April 14, 2012

1 introduction

We consider closed inner orbits connecting edges of acute angled triangles in the hyperbolic plane and bound from above the ratio between the Fangano orbit -the shortest orbit connecting the edges of a triangle and the the triangle's circumference by $\frac{1}{2}$ - a ratio that is only achieved in the equilateral Euclidean triangle and is naturally the solution to the Euclidean version of the same problem [1]. In fact we show that the shortest orbit connecting all three edges coincides with the shortest inner billiard and with the more constructive inner triangle connecting the intersections of the altitudes and the edges.

Since the domain of our problem is acute angled hyperbolic triangles we limit ourselves to the three dimensional closed* polytop:

$$\mathcal{P} = \left\{ \alpha, \beta, \gamma | \alpha + \beta + \gamma <= \pi, 0 <= \alpha, \beta, \gamma <= \tfrac{\pi}{2} \right\}$$

We use section 1 to prove the three closed orbits indeed coincide for every triangle in our domain of interest, section 2 to express the length of the orbit and the circumference as a function of the angles, section 3 to prove the ratio between the two is bounded by $\frac{1}{2}$ on the interior of the polytop and section 4 to show the same holds on the boundary.

*We allow ourselves to consider the closed polytop even though formally $\alpha+\beta+\gamma < \pi$ for triangle on the hyperboloc plane and equality is never achived. We do so since all of our expressions can be shown to be continuous with respect to the Euclidean case in the sense that their values tend to the natural Euclidean version of the expression as the sum of the angles approaches π. We formulate and prove the above in appendix A.

2 Notation

All through the text, points in the hyperbolic plane are marked with Latin capitals. When using a letter, say A in a specific model it may refer either to the point in the hyperbolc plane or to the point representing it in the model, which is used can be deduced by context.

For two different points A and B, AB marks the segment connecting point A and point B, or the model specific representation of the segment - which need

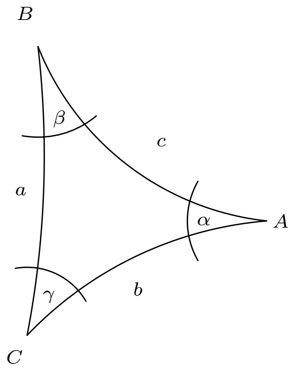

Figure 1: Notation for vertices, adges and angles

not be an Euclidean segment. It is important to note that in Euclidean embeded models the Euclidean segment and hyperbolic segment do not necessarily coincide, in fact they usually do not, in those cases AC refers to the hyperbolic segment and not the Euclidean segment.

In some cases AB may refer to the geodesic on which segment AB is part of. In those cases it is specifically noted by refering to it as 'the geodesic AB'.

Triangle vertices are marked with the capital Latins A, B and C. Hyperbolic lengths of edges are marked a, b and c, each refering to the length of the edge opposite to the matching vertex, a for example refers to $d\left(B,C\right)$ Angles α, β and γ refer to $\alpha = \angle BAC$, $\alpha = \angle CBA$ and $\alpha = \angle ACB$ respectfully (see Figure 1).

The terms Geodesics, Hyperbolic distances, segment lenghs angles and triangles are all defined and explained in section 3.

3 Hyperbolic Geometry

3.1 Short introduction to Hyperbolic Geometry

3.2 Hyperbolic Trigonometry

Theorem 3.1. *First Hyperbolic law of cosines*

Theorem 3.2. *First Hyperbolic law of cosines*

Definition 3.1.

Theorem 3.3. *The heights of an acute angled hyperbolic triangle are concurent*

Theorem 3.4. *In an acute angled triangle ABC, the height leaving A crosses BC.*

3.3 The half plane model

Note: in the following text, half plane points are sometimes parameterized by the value of their x coordinate and their suqare norm, for example:

$$P = \left(x, \sqrt{s - x^2}\right)$$

In which cases we always implicitly assume x is in $\mathbb{R}$ and s, the square norm, is in $\mathbb{R}_+$.

