CSCI 420 Computer Graphics

Lecture 5

Transformations

Vector Spaces
Euclidean Spaces
Frames
Homogeneous Coordinates
Transformation Matrices
[Angel, Ch. 3]

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A Preview of an OpenGL Application

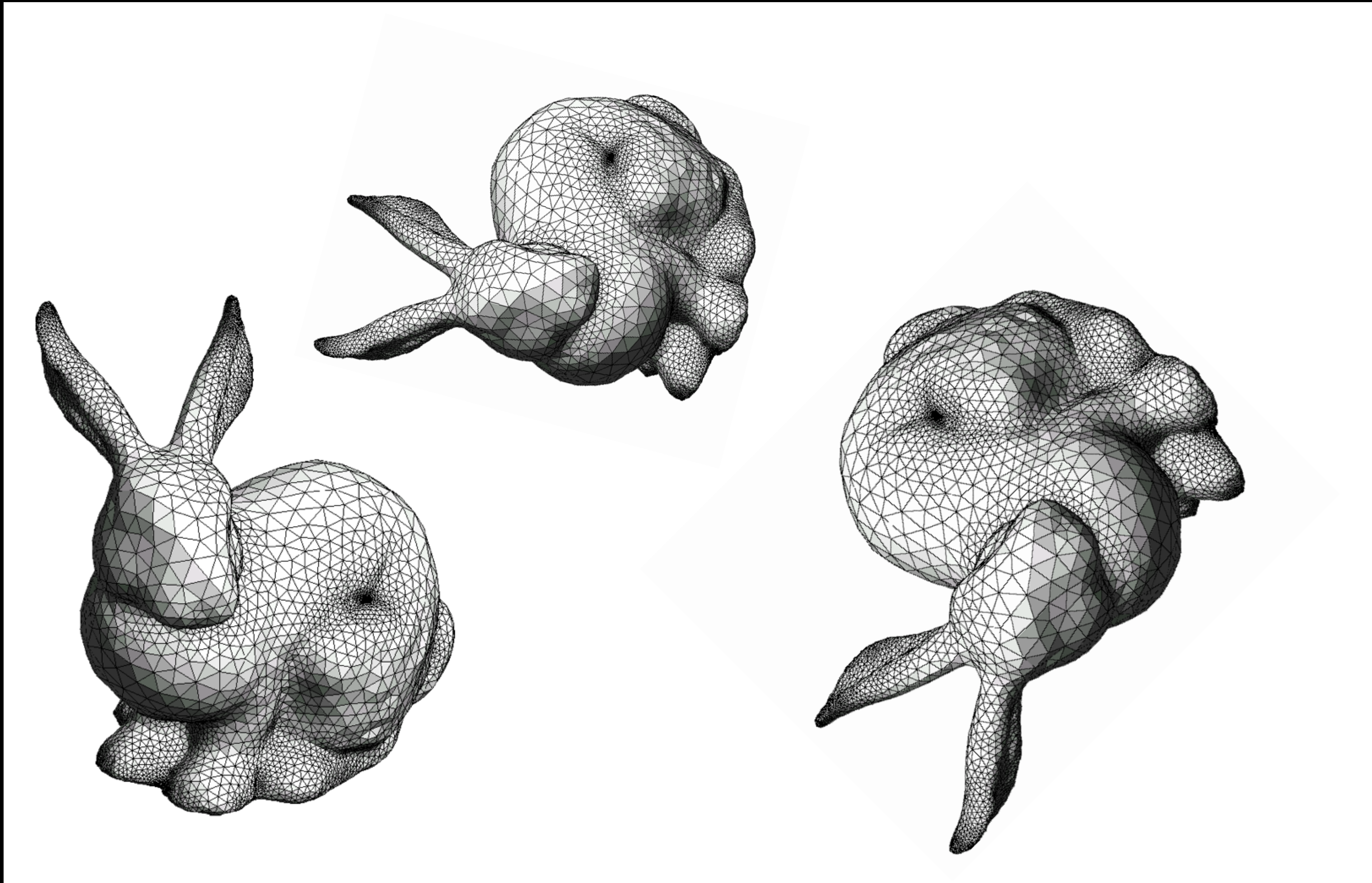
- Some parts will not make sense yet
    - In the future: shaders

- HW1 code is still being updated

- Might help you have something more concrete to look at!

- Will be uploaded to course website.

OpenGL Transformations



OpenGL Transformation Matrices

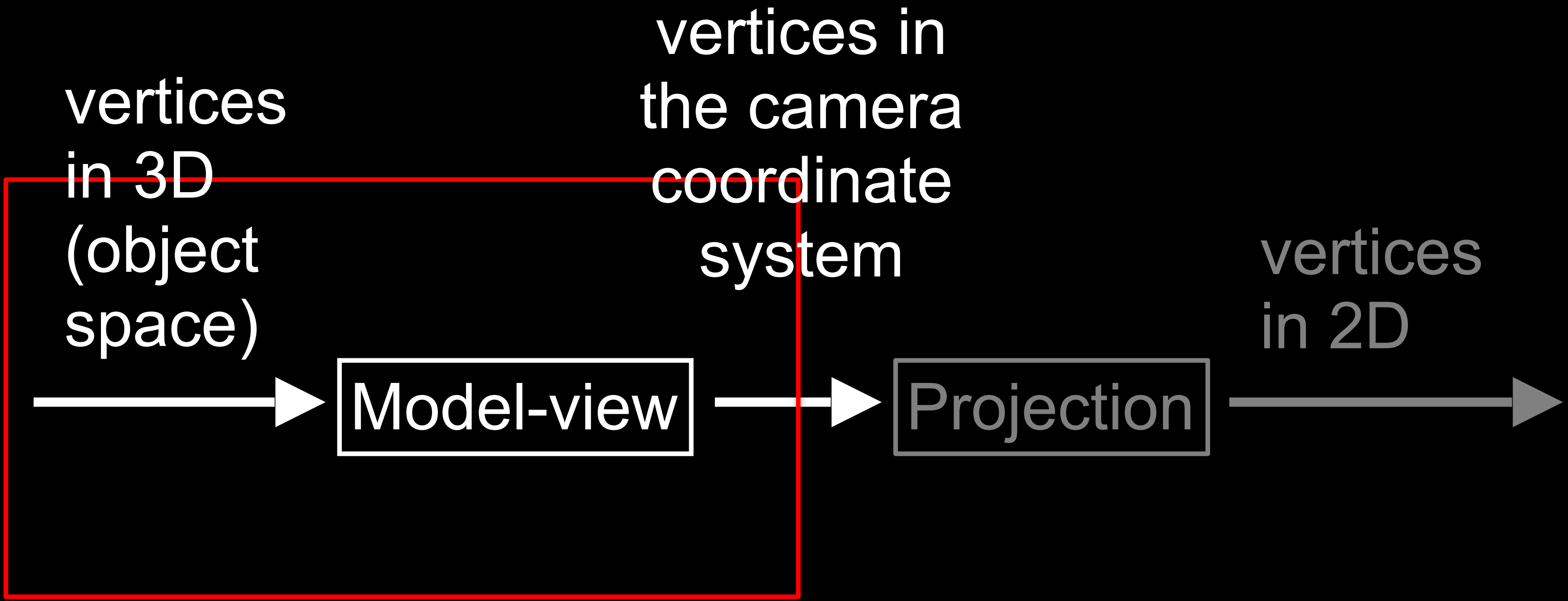
- Model-view matrix (4x4 matrix)
- Projection matrix (4x4 matrix)

vertices in 3D (object space) $\rightarrow$ $\boxed{\text{Model-view}}$ $\xrightarrow{\text{vertices in the camera coordinate system}}$ $\boxed{\text{Projection}}$ $\rightarrow$ vertices in 2D

linear maps

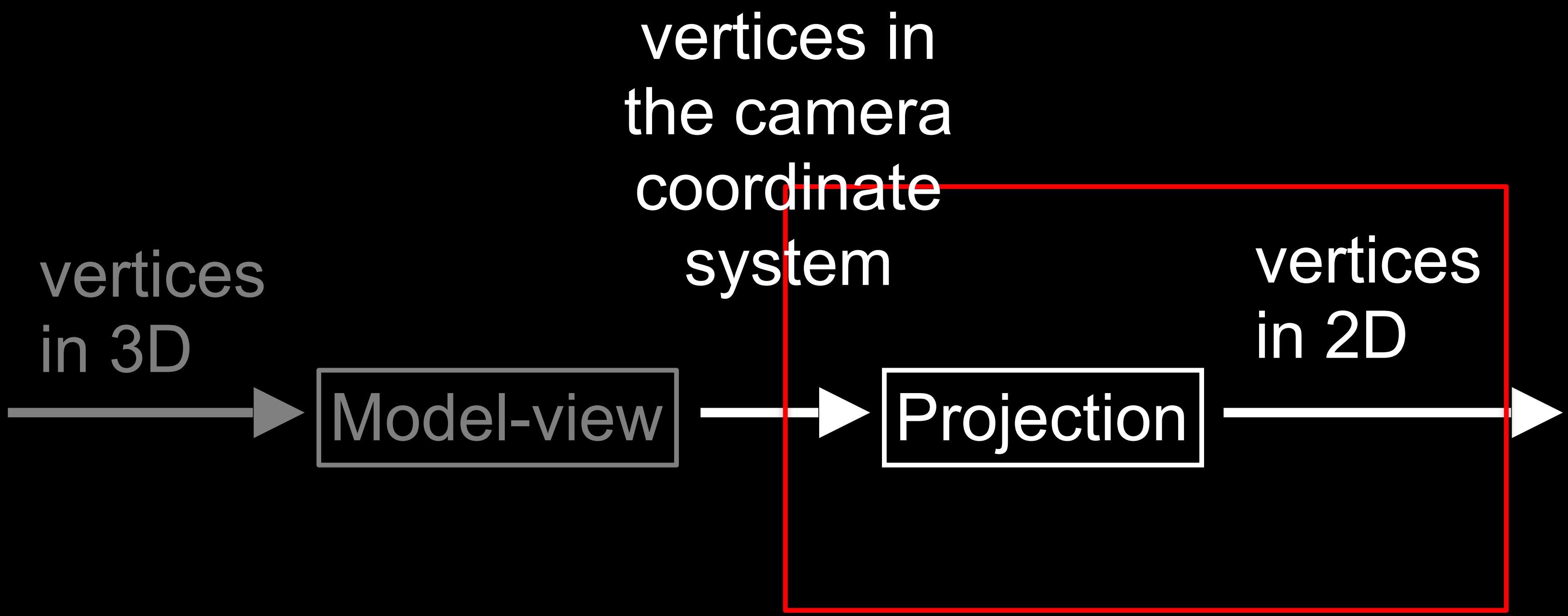
4x4 Model-view Matrix (this lecture)

