = [(y_1, x_1) \dots (y_n, x_n)]$  with outcome and regressors for each individual

### Methodology

Quite similar but main difference that it is residual bootstrap but keep regressors fixed

1. Estimate  $\hat{u}_i = y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i$  for all  $i = 1, \dots, n$
2. Randomly create a bootstrap residual (weights)

$$\text{weights: } w_i = \begin{cases} 1, & \text{with probability } \frac{1}{2} \\ -1, & \text{with probability } \frac{1}{2} \end{cases}$$

Bootstrap residuals  $\hat{u}_i^* = w_i \hat{u}_i$ , for all  $i = 1, \dots, n$

3. Compute the bootstrap dependent variables (essentially changed sign of original residual )

$$Y_i^* = \hat{\beta}_0 + \hat{\beta}_1 x_i + \hat{u}_i^*$$

For all  $i = 1, \dots, n$  Gives : a single bootstrap sample :  $z_1^* = [(y_{11}^*, x_1), (y_{n1}^*, x_n)]$

Repeat 1-3 B times to obtain B wild bootstrap samples :  $z_b^* = [(y_{1b}^*, x_1), \dots, (y_{nb}^*, x_n)]$  for  $b = 1, \dots, B$

### Clustered Bootstrap

Under similar randomisation, you do not preserve the dependence (unobserved factors relating to micro-data) structure in the data

To fix this, we draw blocks of data defined by the groups g. Say block bootstrap by re sampling entire classes instead of individual students, to keep structure of correlation intact.

Can also have cluster 1 bootstrap, maybe you have stratified sampling such that while you sampled you made sure have say gender quota or certain subset, we would need to do bootstrap for this.

The way you sample data structure, try to mimic through the bootstrap exactly this structure. That is, replicate DGP as close as you can (provided we know about it)

**Exercise 2 (Algorithm to obtain SE using clustered bootstrap).**

Set the seed

If we have 1 village, should we have randomly different households in village 1? No

Village 1, 2, 3 and households a, b, c (1), def, (2), ghi (3). Everything within the villages is kept fixed

We then draw 1 (abc) , 3(ghi) , 3 (ghi) for 1-st bootstrap

Then 2, (ghi) , 1 (abc) , 1 (abc) for 2-nd bootstrap

Need large sample.

Based on practice not theory

Advantages - as opposed to random draw, forget about cluster, end up doing randomly a,d,f,c and estimate OLS  $\hat{\beta}_1$  and g,h,c,f and estimate  $\hat{\beta}_2$ , you have no attachment to group and lose correlation within each group

Need to re merge together 'blocks' into single data set since we have individually sampled blocks

Have to store estimator so on

**Exercise 3 (Effect of schooling on wages, use father educ as instrument for years of educ).**

Conditions of good instrument?

Exclusion restriction (orthogonality - instrument cannot be correlated with error term), Relevance restriction ( $Cov(x_1, z) = 0$ )

Maybe since there is push to education, maybe with time this effect is fading, but likely still relevant, but maybe in other countries this is deterministic and is something we can test

Exclusion restriction - 1. We can control for this, if this is not part of the model. 2. Might be violated if we can argue ability for singers, parents can sing, inherit singing talent so opera hires, this might be correlated with number of years taking singing lessons but choosing to take singing lessons due to natural talent suggests violation of exclusion criteria

**Exercise 4 (Is month of birth good instrument for years of education).**

Exclusion Restriction - it is pretty random when you are born, there is no reason to believe the error term left over when explaining wages is correlated with when you are born

Some spikes of the births (seasonality etc) though, could this be an issue?

Relevance condition - Structure of Education System : cutoff for year of schooling in the year, therefore matters for when say leave school at 16

Usa : start in September, able to leave when 16, people born earlier get extra months of schooling

Does extra month of education have strong effect? No

F stat - very low in this case, but we want a large F-stat



If we have a small f stat we have very low relevance, but also  $\beta = \text{cov}(y, x) / \text{cov}(x, z)$

Bias end up having depends on the covariance between y and u and x and z

If low relevance, then  $\frac{\text{cov}(z, u)}{\text{cov}(x, z)}$  'explodes'

To test, there is no proper overall applicable test for exclusion restriction, it is something you have to argue for.

### 3 Instrumental Variables

[L3-IV]

#### Motivation

Let's say we are interested in identifying the causal effect of years of schooling on wage, we estimate the model  $y = \beta_0 + \beta_1 x_1 + \dots + \beta_k x_k + u$

- One of the key assumptions for unbiasedness is the *homogeneity of regressors* :  $E[u|x_1, \dots, x_k] = 0$
- Indeed, the problem arises when the regression error is correlated with a regressor : ie  $E[u|x_k] \neq 0$
- There are three broad reasons for *Endogeneity* :
  1. Omitted variable bias
  2. Measurement error
  3. Simultaneous equations
- In our example,  $x_k$  is said to be endogenous, meaning the years of schooling might be correlated with innate and unobserved ability
- The OLS estimator  $\beta_k$  is *biased* and *inconsistent*
- One approach to deal with this issue is to use instrumental variables

#### 3.1 Forms of Endogeneity

- Omitted variables
  1. Arises in cases when one fails to control for a regressor that is correlated with other regressors
  2. Often due to self selection : if an agent chooses the value of the regressor, this might depend on factors that we cannot observe
  3. That is, *unobserved heterogeneity*
- Measurement Error
  1. Occurs when we can only observe an imperfect measure of a variable
  2. Depending on how the observed and true variable are related, we might have endogeneity
- Simultaneity
  1. Occurs when dependent and independent variables are simultaneously determined
  2. If x is partially determined by y, then the error might be correlated with x

Though this is not to say there exist sharp distinctions

**Example.** Effect of alcohol consumption on worker productivity (measured by wages )

Alcohol usage correlated with unobserved factors such as family background, which may also have an effect on wage. Leading to an Omitted variable problem

