Curvature

One of the main differences between Euclidean space $\mathbb{R}^d$ and a surface $\Omega \subseteq \mathbb{R}^k$ is that Euclidean space is *flat* and surfaces can be *curved*. We saw one effect of this in our chapter on distances: The curvature of surfaces has the effect that, even locally, shortest paths between two points are not lines, but may be curves.

In this chapter we will define the concept of curvature, learn to compute it, and look at the implications curvature has on facts that we know to be true in flat Euclidean space.

Curvature of a curve

The case of curvature is much simples for one-dimensional curves than it is for two-dimensional surfaces. We will start by looking at the curvature of one-dimensional structures.

Let us start by revisiting the definition of a curve, specifically an arc-length parametrized curve.

Definition 1: Curve

A curve in 2D is a continuous function

$$\gamma : [0, l] \to \mathbb{R}^2 \,. \tag{1}$$

γ is *parametrized by arc length* if

$$\int_0^l \left\| \frac{\partial \gamma(t)}{\partial t} \right\| = 1 \,, \tag{2}$$

i.e., the velocity of the map is always 1.

An arc-length parametrization moves at constant speed along the curve. We know that it must exist for all curves, but computing it can be a bit annoying.

Derivative interpretation

We know that the derivative of the function γ is the tangent vector of the curve. Rotate it by $-\frac{\pi}{2}$, and we get the normal vector[1]. For $x = \gamma(t)$,

$$\begin{aligned} \boldsymbol{t}(x) &= \frac{\partial \gamma(t)}{\partial t} \\ \boldsymbol{n}(x) &= -R_{\frac{\pi}{2}} \boldsymbol{t}(x) \,. \end{aligned} \tag{3}$$

The tangent vector is first-order approximation of the curve. How do we compute a second-order approximation? Let us see how the tangent vector $\boldsymbol{t}$ changes over time. If we restrict ourselves to arc-length curves, then $\|\boldsymbol{t}\| = 1$, so the tangent vector can not change at all in the tangent direction:

$$\frac{\partial \boldsymbol{t}(\gamma(t))}{\partial t} \cdot \boldsymbol{t}(x) = 0 \,. \tag{4}$$

The tangent vector's change in the normal direction is what we call curvature.

[1] I made the choice here to define the normal vector as rotated by $-\frac{\pi}{2}$, and not $\frac{\pi}{2}$. This is so that a positive oriented curve in 2D is counter-clockwise, and the normal points out. You will often also find a normal that points in, or a curve that is oriented clockwise - there are lots of conventions.

Definition 2: Curvature (derivative definition)

Let $\gamma : [0, l] \to \mathbb{R}^2$ be an arc length parametrized curve. Let $\boldsymbol{t}$ be the unit tangent vector, and $\boldsymbol{n}$ the oriented unit normal vector.

Let $x = \gamma(t)$ be a point on the curve. The *curvature* κ of the curve γ is the normal component of the derivative of $\boldsymbol{t}$:

$$\kappa(x) = -\frac{\partial \boldsymbol{t}(\boldsymbol{x})}{\partial t} \cdot \boldsymbol{n}(x) . \tag{5}$$

It holds that

$$|\kappa(x)| = \left\| \frac{\partial^2 \gamma(t)}{\partial t^2} \right\| . \tag{6}$$

See the middle of Figure 1 for an illustration of the curvature defined as the infinitesimal change in the unit tangent vector. Note that this definition is *signed*: a change to the left is positive curvature, and a change to the right is negative curvature.