Theorem 3.5. *Let*

$$P_i = \left(x_i, \sqrt{s_i - x_i^2}\right), i \in \{0, 1, 2\}$$

Be three distinct points on the same geodesic, then:

$$(x_2 - x_1)(s_3 - s_2) = (x_3 - x_2)(s_2 - s_1)$$

Proof. ... □

Theorem 3.6. *Let*

$$P_i = \left(x_i, \sqrt{s_i - x_i^2}\right), i \in \{1, 2, 3, 4\}$$

Be four distinct points such that the geodesics P_1P_2 and P_3P_4 intersect the y-axis at points $(0, \sqrt{u})$ and $(0, \sqrt{v})$ (need not be unique, geodesics may coincide with the y-axis) then the two geodescis intersect each other and are perpendicular if and only if

$$2(x_1 - x_2)(x_3 - x_4)(u + v) + (s_1 - s_2)(s_3 - s_4) = 0$$

Proof. ... □

Note: A condition for the fact that the two geodescis meet (not necesseraily being perpendiular) is not as crisp and is not needed in the text and for that reason not detailed.

4 Closed orbits of acute angled hyperbolic triangles

We start by showing a Hyperbolic version of the three relatively trivial claims:

(I) There is a unique inner billiard connecting all three edges.

(II) The shortest inner billiard (Fangano orbit) is the above unique billiard.

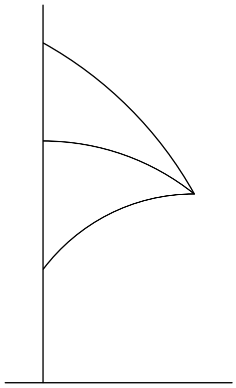

Figure 2: Triangle modeled on the Poincare Upper half plane.

(III) The shortest closed inner orbit passing throgh all edges is a billiard

And will continue on to show a less trivial fact:

(IV) The orthic triangle (The triangle connecting the base points of the three altitudes is an inner billiard.

Proof. Let D, E, F be the unique points on BC, CA, AB such that AD is perpendicular to BC, BE is perpendicular to CA and FC is perpendicular to AB (Exist and unique due to 3.2).

A to BC, E be the projection of point B to AC, F be the projection of point C to AB. Let Q be the reflection of D with respect to AB. Let J be

Let G be the intersection of the y axis and EQ.

Claim: G coincides with F.

Proof of this claim prooves $\angle EDA = \angle FDB$, and by symmetry prooves the mirror property of all vertices of the orbit.

Proof of Claim:

We fomulate the proof in the upper half plane model, relying of the well known fact that the three altitudes of a triangle meet at one point. In an effort to simplify our calculations, we parameterize our points by their x coordinate and their squared distance from the origin:

$$\begin{aligned}
A &= & (0, \sqrt{a}) \; B &= & \left(0, \sqrt{b}\right) \\
C &= & \left(c, \sqrt{r - c^2}\right) D &= & \left(x, \sqrt{s - x^2}\right) \\
Q &= & \left(-x, \sqrt{s - x^2}\right) E &= & \left(y, \sqrt{t - y^2}\right) \\
F &= & \left(0, \sqrt{f}\right) G &= & (0, \sqrt{g})
\end{aligned}$$

Let us show that $f = g$.

We will rely on two half plane identities:

Let us now proove $f = g$. We will start by expressing x and s by a,b,c and r, we know that:

$$b = \frac{cs - xr}{c - x}$$
$$\frac{s-a}{2x}\frac{r-b}{2c} = -\frac{a+b}{2}$$

Solving for x, s:

$$x = \frac{c\left(r-a\right)\left(b-a\right)}{\left(r-a\right)^2 + 2c^2\left(a+b\right)}$$
$$s = \frac{b\left(r-a\right)^2 + 2c^2\left(a+b\right)a}{\left(r-a\right)^2 + 2c^2\left(a+b\right)}$$

And by symmetry:

$$y = \frac{c\left(r-b\right)\left(a-b\right)}{\left(r-b\right)^2 + 2c^2\left(b+a\right)}$$
$$t = \frac{a\left(r-b\right)^2 + 2c^2\left(b+a\right)b}{\left(r-b\right)^2 + 2c^2\left(b+a\right)}$$