Alcohol demand can depend on income, leading to the simultaneity problem

There also exists possibility of mismeasurement of alcohol consumption

### Omitted Variable Bias (OVB)

- Long regression - true model is :  $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + e$
- Then, assuming we cannot observe  $x_2$  but only  $x_1$
- Or, in short :  $y = \delta_0 + \delta_1 x_1 + u$  with  $u = \beta_2 x_2 + e$
- We know the population parameter can be expressed as :

$$\delta_1 = \frac{\text{cov}(y, x_1)}{v[x_1]}$$

replacing y from the true model:

$$\begin{aligned} &= \frac{\text{Cov}(\beta_0 + \beta_1 x_1 + \beta_2 x_2 + e, x_1)}{V[x_1]} \\ &= \beta_1 v[x_1] + \beta_2 \text{Cov}(x_2, x_1) + \text{cov}(e, x_1) / V[x_1] \\ &= \beta_1 + \beta_2 \frac{\text{cov}(x_2, x_1)}{v(x_1)} \end{aligned}$$

Defining  $\tau_1$  as the parameter in the population model that relates  $x_1$  to  $x_2$  :

$$X_2 = \tau_0 + \tau_1 x_1 + \text{'error'}$$

We therefore estimate  $\delta_1 = \beta_1 + \beta_2 \tau_1$

However, our OLS estimate is *inconsistent* (asymptotically biased)

$$\text{plim}_{n \rightarrow \infty} \hat{\delta}_1 - \beta_1 = \beta_2 \frac{\text{Cov}(x_1, x_2)}{v(x_1)} = \beta_2 \tau_1$$

With

$$\text{Bias}(\hat{\delta}_1) = E[\hat{\delta}_1] - \beta_1 = \beta_2 \hat{\tau}_1$$

Where thinking about the direction of the correlation helps us think about the direction of the bias

Essentially, if the omitted variable is related to the included regressor, then the parameter in the short regression will not identify the parameter in the long regression.

With more regressors, the formula changes but the principle remains the same

**Example.** omitted variable bias

Let  $y$  be the wages,  $x_1$  years of education and  $x_2$  ability

Regressing wages on years of education alone delivers a biased estimate  $\hat{\delta}_1$

We would expect both years of education and ability to have a positive impact on average earnings (that is  $\{\beta_{1/2} > 0, \}$ )

But we also expect both regressors to be *positively correlated* as individuals with more *innate ability* tend to choose / acquire more education ( $\tau_1 > 0$ )

Therefore,  $\hat{\tau}_1$  likely overestimates the value of education, since in our education regressor we have not controlled for the correlation with ability and thus include more than the effect of education in this estimate, here thinking about the direction of the correlation has helped us to identify the sign of the bias

**Measurement Error in  $y$** 

**Situation 1 :** Measurement error in the dependent variable ( $y$ )

True model  $y = \beta_0 + \beta_1 x_1 + \varepsilon$ ,  $E[\varepsilon|x_1] = 0$

We can only observe  $\tilde{y}$  which measures the unobserved  $y$  with an error  $\tilde{y} = y + e$

We regress  $\tilde{y} = \tilde{\beta}_0 + \tilde{\beta}_1 x_1 + \tilde{\varepsilon}$

$$\tilde{\beta}_1 = \frac{Cov(\tilde{y}, x_1)}{V[x_1]} = \frac{Cov(y + e, x_1)}{V[x_1]} = \beta_1 + \frac{Cov(e, x_1)}{V(x_1)}$$

$$\tilde{\beta}_0 = E(\tilde{y}) - \tilde{\beta}_1 E(x_1) = E[y] + E[e] - \beta_1 E[x_1] = \beta_0 + E(e)$$

However, this can cause bias and inconsistency. Although it vanishes if the measurement error is statistically independent of each explanatory variable. We note the usual OLS inference procedures are asymptotically valid.

**Situation 2 :** Measurement error in the regressor ( $x$ )

True model  $y = \beta_0 + \beta_1 x_1 + \varepsilon$ ,  $E[\varepsilon|x_1] = 0$

Where we can only observe  $\tilde{x}_1$ , a measure of the unobserved  $x_1$  with an error  $\tilde{x}_1 = x_1 + \varepsilon$

We then regress  $y = \tilde{\beta}_0 + \tilde{\beta}_1 \tilde{x}_1 + \tilde{\varepsilon}$

$$\tilde{\beta}_1 = \frac{Cov(y, \tilde{x}_1)}{V[\tilde{x}_1]} = \frac{Cov(\beta_0 + \beta_1 x_1 + \varepsilon, x_1 + \varepsilon)}{V[\tilde{x}_1]} = \beta_1 \frac{V(x_1)}{V(\tilde{x}_1)}$$

$$\tilde{\beta}_0 = E(\tilde{y}) - \tilde{\beta}_1 E(\tilde{x}_1) = E[y] + E[e] - \beta_1 E[x_1] = \beta_1 \frac{V(x_1)}{V(x_1) + V(e)} = \beta_1 \lambda$$

With the key assumptions that  $Cov(e, x_1) = 0$ ,  $cov(e, \varepsilon) = 0$ , and  $E[e] = 0$

In which we can show  $plim_{n \rightarrow \infty} \hat{\beta}_1 = \beta_1 \lambda$ , where  $\lambda \in \{0, 1\}$  :  $\hat{\beta}_1$  underestimates  $\beta_1$ , this is attenuation bias. Though as  $V(e)$  shrinks relative to  $V(x_1)$ , the attenuation bias disappears.

In the general model, it is not the variance of the true regressor that affects the consistency but the variance in the true regressor after netting out the other explanatory variables

### Simultaneity / Reverse Causality

Problem :  $y_1$  and  $y_2$  are simultaneously determined.

$$\begin{aligned} Y_1 &= \alpha_1 y_2 + \beta_1 z_1 + u_1, E[z_1|u_1] = 0 \\ Y_2 &= \alpha_2 y_1 + \beta_2 z_2 + u_2, E[z_2|u_2] = 0 \end{aligned}$$

A classic example is when  $y_1$  is price and  $y_2$  is the quantity and both equations are demand and supply. But note the intercept is suppressed for simplicity.