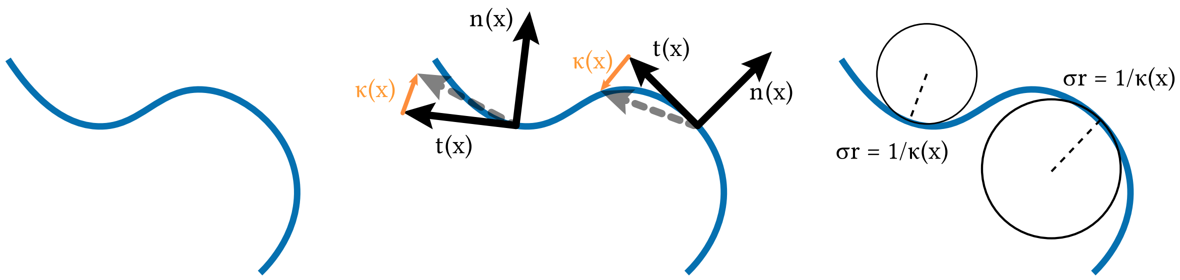

Figure 1: A curve in the plane *(left)*. There are two equivalent definitions of curvature: the rate of change of the unit tangent vector *(middle)*, and the signed radius of the osculating circle *(right)*.

The derivative interpretation of curvature makes it clear that the curvature measures the deviation of γ from being a straight line, since as a graph, all straight lines have zero second derivative.

It is important to note that Definition 2 can be reformulated, equivalently, to understand curvature as the change of the *normal* vector instead of the change of the tangent vector.

Osculating circle interpretation

A different way to interpret the curvature of a curve is the *osculating circle*. In this interpretation, we define the curvature of a circle and then apply it to an arbitrary curve.

Think about what circles have more curvature when you ride on a bike. Which circles are easier to ride along - circles with large radii, or circles with small radii? We know that it is much harder to take tight turns (small radii) than wide turns (large radii). Hence, to match our intuitive understanding of curvature, we say that the curvature of a circle is the inverse of its radius. This inverse radius definition also matches our intuition of a straight line having no curvature at all: A straight line is a circle with infinite radius. The curvature of a circle can be generalized to arbitrary curves by defining the curvature of a curve as the curvature of its tangent circle.

It remains to define a sign for this circular curvature. To match our previous definition from Definition 2, we will designate an inside and an outside of the curve (determined by the normal vector), and use this to sign the curvature of the circle.

Definition 3: Curvature (osculating circle definition)

Let $\gamma : [0, l] \to \mathbb{R}^2$ be a planar curve. Let x be a point on the curve.

The *osculating circle* at the point x is the circle that is tangent to the curve γ at point x and approximates it up to second order.[2]

The *curvature* of γ at the point x is defined as

$$\kappa(x) = \sigma(x) \frac{1}{r(x)} \,, \tag{7}$$

where $r(x)$ is the radius of the osculating circle, and $\sigma(x)$ is 1 if the circle is *inside* the curve (i.e., on the opposide side of the normal vector), and -1 if the circle is *outside* the curve (i.e., on the same side as the normal vector).

The osculating circle definition (visualized on the right of Figure 1) is easier to understand intuitively, but it is harder to actually use in practice than Definition 2. How would you actually compute the osculating circle that fits a curve up to second degree without knowing the derivatives of the curve (at which point you could just use Definition 2). But it is useful as a visualization tool, and for certain proofs.

Curvature of a surface

When it comes to surface curvature, we can generalize both definitions from one-dimensional curves - the change in the normal vector, and the radius of the osculating circle.

The Gauss map

We have encountered the normal vector in many iterations now. But we have not yet used the fact that the normal vector is, in fact, a map itself: It maps a surface to the unit sphere.

Definition 4: Gauss map

Let $\Omega \subseteq \mathbb{R}^3$ be a surface.

Let S^2 be the two-dimensional *unit sphere*: The sphere of radius one in $\mathbb{R}^3$ centered at the origin.

The map that maps a point in Ω to its normal vector is called the *Gauss map*:

$$\boldsymbol{n} : \Omega \to S^2 \,. \tag{8}$$

It is clear that the normal vector is in $\mathbb{R}^3$, since the surface Ω itself is in $\mathbb{R}^3$. But since the normal vector always has unit length, we know that it must, in fact, be in S^2 - the unit sphere.