Now Let us find F as the intersection of QE and the y axis:

$$\begin{aligned}
g &= \frac{ys-(-x)t}{y-(-x)} = \frac{ys+xt}{y+x} \\
&= \frac{\frac{c(r-b)(a-b)}{(r-b)^2+2c^2(b+a)}\frac{b(r-a)^2+2c^2(a+b)a}{(r-a)^2+2c^2(a+b)} + \frac{c(r-a)(b-a)}{(r-a)^2+2c^2(a+b)}\frac{a(r-b)^2+2c^2(b+a)b}{(r-b)^2+2c^2(b+a)}}{\frac{c(r-b)(a-b)}{(r-b)^2+2c^2(b+a)} + \frac{c(r-a)(b-a)}{(r-a)^2+2c^2(a+b)}} \\
&= \frac{c\left(r-b\right)\left(a-b\right)\left(b\left(r-a\right)^2 + 2c^2\left(a+b\right)a\right) + c\left(r-a\right)\left(b-a\right)\left(a\left(r-b\right)^2 + 2c^2\left(b+a\right)b\right)}{c\left(r-b\right)\left(a-b\right)\left(\left(r-a\right)^2 + 2c^2\left(a+b\right)\right) + c\left(r-a\right)\left(b-a\right)\left(\left(r-b\right)^2 + 2c^2\left(b+a\right)\right)} \\
&= \frac{\left(r-b\right)\left(b\left(r-a\right)^2 + 2c^2\left(a+b\right)a\right) - \left(r-a\right)\left(a\left(r-b\right)^2 + 2c^2\left(b+a\right)b\right)}{\left(r-b\right)\left(\left(r-a\right)^2 + 2c^2\left(a+b\right)\right) - \left(r-a\right)\left(\left(r-b\right)^2 + 2c^2\left(b+a\right)\right)} \\
&= \frac{2c^2\left(a+b\right)\left(a\left(r-b\right) - b\left(r-a\right)\right) + \left(r-a\right)\left(r-b\right)\left(b\left(r-a\right) - a\left(r-b\right)\right)}{2c^2\left(a+b\right)\left(r-b-r+a\right) + \left(r-a\right)\left(r-b\right)\left(\left(r-a\right) - \left(r-b\right)\right)} \\
&= \frac{2c^2\left(a+b\right)r\left(a-b\right) + \left(r-a\right)\left(r-b\right)r\left(b-a\right)}{2c^2\left(a+b\right)\left(a-b\right) + \left(r-a\right)\left(r-b\right)\left(b-a\right)} \\
&= r
\end{aligned}$$

$\square$

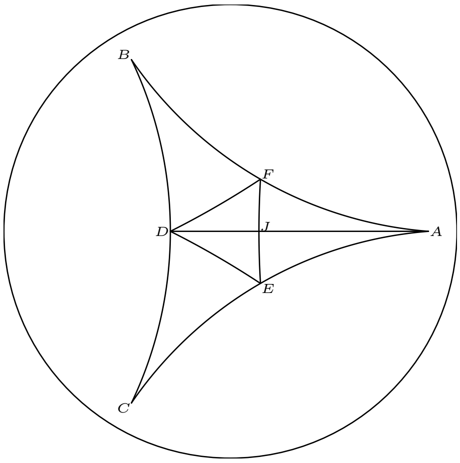

Figure 3: Equilaterlal Triangle modeled on the Poincare Disc.

5 Equilateral Triangles

The family of hyperbolic equilateral triangles is a single parameter family. The parameter can be the edge length or the angle size, we will choose to parameterize the family by the angle, α, that can be any where in the half open interval $\left[0, \frac{\pi}{3}\right)$.

Claim: The ratio between the Fangano orbit length and the perimiter of the triangle is monotone with respect to α.

Corollary: The ratio between the Fangano orbit length and the perimiter is smaller than 2.

We will show the two following identities:

$$\cosh \frac{f}{3} = \cos \alpha + \frac{1}{2} \tag{1}$$

$$\cosh \frac{p}{3} = \frac{\cos \alpha}{1 - \cos \alpha} \tag{2}$$

$$\tag{3}$$

Where f is the Fangano orbit length and p is the perimiter.

Proof:

$$
\begin{aligned}
&\sinh\frac{p}{6} = \sinh BD = \sinh AB \sin\frac{\alpha}{2} = \sinh\frac{p}{3}\sin\frac{\alpha}{2}\\
&\qquad\Downarrow\\
&\sinh\frac{p}{6} = \sinh 2\frac{p}{6}\sin\frac{\alpha}{2}\\
&\qquad\Downarrow\\
&\sinh\frac{p}{6} = 2\sinh\frac{p}{6}\cosh\frac{p}{6}\sin\frac{\alpha}{2}\\
&\qquad\Downarrow\\
&\cosh\frac{p}{6} = \frac{1}{2\sin\frac{\alpha}{2}}\\
&\qquad\Downarrow\\
&\cosh\frac{p}{3} = \frac{2}{4\sin^2\frac{\alpha}{2}} - 1 = \frac{1}{2\sin^2\frac{\alpha}{2}} - 1\\
&\qquad = \frac{1}{1-\cos\alpha} - 1 = \frac{\cos\alpha}{1-\cos\alpha}
\end{aligned}
$$

