MANIPAL UNIVERSITY JAIPUR

(University under Section 2(f) of the UGC Act)

FIRST STATE PRIVATE UNIVERSITY IN RAJASTHAN
ACCREDITED WITH GRADE A+ NAAC
3.28 SCORE

B.TECH SECOND YEAR

ACADEMIC YEAR: 2020-2021

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MANIPAL UNIVERSITY JAIPUR

VISION

Global Leadership in Higher Education and Human Development

MISSION

- Be the most preferred University for innovative and interdisciplinary learning
- Foster academic, research and professional excellence in all domains
- Transform young minds into competent professionals with good human values

VALUES

Integrity, Transparency, Quality,
Team Work, Execution with Passion, Humane Touch

SESSION OUTCOME

" TO UNDERSTAND THE CONCEPT OF ODE AND THEIR APPLICATIONS AND SOLVE THE PROBLEM"

ASSIGNMENT

QUIZ

MID TERM EXAMINATION –I & II

END TERM EXAMINATION

ASSESSMENT CRITERIA'S

PROGRAM OUTCOMES MAPPING WITH CO1

[PO1]

ENGINEERING KNOWLEDGE: APPLY THE KNOWLEDGE OF MATHEMATICS, SCIENCE, ENGINEERING FUNDAMENTALS, AND AN ENGINEERING SPECIALIZATION TO THE SOLUTION OF COMPLEX ENGINEERING PROBLEMS.

Algebraic Structures

- Algebraic systems Examples and general properties
- Semi groups
- Monoids
- **Groups**
- Subgroups