Gauss maps tell us a lot about the nature of a surface, and they can be visualized in a manner similar to Figure 2.

[2] The tangent plane was the plane approximating the curve up to first order - the osculating circle is the second-order version of that.

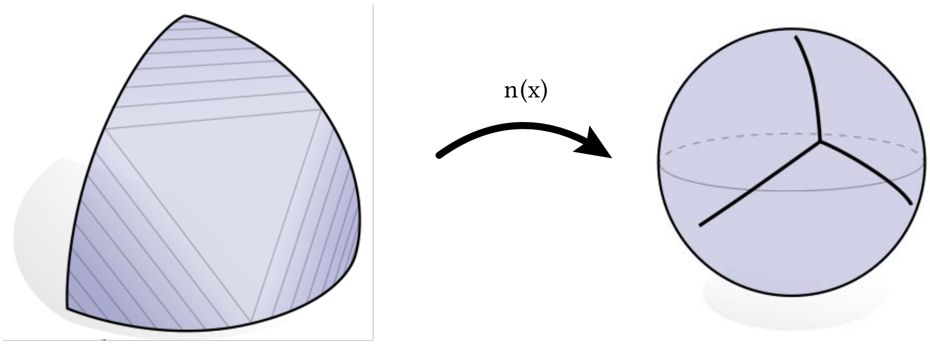

Figure 2: The Gauss map maps every point on the surface to its normal vector in S^2 [2].

Normal curvature

There is no direct way to measure the change of a tangent vector on a surface, since there is no unique tangent vector at any point. There is also no directly measure the radius of an osculating circle on a surface, since there are many different osculating circles at any point. The first kind of curvature on a surface we will learn about is an attempt to resolve this ambiguity.

One can resolve the ambiguity of tangent vectors by just picking one particular tangent vector, and defining curvature based on this tangent vector. This leads us to the definition of *normal curvature*, a directional surface curvature.

Definition 5: Normal curvature

Let $\Omega \subseteq \mathbb{R}^3$ be a surface. Let $x \in \Omega$, and let $\boldsymbol{n}$ be the normal vector of Ω at x.

Let $\boldsymbol{X}$ be a tangent vector of Ω at x (i.e., $\boldsymbol{X} \cdot \boldsymbol{n} = 0$). Let P be the plane defined by the two vectors $\boldsymbol{n}$ and $\boldsymbol{X}$. Let γ be the curve defined by the intersection of Ω with the plane P. γ is oriented so that the normal $\boldsymbol{n}$ of the surface points outside.

The *normal curvature* $\kappa_n(\boldsymbol{X})$ of Ω at x in the direction $\boldsymbol{X}$ is the curvature of the curve $\gamma_{\boldsymbol{X}}$ at the point x.

Figure 3 shows a visualization of normal curvature for a particular direction $\boldsymbol{X}$.

Normal curvature carries some of the intuition we have for the word “curvature”, similar to what we have seen for one-dimensional curves. For example, a surface that is completely flat (a plane) will have zero normal curvature, no matter which direction $\boldsymbol{X}$ we pick.

Principal curvatures

Actually computing the normal curvature $\kappa_n(\boldsymbol{X})$ for *every* possible direction $\boldsymbol{X}$ is difficult and, it turns out, also redundant. We can get all the interesting information we need purely from a surface’s *principal curvatures*.

Definition 6: Principal curvatures

Let $\Omega \subseteq \mathbb{R}^3$ be a surface. Let $x \in \Omega$.

