1 Exercise 1: Stability

1.1 Question 1.1

We have $x(t) = x_0$ and $\Theta(t) = \Theta^*$.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\omega &= \eta * x_0 * (y^2 - y\Theta) \\ \frac{\mathrm{d}}{\mathrm{d}x}\omega &= \eta * x_0 * (\omega^2 * x_0^2 - \omega * x_0 * \Theta^*) \\ \frac{\mathrm{d}}{\mathrm{d}x}\omega &= \eta * x_0^2 * \omega * (\omega * x_0 - \Theta^*)\end{aligned}$$

Fix point when $\frac{\mathrm{d}}{\mathrm{d}x}\omega = 0$

$$\begin{aligned}0 &= \eta * x_0^2 * \omega * (\omega * x_0 - \Theta^*) \\ \Theta^* &= \omega * x_0 = y\end{aligned}$$

From this we can see that the ω is not bounded and diverges.

$$\begin{aligned}\Theta(t) &= \frac{1}{\tau} * \int_{-\infty}^{t} y^p(s)\exp(-\frac{t-s}{\tau})\mathrm{d}s \\ \dot{\omega} &= \eta * x * (y^2 - y * \Theta) \\ \tau * \dot{\Theta}_M &= -\Theta_M + y^p\end{aligned}$$

Fix points if $\frac{\mathrm{d}\omega}{\mathrm{d}t} = 0$ and $\frac{\mathrm{d}\Theta_M}{\mathrm{d}t} = 0$

$$\begin{aligned}\Theta_M &= y^p - \tau * \frac{\mathrm{d}\Theta_M}{\mathrm{d}t} \\ 0 &= \eta * x_0 * (y^2 - y * y^p) \\ 0 &= \eta * x_0 * y^2 * (1 - y^{p-1}) \\ 0 &= \eta * x_0 * y^2 * (1 - (\omega * x_0)^{p-1}) \\ \frac{1}{x_0^{p-1}} &= \omega^{p-1}\end{aligned}$$

And we see that for $p > 1$, ω converges to $\frac{1}{x_0}$, whereas for $p = 1$, ω converges to 0.

1.2 Question 1.2

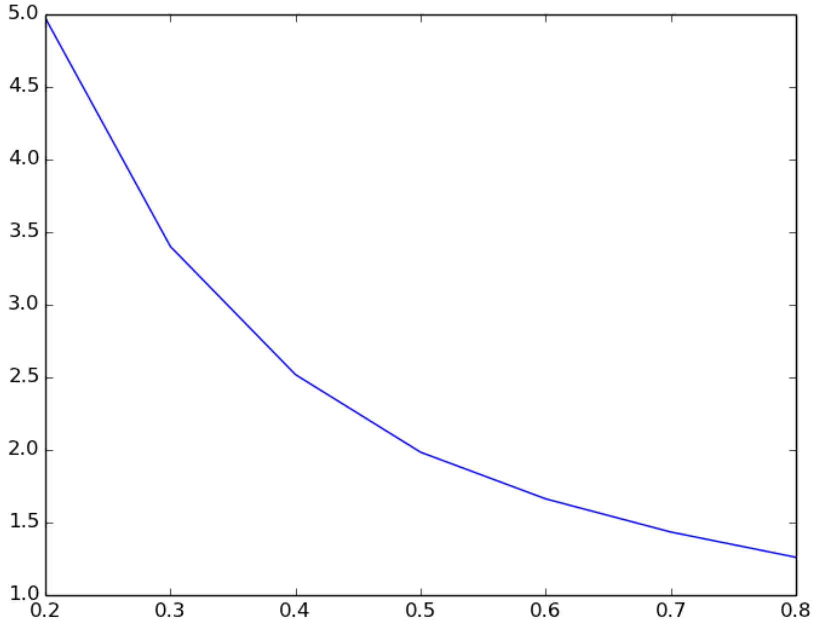

Figure 1: After convergence, the weight is adjusted such that applying x_0, $y = \omega$ and we observe that $\omega = \frac{1}{z}$, whereas z is the probability of $x_0 = 1$

1.3 Question 1.3

We have a linear neuron $y = \omega * x$

1.3.1 Show that $\Theta = \langle y^2 \rangle_x = z * y_0^2$

$$\Theta(t) = \frac{1}{\tau} * \int_{-\infty}^{t} y^p(s) \exp(-\frac{t-s}{\tau}) \mathrm{d}s$$

$$\Theta(t) \approx \langle y^p \rangle_t \approx \langle y^p \rangle_x$$

$$\Theta(t) = \frac{1}{\tau} * y^2 \int_{-\infty}^{t} exp(-\frac{t-s}{\tau}) \mathrm{d}s$$

$$\Theta(t) = \frac{1}{\tau} * y^2 * [\frac{\tau}{s} * \exp(-\frac{t-s}{\tau})]_{-\infty}^{t}$$

$$\Theta(t) = \frac{1}{t} * y^2 = \langle y^2 \rangle = \langle (\omega * x)^2 \rangle$$

When applying an input x_0 with a probability of z, then the average at the output is $z * y_0$ where $y_0 = \omega * x_0$, as we can see in the last equation.