- Translate, rotate, scale objects
- Position the camera



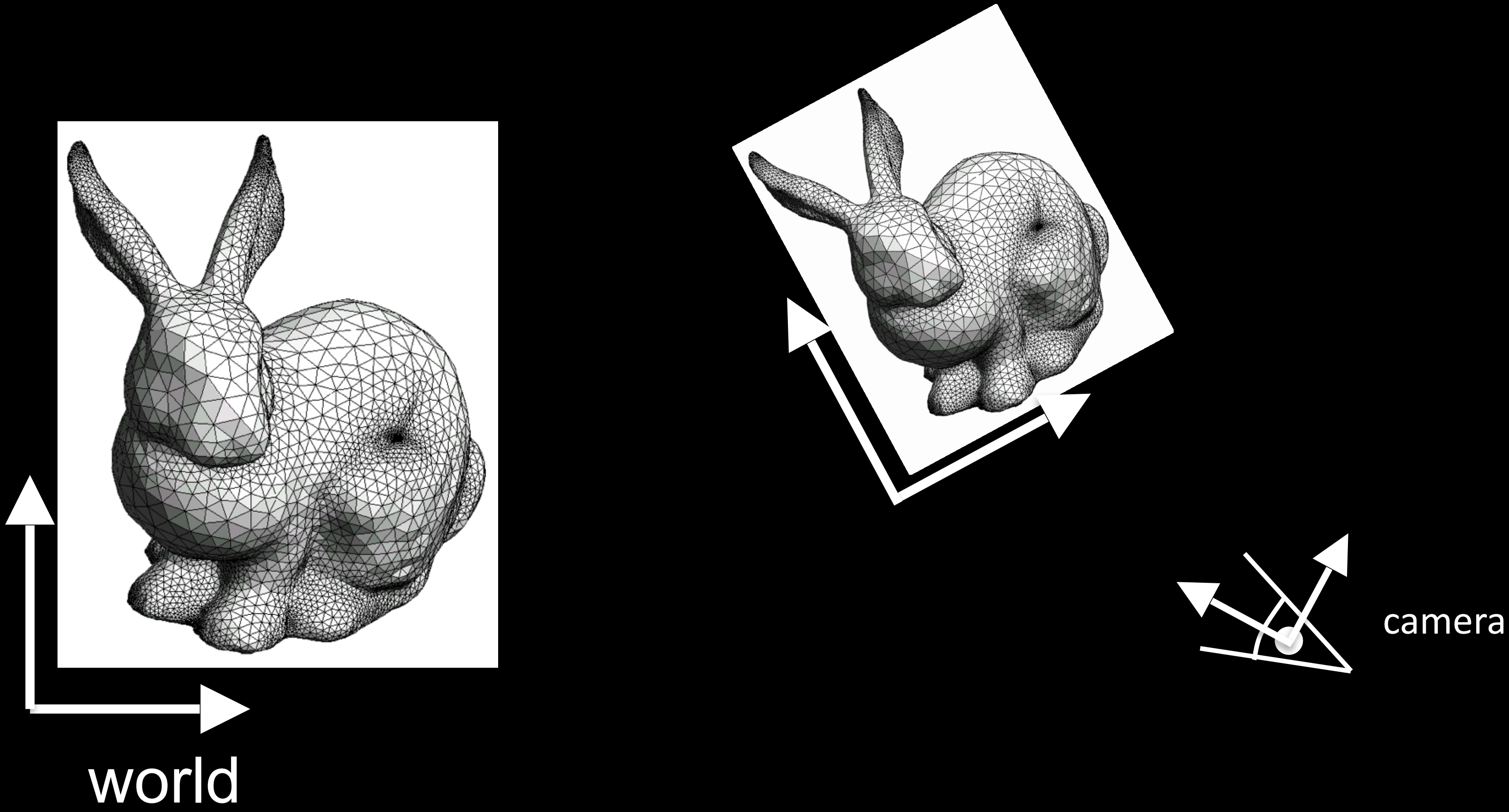
4x4 Projection Matrix (next lecture)

- Project from 3D to 2D



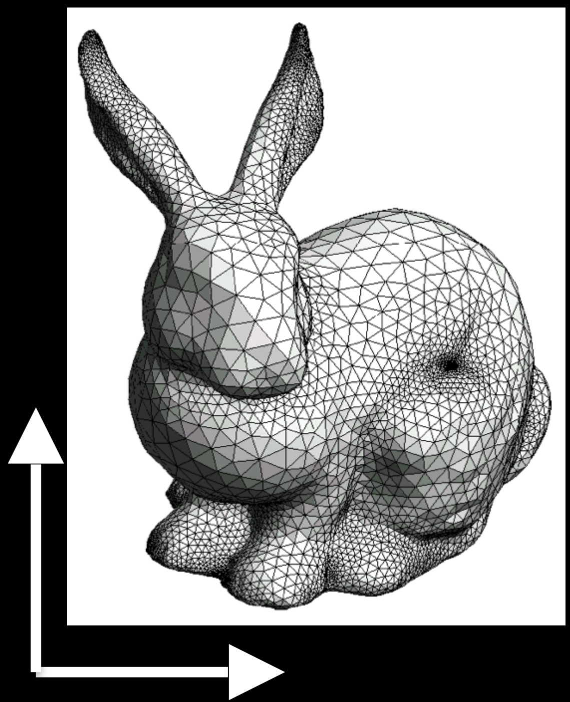
4x4 Model-view Matrix (this lecture)

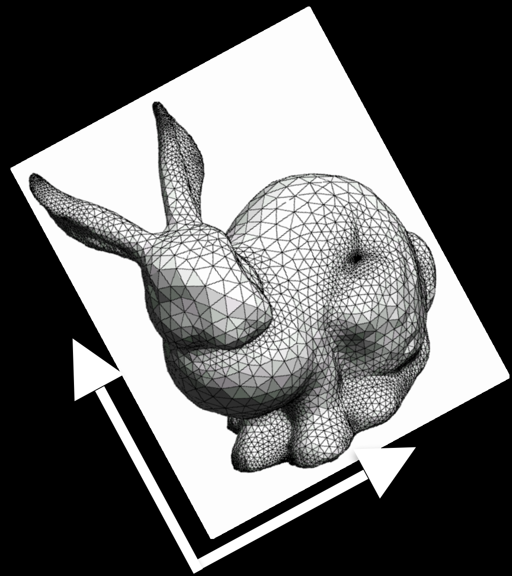
- Translate, rotate, scale objects in world space
- Position and orient the camera



4x4 Model Matrix

- Translate, rotate, scale objects in world space





world

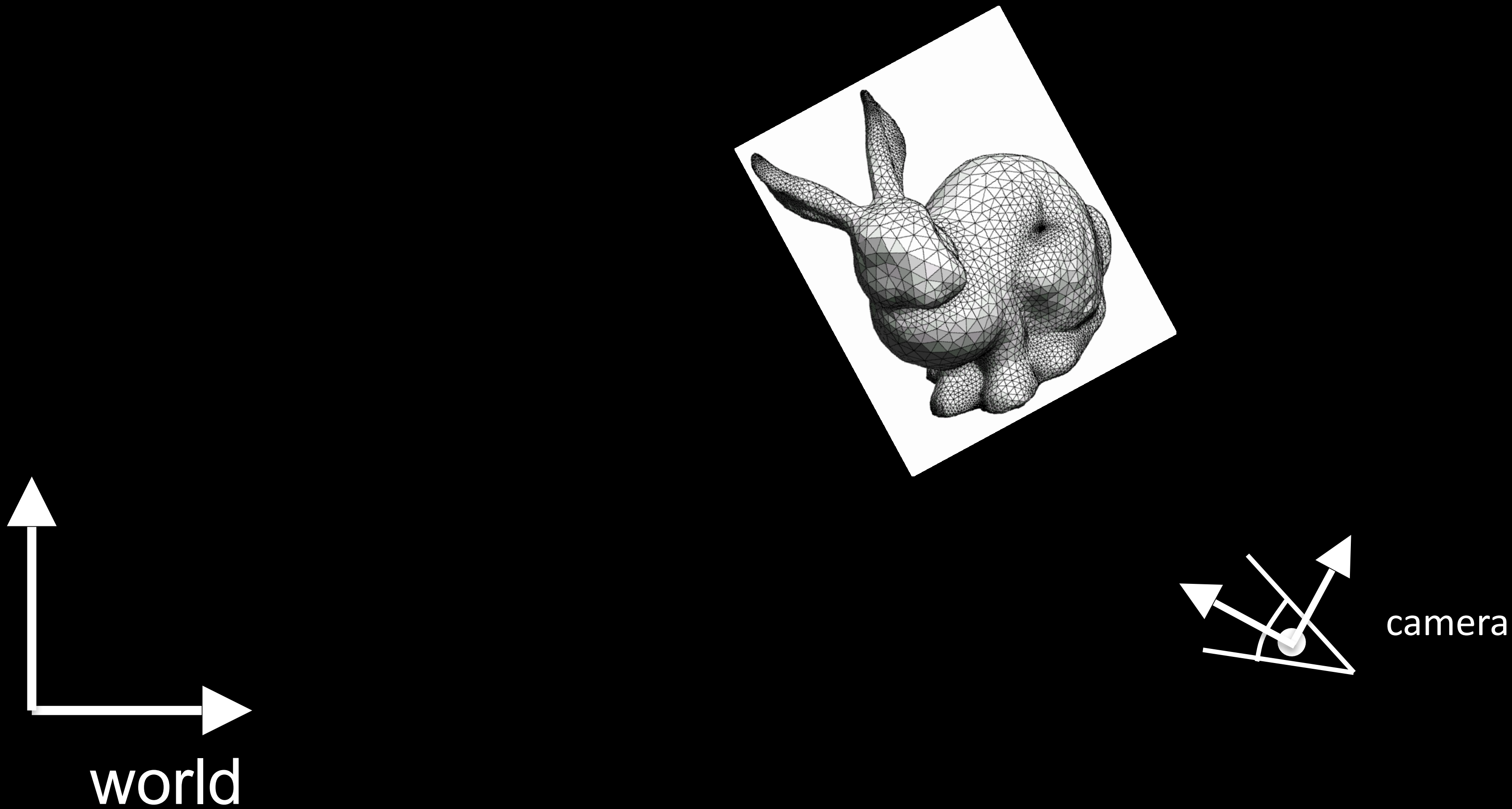
4x4 Model Matrix

- Translate, rotate, scale objects in world space

- The user rotated the object, but we don't want to reload the object.

4x4 View Matrix

- Position and orient the camera
- From world space to camera space



4x4 Model-view Matrix

• These two do more or less the same thing.

• Rigid maps in 3D that position things.

• Difference between "model" (moving around objects) and "view" (moving around the camera) is purely a convention.

• In the OpenGL model, they are often not distinguished at all.

OpenGL Transformation Matrices

→ Model-view → Projection →

- Manipulated separately in OpenGL

- Core profile: set them directly
  - You will load the matrices into memory, and apply them yourself in a shader.

- Compatibility profile: must set matrix mode

```
glMatrixMode (GL_MODELVIEW);
glMatrixMode (GL_PROJECTION);
```

Setting the Modelview Matrix: Core Profile

- Set identity:

```
openGLMatrix.SetMatrixMode(OpenGLMatrix::ModelView);
openGLMatrix.LoadIdentity();
```

- Use our openGLMatrix library functions:

```
openGLMatrix.Translate(dx, dy, dz);
openGLMatrix.Rotate(angle, vx, vy, vz);
openGLMatrix.Scale(sx, sy, sz);
```

- Upload m to the GPU:

```
float m[16]; // column-major
openGLMatrix.GetMatrix(m);
GLboolean isRowMajor = GL_FALSE;
pipelineProgram->SetUniformVariableMatrix4fv(
      "modelViewMatrix", isRowMajor, m);
```

Setting the Modelview Matrix: Core Profile

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openGLMatrix.SetMatrixMode(OpenGLMatrix::ModelView);
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float m[16]; // column-major
openGLMatrix.GetMatrix(m);
GLboolean isRowMajor = GL_FALSE;
pipelineProgram->SetUniformVariableMatrix4fv(
      "modelViewMatrix", isRowMajor, m);
```

what is this?

Setting the Modelview Matrix: Core Profile

- ALTERNATIVELY: do your own linear algebra!