The *principal curvatures* of Ω at x are the smallest and largest normal curvatures,

$$\begin{aligned} \kappa_1 &= \min_{\boldsymbol{X}} \kappa_n(\boldsymbol{X}) \\ \kappa_2 &= \max_{\boldsymbol{X}} \kappa_n(\boldsymbol{X}) \,. \end{aligned} \tag{9}$$

The principal curvature directions are the directions $\boldsymbol{X}$ used to minimize / maximize the normal curvature,

$$\begin{aligned} \boldsymbol{X}_1 &= \operatorname*{argmin}_{\boldsymbol{X}} \kappa_n(\boldsymbol{X}) \\ \boldsymbol{X}_2 &= \operatorname*{argmax}_{\boldsymbol{X}} \kappa_n(\boldsymbol{X}) \,. \end{aligned} \tag{10}$$

Sadly, there is no convention which one of the principal curvatures is κ_1, and which one is κ_2. Some times the smaller curvature comes first, sometimes the larger one does, sometimes they are sorted by absolute value, and so on. Figure 3 shows a surface's principal curvatures.

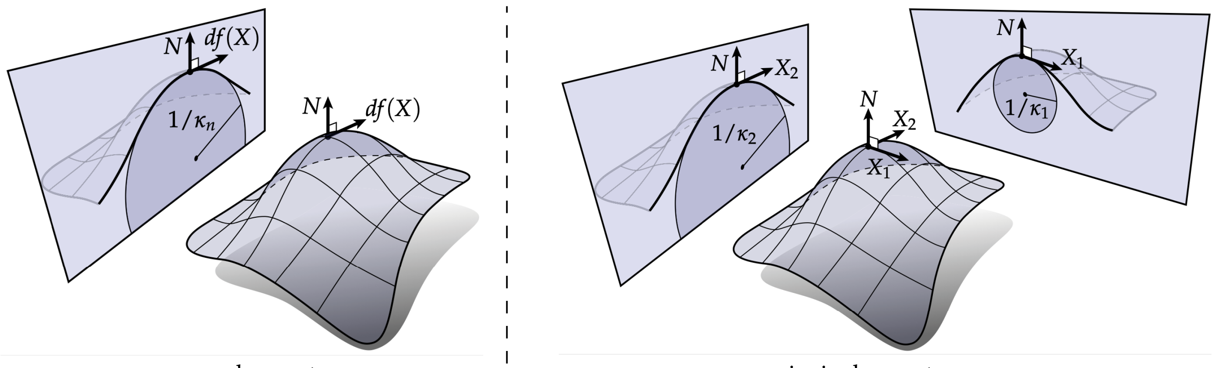

Figure 3: Normal curvature is the one-dimemsional curvature of the curve formed by the intersection of the surface and the plane formed by the normal vector and a directional vector $\boldsymbol{X}$. The principal curvatures are the smallest and largest normal curvatures formed by the directions $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ [1].

It might seem like a lot of information was lost going from the normal curvature $\kappa_n \boldsymbol{X}$ to the principal curvatures κ_1 and κ_2, since there is an infinite choice of $\boldsymbol{X}$ but only two principal curvatures. This is not actually true - we know all the normal curvatures from the principal curvatures alone using the following formula:

$$\kappa_n(\boldsymbol{X}) = \kappa_1 \cos(\varphi) + \kappa_2 \sin(\varphi) \,, \tag{11}$$

where φ is the (oriented) angle between $\boldsymbol{X}_1$ and $\boldsymbol{X}$. This seems almost too good to be true - let us investigate why this is the case.

The shape operator

When talking about the curvature of one-dimensional curves, we briefly remarked on the fact that instead of measuring the change of the tangent vector, we could also have measured the change of the normal vector. This did not seem like a big deal back then, because the tangent and normal vectors of a curve can easily be transformed into one another. This is not the case for surfaces: There is no unique tangent vector, but there *is* a unique normal vector. For surfaces, the object that measures the change of the normal is called the *shape operator*.

Definition 7: Shape operator

Let $\Omega \subseteq \mathbb{R}^3$ be a surface. Let $x \in \Omega$.

