Mappings & Jacobians

We are next going to learn about deforming shapes and computing parametrizations of shapes. In order to better understand the theory behind these two subjects, we will explore the concept of *mappings* in greater detail here.

This is a very rough recap of topics that you might have heard in multivariable calculus with a few sprinkles of differential geometry thrown in.

Maps on surfaces

For this entire chapter, Ω will be a d-dimensional manifold embedded in $\mathbb{R}^{d+1}$.

Definition 1: Map

> A map is a differentiable function $\varphi : \Omega \to \mathbb{R}^k$.
>
> k is usually either $d+1$ (for example, when the map deforms the mesh), or d (when the map is used to parametrize Ω).

Maps, of course, are functions. But they are not scalar functions that assign a value to every point on the surface. Maps can be thought of as transformations of the surface. They can be used to deform a surface, to embed it in a different space, to move it, and so on.

In order to be useful in transforming a surface, the map needs to preserve some properties of the surface it is defined on.

Definition 2: Diffeomorphism

> Let $\varphi : \Omega \to \mathbb{R}^k$ be a an *injective* map (i.e., for $x \neq y$ it holds hat $\varphi(x) \neq \varphi(y)$).
>
> Let $I = \varphi(\Omega)$ be the image of φ. The restriction of the image of φ to I is surjective (i.e., for every $y \in I$ there is an $x \in \Omega$ such that $\varphi(x) = y$).
>
> In an abuse of notation, we call the map that inverts φ just on its inverse φ^{-1}. If φ^{-1} is differentiable, then we call φ a diffeomorphism.

If a map is a diffeomorphism, it preserves the most essential characters of a surface at every point. A diffeomorphism, for example, can not change the *genus* of a surface (how many holes it has), add or remove boundaries, make a non-orientable surface orientable, and so on (see Figure 1).

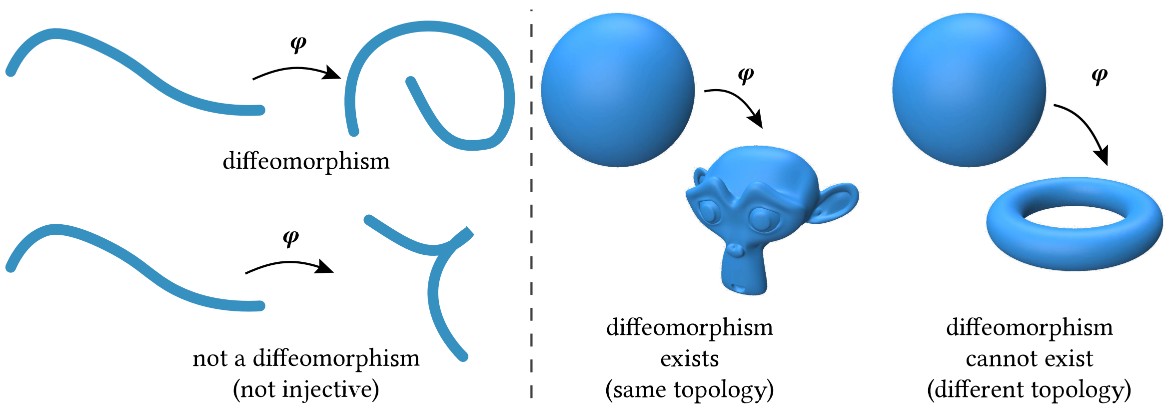

Figure 1: A diffeomorphism is a differentiable injective map that is invertible when restricted onto its image, and whose inverse is also injective on its image *(left)*. Diffeomorphisms preserve topological and smooth characteristics of surfaces *(right)*.

Diffeomorphisms might still change the *geometry* of a surface. They can, for example, increase or shrink its area, its angles, and so on (they are not rigid maps). We will later learn to quantify how much diffeomorphisms change geometry.

Jacobians

You might remember our previous discussion of Jacobians where we computed the Jacobians using a parametrization of the surface. Here we will look at a different way of computing a Jacobian assuming we do not have such a parameterization: By using the tangent space of a surface.

Definition 3: Tangent space

Let Ω be a d-dimensional surface in $\mathbb{R}^{d+1}$.

A curve on Ω is a smooth function

$$\gamma : [0, 1] \to \Omega \,. \tag{1}$$

It has the tangent vector

$$\frac{\partial}{\partial t}\gamma(t) \tag{2}$$

at the point $x = \gamma(t)$.

The *tangent space* of Ω at the point $x \in \Omega$ is defined as the plane contaning the tangent vectors of all curves passing through x at the point x.