Focusing on the 1-st equation, to show that  $Cov(y_2, u_1) \neq 0$  :

$$\begin{aligned} Y_2 &= \alpha_2 [\alpha_1 y_2 + \beta_1 z_1 + u_1] + \beta_2 z_2 + u_2 \\ &= \frac{\beta_1 \alpha_2}{1 - \alpha_1 \alpha_2} z_2 + \frac{\alpha_2}{1 - \alpha_1 \alpha_2} u_1 + \frac{1}{1 - \alpha_1 \alpha_2} u_2 \end{aligned}$$

Assuming that  $\alpha_1 \alpha_2 \neq 0$ ,  $Cov(u_1, u_2) = 0$  and  $cov(z_2, u_1) = cov(z_2, u_2) = 0$ , thus violating exogeneity

$$Cov(y_2, u_1) = \frac{\alpha_2}{1 - \alpha_1 \alpha_2} V(u_1 \neq 0)$$

Although, without additional controls ( $z_1$ ), it can be shown that the consistency has the same sign as  $\frac{\alpha_2}{1 - \alpha_1 \alpha_2}$

## 3.2 IV Estimator

### Single regressor model

**Properties of an instrument to overcome Endogeneity** Model :  $y = \beta_0 + \beta_1 x_1 + u$  with  $E[u|x_1] \neq 0$

We need an instrumental variable (IV)  $z$  with the following properties

- Exclusion Restriction :  $Cov(u, z) = 0$
- Relevance :  $Cov(x_1, z) \neq 0$

The instrument cannot be correlated with any of the omitted variables for egg, but it does need to be correlated with the endogenous regressor

1. We can test the relevance assumption
2. However we cannot generally test the exclusion restriction. One needs to carefully apply common sense and economic theory to convince the audience about the validity of the instrument

**Testing for Relevance** First stage regression

$$X_1 = \pi_0 + \pi_1 z + v$$

Where we regress endogenous regressor on instrument,  $\pi_1$  is equal to covariance of instrument and endow regressor / variance of endow regressor

Recall,  $\pi_1 = \frac{Cov(z, x_1)}{V(x_1)}$

Test

$$H_0 = \pi_1 = 0 \text{ vs } H_0 : \neq 0$$

To understand whether covariance is zero or not

Where we should be able to reject at a small significance level. In this case we can be confident that the relevance condition holds

### How to find IVs?

- It might be hard to think of a valid instrument and or to have data on them
- Instruments come from
  - Economic theory
  - Exogenous sources of variation in the endogenous regressor arising from a random phenomenon such as whether events, or exogenous policies (cutting class sizes to increase grades)

**Example.** Wages Where ability causes the regressor of years of educating to be endogenous One could think of

- Family background variables
- Proximity to school / college
- Month of birth

As potential IIs. Whether these would work requires scrutiny

### Identification

Model :  $y = \beta_0 + \beta_1 x_1 + u$ , with  $E(u|x_1) \neq 0$  Instrument :  $Cov(x_1, z) \neq 0$  and  $Cov(z, u) = 0$  (satisfying exclusion and relevance)

Identification in this context : we can write  $\beta_1$  (parameter of interest) in terms of population moments that can be estimated

We write  $\beta_1$  in terms of population covariances

$$\begin{aligned} Cov(z, y) &= \beta_1 Cov(z, x_1) + Cov(z, u) \\ \text{Since } cov(z, u) &= 0 \text{ and } cov(z, x_1) \neq 0 \text{ (assumptions)} \\ \beta_1 &= \frac{Cov(z, y)}{cov(z, x_1)} \end{aligned}$$

However, this fails if  $Cov(z, x_1) = 0$ , that is the relevance condition doesn't hold. This is an expression we can estimate using a random sample.

### IV estimator

- Given random sampling, we estimate the moments by the sample analogs :

$$\hat{\beta}_1^{IV} = \frac{\sum_{i=1}^n (z_i - \bar{z})(y_i - \bar{y})}{\sum_{i=1}^n (z_i - \bar{z})(x_{i,1} - \bar{x}_1)}, \text{ and } \hat{\beta}_0^{IV} = \bar{y} - \hat{\beta}_1^{IV} \bar{x}_1$$

- When  $z = x_1$  then the IV estimator reduces to the OLS estimator
- Using the Law of Large Numbers, we can show that  $\hat{\beta}_1^{IV}$  is consistent under the assumptions :

$$\text{plim}_{n \rightarrow \infty} \hat{\beta}_1^{IV} = \beta_1$$

- However, the IV estimator is biased
- Requiring large samples

We note that if we divide the denominator and numerator by  $V(z)$  :

$$\beta_1 = \frac{\text{Cov}(z, y)/V(z)}{\text{Cov}(z, x_1)/V(z)}$$

Where  $\beta_1$  is the ratio of the population regression of the reduced form over the first stage.

### Wald Estimator - Binary Instrument

Recalling that for a regression on a binary variable, the resulting slope estimate is the difference between both groups averages. Then, under the IV assumptions, the  $\beta_1$  can be represented as the ratio of the two OLS estimands, in case of a binary IV:

$$\beta_1 = \frac{\text{Cov}(y, z)V(z)}{\text{Cov}(x_1, z)/V(z)} = \frac{E[y|z=1] - E[y|z=0]}{E[x_1|z=1] - E[x_1|z=0]}$$

Then, taking the sample analog :  $\hat{\beta}_1^{IV} = \frac{\bar{y}_1 - \bar{y}_0}{\bar{x}_{1,1} - \bar{x}_{1,0}}$  Where  $\bar{y}_1$  and  $\bar{x}_{1,1}$