The shape operator is the Jacobian of the Gauss map,

$$S(x) = J_{\boldsymbol{n}(x)}(x)\,. \tag{12}$$

The (x) is usually omitted.

You can find the shape operator defined either as an intrinsic 2x2 matrix or as an extrinsic 3x3 matrix. For purposes of defining curvature, we will stick to the intrinsic definition for now (although it is usually easier to calculate as a 3x3 matrix).

The shape operator as an object contains all curvature information, such as the normal curvature from Definition 5. Let $\boldsymbol{X}$ be be a direction in the tangent space of a surface Ω at a point $x \in \Omega$. The change of the tangent vector of a curve in 2D is the same as the change of a normal vector We can restrict the shape operator to the plane spanned by the normal vector $\boldsymbol{n}$ and the vector $\boldsymbol{X}$ by premultiplying it with the transpose of $\boldsymbol{X}$, and we can then measure the magnitude of the change in the direction of $\boldsymbol{X}$ by multiplying the resulting row vector, again, with $\boldsymbol{X}$. The result is the normal curvature:

$$\kappa_n(\boldsymbol{X}) = \boldsymbol{X}^{\mathrm{T}} S \boldsymbol{X}\,. \tag{13}$$

The principal curvatures of Definition 6 are the smallest and largest normal curvatures. Understood through the lens of the shape operator, using Lagrange multipliers

$$\begin{aligned}
&\kappa_1 = \min_{\boldsymbol{X}} \boldsymbol{X}^{\mathrm{T}} S \boldsymbol{X}, \quad \|\boldsymbol{X}\| = 1 \\
&\min_{\boldsymbol{X},\lambda} \boldsymbol{X}^{\mathrm{T}} S \boldsymbol{X} - \lambda\Big(\|\boldsymbol{X}\|^2 - 1\Big) \\
&S\boldsymbol{X} = \lambda \boldsymbol{X}, \quad \|X\| = 1\,.
\end{aligned} \tag{14}$$

We see that (14) is an eigenvalue problem. The principal curvatures κ_1, κ_2 correspond to the smaller and larger eigenvalues of the matrix S, and the principal directions $\boldsymbol{X}_1$, $\boldsymbol{X}_2$ are the corresponding eigenvectors.[3]

In hindsight, (11) makes sense now: It is simply plugging in the two orthogonal eigendirections into the formula (13).

Mean curvature

While the principal curvatures tell us more or less everything we need to know about the curvature of a surface (like the scalar curvature did for 1D-curves), there are two special kinds of curvatures that have a geometric meaning. The first of these is the *mean curvature*.

[3] This even works with the extrinsic shape operator - you just have to make sure to ignore the eigenvector that points in the normal direction.

Definition 8: Mean curvature

Let $\Omega \subseteq \mathbb{R}^3$ be a surface.

The *mean curvature* of Ω is the pointwise quantity

$$H = \frac{1}{2}(\kappa_1 + \kappa_2)\,. \tag{15}$$

The mean curvature, the average of the principal curvatures, can also be written using the trace of the shape operator,

$$H = \frac{1}{2}\,\mathrm{tr}(S)\,. \tag{16}$$

The mean curvature measures how far away a surface is from being a *minimal surface*, i.e., a locally area-minimizing surface. Minimizing surfaces are always saddles - their largest curvatures are equal to minus their smallest curvatures, and they thus have zero mean curvature. Figure 4 shows the mean curvature on a surface.

Gauss curvature

The second special kind of curvature we will learn about is the Gauss curvature.

Definition 9: Gauss curvature

Let $\Omega \subseteq \mathbb{R}^3$ be a surface.

The *Gauss curvature* of Ω is the pointwise quantity

$$H = \kappa_1 \kappa_2\,. \tag{17}$$

The Gauss curvature, the product of the principal curvatures, can also be written using the determinant of the shape operator,

$$K = \det(S)\,. \tag{18}$$

Figure 4 shows the Gauss curvature on a surface.