```
float m[16]; // column-major
m[0] = ...
...
m[15] = ...
GLboolean isRowMajor = GL_FALSE;
pipelineProgram->SetUniformVariableMatrix4fv(
      "modelViewMatrix", isRowMajor, m);
```

- You can multiply with any matrix here.
  - Do more than just boring old rigid maps.

Setting the Modelview Matrix: Compatibility Profile

- Load or post-multiply

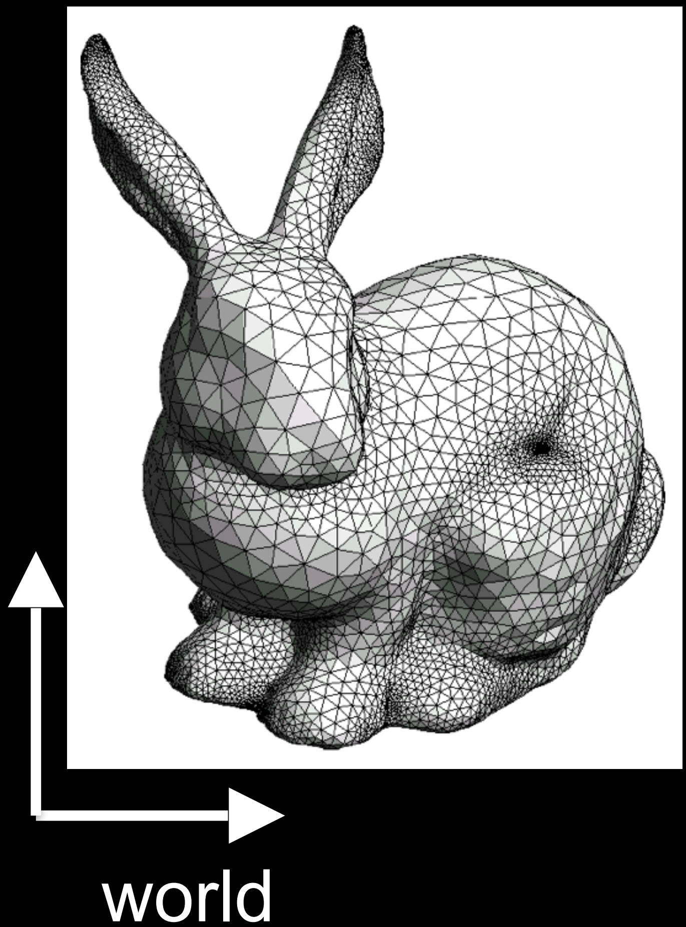
```
glMatrixMode (GL_MODELVIEW);
glLoadIdentity(); // very common usage
float m[16] = { … };
glLoadMatrixf(m); // rare, advanced
glMultMatrixf(m); // rare, advanced
```

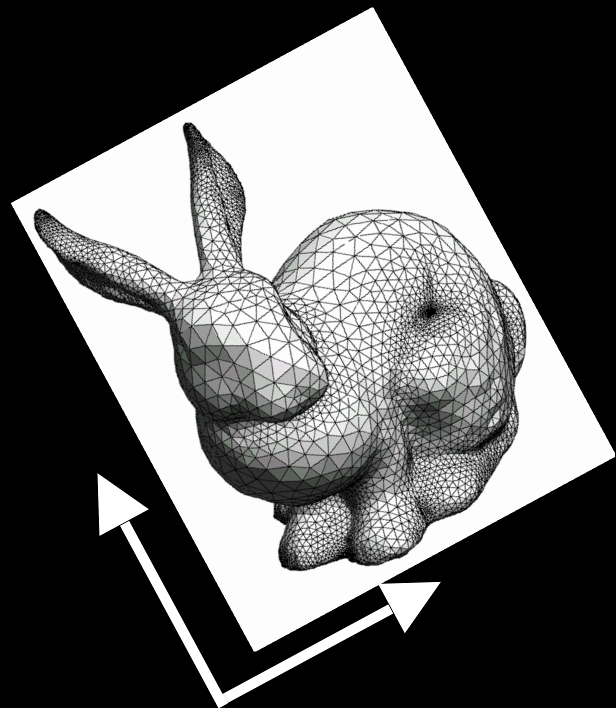
- Use library functions