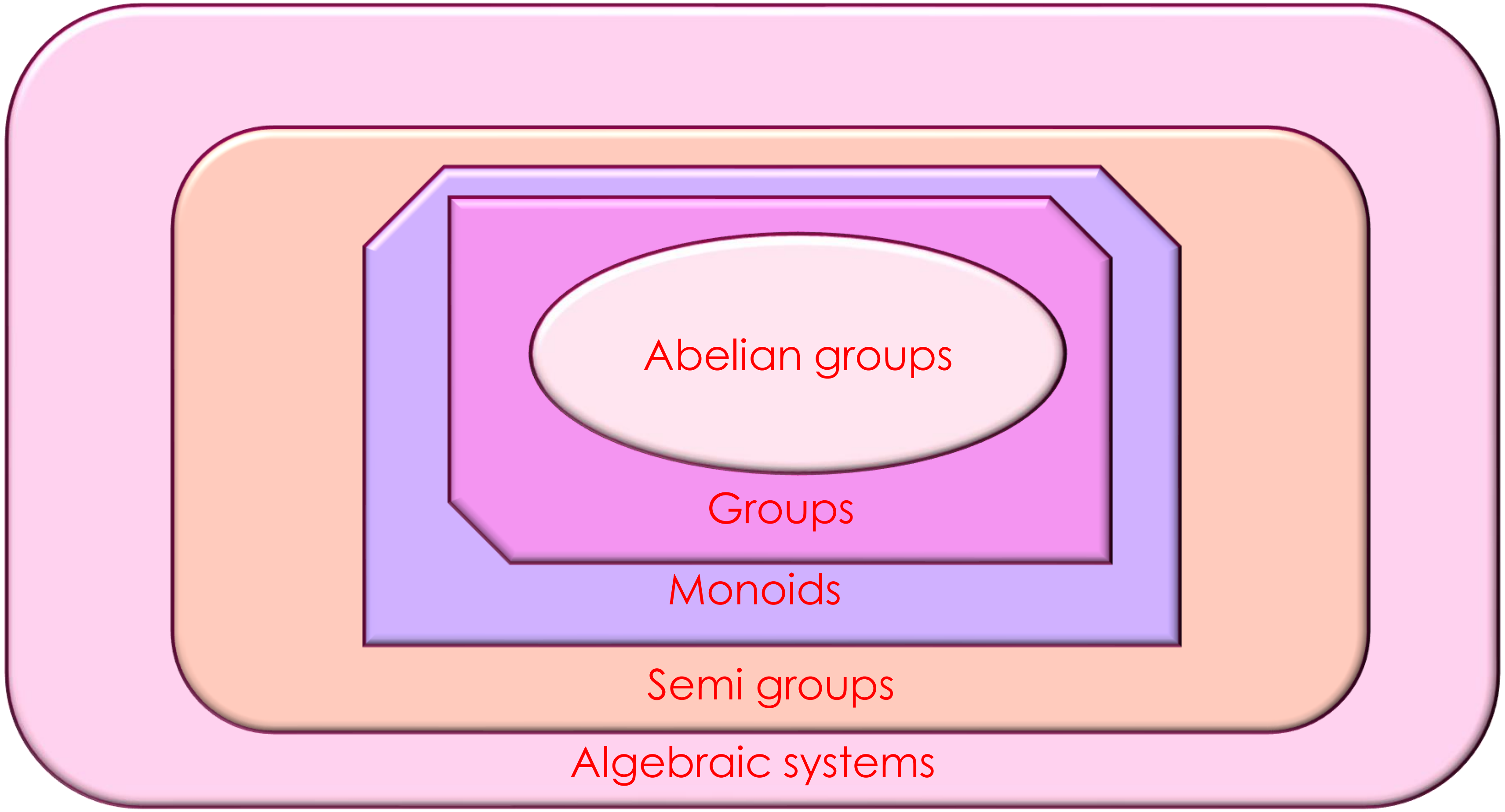
Group

- **Group:** An algebraic system (G, *) is said to be a **group** if the following conditions are satisfied.
  1) * is a closed operation.
  2) * is an associative operation.
  3) There is an identity in G.
  4) Every element in G has inverse in G.

- **Abelian group (Commutative group):** A group (G, *) is said to be ***abelian*** (or ***commutative***) if

$$a * b = b * a \quad \text{for all } a, b \text{ in } G.$$

Algebraic systems



Theorem

- **In a Group (G, *) the following properties hold good**

1. Identity element is unique.
2. Inverse of an element is unique.
3. Cancellation laws hold good

$$a * b = a * c \Rightarrow b = c$$ (left cancellation law)

$$a * c = b * c \Rightarrow a = b$$ (Right cancellation law)

4. $(a * b)^{-1} = b^{-1} * a^{-1}$

- In a group, the identity element is its own inverse.

Theorem

- ***Order of a group***: The number of elements in a group is called order of the group.

- **Finite group:** If the order of a group G is finite, then G is called a finite group.

Ex. Show that, the set of all integers is a group with respect to addition.

- **Solution:** Let Z = set of all integers.

  Let a, b, c are any three elements of Z.

1. **Closure property:** We know that, Sum of two integers is again an integer.

   i.e., $a + b \in Z$ for all $a,b \in Z$

2. **Associativity:** We know that addition of integers is associative.

   i.e., $(a+b)+c = a+(b+c)$ for all $a,b,c \in Z$.

Contd.,

3. Identity: We have $0 \in Z$ and $a + 0 = a$ for all $a \in Z$.

$\therefore$ Identity element exists, and '0' is the identity element.

4. Inverse: To each $a \in Z$, we have $-a \in Z$ such that

$a + (-a) = 0$

Each element in Z has an inverse.

5. Commutativity: We know that addition of integers is commutative.

i.e., $a + b = b + a$ for all $a, b \in Z$.

Hence, $(Z, +)$ is an abelian group.

Ex. Show that set of all non zero real numbers is a group with respect to multiplication.

- **Solution:** Let R^* = set of all nonzero real numbers.

  Let a, b, c are any three elements of R^* .

1. **Closure property:** We know that, product of two nonzero real numbers is again a nonzero real number .

   i.e., $a . b \in R^*$ for all $a,b \in R^*$.

2. **Associativity:** We know that multiplication of real numbers is associative.

   i.e., $(a.b).c = a.(b.c)$ for all $a,b,c \in R^*$.

3. **Identity:** We have $1 \in R^*$ and $a .1 = a$ for all $a \in R^*$.

   $\therefore$ Identity element exists, and '1' is the identity element.

4. **Inverse:** To each $a \in R^*$, we have $1/a \in R^*$ such that

   $a .(1/a) = 1$ i.e., Each element in R^* has an inverse.

Contd.,

- **5. Commutativity:** We know that multiplication of real numbers is commutative.

  i.e., $a . b = b . a$ for all $a,b \in R^*$.

  Hence, $(R^* , .)$ is an abelian group.

- **Ex:** Show that set of all real number's 'R' is not a group with respect to multiplication.
- **Solution:** We have $0 \in R$.

  The multiplicative inverse of 0 does not exist.

  Hence. R is not a group.

Example

- **Ex. Let (Z, *) be an algebraic structure, where Z is the set of integers and the operation * is defined by $n * m = \text{maximum of } (n, m)$.**

Show that (Z, *) is a semi group.

Is (Z, *) a monoid ?. Justify your answer.

- **Solution:** Let a , b and c are any three integers.

Closure property: Now, $a * b = \text{maximum of } (a, b) \in Z$ for all $a, b \in Z$

Example (Contd.)