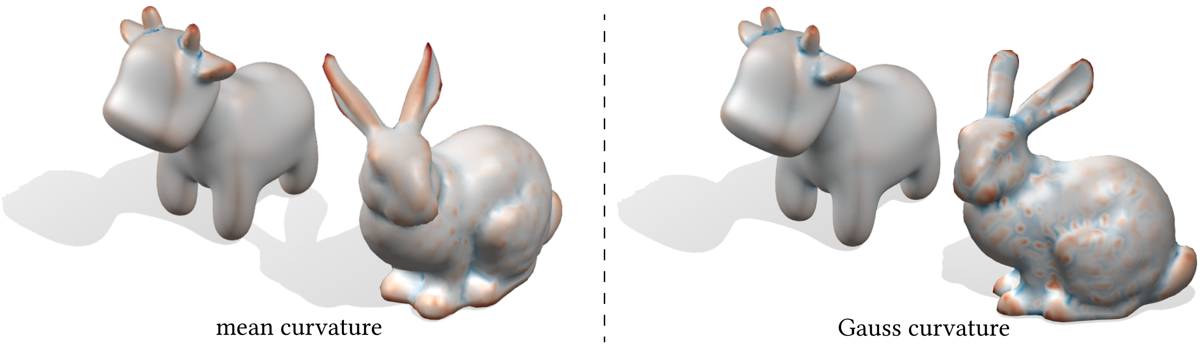

Figure 4: The mean curvature of a surface *(left)* measures the deviation of a surface from being locally a minimal surface. The Gauss curvature *(right)* measures how much more or less area there is in the neighborhood of each point compared to a flat plane. Red is > 0, blue is < 0, and grey is 0.

The Gauss curvature measures how much area there is, locally, around each point. If there is more area than there would be on a flat plane, the Gauss curvature is negative. If there is less area than there would be on a flat plane, the Gauss curvature is negative. If there is zero Gauss curvature, then there is locally just as much area as there would be on a flat plane, and the surface can be flattened without distortion (see Figure 5).

Definition 10: Developable surface

A surface Ω is called *developable* if its Gauss curvature K is 0 at every point.

Developable surfaces can be locally flattened, i.e. deformed into a flat surface, without any distortion: There exists a local isometry

$$\varphi : \Omega \to \mathbb{R}^2 \,. \tag{19}$$

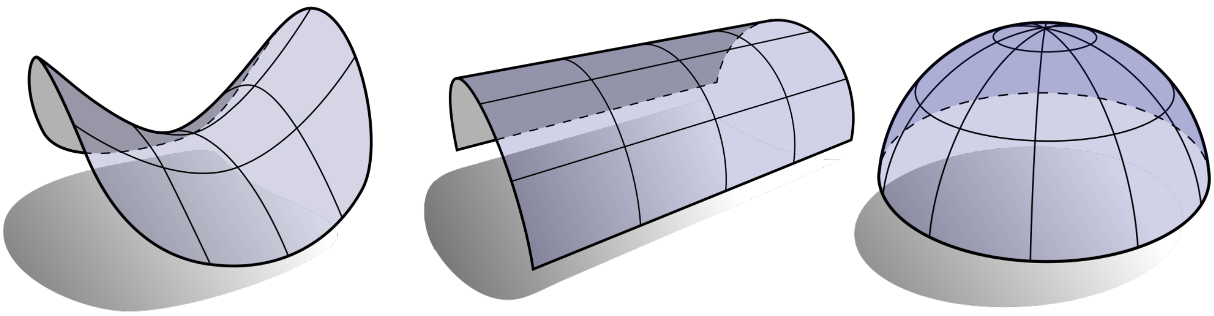

Figure 5: A surface with locally too much area compared to a flat plane has negative Gauss curvature *(left)*. A surface with locally just as much area as a flat plane has zero Gauss curvature and is called a developable surface *(center)* [1]. A surface with locally not enough area compared to a flat plane has positive Gauss curvature *(right)*.