Equivalently, the tangent space at point x, T_x, is also defined as the plane passing through x that is orthogonal to the normal vector $\boldsymbol{n}_x$.

The words tangent space and tangent plane are often used interchangeably. The tangent space at every point is a plane, so it has an orthogonal basis.

Definition 4: Basis of a tangent space

Let Ω be a d-dimensional surface in $\mathbb{R}^{d+1}$. Let $x \in \Omega$, and let T_x be the tangent space at x.

Then the linear space T_x has an orthogonal basis $\boldsymbol{t}_1, ..., \boldsymbol{t}_d$ of linearly independent tangent vectors that are the derivatives of curves $\gamma_1, ..., \gamma_d : [0, 1] \to \Omega$ passing through x at the point x.

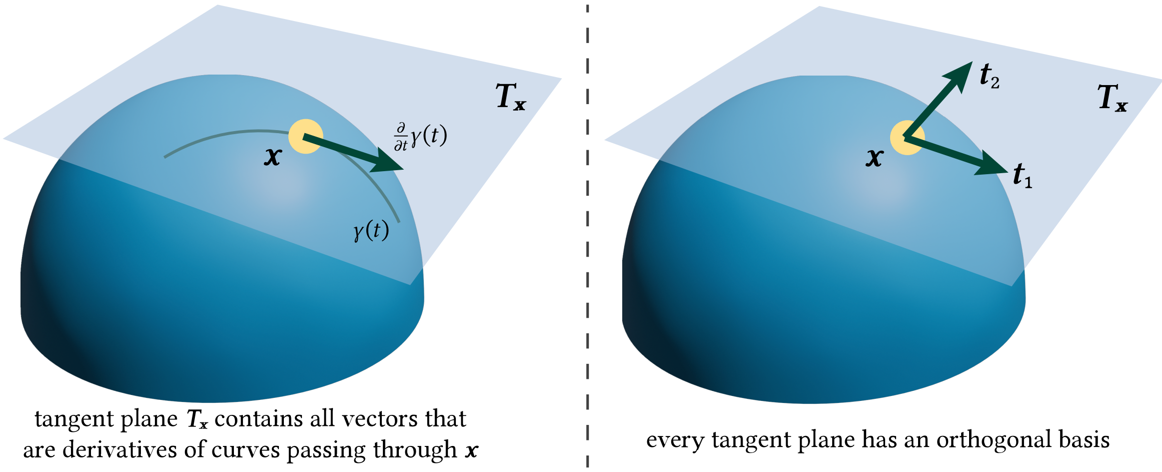

Figure 2: The tangent plane T_x at the point x is the plane containing all derivatives of curves passing through the point x *(left)*. This tangent plane has a basis of d orthogonal tangent vectors *(right)*.

We can take directional derivatives of scalar functions, and the derivatives will be an element of this tangent plane expressable by basis vectors.

Definition 5: Directional derivative

Let Ω be a d-dimensional surface in $\mathbb{R}^{d+1}$. Let $x \in \Omega$, and let T_x be the tangent space at x. Let $u : \Omega \to \mathbb{R}$ be a scalar function on Ω.

Then the directional derivative of u at x in the direction $\boldsymbol{v} \in T_x$ is given by

$$\frac{\partial}{\partial \boldsymbol{v}} u(x) = \frac{\partial}{\partial t}(u \circ \gamma)(0)\,, \tag{3}$$

where $\boldsymbol{v}$ is the derivative of a curve $\gamma : [0,1] \to \Omega$ at x, i.e., $\gamma(0) = x$ and $\boldsymbol{v} = \frac{\partial}{\partial t}\gamma(0)$.

If we have an orthogonal basis $\boldsymbol{t}_1, \ldots, \boldsymbol{t}_d$ of T_x, we write

$$\frac{\partial}{\partial x_i} u(x) = \frac{\partial}{\partial \boldsymbol{t}_i} u(x)\,. \tag{4}$$

This is a proper principled way to define derivatives and gradients of functions on surfaces. We will use this for a definition of the Jacobian based purely on tangent vector language.

Definition 6: Jacobian

Let Ω be a d-dimensional surface in $\mathbb{R}^{d+1}$. Let $x \in \Omega$, and let T_x be the tangent space at x. Let $\varphi : \Omega \to \mathbb{R}^k$ be a map.