```
glTranslatef(dx, dy, dz);
glRotatef(angle, vx, vy, vz);
glScalef(sx, sy, sz);
```

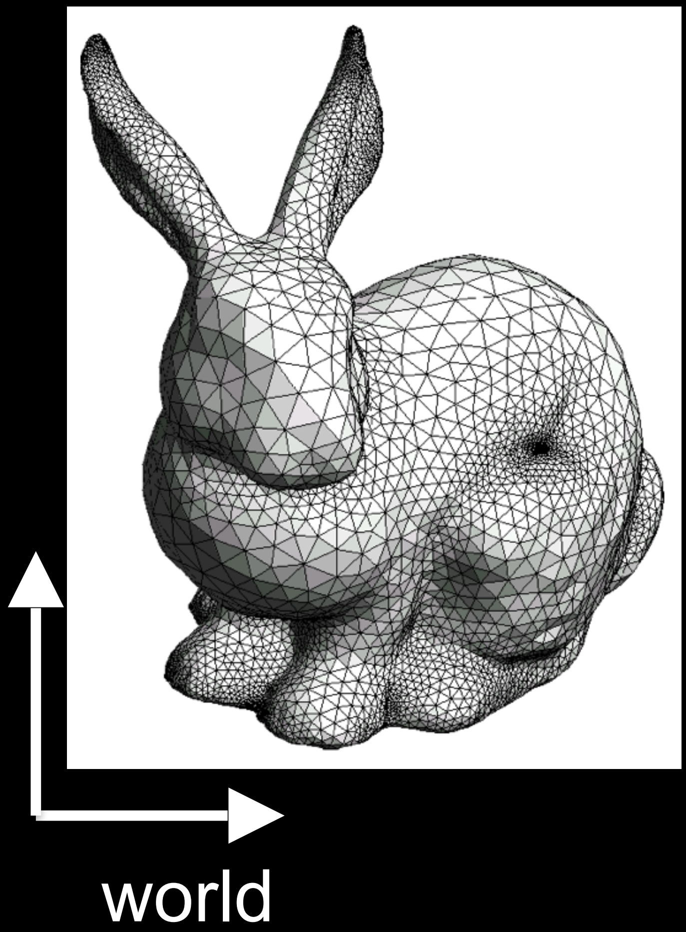
  - More intuitive at first, but causes headaches later.
  - No control over matrix.

Translated, rotated, scaled object





The *rendering* coordinate system

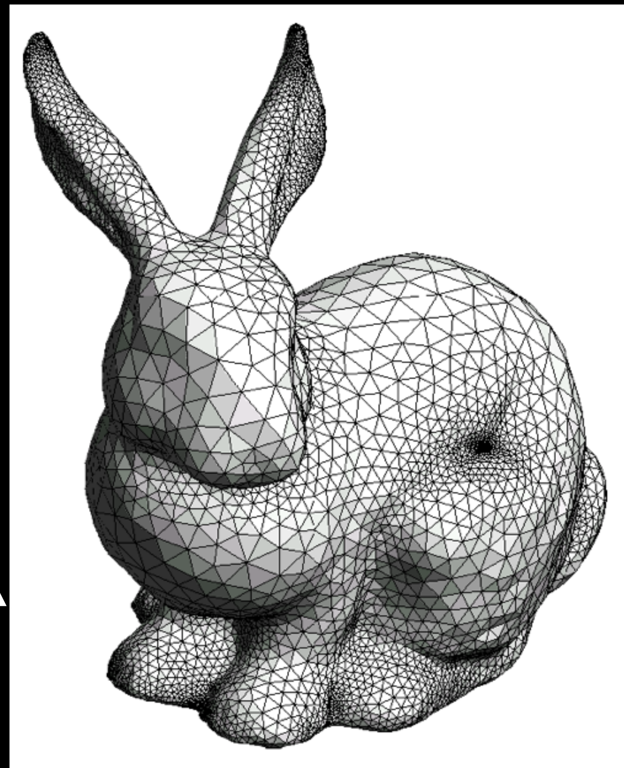


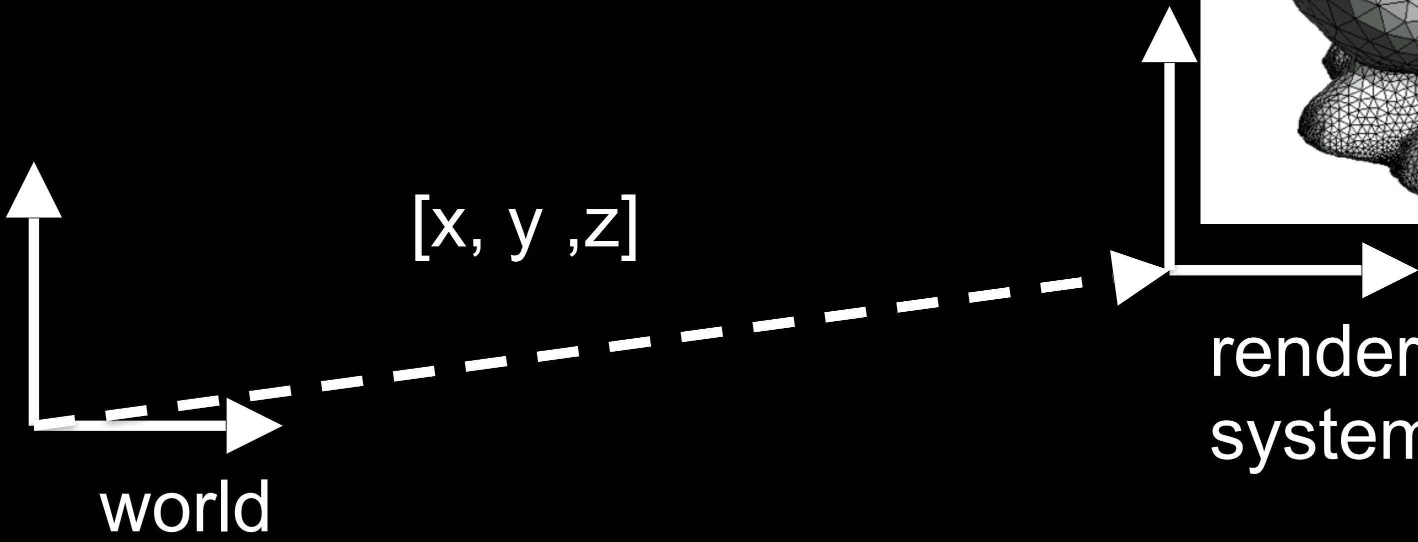
Initially (after `LoadIdentity()`) :

rendering coordinate system = world coordinate system

The *rendering* coordinate system

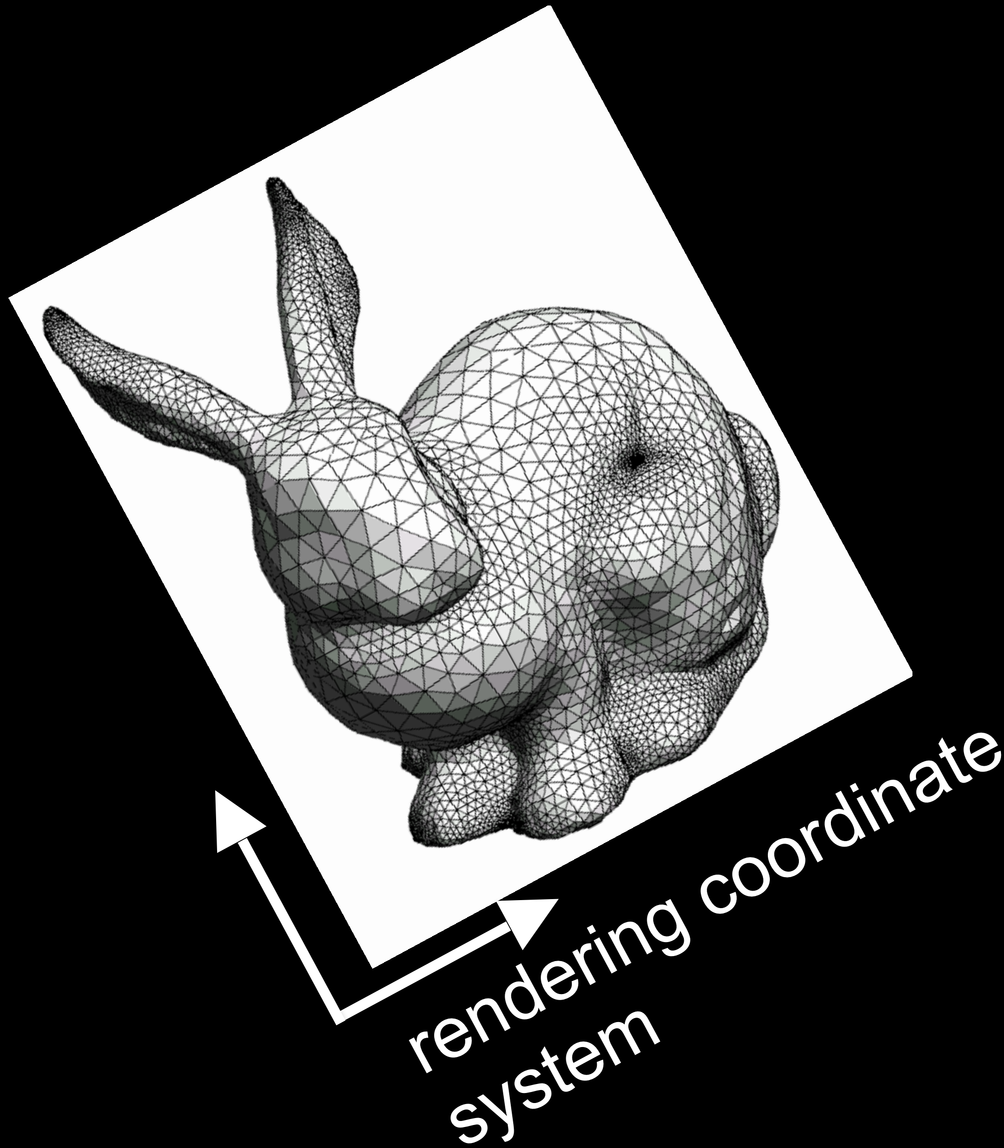
```
Translate(x, y, z);
```

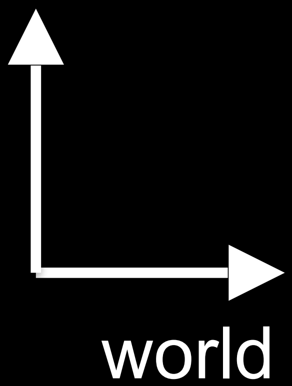




The *rendering* coordinate system

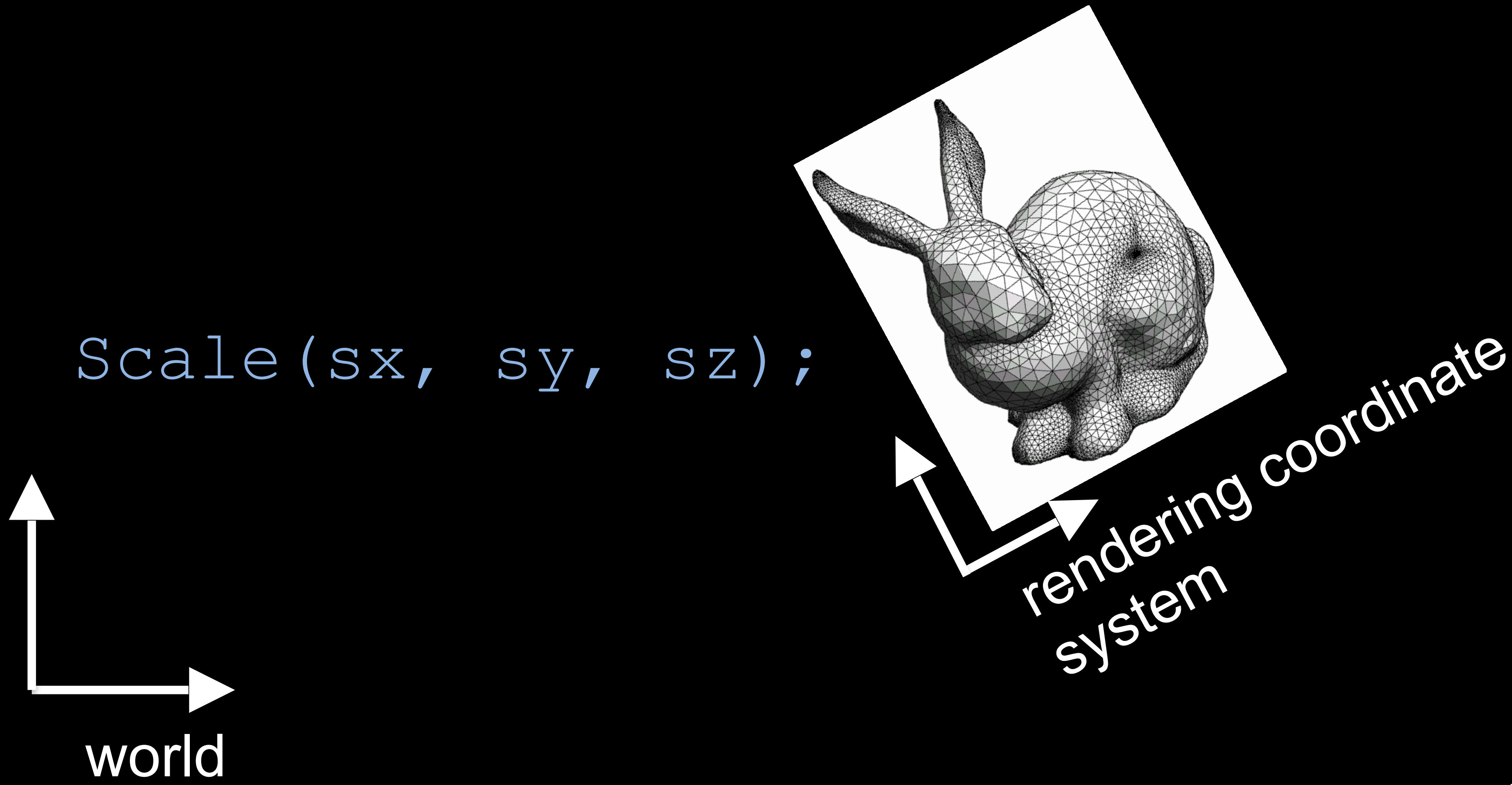
```
Rotate(angle,
  ax, ay, az);
```





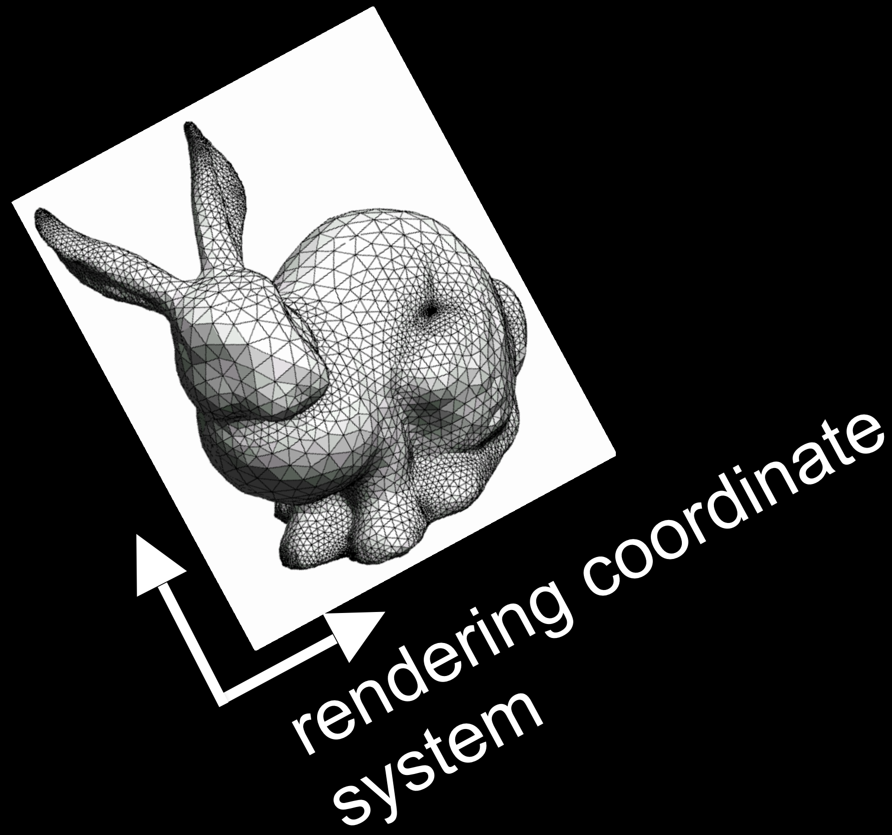
The *rendering* coordinate system

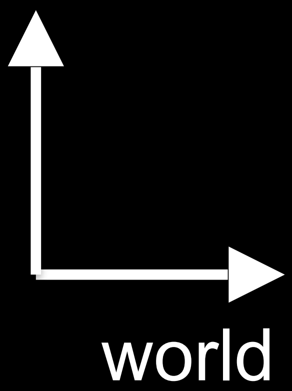
```
Scale(sx, sy, sz);
```



OpenGL pseudo-code

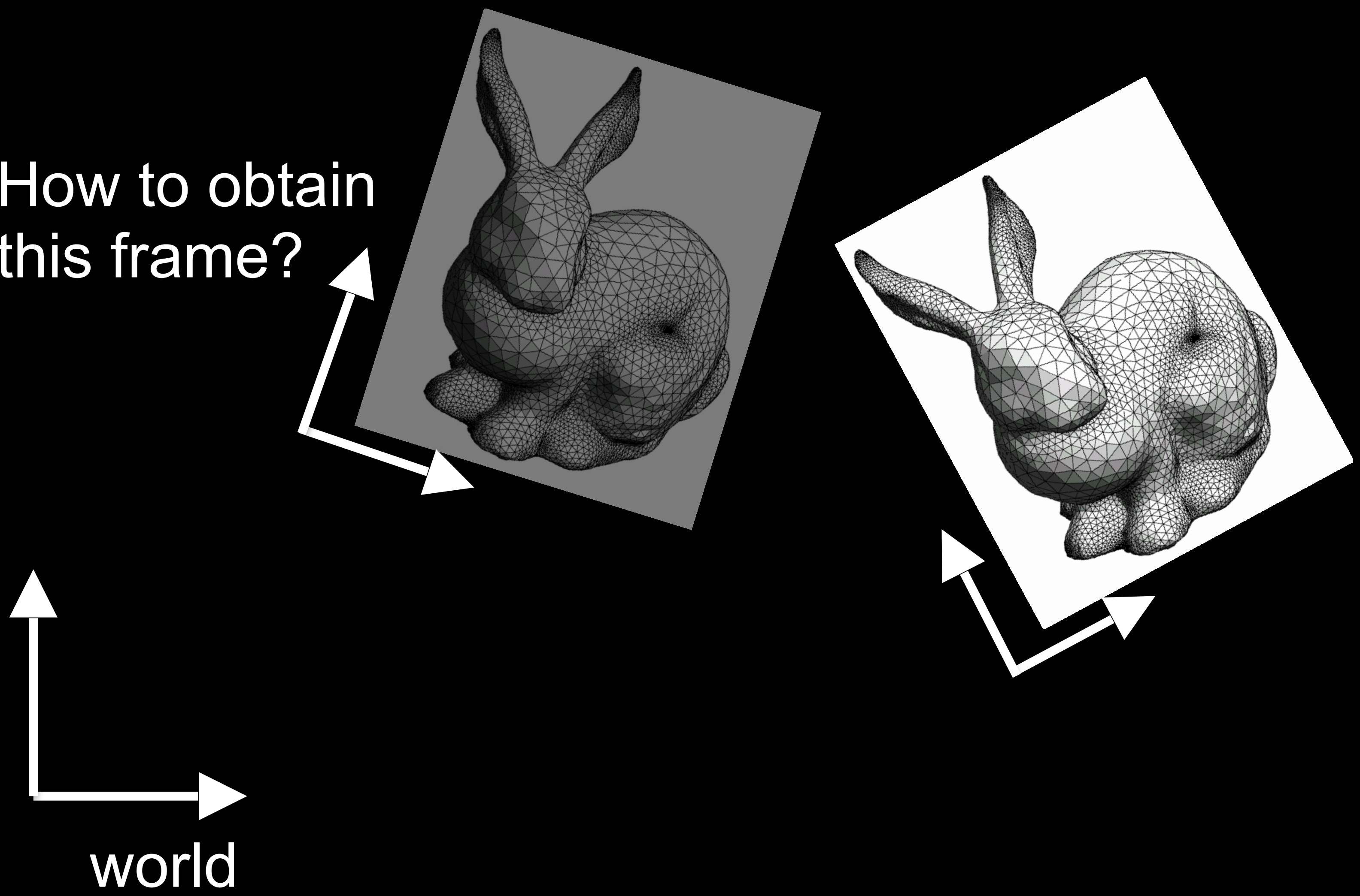
```