## Lecture 4: IV continued

Tue 20 Feb 16:07

[L4-IV] Model  $y = x\beta + u$  with  $x$  a vector of  $k$  exogenous and endogenous regressors and  $z$  a vector of  $m$  IV's (including the exogenous variable)

1.  $M = k$  : the model is just identified ,we have an instrument for each endogenous variable  $\Rightarrow$  use IV
2.  $M < k$  : the model is not identified, we do not have enough IVs
3.  $M > k$  : the model is over-identified  $\rightarrow$  we have too many IVs. Use GIVE / 2SLS

### 3.3 Case : $\text{length}(z) = \text{length}(x)$

Model :  $y = x\beta + u$ ,  $x = (q, x_2, \dots, x_k)$  and  $z = (1, x_2, \dots, x_{k-1}, z_1)$ . We know  $\text{Cov}(x_j, u) = 0$  for  $j = 2, \dots, k-1$  and  $\text{Cov}(x_k, u) \neq 0$

We have an instrument for  $x_k$  :

- Exogenous  $\text{Cov}(z_1, u) = 0$
- Partial Correlation :  $\theta_1 \neq 0$  in  $x_k = \delta_1 + \delta_2 x_2 + \dots + \delta_{k-1} x_{k-1} + \theta_1 z_1 + r_k$

Where the moment conditions imply :

$$E[z'u] = E[z'(y - x\beta)] = 0$$

We have one instrument at our disposal for this endogenous regressor, we include the constant and all the exogenous regressors because they can be used for instruments for themselves.

Partial correlation best seen by regressing endogenous regressor  $x_k$  on all exogenous variables plus the instrument for  $x_k$  and we need the parameter on the instrument  $\theta_1$  not to be 0, thus partial correlation, the correlation cannot be 0 after the other effects have been 'netted' out. Different to simple case where sufficient to have covariance between endogenous regressor and instrument  $\neq 0$

Exogeneity leads to above expression, plugging in expression for  $u$ .

Multiplying the model through with  $z'$ , taking expectation and using the moment condition :

$$\begin{aligned} E[z'y] &= E[z'x]\beta \\ \text{if rank } E[z'x] &= k \\ \beta &= [E[z'x]]^{-1}E[z'y] \end{aligned}$$

There is a *unique solution only under full rank* and it can be shown that if we rule out perfect collinearity in  $z$ , full rank holds iff  $\theta_1 \neq 0$

Given a random sample, we can estimate consistently :

$$\hat{\beta}^{IV} = \left( \frac{1}{n} \sum_{i=1}^n z_i' x_i \right)^{-1} \left( \frac{1}{n} \sum_{i=1}^n z_i' y_i \right) = (Z'X)^{-1} Z'Y$$

Where  $Z$  and  $X$  are  $n \times k$  data matrices and  $y$  is  $N \times 1$

Given the assumptions, this estimator is consistent

### 3.4 2SLS/GIVE

**Case:  $\text{length}(z) > \text{length}(x)$ :**

The idea is to use the fitted values from the first stage regression of the endogenous regressor on all the exogenous variables (including the instruments) and use them as "instruments" in the IV estimator

$$Z = (1, x_1, \dots, x_{k-1}, z_1, \dots, z_l) - m = k + I \text{ vector for } x_k$$

1. Fitted values from the first stage  $\hat{x}_i = (1, x_1, \dots, x_{k-1}, \hat{x}_k)$

$$\begin{aligned} \hat{x}_{ik} &= \hat{\delta}_0 + \hat{\delta}_1 x_{i1} + \dots + \hat{\delta}_{k-1} x_{i,k-1} + \hat{\theta}_1 z_{i1} + \dots + \hat{\theta}_l z_{il} \\ \hat{x}_i &= z_i \left( \sum z_i' z_i \right)^{-1} z_i' x_i \\ \hat{X} &= Z(Z'Z)^{-1} Z'X \end{aligned}$$

For this endogenous regressor you have several potential instruments at your disposal, you would then regress on exogenous variables from initial model and instruments. That gives you a vector of instruments that is equal to (including all potential instruments).

Then you start with obtaining fitted values from First Stage (FS) regressing  $x_k$  on exogenous regressors  $\delta$  and instruments  $z$

Using the fitted values as instruments :

$$\hat{\beta}^{IV} = \left( \frac{1}{n} \sum_{i=1}^n \hat{x}_i' x_i \right)^{-1} \left( \frac{1}{n} \sum_{i=1}^n \hat{x}_i' y_i \right) = (\hat{X}' X)^{-1} \hat{X}' Y$$

Then, using calculus, we can show that  $\hat{X}' X = \hat{X}' \hat{X}$  and hence

$$\hat{\beta}^{IV} = (\hat{X}' \hat{X})^{-1} \hat{X}' Y$$

Which is the GIVE / 2SLS estimator since it equals the OLS estimator on the fitted values from the first stage

Where we have simply replaced with fitted values, and then replaced in matrix form. We can essentially show this is the OLS estimator on the FS using the fitted values. Then we use this to plug in the IV estimator to obtain the OLS estimator on the fitted values

**To obtain the  $\beta$**

1. First Stage: Obtain the fitted values  $\hat{x}_k$  from the regression  $x_k$  on  $1, x_1, \dots, x_{k-1}, z_1, \dots, z_l$
2. Second Stage: Run the OLS regression:  $y$  on  $1 + x_1, \dots, x_{k-1} + \hat{x}_k$ 
  - However, omitting the exogenous regressors in the first stage is easily done and will lead to inconsistency
  - And, SE obtained from the second step are incorrect

Testing for rank condition :  $H_0 : \theta_1 = \dots = \theta_l = 0$  vs at least one  $\theta_s$  for  $s = 1, \dots, l$  is non zero.