1.3.2 Show that $\langle \frac{\mathrm{d}\omega}{\mathrm{d}t} \rangle = \eta * z * x_0 * y_0^2 * (1 - z * y_0)$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \eta * x * (y^2 - y * \Theta)$$
$$\langle \frac{\mathrm{d}\omega}{\mathrm{d}t} \rangle = \langle \eta * x * (y^2 - y * \Theta) \rangle$$
$$\langle \frac{\mathrm{d}\omega}{\mathrm{d}t} \rangle = \eta * \langle x * (y^2 - y * z * y_0^2) \rangle$$
$$\langle \frac{\mathrm{d}\omega}{\mathrm{d}t} \rangle == \eta * z * x_0 * y_0^2 * (1 - z * y_0)$$

2 Question 2

2.1 Question 2.1

$$F(\omega) = \langle (\frac{y}{\sigma_y})^3 \rangle$$
$$\frac{\mathrm{d}F}{\mathrm{d}\omega}(\omega) = \langle \frac{\mathrm{dd}\omega}{(}\frac{y}{\sigma_y})^3 \rangle$$
$$= \langle 3 * (\frac{y}{\sigma_y})^2 * \frac{\frac{\mathrm{d}y}{\mathrm{d}\omega} * \sigma_y - y * \frac{\mathrm{d}}{\mathrm{d}*\omega}\sigma_y}{\sigma_y^2} \rangle$$
$$= \langle 3 * (\frac{y^2}{\langle y^2 \rangle}) * \frac{x * \sigma_y - y * \frac{1}{\sigma_y} * \langle 2 * y * x \rangle}{\sigma_y^2} \rangle$$
$$= \langle \frac{3 * y^2 * x}{\sigma_y^3} - \frac{3 * y^3 * \langle y * x \rangle}{\sigma_y^5} \rangle$$
$$= \langle \frac{3 * y^2 * x}{\sigma_y^3} \rangle - \frac{\langle y^3 \rangle}{\langle y^2 \rangle} * \frac{\langle y * x \rangle}{\sigma_y^3}$$

Θ is equal to $\frac{\langle y^3 \rangle}{\langle y^2 \rangle}$, therefore we can write:

$$= \frac{3}{\sigma_y^3} * \langle y^2 * x \rangle - \Theta * \langle x * y \rangle$$
$$= \frac{3}{\sigma_y^3} * \frac{1}{n} * \sum_i^n y^2 * x - \Theta * \frac{1}{n} * \sum_i^n x * y$$

This is the batch rule. We can transform this to an online rule:

$$= \frac{3}{\sigma_y^3} * y^2 * x - \Theta * x * y$$

From which we ca dedude that $\frac{\mathrm{d}F}{\mathrm{d}\omega} \propto xy^2 - xy\Theta$