Then the *Jacobian* of phi is defined as the matrix

$$J_\varphi = \begin{pmatrix} \frac{\partial \varphi_1}{\partial x_1} & \cdots & \frac{\partial \varphi_1}{\partial x_d} \\ \cdots & & \cdots \\ \frac{\partial \varphi_k}{\partial x_1} & \cdots & \frac{\partial \varphi_k}{\partial x_d} \end{pmatrix}. \tag{5}$$

Definition 6 is the same as the definition that we used in a previous chapter, but now in the language of tangent spaces instead of the language of embeddings. It depends on the choice of orthogonal basis $\boldsymbol{t}_1, ..., \boldsymbol{t}_d$ of T_x (which we sometimes write as $\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_d}$, because they denote the directions of partial derivatives).

You will often see the Jacobian expressed as a $(d+1) \times k$ matrix instead of a $d \times k$ matrix. This happens when we embed the orthogonal basis $\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_d}$ of the tangent space T_x in the Euclidean space $\mathbb{R}^{d+1}$ using a linear embedding E_x. This embedding matrix E_x is then simply multiplied with the Jacobian J_φ to form the embedded Jacobian:

$$\widetilde{J}_\varphi(x) = E_x J_\varphi(x) \,. \tag{6}$$

We will adopt this convention - the Jacobian J_φ can refer to either the object defined in Definition 6, or the embedded version $\widetilde{J}_\varphi$ (and we might just drop the ~).

The Jacobian J_φ is a linear approximation of φ at the point $x \in \Omega$. If $\Omega = \mathbb{R}^d$, and the function φ is an affine map

$$\varphi(x) = Ax + \boldsymbol{t} \,, \tag{7}$$

then the Jacobian of φ is simply the matrix

$$J_\varphi(x) = A \,. \tag{8}$$

Jacobians on a triangle mesh

Let us consider the special case where our surface Ω is a triangle mesh, and our map $\varphi : \Omega \to \mathbb{R}^k$ is a continuous, piecewise linear function.[1] So,

$$\varphi(x) = \sum_{i=1}^{n} \begin{pmatrix} p_{i1} \\ \cdots \\ p_{ik} \end{pmatrix} \eta_i(x) \,, \tag{9}$$

where $\eta_1, ..., \eta_n$ are the finite element basis functions for the mesh with n vertices, and $p_{ij} \in \mathbb{R}$ are scalar coefficients that are the degrees of freedom of φ.

We know the gradients of η_i, they are piecewise constant functions on triangles. We can thus write

$$J_\varphi(x) = \sum_{i=1}^{n} \begin{pmatrix} p_{i1} \\ \cdots \\ p_{ik} \end{pmatrix} \left(\nabla \eta_i(x)\right)^T \,, \tag{10}$$

which is a piecewise constant function (on triangles).

If we have a gradient matrix G that maps from the space of scalar functions on the mesh to the space of constant per-face vectors, then we can compute the Jacobian $J_\varphi(x)$ by simply applying the matrix G to each component of φ. This gradient matrix G exists in many geometry processing libraries (for example libigl or gpytoolbox). It can exist in multiple versions: an *extrinsic* version (usually just named grad) that represents the faces as embedded in R^{d+1} (which leads to the Jacobian we named $\widetilde{J}_\varphi$ above), and an *intrinsic* version that has a separate basis for each tangent space on each triangle (leading to the Jacobian we named just J_φ).

"Something to think about" 1: Intrinsic and extrinsic Jacobians

When would you want to use the intrinsic Jacobian, and when would you want to use the extrinsic Jacobian? What are the advantages / disadvantages of each?

[1] Note that this technically violates our requirement that φ be a differentiable function, but since it is continuous and piecewise differentiable we will pretend that this is fine.

The inverse function theorem

We have learned about maps, diffeomorphisms, and the Jacobian. We will now use the Jacobian to find out when a map is a diffeomorphism.

One difficulty of Definition 2 is that it is quite difficult to check whether a map is bijective on its image from just its definition alone. What if we could use the Jacobian to check?

Theorem 1: Inverse function theorem

> Let Ω be a d-dimensional surface in $\mathbb{R}^{d+1}$. Let $\varphi : \Omega \to \mathbb{R}^k$ be a map.
>
> If the Jacobian $J_\varphi(x)$ has rank d, then there exists an open set $U \subseteq \Omega$, $x \in U$, such that $\varphi|_U$ is a diffeomorphism.
>
> If the Jacobian is written in terms of intrinsic bases on both the domain and image, then the Jacobian of φ^{-1} is given by the inverse of the Jacobian of φ. Let $y = \varphi(x)$. Then
>
> $$J_{\varphi^{-1}}(y) = \left(J_\varphi(x)\right)^{-1}. \tag{11}$$

Theorem 1 is remarkable: as long as the Jacobian of a function at a certain point has a high enough rank, it is invertible in a neighborhood of that point. This means, if the Jacobian has high enough rank *everywhere* (i.e., the intrinsic Jacobian is invertible everywhere), the mep φ is guaranteed to be a diffeomorphism.