MatrixMode(ModelView);
LoadIdentity();
Translate(x, y, z);
Rotate(angle, ax, ay, az);
Scale(sx, sy, sz);
glUniformMatrix4fv(…);
renderBunny();
```





Rendering more objects

How to obtain this frame?

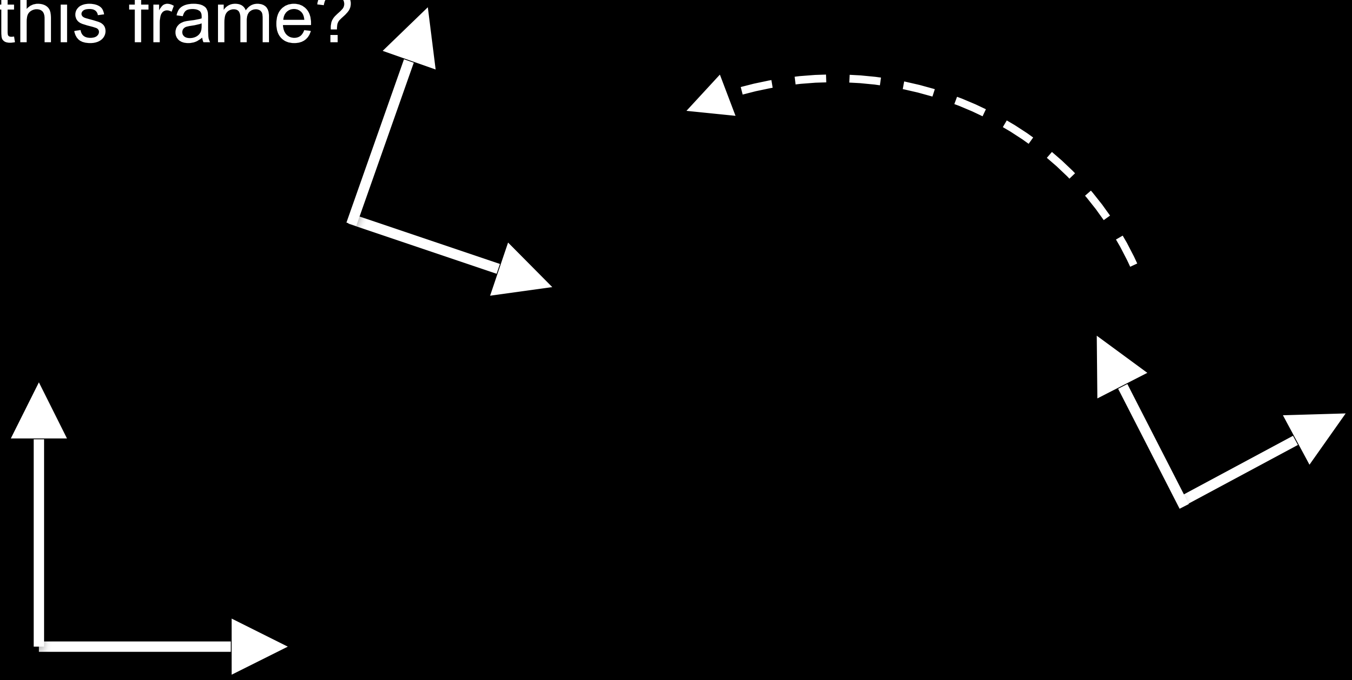


Rendering more objects

Solution 1:

Find `Translate(…), Rotate(…), Scale(…)`

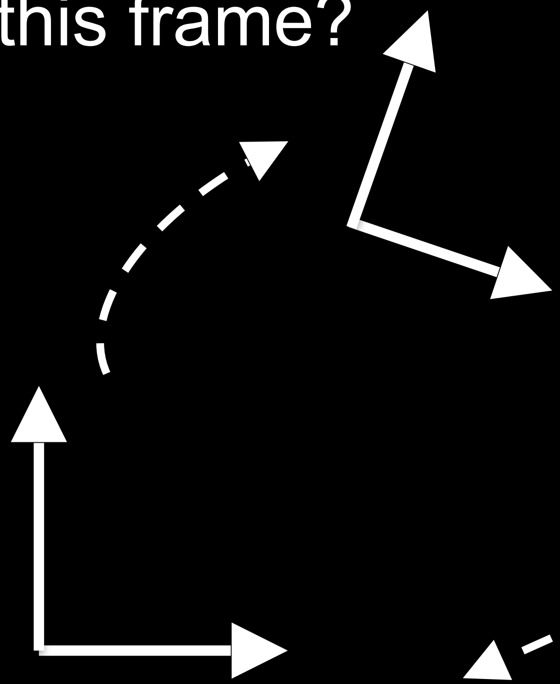
How to obtain this frame?

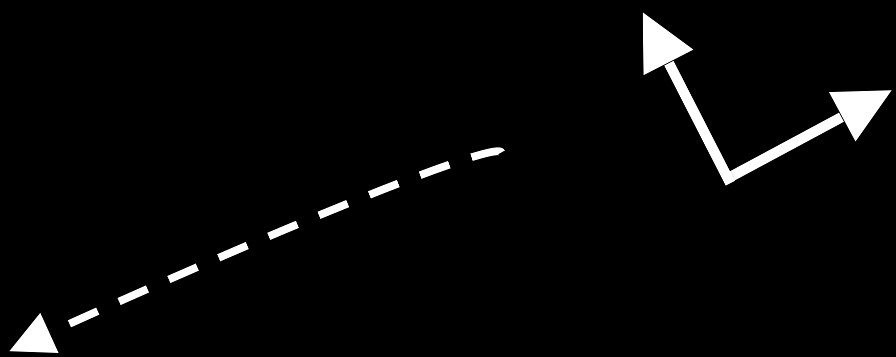


world

Rendering more objects

How to obtain this frame?





world

Solution 2:

Find `Translate(…)`, `Rotate(…)`,
`Scale(…)`

3D Math Review

3D Math Review

(linear algebra with 3x3 and 4x4 matrices)

Scalars

- Scalars α, β, γ from a *scalar field*
- Operations $\alpha+\beta$, $\alpha \cdot \beta$, 0, 1, $-\alpha$, $(\ )^{-1}$
- "Expected" laws apply
- Examples: rationals or reals with addition and multiplication

$$\mathbb{R}$$

Scalars

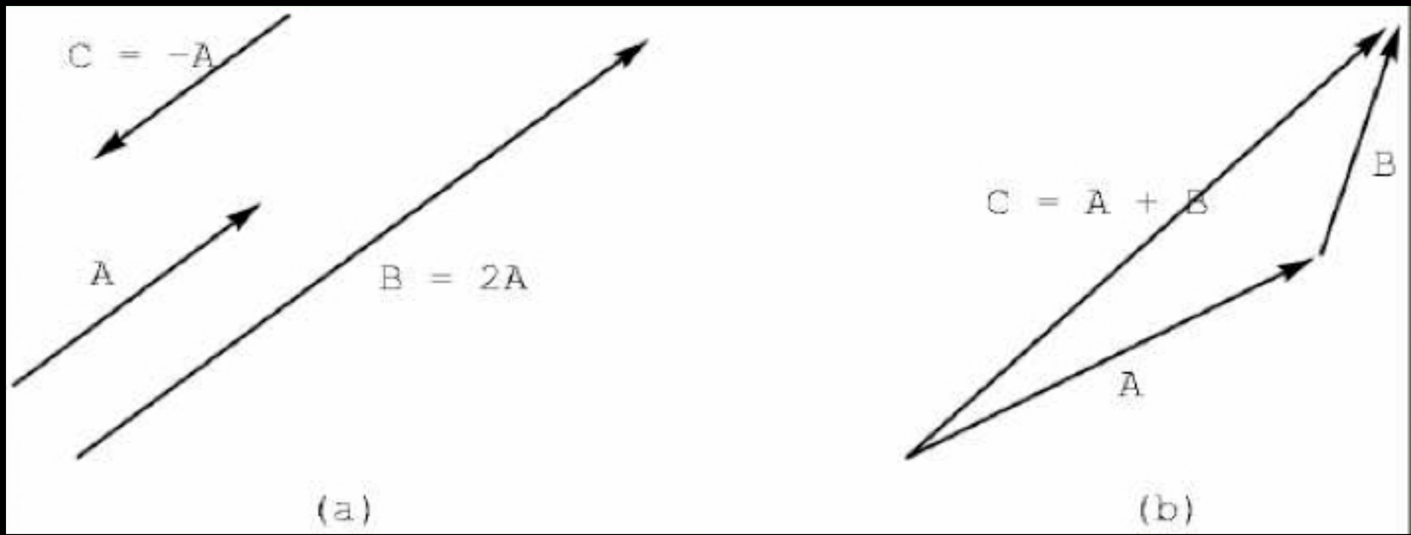
Scalars are numbers