Gauss-Bonnet

The Gaussian curvature also features in a very important theorem of differential Geometry called the Gauss-Bonnet theorem.

Theorem 1: Gauss-Bonnet theorem

Let Ω be a surface without boundary. Let χ be the Euler characteristic of the surface.[4]

Then

$$\int_\Omega K \, dx = 2\pi\chi \,. \tag{20}$$

What you should intuitively be taking away from this theorem is the fact that, if a surface has no boundaries, there should be more positive curvature on it than negative curvature, since the Euler characteristic is always greater than zero.

Discrete curvature

How do we discretize curvature? This is a difficult problem. As we have seen curvature is inherently a second derivative quantity, but our curves (polylines) and surfaces (triangle meshes) are only $C^1_{\text{a.e.}}$. We can not directly apply the many definitions from this chapter.

[4]Remember, this was the topic of the first exercise - for discrete surfaces, it is vertices minus edges plus faces. For smooth surfaces, it can be computed by triangulating the surface, or by counting the number of *holes* in the shape, where $\chi = 2 - 2\#\text{holes}$.

Polyline

A polyline is made up of piecewise straight line segments. These straight line segments are flat, and thus have zero curvature. Whatever curvature occurs must happen at the vertices only. Of course, since the curve is not differentiable at each vertex, the curvature as per Definition 2 is ∞. But what about the integrated curvature?

At each vertex we can measure the change in the tangent vector that occurs at that vertex and spread this change out over the two line segments before and after the vertex. In the same way that we can approximate the infinitesimal derivative $\lim_{h \to 0} \frac{u(x+h)-u(x)}{h}$ by $\frac{u(x+h)-u(x)}{h}$ using finite differences by fixing a small h, we can replace the derivative of the tangent vector in Definition 2 by the *difference* in tangent vectors at the vertex, and divide it by the length of the triangle segments. Or, we can just choose to discretize the integrated curvature, obsoleting the integration procedure.

Definition 11: Curvature of a polyline

> Let $v_1, \ldots, v_n$ be the vertices of a closed polyline.
>
> Let $\boldsymbol{e}_i = \frac{v_{i+1}-v_i}{\|v_{i+1}-v_i\|}$ and $\boldsymbol{e}_{i-1} = \frac{v_i - v_{i-1}}{\|v_i - v_{i-1}\|}$. The integrated discrete curvature at v_i is the *signed* angle θ_i between $\boldsymbol{e}_{i-1}$ and $\boldsymbol{e}_i$.

It turns out that if a discrete polyline approximates a smooth curve in a reasonable way, the discrete integrated curvature from Definition 11 converges to the curvature from Definition 2, integrated over a small local neighborhood.

Surface

In this chapter we will look at two ways to discretize specifically the mean and Gauss curvatures. There are approaches for discretizing the shape operator and the principal curvatures as well, and the assignment features an insight into some of these as well.

Discrete mean curvature

The mean curvature is hard to discretize from just the vertices and faces alone. Luckily, we can use a simple trick from back when we defined our discrete Laplacian: The Laplacian is minimized for minimal surfaces, and the mean curvature measures the deviation of a surface from being a minimal surface. It turns out that the Laplacian of the position function of the mesh (the coordinates of each vertex) is related to the discrete main curvature. This is because the Laplacian can be written as the sum of second derivatives in any coordinate system. So, for a coordinate system in the two principal curvature directions $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$,

$$\Delta x = \frac{\partial^2 x}{\partial \boldsymbol{X}_1^2} + \frac{\partial^2 x}{\partial \boldsymbol{X}_2^2} = \kappa_1 \boldsymbol{n}(x) + \kappa_2 \boldsymbol{n}(x) = 2H\boldsymbol{n}\,. \tag{21}$$

Definition 12: Tip angles

> Let v be a vertex in a triangle mesh. Let $\boldsymbol{n}_v$ be a vertex normal at x.[5] Let Δv be the cotangent Laplacian applied to the position function of the vertex v.
>
> Then the discrete mean curvature H is defined by the equation
>
> $$\Delta v = 2H\boldsymbol{n}_v\,. \tag{22}$$

H is a signed quantity that converges to the mean curvature as the triangle mesh is refined.