### 3.5 Properties

Here  $x$  is  $1 \times k$  and generally includes unity, several elements of  $x$  may be endogenous, while  $z$  includes any exogenous variable

**Assumption 11** 2SLS.1

For some  $1 \times m$ -vector  $z$ ,  $E[z'u] = 0$

**Assumption 12** 2SLS.2

1.  $\text{rank } E[z'z] = m$
2.  $\text{rank } E[z'x] = k$

Under the above 2 assumptions, the 2SLS estimator obtained from a random sample is *consistent* for  $\beta$

### 3.6 Group Mean Estimator

[L4-GM] In some situations, have instruments that can be changed into 2 groups, water (of birth/financial year). 'Chop instrument into groups' like Moulton problem/structure.

It can be shown that group mean estimator is IV, a weighted least squares regression, where it is sufficient to know size of groups and means, do regression and obtain estimator that is equivalent to an IV estimator, that is consistent despite the fact we have an endogenous variable.

- Where  $x_{ig}$  is endogenous



- We have  $g$  *moment conditions*, if this is IV, we know exogeneity must hold, whether group 1 or 2, the error term conditional on this group needs to be equal to 0, this must hold for all groups. Essentially, we have  $g$  different groups this is really  $E[y_{ig}|z_g = I] = \beta_0 + \beta_1 E[x_{ig}|i : z_g = I]$
- To estimate an expectation, we replace with an average, since this is conditional, to estimate the expectation for the first group (born in the 1-st quarter), we take the average for the first group (conditional average by restricting to the first group) - taking the means of all the groups
- We do the same for  $\bar{x}$  and intuitively obtain

$$\bar{y}_g = \beta_0 + \beta_1 \bar{x}_g + \bar{u}_g$$

Doing this is the same as using dummy variables for quarter of birth in 2SLS regression, **thus** group means are consistent.

**Exercise 5.** Group mean estimator - what happens if the number of groups = 2 If we have dummy variable, we obtain the Wald estimator (last week). Our instrument, we obtain the same expression In order to derive,

1. Regress  $x$  on dummies,  $x$  can only belong to 1, so fitted values are sample means of dependent variable  $x_{ig}$ . Fitted values  $\hat{x}_{ig}$  are means  $\bar{x}_g$
2. Apply OLS on this after we have found fitted values, the predicted values are our sample means

### Angrist and Krueger

- Does compulsory school attendance affect schooling and earnings
- Using *quarter of birth* as an instrument
  1. Exclusion : Season of birth is a natural experiment and hence unrelated to innate ability, motivation or family connections
  2. Relevance : In the US, children were allowed to drop out at 16. Since the age of starting school differs, children have different lengths of schooling when they turn 16
- Potentially weak instrument and potential reasons why quarter of birth might be somewhat correlated with the error

TABLE 4.1.2  
Wald estimates of the returns to schooling using  
quarter-of-birth instruments

	(1) Born in 1st Quarter of Year	(2) Born in 4th Quarter of Year	(3) Difference (Std. Error) (1) – (2)
ln (weekly wage)	5.892	5.905	-.0135 (.0034)
Years of education	12.688	12.839	-.151 (.016)
Wald estimate of return to education			.089 (.021)
OLS estimate of return to education			.070 (.0005)

*Notes:* From Angrist and Imbens (1995). The sample includes native-born men with positive earnings from the 1930–39 birth cohorts in the 1980 census 5 percent file. The sample size is 162,515.

Downloaded from <https://www.cambridge.org/core>. University of Cambridge, on 02 Dec 2019 at 10:00:00, subject to the Cambridge Core terms of use, available at <https://www.cambridge.org/core/terms>. <https://doi.org/10.1017/9781009100000.004>

Figure 12: Wald estimates of IV - weak/exclusion violated?

Can't test by how much IV exclusion is violated, it might be best to use OLS, but in the same sense it may be incorrect - can we search for better instrument? Or,

TABLE 4.1.1 2SLS estimates of the economic returns to schooling

	OLS		2SLS					
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Years of education	0.071 (.0004)	0.067 (.0004)	0.102 (0.024)	0.13 (0.020)	0.104 (0.026)	0.108 (0.020)	0.087 (0.019)	0.057 (0.029)
<i>Exogenous Covariates</i>								
Age (in qtrs)								✓
Age (in qtrs) squared								✓
9 yr-birth dummies					✓	✓	✓	✓
50 state-of-birth dummies		✓			✓	✓	✓	✓
<i>Extra Instruments</i>								
Dummy for QoB=1			✓	✓	✓	✓	✓	✓
Dummy for QoB=2			✓	✓	✓	✓	✓	✓
Dummy for QoB=3			✓	✓	✓	✓	✓	✓
QoB dummies interacted with year-of-birth dummies							✓	✓
No covariates <i>k</i>	2	61	2	2	61	61	61	63
No instruments <i>m</i>	2	61	2	4	61	63	93	95

*Notes:* The table reports OLS and 2SLS estimates of the returns to schooling using the the Angrist and Krueger (1991) 1980 Census sample. This sample includes native-born men, born 1930–1939, with positive earnings and non-allocated values for key variables. The sample size is 329,509. Robust standard errors are reported in parentheses. QoB denotes quarter of birth.

Figure 13: 2SLS estimates of economic returns to schooling

Inflated standard errors :  $0 \cdot \frac{0.021}{0.0005} = 42$ , even though the estimate is significant due to a large sample size, the 9% CI is large.

Problem also of a small  $R^2_{x,z}$  : the *instruments are weak*, it might be better to use OLS instead of IV

Including more instruments and covariates

- Reduces SE but comes to the cost of potentially having a weak instrument
- Col 3 : just identified 1 instrument
- Col 4 : over identified (3 QoB instruments)
- Col 5/6 : + 59 covariates to 3/4 :  $m - k = 0$  (or 2 resp)
- Col 7 : + 30 Ifs ( $m - k = 32$ )
- Col 8 : + age and  $age^2$  to  $x$  and  $z$  ( $m - k = 32$ )

But there is potential to test for over identifying restrictions, using the Sargan test

Why  $\beta$  larger? Asymptotic variance depends on error variance, depends on the  $R^2_{x,z}$  from first stage regression (x on instruments, will be higher if instruments highly relevant and vice versa).