```
double, float, ...
```

$$\alpha = 3$$

Vectors

- Vectors *u, v, w* from a *vector space*
- Vector addition *u + v* , subtraction *u - v*
- Zero vector **0**
- Scalar multiplication α *v*



Euclidean Space

- Vector space over real numbers

$$\mathbb{R}^n$$

mostly n=3 or n=4

Euclidean Space

- Vector space over real numbers
- On CPU:
  - `float[3], double[3], ...`
- On GPU:
  - `vec3f, ...`

$$\mathbf{v} = (1,5,3)$$

Euclidean Space

$$\mathbf{v} = \left(v_1, v_2, v_3\right)$$

$$\mathbf{w} = \left(w_1, w_2, w_3\right)$$

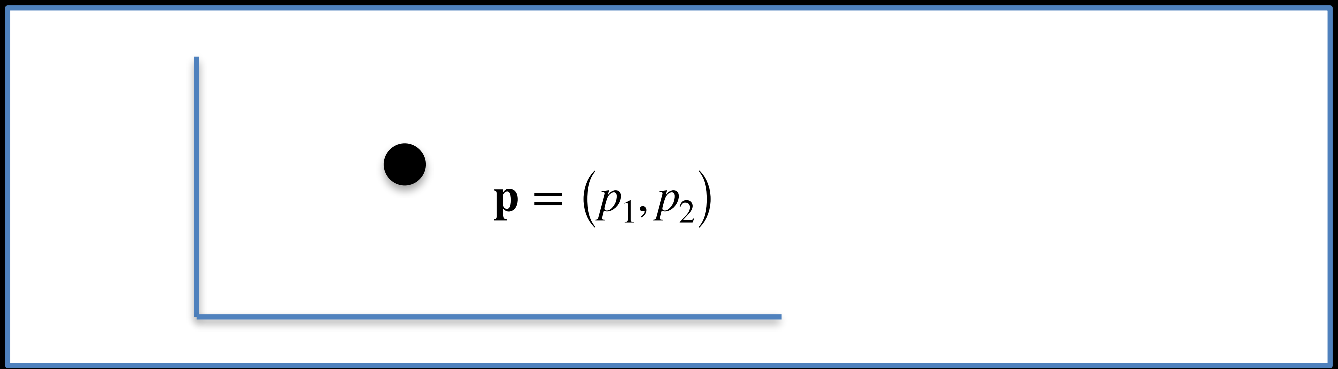
$$\mathbf{v} + \mathbf{w} = \left(v_1 + w_1, v_2 + w_2, v_3 + w_3\right)$$

Euclidean Space

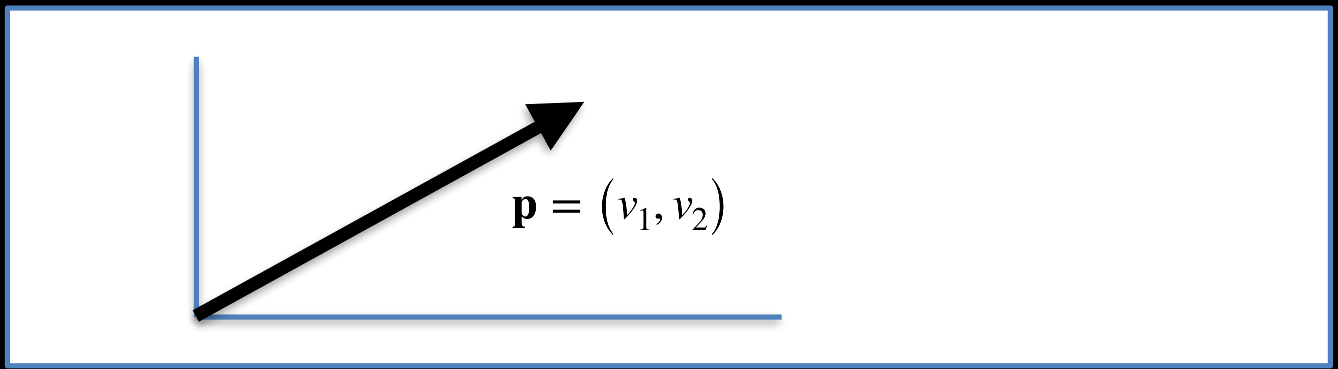
- Vector space over real numbers
- Three-dimensional in computer graphics
- Dot product: $\alpha = u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3$
- $\mathbf{0} \cdot \mathbf{0} = 0$
- u, v are *orthogonal* if $u \cdot \mathrm{v} = 0$
- $|v|^2 = v \cdot v$ defines $|v|$, the *length* of v
- If $|v| = 1$, the vector is *normal*

Two interpretations of vectors

- Just a point in space



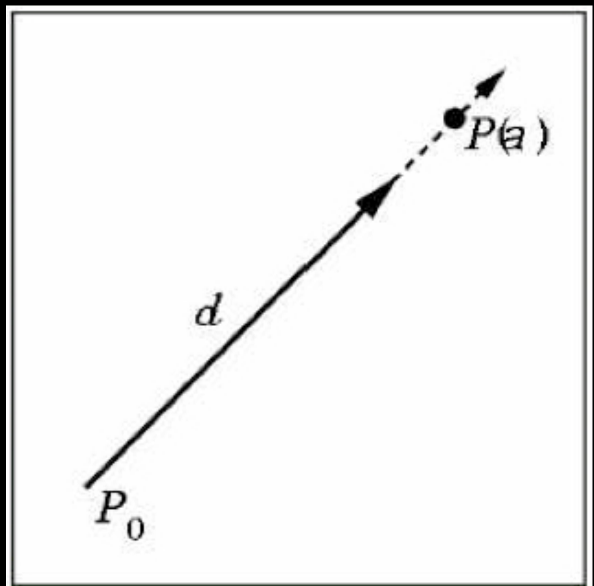
- An arrow pointing in a direction of a certain length

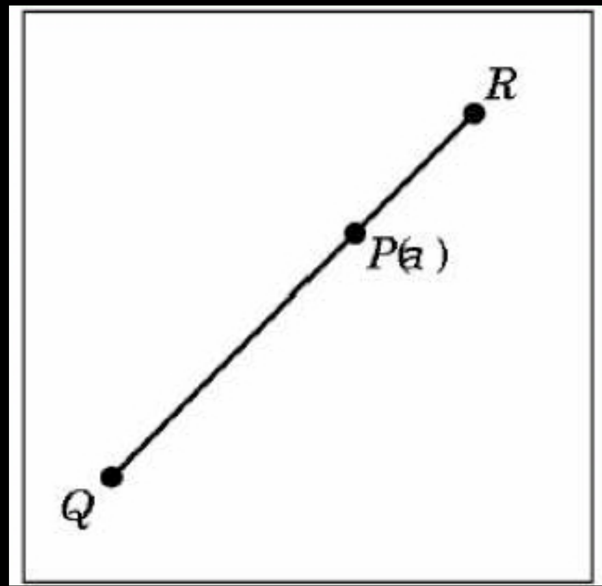


(some might say only the second is a vector,
but in CG we represent both as vectors)

Lines and Line Segments

- Parametric form of line: $P(\alpha) = P_0 + \alpha\, d$



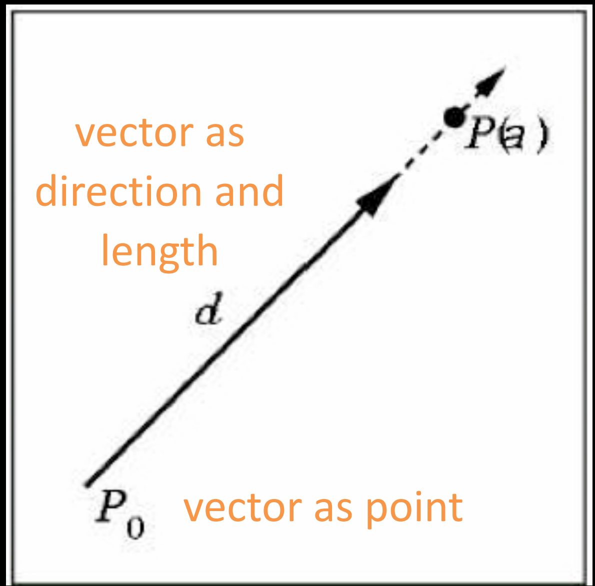


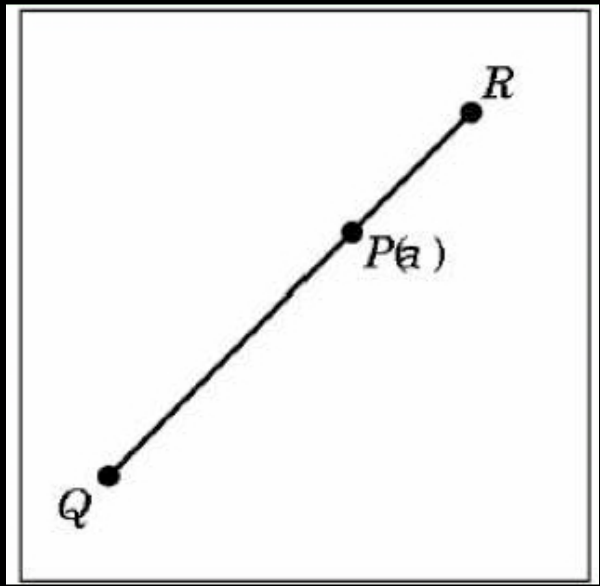
- Line segment between Q and R:

  $P(\alpha) = (1-\alpha)\, Q + \alpha\, R$ for $0 \leq \alpha \leq 1$

Lines and Line Segments

- Parametric form of line: $P(\alpha) = P_0 + \alpha\, d$



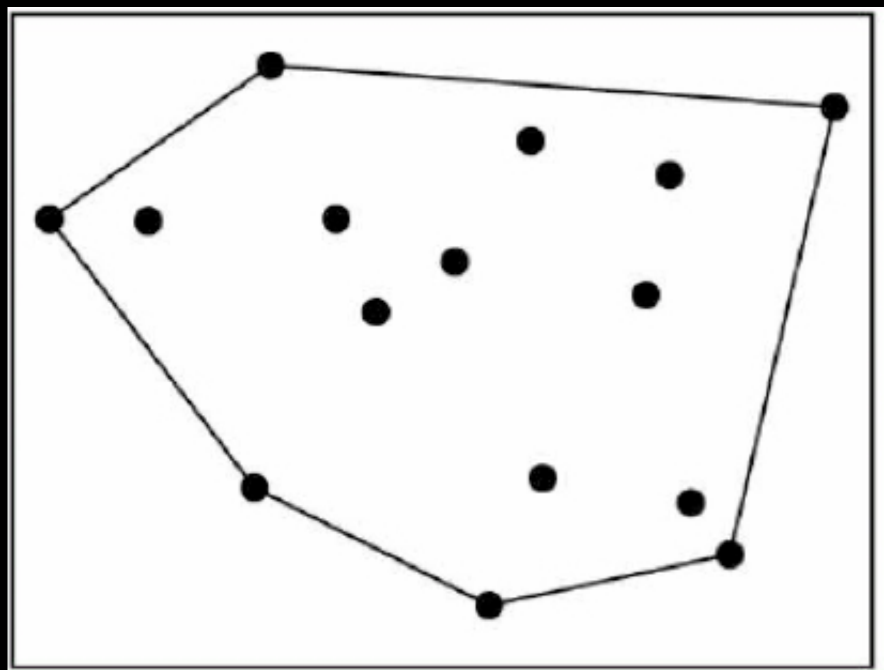


- Line segment between Q and R:
  $P(\alpha) = (1-\alpha)\, Q + \alpha\, R$ for $0 \leq \alpha \leq 1$

Convex Hull

- Convex hull defined by

$$P = \alpha_1 P_1 + \ldots + \alpha_n P_n$$
for $\alpha_1 + \ldots + \alpha_n = 1$
and $0 \leq \alpha_i \leq 1, \; i = 1, \ldots, n$

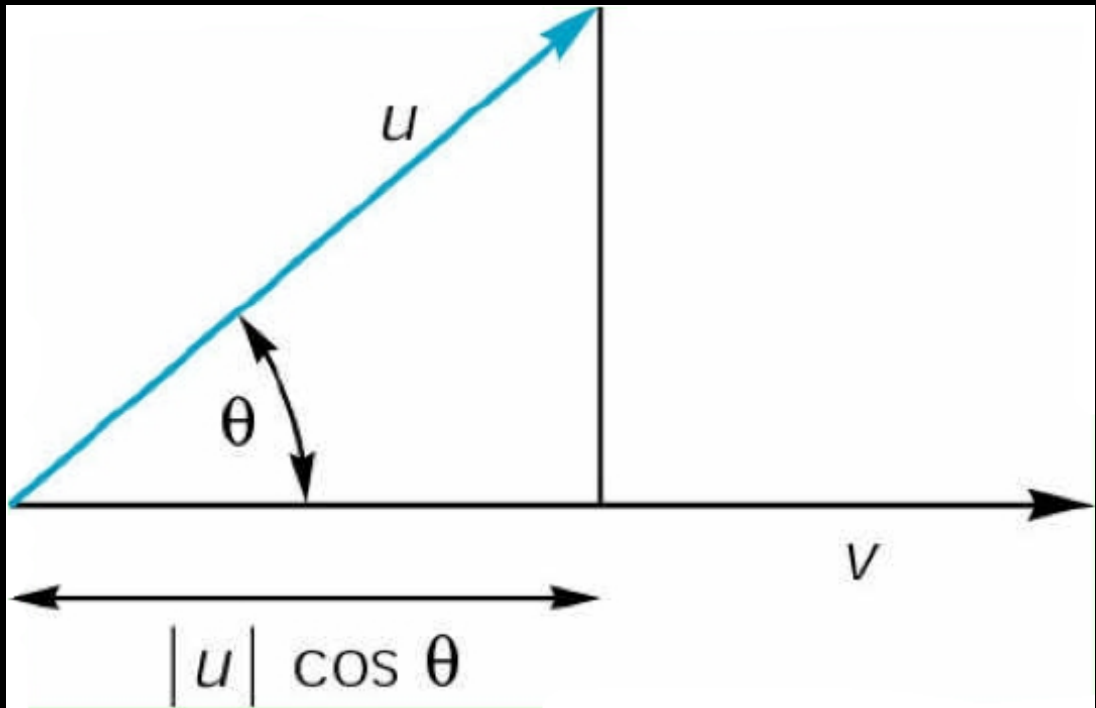


Projection

- Dot product projects one vector onto another vector

$$u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3 = |u|\ |v| \cos(\theta)$$

$$pr_v\, u = (u \cdot v)\, v / |v|^2$$



Orthogonal Projection

- Dot product projects one vector onto another vector

$$u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3 = |u|\ |v| \cos(\theta)$$

$$pr_v\, u = (u \cdot v)\, v \,/\, |v|^2$$

- Why is this called an orthogonal projection?

Orthogonal Projection

- Dot product projects one vector onto another vector

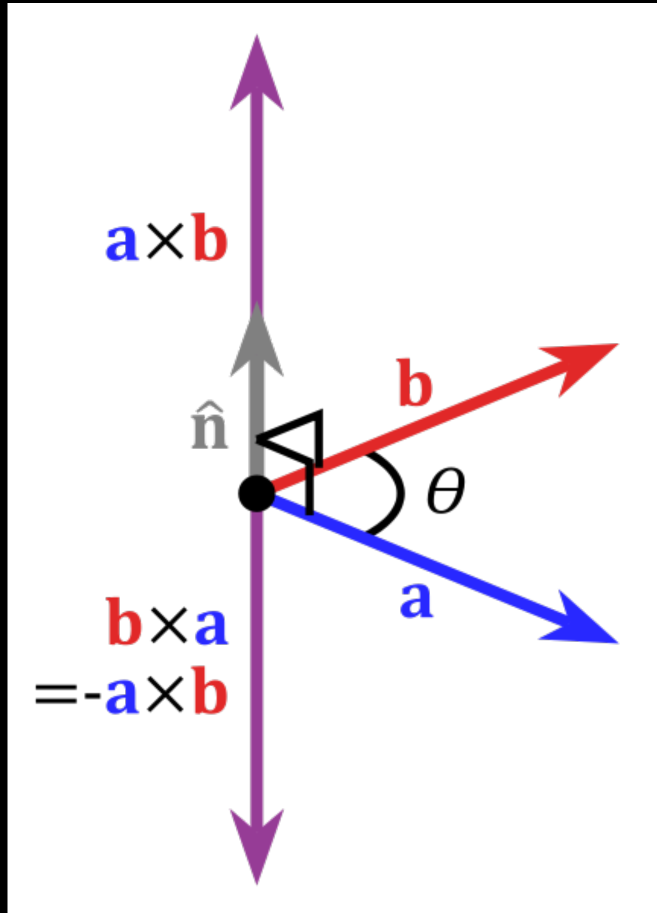
  $u \cdot v = u_1\, v_1 + u_2\, v_2 + u_3\, v_3 = |u|\, |v| \cos(\theta)$

  $pr_v\, u = (u \cdot v)\, v\, /\, |v|^2$

- Why is this called an orthogonal projection?
  - Because $(u - pr_v\, u) \cdot v = (u \cdot v) - (u \cdot v)\, (v \cdot v)\, /\, |v|^2$
    $= (u \cdot v) - (u \cdot v) = 0$

Cross Product

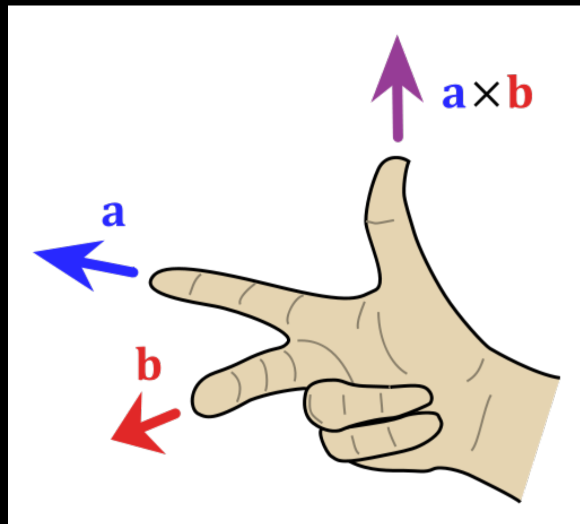
$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$