We can check this property with a very simple map $\varphi : [0,1]^2 \to \mathbb{R}^2$:

$$\varphi(x,y) = \begin{pmatrix} e^y \\ (x+1)^2 \end{pmatrix}. \tag{12}$$

The Jacobian for φ is

$$J_\varphi(x,y) = \begin{pmatrix} 0 & e^y \\ 2(x+1) & 0 \end{pmatrix}. \tag{13}$$

This matrix is invertibly for all $(x,y) \in [0,1]^2$. Thus, φ must be a diffeomorphism (in fact, its inverse is $\varphi^{-1}(x,y) = \begin{pmatrix} \sqrt{y}-1 \\ \log x \end{pmatrix}$.

On the other hand, if our map is $\varphi : [0,1]^2 \to \mathbb{R}^2$:

$$\varphi(x,y) = \begin{pmatrix} x+y \\ x+y \end{pmatrix}, \tag{14}$$

then its Jacobian is

$$J_\varphi(x,y) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{15}$$

This matrix does not have rank 2. Thus φ is not a diffeomorphism.

We conclude this section by reiterating that maps that are diffeomorphisms preserve the character of a shape (such as its topology). This coincides with the Jacobian of a map having full rank at every point. You could say that the rank of the Jacobian tells you exactly how much information is being lost: If there is a rank defect of 1, then two linearly independent vectors from the source tangent space probably collapse to the same vector in the image tangent space, losing information.

Integration by substitution

Jacobians feature prominently in the computation of integrals. When computing an integral on a surface on which we have a diffeomorphism, $\varphi : \Omega \to \mathbb{R}^2$, it can sometimes be easier to follow the diffeomorphism to the image surface and evaluate the integral there. We can do this via integration by substitution.

Theorem 2: Integration by substitution

Let Ω be a d-dimensional surface in $\mathbb{R}^{d+1}$. Let $\varphi : \Omega \to \mathbb{R}^k$ be a diffeomorphism. Let $u : \varphi(\Omega) \to \mathbb{R}$ be a scalar function.

Then

$$\int_{\varphi(\Omega)} u(x)\,dx = \int_{\Omega} u(\varphi(y)) \sqrt{\left| \left(J_\varphi(y)\right)^T J_\varphi(y) \right|}\,dy\,. \tag{16}$$

The matrix $\left(J_\varphi(y)\right)^T J_\varphi(y)$ is sometimes also called the *metric tensor* G. It measures the distortion of vectors under the map φ, and allows us to compute the scalar product of two tangent vectors in the image domain by computing the matrix-vector product in the source domain. I.e., if $\boldsymbol{v}_i$ transform to $\boldsymbol{w}_i$ under φ,

$$\boldsymbol{w}_1 \cdot \boldsymbol{w}_2 = \boldsymbol{v}_1^T G \boldsymbol{v}_2\,. \tag{17}$$

The metric tensor measures the *distortion* introduced by the map φ. We will use this in future chapters to compute distortion-minimizing maps.

Integration theorems

The last two important vector calculus concepts we will repeat in this chapter are integration theorems: Green's theorem for surfaces in 2D, and the divergence theorem for volumes in 3D. These theorems allow us to reduce integrals on the interiod to integrals on the boundary.

Theorem 3: Green's theorem

Let $\boldsymbol{u} : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector-valued function.

Then

$$\int_{\partial\Omega} \boldsymbol{u} \cdot \boldsymbol{\nu}\,dx = \int_{\Omega} \nabla \times \boldsymbol{u}\,dx\,, \tag{18}$$

where $\boldsymbol{\nu}$ is the boundary normal of the boundary $\partial\Omega$.

Theorem 4: Divergence theorem

Let $u : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector-valued function.

Then

$$\int_{\partial\Omega} \boldsymbol{u} \cdot \boldsymbol{\nu}\,dx = \int_{\Omega} \nabla \cdot \boldsymbol{u}\,dx\,. \tag{19}$$

Theorem 3 and Theorem 4 are cut from the same cloth. They are special cases of a more general family of theorems called Stokes' theorems, which allow us to reduce more kinds of integrals than just curls and divergences to the boundaries of domains.

Armed with knowledge of the Jacobian, the inverse function theorem, Green's theorem, and the divergence theorem, we can now move on to characterizing surface deformations and minimizing the distortion of surface maps.