**Exercise 6.** Consequences of weak instruments

1. High SE
2. Slight violation of exclusion restrictions leads to large bias

**Exercise 7.** Discuss intuition behind Hausman Test Exogenous regressor then both (testing for whether x endogenous) OLS and IV consistent (if we have valid instrument) If we have an endogenous regressor and OLS is not consistent, difference does not converge to 0 any more, test stat follows chi-squared distribution with k degrees of freedom, to test for endogeneity 2 main assumptions, 1 test for, 1 assume

- Relies on valid instrument, otherwise  $\hat{\beta}$  would not converge at all, this is almost critical assumption in Hausman test

**Tutorial 1.** IV and simultaneity bias We have expression for  $y_1$  and  $y_2$ , we are going to replace this equation, since our asymptotic bias will sum to .? Asymptotic bias =  $\frac{Cov(u_1, y_2)}{V(y_2)}$  Then we plug long covariance into  $Cov(u_1, y_2)$  We know  $cov(u_1, u_2) = 0$  and  $cov(u_1, z_2) = 0$ , but the problem is the variance is typically positive But depending on assumptions we can determine direction of bias based upon  $\alpha_2$  We show that in the formula we replace by  $y_1$

**Tutorial 2.** IV is asymptotically unbiased That is  $plim \tilde{\alpha}_1 = \alpha_1$  (the IV estimator)

**Tutorial 3.** why can  $z_2$  not be used as an instrument for  $y_1$  to estimate  $\alpha_2$  the slope of the supply curve It is under-identified, we don't have an endogenous variable at our disposal, we don't have a shock for  $y_1 \rightarrow$  we don't have an instrument, two endogenous variables require 2 instruments

**Tutorial 4.** Exercise 1 (tut3) Regression of log wage on education, estimate using OLS, internet, do you expect OLS to be trustworthy? Education on wage includes ability and motivation etc explaining the wages, that are correlate with education  $\rightarrow$  OVB. We expect a positive omitted variable bias, since ability is likely correlated with log wages Testing relevance condition by FS regression : running education on number of siblings, this is significantly different from 0 and f-stat  $> 10$  Then running IV regression, we find the instrument has strong enough F-stat, but it could be that the exclusion restriction is violated, before we found coefficient of 0.059, with IV we find 0.122 (12%), which is higher than OLS, revealing inconsistency already, perhaps our assumption about *exogeneity* is not fulfilled.

**Tutorial 5.** Exercise 2 Using sibs as iv is not same as plugging sibs into education (as in proxy), we find very different result from our IV estimator, that is big diff from 0 controlling for. Education and birth quarter negatively correlated? B) C) again, we get an increase than the OLS estimator, and larger than when we used siblings as IV. But do we have similar concerns now using birth order Is birth order endogenous? Like the number of siblings? The decision to have children might be

related to budget constraints etc. D) identification assumption  $\log(wage) = \beta_0 + \beta_1$  Test whether  $\pi_2$  is significantly different from 0, if we estimate our IV, we need to include all exogenous variables as instruments, we estimate a different coefficient.

## Lecture 5: RCT

Tue 27 Feb 16:02

[L5-RCT]

## 4 Randomised Experiments

### Motivation : Program Evaluation

Binary treatment on a set of outcomes, lets say the effect of having internet at home on school grades, however this would of course run into selection bias since the decision to have internet at home might depend on other unobserved factors (income etc).

Thus, program evaluation is often about how to overcome the problem of selection bias, using the potential outcomes framework allows us to illustrate this.

**Exercise 8.** Do hospitals make people healthier? Q : Do hospitals make people healthier? If we have data on the following questions :

1. In the last 12 months have you spent a night in hospital?
2. What would you rate you health 1-5 (being excellent)

Group	Sample Size	Mean Health	SE
Hospital	7,774	3.21	0.014
No hospital	90,049	3.93	0.003

Figure 14: Naive Hospital Comparison

This naive comparison of individuals hospitalised and not, a difference of 0.72 suggest that non hospitalised people are healthier Thus, can we ask does going to the hospital make people sick? Maybe in some cases, but the main problem is *self selection*

- People who decide to go to the hospital are less healthy to begin with
- Even if the treatment works, such individuals won't be healthier than those who do not go to the hospital

We can formalise this with the *Potential Outcomes framework*

### Treatment Allocation and Outcomes

- Start with single unit I
- Denote the outcome of interest by Y and treatment variable D
  - D = 1 the individual is *treated*
  - D = 0 the individual is *not treated* (control)

- Typical assumption is that one individual can have 2 states
  1.  $Y(1)$  - the potential outcome if I receives treatment
  2.  $Y(0)$  - the potential outcome if I 'would not' receive the treatment (control)
- Individual Causal Effect of the treatment for observation I:

$$Y(1) - Y(0)$$

- The *problem of causal inference* - is that it is impossible to observe **both** potential outcomes at the same time, only one is realised  $\rightarrow$  thus it is impossible to observe the causal effect

### Stable Unit Treatment Value Assumption

- Generalisation to  $n$  units  $i = 1, 2, \dots, n$
- Let  $D_i$  be the treatment for unit  $i$
- Each unit can be exposed to the two treatments : *the problem is that in principle the potential outcomes can depend on the treatment of all units*
- *thus we make the assumption* that the potential outcome for unit  $i$  depends only on the treatment received by unit  $i$  and not on the allocation of other individuals
- Denote  $D_{-i} = (D_j) : j \neq i$  treatment status of all other individuals in the population. Then SUTVA states

$$[Y_i(1), Y_i(0) \perp D_{-i}]$$

- Aka the 'no interference assumption'
- However, this might be violated if individuals interact
- There cannot be contagion between individuals

## 4.1 RCTs - imperfect compliance

### Encouragement design

$$Y = Y(0) + D[Y(1) - Y(0)]$$

$$D = D(0) + Z$$

$$D = 1AT + C \quad D = 0$$

### LATE

Let  $Y_i(z, d)$  be the potential outcome for individual  $i$  with treatment status  $D_i = d$  and the assignment  $Z_i = z, z, d, \in \{0, 1\}$

1. Independence  $[Y_i(z, d) \forall d, z, D_i(1), D_i(0)] \perp Z_i$
2. Exclusion Restriction  $Y_i(d, 0) = Y_i(d, 1) = Y_i(d)$
3. First Stage  $P(D_i = 1 | Z_i = 1) - P(D_i = 1 | Z_i = 0) > 0$
4. Monotonicity  $D_i(1) - D_i(0) \geq 0$  for all  $i$

Exclusion restriction in RCTs can be violated.