Associativity: $(a * b) * c$ = maximum of $\{a, b, c\}$ = $a * (b * c)$

$\therefore$ $(Z, *)$ is a semi group.

Identity: There is no integer x such that

$a * x$ = maximum of $(a, x) = a$ for all $a \in Z$

$\therefore$ Identity element does not exist. Hence, $(Z, *)$ is not a monoid.

Example

- **Ex. Show that the set of all strings 'S' is a monoid under the operation 'concatenation of strings'.**

  Is S a group w.r.t the above operation? Justify your answer.

- **Solution:** Let us denote the operation 'concatenation of strings' by + .

  Let s_1, s_2, s_3 are three arbitrary strings in S.

Closure property: Concatenation of two strings is again a string.

i.e., $s_1+s_2 \in S$

Associativity: Concatenation of strings is associative.

$$(s_1+ s_2) + s_3 = s_1+ (s_2 + s_3)$$

Contd.,

- **Identity:** We have null string , $\lambda \in S$ such that $s_1 + \lambda = S$.
- $\therefore$ S is a monoid.
- **Note:** S is not a group, because the inverse of a nonempty string does not exist under concatenation of strings.

Example

- **Ex.** Let S be a finite set and let F(S) be the collection of all functions $f: S \to S$ under the operation of composition of functions, then show that F(S) is a monoid.

  Is S a group w.r.t the above operation? Justify your answer.

- **Solution:** Let f_1, f_2, f_3 are three arbitrary functions on S.

Closure property: Composition of two functions on S is again a function on S.

i.e., $f_1 o\, f_2 \in F(S)$

Associativity: Composition of functions is associative.

i.e., $(f_1 \circ f_2) \circ f_3 = f_1 \circ (f_2 \circ f_3)$

Contd.,

- **Identity:** We have identity function $I : S \to S$
  such that $f_1 \circ I = f_1$.
  $\therefore$ F(S) is a monoid.

- **Note:** F(S) is not a group, because the inverse of a non bijective function on S does not exist.

Example

Ex. If M is set of all non-singular matrices of order 'n x n'. Then show that M is a group w.r.t. matrix multiplication.

Is (M, *) an abelian group?. Justify your answer.

- **Solution**: Let $A,B,C \in M$.

1. **Closure property:** Product of two non-singular matrices is again a non-singular matrix, because

$|AB| = |A| \cdot |B| \neq 0$ (Since, A and B are nonsingular)

i.e., $AB \in M$ for all $A,B \in M$.

2. **Associativity:** Matrix multiplication is associative.

i.e., $(AB)C = A(BC)$ for all $A,B,C \in M$.

Contd.,

3. Identity: We have $I_n \in M$ and $A\, I_n = A$ for all $A \in M$.

$\therefore$ Identity element exists, and 'I_n' is the identity element.

4. Inverse: To each $A \in M$, we have $A^{-1} \in M$ such that

$A\, A^{-1} = I_n$ i.e., Each element in M has an inverse.

Contd.,

- $\therefore$ M is a group w.r.t. matrix multiplication.

We know that, matrix multiplication is not commutative.

Hence, M is not an abelian group.

Example

Ex. Show that the set of all positive rational numbers forms an abelian group under the composition * defined by

a * b = (ab)/2 .

- **Solution:** Let A = set of all positive rational numbers.

  Let a,b,c be any three elements of A.

1. **Closure property:** We know that, Product of two positive rational numbers is again a rational number.

   i.e., $a * b \in A$ for all $a,b \in A$.

2. **Associativity:** $(a*b)*c = (ab/2) * c = (abc) / 4$

   $a*(b*c) = a * (bc/2) = (abc) / 4$

Contd.,

3. **Identity:** Let e be the identity element.

We have $a*e = (a\ e)/2$ ...(1)

By the definition of *

again, $a*e = a$ .....(2)

Since e is the identity.

From (1)and (2), $(a\ e)/2 = a \Rightarrow e = 2$ and $2 \in A$.

$\therefore$ Identity element exists, and '2' is the identity element in A.

Contd.,

- **4. Inverse:** Let $a \in A$

  let us suppose b is inverse of a.

  Now, $a * b = (a\,b)/2$ ....(1) (By definition of inverse)

  Again, $a * b = e = 2$ .....(2) (By definition of inverse)

  From (1) and (2), it follows that

  $(a\,b)/2 = 2$

  $\Rightarrow$ $b = (4 / a) \in A$

$\therefore$ (A ,*) is a group.

- **Commutativity:** $a * b = (ab/2) = (ba/2) = b * a$

Hence, (A,*) is an abelian group.

Theorem

In a group (G, *), Prove that the identity element is unique.