Which proves (2)

$$
\begin{aligned}
\cosh\frac{f}{3} &= \cosh 2\frac{f}{6}\\
&= 2\sinh^2\frac{f}{6} + 1 = 2\sinh^2 FJ + 1\\
&= 2\sinh^2 FA \sin^2\frac{\alpha}{2} + 1 = (\cosh 2FA - 1)\,\frac{1-\cos\alpha}{2} + 1\\
&= \left(\cosh\frac{p}{3} - 1\right)\frac{1-\cos\alpha}{2} + 1 = \left(\frac{\cos\alpha}{1-\cos\alpha} - 1\right)\frac{1-\cos\alpha}{2} + 1\\
&= \frac{1}{2}\left(\cos\alpha - 1 + \cos\alpha\right) + 1 = \cos\alpha + \frac{1}{2}
\end{aligned}
$$

Which proves (1).

For $\alpha = 0$, the perimiter is infinite while the orbit is finite, so $\frac{f}{a}$ is zero. Inorder to show that the ratio $\frac{f}{a}$ is monotone with respect to α it is enough to show that:

$$\frac{f_\alpha}{a_\alpha} > \frac{f}{a}$$

Where f_α and a_α are the deriviatives of f and a with respect to α.

Let $t = \cos\alpha$ and $s = \sin\alpha$:

$$\frac{f_\alpha}{a_\alpha} = \frac{\frac{s}{\sqrt{\left(t+\frac{1}{2}\right)^2-1}}}{\frac{s}{\sqrt{\left(\frac{t}{1-t}\right)^2-1}}} = \sqrt{\frac{\left(\frac{t}{1-t}\right)^2-1}{\left(t+\frac{1}{2}\right)^2-1}} = \sqrt{\frac{\left(\frac{1}{1-t}\right)\left(\frac{2t-1}{1-t}\right)}{\left(t-\frac{1}{2}\right)\left(t+\frac{3}{2}\right)}}$$
$$= \sqrt{\frac{2\left(t-\frac{1}{2}\right)}{(1-t)^2\left(t-\frac{1}{2}\right)\left(t+\frac{3}{2}\right)}} = \sqrt{\frac{2}{(1-t)^2\left(t+\frac{3}{2}\right)}}$$

6 The general case - Expressing lengths as functions of angles

In this section we will show the two following identities:

$$\sinh\frac{f}{2} = \Phi \tag{4}$$
$$\sinh\frac{s}{2} = \frac{\Phi}{4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}} \tag{5}$$
$$\tanh r = \frac{\Phi}{4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}} \tag{6}$$

Where f is the length of the Fangano orbit, s the perimiter, r the radius of the inscribed (hyperbolic) circle and:

$$\Phi^2 = 2\cos\alpha\cos\beta\cos\gamma + \cos^2\alpha + \cos^2\beta + \cos^2\gamma - 1$$

6.1 Proof

We consider the acute angled hyperbolic triangle $\triangle ABC$. Let a, b and c be the lengths of BC, CA, AB respectfully and α, β and γ the angles $\angle BAC$, $\angle ABC$, $\angle BCA$. Let D be the projection of point A to BC, and d be the length of AD. Let P be the reflection of D with respect to AC and Q the reflection of D with respect to AB. (The construction is demonstrated in Figure 4).

First, since DEF is an inner billiard, and since P and Q are both reflections of D, it easy to see that P, F, E and Q are on a common geodesic and that $DF = PF$ and $ED = EQ$. So:

$$\begin{aligned} 2f &= \overline{DF} + \overline{FE} + \overline{ED} \\ &= \overline{PF} + \overline{FE} + \overline{EQ} \\ &= \overline{PQ} \end{aligned}$$

Second, from the same exact properties, $\angle PAF = \angle DAF$ and $\angle QAE = \angle DAE$, and thus:

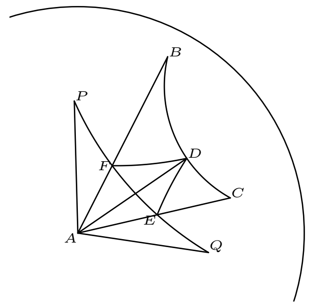

Figure 4: Triangle modeled on the Poincare Disc.

$$
\begin{aligned}
\angle PAQ &= \angle PAD + \angle DAQ = 2\angle BAD + 2\angle DAC \\
&= 2\left(\angle BAD + \angle DAC\right) = 2\angle BAC \\
&= 2\alpha
\end{aligned}
$$

Using the hyperbolic law of cosines for $\triangle PAQ$ and right angle identity for $\triangle ADC$ we get the following identities (respectfully):

$$
\begin{aligned}
\cosh 2f &= \cosh^2 d - \sinh^2 d \cos 2\alpha && (7)\\
\sinh d &= \sinh c \sin\beta && (8)
\end{aligned}
$$

Develope (7):

$$
\begin{aligned}
\cosh 2f &= 1 + \sinh^2 d - \sinh^2 d\left(\cos^2\alpha - \sin^2\alpha\right) \\
&= 1 + \sinh^2 d\left(1 - \cos^2\alpha + \sin^2\alpha\right) \\
&= 1 + 2\sinh^2 d \sin^2\alpha
\end{aligned}
$$

Substituting for $\cosh 2f$ and $\sinh^2 d$ (using (8)), we get:

$$
\begin{aligned}
1 + 2\sinh^2 f &= 1 + 2\sinh^2 c \sin^2\beta \sin^2\alpha \\
&= 1 + 2\left(\frac{\cos\alpha\cos\beta + \cos\gamma}{\sin\alpha\sin\beta}^2 - 1\right)\sin^2\beta\sin^2\alpha \\
&= 1 + 2\left(\cos\alpha\cos\beta\cos\gamma + \cos^2\gamma + \cos^2\alpha\cos^2\beta - \sin^2\beta\sin^2\alpha\right) \\
&= 1 + 2\left(\cos\alpha\cos\beta\cos\gamma + \cos^2\gamma + \cos^2\alpha\cos^2\beta - \left(1 - \cos^2\alpha\right)\left(1 - \cos^2\beta\right)\right) \\
&= 1 + 2\left(2\cos\alpha\cos\beta\cos\gamma + \cos^2\gamma + \cos^2\beta + \cos^2\alpha + \cos^2\beta - 1\right) \\
&= 1 + 2\Phi^2
\end{aligned}
$$

Which proves (4):

$$\sinh f = \Phi$$

...

7 Bounding ratios between invariants on the interior

In this section we will show that $\frac{f}{s}$ has no critical points on the interior of $\mathbf{P}$. We will do so by diving $\mathbf{P}$ to level sets of the perimiter function s, and showing that for a given value of s, the the above ratio achieves its maximum on the boundary of $\mathbf{P}$ or on the set $\alpha = \beta = \gamma$. We will then show that on that set the ratio is monotone with resppect to, say, α.

7.1 Proof

Fixing s, by (5), means fixing

$$\frac{\Phi}{4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}$$

Thus fixing its square:

$$\frac{\Phi^2}{16\cos^2\frac{\alpha}{2}\cos^2\frac{\beta}{2}\cos^2\frac{\gamma}{2}} = \frac{\Phi^2}{2\left(1-\cos\alpha\right)\left(1-\cos\beta\right)\left(1-\cos\gamma\right)}$$

We shall mark the above ratio by ρ.

By (4), $\frac{f}{s}$ achieves its maximum (given a fixed s), when Φ achives its maximum and thus when Φ^2 achives its maximum. Our maximizing problem is thus reduced to maximizing Φ^2, given a fixed value for $\frac{\Phi^2}{\rho}$. Using Lagrange multipliers:

$$\triangle\Phi^2 = \lambda \triangle \frac{\Phi^2}{\rho}$$

$$\Downarrow$$

$$\frac{1}{\lambda}\triangle\left(\Phi^2\right) = \frac{\triangle\left(\Phi^2\right)\rho - \triangle\left(\rho\right)\Phi^2}{\rho^2}$$

$$\Downarrow$$

$$\left(\frac{1}{\lambda} - \frac{1}{\rho}\right)\triangle\left(\Phi^2\right) = -\frac{\Phi^2}{\rho^2}\triangle\left(\rho\right)$$