[5]This can be obtained, for example, by averaging per-face normals, as explained in an earlier chapter.

Discrete Gauss curvature

In a direct analogy to the polyline curvature definition from Definition 11, we can discretize the integrated Gauss curvature by looking at angles at each vertex.

Definition 13: Tip angles

Let v be a vertex in a triangle mesh. Let $f_1, \ldots, f_k$ be the faces of the triangle mesh containing v.

The *tip angles* $\theta_1, \ldots, \theta_k$ are the angles of the triangles $f_1, \ldots, f_k$ at the vertex v within each triangle (see Figure 6).

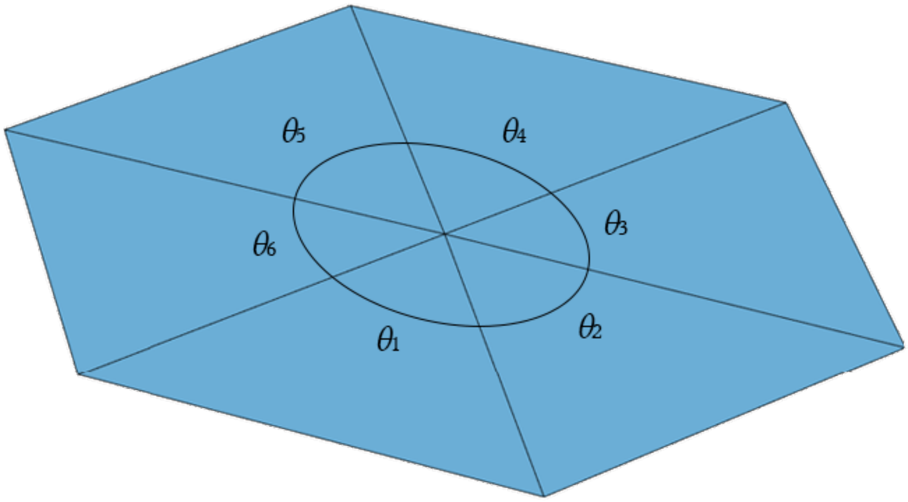

Figure 6: The tip angles at a vertex are the angles of each triangle containing the vertex at the position of that vertex. If a surface is a plane or locally flattenable at a vertex, they will sum to 2π.

We know from basic Euclidean geometry that, on the plane $\mathbb{R}^2$, the sum of all tip angles at a vertex has to equal 2π. If we have less area around each vertex than in a flat plane, then the local integrated Gauss curvature should be positive. If we have more area, it should be negative (recall Figure 5). This leads us to the definition of the discrete Gaussian curvature via angle defect.

Definition 14: Discrete Gauss curvature

Let v be a vertex in a triangle mesh. Let $\theta_1, \ldots, \theta_k$ be its tip angles.

The discrete integrated Gauss curvature at v is given by the *angle defect*

$$2\pi - \sum_i \theta_i \,. \tag{23}$$

Acknowledgements

Many of the figures and some of the structure of this document are taken from the excellent discrete differential geometry course of Crane et al. [1].

Bibliography

[1] Mathieu Desbrun, Peter Schröder Keenan Crane Fernando de Goes. 2013. Digital Geometry Processing with Discrete Exterior Calculus. In *ACM SIGGRAPH 2013 courses (SIGGRAPH '13)*, 2013.

[2] Oded Stein, Eitan Grinspun, and Keenan Crane. 2018. Developability of Triangle Meshes. *ACM Trans. Graph.* 37, 4 (2018).