Proof :

a) Let e_1 and e_2 are two identity elements in G.

Now, $e_1 * e_2 = e_1$ ...(1) (since e_2 is the identity)

Again, $e_1 * e_2 = e_2$ ...(2) (since e_1 is the identity)

From (1) and (2), we have

$$e_1 = e_2$$

$\therefore$ Identity element in a group is unique.

Theorem

In a group (G, *) , Prove that the inverse of any element is unique.

Proof:

- Let $a, b, c \in G$ and e is the identity in G.
- Let us suppose, Both b and c are inverse elements of a .
- Now, $a * b = e$ …(1) (Since, b is inverse of a)
- Again, $a * c = e$ …(2) (Since, c is also inverse of a)
- From (1) and (2), we have
- $a * b = a * c$
- $\Rightarrow$ $b = c$ (By left cancellation law)
- **In a group, the inverse of any element is unique.**

Theorem

In a group (G, *) , Prove that $(a * b)^{-1} = b^{-1} * a^{-1}$ for all $a,b \in G$.

Proof : Consider,

- $(a * b) * (b^{-1} * a^{-1})$
- $= (a * (b * b^{-1}) * a^{-1})$ (By associative property)
- $= (a * e * a^{-1})$ (By inverse property)
- $= (a * a^{-1})$ (Since, e is identity)
- $= e$ (By inverse property)
- Similarly, we can show that
- $(b^{-1} * a^{-1}) * (a * b) = e$
- Hence, $(a * b)^{-1} = b^{-1} * a^{-1}$.

Ex. **If (G, *) is a group and a $\in$ G such that a * a = a, then show that a = e , where e is identity element in G.**

Proof: Given that, $a * a = a$

- $\Rightarrow a * a = a * e$ (Since, e is identity in G)
- $\Rightarrow a = e$ (By left cancellation law)
- Hence, the result follows.

Ex. **If every element of a group is its own inverse, then show that the group must be abelian .**

Proof: Let (G, *) be a group.

- Let a and b are any two elements of G.
- Consider the identity,
- $(a * b)^{-1} = b^{-1} * a^{-1}$
- $\Rightarrow$ $(a * b) = b * a$ (Since each element of G is its own inverse)
- Hence, G is abelian.

Note: $a^2 = a * a$

$a^3 = a * a * a$ etc.

Ex. In a group (G, *), if $(a * b)^2 = a^2 * b^2 \quad \forall a, b \in G$ then show that G is abelian group.

Proof: Given that $(a * b)^2 = a^2 * b^2$

- $\Rightarrow (a * b) * (a * b) = (a * a) * (b * b)$
- $\Rightarrow a * (b * a) * b = a * (a * b) * b$ (By associative law)
- $\Rightarrow (b * a) * b = (a * b) * b$ (By left cancellation law)
- $\Rightarrow (b * a) = (a * b)$ (By right cancellation law)
- Hence, G is abelian group.

Finite groups

Ex. Show that G = {1, -1} is an abelian group under multiplication.

Solution: The composition table of G is

$\cdot$	1	-1
1	1	-1
-1	-1	1

1. **Closure property:** Since all the entries of the composition table are the elements of the given set, the set G is closed under multiplication.
2. **Associativity:** The elements of G are real numbers, and we know that multiplication of real numbers is associative.
3. **Identity:** Here, 1 is the identity element and $1 \in G$.

Contd.,

4. **Inverse:** From the composition table, we see that the inverse elements of 1 and -1 are 1 and -1 respectively.

Hence, G is a group w.r.t multiplication.

5. **Commutativity:** The corresponding rows and columns of the table are identical. Therefore the binary operation . is commutative.

Hence, G is an abelian group w.r.t. multiplication.

Ex. Show that G = {1, ω, ω^2} is an abelian group under multiplication. Where 1, ω, ω^2 are cube roots of unity.

Solution: The composition table of G is

.	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

1. **Closure property:** Since all the entries of the composition table are the elements of the given set, the set G is closed under multiplication.
2. **Associativity:** The elements of G are complex numbers, and we know that multiplication of complex numbers is associative.

Contd.,

3. **Identity:** Here, 1 is the identity element and $1 \in G$.
4. **Inverse:** From the composition table, we see that the inverse elements of 1, ω, ω^2 are 1, ω^2, ω respectively.

- Hence, G is a group w.r.t multiplication.

- 5. **Commutativity:** The corresponding rows and columns of the table are identical. Therefore the binary operation . is commutative.
- Hence, G is an abelian group w.r.t. multiplication.

Ex. Show that $G = \{1, -1, i, -i\}$ is an abelian group under multiplication.