2.2 Question 2.2

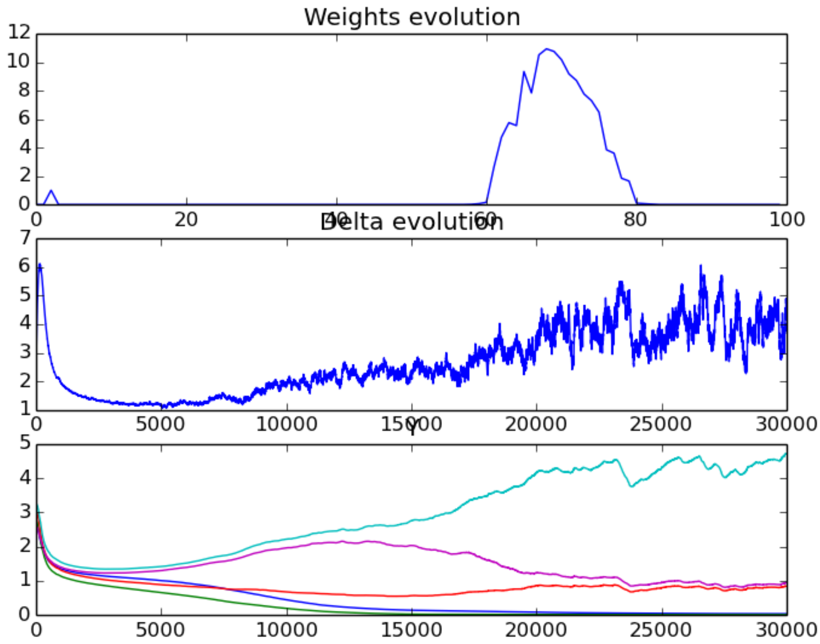

Figure 2

2.3 Question 2.3

3 Question 3

Figure ?? and ?? are two examples of a receptive field formation in a BCM neuron. In both examples, it is clearly visible to which of the pixels the neuron is specialized.
More discussion needed...

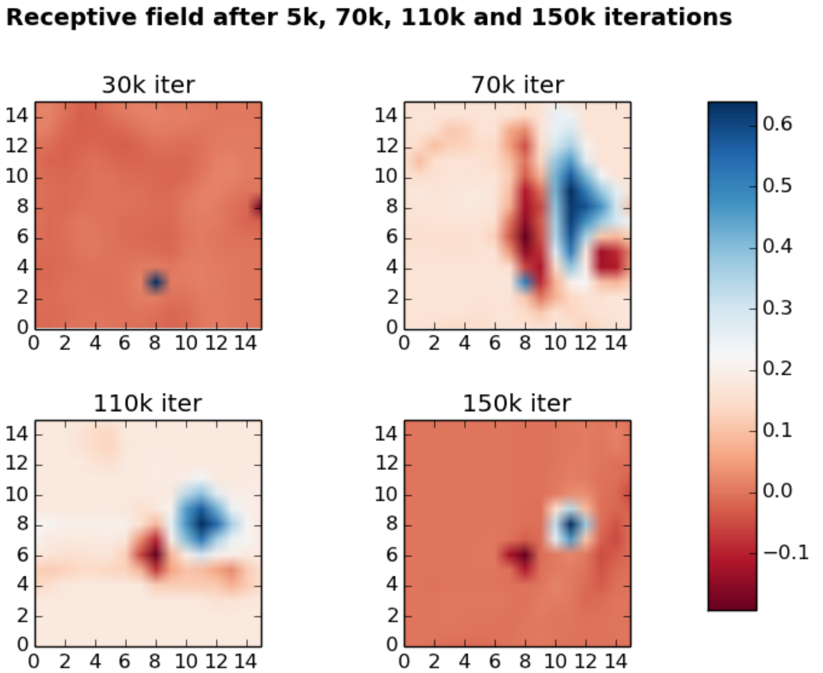

Figure 3

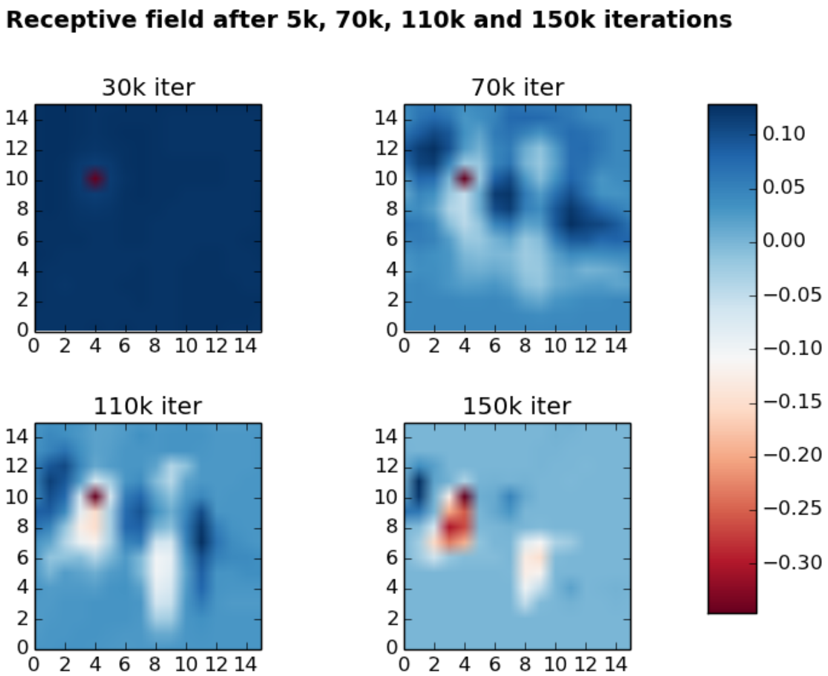

Figure 4