- |a x b| = |a| |b| |sin(θ)|

- Cross product is perpendicular to both a and b

- Right-hand rule



Source: Wikipedia

Plane

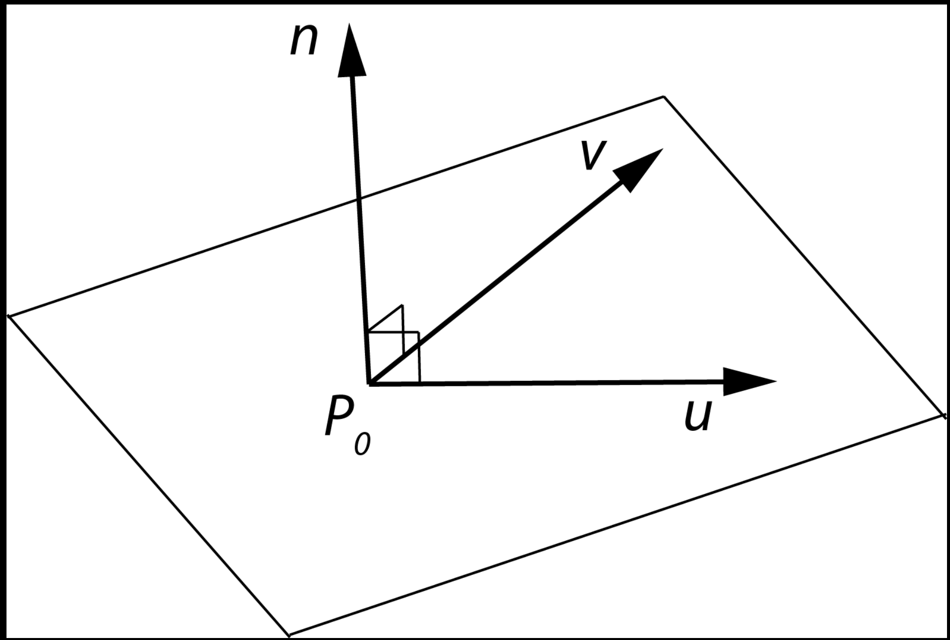
- Plane defined by point P_0 and vectors u and v

- u and v should not be parallel

- Parametric form:

  $T(\alpha, \beta) = P_0 + \alpha\, u + \beta\, v$

  (α and β are scalars)



- $n = u \times v \,/\, |u \times v|$ is the normal

- $n \cdot (P - P_0) = 0$ if and only if P lies in plane

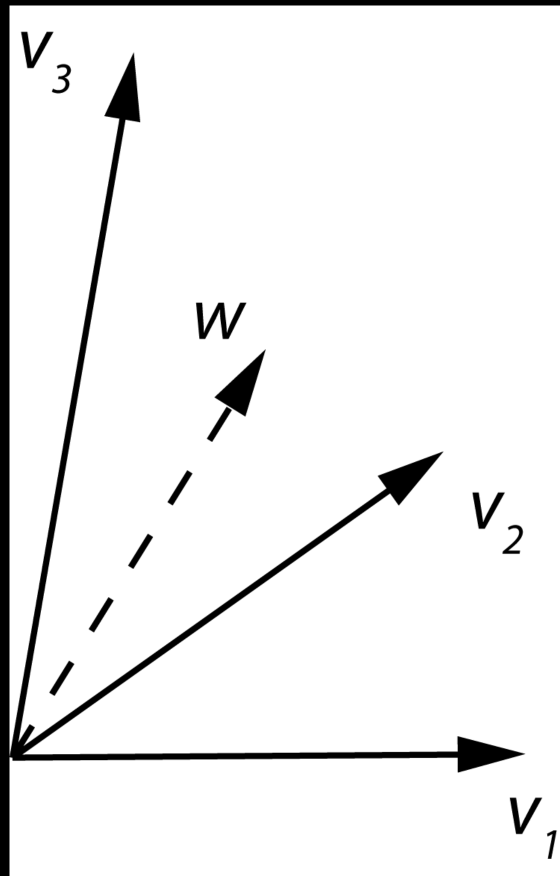
Coordinate Systems

- Let v_1, v_2, v_3 be three linearly independent vectors in a 3-dimensional vector space

- Can write *any* vector w as

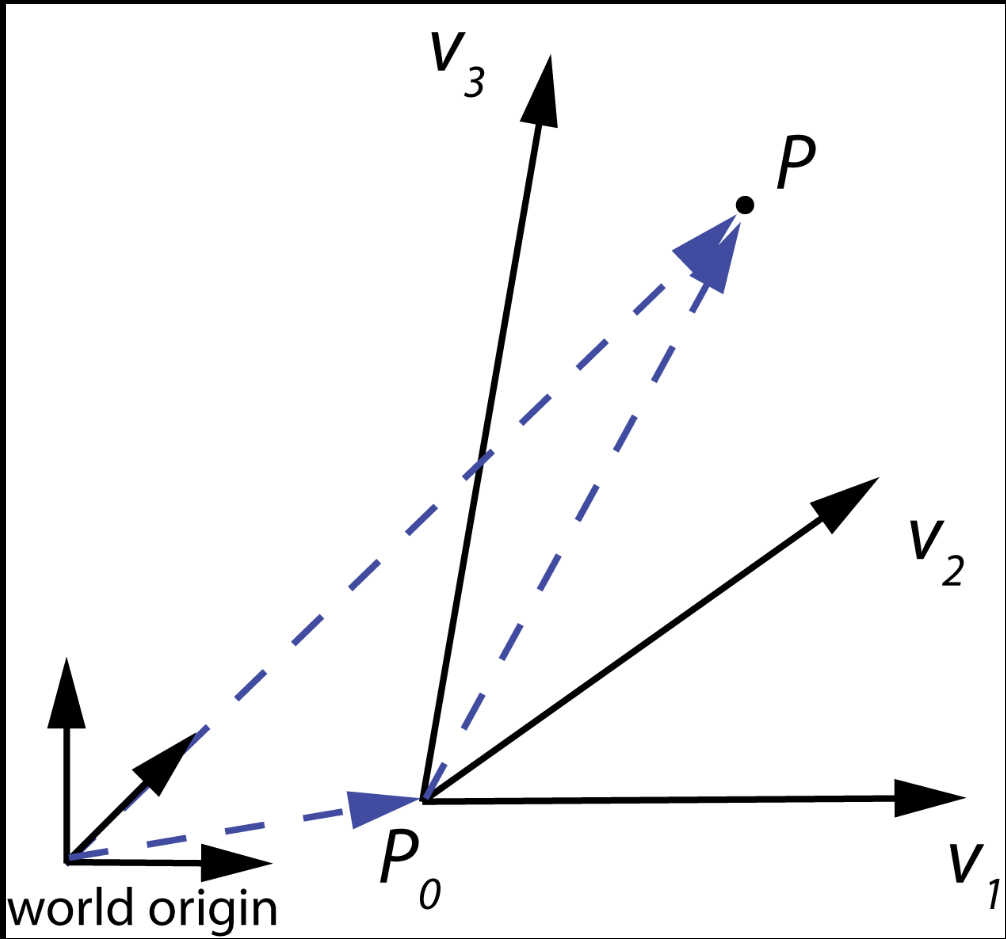
$$w = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$$

for some scalars α_1, α_2, α_3

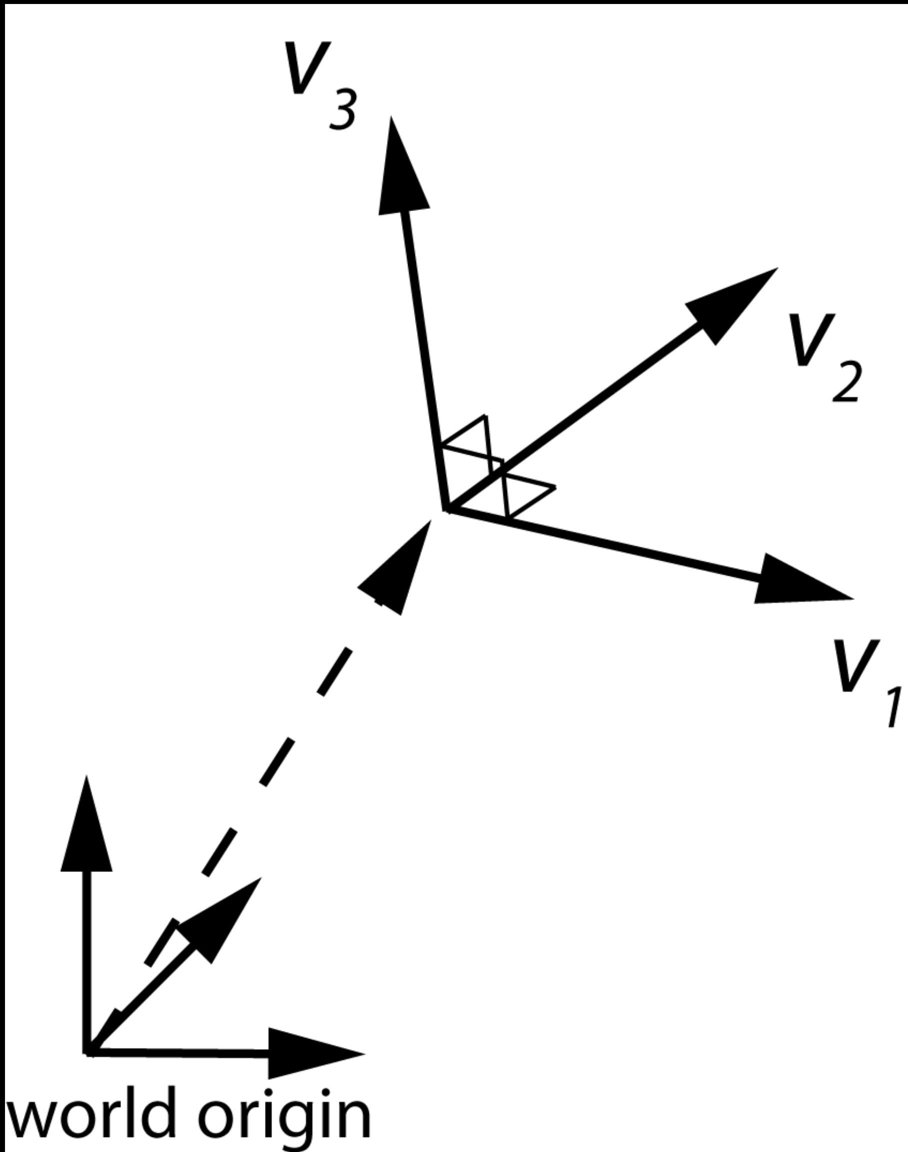


Frames

- Frame = origin P_0 + coordinate system
- Any point $P = P_0 + \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$



In Practice, Frames are Usually Orthonormal



$v_1 \cdot v_2 = 0$
$v_2 \cdot v_3 = 0$
$v_1 \cdot v_3 = 0$

$|| v_1 || = || v_2 || = || v_3 || = 1$

Orthonormality makes life easier

Change of Coordinate System

- Bases $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$
- Express basis vectors u_i in terms of v_j

$$
\begin{aligned}
u_1 &= \gamma_{11}v_1 + \gamma_{12}v_2 + \gamma_{13}v_3 \\
u_2 &= \gamma_{21}v_1 + \gamma_{22}v_2 + \gamma_{23}v_3 \\
u_3 &= \gamma_{31}v_1 + \gamma_{32}v_2 + \gamma_{33}v_3
\end{aligned}
$$

- Represent in matrix form:

$$
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = M \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \qquad M = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix}
$$

Matrices

- matrices represent linear transformations

  a = Mb

$$M = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix}$$

$$\mathbb{R}^{3x3}$$

- On CPU:
  - float[3][3], double[3][3], ...
- On GPU:
  - mat3f, ...

Matrices

- matrices can be added and multiplied by a scalar like a vector

- matrices can be multiplied with each other
  - composition of functions

- Determinant

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = aei + bfg + cdh - ceg - bdi - afh$$

Matrices

- Inverse

$$A\mathbf{x} = y$$

$$\mathbf{x} = A^{-1}y$$

- Not all matrices can be inverted
  - Only if $\det A \neq 0$

- Don't compute them explicitly - our helper libraries and your GPU can computer them.

Matrices

- Orthogonal matrices:
  - $A = (a_1\ a_2\ a_3)$ is orthogonal if...
    - $A^T A = A A^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$
    - $a_1 \cdot a_2 = a_2 \cdot a_3 = a_1 \cdot a_3 = 0$ and $\det A = 1$

Matrices

- Can represent rotation, scaling, and reflection
  - Rotation: SO(3) = {3x3 orthogonal matrices with determinant 1}
  - Scaling: Diag(3) = {3x3 matrices with positive entry on diagonal only}
  - Reflection: Identity matrices where one diagonal element is -1
  - Rotation + Reflection: {3x3 matrices with | determinant 1 or -1}

- Cannot represent translation
  - $u_{trans} = u + t$