FS estimator is equivalent to Wald in IV, gives share of compliers in this case since under monotonicity, we assume everybody reacts to treatment in same way - ruling out the existence of defiers.

So the difference between  $Z = 0, D=1 = D+AT$  and  $Z = 1, D=1 = AT + C$  is

Monotonicity is rather harsh, it has to hold for all individuals. Randomising vaccine may not give to those who want, thus they are defiers.

Monotonicity - it can only be positive in binary world, if end up having defiers fg

## Estimand

### 4.2 Late and Bloom Result

**Exercise 9. [Late-bloom]** Starting from monotonicity, we can write total variation of binary variable, condition this equal to 1 plus same thing conditional on 0, then this term doesn't exist any more, we get rid and our numerator is equal to The switching function exists only due to the exclusion restriction, otherwise we would need to write  $y$  as a function observed by both variables **The denominator** - same thing but for FS, replace now here

IV estimator provides more meaningful interpretation, provides average treatment effect for average treated people We take the numerator again Replace the observed outcome with switching equation via the switching equation, then replace switching equation since the exclusion restriction holds We also know only the non-treated PO is realised, By independence Since  $D_i$  is binary

This is a general framework for a binary IV estimator too. The LATE/ATT is obtained.

**Exercise 10.** Why is administrative data good (esp for RCTs) In RCT can have access to social security info if firms agree to connect, can obtain nationality of recent hires etc - why is this useful?

Attrition - have reliable data, don't have to trust accounts of data - say measurement error, Can undertake longer term RCTs too

## Lecture 6: Panel Data

Wed 06 Mar 12:07

[L6]

## 5 Panel Data Methods

Another way of dealing with problem of endogeneity

### Panel Data

So far we have only seen a cross section of individuals (i)

Now we introduce panel which combines individual and time dimension (t).

We assume the cross section model (t=1) as :

$$Y_{i,1} = \beta_0 + x_{i,1}\beta + \alpha_i + u_{i,1} \quad (2)$$

Where we have previously estimated the impact of education on wages where *ability* is typically an unobserved omitted variable, our solutions so far have been to (a) find an IV and (b) randomise

The additional time dimension gives us new tools.

There are essentially 3 ways of getting rid of  $a_i$ , they are

1. First differences
2. Fixed effects estimator
3. Dummy variable regression

Then, assuming the FE is uncorrelated with the regressor we can use random effects approach or pooled OLS.

**Intuition FD** Assuming that ability is constant over time and does not change, and that we observe the same individual (i) from eq. (2) ( $t=2$ )

$$y_{i,2} = \beta_0 + x_{i,2}\beta + a_i + u_{i,2} \quad (3)$$

The idea is to take the difference between both periods of time, to get rid of the *unobserved* constant effect

Taking the difference between both periods

$$y_{i,2} - y_{i,1} = (x_{i,2} - x_{i,1})\beta + u_{i,2} - u_{i,1} \quad (4)$$

Then, if unobserved ability is constant over time, we can get rid of it.

### First Differences

Typical panel data structure for  $T=2$  (two time periods)

Assuming random sample of individuals that we observe twice at  $t = 1$  and  $t = 2$

We also observe an outcome at both time periods, the outcome of individual 1 in both periods and so on. We also have an intercept, and also 2 binary variables that indicate the relevant time periods. Switching "on" for each period, exactly the opposite of each other.

We also have fixed effects  $a_i$  that only vary with i, so  $a_1, a_2, \dots, a_n$ .

It is important to note we do not observe all of the *fixed effects*.

$$y_{i,t} = \beta_0 + \beta_1 x_{i,t} + \delta_0 d_t^2 + a_i + u_{i,t}$$

Where

1.  $y_{it}$  is the outcome of interest, varies over i and t
2.  $x_{it}$  is the observed regressor, varying over i and t
3.  $M d_t^1$  (resp  $d_t^2$ ) is a period 1 (2) dummy varying over t (but only one enters regression - dummy var trap)
4. The unobserved fixed effect  $a_i$  only varies over i

5.  $u_{i,t}$  is an unobserved idiosyncratic error
6. The time dummy ensures a time varying intercept

*Pooled OLS* is to estimate a composite error term since we don't observe  $a_i$

$$y_{i,t} = \beta_0 + \beta_1 x_{i,t} \delta_0 d_t^2 + \underbrace{v_{i,t}}_{a_i + u_{i,t}}, \quad t = 1, 2$$

Then, *taking first differences* to difference out  $a_i$

$$\begin{aligned} y_{i,2} - y_{i,1} &= \delta_0 + \beta_1(x_{i,2} - x_{i,1}) + u_{i,2} - u_{i,1} \\ \Delta y_i &= \delta_0 + \beta_1 \Delta x_i + \Delta u_i \end{aligned}$$

[L6-LaggedDV]

### Tutorial

[Tutorial 5]

1. Interpretation of coefficient, 0.38% log-log
- 2.
- 3.
4. Use fixed effects since ... Demean

## Lecture 7: Differences-in-Differences

Thu 07 Mar 15:03

[L7-DID]

## 6 Differences-in-Differences

To compare the difference between two groups before and after a change to establish causality

**Example.** John Snow Cholera London

Cholera epidemic in London, he wanted to establish that Cholera was transmitted through contaminated drinking water. Since districts were served by two water companies, To compare the difference between 2 groups before and after a change to establish causality. In 1849, both companies obtained their water supply from the dirty Thames. In 1852 Lambeth company moved its water works upriver to an area less contaminated with sewage. Death rates fell in districts supplied by Lambeth compared to the change in death rates in districts supplied by Southwark and Vauxhall.