In other words, $\triangle\left(\Phi^2\right)$ and $\triangle(\rho)$ are linearly dependant, meaning the ratios between the directional deriviatives are respectfully equal:

$$\frac{\Phi^2_\alpha}{\rho_\alpha}=\frac{\Phi^2_\beta}{\rho_\beta}=\frac{\Phi^2_\gamma}{\rho_\gamma}$$
$$\Downarrow$$
$$\frac{-2\left(\cos\beta\cos\gamma+\cos\alpha\right)\sin\alpha}{2\left(1-\cos\beta\right)\left(1-\cos\gamma\right)\sin\alpha}=\frac{-2\left(\cos\gamma\cos\alpha+\cos\beta\right)\sin\beta}{2\left(1-\cos\gamma\right)\left(1-\cos\alpha\right)\sin\beta}=\frac{-2\left(\cos\alpha\cos\beta+\cos\gamma\right)\sin\gamma}{2\left(1-\cos\alpha\right)\left(1-\cos\beta\right)\sin\gamma}$$

And Specifically:

$$\frac{\left(\cos\beta\cos\gamma+\cos\alpha\right)}{\left(1-\cos\beta\right)\left(1-\cos\gamma\right)}=\frac{\left(\cos\gamma\cos\alpha+\cos\beta\right)}{\left(1-\cos\gamma\right)\left(1-\cos\alpha\right)}$$
$$\Downarrow$$
$$\frac{\left(\cos\beta\cos\gamma+\cos\alpha\right)}{\left(1-\cos\beta\right)}=\frac{\left(\cos\gamma\cos\alpha+\cos\beta\right)}{\left(1-\cos\alpha\right)}$$
$$\Downarrow$$
$$\cos\beta\cos\gamma+\cos\alpha-\cos\alpha\cos\beta\cos\gamma-\cos^2\alpha=\cos\alpha\cos\gamma+\cos\beta-\cos\alpha\cos\beta\cos\gamma-\cos^2\beta$$
$$\Downarrow$$
$$\cos\beta\cos\gamma+\cos\alpha-\cos^2\alpha=\cos\alpha\cos\gamma+\cos\beta-\cos^2\beta$$
$$\Downarrow$$
$$\cos\gamma\left(\cos\beta-\cos\alpha\right)-\left(\cos^2\alpha-\cos^2\beta\right)+\cos\beta-\cos\alpha=0$$
$$\Downarrow$$
$$\left(\cos\beta-\cos\alpha\right)\left(\cos\gamma-\cos\alpha-\cos\beta+1\right)=0$$

Similraly we derive two more equations and arrive with:

$$\left(\cos\beta-\cos\alpha\right)\left(\cos\gamma-\cos\alpha-\cos\beta+1\right)=0$$
$$\left(\cos\gamma-\cos\beta\right)\left(\cos\alpha-\cos\beta-\cos\gamma+1\right)=0$$
$$\left(\cos\alpha-\cos\gamma\right)\left(\cos\beta-\cos\gamma-\cos\alpha+1\right)=0$$

Which has only one internal solution $\alpha=\beta=\gamma$.

Explanation:

If, say, $\alpha=\beta$ Then by the second equation (and the fact the we looking for internal solutions), γ must be equal to β, so a solution other than $\alpha=\beta=\gamma$ must include three distinct angles, meaning:

$$\left(\cos\gamma-\cos\alpha-\cos\beta+1\right)=0$$
$$\left(\cos\alpha-\cos\beta-\cos\gamma+1\right)=0$$
$$\left(\cos\beta-\cos\gamma-\cos\alpha+1\right)=0$$

Of which the only solution is $\cos\alpha=\cos\beta=\cos\gamma=1$. We conclude that any internal critical point must lay on the line $\alpha=\beta=\gamma$.

Let us now look at the single variable function:

8 Bounding the ratio on the boundary

Up to symmetry, the boundary is assembled by sets of one of the types: $\{\alpha = 0\}$, $\left\{\alpha = \frac{\pi}{2}\right\}$, and $\{\alpha + \beta + \gamma = 0\}$. The first is trivial since on it the circumference is infinite and the orbit is finite (since it connects three internal points), the last reduces to the known Euclidean case, which leaves us with the case that one of the angles is a right angle (since we covered the case where one angle is zero we can assume that only one angle is a right angle).

** TODO: complete.

References

[1] Alan F. Beardon The Geometry of Discrete Groups With 93 Illustrations