  - There is no M such that $u_{trans} = M\ u$

In order to represent rotations, scales AND translations: Homogeneous Coordinates

- Augment $[\alpha_1 \;\; \alpha_2 \;\; \alpha_3]^T$ by adding a fourth component (1):

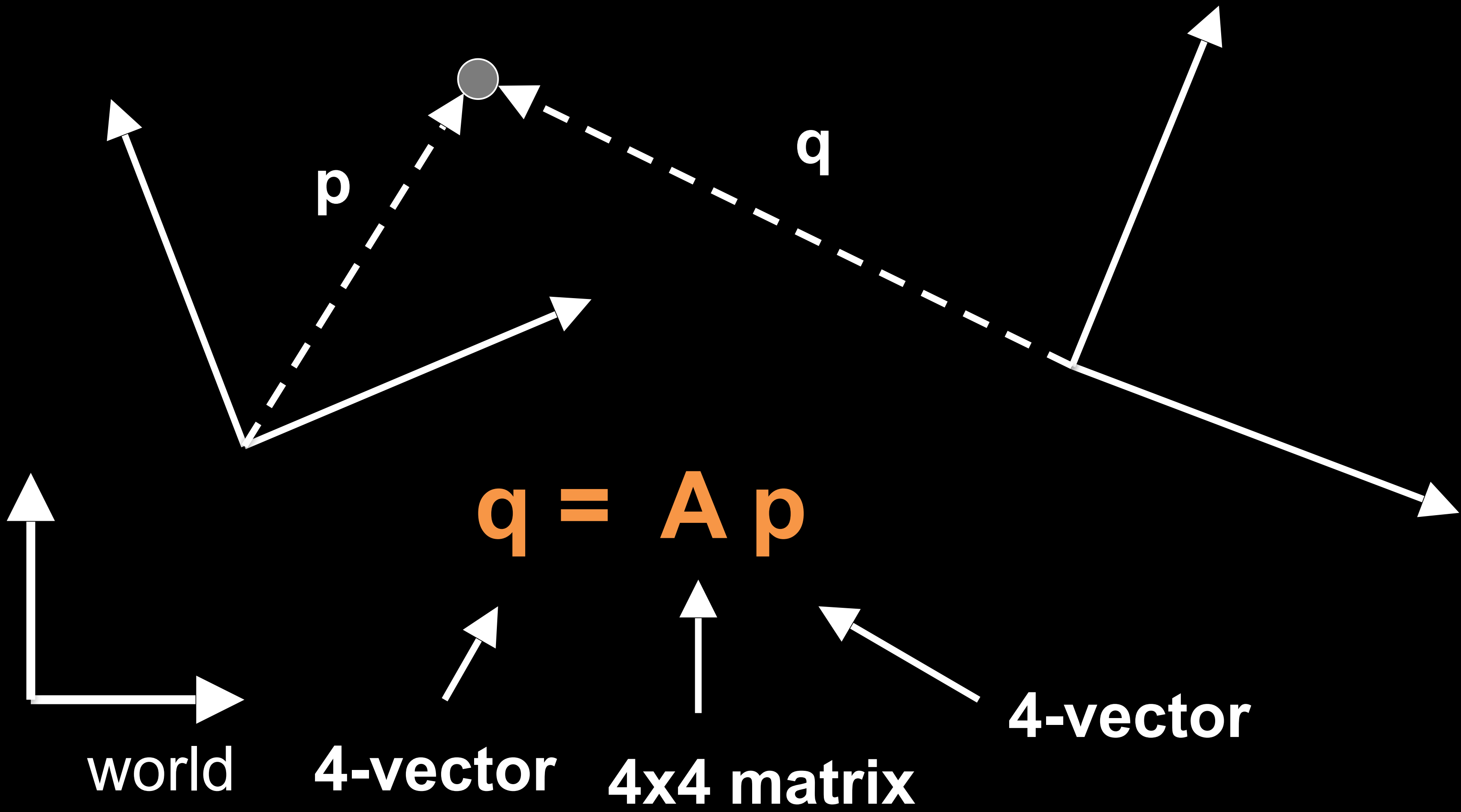
  $\mathbf{p} = [\alpha_1 \;\; \alpha_2 \;\; \alpha_3 \;\; 1]^T$

- Homogeneous property:

  $\mathbf{p} = [\alpha_1 \;\; \alpha_2 \;\; \alpha_3 \;\; 1]^T = [\beta\alpha_1 \;\; \beta\alpha_2 \;\; \beta\alpha_3 \;\; \beta]^T$,

  for any scalar $\beta \neq 0$

Homogeneous coordinates are transformed by 4x4 matrices



Affine Transformations (4x4 matrices)

- Translation
- Rotation
- Scaling
- Any composition of the above
- Later: projective (perspective) transformations
  - Also expressible as 4 x 4 matrices!

Translation

- $\mathbf{q} = \mathbf{p} + \mathbf{d}$ where $\mathbf{d} = [\alpha_x \ \ \alpha_y \ \ \alpha_z \ \ 0]^T$
- $\mathbf{p} = [x \ \ y \ \ z \ \ 1]^T$
- $\mathbf{q} = [x' \ \ y' \ \ z' \ \ 1]^T$
- Express in matrix form **q** = **T p** and solve for **T**

$$T = \begin{bmatrix} 1 & 0 & 0 & \alpha_x \\ 0 & 1 & 0 & \alpha_y \\ 0 & 0 & 1 & \alpha_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Scaling

- $x' = \beta_x\ x$
- $y' = \beta_y\ y$
- $z' = \beta_z\ z$
- Express as **q** = **S** **p** and solve for **S**

$$S = \begin{bmatrix} \beta_x & 0 & 0 & 0 \\ 0 & \beta_y & 0 & 0 \\ 0 & 0 & \beta_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation in 2 Dimensions

- Rotation by θ about the origin
- x' = x cos θ – y sin θ
- y' = x sin θ + y cos θ

- Express in matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Note that the determinant is 1

Rotation in 3 Dimensions

- Orthogonal matrices:

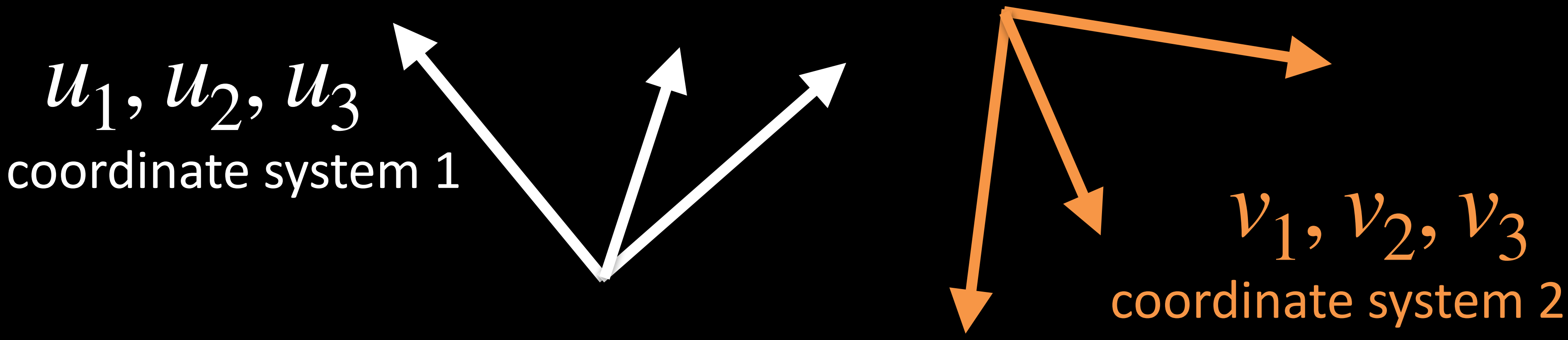
$$RR^{\mathrm{T}} = R^{\mathrm{T}}R = I$$
$$\det(R) = 1$$

- As a 4x4 homogeneous matrix:

$$A = \begin{bmatrix} R_{11} & R_{12} & R_{13} & 0 \\ R_{21} & R_{22} & R_{23} & 0 \\ R_{31} & R_{32} & R_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

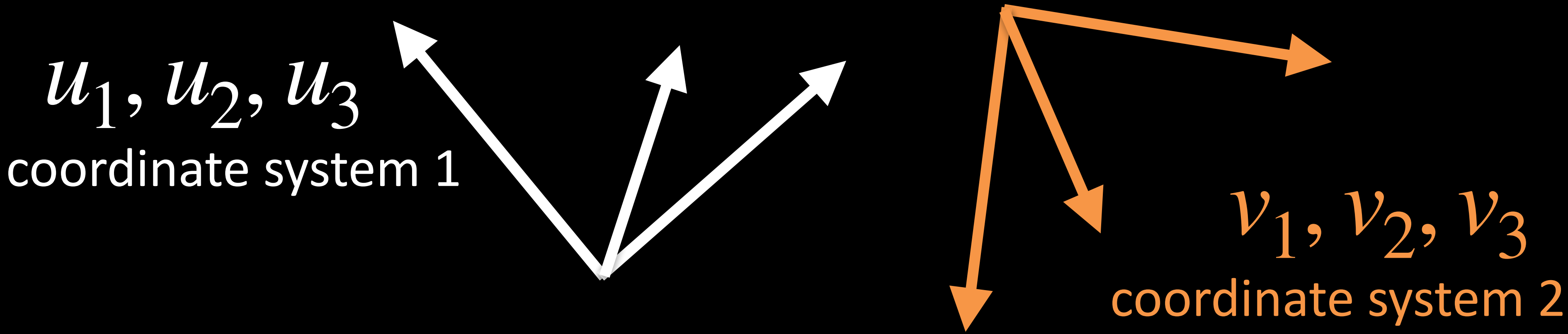
Rotation in 3 Dimensions

- How do we rotate between two orthonormal coordinate systems?



Rotation in 3 Dimensions

- How do we rotate between two orthonormal coordinate systems?



- $U = \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix}$ transforms system 1 to canonical system
- $V = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$ transforms system 2 to canonical system
- To transform $\mathbf{x}$ from system 1 to system 2, $\mathbf{x} \mapsto V^{-1}U\mathbf{x}$

Affine Matrices are Composed by Matrix Multiplication

- $\mathbf{A = A_1 A_2 A_3}$
- Applied from right to left
- $\mathbf{A\ p} = (\mathbf{A}_1\ \mathbf{A}_2\ \mathbf{A}_3)\ \mathbf{p} = \mathbf{A}_1\ (\mathbf{A}_2\ (\mathbf{A}_3\ p))$
- Compatibility mode:
  When calling glTranslate3f, glRotatef, or glScalef,
  OpenGL forms the corresponding 4x4 matrix,
  and multiplies the current modelview matrix with it.

Summary

- OpenGL Transformation Matrices
- Vector Spaces
- Frames
- Homogeneous Coordinates
- Transformation Matrices