Solution: The composition table of G is

.	1	−1	i	-i
1	1	-1	i	- i
-1	-1	1	-i	i
i	i	-i	-1	1
-i	-i	i	1	-1

1. Closure property: Since all the entries of the composition table are the elements of the given set, the set G is closed under multiplication.

Contd.,

2. **Associativity:** The elements of G are complex numbers, and we know that multiplication of complex numbers is associative.
3. **Identity:** Here, 1 is the identity element and $1 \in G$.

- **4. Inverse:** From the composition table, we see that the inverse elements of 1 -1, i, -i are 1, -1, -i, i respectively.
- **5. Commutativity:** The corresponding rows and columns of the table are identical. Therefore the binary operation . is commutative. Hence, (G, .) is an abelian group.

Modulo Systems

Addition modulo m $(+_m)$

- let m is a positive integer. For any two positive integers a and b
- $a +_m b = a + b$ if $a + b < m$
- $a +_m b = r$ if $a + b \geq m$ where r is the remainder obtained by dividing (a+b) with m.

Multiplication modulo p $(\times_p)$

- let p is a positive integer. For any two positive integers a and b
- $a \times_p b = a\,b$ if $a\,b < p$
- $a \times_p b = r$ if $a\,b \geq p$ where r is the remainder obtained
- by dividing (ab) with p.
- **Ex.** $3 \times_5 4 = 2$, $5 \times_5 4 = 0$, $2 \times_5 2 = 4$

Ex. The set G = {0,1,2,3,4,5} is a group with respect to addition modulo 6.

Solution: The composition table of G is

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

- 1. <u>**Closure property:**</u> Since all the entries of the composition table are the elements of the given set, the set G is closed under $+_6$.

Contd.,

- **2. Associativity:** The binary operation $+_6$ is associative in G.

  for ex. $(2 +_6 3) +_6 4 = 5 +_6 4 = 3$ and

  $2 +_6 (3 +_6 4) = 2 +_6 1 = 3$

- **3. Identity:** Here, The first row of the table coincides with the top row. The element heading that row , i.e., 0 is the identity element.
- **4. Inverse:** From the composition table, we see that the inverse elements of 0, 1, 2, 3, 4. 5 are 0, 5, 4, 3, 2, 1 respectively.
- **5. Commutativity:** The corresponding rows and columns of the table are identical. Therefore the binary operation $+_6$ is commutative.
- **Hence, $(G, +_6)$ is an abelian group.**

Ex. The set G = {1,2,3,4,5,6} is a group with respect to multiplication modulo 7.

Solution: The composition table of G is

$\times_7$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

- 1. **Closure property:** Since all the entries of the composition table are the elements of the given set, the set G is closed under $\times_7$.

Contd.,

- **2. Associativity:** The binary operation $\times_7$ is associative in G.

  for ex. $(2 \times_7 3) \times_7 4 = 6 \times_7 4 = 3$ and

  $2 \times_7 (3 \times_7 4) = 2 \times_7 5 = 3$
- **3. Identity :** Here, The first row of the table coincides with the top row. The element heading that row , i.e., 1 is the identity element.
- **4. Inverse:** From the composition table, we see that the inverse elements of 1, 2, 3, 4. 5 ,6 are 1, 4, 5, 2, 5, 6 respectively.
- **5. Commutativity:** The corresponding rows and columns of the table are identical. Therefore the binary operation $\times_7$ is commutative.
- **Hence, $(G, \times_7)$ is an abelian group.**

More on finite groups

- In a group with 2 elements, each element is its own inverse.
- In a group of even order there will be at least one element (other than identity element) which is its own inverse.
- The set $G = \{0,1,2,3,4,\ldots..m-1\}$ is a group with respect to addition modulo m.
- The set $G = \{1,2,3,4,\ldots.p-1\}$ is a group with respect to multiplication modulo p, where p is a prime number.
- **Order of an element of a group:**
- Let (G, *) be a group. Let 'a' be an element of G. The smallest integer n such that $a^n = e$ is called order of 'a'. If no such number exists then the order is infinite.

Examples

Ex. G = {1, -1, i, -i } is a group w.r.t multiplication. The order –i is

a) 2 b) 3 c) 4 d) 1

Ex. Which of the following is not true.

- a) The order of every element of a finite group is finite and is a divisor of the order of the group.

  b) The order of an element of a group is same as that of its inverse.
- c) In the additive group of integers the order of every element except 0 is infinite
- d) In the infinite multiplicative group of nonzero rational numbers the order of every element except 1 is infinite.
- **Ans. d**

THANK YOU

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