### Canonical DiD

- Data with a time dimension, at least 2 repeated cross sections
- An exogenous treatment : no self selection into the event and no change of behaviour in anticipation of the event
- Measure the outcomes of interest before and after the event



- Two groups, one impacted by the event and the other not

One can introduce more groups, time periods and covariates to the *canonical* 'simple 2x2' model.

### Simple Case

Follows frolick and sperling (2019)

Considering the arrival of a large number of refugees in one city. The idea is to *estimate the impacts of this increase in refugees on local markets*, employment. Supposing we have data on an outcome variable  $Y$  for a time period  $t$  after the influx of refugees for a time period  $t - 1$ , before the influx of refugees. The immigrants arrive some time between  $t - 1$  and  $t$ . Thus the before-after-difference is  $Y_t - Y_{t-1}$

Then, if the time periods are far apart, it may be that other changes have an impact during this time. Then, we can subtract the time trend that *would have happened* if no influx of refugees had occurred. With unaffected neighbouring regions helping us to identify this *unobserved trend*.

We have data for city A (arrival) and B (no arrival)

We compare the differences:

$$\begin{aligned} \Delta Y_{t,A} - \Delta Y_{t,B} &= \underbrace{(Y_{t,A} - Y_{t-1,A})}_{\text{diff over time}} - \underbrace{(Y_{t,B} - Y_{t-1,B})}_{\text{diff over time}} \\ &= \underbrace{(Y_{t,A} - Y_{t,B})}_{\text{diff between cities}} - \underbrace{(Y_{t-1,A} - Y_{t-1,B})}_{\text{diff between cities}} \end{aligned} \quad (5)$$

Where taking the differences in the 'differences' over time is the same as taking the differences in the differences between cities

The idea is to use the changes of the outcomes in the control groups to construct the counterfactual outcome for the treated

**We assume** the *common trends* and *SUTVA* assumptions.

**Potential Outcomes** Define  $D = 1$ , if city A and  $D = 0$  if city B. Let  $d_t = 1$  if  $t = 1$  and  $d_t = 0$  if  $t = 0$ . We denote the potential outcomes  $Y_t(1)$  or  $Y_t(0)$ , where an observation is treated if  $D = 1$  and  $d_t = 1$ . In  $t = 0$ , both groups do not receive treatment.

The observed outcome is a linear function of  $t$ ,  $D$  and the Potential Outcomes:

$$Y_t = Y_t(1) \cdot d_t + Y_t(0) (1 - D \cdot d_t) \quad (6)$$

$$= \begin{cases} y_{t=1} = Y(1)_{t=1}D + Y(0)_{t=1}(1 - D), & \text{if } t = 1 \\ Y_{t=0} = Y(0)_{t=0} & \text{if } t = 0 \end{cases} \quad (7)$$

### Assumption 13 Common Trends Assumption (CT)

During the period  $[t - 1, t]$ , the potential non-treatment outcomes  $Y(0)$  followed the same linear trend in the treatment group as in the control group:

$$E[Y(0)_{t=1} - Y(0)_{t=0} | D = 1] = E[Y(0)_{t=1} - Y(0)_{t=0} | D = 0]$$

or the *Parallel trend* or *Parallel Path*

**Proof.** DiD identifies ATT

Recalling that we have only treated individuals in  $t = 1$  therefore in  $t = 1$ ,

$$\tau = E[Y(1)_{t=1} - Y(0)_{t=1} | D = 1]$$

while we can also identify  $E[Y(1)_{t=1} | D = 1] = E[Y_{t=1} | D = 1]$ , the CT assumption helps us to identify  $E[Y(0)_{t=1} | D = 1]$ . Then, rearranging CT:

$$E[Y(0)_{t=1} | D = 1] = E[Y(0)_{t=0} | D = 1] + E[Y(0)_{t=1} - Y(0)_{t=0} | D = 0]$$

then, by equation 1

$$= E[Y_{t=0} | D = 1] + E[Y_{t=1} - Y_{t=0} | D = 0]$$

hence,

$$\tau = E[Y_{t=1} - Y_{t=0} | D = 1] - E[Y_{t=1} - Y_{t=0} | D = 0]$$

[Slide]

□

Toy proof?

Then, taking the sample analog of the conditional expectations:

$$\begin{aligned} \hat{\tau} &= \hat{E}[Y_{t=1} - Y_{t=0} | D = 1] - \hat{E}[Y_{t=1} - Y_{t=0} | D = 0] \\ &= \hat{E}[Y | D = 1, t = 1] - \hat{E}[Y | D = 1, t = 0] - \left\{ \hat{E}[Y | D = 0, t = 1] - \hat{E}[Y | D = 0, t = 0] \right\} \end{aligned} \quad (8)$$

That is, to estimate the DiD estimator, we only need 4 data points.

**Example.** Policy where poorly performing schools are given additional financial resources

DiD compares average school outcomes between treated and control schools before and after the intervention, the school outcomes are measured at the same time before and after the intervention (different pupils), all schools that are below a threshold are selected according to the average performance of their pupils

positive or negative treatment effect? Positive, Control group, upwards sloping students get better, we have observation at  $t=0$ , extrapolating the trend until  $t=1$  where measured. We also know that the treatment group had the same trend, but just before the treatment average treatment dropped. But after the treatment they should be similar on average, drawing the line would misidentify the treatment effect

here our parallel trends assumption is violated just before the treatment,

is the common trend assumption likely to hold?

**Lecture 8: Regression Discontinuity Designs**

Tue 12 Mar 16:19

[L8-RDD] [cont]